Mathematical Foundations of Data Science
Assignment 3

Trimester 1, 2023

1. Consider the following distribution for a discrete random variable:

x	1	2	4	5	8
$P(X = x)$	0.05	0.25	0.3	0.1	0.3

   (a) Find the following probabilities:
      i. $P(X > 1)$.
      ii. Find $P(X > 1|X < 5)$.
   (b) Calculate the expected value, $E[X]$.
   (c) Calculate the variance, $\text{var}(X)$, using the method $E[(X-E[X])^2]$, showing working.
   (d) Confirm your answer to part (c) by calculating $E[X^2] - E[X]^2$. Since you are expecting to get the same answer, you will not get marks for just having the answer. Your working must clearly show that you have independently calculated the variance using both methods.

   Where necessary, please round your answers to 4 decimal places.

2. Suppose that repairs to an old machine are modelled with a Poisson distribution. It is estimated that the machine requires significant repairs once every 14 days. Recall that the Poisson distribution measures the probability of k events when the event frequency is λ as $\dfrac{e^{-\lambda}\lambda^k}{k!}$.
   (a) What is the probability that the machine requires more than 2 significant repairs in the next 14 days?
   (b) Explain why the probability of no significant repairs in the next 35 days is $e^{-5/2}$.
   (c) How many days are required for the probability of at least one significant repair to exceed 0.5?

3. We know that matrix multiplication is usually not commutative. That is, if X and Y are square matrices of the same size, XY is generally not the same as YX. We say that X and Y *commute* if $XY = YX$. Consider a general 2×2 matrix, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.
   (a) For which values of a, b, c, and d does the above matrix commute with $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$? Note, this question is not asking for specific numbers. Your answer should explain the general conditions on a, b, c, and d that describe all possible solutions.

(b) For which values of a, b, c, and d will a general 2×2 matrix commute with both $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$?

(c) Explain why your matrix from part (b) also commutes with $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

4. Consider the following matrices:

$D = \begin{bmatrix} 2 & p & -4 \\ 0 & 3 & q \end{bmatrix}$, $E = \begin{bmatrix} 1 & -1 \\ r & 0 \end{bmatrix}$, $F = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, where p, q, and r are unknown real-valued parameters. Calculate $-2E^2 + DFD^T$, showing all working.