Mathematical Foundations of Data Science
Assignment 5

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1. Consider the following augmented matrix:

$$\left[\begin{array}{ccc|c} x-2 & 0 & 0 & 3 \\ 0 & x^2-4 & 0 & x+2 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

Determine the number of solutions to this system for all possible values of x ∈ R, giving justification.

Answer:

$$A = \begin{bmatrix} x-2 & 0 & 0 \\ 0 & x^2-4 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ b = \begin{bmatrix} 3 \\ x+2 \\ 4 \end{bmatrix}$$

1). No solution (meet either case):

Case 1 :

when $x - 2 = 0$, so $x = 2$, beacause $0 \cdot x_1 \neq 3$

Case 2 :

when $x^2 - 4 = 0$, while $x + 2 \neq 0$, so $x = 2$, beacause $0 \cdot x_2 \neq x + 2$, where $x + 2 \neq 0$

Thus:

when $x = 2$: This equation has no solution.

2). Have solution (meet all condition):

Condition 1:

when $x - 2 \neq 0$, so $x \neq 2$

Condition 2 (meet either case):

Case 1: when $x^2 - 4 = 0$, while $x + 2 = 0$, so $x = -2$

Case 2: or when $x^2 - 4 \neq 0$, while $x + 2 \neq 0$, so $x \neq \pm 2$

Thus, in condition 2, $x \neq 2$

Thus, in all case, if $x \neq 2$, that can make this equation have solution

when $x \neq \pm 2$: Solution Set $(x_1, x_2, x_3) = [\frac{3}{x-2}, \frac{1}{x-2}, 4]^T$, unique solution

when $x = -2$: Solution Set $(x_1, x_2, x_3) = [\frac{3}{x-2}, t, 4]^T$, $\forall t \in R$, infinit solution

2. Let $K = \begin{bmatrix} 2 & a & 1 \\ 0 & 1 & 1/a \\ 1 & 1 & 0 \end{bmatrix}$ for some $a \neq 0$. Use row reduction to find K^{-1}, showing all details of your row operations.

Note: $E_{ij}(k)$ is the math symbol refers to Elememtary Row Transformation.

E_{ij}: Exchange two rows.
$E_i(k)$: Let row i times k.
$E_{ij}(k)$: Let k times row j add to row i.

Answer:

$$[K \mid I] = \left[\begin{array}{ccc|ccc} 2 & a & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{a} & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{E_{13}} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & \frac{1}{a} & 0 & 1 & 0 \\ 2 & a & 1 & 1 & 0 & 0 \end{array}\right] \xrightarrow{E_{31}(-2)}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & \frac{1}{a} & 0 & 1 & 0 \\ 0 & a-2 & 1 & 1 & 0 & -2 \end{array}\right]$$

$$\xrightarrow{E_{32}(2-a)} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & \frac{1}{a} & 0 & 1 & 0 \\ 0 & 0 & \frac{2}{a} & 1 & 2-a & -2 \end{array}\right] \xrightarrow{E_{23}(-1/2)} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{a}{2} & 1 \\ 0 & 0 & \frac{2}{a} & 1 & 2-a & -2 \end{array}\right]$$

$$\xrightarrow{E_{12}(-1)} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{a}{2} & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{a}{2} & 1 \\ 0 & 0 & \frac{2}{a} & 1 & 2-a & -2 \end{array}\right] \xrightarrow{E_3(a/2)}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{a}{2} & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{a}{2} & 1 \\ 0 & 0 & 1 & \frac{a}{2} & -\frac{a^2}{2} + a & -a \end{array}\right] = [I|K^{-1}]$$

Thus, $K^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{a}{2} & 0 \\ -\frac{1}{2} & \frac{a}{2} & 1 \\ \frac{a}{2} & -\frac{a^2}{2} + a & -a \end{bmatrix}$

3. Consider the vectors

(a) By solving $c_1v_1 + c_2v_2 + c_3v_3 = 0$, determine the unique value k for which $\{v_1, v_2, v_3\}$ is a linearly dependent set.

Answer:

We can use **Row Echelon Form(REF)** to determine the **Rank** of $[v_1, v_2, v_3] < 3$, that make v_1, v_2, v_3 are linearly dependent.

$$[v_1, v_2, v_3] = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 3 \\ 2 & 0 & k \end{bmatrix} \xrightarrow{E_{31}(-2)} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 3 \\ 0 & -2 & k+2 \end{bmatrix} \xrightarrow{E_{32}(2)} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & k+8 \end{bmatrix}$$

We don't need to use **Elementary Row Operations** to make the matrix into the **RREF**, just making **REF** like this is enough to determine the **Rank** of this matrix.

So we can draw the conclusion that when $k = -8$, the **Rank** of this matrix must < 3, which makes $\{v_1, v_2, v_3\}$ is a linearly dependent set.

(b) For the value of k above, write one of the vectors as a linear combination of the others.

Answer:

From 3(a) we know that:

$$\{v_1, v_2, v_3\} = \{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -8 \end{bmatrix} \}$$

We know that $v_3 = k_1 \cdot v_1 + k_2 \cdot v_2$, because v_i ($\forall i \in \{1,2,3\}$) can be represented by the linear combination of the other vectors v_j and v_k ($j, k \in \{1,2,3\}$, and $i \neq j \neq k$)

So we have $\begin{bmatrix} -1 \\ 3 \\ -8 \end{bmatrix} = \begin{bmatrix} k_1 \\ 0 \\ 2k_1 \end{bmatrix} + \begin{bmatrix} k_2 \\ k_2 \\ 0 \end{bmatrix} = \begin{bmatrix} k_1 + k_2 \\ k_2 \\ 2k_1 \end{bmatrix}$

$2k_1 = -8,\ k_2 = 3,\ k_1 + k_2 = -1$

So we have $k_1 = -4,\ k_2 = 3,$ that make $v_3 = -4v_1 + 3v_2$

4. Find the eigenvalues and eigenspaces of $A = \begin{bmatrix} 8 & 2 \\ -2 & 3 \end{bmatrix}$, showing all working.

Answer:

We can solve this equation $|\lambda I - A| = 0$ to find the **eigenvalues** λ_i of A

$$|\lambda I - A| = \det \begin{bmatrix} \lambda - 8 & -2 \\ 2 & \lambda - 3 \end{bmatrix} = 0$$

$$(\lambda - 8)(\lambda - 3) + 4 = \lambda^2 - 11\lambda + 28 = (\lambda - 4)(\lambda - 7) = 0$$

So **eigenvalues** of A are $\lambda_1 = 4$ and $\lambda_2 = 7$

when $\lambda = 4$, $[\lambda I - A] = \begin{bmatrix} -4 & -2 \\ 2 & 1 \end{bmatrix} \xrightarrow{E_{12}(2)} \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}$, so $2x_1 + x_2 = 0$, **eigenvector** $v_1 = (x_1, x_2) = [t, -2t]^T, t \in R$

when $\lambda = 7$, $[\lambda I - A] = \begin{bmatrix} -1 & -2 \\ 2 & 4 \end{bmatrix} \xrightarrow{E_{21}(2)} \begin{bmatrix} -1 & -2 \\ 0 & 0 \end{bmatrix}$, so $-x_1 - 2x_2 = 0$, **eigenvector** $v_2 = (x_1, x_2) = [2t, -t]^T, t \in R$

In summary, the **eigenvalues** $\lambda = 4$ correspondes to the **eigenspace** $E_4 = \{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} t | t \in R\}$,

and the **eigenvalues** $\lambda = 7$ correspondes to the **eigenspace** $E_7 = \{ \begin{bmatrix} 2 \\ -1 \end{bmatrix} t | t \in R\}$