Practice Questions (week 9)

1. Let A be an $n \times n$ matrix.

   (a) Define what it means for $\mathbf{x} \in \mathbb{R}^n$ to be an eigenvector of A with eigenvalue λ.

   (b) Define the characteristic polynomial of A.

2. Find all eigenvalues and eigenvectors for the following matrices.

   (a) $A_1 = \begin{bmatrix} 0 & 3 \\ 6 & -3 \end{bmatrix}$

   (b) $A_2 = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

   (c) $A_3 = \begin{bmatrix} 1 & 5 & 5 \\ 0 & 3 & 2 \\ 0 & 2 & 3 \end{bmatrix}$

3. Show that if A has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ then

   (a) cA has eigenvalues $c\lambda_1, c\lambda_2, \ldots, c\lambda_n$ for any constant $c \in \mathbb{R}$.

   (b) If A^{-1} exists, then A^{-1} has eigenvalues $1/\lambda_1, 1/\lambda_2, \ldots, 1/\lambda_n$

   (c) A^m has eigenvalues $\lambda_1^m, \lambda_2^m, \ldots, \lambda_n^m$ for $m = 1, 2, 3, \ldots$.

4. (a) Show that $\lambda = 0$ is an eigenvalue of A if and only if A is not invertible.

   (b) Without calculation find one eigenvalue and two linearly independent eigenvectors of

   $$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}.$$

5. Suppose that for some 3×3 matrix A, we have

   $$A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \quad \text{and} \quad A \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

   (a) Give one eigenvalue of A.

   (b) Explain why $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is an eigenvector for this eigenvalue and hence find $A \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

(c) If det(A)=12, then what is the multiplicity of the eigenvalue from (a)?

6. Let $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$.

(a) Verify that $\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of A. What is the corresponding eigenvalue λ_1?

(b) Use the trace of A to find a second eigenvalue λ_2.

(c) Find the eigenspace for λ_2.