

SAMPLE Examination in the School of Computer and
Mathematical Sciences
Trimester 1, 2023

110010 MATHS7027

**SAMPLE EXAM
Mathematical Foundations
of Data Science**

Instructions:

- Attempt all questions.
- Start your answer to each question on a new page.
- Clearly label your answers by question and part.
- You must show working in your answers to every question. Full marks will not be awarded without supporting working.
- This is a closed book exam. You are only permitted access to the materials listed below.

Permitted Materials:

- A formula sheet, provided at the end of this exam paper.
- A scientific calculator. Calculators with graphics, remote communication, or CAS capabilities are NOT permitted.
- One hard copy of an English language translation dictionary.

For students sitting an online exam only:

- A formula sheet will be provided in the exam course instead of at the end of the exam paper.
- You are permitted blank paper to write on (you will not need to bring your own blank paper if you are sitting an in-person exam).

Duration:

Writing Time: 150 minutes

Scan and Upload Time: 20 minutes

This exam has 6 question(s), for a total of 60 marks.

Question:	1	2	3	4	5	6	Total
Marks	10	10	10	10	10	10	60

**DO NOT OPEN THIS BOOKLET OR COMMENCE WRITING
UNTIL INSTRUCTED TO DO SO. STOP WRITING
IMMEDIATELY WHEN INSTRUCTED.**

Please turn over for the questions...

3 marks

1. (a) Given $A = \{1, 4, \alpha, H\}$, $B = \{4, u, 2, -2\}$, and $C = \{u\}$, determine each of the following sets:

(i) $(A \cup B) \cap C$

(ii) $B \cup C$

(iii) $A \setminus B$

2 marks

- (b) Let $f(x) = \begin{cases} 0, & x < 0 \\ -1, & x \geq 0 \end{cases}$ and let $g(x) = \begin{cases} x + 1, & x < 1 \\ x^2, & x \geq 1 \end{cases}$.

Evaluate the following:

(i) $f(g(0))$

(ii) $g(g(1/2))$

2 marks

- (c) Consider the function $h(x) = x^2$ for $x \in [-1, \infty)$. Explain why this function does not have an inverse, and give the largest subset of the domain on which the inverse may be defined.

3 marks

- (d) Show that

$$\sum_{i=2}^n (1-i)^2 = \frac{n^3}{3} - \frac{n^2}{2} + \frac{n}{6}$$

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4 marks

2. (a) A binary sequence is an ordered sequence of ones and zeros.
- (i) How many binary sequences are there of length 8?
 - (ii) How many binary sequences of length 8 contain exactly 5 ones?
 - (iii) What is the probability that a binary sequence of length 8 contains exactly 5 ones given that the first digit is a one.

3 marks

- (b) Consider the following discrete probability distribution

x	1	2	3
$P(X = x)$	0.1	0.6	a

for some unknown a .

- (i) What value of a makes X a valid discrete distribution?
- (ii) What is the expected value of X ?
- (iii) What is the variance of X ?

3 marks

- (c) Central Adelaide comprises two electoral divisions, A and B . The population of division A is 107,000, and the population of division B is 105,000. At the 2019 Federal Election, the probability that a randomly selected voter voted for The Greens party was 0.16 in division A and 0.07 in division B .
- (i) What is the probability that a randomly chosen voter in Central Adelaide is from division A ?
 - (ii) What is the probability that a randomly chosen voter in Central Adelaide voted for The Greens party? (Use the law of total probability to find the probability of selecting a Greens voter at random from central Adelaide.)
 - (iii) What is the probability that a randomly chosen voter in Central Adelaide is from division A given that they voted for The Greens party?

5 marks

3. (a) Consider the following matrices

$$X = \begin{bmatrix} 4 & -2 \\ -6 & 3 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -5 & 3 \end{bmatrix}$$

If possible, calculate each of the following. If not possible, explain why.

- (i) $X + Y$
- (ii) XY
- (iii) YX
- (iv) Y^T
- (v) $|X|$

3 marks

- (b) Suppose that Alice buys packets of chips, nuts, and dried fruit for a party. Each packet of chips costs \$3, each packet of nuts costs \$5, and each packet of dried fruit costs \$6.

In total, Alice spends \$74. She buys twice as many packets of chips as packets of nuts, and she buys one more packet of dried fruit than she buys packets of nuts.

Formulate a system of linear equations which could be used to determine the number of each type of snack that Alice buys.

You do NOT need to put the equations in matrix form or solve them, just give the equations with clear definitions for your variables.

2 marks

- (c) Consider the matrix

$$A = \begin{bmatrix} 0 & 0 & 2 \\ -1 & \alpha & \alpha \\ 1 & 5 & 0 \end{bmatrix}$$

Find the determinant of A , showing all working, and thus determine the value(s) of α for which A is invertible.

2 marks

4. (a) Suppose that a system of equations $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \mathbf{b}$ has a corresponding reduced row echelon form

$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 4 \end{array} \right]$$

Find the set of solutions to this system of equations.

6 marks

- (b) Consider the system of equations

$$x + 2y + 2z = 1$$

$$2x + y - 2z = 0$$

$$2x - 2y + z = 1$$

- (i) Write down the matrix A and vector \mathbf{b} such that the above system of equations can be written in matrix form $A\mathbf{x} = \mathbf{b}$.
- (ii) After reducing the augmented system $[A|I]$ to row-echelon form, we obtain

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 6/9 & -3/9 & 0 \\ 0 & 0 & 1 & 2/9 & -2/9 & 1/9 \end{array} \right]$$

Perform further row operations to find A^{-1} . Please show details of all row operations you use.

- (iii) Hence, find the solution \mathbf{x} to the equation in part (i).

2 marks

- (c) Suppose the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent. Prove that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly dependent for any vector \mathbf{v}_4 .

4 marks

5. (a) Consider the matrix

$$A = \begin{bmatrix} -1 & 4 \\ 4 & 5 \end{bmatrix}$$

(i) Solve the characteristic equation to show that the eigenvalues of A are $\lambda = 7$ and $\lambda = -3$.

(ii) Hence, find the eigenspaces of A .

2 marks

(b) Suppose that a 2×2 matrix has trace 1 and determinant $-4/9$. Determine its eigenvalues, showing all working.

3 marks

(c) Suppose that a matrix B satisfies $B \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $B \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

(i) Give a matrix P which diagonalises B and the corresponding diagonal matrix D .

(ii) Use this diagonalisation to prove that $B^{2k} = I$ for any positive integer k .

1 mark

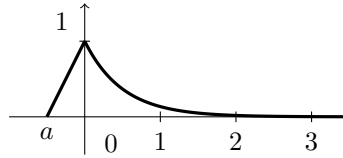
(d) Consider a principal component analysis of some dataset X . The dataset has covariance matrix

$$C = \begin{bmatrix} -1 & 3 & -1 \\ -3 & 5 & -1 \\ -3 & 3 & 3 \end{bmatrix}$$

Show that the vector $\mathbf{v} = (-1, -1, 1)$ is a principal component of the dataset X .

5 marks

6. (a) Consider the following graph of a probability density function. For x -values greater than zero, the function is defined by $f(x) = e^{-2x}$. For x -values less than zero, the function is a straight line that meets the x -axis at some point $a < 0$.



- (i) Determine the value of a .
- (ii) Consider the continuous random variable X defined by this probability density function. What is the probability that X lies between 1 and 2?
- (iii) What is the probability that X is greater than 2?

3 marks

- (b) Let $f(x) = x^{5/2}$.

- (i) Find the degree 3 Taylor polynomial for $f(x)$ about the point $a = 1$, showing all working.
- (ii) Use your Taylor polynomial to find an approximation for $2^{5/2}$.

2 marks

- (c) Calculate the first two steps (i.e., m_1, m_2) of the gradient descent algorithm with an initial point $m_0 = 0$ and a learning rate of $\eta = 0.1$ for the function $f(x) = x^5 - x^3 + x^2 - x - 1$.