Mathematical Foundations of Data Science
Assignment 2

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1. Consider the following distribution for a discrete random variable:

x	1	2	4	5	8
$P(X=x)$	0.05	0.25	0.3	0.1	0.3

(a) Find the following probabilities:
i.$P(X > 1)$.
ii.Find $P(X > 1 | X < 5)$

Answer:

(i) $P(X > 1) = 1 - P(X = 1) = 1 - 0.05 = 0.9500$

(ii) $P(X > 1 | X < 5) = \frac{P(1<X<5)}{P(X<5)} = \frac{(P(X=2) + P(X=4))}{P(X=1) + P(X=2) + P(X=4)} = 0.9167$

(b) Calculate the expected value, $E[X]$.

Answer:
$E(X) = \sum_x x \cdot P(X = x) = 1 \times 0.05 + 2 \times 0.25 + 4 \times 0.3 + 5 \times 0.1 + 8 \times 0.3 = 4.6500$

(c) Calculate the variance, var(X), using the method $E[(X - E[X])^2]$, showing working.

Answer:

$Var(X) = \sum_x (x - E(X))^2 \cdot P(X = x) = (1 - 4.65)^2 \times 0.05 + (2 - 4.65)^2 \times 0.25 + (4 - 4.65)^2 \times 0.3$
$+(5 - 4.65)^2 \times 0.1 + (8 - 4.65)^2 \times 0.3 = 5.9275$

(d) Confirm your answer to part (c) by calculating $E[X^2] - E[X]^2$. Since you are expecting to get the same answer, you will not get marks for just having the answer. Your working must clearly show that you have independently calculated the variance using both methods.

Answer:

$E(X^2) = \sum_x x^2 \cdot P(X = x) = 1^2 \times 0.05 + 2^2 \times 0.25 + 4^2 \times 0.3 + 5^2 \times 0.1 + 8^2 \times 0.3 = 27.5500$

$E(X)^2 = 4.65^2 = 21.6225$

$Var(X) = E(X^2) - E(X)^2 = 27.5500 - 21.6225 = 5.9275$

The Same result as (c)

Where necessary, please round your answers to 4 decimal places.

2. Suppose that repairs to an old machine are modelled with a Poisson distribution. It is estimated that the machine requires significant repairs once every 14 days. Recall that the Poisson distribution measures the probability of k events when the event frequency is λ as $\frac{e^{-\lambda}\lambda^k}{k!}$.

(a) What is the probability that the machine requires more than 2 significant repairs in the next 14 days?

Answer:

Because the expectation couont of repair occurs equals 1 in 14 days, so $\lambda = 1$

$P(k > 2) = 1 - \sum_{k=0}^{2} e^{1} \cdot \frac{1^k}{k!} = 1 - e^{-1} \cdot \frac{1^0}{0!} - e^{-1} \cdot \frac{1^1}{1!} - e^{-1} \cdot \frac{1^2}{2!} = 0.0803$

(b) Explain why the probability of no significant repairs in the next 35 days is $e^{-\frac{5}{2}}$.

Answer:

Because the length of observation is 2.5 times of 14 days which is 35 days, so $E(X)$ is also 2.5 times of 1, which is 2.5, so λ = 2.5 $P(k = 0) = e^{-\frac{5}{2}} \cdot \frac{\frac{5}{2}^0}{0!} = e^{-\frac{5}{2}}$

(c) How many days are required for the probability of at least one significant repair to exceed 0.5?

Answer:

When the observation length = n days, Expect repair occurence $Ex = \frac{n}{14}$ for n days, so $\lambda = \frac{n}{14}$

We should let $P(k \geq 1) > 0.5$ when the observation length = n days, then solve the inequation about n

$P(k \geq 1) = 1 - P(k = 0) = 1 - e^{-\frac{n}{14}} \cdot \frac{\frac{n}{14}^0}{0!} > 0.5$

$\iff e^{-\frac{n}{14}} < 0.5$

$\iff ln(e^{-\frac{n}{14}}) < ln(0.5)$

$\iff -\frac{n}{14} < -ln(2)$

$\iff \frac{n}{14} > ln(2)$

$\iff n > 14 \times ln(2) = 9.7041$

If you allow the number of days to be a continuous value, 9.7041 is a critical value, any days greater than this value can ensure the probability of at least one significant repair to exceed 0.5

Otherwise, if you allow the number of days to be an disceret value we need at least 10 days for the probability of at least one significant repair to exceed 0.5

3. We know that matrix multiplication is usually not commutative. That is, if X and Y are square matrices of the same size, XY is generally not the same as YX. We say that X and Y commute if $XY = YX$.

Consider a general 2 × 2 matrix, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

(a) For which values of a, b, c, and d does the above matrix commute with $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$? Note, this question is not asking for specific numbers. Your answer should explain the general conditions on a, b, c, and d that describe all possible solutions.

Answer:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\Longleftrightarrow \begin{bmatrix} a \times 0 + b \times 1 & a \times 1 + b \times 0 \\ c \times 0 + d \times 1 & c \times 1 + d \times 0 \end{bmatrix} = \begin{bmatrix} 0 \times a + 1 \times c & 0 \times b + 1 \times d \\ 1 \times a + 0 \times c & 1 \times b + 0 \times d \end{bmatrix}$$

$$\Longleftrightarrow \begin{bmatrix} b & a \\ d & c \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

$$\Longleftrightarrow b = c, a = d$$

In order to let matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ commute with $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, we should have the condition: $b = c, a = d$,

(b) For which values of a, b, c, and d will a general 2×2 matrix commute with both $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$?

Answer:

According to question (a) and question (b), Let:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & c \\ c & a \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

The question equals to prove $AB = BA,\ AC = CA$

We have already knows $AB = BA$, so we only need to let $AC = CA$ can find which value can ensure it

$$AC = \begin{bmatrix} a & c \\ c & a \end{bmatrix} \times \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix}$$

$$CA = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} a & c \\ c & a \end{bmatrix}$$

$$= \begin{bmatrix} c & a \\ 0 & 0 \end{bmatrix}$$

In order to let A can commute with both $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, we should have the condition:

$a = d,\ c = b = 0,$

(c) Explain why your matrix from part (b) also commutes with $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

Answer:

From question (b), we can have:

$$A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = B - C$$

$$Because: \ AB = BA, AC = CA$$

$$Then: \ AD = A(B-C) = AB - AC = BA - CA = (B-C)A = DA$$

QED.

4. Consider the following matrices:

$D = \begin{bmatrix} 2 & p & -4 \\ 0 & 3 & q \end{bmatrix}$, $E = \begin{bmatrix} 1 & -1 \\ r & 0 \end{bmatrix}$, $F = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, where p, q, and r are unknown real-valued parameters. Calculate $-2E^2 + DFD^T$, showing all working.

Answer:

$$-2E^2 = -2\begin{bmatrix} 1 & -1 \\ r & 0 \end{bmatrix} \times \begin{bmatrix} 1 & -1 \\ r & 0 \end{bmatrix} = -2\begin{bmatrix} 1-r & -1 \\ r & -r \end{bmatrix} = \begin{bmatrix} 2r-2 & 2 \\ -2r & 2r \end{bmatrix}$$

$$D^T = \begin{bmatrix} 2 & p & -4 \\ 0 & 3 & q \end{bmatrix}^T = \begin{bmatrix} 2 & 0 \\ p & 3 \\ -4 & q \end{bmatrix}$$

$$DFD^T = \begin{bmatrix} 2 & p & -4 \\ 0 & 3 & q \end{bmatrix} \times \begin{bmatrix} 2 & 0 & 1 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix} \times \begin{bmatrix} 2 & 0 \\ p & 3 \\ -4 & q \end{bmatrix} = \begin{bmatrix} 4 & 4p & -10 \\ 0 & 12 & 3q \end{bmatrix} \times \begin{bmatrix} 2 & 0 \\ p & 3 \\ -4 & q \end{bmatrix}$$

$$= \begin{bmatrix} 48+4p^2 & 12p-10q \\ 12p-12q & 36+3q^2 \end{bmatrix}$$

$$-2E^2 + DFD^T = \begin{bmatrix} 2r-2 & 2 \\ -2r & 2r \end{bmatrix} + \begin{bmatrix} 48+4p^2 & 12p-10q \\ 12p-12q & 36+3q^2 \end{bmatrix} = \begin{bmatrix} 46+4p^2+2r & 12p-10q+2 \\ 12p-12q-2r & 36+3q^2+2r \end{bmatrix}$$

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