Practice Questions (week 6)

1. Which of the following equations are linear?
   (a) $x_1 - 5x_2 + x_3^2 = 0$
   (b) $x_1 - 2x_3 = 7 - x_2$
   (c) $x_3 = 0$
   (d) $x_1 - x_2x_3 = 1$
   (e) $0 = 8$

2. (a) Solve the following two linear systems.

   (i) $\begin{aligned} 2x + y &= 4 \\ -x + 3y &= 5 \end{aligned}$ (ii) $\begin{aligned} x + y &= -1 \\ 4x + y &= 2 \end{aligned}$

   (b) What does it mean to say that two linear systems (of the same size) are equivalent?

   (c) Are the two systems in (a) equivalent?

   (d) Can we transform one of the two systems into the other by elementary operations?

3. Solve the following systems of linear equations.

   (a) $\begin{aligned} 2x_1 + 6x_2 &= 12 \\ 4x_1 - 3x_2 &= 19 \end{aligned}$ (b) $\begin{aligned} x_1 - 3x_2 + 2x_3 &= -10 \\ 2x_2 - x_3 &= 6 \\ x_1 + x_3 &= 0 \end{aligned}$

4. Solve the following systems of linear equations.

   (a) $\begin{aligned} 2x_1 + 6x_2 &= 3 \\ x_1 + 3x_2 &= 1 \end{aligned}$ (b) $\begin{aligned} x_1 + 2x_2 &= 4 \\ x_2 - x_3 &= 2 \end{aligned}$

5. Consider the following system of linear equations.

$$\begin{aligned} x_1 - 2x_2 &= 1 \\ 3x_1 + ax_2 &= 4 \end{aligned}$$

   (a) If the system has the unique solution $(x_1, x_2) = (5, 2)$, what is a?
   (b) If the system has no solutions, what is a?

6. Calculate the following determinants. Your answers should be formulas in terms of a and b.

   (a) $\begin{vmatrix} a & 1 & 1 \\ 1 & a & 0 \\ a & 1 & 0 \end{vmatrix}$ (b) $\begin{vmatrix} 0 & a & b \\ b & a & 0 \\ -a & -b & -a \end{vmatrix}$

7. Given that $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 5$, find:

(i) $\begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix}$ (ii) $\begin{vmatrix} d & e & f \\ 2a+d & 2b+e & 2c+f \\ g & h & i \end{vmatrix}$ (iii) $\begin{vmatrix} a-4g & d & g \\ b-4h & e & h \\ c-4i & f & i \end{vmatrix}$

8. Suppose A and B are $n \times n$ matrices with $\det A = 4$ and $\det B = -3$. Find each of the following determinants.

   (a) $\det(AB)$

   (b) $\det(A^2)$

   (c) $\det(B^{-1}A)$

   (d) $\det(2A)$

9. What can you say about $\det(A)$ if the square matrix A satisfies:

   (a) $A^2 = A$ (such a matrix is called idempotent).

   (b) $A^m = O$ for some $m > 1$ (such a matrix is called nilpotent).

10. Find the values of c for which the matrix $A = \begin{bmatrix} c & c & 0 \\ c^2 & 2 & c \\ 0 & c & c \end{bmatrix}$ is invertible.