Mathematical Foundations of Data Science
Week 2 Workshop

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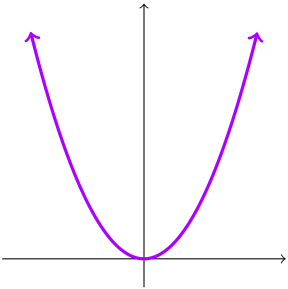
What is a Function?

$f(x) = x^2$? A graph?



$f(t) = t^2$ is the same as $f(x) = x^2$.

A function is a rule that takes a number and does something to it.

Domain of a Function

Every function has a domain, $\mathcal{D}$.

The domain is the numbers we are allowed to apply our rule to.

Suppose $f(x) = x^2$. What is $\mathcal{D}(f)$? $\mathcal{D}(f) = \mathbb{R}$.

The domain can be smaller, e.g. $f(t) = t^2,\ t \geq 0$.

Suppose $g(x) = \frac{1}{x}$. What is $\mathcal{D}(g)$? $\mathcal{D}(g) = \mathbb{R} \setminus \{0\}$.

If no domain is given, assume $\mathcal{D}$ is as large as possible.

Range of a Function

Every function also has a range, $\mathcal{R}$.

The range is the numbers we can get if we apply our function to $\mathcal{D}$.

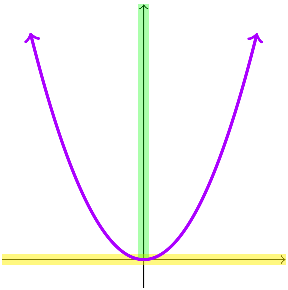
Suppose $f(x) = x^2$. What is $\mathcal{R}(f)$? $\mathcal{R}(f) = \{y \in \mathbb{R} \mid y \geq 0\}$.

Suppose $g(x) = x + 1$. What is $\mathcal{R}(g)$? $\mathcal{R}(g) = \mathbb{R}$

Relationship Between $\mathcal{D}$ and $\mathcal{R}$

A graph can be a useful way of visualising the relationship between a function's domain and range.

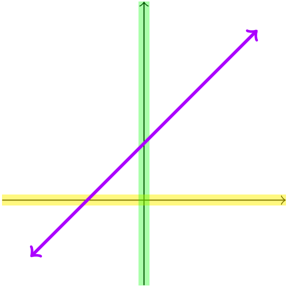
Consider $f(x) = x^2$. $\mathcal{D}(f) = \mathbb{R}$, $\mathcal{R}(f) = \{y \in \mathbb{R} \mid y \geq 0\}$.



Relationship Between $\mathcal{D}$ and $\mathcal{R}$

A graph can be a useful way of visualising the relationship between a function's domain and range.

Consider $g(x) = x + 1$. $\mathcal{D}(g) = \mathbb{R}$, $\mathcal{R}(g) = \mathbb{R}$.



Domain of a Function - Examples

Find $\mathcal{D}$ for the following functions:

(a) $f(x) = \dfrac{\sqrt{x-2}}{x+1}$
What values are not allowed on the numerator? $x < 2$
What values are not allowed on the denominator? $x = -1$
Thus, $\mathcal{D}(f) = [2, \infty)$, or $\{x \in \mathbb{R} \mid x \geq 2\}$.

(b) $g(x) = \dfrac{\sqrt{x+1}}{x-2}$
What values are not allowed on the numerator? $x < -1$
What values are not allowed on the denominator? $x = 2$
Thus, $\mathcal{D}(g) = [-1, 2) \cup (2, \infty)$

The Floor Function

$\lfloor x \rfloor$ is the *greatest integer* function or *floor* function.

$\lfloor x \rfloor$ rounds x to the largest integer less than or equal to x.

e.g., $\lfloor 2.7 \rfloor = 2, \quad \lfloor 0.45 \rfloor = 0, \quad \lfloor -1.2 \rfloor = -2, \quad \lfloor 8 \rfloor = 8.$

If $f(x) = \lfloor x \rfloor$, then:

- $\mathcal{D}(f) = \mathbb{R}$
- $\mathcal{R}(f) = \mathbb{Z}$

Compositions of Functions

Consider $g_1(x) = 2\lfloor x \rfloor$.

$\lfloor x \rfloor$ always returns an integer, then 2 doubles it.

$\mathcal{D}(g_1) = \mathbb{R}$ (same as f). What about $\mathcal{R}(g_1)$?

$\mathcal{R}(g_1) = \{2n \mid n \in \mathbb{Z}\}$ (the *even* integers).

Compositions of Functions

Consider $g_2(x) = \lfloor \frac{1}{2}x \rfloor$.

$\frac{1}{2}$ halves x, then $\lfloor * \rfloor$ returns an integer.

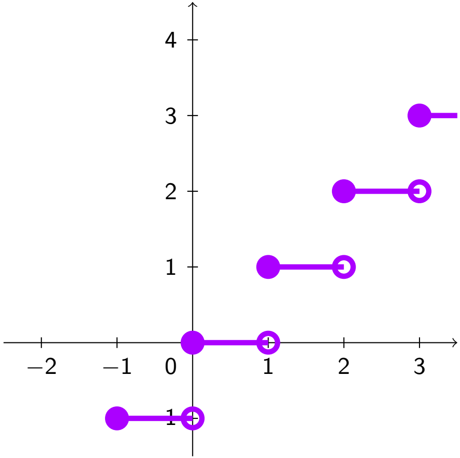
$\mathcal{D}(g_2) = \mathbb{R}$ (same as f and g_1). What about $\mathcal{R}(g_2)$?

e.g., $g_2(0) = \lfloor 0 \rfloor = 0, \quad g_2(1) = \lfloor \frac{1}{2} \rfloor = 0, \quad g_2(2) = \lfloor 1 \rfloor = 1,$
$g_2(3) = \lfloor \frac{3}{2} \rfloor = 1, \ldots$

So $\mathcal{R}(g_2) = \mathbb{Z}$.

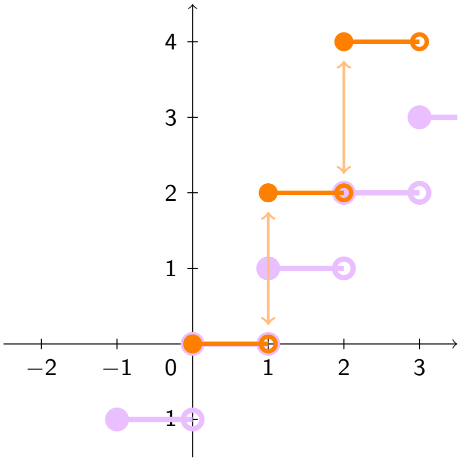
Compositions of Functions

$f(x) = \lfloor x \rfloor$



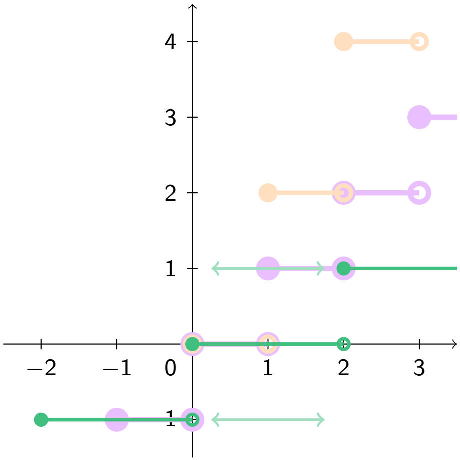
Compositions of Functions

$f(x) = \lfloor x \rfloor \qquad g_1(x) = 2\lfloor x \rfloor$



Compositions of Functions

$f(x) = \lfloor x \rfloor$ $\quad$ $g_1(x) = 2\lfloor x \rfloor$ $\quad$ $g_2(x) = \lfloor \frac{1}{2}x \rfloor$



Compositions of Functions

It can be helpful to think of a function as a machine or a process.

- What can you put in ($\mathcal{D}$)?
- What can you get out ($\mathcal{R}$)?

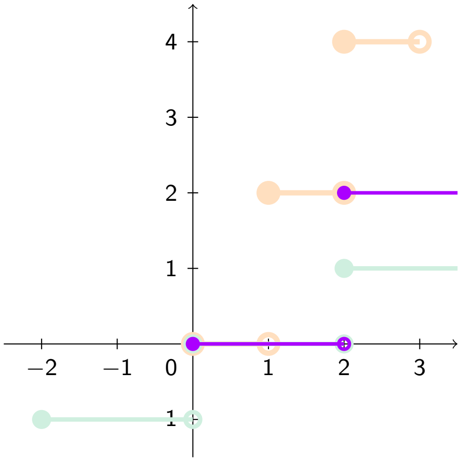
Consider $h(x) = 2\lfloor \frac{1}{2}x \rfloor$ (composing g_1 and g_2).

e.g., $h(0) = 2\lfloor 0 \rfloor = 0, \quad h(1) = 2\lfloor \frac{1}{2} \rfloor = 0, \quad h(2) = 2\lfloor 1 \rfloor = 2,$
$h(3) = 2\lfloor \frac{3}{2} \rfloor = 2, \ldots$

Compared to $f(x) = \lfloor x \rfloor$, $h(x)$ is stretched horizontally and vertically.

Compositions of Functions

$g_1(x) = 2\lfloor x \rfloor$ $\quad$ $g_2(x) = \lfloor \frac{1}{2}x \rfloor$ $\quad$ $h(x) = 2\lfloor \frac{1}{2}x \rfloor$



Inverse Functions

Can you "undo" a function?

Consider $f(x) = 2x$. f takes a value $x \in \mathbb{R}$ and doubles it.

For which values of x does $f(x) = 10$? $x = 5$.

Can we "undo" this? Can we find a function $f^{-1}(x)$?

$f^{-1}(x) = \frac{x}{2}$; $\quad f^{-1}(10) = \frac{10}{2} = 5$.

Inverse Functions

Consider $g_1(x) = 2x + 1$.

What are the steps of this function?

First, double x. Then, add 1.

Consider $g_2(x) = 2(x + 1)$.

What are the steps of this function?

First, add 1 to x. Then, double the result.

g_1 and g_2 look similar, but they are different, and will have different inverses.

Inverse Functions

To "undo" or *invert* a function, we need to retrace our steps in reverse.

Take the last thing you did, and undo that thing first.

e.g., $g_1(x) = 2x + 1$, so $g_1^{-1}(x) = \dfrac{x-1}{2}$.

Subtract 1 from x, then halve the result.

e.g. $g_2(x) = 2(x+1)$, so $g_2^{-1}(x) = \frac{x}{2} - 1$.

Halve x, then subtract 1.

Inverse Functions

We can compose functions and their inverses.

e.g., $g_1(x) = 2x + 1$, $g_1^{-1}(x) = \dfrac{x-1}{2}$.

Then $g_1^{-1}(g_1(x)) = \dfrac{(2x+1)-1}{2} = \frac{2x}{2} = x$.

e.g., $g_2(x) = 2(x+1)$, $g_2^{-1}(x) = \frac{x}{2} - 1$.

Then $g_2^{-1}(g_2(x)) = \dfrac{2(x+1)}{2} - 1 = (x+1) - 1 = x$.

We can use inverses to answer questions, e.g., "find x such that $f(x) = 10$" is really just "$f^{-1}(10)$".

Inverse Functions

Do inverse functions always exist?

Consider $f(x) = x^2$. Can we find $f^{-1}(x)$?

What values of x give $f(x) = 1$? $x = -1$, 1.

But, a function must take 1 input and return 1 output.

We cannot have $f^{-1}(1) = -1, 1$.

$f^{-1}(x)$ exists if and only if $f(x)$ is *one-to-one*.

Inverse Functions

What can we do about this problem?

Restrict the domain of your function to make it 1-1.

e.g., $f(x) = x^2$ has $\mathcal{D}(f) = \mathbb{R}$, but f is not 1-1 on this domain.

Consider $f_1(x) = x^2$, $x \geq 0$. Then $\mathcal{D}(f_1) = \{x \in \mathbb{R} \mid x \geq 0\}$.

Now f_1 is 1-1, so we can invert it.

$f_1^{-1}(x) = \sqrt{(}x)$.