2.2 Electrical Circuits

Ohm's law is a linear relation between current and voltage. It is still true for monofrequency alternating currents and alternating voltages, if capacitors and inductive coils are represented by complex impedances. The equations even remain valid for the general case of passive electrical networks, if one chooses an appropriate representation for currents and voltages in complex space. For a given frequency, they can be solved relatively easily by computer, and with the Fourier transformation, the output voltage can then be calculated for every input signal.

Physics

We consider an alternating voltage $V(t)$ and the corresponding alternating current $I(t)$ and express both as complex-valued functions:

$$\begin{aligned} V(t) &= V_0 e^{i\omega t} , \\ I(t) &= I_0 e^{i\omega t} , \end{aligned} \tag{2.9}$$

where V_0 and I_0 are complex quantities whose phase difference indicates how much the current oscillation precedes or lags behind the voltage oscillation. Strictly speaking, only the real part of (2.9) has a physical meaning, but the phase relations are especially easy to formulate in complex notation. With this, Ohm's law becomes

$$V_0 = Z I_0 \tag{2.10}$$

with a complex impedance Z. For an ohmic resistance R, for a capacitance C, and for an inductance L, Z is given by

$$Z = R \; , \; Z = \frac{1}{i\omega C} \; , \; Z = i\omega L \; , \tag{2.11}$$

where R, C, and L are real quantities, measured for example in the units ohm, farad, and henry. In an electric network, the following conservation laws, also known as Kirchhoff's laws, are valid:

1. Owing to charge conservation, the sum of the incoming currents is equal to the sum of the outgoing currents at every node.
2. Along any path, the partial voltage drops across each element add up to the total voltage over the path.

Together with Ohm's law, these two conditions yield a system of equations which determines all unknown currents and voltages.

As a simple example, we consider an L–C oscillatory circuit which is connected in series with a resistance R (Fig. 2.4). Let V_i and V_o be the complex amplitudes of the input and output voltages with the angular frequency ω, and let I_R, I_C, and I_L be the amplitudes of the currents, which, after the

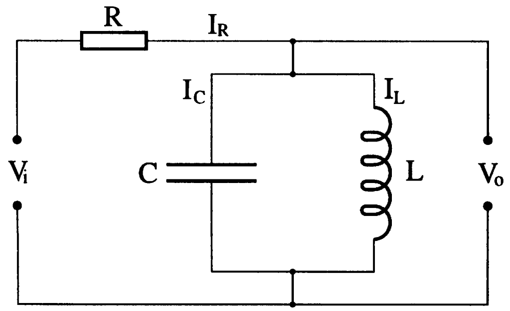

Fig. 2.4. Series connection of resistance and *L–C* oscillatory circuit

transient, have the same frequency as the input voltage. In this case, the following equations hold:

$$\begin{aligned} \text{Voltage addition:}\quad & V_\mathrm{R} + V_\mathrm{o} = V_\mathrm{i}\ , \\ \text{Current conservation:}\quad & I_\mathrm{R} = I_\mathrm{C} + I_\mathrm{L}\ , \\ \text{Ohm's law:}\quad & V_\mathrm{R} = R\, I_\mathrm{R}\ , \\ & V_\mathrm{o} = \frac{1}{\mathrm{i}\omega C}\, I_\mathrm{C}\ , \\ & V_\mathrm{o} = \mathrm{i}\omega L\, I_\mathrm{L}\ . \end{aligned} \tag{2.12}$$

For a given input voltage V_i, these five equations, which in this simple case are easily solved without a computer, determine the five unknowns V_R, V_o, I_R, I_C, and I_L. Independently of R, the magnitude of the output voltage V_o always reaches a maximum at $\omega = 1/\sqrt{LC}$; at this frequency the impedance of the oscillatory circuit is infinite.

In Fig. 2.5 we have expanded the network by adding a series circuit. If a capacitor C and an inductance L are connected in series, then the impedance at the frequency $\omega = 1/\sqrt{LC}$ is minimal. Consequently, we expect a maximal output voltage for this circuit at this frequency. This network is described by the following two equations:

$$\text{Voltage addition:}\quad I_\mathrm{R}\left(R + \frac{1}{\mathrm{i}\omega C} + \mathrm{i}\omega L\right) + V_\mathrm{o} = V_\mathrm{i}\ ,$$

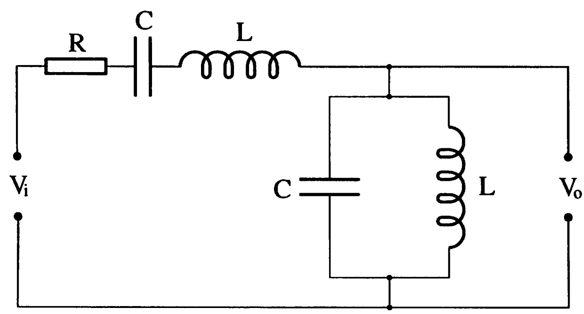

Fig. 2.5. Series connection of *R–C–L* combination and *L–C* oscillatory circuit

$$\text{Current conservation:}\ I_{\mathrm{R}} = \left(\mathrm{i}\omega C + \frac{1}{\mathrm{i}\omega L}\right) V_{\mathrm{o}}\ , \tag{2.13}$$

where Ohm's law has already been inserted. Surprisingly, the result for $V_{\mathrm{o}}(\omega)$ is entirely different from what we (not being experts on electronics) expected. Two new resonances appear above and below $\omega = 1/\sqrt{LC}$; there, for small resistances R, V_{o} becomes much larger than V_{i}. This can be made plausible by the following consideration: For low frequencies, the behavior of the parallel circuit is dominated by the inductance. If we completely neglect the capacitor in the parallel circuit for the time being, then we are dealing with a series connection consisting of the elements R–C–L–L. Because the overall inductance is now $2L$, we get a resonance at $\omega_1 = 1/\sqrt{2LC}$. If, on the other hand, for high frequencies only the capacitance in the parallel circuit is considered, then the result is an R–C-–L–C series connection with a total capacitance of $C/2$ and a corresponding resonance frequency $\omega_2 = \sqrt{2/(LC)}$. In the results section we will compare this approximation with the exact results.

Up to this point, we have only considered monofrequency sinusoidal voltages and currents. Now we want to apply an arbitrary periodic input signal $V_{\mathrm{i}}(t)$ to the circuit. Because the network is linear, from a superposition of input signals one obtains the corresponding superposition of the output voltages. In particular, one can expand any periodic input voltage $V_{\mathrm{i}}(t)$ with period T into a Fourier series:

$$V_{\mathrm{i}}(t) = \sum_{n=-\infty}^{\infty} V_{\mathrm{i}}^{(n)} \exp\left(2\pi \mathrm{i} n \frac{t}{T}\right) \ . \tag{2.14}$$

For every term with the amplitude $V_{\mathrm{i}}^{(n)}$ one can use (2.13) for the frequency $\omega_n = 2\pi n/T$ to obtain an output voltage $V_{\mathrm{o}}(\omega_n) = V_{\mathrm{o}}^{(n)}$, so that the total output signal is given by

$$V_{\mathrm{o}}(t) = \sum_{n=-\infty}^{\infty} V_{\mathrm{o}}^{(n)} \exp\left(2\pi \mathrm{i} n \frac{t}{T}\right) \ . \tag{2.15}$$

Algorithm

In *Mathematica*, (2.12) and (2.13) can be entered directly. Although one can immediately solve both systems manually, we still want to demonstrate the principle using these simple examples. With the normalization $V_{\mathrm{i}} = 1$, the system (2.12) becomes

```
eq1 = {vr + vo == 1, ir == ic + il, vr == ir r,
       vo == ic/(I omega c), vo == I omega L il}
```

One should note that the first equals sign, =, indicates an assignment, while == yields a logical expression. Consequently, a list of equations is assigned to the variable `eq1`. With

```
Solve[eq1, {vo, vr, ir, ic, il}]
```

the system of equations is solved for the specified variables. Because systems of equations generally have several solutions, `Solve` returns a list with lists of rules. Since there is only one solution in this case, though, we extract the first – and in this case the only one – of them via `Solve[...][[1]]`.

As an example of a non-sinusoidal input voltage $V_\mathrm{i}(t)$ we choose a saw-tooth voltage with the period T, which we probe at N equidistant points in time, in order to be able to use the discrete Fourier transformation. Thus we define discrete voltage values by

$$a_r = V_\mathrm{i}\,(t_r) \equiv V_\mathrm{i}\left(\frac{(r-1)\,T}{N}\right)\ , \quad r = 1,\ldots,N\ ,$$

and by using the inverse Fourier transformation we obtain the coefficients b_s with the property

$$a_r = \frac{1}{\sqrt{N}}\sum_{s=1}^{N} b_s \exp\left[2\pi\mathrm{i}\frac{(s-1)\,(r-1)}{N}\right]$$

or

$$V_\mathrm{i}\,(t_r) = \frac{1}{\sqrt{N}}\sum_{s=1}^{N} b_s \exp\left[2\pi\mathrm{i}\frac{(s-1)\,t_r}{T}\right]\ .$$

We cannot use the Fourier transformation itself here, but have to use the inverse transformation instead, so that the sign in the argument of the exponential function agrees with (2.9). Although the authors of *Mathematica* declared that they wanted to follow the physicists' convention as far as this choice of sign in the Fourier transformation is concerned, they have realized exactly the opposite.

The amplitude $b_s/\sqrt{N}$ thus belongs to the frequency $\omega_s = 2\pi(s-1)/T$. At the output, every amplitude b_s is multiplied by the output voltage $V_\mathrm{o}(\omega_s)$, which we obtained from the above equations, by using `Solve[...]`. This is only valid for $s = 1,\ldots,N/2$, however. Higher frequencies result in an inaccurate approximation for $V_\mathrm{i}(t)$, as shown in detail in Sect. 1.3. One must shift the higher frequencies to low negative ones, using $b_s = b_{s-N}$, before transforming the $V_\mathrm{o}(\omega_s)$. The transformed Fourier coefficients are

$$b_s^\mathrm{t} = b_s\ V_\mathrm{o}\,(\omega_s)\ , \quad s = 1,\ldots,\frac{N}{2}\ ,$$

$$b_s^\mathrm{t} = b_s\ V_\mathrm{o}\,(\omega_{s-N})\ , \quad s = \frac{N}{2}+1,\ldots,N\ .$$

The inverse transformation, which in this case is the Fourier transformation itself, then yields the output signal.

Results

The solution of the system of equations (2.12), which describes the series connection (shown in Fig. 2.4) of ohmic resistance and L–C oscillatory circuit, is contained in the variable **vos** in the *Mathematica* program. A readable form of the solution is

$$V_{\mathrm{o}}(\omega) = \frac{-\mathrm{i}\omega L}{-\mathrm{i}\omega L - R + RCL\omega^2} \,.$$

Obviously, the network has a resonance at $\omega_{\mathrm{r}} = 1/\sqrt{LC}$, as can also be seen in Fig. 2.6 for different values of R.

We have chosen $L = 1$ mH and $C = 1$ μF, which yields a resonance frequency ω_{r} of 31 622.8 s^{-1}. For $R = 0$, one obtains $V_{\mathrm{o}}(\omega) = 1$, i.e., no resonance, whereas for $R \to \infty$, a sharp voltage maximum appears at the position of the resonance frequency. At ω_{r} the phase of the output voltage changes from $+\pi/2$ to $-\pi/2$. If we now apply a sawtooth voltage $V_{\mathrm{i}}(t)$ to this filter, then the result will, of course, depend on the sharpness of the resonance at ω_{r}, and on the ratio of the fundamental frequency of the sawtooth voltage to the resonance frequency. Therefore, we parametrize the period T of $V_{\mathrm{i}}(t)$ in the form $2\pi/T = f\,\omega_{\mathrm{r}}$; f thus specifies the ratio of the input fundamental frequency to the resonance frequency. For $f = 1$ and a correspondingly narrow resonance curve the filter should convert the sawtooth-shaped oscillation to just a sinusoidal oscillation with the frequency $\omega = \omega_{\mathrm{r}}$.

Basically, for $f < 1/2$ and $R = 200\,\Omega$, for example, only the multiple of $2\pi/T$ closest to ω_{r} will be filtered out. By contrast, for $f > 1$ and a broad resonance, all harmonics of $2\pi/T$ are included and a distorted copy of the input signal appears. This expectation is confirmed in Figs. 2.7 and 2.8.

For the second example, the series connection of R–C–L combination and L–C oscillatory circuit (Fig. 2.5), *Mathematica*, again with $V_{\mathrm{i}} = 1$, gives the solution

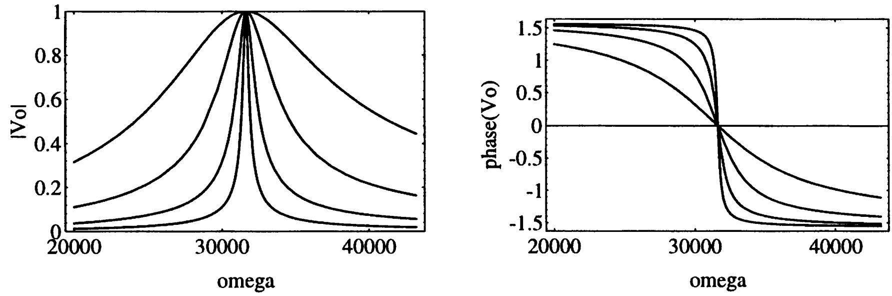

Fig. 2.6. Frequency dependence of the magnitude (*left*) and phase (*right*) of the output voltage $V_{\mathrm{o}}(\omega)$ for the network shown in Fig. 2.4. The curves correspond to the resistances $R = 100\,\Omega$, $300\,\Omega$, $900\,\Omega$, and $2700\,\Omega$. The larger the resistance, the sharper the resonance

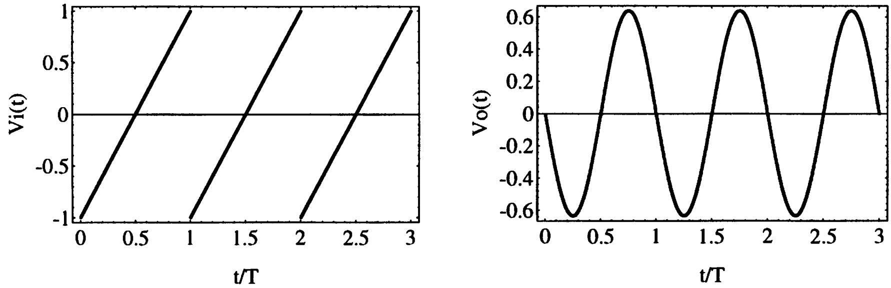

Fig. 2.7. The original sawtooth voltage $V_i(t)$ (*left*) and the voltage $V_o(t)$ at the output of the filter for $f = 1$ and $R = 2700\,\Omega$ (*right*)

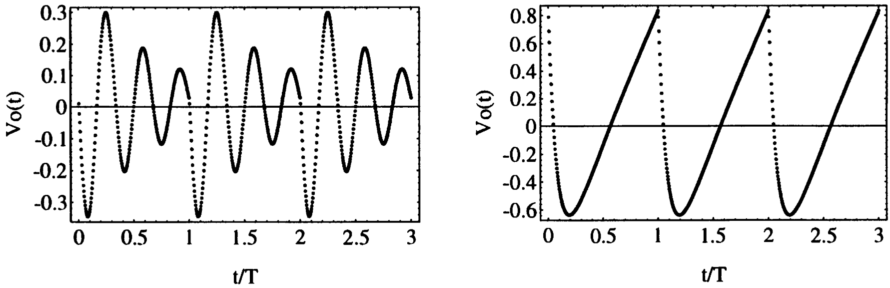

Fig. 2.8. The output voltage for $f = 1/3$ and $R = 200\,\Omega$ (*left*), and the distorted copy of the input signal that results for $f = 3$ and $R = 5\,\Omega$ (*right*)

$$V_o = \frac{CL\omega^2}{CL\omega^2 + (CL\omega^2 - 1)(1 - CL\omega^2 + iCR\omega)} .$$

For $R = 0$, i.e., if the ohmic resistance is equal to zero and the circuit is therefore loss-free, the denominator vanishes at

$$\omega = \sqrt{\frac{3 \pm \sqrt{5}}{2LC}} .$$

Then, with the values $L = 1$ mH and $C = 1\ \mu$F, V_o diverges at the frequencies $\omega = 19\,544\ \mathrm{s}^{-1}$ and $\omega = 51\,166.7\ \mathrm{s}^{-1}$. We can see that our previous estimate for the two resonances, $\omega_1 = 1/\sqrt{2LC} = 22\,360\,\mathrm{s}^{-1}$ and $\omega_2 = \sqrt{2/(LC)} = 44\,720\,\mathrm{s}^{-1}$, was not entirely incorrect.

Figure 2.9 shows $V_o(\omega)$ for $R = 10$, 30, and $90\,\Omega$. For any value of the resistance R, the output voltage at the frequency $\omega = 1/\sqrt{LC}$ is equal to the input signal, $V_o = V_i$. The two resonances are visible only if the resistances are small.

In closing, we want to calculate the power that is converted to heat by the resistor R at the frequency ω. Here, too, the complex representation of current and voltage proves to be advantageous. The calculation – done at first for a general element with a complex impedance Z, through which a

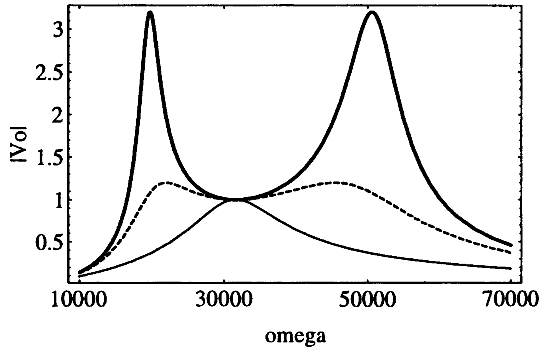

Fig. 2.9. Magnitude of the output voltage for the network of series and parallel circuit. The corresponding values of the resistances are (from top to bottom) $R = 10\,\Omega$, $30\,\Omega$, and $90\,\Omega$

current $I(t) = I_Z \exp(\mathrm{i}\omega t)$ flows and across which there is a voltage drop $V(t) = V_Z \exp(\mathrm{i}\omega t)$ – goes as follows: the power P is the time average of the product of the real parts of current and voltage, $P = \overline{\mathrm{Re}\, I(t) \mathrm{Re}\, V(t)}$. Owing to the purely harmonic time dependence, however, we have $\mathrm{Re}\, I(t) = \mathrm{Im}\, I(t + \pi/(2\omega))$, and correspondingly for $V(t)$. Thus we obtain

$$\begin{aligned} P &= \frac{1}{2}\left(\overline{\mathrm{Re}\, I(t)\, \mathrm{Re}\, V(t)} + \overline{\mathrm{Im}\, I(t)\, \mathrm{Im}\, V(t)}\right) \\ &= \frac{1}{4}\left(\overline{I(t)\, V(t)^*} + \overline{I(t)^*\, V(t)}\right) \\ &= \frac{1}{4}\left(I_Z V_Z^* + I_Z^* V_Z\right) \\ &= \frac{1}{2} |I_Z|^2\, \mathrm{Re}\, Z\,, \end{aligned} \tag{2.16}$$

where we have set $V_Z = Z I_Z$ at the end. For the ohmic resistance R, the power is accordingly $P(\omega) = |I_R(\omega)|^2 R/2$. We compare this to the power P_0, which results if all coils and capacitors in our network are short-circuited, i.e., if the input voltage V_i is connected directly to the resistor R, so that one gets $I_R = V_i/R$, and therefore $P_0 = |V_i|^2/(2R)$. The power ratio $P(\omega)/P_0$ is shown in Fig. 2.10 for $R = 10\,\Omega$. At both resonance frequencies the coils and capacitors appear to be completely conductive, while at $\omega = 1/\sqrt{LC}$ the parallel circuit offers an infinite resistance.

Exercises

In the second R–C–L network (Fig. 2.5), add a resistor R to the parallel L–C circuit, parallel to the capacitor and the coil.

1. Calculate and draw $V_o(\omega)$.
2. What form does a periodic rectangular voltage have after passing through this filter?
3. What is the power dissipation in the two resistances as a function of ω?

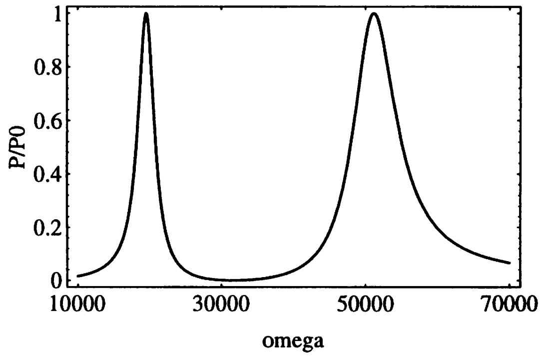

Fig. 2.10. The power $P(\omega)$ dissipated in the ohmic resistance $R = 10\,\Omega$, relative to the power P_0 which is dissipated if the input voltage is connected directly to R

Literature

Crandall R.E. (1991) Mathematica for the Sciences. Addison-Wesley, Redwood City, CA

2.3 Chain Vibrations

It is a well known fact that the motion of a particle in a quadratic potential is described by an especially simple linear differential equation. If several particles interact with each other through such linear forces, their motion can be calculated by linear equations as well. However, one then has several such equations of motion which are coupled to each other. This system of linear equations can be solved by diagonalizing a matrix. A good example of this is the linear chain with different masses. It is a simple model for the calculation of lattice vibrations in a crystal. The eigenvalues of a matrix specify the energy bands of the phonons, while the eigenvectors provide information about the vibration modes of the crystal elements. Every possible motion of the model solid can be represented by superposition of such eigenmodes.

Physics

We consider a chain consisting of pointlike masses m_1 and m_2, which we designate as *light* and *heavy* atoms respectively, for sake of simplicity. The particles are to be arranged in such a way that one heavy atom follows three light ones. The unit cell of length a thus contains four atoms. Only nearest neighbors shall interact with one another. We limit our considerations to small displacements, i.e., the forces are to be linear functions of the shifts of the masses, as indicated in the spring model shown in Fig 2.11.

To describe the longitudinal oscillations, we number the unit cells sequentially and consider the cell with the number n. Within this cell, let r_n, s_n, and t_n be the displacements of the light atoms from their rest positions, and