Duality in Condensed Matter Physics—From Lattice Models to Particle-Vortex Duality and Beyond

Xiaodong Hu*
Department of Physics, Boston College, MA 02135, USA
(Dated: November 17, 2019)

This is a research note documenting duality in condensed matter physics.

Contents

I. WARM-UP: CLASSICAL AND QUANTUM ISING DUALITY

A. $d = 0$ Quantum Ising Model

$d = 0$ Quantum Ising Model[1] is special because there is no interactive term in Hamiltonian

$$H_q = -g \sum_i \sigma_i^x. \tag{I.1}$$

To compute the partition function of (I.1), we need to slice the temperature (imaginary time) to N segments and let N tends to infinity. Namely,

$$\begin{aligned}
\mathcal{Z}_q = \operatorname{tr} e^{-\beta H_q} &= \lim_{N\to\infty} \sum_{s_1,\cdots,s_N} \langle s_N | e^{+\Delta\tau g \sigma_1^x} | s_1 \rangle \langle s_1 | e^{+\Delta\tau g \sigma_2^x} | s_2 \rangle \cdots \langle s_{N-1} | e^{+\Delta\tau g \sigma_N^x} | s_N \rangle \\
&= \lim_{N\to\infty} \sum_{s_1,\cdots,s_N} \prod_{i=0}^{N} \langle s_i | 1 + \Delta\tau g \sigma_j^x | s_j \rangle,
\end{aligned} \tag{I.2}$$

where $s_0 \equiv s_N$ (PBC is applied naturally). Equation (I.2) shares the same structure as *transferring matrix* methods in 1-d *classical* Ising model. In fact, ansatzing

$$\langle s_i | 1 + \Delta\tau g \sigma_j^x | s_j \rangle = \langle s_i | A e^{B \sigma_i^z \sigma_j^z} | s_j \rangle$$

and let $s_i = s_j = 1$ and $s_i = -s_j = 1$, one can immediately show that

$$A^2 = \Delta\tau g, \quad e^{-2B} = \Delta\tau g. \tag{I.3}$$

Therefore, partition function of $0d$-quantum Ising Model can be re-written as

$$\mathcal{Z}_q = A^N \operatorname{tr} \exp(-\beta H_c), \tag{I.4}$$

---

*Electronic address: `xiaodong.hu@bc.edu`
[1] This is pure *statistical model* so has *no* time dimensionality.

where

$$H_c = B\sum_{\langle ij\rangle}\sigma_i^z\sigma_j^z \tag{I.5}$$

is the Hamiltonian of $1d$ classical Ising model.

B. $d > 1$ Quantum Ising Model

Hamiltonian of $d > 1$ Quantum Ising Model is

$$H_q = -J\sum_{\langle ij\rangle}\sigma_i^z\sigma_j^z - g\sum_i\sigma_i^x. \tag{I.6}$$

Still clues of the duality theory can be found from its partition function

$$\langle s_i|e^{\Delta\tau(J\sum_{\langle ij\rangle}\sigma_i^z\sigma_j^z+g\sum_i\sigma_i^x)+\frac{Jg}{2!}\mathcal{O}(\Delta\tau^2)}|s_j\rangle = \sum_{s_k}\underbrace{\langle s_i|e^{\Delta\tau J\sum_{\langle ij\rangle}\sigma_i^z\sigma_j^z}|s_k\rangle}_{\text{d-dim Classical Ising Model}}\overbrace{\langle s_k|e^{\Delta\tau\sum_i\sigma_i^x}|s_j\rangle}^{\text{0-dim Quantum Ising Model}}, \tag{I.7}$$

where B-C-H formula is utilized

$$e^Ae^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}[A,[A,B]]+\cdots}.$$

So we say [1] **Partition function of d-dim quantum statistical model is equivalent to (d+1)-dim classical statistical model**.

II. $\mathbb{Z}_2$ GAUGE THEORY

$\mathbb{Z}_2$ gauge field describe the fluctuation of gauge freedoms, or visually fluctuation of loops on *infinitely large* square lattice. Spins live on the links of sites,

$$H_{\mathbb{Z}_2} = -K\sum_{\square}\prod_{\ell\in\square}\sigma_\ell^z - g\sum_\ell\sigma_\ell^x. \tag{II.1}$$

A. Kramers-Wannier-Wegner Duality

Consider an *infinite* square lattice, on the links of which spins are placed such that the Hamiltonian

[1] T. H. Hsieh, Student review,(2) pp. 1–4 (2016).