Hydrodynamic Methods in Condensed Matter

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In this note I will review modern methods of hydrodynamics in condensed matter systems (unlike old one like superfluids and spin waves with orderes phase), from quantum critical Nernst effects to fractional quantum Hall system

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Contents

I. HYDRODYNAMICS IN SYMMETRY-BREAKING PHASE — SPIN WAVES

A. $S = 1/2$ XXZ Model and Conservation Law

We will consider $S = 1/2$ XXZ model in a uniform magnetic field H_z in this section

$$\mathcal{H} = \mathcal{H}_{XXZ} - H_z \sum_i S_i^z \equiv -\sum_{\langle ij\rangle} J_\perp (S_i^x S_j^x + S_i^y S_j^y) - \sum_{\langle ij\rangle} J_z S_i^z S_j^z - H_z \sum_i S_i^z. \tag{1}$$

Whatever the phase is (*planar ferromagnetic* $J_\perp > |J_z| > 0$, *planar antiferromagnetic* $J_\perp < -|J_z| < 0$, or *planar paramagnetic* $T > T_c$ when $M_\perp e^{i\varphi} \equiv \langle S_i^x\rangle + i\langle S_i^y\rangle$ vanishes [1, 2]), Hamiltonian (1) is clearly invariant under the global spin rotation along $\hat{S}^z$ axis. So we always have a conservation of S^z operator, or coarse-grained z-component magnetization $M_z \equiv \langle S_i^z\rangle$ under ensemble average, namely

$$\partial_t m_z + \nabla \cdot \boldsymbol{j}^{m_z} = 0, \tag{2}$$

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where we introduce the intensive magnetization density m_z and corresponding spin current $\boldsymbol{j}^{m_z}$. Apparently total energy density $E \equiv \langle \mathcal{H}_{XXZ} \rangle$, or energy density ε is also conserved, giving

$$\partial_t \varepsilon + \nabla \cdot \boldsymbol{j}^{\varepsilon} = 0. \tag{3}$$

But given M_z and E, one still cannot determine the nonuniform states out of equilibrium like ferromagnetic phase below T_c. In fact, one must also specify the direction of alignment of perpendicular magnetization φ, which is locally defined as

$$m_x(\boldsymbol{r}) + i m_y(\boldsymbol{r}) = m_\perp(\boldsymbol{r}) e^{i\varphi(\boldsymbol{r})}.$$

In hydrodynamic regime, relevant hydro-variables are those whose long-wavelength variations vary slowly in time compared with the characteristic microscopic relaxation time of the scattering processes. In symmetry breaking phase, such variables are of two classes:

- densities of conserved variables (here is $m_z(\boldsymbol{r})$ and $\varepsilon(\boldsymbol{r})$);
- symmetry-breaking elastic variables (here is $\varphi(\boldsymbol{r})$).

Thus we *assume* that **each point of the system reaches almost thermodynamic equilibrium at each instant of time, so that the system is *completely* determined by the hydro-variables, even though they vary in time**. Particularly, the above undefined current $\boldsymbol{j}^{e}$ and $\boldsymbol{j}^{m_z}$ should be functional of these hydro-variables. Ditto for the symmetry-breaking field if we introduce a scalar valued functional ψ of hydro-variables

$$\partial_t \varphi(\boldsymbol{r}) + \psi(\boldsymbol{r}) = 0. \tag{4}$$

Noting that it is the fluctuation of $\nabla\varphi$ rather than φ that contributes to physical properties such as entropy like ε and m_z. So they are in the same order of fluctuation and we will take $\boldsymbol{v}(\boldsymbol{r}) \equiv \nabla\varphi(\boldsymbol{r})$ as a replacement of fundamental hydro-variables ($\boldsymbol{v}$ is in anolagous of superfluid velocity $\boldsymbol{v}_s$ in liquid helium), satisfying

$$\partial_t \boldsymbol{v}(\boldsymbol{r}) + \nabla\psi(\boldsymbol{r}) = 0. \tag{5}$$

Equation (2), (3), and (5) consist of the complete conservation law of our system.

B. Constitutive Relations

Equations relating coarse-grained current with hydro-variables are called *constitutive relations*. Introducing the conjugate field $\boldsymbol{x}$ of $\boldsymbol{v}$, and h of m_z, the first law of thermodynamics (in true thermodynamic eqiuilibrium) tells us

$$T\,\mathrm{d}s \equiv \mathrm{d}\varepsilon - h\,\mathrm{d}m_z - \boldsymbol{x} \cdot \mathrm{d}\boldsymbol{v}, \tag{6}$$

where $h \equiv -T\partial s/\partial m_z|_S$ and $\boldsymbol{x} \equiv -T\partial s/\partial \boldsymbol{v}|_S$.

1. Zeroth Order (Non-dissipative)

In this subsection, we will **use (assume) the conservation of local entropy to constrain the *non-dissipative* part of "current operators" defined above**.

To the zeroth order of fluctuations, i.e., spatial derivatives of hydro-variables, the most general form of current operator we can write is

$$\boldsymbol{j}^{\varepsilon} \equiv A(\boldsymbol{r})\boldsymbol{v}(\boldsymbol{r}), \quad \boldsymbol{j}^{m_z} \equiv B(\boldsymbol{r})\boldsymbol{v}(\boldsymbol{r}), \quad \psi \equiv C(\boldsymbol{r}), \tag{7}$$

where coefficients A, B and C are scalar functional of $\varepsilon(\boldsymbol{r})$ and $m_z(\boldsymbol{r})$ waiting to be determined. Fortunately, as we will see below, $C(\boldsymbol{r})$ can be obtained in advance without entropy argument [1, 2].

Given ε, m_z and $\boldsymbol{v} = 0$, we know from (5) that the equilibrium state *without* external magnetic field allows a constant precession rate $\mathrm{d}\varphi/\,\mathrm{d}t = -\psi_0(\varepsilon, m_z)$. Let us prepare such stationary state and turn on H_z. The time evolution of $M_\perp$ is now dominated by the Hamiltonian $\mathcal{H} \equiv \mathcal{H}_{XXZ} + \mathcal{H}_{ext}$. Since $\mathcal{H}_{ext} \equiv -H_z \sum_i S_i^z$ commutes with $\mathcal{H}_{XXZ}$, time evolution of $m_\perp(t) \equiv \langle S_i^+(t) \rangle$ gives

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = -(\psi_0(\varepsilon, m) + H_z).$$

However, in *true* thermodynamic equilibrium (with non-vanishing H_z), where the system energy $E - M_z H_z$ is minimized so that

$$\frac{\partial E}{\partial M_z} = \frac{\partial \varepsilon}{\partial m_z} = H_z, \tag{8}$$

the procession rate $\mathrm{d}\varphi/\,\mathrm{d}t$ must be zero, otherwise the rotating $m_\perp$ will radiate and lose energy [1]. Thus ψ_0 coincides with H_z in *true* equilibrium. On the other hand, the fisrt law of thermodynamics (6) tells us the left hand side of (8) is nothing but conjugate field $h(\varepsilon, m_z) \equiv \frac{\partial \varepsilon}{\partial m_z}$. Therefore, we conclude for general *non-dissipative* stationary state

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = -(h - H_z), \tag{9}$$

or

$$C(\boldsymbol{r}) = H_z - h(\varepsilon(\boldsymbol{r}), m_z(\boldsymbol{r})). \tag{10}$$

The left work is to determine $A(\boldsymbol{r})$ and $B(\boldsymbol{r})$ in (7) with the help of entropy conservation (at zeroth order).

Hydrodynamic assumption ensures local entropy density to be the functional of hydro-variables $s = s(\varepsilon, m_z, \boldsymbol{v})$. And at each slice of time, each point of the system is assumed to reach local thermodynamic eqiuilibrium. Therefore we can expand entropy density around its equilibrium value s_0 (which must be the maximum of s due to the second law of thermodynamics) that

$$s \simeq s_0(\varepsilon) - \frac{1}{2T}\chi_s^{-1} m_z^2 - \frac{\rho_s}{2T} v^2, \tag{11}$$

where temperature $T^{-1} \equiv \partial s_0/\partial \varepsilon$ is well-defined in equilibrium and inserted by convention. Magnetic suseptibility χ_s and the magnetic version of “superfluid density” ρ_s may also be functional of hydro-variables, but up to the second order of disturbance, we can safely treat them as constants[1]. Therefore the two conjugate fields we introduce above takes the form of $h \equiv -T\partial s/\partial m_z = \chi_s^{-1} m_z$ and $\boldsymbol{x} \equiv -T\partial s/\partial \boldsymbol{v} = \rho_s \boldsymbol{v}$.

Taking time derivative of the first law of thermodynamics and substituting conservation laws (3), (4), (5) and the zeroth order of non-dissipative currents (7) and (10), we have

$$\begin{aligned} T\frac{\partial s}{\partial t} &= \frac{\partial \varepsilon}{\partial t} - h\frac{\partial m_z}{\partial t} - \boldsymbol{x}\cdot\frac{\partial \boldsymbol{v}}{\partial t} = -\nabla\cdot\boldsymbol{j}^\varepsilon + h\nabla\cdot\boldsymbol{j}^{m_z} - \boldsymbol{x}\cdot\nabla(h - H_z) \\ &= -\nabla\cdot(A\boldsymbol{v}) - h\nabla\cdot(B\boldsymbol{v}) - \boldsymbol{x}\cdot\nabla h \\ &= -\nabla\cdot\Big((A + Bh)\boldsymbol{v}\Big) + (B\boldsymbol{v} - \boldsymbol{x})\cdot\nabla h. \end{aligned} \tag{12}$$

Writing $\boldsymbol{Q} \equiv (A + Bh)\boldsymbol{v}$ and use the identity

$$\nabla\cdot\boldsymbol{Q} \equiv T\nabla\cdot\left(\frac{\boldsymbol{Q}}{T}\right) + \boldsymbol{Q}\cdot\left(\frac{\nabla T}{T}\right),$$

Equation (12) becomes

$$T\frac{\mathrm{d}s}{\mathrm{d}t} \equiv T\left(\frac{\partial s}{\partial t} + \nabla\cdot\left(\frac{\boldsymbol{Q}}{T}\right)\right) = -\boldsymbol{Q}\cdot\left(\frac{\nabla T}{T}\right) - (B\boldsymbol{v} - \boldsymbol{x})\cdot\nabla h \tag{13}$$

if we identify $\boldsymbol{Q}$ as the *entropy current density.* In the absence of dissipation, as is assumed to the zeroth order disturbance, entropy is conserved. Thus

$$\begin{cases} \boldsymbol{Q} \equiv (A + Bh)\boldsymbol{v} = 0, \\ B\boldsymbol{v} - \boldsymbol{x} = 0 \end{cases} \implies \begin{cases} A = -\frac{\rho_s}{\chi_s} m_z, \\ B = -\rho_s \end{cases}, \tag{14}$$

where the expression of h and $\boldsymbol{v}$ is used.

Finally we obtain the *non-dissipative* linearized equation of motion up to the zeroth order of gradients of hydro-variables

$$\partial_t \varepsilon = \tag{15}$$

[1] Apparently χ_s and ρ_s must be nonnegative for stability of the system.

2. First Order (Dissipative)

To the first order, howevere, we will use **the positivity of local entropy production** to constrain further the dissipative part of

C. Hydrodynamic Modes

D. Linear Response

II. TRANSPORT NEAR QUANTUM CRITICALITY

In this section, we focus on *Lorentz-invariant* quantum critical points in the *hydrodynamics region* that external electromagnetic frequencies satisfy $\hbar\omega \ll k_B T$. This condition is widely (actually, almost all) performed in experiments but mismatch the assumption to many theoretical calculation as well as numerical simulation in, for example, DC conductivity near superfluid-insulator phase transition [3].

The fundamental ingredients of hydrodynamic analysis are the conserved quantities and their equations of motion. But before preceeding, let me pause and comment on the old hydrodyamics of Lifshitz superfluids and spin-waves [1, 4]. In hydrodynamics region, $k_B T/\hbar$ naturally plays the role of low-energy frequency characterizing the relation time that we are interested in, while in old prescription the relaxation time just diverges (so the energy scale may not be correct) [5].

A. General Setup

Our system subject to external magnetic field for transport study. Invariance of *gauge transformation* $A_\mu \mapsto A_\mu + \partial_\mu f$ and *diffeomorphism transformation* $x^\mu \mapsto x_\mu + \xi_\mu$ of action gives[2]

$$\partial_\mu J^\mu = 0, \tag{16}$$
$$\partial_\mu T^{\mu\nu} = F^{\mu\nu} J_\mu. \tag{17}$$

respectively.

B. Hydrodynamic Modes and Linear Response

C. Perspetive from Holography

III. NON-RELATIVISTIC TRANSPORT — FRACTIONAL QUANTUM HALL SYSTEM

A. Spacetime Background

B. Response

[1] B. Halperin and P. Hohenberg, Physical Review **188**, 898 (1969).
[2] P. M. Chaikin and T. C. Lubensky, *Principles of condensed matter physics*, vol. 1 (Cambridge university press Cambridge, 2000).
[3] K. Damle and S. Sachdev, Physical Review B **56**, 8714 (1997).
[4] P. C. Hohenberg and B. I. Halperin, Reviews of Modern Physics **49**, 435 (1977).

[2] See, for example, section II D of [6]. The result can also be proved holographically, see [7].

[5] S. A. Hartnoll, P. K. Kovtun, M. Müller, and S. Sachdev, Physical Review B **76**, 144502 (2007).
[6] C. P. Herzog, Journal of Physics A: Mathematical and Theoretical **42**, 343001 (2009).
[7] J. Lindgren, I. Papadimitriou, A. Taliotis, and J. Vanhoof, Journal of High Energy Physics **2015**, 94 (2015).