Nonlinear Field Theory of Large-Spin 1d Heisenberg Antiferromagnets

NLσ Model, Semiclassical Soliton Quantization, and Mass Gap Conjecture

F. D. M. Haldane[1] Xiaodong Hu[2]

[1]Princeton University

[2]Boston College

Phys. Rev. Lett. **50**, 1153 (1983)

Contents

1. Introduction
   - Previous Work
   - Question We Address
2. Low-energy Effective Field Theory of AFM
   - Angle-variable Representation
   - Semi-classical Approximation
   - Non-linear Sigma Model
3. Haldane's Conjecture
   - Topological Soliton Solution
4. Summary
   - Haldane's Conjecture
   - Acknowledgement

Contents

1. Introduction
   - Previous Work
   - Question We Address
2. Low-energy Effective Field Theory of AFM
   - Angle-variable Representation
   - Semi-classical Approximation
   - Non-linear Sigma Model
3. Haldane's Conjecture
   - Topological Soliton Solution
4. Summary
   - Haldane's Conjecture
   - Acknowledgement

Néel Order and Spin Waves in Classical AFM

Classically, isotropic **antiferromagnetic** Heisenberg model

$$H = J \sum_n \boldsymbol{S}_n \cdot \boldsymbol{S}_{n+1}$$

in $d = 1$ has

- A doubly degenerate ground state with anti-parallel alignment (**long-range Néel order**).
- A low-energy **gapless excitation** of spin wave (Halperin&Hohenberg, *Phys. Rev.*, **188**, 898 (1969))

$$\omega_k = \omega \sin ka, \quad \omega \equiv 2JS$$

Holstein-Primakoff Transformation

Representing the $\mathfrak{su}(2)$ operator with bosonic creation and annihilation operators ($n_b \equiv b^\dagger b$)

$$\begin{cases} S^z = S - n_b \\ S^+ = \left(\sqrt{2S - n_b}\right) b \\ S^- = b^\dagger \left(\sqrt{2S - n_b}\right) \end{cases}$$

where $[b, b^\dagger] \equiv 1$, and expanding in the order of $1/S$.

Néel Order and Spin Waves in Classical AFM

Classically, isotropic **antiferromagnetic** Heisenberg model

$$H = J \sum_n \boldsymbol{S}_n \cdot \boldsymbol{S}_{n+1}$$

in $d = 1$ has

- A doubly degenerate ground state with anti-parallel alignment (**long-range Néel order**).
- A low-energy **gapless excitation** of spin wave (Halperin&Hohenberg, *Phys. Rev.*, **188**, 898 (1969))

$$\omega_k = \omega \sin ka, \quad \omega \equiv 2JS$$

Holstein-Primakoff Transformation

Representing the $\mathfrak{su}(2)$ operator with bosonic creation and annihilation operators ($n_b \equiv b^\dagger b$)

$$\begin{cases} S^z = S - n_b \\ S^+ = \left(\sqrt{2S - n_b}\right) b \\ S^- = b^\dagger \left(\sqrt{2S - n_b}\right) \end{cases}$$

where $[b, b^\dagger] \equiv 1$, and expanding in the order of $1/S$.

Will These Two Properties Hold for Full Quantum Heisenberg Model?

Mermin-Wagner Theorem

- The is NO spontaneous symmetry breaking for Heisenberg models in dimension $d = 1$.
- Quantum fluctuation will destroy the long-range order.

So both of them cannot survive.

Will These Two Properties Hold for Full Quantum Heisenberg Model?

Mermin-Wagner Theorem

- The is NO spontaneous symmetry breaking for Heisenberg models in dimension $d = 1$.
- Quantum fluctuation will destroy the long-range order.

So both of them cannot survive.

Will These Two Properties Hold for Full Quantum Heisenberg Model?

Mermin-Wagner Theorem

- The is NO spontaneous symmetry breaking for Heisenberg models in dimension $d = 1$.
- Quantum fluctuation will destroy the long-range order.

So both of them cannot survive.

- **The ground state cannot be SYB ordered**.
- **The spin-wave description of elementary excitation is not enough**.

Will These Two Properties Hold for Full Quantum Heisenberg Model?

Mermin-Wagner Theorem

- The is NO spontaneous symmetry breaking for Heisenberg models in dimension $d = 1$.
- Quantum fluctuation will destroy the long-range order.

So both of them cannot survive.

- **The ground state cannot be SYB ordered**.
- **The spin-wave description of elementary excitation is not enough**.

Know Result

Spin-1/2 antiferromagnetic Heisenberg model is exactly solvable!

- The ground state is found by Bethe ansatz (Bethe, *Zeitschrift für Physik*, **71**.3-4 (1931)).
- The **gapless** excitation is obtained by studying of $d = 1$ delta-repulsion bosonic systems (Yang&Yang, J. Math. Phys., **10**.7 (1969)).

The Question We Adress

- What is the ground state for $S = 1$ or other larger spin-magnitude antiferromagnetic Heisenberg model?
- What is the correct low-energy effective theory of antiferromagnetic Heisenberg model?
- Will the **gapless excitation** still be true for $S = 1$ or other larger spin magnitudes?

Contents

1. Introduction
    - Previous Work
    - Question We Address
2. Low-energy Effective Field Theory of AFM
    - Angle-variable Representation
    - Semi-classical Approximation
    - Non-linear Sigma Model
3. Haldane's Conjecture
    - Topological Soliton Solution
4. Summary
    - Haldane's Conjecture
    - Acknowledgement

Classical Equations of Motion

Parameterizing each spin of the lattice on the Bloch sphere

$$\boldsymbol{S}_n = (-1)^n(\sin\theta_n \cos\varphi_n, \sin\theta_n \sin\varphi_n, \cos\theta_n),$$

then the classical equation of motion reads ($\omega \equiv 2JS$)

Classical EOM

$$\dot{\theta}_n = -\frac{\omega}{2}(-1)^n \sum_{\lambda=\pm} \sin\theta_{n\pm\lambda} \sin(\varphi_{n\pm\lambda} - \varphi_n),$$

$$\dot{\varphi}_n = -\frac{\omega}{2}(-1)^n \sum_{\lambda=\pm} \Big(\cos\theta_{n\pm\lambda} - \cot\theta_n \sin\theta_{n\pm\lambda}\cos(\varphi_{n\pm\lambda} - \varphi_n)\Big).$$

if we assign the canonical conjugate pair with the Poisson-backect

$$\{\varphi_n, S^z_{n'}\} \equiv \delta_{nn'}.$$

Low-energy Gradient Expansion

- We focus on the physics in long-wavelength (low-energy) regime $(a \ll 1)$, so all field operators should vary **slowly** with lattice.
- To keep all nonlinear effects, the fluctuation of **both** branches of low-energy spin wave mods $(k = 0$ and $k = \pi)$ should be taken into account.

So we write

$$\theta_n = \theta(x) + a(-1)^n \alpha(x), \quad \varphi_n \equiv \varphi(x) + a(-1)^n \beta(x).$$

for $x = na$ and expand the EOM with spatial gradient up to quadratic order.

Low-energy Gradient Expansion

- We focus on the physics in long-wavelength (low-energy) regime ($a \ll 1$), so all field operators should vary **slowly** with lattice.
- To keep all nonlinear effects, the fluctuation of **both** branches of low-energy spin wave mods ($k = 0$ and $k = \pi$) should be taken into account.

So we write

$$\theta_n = \theta(x) + a(-1)^n \alpha(x), \quad \varphi_n \equiv \varphi(x) + a(-1)^n \beta(x).$$

for $x = na$ and expand the EOM with spatial gradient up to quadratic order.

Low-energy Gradient Expansion

- We focus on the physics in long-wavelength (low-energy) regime $(a \ll 1)$, so all field operators should vary **slowly** with lattice.
- To keep all nonlinear effects, the fluctuation of **both** branches of low-energy spin wave mods ($k = 0$ and $k = \pi$) should be taken into account.

So we write

$$\theta_n = \theta(x) + a(-1)^n \alpha(x), \quad \varphi_n \equiv \varphi(x) + a(-1)^n \beta(x).$$

for $x = na$ and expand the EOM with spatial gradient up to quadratic order.

Low-energy Classical EOM

$$\dot{\theta}(x) = \omega a \beta \sin\theta, \quad \dot{\alpha}(x) \sin\theta = -\omega a \left[\nabla \left(\sin^2\theta \cdot \nabla\varphi \right) + \sin 2\theta \cdot \alpha\beta \right],$$

$$\dot{\varphi}(x) = -\frac{\omega a}{\sin\theta}\alpha, \quad \dot{\beta}(x) \sin\theta = \frac{\omega a}{2}(\nabla\theta)^2 - \frac{\omega a}{4}\sin 2\theta \left[(\nabla\varphi)^2 - \left(\frac{2\alpha}{\sin\theta} \right)^2 + 4\beta^2 \right].$$

Recognizing Conjugate Variables

Low-energy Classical EOM

$$\dot{\theta}(x) = \omega a \beta \sin\theta, \quad \dot{\alpha}(x)\sin\theta = -\omega a\left[\nabla\left(\sin^2\theta \cdot \nabla\varphi\right) + \sin 2\theta \cdot \alpha\beta\right],$$

$$\dot{\varphi}(x) = -\frac{\omega a}{\sin\theta}\alpha, \quad \dot{\beta}(x)\sin\theta = \frac{\omega a}{2}(\nabla\theta)^2 - \frac{\omega a}{4}\sin 2\theta\left[(\nabla\varphi)^2 - \left(\frac{2\alpha}{\sin\theta}\right)^2 + 4\beta^2\right].$$

The low-energy EOM enlighten us to define the conjugate variables ($g \equiv 2/S$)

$$L(x) \equiv -2g^{-1}\alpha\sin\theta, \quad \Pi_\theta(x) \equiv 2g^{-1}\beta\sin\theta,$$

and assign the Poisson bracket for them

$$\{\theta(x), \Pi_\theta(x')\} = \{\varphi(x), L(x')\} = \delta(x - x').$$

Recognizing Conjugate Variables

Low-energy Classical EOM

$$\dot{\theta}(x) = \omega a \beta \sin\theta, \quad \dot{\alpha}(x)\sin\theta = -\omega a \left[\nabla\left(\sin^2\theta \cdot \nabla\varphi\right) + \sin 2\theta \cdot \alpha\beta\right],$$

$$\dot{\varphi}(x) = -\frac{\omega a}{\sin\theta}\alpha, \quad \dot{\beta}(x)\sin\theta = \frac{\omega a}{2}(\nabla\theta)^2 - \frac{\omega a}{4}\sin 2\theta\left[(\nabla\varphi)^2 - \left(\frac{2\alpha}{\sin\theta}\right)^2 + 4\beta^2\right].$$

The low-energy EOM enlighten us to define the conjugate variables $(g \equiv 2/S)$

$$L(x) \equiv -2g^{-1}\alpha\sin\theta, \quad \Pi_\theta(x) \equiv 2g^{-1}\beta\sin\theta,$$

and assign the Poisson bracket for them[1]

$$\{\theta(x), \Pi_\theta(x')\} = \{\varphi(x), L(x')\} = \delta(x - x').$$

[1] **Given a Hamiltonian, the EOM cannot be uniquely defined unless all conjugate variables are recognized!**

Effective Lagrangian

Then the above four nonlinear EOM can be derived from

Effective Hamiltonian

$$H = \frac{\omega a}{2} \int \mathrm{d}x \left\{ g \left(\Pi_\theta^2 + \frac{L^2}{\sin^2\theta} \right) + \frac{1}{g} \left[(\nabla\theta)^2 + (\nabla\varphi)^2 \sin^2\theta \right] \right\}.$$

Effective Lagrangian

Then the above four nonlinear EOM can be derived from

Effective Hamiltonian

$$H = \frac{\omega a}{2} \int \mathrm{d}x \left\{ g \left(\Pi_\theta^2 + \frac{L^2}{\sin^2\theta} \right) + \frac{1}{g} \left[(\nabla\theta)^2 + (\nabla\varphi)^2 \sin^2\theta \right] \right\}.$$

And the corresponding Lagrangian density has a extreme neat form in terms of the **unit** vector $\boldsymbol{n} \equiv (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$

Effecitve Lagrangian

$$\mathcal{L} = \frac{1}{2g} \left[\frac{1}{c} (\partial_t \boldsymbol{n})^2 - c(\nabla \boldsymbol{n})^2 \right].$$

where $c \equiv \omega a$. This is nothing but $O(3)$ **Nonlinear σ-Model**.

Contents

1 Introduction
  Previous Work
  Question We Address

2 Low-energy Effective Field Theory of AFM
  Angle-variable Representation
  Semi-classical Approximation
  Non-linear Sigma Model

3 Haldane's Conjecture
  Topological Soliton Solution

4 Summary
  Haldane's Conjecture
  Acknowledgement

The Fate of Vacua

Previous work:

- In the aspect of CMP, **the nonlinear EOM of NLσM support the existence of topological soliton solutions**
  (Skyrme *Proc. Roy. Soc. London A* **260** (1961); Belavin&Polyakov, *JEPT Lett.* **22**, 10 (1961)).
- In the aspect of HEP, **different topological sectors of vacuum will tunnel to each other and result in the "decay of vacua"**.
  (Berlavin&Polyakov et al. *Phys. Lett. B*, **59**, 1 (1975))

The Fate of Vacua

Previous work:

- In the aspect of CMP, **the nonlinear EOM of NLσM support the existence of topological soliton solutions**
  (Skyrme *Proc. Roy. Soc. London A* **260** (1961); Belavin&Polyakov, *JEPT Lett.* **22**, 10 (1961)).
- In the aspect of HEP, **different topological sectors of vacuum will tunnel to each other and result in the "decay of vacua"**.
  (Berlavin&Polyakov et al. *Phys. Lett. B*, **59**, 1 (1975))

Indication on our case:

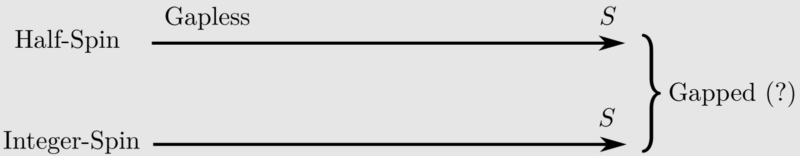
The Fate of False Ground State

- The **trivial** ordered ground state (spin wave) with gapless excitation will be unstable under the nonlinear effects — proliferation of solitons. In such case the **large-S limit** is suggested to be **gapped**.
- We CANNOT rule out the possibility of other **nontrivial** stable phase with **gapless** excitations.

Contradiction?

The Fate of False Ground State

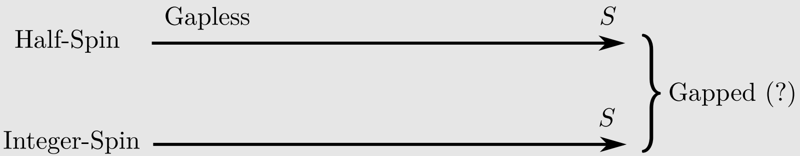
- The **trivial** ordered ground state (spin wave) with gapless excitation will be unstable under the nonlinear effects — proliferation of solitons. In such case the **large-S limit** is suggested to be **gapped**.
- We CANNOT rule out the possibility of other **nontrivial** stable phase with **gapless** excitations.



Contradiction?

The Fate of False Ground State

- The **trivial** ordered ground state (spin wave) with gapless excitation will be unstable under the nonlinear effects — proliferation of solitons. In such case the **large-S limit** is suggested to be **gapped**.
- We CANNOT rule out the possibility of other **nontrivial** stable phase with **gapless** excitations.



RG analysis is necessary!

What's The Connection of Elementary Excitation with the Spin Magnitudes?

RG of NLσM

The β function for coupling constant g is obtained by $(2+\varepsilon)$-dimension expansion (Brézin&Zinn-Justin, *PRL.*, **36**, 691 (1976); Polyakov, *Phys. Lett. B*, **59**.1 (1975))

$$\beta(g) \equiv \frac{\mathrm{d}g}{\mathrm{d}\ln a} = \frac{g^2}{2\pi} + \frac{g^3}{(2\pi)^2} + \cdots \implies \text{Asymptotic Freedom}$$

where a is the ultraviolate cutoff.

Since the physical correlation length remains invariant under RG procedure $\beta(\xi) = 0$, we can connect the correlation length on both sides. Particularly, in the **small-S (strong-coupling) limit**

$$\text{gap}^{-1} \simeq \xi = ae^{\pi S}.$$

Contents

1 Introduction
Previous Work
Question We Address

2 Low-energy Effective Field Theory of AFM
Angle-variable Representation
Semi-classical Approximation
Non-linear Sigma Model

3 Haldane's Conjecture
Topological Soliton Solution

4 Summary
Haldane's Conjecture
Acknowledgement

Haldane's Conjecture

Let me summarizes our result:

- We obtain the low-energy effective field theory of $d = 1$ antiferromagnetic Heisenberg model.
- We study the possible influence brought by the topological soliton solution.
- We try to connect the behavior of elementary excitation in the large-S regime and small-S regime, getting the result

	Small-S		Large-S
Half-Integer	Gapless	$\Longleftrightarrow$	
Integer		$\Longleftrightarrow$	Gapped (possible)

Haldane's Conjecture

- Half-integer spin antiferromagnetic Heisenberg models in $d = 1$ is **gappless**.
- Integer spin antiferromagnetic Heisenberg models in $d = 1$ is **gapped**.

Haldane's Conjecture

Let me summarizes our result:

- We obtain the low-energy effective field theory of $d = 1$ antiferromagnetic Heisenberg model.
- We study the possible influence brought by the topological soliton solution.
- We try to connect the behavior of elementary excitation in the large-S regime and small-S regime, getting the result

	Small-S		Large-S
Half-Integer	Gapless	$\Longleftrightarrow$	
Integer		$\Longleftrightarrow$	Gapped (possible)

Haldane's Conjecture

- Half-integer spin antiferromagnetic Heisenberg models in $d = 1$ is **gappless**.
- Integer spin antiferromagnetic Heisenberg models in $d = 1$ is **gapped**.

Thanks!