Algebraic Theory of Anyons

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This note plays a role as a supplementary material for the hidden but deep demanded mathematical backgrounds of the brilliant paper of Kitaev [1]. Personal understanding and comments are inclued as well.

仰之弥高，钻之弥坚，瞻之在前，忽焉在后。
——「论语 · 子罕第九」

Contents

I. ALGEBRAIC THEORY OF ANYONS

A. Category Theory: General Preliminaries

See my personal note of category theory. The missing Yoneda Lemma may also be extremely important for understanding the logic of emergentism in condensed matter physics.

B. Monoidal Categories

Definition 1. (Monoidal Category) A monoidal category is a quintuple $(\mathcal{C}, \otimes, \alpha, \mathbb{1}, \iota)$ with
1) a category $\mathcal{C}$,
2) a *bifunctor* of *product category* $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called *tensor product*,
3) a *natural isomorphism* α from the functor $(\bullet \otimes \bullet) \otimes \bullet : \mathcal{C} \times \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ to the functor $\bullet \otimes (\bullet \otimes \bullet) : \mathcal{C} \times \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

$$a_{XYZ} : (X \otimes Y) \otimes Z \overset{\sim}{\mapsto} X \otimes (Y \otimes Z)$$

called *associaticity constraint.*
4) an object $\mathbb{1} \in \mathrm{Obj}(\mathcal{C})$,
5) a *natural isomorphism* ι from the functor $\mathbb{1} \otimes \bullet : \mathcal{C} \to \mathcal{C}$ to the identical functor $id_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$

$$\iota_X : \mathbb{1} \otimes X \overset{\sim}{\mapsto} X$$

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called *unitality constraint*[1] such that the *pentagon coherence*

$$\begin{tikzcd}
& (X \otimes Y) \otimes (Z \otimes W) \arrow[dr, "\alpha_{X,Y,Z\otimes W}"] & \\
((X \otimes Y) \otimes Z) \otimes W \arrow[ur, "\alpha_{X\otimes Y,Z,W}"] \arrow[d, "\alpha_{X,Y,Z}\otimes id_W"'] & & X \otimes (Y \otimes (Z \otimes W)) \\
(X \otimes (Y \otimes Z)) \otimes W \arrow[rr, "\alpha_{X,Y\otimes Z,W}"'] & & X \otimes ((Y \otimes Z) \otimes W) \arrow[u, "id_X \otimes \alpha_{Y,Z,W}"']
\end{tikzcd}$$

and the *triangle coherence*

$$\begin{tikzcd}
& X \otimes Y & \\
(X \otimes \mathbb{1}) \otimes Y \arrow[ur, "\iota_X \otimes id_Y"] \arrow[rr, "\alpha_{X,id_{\mathbb{1}},Y}"'] & & X \otimes (\mathbb{1} \otimes Y) \arrow[ul, "id_X \otimes \iota_Y"']
\end{tikzcd}$$

commute.

C. Graphic Calculus

Drawn by hands.

D. Tensor Categories

E. 6j-Symbols

[1] A. Kitaev, Annals of Physics **321**, 2 (2006).
[2] V. G. Turaev and A. Virelizier, *Monoidal categories and topological field theory*, vol. 322 (Springer, 2017).

[1] We do not distinguish left and right unitality constraints as [2] does.