Thermal Hall Transport of Bosonic System with Pairing Hamiltonian — from Magnon to Topological Superconductor

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In this note the recent new approach of Kapustin [1] is applied to calculate the transport thermal Hall coefficient of magnon with pairing Hamiltonian. The result is consistent with the previous result of [2].

一点浩然气，千里快哉风。

—— 苏轼「水调歌头」

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I. THERMAL HALL EFFECT OF MAGNON

A. On-site Quadratic Hamiltonian

A general quadratic bosonic Hamiltonian with N degree of freedom (both sublattice and orbital degree of freedom) per unit cell takes the form of

$$\begin{aligned}\mathcal{H} &= \frac{1}{2}\sum_{\{m,n\}\in\text{lattice}} \boldsymbol{b}_m^\dagger \boldsymbol{A}_{mn}\boldsymbol{b}_n + \boldsymbol{b}_m^\dagger \boldsymbol{B}_{mn}\boldsymbol{b}_n^\dagger + \boldsymbol{b}_m \boldsymbol{C}_{mn}\boldsymbol{b}_n + \boldsymbol{b}_m \boldsymbol{D}_{mn}\boldsymbol{b}_n^\dagger \\ &= \frac{1}{2}\sum_{\{m,n\}\in\text{lattice}} \begin{pmatrix} \boldsymbol{b}_m^\dagger & \boldsymbol{b}_n \end{pmatrix}\begin{pmatrix} \boldsymbol{A}_{mn} & \boldsymbol{B}_{mn} \\ \boldsymbol{C}_{mn} & \boldsymbol{D}_{mn}\end{pmatrix}\begin{pmatrix} \boldsymbol{b}_n \\ \boldsymbol{b}_n^\dagger\end{pmatrix} \end{aligned} \tag{1}$$

where $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$ **are complex coefficient** N **by** N **matrix and** $\boldsymbol{b}_m$ **is a** N**-tuple vector on site** m (we explicitly keep the site label to avoid the ambiguity). Hermicity of Hamiltonian $\mathcal{H}^\dagger \equiv \mathcal{H}$ requires

$$\boldsymbol{A}_{mn}^\dagger \equiv \boldsymbol{A}_{nm}, \quad \boldsymbol{D}_{mn}^\dagger \equiv \boldsymbol{D}_{nm}, \quad \boldsymbol{B}_{mn}^\dagger \equiv \boldsymbol{C}_{nm}.$$

C.C.R. of bosonic field operators $[(b_m)_i, (b_n^\dagger)_j] \equiv \delta_{ij}\delta_{mn}$ and $[(b_m)_i, (b_n)_j] \equiv 0$ relates the term $\boldsymbol{b}_m^\dagger \boldsymbol{A}_{mn}\boldsymbol{b}_n$ with $\boldsymbol{b}_m \boldsymbol{D}_{mn}\boldsymbol{b}_n^\dagger$, giving $\boldsymbol{A}_{mn} \equiv \boldsymbol{D}_{mn}^*$, where i, j are sublattice or orbital labels (do not mix them with the site label!). Ditto for the second and third terms in Hamiltonian so that $\mathbf{B}_{mn} = \mathbf{B}_{nm}^T$. So Hamiltonian (1) becomes

$$\mathcal{H} = \frac{1}{2}\sum_{\{m,n\}\in\text{lattice}} \begin{pmatrix} \boldsymbol{b}_m^\dagger & \boldsymbol{b}_n \end{pmatrix}\begin{pmatrix} \boldsymbol{A}_{mn} & \boldsymbol{B}_{mn} \\ \boldsymbol{B}_{nm}^\dagger & \boldsymbol{A}_{mn}^*\end{pmatrix}\begin{pmatrix} \boldsymbol{b}_n \\ \boldsymbol{b}_n^\dagger\end{pmatrix} =: \frac{1}{2}\sum_{mn}\psi_m^\dagger H_{mn}\psi_n, \tag{2}$$

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where $2N$-tuple vector $\boldsymbol{\psi}_n$ is introduced.

The on-site Hamiltonian $\mathcal{H}_{\boldsymbol{r}}$ s.t. $\mathcal{H} \equiv \sum_{\boldsymbol{r}} \mathcal{H}_{\boldsymbol{r}}$ should be Hermitian as well, so we can make a symmetric choice[1] of the form that

$$\mathcal{H}_{\boldsymbol{r}} = \frac{1}{4} \sum_{\boldsymbol{r}'} \psi_{\boldsymbol{r}}^\dagger H_{\boldsymbol{r},\boldsymbol{r}'} \psi_{\boldsymbol{r}'} + \psi_{\boldsymbol{r}'}^\dagger H_{\boldsymbol{r}',\boldsymbol{r}} \psi_{\boldsymbol{r}}. \tag{3}$$

because

$$H_{\boldsymbol{r}\boldsymbol{r}'}^\dagger \equiv \begin{pmatrix} \boldsymbol{A}_{\boldsymbol{r}\boldsymbol{r}'}^\dagger & \boldsymbol{B}_{\boldsymbol{r}'\boldsymbol{r}} \\ \boldsymbol{B}_{\boldsymbol{r}\boldsymbol{r}'}^\dagger & (\boldsymbol{A}_{\boldsymbol{r}\boldsymbol{r}'}^\dagger)^* \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}_{\boldsymbol{r}'\boldsymbol{r}} & \boldsymbol{B}_{\boldsymbol{r}'\boldsymbol{r}} \\ \boldsymbol{B}_{\boldsymbol{r}\boldsymbol{r}'}^\dagger & \boldsymbol{A}_{\boldsymbol{r}'\boldsymbol{r}}^* \end{pmatrix} \equiv H_{\boldsymbol{r}'\boldsymbol{r}}. \tag{4}$$

Another useful identity is that

$$\sigma_1 H_{\boldsymbol{r}\boldsymbol{r}'} \sigma_1 \equiv \begin{pmatrix} & \mathbf{1} \\ \mathbf{1} & \end{pmatrix} \begin{pmatrix} \boldsymbol{A}_{\boldsymbol{r}\boldsymbol{r}'} & \boldsymbol{B}_{\boldsymbol{r}\boldsymbol{r}'} \\ \boldsymbol{B}_{\boldsymbol{r}'\boldsymbol{r}}^\dagger & \boldsymbol{A}_{\boldsymbol{r}\boldsymbol{r}'}^* \end{pmatrix} \begin{pmatrix} & \mathbf{1} \\ \mathbf{1} & \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}_{\boldsymbol{r}\boldsymbol{r}'}^* & \boldsymbol{B}_{\boldsymbol{r}'\boldsymbol{r}}^\dagger \\ \boldsymbol{B}_{\boldsymbol{r}\boldsymbol{r}'} & \boldsymbol{A}_{\boldsymbol{r}\boldsymbol{r}'} \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}_{\boldsymbol{r}'\boldsymbol{r}}^T & \boldsymbol{B}_{\boldsymbol{r}'\boldsymbol{r}} \\ \boldsymbol{B}_{\boldsymbol{r}'\boldsymbol{r}}^T & \boldsymbol{A}_{\boldsymbol{r}'\boldsymbol{r}}^\dagger \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}_{\boldsymbol{r}'\boldsymbol{r}} & \boldsymbol{B}_{\boldsymbol{r}'\boldsymbol{r}} \\ \boldsymbol{B}_{\boldsymbol{r}\boldsymbol{r}'}^* & \boldsymbol{A}_{\boldsymbol{r}'\boldsymbol{r}}^* \end{pmatrix}^T \equiv H_{\boldsymbol{r}'\boldsymbol{r}}^T. \tag{5}$$

We are interested in translate-invaraint system (to define bands) $H_{\boldsymbol{r}\boldsymbol{r}'} \equiv H_{\boldsymbol{\delta}}$, so the above properties can be reexpressed as

$$H_{\boldsymbol{\delta}}^\dagger \equiv H_{-\boldsymbol{\delta}}, \tag{6}$$

and

$$\sigma_1 H_{\boldsymbol{\delta}} \sigma_1 = H_{-\boldsymbol{\delta}}^T. \tag{7}$$

B. Energy Current

By [1, 3] the energy current on lattice is defined as 1-chain

$$J_{ab}^E \equiv -i[\mathcal{H}_a, \mathcal{H}_b] \equiv -\frac{i}{16} \sum_{cd} [\psi_a^\dagger H_{ac} \psi_c + \psi_c^\dagger H_{ca} \psi_a, \psi_b^\dagger H_{bd} \psi_d + \psi_d^\dagger H_{db} \psi_b]. \tag{8}$$

Taking $\sum_{cd} [\psi_a^\dagger H_{ac} \psi_c, \psi_b^\dagger H_{bd} \psi_d]$ as an example

$$\begin{aligned}
&\sum_{cd} \sum_{ijk\ell} [(\psi_a^\dagger)_i (H_{ac})_{ij} (\psi_c) j, (\psi_b^\dagger)_k (H_{bd})_{k\ell} (\psi_d)_\ell] \\
&= \sum_{cd} \sum_{ijk\ell} \Big\{ (\psi_a^\dagger)_i (H_{ac})_{ij} [(\psi_c)_j, (\psi^\dagger)_k] (H_{bd})_{k\ell} (\psi_d)_\ell + (\psi_a^\dagger)_i (H_{ac})_{ij} (\psi_b^\dagger)_k (H_{bd})_{k\ell} [(\psi_c)_j, (\psi_d)_\ell] \\
&\qquad + (\psi_b^\dagger)_k (H_{bd})_{k\ell} [(\psi_a^\dagger)_i, (\psi_d)_\ell] (H_{ac})_{ij} (\psi_c)_j + [(\psi_a^\dagger)_i, (\psi_b^\dagger)_k] (H_{bd})_{k\ell}) (\psi_d)_\ell (H_{ac})_{ij} (\psi_c)_j \Big\} \\
&= \sum_{cd} \sum_{ijk\ell} \Big\{ (\psi_a^\dagger)_i (H_{ac})_{ij} (\sigma_3)_{jk} (H_{bd})_{k\ell} (\psi_d)_\ell \delta_{bc} + i(\psi_a^\dagger)(H_{ac})_{ij} (\psi_b^\dagger)_k (H_{bd})_{k\ell} (\sigma_2)_{j\ell} \delta_{cd} \\
&\qquad - (\psi_b^\dagger)_k (H_{bd})_{k\ell} (\sigma_3)_{i\ell} (H_{ac})_{ij} (\psi_c)_j \delta_{ad} - i(\sigma_2)_{ik} (H_{bd})_{k\ell} (\psi_d)_\ell (H_{ac})_{ij} (\psi_c)_j \delta_{ab} \Big\} \equiv ① + ② + ③ + ④,
\end{aligned}$$

where in the last line we utilize the C.C.R. of 2-N tuple vector ψ

$$[(\psi_a)_i, (\psi_b^\dagger)_j] = (\sigma_3)_{ij} \delta_{ab}, \tag{9a}$$
$$[(\psi_a)_i, (\psi_b)_j] = i(\sigma_2)_{ij} \delta_{ab}, \tag{9b}$$

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[1] Actually **it is the ambiguity on the choice of $\mathcal{H}$ that gives rise to operator modification and then magnetization**.

$$[(\psi_a^\dagger)_i, (\psi_b^\dagger)_j] = -i(\sigma_2)_{ij}\delta_{ab}. \tag{9c}$$

Using $\sigma_2 \equiv i\sigma_1\sigma_3 \equiv -i\sigma_3\sigma_1, \sigma_1\psi_a \equiv \psi_a^\dagger$ and $\sigma_1\psi_a^\dagger \equiv \psi_a$, we have

$$1 = \sum_{ad} \psi_a^\dagger H_{ac}\sigma_3 H_{bd}\psi_d\delta_{ac},$$

and

$$\begin{aligned}
2 &= \sum_{cd}\sum_{ijk\ell mn} i(\psi_a^\dagger)_i(H_{ac})_{ij}(-i)(\sigma_3)_{jm}(\sigma_1)_{m\ell}(H_{bd})_{k\ell}(\sigma_1)_{kn}(\psi_b)_n\delta_{cd} \\
&= \sum_{cd}\sum_{ijk\ell mn} (\psi_a^\dagger)_i(H_{ac})_{ij}(\sigma_3)_{jm}(\sigma_1)_{m\ell}(H_{bd}^T)_{\ell k}(\sigma_1)_{kn}(\psi_b)_n\delta_{cd} \\
&= \sum_{cd}\sum_{ijmn} (\psi_a^\dagger)_i(H_{ac})_{ij}(\sigma_3)_{jm}(H_{db})_{mn}(\psi_b)_n\delta_{cd} \equiv \sum_{cd}\psi_a^\dagger H_{ac}\sigma_3 H_{db}\psi_b\delta_{cd},
\end{aligned}$$

where in the second line we use (7). Similarly

$$3 = -\sum_{cd}\psi_b^\dagger H_{bd}\sigma_3 H_{ac}\psi_c\delta_{ad},$$

and

$$\begin{aligned}
4 &= \sum_{cd}\sum_{ijk\ell mn} -i(\sigma_3)_{im}(\sigma_1)_{mk}(H_{bd})_{k\ell}(\sigma_1)_{\ell n}(\psi_d^\dagger)_n(H_{ac})_{ij}(\psi_c)_j\delta_{ab} \\
&= -\sum_{cd}\sum_{ijmn}(\sigma_3)_{im}(H_{db}^T)_{mn}(\psi_d^\dagger)_n(H_{ac})_{ij}(\psi_c)_j\delta_{ab} \\
&= -\sum_{cd}\sum_{ijmn}(\psi_d^\dagger)_m(H_{db})_{nm}(\sigma_3)_{mi}(H_{ac})_{ij}(\psi_c)_j\delta_{ab} \equiv -\sum_{cd}\psi_c^\dagger H_{db}\sigma_3 H_{ac}\psi_c\delta_{ab}.
\end{aligned}$$

The other three terms in (8) can be immediately obtained by cyclying the labels of sites. Since energy current only possess anti-symmtric components, all terms containing δ_{ab} is gone in J^E_{ab}. Thus we have

$$\begin{aligned}
J^E_{ab} = \frac{-i}{8}\sum_c \Big\{ &\psi_a^\dagger H_{ab}\sigma_3 H_{bc}\psi_c + \psi_a^\dagger H_{ac}\sigma_3 H_{cb}\psi_b - \psi_b^\dagger H_{ba}\sigma_3 H_{ac}\psi_c \\
&+ \psi_c^\dagger H_{ca}\sigma_3 H_{ab}\psi_b - \psi_b^\dagger H_{bc}\sigma_3 H_{ca}\psi_a - \psi_c^\dagger H_{cb}\sigma_3 H_{ba}\psi_a \Big\}.
\end{aligned} \tag{10}$$

When evaluating with an arbitary 1-cochain δf,

$$J^E(\delta f) \equiv \frac{1}{2}\sum_{ab} J^E_{ab}(f_a - f_b) = \frac{-i}{4}\psi^\dagger(fH\sigma_3 H - H\sigma_3 H)f = \frac{-i}{4}\psi^\dagger[f, H\sigma_3 H]\psi, \tag{11}$$

where we omit the summation over spatial labels. This result is half of the free fermionic case in [1], as expected.

C. Bogoliubov Transformation

For translate-invariant system, let us writing the fields operator in momentum representation, i.e.,

$$b_{\boldsymbol{r}} \equiv \frac{1}{\sqrt{N}}\sum_{\boldsymbol{k}} e^{\boldsymbol{k}\cdot\boldsymbol{r}} b_{\boldsymbol{k}}, \quad b_{\boldsymbol{r}}^\dagger \equiv \frac{1}{\sqrt{N}}\sum_{\boldsymbol{k}} e^{-\boldsymbol{k}\cdot\boldsymbol{r}} b_{\boldsymbol{k}},$$

then the original Hamiltonian (2) becomes

$$\mathcal{H} = \sum_{\boldsymbol{k}} \begin{pmatrix} \boldsymbol{b}_{\boldsymbol{k}}^\dagger & \boldsymbol{b}_{-\boldsymbol{k}} \end{pmatrix} H_{\boldsymbol{k}} \begin{pmatrix} \boldsymbol{b}_{\boldsymbol{k}} \\ \boldsymbol{b}_{-\boldsymbol{k}}^\dagger \end{pmatrix}, \tag{12}$$

where $H_{\boldsymbol{k}} \equiv \sum_{\boldsymbol{k}} H_{\boldsymbol{\delta}} e^{i\boldsymbol{k}\cdot\boldsymbol{\delta}}$. As is shown in [4], $H_{\boldsymbol{k}}$ can diagonalized by a *paraunitary matrix*[2] $T_{\boldsymbol{k}}$ such that

$$\psi_{\boldsymbol{k}} \equiv \begin{pmatrix} b_{\boldsymbol{k}} \\ b^\dagger_{-\boldsymbol{k}} \end{pmatrix} \equiv T_{\boldsymbol{k}} \begin{pmatrix} \gamma_{\boldsymbol{k}} \\ \gamma^\dagger_{-\boldsymbol{k}} \end{pmatrix},$$

or in component

$$(\psi_{\boldsymbol{k}})_i \equiv \sum_{n=1}^{N} \Big((T_{\boldsymbol{k}})_{i,n} \gamma_{\boldsymbol{k},n} + (T_{\boldsymbol{k}})_{i,N+n} \gamma^\dagger_{-\boldsymbol{k},N+n} \Big), \tag{13}$$

and

$$\mathcal{H} = \sum_{\boldsymbol{k}} \begin{pmatrix} \gamma^\dagger_{\boldsymbol{k}} & \gamma_{-\boldsymbol{k}} \end{pmatrix} \mathcal{E}_{\boldsymbol{k}} \begin{pmatrix} \gamma_{\boldsymbol{k}} \\ \gamma^\dagger_{-\boldsymbol{k}} \end{pmatrix} \equiv \sum_{\boldsymbol{k}} \sum_{n=1}^{N} \varepsilon_{n\boldsymbol{k}} \left(\gamma^\dagger_{n\boldsymbol{k}} \gamma_{n\boldsymbol{k}} + \frac{1}{2} \right), \tag{14}$$

where the N bands energy

$$\mathcal{E}_{\boldsymbol{k}} \equiv T^\dagger_{\boldsymbol{k}} H_{\boldsymbol{k}} T_{\boldsymbol{k}} \equiv \left(\begin{array}{ccc|ccc} \varepsilon_{1\boldsymbol{k}} & & & & & \\ & \ddots & & & & \\ & & \varepsilon_{N\boldsymbol{k}} & & & \\ \hline & & & \varepsilon_{1,-\boldsymbol{k}} & & \\ & & & & \ddots & \\ & & & & & \varepsilon_{N,-\boldsymbol{k}} \end{array} \right).$$

D. Kubo Part

Kubo part of thermal Hall coefficient is given by Kubo pair of energy current [5]

$$\kappa_{\text{Kubo}}(\alpha, \gamma) := \beta^2 \int_0^\infty \mathrm{d}t \, e^{-0^+ t} \langle\langle J^E(\delta\alpha, t); J^E(\delta\gamma) \rangle\rangle, \tag{15}$$

where

$$\begin{aligned} \langle\langle J^E(\delta\alpha, t); J^E(\delta\gamma) \rangle\rangle &\equiv \frac{-1}{\beta} \int_0^\beta \mathrm{d}\tau \langle e^{\tau H} J^E(\delta\alpha, t) e^{-\tau H} J^E(\delta\gamma) \rangle - \langle J^E(\delta\alpha, t) \rangle \langle J^E(\delta\gamma) \rangle \\ &= \frac{-1}{\beta} \int_0^\beta \mathrm{d}\tau \langle e^{\tau H} e^{iHt} J^E(\delta\alpha) e^{-iHt} e^{-\tau H} J^E(\delta\gamma) \rangle \\ &= \frac{-1}{\beta} \int_0^\beta \mathrm{d}\tau \langle J^E(\delta\alpha, t - i\tau) J^E(\delta\gamma) \rangle, \end{aligned} \tag{16}$$

the second term in the first line is ignored due to energy current version of Bloch theorem [6, 7]. Inserting energy-current in (11), there are two kinds of connnected contractions, i.e.,

$$\begin{aligned} \langle J^E(\delta\alpha, t - i\tau) J^E(\delta\gamma) \rangle &= \frac{-1}{16} \sum_{\boldsymbol{p},\boldsymbol{q}} \sum_{ijmn}^{N} \left\langle \psi^\dagger_{i\boldsymbol{p}}(t - i\tau) (A_\alpha)_{ij} \psi_{j\boldsymbol{p}}(t - i\tau) \psi^\dagger_{m\boldsymbol{q}} (A_\gamma)_{mn} \psi_{n\boldsymbol{q}} \right\rangle \\ &= \frac{-1}{16} \sum_{\boldsymbol{k}} \sum_{ijmn} \Big(\langle \psi^\dagger_{i\boldsymbol{k}}(t - i\tau) \psi^\dagger_{m\boldsymbol{k}} \rangle \langle \psi_{j\boldsymbol{k}}(t - i\tau) \psi_{n\boldsymbol{k}} \rangle + \langle \psi^\dagger_{i\boldsymbol{k}}(t - i\tau) \psi_{n\boldsymbol{k}} \rangle \langle \psi_{j\boldsymbol{k}}(t - i\tau) \psi^\dagger_{m\boldsymbol{k}} \rangle \Big) \end{aligned}$$

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[2] Paraunitarity means

$$T^\dagger_{\boldsymbol{k}} \sigma_3 T_{\boldsymbol{k}} \equiv \sigma_3, \quad T_{\boldsymbol{k}} \sigma_3 T^\dagger_{\boldsymbol{k}} \equiv \sigma_3,$$

$$\times (A_\alpha)_{ij}(A_\gamma)_{mn}, \tag{17}$$

where $A_\alpha \equiv [\alpha, H_{\boldsymbol{k}}\sigma_3 H_{\boldsymbol{k}}]$. Thermal average of the occupation number of diagonal basis in (14) gives

$$\langle \gamma^\dagger_{n\boldsymbol{k}}\gamma_{m\boldsymbol{k}}\rangle = \delta_{mn} g(\varepsilon_{m\boldsymbol{k}}), \quad \langle \gamma_{n\boldsymbol{k}}\gamma^\dagger_{m\boldsymbol{k}}\rangle = -\delta_{mn}(1-g(\varepsilon_{m\boldsymbol{k}})) \equiv -\delta_{m,n} g(-\varepsilon_{m\boldsymbol{k}}),$$

then from (13)

$$\langle \psi^\dagger_{i\boldsymbol{k}}(t-i\tau)\psi_{j\boldsymbol{k}}\rangle = \sum_{\boldsymbol{k}}\sum_{m=1}^{N}\left(g(\varepsilon_{m\boldsymbol{k}})(T^\dagger_{\boldsymbol{k}})_{m,i}(T_{\boldsymbol{k}})_{j,m} e^{i\varepsilon_{m\boldsymbol{k}}(t-i\tau)} - g(-\varepsilon_{m,-\boldsymbol{k}})(T^\dagger_{\boldsymbol{k}})_{m+N,i}(T_{\boldsymbol{k}})_{j,m+N} e^{-i\varepsilon_{m,-\boldsymbol{k}}(t-i\tau)}\right),$$

$$\langle \psi_{i\boldsymbol{k}}(t-i\tau)\psi^\dagger_{j\boldsymbol{k}}\rangle = \sum_{\boldsymbol{k}}\sum_{m=1}^{N}\left(-g(-\varepsilon_{m\boldsymbol{k}})(T_{\boldsymbol{k}})_{i,m}(T^\dagger_{\boldsymbol{k}})_{j,m} e^{-i\varepsilon_{m\boldsymbol{k}}(t-i\tau)} + g(\varepsilon_{m,-\boldsymbol{k}})(T_{\boldsymbol{k}})_{i,m+N}(T^\dagger_{\boldsymbol{k}})_{m+N,j} e^{i\varepsilon_{m,-\boldsymbol{k}}(t-i\tau)}\right).$$

It can be easily shown that the other two contractions $\langle \psi^\dagger_{i\boldsymbol{k}}(t-i\tau)\psi^\dagger_{j\boldsymbol{k}}\rangle$ and $\langle \psi_{i\boldsymbol{k}}(t-i\tau)\psi_{j\boldsymbol{k}}\rangle$ have exactly the same result. So the above current-current correlation (17) consists of four parts

$$\begin{aligned}\langle J^E(\delta\alpha, t-i\tau)J^E(\delta\gamma)\rangle = \frac{-1}{8}\sum_{\boldsymbol{k}}\sum_{m,n=1}^{N}\Big\{ & g(\varepsilon_{m\boldsymbol{k}})(B_{\alpha\boldsymbol{k}})_{m,n} g(-\varepsilon_{n\boldsymbol{k}})(B_{\gamma\boldsymbol{k}})_{nm} e^{i(\varepsilon_{m\boldsymbol{k}}-\varepsilon_{n\boldsymbol{k}})(t-i\tau)} \\ & - g(\varepsilon_{m\boldsymbol{k}})(B_{\alpha\boldsymbol{k}})_{m,n+N} g(\varepsilon_{n,-\boldsymbol{k}}) B_{n+N,m} e^{i(\varepsilon_{m\boldsymbol{k}}+\varepsilon_{n,-\boldsymbol{k}})(t-i\tau)} \\ & - g(-\varepsilon_{m,-\boldsymbol{k}})(B_{\alpha\boldsymbol{k}})_{m+N,n} g(-\varepsilon_{n,\boldsymbol{k}}) B_{n,m+N} e^{-i(\varepsilon_{m,-\boldsymbol{k}}+\varepsilon_{n,-\boldsymbol{k}})(t-i\tau)} \\ & + g(-\varepsilon_{m,-\boldsymbol{k}})(B_{\alpha\boldsymbol{k}})_{m+N,n+N} g(\varepsilon_{n,-\boldsymbol{k}}) B_{n+N,m+N} e^{-i(\varepsilon_{m,-\boldsymbol{k}}-\varepsilon_{n,-\boldsymbol{k}})(t-i\tau)}\Big\},\end{aligned} \tag{18}$$

where $B_{\alpha\boldsymbol{k}} \equiv T^\dagger_{\boldsymbol{k}} A_\alpha T_{\boldsymbol{k}}$. Substituting (18) back to (15) and integrate over temperature β and time t, we get (after some rearrangement)

$$\begin{aligned}\kappa_{\mathrm{Kubo}}(\alpha,\gamma) = \frac{i}{8\beta}\sum_{\boldsymbol{k}}\sum_{m,n=1}^{N}\Big\{ & (B_{\alpha\boldsymbol{k}})_{mn}(B_{\gamma\boldsymbol{k}})_{nm}\frac{g(\varepsilon_{m\boldsymbol{k}})-g(\varepsilon_{n\boldsymbol{k}})}{(\varepsilon_{m\boldsymbol{k}}-\varepsilon_{n\boldsymbol{k}})^2} - (B_{\alpha\boldsymbol{k}})_{m,n+N}(B_{\gamma\boldsymbol{k}})_{n+N,m}\frac{g(\varepsilon_{m\boldsymbol{k}})-g(-\varepsilon_{n,-\boldsymbol{k}})}{(\varepsilon_{m\boldsymbol{k}}+\varepsilon_{n,-\boldsymbol{k}})^2} \\ - (B_{\alpha\boldsymbol{k}})_{m+N,n}(B_{\gamma\boldsymbol{k}})_{n,m+N} & \frac{g(-\varepsilon_{m,-\boldsymbol{k}})-g(\varepsilon_{n\boldsymbol{k}})}{(\varepsilon_{m,-\boldsymbol{k}}+\varepsilon_{n\boldsymbol{k}})^2} + (B_{\alpha\boldsymbol{k}})_{m+N,n+N}(B_{\gamma\boldsymbol{k}})_{n+N,m+N}\frac{g(-\varepsilon_{m,-\boldsymbol{k}})-g(-\varepsilon_{n,-\boldsymbol{k}})}{(\varepsilon_{m,-\boldsymbol{k}}-\varepsilon_{n,-\boldsymbol{k}})^2}\Big\}.\end{aligned} \tag{19}$$

Minus signs in (19) can be absorbed by insertion of σ_3. Finally we come to the neat expression

$$\begin{aligned}\kappa_{\mathrm{Kubo}}(\alpha,\gamma) &= \frac{i}{8\beta}\sum_{\boldsymbol{k}}\sum_{mn=1}^{2N}(\sigma_3)_{mm}(B_{\alpha\boldsymbol{k}})_{mn}(\sigma_3)_{nn}(B_{\gamma\boldsymbol{k}})_{nm}\frac{g(\sigma_3\mathscr{E}_{\boldsymbol{k}})_{mm}-g(\sigma_3\mathscr{E}_{\boldsymbol{k}})_{nn}}{\big((\sigma_3\mathscr{E}_{\boldsymbol{k}})_{mm}-(\sigma_3\mathscr{E}_{\boldsymbol{k}})_{nn}\big)^2} \\ &= \frac{i}{8\beta}\beta\int \mathrm{d}z\, g(z)\mathbf{Tr}\left\{\delta(z-\sigma_3\mathscr{E})\sigma_3 B_\alpha\frac{1}{(z-\sigma_3\mathscr{E})^2}\sigma_3 B_\gamma - \frac{1}{(\sigma_3\mathscr{E}-z)^2}\sigma_3 B_\alpha\delta(z-\sigma_3\mathscr{E})\sigma_3 B_\gamma\right\} \\ &= \frac{-\beta}{16\pi}\int \mathrm{d}z\, g(z)\mathbf{Tr}\Big\{(G_+-G_-)\sigma_3 B_\alpha G_+^2\sigma_3 B_\gamma - (G_+-G_-)\sigma_3 B_\gamma G_-^2\sigma_3 B_\alpha\Big\},\end{aligned} \tag{20}$$

where the trace $\mathbf{Tr}$ runs over *both* momentum and N-subband or orbital degree of freedom, Green function

$$G_\pm(z) := \frac{1}{z-\sigma\mathscr{E}\pm i\delta},$$

and we use the representation of Dirac delta function

$$\lim_{\delta\to 0}\left(\frac{1}{z-\sigma\mathscr{E}+i\delta} - \frac{1}{z-\sigma\mathscr{E}-i\delta}\right) \equiv -2\pi i\delta(z-\sigma_3\mathscr{E}).$$

Note that although we start from the specific form of two-body operator $A_\alpha \equiv [\alpha, H_{\boldsymbol{k}}\sigma_3 H_{\boldsymbol{k}}]$, the above derivation has nothing to do with that. So generally for arbitary *equal-time* two-body operators $\hat{O}_1 = \psi^\dagger A\psi$ and $\hat{O}_2 = \psi^\dagger B\psi$ (of course with the requirement of Bloch theorem), we have similar results of (20) that

$$\langle\langle \psi^\dagger A\psi, \psi^\dagger B\psi\rangle\rangle = \frac{-2}{\beta}\sum_{\boldsymbol{k}}\sum_{mn=1}^{2N}(\sigma_3)_{mm}(\widetilde{A}_{\boldsymbol{k}})_{mn}(\sigma_3)_{nn}(\widetilde{B}_{\boldsymbol{k}})_{nm}\frac{g(\sigma_3\mathscr{E}_{\boldsymbol{k}})_{mm}-g(\sigma_3\mathscr{E}_{\boldsymbol{k}})_{nn}}{(\sigma_3\mathscr{E}_{\boldsymbol{k}})_{mm}-(\sigma_3\mathscr{E}_{\boldsymbol{k}})_{nn}}$$

$$
\begin{aligned}
&= \frac{-2}{\beta}\int \mathrm{d}z\, g(z)\mathbf{Tr}\left\{\delta(z-\sigma_3\mathscr{E})\sigma_3\widetilde{A}\frac{1}{z-\sigma_3\mathscr{E}}\sigma_3\widetilde{B} - \frac{1}{\sigma_3\mathscr{E}-z}\sigma_3\widetilde{A}\delta(z-\sigma_3\mathscr{E})\sigma_3\widetilde{B}\right\}\\
&= \frac{1}{\pi i\beta}\int \mathrm{d}z\, g(z)\mathbf{Tr}\Big\{(G_+-G_-)\sigma_3\widetilde{A}G_+\sigma_3\widetilde{B} + G_-\sigma_3\widetilde{A}(G_+-G_-)\sigma_3\widetilde{B}\Big\}\\
&= \frac{1}{\pi i\beta}\int \mathrm{d}z\, g(z)\mathbf{Tr}\Big\{G_+\sigma_3\widetilde{A}G_+\sigma_3\widetilde{B} - G_-\sigma_3\widetilde{A}G_-\sigma_3\widetilde{B}\Big\}, \qquad (21)
\end{aligned}
$$

where again $\widetilde{A} \equiv T^\dagger A T$.

Paraunitary matrix $T_{\boldsymbol{k}}$ is annoying here, but fortunately we can absorb them through the identity

$$
T_{\boldsymbol{k}}^{-1} f(\sigma_3 H_{\boldsymbol{k}}) T_{\boldsymbol{k}} = f(\sigma_3 \mathscr{E}_{\boldsymbol{k}}) \qquad (22)
$$

so that for example

$$
\begin{aligned}
\mathbf{Tr}\Big\{G_+\sigma_3\widetilde{A}G_+\sigma_3\widetilde{B}\Big\} &\equiv \sum_{\boldsymbol{k}} \mathbf{Tr}\Big\{\frac{1}{z-\sigma_3\mathscr{E}+i\delta}\sigma_3(T_{\boldsymbol{k}}^\dagger A_{\boldsymbol{k}} T_{\boldsymbol{k}})\frac{1}{z-\sigma_3\mathscr{E}+i\delta}\sigma_3(T_{\boldsymbol{k}}^\dagger B_{\boldsymbol{k}} T_{\boldsymbol{k}})\Big\}\\
&= \sum_{\boldsymbol{k}} \mathbf{Tr}\Big\{\left(T_{\boldsymbol{k}}^{-1}\frac{1}{z-\sigma_3 H_{\boldsymbol{k}}+i\delta}T_{\boldsymbol{k}}\right)\sigma_3(T_{\boldsymbol{k}}^\dagger A_{\boldsymbol{k}} T_{\boldsymbol{k}})\left(T_{\boldsymbol{k}}^{-1}\frac{1}{z-\sigma_3 H_{\boldsymbol{k}}+i\delta}T_{\boldsymbol{k}}\right)\sigma_3(T_{\boldsymbol{k}}^\dagger B_{\boldsymbol{k}} T_{\boldsymbol{k}})\Big\}\\
&= \sum_{\boldsymbol{k}} \mathbf{Tr}\Big\{\frac{1}{z-\sigma_3 H_{\boldsymbol{k}}+i\delta}\sigma_3 A_{\boldsymbol{k}}\frac{1}{z-\sigma_3 H_{\boldsymbol{k}}+i\delta}\sigma_3 A_{\boldsymbol{k}}\Big\}.
\end{aligned}
$$

Therefore (20) and (21) can be written as

$$
\kappa_{\text{Kubo}}(\alpha,\gamma) = \frac{-\beta}{16\pi}\int \mathrm{d}z\, g(z)\mathbf{Tr}\Big\{(\mathcal{G}_+-\mathcal{G}_-)[\alpha,h^2]\mathcal{G}_+^2[\gamma,h^2] - (\mathcal{G}_+-\mathcal{G}_-)[\gamma,h^2]\mathcal{G}_-^2[\alpha,h^2]\Big\}, \qquad (23)
$$

$$
\langle\langle \psi^\dagger A\psi, \psi^\dagger B\psi\rangle\rangle = \frac{1}{\pi i\beta}\int \mathrm{d}z\, g(z)\mathbf{Tr}\Big\{\mathcal{G}_+\sigma_3 A\mathcal{G}_+\sigma_3 B - \mathcal{G}_-\sigma_3 A\mathcal{G}_-\sigma_3 B\Big\}, \qquad (24)
$$

where $h \equiv \sigma_3 H$, and $\mathcal{G}_\pm \equiv 1/(z-\sigma_3 H \pm i\delta)$.

E. Energy Magnetization

Energy Magnetization has a canonical definition as a 1-form valued 2-chain [1, 3] such that

$$
\mu^E_{pqr} \equiv -\beta\langle\langle \mathrm{d}\mathcal{H}_p, J^E_{qr}\rangle\rangle - \beta\langle\langle \mathrm{d}\mathcal{H}_r, J^E_{pq}\rangle\rangle - \beta\langle\langle \mathrm{d}\mathcal{H}_q, J^E_{rp}\rangle\rangle. \qquad (25)
$$

After evaluation on 2-cochain we can separate $\mu^E(\delta\alpha\cup\delta\gamma)$ into two parts

$$
\begin{aligned}
\mu^E(\delta\alpha\cup\delta\gamma) &\equiv \frac{1}{3!}\sum_{pqr}\mu^E_{pqr}\cdot\delta\alpha\cup\delta\gamma(p,q,r)\\
&\equiv \frac{1}{3!}\sum_{pqr}\mu^E_{pqr}\cdot\frac{1}{3!}\Big(\delta\alpha_{pq}\delta\gamma_{qr} - \delta\alpha_{pr}\delta\gamma_{rq} - \delta\alpha_{qp}\delta\gamma_{pr} + \delta\alpha_{qr}\delta\gamma_{rp} + \delta\alpha_{rp}\delta\gamma_{pq} - \delta\alpha_{rq}\delta\gamma_{qp}\Big)\\
&= \frac{1}{12}\sum_{pqr}\mu^E_{pqr}\Big(\alpha_p\delta\gamma_{qr} + \alpha_q\delta_{rp} + \alpha_r\delta\gamma_{rp}\Big) \equiv \frac{-\beta}{12}\sum_{pqr}\Big(\langle\langle \mathrm{d}H_p; J^E_{qr}\rangle\rangle + \text{cycle}\Big)\Big(\alpha_p\delta\gamma_{qr} + \text{cycle}\Big)\\
&= \frac{\beta}{12}\sum_{pqr}\Big(\langle\langle \mathrm{d}\mathcal{H}_p, J^E_{qr}\rangle\rangle + \text{cycle}\Big)(\alpha_p\delta\gamma_{qr} + \text{cycle})\\
&= \frac{-\beta}{12}\left(\left\langle\left\langle \sum_p H_p\alpha_p; \sum_{qr} J^E_{qr}\delta\gamma_{qr}\right\rangle\right\rangle + \text{cycle}\right) + \frac{-\beta}{12}\sum_{pqr}\Big(\langle\langle H_p; J^E_{qr}\rangle\rangle(\alpha_p\delta\gamma_{rp} + \alpha_r\delta\gamma_{pq}) + \text{cycle}\Big)\\
&= \cdots\\
&= \frac{-\beta}{2}\Big(\langle\langle \mathrm{d}\mathcal{H}(\alpha); J^E(\delta\gamma)\rangle\rangle - (\alpha\leftrightarrow\gamma)\Big) + \frac{-\beta}{6}\left(\sum_{pqr}\langle\langle \mathrm{d}\mathcal{H}_p, J^E_{qr}\rangle\rangle\alpha_q\gamma_r + \text{cycle}\right) \equiv \text{PART}_1 + \text{PART}_2.
\end{aligned}
\qquad (26)
$$

For the first part, using (22) we immediately have

$$\begin{aligned}\langle\langle \mathrm{d}H(\alpha), J^E(\delta\gamma)\rangle\rangle &= \left\langle\left\langle \frac{1}{4}\psi^\dagger(\alpha\,\mathrm{d}H + \mathrm{d}H\alpha)\psi; \frac{-i}{4}\psi^\dagger[\gamma, H\sigma_3 H]\psi\right\rangle\right\rangle \\ &= \frac{-1}{16\pi\beta}\sum_{\boldsymbol{k}}\int \mathrm{d}z\, g(z)\mathbf{Tr}\Big\{\mathcal{G}_+\sigma_3(\alpha\,\mathrm{d}H_{\boldsymbol{k}} + \mathrm{d}H_{\boldsymbol{k}}\alpha)\mathcal{G}_+\sigma_3[\gamma, H_{\boldsymbol{k}}\sigma H_{\boldsymbol{k}}] - (\mathcal{G}_+ \leftrightarrow \mathcal{G}_-)\Big\}\end{aligned} \tag{27}$$

Using the identity that

$$\begin{aligned}\mathcal{G}_\pm[\sigma_3 H, f]\mathcal{G}_\pm &\equiv \frac{1}{z-\sigma_3 H \pm i\delta}(\sigma_3 H f - f H\sigma_3)\frac{1}{z-\sigma H \pm i\delta} \\ &= \left(-1+\frac{z}{z-\sigma_3 H \pm i\delta}\right) f\frac{1}{z-\sigma_3 H\pm i\delta} + \frac{1}{z-\sigma_3 H \pm i\delta} f\left(1-\frac{z}{z-\sigma_3 H \pm i\delta}\right) \equiv [\mathcal{G}_\pm, f],\end{aligned}$$

then

$$\begin{aligned}\mathcal{G}_+\sigma_3\alpha\,\mathrm{d}H\mathcal{G}_+ &\equiv \Big(\alpha\mathcal{G}_+ + \mathcal{G}_+[\sigma_3 H,\alpha]\mathcal{G}_+\Big)\sigma_3\,\mathrm{d}H\mathcal{G}_+, \\ \mathcal{G}_+\sigma_3\,\mathrm{d}H\alpha\mathcal{G}_+ &\equiv \mathcal{G}_+\sigma_3\,\mathrm{d}H\Big(\mathcal{G}_+\alpha - \mathcal{G}_+[\sigma_3 H,\alpha]\mathcal{G}_+\Big),\end{aligned}$$

and for example

$$\begin{aligned}&\mathbf{Tr}\Big\{\mathcal{G}_+\sigma_3(\alpha\,\mathrm{d}H_{\boldsymbol{k}} + \mathrm{d}H_{\boldsymbol{k}}\alpha)\mathcal{G}_+\sigma_3[\gamma, H_{\boldsymbol{k}}\sigma H_{\boldsymbol{k}}]\Big\} \\ &\quad = \mathbf{Tr}\Big\{\mathcal{G}_+\sigma_3\,\mathrm{d}H_{\boldsymbol{k}}\mathcal{G}_+\Big(\sigma_3[\gamma, H_{\boldsymbol{k}}\sigma_3 H_{\boldsymbol{k}}]\alpha + \sigma_3[\gamma, H_{\boldsymbol{k}}\sigma_3 H_{\boldsymbol{k}}]\mathcal{G}_+[\sigma_3 H_{\boldsymbol{k}},\alpha] + \alpha\sigma_3[\gamma, H_{\boldsymbol{k}}\sigma_3 H_{\boldsymbol{k}}] - [\sigma_3 H_{\boldsymbol{k}},\alpha]\mathcal{G}_+\sigma_3[\gamma, H_{\boldsymbol{k}}\sigma_3 H_{\boldsymbol{k}}]\Big)\Big\}.\end{aligned}$$

Therefore

$$\begin{aligned}\mathrm{PART}_1 = \frac{1}{32\pi}\int \mathrm{d}z g(z)\mathbf{Tr}\Big\{&\mathcal{G}_+\sigma_3\,\mathrm{d}H_{\boldsymbol{k}}\mathcal{G}_+\Big(\sigma_3[\gamma, H_{\boldsymbol{k}}\sigma_3 H_{\boldsymbol{k}}]\alpha + \sigma_3[\gamma, H_{\boldsymbol{k}}\sigma_3 H_{\boldsymbol{k}}]\mathcal{G}_+[\sigma_3 H_{\boldsymbol{k}},\alpha] + \alpha\sigma_3[\gamma, H_{\boldsymbol{k}}\sigma_3 H_{\boldsymbol{k}}] \\ &- [\sigma_3 H_{\boldsymbol{k}},\alpha]\mathcal{G}_+\sigma_3[\gamma, H_{\boldsymbol{k}}\sigma_3 H_{\boldsymbol{k}}] - (\alpha\leftrightarrow\gamma)\Big) - (\mathcal{G}_+\leftrightarrow\mathcal{G}_-)\Big\} \\ = \frac{1}{32\pi}\int \mathrm{d}z g(z)\mathbf{Tr}\Big\{&\mathcal{G}_+\sigma_3\,\mathrm{d}H_{\boldsymbol{k}}\mathcal{G}_+\Big(\sigma_3[\alpha, H_{\boldsymbol{k}}\sigma_3 H_{\boldsymbol{k}}]\mathcal{G}_+[\gamma,\sigma_3 H_{\boldsymbol{k}}] - \sigma_3[\gamma, H_{\boldsymbol{k}}\sigma_3 H_{\boldsymbol{k}}]\mathcal{G}_+[\alpha,\sigma_3 H_{\boldsymbol{k}}] \\ &+ [\alpha,\sigma_3 H_{\boldsymbol{k}}]\mathcal{G}_+\sigma_3[\gamma, H_{\boldsymbol{k}}\sigma_3 H_{\boldsymbol{k}}] - [\gamma,\sigma_3 H_{\boldsymbol{k}}]\mathcal{G}_+\sigma_3[\alpha, H_{\boldsymbol{k}}\sigma_3 H_{\boldsymbol{k}}] + \Gamma_{\boldsymbol{k}}\Big) - (\mathcal{G}_+\leftrightarrow\mathcal{G}_-)\Big\},\end{aligned} \tag{28}$$

where

$$\Gamma_{\boldsymbol{k}} \equiv \sigma_3[\gamma, H_{\boldsymbol{k}}\sigma_3 H_{\boldsymbol{k}}]\alpha + \sigma_3\alpha[\gamma, H_{\boldsymbol{k}}\sigma_3 H_{\boldsymbol{k}}] - \sigma_3[\alpha, H_{\boldsymbol{k}}\sigma_3 H_{\boldsymbol{k}}]\gamma - \sigma_3\gamma[\alpha, H_{\boldsymbol{k}}\sigma_3 H_{\boldsymbol{k}}]. \tag{29}$$

For the second part, using the on-site current J^E_{ab} in (8), we know

$$\sum_{ab} J^E_{ab}\alpha_a\gamma_b = -\frac{i}{8}\psi^\dagger\Big(\alpha H\sigma_3\gamma H + \alpha H\sigma_3 H\gamma - \gamma H\sigma_3\alpha H + H\alpha\sigma_3 H\gamma - \gamma H\sigma_3 H\alpha - H\gamma\sigma_3 H\alpha\Big)\psi \equiv \frac{-i}{8}\psi^\dagger\sigma_3\Lambda\psi, \tag{30}$$

where we explicitly extract σ_3 in definition of Λ for further simplification. Thus

$$\begin{aligned}\mathrm{PART}_2 &\equiv \frac{-\beta}{6}\left\langle\left\langle\sum_a \mathrm{d}H_a; \sum_{bc} J^E_{bc}\alpha_b\gamma_c\right\rangle\right\rangle + \text{cycle} = \frac{-\beta}{2}\left\langle\left\langle\frac{1}{2}\psi^\dagger\,\mathrm{d}H\psi; \frac{-i}{8}\psi^\dagger\sigma_3\Lambda\psi\right\rangle\right\rangle \\ &= \frac{1}{32\pi}\int \mathrm{d}z g(z)\mathbf{Tr}\Big\{\mathcal{G}_+\sigma_3\,\mathrm{d}H\mathcal{G}_+\Lambda - (\mathcal{G}_+\leftrightarrow\mathcal{G}_-)\Big\}.\end{aligned} \tag{31}$$

Because neither 0-cochain α, γ contains a momentum label or sublattice (or orbital d.o.f.) label, they must commutes with σ_3. Thus we can re-express Γ and Λ in a more suggestive form:

$$\Gamma \equiv [\gamma, h^2]\alpha + \alpha[\gamma, h^2] - [\alpha, h^2]\gamma - \gamma[\alpha, h^2], \tag{32}$$

$$\Lambda \equiv \alpha h\gamma h + \alpha h^2\gamma - \gamma h\alpha h + h\alpha h\gamma - \gamma h^2\alpha - h\gamma h\alpha, \tag{33}$$

where again $h \equiv \sigma_3 H$. Thus in combination of (28) and (31), energy magnetization is

$$\begin{aligned}
\mu^E(\delta\alpha \cup \delta\gamma) &= \frac{1}{32\pi}\int \mathrm{d}z g(z)\mathbf{Tr}\Big\{\mathcal{G}_+\,\mathrm{d}h_{\boldsymbol{k}}\mathcal{G}_+\Big((\Gamma_{\boldsymbol{k}} + \Lambda_{\boldsymbol{k}}) + [h^2_{\boldsymbol{k}},\alpha]\mathcal{G}_+[h_{\boldsymbol{k}},\gamma] + [h_{\boldsymbol{k}},\alpha]\mathcal{G}_+[\gamma,h^2_{\boldsymbol{k}}] \\
&\qquad - [h^2_{\boldsymbol{k}},\gamma]\mathcal{G}_+[h_{\boldsymbol{k}},\alpha] - [h_{\boldsymbol{k}},\gamma]\mathcal{G}_+[\alpha,h^2_{\boldsymbol{k}}]\Big) - (\mathcal{G}_+ \leftrightarrow \mathcal{G}_-)\Big\} \\
&= \cdots \\
&= \frac{1}{32\pi}\int \mathrm{d}z g(z)\mathbf{Tr}\Big\{\mathcal{G}_+\,\mathrm{d}h_{\boldsymbol{k}}\mathcal{G}_+\Big([[h_{\boldsymbol{k}},\alpha],[h_{\boldsymbol{k}},\gamma]] + [h^2_{\boldsymbol{k}},\alpha]\mathcal{G}_+[h_{\boldsymbol{k}},\gamma] + [h_{\boldsymbol{k}},\alpha]\mathcal{G}_+[\gamma,h^2_{\boldsymbol{k}}] \\
&\qquad - [h^2_{\boldsymbol{k}},\gamma]\mathcal{G}_+[h_{\boldsymbol{k}},\alpha] - [h_{\boldsymbol{k}},\gamma]\mathcal{G}_+[\alpha,h^2_{\boldsymbol{k}}]\Big) - (\mathcal{G}_+ \leftrightarrow \mathcal{G}_-)\Big\}.
\end{aligned} \tag{34}$$

F. Transport Thermal Hall Coefficient

Kapustin discussed the appropriate definition on the topological invariant thermal Hall transport coefficient. He claimed that the *transport* thermal Hall coefficient, given by temperature integral of 1-form

$$\frac{\mathrm{d}}{\mathrm{d}T}\left(\frac{\kappa_{\mathrm{tr}}(\alpha,\gamma)}{T}\right) = \frac{\mathrm{d}}{\mathrm{d}T}\left(\frac{\kappa_{\mathrm{Kubo}}(\alpha,\gamma)}{T}\right) - \frac{2}{T^3}\tau^E(\delta\alpha\cup\delta\gamma), \tag{35}$$

where 2-chain $\tau^E \equiv -\mu^E(\mathrm{d}H_p \mapsto H_p)$, **is topological only in the sense of path independence of the integral over parameter space**. In other words, one can only tell the *relative* thermal Hall coefficient to some specific temperature.

After cancellation of many term on the RHS[3] of (35), we are left with (**we are interested in finite-temperature thermal Hall coefficient relative to infinite temperature (assuming vanishing)**)

$$\begin{aligned}
\int_{T_0}^{\infty} \mathrm{d}\left(\frac{\kappa_{\mathrm{tr}}(\alpha,\gamma)}{T}\right) &= \int_{T_0}^{\infty}\mathrm{d}T\,\frac{1}{4\pi T^3}\int \mathrm{d}z\, g'(z)z^3\mathbf{Tr}\Big\{[h,\alpha]\mathcal{G}_+^2[h,\gamma]z^3(\mathcal{G}_+ - \mathcal{G}_-) - [h,\gamma]\mathcal{G}_-^2[h,\gamma](\mathcal{G}_+ - \mathcal{G}_-)\Big\} \\
&= \frac{1}{4\pi}\int \mathrm{d}z\left(\int_{\infty}^{u_0(z)} u^2\frac{\mathrm{d}}{\mathrm{d}u}g(u)\,\mathrm{d}u\right)\times\mathbf{Tr}\Big\{[h,\alpha]\mathcal{G}_+^2[h,\gamma](\mathcal{G}_+ - \mathcal{G}_-) - [h,\gamma]\mathcal{G}_-^2[h,\gamma](\mathcal{G}_+ - \mathcal{G}_-)\Big\},
\end{aligned} \tag{36}$$

where we use the identity

$$\frac{\mathrm{d}}{\mathrm{d}z}g(z) = \frac{-e^{z/T}}{T(e^{z/T}-1)^2} \equiv -\frac{T}{z}\frac{\mathrm{d}}{\mathrm{d}T}g(z)$$

and change integral variable from T to $u \equiv \frac{z}{T}$ so that

$$\begin{aligned}
\int_{T_0}^{\infty}\frac{1}{T^3}\frac{\mathrm{d}}{\mathrm{d}z}g(z)\,\mathrm{d}T &\equiv \frac{-1}{z^3}\int_{\infty}^{u_0(z)} u^2\frac{\mathrm{d}}{\mathrm{d}u}g(u)\,\mathrm{d}u \\
&= \frac{-1}{z^3}\left[\frac{e^{u_0(z)}a^2}{e^{u_0(z)}-1} - 2\mathrm{Li}_2(e^{u_0(z)}) - 2a\ln(1-e^{u_0(z)}) + \frac{2\pi^2}{3}\right] \equiv \frac{-2}{z^3}\left(c_2(g(z)) - \frac{\pi^2}{3}\right).
\end{aligned}$$

Thus the left work is to simplify the trace

$$\left.\frac{\kappa_{\mathrm{tr}}(\alpha,\gamma)}{T}\right|_{T_0}^{\infty} = \frac{-1}{2\pi}\int \mathrm{d}z\left(c_2(g(z)) - \frac{\pi^2}{3}\right)\times\mathbf{Tr}\Big\{[h,\alpha]\mathcal{G}_+^2[h,\gamma](\mathcal{G}_+ - \mathcal{G}_-) - [h,\gamma]\mathcal{G}_-^2[h,\gamma](\mathcal{G}_+ - \mathcal{G}_-)\Big\}, \tag{37}$$

---

[3] See my notes for more caluculation details

where explicitly

$$\begin{aligned}\mathbf{Tr}\{\cdots\} &\equiv -2\pi i\sum_{\boldsymbol{k}}\mathrm{Tr}\Big\{\delta(z-\sigma_3 H)\sigma_3 V_{k_\alpha}\mathcal{G}_+^2\sigma_3 V_{k_\gamma}-\delta(z-\sigma_3 H)\sigma_3 V_{k_\gamma}\mathcal{G}_-^2\sigma_3 V_{k_\alpha}\Big\}\\ &=-2\pi i\sum_{\boldsymbol{k}}\sum_{mn}^{2N}\delta(z-(\sigma_3\mathscr{E})_{mm})\Big\{(\sigma_3)_{mm}(T_{\boldsymbol{k}}^\dagger V_{k_\alpha}T_{\boldsymbol{k}})_{mn}(G_+^2)_{nn}(\sigma_3)_{nn}(T_{\boldsymbol{k}}^\dagger V_{k_\gamma}T_{\boldsymbol{k}})_{nn}\\ &\qquad -(\sigma_3)_{mm}(T_{\boldsymbol{k}}^\dagger V_{k_\gamma}T_{\boldsymbol{k}})_{mn}(G_-^2)_{nn}(\sigma_3)_{nn}(T_{\boldsymbol{k}}^\dagger V_{k_\alpha}T_{\boldsymbol{k}})_{nn}\Big\}.\end{aligned}\tag{38}$$

Only *off-diagonal* element of operator $(T_{\boldsymbol{k}}^\dagger V_{k_f}T_{\boldsymbol{k}})$ contrbute to the asymmetric transport thermal Hall coefficient. So it can be simplified as following: Because $\mathscr{E}_{\boldsymbol{k}}$ is diagonal,

$$(T_{\boldsymbol{k}}^\dagger V_{k_f}T_{\boldsymbol{k}})_{mn}\equiv\Big\{\partial_{k_f}\mathscr{E}_{\boldsymbol{k}}-(\partial_{k_f}T_{\boldsymbol{k}}^\dagger)H_{\boldsymbol{k}}T_{\boldsymbol{k}}-T_{\boldsymbol{k}}^\dagger H_{\boldsymbol{k}}(\partial_{k_f}T_{\boldsymbol{k}})\Big\}_{mn}=-\Big\{(\partial_{k_f}T_{\boldsymbol{k}}^\dagger)H_{\boldsymbol{k}}T_{\boldsymbol{k}}+T_{\boldsymbol{k}}^\dagger H_{\boldsymbol{k}}(\partial_{k_f}T_{\boldsymbol{k}})\Big\}_{mn}.$$

Paraunitarity condition tells us

$$T_{\boldsymbol{k}}^\dagger\sigma_3 T_{\boldsymbol{k}}\equiv\sigma_3\implies\begin{cases}\partial_{k_f}T_{\boldsymbol{k}}=-\sigma_3(T_{\boldsymbol{k}}^\dagger)^{-1}(\partial_{k_f}T_{\boldsymbol{k}}^\dagger)\sigma_3 T_{\boldsymbol{k}},\\ \partial_{k_f}T_{\boldsymbol{k}}^\dagger=-T_{\boldsymbol{k}}^\dagger\sigma_3(\partial_{k_f}T_{\boldsymbol{k}})T_{\boldsymbol{k}}^{-1}\sigma_3.\end{cases}$$

And using the fact that

$$\sigma_3 H_{\boldsymbol{k}}\sigma_3=T_{\boldsymbol{k}}\sigma_3 T_{\boldsymbol{k}}^\dagger H_{\boldsymbol{k}}T_{\boldsymbol{k}}\sigma_3 T_{\boldsymbol{k}}^\dagger=T_{\boldsymbol{k}}\sigma_3\mathscr{E}_{\boldsymbol{k}}\sigma_3 T_{\boldsymbol{k}}^\dagger=T_{\boldsymbol{k}}\mathscr{E}_k T_{\boldsymbol{k}}^\dagger,$$

one immediately obtains

$$(T_{\boldsymbol{k}}^\dagger V_{k_f}T_{\boldsymbol{k}})_{mn}=\begin{cases}\Big[(\sigma_3\mathscr{E}_{\boldsymbol{k}})_{mm}-(\sigma_3\mathscr{E}_{\boldsymbol{k}})_{nn}\Big]\Big((\partial_{k_f}T_{\boldsymbol{k}}^\dagger)\sigma_3 T_{\boldsymbol{k}}\Big)_{mn},\\ \Big[(\sigma_3\mathscr{E}_{\boldsymbol{k}})_{nn}-(\sigma_3\mathscr{E}_{\boldsymbol{k}})_{mm}\Big]\Big(T_{\boldsymbol{k}}^\dagger\sigma_3(\partial_{k_f}T_{\boldsymbol{k}})\Big)_{mn}.\end{cases}\tag{39}$$

Inserting (39) back into (38) and (37) and performing the integral of Dirac delta function (note that the square of $((\sigma_3\mathscr{E}_{\boldsymbol{k}})_{mm}-(\sigma_3\mathscr{E}_{\boldsymbol{k}})_{nn})$ cancel exactly either G_+^2 or G_-^2), we finally get

$$\begin{aligned}0-\frac{\kappa_{\mathrm{tr}}(\alpha,\gamma)}{T_0}&=i\sum_{\boldsymbol{k}}\sum_{mn}\left(c_2(g((\sigma_3\mathscr{E})_{mm}))-\frac{\pi^2}{3}\right)\times\Big\{(\sigma_3)_{mn}\Big((\partial_{k_\alpha}T_{\boldsymbol{k}}^\dagger)\sigma_3 T_{\boldsymbol{k}}\Big)_{mn}(\sigma_3)_{nn}\Big(T_{\boldsymbol{k}}^\dagger\sigma_3(\partial_{k_\gamma}T_{\boldsymbol{k}})\Big)_{nm}-(\alpha\leftrightarrow\gamma)\Big\}\\ &=i\sum_{\boldsymbol{k}}\sum_{m}^{2N}\left(c_2(g(\varepsilon_{m\boldsymbol{k}}))-\frac{\pi^2}{3}\right)\mathrm{Tr}\Big\{\Gamma_m\sigma_3\frac{\partial T_{\boldsymbol{k}}^\dagger}{\partial k_\alpha}\sigma_3\frac{\partial T_{\boldsymbol{k}}}{\partial k_\gamma}-(\alpha\leftrightarrow\gamma)\Big\}\equiv-\sum_{\boldsymbol{k}}\sum_{m}^{2N}\left(c_2(g(\varepsilon_{m\boldsymbol{k}}))-\frac{\pi^2}{3}\right)\Omega_{m\boldsymbol{k}},\end{aligned}\tag{40}$$

which is unsurprisingly the same as the result in [2]. In the last line we recognize the integral of the trace part as Berry curvature of the magnon mth-bands by introducing the projection operator

$$\hat{P}_m\equiv\Gamma_m T_{\boldsymbol{k}}\sigma_3 T_{\boldsymbol{k}}^\dagger\sigma_3,$$

so that by [8]

$$c_m=\frac{1}{2\pi}\int_{\mathrm{BZ}}\mathrm{Tr}\Big\{\hat{P}_m\,\mathrm{d}\hat{P}_m\wedge\mathrm{d}\hat{P}_m\Big\}.\tag{41}$$

where Γ_m is a diagonal matrix taking $+1$ for the mth diagonal component and zero otherwise [9].

II. APPLICATION TO TOPOLOGICAL SUPERCONDUCTORS?

[1] A. Kapustin and L. Spodyneiko, arXiv preprint arXiv:1905.06488 (2019).

[2] R. Matsumoto, R. Shindou, and S. Murakami, Physical Review B **89**, 054420 (2014).
[3] A. Kitaev, Annals of Physics **321**, 2 (2006).
[4] J. Colpa, Physica A: Statistical Mechanics and its Applications **93**, 327 (1978).
[5] R. Kubo, M. Toda, and N. Hashitsume, *Statistical physics II: nonequilibrium statistical mechanics*, vol. 31 (Springer Science & Business Media, 2012).
[6] H. Watanabe, arXiv preprint arXiv:1904.02700 (2019).
[7] A. Kapustin and L. Spodyneiko, arXiv preprint arXiv:1904.05491 (2019).
[8] J. E. Avron, R. Seiler, and B. Simon, Physical review letters **51**, 51 (1983).
[9] R. Shindou, R. Matsumoto, S. Murakami, and J.-i. Ohe, Physical Review B **87**, 174427 (2013).