Fiber bundles, Guage Transformation, and Chern-Simons Fields

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We review the memory matrix formalism with the example of parity-preserving transport, then switching to the parity-violating case (time-reversal symmetry is preserved) where anomalous Hall effect is expected to emerge. Such terms have already been revealed from pure hydrodynamic analysis in high energy physics. As another independent tool, memory matrix formalism is believe to provide the same result for the overlapping regime with hydrodynamics. In this letter, we will show the accodance of them.

流成笔下春风瓣，吹散弦上秋草声。

—— 雨楼清歌「一瓣河川」

Contents

I. A RUSH COURSE ON FIBER BUNDLES

A. Fiber Bundles, Principal Bundles, and Associated Bundles

The best language to interpret the quantum field theory is fiber bundles. We begine with the most general cases:

__**Definition 1.**__ **(Differentiable Fiber Bundle)** A *differentiable fiber bundle* is a tuple (E, π, M, F) of three differentiable manifolds E, M and F and a smooth and surjective projection $\pi : E \to M$ such that

1) For any $p \in M$, the inverse image $\pi^{-1}(\{p\}) = F_p \subset E$ is diffeomorphic to F.

2) For any $p \in M$, there exists a neighborhood U_i of p and a diffeomorphism $\phi_i : \pi^{-1}(U_i) \to U_i \times F$ called *local trivialization* such that the diagram

$$\begin{tikzcd}
U_i \times F \arrow[d, "\pi"'] & \pi^{-1}(U_i) \arrow[l, "\phi_i"'] \arrow[dl, "\pi"] \\
U_i &
\end{tikzcd}$$

commutes. M is called the *base space*, E is called the *total space* and F is called the *typical fiber*.

For each non-empty intersection of two neighborhoods $U_{ij} = U_i \bigcap U_j$ and $p \in U_{ij}$, we have two local trivializations ϕ_i and ϕ_j giving any point of the pre-image $v \in \pi^{-1}(\{p\})$ two coordinate representations

$$\phi_i(v) = (p, f_i), \quad \phi_j(v) = (p, f_j).$$

As elements on the typical fiber F, f_i and f_j are naturally connected as a *left*[1] action $f_i = g_{ij} f_j$ by introducing the diffeomorphic *transition function* $g_{ij} : U_{ij} \to \mathrm{Aut}(F), g_{ij}(\{p\}) \equiv \phi_{i,p} \circ \phi_{j,p}^{-1}$, where $\phi_{i,p}^{-1}(f_i) \equiv \phi_i^{-1}(p, f_i)$. Clearly g_{ij}

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[1] Here left action is consistent with the traditional convention because as an operator the automorphism $\mathrm{Aut}(F)$ is always assumed to act on F on the left. But in principle you can define it as a right action as well. There is no other structure conflicting with this.

satisfies the properties of a group

$$g_{ii} = \mathrm{id}, \quad g_{ij} = g_{ji}^{-1}, \quad g_{ij} \cdot g_{jk} = g_{ik},$$

which we call *diffeomorephism group* $\mathrm{Diff}(F)$. But this goup is too large that we cannot gain any information about the fiber bundle[2]. Thus to have a richer structure, we prefer to limit the domain of transition functions to a smaller subgroup as following.

Note. ⌈ It is local trivialization that determines the fiber bundle. In fact, given the bundle atlas $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$ with U_α the open covering of M and ϕ_α the diffeomorphism, and assign the fiber space F, we can reconstruct the fiber bundle as following:

- Bundle space $\widetilde{E} = \coprod_\alpha U_\alpha \times F/\sim$, where $(x_\alpha \in U_\alpha, f_\alpha) \sim (x_\beta \in U_\beta, f_\beta)$ if $x_\alpha = x_\beta \in U_\alpha \bigcap U_\beta$ and $f_\alpha = g_{\alpha\beta}(x_\alpha) f_\beta$. Here $g_{\alpha\beta} \equiv \phi_\alpha \circ \phi_\beta^{-1}$ is the transition function.
- Projection is trivial $\widetilde{\pi} : E \to M, (x_\alpha, f) \mapsto x_\alpha$.
- Diffeomorphism that satisfies the universal property is also trivial $\widetilde{\phi}_\alpha : \widetilde{\pi}^{-1}(U_\alpha) \to U_\alpha \times F : (x_\alpha, f) \mapsto (x_\alpha, f)$.

⌋

Definition 2. **(Fiber Bundle with Structure Group)** A differentiable fiber bundle (E, π, M, F) is said to equip with a *structure group* G if it admits a bundle atlas $\mathcal{A}^G = \{(U_\alpha, \phi_\alpha)\}$ such that the transition function is defined by the *left group action* $g_{\alpha\beta} : U_{\alpha\beta} \to G \subset \mathrm{Aut}(F)$

Example 1. A *vecter* or *tensor bundle* is the differentiable bundle in which $F = V$ or $F = \bigotimes_i V_i$. To have an appropriate action on them, the structure group should take value in $\mathrm{Aut}(F) = \mathrm{GL}(n)$. Particularly, if the typical fiber of each point $x \in M$ is exactly the tagent space $T_x M$ or cotagent space $T_x^* M$, then we call it *tagent bundle* TM or *cotangent bundle* T^*M.

Definition 3. **(Principal G-bundle)** A *principal G-bundle* (E, π, M, F, G) is a differentiable fiber bundle with structure group G together with a continuous *right group action* on the bundle $E \times G \to E, (v, g) \mapsto v \cdot g$ preserving the fibers $\pi(v \cdot g) = \pi(v)$ and acting *freely*[3] and *transitively*[4] on them.

Note. ⌈ Unlike the situation before, since we have fixed the action of transtion function on the left as a convention, there is no other choice except the right action to place the newly defined action of the structure group. This is because for a compatible structure these two action have to commute with each other $g \circ g_{ij} = g_{ij} \circ g, (x, f) \mapsto (x, g_{ij}(x) \cdot f \cdot g)$, while in general they DO NOT if acting on the same side. See `https://mathoverflow.net/questions/50473` for more discussions. ⌋

Note. ⌈ Such free and transitive left group action gaurantes that the mapping $G \to F$ is bijective. So $F \cong G$ and equivalently we can delete the redundant information about the fiber space F in our definition and replace it with the structure group G itself (so the right action is nothing but the group multiplication of G, which is obvious to be free and transitive). This is why in common literatures principal G-bundle is simply denoted as $E(M, G)$, and defined as "**principal G-bundle is the differentiable fiber bundle with the fiber space coincides with the structure group** G". ⌋

Example 2. A *frame bundle* is constructed by assigning each point $p \in M$ an *ordered* basis of the same linear space (for example, tagent space $T_x M$). The set of all ordered frames F_x admits naturally a free and transitive right action by $GL(n)$ (following from the standard linear algebra result that there is a unique invertible linear transformation sending one basis onto another). So **frame bundle is a principal bundle**. As a contrast, for vector (or tensor) bundle, since every vector space (or tensor space) contains the null vector (tensor) $\mathbf{0}$, the action of the structure group $GL(n)$ or its subgroups can never be free nor transitive. Namely, **vector (tensor) bundle is NOT a principal bundle**.

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[2] For example, to study a general linear space we should use group $\mathrm{GL}(n;\mathbb{C})$, but to study the inner product of a space we should work with unitary group $\mathrm{U}(n) \subset \mathrm{GL}(n;\mathbb{C})$.

[3] By free we mean $\forall x \in F$ and $g, h \in G$, $x \cdot g = x \cdot h \implies g = h$.

[4] By transitive we mean $\forall x, y \in F, \exists g \in G$ s.t. $x = y \cdot g$.

<u>Definition 4.</u> (Section) Given a fiber bundle (E, π, M, F, G), suppose U is one open set of M, then the smooth function $s : U \to E$ is called a *local section*, if

$$\pi \circ s = \mathrm{id}_M.$$

The differences between fiber bundle and principal bundle can be seen even more clearly from

<u>Theorem 1.</u> A principal bundle is trivial if and only if it admits a global section. As a contrast, vector bundle always admits a global zero section by locally mapping every element $p \in U$ to the zero vector of $\pi^{-1}(\{p\})$.

<u>Definition 5.</u> (Associated Bundle of a Principal Bundle) Let $P(M, G)$ be a principle G-bundle and F a differentiable manifold on which the structure group G has a continuous action as representation $\rho : G \to \mathrm{Homeo}(F)$ on the *left*. Then we can construct a associated fiber bundle $Q = P \times_\rho F$ with G the structrue group and F the typical fiber as following:

Utilizing the right action of G on P (as group multiplication) and left action on F (as representation), we can define a *left* action on the product manifold $P \times F$ by $g(u, f) \mapsto (u \cdot g, \rho(g^{-1})f)$. Then the quotient space $Q = P \times F/G$ is defined by assigning the equivalent relation $(u, f) \sim (u \cdot g, \rho(g^{-1})f)$. The surjective projection $\pi_Q : Q \to M$ is naturally induced by the canonical projection and the surjective projection of the original principal bundle $\pi : P \to M$ as following

$$\pi_Q : \ Q = P \times F/G \xrightarrow{\tau} P \times F \xrightarrow{\sigma} P \xrightarrow{\pi} M$$

So $Q = P \times F/G$ furnishes as a bundle space over M with typical fiber F and structural group G, called the *associated fiber bundle* of the principal bundle.

Note. ⌈ The diffeomorphic local trivialization ψ_α of (Q, π_Q, M, F) can also be induced by that of principal bundle as the commutative diagram

$$\begin{tikzcd}
 & \pi_Q^{-1}(U_\alpha) \arrow[dl, "\psi_\alpha"'] \arrow[d, "\tau\circ\sigma"] \arrow[dd, bend left=60, "\pi_Q"] \\
U_\alpha \times F \arrow[dr, "\pi{,}\pi_Q"'] & \pi^{-1}(U_\alpha) \subset P \arrow[l, "\phi_\alpha"'] \arrow[d, "\pi"] \\
 & U_\alpha
\end{tikzcd}$$

so for $x \in U_\alpha$, the pre-image $\pi_Q^{-1}(\{x \in M\}) = (x, f)$ is diffeomorphic to the fiber F, and we indeed define a fiber bundle. ⌋

B. Connection and Characteristic Class

II. CHERN-SIMONS FIELD