Conformal Field Theory and Applications in Condensed Matter Physics

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This is a research note about applied CFT. For example, we will discuss the single-loop RG equation by OPE and the boundary CFT of bulk FQHE.

Contents

I. GENERAL CONFORMAL TRANSFORMATION

Let us consider a local field theory defined on a D-dimensional spacetime with *Euclidean* metric $g_{\mu\mu} \equiv \eta_{\mu\nu}$ and signature (p, q). A general diffeomorphism transforms the metric to

$$g_{\mu\nu} \mapsto \widetilde{g}_{\mu\nu}(\widetilde{x}) \equiv \frac{\partial x_\mu}{\partial \widetilde{x}_\alpha}\frac{\partial x_\alpha}{\partial \widetilde{x}_\beta} g_{\alpha\beta}.$$

<u>Definition 1.</u> (Conformal Group) Conformal group is the subgroup of diffeomorphism group leaving the metric tensor unchcanged up to a scaling factor

$$\widetilde{g}_{\mu\nu}(\widetilde{x}) = \Omega(x) g_{\mu\nu}(x). \tag{1}$$

Clearly Poincaré group ($\Omega \equiv 1$) is the subgroup of confromal group.

The full generators of conformal transformation is obtained by solving the allowed infinitesimal transformation $\widetilde{x}_\mu \equiv x_\mu + \varepsilon_\mu + \mathcal{O}(\varepsilon^2)$ such that

$$\widetilde{g}_{\mu\nu} \equiv (\delta_\mu^\alpha + \partial_\mu \varepsilon^\alpha)(\delta_\nu^\beta + \partial_\nu \varepsilon^\beta) g_{\alpha\beta} = g_{\mu\nu} + \partial_\mu \varepsilon^\alpha g_{\alpha\nu} + \partial_\nu \varepsilon^\beta g_{\mu\beta} + \mathcal{O}(\varepsilon^2)$$

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$$= g_{\mu\nu} + (\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu) + \mathcal{O}(\varepsilon^2)$$

from the constraint (1)

$$(\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu) = (\Omega(x) - 1) g_{\mu\nu} \equiv f(x) g_{\mu\nu}. \tag{2}$$

Tracing both sides with $g^{\mu\nu}$

$$2\partial^\mu \varepsilon_\mu = f(x) D,$$

we then can eliminate the newly defined scaling factor $f(x)$ by inserting back to equation (2)

$$\boxed{(\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu) = \frac{2}{D}(\partial \cdot \varepsilon) g_{\mu\nu}.} \tag{3}$$

Since equation (3) degenerates for 1d spacetime (so the theory is trivial), we will focus on the case when $D \geq 2$.

- For $D = 2$, equation (3) gives the celebrated *Cauchy-Riemann equation*

$$\partial_1 \varepsilon_2 \equiv -\partial_2 \varepsilon_1, \quad \partial_1 \varepsilon_1 \equiv \partial_2 \varepsilon_2. \tag{4}$$

- But for $D > 2$, we have to to re-arrange (3) into a more explicit form. Applying ∂^ν with ∂_ν on (3) gives

$$\partial_\mu \partial_\nu (\partial \cdot \varepsilon) + \partial^2 \partial_\nu \varepsilon_\mu = \frac{2}{D} \partial_\mu \partial_\nu (\partial \cdot \varepsilon). \tag{5}$$

  Similarly for the twice derivatives of ∂_μ and ∂^μ

$$\partial^2 \partial_\mu \varepsilon_\nu + \partial_\nu \partial_\mu (\partial \cdot \varepsilon) = \frac{2}{D} \partial_\nu \partial_\mu (\partial \cdot \varepsilon). \tag{6}$$

  Then by adding up (5) and (6) and inserting (3) to replace $(\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu)$, we obtain the simple differential equation of $(\partial \cdot \varepsilon)$

$$2\partial_\mu \partial_\nu (\partial \cdot \varepsilon) + \partial^2 \left(\frac{2}{D} (\partial \cdot \varepsilon) g_{\mu\nu} \right) = \frac{4}{D} \partial_\mu \partial_\nu (\partial \cdot \varepsilon) \implies (g_{\mu\nu} \partial^2 + (D-2) \partial_\mu \partial_\nu)(\partial \cdot \varepsilon) = 0 \implies (D-1)\partial^2 (\partial \cdot \varepsilon) = 0. \tag{7}$$

A. Conformal Transformation in $D > 2$

For $D > 2$ the constraint (7) implies the infinitesimal ε_μ to be at most *quadratic* in coordinate

$$\varepsilon_\mu = a_\mu + b_{\mu\nu} x^\nu + c_{\mu\nu\rho} x^\nu x^\rho$$

with the symmetric $c_{\mu\nu\rho} \equiv c_{\mu\rho\nu}$.

1) Clearly $\varepsilon_\mu = a_\mu$ represents the (infinitesimal) **space-time translation** $x'_\mu \mapsto x_\mu + a_\mu$ as ususal.

2) By inserting $\varepsilon_\mu = b_{\mu\nu} x^\nu$ back into (3) we have

$$b_{\mu\nu} + b_{\nu\mu} = \frac{2}{D} (b_{\alpha\beta} \partial^\alpha x^\beta) g_{\mu\nu} = \frac{2}{D} b_\alpha{}^\alpha g_{\mu\nu}.$$

   Separating $b_{\mu\nu}$ into symmetric part $b^s_{\mu\nu} \equiv b^s_{\mu\nu}$ and anti-symmetric part $b^a_{\mu\nu} \equiv -b^a_{\mu\nu}$, then the anti-symmetric part represents the familiar (infinitesimal) **space-time rotation** $b^a_{\mu\nu} \equiv \omega_{\mu\nu}$, while the symmetric part $b^s_{\mu\nu} = \frac{1}{D}(b^s)_\alpha{}^\alpha g_{\mu\nu}$ represents the (infinitesimal) **space-time dilation** since

$$x'_\mu = x_\mu + (b^s)_{\mu\nu} x^\nu = \left(1 + \frac{1}{D}(b^s)_\alpha{}^\alpha\right) x_\mu \equiv \lambda x_\mu.$$

Translation	$P^{\alpha}=-i\frac{\delta}{\delta a_{\alpha}}(x_{\mu}+a_{\mu})\partial^{\mu}=-i\partial^{\alpha}$
Dilation	$D=-i\frac{\delta}{\delta\lambda}\lambda x_{\mu}\partial^{\mu}=-ix^{\mu}\partial_{\mu}$
Rotation	$L^{\alpha\beta}=-i\frac{\delta}{\delta\omega_{\alpha\beta}}(x_{\mu}+\omega_{\mu\nu}x^{\nu})\partial^{\mu}=-i(\delta_{\alpha\mu}\delta_{\beta\nu}-\delta_{\alpha\nu}\delta_{\beta\mu})x^{\nu}\partial^{\mu}=i(x^{\alpha}\partial^{\beta}-x^{\beta}\partial^{\alpha})$
SCT	$K^{\alpha}=-i\frac{\delta}{\delta b_{\alpha}}(x_{\mu}+2(b\cdot x)x_{\mu}-b_{\mu}x^2)\partial^{\mu}=-i(2x^{\alpha}(x\cdot\partial)-x^2\partial^{\alpha})$

TABLE I: Corresponding four generators of the infinitesimal conformal transformations.

3) By inserting $\varepsilon_{\mu}=c_{\mu\alpha\beta}x^{\alpha}x^{\beta}$ back into (3) we have

$$c_{\mu\nu\beta}x^{\beta}+c_{\nu\mu\beta}x^{\beta}=\frac{2}{D}c^{\alpha}{}_{\alpha\beta}x^{\beta}g_{\mu\nu}.$$

To express $c_{\mu\nu\beta}$ explicitly, we have to cycle all the indices, summing and subtracting them with symmetric condition on the latter two indices of $c_{\mu\nu\beta}$, which yields the **special conformation transformation** (SCT)

$$c_{\mu\nu\beta}=\frac{1}{D}(c^{\alpha}{}_{\alpha\beta}g_{\mu\nu}+c^{\alpha}{}_{\alpha\nu}g_{\mu\beta}-c^{\alpha}{}_{\alpha\mu}g_{\nu\beta})\equiv b_{\beta}g_{\mu\nu}+b_{\nu}g_{\mu\beta}-b_{\mu}g_{\nu\beta}$$

with $b_{\beta}\equiv c^{\alpha}{}_{\alpha\beta}/D$. So the coordinate transforms (infinitesimally) as

$$x'_{\mu}=x_{\mu}+2(b\cdot x)x_{\mu}-b_{\mu}x^2.$$

The full four generators for these *infinitesimal* conformal transformations can be directly read out as in table I because on the one side, the infinitesimal transformation parameterized by ω_a can be written as $x'_{\mu}(x)\simeq x_{\mu}+\frac{\delta x_{\mu}}{\delta\omega_a}\omega_a=\left(1+\frac{\delta x^{\nu}}{\delta\omega_a}\omega_a\partial_{\nu}\right)x_{\mu}$, on the other side by definition of the generator $x'_{\mu}(x)\equiv e^{iG^a\omega_a}x_{\mu}\simeq(1+iG^a\omega_a)x_{\mu}$. After straightforward but tedious calculations, we can write down the full conformal algebra as following:

$$\begin{aligned}
[D,D]=0,\quad [D,P_{\mu}]=iP_{\mu},\quad [D,K_{\mu}]=-iK_{\mu},\quad [D,L_{\mu\nu}]=0,\\
[P_{\mu},P_{\nu}]=0,\quad [K_{\mu},K_{\nu}]=0,\quad [K_{\mu},P_{\nu}]=2i(g_{\mu\nu}D-L_{\mu\nu}),\\
[L_{\mu\nu},L_{\rho\sigma}]=-i(L_{\mu\rho}g_{\nu\sigma}-L_{\mu\sigma}g_{\nu\rho}-L_{\nu\rho}g_{\mu\sigma}+L_{\nu\sigma}g_{\mu\rho}),\\
[L_{\mu\nu},P_{\rho}]=-i(g_{\mu\rho}P_{\nu}-g_{\nu\rho}P_{\mu}),\quad [L_{\mu\nu},K_{\rho}]=-i(g_{\mu\rho}K_{\nu}-g_{\nu\rho}K_{\mu})
\end{aligned}\tag{8}$$

The four *finite* conformal transformations (namely, the flow) can also be obtained by finding out the integral curve of these generators

$$\dot{x}^{\mu}=iG^a\omega_a x^{\mu}.$$

Let us take the treaky SCT as an example. The nonlinear ODE

$$\dot{x}^{\mu}=2(b\cdot x)x_{\mu}-b_{\mu}x^2$$

is solved by a certain change of variable $I:x^{\mu}\mapsto y^{\mu}\equiv x^{\mu}/x^2$ such that

$$\frac{x^{\mu}(t)}{x^2(t)}=\frac{x^{\mu}(0)}{x^2(0)}-tb,\quad\text{or}\quad x^{\mu}(t)=\frac{x^{\mu}-tb^{\mu}x^2}{1-2tb\cdot x+(tb)^2x^2}.\tag{9}$$

The above change of variable is nothing but the inversion transformation satisfying $I^2\equiv\mathbb{1}$. So finite SCT (8) can be understood as the inversion with translation and inversion again. The full results are listed as in table II.

Translation	$x'_\mu = x_\mu + a_\mu$
Dilation	$x'_\mu = x_\mu + \frac{2}{D}(b^s)^\alpha{}_\alpha g_{\mu\nu} x^\nu \equiv \lambda x_\mu$
Rotation	$x'_\mu = x_\mu + \omega_{\mu\nu} x^\nu$
SCT	$x'_\mu = x_\mu + (b_\beta g_{\mu\nu} + b_\nu g_{\mu\beta} - b_\mu g_{\nu\beta}) x^\nu x^\beta \equiv x_\mu + 2(b\cdot x)x_\mu - b_\mu x^2$

TABLE II: Four finite conformal transformations.

B. Conformal Transformation in $D = 2$

For $D = 2$, the constraint equation for infinitesimal conformal transformation reduces to beautiful *Cauchy-Riemann equation* (4) appearing in complex analysis. So it is natural to introduce the complex variable $z \equiv x_1 + ix_2$ and $\bar{z} \equiv x_1 - ix_2$ with the assignment (such that $\partial z = \bar{\partial}\bar{z} = 1$)

$$\partial \equiv \partial_z = \frac{1}{2}(\partial_1 - i\partial_2), \quad \bar{\partial} \equiv \partial_{\bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2)$$

and also the complex function $\varepsilon(z,\bar{z}) \equiv (\varepsilon_1 + i\varepsilon_2)$, $\bar{\varepsilon}(z,\bar{z}) = (\varepsilon_1 - i\varepsilon_2)$, to decouple the Cauchy-Riemann equation into *holomorphic* and *anti-holomorphic* modes (or *left-movers* and *right-movers*)

$$\begin{aligned} \bar{\partial}\varepsilon = 0 &\implies \varepsilon(z,\bar{z}) = \varepsilon(z), \\ \partial\bar{\varepsilon} = 0 &\implies \bar{\varepsilon}(z,\bar{z}) = \bar{\varepsilon}(\bar{z}). \end{aligned}$$

Thus two dimensional conformal transformation coincides with the analytic coordinate transformation

$$z \mapsto f(z), \quad \bar{z} \mapsto \bar{f}(\bar{z}).$$

Unlike the four kinds of conformal transformation in $D > 2$, here by the analyticity of $\varepsilon(z)$ and $\bar{\varepsilon}(\bar{z})$, we can always expand them into Laurent series[1]

$$\varepsilon(z) = \sum_{n=-\infty}^{\infty} \alpha_n z^{n+1}, \quad \bar{\varepsilon}(\bar{z}) = \sum_{m=-\infty}^{\infty} \beta_m \bar{z}^{m+1}$$

with *infinite* number of infinitesimal generators

$$\ell_n \equiv -i\frac{\delta\varepsilon_n}{\delta\alpha_n}\partial_z = -iz^{n+1}\partial_z, \quad \bar{\ell}_m \equiv -i\frac{\delta\bar{\varepsilon}_m}{\delta\beta_m}\partial_{\bar{z}} = -i\bar{z}^{n+1}\partial_{\bar{z}}. \tag{10}$$

They can be easily shown to satisfy the *Witt algebra* $\mathcal{W}$

$$[\ell_m, \ell_n] = (m-n)\ell_{m+n}, \quad [\bar{\ell}_m, \bar{\ell}_n] = (m-n)\ell_n, \quad [\ell_m, \bar{\ell}_n] = 0, \tag{11}$$

which is clearly the direct sum of two independent subalgebras $\mathcal{W} = \mathcal{A} \bigoplus \bar{\mathcal{A}}$. So let us narrow our discusion to one branch of it.

Claim 1. **Although the dimension of $\mathcal{A}$ is infinite, not all the generators are *globally* well defined.**

▷ To reveal this, we can analytically continue the original domain of arbitrary analytic functions

$$\mathrm{Dom}(f) = \{(z,\bar{z})\} = \{(x_1,x_2)\} = \mathbb{R}^2$$

to $\mathbb{C}^2$ with the "physical constraint" $\bar{z} \equiv z^*$. Then the domain of each branch of conformal transformation, or equivalently analytic coordinate transformation, is now extended form $\mathbb{R}$ to $S^2 = \mathbb{C}\bigcup\infty$. All holomorphic conformal transformations are generated from the (tagent) vector field

$$X \equiv X^n\partial_n = \sum_n \alpha_n \ell_n \equiv \sum_n (-i)\alpha_n z^{n+1}\frac{\partial}{\partial z^n}.$$

[1] We write the exponent as $n+1$ instead of n just for a convention regarding the components that are well-defined locally. This will not do harm to the structure of Witt Algebra.

Clearly the non-singularity of X as $z \to 0$ demands $n \geq -1$. But to impose the non-singularity of X as $z \to \infty$, we have to change the variable $z \mapsto 1/w$ because the tangent field is only defined in the neighbor of the origin. Then the tagent vector field becomes

$$\widetilde{X} \equiv \widetilde{X}^n \widetilde{\partial}_n = \sum_n \frac{\partial w}{\partial z} X^n \frac{\partial}{\partial w^n} = \sum_n (-w^2)(-i)\alpha_n \left(\frac{1}{w}\right)^{n+1} \frac{\partial}{\partial w^n} = \sum_n \alpha_n \left(\frac{i}{w}\right)^{n-1} \frac{\partial}{\partial w^n}.$$

This time the non-singularity of $\widetilde{X}$ as $w \to \infty$ requires $n \leq 1$. Plus the similar argument from another branch, we conclude that **only the conformal transformations generated by $\{\ell_{-1}, \ell_0, \ell_1\} \bigcup \{\bar{\ell}_{-1}, \bar{\ell}_0, \bar{\ell}_1\}$ are globally well-defined**. □

II. ENERGY MOMENTUM TENSOR

III. RADIAL QUANTIZATION

A. Virasora Algebra

IV. TRACE ANOMALY

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VIII. APPENDIX

A. Infinitesimal Symmetric Transformation

Geometrically, a classical tensor field $\mathbf{\Phi}(x)$ is an element of tensor bundle. Positive transformation on the base space-time manifold[2] $f : M \to N, \boldsymbol{x} \mapsto \boldsymbol{x'}$ will be lifted to the fibers as well

$$\mathbf{\Phi'}(\boldsymbol{x'}) \equiv \mathscr{F}(\Phi(\boldsymbol{x})), \tag{12}$$

or more clearly in the commutative diagram FIG. 1.

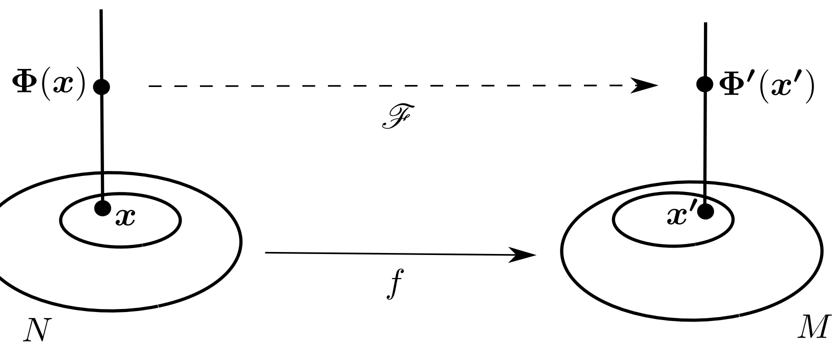

FIG. 1: **General Transformation**

[2] Here without loss of generality we take this transformation to be a mapping between two distinct manifolds. This way of looking is called *positive*. In constrast if one takes such transformation as just a coordinate transformation between two patches of the same manifold, then we call such viewpoint *negative*.

For infinitesimal transformation parameterized by some vector $\{\omega_a\}$

$$x'^{\mu} = x^{\mu} + \frac{\delta x^{\mu}}{\delta \omega_a}\omega_a + \mathcal{O}(\omega^2),$$

identity (12) tells

$$\boldsymbol{\Phi'}(\boldsymbol{x'}) = \mathscr{F}(\boldsymbol{\Phi}(\boldsymbol{x})) = \boldsymbol{\Phi}(\boldsymbol{x}) + \frac{\delta \mathscr{F}}{\delta \omega_a}\omega_a + \mathcal{O}(\omega^2) \tag{13}$$

B. Noether Theorem and Energy-momentum Tensor