Ten-fold Classification of Fermionic Topological Insulators and Topological Superconductors

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In this note, we will review the ten-fold classfication theory of non-interaction fermionic systems, where the non-intrinsic topological orders are protected simply by non-unitary representation of groups. This is the note of the 2016 PCCM summer school by A. Ludwig[1].

流成笔下春风瓣，吹散弦上秋草声。

—— 雨楼清歌「一瓣河川」

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A. Replica Field Theory and Anderson Localization

Let us with a general action $S \equiv S_0 + S_{\mathrm{int}} + S_J$ where S_{int} is the electron-electron interacting part, S_J is the terms coupling with external source field J, and

$$S_0 = \sum_n \int \mathrm{d}\boldsymbol{r}\, \bar{\psi} \left(-i\omega_n + \frac{\nabla^2}{2m} - \mu_F + V(\boldsymbol{r}) \right) \psi. \tag{1}$$

(Random) impurity potential $V_{\mathrm{imp}}(\boldsymbol{r})$ enters in $V(\boldsymbol{r})$ in the action (1). We want to take the average of them among the enire sample, which, in path integral formalism, is represented as a weight $P[V]$ such that

$$\langle \cdots \rangle_{\mathrm{dis}} \equiv \int \mathcal{D}V\, P[V](\cdots).$$

Under theomodynamic limit, we can safely consider the Gaussian distribution of impurity potentials

$$P[V] \equiv \exp\left[-\frac{1}{2\gamma^2} \int \mathrm{d}\boldsymbol{r}\, \mathrm{d}\boldsymbol{r'}\, V(\boldsymbol{r}) K(\boldsymbol{r} - \boldsymbol{r'}) V(\boldsymbol{r'}) \right],$$

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[1] See `https://www.youtube.com/watch?v=i0WGo1ZHTGQ&t=2128s`.

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where K describe the spatial correlation

$$\langle V(\boldsymbol{r})V(\boldsymbol{r'})\rangle_{\text{dis}} = \gamma^2 K(\boldsymbol{r}-\boldsymbol{r'}).$$

In most cases we can simply set $K(\boldsymbol{r}) \equiv \delta(\boldsymbol{r})$. So the disorder weight reduces to

$$P[V] \equiv \exp\left[-\frac{1}{2\gamma^2}\int \mathrm{d}\boldsymbol{r}\, V^2(\boldsymbol{r})\right]. \tag{2}$$

However, the disorder average of an arbitrary operator

$$\langle\langle\mathcal{O}\rangle\rangle_{\text{dis}} \equiv -\left\langle \frac{\delta}{\delta J}\bigg|_{J=0} \ln\mathcal{Z}\right\rangle = -\int \mathcal{D}V\, P[V]\frac{1}{\mathcal{Z}[V,J=0]}\,\frac{\delta}{\delta J}\bigg|_{J=0}\mathcal{Z}[V,J]$$

becomes largely intractable due to the appearance of V on *both* denomenator and numerator (in comparison with the auxilary field J). The only way to circumvent such difficulty is to try to find another tractable object in replacement of the annoying logarithm. It is so-called *replica trick* [1] that achieve this

$$\langle\mathcal{O}\rangle \equiv -\frac{\delta}{\delta J}\ln\mathcal{Z}[J] = -\frac{\delta}{\delta J}\lim_{R\to 0}\frac{1}{R}(e^{R\ln\mathcal{Z}}-1) = -\frac{\delta}{\delta J}\lim_{R\to 0}\frac{1}{R}\mathcal{Z}^R.$$

To understand the above identity, one must be aware that we first consider the n-th copy of the system (n an integer), then recover the original disorder average by analytical continuation (like dimensional regularization). Therefore, the disoder average now becomes simply

$$\langle\langle\mathcal{O}\rangle\rangle_{\text{dis}} = -\frac{\delta}{\delta J}\lim_{R\to 0}\frac{1}{R}\int \mathcal{D}V\, P[V]\mathcal{Z}^R[V,J]. \tag{3}$$

Input the action we give at the very beginning, and perform the path integral over impurity potentials, we get

$$\langle\mathcal{Z}^R[J]\rangle_{\text{dis}} = \int \mathcal{D}(\bar\psi,\psi)\, \exp\left[-\sum_{a=1}^{R} S_{\text{clean}}[\bar\psi^a,\psi^a] - \sum_{a,b=1}^{R} S^{\text{eff}}_{\text{dis}}[\bar\psi^a,\psi^a,\bar\psi^b,\psi^b]\right],$$

where S_{clean} is the action of non-disorder system (containing both free and electron-electron interacting parts), and

$$S^{\text{eff}}_{\text{dis}} = -\frac{\gamma^2}{2}\sum_{m,n}\int \mathrm{d}\boldsymbol{r}\,\bar\psi^a_m(\boldsymbol{r})\psi^a_m(\boldsymbol{r})\bar\psi^b_n(\boldsymbol{r})\psi_n(\boldsymbol{r}) \tag{4}$$

the effective short-range contact interaction. In comparison with the usual four-fermion interacting term, (4) mixes different replicas (or species) of fermions. It is such behavior that brings in complexities.

The standard tool to obtain the low-energy degree of freedom of the four-fermion interacting system is Hubbard-Stratonovich transformation. Usually there are three channels of contribution: *direct (density) channel*, *cooper channel*, and *exchange (diffusive) channel*. But since replica trick requirs the limit $R\to 0$ after path integral, the direct channel (introducing the density field $\rho_m(\boldsymbol{r}) \equiv \sum_a \bar\psi^a_m(\boldsymbol{r})\psi^a_m(\boldsymbol{r}) = R\bar\psi^a_m(\boldsymbol{r})\psi^a_m(\boldsymbol{r})$) will not contribute. For the left two channels of contribution, we will reveal an emergent symmetry that enable us to combine them into one single matrix-valued field.

To see this, let us introduce two infinite (but countable)-dimensional matrix-valued fields $\boldsymbol{\rho}_d(\boldsymbol{r}) \equiv \boldsymbol{\psi}(\boldsymbol{r})\bar{\boldsymbol{\psi}}(\boldsymbol{r})$ and $\boldsymbol{\rho}_c(\boldsymbol{r}) \equiv \boldsymbol{\psi}(\boldsymbol{r})\boldsymbol{\psi}(\boldsymbol{r})$, where $\boldsymbol{\psi}(\boldsymbol{r}) \equiv \{\psi^a_m(\boldsymbol{r})\}$, and split (4) into two channels explicitly

$$S^{\text{eff}}_{\text{dis}} = -\frac{\gamma^2}{4}\int \mathrm{d}\boldsymbol{r}\ \mathrm{tr}\left\{\boldsymbol{\rho}_d^2 + \boldsymbol{\rho}_c^\dagger\boldsymbol{\rho}_c\right\},$$

where the trace runs over both replica copies and Matsubara frequencies, then Hubbard-Stratonovich transformation gives

$$\begin{aligned} e^{-S^{\text{eff}}_{\text{dis}}} &= \int \mathcal{D}\boldsymbol{d}\ \exp\left\{-\int \mathrm{d}\boldsymbol{r}\left[\frac{1}{2\gamma^2}\,\mathrm{tr}\,\boldsymbol{d}^2 - i\,\mathrm{tr}(\boldsymbol{d}\boldsymbol{\rho}_d)\right]\right\}\times\int \mathcal{D}\boldsymbol{c}\exp\left\{-\int \mathrm{d}\boldsymbol{r}\left[\frac{1}{2\gamma^2}\,\mathrm{tr}(\boldsymbol{c}\boldsymbol{c}^\dagger) - \frac{i}{2}\,\mathrm{tr}(\boldsymbol{\rho}_c^\dagger\boldsymbol{c}+\boldsymbol{\rho}_c\boldsymbol{c}^\dagger)\right]\right\} \\ &\equiv \int \mathcal{D}\boldsymbol{d}\ \exp\left\{-\int \mathrm{d}\boldsymbol{r}\left[\frac{1}{2\gamma^2}\,\mathrm{tr}\,\boldsymbol{d}^2 - i\bar{\boldsymbol{\psi}}^T\boldsymbol{d}\boldsymbol{\psi}\right]\right\} \end{aligned}$$

$$\times \int \mathcal{D}\boldsymbol{c} \exp\left\{-\int \mathrm{d}\boldsymbol{r}\left[\frac{1}{2\gamma^2}\operatorname{tr}(\boldsymbol{c}\boldsymbol{c}^\dagger) - \frac{i}{2}(\bar{\boldsymbol{\psi}}^T\boldsymbol{c}\bar{\boldsymbol{\psi}} - \boldsymbol{\psi}^T\boldsymbol{c}^\dagger\boldsymbol{\psi})\right]\right\}. \tag{5}$$

Here we write the trace of coupling term in terms of matrix multiplication[1], so an extra minus sign arises in the last term due to the transposition of two fermionic operators. The newly-involved *diffuson field* $\boldsymbol{d}$ and *cooperon field* $\boldsymbol{c}$ must shares the same symmetry as $\boldsymbol{\rho}_d$ and $\boldsymbol{\rho}_c$ (otherwise the two quadratic terms in each square bracket become inconsistent). Namely,

$$\boldsymbol{d}^\dagger = \boldsymbol{d}, \quad \boldsymbol{c}^T = -\boldsymbol{c}. \tag{6}$$

Noting that

$$\operatorname{tr}\boldsymbol{\sigma}^2 \equiv \operatorname{tr}\left\{\begin{pmatrix} \boldsymbol{d} & \boldsymbol{c} \\ \boldsymbol{c}^\dagger & \boldsymbol{d}^T \end{pmatrix}^2\right\} = 2\operatorname{tr}\boldsymbol{d}^2 + 2\operatorname{tr}(\boldsymbol{c}\boldsymbol{c}^\dagger).$$

Then by defining

$$\boldsymbol{\Psi} \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} \boldsymbol{\psi} \\ \bar{\boldsymbol{\psi}} \end{pmatrix}, \quad \bar{\boldsymbol{\Psi}} \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} \bar{\boldsymbol{\psi}}^T & -\boldsymbol{\psi}^T \end{pmatrix}, \tag{7}$$

we have simply

$$\bar{\boldsymbol{\psi}}^T\boldsymbol{d}\boldsymbol{\psi} + \frac{1}{2}(\bar{\boldsymbol{\psi}}^T\boldsymbol{c}\bar{\boldsymbol{\psi}} + \boldsymbol{\psi}^T\boldsymbol{c}^\dagger\boldsymbol{\psi}) \equiv \bar{\boldsymbol{\Psi}}\boldsymbol{\sigma}\boldsymbol{\Psi}.$$

One should be aware that trick (8) is just a rewriting of the effective action. **In fact, we have NEVER doubled the underlying physical degree of freedom. So in sharp comparison of the original fields $\boldsymbol{\psi}$ and $\bar{\boldsymbol{\psi}}$, the "doubled" fields $\boldsymbol{\Psi}$ and $\bar{\boldsymbol{\Psi}}$ cannot be independent**

$$\boldsymbol{\Psi} \equiv i\sigma_2\bar{\boldsymbol{\Psi}}^T. \tag{8}$$

The above operation also do not affect the symmetry of the action. But with the help of (6), we can observe an emergent *time-reversal symmetry*[2] on the newly-defined space

$$\sigma_2\boldsymbol{\sigma}^*\sigma_2 = \sigma_2\begin{pmatrix} \boldsymbol{d}^T & -\boldsymbol{c}^\dagger \\ -\boldsymbol{c} & \boldsymbol{d} \end{pmatrix}\sigma_2 = \boldsymbol{\sigma}. \tag{9}$$

This discovery is not surprising because our microscopic interacting term (4) does not include the self-dynamics of disorders. It only describes the static scattering off impurities where there is no frequency transfer process (no term like ψ_{m-n}) but only momentum transfer processes, so is energy conserved (and equivalently time-reversal symmetric).

Go back to the partition function (we omit the electron-electron interaction for simplicity)

$$\begin{aligned}\langle\mathcal{Z}^R\rangle_{\text{dis}} &= \int \mathcal{D}\boldsymbol{\Psi}\mathcal{D}\boldsymbol{\sigma}\,\exp\left\{-\int \mathrm{d}\boldsymbol{r}\left[\frac{1}{4\gamma^2}\operatorname{tr}\boldsymbol{\sigma}^2 - \frac{1}{2}\bar{\boldsymbol{\Psi}}\left(-i\hat{\omega} - \frac{\nabla^2}{2m} - \mu_F - 2i\boldsymbol{\sigma}\right)\boldsymbol{\Psi}\right]\right\} \\ &= \int \mathcal{D}\boldsymbol{\sigma}\exp\left[-\frac{1}{4\gamma^2}\int \mathrm{d}\boldsymbol{r}\ \operatorname{tr}\boldsymbol{\sigma}^2 + \frac{1}{2}\operatorname{tr}\ln\hat{G}^{-1}[\boldsymbol{\sigma}]\right].\end{aligned} \tag{10}$$

III. CLASSIFICATION BASED ON ANOMALIES

IV. CLASSIFICATION BASED ON K-THOERY

[1] S. F. Edwards and P. W. Anderson, Journal of Physics F: Metal Physics **5**, 965 (1975).

[1] For example, $\operatorname{tr}(\boldsymbol{\rho}_c\boldsymbol{c}) \equiv \sum_{a,b}(\boldsymbol{\psi}\boldsymbol{\psi})_{ab}(\boldsymbol{c}^\dagger)_{ba} = -\sum_{a,b}\psi_b(\boldsymbol{c}^\dagger)_{ba}\psi_a$.

[2] Recall that a system possesses time-reversal symmetry if its Hamiltonian matrix satisfies $U_T^\dagger H^* U_T = H$.