Conformal Field Theory and Applications in Condensed Matter Physics

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This is a research note about applied CFT. For example, we will discuss the single-loop RG equation by OPE and the boundary CFT of bulk FQHE.

Contents

I. CONFORMAL FIELD THEORY IN $D > 2$

A. General Conformal Transformation

Let us consider a local field theory defined on a D-dimensional spacetime with *Euclidean* metric $g_{\mu\mu} \equiv \eta_{\mu\nu}$ and signature (p, q). A general diffeomorphism transforms the metric to

$$g_{\mu\nu} \mapsto \widetilde{g}_{\mu\nu}(\widetilde{x}) \equiv \frac{\partial x_\mu}{\partial \widetilde{x}_\alpha}\frac{\partial x_\alpha}{\partial \widetilde{x}_\beta} g_{\alpha\beta}.$$

Definition 1. (Conformal Group) Conformal group is the subgroup of diffeomorphism group leaving the metric tensor unchcanged up to a scaling factor

$$\widetilde{g}_{\mu\nu}(\widetilde{x}) = \Omega(x) g_{\mu\nu}(x). \tag{1}$$

Clearly Poincaré group ($\Omega \equiv 1$) is the subgroup of confromal group.

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The full generators of conformal transformation is obtained by solving the allowed infinitesimal transformation $\widetilde{x}_\mu \equiv x_\mu + \varepsilon_\mu + \mathcal{O}(\varepsilon^2)$ such that

$$\begin{aligned}\widetilde{g}_{\mu\nu} &\equiv (\delta^\alpha_\mu + \partial_\mu\varepsilon^\alpha)(\delta^\beta_\nu + \partial_\nu\varepsilon^\beta)g_{\alpha\beta} = g_{\mu\nu} + \partial_\mu\varepsilon^\alpha g_{\alpha\nu} + \partial_\nu\varepsilon^\beta g_{\mu\beta} + \mathcal{O}(\varepsilon^2)\\ &= g_{\mu\nu} + (\partial_\mu\varepsilon_\nu + \partial_\nu\varepsilon_\mu) + \mathcal{O}(\varepsilon^2)\end{aligned}$$

from the constriant (1)

$$(\partial_\mu\varepsilon_\nu + \partial_\nu\varepsilon_\mu) = (\Omega(x) - 1)g_{\mu\nu} \equiv f(x)g_{\mu\nu}. \tag{2}$$

Tracing both sides with $g^{\mu\nu}$ we get

$$2\partial^\mu\varepsilon_\mu = f(x)D.$$

Then can eliminate the newly defined scaling factor $f(x)$ by inserting back to equation (2)

$$\boxed{(\partial_\mu\varepsilon_\nu + \partial_\nu\varepsilon_\mu) = \frac{2}{D}(\partial\cdot\varepsilon)g_{\mu\nu}.} \tag{3}$$

Since equation (3) degenerates for 1d spacetime (so the theory is trivial), we will focus on the case when $D \geq 2$.

- For $D = 2$, equation (3) gives the celebrated *Cauchy-Riemann equation*

$$\partial_1\varepsilon_2 \equiv -\partial_2\varepsilon_1, \quad \partial_1\varepsilon_1 \equiv \partial_2\varepsilon_2. \tag{4}$$

  It is special and we will leave the discuss to the next chapter.

- For $D > 2$, it is helpful to re-arrange (3) into a more explicit form. Applying ∂^ν with ∂_ν on (3) gives

$$\partial_\mu\partial_\nu(\partial\cdot\varepsilon) + \partial^2\partial_\nu\varepsilon_\mu = \frac{2}{D}\partial_\mu\partial_\nu(\partial\cdot\varepsilon). \tag{5}$$

  Similarly for the twice derivatives of ∂_μ and ∂^μ

$$\partial^2\partial_\mu\varepsilon_\nu + \partial_\nu\partial_\mu(\partial\cdot\varepsilon) = \frac{2}{D}\partial_\nu\partial_\mu(\partial\cdot\varepsilon). \tag{6}$$

  Then by adding up (5) and (6) and inserting (3) to replace $(\partial_\mu\varepsilon_\nu + \partial_\nu\varepsilon_\mu)$, we come to the simple differential equation of $(\partial\cdot\varepsilon)$

$$2\partial_\mu\partial_\nu(\partial\cdot\varepsilon) + \partial^2\left(\frac{2}{D}(\partial\cdot\varepsilon)g_{\mu\nu}\right) = \frac{4}{D}\partial_\mu\partial_\nu(\partial\cdot\varepsilon) \implies (g_{\mu\nu}\partial^2 + (D-2)\partial_\mu\partial_\nu)(\partial\cdot\varepsilon) = 0 \implies (D-1)\partial^2(\partial\cdot\varepsilon) = 0. \tag{7}$$

B. Conformal Transformation in $D > 2$

For $D > 2$ the constraint (7) implies that the infinitesimal ε_μ must be at most *quadratic* in coordinate

$$\varepsilon_\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho$$

with the symmetric $c_{\mu\nu\rho} \equiv c_{\mu\rho\nu}$.

1) Clearly $\varepsilon_\mu = a_\mu$ represents the (infinitesimal) **space-time translation** $x'_\mu \mapsto x_\mu + a_\mu$ as ususal.

2) By inserting $\varepsilon_\mu = b_{\mu\nu}x^\nu$ back into (3) we have

$$b_{\mu\nu} + b_{\nu\mu} = \frac{2}{D}(b_{\alpha\beta}\partial^\alpha x^\beta)g_{\mu\nu} = \frac{2}{D}b_\alpha{}^\alpha g_{\mu\nu}.$$

   Separating $b_{\mu\nu}$ into symmetric part $b^s_{\mu\nu} \equiv b^s_{\mu\nu}$ and anti-symmetric part $b^a_{\mu\nu} \equiv -b^a_{\mu\nu}$, then the anti-symmetric part represents the familiar (infinitesimal) **space-time rotation** $b^a_{\mu\nu} \equiv \omega_{\mu\nu}$, while the symmetric part $b^s_{\mu\nu} = \frac{1}{D}(b^s)_\alpha{}^\alpha g_{\mu\nu}$ represents the (infinitesimal) **space-time dilation** since

$$x'_\mu = x_\mu + (b^s)_{\mu\nu}x^\nu = \left(1 + \frac{1}{D}(b^s)_\alpha{}^\alpha\right)x_\mu \equiv \lambda x_\mu.$$

Translation	$P^\alpha = -i\dfrac{\delta}{\delta a_\alpha}(x_\mu + a_\mu)\partial^\mu = -i\partial^\alpha$
Dilation	$D = -i\dfrac{\delta}{\delta\lambda}\lambda x_\mu\partial^\mu = -ix^\mu\partial_\mu$
Rotation	$L^{\alpha\beta} = -i\dfrac{\delta}{\delta\omega_{\alpha\beta}}(x_\mu + \omega_{\mu\nu}x^\nu)\partial^\mu = -i(\delta_{\alpha\mu}\delta_{\beta\nu} - \delta_{\alpha\nu}\delta_{\beta\mu})x^\nu\partial^\mu = i(x^\alpha\partial^\beta - x^\beta\partial^\alpha)$
SCT	$K^\alpha = -i\dfrac{\delta}{\delta b_\alpha}(x_\mu + 2(b\cdot x)x_\mu - b_\mu x^2)\partial^\mu = -i(2x^\alpha(x\cdot\partial) - x^2\partial^\alpha)$

TABLE I: Corresponding four generators of the infinitesimal conformal transformations.

3) By inserting $\varepsilon_\mu = c_{\mu\alpha\beta}x^\alpha x^\beta$ back into (3) we have

$$c_{\mu\nu\beta}x^\beta + c_{\nu\mu\beta}x^\beta = \frac{2}{D}c^\alpha{}_{\alpha\beta}x^\beta g_{\mu\nu}.$$

To express $c_{\mu\nu\beta}$ explicitly, we have to cycle all the indices, summing and subtracting them with symmetric condition on the latter two indices of $c_{\mu\nu\beta}$, which yields the **special conformation transformation** (SCT)

$$c_{\mu\nu\beta} = \frac{1}{D}(c^\alpha{}_{\alpha\beta}g_{\mu\nu} + c^\alpha{}_{\alpha\nu}g_{\mu\beta} - c^\alpha{}_{\alpha\mu}g_{\nu\beta}) \equiv b_\beta g_{\mu\nu} + b_\nu g_{\mu\beta} - b_\mu g_{\nu\beta}$$

with $b_\beta \equiv c^\alpha{}_{\alpha\beta}/D$. So the coordinate transforms (infinitesimally) as

$$x'_\mu = x_\mu + 2(b\cdot x)x_\mu - b_\mu x^2.$$

The full four generators corresponding to these *infinitesimal* conformal transformations can be directly read out as in table I because on the one side, the infinitesimal transformation parameterized by ω_a can be written as $x'_\mu(x) \simeq x_\mu + \dfrac{\delta x_\mu}{\delta\omega_a}\omega_a = \left(1 + \dfrac{\delta x^\nu}{\delta\omega_a}\omega_a\partial_\nu\right)x_\mu$, on the other side by definition of the generator $x'_\mu(x) \equiv e^{iG^a\omega_a}x_\mu \simeq (1+iG^a\omega_a)x_\mu$. After straight-forward but tedious calculations, we can write down the full conformal algebra as following:

$$\begin{aligned}
[D, D] &= 0, \quad [D, P_\mu] = iP_\mu, \quad [D, K_\mu] = -iK_\mu, \quad [D, L_{\mu\nu}] = 0,\\
[P_\mu, P_\nu] &= 0, \quad [K_\mu, K_\nu] = 0, \quad [K_\mu, P_\nu] = 2i(g_{\mu\nu}D - L_{\mu\nu}),\\
[L_{\mu\nu}, L_{\rho\sigma}] &= -i(L_{\mu\rho}g_{\nu\sigma} - L_{\mu\sigma}g_{\nu\rho} - L_{\nu\rho}g_{\mu\sigma} + L_{\nu\sigma}g_{\mu\rho}),\\
[L_{\mu\nu}, P_\rho] &= -i(g_{\mu\rho}P_\nu - g_{\nu\rho}P_\mu), \quad [L_{\mu\nu}, K_\rho] = -i(g_{\mu\rho}K_\nu - g_{\nu\rho}K_\mu)
\end{aligned} \tag{8}$$

The four *finite* conformal transformations (namely, the flow) can also be obtained by finding out the integral curve of these generators

$$\dot{x}^\mu = iG^a\omega_a x^\mu.$$

Let us take the treaky SCT as an example. The nonlinear ODE

$$\dot{x}^\mu = 2(b\cdot x)x_\mu - b_\mu x^2$$

is solved by a certain change of variable $I : x^\mu \mapsto y^\mu \equiv x^\mu/x^2$ such that

$$\frac{x^\mu(t)}{x^2(t)} = \frac{x^\mu(0)}{x^2(0)} - tb, \quad \text{or} \quad x^\mu(t) = \frac{x^\mu - tb^\mu x^2}{1 - 2tb\cdot x + (tb)^2x^2}. \tag{9}$$

The above change of variable is nothing but the inversion transformation satisfying $I^2 \equiv \mathbb{1}$. So finite SCT (8) can be understood as the inversion with translation and inversion again. The full results are listed as in table II.

C. Representation of Conformal Group

The Lie algebra of conformal group in $d > 2$ is already obtained in (8). But to investigate their action on quantum fields, we also need the information of the representation of such Lie algebra on the Hilbert space. A similar problem is encountered in $d = 4$ Lorentz invariant QFT, where we solve it by fixing the standard momentum and constructing the *little group representation* [1]. The full representation is recovered by spacetime boosting.

Let us start with the conformal transformation leaving the origin of field unchanged. The full transformation can be obtained by spatial translation. A familiar transformation is

Translation	$x'_\mu = x_\mu + a_\mu$
Dilation	$x'_\mu = x_\mu + \frac{2}{D}(b^s)^\alpha{}_\alpha g_{\mu\nu}x^\nu \equiv \lambda x_\mu$
Rotation	$x'_\mu = x_\mu + \omega_{\mu\nu}x^\nu$
SCT	$x'_\mu = x_\mu + (b_\beta g_{\mu\nu} + b_\nu g_{\mu\beta} - b_\mu g_{\nu\beta})x^\nu x^\beta \equiv x_\mu + 2(b\cdot x)x_\mu - b_\mu x^2$

TABLE II: Four finite conformal transformations.

D. Energy Momentum Tensor

II. 2D CONFORMAL FIELD THEORY

A. Conformal Transformation in $D = 2$

For $D = 2$, the constraint equation for infinitesimal conformal transformation reduces to beautiful *Cauchy-Riemann equation* (4) appearing in complex analysis. So it is natural to introduce the complex variable $z \equiv x_1 + ix_2$ and $\bar{z} \equiv x_1 - ix_2$ with the assignment (such that $\partial z = \bar{\partial}\bar{z} = 1$)

$$\partial \equiv \partial_z = \frac{1}{2}(\partial_1 - i\partial_2), \quad \bar{\partial} \equiv \partial_{\bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2)$$

and also the complex function $\varepsilon(z,\bar{z}) \equiv (\varepsilon_1 + i\varepsilon_2)$, $\bar{\varepsilon}(z,\bar{z}) = (\varepsilon_1 - i\varepsilon_2)$, to decouple the Cauchy-Riemann equation into *holomorphic* and *anti-holomorphic* modes (or *left-movers* and *right-movers*)

$$\bar{\partial}\varepsilon = 0 \implies \varepsilon(z,\bar{z}) = \varepsilon(z),$$
$$\partial\bar{\varepsilon} = 0 \implies \bar{\varepsilon}(z,\bar{z}) = \bar{\varepsilon}(\bar{z}).$$

Thus two dimensional conformal transformation coincides with the analytic coordinate transformation

$$z \mapsto f(z), \quad \bar{z} \mapsto \bar{f}(\bar{z}).$$

Unlike the four kinds of conformal transformation in $D > 2$, here by the analyticity of $\varepsilon(z)$ and $\bar{\varepsilon}(\bar{z})$, we can always expand them into Laurent series[1]

$$\varepsilon(z) = \sum_{n=-\infty}^{\infty} \alpha_n z^{n+1}, \quad \bar{\varepsilon}(\bar{z}) = \sum_{m=-\infty}^{\infty} \beta_m \bar{z}^{m+1}$$

with *infinite* number of infinitesimal generators

$$\ell_n \equiv -i\frac{\delta\varepsilon_n}{\delta\alpha_n}\partial_z = -iz^{n+1}\partial_z, \quad \bar{\ell}_m \equiv -i\frac{\delta\bar{\varepsilon}_m}{\delta\beta_m}\partial_{\bar{z}} = -i\bar{z}^{n+1}\partial_{\bar{z}}. \tag{10}$$

They can be easily shown to satisfy the *Witt algebra* $\mathcal{W}$

$$[\ell_m, \ell_n] = (m-n)\ell_{m+n}, \quad [\bar{\ell}_m, \bar{\ell}_n] = (m-n)\ell_n, \quad [\ell_m, \bar{\ell}_n] = 0, \tag{11}$$

which is clearly the direct sum of two independent subalgebras $\mathcal{W} = \mathcal{A} \bigoplus \bar{\mathcal{A}}$. So let us narrow our discusion to one branch of it.

Claim 1. **Although the dimension of $\mathcal{A}$ is infinite, not all the generators are *globally* well defined.**

$\triangleright$ To reveal this, we can analytically continue the original domain of arbitrary analytic functions

$$\mathrm{Dom}(f) = \{(z,\bar{z})\} = \{(x_1,x_2)\} = \mathbb{R}^2$$

[1] We write the exponent as $n+1$ instead of n just for a convention regarding the components that are well-defined locally. This will not do harm to the structure of Witt Algebra.

to $\mathbb{C}^2$ with the "physical constraint" $\bar{z} \equiv z^*$. Then the domain of each branch of conformal transformation, or equivalently analytic coordinate transformation, is now extended form $\mathbb{R}$ to $S^2 = \mathbb{C} \bigcup \infty$. All holomorphic conformal transformations are generated from the (tagent) vector field

$$X \equiv X^n \partial_n = \sum_n \alpha_n \ell_n \equiv \sum_n (-i)\alpha_n z^{n+1} \frac{\partial}{\partial z^n}.$$

Clearly the non-singularity of X as $z \to 0$ demands $n \geq -1$. But to impose the non-singularity of X as $z \to \infty$, we have to change the variable $z \mapsto 1/w$ because the tangent field is only defined in the neighbor of the origin. Then the tagent vector field becomes

$$\widetilde{X} \equiv \widetilde{X}^n \widetilde{\partial}_n = \sum_n \frac{\partial w}{\partial z} X^n \frac{\partial}{\partial w^n} = \sum_n (-w^2)(-i)\alpha_n \left(\frac{1}{w}\right)^{n+1} \frac{\partial}{\partial w^n} = \sum_n \alpha_n \left(\frac{i}{w}\right)^{n-1} \frac{\partial}{\partial w^n}.$$

This time the non-singularity of $\widetilde{X}$ as $w \to \infty$ requires $n \leq 1$. Plus the similar argument from another branch, we conclude that **only the conformal transformations generated by $\{\ell_{-1}, \ell_0, \ell_1\} \bigcup \{\bar{\ell}_{-1}, \bar{\ell}_0, \bar{\ell}_1\}$ are globally well-defined**. □

B. Radial Quantization

III. TRACE ANOMALY

IV. CATEGORIZATION AND CLASSIFICATION OF TOPOLOGICAL PHASES

V. MODULAR INVARIANCE, MODULI SPACE, AND FQHE

VI. RELATION TO ENTANGLEMENT ENTROPY

VII. APPENDIX

A. Infinitesimal Symmetric Transformation

Geometrically, a classical tensor field $\mathbf{\Phi}(\boldsymbol{x})$ is a section of tensor bundle $E \xrightarrow{\pi} M$. And because the base space M is assumed to be Euclidean (so *contractible*), the bundle is trivial and the *Eular class* is zero[2]. So section $\mathbf{\Phi} : M \to E$ can be safely extended to a global one[3].

A physical symmetry transformation is a coordinate transformation leaving the (global) section unchanged. Denoting such group action on the two copies of base manifolds as $g : M \to M$, $\boldsymbol{x} \mapsto g \cdot \boldsymbol{x}$, we can naturally induce a lifted group action on the bundle space as well

$$\mathbf{\Phi}'(\boldsymbol{x}') \equiv \mathscr{F}(\mathbf{\Phi}(\boldsymbol{x})). \tag{12}$$

But since the global section is not affected, in the view point of the new charts, up to a transformation on the typical fiber (writing explicitly as a representation $\rho : G \to GL(n, \mathbb{R})$), each point on the bundle looks exactly as that of the original charts, but with an *inverse* coordinate transformation, i.e.,

$$\mathbf{\Phi}'(\boldsymbol{x}') = \rho(g)\mathbf{\Phi}(g^{-1} \cdot \boldsymbol{x}). \tag{13}$$

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[2] The Eular class of $2l$-dimensional manifold M is defined as (it vanishes for odd dimensions)

$$e(M) := \mathrm{Pf}\left(\frac{\mathcal{R}}{2\pi}\right) \equiv \frac{(-1)^l}{(4\pi)^l l!} \sum_{\text{Perm } P} \mathrm{sgn}(P) \mathcal{R}_{P(1)P(2)} \cdots \mathcal{R}_{P(2l-1)P(2l)},$$

where $\mathcal{R}$ is the curvature 2-form. $\mathbb{R}^4$ admits a flat connection, so $\mathcal{R} = 0$ and $e(\mathbb{R}^4) = 0$.

[3] Eular class is the only obstruction for a vector bundle to define a non-trivial global sections

For infinitesimal transformation parameterized by some vector $\{\omega_a\}$

$$\widetilde{x}^{\mu} = x^{\mu} + \frac{\delta x^{\mu}}{\delta \omega_a}\omega_a + \mathcal{O}(\omega^2),$$

identity (12) tells

$$\boldsymbol{\Phi'}(\widetilde{\boldsymbol{x}'}) \equiv \mathscr{F}(\boldsymbol{\Phi}(\widetilde{\boldsymbol{x}})) = \boldsymbol{\Phi}(\boldsymbol{x}) + \frac{\delta \mathscr{F}}{\delta \omega_a}\omega_a + \mathcal{O}(\omega^2) \tag{14}$$

[1] S. Weinberg, *The quantum theory of fields*, vol. 1 (Cambridge university press, 1995).