

# Hydrodynamic Methods in Condensed Matter

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In this note I will review modern methods of hydrodynamics in condensed matter systems (unlike old one like superfluids and spin waves with ordered phase), from quantum critical Nernst effects to fractional quantum Hall system

云行雨施，品物流形。

——「易经·象」

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## I. HYDRODYNAMICS IN SYMMETRY-BREAKING PHASE — SPIN WAVES

### A. $S = 1/2$ XXZ Model

As a practice, let us consider the  $S = 1/2$  XXZ model in a uniform magnetic field  $H_z$

$$\mathcal{H} = \mathcal{H}_{XXZ} - H_z \sum_i S_i^z \equiv - \sum_{\langle ij \rangle} J_{\perp} (S_i^x S_j^x + S_i^y S_j^y) - \sum_{\langle ij \rangle} J_z S_i^z S_j^z - H_z \sum_i S_i^z. \quad (1)$$

The nature of the ordered phases are determined by the sign and relative magnitude of  $J_{\perp}$  and  $J_z$ . Let me list as following [1]:

- 1) *Ferromagnetic Heisenberg Model*  $J_{\perp} = J_z = J > 0$ . The order parameter is a uniform magnetization  $\mathbf{m}$ .
- 2) *Antiferromagnetic Heisenberg Model*  $J_{\perp} = J_z = J < 0$ . The order parameter (Néel order) is a staggered magnetization  $\mathbf{N}$ .
- 3) *Planar ferromagnet*  $J_{\perp} > |J_z| > 0$ . The energy is minimized by parallel alignment of spins in the  $xy$ -plane. So the order parameter is  $\mathbf{m} = (m_x, m_y)$ , or equivalently the complex field  $m_{\perp}(\mathbf{r})e^{i\varphi(\mathbf{r})} \equiv m_x(\mathbf{r}) + im_y(\mathbf{r})$ .

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- 4) *Planar antiferromagnet*  $J_\perp < -|J_z| < 0$ . The energy is minimized by anti-parallel alignment of spins in the  $xy$ -plane. So the order parameter is the two-component staggered magnetization  $\mathbf{N} \equiv (N_x, N_y)$ .
- 5) *Ferromagnetic/Antiferromagnetic Ising model*  $|J_z| < |J_\perp|$ .

Clearly for isotropic models 1) and 2), the entire spin operator  $S \equiv \sum_i \mathbf{S}_i$  commutes with the Hamiltonian so is conserved. While for anisotropic models 3), 4), and 5), only  $S^z \equiv \sum_i S_i^z$  is conserved. Let us start with the spin wave theory in planar ferromagnets.

## B. Planar Ferromagnets

Operator  $S^z$  is conserved in planar ferromagnets, so for the coarse-grained  $z$ -component magnetization  $M_z \equiv \langle S_i^z \rangle$ , or magnetization density  $m_z(\mathbf{r}) \equiv M_z/V$ , we have

$$\partial_t m_z + \nabla \cdot \mathbf{j}^{m_z} = 0, \quad (2)$$

where we introduce the spin current density  $\mathbf{j}^{m_z}$ . Another conserved quantity is the total energy density  $E \equiv \langle \mathcal{H}_{XXZ} \rangle$ , or energy density  $\varepsilon$ , satisfying

$$\partial_t \varepsilon + \nabla \cdot \mathbf{j}^\varepsilon = 0. \quad (3)$$

But given  $M_z$  and  $E$ , one still cannot determine the spin configuration of ordered states in planar ferromagnets. In fact, as is mentioned above, one must also specify the direction of alignment of perpendicular magnetization (labeled by  $\varphi(\mathbf{r})$ ), which is locally defined as

$$m_x(\mathbf{r}) + im_y(\mathbf{r}) = m_\perp(\mathbf{r})e^{i\varphi(\mathbf{r})}.$$

**In hydrodynamic regime, relevant hydro-variables are those whose long-wavelength variations vary slowly in time compared with the characteristic microscopic relaxation time of the scattering processes.** In symmetry breaking phase, such variables are of two classes:

- densities of conserved variables (here is  $m_z(\mathbf{r})$  and  $\varepsilon(\mathbf{r})$ );
- symmetry-breaking elastic variables (here is  $\varphi(\mathbf{r})$ ).

Thus we *assume* that **each point of the system reaches almost thermodynamic equilibrium at each instant of time, so that the system is *completely* determined by the hydro-variables, even though they vary in time.** Particularly, the above unspecified current  $\mathbf{j}^\varepsilon$  and  $\mathbf{j}^{m_z}$  should be functional of these hydro-variables. Ditto for the symmetry-breaking field if we introduce a scalar valued functional  $\psi$  of hydro-variables

$$\partial_t \varphi(\mathbf{r}) + \psi(\mathbf{r}) = 0. \quad (4)$$

Noting that it is the fluctuation of  $\nabla\varphi$  rather than  $\varphi$  that contributes to physical observables in the same order as  $\varepsilon$  and  $m_z$ . So we will take  $\mathbf{v}(\mathbf{r}) \equiv \nabla\varphi(\mathbf{r})$  as a replacement of fundamental hydro-variables ( $\mathbf{v}$  is in anolagous of superfluid velocity  $\mathbf{v}_s$  in liquid helium), satisfying

$$\partial_t \mathbf{v}(\mathbf{r}) + \nabla\psi(\mathbf{r}) = 0. \quad (5)$$

Equation (2), (3), and (5) consist of the complete “conservation law” of our system.

## C. Constitutive Relations

Equations relating coarse-grained current densities with hydro-variables are called *constitutive relations*. Introducing the conjugate field  $\mathbf{x}$  of  $\mathbf{v}$ , and  $h$  of  $m_z$ , as generalized forces, the first law of thermodynamics (in true thermodynamic equilibrium) tells

$$T ds \equiv d\varepsilon - h dm_z - \mathbf{x} \cdot d\mathbf{v}, \quad (6)$$

where  $h \equiv -T\partial s/\partial m_z|_S$  and  $\mathbf{x} \equiv -T\partial s/\partial \mathbf{v}|_S$ .

### 1. Zeroth Order (Non-dissipative)

In this subsection, we will **use (assume) the conservation of local entropy to obtain the *non-dissipative* part of “current operators” introduced above.**

To the zeroth order of fluctuations, i.e., no spatial derivatives of hydro-variables, the most general form of current operator we can write is

$$\mathbf{j}^\varepsilon \equiv A(\mathbf{r})\mathbf{v}(\mathbf{r}), \quad \mathbf{j}^{m_z} \equiv B(\mathbf{r})\mathbf{v}(\mathbf{r}), \quad \psi \equiv C(\mathbf{r}), \quad (7)$$

where coefficients  $A, B$  and  $C$  are scalar functional of  $\varepsilon(\mathbf{r})$  and  $m_z(\mathbf{r})$  waiting to be determined. In general, coefficients  $A$  and  $B$  may be even chosen as rank-2 tensors of functionals of  $\varepsilon$  and  $m_z$ , such that the two vector-valued current operators  $\mathbf{j}^\varepsilon$  and  $\mathbf{j}^{m_z}$  are not in the same direction as  $\mathbf{v}$ . **But here we shall assume lattice *cubic symmetry* [2] to forbid such possibility.** Similarly, one can contract vector  $\mathbf{v}$  with a rank-2 tensor to form a scalar functional allowed in the expression of  $\psi$ . **But we shall assume *reflection symmetry* to forbid such terms for convenience.**

Fortunately, as we will see below,  $C(\mathbf{r})$  can be obtained in advance without entropy argument [1, 2].

Our aim (spin wave mode) is to find a *stationary state* (in comparison with true thermodynamic equilibrium) with a non-vanishing constant precession rate from (4) that  $d\varphi/dt = -\psi_0$ , which is determined by conserved hydro-variables  $\varepsilon, m_z$ , and  $\mathbf{v}$  if it is nonzero.

Let us assume the existence of such stationary state *without* external magnetic field, and turn on  $H_z$ . The time evolution of  $m_\perp(t, H_z) \equiv \langle S_i^+(t) \rangle / V$  is now dominated by the Hamiltonian  $\mathcal{H} \equiv \mathcal{H}_{XXZ} + \mathcal{H}_{ext}$ . And because  $\mathcal{H}_{ext} \equiv -H_z \sum_i S_i^z$  commutes with  $\mathcal{H}_{XXZ}$ ,

$$m_z(t, H_z) = \frac{1}{V} \left\langle e^{i\mathcal{H}t} S_i^+(0) e^{-i\mathcal{H}t} \right\rangle = \frac{1}{V} \left\langle e^{i\mathcal{H}_{XXZ}t} e^{-iH_z \sum_i S_i^z t} S_i^+(0) e^{iH_z \sum_i S_i^z t} e^{-i\mathcal{H}_{XXZ}t} \right\rangle = e^{-iH_z t} m_z(t, 0)$$

gives

$$\frac{d\varphi}{dt} = -(\psi_0(\varepsilon, m) + H_z).$$

However, in *true* thermodynamic equilibrium (with non-vanishing  $H_z$ ), where the system energy  $E - M_z H_z$  is minimized so that

$$\frac{\partial E}{\partial M_z} = \frac{\partial \varepsilon}{\partial m_z} = H_z, \quad (8)$$

the precession rate  $d\varphi/dt$  must be zero, otherwise the rotating  $m_\perp$  will radiate and lose energy [2]. Thus  $\psi_0$  coincides with  $H_z$  in *true* equilibrium. On the other hand, the first law of thermodynamics (6) tells us the left hand side of (8) is nothing but conjugate field  $h(\varepsilon, m_z) \equiv \frac{\partial \varepsilon}{\partial m_z}$ . Therefore, we conclude for general *non-dissipative* stationary state

$$\frac{d\varphi}{dt} = -(h - H_z), \quad (9)$$

or

$$C(\mathbf{r}) = H_z - h(\varepsilon(\mathbf{r}), m_z(\mathbf{r})). \quad (10)$$

The left work is to determine  $A(\mathbf{r})$  and  $B(\mathbf{r})$  in (7) with the help of entropy conservation (at zeroth order).

Hydrodynamic assumption ensures local entropy density to be the functional of hydro-variables  $s = s(\varepsilon, m_z, \mathbf{v})$ . And at each slice of time, each point of the system is assumed to reach local thermodynamic equilibrium. Therefore we can expand entropy density around its equilibrium value  $s_0$  (which must be the maximum of  $s$  due to the second law of thermodynamics) that

$$s \simeq s_0(\varepsilon) - \frac{1}{2T} \chi_s^{-1} m_z^2 - \frac{\rho_s}{2T} v^2, \quad (11)$$

where temperature  $T^{-1} \equiv \partial s_0 / \partial \varepsilon$  is well-defined in equilibrium and inserted by convention. Magnetic susceptibility  $\chi_s$  and the magnetic version of “superfluid density”  $\rho_s$  may also be functional of hydro-variables, but up to the second order of disturbance, we can safely treat them as constants<sup>1</sup>. Therefore the two conjugate fields we introduce above takes the form of  $h \equiv -T \partial s / \partial m_z = \chi_s^{-1} m_z$  and  $\mathbf{x} \equiv -T \partial s / \partial \mathbf{v} = \rho_s \mathbf{v}$ .

<sup>1</sup> Apparently  $\chi_s$  and  $\rho_s$  must be nonnegative for stability of the system.

Taking time derivative of the first law of thermodynamics and substituting conservation laws (3), (4), (5) and the zeroth order of non-dissipative currents (7) and (10), we have

$$\begin{aligned} T \frac{\partial s}{\partial t} &= \frac{\partial \varepsilon}{\partial t} - h \frac{\partial m_z}{\partial t} - \mathbf{x} \cdot \frac{\partial \mathbf{v}}{\partial t} = -\nabla \cdot \mathbf{j}^\varepsilon + h \nabla \cdot \mathbf{j}^{m_z} - \mathbf{x} \cdot \nabla (h - H_z) \\ &= -\nabla \cdot (A\mathbf{v}) + h \nabla \cdot (B\mathbf{v}) - \mathbf{x} \cdot \nabla h \\ &= -\nabla \cdot \left( (A - Bh)\mathbf{v} \right) - (B\mathbf{v} + \mathbf{x}) \cdot \nabla h. \end{aligned} \quad (12)$$

Writing the *heat current*  $\mathbf{Q} \equiv (A - Bh)\mathbf{v}$  and using the identity

$$\nabla \cdot \mathbf{Q} \equiv T \nabla \cdot \left( \frac{\mathbf{Q}}{T} \right) + \mathbf{Q} \cdot \left( \frac{\nabla T}{T} \right),$$

Equation (12) becomes

$$T \frac{ds}{dt} \equiv T \left( \frac{\partial s}{\partial t} + \nabla \cdot \left( \frac{\mathbf{Q}}{T} \right) \right) = -\mathbf{Q} \cdot \left( \frac{\nabla T}{T} \right) - (B\mathbf{v} + \mathbf{x}) \cdot \nabla h \quad (13)$$

if we identify  $\mathbf{Q}$  as the *entropy current density*. In the absence of dissipation, as is assumed to the zeroth order disturbance, entropy is conserved  $\frac{ds}{dt} = 0$ . Thus

$$\begin{cases} \mathbf{Q} \equiv (A - Bh)\mathbf{v} = 0 \\ B\mathbf{v} + \mathbf{x} = 0 \end{cases} \implies \begin{cases} A\mathbf{v} = -\rho_s \mathbf{x} \\ B\mathbf{v} = -\mathbf{x} \end{cases}. \quad (14)$$

Inserting the expression of  $h$  and  $\mathbf{v}$ , finally we come to the *non-dissipative* linearized equation of motion up to the zeroth order of gradients of hydro-variables

$$\partial_t \varepsilon = \frac{\rho_s}{\chi_s} \nabla \cdot (m_z \mathbf{v}) \quad (15)$$

$$\partial_t m_z = \rho_s \nabla \cdot \mathbf{v} \quad (16)$$

$$\partial_t \mathbf{v} = \frac{1}{\chi_s} \nabla m_z. \quad (17)$$

Taking the time derivative of (16) and substituting (17), one immediately have

$$\frac{\partial^2 m_z}{\partial t^2} = -\frac{\rho_s}{\chi_s} \nabla^2 m_z, \quad (18)$$

which predicts a *undamped spin-wave mode*  $\omega = \pm ck$  with  $c \equiv (\rho_s/\chi_s)^{1/2}$ .

## 2. First Order (Dissipative)

To the first order gradients of hydro-variables, however, we will use **the positivity of local entropy production** to constrain the form of *dissipative* part of currents

$$\mathbf{j}^\varepsilon \equiv -\rho_s \mathbf{x} + \mathbf{j}^{\varepsilon'}, \quad \mathbf{j}^{m_z} \equiv -\mathbf{x} + \mathbf{j}^{m_z'}, \quad \psi \equiv H_z - h + \psi'. \quad (19)$$

Here the zeroth-order result has been used. Similar to the procedure we have done above, introducing the first-order *heat current*  $\mathbf{Q}' \equiv \mathbf{j}^{\varepsilon'} - h \mathbf{j}^{m_z'} + \rho_s \mathbf{v} \psi'$ , the local entropy production now takes the form of

$$T \frac{ds}{dt} \equiv T \left( \frac{\partial s}{\partial t} + \nabla \cdot \mathbf{Q}' \right) = -\mathbf{Q}' \cdot \left( \frac{\nabla T}{T} \right) - \mathbf{j}^{m_z'} \cdot \nabla h - \psi' \nabla \cdot \mathbf{x}. \quad (20)$$

The most general constitutive relations up to the *first order* we can construct are

$$\begin{aligned} \mathbf{Q}' &= -K_{11} \nabla T - K_{12} \nabla h \\ \mathbf{j}^{m_z'} &= -K_{21} \nabla T - K_{22} \nabla h \end{aligned}$$

$$\psi' = K_{33}\rho_s \nabla \cdot \mathbf{v}.$$

In general both  $\mathbf{Q}'$  and  $\mathbf{j}^{m_z'}$  may also contain terms proportional to spacial fluctuation of  $\mathbf{v}$  like  $\nabla \times \mathbf{v}$ . But the definition  $\mathbf{v} \equiv \nabla \varphi$  forbids such terms. There may also be possibility that  $\mathbf{Q}', \mathbf{j}^{m_z'}$  be propotional to the contraction of the gradient of  $T$  and  $h$  with a rank-2 tensor, but the **cubic symmetry** prohibit such terms. In addition,  $\psi'$  may also be propotional to terms like  $\mathbf{n} \cdot \nabla T$  and  $\mathbf{n} \cdot \nabla h$  with some specific vector, but again the **cubic symmetry** prohibit them.

Moreover, since we are interested in the fluctuation near true thermodynamic equilibrium where  $M_z = 0$ , **whatever the direction of time arrow is, the entropy production  $\frac{ds}{dt}$  by assumption, will never alters its sign but keep positive**. Therefore **all term on the RHS of (20) must be even under time-reversal transformation**. In light of the fact that  $T$  is time-reversal even and  $h$  is time-reversal odd, both off-diagonal couplings  $K_{12}$  and  $K_{21}$  should be zero, leaving

$$\mathbf{Q}' = -\kappa \nabla T, \quad \mathbf{j}^{m_z'} = -K_{22} \nabla h, \quad \psi' = -K_{33} \rho_s \nabla \cdot \mathbf{v}, \quad (21)$$

where we interpret  $K_{11}$  as the *thermal conductivity*  $\kappa$ .

To make equations (21) close, let us restrict ourselves to the case where magnetization  $M_z = 0$ . This is appropriate for fluctuations about true thermodynamic equilibrium without external magnetic fields. In this case, we have

$$dT = C^{-1} d\varepsilon$$

if we introduce specific heat  $C$ . Inserting back the expression of dissipative currents to conservation laws, we come to the *dissipative* linearized<sup>2</sup> equation of motion up to the first-order fluctuation of hydro-variables

$$\partial_t \varepsilon = \nabla \cdot \left( \kappa \nabla T + K_{22} h \nabla h - K_{33} \rho_s^2 \mathbf{v} \nabla \cdot \mathbf{v} \right) \simeq \kappa C^{-1} \nabla^2 \varepsilon \quad (22)$$

$$\partial_t m_z = \rho_s \nabla \cdot \mathbf{v} + \chi_s^{-1} K_{22} \nabla^2 m_z \quad (23)$$

$$\partial_t \mathbf{v} = \chi_s^{-1} \nabla m_z + K_{33} \rho_s \nabla (\nabla \cdot \mathbf{v}). \quad (24)$$

Clearly solution of (22) yields a heat *diffusion mode*

$$\omega_\varepsilon(\mathbf{k}) = -i\kappa C^{-1} k^2 =: -iD_T k^2. \quad (25)$$

Another *damped spin-wave mode* from coupled equations (23) and (24) can be obtained by diagonalizing the coefficients matrix in momentum space and expanding up to  $k^2$ , giving

$$\omega_\pm(\mathbf{k}) = \pm ck - \frac{1}{2} i D k^2, \quad (26)$$

where  $D \equiv \chi_s^{-1} K_{22} + \rho_s K_{33}$ .

## D. Linear Response

## II. TRANSPORT NEAR QUANTUM CRITICALITY

In this section, we focus on *Lorentz-invariant* quantum critical points in the *hydrodynamics region* that external electromagnetic frequencies satisfy  $\hbar\omega \ll k_B T$ . This condition is widely (actually, almost all) performed in experiments but mismatch the assumption to many theoretical calculation as well as numerical simulation in, for example, DC conductivity near superfluid-insulator phase transition [3].

**The fundamental ingredients of hydrodynamic analysis are the conserved quantities and their equations of motion.** But before preceeding, let me pause and comment on the old hydrodynamics of Lifshitz superfluids and spin-waves [2, 4]. **In hydrodynamics region,  $k_B T / \hbar$  naturally plays the role of low-energy frequency characterizing the relation time that we are interested in, while in old prescription the relaxation time just diverges (so the energy scale may not be correct) [5].**

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<sup>2</sup> By linearity, we mean the equation of motion is expanded to the first order of hydro-variables.

### A. General Setup

Our system subject to external magnetic field for transport study. Invariance of *gauge transformation*  $A_\mu \mapsto A_\mu + \partial_\mu f$  and *diffeomorphism transformation*  $x^\mu \mapsto x_\mu + \xi_\mu$  of action gives<sup>3</sup>

$$\partial_\mu J^\mu = 0, \quad (27)$$

$$\partial_\mu T^{\mu\nu} = F^{\mu\nu} J_\mu. \quad (28)$$

respectively.

### B. Hydrodynamic Modes and Linear Response

### C. Perspective from Holography

## III. NON-RELATIVISTIC TRANSPORT — FRACTIONAL QUANTUM HALL SYSTEM

### A. Spacetime Background

### B. Response

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<sup>3</sup> See, for example, section II D of [6]. The result can also be proved holographically, see [7].