Conformal Field Theory and Applications in Condensed Matter Physics

Xiaodong Hu*
Department of Physics, Boston College
(Dated: June 4, 2019)

This is a research note about CFT. Particularly, we are interested the application beyond critical points around phase transitions.

Contents

I. GENERAL CONFORMAL TRANSFORMATION

Let us consider a local field theory defined on a D-dimensional spacetime with *Euclidean* metric $g_{\mu\mu} \equiv \eta_{\mu\nu}$ and signature (p, q). A general diffeomorphism transform the metric to

$$g_{\mu\nu} \mapsto \widetilde{g}_{\mu\nu}(\widetilde{x}) \equiv \frac{\partial x_\mu}{\partial \widetilde{x}_\alpha}\frac{\partial x_\alpha}{\partial \widetilde{x}_\beta} g_{\alpha\beta}.$$

Definition 1. (Conformal Group) Conformal transformation forms the subgroup of diffeomorphism group leaving the metric tensor unchcanged up to a scale factor

$$\widetilde{g}_{\mu\nu}(\widetilde{x}) = \Omega(x) g_{\mu\nu}(x). \tag{1}$$

Clearly Poincare group ($\Omega \equiv 1$) is the subgroup of our confromal group.

To find out all generators of conformal transformation, we should look at the infinitesimal transformation $\widetilde{x}_\mu \equiv x_\mu + \varepsilon_\mu + \mathcal{O}(\varepsilon^2)$ so that

$$\begin{aligned}\widetilde{g}_{\mu\nu} &\equiv (\delta^\alpha_\mu + \partial_\mu\varepsilon^\alpha)(\delta^\beta_\nu + \partial_\nu\varepsilon^\beta) g_{\alpha\beta} = g_{\mu\nu} + \partial_\mu\varepsilon^\alpha g_{\alpha\nu} + \partial_\nu\varepsilon^\beta g_{\mu\beta} + \mathcal{O}(\varepsilon^2)\\ &= g_{\mu\nu} + (\partial_\mu\varepsilon_\nu + \partial_\nu\varepsilon_\mu) + \mathcal{O}(\varepsilon^2)\end{aligned}$$

Definition (1) demands

$$(\partial_\mu\varepsilon_\nu + \partial_\nu\varepsilon_\mu) = (\Omega(x) - 1) g_{\mu\nu} \equiv K(x) g_{\mu\nu}, \tag{2}$$

mutiplying $g^{\mu\nu}$ on the left we have

$$2\partial^\mu\varepsilon_\mu = K(x) D.$$

*Electronic address: `xiaodong.hu@bc.edu`

In combination of (2) again we can cancel the undetermined scaling factor $\Omega(x)$ and obtain the constraint equation of conformal generators

$$\boxed{(\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu) = \frac{2}{D}(\partial \cdot \varepsilon) g_{\mu\nu}.} \tag{3}$$

To reveal the particularity of dimensionality-dependence we need to re-arrange equation (3) to a more explicit form. Applying ∂^ν and ∂_ν on (3) gives

$$\partial_\mu \partial_\nu (\partial \cdot \varepsilon) + \partial^2 \partial_\nu \varepsilon_\mu = \frac{2}{D} \partial_\mu \partial_\nu (\partial \cdot \varepsilon). \tag{4}$$

Similarly for twice derivatives of ∂_μ and ∂^μ we have

$$\partial^2 \partial_\mu \varepsilon_\nu + \partial_\nu \partial_\mu (\partial \cdot \varepsilon) = \frac{2}{D} \partial_\nu \partial_\mu (\partial \cdot \varepsilon). \tag{5}$$

Adding (4) and (5) up and utilize (3) we get the differential equation of $(\partial \cdot \varepsilon)$

$$2\partial_\mu \partial_\nu (\partial \cdot \varepsilon) + \partial^2 \left(\frac{2}{D} (\partial \cdot \varepsilon) g_{\mu\nu} \right) = \frac{4}{D} \partial_\mu \partial_\nu (\partial \cdot \varepsilon) \implies (g_{\mu\nu} \partial^2 + (D-2) \partial_\mu \partial_\nu)(\partial \cdot \varepsilon) = 0, \tag{6}$$

or (if $D \neq 2$)

$$(D-1)\partial^2 (\partial \cdot \varepsilon) = 0$$

if we multiply $g^{\mu\nu}$ on the left hand side.

Clearly if $D = 1$ the above constraint equation degenerates and the theory is trivial. So we only focus on the case when $D \geq 2$.

A. Conformal Transformation in $D > 2$

For $D > 2$ constraint (6) implies that ε_μ is at most *quadratic* in coordinate

$$\varepsilon_\mu = a_\mu + b_{\mu\nu} x^\nu + c_{\mu\nu\rho} x^\nu x^\rho$$

with $c_{\mu\nu\rho} \equiv c_{\mu\rho\nu}$.

1) Clearly $\varepsilon_\mu = a_\mu$ represents space-time **translation** $x'_\mu \mapsto x_\mu + a_\mu$ as ususal.

2) By inserting $\varepsilon_\mu = b_{\mu\nu} x^\nu$ back into (3) we obtain

$$b_{\mu\nu} + b_{\nu\mu} = \frac{2}{D} (b_{\alpha\beta} \partial^\alpha x^\beta) \eta_{\mu\nu}.$$

Separating $b_{\mu\nu}$ into symmetric part $b^s_{\mu\nu} \equiv b^s_{\mu\nu}$ and anti-symmetric part $b^a_{\mu\nu} \equiv -b^a_{\mu\nu}$, one can immediately finds out that $b^s_{\mu\nu}$ must be a pure trace

$$2b^s_{\mu\nu} = \frac{2}{D} (b^s)_\alpha{}^\alpha \eta_{\mu\nu} \implies b^s_{\mu\nu} \propto \eta_{\mu\nu},$$

which is called space-time **dilation**, while we have no constraint on space-time **rotation** $b^a_{\mu\nu} \equiv \omega_{\mu\nu}$.

3) By inserting $\varepsilon_\mu = c_{\mu\alpha\beta} x^\alpha x^\beta$ back into (3) we have

$$c_{\mu\nu\beta} x^\beta + c_{\nu\mu\beta} x^\beta = \frac{2}{D} c^\alpha{}_{\alpha\beta} x^\beta g_{\mu\nu}.$$

To express $c_{\mu\nu\beta}$ explicitly, we have to cycle all the indices, summing and subtracting them with symmetric condition on the latter two indices of $c_{\mu\nu\beta}$, which yields the **special conformation transformation** (SCT)

$$c_{\mu\nu\beta} = \frac{1}{D} (c^\alpha{}_{\alpha\beta} g_{\mu\nu} + c^\alpha{}_{\alpha\nu} g_{\mu\beta} - c^\alpha{}_{\alpha\mu} g_{\nu\beta}) \equiv b_\beta \eta_{\mu\nu} + b_\nu \eta_{\mu\beta} - b_\mu \eta_{\nu\beta},$$

where $b_\beta \equiv c^\alpha{}_{\alpha\beta}/D$.

Translation	$x'_\mu = x_\mu + a_\mu$
Dilation	$x'_\mu = x_\mu + \frac{2}{D}(b^s)^\alpha{}_\alpha \eta_{\mu\nu} x^\nu \equiv (1+\alpha)x_\mu$
Rotation	$x'_\mu = x_\mu + \omega_{\mu\nu} x^\nu$
SCT	$x'_\mu = x_\mu + (b_\beta \eta_{\mu\nu} + b_\nu \eta_{\mu\beta} - b_\mu \eta_{\nu\beta}) x^\nu x^\beta \equiv x_\mu + 2(b\cdot x)x_\mu - b_\mu x^2$

TABLE I: Four kinds of finite conformal transformations.

Translation	$P^\alpha = -i\frac{\delta}{\delta a_\alpha}(x_\mu + a_\mu)\partial^\mu = -i\partial^\alpha$
Dilation	$D = -i\frac{\delta}{\delta\alpha}(1+\alpha)x_\mu\partial^\mu = -ix^\mu\partial_\mu$
Rotation	$L^{\alpha\beta} = -i\frac{\delta}{\delta\omega_{\alpha\beta}}(x_\mu + \omega_{\mu\nu}x^\nu)\partial^\mu = -i(\delta_{\alpha\mu}\delta_{\beta\nu} - \delta_{\alpha\nu}\delta_{\beta\mu})x^\nu\partial^\mu = i(x^\alpha\partial^\beta - x^\beta\partial^\alpha)$
SCT	$K^\alpha = -i\frac{\delta}{\delta b_\alpha}(x_\mu + 2(b\cdot x)x_\mu - b_\mu x^2)\partial^\mu = -i(2x^\alpha(x\cdot\partial) - x^2\partial^\alpha)$

TABLE II: Corresponding four generators of four kinds of conformal transformation.

In summary, we get four kinds of finite conformal transformation as table I. And the corresponding four kinds of infinitesimal transformation is also shown in table II by taking the *tagent mapping*[1] of $x'_\mu(x)$.

B. Conformal Transformation in $D = 2$

For $D = 2$, constraint equation (3) reduces to simple

$$\partial_1\varepsilon_2 \equiv -\partial_2\varepsilon_1, \quad \partial_1\varepsilon_1 \equiv \partial_2\varepsilon_2. \tag{7}$$

which is nothing but celebrated *Cauchy-Riemann equation* appearing in complex analysis. Particularly, if we introduce two conjugate complex variables $z \equiv x_1 + ix_2$ and $\bar{z} \equiv x_1 - ix_2$, constraint equation (7) tells us that two-dimensional conformal transformation decouple each other and are generated by nothing but *holomorphic* and *anti-holomorphic* functions $\varepsilon \equiv \varepsilon_1 + i\varepsilon_2$ and $\bar{\varepsilon} = \varepsilon_1 - i\varepsilon_2$ as following

$$z \mapsto z' = z + \varepsilon(z), \quad \bar{z} \mapsto \bar{z}' = \bar{z} + \bar{\varepsilon}(\bar{z}),$$

where generally $\varepsilon(z) = \sum_n \alpha_n z^{n+1}$ and $\varepsilon(\bar{z}) = \sum_n \beta_n z^{n+1}$. Therefore infinitesimal generators of these two conformal transformations are

$$l_n = -i\frac{\delta}{\delta\alpha_n}\varepsilon_n\partial_z = -iz^{n+1}\partial_z, \quad \bar{l}_n = -i\frac{\delta}{\delta\beta_n}\bar{\varepsilon}_n\partial_{\bar{z}} = -i\bar{z}^{n+1}\partial_{\bar{z}}, \tag{8}$$

---

[1] Generally the generator for infinitesimal transformation keeping field configuration unchanged $\Phi'(x') = \Phi(x)$ and parameterized by ω_a is given by

$$iG_a \equiv \frac{\delta x^\mu}{\delta\omega_a}\partial_\mu.$$

II. ENERGY MOMENTUM TENSOR

III. RADIAL QUANTIZATION

A. Verosora Algebra

IV. APPENDIX

A. Infinitesimal Symmetric Transformation

Geometrically, a classical tensor field $\mathbf{\Phi}(x)$ is an element of tensor bundle. Transformation on the base space-time manifold[2] $f : M \to N, \boldsymbol{x} \mapsto \boldsymbol{x'}$ will be lifted to the fibers as well

$$\mathbf{\Phi'}(\boldsymbol{x'}) \equiv \mathscr{F}(\Phi(\boldsymbol{x})), \tag{9}$$

or more clearly in commutative diagram FIG. 1.

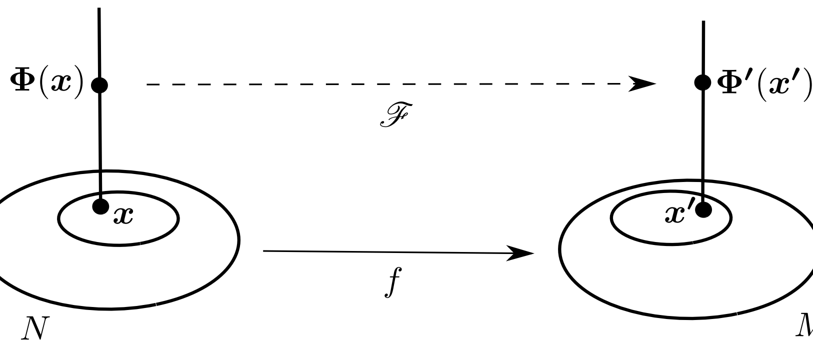

FIG. 1: **General Transformation**

For infinitesimal transformation parameterized by some vector $\{\omega_a\}$

$$x'^\mu = x^\mu + \frac{\delta x^\mu}{\delta \omega_a}\omega_a + \mathcal{O}(\omega^2),$$

identity (9) tells

$$\mathbf{\Phi'}(\boldsymbol{x'}) = \mathscr{F}(\mathbf{\Phi}(\boldsymbol{x})) = \mathbf{\Phi}(\boldsymbol{x}) + \frac{\delta \mathscr{F}}{\delta \omega_a}\omega_a + \mathcal{O}(\omega^2) \tag{10}$$

B. Noether Theorem and Energy-momentum Tensor

[2] Here for generality we take this transformation as a mapping between two distinct manifolds. This way of looking is called *positive*. In constrast if one takes this transformation as just a coordinate transformation between two patches on the same manifold, then we call such viewpoint *negative*.