Duality in Condensed Matter Physics—From Lattice Models to Particle-Vortex Duality and Beyond

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This is a research note documenting duality in condensed matter physics.

Contents

I. EXACT DUALITIES OF LATTICE MODEL

A. Quantum-Classical Mapping

As a warm-up, let us start with the simplest $d = 0$ *quantum Ising model*[1], namely only one quantum spin with the Hamiltonian

$$H_Q = -g\sigma^x. \tag{I.1}$$

The partition function of (I.1) can be evaluated in path integral formalism by slicing the temperature (imaginary time) into N segments, inserting intermediate states, and let N goes to infinity

$$\begin{aligned}\mathcal{Z}_Q \equiv \operatorname{tr} e^{-\beta H_Q} &\equiv \lim_{N\to\infty}\sum_{s_1,\cdots,s_N}\langle s_N|e^{\Delta\tau g\sigma^x}|s_1\rangle\langle s_1|e^{\Delta\tau g\sigma^x}|s_2\rangle\cdots\langle s_{N-1}|e^{\Delta\tau g\sigma^x}|s_N\rangle\\ &= \lim_{N\to\infty}\sum_{s_1,\cdots,s_N}\prod_{i=0}^{N}\langle s_i|1+\Delta\tau g\sigma^x|s_{i+1}\rangle,\end{aligned} \tag{I.2}$$

where $s_0 \equiv s_N$ (PBC is applied) and $\Delta\tau \equiv \beta/N \to 0$ such that β is fixed.

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[1] Or *transverse field Ising model* if you like. Note that statistical model has *no* time dimensionality.

Equation (I.2) is an reminiscent of the structure of *transferring matrix* in 1-d *classical* Ising model. In fact, by identifying each intermediate *quantum* state (which is adding by hand in path integral formalism) with *classical* degree of freedom $s_i = \{\pm 1\}$ on the physical N-site lattice (still with PBC), and ansatzing

$$\langle s_i | 1 + \Delta\tau g \sigma^x | s_{i+1} \rangle \equiv \langle s_i | A e^{B s_i s_{i+1}} | s_{i+1} \rangle,$$

or in $\hat{\sigma}^z$ eigenstates

$$\begin{pmatrix} 1 & -ig\Delta\tau \\ ig\Delta\tau & 1 \end{pmatrix} \equiv \begin{pmatrix} Ae^{B} & Ae^{-B} \\ Ae^{-B} & Ae^{B} \end{pmatrix},$$

one immediately have

$$A = \sqrt{\Delta\tau g}, \quad B = -\frac{1}{2}\ln(\Delta\tau g) \to \infty. \tag{I.3}$$

Therefore, the partition function of quantum Ising model can be re-written as

$$\mathcal{Z}_Q = \lim_{N\to\infty} A^N \operatorname{tr}\exp\left(-\beta H_c\right), \tag{I.4}$$

where

$$-\beta H_c = B \sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z \tag{I.5}$$

is exactly the Hamiltonian of $d = 1$ classical Ising model.

The above result can be easily generalized to higher dimensions. Hamiltonian of $d > 1$ Quantum Ising Model is

$$H_q = -J \sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z - g \sum_i \sigma_i^x. \tag{I.6}$$

Still clues of the duality theory can be found from its partition function

$$\langle s_i | e^{\Delta\tau (J \sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z + g \sum_i \sigma_i^x) + \frac{Jg}{2!}\mathcal{O}(\Delta\tau^2)} | s_j \rangle = \sum_{s_k} \underbrace{\langle s_i | e^{\Delta\tau J \sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z} | s_k \rangle}_{\text{d-dim Classical Ising Model}} \overbrace{\langle s_k | e^{\Delta\tau \sum_i \sigma_i^x} | s_j \rangle}^{\text{0-dim Quantum Ising Model}}, \tag{I.7}$$

where B-C-H formula is utilized

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}[A,[A,B]]+\cdots}.$$

So we say [1] **Partition function of d-dim quantum statistical model is equivalent to (d+1)-dim classical statistical model**.

B. Kramers-Wannier Duality

C. $\mathbb{Z}_2$ Gauge Theory

$\mathbb{Z}_2$ gauge field describe the fluctuation of gauge freedoms, or visually fluctuation of loops on *infinitely large* square lattice. Spins live on the links of sites,

$$H_{\mathbb{Z}_2} = -K \sum_{\square} \prod_{\ell\in\square} \sigma_\ell^z - g \sum_\ell \sigma_\ell^x. \tag{I.8}$$

D. Jordan-Wigner Transformation to Anyons

Consider an *infinite* square lattice, on the links of which spins are placed such that the Hamiltonian

II. IR DUALITIES OF CONTINUOUS FIELD THEORY

A. Bosonic Particle/Vortex Duality

B. Fermionic Particle/Vortex Duality

C. Boson-Fermion Duality and Duality Web

III. APPLICATION OF DUALITY

IV. SUBTLETIES ON DUALITY

A. Does Duality Keep Entanglement Entropy?

Thanks to the deep discussion on `https://physics.stackexchange.com/questions/135098`.

[1] T. H. Hsieh, Student review,(2) pp. 1–4 (2016).