Symmetry Protected Topological Phases and Group cohomology*

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Notes of SPT.

Contents

I. GROUP COHOMOLOGY

A. Preliminary

Definition 1. (Ring) A *ring* (with identity) $(R, +, \cdot)$ is an abelian group of $(R, +)$ and monoid[1] of $(R, \cdot)$ such that the multiplication of monoid is distributive. Particularly if $a \cdot b \equiv b \cdot a$, then R is said to be *commutative*

Example 1. (Group Ring) A *group ring* or $\mathbb{Z}$-*group ring* $\mathbb{Z}[G]$ is the set of *finite sum*

$$\mathbb{Z}[G] := \left\{ \sum_{g\in G} a_g g \;\middle|\; a_g \in \mathbb{Z}, \text{ almost all } a_g = 0 \right\}$$

on which the sum and multiplication are naturally defined as

$$\sum_g a_g g + \sum_h a_h h = \sum_{g\in G} (a_g + b_g) g,$$

and

$$\left(\sum_g a_g g\right) \cdot \left(\sum_h a_h h\right) \equiv \sum_{g,h} (a_g \cdot b_h) gh = \sum_{g,gh} (a_g \cdot b_{g^{-1}gh}) gh = \sum_{g,k} (a_g \cdot b_{g^{-1}k}) k,$$

where we replace dummy group index h by gh since it suns over the entire group G.

Definition 2. (Left R-module) A *left R-module* denoted as M over a ring $(R, +, \cdot)$ consists of an *abelian group* $(M, \times)$ and a ring homomorphism[2] $\sigma : R \to \mathrm{End}_{\mathsf{Ab}}(M), \sigma(r)(m) \mapsto rm \in M$ called scalar multiplication such that this mapping is associative and distributive for $(M, +)$.

Example 2. Denoting F as field, an F-module is a F-vector space. So module can be regarded as the "vector space" over a ring.

Example 3. $\mathbb{Z}$-module is an abelian group.

Example 4. (G-Module) A $\mathbb{Z}[G]$-module (or simply referred as *G-module*) is an abelian group $(A, +)$ with the ring homomorphism $\sigma : \mathbb{Z}[G] \to \mathrm{End}_{\mathsf{Ab}}(A)$ compatible with the abelian group multiplication. Studying the representation of a group can be *equivalently* converted to study the module over its group ring.

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*This is a note of

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[1] A monoid is a semi-group with an identity 1_R (called two-sided multiplication identity).

[2] A ring homomorphism $f \in \mathrm{End}(R)$ is an addition, multiplication, and multiplication identity preserving mapping

$$f(a+b) = f(a) + f(b), \quad f(a \cdot b) = f(a) \cdot f(b), \quad f(1_R) = 1_R.$$

B. Algebraic Definition of Group Cohomology

Given an G-module M and an arbitrary function (called *n-cochain*) $\omega : \underbrace{G \times \cdots \times G}_{n} \to M$, we can naturally assgin an abelian group multiplication on the collection of these functions $\mathcal{C}^n(G, M) \equiv \{\omega | G^n \to M\}$ from the ablian group structure of $(M, +)$ by

$$(\omega * \omega')(g_1, \cdots, g_n) := \omega(g_1, \cdots, g_n) + \omega'(g_1, \cdots, g_n).$$

If we define an n-th differential $d^n : C^n(G, M) \to C^{n+1}(G, M)$ as usual

$$d^n\omega(g_0, g_1, \cdots, g_n) := g_0 \cdot \omega(g_1, \cdots, g_n) + \tag{1}$$