Duality in Condensed Matter Physics—From Lattice Models to Particle-Vortex Duality and Beyond

Xiaodong Hu*
Department of Physics, Boston College, MA 02135, USA
(Dated: November 28, 2019)

This is a research note documenting duality in condensed matter physics.

Contents

I. EXACT DUALITIES OF LATTICE MODEL

A. Quantum-Classical Mapping

As a warm-up, let us start with the simplest $d = 0$ *quantum Ising model*[1], namely a quantum spin plased on one site, with the Hamiltonian

$$H_Q = -g\sigma^x. \tag{I.1}$$

One can always express the finite-T partition function in terms of path integral by slicing the temperature (treating temperature as imaginary time)

$$H_Q = -g\sum_i \sigma_i^x. \tag{I.2}$$

To compute the partition function of (I.2), we need to slice the temperature (imaginary time) to N segments and let N tends to infinity. Namely,

$$\begin{aligned}\mathcal{Z}_q = \operatorname{tr} e^{-\beta H_q} &= \lim_{N\to\infty}\sum_{s_1,\cdots,s_N}\langle s_N|e^{+\Delta\tau g\sigma_1^x}|s_1\rangle\langle s_1|e^{+\Delta\tau g\sigma_2^x}|s_2\rangle\cdots\langle s_{N-1}|e^{+\Delta\tau g\sigma_N^x}|s_N\rangle\\ &= \lim_{N\to\infty}\sum_{s_1,\cdots,s_N}\prod_{i=0}^{N}\langle s_i|1+\Delta\tau g\sigma_j^x|s_j\rangle,\end{aligned} \tag{I.3}$$

---

*Electronic address: `xiaodong.hu@bc.edu`
[1] Or *transverse field Ising model* if you like. Note that statistical model has *no* time dimensionality.

where $s_0 \equiv s_N$ (PBC is applied naturally). Equation (I.3) shares the same structure as *transferring matrix* methods in 1-d *classical* Ising model. In fact, ansatzing

$$\langle s_i|1+\Delta\tau g\sigma_j^x|s_j\rangle = \langle s_i|Ae^{B\sigma_i^z\sigma_j^z}|s_j\rangle$$

and let $s_i = s_j = 1$ and $s_i = -s_j = 1$, one can immediately show that

$$A^2 = \Delta\tau g, \quad e^{-2B} = \Delta\tau g. \tag{I.4}$$

Therefore, partition function of $0d$-quantum Ising Model can be re-written as

$$\mathcal{Z}_q = A^N \operatorname{tr}\exp\left(-\beta H_c\right), \tag{I.5}$$

where

$$H_c = B\sum_{\langle ij\rangle}\sigma_i^z\sigma_j^z \tag{I.6}$$

is the Hamiltonian of $1d$ classical Ising model.

The above result can be easily generalized to higher orders. Hamiltonian of $d>1$ Quantum Ising Model is

$$H_q = -J\sum_{\langle ij\rangle}\sigma_i^z\sigma_j^z - g\sum_i\sigma_i^x. \tag{I.7}$$

Still clues of the duality theory can be found from its partition function

$$\langle s_i|e^{\Delta\tau(J\sum_{\langle ij\rangle}\sigma_i^z\sigma_j^z+g\sum_i\sigma_i^x)+\frac{Jg}{2!}\mathcal{O}(\Delta\tau^2)}|s_j\rangle = \sum_{s_k}\underbrace{\langle s_i|e^{\Delta\tau J\sum_{\langle ij\rangle}\sigma_i^z\sigma_j^z}|s_k\rangle}_{\text{d-dim Classical Ising Model}}\overbrace{\langle s_k|e^{\Delta\tau\sum_i\sigma_i^x}|s_j\rangle}^{\text{0-dim Quantum Ising Model}}, \tag{I.8}$$

where B-C-H formula is utilized

$$e^Ae^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}[A,[A,B]]+\cdots}.$$

So we say [1] **Partition function of d-dim quantum statistical model is equivalent to (d+1)-dim classical statistical model**.

B. Kramers-Wannier Duality

C. $\mathbb{Z}_2$ Gauge Theory

$\mathbb{Z}_2$ gauge field describe the fluctuation of gauge freedoms, or visually fluctuation of loops on *infinitely large* square lattice. Spins live on the links of sites,

$$H_{\mathbb{Z}_2} = -K\sum_{\square}\prod_{\ell\in\square}\sigma_\ell^z - g\sum_\ell\sigma_\ell^x. \tag{I.9}$$

D. Jordan-Wigner Transformation to Anyons

Consider an *infinite* square lattice, on the links of which spins are placed such that the Hamiltonian

II. IR DUALITIES OF CONTINUOUS FIELD THEORY

A. Bosonic Particle/Vortex Duality

B. Fermionic Particle/Vortex Duality

C. Duality Web

III. APPLICATION OF DUALITY

IV. SUBTLETIES ON DUALITY

A. Does Duality Keep Entanglement Entropy?

Thanks to the deep discussion on `https://physics.stackexchange.com/questions/135098`.

---

[1] T. H. Hsieh, Student review,(2) pp. 1–4 (2016).