Hydrodyanmic EOM from Boltzmann Equation

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In this note I will first introduce the correction of phase space volume element from Berry curvature term in semi-classic EOM, then combine it with Boltzmann Kinetic equation to obtain both the constitution relation and hydrodynamic equation of motion.

流成笔下春风瓣，吹散弦上秋草声。

—— 雨楼清歌「一瓣河川」

Contents

I. CORRECTION OF PHASE SPACE VOLUME ELEMENT

A. Symplectic Manifold

Let us first recall the *symplectic structure* of Hamiltonian dynamics. Physically, a ($6N$-dimensional) phase space is a *symplectic manifold* (M, ω) consisting of a smooth manifold M and a closed non-degenerate[1] differential 2-form ω, which in general takes the form of

$$\omega = \frac{1}{2}\omega_{\alpha\beta}\,\mathrm{d}\xi^\alpha \wedge \mathrm{d}\xi^\beta$$

in local coordinates $\{\xi^a\}$. Given a simplectic space $V \in T_pM$, we can always define a linear map $\omega^\sharp : V \to V^*$ by $\boldsymbol{v} \mapsto \omega(\cdot, \boldsymbol{v})$ since ω is bilinear. Clearly simplectic 2-form ω is non-degenerate iff $\omega^\sharp$ is one-to-one[2], so $\omega^\sharp$ is an isomorphism and has an inverse $(\omega^\sharp)^{-1}$.

Equipped with $\omega^\sharp$, we can then define the *Hamiltonian vector field* X_f for any differential function $f \in C^\infty(M)$ by $X_f := (\omega^\sharp)^{-1}\,\mathrm{d}f$. Specifically, in the local coordinates $\{\xi^\alpha\}$, we have

$$\omega^\sharp\left(\frac{\partial}{\partial\xi^\gamma}\right) \equiv \iota_{\frac{\partial}{\partial\xi^\gamma}}\omega \equiv \frac{1}{2}\omega^{\alpha\beta}\,\mathrm{d}\xi^\alpha \wedge \mathrm{d}\xi^\beta\left(\frac{\partial}{\partial\xi^\gamma}\right) = \omega_{\gamma\beta}\,\mathrm{d}\xi^\beta \implies (\omega^\sharp)^{-1}(\mathrm{d}\xi^\alpha) = \omega^{\alpha\gamma}\frac{\partial}{\partial\xi^\gamma}$$

(where we introduce the inverse matrix $\omega^{\alpha\gamma}$ such that $\omega^{\alpha\gamma}\omega_{\gamma\beta} \equiv \delta^\alpha_\beta$) and so

$$X_f = \partial_\alpha f(\omega^\sharp)^{-1}(\mathrm{d}\xi^\alpha) = \omega^{\alpha\gamma}\partial_\alpha f\frac{\partial}{\partial\xi^\gamma}.$$

Then Poisson bracket for two function $f, g \in C^\infty(M)$ can be defined through the two corresponding Hamiltonian vector fields

$$\{f, g\} := \omega(X_f, X_g). \tag{1}$$

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[1] By close we mean $\mathrm{d}\omega \equiv 0$, and by non-degenerate we mean for any $\boldsymbol{v}$, $\omega(\boldsymbol{u}, \boldsymbol{v}) = 0 \implies \boldsymbol{u} = \boldsymbol{0}$.
[2] For a linear map $\omega^\sharp$, one-to-one is equivalent to $\mathrm{Ker}(\omega^\sharp) = \{\boldsymbol{0}\}$, namely $\omega^\sharp(\boldsymbol{u}) = \boldsymbol{0} \implies \boldsymbol{u} = \boldsymbol{0}$. But $\omega^\sharp(\boldsymbol{u}) = \boldsymbol{0} \implies \forall \boldsymbol{v} \in V, \omega^\sharp(\boldsymbol{u})(\boldsymbol{v}) \equiv \omega(\boldsymbol{u}, \boldsymbol{v}) = 0 \implies \boldsymbol{u} = \boldsymbol{0}$ by the non-degeneracy of ω.

In the local coordinates, we immediately have

$$\{f, g\} = \omega^{\alpha\beta} \partial_\alpha f \partial_\beta g. \tag{2}$$

In QM, the eigenvalues of a Hamiltonian operator serves as a differential function of local coordinates $h(\xi) \in C^\infty(M)$, so the semi-classical *Hamiltonian equation* (as the second ingredients of Hamiltonian dynamics) reads

$$\dot{\xi}^\alpha = \{h, \xi^\alpha\} = \omega^{\alpha\beta} \partial_\beta h, \quad \text{or} \quad \omega_{\alpha\beta} \dot{\xi}^\beta = \partial_\alpha h. \tag{3}$$

B. Hamiltonian Dynamics with Berry Curvature

Semi-classical EOM of electronic wave-packets reads

$$\dot{\boldsymbol{r}} = \frac{1}{\hbar} \frac{\partial \varepsilon_n}{\partial \boldsymbol{k}} - \dot{\boldsymbol{k}} \times \Omega(\boldsymbol{k}), \tag{4}$$

$$\hbar \dot{\boldsymbol{k}} = -e\boldsymbol{E} - e\dot{\boldsymbol{r}} \times \boldsymbol{B}. \tag{5}$$

In the presence of both electric magnetic fields, the Hamiltonian function of such quasiparticle is $h = \varepsilon_n - eV$. We can re-arrange the kinetic EOM in terms of standard Hamiltonian equation (3) so that

$$\omega_{\alpha\beta} \dot{\xi}^\beta \equiv \begin{pmatrix} -e\varepsilon_{ijk} B^k & \delta_{jk} \\ -\delta_{jk} & \varepsilon_{ijk} \dfrac{\Omega^k}{\hbar} \end{pmatrix} \begin{pmatrix} \dot{r}_j \\ \dot{p}_j \end{pmatrix} = \begin{pmatrix} \partial_{p_j} \varepsilon \\ eE_j \end{pmatrix} \equiv \partial_\alpha h. \tag{6}$$

The Poisson brackets can be directly obtained from the inverse matrix.

As a bonus, we can also get the *invariant volume element* for the symplectic manifold, which is defined to have a similar form on a manifold with metric structure

$$\mathrm{d}V := \sqrt{|\det(\omega_{\alpha\beta})|} \prod_{\alpha=1}^{2d} \mathrm{d}\xi^\alpha. \tag{7}$$

Form (7) is clearly well-defined under local coordinate transformation (Jacobian cancels exactly). However, **one must be aware of the SHARP differences between symplectic manifolds and Riemannian manifolds. For example, there is no Darboux-like theorem for a Riemann manifold so we cannot locally trivialize a metric tensor by finding out the canonical variables, and there are infinite torsion-free metric compatible connection on symplectic manifolds (while on Riemannian manifold such connection is unique). See `https://physics.stackexchange.com/questions/136182` for more details**.

Particularly in our cases, up to the first order of magnetic fields, the volume element takes the form of

$$\mathrm{d}V = \sqrt{1 + \frac{2e}{\hbar} \boldsymbol{B} \cdot \boldsymbol{\Omega}}\, \mathrm{d}\boldsymbol{r}\, \mathrm{d}\boldsymbol{p} \simeq \left(1 + \frac{e}{\hbar} \boldsymbol{B} \cdot \boldsymbol{\Omega}\right) \mathrm{d}\boldsymbol{r}\, \mathrm{d}\boldsymbol{p}. \tag{8}$$

II. HYDRODYNAMIC EOM FROM BOLTZMANN KINETIC EQUATION