## Algebraic Theory of Anyons

Xiaodong Hu\*

Department of Physics, Boston College
(Dated: September 24, 2019)

This note plays a role as a supplementary material for the hidden but deep demanded mathematical backgrounds of the brilliant paper of Kitaev [1]. Personal understanding and comments are included as well.

仰之弥高,钻之弥坚,瞻之在前,忽焉在后。

—— 「论语·子罕第九」

#### Contents

Ι.	Algebraic Theory of Anyons	1
	A. Category Theory: General Preliminaries	1
	B. Monoidal Categories	1
	C. Graphic Calculus	2
	D. Tensor Categories	2
	E. 6 <i>j</i> -Symbols	2
	References	2

### I. ALGEBRAIC THEORY OF ANYONS

### A. Category Theory: General Preliminaries

See my personal note of category theory. The missing Yoneda Lemma may also be extremely important for understanding the logic of emergentism in condensed matter physics.

# B. Monoidal Categories

**Definition 1.** (Monoidal Category) A monoidal category is a quintuple  $(\mathcal{C}, \otimes, \alpha, \mathbb{1}, \iota)$  with

- 1) a category C,
- 2) a bifunctor of product category  $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$  called tensor product,
- 3) a natural isomorphism  $\alpha$  from the functor  $(\bullet \otimes \bullet) \otimes \bullet : \mathcal{C} \times \mathcal{C} \times \mathcal{C} \to \mathcal{C}$  to the functor  $\bullet \otimes (\bullet \otimes \bullet) : \mathcal{C} \times \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

$$a_{XYZ}: (X \otimes Y) \otimes Z \stackrel{\sim}{\mapsto} X \otimes (Y \otimes Z)$$

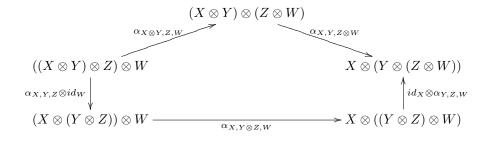
called  $associaticity\ constraint.$ 

- 4) an object  $\mathbb{1} \in \text{Obj}(\mathcal{C})$ ,
- 5) a natural isomorphism  $\iota$  from the functor  $\mathbb{1} \otimes \bullet : \mathcal{C} \to \mathcal{C}$  to the identical functor  $id_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$

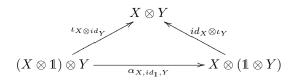
$$\iota_X: \mathbb{1} \otimes X \stackrel{\sim}{\mapsto} X$$

\*Electronic address: xiaodong.hu@bc.edu

called  $unitality \ constraint^1$  such that the  $pentagon \ coherence$ 



and the  $triangle\ coherence$ 



commute.

C. Graphic Calculus

Drawn by hands.

D. Tensor Categories

E. 6j-Symbols

[1] A. Kitaev, Annals of Physics  $\bf 321,\,2$  (2006).

[2] V. G. Turaev and A. Virelizier, Monoidal categories and topological field theory, vol. 322 (Springer, 2017).

<sup>&</sup>lt;sup>1</sup> We do not distinguish left and right unitality constraints as [2] does.