Asymptotic Behaviors about the Self-energy of Electron-Phonon Interaction

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Landau Fermi-liquid theory is successful in explanation of the power of free electron gas model and robustness of the Fermi surface in solids, where electron and phonons are coupled. But to prove that Landau Fermi-liquid theory does holds in solids, one have to compute the lifetime of low energy elementary excitation – quasi-particles and quasi-holes, then check that the fields renormalization factor Z is not zero. So in this paper, we compute the asymptotic behaviors of the imaginary part of self-energy diagram in details at RPA approximation and low temperature. The results support Landau Fermi-liquid theory.

We mainly follow the derivation of [1], but with different computation techniques.

This paper is divided by three sections. In the first section, we compare the notation of two typical factions to eliminate the potential unnecessary distraction due to different form of results. In the next two sections we compute the finite temperature and zero temperature self-energy under RPA approximation, respectively. Little discussion is applied to fit with various given conditions. In the last section, we check the validity of Landau Fermi-liquid theory be computing its field renormalization factor, as is taught on classes.

EQUIVALENCE OF THE NOTATION

Before our mainline derivation, we are to prove the equivalence of the different notation used in [1, 2] and the much cumbersome one used in [3], i.e., the notation taught on classes. The reason we do this not only just to show the theoretical consistency, but we also concern about all the conditions we have utilized here, which crucially influence the future divergent integral approximation. All our discussion is confined to the single-loop self-energy (RPA approximation) in this article, as is illustrated in FIG. 1, so dose the proof of equivalence.
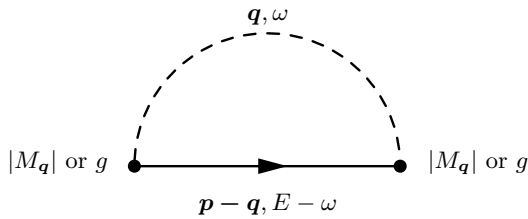

FIG. 1: **Self-Energy of Electron-phonon Interaction**

Starting with a phenomenological *continuum model,*

[2, 4] obtained the phonon field operator through quantization the elastic wave

$$\varphi(\boldsymbol{r}, t) := i \sum_{\boldsymbol{k}} \sqrt{\frac{\omega_{\boldsymbol{k}}}{2V}} \left(a_{\boldsymbol{k}} + a_{-\boldsymbol{k}}^{\dagger} \right) e^{-(i\omega_{\boldsymbol{k}} t - \boldsymbol{r} \cdot \boldsymbol{k})}. \quad (1)$$

With a slightly different definition of Green function (in accordance with that in HEP)

$$D(x, x') := \langle T\varphi(x)\varphi^{\dagger}(x')\rangle,$$

they got a different form of $D^{(0)}$ in the momentum space

$$D^0(\boldsymbol{q}, \omega) = \frac{\omega_{\boldsymbol{q}}^2}{\omega^2 - \omega_{\boldsymbol{q}}^2 + i\delta}. \quad (2)$$

Still in this model, they considered the simplest *deformation potential*[4]: in an *isotropic lattice*, only the *longitudinal acoustic* phonons interact with electrons. Under the lowest order approximation, they still got the coupling term

$$H_{\text{int}} = g \int d\boldsymbol{r}\, \psi^{\dagger}(\boldsymbol{r})\psi(\boldsymbol{r})\varphi(\boldsymbol{r}), \quad (3)$$

or in momentum space,

$$H_{\text{int}} = ig \sum_{\boldsymbol{k},\boldsymbol{q}} \sqrt{\frac{\omega_{\boldsymbol{q}}}{2V}}\, C_{\boldsymbol{k}+\boldsymbol{q}}^{\dagger} C_{\boldsymbol{k}} (a_{\boldsymbol{q}} + a_{-\boldsymbol{q}}^{\dagger}), \quad (4)$$

with g a coupling constant. So in this notation, the amplitude of Feynman diagram FIG. 1 can be expressed as

$$\Sigma(\boldsymbol{p}, E) = ig^2 \int \frac{d\boldsymbol{q}\, d\omega}{(2\pi)^4}\, G^{(0)}(\boldsymbol{p} - \boldsymbol{q}, E - \omega) \frac{\omega_{\boldsymbol{q}}^2}{\omega^2 - \omega_{\boldsymbol{q}}^2 + i\delta}. \quad (5)$$

In contrast, starting with the rigorous consideration of perturbation of lattice vibration, [3] gives a general $\boldsymbol{q}$-dependent interactive Hamiltonian

$$H_{\text{e-p}} = \sum_{\boldsymbol{q},\boldsymbol{k}} M_{\boldsymbol{q}} C_{\boldsymbol{k}+\boldsymbol{q}}^{\dagger} C_{\boldsymbol{k}} (a_{\boldsymbol{q}} + a_{-\boldsymbol{q}}^{\dagger}) \quad (6)$$

with (ignoring all spin labels)

$$M_{\boldsymbol{q}} \equiv -i \left(\frac{N\hbar}{2M\omega_{\boldsymbol{q}}} \right)^{1/2} V_{\text{e-i}}(\boldsymbol{q})(\boldsymbol{e}_{\boldsymbol{q}} \cdot \boldsymbol{q}).$$

FIG. 1 is now expressed as

$$\Sigma(\boldsymbol{p}, E) = i \int \frac{\mathrm{d}\boldsymbol{q}\,\mathrm{d}\omega}{(2\pi)^4} |M_{\boldsymbol{q}}|^2 G^{(0)}(\boldsymbol{p}-\boldsymbol{q}, E-\omega) \frac{2\omega_{\boldsymbol{q}}}{\omega^2 - \omega_{\boldsymbol{q}}^2 + i\delta} \tag{7}$$

instead. And we are to show that (5) and (7) are equivalent under some specific conditions here. In fact, if we suppose that

- Only the *long-wavelength longitudinal acoustic phonons* have contribution to the coupling with electrons in deformation;

- The electron-ion potential $V_{\text{e-i}}(q)$ is *approximately* taken to be a *deformation constant D* at $q \to 0$;

Then $|M_{\boldsymbol{q}}|$ reduces to simple

$$|M_{\boldsymbol{q}}|^2 = \frac{N\hbar}{2Mc|q|} Dq^2,$$

and (7) becomes

$$\Sigma(\boldsymbol{p}, E) = ig^2 \int \frac{\mathrm{d}\boldsymbol{q}\,\mathrm{d}\omega}{(2\pi)^4} G^{(0)}(\boldsymbol{p}-\boldsymbol{q}, E-\omega) \frac{c^2 k^2}{\omega^2 - c^2 k^2 + i\delta}, \tag{8}$$

with $g^2 \equiv N\hbar D/4Mc^2$, which is exactly (5). So we are done.

ASYMPTOTIC BEHAVIOR OF SELF-ENERGY AT FINITE TEMPERATURE

In Matsubara formalism, the imaginary-time self-energy in FIG. 1 is expressed by

$$\Sigma(\boldsymbol{p}, i\omega_n) = \frac{-g^2}{(2\pi)^3\beta} \sum_m \int \frac{\mathrm{d}\boldsymbol{q}}{i(\omega_n - \omega_m) - \xi_{\boldsymbol{p}-\boldsymbol{q}}} \frac{-\omega_{\boldsymbol{q}}^2}{\omega_m^2 + \omega_{\boldsymbol{q}}^2}, \tag{9}$$

where the minus sign at the front comes from the definition of imaginary-time Green function and counts of contraction pairs.

Denote the complex function

$$f(z) \equiv \frac{\omega_{\boldsymbol{q}}^2}{z^2 - \omega_{\boldsymbol{q}}^2} \frac{1}{i\omega_n - z - \xi_{\boldsymbol{p}-\boldsymbol{q}}},$$

then we can perform the frequency summation in (9) in advance that

$$S(\boldsymbol{p}, i\omega_n; \boldsymbol{q}) = -\frac{1}{\beta} \sum_m f(i\omega_m). \tag{10}$$

Now consider the integral on the large circle $\mathcal{C}_R$

$$I = \lim_{R\to\infty} \oint_{\mathcal{C}_R} \frac{\mathrm{d}z}{2\pi i} f(z) n_B(z), \tag{11}$$

where

$$n_B(z) \equiv \frac{1}{e^{\beta z} - 1}.$$

Since $g(z) \equiv f(z) n_B(z)$ is continuous on $z = Re^{i\theta}$, and the limit $\lim_{R\to\infty} zg(z)$ exists, by a lemma in complex analysis, we alway have

$$\lim_{R\to\infty} \oint_{\mathcal{C}_R} g(z)\,\mathrm{d}z = 0.$$

That is, $I = 0$.

On the other hand, by the residue theorem, we have

$$I = 2\pi i \sum_k \text{Res}(z_k; g).$$

All the residues come from three parts: boson distribution n_B, free phonon propagator $\mathcal{D}^{(0)}(\boldsymbol{q}, i\omega_{\boldsymbol{q}})$, and free electron propagator $\mathcal{G}^{(0)}(\boldsymbol{p}-\boldsymbol{q}, i\omega_n - i\omega_m)$. For the first part, let $e^{\beta z} = 1$, one gets $z_m = i\frac{2m\pi}{\beta} = i\omega_m$, so

$$\begin{aligned}
\text{Res}_1(i\omega_m; g) &= \lim_{z\to i\omega_m} (z - i\omega_m) \frac{1}{e^{\beta z} - 1} \frac{\omega_{\boldsymbol{q}}^2}{z^2 - \omega_{\boldsymbol{q}}^2} \frac{1}{i\omega_n - z - \xi_{\boldsymbol{p}-\boldsymbol{q}}}, \\
&= \lim_{z\to i\omega_m} (z - i\omega_m) \frac{1}{e^{\beta(z - i\omega_m)} - 1} \frac{\omega_{\boldsymbol{q}}^2}{z^2 - \omega_{\boldsymbol{q}}^2} \frac{1}{i\omega_n - z - \xi_{\boldsymbol{p}-\boldsymbol{q}}}, \\
&= \lim_{z\to i\omega_m} (z - i\omega_m) \frac{1}{1 + \beta(z - i\omega_m) + \mathcal{O}(\beta^2(z - i\omega_m)^2) - 1} \frac{\omega_{\boldsymbol{q}}^2}{z^2 - \omega_{\boldsymbol{q}}^2} \frac{1}{i\omega_n - z - \xi_{\boldsymbol{p}-\boldsymbol{q}}}, \\
&= \frac{1}{\beta} \frac{\omega_{\boldsymbol{q}}^2}{-\omega_m^2 - \omega_{\boldsymbol{q}}^2} \frac{1}{i(\omega_n - \omega_m) - \xi_{\boldsymbol{p}-\boldsymbol{q}}} = \frac{1}{\beta} f(i\omega_m),
\end{aligned}$$

where in the second line we insert the identity $e^{i\beta\omega_m} = 1$.

As for the residues from $\mathcal{D}^{(0)}(\boldsymbol{q}, i\omega_{\boldsymbol{q}})$, obviously it has

singularity at $z_k = \pm\omega_{\boldsymbol{q}}$. Then

$$\text{Res}_2(+\omega_{\boldsymbol{q}}; g) = \frac{1}{e^{\beta\omega_{\boldsymbol{q}}} - 1} \frac{\omega_{\boldsymbol{q}}^2}{2\omega_{\boldsymbol{q}}} \frac{1}{i\omega_n - \omega_{\boldsymbol{q}} - \xi_{\boldsymbol{p}-\boldsymbol{q}}}$$
$$= \frac{\omega_{\boldsymbol{q}}}{2} \frac{n_B(\omega_{\boldsymbol{q}})}{i\omega_n - \omega_{\boldsymbol{q}} - \xi_{\boldsymbol{p}-\boldsymbol{q}}},$$

and

$$\text{Res}_2(-\omega_{\boldsymbol{q}}; g) = \frac{1}{e^{-\beta\omega_{\boldsymbol{q}}} - 1} \frac{\omega_{\boldsymbol{q}}^2}{-2\omega_{\boldsymbol{q}}} \frac{1}{i\omega_n + \omega_{\boldsymbol{q}} - \xi_{\boldsymbol{p}-\boldsymbol{q}}}$$
$$= \frac{\omega_{\boldsymbol{q}}}{2} \frac{n_B(\omega_{\boldsymbol{q}}) + 1}{i\omega_n + \omega_{\boldsymbol{q}} - \xi_{\boldsymbol{p}-\boldsymbol{q}}}.$$

Similarly, for singularities coming from $\mathcal{G}^{(0)}(\boldsymbol{p} - \boldsymbol{q}, i\omega_n - i\omega_m)$, we have

$$\text{Res}_3(i\omega_n - \xi_{\boldsymbol{p}-\boldsymbol{q}}; g) = \frac{-1}{e^{\beta(i\omega_n - \xi_{\boldsymbol{p}-\boldsymbol{q}})} - 1} \frac{\omega_{\boldsymbol{q}}^2}{(i\omega_n - \xi_{\boldsymbol{p}-\boldsymbol{q}})^2 - \omega_{\boldsymbol{q}}^2}$$

$$= \frac{1}{e^{-\beta\xi_{\boldsymbol{p}-\boldsymbol{q}}} + 1} \frac{\omega_{\boldsymbol{q}}^2}{(i\omega_n - \xi_{\boldsymbol{p}-\boldsymbol{q}})^2 - \omega_{\boldsymbol{q}}^2}$$
$$= \frac{n_F(-\xi_{\boldsymbol{p}-\boldsymbol{q}})\omega_{\boldsymbol{q}}^2}{(i\omega_n - \xi_{\boldsymbol{p}-\boldsymbol{q}})^2 - \omega_{\boldsymbol{q}}^2},$$

where in the second line we utilize the condition that fermions only have the contribution of the summation of odd integer $\omega_n = \frac{(2n+1)\pi}{\beta}$. Rewriting the last residue in partial fractions

$$\text{Res}_3 = \frac{-\omega_{\boldsymbol{q}}}{2} \left(\frac{n_F(-\xi_{\boldsymbol{p}-\boldsymbol{q}})}{i\omega_n - \xi_{\boldsymbol{p}-\boldsymbol{q}} + \omega_{\boldsymbol{q}}} - \frac{n_F(-\xi_{\boldsymbol{p}-\boldsymbol{q}})}{i\omega_n - \xi_{\boldsymbol{p}-\boldsymbol{q}} - \omega_{\boldsymbol{q}}} \right),$$

then the integral 11 becomes

$$0 = \sum_m \frac{1}{\beta} f(i\omega_m) + \frac{\omega_{\boldsymbol{q}}}{2} \left(\frac{n_F(-\xi_{\boldsymbol{p}-\boldsymbol{q}}) + n_B(\omega_{\boldsymbol{q}})}{i\omega_n - \xi_{\boldsymbol{p}-\boldsymbol{q}} - \omega_{\boldsymbol{q}}} + \frac{-n_F(-\xi_{\boldsymbol{p}-\boldsymbol{q}}) + n_B(\omega_{\boldsymbol{q}}) + 1}{i\omega_n - \xi_{\boldsymbol{p}-\boldsymbol{q}} + \omega_{\boldsymbol{q}}} \right).$$

Hence

$$\Sigma(\boldsymbol{p}, i\omega_n) \equiv g^2 \int \frac{d\boldsymbol{q}}{(2\pi)^3} S(\boldsymbol{p}, i\omega_n; \boldsymbol{q}) = \frac{g^2}{2} \int \frac{d\boldsymbol{q}}{(2\pi)^3} \frac{\omega_{\boldsymbol{q}}}{2} \left(\frac{n_F(-\xi_{\boldsymbol{p}-\boldsymbol{q}}) + n_B(\omega_{\boldsymbol{q}})}{i\omega_n - \xi_{\boldsymbol{p}-\boldsymbol{q}} - \omega_{\boldsymbol{q}}} + \frac{-n_F(-\xi_{\boldsymbol{p}-\boldsymbol{q}}) + n_B(\omega_{\boldsymbol{q}}) + 1}{i\omega_n - \xi_{\boldsymbol{p}-\boldsymbol{q}} + \omega_{\boldsymbol{q}}} \right). \tag{12}$$

Standardized *analysis continuation* of (12), $i\omega_n \to E + i\delta$, directly gives the *retarded real-time* self-energy $\Sigma^R(\boldsymbol{p}, E)$

$$\Sigma^R(\boldsymbol{p}, E) = g^2 \int \frac{d\boldsymbol{q}}{(2\pi)^3} \frac{\omega_{\boldsymbol{q}}}{2} \left(\frac{n_F(-\xi_{\boldsymbol{p}-\boldsymbol{q}}) + n_B(\omega_{\boldsymbol{q}})}{E - \xi_{\boldsymbol{p}-\boldsymbol{q}} - \omega_{\boldsymbol{q}} + i\delta} + \frac{-n_F(-\xi_{\boldsymbol{p}-\boldsymbol{q}}) + n_B(\omega_{\boldsymbol{q}}) + 1}{E - \xi_{\boldsymbol{p}-\boldsymbol{q}} + \omega_{\boldsymbol{q}} + i\delta} \right). \tag{13}$$

Now that we only concern about the asymptotic behaviors of the *imaginary part* of $\Sigma^R(\boldsymbol{p}, E)$, we can simplify the complex integral (13) by taking the imaginary part before integration. In fact, by Sokhotsky-Wiestrass formula $\frac{1}{x \pm i\delta} = \mathcal{P}\frac{1}{x} \mp i\delta(x)$, (13) becomes

$$\text{Im}\,\Sigma^R(\boldsymbol{p}, E) = g^2 \int \frac{d\boldsymbol{q}}{(2\pi)^3} \frac{\omega_{\boldsymbol{q}}}{2} \left[\left(-n_F(-\xi_{\boldsymbol{p}-\boldsymbol{q}}) + n_B(\omega_{\boldsymbol{q}}) + 1 \right) \cdot \delta\left(E - \xi_{\boldsymbol{p}-\boldsymbol{q}} + \omega_{\boldsymbol{q}}\right) \right.$$
$$\left. + \left(n_F(-\xi_{\boldsymbol{p}-\boldsymbol{q}}) + n_B(\omega_{\boldsymbol{q}}) \right) \cdot \delta\left(E - \xi_{\boldsymbol{p}-\boldsymbol{q}} - \omega_{\boldsymbol{q}}\right) \right]$$
$$= g^2 \int \frac{d\boldsymbol{q}}{(2\pi)^3} \frac{\omega_{\boldsymbol{q}}}{2} \left[\left(-n_F(-\xi_{\boldsymbol{p}-\boldsymbol{q}}) + n_B(\omega_{\boldsymbol{q}}) + 1 \right) \cdot \delta\left(E - \frac{p^2 - 2\boldsymbol{p}\cdot\boldsymbol{q} + \mathcal{O}(q^2) - p_F^2}{2m} + cq\right) \right.$$
$$\left. + \left(n_F(-\xi_{\boldsymbol{p}-\boldsymbol{q}}) + n_B(\omega_{\boldsymbol{q}}) \right) \cdot \delta\left(E - \frac{p^2 - 2\boldsymbol{p}\cdot\boldsymbol{q} + \mathcal{O}(q^2) - p_F^2}{2m} - cq\right) \right]$$

$$= g^2 \int \frac{q^2 \, \mathrm{d}q \, \mathrm{d}\cos\theta}{(2\pi)^2} \frac{\omega_{\boldsymbol{q}}}{2} \left[\left(-n_F(-\xi_{\boldsymbol{p}-\boldsymbol{q}}) + n_B(\omega_{\boldsymbol{q}}) + 1 \right) \cdot \delta\left(E - \xi_{\boldsymbol{p}} + \frac{pq}{m}\cos\theta + cq \right) \right.$$

$$\left. + \left(n_F(-\xi_{\boldsymbol{p}-\boldsymbol{q}}) + n_B(\omega_{\boldsymbol{q}}) \right) \cdot \delta\left(E - \xi_{\boldsymbol{p}} + \frac{pq}{m}\cos\theta - cq \right) \right]$$

$$= g^2 \int \frac{q^2 \, \mathrm{d}q \, \mathrm{d}\cos\theta}{(2\pi)^2} \frac{\omega_{\boldsymbol{q}}}{2} \frac{1}{\left| \frac{pq}{m} \right|} \left[\left(-n_F(-\xi_{\boldsymbol{p}-\boldsymbol{q}}) + n_B(\omega_{\boldsymbol{q}}) + 1 \right) \cdot \delta\left(\cos\theta - \frac{m}{pq}(\xi_{\boldsymbol{p}} - E - cq) \right) \right.$$

$$\left. + \left(n_F(-\xi_{\boldsymbol{p}-\boldsymbol{q}}) + n_B(\omega_{\boldsymbol{q}}) \right) \cdot \delta\left(\cos\theta - \frac{m}{pq}(\xi_{\boldsymbol{p}} - E + cq) \right) \right]$$

$$= g^2 \int \frac{q^2 \, \mathrm{d}q \, \mathrm{d}\cos\theta}{(2\pi)^2} \frac{\omega_{\boldsymbol{q}} m}{2pq} \left\{ \left[-n_F\left(-\xi_{\boldsymbol{p}} + \frac{pq}{m}\cos\theta \right) + n_B(cq) + 1 \right] \cdot \delta\left(\cos\theta - \frac{m}{pq}(\xi_{\boldsymbol{p}} - E - cq) \right) \right.$$

$$\left. + \left[n_F\left(-\xi_{\boldsymbol{p}} + \frac{pq}{m}\cos\theta \right) + n_B(cq) \right] \cdot \delta\left(\cos\theta - \frac{m}{pq}(\xi_{\boldsymbol{p}} - E + cq) \right) \right] \right\}$$

$$= g^2 \int \frac{\mathrm{d}q}{(2\pi)^2} \frac{mq\omega_{\boldsymbol{q}}}{2p} \left[-n_F(-E - cq) + 2n_B(cq) + 1 + n_F(-E + cq) \right]$$

$$= \frac{g^2 mc}{2p(2\pi)^2} \int \mathrm{d}q \, q^2 \left[\frac{1}{e^{-\beta(E+cq)} + 1} + \frac{1}{e^{\beta(-E+cq)} + 1} + 1 + \frac{2}{e^{-\beta cq} - 1} \right]$$

$$= \frac{g^2 mc}{2p(2\pi)^2} \int \mathrm{d}q \left[\frac{q^2}{e^{\beta cq} + e^{\beta E}} + \frac{q^2}{e^{\beta(-E+cq)} + 1} + \frac{2q^2}{e^{-\beta cq} - 1} \right]$$

$$= \frac{g^2 mc}{2p(2\pi)^2} \int \mathrm{d}q \left[-\frac{2 \operatorname{Li}_3(-e^{-\beta E})}{\beta^3 c^3} - \frac{2 \operatorname{Li}_3(-e^{\beta E})}{\beta^3 c^3} + \frac{4\zeta(3)}{c^3 \beta^3} \right],$$

where in the second equal sign we utilize the conditions mentioned in section one that only *long-wavelength longitudinal acoustic phonons* couple with electrons, so $q \to 0$ and $\omega_{\boldsymbol{q}} = cq$. And in the last line we introduce the *polylogarithm function* and the celebrated *Riemann Zeta function*

$$\operatorname{Li}_n(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^n}, \quad \zeta(s) := \sum_{k=1}^{\infty} k^{-s}.$$

At the *low temperature limit* $\beta \to \infty$, the first part

$$\frac{-2 \operatorname{Li}_3(-e^{-\beta\epsilon})}{\beta^3 c^3} \sim \frac{2e^{-\beta E}}{c^3 \beta^3} + \mathcal{O}(\beta^{-3} e^{-2\beta E})$$

converges exponentially so can be dropped out, while

$$\frac{-2 \operatorname{Li}_3(-e^{\beta\epsilon})}{\beta^3 c^3} \sim \frac{3E^3}{c^3} + \mathcal{O}(\beta^{-3} e^{-\beta E}).$$

Therefore, electron damping due to electron-phonon interaction for $E \ll \omega_D$ and $T \ll \omega_D$ can be written in a unified form that

$$\operatorname{Im} \Sigma^R(\boldsymbol{p}, E) \sim \max\left[\frac{4\zeta(3)}{c^3} T^3, \frac{3}{c^3} E^3 \right]. \tag{14}$$

Particularly, at zero temperature, we expect the asymptotic behavior of self-energy as (still under the condition that $E \ll \omega_D$)

$$\operatorname{Im} \Sigma^R(\boldsymbol{p}, E) \sim E^3. \tag{15}$$

This relation will be independently checked in the next section.

ASYMPTOTIC BEHAVIOR OF SELF-ENERGY AT ZERO TEMPERATURE

In this section, we will prove the self-consistency of the computation we perform above through direct computation of asymptotic behaviors of electron self-energy.

We start with re-writing (8) explicitly

$$\Sigma(\boldsymbol{p}, E) = ig^2 \int \frac{\mathrm{d}\boldsymbol{q} \, \mathrm{d}\omega}{(2\pi)^4} \frac{1}{E - \omega - \xi_{\boldsymbol{p}-\boldsymbol{q}} + i\delta_{\boldsymbol{p}-\boldsymbol{q}}}$$
$$\times \frac{c^2 q^2}{\omega^2 - c^2 q^2 + i\delta}, \tag{16}$$

where $i\delta_{\boldsymbol{p}-\boldsymbol{q}} \equiv i\delta \operatorname{sgn}(\xi_{\boldsymbol{p}-\boldsymbol{q}})$ and we substituted the long wave-length limit $\omega_{\boldsymbol{k}} = ck$. Besides, it is worth mentioning that the two infinitesimal δ on two sides of times sign, though are written in the same symbol, is always *independent*. One can easily observe that singularities $\omega_1 = E - \xi_{\boldsymbol{p}-\boldsymbol{q}} + i\delta_{\boldsymbol{p}-\boldsymbol{q}}, \omega_{2,3} = \pm(cq - i\delta)$ settle on the complex frequency plane. So the integral (21) can be splitted into two parts in terms of the sign of $\delta_{\boldsymbol{p}-\boldsymbol{q}}$: when $\operatorname{sgn}(\xi_{\boldsymbol{p}-\boldsymbol{q}}) > 0$, we choose the *upper* semi-circle as the integral contour, in which merely *one* singularity ω_3 exists. Ditto for another case $\operatorname{sgn}(\xi_{\boldsymbol{p}-\boldsymbol{q}}) > 0$. Note that the power of integrand is q^{-3}, so the contribution from the arc vanishes as the radius of semicircle tends to infinity. Hence by residue theorem the real integral (21) equals to the residues in two cases.

$$\Sigma(\boldsymbol{p}, E) = \frac{ig^2}{(2\pi)^4} \times \int d\boldsymbol{q}\left(2\pi i \sum_{j=2,3} \mathrm{Res}(\omega_j)\right)$$

$$= \frac{ig^2}{(2\pi)^4}\left(\int_{\xi_{\boldsymbol{p}-\boldsymbol{q}}>0} (-2\pi i) \times \frac{1}{E-(cq-i\delta)-\xi_{\boldsymbol{p}-\boldsymbol{q}}+i\delta} \times \frac{1}{2(cq-i\delta)}\right.$$

$$\left. + \int_{\xi_{\boldsymbol{p}-\boldsymbol{q}}<0} 2\pi i \times \frac{1}{E+(cq-i\delta)-\xi_{\boldsymbol{p}-\boldsymbol{q}}-i\delta} \times \frac{1}{-2(cq-i\delta)}\right) \cdot c^2 q^2 \ d\boldsymbol{q}$$

$$= \frac{g^2}{(2\pi)^3}\left(\int_{\xi_{\boldsymbol{p}-\boldsymbol{q}}>0} \frac{1}{E-cq-\xi_{\boldsymbol{p}-\boldsymbol{q}}+2i\delta} \times \frac{1}{2cq-2i\delta} + \int_{\xi_{\boldsymbol{p}-\boldsymbol{q}}<0} \frac{1}{E+cq-\xi_{\boldsymbol{p}-\boldsymbol{q}}-2i\delta} \times \frac{1}{2cq-2i\delta}\right) \cdot c^2 q^2 \ d\boldsymbol{q}$$

$$= \frac{cg^2}{2(2\pi)^3}\left(\int_{\xi_{\boldsymbol{p}-\boldsymbol{q}}>0} \frac{q}{E-cq-\xi_{\boldsymbol{p}-\boldsymbol{q}}+i\delta'} + \int_{\xi_{\boldsymbol{p}-\boldsymbol{q}}<0} \frac{q}{E+cq-\xi_{\boldsymbol{p}-\boldsymbol{q}}-i\delta'}\right) d\boldsymbol{q}, \tag{17}$$

where in the last line we redefine the infinitesimal term by $\delta' = 2\delta$ and it *seems* that we have *partially drop the crucial infinitesimal term* $-2i\delta$ *in each second fraction.* This operation is abused in [1], but is entirely mathematically well-grounded! We will give a short interpretation on this dropping operation in the appendix.

Again, since we only concern about the imaginary part of $\Sigma(\boldsymbol{p}, E)$, Sokhotsky-Wiestrass formula can also be applied to extract the *imaginary part* of (22)

$$\mathrm{Im}\,\Sigma(\boldsymbol{p}, E) = \frac{\pi cg^2}{2(2\pi)^3}\left[\int_{\xi_{\boldsymbol{p}-\boldsymbol{q}}>0} -q\delta\left(E-cq-\xi_{\boldsymbol{p}-\boldsymbol{q}}\right) + \int_{\xi_{\boldsymbol{p}-\boldsymbol{q}}<0} q\delta\left(E+cq-\xi_{\boldsymbol{p}-\boldsymbol{q}}\right)\right] d\boldsymbol{q}$$

$$= \frac{cg^2}{8\pi}\int dq \left\{\int_{\cos\theta<\frac{m\xi_{\boldsymbol{p}}}{pq}} d\cos\theta \left[-q^3\delta\left(E-cq-\xi_{\boldsymbol{p}}+\frac{pq}{m}\cos\theta\right)\right]\right.$$

$$\left. + \int_{\cos\theta>\frac{m\xi_{\boldsymbol{p}}}{pq}} d\cos\theta \left[q^3\delta\left(E+cq-\xi_{\boldsymbol{p}}+\frac{pq}{m}\cos\theta\right)\right]\right\}$$

$$= \frac{cg^2}{8\pi}\int dq \left\{\int_{\cos\theta<\frac{m\xi_{\boldsymbol{p}}}{pq}} d\cos\theta \left[\frac{-q^3}{\left|\frac{pq}{m}\right|}\delta\left(\cos\theta - \frac{m}{pq}(\xi_{\boldsymbol{p}}+cq-E)\right)\right]\right.$$

$$\left. + \int_{\cos\theta>\frac{m\xi_{\boldsymbol{p}}}{pq}} d\cos\theta \left[\frac{q^3}{\left|\frac{pq}{m}\right|}\delta\left(\cos\theta - \frac{m}{pq}(\xi_{\boldsymbol{p}}-E-cq)\right)\right]\right\}$$

where in the second line we again utilize the *long wavelength* condition that $\xi_{\boldsymbol{p}-\boldsymbol{q}} = \xi_{\boldsymbol{p}} - \frac{pq}{m}\cos\theta + \mathcal{O}(q^2)$. The first term is interpreted as the *lifetime of quasi-electrons* and another term as *lifetime of quasi-holes*. In fact, consider the cases $E \ll \omega_D$. If $E > 0$, or the momentum of phonons are limited by $q < \frac{E}{c}$, then the second term has no contribution since $(\xi_{\boldsymbol{p}} - E - cq) < \xi_{\boldsymbol{p}}$. So

$$\mathrm{Im}\,\Sigma(\boldsymbol{p}, E) = \frac{cg^2}{8\pi}\int_0^{E/c} dq \frac{mq^2}{p} = \frac{g^2 c}{24\pi v_F c^3}E^3 \sim E^3, \tag{18}$$

where we take the Fermi surface approximation $\frac{p}{m} = v_F$. Likewise, if $E < 0$, since $(\xi_{\boldsymbol{p}} + cq - E) > \xi_{\boldsymbol{p}}$, only the second term contributes to the integral, so we still have

$$\mathrm{Im}\,\Sigma(\boldsymbol{p}, E) = \frac{cg^2}{8\pi}\int_0^{|E|/c} dq \frac{-mq^2}{p} = \frac{g^2 c}{24\pi v_F c^3}E^3 \sim E^3. \tag{19}$$

As for the case $E \gg \omega_D > 0$, only the first term contributes to the integral and the phonon momentum is limited by k_D this time, so

$$\mathrm{Im}\,\Sigma(\boldsymbol{p}, E) = \frac{cg^2}{8\pi}\int_0^{k_D} dq \frac{mq^2}{p} = \frac{g^2 c}{24\pi v_F c^3}k_D^3. \tag{20}$$

FIELD RENORMALIZATION FACTOR

By the celebrated Kramers–Kronig relation, real part of self-energy at about zero temperature can be obtained

by

$$\operatorname{Re}\Sigma^R(\boldsymbol{p}, E) = \mathcal{P}\int \mathrm{d}E' \, \frac{\operatorname{Im}\Sigma^R(\boldsymbol{p}, E')}{E - E'}$$

$$\sim \mathcal{P}\int \mathrm{d}E' \, \frac{E'^3}{E - E'}$$

$$\sim E_c^2 E - E^3 \log\left(\frac{E^2}{E_c^2 - E^2}\right). \qquad (21)$$

Hence the field renormalization factor

$$Z = \left(1 - \left.\frac{\partial}{\partial E}\right|_{E=0} \operatorname{Re}\Sigma^R(\boldsymbol{p}, E)\right)^{-1} = \frac{1}{1 - E_c^2} \neq 0. \qquad (22)$$

Suppose $f(x)$ is *continuous* on the $\mathbb{R}$ and *compactly supported*, then (under the meaning of distribution)

$$\lim_{\delta \to 0^+}\int \frac{f(x)}{x \pm i\delta}\,\mathrm{d}x \equiv \mp i\pi \lim_{\delta \to 0^+}\int \frac{\delta}{\pi(x^2 + \delta^2)}f(x)\,\mathrm{d}x + \lim_{\delta \to 0^+}\int \frac{x^2}{x^2 + \delta^2}\frac{f(x)}{x}\,\mathrm{d}x$$

$$= \mp i\pi f(0) + \mathcal{P}\int \frac{f(x)}{x}\,\mathrm{d}x,$$

where in the first term we reverse the order of limit and integral, which holds for any distribution – and here is Dirac delta function. As for the second term, since $f(x)$ is well-defined, so is the whole integrand. And Cauchy principle value comes from the fact that the integrand is symmetric about $x = 0$.

Now switch to our case and concentrate on the phonon momentum integral, i.e., $\mathrm{d}\boldsymbol{q} \equiv 2\pi \, \mathrm{d}\cos\theta \, q^2 \, \mathrm{d}q$, then

$$f(\boldsymbol{p}, \omega; \boldsymbol{q}) \sim \frac{q^4}{E - \omega - \xi_{\boldsymbol{p}-\boldsymbol{q}} + i\delta_{\boldsymbol{p}-\boldsymbol{q}}}$$

is well-defined on the real momentum axis with the help of up and down shift $i\delta_{\boldsymbol{p}-\boldsymbol{q}}$ and the proof above can be applied to the penultimate term in (22) (again we emphasize that the two infinitesimals δ are *independent*) and we are done.

and Landau Fermi-liquid theory holds in solids.

Appendix

In this section, we will give the interpretation on the seemingly partial dropping operation of the infinitesimal term in equation (22).

Actually, dropping infinitesimal is *superficiality*. To understand this we would better explore another explicit way to reach (22). This can be done by invoking the standard proof of Sokhotsky-Wiestrass formula as following:

[1] M. V. Sadovskii, *Diagrammatics: lectures on selected problems in condensed matter theory* (World Scientific, 2006).

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[3] G. D. Mahan, *Many-particle physics* (Springer Science & Business Media, 2013).

[4] E. M. Lifshitz and L. P. Pitaevskii, *Statistical physics: theory of the condensed state*, vol. 9 (Elsevier, 2013).