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Assignment - Parameter Estimation

Submitted by : CHELSI
Group : 3CS6
Roll No : 102117161

Q1. Let $(X_1, X_2, \ldots)$ be a random sample of size n taken from a normal population with parameters: mean $= \theta_1$ & variance $= \theta_2$. Find MLE of these 2 parameters.

Soln- For normal population,

$$\text{pmf} : \frac{1}{\sqrt{2\pi\theta_2}} e^{-\frac{1}{2}\left(\frac{x-\theta_1}{\sqrt{\theta_2}}\right)^2}$$

Likelihood function : $L(\theta_1, \theta_2)$

$$L(\theta_1, \theta_2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\cdot\sqrt{\theta_2}} e^{-\frac{1}{2}\frac{(x_i-\theta_1)^2}{\theta_2}}$$

$$\ln L(\theta_1, \theta_2) = \sum_{i=1}^{n}\left[\ln\frac{1}{\sqrt{2\pi}} + \ln\frac{1}{\sqrt{\theta_2}} - \frac{1}{2}\frac{(x_i-\theta_1)^2}{\theta_2}\right]$$

$$\ln L(\theta_1, \theta_2) = \sum_{i=1}^{n}\left[-\ln\sqrt{2\pi} - \ln\sqrt{\theta_2} - \frac{(x_i-\theta_1)^2}{2\theta_2}\right]$$

$$\ln L(\theta_1, \theta_2) = -n\ln\sqrt{2\pi} - n\ln\sqrt{\theta_2} - \frac{\sum_{i=1}^{n}(x_i-\theta_1)^2}{2\theta_2}$$

Taking partial derivative wrt θ_1

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$$\frac{\partial}{\partial \theta_1} \ln L(\theta_1, \theta_2) = \frac{\sum_{i=1}^{n} 2(x_i - \theta_1)}{2\theta_2}$$

$$\frac{\partial}{\partial \theta_1} \ln L(\theta_1, \theta_2) = \frac{\sum_{i=1}^{n} (x_i - \theta_1)}{\theta_2}$$

For getting max value, $\frac{\partial}{\partial \theta_1} \ln L(\theta_1, \theta_2) = 0$

$$\Rightarrow \sum x_i - \sum \theta_1 = 0$$

$$\sum x_i = n\theta_1$$

$$\theta_1 = \frac{\sum x_i}{n}$$

$$\boxed{\hat{\theta}_1 = \bar{x}} \quad \text{(mean)}$$

Taking double derivative,

$$\frac{\partial^2}{\partial \theta_1^2} \ln L(\theta_1, \theta_2) = \frac{\sum(-1)}{\theta_2} = -\frac{n}{\theta_2} < 0$$

Since, it is < 0 So, obtained value is maximum.

Now, taking partial derivative w.r.t. $\sqrt{\theta_2}$.

(Say $\sqrt{\theta_2} = \sigma$)

$$\frac{\partial}{\partial \sigma} \ln L(\theta_1, \sigma^2) = -\frac{n}{\sigma} + \frac{\sum (x_i - \theta_1)^2}{\sigma^3}$$

To get max value, $\frac{\partial}{\partial \sigma} \ln L(\theta_1, \sigma^2) = 0$.

$$\Rightarrow \frac{n}{\sigma} = \frac{\sum (x_i - \theta_1)^2}{\sigma^3}$$

$$\sigma^2 = \frac{\sum (x_i - \theta_1)^2}{n}$$

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$$\Rightarrow \theta_2 = \frac{\Sigma(x_i - \theta_1)^2}{n}$$

Putting $\theta_1 = \bar{X}$,

$$\boxed{\theta_2 = \frac{\Sigma(x_i - \bar{X})^2}{n}}$$ (Variance)

Taking double derivative,

$$\frac{\partial^2}{\partial \theta_2} \ln L(\theta_1, \theta_2) = \frac{n}{\theta_2} - \frac{3}{\theta_2^2} \Sigma(x_i - \theta_1)^2$$

$$= \frac{n}{\theta_2} - \frac{3}{\theta_2^2} n\theta_2$$

$$= -\frac{2n}{\theta_2} < 0$$

Since double derivative is < 0. So, obtained value is maximum.

Overall, $\hat{\theta_1} = \bar{X}$

$$\hat{\theta_2} = \frac{\Sigma(x_i - \theta_1)^2}{n}$$

Ans.

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Q2. Let $X_1, X_2, \ldots X_n$ be random sample from $B(m, \theta)$ distribution where $\theta \in \Theta = (0,1)$ is unknown and m is known positive integer. Compute value of θ using MLE.

Soln: For binomial distribution,

$$\text{pmf}: \; {}^{m}C_{x_i}\, \theta^{x_i}(1-\theta)^{m-x}$$

Likelihood function,

$$L(\theta) = \prod_{i=1}^{n}\left[{}^{m}C_{x_i}\, \theta^{x_i}(1-\theta)^{m-x_i}\right]$$

$$L(\theta) = \prod_{i=1}^{n}\left[{}^{m}C_{x_i}\right] \cdot \theta^{\sum_{i=1}^{n} x_i} \cdot (1-\theta)^{\sum_{i=1}^{n}(m-x_i)}$$

$$L(\theta) = \left(\prod {}^{m}C_{x_i}\right) \cdot \theta^{\Sigma x_i} \cdot (1-\theta)^{nm-\Sigma x_i}$$

$$\ln L(\theta) = \sum_{i=1}^{n} \ln {}^{m}C_{x_i} + \Sigma x_i \ln\theta + (nm - \Sigma x_i)\ln(1-\theta)$$

Taking derivative wrt θ.

$$\frac{\partial}{\partial\theta}\ln L(\theta) = \frac{\Sigma x_i}{\theta} + \frac{nm - \Sigma x_i}{1-\theta}\cdot(-1)$$

$$\frac{\partial}{\partial\theta}\ln L(\theta) = \frac{\Sigma x_i - \theta\Sigma x_i - nm\theta + \theta\Sigma x_i}{\theta(1-\theta)}$$

$$\frac{\partial}{\partial\theta}\ln L(\theta) = \frac{\Sigma x_i - nm\theta}{\theta(1-\theta)}$$

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To get max value, $\frac{\partial}{\partial\theta} \ln h(\theta) = 0$

$$\Rightarrow \quad \theta = \frac{\Sigma x_i}{nm} = \frac{\bar{X}}{m}$$

$$\Rightarrow \boxed{\hat{\theta} = \frac{\bar{X}}{m}}$$

Taking double derivative,

$$\frac{\partial^2}{\partial\theta^2} \ln h(\theta) = -\frac{\Sigma x_i}{\theta^2} - \frac{(nm - \Sigma x_i)}{(1-\theta)^2}$$

$$\frac{\partial^2}{\partial\theta^2} \ln h(\theta) = -\frac{\Sigma x_i - \theta^2 \Sigma x_i + 2\theta \Sigma x_i - \theta^2 nm + \Sigma x_i \theta^2}{\theta^2 (1-\theta)^2}$$

$$= -\left[\frac{\Sigma x_i + \theta^2 nm - 2\theta \Sigma x_i}{\theta^2 (1-\theta)^2}\right]$$

$$= -nm \left[\frac{\theta + \theta^2 - 2\theta^2}{\theta^2 (1-\theta)^2}\right]$$

$$= \frac{-nm (1-\theta)\theta}{\theta (1-\theta)^2}$$

$$\frac{\partial^2}{\partial\theta^2} \ln h(\theta) = -\frac{nm}{\theta(1-\theta)} < 0 \qquad [\theta \in (0,1)]$$

Since double derivative is < 0 so, obtained value is maximum.

$$\therefore \boxed{\hat{\theta} = \frac{\bar{X}}{m}} \quad \text{Ans.}$$