Modelling With Differential Equations

August 2019

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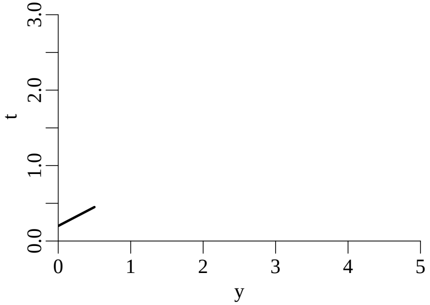
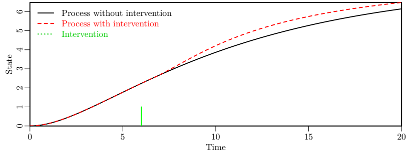
  - Complexity – state dependent parameters, random effects, etc.

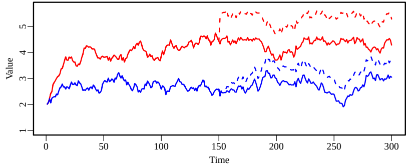
```
y0 = .2
t=4
b=.5
y= y0 + b * t
print(y)
[1] 2.2
```

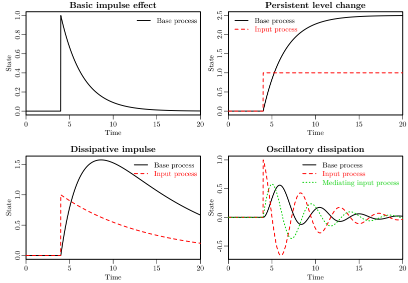
- $\frac{\mathrm{d}y}{\mathrm{d}t} = b$
- $\dot{y} = b$
- $y(t) = y_{t0} + bt$

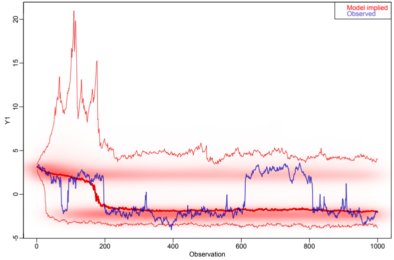


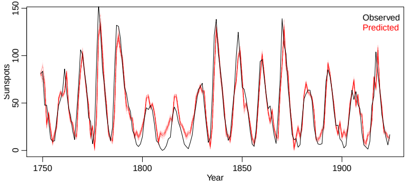
How can we use differential equations?





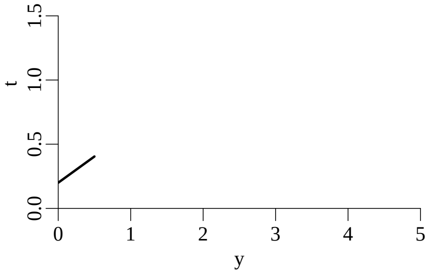






```
y0 = 0.2
t= 4
b= 0.5
A = -0.3
y= exp(A * t) * y0 + (exp(A*t)-1)*(1/A)*b
print(y)
[1] 1.224915
```

- $\frac{\mathrm{d}y}{\mathrm{d}t} = Ay + b$
- $\dot{y} = Ay + b$
- $y(t) = e^{At} y_{t0} + (e^{At} - 1)A^{-1}b$



$$\underbrace{\mathrm{d}\begin{bmatrix}\text{eta1}\\\text{eta2}\end{bmatrix}(t)}_{\mathrm{d}\boldsymbol{\eta}(t)} = \left(\underbrace{\begin{bmatrix}-0.3 & 0\\0.4 & -0.1\end{bmatrix}}_{\mathbf{A}}\underbrace{\begin{bmatrix}\text{eta1}\\\text{eta2}\end{bmatrix}(t)}_{\boldsymbol{\eta}(t)} + \underbrace{\begin{bmatrix}1\\0\end{bmatrix}}_{\mathbf{b}}\right)\mathrm{d}t$$

$$\underbrace{\begin{bmatrix}\text{Y1}\\\text{Y2}\\\text{Y3}\end{bmatrix}(t)}_{\mathbf{Y}(t)} = \underbrace{\begin{bmatrix}1 & 0\\0.8 & 0\\0 & 1\end{bmatrix}}_{\boldsymbol{\Lambda}}\underbrace{\begin{bmatrix}\text{eta1}\\\text{eta2}\end{bmatrix}(t)}_{\boldsymbol{\eta}(t)} + \underbrace{\begin{bmatrix}1.5\\0\\-2\end{bmatrix}}_{\boldsymbol{\tau}} + \underbrace{\begin{bmatrix}\epsilon_1\\\epsilon_2\\\epsilon_3\end{bmatrix}(t)}_{\boldsymbol{\epsilon}(t)}$$

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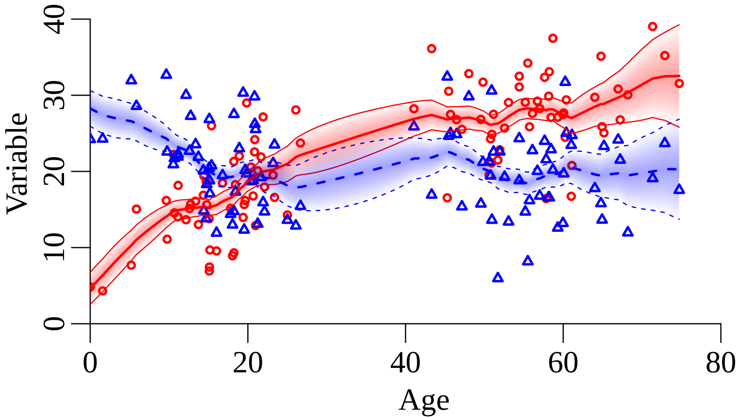
- In multivariate systems, zeroes only imply 'no effect' in differential equation formulation.
- Coupled with appropriate estimation routines – allows relatively unconstrained theory specification and testing.

- Useful for modelling change over any continuous variable.



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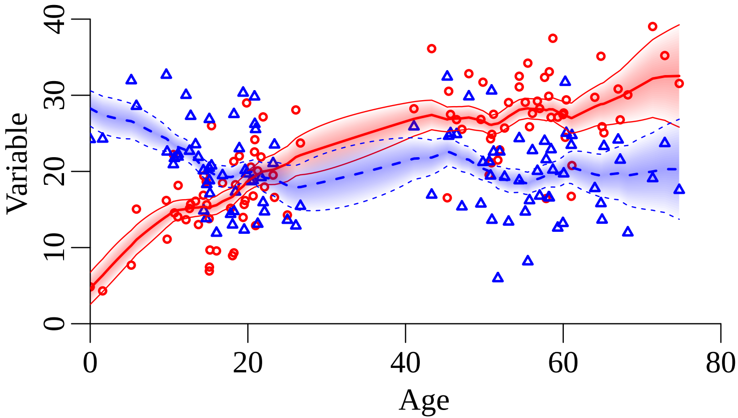
- Besides time, age is an obvious candidate for us.



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- Good reasons to use: More flexible than typical regression / SEM approaches. Random / time varying effects on any parameter. Estimation works under problematic conditions, many parameters etc.
- Good reasons to avoid: More flexible than typical regression / SEM approaches. Less developed workflow, doesn't cope with high dimensions in variable domain.