Approximation by Superpositions of a Sigmoidal Function

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Introduction

In many Machine Learning applications the goal is to approximate a continuous function $f : \mathbf{R}^n \to \mathbf{R}$ by an expression of the form

$$\sum_{j=1}^{N} \alpha_j \sigma(y_j^{\mathsf{T}} x + \theta_j), \qquad (\star)$$

where $y_j \in \mathbf{R}^n$ and $\alpha_j, \theta_j \in \mathbf{R}$ are fixed.
In Machine Learning terminology, the y represent weights and the θ biases.
Equation $(\star)$ models a fully connected network with one hidden layer.

The Main Result

The author demonstrates that, theoretically, such an architecture is sufficient to approximate all continuous functions on the unit cube. That is,

Theorem 1 (Theorem 2 in [C2])

For σ a continuous sigmoidal function, finite sums of the form

$$\sum_{j=1}^{N} \alpha_j \sigma(y_j^{\mathsf{T}} x + \theta_j),$$

are dense in $C(I_n)$.

Sigmoidal Functions are Universal Approximators

The proof will rely on the following more general result

Theorem 2 (Theorem 1 in [C2])

Let σ be continuous and discriminatory. Then the set of finite sums of the form

$$\sum_{j=1}^{N} \alpha_j \sigma(y_j^{\mathsf{T}} x + \theta_j),$$

is dense in $C(I_n)$.

With this theorem at our disposal we need only show that sigmoidal functions are discriminatory; that is, we will show that

Lemma 3 (Lemma 1 in [C2])

Any continuous sigmoidal function is discriminatory.

A Corollary of the Hahn–Banach Theorem

To prove Theorem 2 we will need a little bit of Functional Analysis. In particular, we will need the following corollary of the Hahn–Banach Theorem

Corollary 4

If V is a normed vector space, W a subspace of V, and $v_0 \in V \setminus W$ with $\operatorname{dist}(v_0, W)$ nonzero, then there exists a bounded linear functional L such that $L(v_0) = 1$, but $L(v) = 0$ for all $v \in W$.

Proof.

The proof follows from the Hahn–Banach Theorem applied to the quotient V/W. For details, see Corollary 6.8 in [C1], p. 79. $\square$

The Riesz Representation Theorem

We will also be needing use of the Riesz Representation Theorem

Theorem 5 (Riesz Representation Theorem)

If X is locally compact and μ a measure on X, the map

$$\mu \longmapsto \int_X f \, d\mu$$

is an isometric isomorphism from the space of measures on X, $M(X)$, to the dual of $C_0(X)$.

Proof.

See [C1], Theorem 5.7. □

What the Riesz Representation Theorem is saying is that every bounded linear functional L on $C_0(X)$ is of the form $L(f) = \int_X f \, d\mu$ for some measure μ on X.

Discriminatory and Sigmoidal Functions

Let's define a couple of these terms.

Definition 6 (Discriminatory functions)

We say σ is *discriminatory* if given a measure μ on I_n

$$\int_{I_n} \sigma(y^\mathsf{T} x + \theta)\, d\mu(x) = 0$$

for all $y \in \mathbf{R}^n$ implies $\mu = 0$.

And of course.

Definition 7 (Sigmoidal functions)

We say that σ is sigmoidal if

$$\begin{cases} \sigma(t) \to 1 & \text{as } t \to +\infty, \\ \sigma(t) \to 0 & \text{as } t \to -\infty. \end{cases}$$

Proof of Theorem 1, Part 1

- Let $S = \left\{ \sum_{j=1}^{N} \alpha_j \sigma(y_j^{\mathsf{T}} x + \theta_j) \right\} \subset C(I_n)$. Clearly S is a subspace of $C(I_n)$.
- The author proceeds by showing that the closure, R, of S is $C(I_n)$:
- By Corollary 4, there exists a functional L which is zero on S but nonzero on $C(I_n) \setminus S$.
- By the Riesz Representation Theorem, L is of the form

$$L(h) = \int_{I_n} h(x)\, d\mu(x),$$

for some μ measure on I_n.

Proof of Theorem 1, Part 2

- Since $\sigma(y^\mathsf{T}x + \theta)$ is in R for all y, θ,
$$\int_{I_n} \sigma(y^\mathsf{T}x + \theta)\, d\mu(x) = 0,$$
for all y, θ.
- However, since σ was assumed to be discriminatory, $\mu = 0$.
- This contradicts assumption that L is nonzero.
- Therefore, $R = C(I_n)$.

Sigmoidal Functions are Discriminatory

Next the author shows that the sigmoidal functions that we care about are in fact discriminatory. More generally, he shows that

Lemma 8

Any bounded, measurable sigmoidal function is discriminatory. In particular, any continuous sigmoidal function is discriminatory.

Proof of Lemma 3, Part 1

- First note that

$$\sigma(\lambda(y^{\mathsf{T}}x+\theta)+\phi)\begin{cases}\to 1 & \text{for } y^{\mathsf{T}}x+\theta>0 \text{ as } \lambda\to+\infty,\\ \to 0 & \text{for } y^{\mathsf{T}}x+\theta<0 \text{ as } \lambda\to+\infty,\\ =\sigma(\phi) & \text{for } y^{\mathsf{T}}x+\theta=0 \text{ for all } \lambda.\end{cases}$$

- So $\sigma_\lambda(x) := \sigma(\lambda(y^{\mathsf{T}}x+\theta)+\phi)$ converges pointwise and boundedly to the function

$$\gamma(x)=\begin{cases}1 & \text{for } y^{\mathsf{T}}x+\theta>0,\\ 0 & \text{for } y^{\mathsf{T}}x+\theta<0,\\ \sigma(\phi) & \text{for } y^{\mathsf{T}}x+\theta=0,\end{cases}$$

as $\lambda\to\infty$.

Proof of Lemma 3, Part 2

- Let

$$\Pi_{y,\theta} = \{x\colon y^{\mathsf{T}}x + \theta = 0\},$$
$$H_{y,\theta} = \{x\colon y^{\mathsf{T}}x + \theta > 0\}.$$

- By the Lebesgue Bounded Convergence Theorem,

$$\begin{aligned} 0 &= \int_{I_n} \sigma_\lambda(x)\, d\mu(x) \\ &= \int_{I_n} \gamma(x)\, d\mu(x) \\ &= \sigma(\phi)\mu(\Pi_{y,\theta}) + \mu(H_{y,\theta}) \end{aligned}$$

for all ϕ, θ, y.

Proof of Lemma 3, Part 3

- The author next shows that if $\mu(H_{y,\theta}) = 0$ for all y, θ then $\mu = 0$.
- Fix y and define F by

$$F(h) = \int_{I_n} h(y^T x)\, d\mu(x).$$

  Note that F is a bounded functional on $L^\infty(\mathbf{R})$ since μ is a finite (signed) measure.
- Let $h = \chi_{[\theta,\infty)}$. Then

$$F(h) = \int_{I_n} \chi_{[\theta,\infty)}(y^\mathsf{T} x) = \mu(\Pi_{y,-\theta}) + \mu(H_{y,-\theta}) = 0.$$

- By linearity $L(h) = 0$ for $h = \chi_I$ for I any interval in $\mathbf{R}$.

Proof of Lemma 3, Part 4

- Since the simple functions are dense in $L^\infty(\mathbf{R})$, $F = 0$.
- Let $s(u) = \sin(mu)$ and $c(u) = \cos(mu)$.
- Then

$$
\begin{aligned}
L(c(m^\mathsf{T} x) + is(m^\mathsf{T} x)) &= \int_{I_n} \cos(m^\mathsf{T} x) + i \sin(m^\mathsf{T} x)\, d\mu(x) \\
&= \int_{I_n} \exp(i m^\mathsf{T} x)\, d\mu(x) \\
&= 0
\end{aligned}
$$

  for all m.
- Since the Fourier transform of μ is 0, μ itself must be 0. For further details, consult [R], Ch. 7.
- Therefore, σ is discriminatory.

Conclusion

By Lemma 3 and Theorem 2, the main result follows

Theorem 1 (Theorem 2 in [C2])

For σ a continuous sigmoidal function, finite sums of the form

$$\sum_{j=1}^{N} \alpha_j \sigma(y_j^{\mathsf{T}} x + \theta_j),$$

are dense in $C(I_n)$.

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