# Monopoles with Non-Maximal Symmetry Breaking

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# Contents



## 1 Introduction

These are some background notes about non-Abelian monopoles with non-maximal symmetry breaking. I will start by stating some Lie algebra conventions that we use then reviewing the standard case of maximal symmetry breaking. I will be following a variety of sources here including [3,7–9,14] The review paper [17] also contains some useful discussions about monopole clouds and moduli space metrics in the non-maximal symmetry breaking case.

The idea is that we will understand the example of non-maximal symmetry breaking for $SU(3)$ monopoles discussed in [3, 8], then try to construct our own examples. As part of this I would like to understand a gluing construction for the non-maximal symmetry breaking case. We can also try to understand the Nahm side following [7].

In [6] there is also a description of arbitrary symmetry breaking and how to approach it from the Nahm transform side.

## 2 Lie Algebra Conventions

For the Lie algebra conventions I will follow [11] for a review and to establish consistent conventions. This means that when we take examples from other papers we will need to convert them to these conventions if they are not in the same conventions. What is said here takes the base field to be $\mathbb{C}$ rather than $\mathbb{R}$, for a real Lie algebra we must work with its complexification $\mathfrak{g}^{\mathbb{C}}$ to get a root decomposition. I suppress the complexification here for notational simplicity.

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A Lie group G has Lie algebra $\mathfrak{g}$. A maximal abelian subalgebra of $\mathfrak{g}$ (consisting of semi-simple elements) is called a Cartan subalgebra $\mathfrak{h}$. It is not unique but all Cartan subalgebras are related by automorphisms of $\mathfrak{g}$. All Cartan subalgebras have the same dimension, $\dim\mathfrak{h} = \mathrm{rk}$, the rank of the group.

The generators of $\mathfrak{h}$ are the linearly independent ad-diagonalizable elements of $\mathfrak{g}$ $H^i$, $i = 1, 2, \ldots, \mathrm{rk}$ such that $[H^i, H^j] = 0$. Thus the adjoint maps $\mathrm{ad}_{H^i}$ are simultaneously diagonalisable. Thus $\mathfrak{g}$ is spanned by $y \in \mathfrak{g}$ that are simultaneous eigenvectors of $\mathrm{ad}_h$ $\forall h \in \mathfrak{h}$,

$$[h, y] = \mathrm{ad}_h\left(y\right) = \alpha_y\left(h\right) y, \tag{2.1}$$

with $\alpha_y\left(h\right) \in \mathbb{C}$ depending linearly on $h$ for fixed $y$ *i.e.* $\alpha_y : \mathfrak{h} \to \mathbb{C}$ is a linear functional and thus $\alpha_y \in \mathfrak{h}^*$. We call $\alpha$ a root of the $\mathfrak{g}$ when $\alpha \neq 0$. The roots lead to the root space decomposition of $\mathfrak{g}$ as

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\neq 0} \mathfrak{g}_\alpha, \qquad \mathfrak{g}_\alpha = \left\{x \in \mathfrak{g} \mid [h, x] = \alpha\left(h\right) \cdot x \,\forall h \in \mathfrak{h}\right\}. \tag{2.2}$$

Associated to this decomposition we have a basis $\mathcal{B}$ for $\mathfrak{g}$, which consists of the basis $\{H^i\}$ for $\mathfrak{h}$ and elements $E^\alpha$ such that

$$[H^i, E^\alpha] = \alpha^i E^\alpha. \tag{2.3}$$

The vector $(\alpha^1, \ldots, \alpha^{\mathrm{rk}})$ is called the root vector or just the root. The $E^\alpha$ are called ladder or step operators.

There is an inner product on $\mathfrak{g}$ called the Killing form. It is given by

$$\kappa\left(x, y\right) = \mathrm{Tr}\left(\mathrm{ad}_x \circ \mathrm{ad}_y\right), \tag{2.4}$$

for a matrix Lie algebra in a basis $T^a$ this becomes

$$\kappa^{ab} = \frac{1}{I_{\mathrm{ad}}}\kappa\left(T^a, T^b\right) = \frac{1}{I_{\mathrm{ad}}}\sum_{c,e} f^{bc}_{\ \ e} f^{ae}_{\ \ c}, \tag{2.5}$$

with $I_{\mathrm{ad}}$ a normalisation constant and $f^{bc}_{\ \ e}$ the structure constants such that $[T^b, T^c] = f^{bc}_{\ \ e}T^e$, upto a constant this si just the matrix trace on the basis.

The roots are split into positive, $R^+$, and negative, $R^-$, roots. We also have simple roots given by the positive roots which cannot be obtained as a linear combination of other positive roots with positive coefficients. There are rk simple roots $\alpha_i$. The simple roots do not form an orthonormal basis, with the lack of orthonormality encoded in the Cartan matrix

$$A_{ij} = 2\frac{(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}. \tag{2.6}$$

Here $(,)$ is a non-degenerate inner product on $\mathfrak{h}^*$ defined through $\kappa(,)$.

Associated to the roots are the coroots defined as

$$\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)}, \tag{2.7}$$

with $\alpha_i^\vee$ called a simple coroot[2]. The dual space of the root space is the weight space. Take the coroots to be a basis of the root space then the dual basis elements are the fundamental weights $\varpi_i$ such that $\varpi_i(\alpha_j) = \delta_{ij}$. This basis is called the Dynkin basis and the components of a weight in this basis are the Dynkin labels.

When we restrict to roots given by linear combinations of simple roots with integer coefficients we get the root lattice $L(\mathfrak{g})$. The integer space of simple coroots and fundamental weights are the coroot lattice $L^\vee(\mathfrak{g})$ and the weight lattice $L_W(\mathfrak{g})$ respectively.

When $\mathfrak{g}$ is simple there is a unique highest root called $\theta$ such that the *height* of $\theta$ is larger than any other root and $(\theta,\theta) \geq (\alpha,\alpha)$ for all roots $\alpha$. The height of a root is defined as the sum of the integer coefficients of the simple roots in its decomposition. Conventionally, $(\theta,\theta) = 2$ which fixes the normalisation $I_{\text{ad}}$.

The decomposition of the highest root in terms of simple roots defines the Coxeter numbers $m_i$ and their dual $m_i^\vee$ as

$$\theta = \sum_{i=1}^{\text{rk}} m_i \alpha_i, \qquad \theta^\vee = 2\frac{\theta}{(\theta,\theta)} = \sum_{i=1}^{\text{rk}} m_i^\vee \alpha_i^\vee. \tag{2.8}$$

As the simple roots are linearly independent this implies that

$$\frac{m_i}{m_i^\vee} = \frac{(\theta,\theta)}{(\alpha_i,\alpha_i)} = \frac{2}{(\alpha_i,\alpha_i)}. \tag{2.9}$$

Since $\theta$ is a long root, *e.g.* $(\theta,\theta) = 2$, this implies $m_i^\vee \leq m_i$.

Another important object is the *Weyl vector* $\rho$ also called the half root sum which is given by

$$\rho = \frac{1}{2}\sum_{\alpha\in R^+} \alpha. \tag{2.10}$$

Since $(\rho, \alpha_i^\vee) = 1$ for all simple roots we have that $\rho = \sum_{i=1}^{\text{rk}} \varpi_i$ when expressed in terms of the fundamental weights. We also have that

$$(\rho, \theta^\vee) = \sum_{i=1}^{\text{rk}} m_i^\vee. \tag{2.11}$$

Finally, there is a group action on the root space give by the Weyl group. The Weyl group is the subgroup of the automorphism group of the root space generated by "reflections" in a given root *e.g.*

$$w_\alpha : \beta \mapsto w_\alpha(\beta) = \beta - 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha = \beta - (\beta,\alpha^\vee)\,\alpha. \tag{2.12}$$

Elements of the root space fixed under a Weyl transformation $w_\alpha$ define hyperplanes perpendicular to the roots. Removing these hyperplanes from teh root spaces divides it into open cones known as Weyl chambers. The hyperplanes are often called the walls of the Weyl chambers. Once the simple roots have been chosen there is a distinguished Weyl chamber known as

[2] Note that in some sources, including [3], the factor of 2 is not included when defining the coroots. This is another reason why care needs to be taken when comparing different references.

the dominant or fundamental Weyl chamber or domain where all of the lattice points in the chamber correspond to weights with positive Dynkin labels. Any element of the root system, not on a wall, can be mapped to the fundamental Weyl chamber through the action of a Weyl transformation.

The following computation may prove useful.

$$\begin{aligned}(w_\alpha(\lambda), w_\alpha(\mu)) &= (w_\alpha(\lambda), \mu) - (w_\alpha(\lambda), (\mu, \alpha^\vee)\alpha) \\ &= (\lambda - (\lambda, \alpha^\vee)\alpha, \mu) - (\lambda - (\lambda, \alpha^\vee)\alpha, (\mu, \alpha^\vee)\alpha) \\ &= (\lambda, \mu) - (\lambda, \alpha^\vee)(\mu, \alpha) - (\lambda, \alpha)(\mu, \alpha^\vee) + (\lambda, \alpha)(\mu, \alpha^\vee)(\alpha, \alpha) \\ &= (\lambda, \mu),\end{aligned}$$

where we used the definition of a coroot to move around the $\alpha^\vee$. Next note that for a simple root $\alpha_i$ $w_i(\rho) = \rho - \alpha_i$ since $(\rho, \alpha_i^\vee) = 1$ and that $w_i(\alpha_i) = -\alpha_i$. Finally,

$$\begin{aligned}0 &= (\alpha_i^\vee, \rho) - (\alpha_i^\vee, \rho) \\ &= (\alpha_i^\vee, \rho) - (w_i(\alpha_i^\vee), w_i(\rho)) \\ &= (\alpha_i^\vee, \rho + w_i(\rho)) \\ &= (\alpha_i^\vee, 2\rho - \alpha_i) \\ &= 2\left(\rho^i - 1\right).\end{aligned}$$

Since this is true for every simple root it implies that $\rho = \sum_{i=1}^{\mathrm{rk}} \varpi_i$. Recall that $\rho^i = (\rho, \alpha_i^\vee)$ and that $(\alpha_i^\vee, \alpha_i) = 2\frac{(\alpha_i,\alpha_i)}{(\alpha_i,\alpha_i)} = 2$.

Let us review some simple examples to help secure these conventions in our minds.

**Example 2.1** ($SU(2)$)**.** This is a rank one group so there is one simple root $\alpha$, its associated coroot $\alpha^\vee$, and one fundamental weight $\varpi$. Going back to basics the definition of the group and its Lie algebra as subgroups of complex matrices are

$$SU(2) = \{A \in M(2, \mathbb{C}) \mid AA^* = \mathbb{I}, \det A = 1\}, \tag{2.13}$$

$$\mathfrak{su}(2) = \{A \in M(2, \mathbb{C}) \mid A + A^* = 0, \mathrm{Tr}A = 0\}. \tag{2.14}$$

The Lie algebra is formed of complex anti-hermitian matrices and has as its basis $\frac{\mathrm{i}}{2}$ times the Pauli matrices.

$$T_1 = \frac{\mathrm{i}}{2}\tau_1 = \frac{1}{2}\begin{pmatrix}0 & \mathrm{i} \\ \mathrm{i} & 0\end{pmatrix}, \quad T_2 = \frac{\mathrm{i}}{2}\tau_2 = \frac{1}{2}\begin{pmatrix}0 & -1 \\ 1 & 0\end{pmatrix}, \quad T_3 = \frac{\mathrm{i}}{2}\tau_3 = \frac{1}{2}\begin{pmatrix}\mathrm{i} & 0 \\ 0 & -\mathrm{i}\end{pmatrix} \tag{2.15}$$

They satisfy the commutation relations $[T_i, T_j] = \epsilon_{ij}{}^k T_k$. As mentioned above really we are working with $\mathfrak{su}(2)^{\mathbb{C}} \simeq \mathfrak{sl}(2, \mathbb{C})$ as we will consider complex combinations of generators.

As these generators do not commute the Cartan subalgebra is given by the span of any one of them. The Conventional choice is to take $h = -iT_3$ and the span of $h$ gives $\mathfrak{h}$. Next we need know that there will be one root since $\mathrm{rk} = 1$ for $\mathfrak{su}(2)$. Next take $e_+ = -\mathrm{i}(T_1 + \mathrm{i}T_2)$ and $e_- = -\mathrm{i}(T_1 - \mathrm{i}T_2)$ then

$$[h, e_\pm] = \pm e_\pm, \tag{2.16}$$

$$[e_+, e_-] = 2h. \tag{2.17}$$

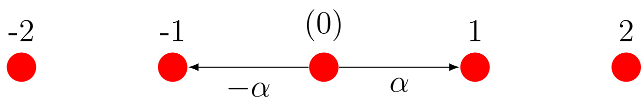


Figure 1: The root spaces of $\mathfrak{su}(2)$. N.B the roots are the weights in the adjoint representation.

This can also be written as

$$\mathrm{ad}_h\left(e_\pm\right) = \pm e_\pm, \tag{2.18}$$
$$\mathrm{ad}_h\left(h\right) = 0. \tag{2.19}$$

Thus $\mathrm{ad}_h$ has been diagonalized with eigenvectors $h, e_\pm$. Thus the root vectors are $\pm 1$ and the $e_\pm$ are the ladder operators.

The simple coroot is related to the generator of the Cartan, $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)} = 2$ which becomes $\mathrm{i}\tau_3$ as a matrix e.g. long coroots have length $\|\alpha^\vee\| = \sqrt{2}$, and the Killing form is normalised so that $\kappa\left(T_3, T_3\right) = -\frac{1}{4}\mathrm{Tr}\left(\mathrm{i}\tau_3\right)^2 = \frac{1}{2}$.

The fundamental Weyl chamber is then the half line $\mathbb{R}_+ = (0, \infty)$ with the wall being the point 0. This should be the simplest example but there are a few things that still need to be clarified.

**Example 2.2** ($SU(3)$)**.** This is the $\mathrm{rk} = 2$ case so now we get two elements in the Cartan $H_1, H_2$ and we will have two roots $\alpha_1, \alpha_2$. Typically the Cartan subalgebra is taken to be that generated by the diagonal Gell–Mann matrices

$$H_1 = \frac{1}{2}\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix}, \qquad H_2 = \frac{1}{2\sqrt{3}}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -2\end{pmatrix}, \tag{2.20}$$

normalised such that $\mathrm{Tr}\left(H_i H_j\right) = \frac{1}{2}\delta_{ij}$ . These are often written as $\vec{H} = (H_1, H_2)$ as a "root space vector"[3] A priori there are three sets of corresponding ladder operators given similarly to the $\mathfrak{su}(2)$ case with roots $\vec{\beta} = (\beta_1, \beta_2)$. In the Cartan-Weyl basis where

$$[H_i, E_{\vec{\beta}}] = \beta_i E_{\vec{\beta}}, \qquad [E_{\vec{\beta}}, E_{-\vec{\beta}}] = \vec{\beta}\cdot\vec{H}, \tag{2.21}$$

the corresponding simple roots are

$$\alpha_1 = (1, 0)\,, \qquad \alpha_2 = \frac{1}{2}\left(-1, \sqrt{3}\right) \tag{2.22}$$

Recall that the relationship between the simple roots and the fundamental weights is given by the Cartan matrix $A_{ij} = 2\frac{(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)}$ for $\mathfrak{su}(3)$ we can compute that

$$A = \begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix}, \tag{2.23}$$

[3]This connects the root vector with the matrix representation of the roots and coroots. e.g. when we write a root (or coroot) as $\alpha = (\alpha_1, \ldots, \alpha_{\mathrm{rk}})$ we are saying that as a matrix $\alpha = \sum_{i=1}^{\mathrm{rk}} \alpha_i H_i$. This relies on some potentially unhelpful identifications of spaces with their duals.

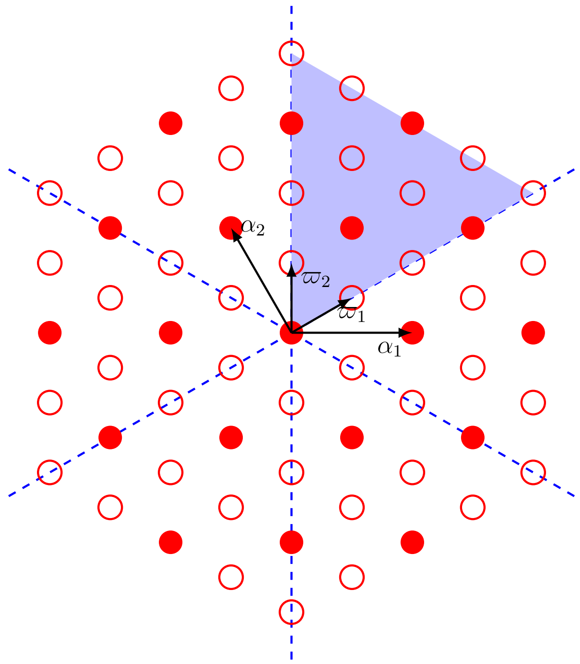


Figure 2: The root spaces of $\mathfrak{su}(3)$. The $\varpi_i$ are the fundamental weights defined by being orthogonal to the simple roots in the root space. From this figure we can see that the root lattice given by the integer span of the $\alpha_i$ is a sublattice of the weight lattice.

and thus $\alpha_1 = 2\varpi_1 - \varpi_2$ and $\alpha_2 = 2\varpi_2 - \varpi_1$.

Note that the $H_i$ that we use in the construction of the Cartan subalgebra and the root decomposition is not an element of $\mathfrak{su}(2)$ since it is hermitian rather than anti-hermitian. Since we are working with the complexification this is fine, however, we should be aware of it.

Now through an abuse of notation we can write an element of $\mathfrak{su}(2)$ as

$$\Phi_\infty = i\vec{h} \cdot \vec{H}, \tag{2.24}$$

where we think of $\vec{h}$ as the vector representation of a root.

## 3 A Brief Review of Maximal Symmetry Breaking

A BPS monopole is a pair $(A, \Phi)$ of a $G$ connection $A$ and an adjoint valued Higgs field $\Phi$ on $\mathbb{R}^3$ which satisfy the BPS equation

$$F_A = \star d_A \Phi \tag{3.1}$$

Check that this matches with $D_i\Phi = B_i$. The boundary conditions are that

$$\lim_{r\to\infty} \|\Phi\|^2 = -\frac{1}{2} \lim_{r\to\infty} \operatorname{Tr}\left(\Phi^2\right) = \frac{1}{2}v^2, \tag{3.2}$$

and that

$$\|B_i\| = \|D_i\Phi\| \sim \mathcal{O}\left(\frac{1}{r^2}\right), \tag{3.3}$$

which set the symmetry breaking scale and enforce finite energy. The form of symmetry breaking is chosen by taking the Higgs field to be

$$\Phi = \Phi_0 - \frac{\kappa}{2r} + \mathcal{O}\left(\frac{1}{r^2}\right), \tag{3.4}$$

in [3] they choose slightly different conditions and work along the $z$-axis so $\Phi(0,0,z) = \Phi_0 - \frac{\kappa}{4\pi z} + \mathcal{O}\left(\frac{1}{z^2}\right)$. Here $\Phi_0$ is a constant element of $\mathfrak{g}$ which is chosen to lie in $\mathfrak{h}$. Abusing notation slightly we write it as $\Phi_0 = i\vec{h}\cdot\vec{H}$.

The two Lie algebra elements $\Phi_0$ and $\kappa$ are called the mass[4] and charge of the monopole. In the vector bundle picture they are given by block diagonal matrices

$$\mu = \mathrm{idiag}\left(\lambda_1, \ldots, \lambda_1, \lambda_2, \ldots, \lambda_2, \ldots, \lambda_n, \ldots, \lambda_n\right) \tag{3.5}$$

$$\kappa = \mathrm{idiag}\left(k_{1,1}, k_{1,2}, \ldots, k_{1,r_1}, k_{2,1}, \ldots, k_{2,r_2}, \ldots, k_{n,1}, \ldots, k_{n,r_n}\right), \tag{3.6}$$

where $r_i$ is the number of times $\lambda_i$ is repeated. For $G = SU(N)$ these are traceless which implies that

$$\sum_j \lambda_j r_j = 0, \qquad \sum_{j=1}^{n} k_j = 0, \tag{3.7}$$

where $k_j = \sum_{l=1}^{j_j} k_{j,l}$. There is also an assumed ordering for the $\lambda_i$, Here we think that the ordering is $\lambda_1 > \lambda_2 > \cdots > \lambda_n$ though this does not agree with that taken in . This also seems to need a condition that $k_{1,j} \leq k_{2,j} \leq \cdots \leq k_{r_j,j}\ \forall j$. The magnetic charges are determined by the $k_{i,j}$ through

$$m_a = \sum_{j=1}^{a} k_j, \tag{3.8}$$

there are $N-1$ of the $m_a's$ as the $N$'th case vanishes due to the traceless condition.

In this context maximal symmetry breaking means that $SU(N)$ breaks to $S\left(U(1)\times U(1)\times\cdots\times U(1)\right)$, while minimal symmetry breaking is $SU(N)$ to $S\left(U(N-1)\times U(1)\right)$ and intermediate symmetry breaking is $SU(N)$ to $S\left(U(r_1)\times \ldots U(r_n)\right)$ *e.g.* there is a factor for each block with the rank of the $U(n_i)$ determined by the size of the block. In the maximal symmetry breaking case the $m_a$ are the first Chern numbers of the asymptotic line bundles.

# 4 Non-Maximal Symmetry Breaking

## 4.1 Configuration and Moduli spaces

Consider now the case of non-maximal symmetry breaking where $SU(N)$ breaks to $S\left(U(r_1)\times\ldots U(r_n)\right)$. More explicitly we will follow [3] and focus on the case of $SU(3) \to S\left(U(2)\times U(1)\right) = U(2)$. The conventions taken there are slightly different from that used in [6], particularly with respect to the ordering of the $\lambda_i$ so care needs to be taken. As above a monopole is a pair $(A, \Phi)$ satisfying Eq. (3.1) with asymptotics as in Eq. (3.4). As we saw in Example. 2.2 above $\mathfrak{su}(3)$ has

[4] The mass is really the norm of this element

two simple roots $\alpha_1, \alpha_2$ and two generators of the Cartan $H_1, H_2$ such that $\mathrm{Tr}\,(H_i H_j) = \frac{1}{2}\delta_{ij}$. In terms of the coroots and the Cartan generators we know that

$$\Phi_0 = \mathrm{i}\vec{h}\cdot\vec{H} = \mathrm{i}vH_2, \tag{4.1}$$

$$\kappa = \mathrm{i}\vec{g}\cdot\vec{H} = \mathrm{i}\left[\left(m_1 - \frac{m_2}{2}\right)H_1 + m_2\frac{\sqrt{3}}{2}H_2\right], \tag{4.2}$$

where $\vec{h} = (0, v)$ is perpendicular to $\alpha_1$ ensuring that $\mathrm{Stab}\,(\Phi_0) = S\,(U(2)\times U(1))$ with the $U(2)$ being that associated with $\mathfrak{su}(2)_{\alpha_1}$ and $H_2$, and the $U(1)$ coming from $\alpha_2$. We have used the Dirac quantisation condition to ensure that

$$\vec{g} = \frac{1}{2}\left(m_1\alpha_1^\vee + m_2\alpha_2^\vee\right), \tag{4.3}$$

with $m_1, m_2 \in \mathbb{Z}$ and the factor of two included so that this agrees with the choice in [3], up to removing the $\frac{4\pi}{e}$ over all factor. With these choices we have $\|\Phi_0\| = \frac{1}{4}v^2$ which is $\frac{1}{2}$ that claimed in [3], to match their conventions it may be better to take $\vec{h} = (0, \sqrt{2}v)$ but this would disagree with what they claim to be taking as then $\Phi_0 = i\sqrt{2}vH_2$.

Since $\kappa$ and $\Phi_0$ are not parallel, unless $m_1 = \frac{m_2}{2}$, $\kappa$ is not invariant under $\mathrm{Stab}\,(\Phi_0) = U(2)$ with the invariant quantity being

$$-\mathrm{Tr}\,(\kappa\Phi_0) = \sqrt{\frac{3}{2}}m_2. \tag{4.4}$$

Thus $m_2$ is the topological charge and is invariant under the asymptotic gauge group while $m_1$ can be changed by the $U(2)$ action. More explicitly the $U(2)$ is given by elements of the form

$$P = e^{\mathrm{i}\chi Y}e^{-\mathrm{i}\alpha I_3}e^{-\mathrm{i}\beta I_2}e^{-\mathrm{i}\gamma I_3}, \tag{4.5}$$

where

$$\begin{aligned}
Y &= \frac{2}{\sqrt{3}}H_2,\\
I_3 &= H_1,\\
I_2 &= \frac{1}{2\mathrm{i}}\left(E_{\alpha_1} - E_{-\alpha_1}\right) = \frac{1}{2\mathrm{i}}\begin{pmatrix}\tau_2 & 0\\ 0 & 0\end{pmatrix},\\
I_1 &= \frac{1}{2}\left(E_{\alpha_1} + E_{-\alpha_1}\right) = \frac{1}{2}\begin{pmatrix}\tau_1 & 0\\ 0 & 0\end{pmatrix},
\end{aligned}$$

generate the $U(2)$.

A computation gives that the orbit of $\kappa$ under the action of this $U(2)$ is

$$\kappa_P = P\kappa P^{-1} = m_2\mathrm{i}\frac{\sqrt{3}}{2}H_2 + \mathrm{i}\vec{k}\cdot\vec{I}, \tag{4.6}$$

where $k = |\vec{k}| = |m_1 - \frac{m_2}{2}|$ and

$$\hat{k} = \begin{pmatrix}\sin\beta\cos\alpha\\ \sin\beta\sin\alpha\\ \cos\beta\end{pmatrix}. \tag{4.7}$$

The $\vec{k}\cdot\vec{I}$ term gives the magnetic orbit as that is the piece being changed by the action of $\mathrm{Stab}\,(\Phi_0)$ and it is explicitly an $S^2 \subset \mathfrak{su}(3)$ with radius $k = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. This magnetic orbit intersects the Cartan $\mathfrak{h}$ when $\vec{k}\cdot\vec{I} \in \mathfrak{h}$ which happens at two points $m_1$ and $m_2 - m_1$. To see this more explicitly write out $\vec{k}\cdot\vec{I}$ and evaluate it at $\beta = 0, \pi$ where you will find that the first entry in $\kappa$ is $m_1, m_2 - m_1$ respectively. These two points are collected together as $[m_1] = \{m_1, m_2 - m_1\}$ and called the holomorphic charge[5].

It is common to use notation $K = m_2$ and $k = |m_1 - \frac{m_2}{2}|$ rather than $m_2, [m_1]$.

Now the framed configuration space is given by

$$\mathcal{C}_{\vec{h}} = \mathcal{A}_{\vec{h}}/\mathcal{G}_0, \tag{4.8}$$

where $\mathcal{G}_0$ are the gauge transformations that tend to the identity asymptotically and $\mathcal{A}_{\vec{h}}$ is the space of pairs satisfying the boundary conditions. The framed moduli space $\mathcal{M}$ is then defined as the space of pairs satisfying the boundary conditions and the BPS equations modulo framed gauge transformations. Taubes' existence theorem for monopoles shows that in the maximal symmetrey breaking case both the configuration space and the moduli space are non-empty and the connected components are labelled by the topological charges[6] While in the non-maximal symmetry breaking case the theorem still tells us the configuration and moduli space are non empty, now the connected components are only labelled by $K$ and these are further split up into pieces labelled by the holomorphic charges. This finer subdivision is not topological in nature but is s stratification with the strata labelled by the holomorphic charge or the radius of the magnetic orbit.

This can be summarised as

$$\mathcal{M}^{\max}_{m_1,m_2} \neq \emptyset, \qquad \mathrm{dim}\mathcal{M}^{\max}_{m_1,m_2} = 4\,(m_1 + m_2)\,, \tag{4.9}$$

$$\mathcal{M}_K \neq \emptyset, \qquad \mathcal{M}_{K,k} \subset \mathcal{M}_K, \tag{4.10}$$

where $\mathcal{M}_{K,k}$ are the strata for a given holomorphic charge. In the maximal symmetry breaking case the dimension of the moduli space is known and there is a well separated region where the parameters are identified as those of $SU(2)$ monopoles in particular $m_1$ monopoles of type $\alpha_1$ and $m_2$ monopoles of type $\alpha_2$. For non-maximal symmetry breaking finding the dimension and identifying what the parameters mean is much harder see [5] for a discussion this, [15] computes the dimension of the moduli spaces by making use of the rational map correspondence.

Include a picture here!

The action of the Weyl group relates $\kappa$ for $m_1$ and $m_1 - m_2$, e.g. the two points on the same magnetic orbit, and thus they correspond to the same moduli space. This is relevant because we can think of each point in the dual lattice as corresponding to a possibly empty component $\mathcal{M}_{K,k}$ of the moduli space, In fact in [15] it is proved that $\mathcal{M}_{K,k} \neq \emptyset$ iff $k \leq \frac{K}{2} > 0$. This means that

$$K \text{ even } \Rightarrow k = 0, 1, 2, \ldots, \frac{K}{2}, \tag{4.11}$$

$$K \text{ odd } \Rightarrow k = \frac{1}{2}, \frac{3}{2}, \ldots, \frac{K}{2}, \tag{4.12}$$

[5] The name is because while the asymptotics determine the topology of the bundle over $S^2_\infty$ its holomorphic structure is changed by the $U(2)$ action so a choice of $[m_1]$ corresponds to a choice of holomorphic structure.

[6] recall that $m_1$ and $m_2$ are both topological for maximal symmetry breaking.

This means that there are only non-empty components of the moduli space inside the cone. Need a picture of this!

The edge of the cone, where $m_1 = 0$ and $k = \frac{m_2}{2}$ is called the small strata. The centre of the cone, which is only allowed for $K$ even, where $m_1 = \frac{m}{2}$ and the magnetic orbit is a point is called the large strata.

### 4.2 Embedding $SU(2)$ monopoles

The small strata can be entirely realised by considering $SU(2)$ monopoles embedded in $SU(3)$ along the root $\alpha_2$. This was proved in [5] using the Nahm data and gives an alternative way to compute the dimension of the small strata as well as giving $\mathcal{M}_{K,k}$ the strutcure of a fibration with fibre $\mathcal{M}_K^{SU(2)}$ and base the magnetic orbit $S^2$.

Consider a charge $K$ $SU(2)$ monopole $(a, \phi)$, where $SU(2)$ has generators $\tau_1, \tau_2, \tau_3$ and the asymptotic $U(1)$ is generated by $\tau_3$. Then in the usual gauge the Higgs field has asymptotics

$$\phi \sim \frac{\omega}{2}\mathrm{i}\tau_3 - \frac{K}{2r}\mathrm{i}\tau_3. \tag{4.13}$$

More generally the Higgs field is $\phi = \phi_{\mathrm{BPS}}\frac{i}{2}\tau_3$ with $\phi_{\mathrm{BPS}} = 2\omega\coth(2\omega r) - \frac{1}{r}$. Here $\omega = 3$ is the mass of the monopole.

This $SU(2)$ is embedded into $SU(3)$ as the $SU(2)$ associated with $\alpha_2$, in other words

$$U_i = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \tau_i \end{pmatrix} \tag{4.14}$$

are the generators of $SU(2) \subset SU(3)$. Then an $SU(3)$ monopole satisfying the boundary conditions of Eq. (3.4) is

$$\Phi^U = \mathrm{i}\phi_{\mathrm{BPS}}U_3 + \mathrm{i}\mathrm{diag}\left(1, -\frac{1}{2}, -\frac{1}{2}\right) \qquad A^U = \sum_{i=1}^{3} a^i U_i. \tag{4.15}$$

The constant piece is added in $\Phi^U$ to make sure that it has the correct asymptotics. To see this note that

$$\Phi^U \sim i2\sqrt{3}H_2 - \frac{K}{2r}U_3, \tag{4.16}$$

so $v = 2\sqrt{3}$ is the vev.

These are not unique as we have $\mathrm{Stab}\,(\Phi_0) = U(2)$ acting as an asymptotic gauge group to give

$$\Phi_P = P\Phi^U P^{-1}, \tag{4.17}$$

but now the is an $U(1)$ sub group that leaves $\Phi^U$ invariant, this is the $U(1)$ generated by $Y + 2I_3$. This generator is

$$Y + 2I_3 = \frac{1}{3}\begin{pmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \tag{4.18}$$

thus the group something looks like

$$\mathrm{diag}\left(e^{\mathrm{i}2\theta}, e^{-\mathrm{i}\theta}, e^{-\mathrm{i}\theta}\right) \tag{4.19}$$

this means that it is the over all $U(1)$ in the $U(2)$ and thus the orbit is

$$U(2)/U(1) = SU(2) = S^3_P. \tag{4.20}$$

The natural coordinates on $SU(2)$ are the Euler angles $(\alpha, \beta, \gamma)$, where $\gamma$ parametrises a $S^1$ known as the electric circle. There is then a Hopf fibration

$$S^1 \hookrightarrow S^3_P \to S^2 \tag{4.21}$$

with fibre the electric circle and, since we can take the projection to be

$$\pi^k : S^3_P \to S^2, \qquad \pi^k(\alpha, \beta, \gamma) \mapsto \vec{k}, \tag{4.22}$$

the base is the magnetic orbit.

From [2] it is known that the moduli space of charge $K$ $SU(2)$ monopoles is of the form

$$\mathcal{M}^{SU(2)}_K = \mathbb{R}^3 \times \frac{S^1 \times \mathcal{M}^0_K}{\mathbb{Z}_k}, \tag{4.23}$$

where $\mathbb{R}^3$ is the centre of mass position, $S^1$ is the electric circle and $\mathcal{M}^0_K$ is the $k$-fold cover of the charge $k$ centred monopole moduli space. This is known to be a hyperKähler manifold of dimension $4K$.

These embedded $SU(2)$ monopoles are charge $K$ $SU(3)$ monopoles on the magnetic orbit of radius $\frac{K}{2}$, *i.e.* they lie in the small strata. The results of [5] show that in fact this gives the whole of the small strata and thus it is given by

$$\mathcal{M}_{K,\frac{K}{2}} = \mathbb{R}^3 \times \frac{S^3_P \times \mathcal{M}^0_K}{\mathbb{Z}_k}, \tag{4.24}$$

where the $\mathbb{Z}_k$ action on the electric circle is $\gamma \mapsto \gamma + \frac{2\pi}{k}$. The number of independent parameters is then $4K + 2$ which agrees with the dimension of the small strata computed in [15]. The fact that we get the small strata as a fibration with fibre the Atiyah-Hitchin manifold and base the magnetic orbit means that, up to scaling, there is a well defined hyperKähler metric on the fibres. There can be a metric on the base, however, as we will see later, this depends on if $K$ is even or odd.

### 4.3 Asymptotic gauge group and collective coordinates

There is a common *lore* in the study of topological defects that the generators of the broken symmetry are related to collective coordinates for the defect. It is known that this is not always true in gauge theories when the unbroken gauge group is non abelian. In particular the case of $SU(5) \to SU(3)_{\text{colour}} \times U(2)_{\text{electroweak}}$ is discussed in [1]. This problem can be referred to as "Is global colour defined?" and the answer is not always. This issue is also discussed in [4, 16].

This is relevant here since the asymptotic gauge group $U(2)$ plays the role of the "global" colour. Consider the Higgs field of the monopole on the two sphere at infinity $S^2_\infty$ in a regular gauge, then one can only define generators of the $U(2)$ which commute with $\Phi_0$ and vary smoothly over $S^2_\infty$ when $K$ is even. I need to read the original papers to understand this as [3] does not discuss this in detail. For odd topological charge $K$ there is a topological obstruction. Apparently the argument is that if $K$ is odd and we try to define the $U(2)$ action at a fixed point, such as one of the poles. Then extending the action to the whole of $S^2_\infty$ changes both the $\frac{1}{2r}$ term in $\Phi_0$ and the $r^0$ term somewhere on $S^2_\infty$. As $U(2)$ changes $\Phi$ in this way we cannot find collective coordinates that are compatible with the Gauss' law constraint. This example needs to be spelt out in more detail.

As a not here collective coordinates mean coordinates whose infinitesimal variation are $L^2$ zero modes satisfying Gauss' law. These are the dynamically relevant coordinates since the gauge directions are removed by imposing Gauss' law.

Note that this issue does not effect the large strata since $\mathcal{M}_{K,0}$ has $K$ even. For the other strata $\kappa = \mathrm{i}\vec{g}\cdot\vec{H}$ obstructs the unbroken $U(2)$ action, *i.e.* $\mathrm{Stab}\,(\Phi_0) \neq \mathrm{Stab}\,(\kappa)$ so only the centraliser $C\,(\kappa) \subset U(2)$ is dynamically relevant. In the small strata we can identify the physically problematic coordinates, they are coordinates of the magnetic orbit which inherits neither a smooth structure nor a metric from the field theory. In other words, motion on the fibre in $\mathcal{M}_{K,\frac{K}{2}}$ is fine physically but motion orthogonal to the fibre is not allowed. This is why the magnetic orbit is often disregarded, particularly in the mathematics literature. However, it is important if one wants to understand dyonic excitations and interactions, as well as the quantum theory which are the goals of [3].

There is a conjecture that all of the strata $\mathcal{M}_{K,k}$, with $k > 0$, are fibred over an $S^2$ parametrising the non-Abelian magnetic charges. I do not know if this conjecture has been proved, or even how much work has been done in this direction since [3].

As an explicit example of dynamical vs non dynamical coordinates and the parameter counting consider the $K = 1$ monopole moduli space $\mathcal{M}_{1,\frac{1}{2}} = \mathbb{R}^3 \times S^3_p$, note that there is only one strata in this case since $k \leq \frac{K}{2} = \frac{1}{2}$. Here there are 4 dynamical coordinates and 3 from the position and 1 from the phase. This is the same as for an $SU(2)$ monopole which makes sense since all monopoles in the small strata come from embedded $SU(2)$ monopoles. There are also 2 non dynamical coordinates coming from the direction of $\vec{g}$ in $\mathfrak{su}(3)$. However, there is an extra subtlety since the electric circle is the centraliser of $\kappa$ in $U(2)$ and is thus not independent of the choice of $\vec{g}$. In other words for a different choice of magnetic direction the electric charge comes from a different $U(1)$.

We can see another consequence by considering the charge $K = 2$ case. Here there are two strata: a small strata $\mathcal{M}_{2,1}$, and a large strata $\mathcal{M}_{2,0}$. To have a configuration in the small strata we need the individual monopoles to have parallel magnetic charges, e.g. they are both $SU(2)$ monopoles embedded along the same root with topological charge $\frac{K}{2} = 1$. Now in the large strata, since $\vec{g}$ needs to align with $H_2$ the individual monopoles need to have "anti parallel" vector magnetic charges[7] Essentially this discussion says that all we need to understand qualitatively are single monopoles.

[7] I understand this to mean something like the $\alpha_1^\vee$ components need to be opposite though this is not quite correct, it is more that we need $\vec{g}\cdot\vec{H}$ to be proportional to $H_2$.

## 5 Monopoles and rational maps

This section is based on Jakob's notes made following [10, 12] andother reference is [13].

To understand monopole moduli spaces we can instead understand the moduli space of based rational maps. This was first explained in [10] where Donaldson proved that

$$\mathcal{M}_K^{\text{Nahm}} = \mathcal{M}_K^{\text{Rat}}, \tag{5.1}$$

where on the left hand side we have the moduli space of Nahm equations corresponding to charge $K$ $SU(2)$ monopoles and on the right hand side is moduli space of based rational maps $f : \mathbb{C}P^1 \to \mathbb{C}P^1$ $f(\infty) = 0$ of degree $K$ where the equivalence relation is that $f(z) \sim e^{i\theta} f(z)$. Recall that a rational map is a complex function $f(z) = \frac{p(z)}{q(z)}$ which is a ration of unique polynomials with $\gcd(p, q) = 1$, $q(z)$ monic and $\deg(q) = K \geq \deg(p) + 1$. Roughly speaking poles of $f(z)$ correspnd to (widely separated) monopoles via Taubes' gluing construction *e.g.* $f(z) = \frac{1}{z^K}$ is "the" axisymmetric charge $K$ BPS monopole.

In [13] this correspondence is generalised to the case of $G$ monopoles broken to the maximal torus $T$ with charge $\rho = (m_1, \ldots, m_{\text{rk}})$ which are equivalent to based rational maps $f : \mathbb{C}P^1 \to G/T$ with $\deg f = \rho$.

The Nahm to rational map direction is easier to understand and proceeds as follows. The Nahm data of a charge $K$ $SU(2)$ monopole is a set of four $K \times K$ matrix functions $T_i : [0, 2] \to M_K(\mathbb{C})$, $i = 0, 1, 2, 3$ (with $T_0 ds$ a connection that is commonly gauged to zero) satisfying:

(1) $\frac{dT_i}{ds} + \frac{1}{2}\sum_{j,k} \varepsilon_{ijk}[T_j, T_k] = 0$

(2) $T_i^\dagger(s) = -T_i(s)$.

(3) $T_i(2 - s) = T_i^T(s)$.

(4) $T_i$ has simple poles at both ends of the interval, *i.e.* it is of order $\frac{1}{s}$ as $s = 0$ and $\frac{1}{s-2}$ at $s = 2$.

(5) The residues of the $T_i$, $\text{Res}(T_i(0)), \text{Res}(T_i(2))$, form $K$-dimensional irreps of $SU(2)$.

The equivalence on the Nahm side is the action of $O(K, \mathbb{C})$.

It is convenient to gauge away $T_0(s)$ and use the complex combinations

$$A_0 = T_1 + \mathrm{i}T_2, \quad A_1 = -2\mathrm{i}T_3, \quad A_2 = T_1 - \mathrm{i}T_2, \tag{5.2}$$

in terms of which the Nahm equations become

$$\frac{dA_0}{ds} = -\frac{1}{2}[A_1, A_0] = 0. \tag{5.3}$$

The fact that the residues of the $T_i(0), T_i(2)$ form a $K$ dimensional irrep of $SU(2)$ imply that the eigenvalues of $\text{Res}(A_1(0))$ are elements of $\{-(K-1), \ldots, K-1\}$. for example in the case of charge $K = 1$ the only eigenvalue is 0 and for $K = 2$ there are three eigenvalues $1, 0, -1$ Are these factors of $K$ correct? or should they be one less *e.g.* so $K = 1$ has three eigenvalues?.

Let $v_{k-1}$ be the eigenvector of $\mathrm{Res}\,(A_1(0))$ with eigenvalue $k-1$ with norm 1. Next take $v(s)$ to be the unique solution of

$$\frac{dv}{ds} - \frac{1}{2}A_1 v(s) = 0, \tag{5.4}$$

with $v : (0,2) \to \mathbb{C}^K$ such that $s^{-\frac{k-1}{2}} v \to v_{k-1}$ as $s \to 0$. Then setting $B = -A_0(1)$ and $w = v(1)$ we get the rational map as

$$f(z) = w^{\perp} \left(z\mathbb{I} - B\right)^{-1} w. \tag{5.5}$$

As a sanity check observe that $\lim_{z\to\infty} f(z) \to 0$ as it should for a based rational map.

To establish the full relationship between monopoles, Nahm data, and rational maps, a useful too is the spectral curve. We can construct it from both the monopole and the Nahm data sides and it also leads to a construction of the rational map. I have a reasonably good qualitative understanding of this correspondence but want to understand the quantitative aspects better.

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