# Topology in Physics
# Some recent applications: 2

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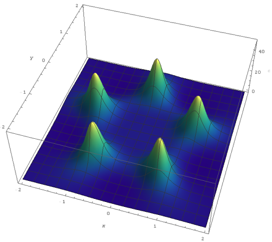


Solitons in chiral magnets

# Outline

# Solitons in real materials

- In the previous lecture we saw several examples of mathematical models with soliton solutions.
- Now we want to see some examples of applications of solitons.
- The theoretical models of magnetic skyrmions originate in the work of Bogdanov and collaborators starting in 1989.

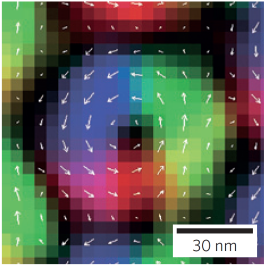



Figure 1: Experimental image of a magnetic skyrmion from Nagaosa and Tokura 2013.

# $O(3)$ sigma model

On Monday we met the $O(3)$ sigma model.

- The static energy is

$$E[m] = \frac{1}{2}\int_{\mathbb{R}^2} d^2x\,(\nabla m)^2\,, \qquad m : \mathbb{R}^2 \to S^2$$

- Finite energy solutions extend to $m : S^2 \to S^2$ with topological charge

$$Q[m] = \frac{1}{4\pi}\int d^2x\,(m \cdot \partial_1 m \times \partial_2 m)\,.$$

- There is a Bogomol'nyi bound

$$E[m] = \frac{1}{2}\int_{\mathbb{R}^2} (\partial_1 m \pm m \times \partial_2 m)^2 + 2\pi |Q[m]|.$$

# O(3) Bogomol'nyi equations

- The minimum energy configurations solve the Bogomol'nyi equations:

$$\partial_1 m \pm m \times \partial_2 m = 0.$$

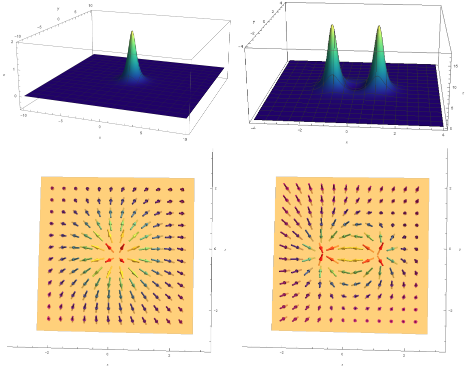
- These are much easier to study using complex stereographic coordinates

$$m_1 + i m_2 = \frac{2w}{1+|w|^2}, \quad m_3 = \frac{1-|w|^2}{1+|w|^2}.$$

The Bogomol'nyi equations, for local $\mathbb{C}$ coord $z = x + iy$, are

$$\partial_z w = 0 \quad \text{or} \quad \partial_{\bar{z}} w = 0.$$

- These are solved by rational maps $w(z) = \frac{z^n + a_1 z^{n-1} + \cdots + a_n}{b_0 z^m + b_1 z^{m-1} + \cdots + b_m}$.



Plots of energy density and $m$ for $w = z$ and $w = (z-1)(z+1)$.

# Real materials

- To describe real materials we need to add extra terms to the energy functional.
- For applications to nuclear matter the 4th order Skyrme term and the 6th order sextic term.
- For magnetic matter a first order term is needed to account for spin orbit interaction (anti-symmetric exchange)

$$E[m] = \int_{\mathbb{R}^2} d^2x \left[ \frac{1}{2} (\nabla m)^2 + \kappa m \cdot \nabla_{-\alpha} \times m + U(m_3) \right]$$

# DM Interaction

Dzyaloshinskii (1958) and Moriya (1960) realised that atomic spin orbit effects lead to a contribution of the form

$$m \cdot \nabla_{-\alpha} \times m = m \cdot \sum_{i=1}^{2} e_i^{-\alpha} \times \partial_i m.$$

$e_i^{-\alpha} = R_3(-\alpha) e_i$ are rotations of the standard basis vectors.

The symmetry of the energy functional is the diagonal subgroup of $SO(2) \times SO(2)$. (Translations are also symmetries)

This most commonly studied DM terms are the $\alpha = 0$ “Bloch” type and the $\alpha = \frac{\pi}{2}$ “Néel” type.

# Boundary term

- From an analysis point of view the DM term can cause issues with the variational problem for $E[m]$.
- This is because varying it leads to the DM term leads to the boundary term

$$BT[m] = -\kappa \int_{\mathbb{R}^2} d^2x \, \nabla \cdot (m \times \delta m)$$

- If the field falls off as $\frac{1}{r}$, like the $O(3)$ lumps, then this term is not set to zero by $\lim_{\vec{x}\to\infty} m \to m_\infty \in \mathcal{V}$, $\lim_{\vec{x}\to\infty} \delta m \to 0$.
- One solution is to subtract this boundary term. Here we want to showcase the solutions so will not subtract it.

# Potential $V(m_3)$ I

The potential has the form

$$V(m_3) = B(1 - m_3) - A\left(1 - m_3^2\right)$$

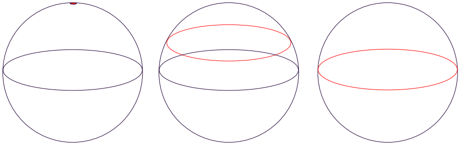
First term is a Zeeman term, minimised by $m_3$ pointing in the positive $z$ direction everywhere. Second term is anisotropy term, depends on the sign of $A$.

- $A < 0$, easy axis potential. This prefers spins to all point in the $+z$ or - $z$ direction.
- $A > 0$, easy plane potential. Prefers spins to lie in the $x - y$ plane.

# Potential $V(m_3)$ II

The ground state depends on the relative sizes of $A$ and $B$. Can assume $B > 0$ w.l.o.g

- $B \geq 2A$ the minimum is $V(m_3) = 0$ at $m_3 = 1$
- $B < 2A$ the minimum is $V(m_3) = B - A - \frac{B^2}{4A}$ at $m_3 = \frac{B}{2A}$.
- The vacuum manifolds are



# The Solvable line $B = 2A$

- The extended $O(3)$ model has exact solutions for the easy plane case with $A = \frac{B}{2}$.
- The potential can be written as

$$V(m_3) = \frac{B}{2}(1 - m_3)^2$$

- and the energy functional is

$$E[m] = \int_{\mathbb{R}^2} d^2x \left[ \frac{1}{2}(\nabla_{-\alpha} m)^2 + \kappa m \cdot (\nabla m \times m) + \frac{B}{2}(1 - m_3)^2 \right]$$

# Axially-symmetric configurations I

- A particularly nice class of field configurations, respecting the $SO(2)$ symmetries of the model are the hedgehog fields

$$m = \begin{pmatrix} \sin\Theta\cos\Phi \\ \sin\Theta\sin\Phi \\ \cos\Theta \end{pmatrix}$$

  with $\Theta = \Theta(r)$ and $\Phi = n\phi + \gamma$.
- They have topological charge $Q[m] = -n$.
- Hedgehogs respect the full $O(2)$ symmetry if $\gamma = \frac{\pi}{2} - \alpha$.
- Configurations with $n > 1$ are unstable!

# Axially-symmetric configurations II

- Searching for solutions of the hedgehog form the equation of motion becomes

$$\frac{d^2\Theta}{dr^2} = -\frac{1}{r}\frac{d\Theta}{dr} + \frac{\sin(2\Theta)}{2r^2} - 2\kappa\frac{\sin^2\Theta}{r} + B\sin\Theta\,(1-\cos\Theta)\,.$$

- For Hedgehog configurations the $O(3)$ Bogomol'nyi equations are

$$\frac{d\Theta}{dr} + \frac{\sin\Theta}{r} = 0$$

solved by

$$\Theta(r) = 2\arctan\left(\frac{2}{\lambda r}\right) \qquad \lambda \in \mathbb{R}.$$

- The EOM for the $O(3)$ model is equivalent to half of the above EOM:

$$\frac{d^2\Theta}{dr^2} = -\frac{1}{r}\frac{d\Theta}{dr} + \frac{\sin(2\Theta)}{2r^2}$$

# Axially-symmetric configurations III

- For $\Theta(r)$ to satisfy

$$2\kappa \frac{\sin^2\Theta}{r} = B \sin\Theta \left(1 - \cos\Theta\right).$$

$\Rightarrow$

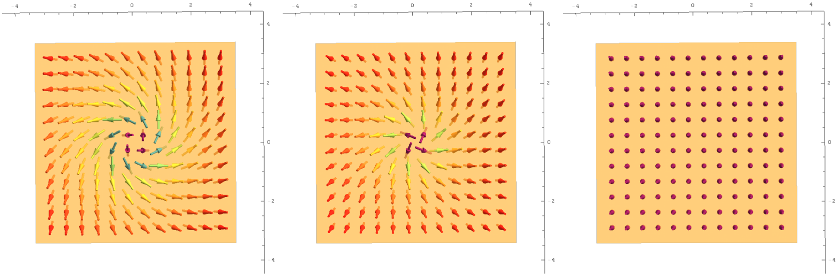
$$\lambda = \frac{B}{\kappa}$$

- Thus there are $Q = -1$ hedgehog configurations with chiral magnets on the solvable line $B = 2A$.
- They have energy

$$E[m] = 4\pi$$

# Axially-symmetric configurations IV

- Unlike in most theories with topological solitons there is a finite energy barrier between the skyrmion and the vacuum.
- To see this consider hedgehog configurations with $\gamma \neq \frac{\pi}{2} - \alpha$. These are not solutions of the EOM but still have energy $4\pi$.
- Examples with $\alpha = 0$ and $\gamma = \frac{\pi}{4}, \gamma = \frac{\pi}{8}, \gamma = 0$.



# Critically coupled model I

- By tuning the coupling of the DM term and the potential to $B = \kappa^2$ we can find a whole family of mutli skyrmion configurations.
- This critically coupled model can be interpreted as a gauged version of the $O(3)$ model.
- The connection and curvature are

$$A_i = -\kappa e_i^{-\alpha}, \qquad F_{12} = \kappa^2 e_3,$$

- The Covariant derivative is

$$D_i m = \partial_i m - \kappa e_i^{-\alpha} m, \qquad e_i^{-\alpha} = R(-\alpha) e_i.$$

- A quick computation gives

$$\frac{1}{2}\left[(D_1 m)^2 + (D_2 m)^2\right] = \frac{1}{2}\left(\nabla_{-\alpha} m\right)^2 + \kappa m \cdot (\nabla m \times m) + \frac{\kappa^2}{2}\left(1 - m_3\right)^2$$

# Critically coupled model II

- In terms of the covariant derivative the energy functional is

$$E[m] = \frac{1}{2}\int_{\mathbb{R}^2} d^2x \left[(D_1 m)^2 + (D_2 m)^2\right]$$

- A useful identity re-expresses the covariant derivative in terms of the topological charge density as

$$\begin{aligned}\frac{1}{2}\left[(D_1 m)^2 + (D_2 m)^2\right] = {} & \frac{1}{2}\left(D_1 m + m \times D_2 m\right)^2 \\ & + m \cdot \partial_1 m \times \partial_2 m + \kappa\left(\partial_1 m_2^\alpha - \partial_2 m_1^\alpha\right)\end{aligned}$$

- This leads to a bound for the energy

$$E[m] \geq 4\pi\left(Q[m] + \Omega[m]\right)$$

# Critically coupled model III

- There is equality when the Bogomol'nyi equations are satisfied,

$$D_1 m + m \times D_2 m = 0.$$

- The quantities in the bound are

$$Q[m] = \frac{1}{4\pi} \int_{\mathbb{R}^2} d^2x \left( m \cdot \partial_1 m \times \partial_2 m \right),$$
$$\Omega[m] = \frac{\kappa}{4\pi} \int_{\mathbb{R}^2} d^2x \left( \partial_1 m_2^\alpha - \partial_2 m_1^\alpha \right)$$

- These are the topological charge and the total vortex strength.

# Critically coupled model IV

- This bound is different from the familiar $O(3)$ sigma model as the $Q[m]$ appears not $|Q[m]|$.
- The integrand of the total vortex strength is $e_3 \cdot (\nabla_{-\alpha} m \times m)$, it is the boundary piece of the DM term.
- For some configurations the integral of the total vortex strength is not well defined and needs to be regularised on a disc with a circular boundary.
- The best way to understand the Bogomol'nyi equations is to work in stereographic coordinates

$$w = \frac{m_1 + i m_2}{1 + m_3}, \qquad \text{and} \;\; v = \frac{1}{w}.$$

# Complex coordinates

- In stereographic coordinates the Bogomol'nyi equations become

$$\partial_{\bar{z}} v = -\frac{i}{2}\kappa e^{i\alpha}.$$

- This has the general solution

$$v = -\frac{i}{2}e^{i\alpha}\bar{z} + f(z)$$

  for an arbitrary holomorphic function $f$.
- When $f$ is rational, $f(z) = \frac{P(z)}{Q(z)}$, with $P, Q$ of degree $p, q$ then

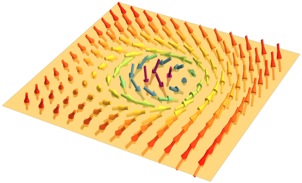
$$E[m] = 4\pi\text{max}(p, q+1)$$

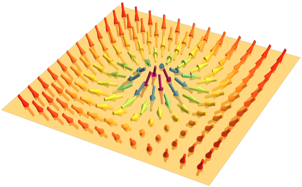
  when $p = q + 1$ this is the regularised energy.

Proving this is a worthwhile computation.

# A zoo of skyrmion configurations

- There are many nice examples of solutions found by picking your favourite holomorphic function.
- The simplest choice of $f(z) = 0$ leads to hedgehog Bloch and Neél skyrmions depending on if $\alpha = 0$ or $\alpha = \frac{\pi}{2}$.





# $E = 4\pi$ I

In this sector there is a four dimensional family of solutions

$$v = -\frac{i}{2}\kappa e^{i\alpha}\bar{z} + az + b, \quad a, b \in \mathbb{C}.$$

By translations and rotations can fix everything but $|a|$.
Changing $|a|$ corresponds to stretching or squeezing the energy density of the solution.

# $E = 4\pi$ II

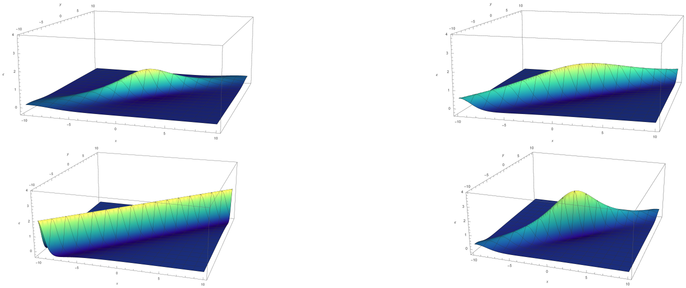


Figure 2: Stretching and squeezing for the configuration $v = -\frac{i}{2}\bar{z} + az$ with $a = 0.3$ (top left), $a = 0.4$ (top right), $a = 0.5$ (bottom left) and $a = 0.7$ (bottom right).

# $E = 4\pi$ III

When $|a| > \frac{1}{2}$, $Q = 1$ and the solutions look like an anti-skyrmions.

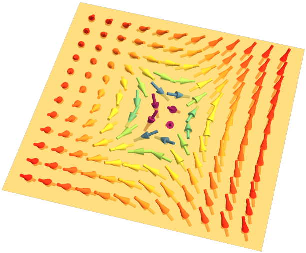


Figure 3: For $v = -\frac{i}{2}\bar{z} + 3iz$ we have an anti skyrmion with $Q[v] = 1$.

# Line defect I

- Within the family of solutions with regularised energy $4\pi$ a particularly interesting type of solution is

$$v = -\frac{i}{2}e^{i\alpha}\left(\bar{z} + e^{i\delta}z\right).$$

- These solutions have a whole line where $n_3 = -1$, $\varphi = -\frac{\delta}{2} \pm \pi$.
- This is an example of a solution which does not extend to a map of spheres.

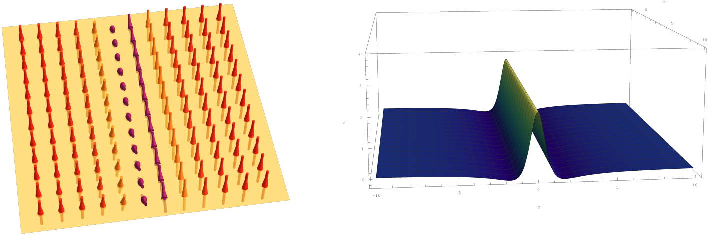


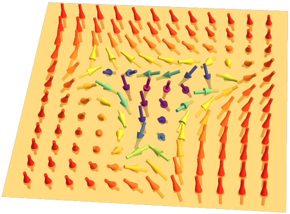
Figure 4: Left to right: magnetisation plot and energy density plot for the solution $v = -\frac{i}{2}\left(\bar{z} - z\right)$

# Line defect III

- A feature of the critically coupled model is that linear solutions can pass through the line defect and change degree.
- This is one of two places where we see a line of south poles, the others are skyrmion bag configurations.
- These solutions are one of the reasons we need to work with the regularised energy.

# $E = 8\pi$ I

- Moving to the next energy sector there is an eight dimensional family of solutions
$$v = -\frac{i}{2}\kappa e^{i\alpha}\bar{z} + \frac{az^2 + bz + c}{dz + e},$$
with $a, b, c, d, e \in \mathbb{C}$, and $(a, b, c, d, e) \sim \lambda(a, b, c, d, e),\ \lambda \in \mathbb{C}^*$.
- In this family we can find solutions which are a combination of skyrmions and anti-skyrmions and solutions which just consist of anti-skyrmions.



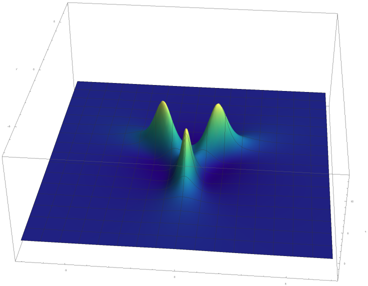


Figure 5: Magnetisation and energy density for $v = -\frac{i}{2}\bar{z} + \frac{1}{2}z^2$. This is an example of a configuration involving a skyrmion and three anti-skyrmions.

# Skyrmion bags I

- An interesting feature that arises at $E = 8\pi$ are the $Q = 0$ skyrmion bags or sacks. These have been seen numerically in the full model by Foster and collaborators (2018) and Rybakov and Kiselev (2018).
- In the critically coupled model they arise when

$$v = -\frac{i}{2}\kappa e^{i\alpha}\left(\bar{z} - \frac{R^2}{z}\right).$$

  with $R \in \mathbb{R}_{>0}$ the radius of the bag.
- Like the basic holomorphic solution these are invariant under spin-isospin rotations.
- In the numerics there are bags with skyrmions inside them but these are not possible in the critically coupled model.
- As bags have $Q = 0$ they are non-topological solitons.

# Skyrmion bags II

An example of a bag is $v = -\frac{i}{2}\left(\bar{z} - \frac{16}{z}\right)$

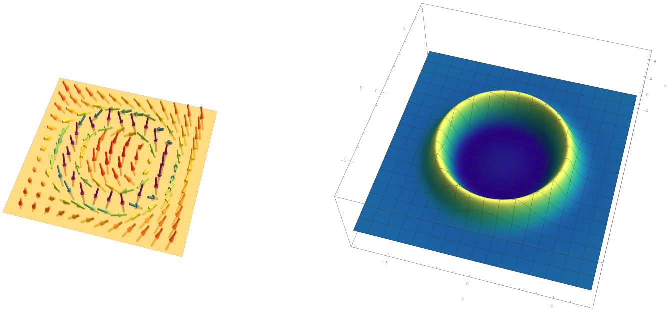


Figure 6: Magnetisation and energy density for the skyrmion bag defined by $v = -\frac{i}{2}\left(\bar{z} - \frac{16}{z}\right)$.

# Higher energy

The higher energy solutions have been less studied but we can find solutions with interesting configurations arising.

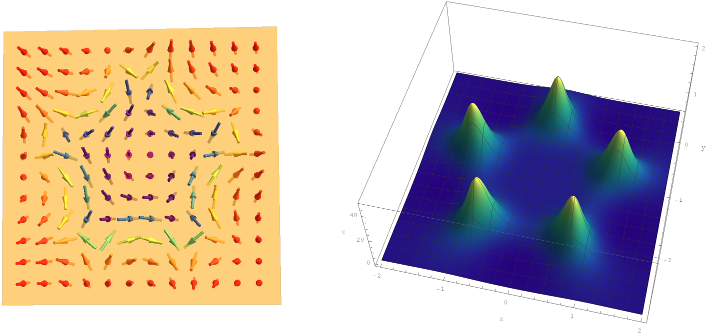



Figure 7: Magnetisation and energy density for $v = -\frac{i}{2}\bar{z} + \frac{1}{2}z^4$. There are five anti-skyrmions surrounding one skyrmion at the centre.

# Sources of more information

The critically coupled model and its generalisations are an active area of study. For more information about magnetic skyrmions check out:

- B. Barton-Singer, CR and B. J. Schroers. Magnetic skyrmions at Critical Coupling. CMP 2020.
- B. J. Schroers. Gauged Sigma Models and Magnetic Skyrmions. Sci Post Phys 2019.
- A. Bogdanov and A. Hubert. Thermodynamically stable magnetic vortex states in magnetic crystals. Journal of Magnetism and Magnetic Materials 1994
- N. Nagaosa and Y. Tokura. Topological properties and dynamics of magnetic skyrmions. Nature nanotechnology 2013.