Statistical field theory reading group:
Computations

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These are some notes for me to keep track of the details of some of the computations from the reading group on statistical field theory following [1]. This will not be a particularly detailed summary of what we discussed but is meant to highlight the details of computations that we may not discuss in the meetings but that I think are worth being aware of.

1 Week 1: The Ising Model

There are a few calculations here that are very similar but I think it is worth showcasing all of them.

Example 1.1 (The saddle point approximation). *Consider the partition function*

$$Z = \int dm e^{-\beta N f(m)}, \tag{1.1}$$

for N very large this will be dominated by the contribution from the minimum of $f(m)$. Taylor expanding the free energy around the minimiser m_0 we have

$$f(m) \simeq f(m_0) + \frac{1}{2}(m - m_0)^2 \frac{\partial^2 f}{\partial m^2}\Big|_{m=m_0} + \dots, \tag{1.2}$$

the linear term is zero as $\frac{\partial f}{\partial m}\big|_{m=m_0} = 0$. This lets us approximate the partition function as

$$Z \simeq e^{-\beta N f(m_0)} \int dm e^{-\frac{\beta N f''(m_0)}{2}(m-m_0)^2} = \sqrt{\frac{2\pi}{\beta N f''(m_0)}} e^{-\beta N f(m_0)}. \tag{1.3}$$

Something important to mention is that while the integral over magnetisations is only from -1 to 1 it is extended to an integral from $-\infty$ to $+\infty$ by using the reasoning that $e^{-\beta N f(m)}$ looks like a bump function centred at $m = m_0$ and so the pieces from outside the interval $[-1, 1]$ will have a negligible contribution.

We then do the usual QFT trick of redeefining the partition function to absorb the prefactor and arrive at

$$Z \simeq e^{-\beta N f(m_0)} \tag{1.4}$$

I can update this discussion if necessary.

Example 1.2 (Relationship between the equilibrium magnetisation and the minimiser if the free energy). *This was another point of contention in the meeting. The key point is that on page 9 of [1] it states that the equilibrium value of the magnetisation m_{eq} is the minimum value, m_0 such that $\frac{\partial f}{\partial m}\big|_{m=m_0}$.*

One way to justify it utilises that the saddle point approximation tells us that the thermal free energy is approximately the same as $Nf(m_0) = F(m_0$ but the thermal free energy is minimised at equilibrium. A discussion of this for the case of energy dependent free energy rather than magnetisation dependent is found at `http://young.physics.ucsc.edu/112/fmin.pdf`.

Example 1.3 (Logarithmic "magic")**.** *We now want to explain the process of rewriting the log of the number of configurations with $N_\uparrow$ up spins and $N_\downarrow = N - N_\uparrow$ down spins. The magnetisation of this configuration is given by*

$$m = \frac{2N_\uparrow - N}{N} = 2\frac{N_\uparrow}{N} - 1, \tag{1.5}$$

and the number of such configurations is

$$\Omega = \frac{N!}{N_\uparrow!\,(N - N_\uparrow)!}. \tag{1.6}$$

Using Stirlings formula we have that

$$\log\Omega \simeq N\log N - N_\uparrow \log N_\uparrow - (N - N_\uparrow)\log(N - N_\uparrow). \tag{1.7}$$

Now consider that

$$(1+m)\log(1+m) + (1-m)\log(1-m) = 2\frac{N_\uparrow}{N}\log\left(2\frac{N_\uparrow}{N}\right) - 2\frac{(N-N_\uparrow)}{N}\log\left(2\frac{N-N_\uparrow}{N}\right), \tag{1.8}$$

$$= 2\Bigg[\frac{N_\uparrow}{N}(\log 2 + \log N_\uparrow - \log N), - \left(1 - \frac{N_\uparrow}{N}\right)(\log 2 + \log(N - N_\uparrow) - \log N)\Bigg], \tag{1.9}$$

$$= 2\left[-\log 2 + \frac{N_\uparrow}{N}\log\left(\frac{N_\uparrow}{N}\right) + \log N - \frac{(N-N_\uparrow)}{N}\log(N - N_\uparrow)\right] \tag{1.10}$$

Thus we have that

$$\frac{\log\Omega}{N} \simeq \log 2 - \frac{1}{2}(1+m)\log(1+m) - \frac{1}{2}(1-m)\log(1-m). \tag{1.11}$$

The next few questions all make use of the Landau free energy which we take to be

$$f(m) \simeq -T\log 2 - Bm + \frac{1}{2}(T - T_c)m^2 + \frac{1}{12}Tm^4, \tag{1.12}$$

We will be concerned with minimising this in both the cases $B = 0$ and $B \neq 0$ as well as using it to compute some physical quantities.

The $-T\log 2$ piece can be discounted because in the partition function it becomes,

$$Z \simeq e^{-\beta N\left(-T\log 2 - Bm_0 + \frac{1}{2}(T-T_c)m_0^2 + \frac{1}{12}Tm_0^4\right)} = 2^N e^{-\beta N\left(-Bm_0 + \frac{1}{2}(T-T_c)m_0^2 + \frac{1}{12}Tm_0^4\right)}, \tag{1.13}$$

and when we take derivatives the 2^N will not contribute.

Example 1.4 (Minimising the free energy when $B = 0$)**.** *This is a straight forward case of minimising the Landau free energy* (1.12)*, where we only consider the quadratic and quartic terms,*

$$\frac{\partial f}{\partial m} = (T - T_c)\, m + \frac{1}{3}Tm^3 = m\left((T - T_c) + \frac{1}{3}Tm^2\right). \tag{1.14}$$

There are now two cases to consider

1. $T > T_c$: *Here the minima is at $m = 0$ as the quadratic piece does not have any real solutions.*
2. $T < T_c$: *Now there are three stationary points: the first is still $m = 0$, now a local maximum, but the quadratic piece now has real solutions and gives the two minima*

$$m = \pm m_0 = \pm\sqrt{\frac{3\,(T_c - T)}{T}}. \tag{1.15}$$

Example 1.5 (Computing the heat capacity of the second order phase transition when $B = 0$)**.** *We know that at a phase transition there will be a discontinuous quantity and in this case it will be the specific heat capacity. In the canonical ensemble the specific heat capacity is given by*

$$C = \beta^2 \frac{\partial^2}{\partial \beta^2} \log Z, \tag{1.16}$$

to compute this we use the saddle point approximation and consider the two cases separately.

1. $T > T_c$: *As the minima $m = 0$ the free energy becomes $f(0) = -T\log 2$ and the partition function is*

$$Z \simeq 2^N. \tag{1.17}$$

   This immediately gives that $C = 0$.

2. $T < T_c$: *Now the minima are at $m = \pm\sqrt{\frac{3(T_c - T)}{T}}$ and the free energy is*

$$f(\pm m_0) = -T\log 2 - \frac{3}{4}\frac{(T_c - T)}{T}, \tag{1.18}$$

   giving the partition function

$$Z \simeq 2^N e^{\frac{3}{4}\beta^2 (T_c - T)^2 N}. \tag{1.19}$$

Now computing the specific heat capacity we have that

$$C = \beta^2 \frac{\partial^2}{\partial \beta^2} \left(N \log 2 + \log e^{\frac{3}{4}\beta^2 (T_c - T)^2 N} \right), \tag{1.20}$$

$$\frac{C}{N} = \frac{3}{4}\beta^2 \frac{\partial^2}{\partial \beta^2} \beta^2 \left(T_c - T\right)^2, \tag{1.21}$$

$$= \frac{3}{2} \left(1 + \beta^2 \left(T_c - T\right)^2 - \beta^{-1} \left(T_c - T\right) \right). \tag{1.22}$$

Combining these two results we have that

$$\frac{C}{N} = \begin{cases} 0 \ \text{as } T \to T_c^+ \\ \frac{3}{2} \ \text{as } T \to T_c^- \end{cases} \tag{1.23}$$

Example 1.6 (Computing the magnetic susceptibility of the first order phase transition when $B \neq 0$)**.** *The magnetic susceptibility is defined as*

$$\chi = \left. \frac{\partial m}{\partial B} \right|_T . \tag{1.24}$$

When $T > T_c$, just keeping the linear and quadratic terms in the Landau free energy,

$$\frac{\partial f}{\partial m} \simeq -B + \left(T - T_c\right) m \tag{1.25}$$

leads to

$$m \simeq \frac{B}{T - T_c} \tag{1.26}$$

and thus to

$$\chi \sim \frac{1}{T - T_c}. \tag{1.27}$$

When $T < T_c$ consider B small such that the minimum is $m_0 + \delta m$ we then have, keeping only terms linear in δm that $m^3 = m_0^3 + 3m_0^2 \delta m$. This means that

$$\frac{\partial f}{\partial m} \simeq -B + \left(T - T_c\right) \delta m + T m_0^2 \delta m + \left. \frac{\partial f}{\partial m} \right|_{B=0, m=m_0}, \tag{1.28}$$

$$\simeq -B + \left(\left(T - T_c\right) + 3\left(T_c - T\right) \right) \delta m, \tag{1.29}$$

$$\simeq -B + 2\left(T_c - T\right) \delta m. \tag{1.30}$$

Now $m + \delta m$ is the minimum so

$$\delta m = \frac{B}{2\left(T_c - T\right)}, \tag{1.31}$$

and thus that

$$\chi \sim \frac{1}{\left(T_c - T\right)}. \tag{1.32}$$

2 Week 2: Ginsburg-Landau Theory

Example 2.1 (Domain Wall's in the Ising model and their free energy)**.** *The Ginsburg-Landau Free energy functional is*

$$F[m(\vec{x})] = \int d^d x \left[\frac{1}{2}\alpha_2(T)m^2 + \frac{1}{4}\alpha_4 m^4 + \frac{1}{2}\gamma(T)\left(\nabla m\right)^2 + \cdots\right], \tag{2.1}$$

where the $\cdots$ *are higher order terms that are irrelevant for our current purposes.*

The Euler-Lagrange equation for this Free energy is

$$\gamma \nabla^2 m = \alpha_2 m + \alpha_4 m^3. \tag{2.2}$$

Assuming that m *is constant this is solved by* $m = 0$ *for* $T > T_c$ *and by* $m = \pm m_0 = \pm\sqrt{-\frac{\alpha_2}{\alpha_4}}$ *when* $T < T_c$*.*

The next simplest case is to assume that $m(\vec{x}) = m(x)$*, that is the magnetisation only varies in one direction. In that case the Euler-Lagrange becomes the ODE*

$$\gamma \frac{d^2 m}{dx^2} = \alpha_2 m + \alpha_4 m^3. \tag{2.3}$$

Assuming that we have a domain wall in the system, that is $m(x)$ *asymptotes to a* $\pm m_0$ *as* $x \to \pm\infty$*, the solution can then be shown to be*

$$m = m_0 \tanh\left(\frac{x - X}{W}\right), \tag{2.4}$$

for X *the position of the domain wall, found by translating the solution, and*

$$W = \sqrt{-\frac{\gamma}{\alpha_2}} \tag{2.5}$$

interpreted as the width. To see that this is a solution consider

$$\begin{aligned}
\frac{d^2 m}{dx^2} &= \frac{d^2}{dx^2} m_0 \tanh\left(\frac{x - X}{W}\right), && (2.6)\\
&= \frac{m_0}{W}\frac{d}{dx}\left(\frac{1}{\cosh^2\left(\frac{x-X}{W}\right)}\right), && (2.7)\\
&= -2\frac{m_0}{W^2}\frac{\tanh\left(\frac{x-X}{W}\right)}{\cosh^3\left(\frac{x-X}{W}\right)}, && (2.8)\\
&= -2m_0\left(\frac{-\alpha_2}{\gamma}\right)\left[\tanh\left(\frac{x-X}{W}\right) - \tanh^3\left(\frac{x-X}{W}\right)\right], && (2.9)\\
&= \frac{\alpha_2}{\gamma}m - \frac{\alpha_2}{\gamma m_0^2}m^3, && (2.10)\\
&= \frac{\alpha_2}{\gamma}m + \frac{\alpha_4}{\gamma}m^3. && (2.11)
\end{aligned}$$

Next we want to compute the free energy of a domain wall, that is we want to plug in the form of $m(x)$ from Equation (2.4) into the Ginsburg-Landau Free energy Equation (2.1). This results in:

$$F[m(x)]_{Wall} = \frac{1}{2}\int d^d x \left[\alpha_2 m^2 + \frac{1}{2}\alpha_4 m^4 - \gamma m \frac{d^2 m}{dx^2}\right], \tag{2.12}$$

$$= -\frac{\alpha_4}{2} m_0^4 \int d^{d-1}x \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \tanh^2\left(\frac{x-X}{W}\right), \tag{2.13}$$

$$\sim L^d + L^{d-1}\sqrt{\frac{\gamma\alpha_2^3}{\alpha_4^2}}, \tag{2.14}$$

This means $F_{Wall} - F_{Vacuum} \sim L^{d-1}\sqrt{\frac{\gamma\alpha_2^3}{\alpha_4^2}}$.

3 Week 3: Example Sheet 1

This week was spent discussing the questions on the first example sheet. I will not tex up everything but will try and included details for some of the exercises that we discussed. We talked about Questions 1,4,5 and 6, I will not include anything about 1 as it is fairly straight forward but will try and tex up most of the details of 4,5 and 6 as it could be useful to reference them.

Example 3.1 (Question 4)**.** *We are given the free energy*

$$f(m) = \alpha_2(T)m^2 + \alpha_4 m^4 + \alpha_6 m^6, \tag{3.1}$$

with $\alpha_4 < 0 < \alpha_6$ and $\alpha_2(T)$ varying with the temperature but starting positive and becoming negative as the temperature is lowered. We giving some sketches of the free energy for three cases, $\alpha_2 > 0$, Figure 1, $\alpha_2 = 0$, Figure 2, and $\alpha_2 < 0$,Figure 3. This shows us that there is a phase transition at some positive value of α_2 where $m = 0$ is joined by two other critical points $m = \pm m_0$. The transition will be first order as the magnetisation will be discontinuous at the transition. For large α_2 the graph will just appear to be a parabola.

To find the value of α_2 at the transition we compute the stationary points of the free energy

$$\frac{\partial f}{\partial m} = 2m\left(\alpha_2 + 2\alpha_4 m^2 + 3\alpha_6 m^4\right), \tag{3.2}$$

This will be zero at $m = 0$ and at

$$m^2 = \frac{-2\alpha_4 \pm \sqrt{4\alpha_4^2 - 12\alpha_2\alpha_6}}{6\alpha_6}, \tag{3.3}$$

From this we read off that the transition will occur when the discriminant vanishes, that

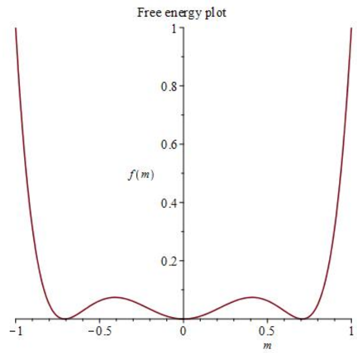

Figure 1: A plot of the free energy for α_2 positive. This is after the phase transition has already happened.

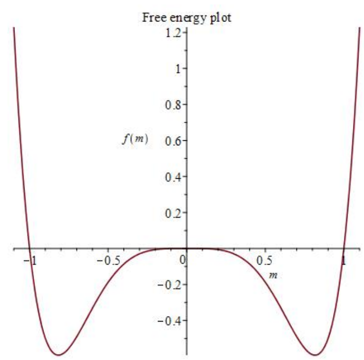

Figure 2: A plot of the free energy for $\alpha_2 = 0$.

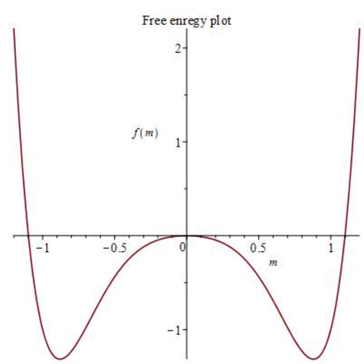

Figure 3: A plot of the free energy for α_2 negative.

is when

$$4\alpha_4^2 - 12\alpha_2\alpha_6 = 0, \tag{3.4}$$

$$\Rightarrow \alpha_2 = \frac{\alpha_4^2}{3\alpha_6}. \tag{3.5}$$

Some plots of the free energy for α_2 either side of the phase transition are included in Figure 4 and Figure 5.

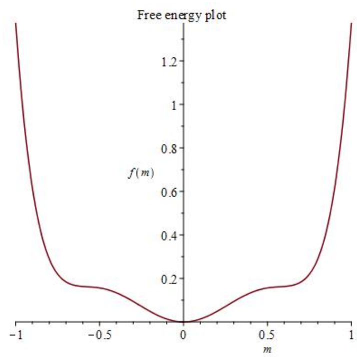

Figure 4: A plot of the free energy for α_2 just above $\frac{\alpha_4^2}{3\alpha_6}$ we see that here there is only one critical point at $m = 0$.

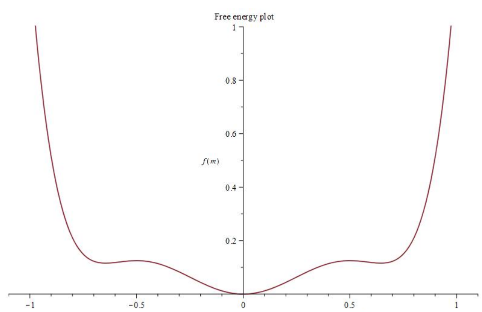

Figure 5: A plot of the free energy for α_2 just below $\frac{\alpha_4^2}{3\alpha_6}$. There is now a minimum at $m = 0$ but two meta stable points at $m = \pm m_0$.

At the phase transition the new metastable points are

$$m = \pm m_0 = \pm\sqrt{\frac{|\alpha_4|}{3\alpha_6}}, \tag{3.6}$$

and m_0 is the jump in the magnetisation.

Next we want a plot of α_2 as a function of α_4 using Equation (3.5) *this is given in Figure 6.*

Finally we are interested in the case $\alpha_4 = 0$ which results in $\alpha_2(T_c) = 0$. We would like to compute the critical exponents for this case.

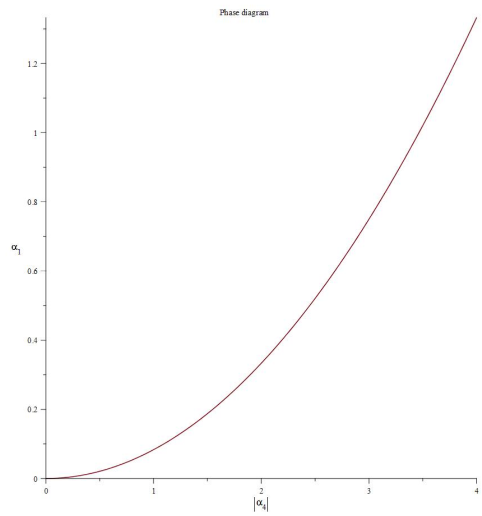

Figure 6: A phase diagram where the solid line is $\alpha_2 = \frac{\alpha_4^2}{3\alpha_6}$. To the left of this is the ordered phase and to the right is the disordered phase.

Example 3.2 (Question 5)**.** *This question involves studying the XY model and seeing that it has the same critical exponents as the Ising model. In terms of a complex scalar field ψ we take the free energy to be*

$$f(\psi) = \alpha_2|\psi|^2 + \alpha_4|\psi|^4. \tag{3.7}$$

When $\alpha_2 < 0$ and $\alpha_4 > 0$ the free energy is as shown in Figure 7, we will also assume that α_2 is analytic in the temperature and the leading order term is such that $\alpha_2 \sim (T - T_c)$.

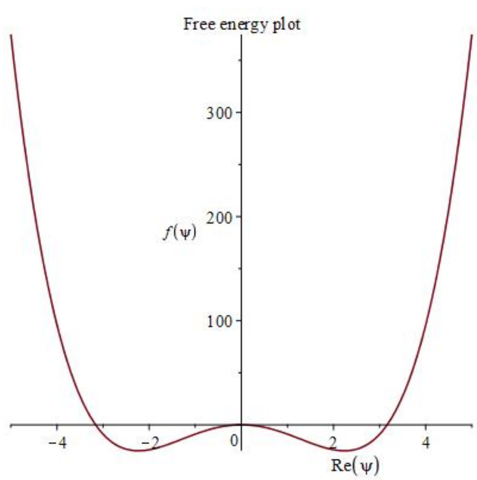

Figure 7: A sketch of the free energy for $\alpha_2 < 0$ and $\alpha_4 > 0$. When $\alpha_2 > 0$ this will just be a quadratic. Rotating this through the complex plane results in the typical sketch of the bottom of a wine bottle which is common in discussions of spontaneous symmetry breaking.

I have sketched it for ψ being real but there will be a flat direction from rotating the *minima through the complex plane. The metastable point at $\psi = 0$ is invariant under the action of $U(1)$, in this case rotation of the phase of the complex number. While there is a whole circle worth of minima coming from $U(1)$ acting on the minima, $\pm\psi_0$, these are distinct points so the minima breaks the $U(1)$ symmetry. The ground states for $\alpha_2 < 0$ will be $\psi = \psi_0 e^{i\theta}$. If we make the substitution $m = |\psi|$ then the XY model free energy, Equation (3.7), becomes the same as that of the Ising model and thus the critical exponents will be the same.*

For use in the next question we note that if we take $\psi = x + iy$ then

$$f(x, y) = \alpha_2 \left(x^2 + y^2\right) + \alpha_4 \left(x^2 + y^2\right)^2. \tag{3.8}$$

Example 3.3 (Question 6)**.** *For this question we bring together the qualitative behaviour that we know the Ising model and the XY model have to explain the phase diagram that we are given. The energy of this lattice model, in terms of the order parameter being a unit vector $\vec{s}$, is*

$$E = -J \sum_{\langle ij \rangle} s_i s_j + g \sum_i \left((s_i^z)^2 - \frac{1}{2} \left((s_i^x)^2 + (s_i^y)^2 \right) \right), \tag{3.9}$$

with $J > 0$.

We want to qualitatively explain the phase diagram of this model. In particular we will consider three regimes: Low T with varying g, $g > 0$ and increasing T and $g < 0$ with T increasing.

1: Low T — *When the temperature is low and g is negative the ground state of this model will be the same as the ground state of the ising model, that is all spins aligned and either pointing up $s^z = 1$ or down $s^z = -1$. However, when g is positive the ground state will be that of the XY model, the spins will all lie in the $x - y$ plane and the configuration will have a $U(1)$ symmetry. As the magnetisation will be discontinuous going between these two states there will be a first order phase transition at $g = 0$, as long as we are below the critical temperature of the Ising and XY model.*

2: $g < 0$ — *When $g < 0$ we have seen that we are in the regime of the Ising model. This means that as the temperature increases we will find a second order phase transition to a disordered state. As g decreases teh energy of the ordered state goes down so the temperature will need to be higher before the disordered state becomes the minima.*

3: $g > 0$ — *The analysis here is basically the same as the step above where now we note that for $g > 0$ we are in the regime of the XY-model which also has a second order phase transition to a disordered state when $T > T_c$. Again as g increases the energy of the ordered state will decrease and so the temperature needs to be higher before the disordered state will have lower energy than the ordered state.*

I may also include other questions that we did not discuss such as 7 and 8 if I get round to looking at them.

4 Week 4: Path Integrals part 1.

It could be worth texing up a discussion of the computation of the partition function in the Gaussian approximation,as well as a computation of the heat capacity. Also some of the details of computing the Ornstein-Zernicke correlation.

5 Week 5: Path Integrals part 2.

Show the relation between the two point function and the Green's function and compute the magnetic susceptibility.

6 Week 6: RG part 1.

There was not anything that I thought needed to be texed up based on our discussion of this section.

7 Week 7: RG part 2.

References

[1] David Tong. *Lectures on Statistical Field Theory.* `http://www.damtp.cam.ac.uk/user/tong/sft.html`