Notes related to the Nahm's equations and hyperkähler quotient constructions reading group

Calum Ross

Last updated June 8, 2022

Hopefully I will tex up notes on some of the topics that we cover during our reading group following [1]. I don't mean to suggest that I will make exhaustive notes or even any notes on most of the discussions. However, I will try and include the important points that we discuss. A webpage containing the details of the reading group and links to further references is available here. Depending how quickly we progress we may also try to understand the hyperkähler quotient construction of the moduli space of solutions to Hitchin's equations following [2], I have previously made some notes on aspects of this topic based on a talk that I gave at the 2017 BIG workshop on Higgs bundles.

1 Week 1: 1/06/2022

We had a general house keeping meeting where we decided that we wanted to read [1] and understand the example of an infinite dimensional hyperkähler quotient construction coming from Nahm's equations. For the 8th, 15th, and 22nd of June I have booked room 707 for us to use.

2 Week 2: 8/06/2022

I will give some general motivation and discuss the dimensional reduction of the ASD equations to give BPS monopole, Hitchin equations, and the Nahm equations. It will roughly follow section 2 of [1].

A sketch of what we discussed is included here, if I have the time I will try to flesh it out in more detail and address some of the comments that I have added.

General Motivation/ Why Quotients

The idea of why we care about quotients is easiest to see in the finite dimensional case where we have a Lie group acting on a manifold $G \circlearrowleft M$, if this action is "nice" then the "unique" configurations are given by the quotient M/G. The natural physics setting for this is gauge theory[1] The ingredients are

- Manifold M,
- Lie group G (assumed compact),
- Principal G-bundle $P \to M$ (and its associated vector bundles $E \to M$ usually the adjoint representation),
- gauge field are connections on P, the space of gauge fields, $\mathcal{A}$, is an affine space modelled on the $\mathfrak{g}$ valued 1-forms (often written $\Omega^1_P(\mathfrak{g})$),

[1] It also shows up in lots of other cases, e.g. translation or rotation symmetries in physical theories. However, these are all finite dimensional examples. The natural source of infinite dimensional group actions on infinite dimensional spaces is in gauge theory.

- Acting on $\mathcal{A}$ is the group of gauge transformations $\mathcal{G}$, these are sections of the Adjoint bundle, when P is trivial

$$\mathcal{G} = \{g : M \to G\} = C^\infty(M, G), \tag{2.1}$$

- $\mathcal{G} \circlearrowleft \mathcal{A}$ as

$$A \mapsto A^g = gAg^{-1} + gdg^{-1}. \tag{2.2}$$

The connections of interest will solve an equation coming from an energy functional: e.g. when $G = SU(2)$ and $M = \mathbb{R}^4$ take

$$E = -\frac{1}{8\pi^2} \int \mathrm{Tr}\,(F \wedge \star F)\,, \tag{2.3}$$

The critical points of this energy are connections A such that $d_A \star F_A = 0$, while the absolute minima are called instantons and satisfy the ASD equations $F_A = - \star F_A$ with energy $E = k \in Z$. If A is an instanton then A^g is also an instanton with the same energy k. (If A satisfies any boundary conditions then they may only be preserved by $g \in \mathcal{G}_0$ e.g. $g \to \mathbb{I}$ asymptotically[2]). Then "physical" configurations are given by

$$\{A \in \mathcal{A} | d_A \star F_A = 0\}/\mathcal{G}_0, \tag{2.4}$$

and "physical" instantons with energy k are in the moduli space

$$\mathcal{M}_k = \{A \in \mathcal{A} | F_A = - \star F_A\}/\mathcal{G}_0. \tag{2.5}$$

The question that we want to answer in this reading group is "How are these quotients computed?" If everything was finite dimensional there is a standard theory that could be applied however, $\dim\mathcal{A} =$ "∞'' $= \dim\mathcal{G}_0$, so how do we know that $\mathcal{M}_k$ is even a manifold?

This can be proved, however it is done in a case by case basis. e.g.

- YM on a Riemann surface [3],
- Hitchin equations on a Riemann surface [2],
- Moduli space of magnetic monopoles on $\mathbb{R}^3$ [4] (this case is really done by making use of the Nahm transform and computing the moduli space of solutions to Nahm's equations which is exactly the problem that we will study in this reading group).

Instantons all the way down

Instantons are solitons in four dimensions by imposing symmetries on them we get other interesting objects, see Table 1 for what happens to translationally invariant instantons.

Manifold	$\mathbb{R}^4$	$\mathbb{R}^3$	$\mathbb{R}^2$	$\mathbb{R}^1$	$\mathbb{R}^0$
Coordinates	(x^0, x^1, x^2, x^3)	(x^1, x^2, x^3)	(x^0, x^1)	(x^0)	None
Invariances	None	x^0	x^2, x^3	x^1, x^2, x^3	x^0, x^1, x^2, x^3
Fields	A	$A, \Phi = A_0$	$A, \Phi = (A_3 - iA_2)\frac{dz}{2}$	$A_0, \vec{A}$	A_i
ASD eqs	$F_A = - \star F_A$	$F_A = \star d_A \Phi$	$\bar{\partial}_A \Phi = F_A + [\Phi, \Phi^*]$	Nahm equations	ADHM equations

Table 1: Instantons and their descendents.

These different dimensional instantons are not independent, some of them are related to each other through the Nahm transform. I may add more on this later depending on if we discuss the Nahm transform more in this reading group.

[2] or at least g is asymptotically in the same connected component as the identity.

Dimensional reductions

In the session I only discussed the Nahm case however, I have notes on the other cases that I will try to tex up.

Instantons

BPS monopoles

Hitchin System

Nahm equations

Consider $\mathbb{R}^4 = M = \mathbb{R} \times \mathbb{R}^3$ and impose invariance on the $\mathbb{R}^3$. An instanton on $\mathbb{R}^4$, and teh associated bundle P descend to $\mathbb{R}$ where the instanton becomes

$$A_0 dx^0 + A_1 dx^1 + A_2 dx^2 + A_3 dx^3 = A + A_i dx^i. \tag{2.6}$$

Since $A_i : \mathbb{R} \to \mathfrak{g}$ combine them into a "Higgs field"

$$\Phi = A_1 i + A_2 j + A_3 k : \mathbb{R} \to \mathfrak{g} + \mathrm{Im}\mathbb{H}. \tag{2.7}$$

Often the notation $T_0, \vec{T}$ is used rather than A_0, A_i but I will try to stick with the notation used in [1].

The ASD equations reduce to

$$\star\, d_A \Phi + \frac{1}{2}[\Phi, \Phi] = 0, \tag{2.8}$$

or

$$\dot{A}_i + [A_0, A_i] + \frac{1}{2}\varepsilon_{ijk}[A_j, A_k] = 0, \tag{2.9}$$

where we sum over j, k and there is an equation for each $i = 1, 2, 3$. Checking this dimensional reduction explicitly is left as an exercise to the especially interested reader.

The action of the group of gauge transformations preserves Eq. (2.9), and $g \in \mathcal{G}$ acts on an element of $\mathcal{A}$ as

$$g \cdot (A_0, \vec{A}) = \left(\mathrm{Ad}_g A_0 - \dot{g} g^{-1}, \mathrm{Ad}_g \vec{A}\right), \tag{2.10}$$

N.B. this matches ths earlier definition of a gauge transformation since $dgg^{-1} = -gdg^{-1}$.

When G is a unitary group e.g. $U(n)$ it is useful to think of the associated rank n vector bundle $E \to I = [0, 1]$. We have swapped to considering Nahm over a compact interval as that is the most commonly studied case in the literature. Following on from Jakob's question this seems to be because monopoles on $\mathbb{R}^3$ correspond to Nahm equations on I, the specifics follow from looking at the details of the Nahm transform if we cover this later I will add the details. Then

$$A_0 \to \nabla : \Omega^0_I(E) \to \Omega^1_I(E) \tag{2.11}$$

becomes a connection on a vector bundle and the A_i are skew adjoint sections of $\mathrm{End}\,(E)$, in trivialisation they are just skew adjoint matrices. If the coordinate on I is t then Nahm's equations become

$$\nabla_t A_i + \frac{1}{2}\varepsilon_{ijk}[A_j, A_k] = 0. \tag{2.12}$$

The group of gauge transformations are now the automorphisms of E which preserve the hermitian metric, $\mathcal{G} = \mathrm{Aut}_0\,(E)$, $g \in \mathcal{G}$ acts by conjugation on ∇_t and on $\vec{A}$.

Example 2.1. Let $G = SU(2)$ and take its basis to be $\sigma_1, \sigma_2, \sigma_3 \in \mathfrak{su}(2)$ such that $[\sigma_i, \sigma_j] = \varepsilon_{ijk}\sigma_k$ e.g. $\sigma_i = -\frac{i}{2}\tau_i$ for τ_i the Pauli matrices.

Then $\left(0, \frac{\vec{\sigma}}{t}\right)$ is a solution to Eq. (2.12) with a first order pole at $t = 0$. More generally, there are solutions in terms of Jacobi elliptic functions given in terms of an elliptic modulus $0 \leq k \leq 1$ and a parameter $0 \leq D \leq 2K(k)$, for $K(k)$ the complete elliptic integral of the second kind. Taking

$$(A_0, A_1, A_2, A_3) = (0, f_1\sigma_1, f_2\sigma_2, f_3\sigma_3) \tag{2.13}$$

Nahm's equations reduce to the Euler-Poinsot equations for a spinning top which are solved by

$$f_1(t) = \frac{D\mathrm{cn}_k\,(Dt)}{\mathrm{sn}_k\,(Dt)}, \tag{2.14}$$

$$f_2(t) = \frac{D\mathrm{dn}_k\,(Dt)}{\mathrm{sn}_k\,(Dt)}, \tag{2.15}$$

$$f_3(t) = \frac{D}{\mathrm{sn}_k\,(Dt)}. \tag{2.16}$$

These solutions also have first order poles at $t = 0$ and for $D \to 0$ these reduce to the initial examples. When $D = 2K(k)$ there is also a pole at $t = 1$, and when $k \to 0$ these become trigonometric

$$f_1(t) = \frac{D\cos(Dt)}{\sin(Dt)}, \tag{2.17}$$

$$f_2(t) = f_3(t) = \frac{D}{\sin(Dt)}. \tag{2.18}$$

Acting on these solutions with the group of gauge transformations,

$$\mathcal{G}_0 = \{g : I \to G | g(0) = \mathbb{I}\} \tag{2.19}$$

and by an $SO(3)$ action known as hyperkähler rotation[3] gives all the $SU(2)$ solutions to Nahm's equations with these poles and residues.

3 Week 3:15/06/2022

4 Week 4: 22/06/2022

References

[1] M. Mayrand. *Nahm's equations in hyperähler geometry*, Lecture notes, 2020.

[2] N. J. Hitchin. The Self-Duality Equations on a Riemann Surface. *Proc. London Math. Soc.*, (3) 55 (1987) 59-126.

[3] M. Atiyah and R. Bott. *The Yang-Mills Equations over Riemann Surfaces*, Phil. Trans. R. Soc. Lond. A. 308 (1983), 523–615.

[4] M. Atiyah and N. Hitchin, *The Geometry and Dynamics of Magnetic Monopoles.* Princeton legacy library. Princeton Univercity Press, Princeton, July 2014.

[3]this is the action $q \in SP(1) \subset \mathbb{H}^*$ $A \mapsto q \cdot A = qAq^{-1}$ which is an $SO(3)$ action since q and $-q$ act the same on A.