

Monopoles with Non-Maximal Symmetry Breaking

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1 Introduction

These are some background notes about non-Abelian monopoles with non-maximal symmetry breaking. I will start by stating some Lie algebra conventions that we use then reviewing the standard case of maximal symmetry breaking. I will be following a variety of sources here including [2, 5–7, 9] The review paper [11] also contains some useful discussions about monopole clouds and moduli space metrics in the non-maximal symmetry breaking case.

The idea is that we will understand the example of non-maximal symmetry breaking for $SU(3)$ monopoles discussed in [2, 6], then try to construct our own examples. As part of this I would like to understand a gluing construction for the non-maximal symmetry breaking case. We can also try to understand the Nahm side following [5].

In [4] there is also a description of arbitrary symmetry breaking and how to approach it from the Nahm transform side.

2 Lie Algebra Conventions

For the Lie algebra conventions I will follow [8] for a review and to establish consistent conventions. This means that when we take examples from other papers we will need to convert them to these conventions if they are not in the same conventions. What is said here takes the base field to be \mathbb{C} rather than \mathbb{R} , for a real Lie algebra we must work with its complexification $\mathfrak{g}^{\mathbb{C}}$ to get a root decomposition. I suppress the complexification here for notational simplicity.

A Lie group G has Lie algebra \mathfrak{g} . A maximal abelian subalgebra of \mathfrak{g} (consisting of semi-simple elements) is called a Cartan subalgebra \mathfrak{h} . It is not unique but all Cartan subalgebras are

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related by automorphisms of \mathfrak{g} . All Cartan subalgebras have the same dimension, $\dim \mathfrak{h} = \text{rk}$, the rank of the group.

The generators of \mathfrak{h} are the linearly independent ad-diagonalizable elements of \mathfrak{g} H^i , $i = 1, 2, \dots, \text{rk}$ such that $[H^i, H^j] = 0$. Thus the adjoint maps ad_{H^i} are simultaneously diagonalisable. Thus \mathfrak{g} is spanned by $y \in \mathfrak{g}$ that are simultaneous eigenvectors of $\text{ad}_h \forall h \in \mathfrak{h}$,

$$[h, y] = \text{ad}_h(y) = \alpha_y(h) y, \quad (2.1)$$

with $\alpha_y(h) \in \mathbb{C}$ depending linearly on h for fixed y *i.e.* $\alpha_y : \mathfrak{h} \rightarrow \mathbb{C}$ is a linear functional and thus $\alpha_y \in \mathfrak{h}^*$. We call α a root of the \mathfrak{g} when $\alpha \neq 0$. The roots lead to the root space decomposition of \mathfrak{g} as

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \neq 0} \mathfrak{g}_\alpha, \quad \mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h) \cdot x \forall h \in \mathfrak{h}\}. \quad (2.2)$$

Associated to this decomposition we have a basis \mathcal{B} for \mathfrak{g} , which consists of the basis $\{H^i\}$ for \mathfrak{h} and elements E^α such that

$$[H^i, E^\alpha] = \alpha^i E^\alpha. \quad (2.3)$$

The vector $(\alpha^1, \dots, \alpha^{\text{rk}})$ is called the root vector or just the root. The E^α are called ladder or step operators.

There is an inner product on \mathfrak{g} called the Killing form. It is given by

$$\kappa(x, y) = \text{Tr}(\text{ad}_x \circ \text{ad}_y), \quad (2.4)$$

for a matrix Lie algebra in a basis T^a this becomes

$$\kappa^{ab} = \frac{1}{I_{\text{ad}}} \kappa(T^a, T^b) = \frac{1}{I_{\text{ad}}} \sum_{c,e} f_e^{bc} f_c^{ae}, \quad (2.5)$$

with I_{ad} a normalisation constant and f_e^{bc} the structure constants such that $[T^b, T^c] = f_e^{bc} T^e$, upto a constant this is just the matrix trace on the basis.

The roots are split into positive, R^+ , and negative, R^- , roots. We also have simple roots given by the positive roots which cannot be obtained as a linear combination of other positive roots with positive coefficients. There are rk simple roots α_i . The simple roots do not form an orthonormal basis, with the lack of orthonormality encoded in the Cartan matrix

$$A_{ij} = 2 \frac{(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}. \quad (2.6)$$

Here $(,)$ is a non-degenerate inner product on \mathfrak{h}^* defined through $\kappa(,)$.

Associated to the roots are the coroots defined as

$$\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)}, \quad (2.7)$$

with α_i^\vee called a simple coroot². The dual space of the root space is the weight space. Take the coroots to be a basis of the root space then the dual basis elements are the fundamental

²Note that in some sources, including [2], the factor of 2 is not included when defining the coroots. This is another reason why care needs to be taken when comparing different references.

weights ϖ_i such that $\varpi_i(\alpha_j) = \delta_{ij}$. This basis is called the Dynkin basis and the components of a weight in this basis are the Dynkin labels.

When we restrict to roots given by linear combinations of simple roots with integer coefficients we get the root lattice $L(\mathfrak{g})$. The integer space of simple coroots and fundamental weights are the coroot lattice $L^\vee(\mathfrak{g})$ and the weight lattice $L_W(\mathfrak{g})$ respectively.

When \mathfrak{g} is simple there is a unique highest root called θ such that the *height* of θ is larger than any other root and $(\theta, \theta) \geq (\alpha, \alpha)$ for all roots α . The height of a root is defined as the sum of the integer coefficients of the simple roots in its decomposition. Conventionally, $(\theta, \theta) = 2$ which fixes the normalisation I_{ad} .

The decomposition of the highest root in terms of simple roots defines the Coxeter numbers m_i and their dual m_i^\vee as

$$\theta = \sum_{i=1}^{\text{rk}} m_i \alpha_i, \quad \theta^\vee = 2 \frac{\theta}{(\theta, \theta)} = \sum_{i=1}^{\text{rk}} m_i^\vee \alpha_i^\vee. \quad (2.8)$$

As the simple roots are linearly independent this implies that

$$\frac{m_i}{m_i^\vee} = \frac{(\theta, \theta)}{(\alpha_i, \alpha_i)} = \frac{2}{(\alpha_i, \alpha_i)}. \quad (2.9)$$

Since θ is a long root, *e.g.* $(\theta, \theta) = 2$, this implies $m_i^\vee \leq m_i$.

Another important object is the *Weyl vector* ρ also called the half root sum which is given by

$$\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha. \quad (2.10)$$

Since $(\rho, \alpha_i^\vee) = 1$ for all simple roots we have that $\rho = \sum_{i=1}^{\text{rk}} \varpi_i$ when expressed in terms of the fundamental weights. We also have that

$$(\rho, \theta^\vee) = \sum_{i=1}^{\text{rk}} m_i^\vee. \quad (2.11)$$

Finally, there is a group action on the root space give by the Weyl group. The Weyl group is the subgroup of the automorphism group of the root space generated by “reflections” in a given root *e.g.*

$$w_\alpha : \beta \mapsto w_\alpha(\beta) = \beta - 2 \frac{(\beta, \alpha)}{(\alpha, \alpha)} \alpha = \beta - (\beta, \alpha^\vee) \alpha. \quad (2.12)$$

Elements of the root space fixed under a Weyl transformation w_α define hyperplanes perpendicular to the roots. Removing these hyperplanes from the root spaces divides it into open cones known as Weyl chambers. The hyperplanes are often called the walls of the Weyl chambers. Once the simple roots have been chosen there is a distinguished Weyl chamber known as the dominant or fundamental Weyl chamber or domain where all of the lattice points in the chamber correspond to weights with positive Dynkin labels. Any element of the root system, not on a wall, can be mapped to the fundamental Weyl chamber through the action of a Weyl transformation.

The following computation may prove useful.

$$\begin{aligned}
(w_\alpha(\lambda), w_\alpha(\mu)) &= (w_\alpha(\lambda), \mu) - (w_\alpha(\lambda), (\mu, \alpha^\vee) \alpha) \\
&= (\lambda - (\lambda, \alpha^\vee) \alpha, \mu) - (\lambda - (\lambda, \alpha^\vee) \alpha, (\mu, \alpha^\vee) \alpha) \\
&= (\lambda, \mu) - (\lambda, \alpha^\vee) (\mu, \alpha) - (\lambda, \alpha) (\mu, \alpha^\vee) + (\lambda, \alpha) (\mu, \alpha^\vee) (\alpha, \alpha) \\
&= (\lambda, \mu),
\end{aligned}$$

where we used the definition of a coroot to move around the α^\vee . Next note that for a simple root α_i $w_i(\rho) = \rho - \alpha_i$ since $(\rho, \alpha_i^\vee) = 1$ and that $w_i(\alpha_i) = -\alpha_i$. Finally,

$$\begin{aligned}
0 &= (\alpha_i^\vee, \rho) - (\alpha_i^\vee, \rho) \\
&= (\alpha_i^\vee, \rho) - (w_i(\alpha_i^\vee), w_i(\rho)) \\
&= (\alpha_i^\vee, \rho + w_i(\rho)) \\
&= (\alpha_i^\vee, 2\rho - \alpha_i) \\
&= 2(\rho^i - 1).
\end{aligned}$$

Since this is true for every simple root it implies that $\rho = \sum_{i=1}^{\text{rk}} \varpi_i$. Recall that $\rho^i = (\rho, \alpha_i^\vee)$ and that $(\alpha_i^\vee, \alpha_i) = 2 \frac{(\alpha_i, \alpha_i)}{(\alpha_i, \alpha_i)} = 2$.

Let us review some simple examples to help secure these conventions in our minds.

Example 2.1 ($SU(2)$). This is a rank one group so there is one simple root α , its associated coroot α^\vee , and one fundamental weight ϖ . Going back to basics the definition of the group and its Lie algebra as subgroups of complex matrices are

$$SU(2) = \{A \in M(2, \mathbb{C}) \mid AA^* = \mathbb{I}, \det A = 1\}, \quad (2.13)$$

$$\mathfrak{su}(2) = \{A \in M(2, \mathbb{C}) \mid A + A^* = 0, \text{Tr} A = 0\}. \quad (2.14)$$

The Lie algebra is formed of complex anti-hermitian matrices and has as its basis $\frac{i}{2}$ times the Pauli matrices.

$$T_1 = \frac{i}{2}\tau_1 = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad T_2 = \frac{i}{2}\tau_2 = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T_3 = \frac{i}{2}\tau_3 = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad (2.15)$$

They satisfy the commutation relations $[T_i, T_j] = \epsilon_{ij}^{k} T_k$. As mentioned above really we are working with $\mathfrak{su}(2)^\mathbb{C} \simeq \mathfrak{sl}(2, \mathbb{C})$ as we will consider complex combinations of generators.

As these generators do not commute the Cartan subalgebra is given by the span of any one of them. The Conventional choice is to take $h = -iT_3$ and the span of h gives \mathfrak{h} . Next we need know that there will be one root since $\text{rk} = 1$ for $\mathfrak{su}(2)$. Next take $e_+ = -i(T_1 + iT_2)$ and $e_- = -i(T_1 - iT_2)$ then

$$[h, e_\pm] = \pm e_\pm, \quad (2.16)$$

$$[e_+, e_-] = 2h. \quad (2.17)$$

This can also be written as

$$\text{ad}_h(e_\pm) = \pm e_\pm, \quad (2.18)$$

$$\text{ad}_h(h) = 0. \quad (2.19)$$

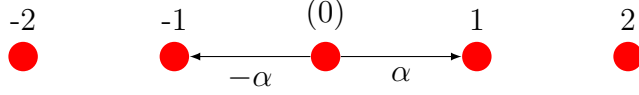


Figure 1: The root spaces of $\mathfrak{su}(2)$. N.B the roots are the weights in the adjoint representation.

Thus ad_h has been diagonalized with eigenvectors h, e_{\pm} . Thus the root vectors are ± 1 and the e_{\pm} are the ladder operators.

The simple coroot is related to the generator of the Cartan, $\alpha^{\vee} = \frac{2\alpha}{(\alpha, \alpha)} = 2$ which becomes $i\tau_3$ as a matrix e.g. long coroots have length $\|\alpha^{\vee}\| = \sqrt{2}$, and the Killing form is normalised so that $\kappa(T_3, T_3) = -\frac{1}{4}\text{Tr}(i\tau_3)^2 = \frac{1}{2}$.

The fundamental Weyl chamber is then the half line $\mathbb{R}_+ = (0, \infty)$ with the wall being the point 0. **This should be the simplest example but there are a few things that still need to be clarified.**

Example 2.2 ($SU(3)$). This is the $\text{rk} = 2$ case so now we get two elements in the Cartan H_1, H_2 and we will have two roots α_1, α_2 . Typically the Cartan subalgebra is taken to be that generated by the diagonal Gell–Mann matrices

$$H_1 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad (2.20)$$

normalised such that $\text{Tr}(H_i H_j) = \frac{1}{2}\delta_{ij}$. These are often written as $\vec{H} = (H_1, H_2)$ as a “root space vector”³ A priori there are three sets of corresponding ladder operators given similarly to the $\mathfrak{su}(2)$ case with roots $\vec{\beta} = (\beta_1, \beta_2)$. In the Cartan-Weyl basis where

$$[H_i, E_{\vec{\beta}}] = \beta_i E_{\vec{\beta}}, \quad [E_{\vec{\beta}}, E_{-\vec{\beta}}] = \vec{\beta} \cdot \vec{H}, \quad (2.21)$$

the corresponding simple roots are

$$\alpha_1 = (1, 0), \quad \alpha_2 = \frac{1}{2}(-1, \sqrt{3}) \quad (2.22)$$

Recall that the relationship between the simple roots and the fundamental weights is given by the Cartan matrix $A_{ij} = 2\frac{(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}$ for $\mathfrak{su}(3)$ we can compute that

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad (2.23)$$

and thus $\alpha_1 = 2\varpi_1 - \varpi_2$ and $\alpha_2 = 2\varpi_2 - \varpi_1$.

Note that the H_i that we use in the construction of the Cartan subalgebra and the root decomposition is not an element of $\mathfrak{su}(2)$ since it is hermitian rather than anti-hermitian. Since we are working with the complexification this is fine, however, we should be aware of it.

³This connects the root vector with the matrix representation of the roots and coroots. e.g. when we write a root (or coroot) as $\alpha = (\alpha_1, \dots, \alpha_{\text{rk}})$ we are saying that as a matrix $\alpha = \sum_{i=1}^{\text{rk}} \alpha_i H_i$. **This relies on some potentially unhelpful identifications of spaces with their duals.**

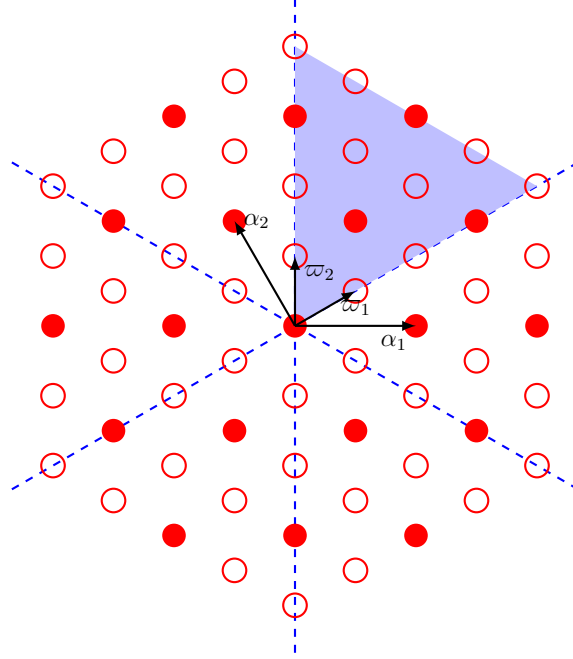


Figure 2: The root spaces of $\mathfrak{su}(3)$. The ϖ_i are the fundamental weights defined by being orthogonal to the simple roots in the root space. From this figure we can see that the root lattice given by the integer span of the α_i is a sublattice of the weight lattice.

Now through an abuse of notation we can write an element of $\mathfrak{su}(2)$ as

$$\Phi_\infty = i\vec{h} \cdot \vec{H}, \quad (2.24)$$

where we think of \vec{h} as the vector representation of a root.

3 A Brief Review of Maximal Symmetry Breaking

A BPS monopole is a pair (A, Φ) of a G connection A and an adjoint valued Higgs field Φ on \mathbb{R}^3 which satisfy the BPS equation

$$F_A = \star d_A \Phi \quad (3.1)$$

Check that this matches with $D_i \Phi = B_i$. The boundary conditions are that

$$\lim_{r \rightarrow \infty} \|\Phi\|^2 = -\frac{1}{2} \lim_{r \rightarrow \infty} \text{Tr} (\Phi^2) = \frac{1}{2} v^2, \quad (3.2)$$

and that

$$\|B_i\| = \|D_i \Phi\| \sim \mathcal{O} \left(\frac{1}{r^2} \right), \quad (3.3)$$

which set the symmetry breaking scale and enforce finite energy. The form of symmetry breaking is chosen by taking the Higgs field to be

$$\Phi = \Phi_0 - \frac{\kappa}{2r} + \mathcal{O} \left(\frac{1}{r^2} \right), \quad (3.4)$$

in [2] they choose slightly different conditions and work along the z -axis so $\Phi(0,0,z) = \Phi_0 - \frac{\kappa}{4\pi z} + \mathcal{O}\left(\frac{1}{z^2}\right)$. Here Φ_0 is a constant element of \mathfrak{g} which is chosen to lie in \mathfrak{h} . Abusing notation slightly we write it as $\Phi_0 = i\vec{h} \cdot \vec{H}$.

The two Lie algebra elements Φ_0 and κ are called the mass⁴ and charge of the monopole. In the vector bundle picture they are given by block diagonal matrices

$$\mu = \text{idiag}(\lambda_1, \dots, \lambda_1, \lambda_2, \dots, \lambda_2, \dots, \lambda_n, \dots, \lambda_n) \quad (3.5)$$

$$\kappa = \text{idiag}(k_{1,1}, k_{1,2}, \dots, k_{1,r_1}, k_{2,1}, \dots, k_{2,r_2}, \dots, k_{n,1}, \dots, k_{n,r_n}), \quad (3.6)$$

where r_i is the number of times λ_i is repeated. For $G = SU(N)$ these are traceless which implies that

$$\sum_j \lambda_j r_j = 0, \quad \sum_{j=1}^n k_j = 0, \quad (3.7)$$

where $k_j = \sum_{l=1}^{j_j} k_{j,l}$. There is also an assumed ordering for the λ_i , [Here we think that the ordering is \$\lambda_1 > \lambda_2 > \dots > \lambda_n\$ though this does not agree with that taken in . This also seems to need a condition that \$k_{1,j} \leq k_{2,j} \leq \dots \leq k_{r_j,j} \forall j\$.](#) The magnetic charges are determined by the $k_{i,j}$ through

$$m_a = \sum_{j=1}^a k_j, \quad (3.8)$$

there are $N - 1$ of the m'_a s as the N 'th case vanishes due to the traceless condition.

In this context maximal symmetry breaking means that $SU(N)$ breaks to $S(U(1) \times U(1) \times \dots \times U(1))$, while minimal symmetry breaking is $SU(N)$ to $S(U(N-1) \times U(1))$ and intermediate symmetry breaking is $SU(N)$ to $S(U(r_1) \times \dots \times U(r_n))$ *e.g.* there is a factor for each block with the rank of the $U(n_i)$ determined by the size of the block. In the maximal symmetry breaking case the m_a are the first Chern numbers of the asymptotic line bundles.

4 Non-Maximal Symmetry Breaking

4.1 Configuration and Moduli spaces

Consider now the case of non-maximal symmetry breaking where $SU(N)$ breaks to $S(U(r_1) \times \dots \times U(r_n))$. More explicitly we will follow [2] and focus on the case of $SU(3) \rightarrow S(U(2) \times U(1)) = U(2)$. The conventions taken there are slightly different from that used in [4], particularly with respect to the ordering of the λ_i so care needs to be taken. As above a monopole is a pair (A, Φ) satisfying Eq. (3.1) with asymptotics as in Eq. (3.4). As we saw in Example. 2.2 above $\mathfrak{su}(3)$ has two simple roots α_1, α_2 and two generators of the Cartan H_1, H_2 such that $\text{Tr}(H_i H_j) = \frac{1}{2} \delta_{ij}$. In terms of the coroots and the Cartan generators we know that

$$\Phi_0 = i\vec{h} \cdot \vec{H} = i\nu H_2, \quad (4.1)$$

$$\kappa = i\vec{g} \cdot \vec{H} = i \left[\left(m_1 - \frac{m_2}{2} \right) H_1 + m_2 \frac{\sqrt{3}}{2} H_2 \right], \quad (4.2)$$

⁴The mass is really the norm of this element

where $\vec{h} = (0, v)$ is perpendicular to α_1 ensuring that $\text{Stab}(\Phi_0) = S(U(2) \times U(1))$ with the $U(2)$ being that associated with $\mathfrak{su}(2)_{\alpha_1}$ and H_2 , and the $U(1)$ coming from α_2 . We have used the Dirac quantisation condition to ensure that

$$\vec{g} = \frac{1}{2} (m_1 \alpha_1^\vee + m_2 \alpha_2^\vee), \quad (4.3)$$

with $m_1, m_2 \in \mathbb{Z}$ and the factor of two included so that this agrees with the choice in [2], up to removing the $\frac{4\pi}{e}$ over all factor. With these choices we have $\|\Phi_0\| = \frac{1}{4}v^2$ which is $\frac{1}{2}$ that claimed in [2], **to match their conventions it may be better to take $\vec{h} = (0, \sqrt{2}v)$ but this would disagree with what they claim to be taking as then $\Phi_0 = i\sqrt{2}vH_2$.**

Since κ and Φ_0 are not parallel, unless $m_1 = \frac{m_2}{2}$, κ is not invariant under $\text{Stab}(\Phi_0) = U(2)$ with the invariant quantity being

$$-\text{Tr}(\kappa\Phi_0) = \sqrt{\frac{3}{2}}m_2. \quad (4.4)$$

Thus m_2 is the topological charge and is invariant under the asymptotic gauge group while m_1 can be changed by the $U(2)$ action. More explicitly the $U(2)$ is given by elements of the form

$$P = e^{i\chi Y} e^{-i\alpha I_3} e^{-i\beta I_2} e^{-i\gamma I_3}, \quad (4.5)$$

where

$$\begin{aligned} Y &= \frac{2}{\sqrt{3}}H_2, \\ I_3 &= H_1, \\ I_2 &= \frac{1}{2i}(E_{\alpha_1} - E_{-\alpha_1}) = \frac{1}{2i} \begin{pmatrix} \tau_2 & 0 \\ 0 & 0 \end{pmatrix}, \\ I_1 &= \frac{1}{2}(E_{\alpha_1} + E_{-\alpha_1}) = \frac{1}{2} \begin{pmatrix} \tau_1 & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

generate the $U(2)$.

A computation gives that the orbit of κ under the action of this $U(2)$ is

$$\kappa_P = P\kappa P^{-1} = m_2 i \frac{\sqrt{3}}{2} H_2 + i \vec{k} \cdot \vec{I}, \quad (4.6)$$

where $k = |\vec{k}| = |m_1 - \frac{m_2}{2}|$ and

$$\hat{k} = \begin{pmatrix} \sin \beta \cos \alpha \\ \sin \beta \sin \alpha \\ \cos \beta \end{pmatrix}. \quad (4.7)$$

The $\vec{k} \cdot \vec{I}$ term gives the magnetic orbit as that is the piece being changed by the action of $\text{Stab}(\Phi_0)$ and it is explicitly an $S^2 \subset \mathfrak{su}(3)$ with radius $k = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$. This magnetic orbit intersects the Cartan \mathfrak{h} when $\vec{k} \cdot \vec{I} \in \mathfrak{h}$ which happens at two points m_1 and $m_2 - m_1$. To see this more explicitly write out $\vec{k} \cdot \vec{I}$ and evaluate it at $\beta = 0, \pi$ where you will find that

the first entry in κ is $m_1, m_2 - m_1$ respectively. These two points are collected together as $[m_1] = \{m_1, m_2 - m_1\}$ and called the holomorphic charge⁵.

It is common to use notation $K = m_2$ and $k = |m_1 - \frac{m_2}{2}|$ rather than $m_2, [m_1]$.

Now the framed configuration space is given by

$$\mathcal{C}_{\vec{h}} = \mathcal{A}_{\vec{h}} / \mathcal{G}_0, \quad (4.8)$$

where \mathcal{G}_0 are the gauge transformations that tend to the identity asymptotically and $\mathcal{A}_{\vec{h}}$ is the space of pairs satisfying the boundary conditions. The framed moduli space \mathcal{M} is then defined as the space of pairs satisfying the boundary conditions and the BPS equations modulo framed gauge transformations. Taubes' existence theorem for monopoles shows that in the maximal symmetry breaking case both the configuration space and the moduli space are non-empty and the connected components are labelled by the topological charges⁶ While in the non-maximal symmetry breaking case the theorem still tells us the configuration and moduli space are non empty, now the connected components are only labelled by K and these are further split up into pieces labelled by the holomorphic charges. This finer subdivision is not topological in nature but is a stratification with the strata labelled by the holomorphic charge or the radius of the magnetic orbit.

This can be summarised as

$$\mathcal{M}_{m_1, m_2}^{\max} \neq \emptyset, \quad \dim \mathcal{M}_{m_1, m_2}^{\max} = 4(m_1 + m_2), \quad (4.9)$$

$$\mathcal{M}_K \neq \emptyset, \quad \mathcal{M}_{K, k} \subset \mathcal{M}_K, \quad (4.10)$$

where $\mathcal{M}_{K, k}$ are the strata for a given holomorphic charge. In the maximal symmetry breaking case the dimension of the moduli space is known and there is a well separated region where the parameters are identified as those of $SU(2)$ monopoles in particular m_1 monopoles of type α_1 and m_2 monopoles of type α_2 . For non-maximal symmetry breaking finding the dimension and identifying what the parameters mean is much harder see [3] for a discussion this, [10] computes the dimension of the moduli spaces by making use of the rational map correspondence.

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The action of the Weyl group relates κ for m_1 and $m_1 - m_2$, e.g. the two points on the same magnetic orbit, and thus they correspond to the same moduli space. This is relevant because we can think of each point in the dual lattice as corresponding to a possibly empty component $\mathcal{M}_{K, k}$ of the moduli space, In fact in [10] it is proved that $\mathcal{M}_{K, k} \neq \emptyset$ iff $k \leq \frac{K}{2} > 0$. This means that

$$K \text{ even} \Rightarrow k = 0, 1, 2, \dots, \frac{K}{2}, \quad (4.11)$$

$$K \text{ odd} \Rightarrow k = \frac{1}{2}, \frac{3}{2}, \dots, \frac{K}{2}, \quad (4.12)$$

This means that there are only non-empty components of the moduli space inside the cone.

Need a picture of this!

⁵The name is because while the asymptotics determine the topology of the bundle over S_∞^2 its holomorphic structure is changed by the $U(2)$ action so a choice of $[m_1]$ corresponds to a choice of holomorphic structure.

⁶recall that m_1 and m_2 are both topological for maximal symmetry breaking.

The edge of the cone, where $m_1 = 0$ and $k = \frac{m_2}{2}$ is called the small strata. The centre of the cone, which is only allowed for K even, where $m_1 = \frac{m}{2}$ and the magnetic orbit is a point is called the large strata.

4.2 Embedding $SU(2)$ monopoles

The small strata can be entirely realised by considering $SU(2)$ monopoles embedded in $SU(3)$ along the root α_2 . This was proved in [3] using the Nahm data and gives an alternative way to compute the dimension of the small strata as well as giving $\mathcal{M}_{K,k}$ the structure of a fibration with fibre $\mathcal{M}_K^{SU(2)}$ and base the magnetic orbit S^2 .

Consider a charge K $SU(2)$ monopole (a, ϕ) , where $SU(2)$ has generators τ_1, τ_2, τ_3 and the asymptotic $U(1)$ is generated by τ_3 . Then in the usual gauge the Higgs field has asymptotics

$$\phi \sim \frac{\omega}{2} i \tau_3 - \frac{K}{2r} i \tau_3. \quad (4.13)$$

More generally the Higgs field is $\phi = \phi_{\text{BPS}} \frac{i}{2} \tau_3$ with $\phi_{\text{BPS}} = 2\omega \coth(2\omega r) - \frac{1}{r}$. Here $\omega = 3$ is the mass of the monopole.

This $SU(2)$ is embedded into $SU(3)$ as the $SU(2)$ associated with α_2 , in other words

$$U_i = \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & \tau_i \end{pmatrix} \quad (4.14)$$

are the generators of $SU(2) \subset SU(3)$. Then an $SU(3)$ monopole satisfying the boundary conditions of Eq. (3.4) is

$$\Phi^U = i \phi_{\text{BPS}} U_3 + i \text{diag} \left(1, -\frac{1}{2}, -\frac{1}{2} \right) \quad A^U = \sum_{i=1}^3 a^i U_i. \quad (4.15)$$

The constant piece is added in Φ^U to make sure that it has the correct asymptotics. To see this note that

$$\Phi^U \sim i 2\sqrt{3} H_2 - \frac{K}{2r} U_3, \quad (4.16)$$

so $v = 2\sqrt{3}$ is the vev.

These are not unique as we have $\text{Stab}(\Phi_0) = U(2)$ acting as an asymptotic gauge group to give

$$\Phi_P = P \Phi^U P^{-1}, \quad (4.17)$$

but now there is an $U(1)$ sub group that leaves Φ^U invariant, this is the $U(1)$ generated by $Y + 2I_3$. This generator is

$$Y + 2I_3 = \frac{1}{3} \begin{pmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad (4.18)$$

thus the group something looks like

$$\text{diag} (e^{i2\theta}, e^{-i\theta}, e^{-i\theta}) \quad (4.19)$$

this means that it is the over all $U(1)$ in the $U(2)$ and thus the orbit is

$$U(2)/U(1) = SU(2) = S_P^3. \quad (4.20)$$

The natural coordinates on $SU(2)$ are the Euler angles (α, β, γ) , where γ parametrises a S^1 known as the electric circle. There is then a Hopf fibration

$$S^1 \hookrightarrow S_P^3 \rightarrow S^2 \quad (4.21)$$

with fibre the electric circle and, since we can take the projection to be

$$\pi^k : S_P^3 \rightarrow S^2, \quad \pi^k(\alpha, \beta, \gamma) \mapsto \vec{k}, \quad (4.22)$$

the base is the magnetic orbit.

From [1] it is known that the moduli space of charge K $SU(2)$ monopoles is of the form

$$\mathcal{M}_K^{SU(2)} = \mathbb{R}^3 \times \frac{S^1 \times \mathcal{M}_K^0}{\mathbb{Z}_k}, \quad (4.23)$$

where \mathbb{R}^3 is the centre of mass position, S^1 is the electric circle and \mathcal{M}_K^0 is the k -fold cover of the charge k centred monopole moduli space. This is known to be a hyperKähler manifold of dimension $4K$.

These embedded $SU(2)$ monopoles are charge K $SU(3)$ monopoles on the magnetic orbit of radius $\frac{K}{2}$, *i.e.* they lie in the small strata. The results of [3] show that in fact this gives the whole of the small strata and thus it is given by

$$\mathcal{M}_{K, \frac{K}{2}} = \mathbb{R}^3 \times \frac{S_P^3 \times \mathcal{M}_K^0}{\mathbb{Z}_k}, \quad (4.24)$$

where the \mathbb{Z}_k action on the electric circle is $\gamma \mapsto \gamma + \frac{2\pi}{k}$. The number of independent parameters is then $4K + 2$ which agrees with the dimension of the small strata computed in [10]. The fact that we get the small strata as a fibration with fibre the Atiyah-Hitchin manifold and base the magnetic orbit means that, up to scaling, there is a well defined hyperKähler metric on the fibres. There can be a metric on the base, however, as we will see later, this depends on if K is even or odd.

4.3 Asymptotic gauge group and collective coordinates

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