$$y_1, y_2, \ldots, y_n \overset{iid}{\sim} N(\mu, \phi)$$

$$P(\mu / \bar{y}) = \int_\phi P(\mu, \phi / \bar{y}) \cdot d\phi$$

$$= \int_\phi P(\bar{y} / \mu, \phi) \frac{P(\mu) P(\phi)}{P(\bar{y})} d\phi$$

$$= \frac{P(\mu)}{P(\bar{y})} \int_\phi P(\bar{y} / \mu, \phi) P(\phi) \cdot d\phi$$

$$\bar{y} / \mu, \phi \sim N(\mu, \phi/n)$$

$$\therefore \int_\phi P(\bar{y} / \mu, \phi) P(\phi) \cdot d\phi \quad \text{①}$$

$$= \int_\phi \frac{1}{\sqrt{2\pi\phi/n}} e^{-\frac{n}{2\phi}(\bar{y}-\mu)^2} P(\phi) \cdot d\phi$$

$$\phi \sim IG(\alpha, \beta) \quad \therefore P(\phi) = \frac{\beta^\alpha}{\Gamma(\alpha)} \phi^{-\alpha-1} e^{-\frac{\beta}{\phi}}$$

$$\therefore \text{①} \rightarrow \sqrt{\frac{n}{2\pi}} \frac{\beta^\alpha}{\Gamma(\alpha)} \int_\phi \phi^{-\frac{1}{2}} e^{-\frac{n}{2\phi}(\bar{y}-\mu)^2} \phi^{-\alpha-1} e^{-\frac{\beta}{\phi}}$$

$$= \sqrt{\frac{n}{2\pi}} \frac{\beta^\alpha}{\Gamma(\alpha)} \times \Gamma(\alpha + \tfrac{1}{2}) / (\beta + n(\bar{y}-\mu)^2)^{\alpha+\frac{1}{2}}$$

$$\int_\phi \frac{(\beta + n(\bar{y}-\mu)^2)^{\alpha+\frac{1}{2}}}{\Gamma(\alpha+\frac{1}{2})} \phi^{-(\alpha+\frac{1}{2})-1} e^{-\frac{1}{\phi}[\beta + \frac{n}{2}(\bar{y}-\mu)^2]}$$

$$\therefore P(\mu/\bar{y}) = \frac{P(\mu)}{P(\bar{y})} \times \sqrt{\frac{n}{2\pi}} \frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha)} \times \frac{\beta^\alpha}{(\beta + \frac{n}{2}(\bar{y}-\mu)^2)^{\alpha+\frac{1}{2}}}$$

$$\therefore P(\mu/\bar{y}) \propto P(\mu) \left[\beta + \frac{n}{2}(\bar{y}-\mu)^2\right]^{-\alpha-\frac{1}{2}}$$

$$P(\mu|y) = \int_\phi P(\mu,\phi|y)\,.d\phi = \int_\phi \frac{P(y|\mu,\phi)\,\overline{p(\mu)\,P(\phi)}}{P(y)}.d\phi$$

$$= \frac{P(\mu)}{P(y)} \frac{\beta^\alpha}{\Gamma(\alpha)} \int_\phi \left\{ \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\phi}} e^{-\frac{1}{2\phi}(y_i-\mu)^2} \right\} \times \phi^{-\alpha-1} e^{-\frac{\beta}{\phi}} \,.d\phi$$

$$= \frac{P(\mu)}{P(y)} \beta^\alpha/\Gamma(\alpha) \, (2\pi)^{-\frac{n}{2}} \int_\phi \phi^{-\frac{n}{2}} e^{-\frac{1}{2\phi}\sum_i (y_i-\mu)^2} \times \phi^{-\alpha-1} e^{-\beta/\phi} \,.d\phi$$

$$= \frac{P(\mu)}{P(y)} \frac{\beta^\alpha}{\Gamma(\alpha)} (2\pi)^{-\frac{n}{2}} \times \Gamma(\alpha+\tfrac{n}{2}) \Big/ \left[\beta + \tfrac{1}{2}\sum_i (y_i-\mu)^2\right]^{\alpha+\frac{n}{2}} \int_\phi \frac{\left[\beta+\frac{1}{2}\sum_i (y_i-\mu)^2\right]^{\alpha+\frac{n}{2}}}{\Gamma(\alpha+\frac{n}{2})} \phi^{-(\alpha+\frac{n}{2})-1} e^{-\frac{1}{\phi}\left[\beta+\frac{1}{2}\sum_i (y_i-\mu)^2\right]} \,.d\phi$$

$$\therefore P(\mu|y) = \frac{P(\mu)}{P(y)} (2\pi)^{-\frac{n}{2}} \frac{\Gamma(\alpha+\frac{n}{2})}{\Gamma(\alpha)} \times \frac{\beta^\alpha}{\left[\beta+\frac{1}{2}\sum_i (y_i-\mu)^2\right]^{\alpha+\frac{n}{2}}}$$

$$\therefore P(\mu|y) \propto P(\mu)\left[\beta + \tfrac{1}{2}\sum_i (y_i-\mu)^2\right]^{-\alpha-\frac{n}{2}}$$