

Pushforward structure to relate geometric cycles

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A joint work with M. Velasquez and P. Carrillo.

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Motivation

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On the Equivalence of Geometric and Analytic K-Homology

Paul Baum, Nigel Higson, and Thomas Schick

Abstract: We give a proof that the geometric K-homology theory for finite CW-complexes defined by Baum and Douglas is isomorphic to Kasparov's K-homology. The proof is a simplification of more elaborate arguments which deal with the geometric formulation of *equivariant* K-homology theory.

Problem of interest

$$\begin{array}{ccc} k_{\text{ev/odd}}(X, Y) & \xrightarrow{\alpha} & K_{\text{ev/odd}}(X, Y) \\ \beta \searrow & & \nearrow \mu \\ & K_{\text{ev/odd}}^{\text{geom}}(X, Y) & \end{array}$$

Figure: Auxiliary homology theory

Basic definitions

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Classifying space: X a proper G -space, then there exists a G -map

$$\phi_X : X \rightarrow \underline{E}G,$$

and any two G -maps from X to $\underline{E}G$ are G -homotopic.

Groupoid and Groupoid actions

- A topological groupoid is a groupoid \mathcal{G} such that Z and \mathcal{G} are Hausdorff and the structure maps source and target $s, t : \mathcal{G} \rightarrow Z$ are continuous and open.
- Let \mathcal{G} be a topological groupoid. A topological space X is called a **\mathcal{G} -space** if there is a continuous map $a : X \rightarrow Z$ called the **anchor** map and a homeomorphism

$$\mathcal{G}_s \times_a X \rightarrow \mathcal{G}_t \times_a X : (g, x) \mapsto (g, g \cdot x),$$

which determines the **action**, and holds associativity and unitality conditions.

Groupoid examples

Example

For a group Γ , consider the groupoid $\mathcal{G} := \Gamma^*$, where $\mathcal{G}^0 := \{*\}$ and $\mathcal{G}^1 := \Gamma$.

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Example

The transformation groupoid $\mathcal{G} \ltimes X$ (see [1], p. 4, Definition 2.2) is defined by X as its object space, morphisms the elements of $\mathcal{G}_s \times_a X$; the range and source maps are defined by $r(g, x) = g \cdot x$ and $s(g, x) = x$ respectively ; and its composition is given by $(g, x) \cdot (h, y) = (gh, y)$.

$(\Gamma^* \ltimes X)$ -bundles

When \mathcal{G} is the transformation groupoid (or action groupoid) of a group, we recover the notion of a Γ -vector bundle. For Γ be a discrete group, and X be a Γ -space, there is a bijection:

$$\{\text{---vector bundles over } X\} \leftrightarrow \{(\Gamma^* \ltimes X)\text{---vector bundles over } X\}$$

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(\leftarrow) : For a $(\Gamma^* \ltimes X)$ -vector bundle $\rho : E \rightarrow X$, we note the Γ -vector bundle structure with action on E given by

$$\gamma e := (\gamma, \rho(e))e,$$

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(\rightarrow) : If $\rho : E \rightarrow X$ is a Γ -vector bundle we note that it has a $(\Gamma^* \ltimes X)$ -vector bundle structure by the anchor map $a := \rho : E \rightarrow X$, and the natural action

$$(\gamma, x)e := \gamma e \quad \text{when } s((\gamma, x)) = a(e), \text{ i.e., } e \in E_x = \rho^{-1}(x)$$

Emerson-Meyer

Let \mathcal{G} be a proper groupoid, let X and Y be smooth \mathcal{G} -manifolds, and let $f : X \rightarrow Y$ be a smooth \mathcal{G} -equivariant map (+).

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- a smooth \mathcal{G} -vector bundle V over X ,
- a smooth \mathcal{G} -vector bundle E over Z ,
- a smooth \mathcal{G} -equivariant, open embedding $\eta_f : V \rightarrow E^Y$,

such that $f = \rho_{E^Y} \circ \eta_f \circ \xi_V$, where $\xi_V : X \rightarrow V$ is the *zero-section* of the vector bundle $\rho_V : V \rightarrow X$.

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$$\begin{array}{ccccc} V & \xrightarrow{\eta_f} & E^Y & & E \\ \xi_V \uparrow & & \downarrow \rho_{E^Y} & & \downarrow \\ X & \xrightarrow{f} & Y & \xrightarrow{a} & \underline{E}\Gamma \end{array}$$

Main definitions

Cohomology Theory: $\mathcal{H}_\Gamma^* : (\text{Proper Pairs}) \rightarrow (R\text{-mod})$ with connection maps $\partial_\Gamma^n : \mathcal{H}_\Gamma^n(A) \rightarrow \mathcal{H}_\Gamma^{n+1}(X, A)$ with axioms

- Γ -homotopy invariance: If f and f' are homotopic then $f^* = f'^*$
- Long exact sequence: For $i : A \rightarrow X$ and $j : X \rightarrow (X, A)$,

$$\mathcal{H}_\Gamma^n(X, A) \xrightarrow{j^n} \mathcal{H}_\Gamma^n(X) \xrightarrow{i^n} \mathcal{H}_\Gamma^n(A) \xrightarrow{\partial_\Gamma^n} \mathcal{H}_\Gamma^{n+1}(X, A)$$

- Excision: For (X, A) a Γ -proper-pair and $f : A \rightarrow B$, the canonical map $F : (X, A) \rightarrow (X \cup_f B, B)$ induces an isomorphism

$$F^* : \mathcal{H}_\Gamma^n(X, A) \rightarrow \mathcal{H}_\Gamma^n(X \cup_f B, B).$$

- Disjoint union: the following map is a group isomorphism:

$$\prod_{i \in I} j_i^* : \mathcal{H}_\Gamma^n(\coprod_{i \in I} X_i) \rightarrow \prod_{i \in I} \mathcal{H}_\Gamma^n(X_i)$$

Definition

For a d -dimensional real Γ -vector bundle $E \rightarrow X$ over a proper Γ -space X , a \mathcal{H}_Γ -orientation for E is a class $\tau \in \mathcal{H}_\Gamma^d(E)$ such that for each proper Γ -map $f : Y \rightarrow X$, multiplication with $f^*\tau \in \mathcal{H}_\Gamma^d(f^*E)$ induces an isomorphism (referred as Thom-isomorphism)

$$- \otimes \tau : \mathcal{H}_\Gamma^*(Y) \rightarrow \mathcal{H}_\Gamma^{*+d}(f^*E).$$

We say that a smooth manifold with boundary M is a \mathcal{H}_Γ -orientable if TM is \mathcal{H}_Γ -orientable. When is chosen a fixed \mathcal{H}_Γ -orientation for a real Γ -vector bundle (or particularly for a manifold), we say that it is a \mathcal{H}_Γ -oriented bundle (or manifold).

Pushforward structure

For $f : (M, \partial M) \rightarrow (N, \partial N)$,

$$f! : \mathcal{H}_\Gamma^*(M) \rightarrow \mathcal{H}_\Gamma^{*+\dim(N)-\dim(M)}(N)$$

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- $(f \circ g)! = f! \circ g!$.
- If f and f' are homotopic then $f! = f'!$
- For $\rho : E \rightarrow M$ is a Γ -vector bundle, if $s : M \rightarrow E$ is the zero section, then the following map is a Thom isomorphism:

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$$s! : \mathcal{H}_\Gamma^*(M) \rightarrow \mathcal{H}_\Gamma^{*+\dim(E)-\dim(M)}(E)$$

- For a collar around ∂N in N , $l! = \partial^*$ and $i^0! = j^*$

$$\cdots \rightarrow \mathcal{H}_\Gamma^{*-1}(\partial N) \xrightarrow{l!} \mathcal{H}_\Gamma^*(N^0) \xrightarrow{i_!^0} \mathcal{H}_\Gamma^*(N) \rightarrow \cdots$$

Equivalence relation

- n -Cycle over (X, A) : $\pi : (M, \partial M) \rightarrow (X, A)$ and $x_M \in \mathcal{H}_\Gamma^n(M)$.

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$$\begin{array}{ccc} W & \xleftarrow{f_N} & N \\ f_M \uparrow & \searrow \pi_W & \downarrow \pi_N \\ M & \xrightarrow[\pi_M]{} & X \end{array}$$

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- Define $\mathcal{H}_*^{pf}(X, A; \Gamma)$ the set of equivalence relations of tuples $(M, \partial M, \pi_M, x_M)$ where $x_M \in \mathcal{H}_\Gamma^{*+\dim(M)}(M)$.

Important Lemmas

Lemma

If $(M, \partial M, \pi_M, x_M) \sim (N, \partial N, \pi_N, x_N)$ by the maps $f_M : M \rightarrow W$ and $f_N : N \rightarrow W$, then we get that $(M, \partial M, \pi_M, x_M) \sim (N, \partial N, \pi_N, x_N)$ by maps $\phi_M : M \rightarrow E^W$ and $\phi_N : N \rightarrow E^W$ with ϕ_N being an embedding and $E^W \rightarrow W$ is a vector bundle.

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Lemma

For any Γ -proper embeddings $i_1 : N \rightarrow W_1$ and $i_2 : N \rightarrow W_2$, there exist Γ -vector bundles $D_1 \rightarrow W_1$ and $D_2 \rightarrow W_2$ such that

$$\nu_{\xi_{D_1} \circ i_1} \cong \nu_{\xi_{D_2} \circ i_2}.$$

Equivalence relation (sketch of proof) I

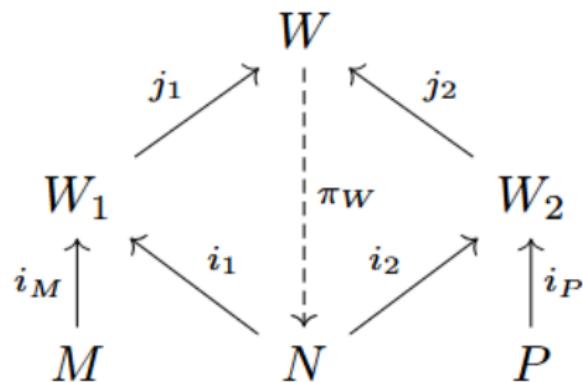


Figure: Initial relations.

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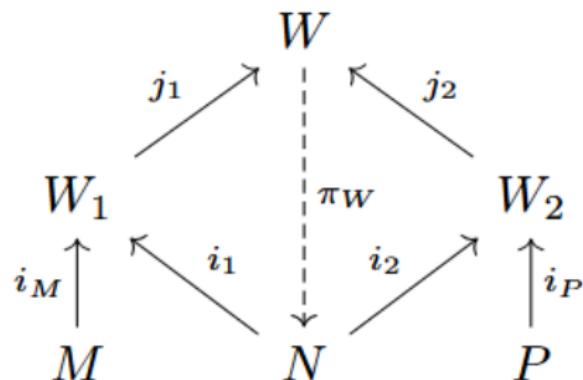


Figure: Initial relations.

$$W := (D_1 \oplus \mathbb{R})_{\nu_1} \#_{\nu_2} (D_2 \oplus \mathbb{R}),$$

Equivalence relation (sketch of proof) II

$$\begin{array}{ccccccc} E_1 & \quad a_1^* E_1 \cong TW_1 \oplus C_1 & & a_2^* E_2 \cong TW_2 \oplus C_2 & & E_2 \\ \downarrow & \downarrow & & \downarrow & & \downarrow \\ Z & \xleftarrow{a_1} & W_1 & \xleftarrow{i_1} & N & \xrightarrow{i_2} & W_2 & \xrightarrow{a_2} & Z \end{array}$$

Figure: Γ -vector bundles

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$$D_1 := a_1^* E_2 \oplus C_1 \rightarrow W_1$$

Equivalence relation (sketch of proof) III

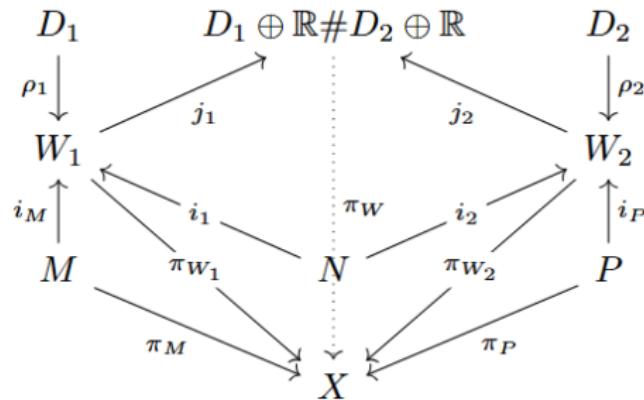


Figure: The desired connection space.

Examples

$[M, \partial M, \pi_M, x_M] \in \mathcal{H}_*^{pf}(X, A; \Gamma)$ with $x_M \in \mathcal{H}_\Gamma^{*+\dim(M)}(M)$

$$\pi_M : M \rightarrow X$$

- $\mathcal{H}_*^{pf}(\{a\}; \Gamma) = \{[M, x_M] : \partial M = \emptyset\}$ ($|\Gamma| < \infty$).

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- $\mathcal{H}_0^{pf}(X, A; \Gamma) \in [S^n, \emptyset, \pi_{S^n}, \alpha]$, where $\pi_{S^n} : S^n \rightarrow X$ is an n -loop and $\alpha \in \mathcal{H}_\Gamma^n(S^n)$.

Examples

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- For $k > 0$, in $\mathcal{H}_k^{pf}(X, A; \Gamma)$, $[S^n, \emptyset, \pi_{S^n}, \alpha] = 0$, by $\mathcal{H}_\Gamma^{n+k}(S^n) = 0$ ($\Gamma = \{e\}$).

Swinging

Remark

If there is a proper smooth, Γ -equivariant, \mathcal{H}_Γ -orientation preserving map $f : (M, \partial M) \rightarrow (N, \partial N)$ such that $\pi_N \circ f = \pi_M$ and $f!(x_M) = x_N$, then

$$[M, \partial M, \pi_M, x_M] = [N, \partial N, \pi_N, x_N],$$

i.e.,

$$[M, \partial M, \pi_N \circ f, x_M] = [N, \partial N, \pi_N, f!(x_M)].$$

Functionality and homotopic invariance

For a map $g : (X, A) \rightarrow (Y, B)$,
 $g_* : \mathcal{H}_*^{pf}(X, A; \Gamma) \rightarrow \mathcal{H}_*^{pf}(Y, B; \Gamma) :$
 $[M, \partial M, \pi, x] \mapsto [M, \partial M, g \circ \pi, x]$

$$\begin{array}{ccc} M & & \\ \pi \downarrow & & \\ X & \xrightarrow{g} & Y \end{array}$$

- ① $(f \circ g)_* = f_* \circ g_*$
- ② Γ -homotopy invariance: if $g_0, g_1 : (X, A) \rightarrow (Y, B)$ are proper Γ -homotopic maps of Γ -proper pairs, then

$$g_{0*} = g_{1*} : \mathcal{H}_n^{pf}(X, A; \Gamma) \rightarrow \mathcal{H}_n^{pf}(Y, B; \Gamma)$$

Long exact sequence

For the sequence of Γ -proper pairs

$$(A, \emptyset) \xrightarrow{j} (X, \emptyset) \xrightarrow{i} (X, A)$$

The following sequence is exact

$$\mathcal{H}_*^{pf}(A; \Gamma) \xrightarrow{j_*} \mathcal{H}_*^{pf}(X; \Gamma) \xrightarrow{i_*} \mathcal{H}_*^{pf}(X, A; \Gamma) \xrightarrow{\partial_*} \mathcal{H}_{*-1}^{pf}(A; \Gamma)$$

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where

- $j_*([M, \emptyset, \pi_M, x_M]) = [M, \emptyset, j \circ \pi_M, x_M]$.
- $i_*([M, \emptyset, \pi_M, x_M]) = [M, \emptyset, \pi_M, x_M]$.
- $\partial([M, \partial M, \pi_M, x_M]) = [\partial M, \emptyset, \pi_M|_{\partial M}, i_{\partial M}^*(x_M)]$, with $i_{\partial M} : \partial M \hookrightarrow M$.

Exactness in $\mathcal{H}_*^{pf}(X; \Gamma)$

$Ker(i_*) \subseteq Im(j_*)$: Let $[M, \emptyset, \pi_M, x_M] = [N, \partial N, \pi_N, 0] \in \mathcal{H}_*^{pf}(X, A; \Gamma)$ with $f_M!(x_M) = 0$ with $f_M : M \rightarrow W$. $f_M!(x_M) \in Ker(i_W^0!)$

$$\mathcal{H}_\Gamma^{*-1}(\partial W) \xrightarrow{\partial} \mathcal{H}_\Gamma^*(W^0) \xrightarrow{i_{W^0}!} \mathcal{H}_\Gamma^*(W)$$

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$$\mathcal{H}_\Gamma^{*-1}(\partial W) \xrightarrow[\partial]{} \mathcal{H}_\Gamma^*(W^0) \xrightarrow[i_{W^0}!]{} \mathcal{H}_\Gamma^*(W)$$

There exist $y \in \mathcal{H}_\Gamma^{*-1}(\partial W)$ such that $\partial(y) = f_M!(x_M)$, and we get

$$\begin{aligned} j_*([\partial W, \emptyset, \pi_W|_{\partial W}, y]) &= [\partial W, \emptyset, j \circ \pi_W|_{\partial W}, y] \\ &= [\partial W, \emptyset, \pi_W \circ l, y] \\ &= [W^0, \emptyset, \pi_W|_{W^0}, l!(y)] \\ &= [W^0, \emptyset, \pi_W|_{W^0}, \partial(y)] \\ &= [W^0, \emptyset, \pi_W|_{W^0}, f_M!(x_M)] \\ &= [M, \emptyset, \pi_W \circ f_M, x_M] \\ &= [M, \emptyset, \pi_M, x_M]. \end{aligned}$$

Exactness in $\mathcal{H}_*^{pf}(X, A; \Gamma)$

$Ker(\partial) \subseteq Im(i_*)$: Let $[M, \partial M, \pi_M, x_M] \in Ker(\partial) \subseteq \mathcal{H}_*^{pf}(X, A; \Gamma)$, then, $[\partial M, \emptyset, \pi_M|_{\partial M}, r_{\partial}(x_M)] = [W, \emptyset, \pi_W, 0]$

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$$M \xleftarrow{i_{M^0}} M^0 \xleftarrow{l} \partial M \xrightarrow[f]{ } W$$

consider the embeddings $i_{M^0} \circ l : \partial M \rightarrow M$ and $f : \partial M \rightarrow W$, to get that there exist Γ -vector bundles $\rho_{D_M} : D_M \rightarrow M$ and $\rho_{D_W} : D_W \rightarrow W$ such that $\nu_M := \nu_{\xi_{D_M} \circ i_{M^0} \circ l} \cong \nu_{\xi_{D_W} \circ f} =: \nu_W$, and

$$T := (D_M \oplus \mathbb{R})_{\nu_M \oplus \mathbb{R}} \#_{\nu_W \oplus \mathbb{R}} (D_W \oplus \mathbb{R}),$$

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$$[M, \partial M, \pi_M, x_M] = [T, \partial T, \pi_T, \iota!(x_M)] = i_*([T^0, \emptyset, \pi_T|_{T^0}, y])$$

Exactness in $\mathcal{H}_*^{pf}(A; \Gamma)$

$$[M, \emptyset, \pi_M, x_M] \in \text{Ker}(j_{*-1}) \subseteq \text{Im}(\partial)$$

$$[M, \emptyset, j \circ \pi_M, x_M] = [W, \emptyset, \pi_W, 0_W] \in \mathcal{H}_*^{pf}(X, \emptyset; \Gamma)$$

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$$\begin{array}{ccc} V & \xrightarrow{\eta_f} & E \\ \xi_V \uparrow & & \downarrow \rho_E \\ M & \xrightarrow{f} & W \end{array}$$

$$T := DNC(E, M)|_{[0,1)} = V \times \{0\} \sqcup E \times (0, 1)$$

Exactness in $\mathcal{H}_*^{pf}(A; \Gamma)$

$$[M, \emptyset, \pi_M, x_M] \in \text{Ker}(j_{*-1}) \subseteq \text{Im}(\partial)$$

$$[M, \emptyset, j \circ \pi_M, x_M] = [W, \emptyset, \pi_W, 0_W] \in \mathcal{H}_*^{pf}(X, \emptyset; \Gamma)$$

$$\begin{array}{ccc} V & \xrightarrow{\eta_f} & E \\ \xi_V \uparrow & & \downarrow \rho_E \\ M & \xrightarrow{f} & W \end{array}$$

$$T := DNC(E, M)|_{[0,1)} = V \times \{0\} \sqcup E \times (0, 1)$$

$$\partial([T, \partial T, \pi_T, y]) = [M, \emptyset, \pi_M, x_M]$$

Excision and future work

For a map $\phi : (X, A) \rightarrow (Y, B)$ such that $Y \cong X \cup_A B$, we get that $\mathcal{H}_*^{pf}(X, A) \cong \mathcal{H}_*^{pf}(X \cup_A B, B)$.

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$$[M, \partial M, \pi_M, x_M] \in \mathcal{H}_*^{pf}(X, A) \mapsto [M, \partial M, \phi \circ \pi_M, x_M] \in \mathcal{H}_*^{pf}(Y, B)$$

with inverse

$$h([M, \partial M, \pi_M, x_M]) = [M_U = \pi_M^{-1}(U), \partial M_U, r \circ \pi_M|_{M_U}, i_{M_U}^* x_M].$$

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$$\begin{aligned}[M_U, \partial M_U, \phi \circ r \circ \pi_M|_{M_U}, i_{M_U}^* x_M] &= [W^0, \emptyset, \pi_{W^0}, i_1!(x_M)] \\ &= [M, \partial M, \pi_M, x_M],\end{aligned}$$

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Thanks a lot :)