Pushforward structure to relate geometric cycles

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Contents

Motivation

Pure and Applied Mathematics Quarterly
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On the Equivalence of Geometric and Analytic K-Homology

Paul Baum, Nigel Higson, and Thomas Schick

Abstract: We give a proof that the geometric K-homology theory for finite *CW*-complexes defined by Baum and Douglas is isomorphic to Kasparov's K-homology. The proof is a simplification of more elaborate arguments which deal with the geometric formulation of *equivariant* K-homology theory.

Problem of interest

$$\begin{tikzcd}
k_{ev\,/\,odd}(X,Y) \arrow[rr, "\alpha"] \arrow[dr, "\beta"'] & & K_{ev\,/\,odd}(X,Y) \\
& K^{geom}_{ev\,/\,odd}(X,Y) \arrow[ur, "\mu"'] &
\end{tikzcd}$$

Figure: Auxiliary homology theory

Basic definitions

A continuous map $f : X \to Y$ is called **proper** if for every compact $K \subseteq Y$, we have that $f^{-1}(K)$ is a compact of X.

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Classifying space: X a proper G-space, then there exists a G-map

$$\phi_X : X \to \underline{E}G,$$

and any two G-maps from X to $\underline{E}G$ are G-homotopic.

Emerson-Meyer

Let $\mathcal{G}$ be a proper groupoid, let X and Y be smooth $\mathcal{G}$-manifolds, and let $f : X \to Y$ be a smooth $\mathcal{G}$-equivariant map (+).

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- a smooth $\mathcal{G}$-vector bundle V over X,
- a smooth $\mathcal{G}$-vector bundle E over Z,
- a smoth $\mathcal{G}$-equivariant, open embedding $\eta_f : V \to E^Y$,

such that $f = \rho_{E^Y} \circ \eta_f \circ \xi_V$, where $\xi_V : X \to V$ is the *zero*-section of the fiber bundle $\rho_V : V \to X$.

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$$\begin{array}{ccccc} V & \xrightarrow{\eta_f} & E^Y & & E \\ \uparrow{\scriptstyle \xi_V} & & \downarrow{\scriptstyle \rho_{E^Y}} & & \downarrow \\ X & \xrightarrow[f]{} & Y & \xrightarrow[a]{} & \underline{E}\Gamma \end{array}$$

Let $\mathcal{G}$ be a topological groupoid[1].

- A topological space X is called a **$\mathcal{G}$-space** if there is a continuous map $a : X \to Z$ called the **anchor** map and a homeomorphism

$$\mathcal{G}_s \times_a X \to \mathcal{G}_t \times_a X : (g, x) \mapsto (g, g \cdot x),$$

which determines the **action**, and holds associativity and unitality conditions.

[1] A topological groupoid is a groupoid $\mathcal{G}$ such that Z and $\mathcal{G}$ are Hausdorff and the structure maps source and target $s, t : \mathcal{G} \to Z$ are continuous and open (see [**?**], p. 2).

Groupoid examples

Example

For a group Γ, consider the groupoid $\mathcal{G} := \Gamma^*$, where $\mathcal{G}^0 := \{*\}$ and $\mathcal{G}^1 := \Gamma$.

Example

The transformation groupoid $\mathcal{G} \ltimes X$ (see [1], p. 4, Definition 2.2) is defined by X as its object space, morphisms the elements of $\mathcal{G}_s \times_a X$; the range and source maps are defined by $r(g,x) = g \cdot x$ and $s(g,x) = x$ respectively ; and its composition is given by $(g,x) \cdot (h,y) = (gh, y)$.

$(\Gamma^* \ltimes X)$-bundles

When $\mathcal{G}$ is the transformation groupoid (or action groupoid) of a group, we recover the notion of a Γ-vector bundle. For Γ be a discrete group, and X be a Γ-space, there is a bijection:

$$\{\Gamma\text{−vector bundles over } X\} \leftrightarrow \{(\Gamma^* \ltimes X)\text{−vector bundles over } X\}$$

$(\leftarrow)$: For a $(\Gamma^* \ltimes X)$-vector bundle $\rho : E \to X$, we note the Γ-vector bundle structure with action on E given by

$$\gamma e := (\gamma, \rho(e))e,$$

where this action and Remark **??** make ρ a Γ-equivariant map

$$\rho(\gamma e) = \rho((\gamma, \rho(e))e) = (\gamma, \rho(e))\rho(e) = \gamma\rho(e).$$

$(\rightarrow)$: If $\rho : E \to X$ is a Γ-vector bundle we note that it has a $(\Gamma^* \ltimes X)$-vector bundle structure by the anchor map $a := \rho : E \to X$, and the natural action

$$(\gamma, x)e := \gamma e \quad when\ s((\gamma, x)) = a(e),\ i.e.,\ e \in E_x = \rho^{-1}(x)$$

where this implies that ρ is a $(\Gamma^* \ltimes X)$-equivariant map by

Main definitions

Cohomology Theory: $\mathscr{H}_\Gamma^* : (Proper\ Pairs) \to (R-mod)$ with connection maps $\partial_\Gamma^n : \mathscr{H}_\Gamma^n(A) \to \mathscr{H}_\Gamma^{n+1}(X, A)$ with axioms

- Γ-homotopy invariance: If f and f' are homotopics then $f^* = f'^*$
- Long exact sequence: For $i : A \to X$ and $j : X \to (X, A)$,

$$\mathscr{H}_\Gamma^n(X, A) \xrightarrow{j^n} \mathscr{H}_\Gamma^n(X) \xrightarrow{i^n} \mathscr{H}_\Gamma^n(A) \xrightarrow{\partial_\Gamma^n} \mathscr{H}_\Gamma^{n+1}(X, A)$$

- Excision: For (X, A) a Γ-proper-pair and $f : A \to B$, the canonical map $F : (X, A) \to (X \cup_f B, B)$ induces an isomorphism

$$F^* : \mathscr{H}_\Gamma^n(X, A) \to \mathscr{H}_\Gamma^n(X \cup_f B, B).$$

- Disjoint union: the following map is a group isomorphism:

$$\prod_{i\in I} j_i^* : \mathscr{H}_\Gamma^n(\coprod_{i\in I} X_i) \to \prod_{i\in I} \mathscr{H}_\Gamma^n(X_i)$$

Pushforward structure

For $f : (M, \partial M) \to (N, \partial N)$,

$$f! : \mathscr{H}_\Gamma^*(M) \to \mathscr{H}_\Gamma^{*+dim(N)-dim(M)}(N)$$

With axioms:

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With axioms:

- $(f \circ g)! = f! \circ g!$.
- If f and f' are homotopic then $f! = f'!$
- For $\rho : E \to M$ is a Γ-vector bundle, if $s : M \to E$ is the zero section, then the following map is a Thom isomorphism:

$$s! : \mathscr{H}^*_\Gamma(M) \to \mathscr{H}^{*+dim(E)-dim(M)}_\Gamma(E)$$

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- For a collar around ∂N in N, $l!=\partial^*$ and $i^0!=j^*$

$$\cdots\to\mathscr{H}^{*-1}_\Gamma(\partial N)\xrightarrow{l_!}\mathscr{H}^*_\Gamma(N^0)\xrightarrow{i^0_!}\mathscr{H}^*_\Gamma(N)\to\cdots$$

Equivalence relation

- n-Cycle over X, A: $\pi : (M, \partial M) \to (X, A)$ and $x_M \in \mathscr{H}_\Gamma^n(M)$.

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- Related cycles: $(M, \partial M, \pi_M, x_M) \sim (N, \partial N, \pi_N, x_N)$ if the following diagram is commutative and $f_M!(x_M) = f_N!(x_N)$:

$$
\begin{tikzcd}
W \arrow[dr, "\pi_W"] & N \arrow[l, "f_N"'] \arrow[d, "\pi_N"] \\
M \arrow[u, "f_M"] \arrow[r, "\pi_M"'] & X
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- Define $\mathscr{H}_*^{pf}(X, A; \Gamma)$ the set of equivalence relations of tuples $(M, \partial M, \pi_M, x_M)$ where $x_M \in \mathscr{H}_\Gamma^{*+dim(M)}(M)$.

Equivalence relation (sketch of proof) I

$$\begin{tikzcd}
& W \arrow[dd, dashed, "\pi_W"] & \\
W_1 \arrow[ur, "j_1"] & & W_2 \arrow[ul, "j_2"'] \\
M \arrow[u, "i_M"] & N \arrow[ul, "i_1"'] \arrow[ur, "i_2"] & P \arrow[u, "i_P"']
\end{tikzcd}$$

Figure: Initial relations.

Equivalence relation (sketch of proof) II

$$\begin{tikzcd}
E_1 \arrow[d] & a_1^*E_1 \cong TW_1 \oplus C_1 \arrow[d] & & a_2^*E_2 \cong TW_2 \oplus C_2 \arrow[d] & E_2 \arrow[d] \\
Z & W_1 \arrow[l, "a_1"'] & N \arrow[l, "i_1"'] \arrow[r, "i_2"'] & W_2 \arrow[r, "a_2"'] & Z
\end{tikzcd}$$

Figure: Γ-vector bundles

Equivalence relation (sketch of proof) III

$$
\begin{tikzcd}
D_1 \arrow[d, "\rho_1"'] & & D_1 \oplus \mathbb{R} \# D_2 \oplus \mathbb{R} \arrow[ddd, dotted, "\pi_W"] & & D_2 \arrow[d, "\rho_2"] \\
W_1 \arrow[urr, "j_1"'] \arrow[ddrr, "\pi_{W_1}"] & & & & W_2 \arrow[ull, "j_2"] \arrow[ddll, "\pi_{W_2}"'] \\
M \arrow[u, "i_M"] \arrow[drr, "\pi_M"'] & & N \arrow[ull, "i_1"] \arrow[urr, "i_2"'] & & P \arrow[u, "i_P"'] \arrow[dll, "\pi_P"] \\
& & X & &
\end{tikzcd}
$$

Figure: The desired connection space.

Examples

$$[M, \partial M, \pi_M, x_M] \in \mathscr{H}_*^{pf}(X, A; \Gamma) \text{ with } x_M \in \mathscr{H}_\Gamma^{*+dim(M)}(M)$$

$$\pi_M : M \to X$$

- $\mathscr{H}_*^{pf}(\{a\}; \Gamma) = \{[M, x_M] : \partial M = \emptyset\}$ $(|\Gamma| < \infty)$.

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- $\mathscr{H}_0^{pf}(X, A; \Gamma) \in [S^n, \emptyset, \pi_{S^n}, \alpha]$, where $\pi_{S^n} : S^n \to X$ is an n-loop and $\alpha \in \mathscr{H}_\Gamma^n(S^n)$.

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- For $k > 0$, in $\mathscr{H}_k^{pf}(X, A; \Gamma)$, $[S^n, \emptyset, \pi_{S^n}, \alpha] = 0$, by $\mathscr{H}_\Gamma^{n+k}(S^n) = 0$ $(\Gamma = \{e\})$.

Functoriality and homotopic invariance

For a map $g : (X, A) \to (Y, B)$,

$$g_* : \mathscr{H}_*^{pf}(X, A; \Gamma) \to \mathscr{H}_*^{pf}(Y, B; \Gamma) :$$
$$[M, \partial M, \pi, x] \mapsto [M, \partial M, g \circ \pi, x]$$

$$\begin{tikzcd} M \arrow[d, "\pi"'] & \\ X \arrow[r, "g"'] & Y \end{tikzcd}$$

1. $(f \circ g)_* = f_* \circ g_*$
2. Γ-homotopy invariance: if $g_0, g_1 : (X, A) \to (Y, B)$ are proper Γ-homotopic maps of Γ-proper pairs, then

$$g_{0*} = g_{1*} : \mathscr{H}_n^{pf}(X, A; \Gamma) \to \mathscr{H}_n^{pf}(Y, B; \Gamma)$$

Long exact sequence

For the sequence of Γ-proper pairs $(A, \emptyset) \xrightarrow{j} (X, \emptyset) \xrightarrow{i} (X, A)$

The following sequence is exact

$$\mathscr{H}_*^{pf}(A; \Gamma) \xrightarrow{j_*} \mathscr{H}_*^{pf}(X; \Gamma) \xrightarrow{i_*} \mathscr{H}_*^{pf}(X, A; \Gamma) \xrightarrow{\partial_*} \mathscr{H}_{*-1}^{pf}(A; \Gamma)$$

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where

- $j_*([M,\emptyset,\pi_M,x_M]) = [M,\emptyset,j\circ\pi_M,x_M]$.
- $i_*([M,\emptyset,\pi_M,x_M]) = [M,\emptyset,\pi_M,x_M]$.
- $\partial([M,\partial M,\pi_M,x_M]) = [\partial M,\emptyset,\pi_M|_{\partial M},i_{\partial M}^*(x_M)]$, with $i_{\partial M}:\partial M \hookrightarrow M$.

Exactness in $\mathscr{H}_*^{pf}(X;\Gamma)$

$Ker(i_*) \subseteq Im(j_*)$: Let $[M, \emptyset, \pi_M, x_M] = [N, \partial N, \pi_N, 0] \in \mathscr{H}_*^{pf}(X, A; \Gamma)$ with $f_M!(x_M) = 0$ with $f_M : M \to W$. $f_M!(x_M) \in Ker(i_W^0!)$

$$\mathscr{H}_\Gamma^{*-1}(\partial W) \xrightarrow[\partial]{} \mathscr{H}_\Gamma^*(W^0) \xrightarrow[i_{W^0}!]{} \mathscr{H}_\Gamma^*(W)$$

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$$\mathscr{H}_\Gamma^{*-1}(\partial W) \xrightarrow[\partial]{} \mathscr{H}_\Gamma^{*}(W^0) \xrightarrow[i_{W^0}!]{} \mathscr{H}_\Gamma^{*}(W)$$

There exist $y \in \mathscr{H}_\Gamma^{*-1}(\partial W)$ such that $\partial(y) = f_M!(x_M)$, and we get

$$\begin{aligned}
j_*([\partial W, \emptyset, \pi_W|_{\partial W}, y]) &= [\partial W, \emptyset, j \circ \pi_W|_{\partial W}, y] \\
&= [\partial W, \emptyset, \pi_W \circ l, y] \\
&= [W^0, \emptyset, \pi_W|_{W^0}, l!(y)] \\
&= [W^0, \emptyset, \pi_W|_{W^0}, \partial(y)] \\
&= [W^0, \emptyset, \pi_W|_{W^0}, f_M!(x_M)] \\
&= [M, \emptyset, \pi_W \circ f_M, x_M] \\
&= [M, \emptyset, \pi_M, x_M].
\end{aligned}$$

Exactness in $\mathscr{H}_*^{pf}(X, A; \Gamma)$

$Ker(\partial) \subseteq Im(i_*)$: Let $[M, \partial M, \pi_M, x_M] \in Ker(\partial) \subseteq \mathscr{H}_*^{pf}(X, A; \Gamma)$, then, $[\partial M, \emptyset, \pi_M|_{\partial M}, r_\partial(x_M)] = [N, \emptyset, \pi_N, 0]$

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$$\begin{tikzcd} V \arrow[r, "\eta_f"] & E \arrow[d, "\rho_E"] \\ \partial M \arrow[u, "\xi_V"] \arrow[r, "f"'] & W \end{tikzcd}$$

$$\begin{aligned} T &= \mathbb{R}^k \times M \times \{0\} \bigcup_{(\phi,0)} DNC(E, \partial M)|_{[0,1]} \\ &= \mathbb{R}^k \times M \times \{0\} \bigcup_{(\phi,0)} (E \times (0,1]) \sqcup (V \times \{0\}), \end{aligned}$$

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$$[M, \partial M, \pi_M, x_M] = [T, \partial T, \pi_T, (\iota \circ \xi_V)!(x_M)] = i_*([T^0, \emptyset, \pi_T|_{T^0}, y])$$

Exactness in $\mathscr{H}_*^{pf}(A;\Gamma)$

$[M, \emptyset, \pi_M, x_M] \in Ker(j_{*-1}) \subseteq Im(\partial)$

$$[M, \emptyset, j \circ \pi_M, x_M] = [W, \emptyset, \pi_W, 0_W] \in \mathscr{H}_*^{pf}(X, \emptyset; \Gamma)$$

Exactness in $\mathscr{H}_*^{pf}(A;\Gamma)$

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$$T := DNC(E, M)|_{[0,1)} = V \times \{0\} \sqcup E \times (0,1)$$

Exactness in $\mathscr{H}_*^{pf}(A;\Gamma)$

$[M,\emptyset,\pi_M,x_M]\in Ker(j_{*-1})\subseteq Im(\partial)$

$$[M,\emptyset,j\circ\pi_M,x_M]=[W,\emptyset,\pi_W,0_W]\in\mathscr{H}_*^{pf}(X,\emptyset;\Gamma)$$

$$\begin{tikzcd} V \arrow[r, "\eta_f"] & E \arrow[d, "\rho_E"] \\ M \arrow[u, "\xi_V"] \arrow[r, "f"'] & W \end{tikzcd}$$

$$T:=DNC(E,M)|_{[0,1)}=V\times\{0\}\sqcup E\times(0,1)$$

$$\partial([T,\partial T,\pi_T,y])=[M,\emptyset,\pi_M,x_M]$$

Excision and future work

Excision: For a map $\phi : (X, A) \to (Y, B)$ such that $Y \cong X \cup_A B$, we get that $\mathscr{H}_*^{pf}(X, A) \cong \mathscr{H}_*^{pf}(X \cup_A B, B)$.

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Idea:

$$[M, \partial M, \pi_M, x_M] \in \mathscr{H}_*^{pf}(X, A) \mapsto [M, \partial M, \phi \circ \pi_M, x_M] \in \mathscr{H}_*^{pf}(Y, B)$$

with inverse

$$h([M, \partial M, \pi_M, x_M]) = [M_X = \pi_M^{-1}(U), \partial M_X, r \circ \pi_M|_{M_X}, i_{M_X}^* x_M].$$

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$$\begin{tikzcd}
M \arrow[d, "\pi_M"'] & M_X \arrow[l, "i_{M_X}"'] \arrow[d, "\pi_M"'] & \\
Y & U \arrow[l, "i_U"'] \arrow[r, "r"'] & X \\
& X \arrow[ul, "\phi"] \arrow[u, "i_X"] &
\end{tikzcd}$$

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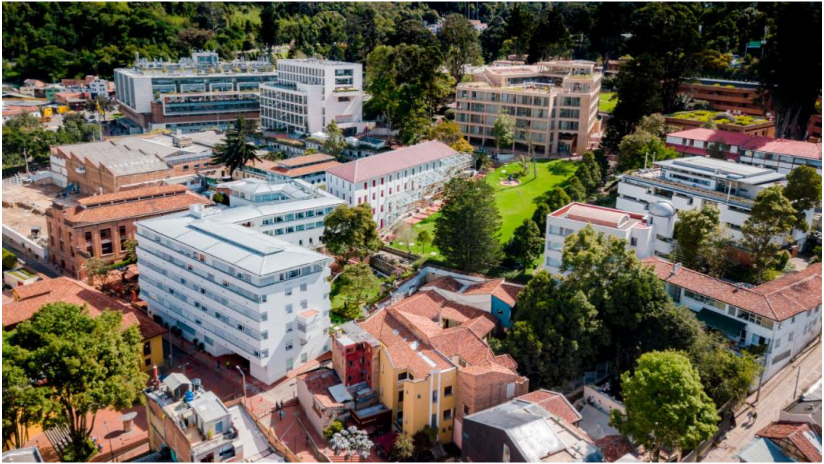

Thanks a lot :)