6/6/05

Derivation of a (scattering length)

Note: $\langle \vec{p}\,'|\hat{V}|\vec{p}\rangle = \frac{4\pi}{m} g\, e^{-p'^2/\Lambda^2} e^{-p^2/\Lambda^2} \equiv V(p',p)$

↳ s-wave potential

$$\hat{1} \equiv \int \frac{d\vec{q}}{(2\pi)^3} |\vec{p}\rangle\langle\vec{p}|$$

Let's solve for $\hat{T}$ where $\hat{T} = \hat{V} + \hat{V}\hat{G}_0\hat{T}$

$$= \hat{V} + \hat{V}\hat{G}_0\hat{V} + \ldots$$

$$\therefore \langle\vec{p}\,'|\hat{T}|\vec{p}\rangle = V(p',p) + \langle\vec{p}\,'|\hat{V}\int\frac{d\vec{q}}{(2\pi)^3}|\vec{q}\rangle\langle\vec{q}|\hat{G}_0\int\frac{d\vec{q}\,'}{(2\pi)^3}|\vec{q}\,'\rangle\langle\vec{q}\,'|\hat{V}|\vec{p}\rangle + \ldots$$

Note: $\langle\vec{q}|G_0|\vec{q}\,'\rangle = \frac{1}{E - q^2/2\mu}\,\delta(\vec{q}-\vec{q}\,')(2\pi)^3 \quad (\mu = \frac{m}{2})$

$$\therefore \langle\vec{p}\,'|\hat{T}|\vec{p}\rangle = V(p',p) + \langle\vec{p}\,'|\hat{V}\int\frac{d\vec{q}}{(2\pi)^3}|\vec{q}\rangle \frac{1}{E-\frac{q^2}{2\mu}}\langle\vec{q}|\hat{V}|\vec{p}\rangle + \ldots$$

$$= V(p',p) + \frac{4\pi}{m}g\,e^{-p'^2/\Lambda^2}\left[\int\frac{dq}{2\pi^2}\,e^{-2q^2/\Lambda^2}\frac{q^2}{E-q^2/2\mu}\right]\frac{4\pi}{m}g\,e^{-p^2/\Lambda^2} + \ldots$$

$$= V(p',p) + V(p',p)\left[\frac{4\pi g}{m}\int\frac{dq}{2\pi^2}e^{-2q^2/\Lambda^2}\frac{q^2}{E-q^2/2\mu}\right] + V(p',p)\left[\frac{4\pi g}{m}\int\frac{dq}{2\pi^2}e^{-2q^2/\Lambda^2}\frac{q^2}{E-q^2/2\mu}\right]\times$$

$$\left[\frac{4\pi g}{m}\int\frac{dq'}{2\pi^2}e^{-2q'^2/\Lambda^2}\frac{q'^2}{E-q'^2/2\mu}\right] + \ldots$$

$$= \frac{V(p',p)}{1-\frac{4\pi}{m}g\int\frac{dq}{2\pi^2}e^{-2q^2/\Lambda^2}\frac{q^2}{E-q^2/2\mu}} = \frac{V(p',p)}{1-\frac{g}{m}\cdot\frac{2}{\pi}\int dq\, e^{-\frac{2q^2}{\Lambda^2}}\frac{q^2}{E-q^2/2\mu}}$$

Let's be explicit with all variables –

$$\langle \vec{p}' | \hat{T}(E) | \vec{p} \rangle \equiv T(p', p; E)$$

$$= \frac{V(p', p)}{1 - \frac{g}{m}\frac{2}{\pi}\int dq\, e^{-\frac{2q^2}{\Lambda^2}} \frac{q^2}{E - \frac{q^2}{2\mu}}}$$

Scattering length is defined at threshold: $\boxed{\langle \vec{p}'=0 | \hat{T}(E=0) | \vec{p}=0 \rangle = \frac{4\pi a}{m}}$

$$\therefore \quad T(0,0;0) = \frac{4\pi a}{m} = \frac{V(0,0)}{1 + g\frac{2}{\pi}\int dq\, e^{-2q^2/\Lambda^2}}$$

$$= \frac{\frac{4\pi g}{m}}{1 + g\frac{2}{\pi}\int dq\, e^{-2q^2/\Lambda^2}}$$

$$\therefore \quad a = \frac{g}{1 + g\frac{2}{\pi}\int_0^\infty e^{-2q^2/\Lambda^2}\, dq}$$

$$= \boxed{\frac{g}{1 + \frac{g\Lambda}{\sqrt{2\pi}}} = a}$$

$$\Rightarrow \boxed{g = \frac{a}{1 - \frac{a\Lambda}{\sqrt{2\pi}}}}$$

Two particles in a HO-trap

Strategy: Solve for $t(E)$ matrix;
Eigenvalues correspond to poles of $t(E)$, or equivalently, $t^{-1}(E)=0$.

Let's first do s-wave case as an example:

$$\hat{V}_0 = C_0\,\hat{\delta}^R$$

$\hat{C}$ tuned to free space scattering length

$$\langle \phi_{n\ell 0} | \hat{V}_0 | \phi_{n'0} \rangle = \frac{C_0(\Lambda)}{2\sqrt{2}}\left[\int \frac{dq}{2\pi^2}\, q^2 R^*_{n0}(q) f_\Lambda(q)\right]\left[\int \frac{dq}{2\pi^2}\, q^2 R_{n'0}(q) f_\Lambda(q)\right]$$

H.O. State

$$\hat{t}(E) = \hat{V}_0 + \hat{V}_0 \frac{1}{E-\hat{H}_0}\hat{V}_0 + \dots$$

$$= \sum_{\alpha\beta} |\phi_{\alpha 0}\rangle\langle\phi_{\alpha 0}|\hat{V}_0|\phi_{\beta 0}\rangle\langle\phi_{\beta 0}| + \sum_{\alpha\beta} |\phi_{\alpha 0}\rangle\langle\phi_{\alpha 0}|\hat{V}_0 \frac{1}{E-\hat{H}_0}\hat{V}_0|\phi_{\beta 0}\rangle\langle\phi_{\beta 0}| + \dots$$

$$= \sum_{\alpha\beta} |\phi_{\alpha 0}\rangle\langle\phi_{\beta 0}| \frac{C_0(\Lambda)}{2\sqrt{2}} R^{(\Lambda)*}_{\alpha} R^{(\Lambda)}_{\beta} + \sum_{\alpha\beta} \frac{C_0(\Lambda)}{2\sqrt{2}} R^{(\Lambda)*}_{\alpha} R^{(\Lambda)}_{\beta}\left\{\frac{C_0(\Lambda)}{2\sqrt{2}}\sum_{\gamma}\frac{|R^{(\Lambda)}_{\gamma 0}|^2}{E-(2\gamma+3/2)\hbar\omega}\right\} + \dots$$

$$= \frac{C_0}{2\sqrt{2}}\sum_{\alpha\beta} |\phi_{\alpha 0}\rangle R^{(\Lambda)*}_{\alpha 0} R^{(\Lambda)}_{\beta 0}\langle\phi_{\beta 0}| \left\{1 + \frac{C_0(\Lambda)}{2\sqrt{2}}\sum_{\gamma}\frac{|R^{(\Lambda)}_{\gamma 0}|^2}{E-(2\gamma+3/2)\hbar\omega} + (\dots)^2 \right\}$$

$$= \frac{\dfrac{C_0}{2\sqrt{2}}\displaystyle\sum_{\alpha\beta} |\phi_{\alpha 0}\rangle R^{(\Lambda)*}_{\alpha 0} R^{(\Lambda)}_{\beta 0}\langle\phi_{\beta 0}|}{1 - \dfrac{C_0(\Lambda)}{2\sqrt{2}}\displaystyle\sum_{\gamma}\frac{|R^{(\Lambda)}_{\gamma 0}|^2}{E-(2\gamma+3/2)\hbar\omega}}$$

Poles occur when $\displaystyle 1 - \frac{C_0(\Lambda)}{2\sqrt{2}}\sum_{\gamma}\frac{|R^{(\Lambda)}_{\gamma 0}|^2}{E-(2\gamma+3/2)\hbar\omega} = 0$ ✓