Goal: Matrix elements of $\vec{\sigma}\cdot q$ for $l=0$ in Bush basis

Busch eigenstates: $\Psi_{\nu} = \frac{1}{2}\pi^{-3/2} A\ \Gamma(-\nu)\ U(-\nu, \frac{3}{2}, r^2)\ e^{-r^2/2}$

Where ~~$A^2 \approx .0897597$ or $A \approx 0.299594$~~ A depends on a_0

and $E = \frac{3}{2} + \nu$ where $\sqrt{2}\,\Gamma(-E/2 + 3/4)/\Gamma(-E/2 + 1/4) = \frac{1}{a_0}$

excluding cm energy

Can also write $\Psi_\nu = \frac{1}{2}\pi^{-3/2} A \sum_{n=0}^{\infty} \frac{L_n^{(1/2)}(r^2)}{n-\nu}\ e^{-r^2/2}$

$$= \frac{1}{2}\pi^{-3/2} A \sum_{n=0}^{\infty} \int_0^\infty \frac{dy}{(1+y)^2}\left(\frac{y}{1+y}\right)^{n-\nu-1} L_n^{(1/2)}(r^2)\, e^{-r^2/2}$$

HO eigenstates are used for $l \neq 0$

$$\phi_{nlm}(\vec{r}) = \sqrt{\frac{2(n)!}{\Gamma(n+l+3/2)}}\ r^l e^{-r^2/2} L_n^{l+1/2}(r^2) \qquad E = 2n + l + 3/2 \quad \text{(excluding cm)}$$

$$\frac{\partial\phi}{\partial r} = \sqrt{\frac{2n!}{\Gamma(n+l+3/2)}}\, e^{-r^2/2}\left(l r^{l-1} L_n^{l+1/2}(r^2) - r^{l+1} L_n^{l+1/2}(r^2) - 2r^{l+1} L_{n-1}^{l+3/2}(r^2)\right)$$

We want to calculate

$$\langle \nu' \, l'=0 \| q \| n\ L=1 \rangle = \frac{1}{\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}} \langle \nu' l'=0\ m'=0 | -i\nabla_0 | n\ l=1\ m=0 \rangle$$

$$= +i\sqrt{3}\,\langle \nu' l'=0\, m'=0 | \nabla_0 | n\ l=1\ m=0 \rangle$$

$$= \frac{i\sqrt{3}}{3} \int r^2 dr\ \Psi_{\nu'}(r)\left(\frac{\partial}{\partial r} + \frac{l+1}{r}\right)\phi_{n1}(r)$$

$$= i\, 3^{-1/2}\, \frac{1}{2}\pi^{-3/2} A \sqrt{\frac{2n!}{\Gamma(n+5/2)}}$$

$$\times \int r^2 dr \left(\sum_{n'=0}^{\infty} \frac{L_{n'}^{(1/2)}(r^2)}{n'-\nu}\right) e^{-r^2}\left(L_n^{(3/2)}(r^2) - r^2 L_n^{(3/2)}(r^2) - 2r^2 L_{n-1}^{5/2}(r^2)\right)$$

$$= \frac{i\,A}{2\sqrt{3}\,\pi^{3/2}} \sqrt{\frac{2n!}{\Gamma(n+5/2)}} \sum_{n'=0}^{\infty} \int_0^\infty dr\ e^{-r^2}\left[\frac{r^2 L_{n'}^{(1/2)}(r^2) L_n^{(3/2)}(r^2)}{n'-\nu} - \frac{r^4 L_{n'}^{(1/2)} L_n^{(3/2)}(r^2)}{n'-\nu} - \frac{2r^4 L_{n'}^{(1/2)}(r^2) L_n^{(3/2)}(r^2)}{n'-\nu}\right]$$

Or we can use sum form of $L_n^{3/2}$

$$= \frac{iA}{2\sqrt{3}\pi^{3/2}} \sqrt{\frac{2n!}{\Gamma(n+5/2)}} \sum_{n'=0}^{\infty} \int r^2 dr \left(\sum_{m'=0}^{n'} \frac{\Gamma(1/2+n'+1) \quad (-r^2)^{m'}}{\Gamma(m'+1)\Gamma(n'-m'+1)\Gamma(1/2+m'+1)} \right) e^{-r^2/2}$$

$$\times \left(\frac{\partial}{\partial r} + \frac{2}{r} \right) e^{-r^2/2} r \sum_{m=0}^{n} \frac{\Gamma(3/2+n+1) \quad (-r^2)^m}{\Gamma(m+1)\Gamma(n-m+1)\Gamma(3/2+m+1)}$$

$$= \frac{iA}{2\sqrt{3}\pi^{3/2}} \sqrt{\frac{2n!\,\Gamma(n+5/2)}{\cancel{\Gamma(n+5/2)}}} \sum_{n'=0}^{\infty} \sum_{m'=0}^{n'} \sum_{m=0}^{n} \int_0^{\infty} r^2 dr \, (-1)^{m+m'} \frac{e^{-r^2/2}}{\Gamma(m'+1)\Gamma(n'-m'+1)\Gamma(m'+3/2)} (r)^{2m'}$$

$$\times \left(\frac{-r^2 e^{-r^2/2} (r)^{2m} + e^{-r^2/2}(2m+3) r^{2m}}{\Gamma(m+1)\Gamma(n-m+1)\Gamma(m+5/2)} \right)$$

$$= \frac{iA}{2\pi^{3/2}\sqrt{3}} \sqrt{2(n)!} \sum_{n'=0}^{\infty} \sum_{m'=0}^{n'} \sum_{m=0}^{n} \frac{(-1)^{m+m'}}{\Gamma(m'+1)\Gamma(n'-m'+1)\Gamma(m'+3/2)\,\Gamma(m+1)\Gamma(n-m+1)\Gamma(m+5/2)}$$

$$\int dr \, e^{-r^2} \left(-r^{2m+2m'+4} + (2m+3) r^{2m+2m'+2} \right)$$

$$\boxed{= \frac{iA\sqrt{n!\,\Gamma(n+\frac{5}{2})}}{\pi^{3/2}\sqrt{6}} \sum_{n'=0}^{\infty} \sum_{m'=0}^{n'} \sum_{m=0}^{n} \frac{(-1)^{m+m'} \frac{1}{4}(3+2m-2m')\Gamma(m+m'+3/2)}{\Gamma(m'+1)\Gamma(n'-m'+1)\Gamma(m'+3/2)\Gamma(m+1)\Gamma(n-m+1)\Gamma(m+5/2)} = \langle \nu' \, l'=0 \| q \| n \; l=1 \rangle}$$

Alternately we can write as a finite sum of generalized hypergeometric functions

$$= \frac{i}{\sqrt{3}} \int r^2 dr \, \frac{1}{2}\pi^{-3/2} A\, \Gamma(-\nu')\, U(-\nu', 3/2, r^2)\, e^{-r^2/2} \left(\frac{\partial}{\partial r} + \frac{l+1}{r}\right)$$

$$\sqrt{\frac{2(n!)}{\Gamma(n+5/2)}}\; \cancel{r}\, e^{-r^2/2} \sum_{m=0}^{n} \frac{\Gamma(n+5/2)\,(r)^{2m+1}\,(-1)^m}{\Gamma(m+1)\,\Gamma(n-m+1)\,\Gamma(m+5/2)}$$

$$= i \sqrt{\frac{(n!)\,\Gamma(n+5/2)}{6\,\pi^3}}\; A\,\Gamma(-\nu') \int r^2 dr\; e^{-r^2/2}\, U(-\nu', 3/2, r^2)$$

$$e^{-r^2/2} \sum_{m=0}^{n} \frac{(-1)^m}{\Gamma(m+1)\,\Gamma(n-m+1)\,\Gamma(m+5/2)} \left(-\, r^{2m+2} + (2m+1)\, r^{2m}\right)$$

$$= i A\, \Gamma(-\nu') \sqrt{\frac{n!\,\Gamma(n+5/2)}{6\pi^3}} \sum_{m=0}^{n} \int dr\; e^{-r^2}\, \frac{U(-\nu', 3/2, r^2)\left[(2m+1)\, r^{2m+2} - r^{2m+4}\right]}{(-1)^n\, \Gamma(m+1)\,\Gamma(n-m+1)\,\Gamma(m+5/2)}$$

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$$= i A\, \Gamma(-\nu') \sqrt{\frac{n!\,\Gamma(n+5/2)}{6\pi^3}} \sum_{m=0}^{n} \frac{(-1)^m\, \pi^{1/2}}{2\,\Gamma(n-m+1)} \left[\frac{(2m+1)\, {}_2F_1(m+1, -\nu'-1/2; \tfrac{1}{2}; 1) - (m+1)\, {}_2F_1(m+2, -\nu'-1/2; 1/2; 1)}{\Gamma(m+5/2)\;\; \Gamma(-\nu')} \right.$$

$$+ \left[2(2+m)(2m+1)(m+1-\nu')\, {}_2F_1(m+\tfrac{3}{2}, -\nu'; \tfrac{1}{2}; 1) \right.$$

$$\left.\left. - (m+1)(m+2-\nu')\, {}_2F_1(m+5/2, -\nu'; 1/2; 1) \right] \Big/ \Gamma(m+3)\, \Gamma(1/2-\nu') \right]$$

$$= \langle \nu'\, l'=0 \,\|\, q \,\|\, n\;\, l=1 \rangle$$