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Goal: Use Busch eigenstates (modifies $l=0$ basis states)
Then the Green's function is diagonal

Basis states:

For $l=0$

$$\psi_\nu = \frac{1}{2}\pi^{-3/2} A\, \Gamma(-\nu)\, U(-\nu, 3/2, r^2)\, e^{-r^2/2}$$

$$A^{-2} = \int r^2 |\psi_\nu|^2 dr = \frac{4^{-2-\nu}}{\pi^{5/2}\Gamma(1-2\nu)} \Gamma(-\nu)^2 \Big[-2\nu\, {}_2F_1(1, -\tfrac{1}{2}-\nu; \tfrac{1}{2}-\nu; 1)$$
$$+ (1+2\nu)\, {}_2F_1(1, -\nu; 1-\nu; 1) \Big]$$

These hypergeometric functions lie on a branch cut for the typical definition of ${}_{q+1}F_q$ functions (see DLMF 16.2(iii))

In Mathematica I defined A using a limiting procedure

$$A^{-2} = \lim_{\varepsilon \to 0^+} \frac{4^{-2-\nu}}{\pi^{5/2}\Gamma(1-2\nu)} \Gamma(-\nu)^2 \Big[-2\nu\, {}_2F_1(1, -\tfrac{1}{2}-\nu; \tfrac{1}{2}-\nu; 1-\varepsilon)$$
$$+ (1+2\nu)\, {}_2F_1(1, -\nu; 1-\nu; 1-\varepsilon) \Big]$$

$$\boxed{A = \left(\frac{\Gamma(-\nu)\left[\psi_0(-\nu) - \psi_0(-\tfrac{1}{2}-\nu)\right]}{8\pi^2\,\Gamma(-1/2-\nu)} \right)^{-1/2}}$$

where $\psi_0(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ is the digamma function

Apologies for repeated notation here...
Numerical integration confirms that this definition of A properly normalizes the state.

We want to determine matrix elts of ∇_0 in analogy with the HO basis approach.

Therefore we need

$$\partial_r \Psi_\nu(r) = \partial_r\, e^{-r^2/2}\, U(-\nu, 3/2, r^2)\, \frac{\Gamma(-\nu) A}{2\pi^{3/2}}$$

$$= \frac{\Gamma(-\nu) A}{2\pi^{3/2}}\, r e^{-r^2/2} \left[2\nu\, U(1-\nu, 5/2, r^2) - U(-\nu, 3/2, r^2) \right]$$

The easier matrix elt is (Edmonds 5.7)

$$\langle n\ \ell{=}1 | \nabla_0 | \nu\ \ell{=}0 \rangle = \frac{1}{\sqrt{3}} \int r^2 dr\ \phi_{n,\ell=1}(r)\, \partial_r \Psi_\nu(r)$$

Writing the HO wavefunction as a finite sum

$$= \frac{1}{\sqrt{3}} \sqrt{\frac{2\ n!}{\Gamma(n+5/2)}} \int r^2 dr\ r e^{-r^2/2} \sum_{m=0}^{n} \frac{(-1)^m\, \Gamma(n+5/2)}{m!\,(n-m)!\,\Gamma(m+5/2)}\, r^{2m}\, \partial_r \Psi_\nu(r)$$

$$= \sqrt{\frac{2\ n!}{3\ \Gamma(n+5/2)}}\ \frac{\Gamma(-\nu)\Gamma(n+5/2) A}{2\ \pi^{3/2}} \sum_{m=0}^{n} \frac{(-1)^m}{m!\,(n-m)!\,\Gamma(m+5/2)}$$

$$\int dr\, e^{-r^2} r^{2m+4} \left[2\nu\, U(1-\nu, 5/2, r^2) - U(-\nu, 3/2, r^2) \right]$$

The 2 integrals of hypergeometric functions are, using DLMF 13.10.7

$$2\nu \int dr\, e^{-r^2} r^{2m+4}\, U(1-\nu, 5/2, r^2)$$

$$= \nu \int dt\, e^{-t}\, t^{m+5/2-1}\, U(1-\nu, 5/2, r^2)$$

$$= \nu\, \Gamma(m+5/2)\Gamma(m+1) \frac{{}_2F_1(1-\nu, \cancel{m+5/2}, 2+m-\nu; 0)}{\Gamma(2+m-\nu)}$$

$$= \nu \frac{\Gamma(m+5/2)\Gamma(m+1)}{\Gamma(2+m-\nu)}$$

And

$$-\int dr\, e^{-r^2} r^{2m+4}\, U(-\nu, 3/2, r^2)$$

$$= -\frac{1}{2} \int dt\, e^{-t}\, t^{m+5/2-1}\, U(-\nu, 3/2, t)$$

$$= -\frac{1}{2} \Gamma(m+5/2)\Gamma(m+2) \frac{{}_2F_1(-\nu, \cancel{m+5/2}, 2+m-\nu; 0)}{\Gamma(2+m-\nu)}$$

$$= -\frac{\Gamma(m+5/2)\Gamma(m+2)}{2\Gamma(2+m-\nu)}$$

So

$$\langle n\ l{=}1 | \nabla_0 | \nu\ l{=}0 \rangle = \sqrt{\frac{n!\,\Gamma(n+5/2)}{6\pi^3}}\ A\,\Gamma(-\nu)$$

$$\sum_{m=0}^{n} \frac{(-1)^m\, \cancel{\Gamma(m+5/2)}}{m!\,(n-m)!\,\cancel{\Gamma(m+5/2)}} \left(\frac{\nu\Gamma(m+1) - \frac{1}{2}\Gamma(m+2)}{\Gamma(2+m-\nu)} \right)$$

$$\boxed{= A\,\Gamma(-\nu) \sqrt{\frac{n!\,\Gamma(n+5/2)}{6\pi^3}} \sum_{m=0}^{n} \frac{(-1)^m}{(n-m)!\,\Gamma(2+m-\nu)} \left(\nu - \frac{m+1}{2} \right)}$$

The reduced matrix element of the momentum operator is therefore

$$\langle n\ \ell{=}1 \| q \| \nu\ \ell{=}0 \rangle = \frac{1}{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}} \langle n\ \ell{=}1 | -i\nabla_0 | \nu\ \ell{=}0 \rangle$$

$$= i\sqrt{3}\, \langle n\ \ell{=}1 | \nabla_0 | \nu\ \ell{=}0 \rangle$$

$$\boxed{\langle n\ \ell{=}1 \| q \| \nu\ \ell{=}0 \rangle = iA\,\Gamma(-\nu) \sqrt{\frac{n!\,\Gamma(n+5/2)}{2\pi^3}} \sum_{m=0}^{n} \frac{(-1)^m}{(n-m)!\,\Gamma(2+m-\nu)} \left(\nu - \frac{m+1}{2}\right)}$$