Using Edmonds 7.1.6 we write

$$\langle n'(l's')j'j_z'|\vec{q}\cdot\vec{\sigma}|n(ls)jj_z\rangle =$$
$$(-1)^{l+s'+j}\delta_{jj'}\delta_{j_zj_z'}\begin{Bmatrix} j & s' & l' \\ 1 & l & s\end{Bmatrix}\sum_{n''}\langle n'l'\|\vec{q}\|n''l\rangle\langle n''s'\|\vec{\sigma}\|ns\rangle$$

Now $\langle n''s'\|\vec{\sigma}\|ns\rangle = \delta_{nn'}\langle s'\|\vec{\sigma}\|s\rangle$
and we specealize to $s=s'=1/2$ where Edmonds 5.4.4 tells us that $\langle 1/2\|\sigma\|1/2\rangle = 2\sqrt{\frac{3}{2}}$ (recall $S=\hbar\sigma/2$) so

$$\langle n'(l's')j'j_z'|\vec{q}\cdot\vec{\sigma}|n(ls)jj_z\rangle =$$
$$(-1)^{l+j+1/2}\delta_{jj'}\delta_{j_zj_z'}\begin{Bmatrix} j & 1/2 & l' \\ 1 & l & 1/2\end{Bmatrix}\langle n'l'\|\vec{q}\|nl\rangle\sqrt{6}$$

The momentum RME takes a bit more effort. From definition of RME (Edmonds 5.4.1) and using $m'=m=q=0$

$$\langle n'l'\|\vec{q}\|nl\rangle = (-1)^{l'}\frac{1}{\begin{pmatrix} l' & 1 & l \\ 0 & 0 & 0\end{pmatrix}}\langle n'l'0|-i\nabla_0|nl0\rangle$$

From Edmonds table 2

$$\begin{pmatrix} l' & 1 & l \\ 0 & 0 & 0\end{pmatrix} = (-1)^{\frac{l+l'+1}{2}}\sqrt{\frac{(l+l'-1)!(1+l-l')!(1+l'-l)!}{(l+l'+2)!}}$$
$$\times\frac{\left(\frac{l+l'+1}{2}\right)!}{\left(\frac{-l+l'+1}{2}\right)!\left(\frac{l-l'+1}{2}\right)!\left(\frac{l+l'-1}{2}\right)!}$$

From Edmonds section 5.7 we know $\langle n'l' | \nabla_0 | n l \rangle$ is nonzero only for $l' = l \pm 1$

For $l' = l+1$

$$\begin{pmatrix} l+1 & 1 & l \\ 0 & 0 & 0 \end{pmatrix} = (-1)^{l+1} \sqrt{\frac{(2l)!\,0!\,2!}{(2l+3)!}} \frac{(l+1)!}{1!\,0!\,l!}$$

$$= (-1)^{l+1} \sqrt{\frac{2}{(2l+3)(2l+2)(2l+1)}} (l+1)$$

For $l' = l-1$

$$\begin{pmatrix} l-1 & 1 & l \\ 0 & 0 & 0 \end{pmatrix} = (-1)^{l} \sqrt{\frac{(2l-2)!\,2!\,0!}{(2l+1)!}} \frac{l!}{0!\,(l-1)!}$$

$$= (-1)^{l} \sqrt{\frac{2}{(2l+1)(2l)(2l-1)}}\, l$$

Following Tom's notes, I try writing

$$\langle n'l' \| q \| n l \rangle = \begin{cases} (-1)^{l'} \left[\sqrt{(2l'+1)(2l+1)} \begin{pmatrix} l' & 1 & l \\ 0 & 0 & 0 \end{pmatrix}\right]^{-1} \Gamma(n'l'; nl) \\ 0 \quad \text{if } |l-l'| \neq 1 \end{cases}$$

It seems he meant to invert? See blue.
No, he wants to bring these factors into Γ?

It seems to me the best way to write things starts by explicitly dividing the 6j/3j symbols

For $l' = l+1$ we use Edmonds table 5 line 2 with $b = 1/2$, $a = j$, $c = l+1$

$$\begin{Bmatrix} j & 1/2 & l+1 \\ 1 & l & 1/2 \end{Bmatrix} = (-1)^{j+l+3/2} \sqrt{\frac{\cancel{2}(j+l+5/2)(l+3/2-j)(j+l+1/2)}{1(\cancel{2})(3)(2l+1)(2l+2)}} \times \sqrt{\frac{(j+1/2-l)}{(2l+3)}}$$

so $\begin{Bmatrix} j & 1/2 & l+1 \\ 1 & l & 1/2 \end{Bmatrix} \Big/ \begin{pmatrix} l+1 & 1 & l \\ 0 & 0 & 0 \end{pmatrix}$

$$= (-1)^{j+\cancel{l}+3/2+\cancel{l}+1} \sqrt{\frac{(j+l+5/2)(l+3/2-j)(j+l+1/2)(j+1/2-l)}{3\cancel{(2l+1)}\cancel{(2l+2)}\cancel{(2l+3)}}} \times \sqrt{\frac{\cancel{(2l+1)}\cancel{(2l+2)}\cancel{(2l+3)}}{2}} \frac{1}{l+1}$$

Because $l' = l+1$ and $j = j'$, $j = l + 1/2$

$$= \frac{(-1)^{l+1}}{\sqrt{6}\,(l+1)} \sqrt{(2l+3)(1)(2l+1)(1)}$$

$$= \frac{(-1)^{l+1}}{\sqrt{6}\,(l+1)} \sqrt{(2l+3)(2l+1)} = \frac{(-1)^{l'} \sqrt{(2l'+1)(2l+1)}}{(l')\ \sqrt{6}}$$

For $\ell' = \ell - 1$

$$\begin{Bmatrix} j & 1/2 & \ell' \\ 1 & \ell & 1/2 \end{Bmatrix} \overset{\text{by } 6.2.4}{=} \begin{Bmatrix} j & \ell' & 1/2 \\ 1 & 1/2 & \ell \end{Bmatrix} \overset{\text{by } 6.2.5}{=} \begin{Bmatrix} j & 1/2 & \ell \\ 1 & \ell' & 1/2 \end{Bmatrix}$$

so

$$\begin{Bmatrix} j & 1/2 & \ell-1 \\ 1 & \ell & 1/2 \end{Bmatrix} = \begin{Bmatrix} j & 1/2 & \ell \\ 1 & \ell-1 & 1/2 \end{Bmatrix}$$

And we again use the previous result taking $\ell' \to \ell - 1$ and $j \to \ell - 1/2$ i.e.
$a = \ell - 1/2$ $b = 1/2$ $c = \ell$ $s = 2\ell$ in Table 5 line 2

$$\begin{Bmatrix} \ell-1/2 & 1/2 & \ell \\ 1 & \ell-1 & 1/2 \end{Bmatrix} = \cancel{(-1)^{2\ell}} \sqrt{\frac{\cancel{2}\cancel{(2\ell+1)}(1)\cancel{(2\ell-1)}(1)}{1\cancel{(2)}\, 3\, \cancel{(2\ell-1)}\, 2\ell \cancel{(2\ell+1)}}}$$

$$= \sqrt{\frac{1}{6\ell}}$$

$$\begin{Bmatrix} \ell-1/2 & 1/2 & \ell \\ 1 & \ell-1 & 1/2 \end{Bmatrix} \Big/ \begin{pmatrix} \ell-1 & 1 & \ell \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \sqrt{\frac{1}{6\ell}} \sqrt{\frac{(2\ell+1)(2\ell-1)}{(2\ \ \ell}} \ (-1)^{\ell}$$

$$= \frac{(-1)^{\ell}}{\sqrt{6}\,\ell} \sqrt{(2\ell+1)(2\ell'+1)}$$

So combining everything for $\ell' = \ell + 1$

$\langle n'(\ell' s')j' j_z' | \vec{q}\cdot\vec{\sigma} | n(\ell s) j j_z \rangle =$

$$(-1)^{\ell+\ell+1} \delta_{jj'} \delta_{j_z j_z'} (-1)^{\ell'} \cancel{\sqrt{6}} (-i)(-1)^{\ell'} \frac{\sqrt{(2\ell'+1)(2\ell+1)}}{\cancel{\sqrt{6}}\, \ell'}$$

$$\times \langle n'\ell' 0 | \nabla_0 | n\ell 0 \rangle$$

$$= i\, \delta_{jj'} \delta_{j_z j_z'} \frac{\sqrt{(2\ell'+1)(2\ell+1)}}{\ell'} \langle n'\ell' 0 | \nabla_0 | n\ell 0 \rangle$$

Using Section 5.7 results

$$= i \delta_{jj'} \delta_{j_z j_z'} \frac{\sqrt{\cancel{(2\ell'+1)}(2\ell+1)}}{\cancel{\ell'}} \frac{\cancel{\ell+1}}{\sqrt{\cancel{(2\ell+1)}(2\ell+3)}} \int_0^\infty dr\, r^2 R_{n'\ell+1}(r) \left(\frac{\partial}{\partial r} - \frac{\ell}{r}\right) R_{n\ell}(r)$$

$$= i \delta_{jj'} \delta_{j_z j_z'} \int_0^\infty r^2 dr\, R_{n'\ell+1}(r) \left(\frac{\partial}{\partial r} - \frac{\ell}{r}\right) R_{n\ell}(r)$$

For $\ell' = \ell - 1$, $j = \ell - 1/2$

$\langle n'(\ell' s')j' j_z' | \vec{q}\cdot\vec{\sigma} | n(\ell s) j j_z \rangle =$

$$(-1)^{2\ell} \delta_{jj'} \delta_{j_z j_z'} \frac{(-1)^{\ell'}(-1)^{\ell}}{\sqrt{6}\, \cancel{\ell}} \sqrt{6}\, \cancel{\sqrt{(2\ell+1)(2\ell'+1)}}$$

$$\times \frac{\cancel{\ell}}{\cancel{\sqrt{(2\ell-1)(2\ell+1)}}} (-i) \int_0^\infty r^2 dr\, R_{n'\ell-1}(r) \left(\frac{\partial}{\partial r} + \frac{\ell+1}{r}\right) R_{n\ell}(r)$$

$$= i\, \delta_{jj'} \delta_{j_z j_z'} \int r^2 dr\, R_{n'\ell-1}(r) \left(\frac{\partial}{\partial r} + \frac{\ell+1}{r}\right) R_{n\ell}(r)$$

$$= i\,\delta_{jj'}\,\delta_{j_z j_z'}\,\frac{\sqrt{(2l'+1)(2l+1)}}{l}\,\langle n'l0|\nabla_0|nl0\rangle$$

For $l'=l+1$ the denominator is

$$l'=l+1 \quad = \frac{l+l'+1}{2}$$

For $l'=l-1$ we have $l=l'+1=\dfrac{l+l'+1}{2}$

so we can write

$$\langle n'(l's')j'j_z'|\vec{\sigma}\cdot\vec{q}\,|n(ls)j\,j_z\rangle =$$

$$i\,\delta_{jj'}\,\delta_{j_z j_z'}\,\frac{\sqrt{(2l'+1)(2l+1)}}{\left(\frac{l+l'+1}{2}\right)}\,\langle n'l'0|\nabla_0|nl0\rangle$$

$$= \delta_{jj'}\,\delta_{j_z j_z'}\,\Gamma(n'l';nl)$$

where

$$\langle n'l'0|\nabla_0|nl0\rangle = \frac{\frac{l+l'+1}{2}}{\sqrt{(2l'+1)(2l+1)}}\int_0^\infty r^2dr\,R_{n'l'}(r)\begin{cases}\frac{\partial}{\partial r}-\frac{l}{r} & l'=l+1\\ \frac{\partial}{\partial r}+\frac{l+1}{r} & l'=l-1\end{cases}\times R_{nl}(r)$$

$$=\frac{(l+l'+1)/2\,(-i)\,\Gamma(n'l';nl)}{\sqrt{(2l'+1)(2l+1)}}$$

Now for the case of 2 identical particles
(suppressing j_z)

$$\langle n'(l's')j_j'\; N'L';(j'L')J' \,|\, \vec{q}\cdot\vec{\sigma} \,|\, n(l s)j_j\; NL;(jL)J \rangle$$

Edmonds 7.1.6

$$= \underbrace{\delta_{NN'}\,\delta_{LL'}}_{\text{does not act on these}}\;\delta_{jj'}\;\underbrace{\delta_{JJ'}}_{\text{If } j \text{ and } L \text{ are diagonal}}\;(-1)^{l+s'+j}\begin{Bmatrix} j & s' & l' \\ 1 & l & s \end{Bmatrix}\langle n'l'\|q\|nl\rangle\,\langle ns'\|\sigma\|ns\rangle$$

~~this must be~~

The rme of q is already determined, but the $6j$ and rme of σ have changed from the 1-body case.

$$\langle ns'\|\sigma\|ns\rangle = \langle s'\|\sigma_1-\sigma_2\|s\rangle = \langle \tfrac{1}{2}\tfrac{1}{2}\,s'\|\,\sigma_1-\sigma_2\,\|\tfrac{1}{2}\tfrac{1}{2}\,s\rangle$$

By Edmonds 7.1.7 and 7.1.8

$$= (-1)^{1/2+1/2+s+1}\left[(2s'+1)(2s+1)\right]^{1/2}\begin{Bmatrix} 1/2 & s' & 1/2 \\ s & 1/2 & 1 \end{Bmatrix}\langle 1/2\|\sigma\|1/2\rangle$$

$$-(-1)^{s'}\left[(2s'+1)(2s+1)\right]^{1/2}\begin{Bmatrix} 1/2 & s' & 1/2 \\ s & 1/2 & 1 \end{Bmatrix}\langle 1/2\|\sigma\|1/2\rangle$$

$$= \left[(-1)^{s}-(-1)^{s'}\right]\sqrt{6(2s'+1)(2s+1)}\begin{Bmatrix} 1/2 & s' & 1/2 \\ s & 1/2 & 1 \end{Bmatrix}$$

The possible values of s, s' are $0, 1$. Mathematica says that the $6j$ is nonzero only when $s' \neq s$, in which case it $= 6^{-1/2}$. We thus rewrite the above as

$$= (s'-s)\; 2\sqrt{3}$$

So

$$\langle n'(l's')j'; N'L'; (j'L')J' | \vec{q}\cdot\vec{\sigma} | n(ls)j; NL; (jL)J \rangle$$

$$= \delta_{NN'}\delta_{LL'}\delta_{jj'}\delta_{JJ'} (-1)^{l+s'+j} \begin{Bmatrix} j & s' & l' \\ 1 & l & s \end{Bmatrix} (s'-s)\, 2\sqrt{3}\, \langle n'l' \| q \| nl \rangle$$

Now since $j'=j$, and $|l'-l|=1$ can we say anything?

s	s'	l'	j	$(-1)^{l+s'+j}$
0	1	$l+1$	l	$(-1)^{l+1+l}$
1	0	$l+1$	$l+1$	$(-1)^{l+l+1}$
0	1	$l-1$	l	$(-1)^{l+1+l}$
1	0	$l-1$	$l-1$	$(-1)^{l+l-1}$

$j = l^{s'} l'^{s}$

$= l\cdot s' + l'\cdot s$

Playing with Mathematica a bit I can see that

$$\begin{Bmatrix} j & s' & l' \\ 1 & l & s \end{Bmatrix} = \frac{1}{\sqrt{3}} \frac{1}{\sqrt{2j+1}}$$

Therefore the original ME is

$$\delta_{NN'}\delta_{LL'}\delta_{jj'}\delta_{JJ'} (-1)^{l+s'+j} \frac{2(s'-s)}{\sqrt{2j+1}} \langle n'l' \| q \| nl \rangle$$

$$= \delta_{NN'}\delta_{LL'}\delta_{jj'}\delta_{JJ'} (-1)^{l+s'+j} \frac{2(s'-s)}{\sqrt{2j+1}} (-i) \langle n'l' \| \nabla_0 \| nl \rangle$$

$$= \delta_{NN'}\delta_{LL'}\delta_{jj'}\delta_{JJ'} \frac{+i\, 2(s'-s)}{\sqrt{2j+1}} \frac{(-1)^{l'}}{\begin{pmatrix} l' & 1 & l \\ 0 & 0 & 0 \end{pmatrix}} \langle n'l'0 | \nabla_0 | nl0 \rangle$$

From ②

$$\begin{pmatrix} l-1 & 1 & l \\ 0 & 0 & 0 \end{pmatrix} = (-1)^{l} \sqrt{\frac{l}{(2l+1)(2l-1)}}$$

$$\begin{pmatrix} l+1 & 1 & l \\ 0 & 0 & 0 \end{pmatrix} = (-1)^{l+1} \sqrt{\frac{(l+1)}{(2l+1)(2l+3)}}$$

So

$$\langle n'(l's')j' ; N'L' ; (j'L')J' | \vec{\sigma}\cdot\vec{q} | n(ls)j ; NL ; (jL)J \rangle$$

$$= \delta_{NN'}\delta_{LL'}\delta_{jj'}\delta_{JJ'}\ 2i(s-s')(-1)^{l'}(-1)^{(l+l'+1)/2} \sqrt{\frac{(2l+1)(2l'+1)}{(2j+1)\left(\frac{l+l'+1}{2}\right)}}$$

$$\langle n'l'0 | \nabla_0 | nl0 \rangle$$

$$= \delta_{NN'}\delta_{LL'}\delta_{jj'}\delta_{JJ'}\ 2^{3/2} i(s-s')(-1)^{\frac{l-l'+1}{2}} \sqrt{\frac{(2l+1)(2l'+1)}{(2j+1)(l+l'+1)}}$$

$$\langle n'l'0 | \nabla_0 | nl0 \rangle$$

$$= \delta_{NN'}\delta_{LL'}\delta_{jj'}\delta_{JJ'}\ (s'-s)(-1)^{l-l'+1/2} \sqrt{\frac{2(l+l'+1)}{(2j+1)}}\ \Gamma(n'l';nl)$$

To calculate matrix elements of $\vec{\Sigma}\cdot\vec{Q}$ we make a change of basis

Since $\delta\mathcal{J}=0$ we can assume $\mathcal{J}=\mathcal{J}'$ immediately

$$\langle n'(\ell's')j';N'L';(j'L')J|\vec{\Sigma}\cdot\vec{Q}|n(\ell s)j;NL;(jL)J\rangle$$

$$=\sum_{\mathcal{J}\mathcal{J}'}\langle n'(\ell's')j';N'L';(j'L')J|n'\ell';N'(L's')\mathcal{J}';(\mathcal{J}'\ell')J\rangle$$

$$\langle n'\ell';N'(L's')\mathcal{J}';(\ell'\mathcal{J}')J|\vec{\Sigma}\cdot\vec{Q}|n\ell;N(Ls)\mathcal{J};(\ell\mathcal{J})J\rangle$$

$$\langle n\ell;N(Ls)\mathcal{J};(\ell\mathcal{J})J|n(\ell s)j;NL;(jL)J\rangle$$

$$=\sum_{\mathcal{J}}\sum_{\mathcal{J}'}\sqrt{(2j'+1)(2\mathcal{J}'+1)}\,(-1)^{\ell'+s'+L'+J'}\begin{Bmatrix}\ell' & s' & j'\\ L' & J & \mathcal{J}'\end{Bmatrix}$$

$$\langle n'\ell';N'(L's')\mathcal{J}';(\ell'\mathcal{J}')J|\vec{\Sigma}\cdot\vec{Q}|n\ell;N(Ls)\mathcal{J};(\ell\mathcal{J})J\rangle$$

$$\sqrt{(2j+1)(2\mathcal{J}'+1)}\,(-1)^{\ell+s+L+J}\begin{Bmatrix}\ell & s & j\\ L & J & \mathcal{J}\end{Bmatrix}$$

Using Edmonds 7.1.6 yet again

$$\langle n'\ell';N'(L's')\mathcal{J}';(\ell'\mathcal{J}')J|\vec{\Sigma}\cdot\vec{Q}|n\ell;N(Ls)\mathcal{J};(\ell\mathcal{J})J\rangle$$

$$=(-1)^{L+s'+\mathcal{J}}\delta_{\mathcal{J}\mathcal{J}'}\delta_{\mathcal{J}_z\mathcal{J}_z'}\begin{Bmatrix}\mathcal{J} & s' & L'\\ 1 & L & s\end{Bmatrix}\sum_{N''}\langle N'L'\|Q\|N''L\rangle\langle N''s'\|\Sigma\|Ns\rangle$$

$$\times\delta_{nn'}\delta_{\ell\ell'}\delta_{\mathcal{J}\mathcal{J}'}$$

Plugging this into the full ME

$$=\sum_{\mathcal{J}}(2\mathcal{J}+1)\sqrt{(2j+1)(2j'+1)}\,(-1)^{s+s'+L+L'}\begin{Bmatrix}\ell & s' & j'\\ L' & J & \mathcal{J}\end{Bmatrix}$$

$$\begin{Bmatrix}\ell & s & j\\ L & J & \mathcal{J}\end{Bmatrix}(-1)^{L+s'+\mathcal{J}}\delta_{nn'}\delta_{\ell\ell'}\delta_{\mathcal{J}\mathcal{J}'}\begin{Bmatrix}\mathcal{J} & s' & L'\\ 1 & L & s\end{Bmatrix}\langle N'L'\|Q\|NL\rangle \times\langle Ns\|\Sigma\|Ns\rangle$$

$$= (-1)^{\cancel{s+s'}+\cancel{L}+L'+\cancel{L}+\cancel{s'}} \; \overset{=(-1)^{s+L'}}{} \sqrt{(2j+1)(2j'+1)}\, \delta_{nn'}\delta_{\ell\ell'}\delta_{JJ'}$$

$$\sum_{\mathcal{J}} (-1)^{\mathcal{J}} \begin{Bmatrix} \ell & s' & j' \\ L' & J & \mathcal{J} \end{Bmatrix} \begin{Bmatrix} \ell & s & j \\ L & J & \mathcal{J} \end{Bmatrix} \begin{Bmatrix} \mathcal{J} & s' & L' \\ 1 & L & s \end{Bmatrix}$$

$$\langle NL' \| Q \| NL \rangle \langle s' \| \Sigma \| s \rangle$$

For the spin RME, in analogy to that of σ

$$\langle s' \| \Sigma \| s \rangle = \langle s' \| \sigma_1 + \sigma_2 \| s \rangle$$

by E 7.1.7, 7.1.8, 5.4.4
$$= [(-1)^s + (-1)^{s'}] \sqrt{6(2s+1)(2s'+1)} \begin{Bmatrix} 1/2 & s' & 1/2 \\ s & 1/2 & 1 \end{Bmatrix}$$

The first term requires $s'=s$, while the triangle inequalities for the 6j symbol disallow $s=s'=0$. Therefore $s=s'=1$ or the RME is zero.

$$\langle s' \| \Sigma \| s \rangle = \delta_{s,1}\delta_{s',1} \; -2\sqrt{6\cdot\cancel{3}\cdot\cancel{3}} \cdot \frac{-1}{\cancel{3}}$$

$$= + 2\sqrt{6}\, \delta_{s,1}\delta_{s',1}$$

While $\langle NL' \| Q \| NL \rangle$ is the same as the $\vec{g}$ case
Overall ME is now

$$= (-1)^{1+L'}\, 2\sqrt{6(2j+1)(2j'+1)}\, \delta_{nn'}\delta_{\ell\ell'}\delta_{JJ'}\delta_{s,1}\delta_{s',1}$$

$$\langle N'L' \| Q \| NL \rangle \sum_{\mathcal{J}=\max(L-1,L'-1)}^{\min(L+1,L'+1)} (-1)^{\mathcal{J}}(2\mathcal{J}+1) \begin{Bmatrix} \ell & 1 & j' \\ L' & J & \mathcal{J} \end{Bmatrix} \begin{Bmatrix} \ell & 1 & j \\ L & J & \mathcal{J} \end{Bmatrix} \begin{Bmatrix} \mathcal{J} & 1 & L' \\ 1 & L & 1 \end{Bmatrix}$$

For the Rashba term,

5.1.8 $$V_R \propto (\sigma_x k_y - \sigma_y k_x) = [\vec{\sigma} \times \vec{k}]_z = -i\sqrt{2}\,[\sigma \otimes k]_{10}$$

For two particles

$$[\vec{\sigma}_1 \times \vec{k}_1]_z + [\vec{\sigma}_2 \times \vec{k}_2]_z = [(\vec{\sigma}_1 - \vec{\sigma}_2) \times (\vec{k}_1 - \vec{k}_2)]_z/2 + [(\vec{\sigma}_1 + \vec{\sigma}_2) \times (k_1 + k_2)]_z/2$$

$$= ([\vec{\sigma}_1 \times \vec{k}_1]_z - \cancel{[\sigma_2 \times k_1]_z} - \cancel{[\sigma_1 \times k_2]_z} + [\sigma_2 \times k_2]_z + [\sigma_1 \times k_1]_z + \cancel{[\sigma_2 \times k_1]_z} + \cancel{[\sigma_1 \times k_2]_z} + [\sigma_2 \times k_2]_z)/2$$

$$= \frac{1}{\sqrt{2}}\left([\vec{\sigma} \times \vec{q}]_z + [\vec{\Sigma} \times \vec{Q}]_z\right)$$

$$= -i\left([\vec{\sigma} \otimes \vec{q}]_{10} + [\vec{\Sigma} \otimes \vec{Q}]_{10}\right) = i\left([q \otimes \sigma]_{10} + [Q \otimes \Sigma]_{10}\right)$$

For the relative coordinate part we first apply Wigner--Eckart

$$-i\langle n'(l's')j';N'L';(j'L')J'J_z'|[\sigma \otimes q]_{10}|n(ls)j;NL;(jL)JJ_z\rangle$$

$$= -i(-1)^{J'-J_z'}\begin{pmatrix} J' & 1 & J \\ -J_z' & 0 & J_z \end{pmatrix}\langle n'(l's')j';N'L';(j'L')J'\|[\sigma \otimes q]_{10}\|n(ls)j;NL;(jL)J\rangle$$

Edmonds 7.1.7 $$= i(-1)^{J'-J_z'+1}\begin{pmatrix} J' & 1 & J \\ -J_z' & 0 & J_z \end{pmatrix}(-1)^{j'+L+J+1}\sqrt{(2J+1)(2J'+1)}\begin{Bmatrix} j' & J' & L \\ J & j & 1 \end{Bmatrix}\langle n'(l's')j'\|[\sigma \otimes q]_{10}\|n(ls)j\rangle$$

Note that $[\sigma \otimes q]_{10} = -[q \otimes \sigma]_{10}$ and apply Edmonds 7.1.5

$$= i(-1)^{J+J'-J_z'+j'+L+1} \begin{pmatrix} J' & 1 & J \\ -J_z' & 0 & J_z \end{pmatrix} \sqrt{(2j+1)(2j'+1)(2J+1)(2J'+1)3}$$
$$\begin{Bmatrix} j' & J' & L \\ J & j & 1 \end{Bmatrix} \begin{Bmatrix} l' & l & 1 \\ s' & s & 1 \\ j' & j & 1 \end{Bmatrix} \langle n'l' \| q \| nl \rangle \langle s' \| \sigma \| s \rangle$$

We have already derived these reduced matrix elements. Try to simplify the 3j symbol:

$$\begin{pmatrix} J' & 1 & J \\ -J_z' & 0 & J_z \end{pmatrix} = \delta_{J_z J_z'} \begin{pmatrix} J' & 1 & J \\ -J_z & 0 & J_z \end{pmatrix} = \delta_{J_z J_z'} \begin{pmatrix} J & J' & 1 \\ J_z & -J_z & 0 \end{pmatrix}$$

From Edmonds Table 2:

For $J' = J-1 \Rightarrow = \delta_{J_z J_z'} (-1)^{J'-J_z-1} \sqrt{\dfrac{2(J+J_z)(J-J_z)}{(2J+1)2J(2J-1)}}$

$J' = J \Rightarrow = \delta_{J_z J_z'} (-1)^{J-J_z} \dfrac{J_z}{\sqrt{(2J+1)(J+1)J}}$

$J' = J+1 \Rightarrow = \delta_{J_z J_z'} (-1)^{J+J'+1}(-1)^{J+J_z-1} \sqrt{\dfrac{(J+J_z+1)(J-J_z+1)}{(2J+3)(2J+2)(2J+1)}}$

0 if $|J-J'|>1$

Doesn't really simplify... Replacing the spin RME we get

$$\boxed{= 6i(-1)^{J+J'-J_z'+j'+L+1} \delta_{J_z J_z'} \sqrt{(2J+1)(2J'+1)(2j+1)(2j'+1)} \begin{pmatrix} J' & 1 & J \\ -J_z & 0 & J_z \end{pmatrix} \begin{Bmatrix} j' & J' & L \\ J & j & 1 \end{Bmatrix} \begin{Bmatrix} l' & l & 1 \\ s' & s & 1 \\ j' & j & 1 \end{Bmatrix} (s'-s) \langle n'l' \| q \| nl \rangle}$$

For the CM SOC we again re-expand using Edmonds 6.1.5 (with $\delta J = 0$ now, unfortunately)

$$\langle n'(l's')j';N'L';(j'L')J' | i[Q\otimes\Sigma]_{10} | n(ls)j;NL;(jL)J\rangle$$

$$= i\sum_{\mathcal{J}\mathcal{J}'}\sqrt{(2j+1)(2j'+1)(2\mathcal{J}+1)(2\mathcal{J}'+1)}\,(-1)^{l+l'+s+s'+L+L'+J+J'}$$

$$\begin{Bmatrix} l' & s' & j' \\ L' & J' & \mathcal{J}' \end{Bmatrix}\begin{Bmatrix} l & s & j \\ L & J & \mathcal{J} \end{Bmatrix}\langle n'l';N'(L's')\mathcal{J}';(l'\mathcal{J}')J' | [Q\otimes\Sigma]_{10} | nl;N(Ls)\mathcal{J};(l\mathcal{J})J\rangle$$

$l+l'+s+s'$ = even

$$= i(-1)^{L+L'+J+J'}\sqrt{(2j+1)(2j'+1)}\sum_{\mathcal{J}\mathcal{J}'}\sqrt{(2\mathcal{J}+1)(2\mathcal{J}'+1)}$$

$$\begin{Bmatrix} l' & s' & j' \\ L' & J' & \mathcal{J}' \end{Bmatrix}\begin{Bmatrix} l & s & j \\ L & J & \mathcal{J} \end{Bmatrix}\langle n'l';N'(L's')\mathcal{J}';(l'\mathcal{J}')J' | [Q\otimes\Sigma]_{10} | nl;N(Ls)\mathcal{J};(l\mathcal{J})J\rangle$$

Edmonds 5.4.1 + 7.1.8

$$= i(-1)^{L+L'+J+J'}\sqrt{(2j+1)(2j'+1)}\,(-1)^{J'-J_z'}\delta_{J_zJ_z'}\begin{pmatrix} J' & 1 & J \\ -J_z & 0 & J_z \end{pmatrix}$$

$$\sum_{\mathcal{J}\mathcal{J}'}\sqrt{(2\mathcal{J}+1)(2\mathcal{J}'+1)}\begin{Bmatrix} l' & s' & j' \\ L' & J' & \mathcal{J}' \end{Bmatrix}\begin{Bmatrix} l & s & j \\ L & J & \mathcal{J} \end{Bmatrix}(-1)^{l+\mathcal{J}+J'+1}\delta_{ll'}\delta_{nn'}$$

$$\sqrt{(2J+1)(2\mathcal{J}'+1)}\begin{Bmatrix} \mathcal{J}' & J' & l \\ J & \mathcal{J} & 1 \end{Bmatrix}\langle N'(L's')\mathcal{J}' \| [Q\otimes\Sigma]_{10} \| N(Ls)\mathcal{J}\rangle$$

$2J$ = even
Use Edmonds 7.1.5

$$= i(-1)^{L+L'+J+J'+l+1-J_z}\sqrt{(2j+1)(2j'+1)(2J+1)(2J'+1)}$$

$$\delta_{ll'}\delta_{nn'}\delta_{J_zJ_z'}\begin{pmatrix} J' & 1 & J \\ -J_z & 0 & J_z \end{pmatrix}\sum_{\mathcal{J}\mathcal{J}'}(-1)^{\mathcal{J}}\sqrt{(2\mathcal{J}+1)(2\mathcal{J}'+1)}\begin{Bmatrix} l' & s' & j' \\ L' & J' & \mathcal{J}' \end{Bmatrix}$$

$$\begin{Bmatrix} l & s & j \\ L & J & \mathcal{J} \end{Bmatrix}\begin{Bmatrix} \mathcal{J}' & J' & l \\ J & \mathcal{J} & 1 \end{Bmatrix}\sqrt{(2\mathcal{J}+1)(2\mathcal{J}'+1)3}\begin{Bmatrix} L' & L & 1 \\ s' & s & 1 \\ \mathcal{J}' & \mathcal{J} & 1 \end{Bmatrix}$$

$$\langle N'L' \| Q \| NL\rangle\langle s' \| \Sigma \| s\rangle$$

Just rewriting for clarity + subbing for RME of Σ

$$\langle n'(l's')j';N'L';(jL')J' | i[Q\otimes\Sigma]_{10} | n(ls)j;NL;(jL)J\rangle$$

$$= i(-1)^{J+J'+L+L'+l+1-J_z}\sqrt{3(2j+1)(2j'+1)(2J+1)(2J'+1)}\,\delta_{ll'}\delta_{nn'}\delta_{J_zJ_z'}$$

$$\times\begin{pmatrix} J' & 1 & J \\ -J_z & 0 & J_z \end{pmatrix}\sum_{\mathcal{J}\mathcal{J}'}(-1)^{\mathcal{J}}(2\mathcal{J}+1)(2\mathcal{J}'+1)\begin{Bmatrix} l & s' & j' \\ L' & J' & \mathcal{J}' \end{Bmatrix}\begin{Bmatrix} l & s & j \\ L & J & \mathcal{J} \end{Bmatrix}$$

$$\times\begin{Bmatrix} L' & L & 1 \\ s' & s & 1 \\ \mathcal{J}' & \mathcal{J} & 1 \end{Bmatrix}\delta_{s,1}\delta_{s',1}\,2\sqrt{6}\,\langle N'L'\|Q\|NL\rangle\begin{Bmatrix} \mathcal{J}' & J' & l \\ J & \mathcal{J} & 1 \end{Bmatrix}$$

Note $L+L'$ odd from Q RME

$$\boxed{\begin{aligned} &= i(-1)^{J+J'+l-J_z}\,6\sqrt{2}\,\delta_{ll'}\delta_{nn'}\delta_{s,1}\delta_{s',1}\delta_{J_zJ_z'}\begin{pmatrix} J' & 1 & J \\ -J_z & 0 & J_z \end{pmatrix} \\ &\times\sqrt{(2j+1)(2j'+1)(2J+1)(2J'+1)}\,\langle N'L'\|Q\|NL\rangle \\ &\times\sum_{\mathcal{J}}\sum_{\mathcal{J}'}(-1)^{\mathcal{J}}(2\mathcal{J}+1)(2\mathcal{J}'+1)\begin{Bmatrix} \mathcal{J}' & J' & l \\ J & \mathcal{J} & 1 \end{Bmatrix}\begin{Bmatrix} l & 1 & j' \\ L' & J' & \mathcal{J}' \end{Bmatrix}\begin{Bmatrix} l & 1 & j \\ L & J & \mathcal{J} \end{Bmatrix}\begin{Bmatrix} L' & L & 1 \\ 1 & 1 & 1 \\ \mathcal{J}' & \mathcal{J} & 1 \end{Bmatrix}\end{aligned}}$$