Using Edmonds 7.1.6 we write

$$\langle n'(l's')j'j_z'|\vec{q}\cdot\vec{\sigma}|n(ls)jj_z\rangle = (-1)^{l+s'+j}\delta_{jj'}\delta_{j_zj_z'}\begin{Bmatrix} j & s' & l' \\ 1 & l & s\end{Bmatrix}\sum_{n''}\langle n'l'\|\vec{q}\|n''l\rangle\langle n''s'\|\vec{\sigma}\|ns\rangle$$

Now $\langle n''s'\|\vec{\sigma}\|ns\rangle = \delta_{nn'}\langle s'\|\vec{\sigma}\|s\rangle$ and we specealize to $s=s'=1/2$ where Edmonds 5.4.4 tells us that $\langle 1/2\|\sigma\|1/2\rangle = 2\sqrt{\frac{3}{2}}$ (recall $S=\hbar\sigma/2$) so

$$\langle n'(l's')j'j_z'|\vec{q}\cdot\vec{\sigma}|n(ls)jj_z\rangle = (-1)^{l+j+1/2}\delta_{jj'}\delta_{j_zj_z'}\begin{Bmatrix} j & 1/2 & l' \\ 1 & l & 1/2\end{Bmatrix}\langle n'l'\|\vec{q}\|nl\rangle\sqrt{6}$$

The momentum RME takes a bit more effort. From definition of RME (Edmonds 5.4.1) and using $m'=m=q=0$

$$\langle n'l'\|\vec{q}\|nl\rangle = (-1)^{l'}\frac{1}{\begin{pmatrix} l' & 1 & l \\ 0 & 0 & 0\end{pmatrix}}\langle n'l'0|-i\nabla_0|nl0\rangle$$

From Edmonds table 2

$$\begin{pmatrix} l' & 1 & l \\ 0 & 0 & 0\end{pmatrix} = (-1)^{\frac{l+l'+1}{2}}\sqrt{\frac{(l+l'-1)!(1+l-l')!(1+l'-l)!}{(l+l'+2)!}}$$

$$\times\frac{\left(\frac{l+l'+1}{2}\right)!}{\left(\frac{-l+l'+1}{2}\right)!\left(\frac{l-l'+1}{2}\right)!\left(\frac{l+l'-1}{2}\right)!}$$

From Edmonds section 5.7 we know $\langle n'l' | \nabla_0 | nl \rangle$ is nonzero only for $l' = l \pm 1$

For $l' = l+1$

$$\begin{pmatrix} l+1 & 1 & l \\ 0 & 0 & 0 \end{pmatrix} = (-1)^{l+1} \sqrt{\frac{(2l)!\,0!\,2!}{(2l+3)!}} \frac{(l+1)!}{1!\,0!\,l!}$$

$$= (-1)^{l+1} \sqrt{\frac{2}{(2l+3)(2l+2)(2l+1)}} \, (l+1)$$

For $l' = l-1$

$$\begin{pmatrix} l-1 & 1 & l \\ 0 & 0 & 0 \end{pmatrix} = (-1)^{l} \sqrt{\frac{(2l-2)!\,2!\,0!}{(2l+1)!}} \frac{l!}{0!\,!\,(l-1)!}$$

$$= (-1)^{l} \sqrt{\frac{2}{(2l+1)\,(2l)\,(2l-1)}} \, l$$

Following Tom's notes, I try writing

$$\langle n'l' \| q \| nl \rangle = \begin{cases} (-1)^{l'} \left[ \sqrt{(2l'+1)(2l+1)} \begin{pmatrix} l' & 1 & l \\ 0 & 0 & 0 \end{pmatrix} \right]^{-1} \Gamma(n'l'; nl) \\ 0 \qquad\qquad \text{if } |l - l'| \neq 1 \end{cases}$$

It seems he meant to invert? See blue.

No, he wants to bring these factors into $\Gamma$?

It seems to me the best way to write things starts by explicitly dividing the 6j/3j symbols

For $l' = l+1$ we use Edmonds table 5 line 2 with $b = 1/2$, $a = j$, $c = l+1$

$$\begin{Bmatrix} j & 1/2 & l+1 \\ 1 & l & 1/2 \end{Bmatrix} = (-1)^{j+l+3/2} \sqrt{\frac{\cancel{2}(j+l+5/2)(l+3/2-j)(j+l+1/2)}{1(\cancel{2})(3)(2l+1)(2l+2)}}$$

$$\times \sqrt{\frac{(j+1/2-l)}{(2l+3)}}$$

so $\begin{Bmatrix} j & 1/2 & l+1 \\ 1 & l & 1/2 \end{Bmatrix} \Big/ \begin{pmatrix} l+1 & 1 & l \\ 0 & 0 & 0 \end{pmatrix}$

$$= (-1)^{j+\cancel{l}+3/2+\cancel{l}+1} \sqrt{\frac{(j+l+5/2)(l+3/2-j)(j+l+1/2)(j+1/2-l)}{3\cancel{(2l+1)}\cancel{(2l+2)}\cancel{(2l+3)}}}$$

$$\times \sqrt{\frac{\cancel{(2l+1)}\cancel{(2l+2)}\cancel{(2l+3)}}{2}} \qquad \frac{1}{l+1}$$

Because $l' = l+1$ and $j = j'$, $j = l + 1/2$

$$= \frac{(-1)^{l+1}}{\sqrt{6}\,(l+1)} \sqrt{(2l+3)(1)(2l+1)(1)}$$

$$= \frac{(-1)^{l+1}}{\sqrt{6}\,(l+1)} \sqrt{(2l+3)(2l+1)} = \frac{(-1)^{l'} \sqrt{(2l'+1)(2l+1)}}{(l')\,\sqrt{6}}$$

For $\ell' = \ell - 1$

$$\begin{Bmatrix} j & 1/2 & \ell' \\ 1 & \ell & 1/2 \end{Bmatrix} \overset{\text{by } 6.2.4}{=} \begin{Bmatrix} j & \ell' & 1/2 \\ 1 & 1/2 & \ell \end{Bmatrix} \overset{\text{by } 6.2.5}{=} \begin{Bmatrix} j & 1/2 & \ell \\ 1 & \ell' & 1/2 \end{Bmatrix}$$

so

$$\begin{Bmatrix} j & 1/2 & \ell-1 \\ 1 & \ell & 1/2 \end{Bmatrix} = \begin{Bmatrix} j & 1/2 & \ell \\ 1 & \ell-1 & 1/2 \end{Bmatrix}$$

And we again use the previous result taking $\ell' \to \ell-1$ and $j \to \ell - 1/2$ i.e. $a = \ell - 1/2$ $b = 1/2$ $c = \ell$ $s = 2\ell$ in Table 5 line 2

$$\begin{Bmatrix} \ell-1/2 & 1/2 & \ell \\ 1 & \ell-1 & 1/2 \end{Bmatrix} = \cancel{(-1)^{2\ell}} \sqrt{\frac{\cancel{2}\cancel{(2\ell+1)}(1)\cancel{(2\ell-1)}(1)}{1\cancel{(2)}\,3\,\cancel{(2\ell-1)}\,2\ell\cancel{(2\ell+1)}}}$$

$$= \sqrt{\frac{1}{6\ell}}$$

$$\begin{Bmatrix} \ell-1/2 & 1/2 & \ell \\ 1 & \ell-1 & 1/2 \end{Bmatrix} \Big/ \begin{pmatrix} \ell-1 & 1 & \ell \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \sqrt{\frac{1}{6\ell}} \sqrt{\frac{(2\ell+1)(2\ell-1)}{(2\ [illegible]\ \ell}} \quad (-1)^{\ell}$$

$$= \frac{(-1)^{\ell}}{\sqrt{6}\,\ell} \sqrt{(2\ell+1)(2\ell'+1)}$$

So combining everything for $l' = l+1$

$\langle n'(l's')j'j_z' | \vec{q}\cdot\vec{\sigma} | n(ls) j j_z\rangle =$

$$(-1)^{\lambda+l+1}\, \delta_{jj'}\, \delta_{j_z j_z'}\, (-1)^{l'} \cancel{\sqrt{6}}\, (-i)(-1)^{l'} \frac{\sqrt{(2l'+1)(2l+1)}}{\cancel{\sqrt{6}}\; l'}$$

$$\times \langle n'l'0|\nabla_0|nl0\rangle$$

$$= i\,\delta_{jj'}\,\delta_{j_zj_z'} \frac{\sqrt{(2l'+1)(2l+1)}}{l'}\langle n'l'0|\nabla_0|nl0\rangle$$

Using Section 5.7 results

$$= i\,\delta_{jj'}\,\delta_{j_zj_z'} \frac{\cancel{\sqrt{(2l'+1)(2l+1)}}}{\cancel{l'}}\; \frac{\cancel{l+1}}{\cancel{\sqrt{(2l+1)(2l+3)}}} \int_0^\infty dr\, r^2 R_{n'l+1}(r)\left(\frac{\partial}{\partial r}-\frac{l}{r}\right)R_{nl}(r)$$

$$= i\,\delta_{jj'}\,\delta_{j_zj_z'}\int_0^\infty r^2dr\, R_{n'l+1}(r)\left(\frac{\partial}{\partial r} - \frac{l}{r}\right) R_{nl}(r)$$

For $l' = l-1$, $j = l - 1/2$

$\langle n'(l's')j'j_z'|\vec{q}\cdot\vec{\sigma}|n(ls)jj_z\rangle =$

$$(-1)^{2l}\,\delta_{jj'}\,\delta_{j_zj_z'}\, \frac{(-1)^{l'}(-1)^{l}}{\cancel{\sqrt{6}}\;\cancel{l}}\, \cancel{\sqrt{6}}\, \cancel{\sqrt{(2l+1)(2l'+1)}}$$

$$\times \frac{\cancel{l}}{\cancel{\sqrt{(2l-1)(2l+1)}}}\,(-i)\int_0^\infty r^2dr\, R_{n'l-1}(r)\left(\frac{\partial}{\partial r} + \frac{l+1}{r}\right)R_{nl}(r)$$

$$= i\,\delta_{jj'}\,\delta_{j_zj_z'}\int r^2dr\, R_{n'l-1}(r)\left(\frac{\partial}{\partial r} + \frac{l+1}{r}\right)R_{nl}(r)$$

$$= i\delta_{jj'}\,\delta_{j_z j_z'}\,\frac{\sqrt{(2\ell'+1)(2\ell+1)}}{\ell}\,\langle n'\ell 0|\nabla_0|n\ell 0\rangle$$

For $\ell' = \ell+1$ the denominator is

$$\ell' = \ell+1 = \frac{\ell+\ell'+1}{2}$$

For $\ell' = \ell - 1$ we have $\ell = \ell'+1 = \dfrac{\ell+\ell'+1}{2}$

so we can write

$$\langle n'(\ell' s')j' j_z'|\vec{\sigma}\cdot\vec{q}\,|n(\ell s)j\,j_z\rangle =$$

$$i\,\delta_{jj'}\,\delta_{j_z j_z'}\,\frac{\sqrt{(2\ell'+1)(2\ell+1)}}{\left(\frac{\ell+\ell'+1}{2}\right)}\,\langle n'\ell' 0|\nabla_0|n\ell 0\rangle$$

$$= \delta_{jj'}\,\delta_{j_z j_z'}\,\Gamma(n'\ell';n\ell)$$

where

$$\langle n'\ell' 0|\nabla_0|n\ell 0\rangle = \frac{\frac{\ell+\ell'+1}{2}}{\sqrt{(2\ell'+1)(2\ell+1)}}\int_0^\infty r^2 dr\, R_{n'\ell'}(r)\begin{cases}\frac{\partial}{\partial r} - \frac{\ell}{r} & \ell' = \ell+1\\[4pt] \frac{\partial}{\partial r} + \frac{\ell+1}{r} & \ell' = \ell-1\end{cases}$$

$$\times R_{n\ell}(r)$$

$$= \frac{(\ell+\ell'+1)/2\,(-i)\,\Gamma(n'\ell';n\ell)}{\sqrt{(2\ell'+1)(2\ell+1)}}$$

Now for the case of 2 identical particles
(suppressing $j_z$)

$$\langle n'(l's')j_j'\; N'L';(j'L')J' \,|\, \vec{g}\cdot\vec{\sigma} \,|\, n(ls)j_j\; NL;(jL)J \rangle$$

Edmonds 7.1.6

$$= \underbrace{\delta_{NN'}\,\delta_{LL'}}_{\text{does not act on these}}\;\delta_{jj'}\;\underbrace{\delta_{JJ'}}_{\text{If } j \text{ and } L \text{ are diagonal}}\;(-1)^{l+s'+j}\begin{Bmatrix} j & s' & l' \\ 1 & l & s \end{Bmatrix}\langle n'l'\|g\|nl\rangle\,\langle ns'\|\sigma\|ns\rangle$$

~~this must be~~

The rme of $g$ is already determined, but the $6j$ and rme of $\sigma$ have changed from the 1-body case.

$$\langle ns'\|\sigma\|ns\rangle = \langle s'\|\sigma_1-\sigma_2\|s\rangle = \langle \tfrac{1}{2}\tfrac{1}{2}\,s'\|\,\sigma_1-\sigma_2\,\|\tfrac{1}{2}\tfrac{1}{2}\,s\rangle$$

By Edmonds 7.1.7 and 7.1.8

$$= (-1)^{1/2+1/2+s+1}\left[(2s'+1)(2s+1)\right]^{1/2}\begin{Bmatrix} 1/2 & s' & 1/2 \\ s & 1/2 & 1 \end{Bmatrix}\langle 1/2\|\sigma\|1/2\rangle$$

$$-(-1)^{s'}\left[(2s'+1)(2s+1)\right]^{1/2}\begin{Bmatrix} 1/2 & s' & 1/2 \\ s & 1/2 & 1 \end{Bmatrix}\langle 1/2\|\sigma\|1/2\rangle$$

$$= \left[(-1)^{s}-(-1)^{s'}\right]\sqrt{6(2s'+1)(2s+1)}\begin{Bmatrix} 1/2 & s' & 1/2 \\ s & 1/2 & 1 \end{Bmatrix}$$

The possible values of $s, s'$ are $0, 1$. Mathematica says that the $6j$ is nonzero only when $s'\neq s$, in which case it $= 6^{-1/2}$. We thus rewrite the above as

$$= (s'-s)\;2\sqrt{3}$$

So

$$\langle n'(l's')j'; N'L'; (j'L')J' | \vec{q}\cdot\vec{\sigma} | n(ls)j; NL; (jL)J \rangle$$

$$= \delta_{NN'}\delta_{LL'}\delta_{jj'}\delta_{JJ'} (-1)^{l+s'+j} \begin{Bmatrix} j & s' & l' \\ 1 & l & s \end{Bmatrix} (s'-s)\, 2\sqrt{3}\, \langle n'l' \| q \| nl \rangle$$

Now since $j'=j$, and $|l'-l|=1$ can we say anything?

| $s$ | $s'$ | $l'$ | $j$ | $(-1)^{l+s'+j}$ |
|---|---|---|---|---|
| 0 | 1 | $l+1$ | $l$ | $(-1)^{l+1+l}$ |
| 1 | 0 | $l+1$ | $l+1$ | $(-1)^{l+l+1}$ |
| 0 | 1 | $l-1$ | $l$ | $(-1)^{l+1+l}$ |
| 1 | 0 | $l-1$ | $l-1$ | $(-1)^{l+l-1}$ |

$j = l^{s'} l'^{s} = l\cdot s' + l'\cdot s$

Playing with Mathematica a bit I can see that

$$\begin{Bmatrix} j & s' & l' \\ 1 & l & s \end{Bmatrix} = \frac{1}{\sqrt{3}} \frac{1}{\sqrt{2j+1}}$$

Therefore the original ME is

$$\delta_{NN'}\delta_{LL'}\delta_{jj'}\delta_{JJ'} (-1)^{l+s'+j} \frac{2(s'-s)}{\sqrt{2j+1}} \langle n'l' \| q \| nl \rangle$$

$$= \delta_{NN'}\delta_{LL'}\delta_{jj'}\delta_{JJ'} (-1)^{l+s'+j} \frac{2(s'-s)}{\sqrt{2j+1}} (-i) \langle n'l' \| \nabla_0 \| nl \rangle$$

$$= \delta_{NN'}\delta_{LL'}\delta_{jj'}\delta_{JJ'} \frac{+i\, 2(s'-s)}{\sqrt{2j+1}} \frac{(-1)^{l'}}{\begin{pmatrix} l' & 1 & l \\ 0 & 0 & 0 \end{pmatrix}} \langle n'l'0 | \nabla_0 | nl0 \rangle$$

From (2)

$$\begin{pmatrix} l-1 & 1 & l \\ 0 & 0 & 0 \end{pmatrix} = (-1)^l \sqrt{\frac{l}{(2l+1)(2l-1)}}$$

$$\begin{pmatrix} l+1 & 1 & l \\ 0 & 0 & 0 \end{pmatrix} = (-1)^{l+1} \sqrt{\frac{(l+1)}{(2l+1)(2l+3)}}$$

So

$$\langle n'(l's')j';N'L';(j'L')J' | \vec{\sigma}\cdot\vec{q} | n(ls)j;NL;(jL)J \rangle$$

$$= \delta_{NN'}\delta_{LL'}\delta_{jj'}\delta_{JJ'}\; 2i(s-s')(-1)^{l'}(-1)^{(l+l'+1)/2} \sqrt{\frac{(2l+1)(2l'+1)}{(2j+1)\left(\frac{l+l'+1}{2}\right)}}$$

$$\langle n'l'0 | \nabla_0 | nl0 \rangle$$

$$= \delta_{NN'}\delta_{LL'}\delta_{jj'}\delta_{JJ'}\; 2^{3/2} i(s-s')(-1)^{\frac{l-l'+1}{2}} \sqrt{\frac{(2l+1)(2l'+1)}{(2j+1)(l+l'+1)}}$$

$$\langle n'l'0 | \nabla_0 | nl0 \rangle$$

$$= \delta_{NN'}\delta_{LL'}\delta_{jj'}\delta_{JJ'}\; (s'-s)(-1)^{l-l'+1/2} \sqrt{\frac{2(l+l'+1)}{(2j+1)}}\; \Gamma(n'l';nl)$$

To calculate matrix elements of $\vec{\Sigma}\cdot\vec{Q}$ we make a change of basis

Since $\delta_{JJ'}$ we can assume $J = J'$ immediately

$$\langle n'(l's')j'; N'L'; (j'L')J \,|\, \vec{\Sigma}\cdot\vec{Q} \,|\, n(ls)j; NL; (jL)J \rangle$$

$$= \sum_{\mathcal{J}\mathcal{J}'} \langle n'(l's')j'; N'L'; (j'L')J \,|\, n'l'; N'(L's')\mathcal{J}'; (\mathcal{J}'l')J \rangle$$

$$\langle n'l'; N'(L's')\mathcal{J}'; (l'\mathcal{J}')J \,|\, \vec{\Sigma}\cdot\vec{Q} \,|\, nl; N(Ls)\mathcal{J}; (l\mathcal{J})J \rangle$$

$$\langle nl; N(Ls)\mathcal{J}; (l\mathcal{J})J \,|\, n(ls)j; NL; (jL)J \rangle$$

$$= \sum_{\mathcal{J}} \sum_{\mathcal{J}'} \sqrt{(2j'+1)(2\mathcal{J}'+1)}\, (-1)^{l'+s'+L'+J'} \begin{Bmatrix} l' & s' & j' \\ L' & J & \mathcal{J}' \end{Bmatrix}$$

$$\langle n'l'; N'(L's')\mathcal{J}'; (l'\mathcal{J}')J \,|\, \vec{\Sigma}\cdot\vec{Q} \,|\, nl; N(Ls)\mathcal{J}; (l\mathcal{J})J \rangle$$

$$\sqrt{(2j+1)(2\mathcal{J}+1)}\, (-1)^{l+s+L+J} \begin{Bmatrix} l & s & j \\ L & J & \mathcal{J} \end{Bmatrix}$$

Using Edmonds 7.1.6 yet again

$$\langle n'l'; N'(L's')\mathcal{J}'; (l'\mathcal{J}')J \,|\, \vec{\Sigma}\cdot\vec{Q} \,|\, nl; N(Ls)\mathcal{J}; (l\mathcal{J})J \rangle$$

$$= (-1)^{L+s'+\mathcal{J}} \delta_{\mathcal{J}\mathcal{J}'} \delta_{\mathcal{J}_z\mathcal{J}_z'} \begin{Bmatrix} \mathcal{J} & s' & L' \\ 1 & L & s \end{Bmatrix} \sum_{N''} \langle N'L' \| Q \| N''L \rangle \langle N''s' \| \Sigma \| Ns \rangle$$

$$\times\, \delta_{nn'} \delta_{ll'} \delta_{JJ'}$$

Plugging this into the full ME

$$= \sum_{\mathcal{J}} (2\mathcal{J}+1) \sqrt{(2j+1)(2j'+1)}\, (-1)^{s+s'+L+L'} \begin{Bmatrix} l & s' & j' \\ L' & J & \mathcal{J} \end{Bmatrix}$$

$$\begin{Bmatrix} l & s & j \\ L & J & \mathcal{J} \end{Bmatrix} (-1)^{L+s'+\mathcal{J}} \delta_{nn'} \delta_{ll'} \delta_{JJ'} \begin{Bmatrix} \mathcal{J} & s' & L' \\ 1 & L & s \end{Bmatrix} \langle N'L' \| Q \| NL \rangle \times \langle Ns \| \Sigma \| Ns \rangle$$

$$= (-1)^{s+\cancel{s'}+\cancel{L}+L'+\cancel{L}+\cancel{s'}} \sqrt{(2j+1)(2j'+1)}\, \delta_{nn'}\delta_{\ell\ell'}\delta_{JJ'}$$

$$\overset{= (-1)^{s+L'}}{}$$

$$\sum_{\mathcal{J}} (-1)^{\mathcal{J}} \begin{Bmatrix} \ell & s' & j' \\ L' & J & \mathcal{J} \end{Bmatrix} \begin{Bmatrix} \ell & s & j \\ L & J & \mathcal{J} \end{Bmatrix} \begin{Bmatrix} \mathcal{J} & s' & L' \\ 1 & L & s \end{Bmatrix}$$

$$\langle NL' \| Q \| NL \rangle \langle s' \| \Sigma \| s \rangle$$

For the spin RME, in analogy to that of $\sigma$

$$\langle s' \| \Sigma \| s \rangle = \langle s' \| \sigma_1 + \sigma_2 \| s \rangle$$

by E 7.1.7, 7.1.8, 5.4.4

$$= [(-1)^s + (-1)^{s'}] \sqrt{6(2s+1)(2s'+1)} \begin{Bmatrix} 1/2 & s' & 1/2 \\ s & 1/2 & 1 \end{Bmatrix}$$

The first term requires $s'=s$, while the triangle inequalities for the 6j symbol disallow $s=s'=0$. Therefore $s=s'=1$ or the RME is zero.

$$\langle s' \| \Sigma \| s \rangle = \delta_{s,1}\delta_{s',1} \; -2\sqrt{6 \cdot \cancel{3} \cdot \cancel{3}} \cdot \frac{-1}{\cancel{3}}$$

$$= +\, 2\sqrt{6}\, \delta_{s,1}\delta_{s',1}$$

While $\langle NL' \| Q \| NL \rangle$ is the same as the $\vec{\sigma}$ case

Overall ME is now

$$= (-1)^{1+L'}\, 2\sqrt{6(2j+1)(2j'+1)}\, \delta_{nn'}\delta_{\ell\ell'}\delta_{JJ'}\delta_{s,1}\delta_{s',1}$$

$$\langle N'L' \| Q \| NL \rangle \sum_{\mathcal{J}=\max(L-1,L'-1)}^{\min(L+1,L'+1)} (-1)^{\mathcal{J}} (2\mathcal{J}+1) \begin{Bmatrix} \ell & 1 & j' \\ L' & J & \mathcal{J} \end{Bmatrix} \begin{Bmatrix} \ell & 1 & j \\ L & J & \mathcal{J} \end{Bmatrix} \begin{Bmatrix} \mathcal{J} & 1 & L' \\ 1 & L & 1 \end{Bmatrix}$$

For the Rashba term,

5.1.8 $$V_R \propto (\sigma_x k_y - \sigma_y k_x) = [\vec{\sigma} \times \vec{k}]_z = -i\sqrt{2}\,[\sigma \otimes k]_{10}$$

For two particles

$$[\vec{\sigma}_1 \times \vec{k}_1]_z + [\vec{\sigma}_2 \times \vec{k}_2]_z = [(\vec{\sigma}_1 - \vec{\sigma}_2)\times(\vec{k}_1 - \vec{k}_2)]_z/2 + [(\vec{\sigma}_1 + \vec{\sigma}_2)\times(k_1 + k_2)]_z/2$$

$$= ([\vec{\sigma}_1 \times \vec{k}_1]_z - \cancel{[\sigma_2 \times k_1]_z} - \cancel{[\sigma_1 \times k_2]_z} + [\sigma_2 \times k_2]_z + [\sigma_1 \times k_1]_z + \cancel{[\sigma_2 \times k_1]_z} + \cancel{[\sigma_1 \times k_2]_z} + [\sigma_2 \times k_2]_z)/2$$

$$= \frac{1}{\sqrt{2}}\left([\vec{\sigma} \times \vec{q}]_z + [\vec{\Sigma} \times \vec{Q}]_z\right)$$

$$= -i\left([\vec{\sigma} \otimes \vec{q}]_{10} + [\vec{\Sigma} \otimes \vec{Q}]_{10}\right) = i\left([q \otimes \sigma]_{10} + [Q \otimes \Sigma]_{10}\right)$$

For the relative coordinate part we first apply Wigner-Eckart

$$-i\langle n'(l's')j';N'L';(j'L')J'J_z' | [\sigma \otimes q]_{10} | n(ls)j;NL;(jL)JJ_z\rangle$$

$$= -i(-1)^{J'-J_z'}\begin{pmatrix} J' & 1 & J \\ -J_z' & 0 & J_z \end{pmatrix} \langle n'(l's')j';N'L';(j'L')J' \| [\sigma \otimes q]_{10} \| n(ls)j;NL;(jL)J\rangle$$

Edmonds 7.1.7 $$= i(-1)^{J'-J_z'+1}\begin{pmatrix} J' & 1 & J \\ -J_z' & 0 & J_z \end{pmatrix}(-1)^{j'+L+J+1}\sqrt{(2J+1)(2J'+1)}$$

$$\begin{Bmatrix} j' & J' & L \\ J & j & 1 \end{Bmatrix} \langle n'(l's')j' \| [\sigma \otimes q]_{10} \| n(ls)j\rangle$$

Note that $[\sigma\otimes q]_{10} = -[q\otimes\sigma]_{10}$ and apply Edmonds 7.1.5

$$= i(-1)^{J+J'-J_z'+j'+L+1}\begin{pmatrix} J' & 1 & J\\ -J_z' & 0 & J_z\end{pmatrix}\sqrt{(2j+1)(2j'+1)(2J+1)(2J'+1)3}$$
$$\begin{Bmatrix} j' & J' & L\\ J & j & 1\end{Bmatrix}\begin{Bmatrix} l' & l & 1\\ s' & s & 1\\ j' & j & 1\end{Bmatrix}\langle n'l'\|q\|nl\rangle\langle s'\|\sigma\|s\rangle$$

We have already derived these reduced matrix elements. Try to simplify the 3j symbol:

$$\begin{pmatrix} J' & 1 & J\\ -J_z' & 0 & J_z\end{pmatrix} = \delta_{J_z J_z'}\begin{pmatrix} J' & 1 & J\\ -J_z & 0 & J_z\end{pmatrix} = \delta_{J_z J_z'}\begin{pmatrix} J & J' & 1\\ J_z & -J_z & 0\end{pmatrix}$$

From Edmonds Table 2:

For $J' = J-1 \Rightarrow = \delta_{J_z J_z'}(-1)^{J'-J_z-1}\sqrt{\dfrac{2(J+J_z)(J-J_z)}{(2J+1)2J(2J-1)}}$

$J' = J \Rightarrow = \delta_{J_z J_z'}(-1)^{J-J_z}\dfrac{J_z}{\sqrt{(2J+1)(J+1)J}}$

$J' = J+1 \Rightarrow = \delta_{J_z J_z'}(-1)^{J+J'+1}(-1)^{J+J_z-1}\sqrt{\dfrac{(J+J_z+1)(J-J_z+1)}{(2J+3)(2J+2)(2J+1)}}$

$0$ if $|J-J'|>1$

Doesn't really simplify... Replacing the spin RME we get

$$\boxed{= 6i(-1)^{J+J'-J_z'+j'+L+1}\delta_{J_z J_z'}\sqrt{(2J+1)(2J'+1)(2j+1)(2j'+1)}\begin{pmatrix} J' & 1 & J\\ -J_z & 0 & J_z\end{pmatrix}\begin{Bmatrix} j' & J' & L\\ J & j & 1\end{Bmatrix}\begin{Bmatrix} l' & l & 1\\ s' & s & 1\\ j' & j & 1\end{Bmatrix}(s'-s)\langle n'l'\|q\|nl\rangle}$$

For the CM SOC we again re-expand using Edmonds 6.1.5 (with $\delta J=0$ now, unfortunately)

$$\langle n'(l's')j';N'L';(j'L')J'|\,i[Q\otimes\Sigma]_{10}\,|n(ls)j;NL;(jL)J\rangle$$

$$= i\sum_{\mathcal{J}\mathcal{J}'}\sqrt{(2j+1)(2j'+1)(2\mathcal{J}+1)(2\mathcal{J}'+1)}\,(-1)^{l+l'+s+s'+L+L'+\mathcal{J}+\mathcal{J}'}$$

$l+l'+s+s'$ = even

$$\begin{Bmatrix} l' & s' & j' \\ L' & J' & \mathcal{J}' \end{Bmatrix}\begin{Bmatrix} l & s & j \\ L & J & \mathcal{J} \end{Bmatrix}\langle n'l';N'(L's')\mathcal{J}';(l'\mathcal{J}')J'|[Q\otimes\Sigma]_{10}|nl;N(Ls)\mathcal{J};(l\mathcal{J})J\rangle$$

$$= i(-1)^{L+L'+\mathcal{J}+\mathcal{J}'}\sqrt{(2j+1)(2j'+1)}\sum_{\mathcal{J}\mathcal{J}'}\sqrt{(2\mathcal{J}+1)(2\mathcal{J}'+1)}$$

$$\begin{Bmatrix} l' & s' & j' \\ L' & J' & \mathcal{J}' \end{Bmatrix}\begin{Bmatrix} l & s & j \\ L & J & \mathcal{J} \end{Bmatrix}\langle n'l';N'(L's')\mathcal{J}';(l'\mathcal{J}')J'|[Q\otimes\Sigma]_{10}|nl;N(Ls)\mathcal{J};(l\mathcal{J})J\rangle$$

Edmonds 5.4.1 + 7.1.8

$$= i(-1)^{L+L'+\mathcal{J}+\mathcal{J}'}\sqrt{(2j+1)(2j'+1)}\,(-1)^{J'-J_z'}\delta_{J_zJ_z'}\begin{pmatrix} J' & 1 & J \\ -J_z & 0 & J_z \end{pmatrix}$$

$$\sum_{\mathcal{J}\mathcal{J}'}\sqrt{(2\mathcal{J}+1)(2\mathcal{J}'+1)}\begin{Bmatrix} l' & s' & j' \\ L' & J' & \mathcal{J}' \end{Bmatrix}\begin{Bmatrix} l & s & j \\ L & J & \mathcal{J} \end{Bmatrix}(-1)^{l+\mathcal{J}+J'+1}\delta_{ll'}\delta_{nn'}$$

$$\sqrt{(2J+1)(2J'+1)}\begin{Bmatrix} \mathcal{J}' & J' & l \\ J & \mathcal{J} & 1 \end{Bmatrix}\langle N'(L's')\mathcal{J}'\|[Q\otimes\Sigma]_{10}\|N(Ls)\mathcal{J}\rangle$$

$2J$ = even
Use Edmonds 7.1.5

$$= i(-1)^{L+L'+J+\mathcal{J}'+l+1-J_z}\sqrt{(2j+1)(2j'+1)(2J+1)(2J'+1)}$$

$$\delta_{ll'}\delta_{nn'}\delta_{J_zJ_z'}\begin{pmatrix} J' & 1 & J \\ -J_z & 0 & J_z \end{pmatrix}\sum_{\mathcal{J}\mathcal{J}'}(-1)^{\mathcal{J}}\sqrt{(2\mathcal{J}+1)(2\mathcal{J}'+1)}\begin{Bmatrix} l' & s' & j' \\ L' & J' & \mathcal{J}' \end{Bmatrix}$$

$$\begin{Bmatrix} l & s & j \\ L & J & \mathcal{J} \end{Bmatrix}\begin{Bmatrix} \mathcal{J}' & J' & l \\ J & \mathcal{J} & 1 \end{Bmatrix}\sqrt{(2\mathcal{J}+1)(2\mathcal{J}'+1)3}\begin{Bmatrix} L' & L & 1 \\ s' & s & 1 \\ \mathcal{J}' & \mathcal{J} & 1 \end{Bmatrix}$$

$$\langle N'L'\|Q\|NL\rangle\langle s'\|\Sigma\|s\rangle$$

Just rewriting for clarity + subbing for RME of $\Sigma$

$$\langle n'(l's')j';N'L';(j'L')J' | i[Q\otimes\Sigma]_{10} | n(ls)j;NL;(jL)J\rangle$$

$$= i(-1)^{J+J'+L+L'+l+1-J_z}\sqrt{3(2j+1)(2j'+1)(2J+1)(2J'+1)}\,\delta_{ll'}\delta_{nn'}\delta_{J_zJ_z'}$$

$$\times\begin{pmatrix} J' & 1 & J \\ -J_z & 0 & J_z\end{pmatrix}\sum_{\mathcal{J}\mathcal{J}'}(-1)^{\mathcal{J}}(2\mathcal{J}+1)(2\mathcal{J}'+1)\begin{Bmatrix} l & s' & j' \\ L' & J' & \mathcal{J}'\end{Bmatrix}\begin{Bmatrix} l & s & j \\ L & J & \mathcal{J}\end{Bmatrix}$$

$$\times\begin{Bmatrix} L' & L & 1 \\ s' & s & 1 \\ \mathcal{J}' & \mathcal{J} & 1\end{Bmatrix}\delta_{s,1}\delta_{s',1}\,2\sqrt{6}\,\langle N'L'\|Q\|NL\rangle\begin{Bmatrix} \mathcal{J}' & J' & l \\ J & \mathcal{J} & 1\end{Bmatrix}$$

Note $L+L'$ odd from $Q$ RME

$$\boxed{\begin{aligned} &= i(-1)^{J+J'+l-J_z}\,6\sqrt{2}\,\delta_{ll'}\delta_{nn'}\delta_{s,1}\delta_{s',1}\delta_{J_zJ_z'}\begin{pmatrix} J' & 1 & J \\ -J_z & 0 & J_z\end{pmatrix} \\ &\times\sqrt{(2j+1)(2j'+1)(2J+1)(2J'+1)}\,\langle N'L'\|Q\|NL\rangle \\ &\times\sum_{\mathcal{J}}\sum_{\mathcal{J}'}(-1)^{\mathcal{J}}(2\mathcal{J}+1)(2\mathcal{J}'+1)\begin{Bmatrix} \mathcal{J}' & J' & l \\ J & \mathcal{J} & 1\end{Bmatrix}\begin{Bmatrix} l & 1 & j' \\ L' & J' & \mathcal{J}'\end{Bmatrix}\begin{Bmatrix} l & 1 & j \\ L & J & \mathcal{J}\end{Bmatrix}\begin{Bmatrix} L' & L & 1 \\ 1 & 1 & 1 \\ \mathcal{J}' & \mathcal{J} & 1\end{Bmatrix}\end{aligned}}$$