LLNL-JRNL-????

One, two, and three fermions with one-body parity-violating 'spin-orbit' coupling

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Abstract

I present my notes on various fermionic systems undergoing some type of external parity-violating interaction in different dimensions.

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Contents

I. PROBLEM SETUP

In these notes I consider particles undergoing a one-body, parity-violating interaction of the type

$$
\begin{aligned}
&\alpha\, \vec{k}\cdot\vec{\sigma} && && (1)\\
&\alpha_R\,(\sigma_x k_y - \sigma_y k_x) && \text{Rashba} && (2)\\
&-\alpha_{D1}\,(\sigma_x k_x - \sigma_y k_y) && \text{Dresselhaus (linear)} && (3)\\
&\alpha_{D3}\left(\sigma_x k_x k_y^2 - \sigma_y k_x^2 k_y\right) && \text{Dresselhaus (cubic).} && (4)
\end{aligned}
$$

I consider the particles interacting in free space as well as within an oscillator well, both for 2D and 3D systems. I also consider the case where the particles interact via a short-ranged (contact), s-wave interaction,

$$
\frac{4\pi a(\Lambda)}{m}\delta^R|\vec{k}_1;\vec{k}_2\rangle = \frac{4\pi a(\Lambda)}{m} f(|\vec{k}_1-\vec{k}_2|/\Lambda)|\vec{k}_1;\vec{k}_2\rangle \tag{5}
$$

where $f(x)$ refers to some regulator and $a(\Lambda)$ is tuned to produce the s-wave scattering length and is dependent on the form of the regulator $f(x)$.

II. $\vec{k}\cdot\vec{\sigma}$ IN A 3-DIMENSIONAL OSCILLATOR WELL

A. 1-particle spectrum

The one-body Hamiltonian is[1]

$$
\begin{aligned}
H &= \frac{\vec{k}^2}{2m} + \frac{1}{2}m\omega^2 r^2 + \alpha\vec{k}\cdot\vec{\sigma} && (6)\\
&= H_0 + \alpha\vec{k}\cdot\vec{\sigma}\ . && (7)
\end{aligned}
$$

Using the following definitions,

$$
\begin{aligned}
\vec{q} &= b\vec{k} && (8)\\
\vec{x} &= \vec{r}/b && (9)\\
b &= \frac{1}{\sqrt{m\omega}}\ , && (10)
\end{aligned}
$$

we can express eq. 6 as

$$
\begin{aligned}
H &= \omega\left[\frac{\vec{q}^2}{2} + \frac{\vec{x}^2}{2} + \tilde{\alpha}\vec{q}\cdot\vec{\sigma}\right] && (11)\\
&= \omega\left[h_0 + \tilde{\alpha}\vec{q}\cdot\vec{\sigma}\right]\ , && (12)
\end{aligned}
$$

and $\tilde{\alpha} = \alpha\sqrt{m/\omega}$ is dimensionless. In fact, the only dimensional term in the equation above is ω.

[1] I set $\hbar = c = 1$.

1. *Basis states*

We know the eigenvalues and eigenfunctions for h_0–they correspond to the harmonic oscillator and are given by

$$h_0|n(ls)j\ j\rangle = |n(ls)j\ j_z\rangle\ \ (2n + l + 3/2)\ , \tag{13}$$

where s is the spin of the particle and I have chosen to work in a basis where I couple orbital angular momentum l with spin s to make total angular momentum j, i.e.

$$|(ls)j\ j_z\rangle \equiv \sum_{l_z,s_z} \langle l, l_z\ ; s, s_z||j\ j_z\rangle\ \ |l\ l_z\rangle\ |s\ s_z\rangle\ , \tag{14}$$

where the first term on the right-hand side is a Clebsch-Gordan coefficient.

2. *Matrix elements of $\alpha\vec{q}\cdot\vec{\sigma}$*

To solve for the one-body spectrum of this Hamiltonian, I will simply do a brute force large matrix diagonalization to get at the eigenvalues. For this calculation I need to know the following matrix elements:

$$\langle n'(l's')j'j'_z|\tilde{\alpha}\vec{q}\cdot\vec{\sigma}|n(ls)jj_z\rangle\ . \tag{15}$$

There are many *classic* (i.e. old) textbooks on angular momentum that explain how to calculate matrix elements of this form. I use Edmonds [1], and in particular, chapter 7 of Edmonds. Equation 15 becomes

$$\langle n'(l's')j'j'_z|\tilde{\alpha}\vec{q}\cdot\vec{\sigma}|n(ls)jj_z\rangle = \alpha\delta_{j,j'}\delta_{j_z,j'_z}\delta_{s,s'}(-)^{l+s'+j}\begin{Bmatrix} j & s' & l' \\ 1 & l & s \end{Bmatrix}\langle n'l'||q||nl\rangle\langle s||\sigma||s\rangle\ , \tag{16}$$

where $\langle n'l'||q||nl\rangle$ and $\langle s||\sigma||s\rangle$ are reduced matrix elements. Equation 16 shows that this matrix element is diagonal in total angular momentum j (as it should be since $\vec{q}\cdot\vec{\sigma}$ is a rank 0 object) and is also diagonal in spin s of the particle. For the time being, I will set $s = 1/2$, and from Edmonds eq. 5.4.4, we have

$$\langle\frac{1}{2}||\sigma||\frac{1}{2}\rangle = 2\sqrt{\frac{3}{2}}\ . \tag{17}$$

Also note that eq. 16, in addition to being diagnal in j_z, is independent of j_z. This is also due to the fact that this matrix element is a rank 0 object. Furthermore, this implies that for a given j, the spectrum is $2j + 1$ degenerate. Since the spectrum does not depend on j_z, I will omit this term within expressions from now on.

Taking the above points into account, eq. 16 becomes

$$\langle n'\left(l'\frac{1}{2}\right)j|\tilde{\alpha}\vec{q}\cdot\vec{\sigma}|n\left(l\frac{1}{2}\right)j\rangle = (-)^{l+1/2+j}\sqrt{6}\alpha\begin{Bmatrix} j & \frac{1}{2} & l' \\ 1 & l & \frac{1}{2} \end{Bmatrix}\langle n'l'||q||nl\rangle \tag{18}$$

As I show in app. **??**, the remaining reduced matrix element is

$$\langle n'l'||q||nl\rangle = \begin{cases} (-)^{l'}\sqrt{(2l'+1)(2l+1)}\begin{pmatrix} l' & 1 & l \\ 0 & 0 & 0 \end{pmatrix}\Gamma(n',l';n,l) & \text{if } |l'-l| = 1 \\ 0 & \text{otherwise} \end{cases}\ , \tag{19}$$

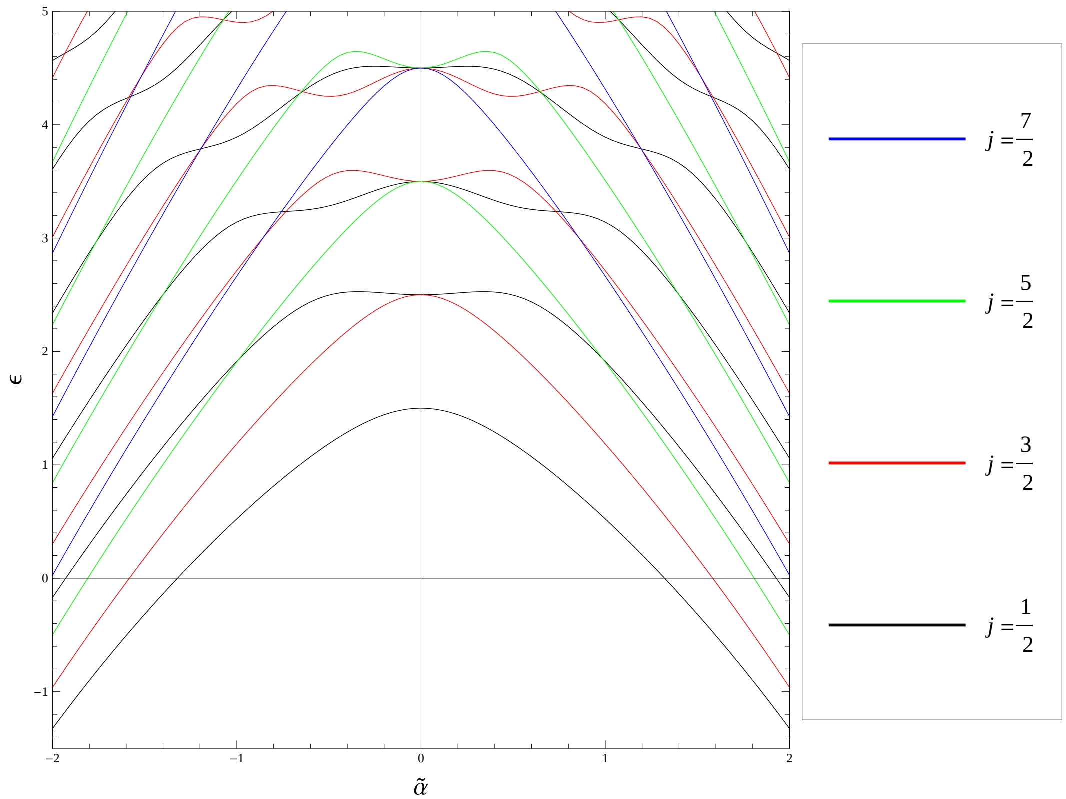

FIG. 1: (Color online) The one-body spectrum of spin-1/2 fermion in HO well with parity violating operator $\tilde{\alpha}\vec{q}\cdot\vec{\sigma}$. Here $\epsilon = E/\omega$. For each j, the spectrum is $2j+1$ degenerate. Note that for $\tilde{\alpha} = 0$, the HO spectrum (and its degeneracies) is recovered.

and $\Gamma(n', l'; n, l)$ is an expression (given in app. **??**) that depends on an integral over the HO radial wavefunctions $R_{nl}(q)$ and $R_{n'l'}(q)$. Not surprisingly, eq. 19 shows that this matrix element couples angular momentum of different parities.

3. Spectrum

Inserting eq. 19 into eq. 18, and using the explicit formulas for the $6j$- and $3j$-symbols (found for example, using Mathematica), eq. 18 undergoes a large amount of simplification:

$$\langle n' \left(l' \frac{1}{2} \right) j | \tilde{\alpha}\vec{q}\cdot\vec{\sigma} | n \left(l \frac{1}{2} \right) j \rangle = \begin{cases} -\tilde{\alpha}\Gamma(n', l'; n, l) & \text{if } |l' - l| = 1 \\ 0 & \text{otherwise} \end{cases} . \tag{20}$$

Therefore the matrix elements of the total Hamiltonian are

$$\langle n' \left(l' \frac{1}{2} \right) j | h_0 + \tilde{\alpha}\vec{q}\cdot\vec{\sigma} | n \left(l \frac{1}{2} \right) j \rangle = \begin{cases} 2n + l + 3/2 & \text{if } n' = n \text{ and } l' = l \\ -\tilde{\alpha}\Gamma(n', l'; n, l) & \text{if } |l' - l| = 1 \\ 0 & \text{otherwise} \end{cases} . \tag{21}$$

I have coded up eq. 21 and calculated the spectrum for different angular momenta j.

The results are shown in fig. 1. The calculation was done for a large N_{max}=50 cutoff[2], where I determined that the lowest eigenvalues (shown in fig. 1) had converged to the sixth significant digit (or higher).

I note that given the one-body spectrum in fig. 1, the spectrum of *many* non-interacting fermions can be deduced. In particular, for total angular momentum $J = 0$ cases, one needs only fill in all degenerate states to make a closed shell, and the energy per particle is simply

$$\frac{E_{J=0}(\alpha)}{N} = \frac{\sum_{j_{\rm shell}} g_j E(\alpha)_j}{\sum_{j_{\rm shell}} g_j} \tag{22}$$

where g_j is the degeneracy of the j^{th} shell and $N = \sum g_j$. Relevant for the next section is the ground state energy of two non-interacting spin-1/2 fermions, which is simply twice the lowest energy in fig. 1:

$$\frac{E_{J=0}(\alpha)}{\omega} = 2\epsilon_{j=1/2}(\alpha)\ . \tag{23}$$

B. 2-particle spectrum *without* relative interaction

As a check of eq. 23, I now solve explicitly the case of two equal mass m, non-interacting spin-1/2 fermions confined to a HO well with the $\alpha\vec{k}\cdot\vec{\sigma}$ external interaction. The Hamiltonian in this case is simply the sum of the one-body Hamiltonian for each particle,

$$H = \sum_{i=1}^{2}\left[\frac{\vec{k}_i^{\,2}}{2m} + \frac{1}{2}m\omega^2 r_i^2 + \alpha\vec{k}_i\cdot\vec{\sigma}_i\right] \tag{24}$$

$$= \sum_{i=1}^{2}\left[H_{0,i} + \alpha\vec{k}_i\cdot\vec{\sigma}_i\right]\ . \tag{25}$$

I now employ the standard trick of recasting the single-particle basis into relative (Jacobi) and center-of-mass (CM) coordinates, using the following transformations:

$$\vec{K} = \frac{\vec{k}_1 + \vec{k}_2}{\sqrt{2}} \tag{26}$$

$$\vec{k} = \frac{\vec{k}_1 - \vec{k}_2}{\sqrt{2}}\ , \tag{27}$$

for the CM and relative momenta, and

$$\vec{R} = \frac{\vec{r}_1 + \vec{r}_2}{\sqrt{2}} \tag{28}$$

$$\vec{r} = \frac{\vec{r}_1 - \vec{r}_2}{\sqrt{2}}\ , \tag{29}$$

for the CM and relative positions. Equation 24 can be expressed in terms of these coordinates,

$$H = \frac{1}{2m}\left(\vec{k}^2 + \vec{K}^2\right) + \frac{m\omega^2}{2}\left(\vec{r}^2 + \vec{R}^2\right) + \frac{\alpha}{\sqrt{2}}\left(\vec{k}\cdot\vec{\sigma} + \vec{K}\cdot\vec{\Sigma}\right)\ , \tag{30}$$

[2] For a given N_{max} cutoff, I only use HO states $|n(ls)j\rangle$ such that $2n + l \leq N_{max}$.

where I have made the following definitions:

$$\vec{\sigma} = \vec{\sigma}_1 - \vec{\sigma}_2 \tag{31}$$

$$\vec{\Sigma} = \vec{\sigma}_1 + \vec{\sigma}_2 \ . \tag{32}$$

I now move to dimensionless variables,

$$\vec{q} = b\vec{k} \tag{33}$$

$$\vec{Q} = b\vec{K} \tag{34}$$

$$\vec{x} = \vec{r}/b \tag{35}$$

$$\vec{X} = \vec{R}/b \tag{36}$$

$$\tilde{\alpha} = \alpha\sqrt{\frac{m}{\omega}} \tag{37}$$

$$b = \frac{1}{\sqrt{m\omega}} \ . \tag{38}$$

The Hamiltonian now becomes

$$H = \omega\left(h_{0,\mathrm{rel}} + \frac{\tilde{\alpha}}{\sqrt{2}}\vec{q}\cdot\vec{\sigma}\right) + \omega\left(h_{0,\mathrm{CM}} + \frac{\tilde{\alpha}}{\sqrt{2}}\vec{Q}\cdot\vec{\Sigma}\right) \ , \tag{39}$$

where

$$h_{0,\mathrm{rel}} = \frac{1}{2}\left(\vec{q}^2 + \vec{x}^2\right) \tag{40}$$

$$h_{0,\mathrm{CM}} = \frac{1}{2}\left(\vec{Q}^2 + \vec{X}^2\right) \ . \tag{41}$$

1. Basis states

There are many possible combinations of complete basis states to choose from. I use

$$|n(ls)j; NL; (jL)J\rangle \ ,$$

where $\{nl\}$ correspond to the relative HO quantum numbers, and $\{NL\}$ the CM HO quantum numbers. The total spin of the two fermions is denoted by s $(= s_1 + s_2)$. For two spin-1/2 particles, s can be either 0 or 1. The spin s is coupled with l to make angular momentum j. Angular momentum j and the CM angular momentum L are then coupled to make a state of total angular momentum J. For the same reasons as stated in the previous section, the interaction is independent of J_z, and so I omit this quantum number. There is another restriction due to Pauli exclusion, which states that $l + s$ must be even to enforce antisymmetry under exchange of particle 1 and 2.

As in the previous section, I do a simple brute force diagonalization to arrive at the two-body spectrum. Since all terms in the Hamiltonian are rank 0 tensors, the total angular momentum J is conserved. For a particular J, I enumerate all basis states up to N_{max} such that $2n+l+2N+L \leq N_{max}$, under the proviso that $l+s$ is even and the angular momentum couplings are realizable. As an example, I give all basis states for a (very small) $N_{max} = 3$ calculation for the $J = 0$ case in tab. I. Note that I include states of both parities, as I

TABLE I: Enumeration of two-body basis states $|i\rangle$ with total angular momentum $J = 0$ up to $N_{shell} = 3$, where $N_{shell} = 2n + l + 2N + L$. The parity of the state is given by $(-1)^{l+L}$.

$\|i\rangle$	$\|n(ls)j; NL; (jL)J\rangle$	N_{shell}	Parity
$\|1\rangle$	$\|0(00)0; 00; (00)0\rangle$	0	+
$\|2\rangle$	$\|0(11)0; 00; (00)0\rangle$	1	−
$\|3\rangle$	$\|0(11)1; 01; (11)0\rangle$	2	+
$\|4\rangle$	$\|0(00)0; 10; (00)0\rangle$	2	+
$\|5\rangle$	$\|1(00)0; 00; (00)0\rangle$	2	+
$\|6\rangle$	$\|0(11)2; 02; (22)0\rangle$	3	−
$\|7\rangle$	$\|0(11)0; 10; (00)0\rangle$	3	−
$\|8\rangle$	$\|1(11)0; 00; (00)0\rangle$	3	−

expect the matrix elements (derived below) to couple states of different parities. It is simple to devise an algorithm that will enumerate all basis states up to some N_{max} for a given J (I did it in Mathematica). I note that the dimensionality of the problem grows very fast with increasing N_{max}.

2. *Matrix Elements*

The matrix element of the oscillator part of the Hamiltonian is trivial,

$$\langle n'(l's')j'; N'L'; (j'L')J|h_{0,\mathrm{rel}} + h_{0,\mathrm{CM}}|n(ls)j; NL; (jL)J\rangle = \delta_{j,j'}\delta_{n,n'}\delta_{l,l'}\delta_{N,N'}\delta_{L,L'}\delta_{s,s'}\left(2n + l + 3/2 + 2N + L + 3/2\right) \ . \quad (42)$$

Let me now consider the matrix element

$$\langle n'(l's')j'; N'L'; (j'L')J|\frac{\tilde{\alpha}}{\sqrt{2}}\vec{q}\cdot\vec{\sigma}|n(ls)j; NL; (jL)J\rangle \ .$$

Again using Edmonds, I find that

$$\langle n'(l's')j'; N'L'; (j'L')J|\frac{\tilde{\alpha}}{\sqrt{2}}\vec{q}\cdot\vec{\sigma}|n(ls)j; NL; (jL)J\rangle = \delta_{N,N'}\delta_{L,L'}\delta_{j,j'}\frac{\tilde{\alpha}}{\sqrt{2}}(-1)^{l+s'+j}\begin{Bmatrix} j & s' & l' \\ 1 & l & s \end{Bmatrix}\langle n'l'||q||nl\rangle\langle s'||\sigma||s\rangle \ . \quad (43)$$

We already know the form of $\langle n'l'||q||nl\rangle$ which is given in eq. 19. Recall that this operator mixes states of different parity, essentially changing the angular momentum l to $l \pm 1$. For

$\langle s'||\sigma||s\rangle$, standard '*Racah*' algebra gives

$$\begin{aligned}
\langle s'||\sigma||s\rangle &= \langle \begin{pmatrix} \frac{1}{2}\frac{1}{2} \end{pmatrix} s'||\sigma_1 - \sigma_2|| \begin{pmatrix} \frac{1}{2}\frac{1}{2} \end{pmatrix} s\rangle \\
&= \left[(-1)^s - (-1)^{s'}\right] \sqrt{6(2s+1)(2s'+1)} \begin{Bmatrix} \frac{1}{2} & s' & \frac{1}{2} \\ s & \frac{1}{2} & 1 \end{Bmatrix} \qquad (44) \\
&= \begin{cases} 2\sqrt{3} & \text{for } s'=1,\ s=0 \\ -2\sqrt{3} & \text{for } s'=0,\ s=1 \\ 0 & \text{otherwise} \end{cases} . \qquad (45)
\end{aligned}$$

The equation above shows that this interaction flips the total spin of the two particles. This is a reassuring result for the following reason: our interaction respects the underlying symmetry of the two particles, i.e. two-particle wavefunctions that are antisymmetric remain antisymmetric (the same is true for symmetric wavefunctions). As mentioned earlier, antisymmetric wavefunctions must satisfy the condition that $l+s$ is even. Since this interaction changes parity of the system by making $l \to l \pm 1$, then to maintain the antisymmetric condition, the spin s must flip, which is evident in eq. 45.

Combining eqs. 19, 44, and 43, I have for the full matrix element of $\tilde{\alpha}\vec{q}\cdot\vec{\sigma}/\sqrt{2}$ to be

$$\begin{aligned}
&\langle n'(l's')j;NL;(jL)J|\frac{\tilde{\alpha}}{\sqrt{2}}\vec{q}\cdot\vec{\sigma}|n(ls)j;NL;(jL)J\rangle = (-1)^{l+l'+s'+j}\times \\
&3\tilde{\alpha} \begin{Bmatrix} j & s' & l' \\ 1 & l & s \end{Bmatrix} \begin{pmatrix} l' & 1 & l \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} \frac{1}{2} & s' & \frac{1}{2} \\ s & \frac{1}{2} & 1 \end{Bmatrix} \sqrt{(2l'+1)(2l+1)} \left[(-1)^s - (-1)^{s'}\right] \Gamma(n',l';n,l) , \qquad (46)
\end{aligned}$$

and I have used the fact that $|s-s'| = 1$ for the matrix element to be non-vanishing. As an example, eq. 46 shows that the non-vanishing matrix elements using the basis states given in tab. I to be $\langle 1|\frac{\tilde{\alpha}}{\sqrt{2}}\vec{q}\cdot\vec{\sigma}|2\rangle$, $\langle 1|\frac{\tilde{\alpha}}{\sqrt{2}}\vec{q}\cdot\vec{\sigma}|8\rangle$, $\langle 2|\frac{\tilde{\alpha}}{\sqrt{2}}\vec{q}\cdot\vec{\sigma}|5\rangle$, $\langle 4|\frac{\tilde{\alpha}}{\sqrt{2}}\vec{q}\cdot\vec{\sigma}|7\rangle$, and $\langle 5|\frac{\tilde{\alpha}}{\sqrt{2}}\vec{q}\cdot\vec{\sigma}|8\rangle$ (as well has their hermitian conjugates)[3].

Next I will consider the matrix element

$$\langle n'(l's')j';N'L';(j'L')J|\frac{\tilde{\alpha}}{\sqrt{2}}\vec{Q}\cdot\vec{\Sigma}|n(ls)j;NL;(jL)J\rangle .$$

In this case it is convenient to expand the basis states in the following manner:

$$\begin{aligned}
&|n(ls)j;NL;(jL)J\rangle = \\
&\qquad (-1)^{l+J}\sqrt{2j+1}\sum_{\mathcal{J}}(-1)^{\mathcal{J}}\sqrt{2\mathcal{J}+1} \begin{Bmatrix} l & s & j \\ L & J & \mathcal{J} \end{Bmatrix} |nl;N(Ls)\mathcal{J};(l\mathcal{J})J\rangle . \qquad (47)
\end{aligned}$$

As the interaction only acts on the CM coordinates and total spin s, it is diagonal in the relative quantum numbers $\{n, l\}$. And since it is a rank 0 tensor, it is also diagonal in $\mathcal{J}$

[3] Turns out that $\langle 1|\frac{\tilde{\alpha}}{\sqrt{2}}\vec{q}\cdot\vec{\sigma}|8\rangle = 0$ since $\Gamma(1,1;0,0) = 0$.

and J. With these constraints, the matrix element is therefore

$$
\begin{aligned}
&\langle n(ls')j'; N'L'; (j'L')J|\frac{\tilde{\alpha}}{\sqrt{2}}\vec{Q}\cdot\vec{\Sigma}|n(ls)j; NL; (jL)J\rangle = \frac{\tilde{\alpha}}{\sqrt{2}}\sqrt{(2j'+1)(2j+1)}\times \\
&\sum_{\mathcal{J}}(2\mathcal{J}+1)\begin{Bmatrix} l & s' & j' \\ L & J & \mathcal{J}\end{Bmatrix}\begin{Bmatrix} l & s & j \\ L & J & \mathcal{J}\end{Bmatrix}\langle nl; N'(L's')\mathcal{J}; (l\mathcal{J})J|\vec{Q}\cdot\vec{\Sigma}|nl; N(Ls)\mathcal{J}; (l\mathcal{J})J\rangle = \\
&\qquad \frac{\tilde{\alpha}}{\sqrt{2}}\sqrt{(2j'+1)(2j+1)}(-1)^{L+s}\langle N'L'||Q||NL\rangle\langle s'||\Sigma||s\rangle\times \\
&\qquad\qquad \left[\sum_{\mathcal{J}}(-1)^{\mathcal{J}}(2\mathcal{J}+1)\begin{Bmatrix} l & s' & j' \\ L' & J & \mathcal{J}\end{Bmatrix}\begin{Bmatrix} l & s & j \\ L & J & \mathcal{J}\end{Bmatrix}\begin{Bmatrix} \mathcal{J} & s' & L' \\ 1 & L & s\end{Bmatrix}\right]. \qquad (48)
\end{aligned}
$$

Now we know the form of the reduced matrix element $\langle N'L'||Q||NL\rangle$ already. The reduced matrix element $\langle s'||\Sigma||s\rangle$ is slightly different from the previous section,

$$
\begin{aligned}
\langle s'||\Sigma||s\rangle &= \langle\begin{pmatrix}\frac{1}{2}\frac{1}{2}\end{pmatrix}s'||\sigma_1+\sigma_2||\begin{pmatrix}\frac{1}{2}\frac{1}{2}\end{pmatrix}s\rangle \\
&= \left[(-1)^s+(-1)^{s'}\right]\sqrt{6(2s+1)(2s'+1)}\begin{Bmatrix}\frac{1}{2} & s' & \frac{1}{2} \\ s & \frac{1}{2} & 1\end{Bmatrix} \qquad (49) \\
&= \begin{cases} 2\sqrt{6} & \text{for } s'=s=1 \\ 0 & \text{otherwise}\end{cases}. \qquad (50)
\end{aligned}
$$

In this case, the interaction is non-trivial for only the case when $s' = s = 1$. In other words, the interaction only occurs with maximal spin $s = 1$. The fact that the spin does *not* flip is a consequence of this interaction *not* changing l, but rather only $L \to L \pm 1$. The antisymmetric condition on the relative wavefunction is therefore respected. Only the parity of the CM angular momentum L is changed, which has no conditions placed on it.

Assuming $s' = s = 1$, the matrix element can be reduced further to

$$
\begin{aligned}
&\langle n(ls)j'; N'L'; (j'L')J|\frac{\tilde{\alpha}}{\sqrt{2}}\vec{Q}\cdot\vec{\Sigma}|n(ls)j; NL; (jL)J\rangle = (-1)^{L+L'+1}\sqrt{3}\,\tilde{\alpha}\times \\
&\qquad \sqrt{(2j'+1)(2j+1)(2L'+1)(2L+1)}\begin{pmatrix} L' & 1 & L \\ 0 & 0 & 0\end{pmatrix}\Gamma(N',L';N,L)\times \\
&\qquad\qquad \left[\sum_{\mathcal{J}}(-1)^{\mathcal{J}}(2\mathcal{J}+1)\begin{Bmatrix} l & s & j' \\ L' & J & \mathcal{J}\end{Bmatrix}\begin{Bmatrix} l & s & j \\ L & J & \mathcal{J}\end{Bmatrix}\begin{Bmatrix} \mathcal{J} & s & L' \\ 1 & L & s\end{Bmatrix}\right]. \qquad (51)
\end{aligned}
$$

Again using the basis states of tab. I as an example, the non-vanishing matrix elements in this case are $\langle 2|\frac{\tilde{\alpha}}{\sqrt{2}}\vec{Q}\cdot\vec{\Sigma}|3\rangle$, $\langle 3|\frac{\tilde{\alpha}}{\sqrt{2}}\vec{Q}\cdot\vec{\Sigma}|6\rangle$, and $\langle 3|\frac{\tilde{\alpha}}{\sqrt{2}}\vec{Q}\cdot\vec{\Sigma}|7\rangle$ (and their hermitian conjugates).

The spectrum of the $J = 0$ system calculated using the matrix elements derived above is shown in fig. 2 by (blue) dots. The (red) lines in fig. 2 come from using two $j = 1/2$ single-particle spectra in fig. 1 to determine the $J = 0$ spectrum. Dots that do not have overlapping lines correspond to two fermion states with other j quantum numbers (e.g. two $j = 7/2$ single-particles, etc. . .). See caption that accompanies fig. 2 for more description.

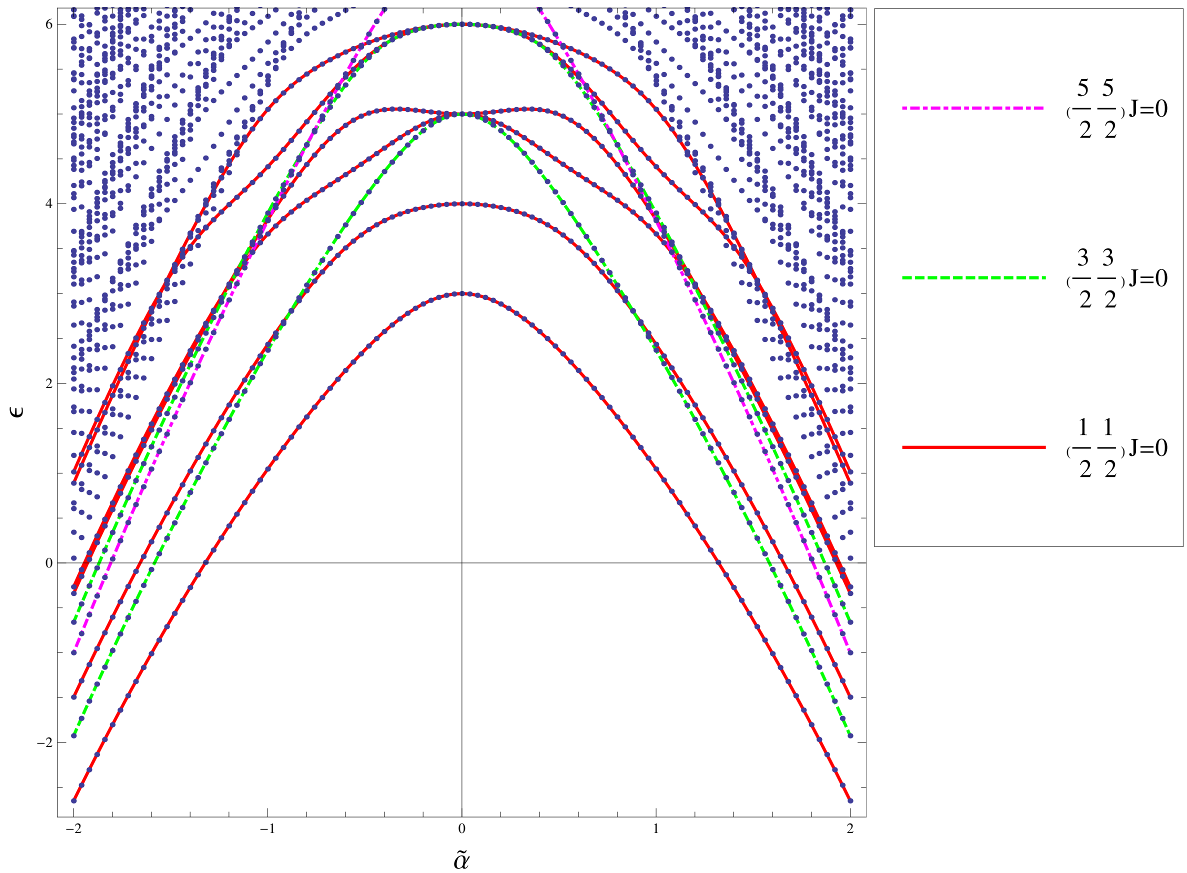

FIG. 2: (Color online) The $J = 0$ spectrum for two spin-1/2 fermions in an HO well with external parity violating interaction $\alpha\vec{k}\cdot\vec{\sigma}$. The (blue) dots correspond to numerical calculations described in sect. II B. The (red) lines are determined using the $j = 1/2$ single particle energies shown in fig. 1. The (green) dashed line corresponds to two $j = 3/2$ single particles, taken from fig. 1. The (purple) dot-dashed line corresponds to two $j = 5/2$ single particles.

C. 2-particle spectrum *with* relative interaction

The Hamiltonian in consideration in this case is

$$H = \omega\left(h_{0,\mathrm{rel}} + \frac{\tilde{\alpha}}{\sqrt{2}}\vec{q}\cdot\vec{\sigma} + \frac{2\sqrt{2}}{\pi}\tilde{a}\delta^{\Lambda}(\vec{x})\right) + \omega\left(h_{0,\mathrm{CM}} + \frac{\tilde{\alpha}}{\sqrt{2}}\vec{Q}\cdot\vec{\Sigma}\right)\,, \tag{52}$$

where $\tilde{a} = a/b$ and a is some constant with dimensions of length[4]. The superscript Λ in the delta function indicates that some form of regulator is to be applied to this interaction. Lastly, this interaction only acts in the relative coordinates.

The strategy I employ in solving this system utilizes the fact that the solution for this system *without* the external parity-violating interaction (i.e. $\tilde{\alpha} = 0$) has been determined

[4] In free space this variable is the scattering length.

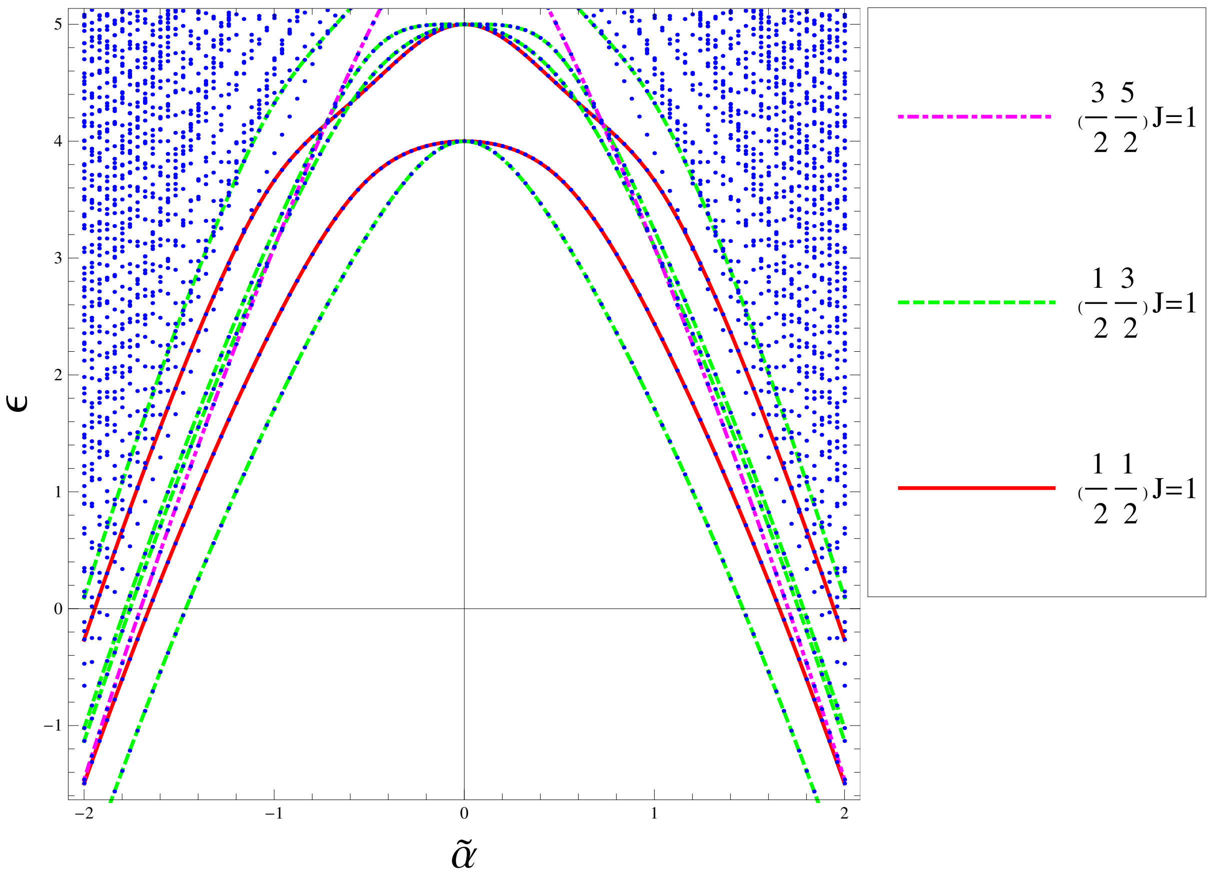

FIG. 3: (Color online) Same as in fig. 2, but with $J = 1$.

exactly by Busch et al. [2]. The eigenvalue equation I want to solve is

$$\frac{1}{\epsilon - h_{0,\mathrm{rel}} - h_{0,\mathrm{CM}} - \frac{2\sqrt{2}}{\pi}\tilde{a}\delta^{\Lambda}(\vec{x})} \frac{\tilde{\alpha}}{\sqrt{2}} \left(\vec{q}\cdot\vec{\sigma} + \vec{Q}\cdot\vec{\Sigma}\right)|\Psi\rangle = \lambda|\Psi\rangle \ . \tag{53}$$

Given input values for $\tilde{a}$ and $\tilde{\alpha}$, one finds the value(s) of ϵ such that $\lambda = 1$[5]. It can be shown that the values of $\{\epsilon, \tilde{a}, \tilde{\alpha}\}$ that satisfy this condition will also satisfy the eigenvalue equation using the Hamiltonian in eq. 52. The benefit of this method comes from the fact that matrix elements

$$\langle n'(l's')j'; N'L'; (j'L')J'| \frac{1}{\epsilon - h_{0,\mathrm{rel}} - h_{0,\mathrm{CM}} - \frac{2\sqrt{2}}{\pi}\tilde{a}\delta^{\Lambda}(\vec{x})} |n(ls)j; NL; (jL)J\rangle$$

are known analytically for any Λ, and in particular $\Lambda = \infty$, due to Busch et al. [2]. This

[5] Actually, it is much easier to determine $\tilde{\alpha}$ given input values for ϵ and $\tilde{a}$.

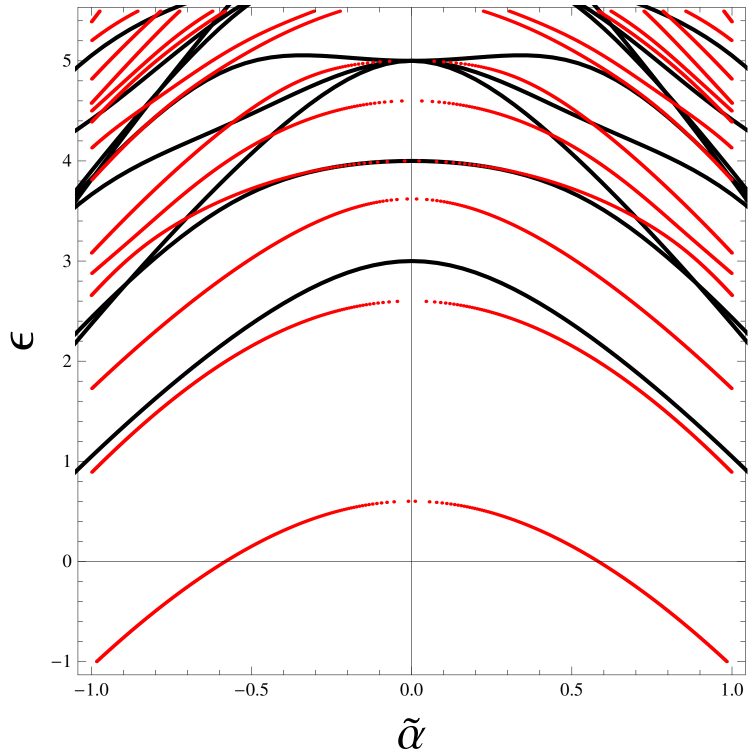

FIG. 4: (Color online) The $J = 0$ spectrum for two spin-1/2 fermions in an HO well with external parity violating interaction $\alpha\vec{k}\cdot\vec{\sigma}$ and contact interaction $2\sqrt{2}\tilde{a}/\pi\delta(x)$. The (black) lines correspond to numerical calculations with $\tilde{a} = 0$. The (red) lines are calculations with $\tilde{a} = 1$.

can be seen by rearranging this operator in the following way:

$$\begin{aligned}\frac{1}{\epsilon - h_{0,\mathrm{rel}} - h_{0,\mathrm{CM}} - \frac{2\sqrt{2}}{\pi}\tilde{a}\delta^{\Lambda}(\vec{x})} &= \frac{1}{\epsilon - h_{0,\mathrm{rel}} - h_{0,\mathrm{CM}}} + \\ &\qquad \frac{1}{\epsilon - h_{0,\mathrm{rel}} - h_{0,\mathrm{CM}}} T_\delta(\epsilon) \frac{1}{\epsilon - h_{0,\mathrm{rel}} - h_{0,\mathrm{CM}}} \\ &= G_0(\epsilon) + G_0(\epsilon) T_\delta(\epsilon) G_0(\epsilon) \ , \end{aligned} \quad (54)$$

where

$$G_0(\epsilon) = \frac{1}{\epsilon - h_{0,\mathrm{rel}} - h_{0,\mathrm{CM}}} \quad (55)$$

and

$$T_\delta(\epsilon) = \frac{2\sqrt{2}}{\pi}\tilde{a}\delta^{\Lambda}(\vec{x}) + G_0(\epsilon)\frac{2\sqrt{2}}{\pi}\tilde{a}\delta^{\Lambda}(\vec{x})G_0(\epsilon) + \ldots \quad (56)$$

is an infinite sum. Note that $G_0(\epsilon)$ is diagonal in all quantum numbers, and its matrix elements are trivial. Furthermore, since δ only acts on relative coordinates, it is diagonal in the CM quantum numbers $\{N, L\}$. Because of this, T_δ is also diagonal in the CM quantum numbers. Further, δ, and by extension T_δ, only acts on s-wave, and therefore l and s must be 0.

The separability of the δ interaction allows T_δ to be summed geometrically. As outlined

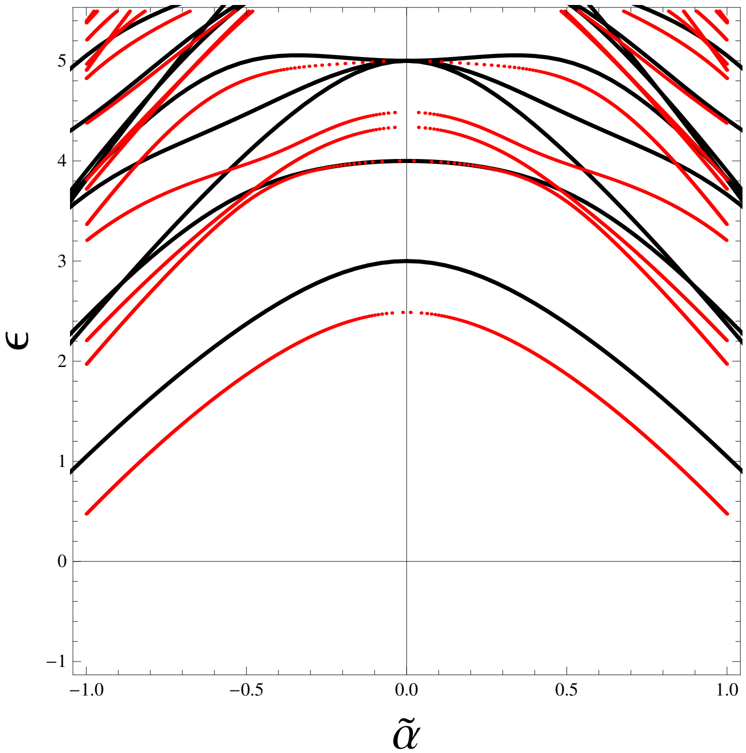

FIG. 5: (Color online) The $J = 0$ spectrum for two spin-1/2 fermions in an HO well with external parity violating interaction $\alpha\vec{k}\cdot\vec{\sigma}$ and contact interaction $2\sqrt{2}\tilde{a}/\pi\delta(x)$. The (black) lines correspond to numerical calculations with $\tilde{a} = 0$. The (red) lines are calculations with $\tilde{a} = -1$.

(someday) in the appendix, the matrix elements of $T_\delta(\epsilon)$ are analytically known

$$\langle n'(l's')j'; N'L'; (j'L')J'|T_\delta(\epsilon)|n(ls)j; NL; (jL)J\rangle =$$
$$\delta_{N',N}\delta_{L',L}\delta_{s',s}\delta_{s,0}\delta_{l',l}\delta_{l,0}\delta_{j',j}\delta_{J',J}\frac{\frac{2\sqrt{2}}{\pi}\sqrt{\frac{\Gamma(n'+\frac{3}{2})\Gamma(n+\frac{3}{2})}{\Gamma(n'+1)\Gamma(n+1)}}}{\frac{1}{\tilde{a}} - \sqrt{2}\frac{\Gamma(\frac{3}{4}-\frac{\epsilon-2N-L-3/2}{2})}{\Gamma(\frac{1}{4}-\frac{\epsilon-2N-L-3/2}{2})}} \tag{57}$$

Busch et al.'s [2] solution should be apparent in the equation above.

These results, and the derivations from earlier sections, allow me to determine the matrix elements of

$$\left[\frac{1}{\epsilon - h_{0,\mathrm{rel}} - h_{0,\mathrm{CM}} - \frac{2\sqrt{2}}{\pi}\tilde{a}\delta^\Lambda(\vec{x})}\right]$$

and

$$\left[\frac{\tilde{\alpha}}{\sqrt{2}}\left(\vec{q}\cdot\vec{\sigma} + \vec{Q}\cdot\vec{\Sigma}\right)\ ,\right]$$

which in turn allow me to determine the spectrum of two fermions under the influence of an external parity violating interaction with relative contact interaction. In figs. 4-6 I show my results, as a function of $\tilde{\alpha}$, for the cases of $\tilde{a} = +1$, -1, and ∞, respectively (as red points). In all figures I compare to the case when $\tilde{a} = 0$, given by the (black) lines.

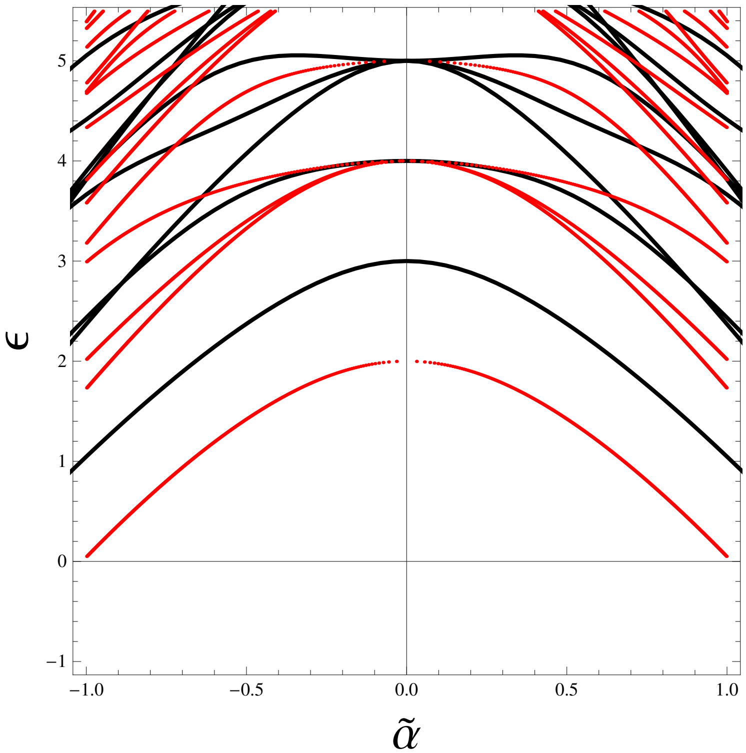

FIG. 6: (Color online) The $J = 0$ spectrum for two spin-1/2 fermions in an HO well with external parity violating interaction $\alpha\vec{k}\cdot\vec{\sigma}$ and contact interaction $2\sqrt{2}\tilde{a}/\pi\delta(x)$. The (black) lines correspond to numerical calculations with $\tilde{a} = 0$. The (red) lines are calculations with $\tilde{a} = \infty$.

D. 3-particle spectrum

[1] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, 1957).

[2] T. Busch, B.-G. Englert, K. Rzazewski, and M. Wilkens, Foundations of Physics **28**, 549 (1998).