# POTENTIALS

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ABSTRACT. I present my notes on how to construct ad hoc potentials given the scattering length $a$, effective range $r$, and two-body binding energy $B_2$.

## 1. Preliminaries

Given a scattering length $a$, an effective range $r$, and two-body binding energy $B_2$, I want to calculate a potential that reproduces these effective range parameters in the threshold limit, while also being amenable to matrix element calculations within the harmonic oscillator (HO) basis. I also want to construct single-particle anti-symmetric two-body matrix elements of this potential to be used in shell-model-like many-body calculations of light nuclei.

In these notes I show how to construct an *ad hoc* potential (i.e. no underlying power-counting) and with this potential calculate the single-particle matrix elements to be used within a slater-determinant HO basis.

## 2. Potential parametrization

I parameterize the potential as

$$\langle \vec{p}'|V|\vec{p}\rangle \equiv V(\vec{p}',\vec{p}) = \frac{4\pi}{m}\left(C_0 + \frac{1}{2}\left(p'^2+p^2\right)C_2\right) f_\Lambda(p')f_\Lambda(p) \ , \tag{1}$$

where $f_\Lambda(p)$ is some regulator with cutoff $\Lambda$ and the coefficients $C_i$ are $\Lambda$-dependent and tuned to reproduce the effective range parameters at threshold. Without loss of generality, I will define $f_\Lambda(p)$ such that $f_\Lambda(0)=1$. Note that the potential is separable in $p'$ and $p$.

### 2.1. **T-matrix.**

To determine the coefficients $C_i$, I calculate the T-matrix,

$$T(E) = V + VG_0(E)T(E) \tag{2}$$
$$= V + VG_0(E)V + VG_0(E)VG_0(E)V + \dots \ , \tag{3}$$

where $G_0(E)$ is the non-interacting Greens function,

$$G_0(E)|\vec{q}\rangle = |\vec{q}\rangle \frac{1}{E - q^2/m + i\epsilon} \ . \tag{4}$$

Taking advantage of the separability of the potential in eqn. 1, I can sum eqn. 2 to all orders to get

$$\langle \vec{p}'|T(E)|\vec{p}\rangle \equiv T(\vec{p}',\vec{p};E) = \frac{4\pi}{m}\frac{\mathcal{C}_0(E)+\frac{1}{2}\left(p'^2+p^2\right)\mathcal{C}_2(E)+p'^2p^2\mathcal{X}_4(E)}{\mathcal{D}(E)} f_\Lambda(p')f_\Lambda(p) \ , \tag{5}$$

where

$$\mathcal{C}_0(E) = C_0 + \frac{1}{4}C_2^2 I_4(E) \tag{6}$$

$$\mathcal{C}_2(E) = C_2\left(1-\frac{1}{2}C_2I_2(E)\right) \tag{7}$$

$$\mathcal{X}_4(E) = \frac{1}{4}C_2^2 I_0(E) \tag{8}$$

$$\mathcal{D}(E) = \left(\frac{\mathcal{C}_2}{C_2}\right)^2 - \mathcal{C}_0 I_0(E) \ . \tag{9}$$

The loop integrals $I_n(E)$ are defined as

$$I_n(E) \equiv \frac{4\pi}{m}\frac{1}{2\pi^2}\int dq \frac{q^{2+n}}{E - q^2/m + i\epsilon} f_\Lambda^2(q) \ . \tag{10}$$

In performing these integrals, appropriate limits must be taken to obtain the correct principal value and imaginary parts. For example,

$$I_0(E) = \frac{4\pi}{m}\frac{1}{2\pi^2}\ P.V. \int dq \frac{q^2}{E - q^2/m} f_\Lambda^2(q) - i\sqrt{mE} f_\Lambda^2(\sqrt{mE}) \ , \tag{11}$$

where $P.V.$ stands for *Principal Value.*

## 3. Matching to threshold

The on-shell T-matrix is related to the phase shift $\delta$ by the following relation,

$$T(p' = p; E = p^2/m) = -\frac{4\pi}{m}\frac{1}{p\mathrm{cot}\delta - ip} \ . \tag{12}$$

Comparing with eqn. 5, I get the following relation for the phase shift,

$$p\mathrm{cot}\delta - ip = -\frac{\mathcal{D}(\frac{p^2}{m})}{\mathcal{C}_0(\frac{p^2}{m}) + p^2\mathcal{C}_2(\frac{p^2}{m}) + p^4\mathcal{X}_4(\frac{p^2}{m})} f_\Lambda^{-2}(p) \ , \tag{13}$$

where the right-hand side (RHS) is evaluated at $E = p^2/m$.

I now expand both sides in powers of $p^2$ and match terms order by order. The left-hand side (LHS) is the familiar effective range expansion,

$$-\frac{1}{a} + \frac{r}{2}p^2 - ip + \mathcal{O}(p^4) \ .$$

The RHS takes a little work. The first thing to note is that the numerator also has a term linear in $p$ that comes from the imaginary part in eq. 11,

$$\mathcal{D}\left(\frac{p^2}{m}\right) f_\Lambda^{-2}(p) \sim \mathcal{D}(0) + ip\ \mathcal{C}_0(0) + p^2\frac{\partial}{\partial p^2}\left[\mathcal{D}(\frac{p^2}{m}) f_\Lambda^{-2}(p)\right] + \mathcal{O}(p^4) \ . \tag{14}$$

Therefore I have

$$\begin{aligned} -&\frac{\mathcal{D}(\frac{p^2}{m}) f_\Lambda^{-2}(p)}{\mathcal{C}_0(\frac{p^2}{m}) + p^2\mathcal{C}_2(\frac{p^2}{m}) + p^4\mathcal{X}_4(\frac{p^2}{m})} \sim \\ &\qquad -\frac{\mathcal{D}(0) + ip\ \mathcal{C}_0(0) + p^2\frac{\partial}{\partial p^2}\left[\mathcal{D}(\frac{p^2}{m}) f_\Lambda^{-2}(p)\right] + \mathcal{O}(p^4)}{\mathcal{C}_0(0) + p^2\left[\mathcal{C}_2(0) + \frac{\partial}{\partial p^2}\mathcal{C}_0(\frac{p^2}{m})\right] + \mathcal{O}(p^4)} \\ &= -\frac{1}{a} - ip + p^2\left[\frac{1}{a}\frac{\mathcal{C}_2(0)}{\mathcal{C}_0(0)} + \frac{\partial}{\partial p^2}\left(\frac{1}{a}\log\mathcal{C}_0\left(p^2/m\right) - \frac{\mathcal{D}(p^2/m)}{\mathcal{C}_0(0)} f_\Lambda^{-2}(p)\right)\right] + \mathcal{O}(p^4) \ , \end{aligned} \tag{15}$$

where in the last line I have made use of the momentum-independent matching condition:

$$\frac{1}{a} = \frac{\mathcal{D}(0)}{\mathcal{C}_0(0)} \ . \tag{16}$$

Let's look a little closer at eqn. 16:

$$\frac{1}{a} = \frac{\mathcal{D}(0)}{\mathcal{C}_0(0)} \tag{17}$$

$$= \frac{1}{\mathcal{C}_0}\left(\frac{\mathcal{C}_2}{C_2}\right)^2 - I_0(0) \tag{18}$$

$$\Rightarrow \frac{1}{a} + I_0(0) = \frac{\mathcal{C}_2}{\mathcal{C}_0}\frac{\mathcal{C}_2}{C_2^2} \tag{19}$$

This relation is used quite often in the notes to follow. Note that in the limit that $C_2 \to 0$ (i.e. no momentum-dependent term in the potential aside from the regulator), the above equation becomes

$$\frac{1}{C_0} = \frac{1}{a} + I_0(0) \qquad [\text{matching when } C_2 = 0]\,. \tag{20}$$

I will make use of this result later.

For $C_2 \neq 0$, the matching condition for the effective range is

$$\frac{r}{2} = \frac{1}{a}\frac{\mathcal{C}_2(0)}{\mathcal{C}_0(0)} + \frac{\partial}{\partial p^2}\left(\frac{1}{a}\log\,\mathcal{C}_0\left(p^2/m\right) - \frac{\mathcal{D}(p^2/m)}{\mathcal{C}_0(0)} f_\Lambda^{-2}(p)\right)\Bigg|_{p^2=0} \tag{21}$$

$$= \frac{1}{a}\left(\frac{1}{a} + I_0(0)\right)\frac{C_2^2}{\mathcal{C}_2} + \frac{\partial}{\partial p^2}\left(\frac{1}{a}\log\,\mathcal{C}_0\left(p^2/m\right) - \frac{\mathcal{D}(p^2/m)}{\mathcal{C}_0(0)} f_\Lambda^{-2}(p)\right)\Bigg|_{p^2=0}, \tag{22}$$

where in the second line I have made use of eqn. 19. I will now derive the different terms separately and combine them together in the end. The first term I consider is

$$\frac{1}{\mathcal{C}_0(0)}\frac{\partial}{\partial p^2}\mathcal{D} f_\Lambda^{-2} = \frac{f_\Lambda^{-2}(0)}{\mathcal{C}_0(0)}\frac{\partial}{\partial p^2}\mathcal{D} + \frac{1}{a}\frac{\partial}{\partial p^2} f_\Lambda^{-2} \tag{23}$$

$$= \frac{1}{\mathcal{C}_0(0)}\frac{\partial}{\partial p^2}\mathcal{D} + \frac{1}{a}\frac{\partial}{\partial p^2} f_\Lambda^{-2} \tag{24}$$

$$= \frac{1}{\mathcal{C}_0(0)}\left(2\left(\frac{\mathcal{C}_2}{C_2^2}\right)\frac{\partial}{\partial p^2}\mathcal{C}_2 - \frac{\partial}{\partial p^2}(\mathcal{C}_0 I_0)\right) + \frac{1}{a}\frac{\partial}{\partial p^2} f_\Lambda^{-2} \tag{25}$$

$$= \left(\frac{1}{a} + I_0(0)\right)\frac{2}{\mathcal{C}_2}\frac{\partial}{\partial p^2}\mathcal{C}_2 - I_0\frac{\partial}{\partial p^2}\log\,\mathcal{C}_0 - \frac{\partial}{\partial p^2} I_0 + \frac{1}{a}\frac{\partial}{\partial p^2} f_\Lambda^{-2} \tag{26}$$

$$= 2\left(\frac{1}{a} + I_0\right)\frac{\partial}{\partial p^2}\log\,\mathcal{C}_2 - I_0\frac{\partial}{\partial p^2}\log\,\mathcal{C}_0 - \frac{\partial}{\partial p^2} I_0 + \frac{1}{a}\frac{\partial}{\partial p^2} f_\Lambda^{-2}\,, \tag{27}$$

where I have made use of the various expressions derived above. Combining this result with eqn 22 gives

$$\frac{r}{2} = \left(\frac{1}{a} + I_0\right)\left(\frac{1}{a}\frac{C_2^2}{\mathcal{C}_2} - 2\frac{\partial}{\partial p^2}\log\,\mathcal{C}_2\right) + \left(\frac{1}{a} + I_0\right)\frac{\partial}{\partial p^2}\log\,\mathcal{C}_0 + \frac{\partial}{\partial p^2} I_0 - \frac{1}{a}\frac{\partial}{\partial p^2} f_\Lambda^{-2}. \tag{28}$$

TABLE 1. Values of various loop integrals $I_n(0)$ for particular regulators $f_\Lambda(p)$.

| $f_\Lambda(p)$ | $f_\Lambda^{-2}(0)$ | $I_0(0)$ | $I_2(0)$ | $I_4(0)$ |
|---|---|---|---|---|
| $e^{-p^2/\Lambda^2}$ | 1 | $-\frac{\Lambda}{\sqrt{2\pi}}$ | $\frac{\Lambda^2}{4}I_0(0)$ | $\frac{3\Lambda^4}{16}I_0(0)$ |
| $e^{-p^4/\Lambda^4}$ | 1 | $-\frac{2^{3/4}\Lambda\Gamma\left(\frac{5}{4}\right)}{\pi}$ | $\frac{\Lambda^2\Gamma\left(\frac{3}{4}\right)}{\sqrt{2}\Gamma\left(\frac{1}{4}\right)}I_0(0)$ | $\frac{\Lambda^4}{8}I_0(0)$ |
| $\frac{1}{1+p^4/\Lambda^4}$ | 1 | $-\frac{3\Lambda}{4\sqrt{2}}$ | $\frac{\Lambda^2}{3}I_0(0)$ | $\frac{\Lambda^4}{3}I_0(0)$ |

TABLE 2. Derivative values of various loop integrals $I_n(0)$ for particular regulators $f_\Lambda(p)$.

| $f_\Lambda(p)$ | $\frac{\partial}{\partial p^2}f_\Lambda^{-2}(p)\big\vert_{p^2=0}$ | $\frac{\partial}{\partial p^2}I_0(p^2/m)\big\vert_{p^2=0}$ | $\frac{\partial}{\partial p^2}I_2(p^2/m)\big\vert_{p^2=0}$ | $\frac{\partial}{\partial p^2}I_4(p^2/m)\big\vert_{p^2=0}$ |
|---|---|---|---|---|
| $e^{-p^2/\Lambda^2}$ | $\frac{2}{\Lambda^2}$ | $-\frac{4}{\Lambda^2}I_0(0)$ | $I_0(0)$ | $I_2(0)$ |
| $e^{-p^4/\Lambda^4}$ | 0 | $-\frac{\sqrt{2}\Gamma\left(\frac{3}{4}\right)}{\Lambda^2\Gamma\left(\frac{5}{4}\right)}I_0(0)$ | $I_0(0)$ | $I_2(0)$ |
| $\frac{1}{1+p^4/\Lambda^4}$ | 0 | $-\frac{5}{3\Lambda^2}I_0(0)$ | $I_0(0)$ | $I_2(0)$ |

Let me rearrange some terms,

$$\left(\frac{1}{a}+I_0\right)\left[\frac{1}{a}\frac{C_2^2}{\mathcal{C}_2}-2\frac{\partial}{\partial p^2}\log\mathcal{C}_2+\frac{\partial}{\partial p^2}\log\mathcal{C}_0\right]=\frac{r}{2}-\frac{\partial}{\partial p^2}I_0+\frac{1}{a}\frac{\partial}{\partial p^2}f_\Lambda^{-2} \tag{29}$$

Now I have that

$$\frac{\partial}{\partial p^2}\log\mathcal{C}_0 = \frac{1}{4}\frac{C_2^2}{\mathcal{C}_0}\frac{\partial}{\partial p^2}I_4 \tag{30}$$

$$\frac{\partial}{\partial p^2}\log\mathcal{C}_2 = -\frac{1}{2}\frac{C_2^2}{\mathcal{C}_2}\frac{\partial}{\partial p^2}I_2\ . \tag{31}$$

Plugging these results into eqn. 29 gives

$$\frac{r}{2}-\frac{\partial}{\partial p^2}I_0+\frac{1}{a}\frac{\partial}{\partial p^2}f_\Lambda^{-2} = \left(\frac{1}{a}+I_0\right)\left[\frac{1}{a}\frac{C_2^2}{\mathcal{C}_2}+\frac{C_2^2}{\mathcal{C}_2}\frac{\partial}{\partial p^2}I_2+\frac{1}{4}\frac{C_2^2}{\mathcal{C}_0}\frac{\partial}{\partial p^2}I_4\right] \tag{32}$$

$$= \left(\frac{1}{a}+I_0\right)\frac{C_2^2}{\mathcal{C}_2}\left[\frac{1}{a}+\frac{\partial}{\partial p^2}I_2+\frac{1}{4}\frac{\mathcal{C}_2}{\mathcal{C}_0}\frac{\partial}{\partial p^2}I_4\right] \tag{33}$$

$$= \left(\frac{1}{a}+I_0\right)\frac{C_2^2}{\mathcal{C}_2}\left[\frac{1}{a}+\frac{\partial}{\partial p^2}I_2+\frac{1}{4}\left(\frac{1}{a}+I_0\right)\frac{C_2^2}{\mathcal{C}_2}\frac{\partial}{\partial p^2}I_4\right]\ . \tag{34}$$

Now to go any further, I finally need to pick a regulator $f_\Lambda$ and perform the integrals $I_n$. In tab. 1 I give some regulators and their associated values of $I_n(0)$ and their derivatives.

For the regulators listed in tab. 2, I can simplify eqn. 34 even further using the results $\frac{\partial}{\partial p^2} I_2 = I_0$ and $\frac{\partial}{\partial p^2} I_4 = I_2$,

$$\frac{r}{2} - \frac{\partial}{\partial p^2} I_0 + \frac{1}{a} \frac{\partial}{\partial p^2} f_\Lambda^{-2} = \left(\frac{1}{a} + I_0\right)^2 \frac{C_2^2}{\mathcal{C}_2} \left[1 + \frac{1}{4} \frac{C_2^2}{\mathcal{C}_2} I_2\right] \ . \tag{35}$$

To simplify things, I make the following definitions:

$$\frac{1}{a(\Lambda)} = \frac{1}{a} + I_0(0) \tag{36}$$

$$r(\Lambda) = r\left(1 - \frac{2}{r}\frac{\partial}{\partial p^2} I_0 + \frac{2}{ar}\frac{\partial}{\partial p^2} f_\Lambda^{-2}\right) \ , \tag{37}$$

and so eqn. 35 becomes

$$\frac{C_2^2}{\mathcal{C}_2}\left[1 + \frac{1}{4}\frac{C_2^2}{\mathcal{C}_2} I_2\right] = a(\Lambda)^2 \frac{r(\Lambda)}{2} \ . \tag{38}$$

From this I get the following solution:

$$\frac{C_2^2}{\mathcal{C}_2} = \frac{C_2}{1 - \frac{1}{2} C_2 I_2(0)} = \frac{-1 \pm \sqrt{1 + a(\Lambda)^2 \frac{r(\Lambda)}{2} I_2(0)}}{\frac{1}{2} I_2(0)} \tag{39}$$

$$\Rightarrow \frac{1}{C_2} = \frac{\frac{1}{2} I_2(0)}{-1 \pm \sqrt{1 + a(\Lambda)^2 \frac{r(\Lambda)}{2} I_2(0)}} + \frac{1}{2} I_2(0) \tag{40}$$

To keep $C_2$ purely real, then I must have that

$$r(\Lambda) < \frac{2}{|a^2(\Lambda) I_2|} \ .$$

Unfortunately this puts severe restrictions on the size of $\Lambda$.

NPLQCD's recent results suggest that at the SU(3) point we have the following values for the scattering length and effective range for the $^1S_0$ and $^3S_1$ channels, respectively,

$$m_\pi a = 9.51 \qquad m_\pi r = 4.76 \tag{41}$$

$$m_\pi a = 7.45 \qquad m_\pi r = 3.71 \ . \tag{42}$$

Using these values, I determine the coefficients $C_0$ and $C_2$ by varying $\Lambda$ so that the coefficients remain real and the shape parameter ($p^4$ term) is minimized. I show the resulting phase shifts in figs. 1 and 2.

## 4. Eliminating the induced shape parameter

In this section I choose yet another form of the regulator,

$$f_\Lambda(p) = e^{-\alpha p^2/\Lambda^2 - p^4/\Lambda^4} \ . \tag{43}$$

The hope is that I can tune $\alpha$ so that the elimination of the shape parameter term is exact, or at least to make it very small, while at the same time maintain reality of $C_0$ and $C_2$. In

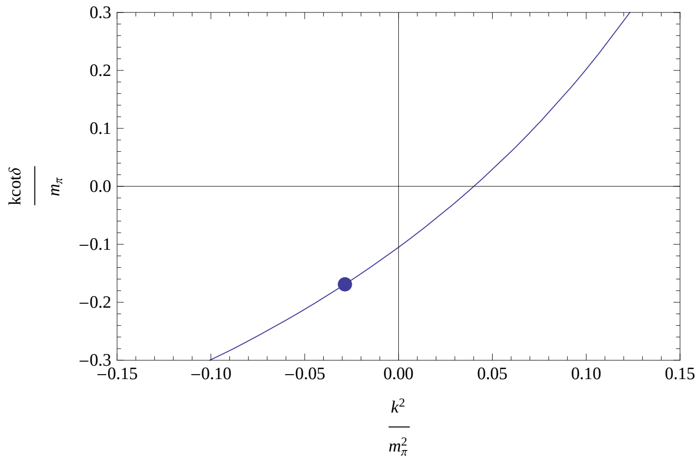


FIGURE 1. Phase shift for the $^1S_0$ channel. The point corresponds to the bound state, determined by solving for the zero of the T-matrix for imaginary momentum. The regulator used was $e^{-p^2/\Lambda^2}$. The optimal cutoff that minimized the shape parameter while enforcing reality of $C_0$ and $C_2$ was $\Lambda/m_\pi = .490249$.

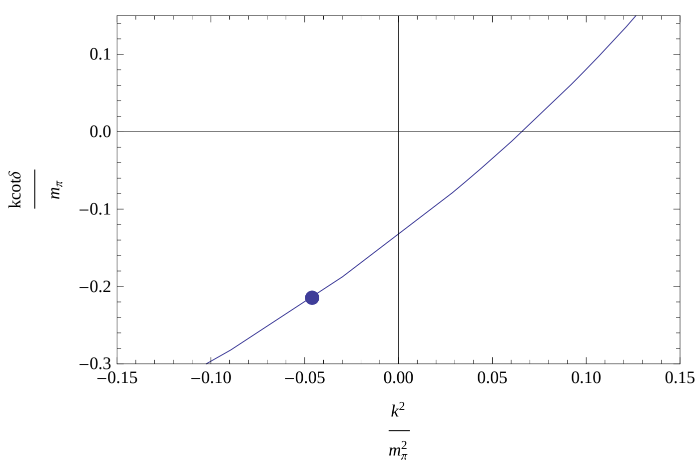


FIGURE 2. Phase shift for the $^3S_1$ channel. The point corresponds to the bound state, determined by solving for the zero of the T-matrix for imaginary momentum. The regulator used was $e^{-p^2/\Lambda^2}$. The optimal cutoff that minimized the shape parameter while enforcing reality of $C_0$ and $C_2$ was $\Lambda/m_\pi = .630846$.

TABLE 3. Values of various loop integrals $I_n(0)$ for the regulator $f_\Lambda(p) = e^{-\alpha p^2/\Lambda^2 - p^4/\Lambda^4}$. Here $K^{\pm}_{\nu}(x) \equiv \frac{\pi}{2}\left(I_{-\nu}(x) \pm I_{\nu}(x)\right)/\sin(\nu\pi)$ are modified Bessel functions and $\pm$ corresponds to whether $\alpha$ is greater or lesser than zero (i.e. $\mathrm{sign}(\alpha) = \pm 1$).

| | |
|---|---|
| $f_\Lambda^{-2}(0)$ | $1$ |
| $I_0(0)$ | $-\frac{e^{\frac{\alpha^2}{4}}\lvert\alpha\rvert^{1/2}\Lambda K^{\mp}_{\frac{1}{4}}\left(\frac{\alpha^2}{4}\right)}{2\pi}$ |
| $I_2(0)$ | $-\frac{e^{\frac{\alpha^2}{4}}\lvert\alpha\rvert^{3/2}\Lambda^3\left(K^{\mp}\left(\frac{3}{4},\frac{\alpha^2}{4}\right)\mp K^{\mp}\left(\frac{1}{4},\frac{\alpha^2}{4}\right)\right)}{8\pi}$ |
| $I_4(0)$ | $-\frac{e^{\frac{\alpha^2}{4}}\lvert\alpha\rvert^{1/2}\Lambda^5\left(\left(\alpha^2+1\right)K^{\mp}\left(\frac{1}{4},\frac{\alpha^2}{4}\right)\mp\alpha^2 K^{\mp}\left(\frac{3}{4},\frac{\alpha^2}{4}\right)\right)}{16\pi}$ |

TABLE 4. Derivative values of various loop integrals $I_n(0)$ for the regulator $f_\Lambda(p) = e^{-\alpha p^2/\Lambda^2 - p^4/\Lambda^4}$.

| | |
|---|---|
| $\frac{\partial}{\partial p^2} f_\Lambda^{-2}(p)\Big\rvert_{p^2=0}$ | $\frac{2\alpha}{\Lambda^2}$ |
| $\frac{\partial}{\partial p^2} I_0(p^2/m)\Big\rvert_{p^2=0}$ | $-\frac{4\alpha I_0(0)}{\Lambda^2} - \frac{8 I_2(0)}{\Lambda^4}$ |
| $\frac{\partial}{\partial p^2} I_2(p^2/m)\Big\rvert_{p^2=0}$ | $I_0(0)$ |
| $\frac{\partial}{\partial p^2} I_4(p^2/m)\Big\rvert_{p^2=0}$ | $I_2(0)$ |

tabs. 3 and 4 I give the values of the loop integrals and their derivatives for this regulator.

The running of $r(\Lambda)$ is now given by

$$r(\Lambda) = r\left(1 + \frac{2}{r}\left(\frac{4\alpha I_0(0)}{\Lambda^2} + \frac{8 I_2(0)}{\Lambda^4}\right) + \frac{2}{ar}\frac{2\alpha}{\Lambda^2}\right) \tag{44}$$

$$= r\left(1 + \frac{4\alpha}{r\Lambda^2}\left(2 I_0(0) + \frac{1}{a}\right) + \frac{16}{r}\frac{I_2(0)}{\Lambda^4}\right) \ . \tag{45}$$

## 5. Harmonic Oscillator Matrix Elements

I now give expressions for harmonic oscillator s-wave matrix elements of the various potentials. I will calculate the matrix elements in the symmetric Jacobi basis,

$$\vec{p}_{rel} \equiv \vec{p} = \frac{1}{\sqrt{2}}\left(\vec{p}_1 - \vec{p}_2\right)$$

$$\vec{P}_{CM} = \frac{1}{\sqrt{2}}\left(\vec{p}_1 + \vec{p}_2\right) \ .$$

The s-wave momentum radial wave function with argument $x = bp$, where $b^2 = 1/m\omega$ is the oscillator parameter, is given by

$$\begin{aligned}\langle \vec{p}|n,0\rangle &= \sqrt{2}\pi b^{3/2}\mathcal{R}_{n,0}(x),\\ \mathcal{R}_{n,0}(x) &= (-1)^n e^{-x^2/2}\sqrt{\frac{2\Gamma(n+1)}{\Gamma(n+3/2)}}L_n^{1/2}(x^2)\ ,\end{aligned}$$

where $L_n^{1/2}(x)$ is the associated Laguerre polynomial. Matrix elements are then[1]

$$\begin{aligned}\langle n',0|V|n,0\rangle &= \int\frac{d\vec{p'}}{(2\pi)^3}\int\frac{d\vec{p}}{(2\pi)^3}\langle n',0|\vec{p'}\rangle V\left(\frac{\vec{p'}}{\sqrt{2}},\frac{\vec{p}}{\sqrt{2}}\right)\langle\vec{p}|n,0\rangle\\ &= \frac{\omega}{2\pi^2}\left(\int dx' dx\ \mathcal{R}_{n',0}(x')x'^2\left[\frac{1}{\omega b^3}\right]V\left(\frac{x'}{\sqrt{2}},\frac{x}{\sqrt{2}}\right)x^2\mathcal{R}_{n,0}(x)\right)\ .\end{aligned}$$

I now define the dimensionless potential $v \equiv \frac{V}{2\pi^2\omega b^3}$ and take advantage of its separability, $v(x',x) = \alpha v_L(x')v_R(x)$. The coefficient $\alpha$ soaks up various constants of $\pi$, etc. This gives

$$\begin{aligned}\langle n',0|V|n,0\rangle &= \omega\ \alpha\left[\int dx\ x^2\mathcal{R}_{n',0}(x)v_L\left(\frac{x}{\sqrt{2}}\right)\right]\ \left[\int dx\ x^2\mathcal{R}_{n,0}(x)v_R\left(\frac{x}{\sqrt{2}}\right)\right]\\ &\equiv \omega\ \alpha\ \Gamma^L_{n',0}\ \Gamma^R_{n,0}\ .\end{aligned}$$

I can express $\Gamma_{n,0}$ in terms of the polynomial expansion of $L_n^{1/2}$,

$$\Gamma^{L,R}_{n,0} = (-1)^n\sqrt{2\Gamma(n+1)\Gamma(n+3/2)}\sum_{m=0}^{n}\frac{(-1)^m\int dx\ x^{2(1+m)}e^{-x^2/2}v_{L,R}(\frac{x}{\sqrt{2}})}{\Gamma(n-m+1)\Gamma(m+3/2)\Gamma(m+1)}\ . \tag{46}$$

To go any further, I need explicit forms of $v_{L,R}$. For example, if I have a regulated contact interaction

$$v\left(\frac{x'}{\sqrt{2}},\frac{x}{\sqrt{2}}\right) = \frac{C_0}{2\sqrt{2}}\frac{4\pi}{2\pi^2 b}f_\Lambda\left(\frac{x'}{\sqrt{2}}\right)f_\Lambda\left(\frac{x}{\sqrt{2}}\right)\ ,$$

then $\alpha = C_0/(\sqrt{2}\pi b)$ and $v_{L,R} = f_\Lambda$. For $f_\Lambda(x/\sqrt{2}) = e^{-x^2/2(b\Lambda)^2}$, the integral and sum in eqn. 46 can be performed to give

$$\Gamma^{L,R}_{n,0} = (-1)^n\ 2\sqrt{\frac{\Gamma(n+3/2)}{\Gamma(n+1)}}\left(\frac{\lambda^2}{1+\lambda^2}\right)^{3/2}\left(\frac{1-\lambda^2}{1+\lambda^2}\right)^n\qquad\text{for } f_\Lambda(x/\sqrt{2}) = e^{-x^2/2\lambda^2}\ , \tag{47}$$

where $\lambda \equiv b\Lambda$. Unfortunately, for all the other regulators considered in these notes, compact analytic results for $\Gamma^{L,R}_{n,0}$ could not be found, and so numerical integration methods must be employed.

[1] Recall that the potentials are defined in the asymmetric Jacobi basis, i.e. $\vec{p}_{rel} = \frac{1}{2}(\vec{p}_1 - \vec{p}_2)$.