BLOCH-HOROWITZ EQUATION FOR TWO-BODY AND THREE-BODY SYSTEMS IN A HARMONIC TRAP

T. LUU

ABSTRACT. I present my notes on the Bloch-Horowitz equation for the two- and three-body systems in harmonic oscillator trap assuming a short-range interaction.

1. Preliminaries

As the interaction V_{ij} I use in these notes depends only on the relative coordinates, I work in the Jacobi basis and I ignore the center of mass (CM) motion. I also assume that the particles have equal mass m. *I need to give notes on the HO potential and my definition of Jacobi coordinates, and re-express things in dimensionless quantities.*

2. Two-body system

The Schrödinger equation for the system consisting of particles 1 and 2 is

$$(h_0 + v_{12})|\psi\rangle = \epsilon|\psi\rangle \ ,$$

where h_0 is the dimensionless harmonic oscillator Hamiltonian, $v_{12} = V_{12}/\omega$ the dimensionless two-body interaction, and $\epsilon = E/\omega$ and $|\psi\rangle$ are the eigenvalue and eigenfunction, respectively. The Bloch-Horowitz (BH) equation for this system is

$$h_{eff}|\psi_P\rangle = (h_0 + v_{eff})\,|\psi_P\rangle = \epsilon|\psi_P\rangle \ , \tag{1}$$

where

$$v_{eff} = v_{12} + v_{12}\frac{1}{\epsilon - h_0 - Qv_{12}}Qv_{12} \ , \tag{2}$$

$$|\psi_P\rangle = P|\psi\rangle \ , \tag{3}$$

and P and Q are projection operators defined such that $P + Q = 1$ and $[h_0, P] = [h_0, Q] = 0$.

Now v_{eff} satisfies the following integral equation,

$$v_{eff} = v_{12} + v_{12}g_0(\epsilon)Qv_{eff} \ , \tag{4}$$

where $g_0(\epsilon) = (\epsilon - h_0)^{-1}$. If I assume the following δ-function interaction,

$$v_{12} = \frac{2\sqrt{2}}{\pi}a\ \delta \ , \tag{5}$$

where a is the dimensionless s-wave scattering length (in units of oscillator parameter b), I can solve this integral equation analytically, giving the solution that I've been using since the beginning of time,

$$v_{eff} \equiv t_{12}(\epsilon, \Lambda) = \frac{2\sqrt{2}}{\pi}a_{eff}(a, \epsilon, \Lambda)\ \delta \ , \tag{6}$$

where Λ is the cutoff scale that defines the demarcation between P and Q spaces and

$$a_{eff}(a, \epsilon, \Lambda) = \left(\frac{1}{a} - \sqrt{2}\frac{\Gamma\left(\frac{3}{4} - \frac{\epsilon}{2}\right)}{\Gamma\left(\frac{1}{4} - \frac{\epsilon}{2}\right)} - \frac{2\sqrt{2}}{\pi}\sum_{i=0}^{\Lambda/2}\frac{\Gamma\left(i + \frac{3}{2}\right)}{\left(2i - \epsilon + \frac{3}{2}\right)\Gamma(i+1)}\right)^{-1} . \tag{7}$$

In deriving eqn. 6 I had to regulate the interaction and take appropriate limits. I have this derived in other notes elsewhere (I should cite these notes).

Two-body matrix elements of eq. 6 are analytical and therefore simple to compute. Note that in the limit $\Lambda = 0$, there is only one matrix element of h_{eff}, and the result is simply Busch *et al.*'s result.

3. Three-Body System

The Schrödinger equation for the the three-body system is

$$(h_0 + v_2 + v_3)|\psi\rangle = \epsilon|\psi\rangle \ ,$$

where

$$v_2 = \sum_{i<j}^{3} v_{ij} \tag{8}$$

is the sum of all possible pairwise interactions and there is the possibility of a pure three-body interaction, v_3. For the rest of this section, I assume that this interaction is not present, i.e. $v_3 = 0$.

As before, the BH equation becomes

$$h_{eff}|\psi_P\rangle = (h_0 + v_{eff})\,|\psi_P\rangle = \epsilon|\psi_P\rangle \ , \tag{9}$$

with

$$v_{eff} = v_2 + v_2 \frac{1}{\epsilon - h_0 - Qv_2} Qv_{12} \ . \tag{10}$$

Equation 10 has both 2-body and *induced* 3-body interactions, all mixed up together. It would be nice to separate out the purely 2-body contribution from the 3-body contribution. I can do this by invoking the Faddeev decompositions.

First note that v_{eff} satisfies the following integral equation:

$$v_{eff} = v_2 + v_2 g_0(\epsilon) Q v_{eff} \ .$$

Therefore any state acting on v_{eff}, such as $v_{eff}|\psi_P\rangle \equiv |\zeta\rangle$, satisfies the following integral equation,

$$|\zeta\rangle = v_2|\psi_P\rangle + v_2 g_0(\epsilon) Q|\zeta\rangle \ . \tag{11}$$

Now define the Faddeev components to be

$$|\zeta\rangle_{ij} = v_{ij}|\psi_P\rangle + v_{ij} g_0(\epsilon) Q|\zeta\rangle \ .$$

This can be inverted to give

$$|\zeta\rangle_{ij} = (1 - v_{ij} g_0(\epsilon) Q)^{-1}\, v_{ij}|\psi_P\rangle + (1 - v_{ij} g_0(\epsilon) Q)^{-1}\, v_{ij} g_0(\epsilon) Q \Pi |\zeta\rangle_{ij} \ . \tag{12}$$

Note that

$$(1 - v_{ij} g_0(\epsilon) Q)^{-1}\, v_{ij} = t_{ij}(\epsilon, \Lambda) \ , \tag{13}$$

which is the same expression in eq. 6, but here both ϵ and Λ are defined in the three-body space. With this identification eq. 12 becomes

$$|\zeta\rangle_{ij} = t_{ij}(\epsilon, \Lambda)|\psi_P\rangle + t_{ij}(\epsilon, \Lambda) g_0(\epsilon) Q \Pi |\zeta\rangle_{ij} \ , \tag{14}$$

which can be iterated *ad infinitum* and re-summed to give

$$|\zeta\rangle_{ij} = t_{ij}(\epsilon,\Lambda)|\psi_P\rangle + t_{ij}(\epsilon,\Lambda)\frac{1}{\epsilon - h_0 - Q\Pi t_{ij}(\epsilon,\Lambda)} Q\Pi t_{ij}(\epsilon,\Lambda)|\psi_P\rangle \ . \tag{15}$$

This implies that

$$v_{eff} = \sum_{i<j}^{3} t_{ij}(\epsilon,\Lambda) + \sum_{i<j}^{3} t_{ij}(\epsilon,\Lambda)\frac{1}{\epsilon - h_0 - Q\Pi t_{ij}(\epsilon,\Lambda)} Q\Pi t_{ij}(\epsilon,\Lambda) \tag{16}$$

$$\equiv v_2(\epsilon,\Lambda) + v_3(\epsilon,\Lambda) \ , \tag{17}$$

where

$$v_2(\epsilon,\Lambda) = \sum_{i<j}^{3} t_{ij}(\epsilon,\Lambda) \tag{18}$$

$$v_3(\epsilon,\Lambda) = \sum_{i<j}^{3} t_{ij}(\epsilon,\Lambda)\frac{1}{\epsilon - h_0 - Q\Pi t_{ij}(\epsilon,\Lambda)} Q\Pi t_{ij}(\epsilon,\Lambda). \tag{19}$$

This re-expression of v_{eff} is convenient in that it splits out the purely 2-body interaction from the (induced) 3-body interaction.

So here's my confusion. My assumption has always been that if one ignores $v_3(\epsilon,\Lambda)$, then the calculation becomes the analog of what Bedaque and others have done in the continuum case when they looked at the Effimov effect. But I'm not sure this is the case anymore.

So I think we just need to solve the full BH equation that includes the induced 3-body term. To do this, it is probably simpler to go back a few steps, and use eq. 14 as a guide. Given two HO states in the P space, say $|\Omega\rangle$ and $|\Omega'\rangle$, the matrix element

$$\langle\Omega'|v_{eff}|\Omega\rangle = \langle\Omega'|(1+\Pi)|\omega\rangle_{12} = 3\langle\Omega'|\omega\rangle_{12} \ , \tag{20}$$

where

$$|\omega\rangle_{12} = t_{12}(\epsilon,\Lambda)|\Omega\rangle + t_{12}(\epsilon,\Lambda)g_0(\epsilon)Q\Pi|\omega\rangle_{12} \ . \tag{21}$$

So I have to solve an integral equation for $|\omega\rangle_{12}$, which brings me back to my thesis.

I should note, however, that in my most recent rendition of the three-body code that uses two-body contact interactions, I do not actually separate out the effective two- and three-body interactions via eqns. 18 and 19, but rather just invert eqn. 14:

$$|\zeta\rangle_{ij} = \left[1 - t_{ij}(\epsilon,\Lambda)g_0(\epsilon)Q\Pi\right]^{-1} t_{ij}(\epsilon,\Lambda)|\psi_P\rangle \ . \tag{22}$$

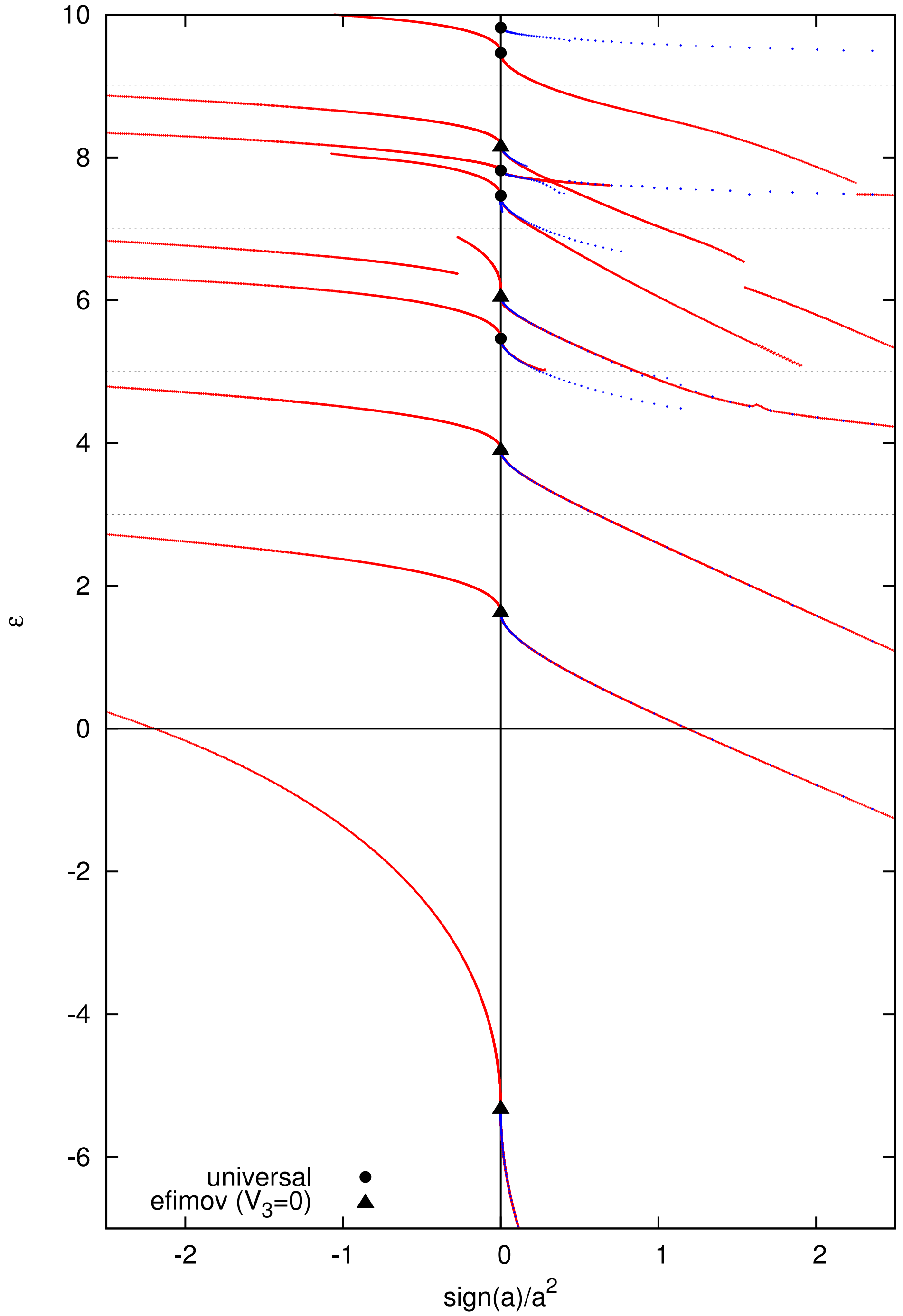

FIGURE 1. The spectrum of three spinless bosons in a trap, as a function of scattering length. A zero-range contact interaction was used in these calculations and no bare three-body interaction was used. The black circles and triangles correspond to Werner and Castin's universal and efimov-like solutions, respectively.

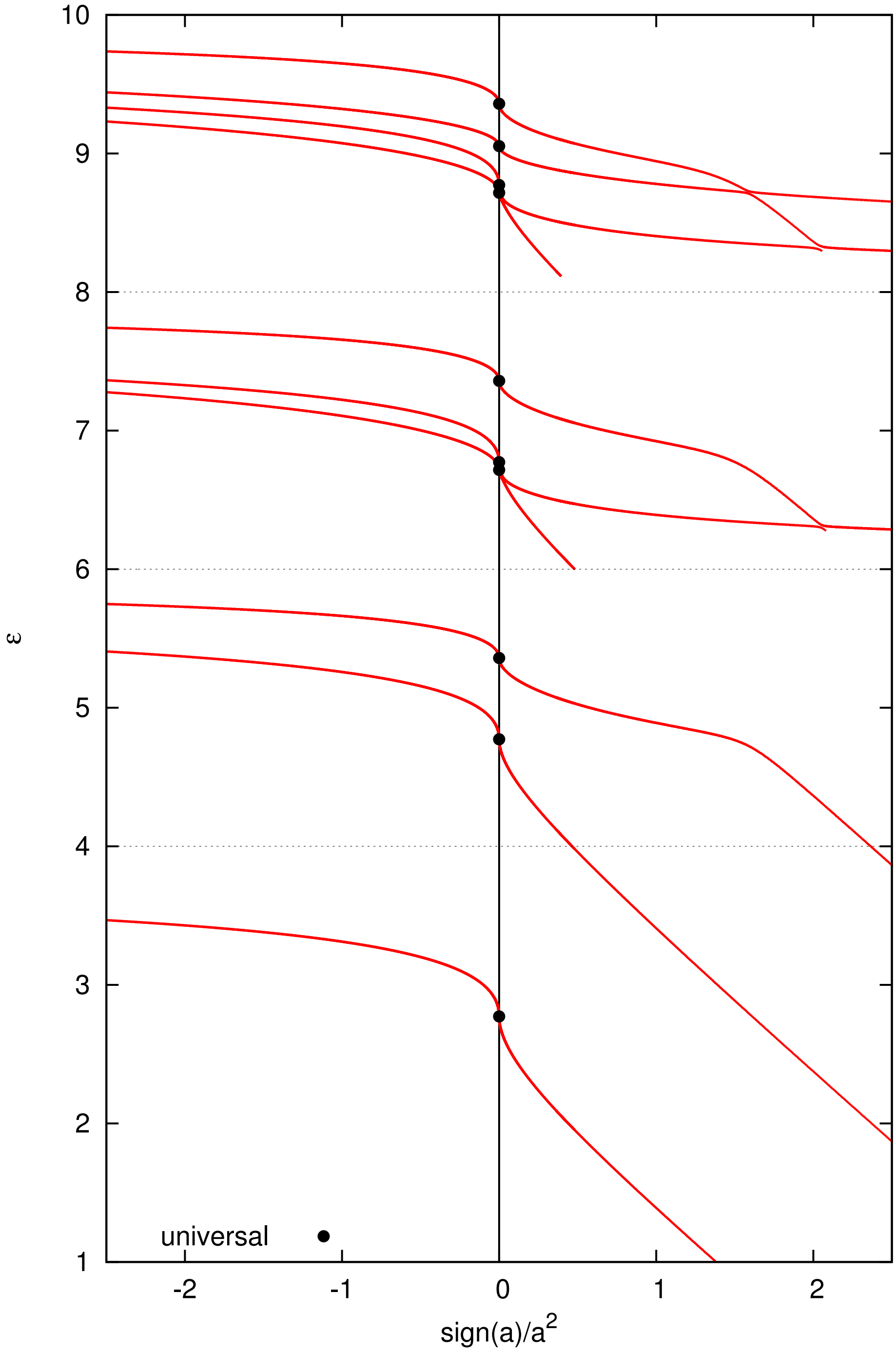

FIGURE 2. The spectrum of three spin 1/2 fermions, coupled to $J = 1$, in a trap, as a function of scattering length. A zero-range contact interaction was used in these calculations and no bare three-body interaction was used. The black circles correspond to Werner and Castin's universal solutions.

3.1. **Adding in a contact three-body force.** So let me now try to include a nonzero v_3. For the moment, let's assume that

$$v_3 = b_3\delta(\vec{r}_{12})\delta(\vec{\rho}_3)\ , \tag{23}$$

where $\vec{r}_{12}$ and $\vec{\rho}_3$ are the two relative Jacobi coordinates. And let me assume that $v_2 = b_2\sum_{i<j}\delta(\vec{r}_{ij})$. Actually, taking advantage of the symmetries of the three-body interaction, I'll express v_3 as

$$v_3 = \frac{b_3}{3}\left(\delta(\vec{r}_{12})\delta(\vec{\rho}_3) + \delta(\vec{r}_{23})\delta(\vec{\rho}_1) + \delta(\vec{r}_{13})\delta(\vec{\rho}_2)\right)\ . \tag{24}$$

I now combine this interaction with v_2, and define a new term

$$\eta = v_2 + v_3 = \sum_{i<j}\eta_{ij}\ , \tag{25}$$

where

$$\eta_{ij} = b_2\delta(\vec{r}_{ij})\left(1 + \frac{b_3}{3b_2}\delta(\vec{\rho}_k)\right)\quad k \neq i,j\ . \tag{26}$$

I can now go through the faddeev decomposition rigamarole to get the equivalent of eq. 12,

$$\begin{aligned}
|\zeta\rangle_{ij} &= \left(1-\eta_{ij}g_0(\epsilon)Q\right)^{-1}\eta_{ij}|\psi_P\rangle + \left(1-\eta_{ij}g_0(\epsilon)Q\right)^{-1}\eta_{ij}g_0(\epsilon)Q\Pi|\zeta\rangle_{ij} &(27)\\
&= \tau^{\Lambda}_{ij}(\epsilon)|\psi_P\rangle + \tau^{\Lambda}_{ij}(\epsilon)g_0(\epsilon)Q\Pi|\zeta\rangle_{ij}\ , &(28)
\end{aligned}$$

where

$$\tau^{\Lambda}_{ij}(\epsilon) = \left(1-\eta_{ij}g_0(\epsilon)Q\right)^{-1}\eta_{ij} \tag{29}$$

is the analog of $t_{ij}(\epsilon,\Lambda)$ but includes the effects of the three-body interaction. It should recover $t_{ij}(\epsilon,\Lambda)$ in the limit that $b_3 \to 0$.

3.1.1. *Derivation of $\tau^{\Lambda}_{ij}(\epsilon)$ in terms of $t_{ij}(\epsilon,\Lambda)$ and v_3.* So now the goal is to express $\tau^{\Lambda}_{ij}(\epsilon)$ in terms of useful quantities, such as $t_{ij}(\epsilon,\Lambda)$. Note the following relationship:

$$\tau^{\Lambda}_{ij}(\epsilon) = \left(1-v_{ij}g_0(\epsilon)Q\right)^{-1}\eta_{ij} + \left(1-v_{ij}g_0(\epsilon)Q\right)^{-1}\frac{v_{ijk}}{3}g_0(\epsilon)Q\tau^{\Lambda}_{ij}(\epsilon) \tag{30}$$

where I have introduced the shorthand expressions $v_{ij} = b_2\delta(\vec{r}_{ij})$ and $v_{ijk} = b_3\delta(\vec{r}_{ij})\delta(\vec{\rho}_k)$. Now I have that

$$\left(1-v_{ij}g_0(\epsilon)Q\right)^{-1} = 1 + t_{ij}(\epsilon,\Lambda)g_0(\epsilon)Q\ . \tag{31}$$

So eqn. 30 becomes

$$\begin{aligned}
\tau^{\Lambda}_{ij}(\epsilon) = t_{ij}(\epsilon,\Lambda) + \left(1+t_{ij}(\epsilon,\Lambda)g_0(\epsilon)Q\right)\frac{v_{ijk}}{3} +&\\
\left(1+t_{ij}(\epsilon,\Lambda)g_0(\epsilon)Q\right)\frac{v_{ijk}}{3}g_0(\epsilon)Q\tau^{\Lambda}_{ij}(\epsilon),&
\end{aligned} \tag{32}$$

which can now be inverted to give

$$\tau_{ij}^{\Lambda}(\epsilon) = \left[1-(1+t_{ij}(\epsilon,\Lambda)g_0(\epsilon)Q)\frac{v_{ijk}}{3}g_0(\epsilon)Q\right]^{-1}\left(t_{ij}(\epsilon,\Lambda)+(1+t_{ij}(\epsilon,\Lambda)g_0(\epsilon)Q)\frac{v_{ijk}}{3}\right) . \tag{33}$$

Let me clean up the notation a little. Let's set

$$\begin{aligned} t_{ij}(\epsilon,\Lambda) &= t_{ij}^{\Lambda}(\epsilon) \\ g_0(\epsilon)Q &= g_0^{\Lambda}(\epsilon) . \end{aligned}$$

So now eqn. 33 becomes

$$\tau_{ij}^{\Lambda}(\epsilon) = \left[1-\left(1+t_{ij}^{\Lambda}(\epsilon)g_0^{\Lambda}(\epsilon)\right)\frac{v_{ijk}}{3}g_0^{\Lambda}(\epsilon)\right]^{-1}\left(t_{ij}^{\Lambda}(\epsilon)+\left(1+t_{ij}^{\Lambda}(\epsilon)g_0^{\Lambda}(\epsilon)\right)\frac{v_{ijk}}{3}\right) . \tag{34}$$

It is obvious from the equation above that $\tau_{ij}^{\Lambda}(\epsilon)$ does indeed become $t_{ij}^{\Lambda}(\epsilon)$ when $b_3 \to 0$.

3.1.2. *Side track: Can I solve* $[\epsilon - h_0 - v_3]^{-1}$*?* Actually, let me define the problem better. I want to solve for t_{ijk}, where

$$t_{ijk} = v_{ijk} + v_{ijk}g_0(\epsilon)t_{ijk} . \tag{35}$$

The separability of v_{ijk} gives me the following relation,

$$t_{ijk} = \frac{v_{ijk}}{1-b_3(\Lambda)\sum_{n,\eta}\frac{f(n,\Lambda)^2 f(\eta,\Lambda)^2}{\epsilon-(2n+2\eta+3)}} , \tag{36}$$

where the poles of the equation provide the spectrum of the three-body system. So I'll use a gaussian regulator. I find that b_3 runs as

$$\frac{1}{b_3(\epsilon,\Lambda)} = \frac{1}{b_3} + \frac{1}{64}\pi\left(\Lambda^4 - 2\left(\epsilon^2-1\right)\log(\Lambda) - 2\Lambda^2\epsilon\right) , \tag{37}$$

where b_3 is tuned to some experimental three-body observable (e.g. binding energy). Note that the renormalized coupling, in addition to being cutoff dependent, depends on energy ϵ. The equation that determines the spectrum of the three-body system is given by

$$\frac{1}{b_3} = \frac{1}{128}\pi(\epsilon+1)\left(2(\epsilon-1)H_{\frac{1}{2}-\frac{\epsilon}{2}} + \epsilon + 3\right) , \tag{38}$$

where H_n represents the n^{th} harmonic number. The right hand side of eq. 38 is shown in fig. 3 as a function of ϵ.

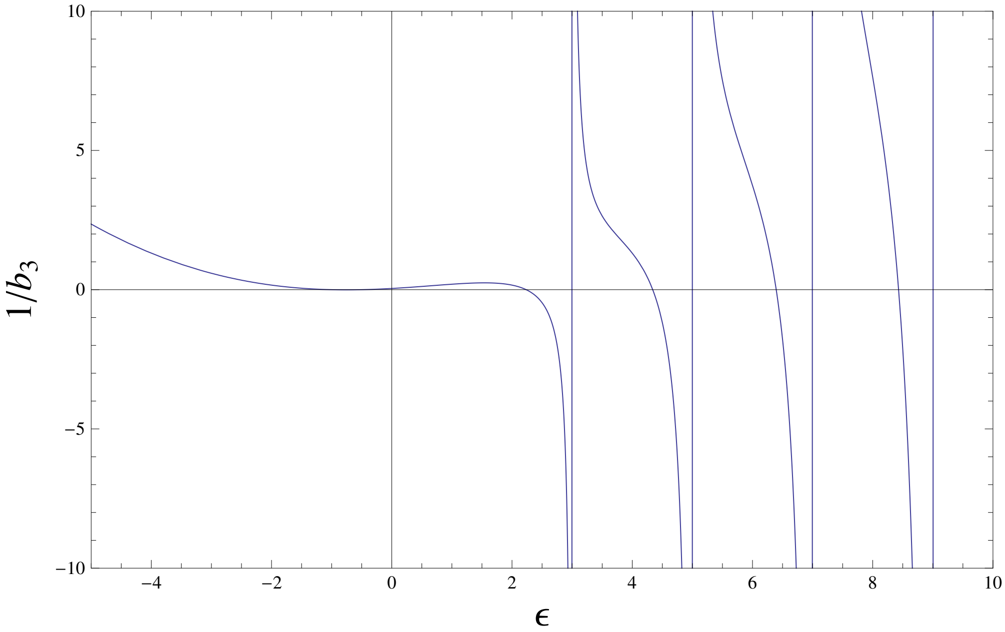

FIGURE 3. The spectrum of three spinless bosons with only an s-wave three-body contact interaction (i.e. NO two-body interaction).