INTEGER CHARACTERIZING SLOPES AND UNKNOTTING NUMBERS

GABRIEL AGOSTINI, SOPHIA CHEN, ALIAKBAR DAEMI, KYLE HAYDEN, CHRISTIAN SERIO, CECILIA WANG, ANTON WU, AND KEXIN WU

Abstract. [Write one!]

1. Introduction

Given a knot $K \subset S^3$, we say that p/q is a *characterizing slope* for K if the oriented homeomorphism type of the manifold obtained by p/q-surgery on K determines K uniquely up to ambient isotopy. In this paper, we denote ambient isotopy by $\simeq$ and oriented homeomorphism by $\cong$. It was a long standing conjecture of Gordon, and was eventually proven by Kronheimer, Mrowka, Ozsváth, and Szabó that every slope is a characterizing slope for the trefoil and the figure eight knot.

In 2018, Picirillo developed a construction for producing isotopic knots for knots with unknotting number one. Through this construction, she was able to prove that the Conway knot is not slice by demonstrating that the produced knot is not slice. Well be adapting Picirillos Construction To (Purpose of the paper) *put an introduction between 1.2 and 1.3 (Describe Picrillos construction in an appendix)

[Introduction with some motivation and introduction of terms like "characterizing slopes." See the papers by Yi and Zhang and by McCoy and by Lackenby on characterizing slopes for ideas to steal. Mention Piccirillo's results for unknotting number one and using this to show that the Conway knot isn't slice. Mention that McCoy's work [McC18], which shows that a hyperbolic knot has only finitely many non-characterizing slopes p/q with $|q| \geq 3$. In a sense, this implies that "most" slopes p/q are characterizing for any given hyperbolic knot K: The probability that a randomly chosen slope p/q is characterizing approaches 1 as $|p| + |q| \to \infty$.]

Theorem 1.1. *If a knot K has unknotting number $u(K) = 1$ and is not a twisted Whitehead double, then K has at most finitely many integer characterizing slopes.*

We worked our way towards Theorem 1.1 by first manually finding knots K_0' with the same 0-surgery as a given knot K with $u(K) = 1$ inspired by Piccirillo's construction and then devising a mechanism to produce all K_n' that shares the same n-surgery with K. Beginning with the former process, we found the following result for knots with low crossing number:

Theorem 1.2. *For knots K with crossing number $c(K) \leq 10$:*

- **(a)** *If K has unknotting number $u(K) = 1$ and K is not a twist knot, then K has at most one integer characterizing slope, namely ± 2.*
- **(b)** *If K is one of the knots 8_4, 8_6, 8_{10}, or 8_{12}, then K has $u(K) > 1$ and has no integer characterizing slopes.*
- **(c)** *If K is the twist knot 8_1, then K has at most one integer characterizing slope, namely 0.*

This theorem specifies the number of integer characterizing slopes for several knots with $c(K) \leq 10$. It is possible that the bound on $c(K)$ can be increased, especially for knots with unknotting number $u(K) = 1$. Similarly, the list in part (b) can probably be expanded, encompassing every knot that fits Piccirilo's construction. Part (c) of this theorem also shows that the assumption in Theorem 1.1 that K is not a twisted Whitehead double is not a necessary condition for the conclusion to hold. Moreover, it was proven by Ozsváth and Szabó 2006, maybe cite that every slope is characterizing for the trefoil and the figure-eight knot. Part (c) shows that this fact does not generalize to twist knots with higher crossing numbers.

Theorem 1.2 affirmatively answers a question posed by Baker and Motegi, who ask if there are knots with crossing number less than eight that have infinitely many non-characterizing slopes [BM18, Question 1.7]. In fact, Theorem 1.2 shows that all knots with crossing number less than eight and unknotting number one which are not twist knots have infinitely many non-characterizing slopes. [See if we can expand this result to be independent of $u(K)$.]

The proof of this theorem relies in part on a computer program to calculate volumes of finitely many surgeries on each knot. We use the software SnapPy, which can triangulate hyperbolic manifolds. The script used is included on maybe put on Appendix?, and the files produced for all knots encompassed can be found on Kyle's website.

Conjecture 1.3 (Baker)**.** *If K and K' are non-isotopic knots in S^3 which yield the same 3-manifold Y under p/q-surgery, then their surgery duals γ and γ' in Y are not homotopic.*

We showed that Baker's conjecture holds for all knots K with unknotting number $u(K) = 1$ and crossing number $c(K) \leq 10$ in the case of zero surgery. The files corresponding to this verification can be found on Kyle's website.

2. Knots with unknotting number one

2.1. **Banded Hopf link diagrams.** A knot K with unknotting number one can be represented by a diagram that consists of a Hopf link with its two components connected by a band. We have the following result:

Theorem 2.1. *If K has unknotting number one, then it is obtained from the Hopf link by a single band move.*

To find the banded Hopf link diagram for a knot K with $u(K) = 1$, we first find the unkotting crossing of K and alternate it. We then draw a meridian adjacent to it [Diagram showing how meridian has to be parallel to crossing] and slide K over the meridian. After simplification, we obtain the banded Hopf link diagram for K.

Twisted Whitehead double knots can be defined in various ways. For our purposes, we use a characterization in terms of band presentations.

Definition 2.2. A knot K is a *twisted Whitehead double* if there exists a band presentation for K in which the band does not cross either component of the Hopf link.

We note that twist knots are twisted Whitehead doubles, but not all twisted Whitehead doubles are twist knots.

2.2. **Baker and Motegi's Construction.** We follow Baker and Motegi [BM18] in constructing knots K'_n with the same n-surgery as a given knot K satisfying certain hypotheses. We prove the following, which strengthens Theorem 1.3 in Baker-Motegi:

Proposition 2.3. *Let K be a knot in S^3. Suppose we can take an unknot c linked with K so that $(0, 0)$ surgery on $K \cup c$ yields S^3 and c is not a meridian of K. Then K has at most finitely many integer characterizing slopes.*

We first recall the definition of a surgery dual to a knot $K \subset S^3$. When we perform p/q-surgery on K, we thicken K to obtain a solid torus, remove this solid torus from S^3, and glue in another solid torus. The core circle of this new solid torus is a new knot γ, the *surgery dual* of K in the manifold $S^3_{p/q}(K)$. The following lemma gives a symmetry condition on the relation of surgery duality.

Lemma 2.4. *Let K be a knot in S^3 and let $n \in \mathbb{Z}$. Let γ be the surgery dual to K in the manifold $S^3_n(K)$. Let K' be the surgery dual to γ in the manifold $S^3_0(\gamma)$. Then $K \simeq K'$ in S^3.*

Proof. In a Kirby diagram for n-surgery on K, we can represent γ by a small meridian to K. Performing 0-surgery on γ in the link $K \cup \gamma$ yields S^3 by a slam dunk move. Now the dual K' of γ can be represented in the Kirby diagram by a meridian to γ. If we slide K' over K in the surgery diagram and remove the cancelling pair $K \cup \gamma$, then we simply obtain another copy of K. Since handle slides give isotopies in the surgered manifold, this proves that $K \simeq K'$. □

We now state a lemma of Baker-Motegi [BM18, Lemma 2.4], with notation adapted.

Lemma 2.5 (Baker-Motegi)**.** *Let $K' \cup c'$ be a two-component link in S^3 such that c' is a meridian of K'. Then $(0, 0)$-surgery on $K' \cup c'$ results in S^3 with its surgery dual link $c \cup K$, for which c is a meridian of K.*

To be clear, c is the surgery dual to K' and K is the dual to c' in S^3. To apply this lemma, we prove the following result:

Lemma 2.6. *Let K be a knot in S^3 and c an unknot linked with K, such that $(0,0)$-surgery on the link $K \cup c$ results in S^3. Define c' and K' to be the surgery duals to K and c in S^3, respectively. Then $(0,0)$-surgery on $K' \cup c'$ yields S^3.*

Proof. By assumption, the manifold obtained after $(0,0)$-surgery on $K \cup c$ is S^3. Thus to perform $(0,0)$-surgery on $K' \cup c'$ in S^3, we can first perform $(0,0)$-surgery on $K \cup c$ and then perform $(0,0)$-surgery on $K' \cup c'$ in the surgered manifold. To do so, we first thicken the two knots K and c to solid tori, remove these solid tori from S^3, and replace them with new solid tori. By definition of surgery duals, the cores of these two new solid tori are c' and K' respectively. We now thicken K' and c' and replace their tubular neighborhoods with solid tori whose core circles are the duals of K' and c'. By Lemma 2.4, these duals are c and K respectively. Since the two solid tori that were initially removed when we performed $(0,0)$-surgery on $K \cup c$ had core circles c and K, this procedure amounts to removing two solid tori from S^3 and gluing them back in the same manner. We therefore obtain S^3 again. □

We record one more result of Baker-Motegi [BM18, Theorem 2.1], which we state as a lemma:

Lemma 2.7 (Baker-Motegi)**.** *Let $K' \cup c'$ be a link in S^3 such that c' is unknotted. Suppose that $(0,0)$-surgery on $K' \cup c'$ results in S^3. Let K be the knot in S^3 which is surgery dual to the image of c' in the surgered S^3, and let K'_n be the knot obtained from K' by twisting n times along c'. In particular, $K'_0 = K'$. Then $S^3_n(K) \cong S^3_n(K'_n)$ for all integers n. Moreover, if c' is not a meridian to K', then $K \not\cong K'_n$ for all but finitely many n.*

We are now equipped to prove Proposition 2.3. Let c' and K' be the surgery duals to K and c as in Lemma 2.6. Then by Lemma 2.6 and the contrapositive of Lemma 2.5, we see that c' is not a meridian to K' under the hypotheses on K and c. It follows from Lemma 2.7 that all but finitely many of the knots K'_n sharing n-surgery with K are not isotopic to K, i.e., all but finitely many integers are non-characterizing slopes for K.

2.3. **Proof of Theorem 1.1.** We show that if K has $u(K) = 1$ and is not a twisted Whitehead double, then we can find an unknot c as in Proposition 2.3. To do so, we consider the band presentation for K, as described in Section 2.1. We begin with a Hopf link $R \cup B$, with both components framed 0. Using the band presentation for K as an instruction, we handle slide R over B. Afterwards, R becomes the knot K, and B becomes an unknot c linked with K. We first show that this link $K \cup c$ satisfies the surgery hypothesis in Proposition 2.3:

Lemma 2.8. *Performing $(0,0)$-surgery on the link $K \cup c$ yields S^3.*

Proof. By construction, $K \cup c$ is obtained by a handle slide from a Hopf link with both components R and B framed 0. After the handle slide, R becomes K with framing ± 2, depending on the sign of the linking number of R and B, and B becomes c. If we adjust the initial framing of R to ∓ 2, then the handle slide yields the same link $K \cup c$, with both K and c having 0-framing. Thus $(0,0)$-surgery on $K \cup c$ is homeomorphic to $(\mp 2, 0)$ surgery on $R \cup B$. It is a well-known fact that for any integer n, $(n,0)$-surgery on a Hopf link yields S^3. □

Since K is not a twisted Whitehead double, in any band presentation for K, the band must cross one component of the Hopf link. The following lemma therefore proves that c is not a meridian to K in the above construction.

Lemma 2.9. *Let $R \cup B$ be a Hopf link, and consider a handle slide of R over B which leaves R a meridian to B. Then there is an equivalent handle slide of R over B along a band which does not cross either R or B.*

Proof. TBD □

Theorem 1.1 now follows from Proposition 2.3.

3. Knots with low crossing number

3.1. **Hyperbolic Dehn surgery.** [Recall and discuss the relevant theorems used in our approach.]

3.2. **Proof of Theorem 1.2.** To prove Theorem 1.2, we need to rule out infinitely many characterizing slopes for each of the knots K we are interested in. In general, this is a two-part algorithm:

(1) We use the theorems of hyperbolic Dehn surgery to find a bound $N \in \mathbb{Z}_{>0}$ such that any slope $n \in \mathbb{Z}$ is not characterizing if $|n| > N$.
(2) For the $2N + 1$ remaining cases, we directly examine a knot K'_n with the same n-surgery as K. If $K'_n \neq K$, we know that n is noncharacterizing.

[First part, based on hyperbolic Dehn surgery theorems. Explain how we found N.]

Finding this bound and examining the volumes of each of $2N+1$ knots in the twist family of each knot K is a tedious computational problem. It was convenient for us to let a computer program run this proof given minimal input.

3.2.1. *Code and Data.* We produced a framed link $L = K \cup R \cup c$ for each of the knots K, where R and c are unknotted components and c is a meridian to R. This link is such that, if the framing on R is zero and the framings on K and c are any integer numbers, an equivalent surgery diagram to L is a link $L' = B \cup R \cup G$ that satisfies Piccirilo's theorem, with K turning into B and c turning into G.

Any link can be described by a partitioned sequence of even numbers called its Dowker-Thistlethwaite code, or DT code. This is provided by the software SnapPy. In the case of a framed link L fitting the description above, we use the following lemma:

Lemma 3.1. *Up to isotopy, L can be uniquely described by its DT code.*

Proof. This lemma is a direct application of Doll and Hoste's Theorem 1.2 [CITE] to our link L. As defined, L will always be a nonsplit link whose orientation is irrelevant for the purposes of integral surgery. R and c are two unknots, so L is a prime link if and only if K is a prime knot. That is the single case we are interested in, since only prime knots are classified along with their invariants and diagrams. Thus, by Doll and Hoste, minimal projections of L are in one-to-one correspondence with valid DT codes. Finally, there is a single solution for the question of which of the components is K, R, and c. We use that c is strictly a meridian to R, so its only crossings are the two they share. Since the entries on the DT code come from crossings either within a single component or between two different ones, we notice that the code corresponding to the component c will have length 1 (since, by definition of a DT code, only even labeled crossings appear). R also does not have crossings with itself, but it has crossings with R and c, so its code has length greater than 1. Moreover, since K is not an unknot, $c(K) \geq 3$, so the code corresponding to K will always be longer than the code corresponding to R. Thus, it is clear that $l(K) > l(R) > l(c) = 1$, where $l(X)$ is the code length of the component X. □

The script `integral_slopes.py`, given the DT code of a link L satisfying the conditions above, outputs a list of possible integral characteristic slopes for any knot K. It relies on writing a text file, running it on SnapPy, saving the output, and interpreting it repeatedly. There are certain system requirements explicit on the code. The script consists of the following steps:

(1) From the DT code, the program writes the file `K [INFO].py`. This file asks for the volume of the knot K, for the volume of the manifold $Z = S^3_0(R)$ obtained by zero-surgery filling the 1-handle component, and for the length of the Seifert longitude in Z.
(2) SnapPy opens, runs, and saves this file, producing a first output file named `[OUT] K [INFO].py`.
(3) The program extracts this information and calculates the bound N using the theorems of hyperbolic Dehn surgery.
(4) The program writes the file `K [TEST].py`. This file asks for the volume of each knot K'_n in the twist family of K, whose exterior is easily obtained by doing n-surgery on the first component of the manifold Z, for all n such that $|n| \leq N$.
(5) SnapPy opens, runs, and saves this file, producing a second output file named `[OUT] K [TEST].py`.

(6) The program verifies if this last file contains all $2N + 1$ cases. This step is included because the run time in SnapPy is sometimes unpredictable.
(7) The program identifies the knots whose volume is less than or equal (up to approximations) to the volume of the knot K. Those cases are singled out in the output file and, if possible, identified by SnapPy.

All files corresponding to the knots listed in Theorem 1.2 can be found on Kyle's website. These files prove the theorem.

Proof. We produced the link L aforementioned for all knots K mentioned in Theorem 1.2. All the DT codes are included in the file `DT_List.txt`, which is correctly formatted for the script `integral_slopes.py`. Running it outputs the `.py` files labeled `[OUT]`, and text that is reproduced in the file `output.txt`. From reading this file, we see that:

(a) For all knots K of crossing number at most 10 with unknotting number $u(K) = 1$ and not twist knots, the program indicates that only $n = 2$ or $n = -2$ could be characterizing.
(b) For K one of the knots 8_4, 8_6, 8_{10}, or 8_{12}, the program indicates that all knots K'_n have different volume than K, so no integral slopes can be characterizing.
(c) For the knot 8_1, the program indicates that only $n = 0$ could be characterizing.

□

3.3. **Possible extensions of Theorem 1.1.** Theorem 1.2 is stronger than Theorem 1.1 for low-crossing number knots, since it is valid for knots K such that the unknotting number $u(K) > 1$ and even some Twisted Whitehead Doubles. It suggests that the previous theorem can be extended to encompass a large number of knots than those with $u(K) = 1$.

To use Piccirillo's construction, the original band presentation is not necessary. Any link $L' = K \cup R$, where R is a one-handled unknot can receive a meridian c to become the link L described in section 3.2.1. This link fits Piccirilo's construction, and will yield a non-trivial twist family of knots K'_n if and only if R is not a meridian to K. As proved, the banded Hopf Link presentation will always yield such L' through a single handle slide, but we have encountered diagrams for this link that do not depend on a band presentation.

We prove parts (b) and (c) of Theorem 1.2 using these diagrams. We could not find an algorithm to produce them, but a few strategies were recurrently successful:

(1) Start with a simple banded Hopf link diagram, then change the framing on the band and perform the handle slide.
(2) Start with a simple banded Hopf link diagram for a knot K and perform the usual handle slide. Then, perform a second handle slide of K over R on a region where K is not parallel to R. Sometimes,

different framings on the second handle slide successfully yielded different links fitting Piccirilo's construction.

(3) Start from a simple diagram $B \cup R \cup G$ that fits Piccirilo's construction. Then, repeatedly slide B over R until it comes off of G.

Such strategies take diagrams for knots we previously find and modify them in order to find analogous for other knots. Whether the two knots related by these operations have any relationship is unknown. If real, such relationship could lead to an algorithm that produces a link L for J with $u(J) > 1$ given the link L for K with $u(K) = 1$. Moreover, using the third strategy we were even successful to find a link L that fits Piccirilo's construction for the twist knot 8_1.

We now pose a question that would allow us to generalize Theorem 1.1 for more knots, dropping the assumptions on the unknotting number and the Twisted Whitehead classification.

[Not sure how to phrase this but I think it could be a good open question from our paper.]

Question 3.2. *For what knots K can a link $L' = K \cup R$, where R is an unknot that fails to be a meridian of K, be produced through handle slides on a Hopf Link?*

4. Additional results

[Any additional findings, including HFK or Khovanov homology findings, or enhanced sliceness obstructions using Piccirillo's technique, can go here.]

References

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