INTEGER CHARACTERIZING SLOPES AND UNKNOTTING NUMBERS

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Abstract. [Write one!]

1. Introduction

[Introduction with some motivation and introduction of terms like "characterizing slopes". See the papers by Yi and Zhang and by McCoy and by Lackenby on characterizing slopes for ideas to steal. Mention Piccirillo's results for unknotting number one and using this to show that the Conway knot isn't slice. Mention that McCoy's work [McC18], which shows that a hyperbolic knot has only finitely many non-characterizing slopes p/q with $|q| \geq 3$. In a sense, this implies that "most" slopes p/q are characterizing for any given hyperbolic knot K: The probability that a randomly chosen slope p/q is characterizing approaches 1 as $|p| + |q| \to \infty$.]

Theorem 1.1. *If a knot K has unknotting number one and is not a twisted Whitehead double, then K has at most finitely many integer characterizing slopes.*

We worked our way towards Theorem 1.1 by manually finding knots K'_n with equal $n-$ surgery as an unknotting number $u(K) = 1$ knot K. Beginning with this process, we found the following result for knots with low crossing number:

Theorem 1.2. *For knots K with crossing number $c(K) \leq 10$:*

- **(a)** *If K has unknotting number $u(K) = 1$ and K is not a twist knot, then K has at most one integer characterizing slope, namely ± 2.*
- **(b)** *If K is one of the knots 8_4, 8_6, 8_{10}, or 8_{12}, then K has $u(K) > 1$ and has no integer characterizing slopes.*
- **(c)** *If K is the twist knot 8_1, then K has at most one integer characterizing slope, namely 0.*

This theorem specifies the number of integer characterizing slopes for several knots with $c(K) \leq 10$. It is possible that the bound on $c(K)$ can be increased, especially for knots with unknotting number $u(K) = 1$. Similarly, the list in part (b) can probably be expanded, encompassing every knot that fits Piccirilo's construction. Part (c) of this theorem also suggests that the assumption in Theorem 1.1 that K is not a Twisted Whitehead Double might not be necessary. But it was proven by Ozsváth and Szabó 2006, maybe cite that every slope is characterizing for K the trefoil and the figure-eight knots, so we should expect part (c) to possibly hold only for twist knots with n twists where n is sufficiently large. Theorem 1.2 affirmatively answers a question proposed in Baker and Motegi:

Question 1.3. *Are there any knots of crossing number less than 8 that have infinitely many noncharacterizing slopes?* [BM18, Question 1.7]

The proof of this theorem relies in part on a computer program to calculate volumes of finitely many surgeries on each knot. The script used is included on maybe put on Appendix?, and the files produced for all knots encompassed can be found on Kyle's website.

1.1. **Surgery duals.** Let Y be a 3-manifold given by p/q-surgery on a knot $K \subset S^3$. Then Y is obtained by thickening K to a solid torus $N(K)$, removing the interior of $N(K)$ from S^3, and gluing a solid torus back into $S^3 \setminus \mathrm{int}(N(K))$ in a different manner. Note that K is the core of the solid torus $N(K)$, i.e., K is identified with $S^1 \times \{0\}$ under the diffeomorphism between $N(K)$ and $S^1 \times D^2$. The core of the new solid torus that is glued into $S^3 \setminus \mathrm{int}(N(K))$ specifies a knot γ in Y, which we call the *surgery dual* of K. In a surgery diagram for Y, γ can be represented by a meridian to K. If $K' \subset S^3$ is another knot which yields the same p/q-surgery as K, then one can ask how the surgery dual γ' of K' in Y is related to γ.

Conjecture 1.4 (Baker)**.** *If K and K' are non-isotopic knots in S^3 which yield the same 3-manifold Y under p/q-surgery, then their surgery duals γ and γ' in Y are not homotopic.*

We showed that Baker's conjecture holds for all knots K with unknotting number $u(K) = 1$ and crossing number $c(K) \leq 10$. The files corresponding to this verification can be found on Kyle's website.

2. Knots with unknotting number one

2.1. **Banded Hopf link diagrams.** [Describe these and how they produce diagrams that fit into Piccirillo's construction. Give examples for a couple knots with unknotting number one, plus an example for a knot with unknotting number two.]

2.2. **Piccirillo's construction.** We have the following theorem due to Piccirillo:

Theorem 2.1 (Piccirillo, 2018)**.** *Let L be a three-component link consisting of disjoint components R, B, and G, giving a surgery diagram such that*

(1) R is a 0-framed unknot,
(2) B and G have integral framings,
(3) If we remove G (resp. B), then R is isotopic to a meridian to B (resp. G).

Let Y be the 3-manifold given by surgery on L. Then for any $n \in \mathbb{Z}$, there are knots $K, K' \subset S^3$ such that $Y \cong S^3_n(K) \cong S^3_n(K')$.

If K is a knot with $u(K) = 1$, then for any integer n, this construction yields another knot K'_n (not necessarily distinct from K) with the same n-surgery as K. To see this, we begin with a banded Hopf link diagram for K. We then take a Hopf link with components R and B, both with framings 0, and we handle slide B over R according to the band presentation for K. This produces a two-component link, with B becoming K and R becoming a 0-framed unknot c linked with K. We adjust the framing of B to n. If we add a meridian G to R and slide B back over R, then we obtain a diagram fitting into Picirillo's construction.

To obtain the knot K'_n, we begin with the two component link $B \cup R$ described above, where B is K with framing n and R is a 0-framed unknot. We add a 0-framed meridian G to K, and we note that if B has nonzero framing, then by a slam dunk, we can change the framing of B to 0 if we add a meridian P to B with framing $-1/n$. We slide B over R, and we isotope the diagram until B and R are both unknots which cross each other twice, i.e., B and R form a Hopf link if G is removed. We then slide G over B until G is no longer linked with R. At this stage, we can remove R using a "lightbulb trick." This leaves us with a two-component link $G \cup P$. The component G is the knot K'_0 with the same 0-surgery as K, and P is an unknot c' linked with K'_0. To obtain K'_n, we twist K along c', n times.

2.3. **Twist families of knots.** [Define what it means to produce a twist family of knots K_n from a two-component link $K \cup C$ where C is an unknot that is linked with K. A picture would help.]

Let K be an unoriented smooth knot in the oriented 3-sphere S^3 and V a solid torus with a preferred framing that contains K in its interior. Define an orientation-preserving homeomorphism f_n of V s.t. f_n(m)=m and f_n(l)=l+nm in $H_1(\partial V)$ where (m, l) is a preferred meridian-longitude pair of V. Call f_n a twisting homeomorphism, which defines the new knot $f_n(K)$ in S^3 and thus a twisting operation on K. Write $K_{V,n} := f_n(K)$.

From a two-component link $K \cup C$ where C is an unknot that is linked with K, we can get the new knot $K_n := K_{V,n}$ where $K_{V,n}$ is with respect to C as the preferred meridian of a solid torus V containing K in its interior (i.e. we obtain K_n from $K \cup C$ by twisting n times along C for K $\subset$ int(V) with V a solid torus with meridian C). A twist family of knots, $K_A := \{K_n | n \in A\}$ where A some index set. Note that $\forall m, n \in A$, K_m is obtained from K_n by a (m-n)-twist of V by Remark 2.2 in "Twisting and Knot Types (Kouno, Morel and Shibuya)".

[Recall Theorem 3.2 from KMS.]

Theorem 2.2. *Let K be a knot in S^3 and V an unknotted solid torus containing K with $w_V(K) \geq 2$. Then there are at most finitely many integers n_i such that $K_{V,n_i} \sim K$.*

2.4. **Proof of Theorem 1.1.** To prove the theorem, we show that under the hypotheses on K, we have $K'_n \simeq K$ for at most finitely many n. If $K'_n \not\simeq K$ for all n, then K has no integer characterizing slopes. If $K'_N \simeq K$ for some N, then we note that K'_n is obtained by twisting K through the unknot c, $n - N$ times. Moreover, c is not a meridian to K'_N if and only if it is not a meridian to K'_0. To see this, note that if c is a meridian to K'_0, then twisting through c does not change K'_0; hence $K'_0 \simeq K'_N$, and c is not a meridian to K'_N. The converse follows by interchanging K'_0 and K'_N. It remains to show that c' is not a meridian to K'_0, for then the result of Kouno, Motegi, and Shibuya [KMS91, Theorem 3.2] described in 2.3 applies to the twist family $\{K'_n\}$ to show that $K'_n \simeq K'_N \simeq K$ for at most finitely many n.

Recall from 2.1 that since K is not a twisted Whitehead double, in any band presentation for K, the band must cross one component of the Hopf link. The following lemma then proves that in the link $K \cup c$ appearing in Piccirillo's construction, the unknot c is not a meridian to K.

Lemma 2.3. *Let $R \cup B$ be a Hopf link, and consider a handle slide of R over B which leaves R a meridian to B. Then there is an equivalent handle slide of R over B along a band which does not cross either R or B.*

Proof. TBD □

We now appeal to a lemma of Baker and Motegi [BM18, Lemma 2.4], with notation adapted:

Lemma 2.4 (Baker-Motegi, 2018)**.** *Let $K'_0 \cup c'$ be a two-component link in S^3 such that c' is a meridian of K'_0. Then $(0,0)$-surgery on $K'_0 \cup c'$ results in S^3 with its surgery dual link $c \cup K$, for which c is a meridian to K.*

Corollary 2.5. *If c is not a meridian to K, then c' is not a meridian to K'_0.*

Proof. We know that c' and K'_0 are surgery duals to K and c respectively. Show that it follows that c and K are dual to K'_0 and c' respectively. □

This completes the proof of the theorem.

3. Knots with low crossing number

3.1. **Possible extensions of Theorem 1.1.** [Note that $u(K) = 1$ is not necessary for Piccirillo's construction. We only need to be able to produce a link $R \cup B$ (with $B \simeq K$ and R an unknot) from handle slides on a Hopf link. Explain this procedure for knots with $u(K) > 1$.]

3.2. **Hyperbolic Dehn surgery.** [Recall and discuss the relevant theorems used in our approach.]

3.3. **Proof of Theorem 1.2.** [Desribe the two-part algorithm we use to rule out integer characterizing slopes, beginning with finding $N \in \mathbb{Z}_{>0}$ such that $\mathrm{vol}(S^3 \setminus K'_n) > \mathrm{vol}(S^3 \setminus K)$ for all $|n| > N$ and then directly examining K'_n for $|n| \leq N$.]

4. Additional results

[Any additional findings, including HFK or Khovanov homology findings, or enhanced sliceness obstructions using Piccirillo's technique, can go here.]

References

[BM18] Kenneth L. Baker and Kimihiko Motegi. Noncharacterizing slopes for hyperbolic knots. *Algebr. Geom. Topol.*, 18(3):1461–1480, 2018.

[KMS91] Masaharu Kouno, Kimihiko Motegi, and Tetsuo Shibuya. Twisting and knot types. *Journal of the Mathematical Society of Japan*, 44(2):199–216, March 1991.

[McC18] Duncan McCoy. On the characterising slopes of hyperbolic knots. Available as arXiv:1807.11099, August 2018.