Knot Surgery and Integer Characterizing Slopes

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Knots and links in the 3-sphere

Definition

A *knot* K is the image of a smooth embedding of the circle S^1 into a 3-manifold, usually the 3-sphere S^3. In particular, K is diffeomorphic to S^1. A *link* L is a disjoint union of knots, which may be knotted together.

Definition

Let M, N be manifolds and $g, h \colon N \to M$ embeddings. An *ambient isotopy* of M carrying g to h is a continuous map $F \colon M \times [0,1] \to M$, such that $F_t = F(\cdot, t)$ is a homeomorphism of M for each $t \in [0,1]$, $F_0 = \mathbb{1}$, and $F_1 \circ g = h$.

- We regard two knots $K, K' \subset S^3$ to be equivalent if they differ by an ambient isotopy of S^3. We write $K \simeq K'$.
- Equivalently, we can view knots as subsets of $\mathbb{R}^3$ rather than S^3.

Knot diagrams

- We can study a knot $K \subset \mathbb{R}^3$ by projecting it onto a hyperplane $\mathbb{R}^2$.
- If $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ is a projection such that $\pi(K)$ is an embedded curve except at finitely many *crossing points*, then $\pi(K)$ is a *diagram* for K.
- The *crossing number* $c(K)$ is the minimum number of crossings in a diagram of K.

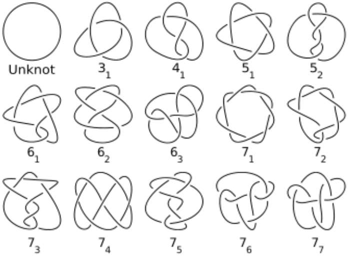


Figure: Knots with $c(K) \leq 7$

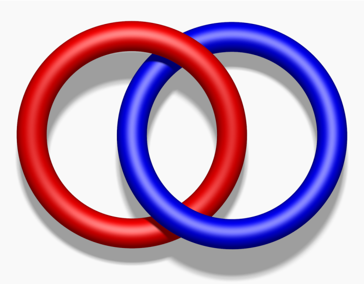
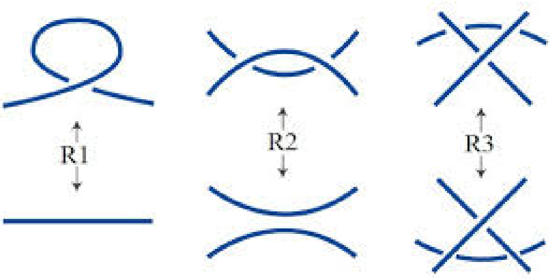


Figure: Hopf link

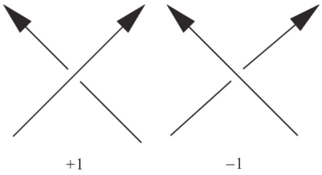
Theorem (Reidemeister)

Two knots K, K' are isotopic if and only if they have diagrams that differ by a sequence of planar isotopies and Reidemeister moves.



Crossings

- If we orient a knot K, then we can define a *sign* for each crossing by the right-hand rule.
- For a two-component link $L = K \cup K'$, the *linking number* $\mathrm{lk}(L)$ is one half the sum of the signs of the crossings between K and K' in a diagram of L.



Unknotting number and band move

Definition

The *unknotting number* $u(K)$ of a knot K is the minimal number of crossing changes that are required to change some diagram of K into a diagram of the unknot.

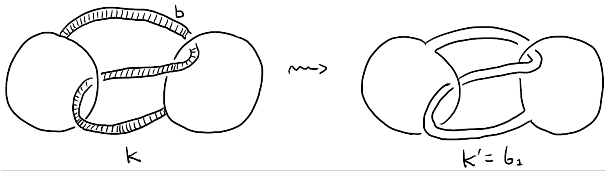
- If $u(K)=0$, then K is an unknot.

Definition

Let K be any knot(or link) in S^3 and let $b \subset S^3$ be any embedded band where one pair of sides lie along arcs in K and where b is otherwise disjoint from K. Join K to itself by deleting the arcs $K \cap b$ and adding the arcs forming the other side of b. The resulting knot(or link) K' is obtained from K by a *band move* along b.

Example of band move and band presentation

- Example of band move:
  The Stevedore knot 6_1 is obtained from a band move on two unknots:



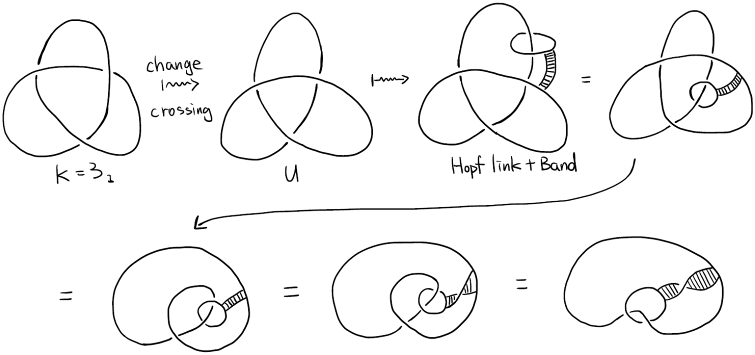
- If K can be obtained from another knot(or link) K' by a single band move, then we say K has a *band presentation*.
- If K can be obtained from a Hopf link by a single band move, then K has a *banded Hopf link presentation*.

Example of banded Hopf link presentation

Theorem

If K has $u(K) = 1$, then it is obtained from the Hopf link by a single band move.

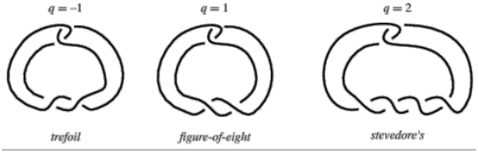
- Example: banded Hopf link presentation of Trefoil:



Twist knots and Twisted whitehead double

Definition

A *twist knot* K is a knot obtained by repeatedly twisting an unknot and linking the ends together.



Definition

A knot K is a *twisted Whitehead double* if there exists a band presentation for K in which the band does not cross either component of the Hopf link.

- Note: twist knots are twisted Whitehead doubles, but not all twisted Whitehead doubles are twist knots.

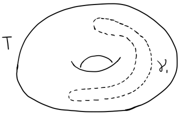
Definition

A s.c.c $\gamma \subset T$ is called *non-separating* if $T - \gamma$ is connected.

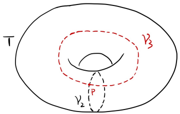
- Every non-sperating s.c.c. in T bounds a disc.
- Any two oriented and non-separating s.c.c's $\gamma, \gamma' \subset T$ are isotopic if and only if their signed intersection, the sum of their local intersections, is 0.

Definition

Any two s.c.c's $\gamma, \gamma' \subset T$ form a *basis* if they are non-separating and intersecting in a single point.



γ_1 = separating

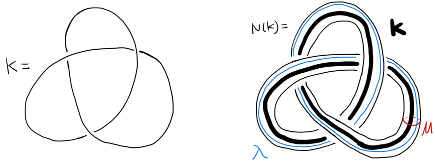


γ_2, γ_3 = non-separating

Preferred Meridian and Longitude Pair for a knot $K \subset S^3$

Definition

Given a knot $K \subset S^3$, $N(K) \cong S^1 \times D^2 \cong$ a solid torus $\mathbb{T}$ where $\partial N(K) \cong$ a torus T. We define a *meridian* μ of K to be an unknot $\partial D^2 \times 1 \subset \partial N(K) \cong T$ that bounds a disc which K crosses exactly once. Then we choose a longitude of K as a circle $\lambda = 1 \times S^1 \subset \partial N(K) \cong T$, which is a parallel copy of K on $\partial N(K)$.



- Note that our choice of the longitude λ for K which intersects the meridian μ non-tangentially in exactly one point and thus (μ, λ) form a basis of T.

Classification of Isotopy Classes of S.C.C's on a Torus

- If a preferred basis of a torus T is (m, l), then any s.c.c $\gamma \subset T$ can be determined (up to isotopy) by two numbers, $p := |n(\gamma \cap m)|$ and $q := |n(\gamma \cap l)|$ where $n(a \cap b)$ is the signed intersection between the two oriented simple closed curves a and b on T.

- Further, isotopy classes s.c.c's on a torus are classified by the set of extended rational numbers $\mathbb{Q} \cup \{\infty\}$.

- Hence, given a knot $K \in S^3$ where $\partial N(K) \cong$ a torus T, any $p/q \in \mathbb{Q} \cup \{\infty\}$ determines a s.c.c β on $\partial N(K)$ that is well-defined up to isotopy. Intuitively, p (and q resp.) is the number of of times that β goes around the meridian (and longitude resp.) that we choose for K.
- We use β to define a 3-manifold $S^3_{p/q}(K)$, called p/q-surgery on K.

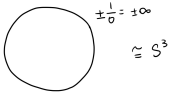
p/q-surgery on a Knot: *drilling* then *filling*

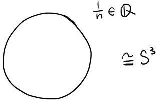
Definition

Given a knot $K \subset S^3$, the *exterior* of K, denoted $X(K)$, is a manifold with boundary obtained by removing the interior of $N(K) = S^1 \times D^2 \cong$ a solid torus $\mathbb{T}$ from S^3. Then $\partial X(K) = \partial N(K) \cong$ a torus T. Let β be the s.c.c on T determined (up to isotopy) by some extended rational number $p/q \in \mathbb{Q} \cup \{\infty\}$. Then the *$p/q$-surgery* on K, a 3-manifold denoted $S^3_{p/q}(K)$, is obtained by gluing another solid torus $\mathbb{T}'$ back to $X(K)$ so that the meridian of $\mathbb{T}'$ is glued to the curve β.

Examples of p/q-surgery on a Knot

- Example: $\pm 1/0$-surgery on a knot K gives back S^3 trivially. Thus any nontrivial surgery on a knot K has rational slopes.
- Example: $S^3_{1/n}(U) \cong S^3$ where U is an unknot and $n \subset \mathbb{Z}$, which we will see why in a second using a Rolfson twist.





Surgery on a Link

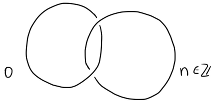
Definition

Similarly, given a link L in S^3 s.t. $\exists p/q \in \mathbb{Q} \cup \{\infty\}$ associated to each component of L, we can do surgery on L by doing surgery along each of its component. Any such link is called a *framed* link.

Theorem

Any oriented, connected 3*-manifold can be obtained by doing surgery along a framed link. So we can think of any* 3*-manifold in terms of a diagram for a framed link. Any such diagram is called a surgery diagram.*

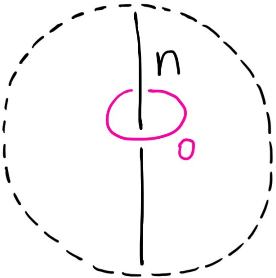
- Example: for any integer n, $(n, 0)$-surgery on a Hopf link K yields S^3, which we will see why in a second by a slam dunk.



Example of Surgery on a Link

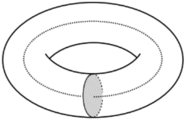
Theorem

For any integer n and any knot K with an unknot c as its meridian, $(n,0)$-surgery on $K \cup c$ yields S^3. Such a pair of K and c forms a cancelling pair that can be deleted without changing the 3-manifold described by the surgery diagram.



A Surgery Dual to a Knot

- Note that K is given by $S^1 \times 0$ inside $N(K) = S^1 \times D^2 \cong$ a solid torus $\mathbb{T}$. i.e. K is the core curve of the solid torus $N(K)$.

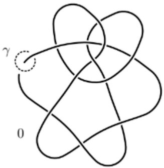


Definition

When doing p/q-surgery on a knot $K \in S^3$, the new solid torus $\mathbb{T}' \cong S^1 \times D^2$ that we glue back in $X(K)$ to produce $S^3_{p/q}(K)$ also has a core curve $S^1 \times 0$, which specifies a knot η in $S^3_{p/q}(K)$. Call η the *surgery dual* of K.

A Surgery Dual to a Knot (Continued)

- For any knot K and any integer n, the *surgery dual* to K in $S^3_n(K)$ can be represented as a meridian to K in the surgery diagram.



$Y = S^3_0(K)$, with $K = 11n38$

Lemma

Surgery duality for knots in S^3 is symmetric.

$$\begin{array}{ccccc} S^3 & n\in\mathbb{Z} & S^3_n(K) & & S^3 \\ \cup & n\text{-surgery} & \cup & 0\text{-surgery} & \cup \\ K & \rightsquigarrow & \gamma & \rightsquigarrow & K \end{array}$$

A Surgery Dual Link to a Framed Link

Definition

A *surgery dual link* to a framed link $L = \bigcup_{i=1}^{N} L_i \in S^3$ is a link $L' = \bigcup_{i=1}^{N} L'_i$, where each component L'_i is the surgery dual to L_i in the surgered manifold Y that we obtain after doing surgery on L.

Piccirillo's construction: Knots with the same surgery

Theorem (Piccirillo 2018)

Let $L = R \cup G \cup B$ be a surgery diagram for some 3-manifold Y such that:

1. *R is a zero-framed unknot, B and G have integral framings.*
2. *Ignoring B, R is isotopic to a meridian of G.*
3. *Ignoring G, R is isotopic to a meridian of B.*
4. *B and G have linking number 0.*

Then, there exist knots K and K' such that $Y \cong S^3_n(K) \cong S^3_n(K')$.

- Piccirillo's construction comes from an older construction, the *dualizable patterns* construction, to produce knots with the same surgery.

Piccirillo's construction (cont.)

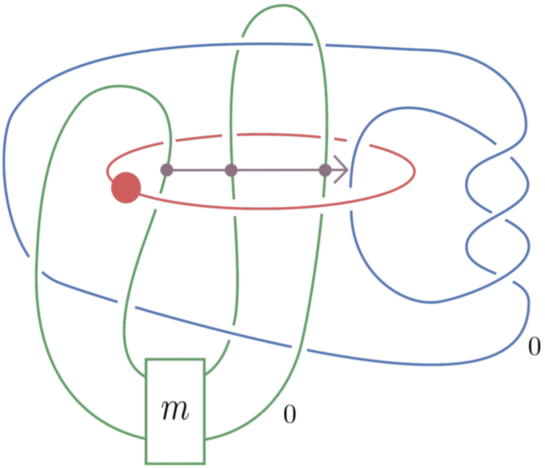


Figure: Diagram of a link L used by Piccirillo on her original paper. The colors (red, blue, and green) match the component letters R, B, G.

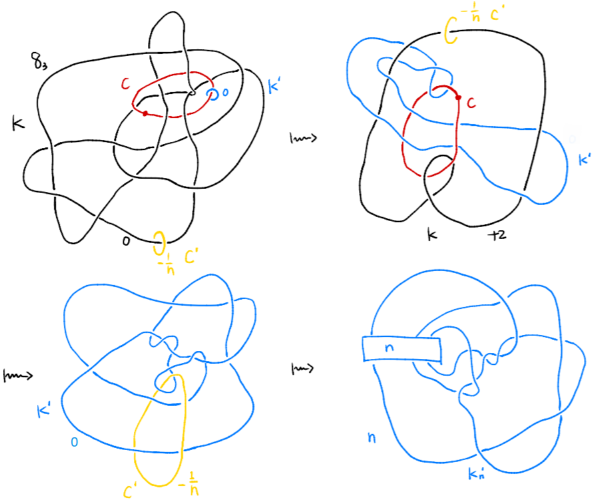
Baker-Motegi: Knots with the same surgery

- We adapted the following construction from Baker and Motegi (2018).
- Let K be a knot, and suppose we can take an unknot c linked with K such that $(0,0)$-surgery on $K \cup c$ is S^3.
- Baker-Motegi present a method for producing knots K_n' with $S^3_n(K) \cong S^3_n(K_n')$ from this link.
- Define K' to be the surgery dual to c in $S^3 = S^3_{(0,0)}(K \cup c)$.
- Then K' has the same 0-surgery as K.

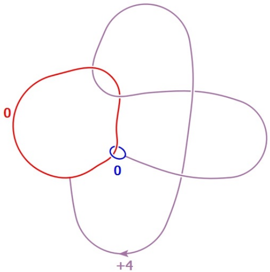
Baker-Motegi (cont.)

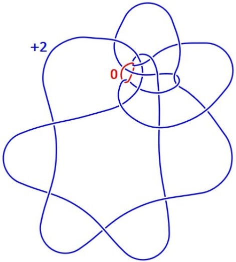
- Also define c' to be the surgery dual to K in S^3.
- After some Kirby calculus, we find that in the surgered manifold $S^3 = S^3_{(0,0)}(K \cup c)$, c' is an unknot linked with K'.
- Let K'_n be the result of twisting K' through c', n times.
- We say that $\{K'_n\}$ forms a *twist family*.
- Then K'_n has the same n-surgery as K for all n.
- Moreover, if c is not a meridian to K, then $K \simeq K'_n$ for at most finitely many n.

Baker-Motegi Illustration



- In the special case where K has unknotting number one, we can use a band presentation for K.
- We start with a Hopf link $R \cup B$ and slide one component over the other according to the band presentation for K.
- Then R remains an unknot, which we rename c, and B becomes the knot K.
- After adding a meridian to R and sliding B over R, we get a diagram that also fits Piccirillo.
- Example: 7_7





$K \cup c$ for $u(K) = 1$ (cont.)

Lemma

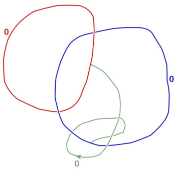
Let $K \cup c$ be the link obtained by the handle slide above. Then $(0, 0)$-surgery on $K \cup c$ gives S^3.

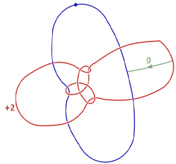
Proof.

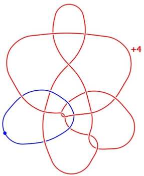
After the handle slide, the framing of c remains 0, and the framing of K changes by ± 2 depending on the linking number of the Hopf link. If we adjust the framing of the blue component of the Hopf link to ∓ 2 before the slide, then we see that $(0, 0)$-surgery on $K \cup c$ is the same as $(\mp 2, 0)$ surgery on a Hopf link. It can be shown that $(n, 0)$-surgery on a Hopf link is S^3 for any $n \in \mathbb{Z}$. $\square$

Obtaining $K \cup c$ when $u(K) > 1$

- We do not have a systematic way of finding a link $K \cup c$ for a given knot K with $u(K) > 1$.
- If we perform multiple slides on a Hopf link, then we obtain a link $K \cup c$ for some knot K with higher unknotting number, but we have no way of predicting what K will be.
- Example: 10_{125}







Applications of the construction

- We have produced a banded Hopf link presentation for all the 78 unknotting number one knots K with $c(K) \leq 10$.
- We followed the above procedure to manually produce a diagram for a knot K' that has the same zero-surgery as K, using a software called KLO which can perform Kirby calculus on link diagrams.
- Using a software called SnapPy, we were able to verify that whenever K was not a twist knot, K' was not isotopic to K. Thus, zero was not a characterizing slope.
- We can use a similar process to check whether any integer slope is characterizing.

- **Question:** Can we classify all integer slopes once we have the link $K \cup c$, with finitely many computations?

Ruling Out Integer Characterizing Slopes

- We developed an algorithm that can be carried out by a computer script to rule out the integer characterizing slopes of some knot K, once we have manually produced the link $L = K \cup c$.

1. Given a link L, run SnapPy commands to find out the volume of the knot K, the volume of the manifold $Z = S^3_0(c)$, and the length of the Seifert longitude in Z.
2. Using these values, apply the theorems of hyperbolic Dehn surgery to find the bound N such that any integer $|n| > N$ is a characterizing slope for K.
3. For the remaining $2N + 1$ cases, verify if the volume of the knot with the same n-surgery as K matches the volume of K.

- We use the *DT code* of the link, which uniquely describes all links that we deal with, up to isotopy.

Findings

Theorem (Low-Crossing Knots)

Regarding the integer slopes of knots K such that $c(K) \leq 10$:

- *If K has unknotting number $u(K) = 1$ and K is not a twist knot, then K has at most one integer characterizing slope, namely ± 2.*
- *If K is the twist knot 8_1, then K has at most one integer characterizing slope, namely 0.*
- *If K is one of the $u(K) = 2$ knots 8_4, 8_6, 8_{10}, 8_{12}, 8_{16}, 10_{148}, 10_{149}, or 10_{150}, K has no possible integer characterizing slope.*
- *If K is one of the $u(K) = 2$ knots 8_3, 10_{125}, or 10_{126}, K has at most one integer characterizing slope.*

Proof of Theorem 1 (cont.)

- Recall that for K with $u(K) = 1$, we could obtain a link $K \cup c$ as in Baker-Motegi with $(0, 0)$-surgery S^3 by sliding over a Hopf link according to the band presentation for K.
- It remains to show that if K is not a twisted Whitehead double, then c is not a meridian to K after the handle slide.
- In this case, in any band presentation for K, the band must cross the disc bounded by one of the components of the Hopf link.

Lemma

Let $R \cup B$ be a Hopf link, and consider a handle slide of R over B yielding a link L in which R remains a meridian to B. Then there exists a handle slide of R over B, yielding a link isotopic to L, along a band that does not cross either of the discs bounded by R or B.

Extension of Theorem 1

- How can we drop the condition on Twisted Whitehead Doubles? And on unknotting number?
- Can we produce an algorithm to find links $L = K \cup c$ from handle slides on a Hopf link?

Conjecture

If K is a knot with unknotting number $u(K) = 1$ and K is not a twist knot, then K has at most one integer characterizing slope: ± 2.

Next Steps

- Rigorously prove the lemma on band presentations.
- Keep experimenting with handle slides in order to expand the list on Theorem 1.2.
- Attempt to use other tools to prove a version of Theorem 1.1 for Twisted Whitehead Doubles.
- Look into our final conjecture.

Proposition 2.3: Strengthening a theorem from Baker-Motegi paper

- We sought to get some condition(s) on a knot $K \subset S^3$ more general than the concrete conditions on K in our theorem 1.1, that ensures K has at most finitely many integer characterizing slopes, in order to apply it to prove theorem 1.1 and gave theoretical grounds that justify our adapted construction that yields K_n' Baker-Motegi.
- We strengthened Theorem 1.1 from Baker-Motegi in which the authors found conditions that ensure a knot $K \subset S^3$ has infinitely many integer non-characterizing slopes. To see that those conditions further ensure K has at most finitely many integer characterizing slopes, we summarize our strengthened result in Proposition 2.3.

Proposition

Let K be a knot in S^3. Suppose we can take an unknot c linked with K so that $(0, 0)$-surgery on $K \cup c$ yields S^3 and c is not a meridian of K. Then K has at most finitely many integer characterizing slopes.

Proof of Proposition 2.3: Identifying Useful Results from Baker-Motegi

- To prove Prop 2.3, observe its hypothesis and conclusion to identify relevant results from Baker-Motegi.
- Hypothesis of Prop 2.3: we can take an unknot c linked with a knot $K \in S^3$ s.t.
- 1) $(0, 0)$-surgery on $K \cup c$ yields S^3
- 2) c is not a meridian of K
- Conclusion of Prop 2.3: K has at most finitely many integer characterizing slopes.
- Theorem 2.1 from Baker-Motegi directly yields our desired conclusion, and we also get knots K'_n that share the same n-surgery with K as a bonus. We cite this theorem as a lemma subsequently.

Lemma (Baker-Motegi)

Let $K' \cup c'$ be a link in S^3 such that c' is unknotted. Suppose that $(0,0)$-surgery on $K' \cup c'$ results in S^3. Let K be the knot in S^3 which is surgery dual to the image of c' in the surgered S^3, and let K'_n be the knot obtained from K' by twisting n times along c'. In particular, $K'_0 = K'$. Then $S^3_n(K) \cong S^3_n(K'_n)$ for all integers n. Moreover, if c' is not a meridian to K', then $K \not\simeq K'_n$ for all but finitely many n.

- Note that if we start with K as in the hypothesis of Prop 2.3 and let $c' \cup K'$ be the surgery dual link as a result of $(0,0)$-surgery on $K \cup c$, we need to show the following to appeal to the above lemma to get both of its conclusions:
- 1) $(0,0)$-surgery on $K' \cup c'$ results in S^3;
- 2) K is the surgery dual to the image of c' in the surgered S^3;
- 3) c' is not a meridian to K'.