

# Knot Surgery and Integer Characterizing Slopes

Gabriel Agostini, Sophia Chen, Christian Serio, Cecilia Wang, Anton Wu, and Kexin Wu

Advisors: Kyle Hayden and Aliakbar Daemi

Columbia University

August 1, 2019

# Knots and links in the 3-sphere

## Definition

A *knot*  $K$  is the image of a smooth embedding of the circle  $S^1$  into a 3-manifold, usually the 3-sphere  $S^3$ . In particular,  $K$  is diffeomorphic to  $S^1$ . A *link*  $L$  is a disjoint union of knots, which may be knotted together.

## Definition

Let  $M, N$  be manifolds and  $g, h: N \rightarrow M$  embeddings. An *ambient isotopy* of  $M$  carrying  $g$  to  $h$  is a continuous map  $F: M \times [0, 1] \rightarrow M$ , such that  $F_t = F(\cdot, t)$  is a homeomorphism of  $M$  for each  $t \in [0, 1]$ ,  $F_0 = \mathbb{1}$ , and  $F_1 \circ g = h$ .

- We regard two knots  $K, K' \subset S^3$  to be equivalent if they differ by an ambient isotopy of  $S^3$ . We write  $K \simeq K'$ .
- Equivalently, we can view knots as subsets of  $\mathbb{R}^3$  rather than  $S^3$ .

# Knot diagrams

- We can study a knot  $K \subset \mathbb{R}^3$  by projecting it onto a hyperplane  $\mathbb{R}^2$ .
- If  $\pi: \mathbb{R}^3 \rightarrow \mathbb{R}^2$  is a projection such that  $\pi(K)$  is an embedded curve except at finitely many *crossing points*, then  $\pi(K)$  is a *diagram* for  $K$ .
- The *crossing number*  $c(K)$  is the minimum number of crossings in a diagram of  $K$ .

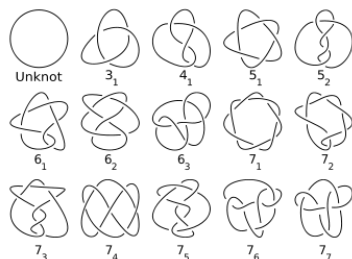


Figure: Knots with  $c(K) \leq 7$

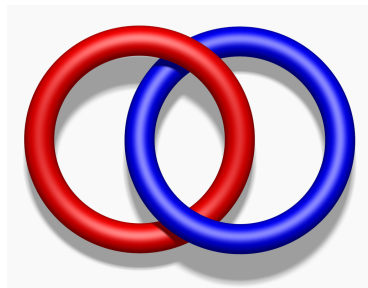
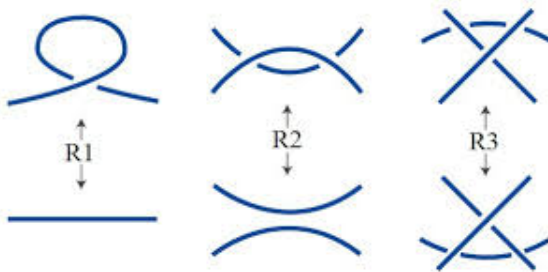


Figure: Hopf link

# Reidemeister moves

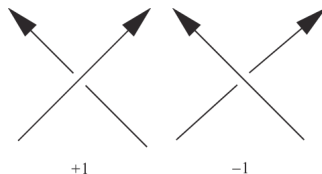
## Theorem (Reidemeister)

*Two knots  $K, K'$  are isotopic if and only if they have diagrams that differ by a sequence of planar isotopies and Reidemeister moves.*



# Crossings

- If we orient a knot  $K$ , then we can define a *sign* for each crossing by the right-hand rule.
- For a two-component link  $L = K \cup K'$ , the *linking number*  $\text{lk}(L)$  is one half the sum of the signs of the crossings between  $K$  and  $K'$  in a diagram of  $L$ .



# Unknotting number and band move

## Definition

The *unknotting number*  $u(K)$  of a knot  $K$  is the minimal number of crossing changes that are required to change some diagram of  $K$  into a diagram of the unknot.

- If  $u(K)=0$ , then  $K$  is an unknot.

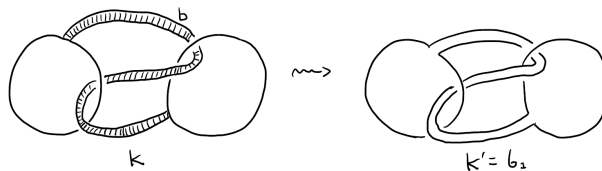
## Definition

Let  $K$  be any knot(or link) in  $S^3$  and let  $b \subset S^3$  be any embedded band where one pair of sides lie along arcs in  $K$  and where  $b$  is otherwise disjoint from  $K$ . Join  $K$  to itself by deleting the arcs  $K \cap b$  and adding the arcs forming the other side of  $b$ . The resulting knot(or link)  $K'$  is obtained from  $K$  by a *band move* along  $b$ .

# Example of band move and band presentation

- Example of band move:

The Stevedore knot  $6_1$  is obtained from a band move on two unknots:



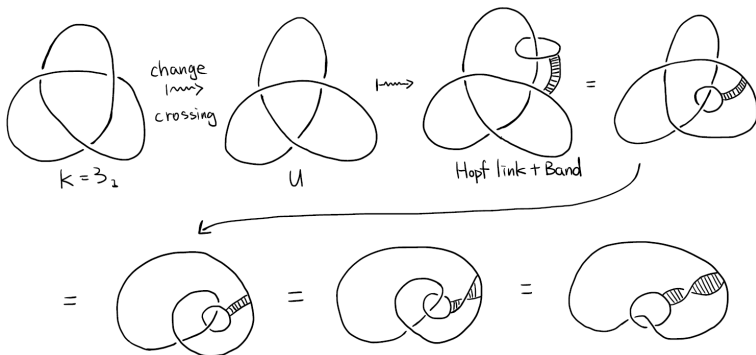
- If  $K$  can be obtained from another knot(or link)  $K'$  by a single band move, then we say  $K$  has a *band presentation*.
- If  $K$  can be obtained from a Hopf link by a single band move, then  $K$  has a *banded Hopf link presentation*.

# Example of banded Hopf link presentation

## Theorem

If  $K$  has  $u(K) = 1$ , then it is obtained from the Hopf link by a single band move.

- Example: banded Hopf link presentation of Trefoil:

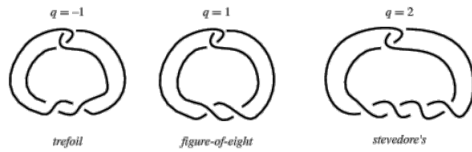




# Twist knots and Twisted whitehead double

## Definition

A *twist knot*  $K$  is a knot obtained by repeatedly twisting an unknot and linking the ends together.



## Definition

A knot  $K$  is a *twisted Whitehead double* if there exists a band presentation for  $K$  in which the band does not cross either component of the Hopf link.

- Note: twist knots are twisted Whitehead doubles, but not all twisted Whitehead doubles are twist knots.

# Simple Closed Curves

## Definition

A *simple closed curve* (s.c.c) is a submanifold of a smooth manifold  $X$ , with itself diffeomorphic to  $S^1$ .



a s.c.c



a non-simple closed curve

- By definition, a s.c.c. is embedded in some smooth manifold and is intuitively a 1-dimensional object diffeomorphic to  $S^1$  without any self-intersection.
- In particular, we can embed s.c.c's in smooth 3-manifolds such as  $S^3$  or a torus  $T$ . An alternate definition for a knot  $K \subset S^3$  is a simple closed curve in it.

# Embedded S.c.c.'s on a Torus

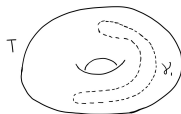
## Definition

A s.c.c.  $\gamma \subset T$  is called *non-separating* if  $T - \gamma$  is connected.

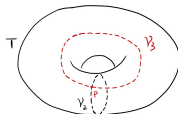
- Every non-separating s.c.c. in  $T$  bounds a disc.
- Any two oriented and non-separating s.c.c.'s  $\gamma, \gamma' \subset T$  are isotopic if and only if their signed intersection, the sum of their local intersections, is 0.

## Definition

Any two s.c.c.'s  $\gamma, \gamma' \subset T$  form a *basis* if they are non-separating and intersecting in a single point.



$\gamma_1$  = separating

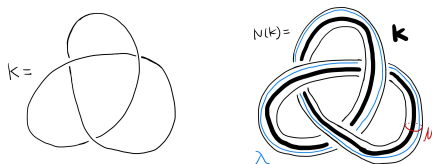


$\gamma_2, \gamma_3$  = non-separating

# Preferred Meridian and Longitude Pair for a knot $K \subset S^3$

## Definition

Given a knot  $K \subset S^3$ ,  $N(K) \cong S^1 \times D^2 \cong$  a solid torus  $\mathbb{T}$  where  $\partial N(K) \cong$  a torus  $T$ . We define a *meridian*  $\mu$  of  $K$  to be an unknot  $\partial D^2 \times 1 \subset \partial N(K) \cong T$  that bounds a disc which  $K$  crosses exactly once. Then we choose a longitude of  $K$  as a circle  $\lambda = 1 \times S^1 \subset \partial N(K) \cong T$ , which is a parallel copy of  $K$  on  $\partial N(K)$ .



- Note that our choice of the longitude  $\lambda$  for  $K$  which intersects the meridian  $\mu$  non-tangentially in exactly one point and thus  $(\mu, \lambda)$  form a basis of  $T$ .

# Classification of Isotopy Classes of S.C.C's on a Torus

- If a preferred basis of a torus  $T$  is  $(m, l)$ , then any s.c.c  $\gamma \subset T$  can be determined (up to isotopy) by two numbers,  $p := |n(\gamma \cap m)|$  and  $q := |n(\gamma \cap l)|$  where  $n(a \cap b)$  is the signed intersection between the two oriented simple closed curves  $a$  and  $b$  on  $T$ .
- Further, isotopy classes s.c.c's on a torus are classified by the set of extended rational numbers  $\mathbb{Q} \cup \{\infty\}$ .
- Hence, given a knot  $K \in S^3$  where  $\partial N(K) \cong$  a torus  $T$ , any  $p/q \in \mathbb{Q} \cup \{\infty\}$  determines a s.c.c  $\beta$  on  $\partial N(K)$  that is well-defined up to isotopy. Intuitively,  $p$  (and  $q$  resp.) is the number of times that  $\beta$  goes around the meridian (and longitude resp.) that we choose for  $K$ .
- We use  $\beta$  to define a 3-manifold  $S^3_{p/q}(K)$ , called  $p/q$ -surgery on  $K$ .

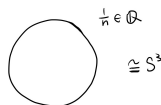
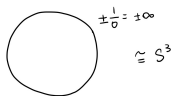
# $p/q$ -surgery on a Knot: *drilling* then *filling*

## Definition

Given a knot  $K \subset S^3$ , the *exterior* of  $K$ , denoted  $X(K)$ , is a manifold with boundary obtained by removing the interior of  $N(K) = S^1 \times D^2 \cong$  a solid torus  $\mathbb{T}$  from  $S^3$ . Then  $\partial X(K) = \partial N(K) \cong$  a torus  $T$ . Let  $\beta$  be the s.c.c on  $T$  determined (up to isotopy) by some extended rational number  $p/q \in \mathbb{Q} \cup \{\infty\}$ . Then the  $p/q$ -surgery on  $K$ , a 3-manifold denoted  $S^3_{p/q}(K)$ , is obtained by gluing another solid torus  $\mathbb{T}'$  back to  $X(K)$  so that the meridian of  $\mathbb{T}'$  is glued to the curve  $\beta$ .

# Examples of $p/q$ -surgery on a Knot

- Example:  $\pm 1/0$ -surgery on a knot  $K$  gives back  $S^3$  trivially. Thus any nontrivial surgery on a knot  $K$  has rational slopes.
- Example:  $S^3_{1/n}(U) \cong S^3$  where  $U$  is an unknot and  $n \in \mathbb{Z}$ , which we will see why in a second using a Rolfsen twist.



# Surgery on a Link

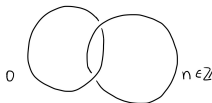
## Definition

Similarly, given a link  $L$  in  $S^3$  s.t.  $\exists p/q \in \mathbb{Q} \cup \{\infty\}$  associated to each component of  $L$ , we can do surgery on  $L$  by doing surgery along each of its component. Any such link is called a *framed link*.

## Theorem

*Any oriented, connected 3-manifold can be obtained by doing surgery along a framed link. So we can think of any 3-manifold in terms of a diagram for a framed link. Any such diagram is called a surgery diagram.*

- Example: for any integer  $n$ ,  $(n, 0)$ -surgery on a Hopf link  $K$  yields  $S^3$ , which we will see why in a second by a slam dunk.

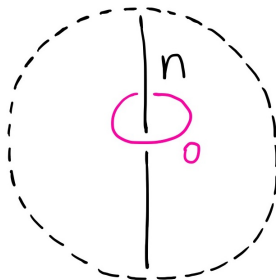




# Example of Surgery on a Link

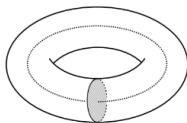
## Theorem

*For any integer  $n$  and any knot  $K$  with an unknot  $c$  as its meridian,  $(n, 0)$ -surgery on  $K \cup c$  yields  $S^3$ . Such a pair of  $K$  and  $c$  forms a cancelling pair that can be deleted without changing the 3-manifold described by the surgery diagram.*



# A Surgery Dual to a Knot

- Note that  $K$  is given by  $S^1 \times 0$  inside  $N(K) = S^1 \times D^2 \cong$  a solid torus  $\mathbb{T}$ . i.e.  $K$  is the core curve of the solid torus  $N(K)$ .

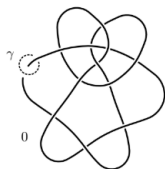


## Definition

When doing  $p/q$ -surgery on a knot  $K \in S^3$ , the new solid torus  $\mathbb{T}' \cong S^1 \times D^2$  that we glue back in  $X(K)$  to produce  $S^3_{p/q}(K)$  also has a core curve  $S^1 \times 0$ , which specifies a knot  $\eta$  in  $S^3_{p/q}(K)$ . Call  $\eta$  the *surgery dual* of  $K$ .

# A Surgery Dual to a Knot (Continued)

- For any knot  $K$  and any integer  $n$ , the *surgery dual* to  $K$  in  $S_n^3(K)$  can be represented as a meridian to  $K$  in the surgery diagram.



$Y = S_0^3(K)$ , with  $K = 11n38$

## Lemma

*Surgery duality for knots in  $S^3$  is symmetric.*

$$\begin{array}{ccccccc}
 S^3 & n \in \mathbb{Z} & S_n^3(K) & & S^3 \\
 U & n\text{-surgery} & U & 0\text{-surgery} & U \\
 K & \rightsquigarrow & \gamma & \rightsquigarrow & K
 \end{array}$$

# A Surgery Dual Link to a Framed Link

## Definition

A *surgery dual link* to a framed link  $L = \bigcup_{i=1}^N L_i \in S^3$  is a link  $L' = \bigcup_{i=1}^N L'_i$ , where each component  $L'_i$  is the surgery dual to  $L_i$  in the surgered manifold  $Y$  that we obtain after doing surgery on  $L$ .

# Piccirillo's construction: Knots with the same surgery

## Theorem (Piccirillo 2018)

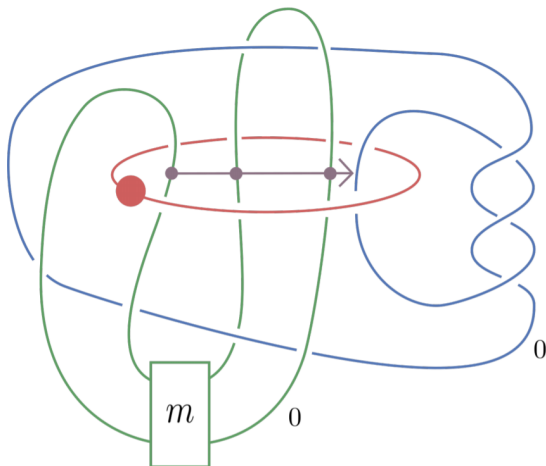
Let  $L = R \cup G \cup B$  be a surgery diagram for some 3-manifold  $Y$  such that:

- 1  $R$  is a zero-framed unknot,  $B$  and  $G$  have integral framings.
- 2 Ignoring  $B$ ,  $R$  is isotopic to a meridian of  $G$ .
- 3 Ignoring  $G$ ,  $R$  is isotopic to a meridian of  $B$ .
- 4  $B$  and  $G$  have linking number 0.

Then, there exist knots  $K$  and  $K'$  such that  $Y \cong S_n^3(K) \cong S_n^3(K')$ .

- Piccirillo's construction comes from an older construction, the *dualizable patterns* construction, to produce knots with the same surgery.

# Piccirillo's construction (cont.)



**Figure:** Diagram of a link  $L$  used by Piccirillo on her original paper. The colors (red, blue, and green) match the component letters  $R$ ,  $B$ ,  $G$ .

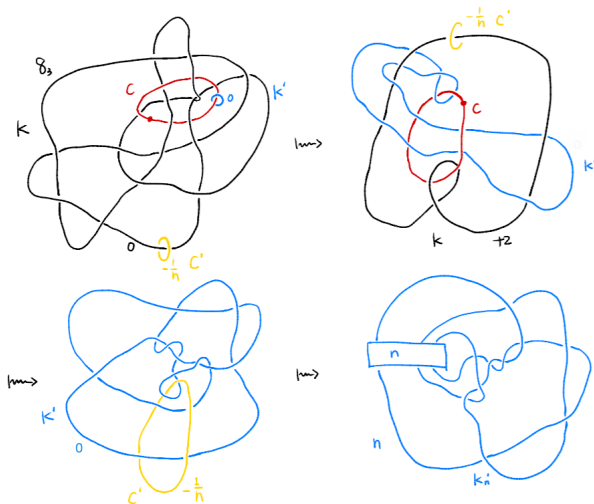
# Baker-Motegi: Knots with the same surgery

- We adapted the following construction from Baker and Motegi (2018).
- Let  $K$  be a knot, and suppose we can take an unknot  $c$  linked with  $K$  such that  $(0,0)$ -surgery on  $K \cup c$  is  $S^3$ .
- Baker-Motegi present a method for producing knots  $K'_n$  with  $S^3_n(K) \cong S^3_n(K'_n)$  from this link.
- Define  $K'$  to be the surgery dual to  $c$  in  $S^3 = S^3_{(0,0)}(K \cup c)$ .
- Then  $K'$  has the same 0-surgery as  $K$ .

- Also define  $c'$  to be the surgery dual to  $K$  in  $S^3$ .
- After some Kirby calculus, we find that in the surgered manifold  $S^3 = S^3_{(0,0)}(K \cup c)$ ,  $c'$  is an unknot linked with  $K'$ .
- Let  $K'_n$  be the result of twisting  $K'$  through  $c'$ ,  $n$  times.
- We say that  $\{K'_n\}$  forms a *twist family*.
- Then  $K'_n$  has the same  $n$ -surgery as  $K$  for all  $n$ .
- Moreover, if  $c$  is not a meridian to  $K$ , then  $K \simeq K'_n$  for at most finitely many  $n$ .

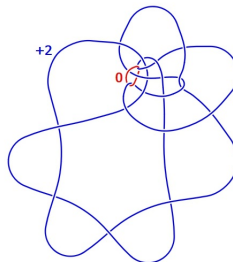
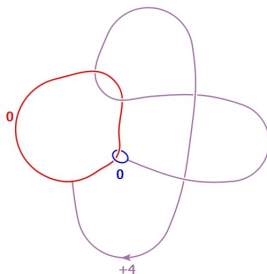


# Baker-Motegi Illustration



# Obtaining a link $K \cup c$ when $u(K) = 1$ (Piccirillo)

- In the special case where  $K$  has unknotting number one, we can use a band presentation for  $K$ .
- We start with a Hopf link  $R \cup B$  and slide one component over the other according to the band presentation for  $K$ .
- Then  $R$  remains an unknot, which we rename  $c$ , and  $B$  becomes the knot  $K$ .
- After adding a meridian to  $R$  and sliding  $B$  over  $R$ , we get a diagram that also fits Piccirillo.
- Example:  $7_7$



# $K \cup c$ for $u(K) = 1$ (cont.)

## Lemma

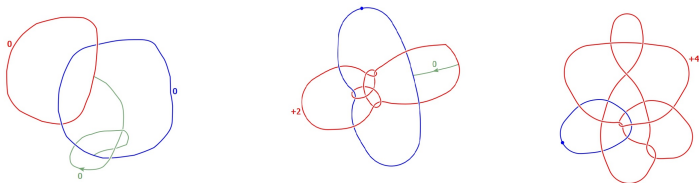
*Let  $K \cup c$  be the link obtained by the handle slide above. Then  $(0,0)$ -surgery on  $K \cup c$  gives  $S^3$ .*

## Proof.

After the handle slide, the framing of  $c$  remains 0, and the framing of  $K$  changes by  $\pm 2$  depending on the linking number of the Hopf link. If we adjust the framing of the blue component of the Hopf link to  $\mp 2$  before the slide, then we see that  $(0,0)$ -surgery on  $K \cup c$  is the same as  $(\mp 2, 0)$  surgery on a Hopf link. It can be shown that  $(n, 0)$ -surgery on a Hopf link is  $S^3$  for any  $n \in \mathbb{Z}$ . □

# Obtaining $K \cup c$ when $u(K) > 1$

- We do not have a systematic way of finding a link  $K \cup c$  for a given knot  $K$  with  $u(K) > 1$ .
- If we perform multiple slides on a Hopf link, then we obtain a link  $K \cup c$  for some knot  $K$  with higher unknotting number, but we have no way of predicting what  $K$  will be.
- Example:  $10_{125}$



# Applications of the construction

- We have produced a banded Hopf link presentation for all the 78 unknotting number one knots  $K$  with  $c(K) \leq 10$ .
- We followed the above procedure to manually produce a diagram for a knot  $K'$  that has the same zero-surgery as  $K$ , using a software called KLO which can perform Kirby calculus on link diagrams.
- Using a software called SnapPy, we were able to verify that whenever  $K$  was not a twist knot,  $K'$  was not isotopic to  $K$ . Thus, zero was not a characterizing slope.
- We can use a similar process to check whether any integer slope is characterizing.
- **Question:** Can we classify all integer slopes once we have the link  $K \cup c$ , with finitely many computations?

# Ruling Out Integer Characterizing Slopes

- We developed an algorithm that can be carried out by a computer script to rule out the integer characterizing slopes of some knot  $K$ , once we have manually produced the link  $L = K \cup c$ .
- ① Given a link  $L$ , run SnapPy commands to find out the volume of the knot  $K$ , the volume of the manifold  $Z = S_0^3(c)$ , and the length of the Seifert longitude in  $Z$ .
- ② Using these values, apply the theorems of hyperbolic Dehn surgery to find the bound  $N$  such that any integer  $|n| > N$  is a characterizing slope for  $K$ .
- ③ For the remaining  $2N + 1$  cases, verify if the volume of the knot with the same  $n$ -surgery as  $K$  matches the volume of  $K$ .
- We use the *DT code* of the link, which uniquely describes all links that we deal with, up to isotopy.

## Theorem (Low-Crossing Knots)

*Regarding the integer slopes of knots  $K$  such that  $c(K) \leq 10$ :*

- If  $K$  has unknotting number  $u(K) = 1$  and  $K$  is not a twist knot, then  $K$  has at most one integer characterizing slope, namely  $\pm 2$ .*
- If  $K$  is the twist knot  $8_1$ , then  $K$  has at most one integer characterizing slope, namely  $0$ .*
- If  $K$  is one of the  $u(K) = 2$  knots  $8_4, 8_6, 8_{10}, 8_{12}, 8_{16}, 10_{148}, 10_{149}$ , or  $10_{150}$ ,  $K$  has no possible integer characterizing slope.*
- If  $K$  is one of the  $u(K) = 2$  knots  $8_3, 10_{125}$ , or  $10_{126}$ ,  $K$  has at most one integer characterizing slope.*

# Proof of Theorem 1 (cont.)

- Recall that for  $K$  with  $u(K) = 1$ , we could obtain a link  $K \cup c$  as in Baker-Motegi with  $(0,0)$ -surgery  $S^3$  by sliding over a Hopf link according to the band presentation for  $K$ .
- It remains to show that if  $K$  is not a twisted Whitehead double, then  $c$  is not a meridian to  $K$  after the handle slide.
- In this case, in any band presentation for  $K$ , the band must cross the disc bounded by one of the components of the Hopf link.

## Lemma

*Let  $R \cup B$  be a Hopf link, and consider a handle slide of  $R$  over  $B$  yielding a link  $L$  in which  $R$  remains a meridian to  $B$ . Then there exists a handle slide of  $R$  over  $B$ , yielding a link isotopic to  $L$ , along a band that does not cross either of the discs bounded by  $R$  or  $B$ .*



# Extension of Theorem 1

- How can we drop the condition on Twisted Whitehead Doubles? And on unknotting number?
- Can we produce an algorithm to find links  $L = K \cup c$  from handle slides on a Hopf link?

## Conjecture

*If  $K$  is a knot with unknotting number  $u(K) = 1$  and  $K$  is not a twist knot, then  $K$  has at most one integer characterizing slope:  $\pm 2$ .*

# Next Steps

- Rigorously prove the lemma on band presentations.
- Keep experimenting with handle slides in order to expand the list on Theorem 1.2.
- Attempt to use other tools to prove a version of Theorem 1.1 for Twisted Whitehead Doubles.
- Look into our final conjecture.