

# Knot Surgery and Integer Characterizing Slopes

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# Knots and links in the 3-sphere

## Definition

A *knot*  $K$  is the image of a smooth embedding of the circle  $S^1$  into a 3-manifold, usually the 3-sphere  $S^3$ . In particular,  $K$  is diffeomorphic to  $S^1$ . A *link*  $L$  is a disjoint union of knots, which may be knotted together.

## Definition

Let  $M, N$  be manifolds and  $g, h: N \rightarrow M$  embeddings. An *ambient isotopy* of  $M$  carrying  $g$  to  $h$  is a continuous map  $F: M \times [0, 1] \rightarrow M$ , such that  $F_t = F(\cdot, t)$  is a homeomorphism of  $M$  for each  $t \in [0, 1]$ ,  $F_0 = \mathbb{1}$ , and  $F_1 \circ g = h$ .

- We regard two knots  $K, K' \subset S^3$  to be equivalent if they differ by an ambient isotopy of  $S^3$ . We write  $K \simeq K'$ .
- Equivalently, we can view knots as subsets of  $\mathbb{R}^3$  rather than  $S^3$ .

# Knot diagrams

- We can study a knot  $K \subset \mathbb{R}^3$  by projecting it onto a hyperplane  $\mathbb{R}^2$ .
- If  $\pi: \mathbb{R}^3 \rightarrow \mathbb{R}^2$  is a projection such that  $\pi(K)$  is an embedded curve except at finitely many *crossing points*, then  $\pi(K)$  is a *diagram* for  $K$ .
- The *crossing number*  $c(K)$  is the minimum number of crossings in a diagram of  $K$ .

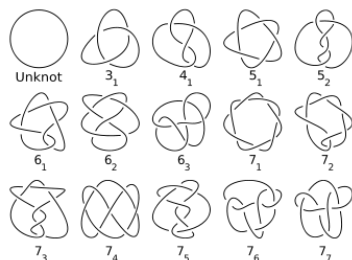


Figure: Knots with  $c(K) \leq 7$

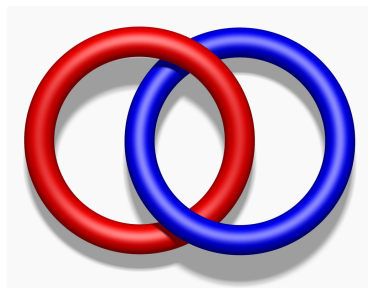
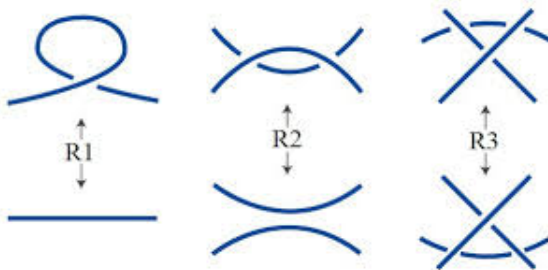


Figure: Hopf link

# Reidemeister moves

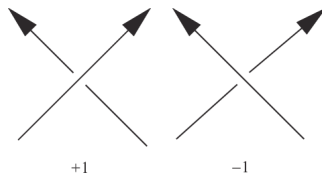
## Theorem (Reidemeister)

*Two knots  $K, K'$  are isotopic if and only if they have diagrams that differ by a sequence of planar isotopies and Reidemeister moves.*



# Crossings

- If we orient a knot  $K$ , then we can define a *sign* for each crossing by the right-hand rule.
- For a two-component link  $L = K \cup K'$ , the *linking number*  $\text{lk}(L)$  is one half the sum of the signs of the crossings between  $K$  and  $K'$  in a diagram of  $L$ .



# Piccirillo's construction: Knots with the same surgery

## Theorem (Piccirillo 2018)

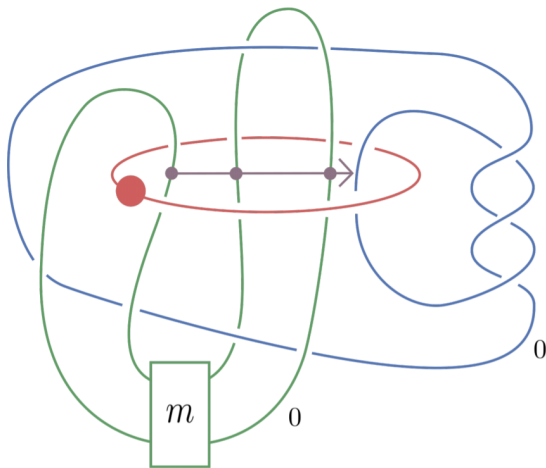
Let  $L = R \cup G \cup B$  be a surgery diagram for some 3-manifold  $Y$  such that:

- ①  $R$  is a zero-framed unknot,  $B$  and  $G$  have integral framings.
- ② Ignoring  $B$ ,  $R$  is isotopic to a meridian of  $G$ .
- ③ Ignoring  $G$ ,  $R$  is isotopic to a meridian of  $B$ .
- ④  $B$  and  $G$  have linking number 0.

Then, there exist knots  $K$  and  $K'$  such that  $Y \cong S_n^3(K) \cong S_n^3(K')$ .

- Piccirillo's construction comes from an older construction, the *dualizable patterns* construction, to produce knots with the same surgery.

# Piccirillo's construction (cont.)



**Figure:** Diagram of a link  $L$  used by Piccirillo on her original paper. The colors (red, blue, and green) match the component letters  $R$ ,  $B$ ,  $G$ .

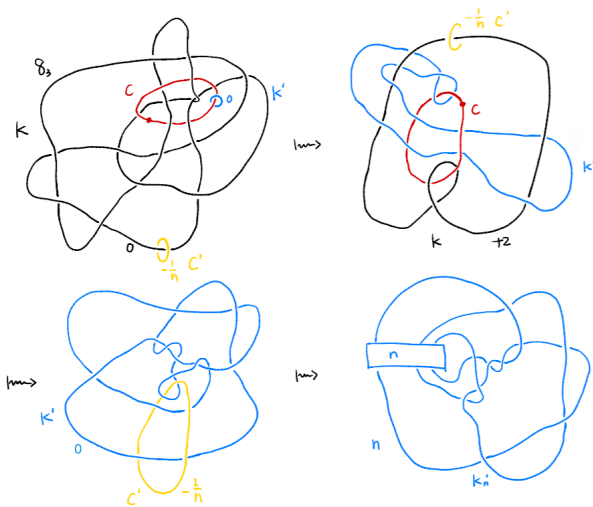
# Baker-Motegi: Knots with the same surgery

- We adapted the following construction from Baker and Motegi (2018).
- Let  $K$  be a knot, and suppose we can take an unknot  $c$  linked with  $K$  such that  $(0,0)$ -surgery on  $K \cup c$  is  $S^3$ .
- Baker-Motegi present a method for producing knots  $K'_n$  with  $S^3_n(K) \cong S^3_n(K'_n)$  from this link.
- Define  $K'$  to be the surgery dual to  $c$  in  $S^3 = S^3_{(0,0)}(K \cup c)$ .
- Then  $K'$  has the same 0-surgery as  $K$ .



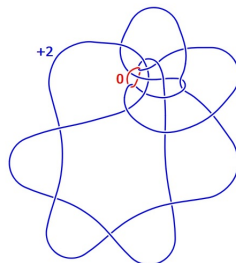
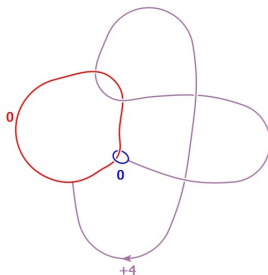
- Also define  $c'$  to be the surgery dual to  $K$  in  $S^3$ .
- After some Kirby calculus, we find that in the surgered manifold  $S^3 = S^3_{(0,0)}(K \cup c)$ ,  $c'$  is an unknot linked with  $K'$ .
- Let  $K'_n$  be the result of twisting  $K'$  through  $c'$ ,  $n$  times.
- Then  $K'_n$  has the same  $n$ -surgery as  $K$  for all  $n$ .
- Moreover, if  $c$  is not a meridian to  $K$ , then  $K \simeq K'_n$  for at most finitely many  $n$ .

# Baker-Motegi Illustration



# Obtaining a link $K \cup c$ when $u(K) = 1$ (Piccirillo)

- In the special case where  $K$  has unknotting number one, we can use a band presentation for  $K$ .
- We start with a Hopf link  $R \cup B$  and slide one component over the other according to the band presentation for  $K$ .
- Then  $R$  remains an unknot, which we rename  $c$ , and  $B$  becomes the knot  $K$ .
- After adding a meridian to  $R$  and sliding  $B$  over  $R$ , we get a diagram that also fits Piccirillo.
- Example:  $7_7$



# $K \cup c$ for $u(K) = 1$ (cont.)

## Lemma

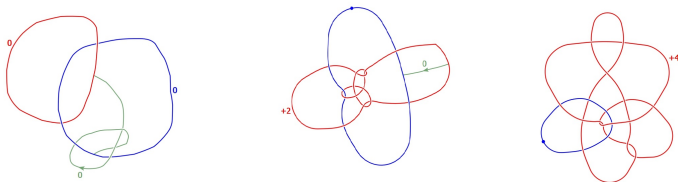
*Let  $K \cup c$  be the link obtained by the handle slide above. Then  $(0,0)$ -surgery on  $K \cup c$  gives  $S^3$ .*

## Proof.

After the handle slide, the framing of  $c$  remains 0, and the framing of  $K$  changes by  $\pm 2$  depending on the linking number of the Hopf link. If we adjust the framing of the blue component of the Hopf link to  $\mp 2$  before the slide, then we see that  $(0,0)$ -surgery on  $K \cup c$  is the same as  $(\mp 2, 0)$  surgery on a Hopf link. It can be shown that  $(n, 0)$ -surgery on a Hopf link is  $S^3$  for any  $n \in \mathbb{Z}$ . □

# Obtaining $K \cup c$ when $u(K) > 1$

- We do not have a systematic way of finding a link  $K \cup c$  for a given knot  $K$  with  $u(K) > 1$ .
- If we perform multiple slides on a Hopf link, then we obtain a link  $K \cup c$  for some knot  $K$  with higher unknotting number, but we have no way of predicting what  $K$  will be.
- Example:  $10_{125}$



# Applications of the construction

- We have produced a banded Hopf link presentation for all the 78 unknotting number one knots  $K$  with  $c(K) \leq 10$ .
- We followed the above procedure to manually produce a diagram for a knot  $K'$  that has the same zero-surgery as  $K$ , using a software called KLO which can perform Kirby calculus on link diagrams.
- Using a software called SnapPy, we were able to verify that whenever  $K$  was not a twist knot,  $K'$  was not isotopic to  $K$ . Thus, zero was not a characterizing slope.
- **Question:** Can we classify all integer slopes once we have the link  $K \cup c$ ?

# Ruling Out Integer Characterizing Slopes

- We developed an algorithm that can be carried out by a computer script to rule out the integer characterizing slopes of some knot  $K$ , once we have manually produced the link  $L = K \cup c$ .
- ① Given a link  $L$ , run SnapPy commands to find out the volume of the knot  $K$ , the volume of the manifold  $Z = S_0^3(c)$ , and the length of the Seifert longitude in  $Z$ .
- ② Using these values, apply the theorems of hyperbolic Dehn surgery to find the bound  $N$  such that any integer  $|n| > N$  is a characterizing slope for  $K$ .
- ③ For the remaining  $2N + 1$  cases, verify if the volume of the knot with the same  $n$ -surgery as  $K$  matches the volume of  $K$ .
- We use the *DT code* of the link, which uniquely describes all links that we deal with, up to isotopy.

## Theorem (Low-Crossing Knots)

*Regarding the integer slopes of knots  $K$  such that  $c(K) \leq 10$ :*

- If  $K$  has unknotting number  $u(K) = 1$  and  $K$  is not a twist knot, then  $K$  has at most one integer characterizing slope, namely  $\pm 2$ .*
- If  $K$  is the twist knot  $8_1$ , then  $K$  has at most one integer characterizing slope, namely  $0$ .*
- If  $K$  is one of the  $u(K) = 2$  knots  $8_4, 8_6, 8_{10}, 8_{12}, 8_{16}, 10_{148}, 10_{149}$ , or  $10_{150}$ ,  $K$  has no possible integer characterizing slope.*
- If  $K$  is one of the  $u(K) = 2$  knots  $8_3, 10_{125}$ , or  $10_{126}$ ,  $K$  has at most one integer characterizing slope.*



# Proof of Theorem 1 (cont.)

- Recall that for  $K$  with  $u(K) = 1$ , we could obtain a link  $K \cup c$  as in Baker-Motegi with  $(0, 0)$ -surgery  $S^3$  by sliding over a Hopf link according to the band presentation for  $K$ .
- It remains to show that if  $K$  is not a twisted Whitehead double, then  $c$  is not a meridian to  $K$  after the handle slide.
- In this case, in any band presentation for  $K$ , the band must cross the disc bounded by one of the components of the Hopf link.

## Lemma

*Let  $R \cup B$  be a Hopf link, and consider a handle slide of  $R$  over  $B$  yielding a link  $L$  in which  $R$  remains a meridian to  $B$ . Then there exists a handle slide of  $R$  over  $B$ , yielding a link isotopic to  $L$ , along a band that does not cross either of the discs bounded by  $R$  or  $B$ .*

# Extension of Theorem 1

- How can we drop the condition on Twisted Whitehead Doubles? And on unknotting number?
- Can we produce an algorithm to find links  $L = K \cup c$  from handle slides on a Hopf link?

## Conjecture

*If  $K$  is a knot with unknotting number  $u(K) = 1$  and  $K$  is not a twist knot, then  $K$  has at most one integer characterizing slope:  $\pm 2$ .*

# Next Steps

- Rigorously prove the lemma on band presentations.
- Keep experimenting with handle slides in order to expand the list on Theorem 1.2.
- Attempt to use other tools to prove a version of Theorem 1.1 for Twisted Whitehead Doubles.
- Look into our final conjecture.