

INTEGER CHARACTERIZING SLOPES AND UNKNOTTING NUMBERS

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ABSTRACT. [Write one!]

1. INTRODUCTION

[Introduction with some motivation and introduction of terms like “characterizing slopes.” See the papers by Yi and Zhang and by McCoy and by Lackenby on characterizing slopes for ideas to steal. Mention Piccirillo’s results for unknotting number one and using this to show that the Conway knot isn’t slice. Mention that McCoy’s work [McC18], which shows that a hyperbolic knot has only finitely many non-characterizing slopes p/q with $|q| \geq 3$. In a sense, this implies that “most” slopes p/q are characterizing for any given hyperbolic knot K : The probability that a randomly chosen slope p/q is characterizing approaches 1 as $|p| + |q| \rightarrow \infty$.]

Theorem 1.1. *If a knot K has unknotting number $u(K) = 1$ and is not a twisted Whitehead double, then K has at most finitely many integer characterizing slopes.*

We worked our way towards Theorem 1.1 by first manually finding knots K'_0 with the same 0-surgery as a given knot K with $u(K) = 1$ inspired by Piccirillo’s construction and then devising a mechanism to produce all K'_n that shares the same n -surgery with K . Beginning with the former process, we found the following result for knots with low crossing number:

Theorem 1.2. *For knots K with crossing number $c(K) \leq 10$:*

- (a) *If K has unknotting number $u(K) = 1$ and K is not a twist knot, then K has at most one integer characterizing slope, namely ± 2 .*
- (b) *If K is one of the knots 8_4 , 8_6 , 8_{10} , or 8_{12} , then K has $u(K) > 1$ and has no integer characterizing slopes.*
- (c) *If K is the twist knot 8_1 , then K has at most one integer characterizing slope, namely 0.*

This theorem specifies the number of integer characterizing slopes for several knots with $c(K) \leq 10$. It is possible that the bound on $c(K)$ can be increased, especially for knots with unknotting number $u(K) = 1$. Similarly, the list in part (b) can probably be expanded, encompassing every knot that fits Piccirillo’s construction. Part (c) of this theorem also shows that the

assumption in Theorem 1.1 that K is not a twisted Whitehead double is not a necessary condition for the conclusion to hold. Moreover, it was proven by Ozsváth and Szabó 2006, maybe cite that every slope is characterizing for the trefoil and the figure-eight knot. Part (c) shows that this fact does not generalize to twist knots with higher crossing numbers.

Theorem 1.2 affirmatively answers a question posed by Baker and Motegi, who ask if there are knots with crossing number less than eight that have infinitely many non-characterizing slopes [BM18, Question 1.7]. In fact, Theorem 1.2 shows that all knots with crossing number less than eight and unknotting number one which are not twist knots have infinitely many non-characterizing slopes. [See if we can expand this result to be independent of $u(K)$.]

The proof of this theorem relies in part on a computer program to calculate volumes of finitely many surgeries on each knot. We use the software SnapPy, which can triangulate hyperbolic manifolds. The script used is included on maybe put on Appendix?, and the files produced for all knots encompassed can be found on Kyle's website.

1.1. Surgery duals. Let Y be a 3-manifold given by p/q -surgery on a knot $K \subset S^3$. Then Y is obtained by thickening K to a solid torus $N(K)$, removing the interior of $N(K)$ from S^3 , and gluing a solid torus back into $S^3 \setminus \text{int}(N(K))$ in a different manner. Note that K is the core of the solid torus $N(K)$, i.e., K is identified with $S^1 \times \{0\}$ under the diffeomorphism between $N(K)$ and $S^1 \times D^2$. The core of the new solid torus that is glued into $S^3 \setminus \text{int}(N(K))$ specifies a knot γ in Y , which we call the *surgery dual* of K . In a surgery diagram for Y , γ can be represented by a meridian to K . If $K' \subset S^3$ is another knot which yields the same p/q -surgery as K , then one can ask how the surgery dual γ' of K' in Y is related to γ .

Conjecture 1.3 (Baker). *If K and K' are non-isotopic knots in S^3 which yield the same 3-manifold Y under p/q -surgery, then their surgery duals γ and γ' in Y are not homotopic.*

We showed that Baker's conjecture holds for all knots K with unknotting number $u(K) = 1$ and crossing number $c(K) \leq 10$ in the case of zero surgery. The files corresponding to this verification can be found on Kyle's website.

2. KNOTS WITH UNKNOTTING NUMBER ONE

2.1. Banded Hopf link diagrams. A knot K with unknotting number one can be represented by a diagram that consists of a Hopf link with its two components connected by a band. We have the following theorem:

Theorem 2.1. *If K has unknotting number one, then it is obtained from the Hopf link by a single band move.*

To find the banded Hopf link diagram for a knot K with $u(K) = 1$, we first find the unknotting crossing of K and alternate it. We then draw a

meridian adjacent to it and slide K over the meridian. After simplification, we obtain the banded Hopf link diagram for K .

Twisted Whitehead double knots can be defined in various ways. For our purposes, we use a characterization in terms of band presentations.

Definition 2.2. A knot K is a *twisted Whitehead double* if there exists a band presentation for K in which the band does not cross either component of the Hopf link.

We note that twist knots are twisted Whitehead doubles, but not all twisted Whitehead doubles are twist knots.

2.2. Piccirillo's construction. We have the following theorem due to Piccirillo:

Theorem 2.3 (Piccirillo, 2018). *Let L be a three-component link consisting of disjoint components R , B , and G , giving a surgery diagram such that*

- (a) R is a 0-framed unknot,
- (b) B and G have integral framings,
- (c) *If we remove G (resp. B), then R is isotopic to a meridian to B (resp. G).*

Let Y be the 3-manifold given by surgery on L . Then for any $n \in \mathbb{Z}$, there are knots $K, K' \subset S^3$ such that $Y \cong S_n^3(K) \cong S_n^3(K')$.

Thus for K a knot with unknotting number $u(K) = 1$, there exists another knot K'_n (not necessarily distinct from K) with the same n -surgery as K for any integer n .

Inspired by Piccirillo's construction, we can produce some form of the link $K \cup c$ for K a knot with $u(K) = 1$ that would serve as an intermediary between n -surgery on K and explicit K'_n . We start with a banded Hopf link diagram for K . We take a Hopf link with components R and B , both with framings 0, and we handle slide B over R according to the band presentation for K . This produces a two-component link where B becomes K and R a 0-framed unknot c linked with K . We then adjust the framing of B to n to correspond with n -surgery on K . Now add a meridian G to $R = c$ and slide B back over R and we hereby obtain a diagram that fits into criteria of Piccirillo's construction that would allow us to produce K'_n that shares the same n -surgery as K by Kirby Calculus and isotopes in the end.

Overall, we seek an explicit construction that would turn n -surgery on K into K'_n , with its initial step inspired by Piccirillo's as above. Next, we take the following concepts developed by Baker and Motegi in [BM18] and Kouno, Motegi, and Shibuya in [KMS91] and adapt them for our use.

2.3. Twist families of knots. We follow Kouno, Motegi, and Shibuya's treatment of a twisting operation on a knot inside a solid torus. Let K be an unoriented smooth knot in the oriented 3-sphere S^3 and V a solid torus with a preferred framing that contains K in its interior. Define an

orientation-preserving homeomorphism f_n of V such that $f_n(m) = m$ and $f_n(l) = l + nm$ in $H_1(\partial V)$ where (m, l) is a preferred meridian-longitude pair of V . This twisting homeomorphism f_n defines a new knot $f_n(K)$ in S^3 , the result of twisting K inside V . Write $K_{V,n} := f_n(K)$.

From a two-component link $K \cup c$ with c an unknot linked with K , we can obtain a new knot $K_n := K_{V,n}$ by taking V to be a solid torus that has preferred meridian c and contains K in its interior. Loosely speaking, we obtain K_n from $K \cup c$ by twisting K for n times along c and in particular, $K_0 = K$. We take after [BM18, Section 2] to define the *twist family* with base knot K and unknot c to be the set $\{K_n | n \in \mathbb{Z}\}$. Note that for all $m, n \in \mathbb{Z}$, K_m is obtained from K_n by an $(m - n)$ -twist along V [KMS91, Remark 2.2], so any knot K_n can be taken as the base of the twist family.

To construct the knot K'_n from n -surgery on K , we first change the framing of K to n and call it B . We then add a cancelling pair $R \cup G$ to B in the exact same place as in the diagram $B \cup R \cup G$ that we would get by following the procedure illustrated near the end of section 2.2 to invoke Piccirillo's theorem. Note that by a slam dunk, we can change the framing of B to 0 if we add a $-1/n$ -framed meridian P to B . Specifically when $n = 0$, such a slam dunk leads to adding P with $-\infty$ framing to B while the framing of B remains unchanged, which amounts to doing nothing to B . We slide B over R and isotope the diagram until B and R are both unknots that cross each other twice (i.e., B and R form a Hopf link if G were removed). Then slide G over B until G is no longer linked with R . Now we can remove R using a "lightbulb trick" that would leave us a two-component link $G \cup P$, with G as the knot K'_0 that shares the same 0-surgery as K (since a slam dunk amounted to not having added any P in the first place) and P as an unknot c' linked with K'_0 . Observe that the framing of P remains unchanged throughout this process and we can therefore do a n -Rolfsen twist on $G \cup P$ to get rid of $P = c'$. As a result, we are left with only one knot, the knot obtained from twisting K'_0 along c' for n times, which is exactly K'_n as desired.

The knots K'_n for $n \in \mathbb{Z}$ clearly form a twist family with base K'_0 (in fact, any K'_n works) and unknot c' , according to our definition above. Thus we aim to adapt a theorem in [KMS91] into our context for our use in the proof of theorem 1.1.

The wrapping number $w_V(K)$ of K in V is defined to be the minimal intersection number of K with a meridional disc in V . Kouno, Motegi, and Shibuya proved that for a twist family $\{K_{V,n}\}$ in a solid torus V with $w_V(K) \geq 2$, there are at most finitely many integers n_i for which $K_{V,n_i} \sim K$ [KMS91, Theorem 3.2]. Here, $K_{V,n_i} \sim K$ means that there exists a homeomorphism of S^3 carrying K_{V,n_i} to K , which is weaker than the existence of an orientation-preserving homeomorphism of S^3 sending K_{V,n_i} to K . Further, the latter existence is equivalent to the existence of an ambient isotopy between K_{V,n_i} and K [KMS91, Introduction]. Moreover, note that $w_V(K) \geq 1$ for c linked with K and $w_V(K) = 1$ if and only if c is a meridian to K . Our

G and $P = c'$ are always linked following the slam dunk in our construction of K'_n , assuming that for $n=0$, we add P with $-\infty$ framing to B . Hence, we can translate [KMS91, Theorem 3.2] into the language of the link $K \cup c$ as follows:

Theorem 2.4 (Kouno-Motegi-Shibuya). *Let K be a knot in S^3 and c an unknot linked with K . Let $\{K_n\}$ be the twist family with base knot K and unknot c . If c is not a meridian to K , then there are at most finitely many integers n_i such that $K_{n_i} \simeq K$.*

Here, the relation \simeq denotes ambient isotopy of knots in S^3 .

2.4. Proof of Theorem 1.1. We show that under the hypotheses on K , $K'_n \simeq K$ for at most finitely many n . If $K'_n \not\simeq K$ for all n , the conclusion holds. If $K'_N \simeq K$ for some N , then we regard K'_N as the base knot of the family $\{K'_n\}$. Note that c is not a meridian to K'_N as long as it is not a meridian to K'_0 . To see this, note that if c' is a meridian to K'_N , then twisting through c' does not change K'_N ; hence by twisting $-N$ times, we see that $K'_N \simeq K'_0$, and c' is a meridian to K'_0 . It remains to show that c' is not a meridian to K'_0 , for then Theorem 2.5 applies to the twist family $\{K'_n\}$ to show that $K'_n \simeq K'_N \simeq K$ for at most finitely many n .

Since K is not a twisted Whitehead double, in any band presentation for K , the band must cross one component of the Hopf link. The following lemma therefore proves that in the link $K \cup c$ appearing in Piccirillo's construction, the unknot c is not a meridian to K .

Lemma 2.5. *Let $R \cup B$ be a Hopf link, and consider a handle slide of R over B which leaves R a meridian to B . Then there is an equivalent handle slide of R over B along a band which does not cross either R or B .*

Proof. TBD □

We now appeal to a lemma of Baker and Motegi [BM18, Lemma 2.4], with notation adapted:

Lemma 2.6 (Baker-Motegi, 2018). *Let $K'_0 \cup c'$ be a two-component link in S^3 such that c' is a meridian of K'_0 . Then $(0,0)$ -surgery on $K'_0 \cup c'$ results in S^3 with its surgery dual link $c \cup K$, for which c is a meridian to K .*

Note that in our context, $(0,0)$ -surgery on $K'_0 \cup c'$ always results in S^3 with its surgery dual link $c \cup K$. We can thus simplify the lemma as: c' being a meridian of K'_0 implies that c is a meridian to K . Hence, we obtain the following corollary as the contrapositive of the simplified lemma.

Corollary 2.7. *If c is not a meridian to K , then c' is not a meridian to K'_0 , with K , c , K'_0 and c' as above.*

Proof. We know that c' and K'_0 are surgery duals to K and c respectively. Show that it follows that c and K are dual to K'_0 and c' respectively. □

This completes the proof of the theorem.

3. KNOTS WITH LOW CROSSING NUMBER

3.1. Hyperbolic Dehn surgery. [Recall and discuss the relevant theorems used in our approach.]

3.2. Proof of Theorem 1.2. To prove Theorem 1.2, we need to rule out infinitely many characterizing slopes for each of the knots K we are interested in. In general, this is a two-part algorithm:

- (1) We use the theorems of hyperbolic Dehn surgery to find a bound $N \in \mathbb{Z}_{>0}$ such that any slope $n \in \mathbb{Z}$ is not characterizing if $|n| > N$.
- (2) For the $2N + 1$ remaining cases, we directly examine a knot K'_n with the same n -surgery as K . If $K'_n \neq K$, we know that n is noncharacterizing.

[First part, based on hyperbolic Dehn surgery theorems. Explain how we found N .]

Finding this bound and examining the volumes of each of $2N + 1$ knots in the twist family of each knot K is a tedious computational problem. It was convenient for us to let a computer program run this proof given minimal input.

3.2.1. Code and Data. We produced a framed link $L = K \cup R \cup c$ for each of the knots K , where R and c are unknotted components and c is a meridian to R . This link is such that, if the framing on R is zero and the framings on K and c are any integer numbers, an equivalent surgery diagram to L is a link $L' = B \cup R \cup G$ that satisfies Piccirillo's theorem, with K turning into B and c turning into G .

Any link can be described by a partitioned sequence of even numbers called its Dowker-Thistlethwaite code, or DT code. This is provided by the software SnapPy. In the case of a framed link L fitting the description above, we use the following lemma:

Lemma 3.1. *Up to isotopy, L can be uniquely described by its DT code.*

Proof. This lemma is a direct application of Doll and Hoste's Theorem 1.2 [CITE] to our link L . As defined, L will always be a nonsplit link whose orientation is irrelevant for the purposes of integral surgery. R and c are two unknots, so L is a prime link if and only if K is a prime knot. That is the single case we are interested in, since only prime knots are classified along with their invariants and diagrams. Thus, by Doll and Hoste, minimal projections of L are in one-to-one correspondence with valid DT codes. Finally, there is a single solution for the question of which of the components is K , R , and c . We use that c is strictly a meridian to R , so its only crossings are the two they share. Since the entries on the DT code come from crossings either within a single component or between two different ones, we notice that the code corresponding to the component c will have length 1 (since, by definition of a DT code, only even labeled crossings appear). R also does not have crossings with itself, but it has crossings with R and c , so its code has

length greater than 1. Moreover, since K is not an unknot, $c(K) \geq 3$, so the code corresponding to K will always be longer than the code corresponding to R . Thus, it is clear that $l(K) > l(R) > l(c) = 1$, where $l(X)$ is the code length of the component X . \square

The script `integral_slopes.py`, given the DT code of a link L satisfying the conditions above, outputs a list of possible integral characteristic slopes for any knot K . It relies on writing a text file, running it on SnapPy, saving the output, and interpreting it repeatedly. There are certain system requirements explicit on the code. The script consists of the following steps:

- (1) From the DT code, the program writes the file `K [INFO].py`. This file asks for the volume of the knot K , for the volume of the manifold $Z = S_0^3(R)$ obtained by zero-surgery filling the 1-handle component, and for the length of the Seifert longitude in Z .
- (2) SnapPy opens, runs, and saves this file, producing a first output file named `[OUT] K [INFO].py`.
- (3) The program extracts this information and calculates the bound N using the theorems of hyperbolic Dehn surgery.
- (4) The program writes the file `K [TEST].py`. This file asks for the volume of each knot K'_n in the twist family of K , whose exterior is easily obtained by doing n -surgery on the first component of the manifold Z , for all n such that $|n| \leq N$.
- (5) SnapPy opens, runs, and saves this file, producing a second output file named `[OUT] K [TEST].py`.
- (6) The program verifies if this last file contains all $2N + 1$ cases. This step is included because the run time in SnapPy is sometimes unpredictable.
- (7) The program identifies the knots whose volume is less than or equal (up to approximations) to the volume of the knot K . Those cases are singled out in the output file and, if possible, identified by SnapPy.

All files corresponding to the knots listed in Theorem 1.2 can be found on [Kyle's website](#). These files prove the theorem.

Proof. We produced the link L aforementioned for all knots K mentioned in Theorem 1.2. All the DT codes are included in the file `DT_List.txt`, which is correctly formatted for the script `integral_slopes.py`. Running it outputs the `.py` files labeled `[OUT]`, and text that is reproduced in the file `output.txt`. From reading this file, we see that:

- (a) For all knots K of crossing number at most 10 with unknotting number $u(K) = 1$ and not twist knots, the program indicates that only $n = 2$ or $n = -2$ could be characterizing.
- (b) For K one of the knots 8_4 , 8_6 , 8_{10} , or 8_{12} , the program indicates that all knots K'_n have different volume than K , so no integral slopes can be characterizing.

- (c) For the knot 8_1 , the program indicates that only $n = 0$ could be characterizing.

□

3.3. Possible extensions of Theorem 1.1. Theorem 1.2 is stronger than Theorem 1.1 for low-crossing number knots, since it is valid for knots K such that the unknotting number $u(K) > 1$ and even some Twisted Whitehead Doubles. It suggests that the previous theorem can be extended to encompass a large number of knots than those with $u(K) = 1$.

To use Piccirillo's construction, the original band presentation is not necessary. Any link $L' = K \cup R$, where R is a one-handed unknot can receive a meridian c to become the link L described in section 3.2.1. This link fits Piccirillo's construction, and will yield a non-trivial twist family of knots K'_n if and only if R is not a meridian to K . As proved, the banded Hopf Link presentation will always yield such L' through a single handle slide, but we have encountered diagrams for this link that do not depend on a band presentation.

We prove parts (b) and (c) of Theorem 1.2 using these diagrams. We could not find an algorithm to produce them, but a few strategies were recurrently successful:

- (1) Start with a simple banded Hopf link diagram, then change the framing on the band and perform the handle slide.
- (2) Start with a simple banded Hopf link diagram for a knot K and perform the usual handle slide. Then, perform a second handle slide of K over R on a region where K is not parallel to R . Sometimes, different framings on the second handle slide successfully yielded different links fitting Piccirillo's construction.
- (3) Start from a simple diagram $B \cup R \cup G$ that fits Piccirillo's construction. Then, repeatedly slide B over R until it comes off of G .

Such strategies take diagrams for knots we previously find and modify them in order to find analogous for other knots. Whether the two knots related by these operations have any relationship is unknown. If real, such relationship could lead to an algorithm that produces a link L for J with $u(J) > 1$ given the link L for K with $u(K) = 1$. Moreover, using the third strategy we were even successful to find a link L that fits Piccirillo's construction for the twist knot 8_1 .

We now pose a question that would allow us to generalize Theorem 1.1 for more knots, dropping the assumptions on the unknotting number and the Twisted Whitehead classification.

[Not sure how to phrase this but I think it could be a good open question from our paper.]

Question 3.2. *For what knots K can a link $L' = K \cup R$, where R is an unknot that fails to be a meridian of K , be produced through handle slides on a Hopf Link?*

4. ADDITIONAL RESULTS

[Any additional findings, including HFK or Khovanov homology findings, or enhanced sliceness obstructions using Piccirillo’s technique, can go here.]

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