

# Knot Surgery and Integer Characterizing Slopes

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# Knots and links in the 3-sphere

## Definition

A *knot*  $K$  is the image of a smooth embedding of the circle  $S^1$  into a 3-manifold, usually the 3-sphere  $S^3$ . In particular,  $K$  is diffeomorphic to  $S^1$ . A *link*  $L$  is a disjoint union of knots, which may be tangled together.

## Definition

Let  $M, N$  be manifolds and  $g, h: N \rightarrow M$  embeddings. An *ambient isotopy* of  $M$  carrying  $g$  to  $h$  is a continuous map  $F: M \times [0, 1] \rightarrow M$ , such that  $F_t = F(\cdot, t)$  is a homeomorphism of  $M$  for each  $t \in [0, 1]$ ,  $F_0 = \mathbb{1}$ , and  $F_1 \circ g = h$ .

- We regard two knots  $K, K' \subset S^3$  to be equivalent if they differ by an ambient isotopy of  $S^3$ . We write  $K \simeq K'$ .
- Equivalently, we can view knots as subsets of  $\mathbb{R}^3$  rather than  $S^3$ .

# Knot diagrams

- We can study a knot  $K \subset \mathbb{R}^3$  by projecting it onto a hyperplane  $\mathbb{R}^2$ .
- If  $\pi: \mathbb{R}^3 \rightarrow \mathbb{R}^2$  is a projection such that  $\pi(K)$  is an embedded curve except at finitely many *crossing points*, then  $\pi(K)$  is a *diagram* for  $K$ .
- The *crossing number*  $c(K)$  is the minimum number of crossings in a diagram of  $K$ .

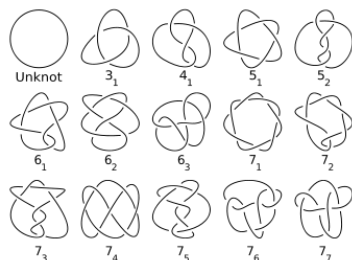


Figure: Knots with  $c(K) \leq 7$

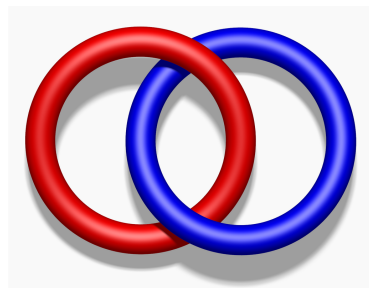
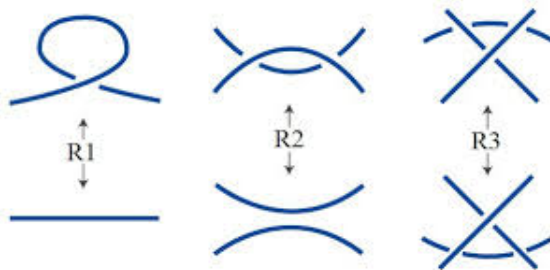


Figure: Hopf link

# Reidemeister moves

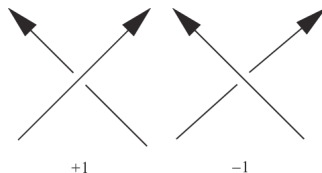
## Theorem (Reidemeister)

*Two knots  $K, K'$  are isotopic if and only if they have diagrams that differ by a sequence of planar isotopies and Reidemeister moves.*



# Crossings

- If we orient a knot  $K$ , then we can define a *sign* for each crossing by the right-hand rule.
- For a two-component link  $L = K \cup K'$ , the *linking number*  $\text{lk}(L)$  is one half the sum of the signs of the crossings between  $K$  and  $K'$  in a diagram of  $L$ .



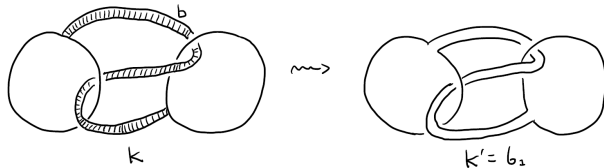
# Unknotting number and band move

## Definition

The *unknotting number*  $u(K)$  of a knot  $K$  is the minimal number of crossing changes that are required to change some diagram of  $K$  into a diagram of the unknot.

- Example of band move:

The Stevedore knot  $6_1$  is obtained from a band move on two unknots:



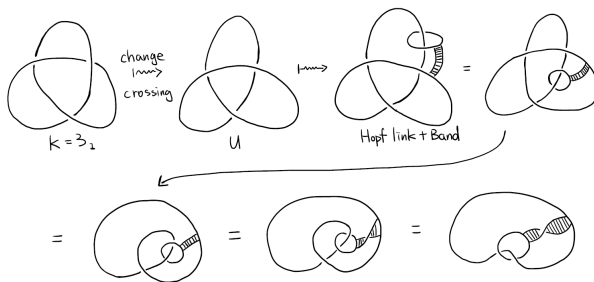
# Example of banded Hopf link presentation

- If  $K$  can be obtained from a Hopf link by a single band move, then  $K$  has a *banded Hopf link presentation*.

## Theorem

If  $K$  has  $u(K) = 1$ , then it is obtained from the Hopf link by a single band move.

- Example: banded Hopf link presentation of trefoil:





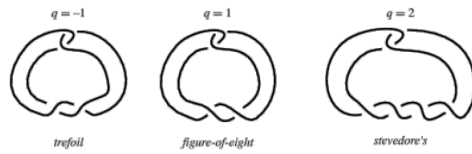
# Twist knots and Twisted whitehead doubles

## Definition

A knot  $K$  is a *twisted Whitehead double* if there exists a band presentation for  $K$  in which the band does not cross either component of the Hopf link.

## Definition

A *twist knot*  $K$  is a knot obtained by repeatedly twisting an unknot and linking the ends together.

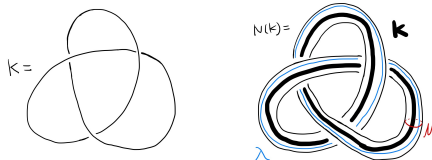


- Note: twist knots are twisted Whitehead doubles, but not all twisted Whitehead doubles are twist knots.

# Preferred meridian and longitude pair for a knot $K \subset S^3$

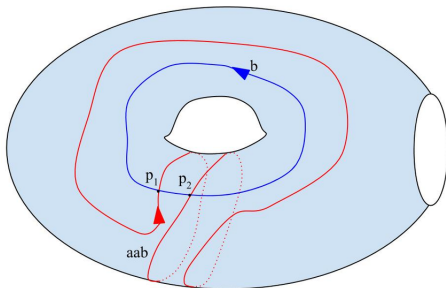
## Definition

Let  $K \subset S^3$  be a knot, and let  $N(K)$  be tubular neighborhood of  $K$ . Then  $N(K)$  is a solid torus  $\mathbb{T}$ , and  $\partial N(K)$  is a torus  $T^2$ . We define a *meridian*  $\mu$  of  $K$  to be an unknot lying on  $\partial N(K)$  that bounds a disc which  $K$  crosses exactly once. A *longitude*  $\lambda$  of  $K$  is a parallel copy of  $K$  on  $\partial N(K)$ .



# Isotopy classes of curves on $T^2$

- Given a knot  $K \in S^3$  where  $\partial N(K) \cong$  a torus  $T^2$ , any  $p/q \in \mathbb{Q} \cup \{\infty\}$  determines a curve  $\beta$  on  $\partial N(K)$  that is well-defined up to isotopy. Intuitively,  $p$  (and  $q$  resp.) is the number of times that  $\beta$  goes around the meridian (and longitude resp.) that we choose for  $K$ .
- We use  $\beta$  to define a 3-manifold  $S^3_{p/q}(K)$ , called  $p/q$ -surgery on  $K$ .



# Surgery on a knot: *drilling* then *filling*

## Definition

Given a knot  $K \subset S^3$ , the *exterior* of  $K$ , denoted  $X(K)$ , is a manifold with boundary obtained by removing the interior of  $N(K) = S^1 \times D^2 \cong$  a solid torus  $\mathbb{T}$  from  $S^3$ . Then  $\partial X(K) = \partial N(K) \cong$  a torus  $T^2$ . Let  $\beta$  be the curve on  $T^2$  determined (up to isotopy) by some extended rational number  $p/q \in \mathbb{Q} \cup \{\infty\}$ . Then the  $p/q$ -surgery on  $K$ , a 3-manifold denoted  $S^3_{p/q}(K)$ , is obtained by gluing another solid torus  $\mathbb{T}'$  back to  $X(K)$  so that the meridian of  $\mathbb{T}'$  is identified with the curve  $\beta$ .

- We call  $p/q$  the *surgery slope*.

# Surgery on a link

## Definition

Similarly, given a link  $L$  in  $S^3$  s.t.  $\exists p/q \in \mathbb{Q} \cup \{\infty\}$  associated to each component of  $L$ , we can do surgery on  $L$  by doing surgery along each of its component. Any such link is called a *framed link*.

## Theorem

*Any closed, oriented, connected 3-manifold can be obtained by doing surgery along a framed link. So we can think of any 3-manifold in terms of a diagram for a framed link. Any such diagram is called a surgery diagram.*

- Suppose that Dehn surgery on two surgery diagrams  $D_1, D_2$  gives two 3-manifolds  $M_1, M_2$ .

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- Then  $M_1$  and  $M_2$  are homeomorphic if and only if  $D_1$  and  $D_2$  are related by a sequence of *Kirby moves*.

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  - Handle-slide
  - Rolfsen twist

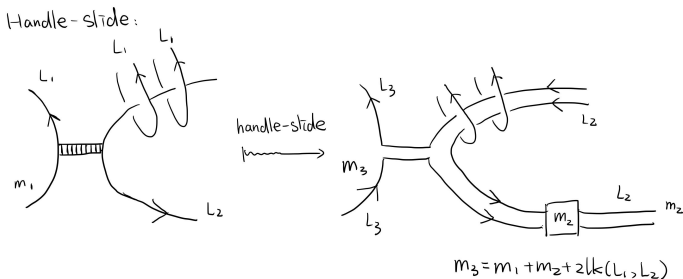
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  - Handle-slide
  - Rolfsen twist
  - Slam dunk

# Handle-slide

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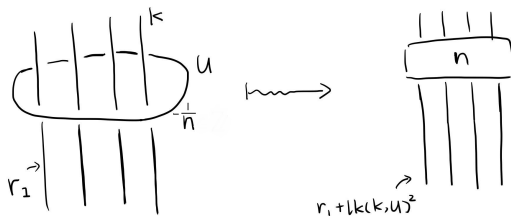
# Rolfsen twist

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Rolfsen twist:

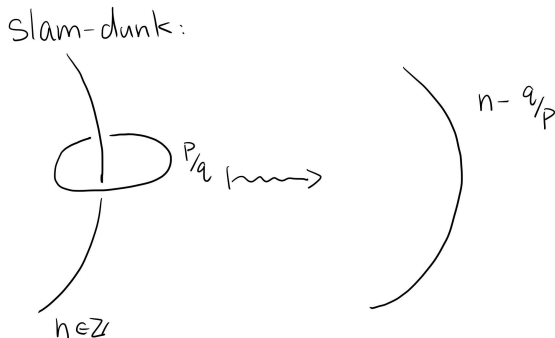


# Slam dunk

- A 0-framed link component  $K$  with a  $(-1/n)$ -framed meridian is equivalent to  $K$  with  $n$ -framing.

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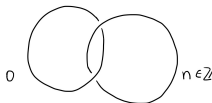
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# Examples of surgery

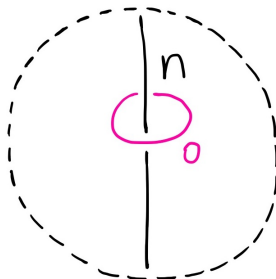
- Example:  $\pm 1/0$ -surgery on a knot  $K$  gives back  $S^3$  trivially. Thus any nontrivial surgery on a knot  $K$  has rational slopes.
- Example:  $S^3_{1/n}(u) \cong S^3$  where  $u$  is an unknot and  $n \in \mathbb{Z}$ .
- Example: for any integer  $n$ ,  $(n, 0)$ -surgery on a Hopf link  $K$  yields  $S^3$ .



# Lightbulb trick

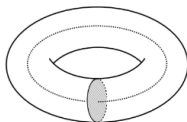
## Theorem

*For any integer  $n$  and any knot  $K$  with an unknot  $c$  as its meridian,  $(n, 0)$ -surgery on  $K \cup c$  yields  $S^3$ . Such a pair of  $K$  and  $c$  forms a cancelling pair that can be deleted without changing the 3-manifold described by the surgery diagram.*



# Surgery dual to a knot

- Note that  $K$  is given by  $S^1 \times 0$  inside  $N(K) = S^1 \times D^2 \cong$  a solid torus  $\mathbb{T}$ . i.e.  $K$  is the core curve of the solid torus  $N(K)$ .

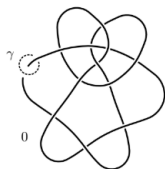


## Definition

When doing  $p/q$ -surgery on a knot  $K \in S^3$ , the new solid torus  $\mathbb{T}' \cong S^1 \times D^2$  that we glue back in  $X(K)$  to produce  $S^3_{p/q}(K)$  also has a core curve  $S^1 \times 0$ , which specifies a knot  $\gamma$  in  $S^3_{p/q}(K)$ . Call  $\gamma$  the *surgery dual* of  $K$ .

# Surgery dual to a knot (cont.)

- For any knot  $K$  and any integer  $n$ , the *surgery dual* to  $K$  in  $S_n^3(K)$  can be represented as a meridian to  $K$  in the surgery diagram.



$Y = S_0^3(K)$ , with  $K = 11n38$

## Lemma

*Surgery duality for knots in  $S^3$  is symmetric.*

$S^3$	$n \in \mathbb{Z}$	$S_n^3(K)$	$S^3$
$U$	$n$ -surgery	$U$	$0$ -surgery $U$
$K$	$\rightsquigarrow$	$\gamma$	$\rightsquigarrow K$

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# Piccirillo's construction: Knots with the same surgery

## Theorem (Piccirillo 2018)

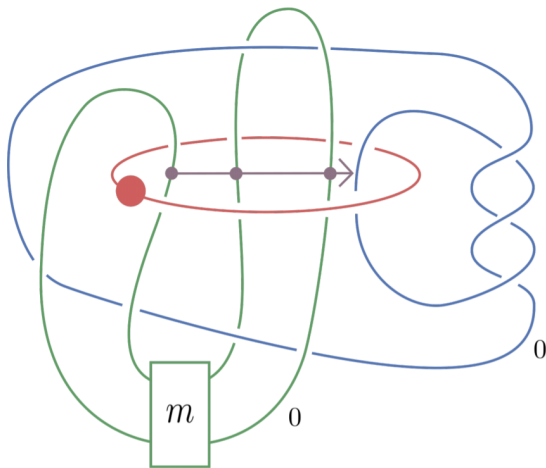
Let  $L = R \cup G \cup B$  be a surgery diagram for some 3-manifold  $Y$  such that:

- ①  $R$  is a zero-framed unknot,  $B$  and  $G$  have integral framings.
- ② Ignoring  $B$ ,  $R$  is isotopic to a meridian of  $G$ .
- ③ Ignoring  $G$ ,  $R$  is isotopic to a meridian of  $B$ .
- ④  $B$  and  $G$  have linking number 0.

Then, there exist knots  $K$  and  $K'$  such that  $Y \cong S_n^3(K) \cong S_n^3(K')$ .

- Piccirillo's construction comes from an older construction, the *dualizable patterns* construction, to produce knots with the same surgery.

# Piccirillo's construction (cont.)



**Figure:** Diagram of a link  $L$  used by Piccirillo on her original paper. The colors (red, blue, and green) match the component letters  $R$ ,  $B$ ,  $G$ .

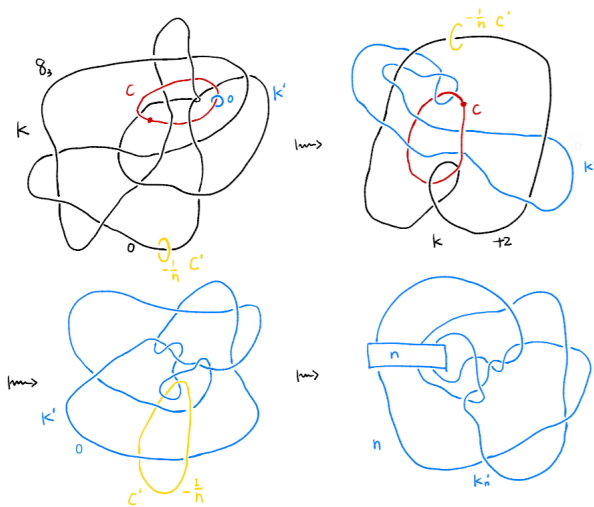
# Baker-Motegi: Knots with the same surgery

- We adapted the following construction from Baker and Motegi (2018).
- Let  $K$  be a knot, and suppose we can take an unknot  $c$  linked with  $K$  such that  $(0,0)$ -surgery on  $K \cup c$  is  $S^3$ .
- Baker-Motegi present a method for producing knots  $K'_n$  with  $S^3_n(K) \cong S^3_n(K'_n)$  from this link.
- Define  $K'$  to be the surgery dual to  $c$  in  $S^3 = S^3_{(0,0)}(K \cup c)$ .
- Then  $K'$  has the same 0-surgery as  $K$ .



- Also define  $c'$  to be the surgery dual to  $K$  in  $S^3$ .
- After some Kirby calculus, we find that in the surgered manifold  $S^3 = S^3_{(0,0)}(K \cup c)$ ,  $c'$  is an unknot linked with  $K'$ .
- Let  $K'_n$  be the result of twisting  $K'$  through  $c'$ ,  $n$  times.
- We say that  $\{K'_n\}$  forms a *twist family*.
- Then  $K'_n$  has the same  $n$ -surgery as  $K$  for all  $n$ .
- Moreover, if  $c$  is not a meridian to  $K$ , then  $K \simeq K'_n$  for at most finitely many  $n$ .

# Baker-Motegi Illustration

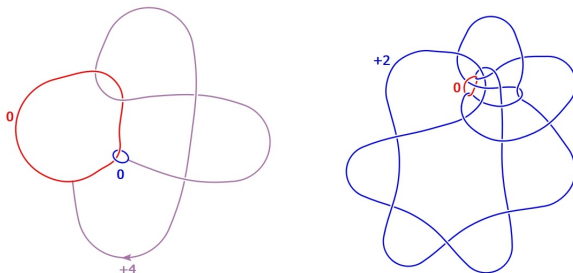


# Obtaining a link $K \cup c$ when $u(K) = 1$ (Piccirillo)

- In the special case where  $K$  has unknotting number one, we can use a band presentation for  $K$ .
- We start with a Hopf link  $R \cup B$  and slide one component over the other according to the band presentation for  $K$ .
- $R$  remains an unknot, which we rename  $c$ , and  $B$  becomes  $K$ .

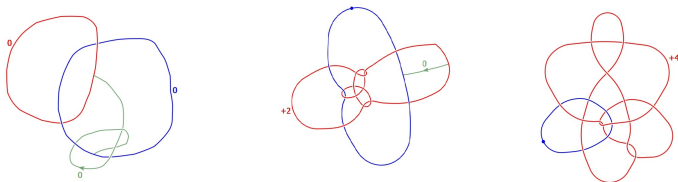
## Lemma

*Let  $K \cup c$  be the link obtained by the handle slide above. Then  $(0, 0)$ -surgery on  $K \cup c$  gives  $S^3$ .*



# Obtaining $K \cup c$ when $u(K) > 1$

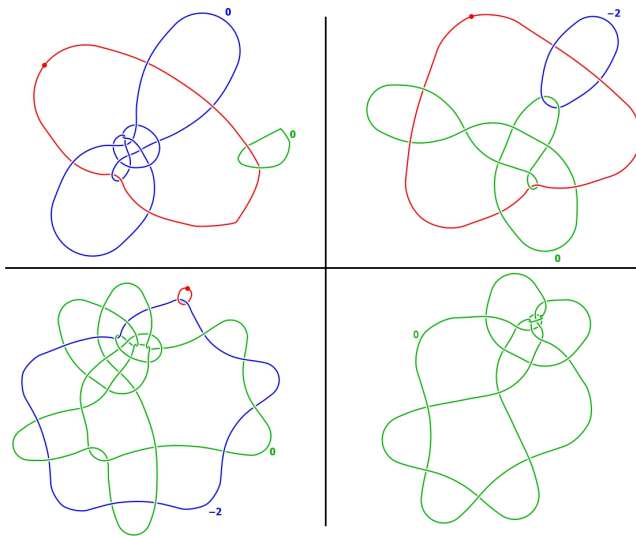
- We do not have a systematic way of finding a link  $K \cup c$  for a given knot  $K$  with  $u(K) > 1$ .
- If we perform multiple slides on a Hopf link, then we obtain a link  $K \cup c$  for some knot  $K$  with higher unknotting number, but we have no way of predicting what  $K$  will be.
- Example:  $10_{125}$



# Applications of the construction

- We have produced a banded Hopf link presentation for all the 78 unknotting number one knots  $K$  with  $c(K) \leq 10$ .
- We followed the above procedure to manually produce a diagram for a knot  $K'$  that has the same zero-surgery as  $K$ , using a software called KLO which can perform Kirby calculus on link diagrams.

# Applications (cont.)



# Applications (cont.)

- Using a software called SnapPy, we were able to verify that whenever  $K$  was not a twist knot,  $K'$  was not isotopic to  $K$ . Thus, zero was not a characterizing slope.
- We can use a similar process to check whether any integer slope is characterizing.

## Definition

Let  $K$  be a knot. Then  $p/q$  is a *characterizing slope* for  $K$  if  $S_{p/q}^3(K')$  is not homeomorphic to  $S_{p/q}^3(K)$  for any knot  $K' \neq K$ .

- **Question:** Can we classify all integer slopes once we have the link  $K \cup c$ , with finitely many computations?

# Known results on characterizing slopes

- The following theorem shows that for a most knot, most rational slopes are characterizing.



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## Theorem (McCoy, 2018)

*If  $K$  is a hyperbolic knot, then  $K$  has only finitely many non-characterizing slopes  $p/q$  with  $|q| \geq 3$ . Moreover, the probability that a randomly chosen slope  $p/q$  is characterizing for  $K$  approaches 1 as  $|p| + |q| \rightarrow \infty$ .*

- Moreover, it was proven by Ozsváth and Szabó (2006) that all slopes are characterizing for the trefoil and figure-eight knot (twist knots).

## Results (cont.)

- However, the next theorem shows that for a certain type of knot, there are infinitely many non-characterizing slopes.

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### Theorem (Baker–Moteği, 2018)

*Let  $K \subset S^3$  be a knot, where there is an unknot  $c$  which is not a meridian to  $K$  such that  $(0, 0)$ -surgery on  $K \cup c$  yields  $S^3$ . Then  $K$  has infinitely many non-characterizing slopes.*

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- Our work shows that for knots satisfying reasonable conditions, most integer slopes are non-characterizing!

## Theorem 1

*Regarding the integer slopes of knots  $K$  such that  $c(K) \leq 10$ :*

- *If  $K$  has unknotting number  $u(K) = 1$  and  $K$  is not a twist knot, then  $K$  has at most one integer characterizing slope, namely  $\pm 2$ .*
- *If  $K$  is the twist knot  $8_1$ , then  $K$  has at most one integer characterizing slope, namely  $0$ .*
- *If  $K$  is one of the  $u(K) = 2$  knots  $8_4, 8_6, 8_{10}, 8_{12}, 8_{16}, 10_{148}, 10_{149}$ , or  $10_{150}$ ,  $K$  has no possible integer characterizing slope.*
- *If  $K$  is one of the  $u(K) = 2$  knots  $8_3, 10_{125}$ , or  $10_{126}$ ,  $K$  has at most one integer characterizing slope.*

# Our results

## Theorem 1

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- If  $K$  is one of the  $u(K) = 2$  knots  $8_3, 10_{125}$ , or  $10_{126}$ ,  $K$  has at most one integer characterizing slope.*

## Theorem 2

*If a knot  $K$  has unknotting number  $u(K) = 1$  and is not a twisted Whitehead double, then  $K$  has at most finitely many integer characterizing slopes.*

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# Hyperbolic Dehn surgery

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- If  $K$  is a hyperbolic knot, then the torus  $\partial N(K)$  inherits a Euclidean metric.
- According to this Euclidean metric, the *length*  $\ell(p/q)$  of a slope  $p/q$  is the length of the shortest curve on the torus  $\partial N(K)$  with slope  $p/q$ .

# Hyperbolic surgery theorems

## Theorem (Gromov-Thurston)

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## Theorem (Futer, et al.)

*Let  $K$  be a hyperbolic knot; let  $X := S^3 \setminus N(K)$ . If  $\ell(p/q) > 2\pi$ , then the volume  $\text{vol}(X_{p/q})$  of the  $(p/q)$ -filling on  $X$  satisfies*

$$\text{vol}(X_{p/q}) \geq \left(1 - \left(\frac{2\pi}{\ell(p/q)}\right)^2\right)^{3/2} \text{vol}(X).$$

## Theorem (Cooper-Lackenby)

*Let  $X$  be a cusped hyperbolic 3-manifold, and suppose  $s_1, s_2$  are two slopes on a torus  $T \subset \partial X$ . Then*

$$\ell(s_1)\ell(s_2) \geq \sqrt{3} \Delta(s_1, s_2),$$

*where  $\Delta(s_1, s_2)$  is the minimum possible number of intersections between a curve with slope  $s_1$  and a curve with slope  $s_2$  on  $T$ .*

# Sketch of proof of Theorem 1

## Theorem 1

*Regarding the integer slopes of knots  $K$  such that  $c(K) \leq 10$ :*

- *If  $K$  has unknotting number  $u(K) = 1$  and  $K$  is not a twist knot, then  $K$  has at most one integer characterizing slope, namely  $\pm 2$ .*
- *If  $K$  is the twist knot  $8_1$ , then  $K$  has at most one integer characterizing slope, namely  $0$ .*
- *If  $K$  is one of the  $u(K) = 2$  knots  $8_4, 8_6, 8_{10}, 8_{12}, 8_{16}, 10_{148}, 10_{149}$ , or  $10_{150}$ ,  $K$  has no possible integer characterizing slope.*
- *If  $K$  is one of the  $u(K) = 2$  knots  $8_3, 10_{125}$ , or  $10_{126}$ ,  $K$  has at most one integer characterizing slope.*



# Proof of Theorem 1 (cont.)

- Following Baker–Moteği, we can produce a link  $(K \cup c) \subset S^3$ , where  $c$  is an unknot which is not a meridian to  $K$  such that  $(0, 0)$ -surgery on  $K \cup c$  yields  $S^3$ .

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$$v'_n \geq \left(1 - \left(\frac{2\pi\ell}{|n|\sqrt{3}}\right)^2\right)^{3/2} v_X > v$$

$$\text{for } |n| > N := \frac{2\pi}{\sqrt{3}} \ell (1 - (v/v_X)^{2/3})^{-1/2}.$$

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- Hence all integer slopes  $|n| > N$  are non-characterizing.

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- Hence all integer slopes  $|n| > N$  are non-characterizing.
- Finally, we check the finitely many remaining cases by computer.



# Ruling out integer characterizing slopes

- We developed an algorithm that can be carried out by a computer script to rule out the integer characterizing slopes of some knot  $K$ , once we have manually produced the link  $L = K \cup c$ .
- ① Given a link  $L$ , run SnapPy commands to find out the volume of the knot  $K$ , the volume of the manifold  $Z = S_0^3(c)$ , and the length of the Seifert longitude in  $Z$ .
- ② Using these values, apply the theorems of hyperbolic Dehn surgery to find the bound  $N$  such that any integer  $|n| > N$  is a characterizing slope for  $K$ .
- ③ For the remaining  $2N + 1$  cases, verify if the volume of the knot with the same  $n$ -surgery as  $K$  matches the volume of  $K$ .
- We use the *DT code* of the link, which uniquely describes all links that we deal with, up to isotopy.

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# Theorem 2

## Theorem 2

*If a knot  $K$  has unknotting number  $u(K) = 1$  and is not a twisted Whitehead double, then  $K$  has at most finitely many integer characterizing slopes.*

# Proof sketch of Theorem 2

## Proposition

*Let  $K$  be a knot in  $S^3$ . Suppose we can take an unknot  $c$  linked with  $K$  so that  $(0,0)$ -surgery on  $K \cup c$  yields  $S^3$  and  $c$  is not a meridian of  $K$ . Then  $K$  has at most finitely many integer characterizing slopes.*

- This is a strengthened form of a result proven by Baker-Motegi, which we proved using symmetry of surgical duality.
- Part of the proof of this proposition guarantees that our adapted version of Baker-Motegi construction yields the desired  $K'_n$  that shares the same  $n$ -surgery as  $K$  from purely theoretical grounds of duality.
- We can apply this proposition to prove our main theorem, which admits more concrete conditions on  $K$  than the hypothesis in the proposition.

# Proof of Theorem 2 (cont.)

- Recall that for  $K$  with  $u(K) = 1$ , we could obtain a link  $K \cup c$  as in Baker-Motegi with  $(0, 0)$ -surgery  $S^3$  by sliding over a Hopf link according to the band presentation for  $K$ .
- It remains to show that if  $K$  is not a twisted Whitehead double, then  $c$  is not a meridian to  $K$  after the handle slide.
- In this case, in any band presentation for  $K$ , the band must cross the disc bounded by one of the components of the Hopf link.

## Lemma

*Let  $R \cup B$  be a Hopf link, and consider a handle slide of  $R$  over  $B$  yielding a link  $L$  in which  $R$  remains a meridian to  $B$ . Then there exists a handle slide of  $R$  over  $B$ , yielding a link isotopic to  $L$ , along a band that does not cross either of the discs bounded by  $R$  or  $B$ .*

# Extension of Theorem 2

- How can we loosen the condition on twisted Whitehead doubles? And on unknotting number?
- Can we produce an algorithm to find links  $L = K \cup c$  from handle slides on a Hopf link?

## Conjecture

*If  $K$  is a knot with unknotting number  $u(K) = 1$  and  $K$  is not a twist knot, then  $K$  has at most one integer characterizing slope:  $\pm 2$ .*

# Next steps

- Rigorously prove the lemma on band presentations.
- Keep experimenting with handle slides in order to expand the list on Theorem 1.
- Attempt to use other tools to prove a version of Theorem 2 for twisted Whitehead doubles.
- Look into our final conjecture.