Connected sums, band moves, and diagrams

Given any pair of knots K, K', we can produce a new knot as follows:

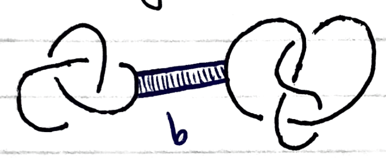
1. Fix disjoint diagrams of K, K' in the plane.

   e.g.

   K  K'

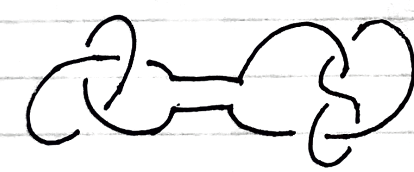
2. Find a band (i.e. rectangle) b in the plane where one pair of sides are arcs lying along K and K', but with b otherwise disjoint from the diagram.

   e.g.

3. Join K and K' by deleting the arcs $b \cap K$ and $b \cap K'$ and adding the arcs forming the other sides of b.

   e.g.

    $K \# K'$

The resulting knot is called the <u>connected sum</u> of K and K', denoted $K \# K'$.

<u>Exercise</u> Show that $K \# \mathcal{U} = K$ for any knot K and $\mathcal{U}$ = unknot.

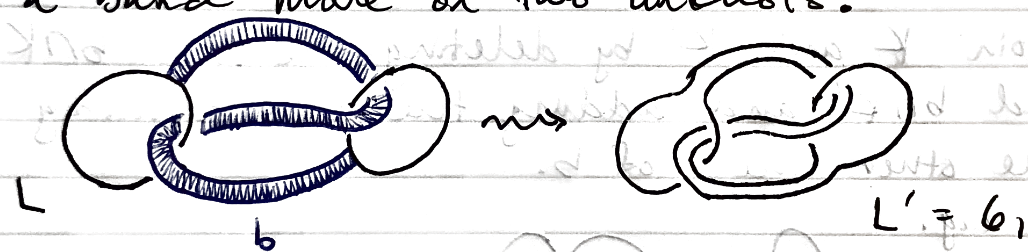
<u>Def</u> K is called <u>prime</u> if, for any presentation of K as a connected sum $K = K_1 \# K_2$, one of $K_i = K$ and the other is $\mathcal{U}$.

We can get much more interesting knots/links by generalizing this:

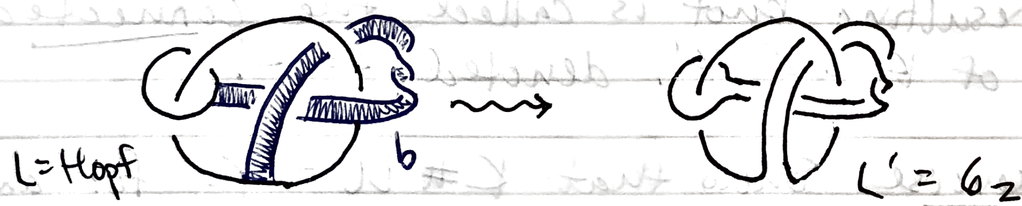
Def Let L be any link in S^3 and let $b \subset S^3$ be any embedded band where one pair of sides lie along arcs in L and where b is otherwise disjoint from L. Join L to itself by deleting the arcs $L \cap b$ and adding the arcs forming the other side of b. The resulting link L' is obtained from L by a band move along b.

Note Any connected sum is a band move. But here are more interesting examples:

EX • The Stevedore knot 6_1 is obtained from a band move on two unknots:



• The knot 6_2 is obtained from the Hopf link by a band move:



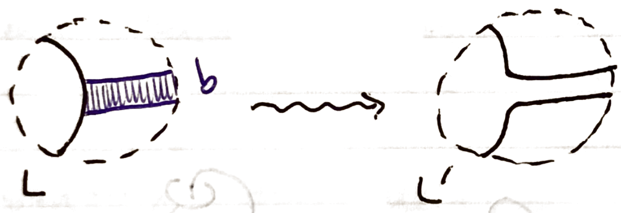
Note: In these examples, L' is a knot, but this isn't always true. And L can be chosen to be a knot, too, if desired.

Thm If K has unknotting number one, then it is obtained from the Hopf link by a single band move.

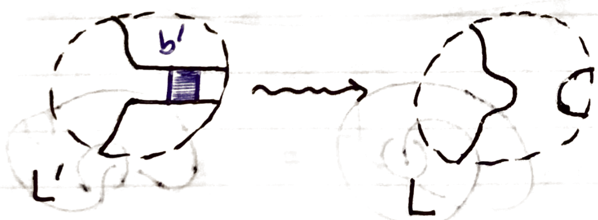
First, a lemma: (Actually, we won't use this lemma, but it is related.)

Lemma If L' is obtained from L by a single band move, then L is also obtained from L' by a single band move.

Pf Consider a nd of one of the sides of b on L, as below. As shown, we can also draw the corresponding nd in the diagram of L'.

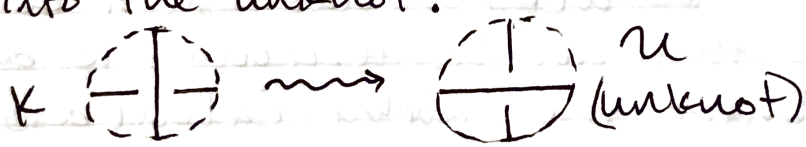


Now performing a band move along b' as below reproduces L:

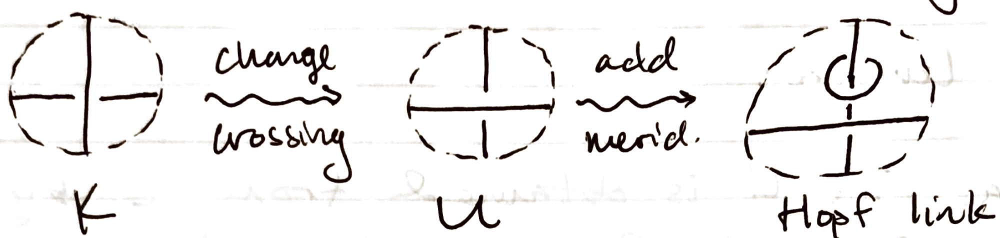


this segment contracts back to the other arc in L where the other end of b was attached. $\square$

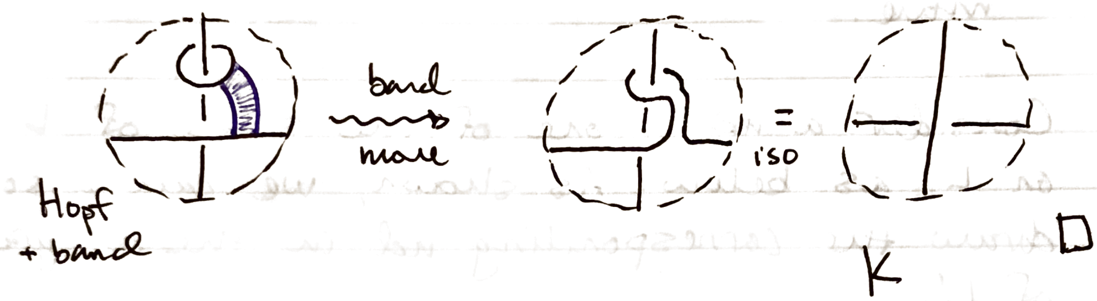
Pf (of thm): Let c denote the unknotting crossing in K. Then changing this nd as below turns K into the unknot:



(Pf, ctd) It follows that we can get a diagram of the Hopf link by adding a meridional unknot after the crossing change:

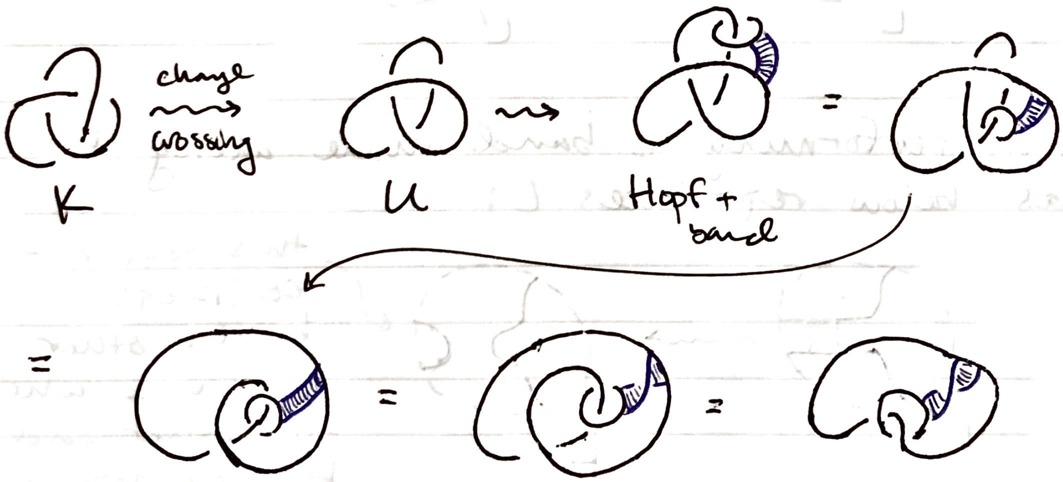


Now we get K back via a band move:



□

EX Trefoil:



We call this a "banded Hopf link presentation" (of the trefoil, in this case). The thm shows that every K with u(K)=1 has such a pres., though they can be tedious to draw by hand. Enter the Kirby Calculator...