Knot Surgery and Integer Characterizing Slopes

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Knots and links in the 3-sphere

Definition

A *knot* K is the image of a smooth embedding of the circle S^1 into a 3-manifold, usually the 3-sphere S^3. In particular, K is diffeomorphic to S^1. A *link* L is a disjoint union of knots, which may be knotted together.

Definition

Let M, N be manifolds and $g, h \colon N \to M$ embeddings. An *ambient isotopy* of M carrying g to h is a continuous map $F \colon M \times [0,1] \to M$, such that $F_t = F(\cdot, t)$ is a homeomorphism of M for each $t \in [0,1]$, $F_0 = \mathbb{1}$, and $F_1 \circ g = h$.

- We regard two knots $K, K' \subset S^3$ to be equivalent if they differ by an ambient isotopy of S^3. We write $K \simeq K'$.
- Equivalently, we can view knots as subsets of $\mathbb{R}^3$ rather than S^3.

Knot diagrams

- We can study a knot $K \subset \mathbb{R}^3$ by projecting it onto a hyperplane $\mathbb{R}^2$.
- If $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ is a projection such that $\pi(K)$ is an embedded curve except at finitely many *crossing points*, then $\pi(K)$ is a *diagram* for K.
- The *crossing number* $c(K)$ is the minimum number of crossings in a diagram of K.

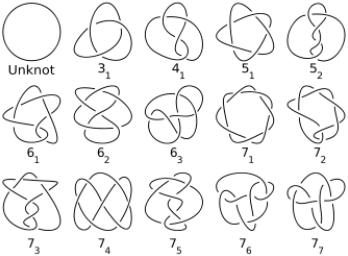


Figure: Knots with $c(K) \leq 7$

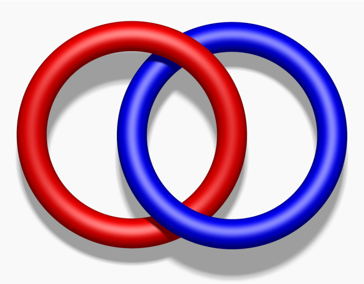
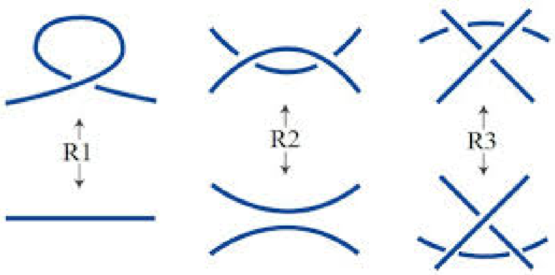


Figure: Hopf link

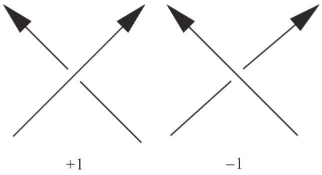
Theorem (Reidemeister)

Two knots K, K' are isotopic if and only if they have diagrams that differ by a sequence of planar isotopies and Reidemeister moves.



Crossings

- If we orient a knot K, then we can define a *sign* for each crossing by the right-hand rule.
- For a two-component link $L = K \cup K'$, the *linking number* $\mathrm{lk}(L)$ is one half the sum of the signs of the crossings between K and K' in a diagram of L.

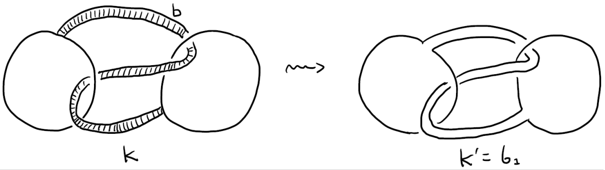


Unknotting number and band move

Definition

The *unknotting number* $u(K)$ of a knot K is the minimal number of crossing changes that are required to change some diagram of K into a diagram of the unknot.

- Example of band move:
  The Stevedore knot 6_1 is obtained from a band move on two unknots:



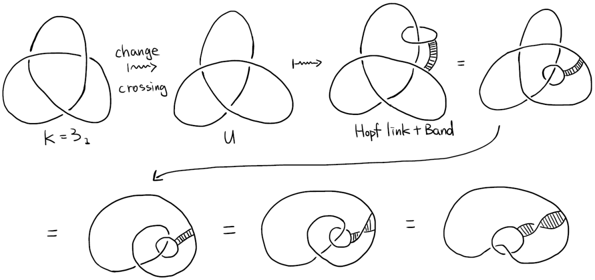
Example of banded Hopf link presentation

- If K can be obtained from a Hopf link by a single band move, then K has a *banded Hopf link presentation*.

Theorem

If K has $u(K) = 1$, then it is obtained from the Hopf link by a single band move.

- Example: banded Hopf link presentation of Trefoil:



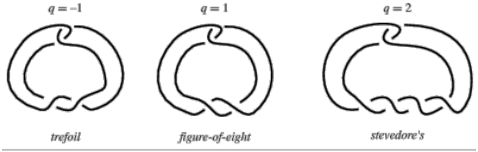
Twist knots and Twisted whitehead double

Definition

A knot K is a *twisted Whitehead double* if there exists a band presentation for K in which the band does not cross either component of the Hopf link.

Definition

A *twist knot* K is a knot obtained by repeatedly twisting an unknot and linking the ends together.

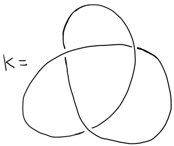


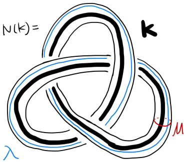
- Note: twist knots are twisted Whitehead doubles, but not all twisted Whitehead doubles are twist knots.

Preferred meridian and longitude pair for a knot $K \subset S^3$

Definition

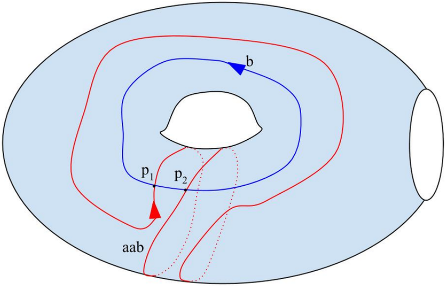
Let $K \subset S^3$ be a knot, and let $N(K)$ be tubular neighborhood of K. Then $N(K)$ is a solid torus $\mathbb{T}$, and $\partial N(K)$ is a torus T^2. We define a *meridian* μ of K to be an unknot lying on $\partial N(K)$ that bounds a disc which K crosses exactly once. A *longitude* λ of K is a parallel copy of K on $\partial N(K)$.





Isotopy classes of curves on T^2

- Given a knot $K \in S^3$ where $\partial N(K) \cong$ a torus T^2, any $p/q \in \mathbb{Q} \cup \{\infty\}$ determines a curve β on $\partial N(K)$ that is well-defined up to isotopy. Intuitively, p (and q resp.) is the number of of times that β goes around the meridian (and longitude resp.) that we choose for K.
- We use β to define a 3-manifold $S^3_{p/q}(K)$, called p/q-surgery on K.



Surgery on a knot: *drilling* then *filling*

Definition

Given a knot $K \subset S^3$, the *exterior* of K, denoted $X(K)$, is a manifold with boundary obtained by removing the interior of $N(K) = S^1 \times D^2 \cong$ a solid torus $\mathbb{T}$ from S^3. Then $\partial X(K) = \partial N(K) \cong$ a torus T^2. Let β be the knot on T^2 determined (up to isotopy) by some extended rational number $p/q \in \mathbb{Q} \cup \{\infty\}$. Then the p/q*-surgery* on K, a 3-manifold denoted $S^3_{p/q}(K)$, is obtained by gluing another solid torus $\mathbb{T}'$ back to $X(K)$ so that the meridian of $\mathbb{T}'$ is identified with the curve β.

- We call p/q the *surgery slope*.

Surgery on a link

Definition

Similarly, given a link L in S^3 s.t. $\exists p/q \in \mathbb{Q} \cup \{\infty\}$ associated to each component of L, we can do surgery on L by doing surgery along each of its component. Any such link is called a *framed* link.

Theorem

Any oriented, connected 3-manifold can be obtained by doing surgery along a framed link. So we can think of any 3-manifold in terms of a diagram for a framed link. Any such diagram is called a surgery diagram.

Kirby calculus

- Suppose that Dehn surgery on two surgery diagrams D_1, D_2 gives two 3-manifolds M_1, M_2.

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    - Handle-slide
    - Rolfsen twist
    - Slam dunk

Handle-slide

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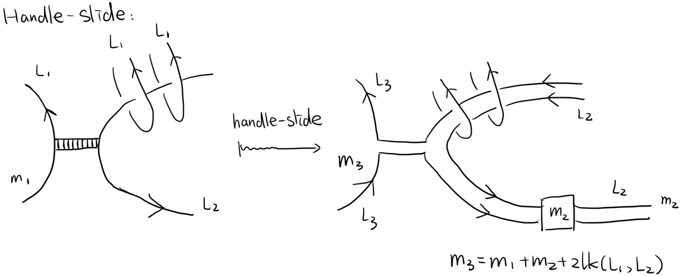

Rolfsen twist

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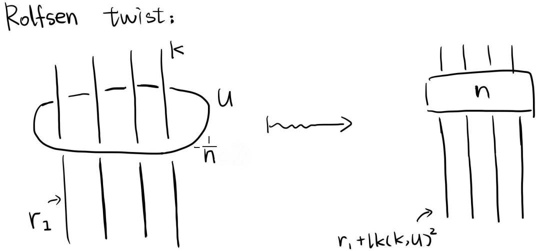

Slam dunk

- A 0-framed linke component K with a $(-1/n)$-framed meridian is equivalent to K with n-framing.

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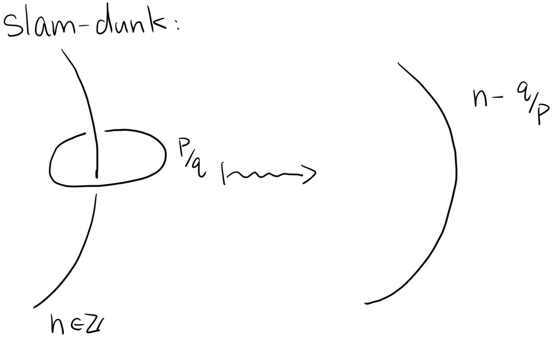
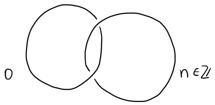


Examples of surgery

- Example: $\pm 1/0$-surgery on a knot K gives back S^3 trivially. Thus any nontrivial surgery on a knot K has rational slopes.

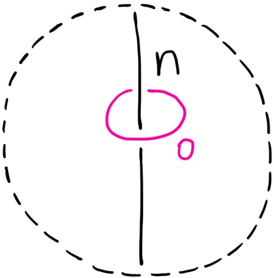
- Example: $S^3_{1/n}(u) \cong S^3$ where u is an unknot and $n \in \mathbb{Z}$.
- Example: for any integer n, $(n, 0)$-surgery on a Hopf link K yields S^3.



Lightbulb trick

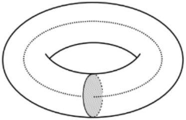
Theorem

For any integer n and any knot K with an unknot c as its meridian, $(n, 0)$-surgery on $K \cup c$ yields S^3. Such a pair of K and c forms a cancelling pair that can be deleted without changing the 3-manifold described by the surgery diagram.



Surgery dual to a knot

- Note that K is given by $S^1 \times 0$ inside $N(K) = S^1 \times D^2 \cong$ a solid torus $\mathbb{T}$. i.e. K is the core curve of the solid torus $N(K)$.

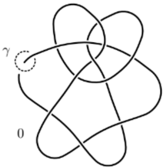


Definition

When doing p/q-surgery on a knot $K \in S^3$, the new solid torus $\mathbb{T}' \cong S^1 \times D^2$ that we glue back in $X(K)$ to produce $S^3_{p/q}(K)$ also has a core curve $S^1 \times 0$, which specifies a knot γ in $S^3_{p/q}(K)$. Call γ the *surgery dual* of K.

Surgery dual to a knot (cont.)

- For any knot K and any integer n, the *surgery dual* to K in $S^3_n(K)$ can be represented as a meridian to K in the surgery diagram.



$Y = S^3_0(K)$, with $K = 11n38$

Lemma

Surgery duality for knots in S^3 is symmetric.

S^3	$n \in \mathbb{Z}$	$S^3_n(K)$		S^3
$\cup$	n-surgery	$\cup$	0-surgery	$\cup$
K	$\rightsquigarrow$	γ	$\rightsquigarrow$	K

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Piccirillo's construction: Knots with the same surgery

Theorem (Piccirillo 2018)

Let $L = R \cup G \cup B$ be a surgery diagram for some 3-manifold Y such that:

1. *R is a zero-framed unknot, B and G have integral framings.*
2. *Ignoring B, R is isotopic to a meridian of G.*
3. *Ignoring G, R is isotopic to a meridian of B.*
4. *B and G have linking number 0.*

Then, there exist knots K and K' such that $Y \cong S^3_n(K) \cong S^3_n(K')$.

- Piccirillo's construction comes from an older construction, the *dualizable patterns* construction, to produce knots with the same surgery.

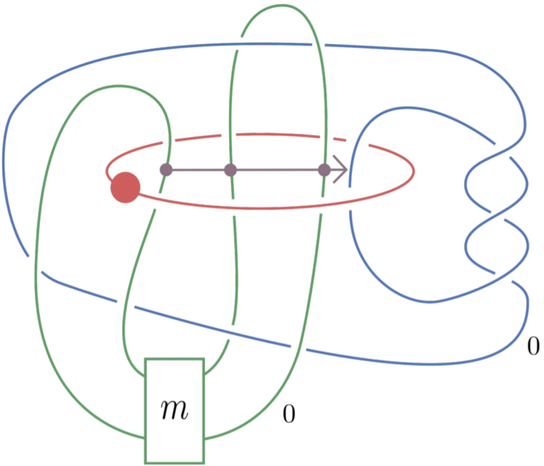


Figure: Diagram of a link L used by Piccirillo on her original paper. The colors (red, blue, and green) match the component letters R, B, G.

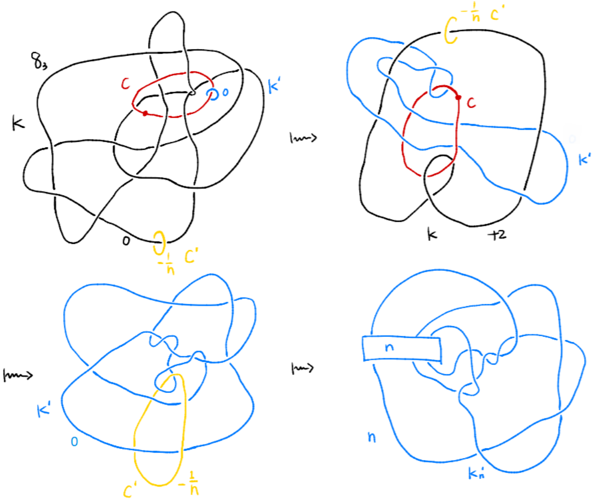
Baker-Motegi: Knots with the same surgery

- We adapted the following construction from Baker and Motegi (2018).
- Let K be a knot, and suppose we can take an unknot c linked with K such that $(0,0)$-surgery on $K \cup c$ is S^3.
- Baker-Motegi present a method for producing knots K'_n with $S^3_n(K) \cong S^3_n(K'_n)$ from this link.
- Define K' to be the surgery dual to c in $S^3 = S^3_{(0,0)}(K \cup c)$.
- Then K' has the same 0-surgery as K.

Baker-Motegi (cont.)

- Also define c' to be the surgery dual to K in S^3.
- After some Kirby calculus, we find that in the surgered manifold $S^3 = S^3_{(0,0)}(K \cup c)$, c' is an unknot linked with K'.
- Let K'_n be the result of twisting K' through c', n times.
- We say that $\{K'_n\}$ forms a *twist family*.
- Then K'_n has the same n-surgery as K for all n.
- Moreover, if c is not a meridian to K, then $K \simeq K'_n$ for at most finitely many n.

Baker-Motegi Illustration

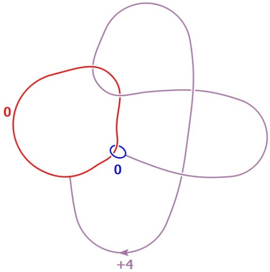


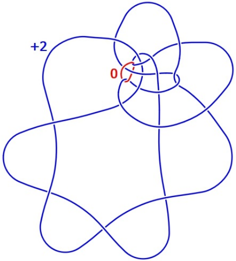
Obtaining a link $K \cup c$ when $u(K) = 1$ (Piccirillo)

- In the special case where K has unknotting number one, we can use a band presentation for K.
- We start with a Hopf link $R \cup B$ and slide one component over the other according to the band presentation for K.
- R remains an unknot, which we rename c, and B becomes K.

Lemma

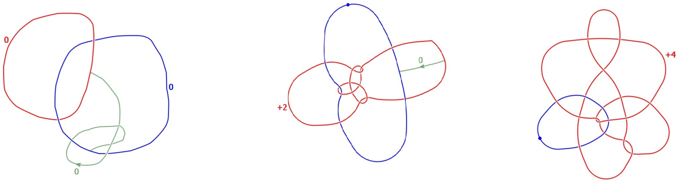
Let $K \cup c$ be the link obtained by the handle slide above. Then $(0,0)$-surgery on $K \cup c$ gives S^3.





Obtaining $K \cup c$ when $u(K) > 1$

- We do not have a systematic way of finding a link $K \cup c$ for a given knot K with $u(K) > 1$.
- If we perform multiple slides on a Hopf link, then we obtain a link $K \cup c$ for some knot K with higher unknotting number, but we have no way of predicting what K will be.
- Example: 10_{125}



Applications of the construction

- We have produced a banded Hopf link presentation for all the 78 unknotting number one knots K with $c(K) \leq 10$.
- We followed the above procedure to manually produce a diagram for a knot K' that has the same zero-surgery as K, using a software called KLO which can perform Kirby calculus on link diagrams.

- Using a software called SnapPy, we were able to verify that whenever K was not a twist knot, K' was not isotopic to K. Thus, zero was not a characterizing slope.
- We can use a similar process to check whether any integer slope is characterizing.

Definition

Let K be a knot. Then p/q is a *characterizing slope* for K if $S^3_{p/q}(K')$ is not homeomorphic to $S^3_{p/q}(K)$ for any knot $K' \neq K$.

- **Question:** Can we classify all integer slopes once we have the link $K \cup c$, with finitely many computations?

Known results on characterizing slopes

- The following theorem shows that for a most knot, most rational slopes are characterizing.

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Theorem (McCoy, 2018)

If K is a hyperbolic knot, then K has only finitely many non-characterizing slopes p/q with $|q| \geq 3$. Moreover, the probability that a randomly chosen slope p/q is characterizing for K approaches 1 as $|p| + |q| \to \infty$.

- Moreover, it was proven by Ozsváth and Szabó (2006) that all slopes are characterizing for the trefoil and figure-eight knot (twist knots).

Results (cont.)

- However, the next theorem shows that for a certain type of knot, there are infinitely many non-characterizing slopes.

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Theorem (Baker–Motegi, 2018)

Let $K \subset S^3$ be a knot, where there is an unknot c which is not a meridian to K such that $(0,0)$-surgery on $K \cup c$ yields S^3. Then K has infinitely many non-characterizing slopes.

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- Our work shows that for knots satisfying reasonable conditions, most integer slopes are non-characterizing!

Our results

Theorem 1

Regarding the integer slopes of knots K such that $c(K) \leq 10$:

- *If K has unknotting number $u(K) = 1$ and K is not a twist knot, then K has at most one integer characterizing slope, namely ± 2.*
- *If K is the twist knot 8_1, then K has at most one integer characterizing slope, namely 0.*
- *If K is one of the $u(K) = 2$ knots 8_4, 8_6, 8_{10}, 8_{12}, 8_{16}, 10_{148}, 10_{149}, or 10_{150}, K has no possible integer characterizing slope.*
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Theorem 2

If a knot K has unknotting number $u(K) = 1$ and is not a twisted Whitehead double, then K has at most finitely many integer characterizing slopes.

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Hyperbolic Dehn surgery

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- If K is a hyperbolic knot, then the torus $\partial N(K)$ inherits a Euclidean metric.
- According to this Euclidean metric, the *length* $\ell(p/q)$ of a slope p/q is the length of the shortest curve on the torus $\partial N(K)$ with slope p/q.

Theorem (Gromov-Thurston)

Let K be a hyperbolic knot. If $\ell(p/q) > 2\pi$, then the (p/q)-filling on $S^3 \setminus N(K)$ is hyperbolic.

Hyperbolic surgery theorems

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Theorem (Futer, et al.)

Let K be a hyperbolic knot; let $X := S^3 \setminus N(K)$. If $\ell(p/q) > 2\pi$, then the volume $\mathrm{vol}(X_{p/q})$ of the (p/q)-filling on X satisfies

$$\mathrm{vol}(X_{p/q}) \geq \left(1 - \left(\frac{2\pi}{\ell(p/q)}\right)^2\right)^{3/2} \mathrm{vol}(X).$$

Hyperbolic sugary theorems

Theorem (Cooper-Lackenby)

Let X be a cusped hyperbolic 3-manifold, and suppose s_1, s_2 are two slopes on a torus $T \subset \partial X$. Then

$$\ell(s_1)\ell(s_2) \geq \sqrt{3}\,\Delta(s_1, s_2),$$

where $\Delta(s_1, s_2)$ is the minimum possible number of intersections between a curve with slope s_1 and a curve with slope s_2 on T.

Sketch of proof of Theorem 1

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Proof of Theorem 1 (cont.)

- Following Baker–Motegi, we can produce a link $(K \cup c) \subset S^3$, where c is an unknot which is not a meridian to K such that $(0, 0)$-surgery on $K \cup c$ yields S^3.

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- Add a meridian m to c; notice that m *is* K'_n, in the non-standard representation of S^3.

Proof of Theorem 1 (cont.)

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$$v'_n \geq \left(1 - \left(\frac{2\pi\ell}{|n|\sqrt{3}}\right)^2\right)^{3/2} v_X > v$$

for $|n| > N := \frac{2\pi}{\sqrt{3}}\,\ell\,\left(1 - \left(v/v_X\right)^{2/3}\right)^{-1/2}$.

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- Hence all integer slopes $|n| > N$ are non-characterizing.
- Finally, we check the finitely many remaining cases by computer.

Ruling out integer characterizing slopes

- We developed an algorithm that can be carried out by a computer script to rule out the integer characterizing slopes of some knot K, once we have manually produced the link $L = K \cup c$.

1. Given a link L, run SnapPy commands to find out the volume of the knot K, the volume of the manifold $Z = S^3_0(c)$, and the length of the Seifert longitude in Z.
2. Using these values, apply the theorems of hyperbolic Dehn surgery to find the bound N such that any integer $|n| > N$ is a characterizing slope for K.
3. For the remaining $2N + 1$ cases, verify if the volume of the knot with the same n-surgery as K matches the volume of K.

- We use the *DT code* of the link, which uniquely describes all links that we deal with, up to isotopy.

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Theorem 2

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If a knot K has unknotting number $u(K) = 1$ and is not a twisted Whitehead double, then K has at most finitely many integer characterizing slopes.

Proof sketch of Theorem 2

Proposition

Let K be a knot in S^3. Suppose we can take an unknot c linked with K so that $(0,0)$-surgery on $K \cup c$ yields S^3 and c is not a meridian of K. Then K has at most finitely many integer characterizing slopes.

- This is a strengthened form of a result proven by Baker-Motegi, which we proved using symmetry of surgical duality.
- Part of the proof of this proposition guarantees that our adapted version of Baker-Motegi construction yields the desired K_n' that shares the same n-surgery as K from purely theoretical grounds of duality.
- We can apply this proposition to prove our main theorem, which admits more concrete conditions on K than the hypothesis in the proposition.

Proof of Theorem 2 (cont.)

- Recall that for K with $u(K) = 1$, we could obtain a link $K \cup c$ as in Baker-Motegi with $(0, 0)$-surgery S^3 by sliding over a Hopf link according to the band presentation for K.
- It remains to show that if K is not a twisted Whitehead double, then c is not a meridian to K after the handle slide.
- In this case, in any band presentation for K, the band must cross the disc bounded by one of the components of the Hopf link.

Lemma

Let $R \cup B$ be a Hopf link, and consider a handle slide of R over B yielding a link L in which R remains a meridian to B. Then there exists a handle slide of R over B, yielding a link isotopic to L, along a band that does not cross either of the discs bounded by R or B.

Extension of Theorem 2

- How can we loosen the condition on twisted Whitehead doubles? And on unknotting number?
- Can we produce an algorithm to find links $L = K \cup c$ from handle slides on a Hopf link?

Conjecture

If K is a knot with unknotting number $u(K) = 1$ and K is not a twist knot, then K has at most one integer characterizing slope: ± 2.

Next Steps

- Rigorously prove the lemma on band presentations.
- Keep experimenting with handle slides in order to expand the list on Theorem 1.2.
- Attempt to use other tools to prove a version of Theorem 1.1 for Twisted Whitehead Doubles.
- Look into our final conjecture.