Knot Surgery and Integer Characterizing Slopes

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Knots and links in the 3-sphere

Definition

A *knot* K is the image of a smooth embedding of the circle S^1 into a 3-manifold, usually the 3-sphere S^3. In particular, K is diffeomorphic to S^1. A *link* L is a disjoint union of knots, which may be knotted together.

Definition

Let M, N be manifolds and $g, h\colon N \to M$ embeddings. An *ambient isotopy* of M carrying g to h is a continuous map $F\colon M \times [0,1] \to M$, such that $F_t = F(\cdot, t)$ is a homeomorphism of M for each $t \in [0,1]$, $F_0 = \mathbb{1}$, and $F_1 \circ g = h$.

- We regard two knots $K, K' \subset S^3$ to be equivalent if they differ by an ambient isotopy of S^3. We write $K \simeq K'$.
- Equivalently, we can view knots as subsets of $\mathbb{R}^3$ rather than S^3.

Knot diagrams

- We can study a knot $K \subset \mathbb{R}^3$ by projecting it onto a hyperplane $\mathbb{R}^2$.
- If $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ is a projection such that $\pi(K)$ is an embedded curve except at finitely many *crossing points*, then $\pi(K)$ is a *diagram* for K.
- The *crossing number* $c(K)$ is the minimum number of crossings in a diagram of K.

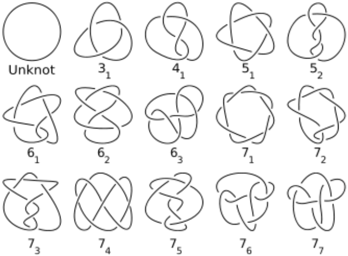


Figure: Knots with $c(K) \leq 7$

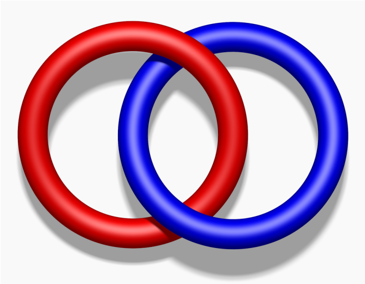
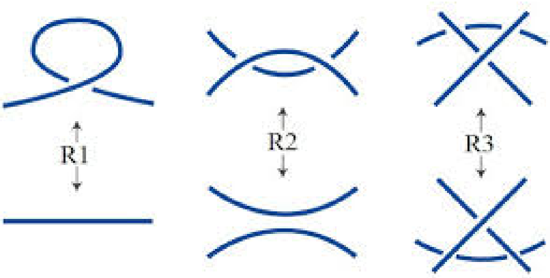


Figure: Hopf link

Reidemeister moves

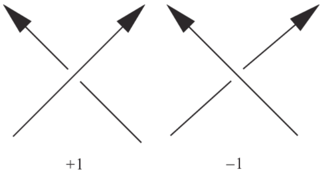
Theorem (Reidemeister)

Two knots K, K' are isotopic if and only if they have diagrams that differ by a sequence of planar isotopies and Reidemeister moves.



Crossings

- If we orient a knot K, then we can define a *sign* for each crossing by the right-hand rule.
- For a two-component link $L = K \cup K'$, the *linking number* $\mathrm{lk}(L)$ is one half the sum of the signs of the crossings between K and K' in a diagram of L.



Unknotting number and band move

Definition

The *unknotting number* $u(K)$ of a knot K is the minimal number of crossing changes that are required to change some diagram of K into a diagram of the unknot.

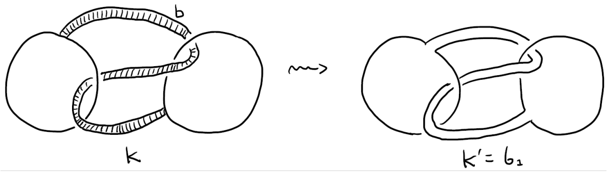
- If $u(K)=0$, then K is an unknot.

Definition

Let K be any knot(or link) in S^3 and let $b \subset S^3$ be any embedded band where one pair of sides lie along arcs in K and where b is otherwise disjoint from K. Join K to itself by deleting the arcs $K \cap b$ and adding the arcs forming the other side of b. The resulting knot(or link) K' is obtained from K by a *band move* along b.

Example of band move and band presentation

- Example of band move:
  The Stevedore knot 6_1 is obtained from a band move on two unknots:



- If K can be obtained from another knot(or link) K' by a single band move, then we say K has a *band presentation*.

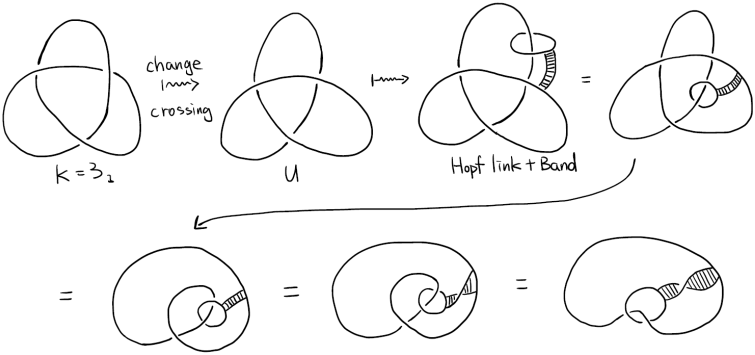
- If K can be obtained from a Hopf link by a single band move, then K has a *banded Hopf link presentation*.

Example of banded Hopf link presentation

Theorem

If K has $u(K) = 1$, then it is obtained from the Hopf link by a single band move.

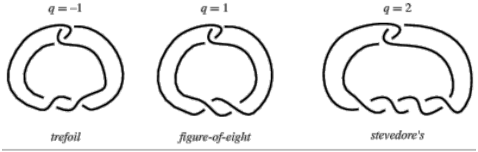
- Example: banded Hopf link presentation of Trefoil:



Twist knots and Twisted whitehead double

Definition

A *twist knot* K is a knot obtained by repeatedly twisting an unknot and linking the ends together.



Definition

A knot K is a *twisted Whitehead double* if there exists a band presentation for K in which the band does not cross either component of the Hopf link.

- Note: twist knots are twisted Whitehead doubles, but not all twisted Whitehead doubles are twist knots.

Piccirillo's construction: Knots with the same surgery

Theorem (Piccirillo 2018)

Let $L = R \cup G \cup B$ be a surgery diagram for some 3-manifold Y such that:

1. *R is a zero-framed unknot, B and G have integral framings.*
2. *Ignoring B, R is isotopic to a meridian of G.*
3. *Ignoring G, R is isotopic to a meridian of B.*
4. *B and G have linking number 0.*

Then, there exist knots K and K' such that $Y \cong S^3_n(K) \cong S^3_n(K')$.

- Piccirillo's construction comes from an older construction, the *dualizable patterns* construction, to produce knots with the same surgery.

Piccirillo's construction (cont.)

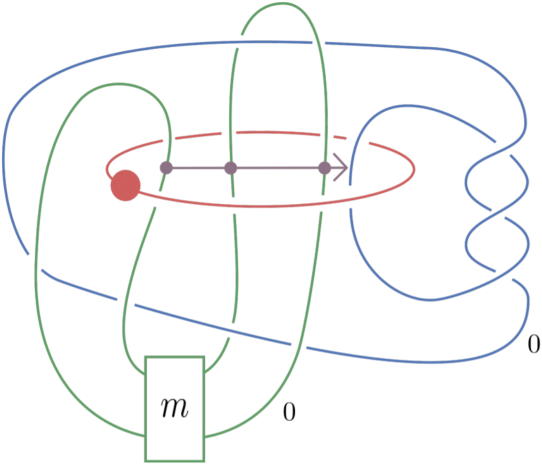


Figure: Diagram of a link L used by Piccirillo on her original paper. The colors (red, blue, and green) match the component letters R, B, G.

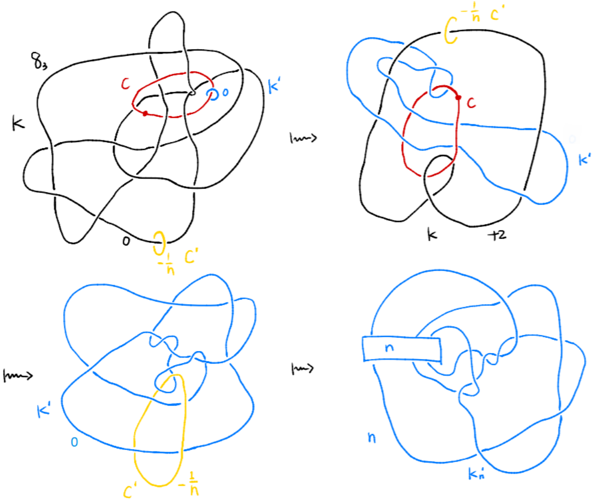
Baker-Motegi: Knots with the same surgery

- We adapted the following construction from Baker and Motegi (2018).
- Let K be a knot, and suppose we can take an unknot c linked with K such that $(0,0)$-surgery on $K \cup c$ is S^3.
- Baker-Motegi present a method for producing knots K'_n with $S^3_n(K) \cong S^3_n(K'_n)$ from this link.
- Define K' to be the surgery dual to c in $S^3 = S^3_{(0,0)}(K \cup c)$.
- Then K' has the same 0-surgery as K.

Baker-Motegi (cont.)

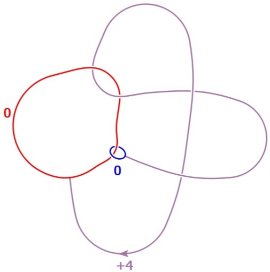
- Also define c' to be the surgery dual to K in S^3.
- After some Kirby calculus, we find that in the surgered manifold $S^3 = S^3_{(0,0)}(K \cup c)$, c' is an unknot linked with K'.
- Let K'_n be the result of twisting K' through c', n times.
- We say that $\{K'_n\}$ forms a *twist family*.
- Then K'_n has the same n-surgery as K for all n.
- Moreover, if c is not a meridian to K, then $K \simeq K'_n$ for at most finitely many n.

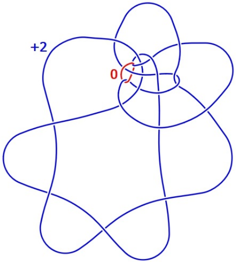
Baker-Motegi Illustration



Obtaining a link $K \cup c$ when $u(K) = 1$ (Piccirillo)

- In the special case where K has unknotting number one, we can use a band presentation for K.
- We start with a Hopf link $R \cup B$ and slide one component over the other according to the band presentation for K.
- Then R remains an unknot, which we rename c, and B becomes the knot K.
- After adding a meridian to R and sliding B over R, we get a diagram that also fits Piccirillo.
- Example: 7_7





$K \cup c$ for $u(K) = 1$ (cont.)

Lemma

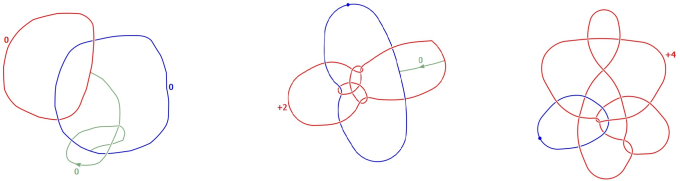
Let $K \cup c$ be the link obtained by the handle slide above. Then $(0,0)$-surgery on $K \cup c$ gives S^3.

Proof.

After the handle slide, the framing of c remains 0, and the framing of K changes by ± 2 depending on the linking number of the Hopf link. If we adjust the framing of the blue component of the Hopf link to ∓ 2 before the slide, then we see that $(0,0)$-surgery on $K \cup c$ is the same as $(\mp 2, 0)$ surgery on a Hopf link. It can be shown that $(n,0)$-surgery on a Hopf link is S^3 for any $n \in \mathbb{Z}$. $\square$

Obtaining $K \cup c$ when $u(K) > 1$

- We do not have a systematic way of finding a link $K \cup c$ for a given knot K with $u(K) > 1$.
- If we perform multiple slides on a Hopf link, then we obtain a link $K \cup c$ for some knot K with higher unknotting number, but we have no way of predicting what K will be.
- Example: 10_{125}



Applications of the construction

- We have produced a banded Hopf link presentation for all the 78 unknotting number one knots K with $c(K) \leq 10$.
- We followed the above procedure to manually produce a diagram for a knot K' that has the same zero-surgery as K, using a software called KLO which can perform Kirby calculus on link diagrams.
- Using a software called SnapPy, we were able to verify that whenever K was not a twist knot, K' was not isotopic to K. Thus, zero was not a characterizing slope.
- We can use a similar process to check whether any integer slope is characterizing.

- **Question:** Can we classify all integer slopes once we have the link $K \cup c$, with finitely many computations?

Ruling Out Integer Characterizing Slopes

- We developed an algorithm that can be carried out by a computer script to rule out the integer characterizing slopes of some knot K, once we have manually produced the link $L = K \cup c$.

1. Given a link L, run SnapPy commands to find out the volume of the knot K, the volume of the manifold $Z = S^3_0(c)$, and the length of the Seifert longitude in Z.
2. Using these values, apply the theorems of hyperbolic Dehn surgery to find the bound N such that any integer $|n| > N$ is a characterizing slope for K.
3. For the remaining $2N + 1$ cases, verify if the volume of the knot with the same n-surgery as K matches the volume of K.

- We use the *DT code* of the link, which uniquely describes all links that we deal with, up to isotopy.

Findings

Theorem (Low-Crossing Knots)

Regarding the integer slopes of knots K such that $c(K) \leq 10$:

- *If K has unknotting number $u(K) = 1$ and K is not a twist knot, then K has at most one integer characterizing slope, namely ± 2.*
- *If K is the twist knot 8_1, then K has at most one integer characterizing slope, namely 0.*
- *If K is one of the $u(K) = 2$ knots 8_4, 8_6, 8_{10}, 8_{12}, 8_{16}, 10_{148}, 10_{149}, or 10_{150}, K has no possible integer characterizing slope.*
- *If K is one of the $u(K) = 2$ knots 8_3, 10_{125}, or 10_{126}, K has at most one integer characterizing slope.*

Proof of Theorem 1 (cont.)

- Recall that for K with $u(K) = 1$, we could obtain a link $K \cup c$ as in Baker-Motegi with $(0, 0)$-surgery S^3 by sliding over a Hopf link according to the band presentation for K.
- It remains to show that if K is not a twisted Whitehead double, then c is not a meridian to K after the handle slide.
- In this case, in any band presentation for K, the band must cross the disc bounded by one of the components of the Hopf link.

Lemma

Let $R \cup B$ be a Hopf link, and consider a handle slide of R over B yielding a link L in which R remains a meridian to B. Then there exists a handle slide of R over B, yielding a link isotopic to L, along a band that does not cross either of the discs bounded by R or B.

Extension of Theorem 1

- How can we drop the condition on Twisted Whitehead Doubles? And on unknotting number?
- Can we produce an algorithm to find links $L = K \cup c$ from handle slides on a Hopf link?

Conjecture

If K is a knot with unknotting number $u(K) = 1$ and K is not a twist knot, then K has at most one integer characterizing slope: ± 2.

Next Steps

- Rigorously prove the lemma on band presentations.
- Keep experimenting with handle slides in order to expand the list on Theorem 1.2.
- Attempt to use other tools to prove a version of Theorem 1.1 for Twisted Whitehead Doubles.
- Look into our final conjecture.