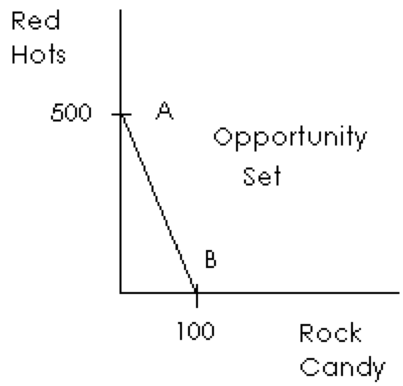
Elton, Gruber, Brown and Goetzmann
Modern Portfolio Theory and Investment Analysis
Selected Solutions to Text Problems

Chapter 1: Problem 1

A. Opportunity Set

With one dollar, you can buy 500 red hots and no rock candies (point A), or 100 rock candies and no red hots (point B), or any combination of red hots and rock candies (any point along the opportunity set line AB).



Algebraically, if X = quantity of red hots and Y = quantity of rock candies, then:

$$0.2X + 1Y = 100$$

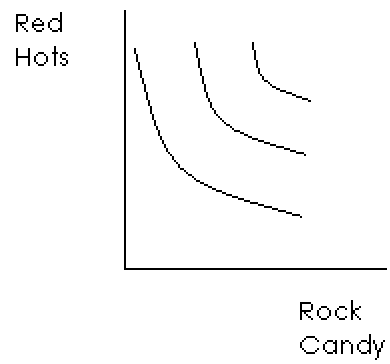
That is, the money spent on candies, where red hots sell for 0.2 cents a piece and rock candy sells for 1 cent a piece, cannot exceed 100 cents ($1.00). Solving the above equation for X gives:

$$X = 500 - 5Y$$

which is the equation of a straight line, with an intercept of 500 and a slope of –5.

B. Indifference Map

Below is one indifference map. The indifference curves up and to the right indicate greater happiness, since these curves indicate more consumption from both candies. Each curve is negatively sloped, indicating a preference of more to less, and each curve is convex, indicating that the rate of exchange of red hots for rock candies decreases as more and more rock candies are consumed. Note that the exact slopes of the indifference curves in the indifference map will depend an the individual's utility function and may differ among students.



Chapter 1: Problem 3

If you consume nothing at time 1 and instead invest all of your time-1 income at a riskless rate of 10%, then at time 2 you will be able to consume all of your time-2 income plus the proceeds earned from your investment:

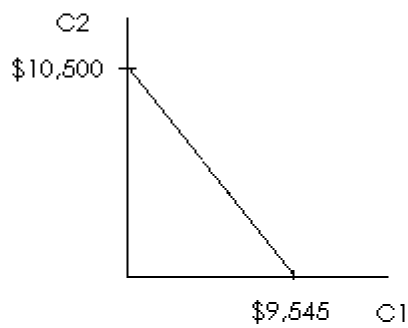
$5,000 + $5,000 (1.1) = $10,500.

If you consume nothing at time 2 and instead borrow at time 1 the present value of your time-2 income at a riskless rate of 10%, then at time 1 you will be able to consume all of the borrowed proceeds plus your time-1 income:

$5,000 + $5,000 ÷ (1.1) = $9,545.45

All other possible optimal consumption patterns of time 1 and time 2 consumption appear on a straight line (the opportunity set) with an intercept of $10,500 and a slope of –1.1:

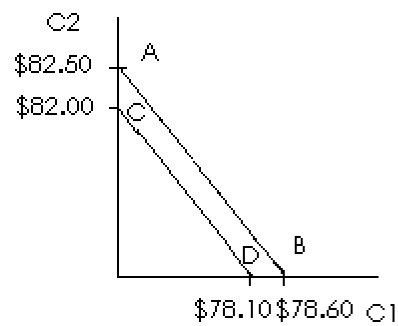
$C_2 = \$5{,}000 + (\$5{,}000 - C_1) \times (1.1) = \$10{,}500 - 1.1C_1$



Chapter 1: Problem 5

With Job 1 you can consume \$30 + \$50 (1.05) = \$82.50 at time 2 and nothing at time 1, \$50 + \$30 ÷ (1.05) = \$78.60 at time 1 and nothing at time 2, or any consumption pattern of time 1 and time 2 consumption shown along the line AB: C_2 = \$82.50 – 1.05C_1.

With Job 2 you can consume \$40 + \$40 (1.05) = \$82.00 at time 2 and nothing at time 1, \$40 + \$40 ÷ (1.05) = \$78.10 at time 1 and nothing at time 2, or any consumption pattern of time 1 and time 2 consumption shown along the line CD: C_2 = \$82.00 – 1.05C_1.



The individual should select Job 1, since the opportunity set associated with it (line AB) dominates the opportunity set of Job 2 (line CD).

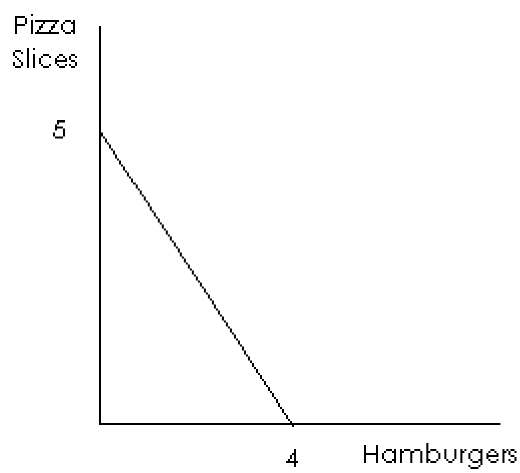
Chapter 1: Problem 9

Let X = the number of pizza slices, and let Y = the number of hamburgers. Then, if pizza slices are \$2 each, hamburgers are \$2.50 each, and you have \$10, your opportunity set is given algebraically by

$$\$2X + \$2.50Y = \$10$$

Solving the above equation for X gives $X = 5 - 1.25Y$, which is the equation for a straight line with an intercept of 5 and a slope of -1.25.

Graphically, the opportunity set appears as follows:



Assuming you like both pizza and hamburgers, your indifference curves will be negatively sloped, and you will be better off on an indifference curve to the right of another indifference curve. Assuming diminishing marginal rate of substitution between pizza slices and hamburgers (the lower the number of hamburgers you have, the more pizza slices you need to give up one more burger without changing your level of satisfaction), your indifference curves will also be convex.

A typical family of indifference curves appears below. Although you would rather be on an indifference curve as far to the right as possible, you are constrained by your $10 budget to be on an indifference curve that is on or to the left of the opportunity set. Therefore, your optimal choice is the combination of pizza slices and hamburgers that is represented by the point where your indifference curve is just tangent to the opportunity set (point A below).

Chapter 4: Problem 1

A. Expected return is the sum of each outcome times its associated probability.

Expected return of Asset 1 = $\overline{R_1} = 16\% \times 0.25 + 12\% \times 0.5 + 8\% \times 0.25 = 12\%$

$\overline{R_2} = 6\%$; $\overline{R_3} = 14\%$; $\overline{R_4} = 12\%$

Standard deviation of return is the square root of the sum of the squares of each outcome minus the mean times the associated probability.

Standard deviation of Asset 1 =

$\sigma_1 = [(16\% - 12\%)^2 \times 0.25 + (12\% - 12\%)^2 \times 0.5 + (8\% - 12\%)^2 \times 0.25]^{1/2} = 8^{1/2}$
$= 2.83\%$

$\sigma_2 = 2^{1/2} = 1.41\%$; $\sigma_3 = 18^{1/2} = 4.24\%$; $\sigma_4 = 10.7^{1/2} = 3.27\%$

B. Covariance of return between Assets 1 and 2 = σ_{12} =

$(16 - 12) \times (4 - 6) \times 0.25 + (12 - 12) \times (6 - 6) \times 0.5 + (8 - 12) \times (8 - 6) \times 0.25$
$= -4$

The variance/covariance matrix for all pairs of assets is:

	1	2	3	4
1	8	– 4	12	0
2	– 4	2	– 6	0
3	12	– 6	18	0
4	0	0	0	10.7

Correlation of return between Assets 1 and 2 = $\rho_{12} = \frac{-4}{2.83 \times 1.41} = -1.$

The correlation matrix for all pairs of assets is:

	1	2	3	4
1	1	– 1	1	0
2	– 1	1	– 1	0
3	1	– 1	1	0
4	0	0	0	1

C. Portfolio — Expected Return

Portfolio	Expected Return
A	$1/2 \times 12\% + 1/2 \times 6\% = 9\%$
B	13%
C	12%
D	10%
E	13%
F	$1/3 \times 12\% + 1/3 \times 6\% + 1/3 \times 14\% = 10.67\%$
G	10.67%
H	12.67%
I	$1/4 \times 12\% + 1/4 \times 6\% + 1/4 \times 14\% + 1/4 \times 12\% = 11\%$

Portfolio	Variance
A	$(1/2)^2 \times 8 + (1/2)^2 \times 2 + 2 \times 1/2 \times 1/2 \times (-4) = 0.5$
B	12.5
C	4.6
D	2
E	7
F	$(1/3)^2 \times 8 + (1/3)^2 \times 2 + (1/3)^2 \times 18 + 2 \times 1/3 \times 1/3 \times (-4) + 2 \times 1/3 \times 1/3 \times 12 + 2 \times 1/3 \times 1/3 \times (-6) = 3.6$
G	2
H	6.7
I	$(1/4)^2 \times 8 + (1/4)^2 \times 2 + (1/4)^2 \times 18 + (1/4)^2 \times 10.7 + 2 \times 1/4 \times 1/4 \times (-4) + 2 \times 1/4 \times 1/4 \times 12 + 2 \times 1/4 \times 1/4 \times 0 + 2 \times 1/4 \times 1/4 \times (-6) + 2 \times 1/4 \times 1/4 \times 0 + 2 \times 1/4 \times 1/4 \times 0 = 2.7$

Chapter 5: Problem 1

From Problem 1 of Chapter 4, we know that:

$\bar{R}_1 = 12\%$	$\bar{R}_2 = 6\%$	$\bar{R}_3 = 14\%$	$\bar{R}_4 = 12\%$		
$\sigma^2_1 = 8$	$\sigma^2_2 = 2$	$\sigma^2_3 = 18$	$\sigma^2_4 = 10.7$		
$\sigma_1 = 2.83\%$	$\sigma_2 = 1.41\%$	$\sigma_3 = 4.24\%$	$\sigma_4 = 3.27\%$		
$\sigma_{12} = -4$	$\sigma_{13} = 12$	$\sigma_{14} = 0$	$\sigma_{23} = -6$	$\sigma_{24} = 0$	$\sigma_{34} = 0$
$\rho_{12} = -1$	$\rho_{13} = 1$	$\rho_{14} = 0$	$\rho_{23} = -1.0$	$\rho_{24} = 0$	$\rho_{34} = 0$

In this problem, we will examine 2-asset portfolios consisting of the following pairs of securities:

Pair	Securities
A	1 and 2
B	1 and 3
C	1 and 4
D	2 and 3
E	2 and 4
F	3 and 4

A. Short Selling Not Allowed

(Note that the answers to part A.4 are integrated with the answers to parts A.1, A.2 and A.3 below.)

A.1

We want to find the weights, the standard deviation and the expected return of the minimum-risk porfolio, also known as the global minimum variance (GMV) portfolio, of a pair of assets when short sales are not allowed.

We further know that the compostion of the GMV portfolio of any two assets *i* and *j* is:

$$X_i^{GMV} = \frac{\sigma_j^2 - \sigma_{ij}}{\sigma_i^2 + \sigma_j^2 - 2\sigma_{ij}}$$

$$X_j^{GMV} = 1 - X_i^{GMV}$$

Pair A (assets 1 and 2):
Applying the above GMV weight formula to Pair A yields the following weights:

$$X_1^{GMV} = \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}} = \frac{2-(-4)}{8+2-(2)(-4)} = \frac{6}{18} = \frac{1}{3} \text{ (or 33.33\%)}$$

$$X_2^{GMV} = 1 - X_1^{GMV} = 1 - \frac{1}{3} = \frac{2}{3} \text{ (or 66.67\%)}$$

This in turn gives the following for the GMV portfolio of Pair A:

$$\bar{R}_{GMV} = \frac{1}{3} \times 12\% + \frac{2}{3} \times 6\% = 8\%$$

$$\sigma_{GMV}^2 = \left(\frac{1}{3}\right)^2 (8) + \left(\frac{2}{3}\right)^2 (2) + (2)\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)(-4) = 0$$

$$\sigma_{GMV} = 0$$

Recalling that $\rho_{12} = -1$, the above result demonstrates the fact that, when two assets are perfectly negatively correlated, the minimum-risk portfolio of those two assets will have zero risk.

Pair B (assets 1 and 3):
Applying the above GMV weight formula to Pair B yields the following weights:

$$X_1^{GMV} = 3 \text{ (300\%) and } X_3^{GMV} = -2 \text{ (–200\%)}$$

This means that the GMV portfolio of assets 1 and 3 involves short selling asset 3. But if short sales are not allowed, as is the case in this part of Problem 1, then the GMV “portfolio” involves placing all of your funds in the lower risk security (asset 1) and none in the higher risk security (asset 3). This is obvious since, because the correlation between assets 1 and 3 is +1.0, portfolio risk is simply a linear combination of the risks of the two assets, and the lowest value that can be obtained is the risk of asset 1.

Thus, when short sales are not allowed, we have for Pair B:

$$X_1^{GMV} = 1 \ (100\%) \text{ and } X_3^{GMV} = 0 \ (0\%)$$

$$\bar{R}_{GMV} = \bar{R}_1 = 12\% \, ; \ \sigma_{GMV}^2 = \sigma_1^2 = 8 \, ; \ \sigma_{GMV} = \sigma_1 = 2.83\%$$

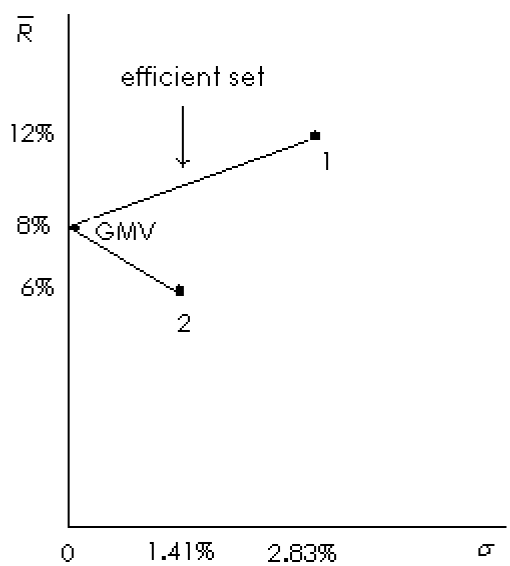
For the GMV portfolios of the remaining pairs above we have:

Pair	X_i^{GMV}	X_j^{GMV}	$\bar{R}_{GMV}$	σ_{GMV}
C (i = 1, j = 4)	0.572	0.428	12%	2.14%
D (i = 2, j = 3)	0.75	0.25	8%	0%
E (i = 2, j = 4)	0.8425	0.1575	6.95%	1.3%
F (i = 3, j = 4)	0.3728	0.6272	12.75%	2.59%

A.2 and A.3

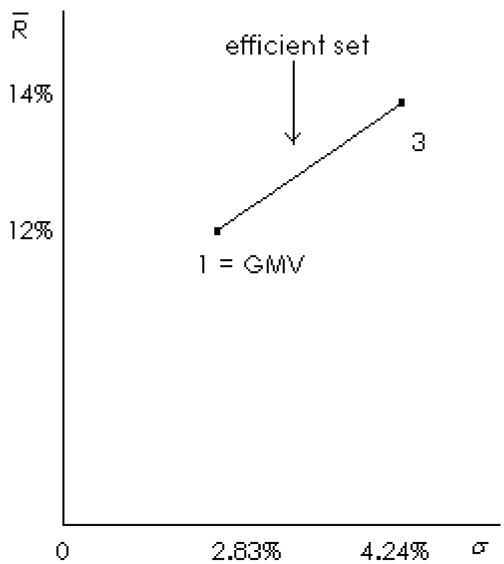
For each of the above pairs of securities, the graph of all possible combinations (portfolios) of the securities (the portfolio possibilties curves) and the efficient set of those portfolios appear as follows when short sales are not allowed:

Pair A



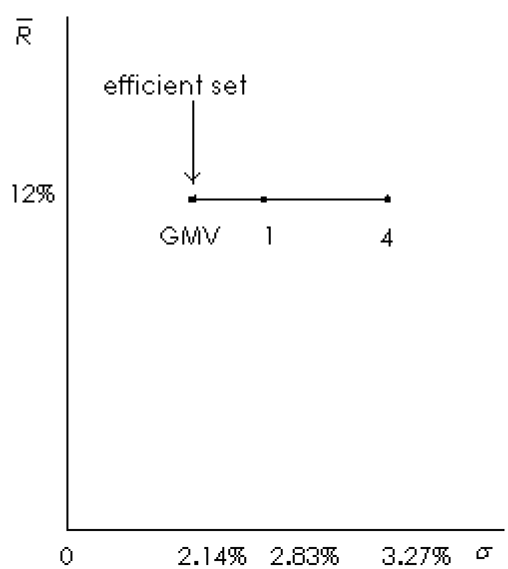
The efficient set is the positively sloped line segment.

Pair B



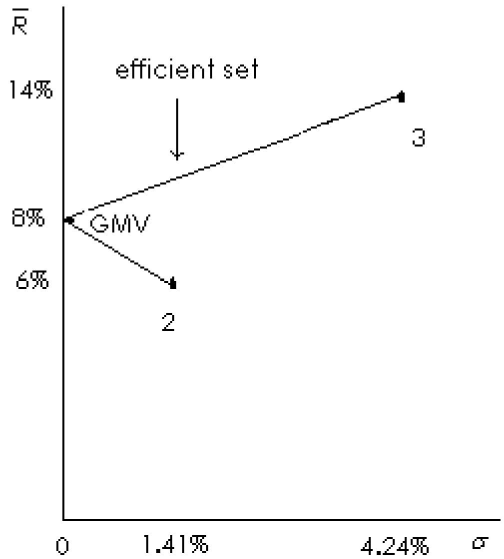
The entire line is the efficient set.

Pair C



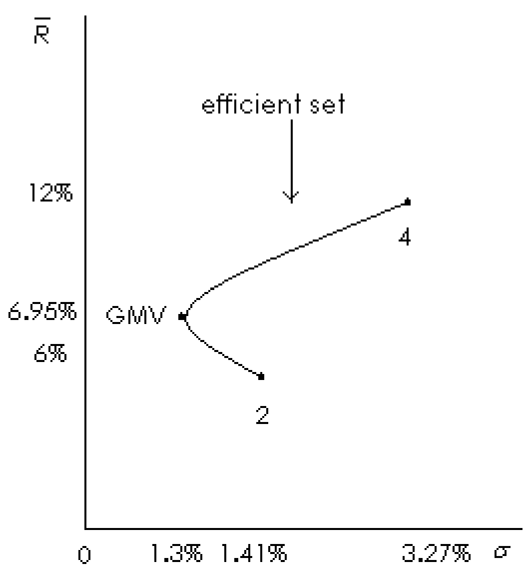
Only the GMV portfolio is efficient.

Pair D



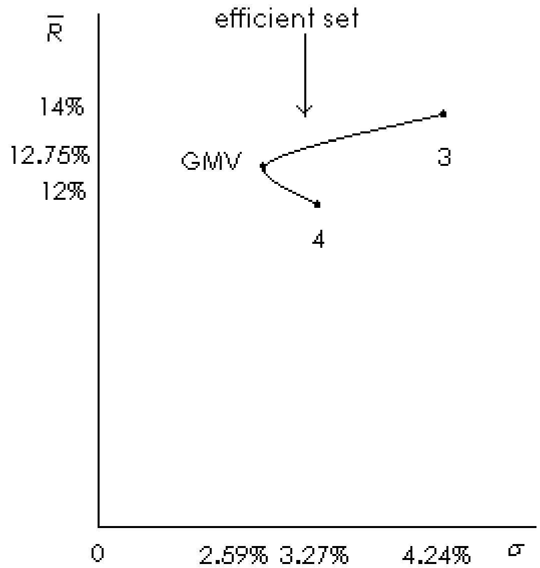
The efficient set is the positively sloped line segment.

Pair E



The efficient set is the positively sloped part of the curve, starting at the GMV portfolio and ending at security 4.

Pair F



The efficient set is the positively sloped part of the curve, starting at the GMV portfolio and ending at security 3.

B. Short Selling Allowed

(Note that the answers to part B.4 are integrated with the answers to parts B.1, B.2 and B.3 below.)

B.1

When short selling is allowed, all of the GMV portfolios shown in Part A.1 above are the same except the one for Pair B (assets 1 and 3). In the no-short-sales case in Part A.1, the GMV "portfolio" for Pair B was the lower risk asset 1 alone. However, applying the GMV weight formula to Pair B yielded the following weights:

$$X_1^{GMV} = 3 \text{ (300\%) and } X_3^{GMV} = -2 \text{ (–200\%)}$$

This means that the GMV portfolio of assets 1 and 3 involves short selling asset 3 in an amount equal to twice the investor's original wealth and then placing the original wealth plus the proceeds from the short sale into asset 1. This yields the following for Pair B when short sales are allowed:

$$\bar{R}_{GMV} = 3 \times 12\% - 2 \times 14\% = 8\%$$

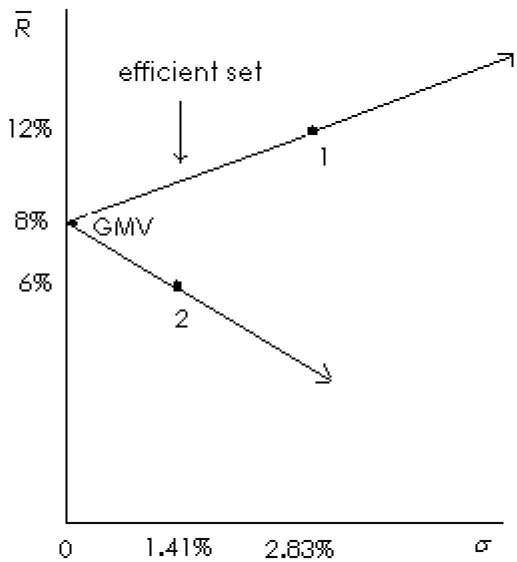
$$\sigma_{GMV}^2 = (3)^2(8) + (-2)^2(18) + (2)(3)(-2)(12) = 0$$

$$\sigma_{GMV} = 0$$

Recalling that ρ_{13} = +1, this demonstrates the fact that, when two assets are perfectly positively correlated and short sales are allowed, the GMV portfolio of those two assets will have zero risk.

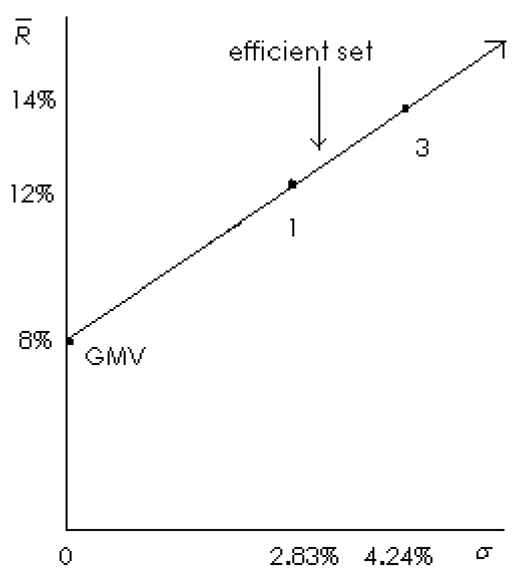
B.2 and B.3
When short selling is allowed, the portfolio possibilities graphs are extended.

Pair A



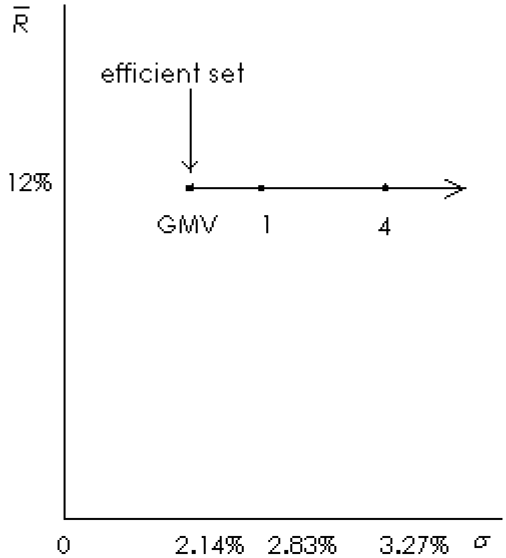
The efficient set is the positively sloped line segment through security 1 and out toward infinity.

Pair B



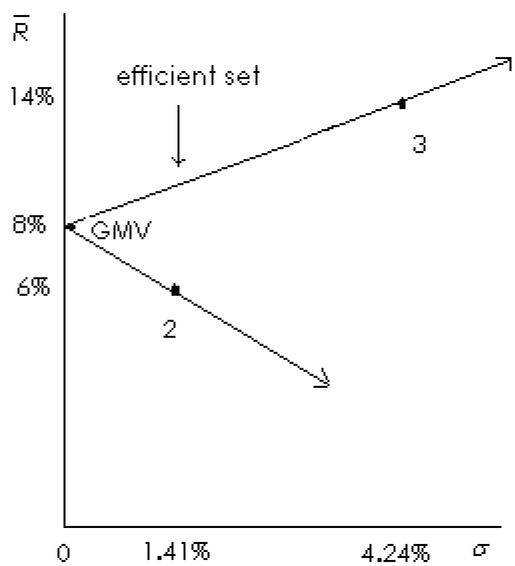
The entire line out toward infinity is the efficient set.

Pair C



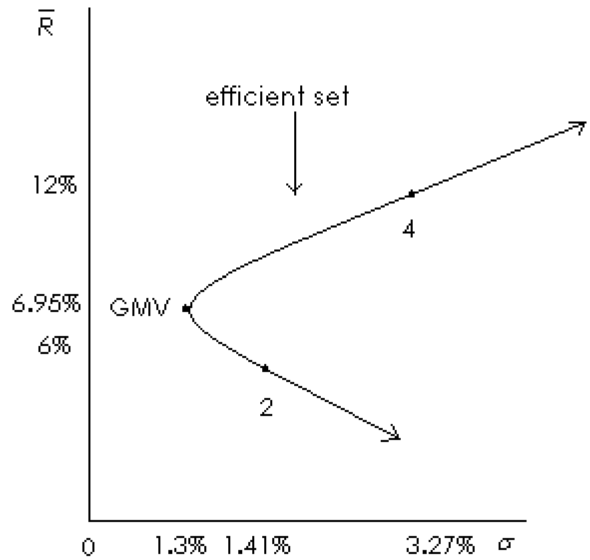
Only the GMV portfolio is efficient.

Pair D



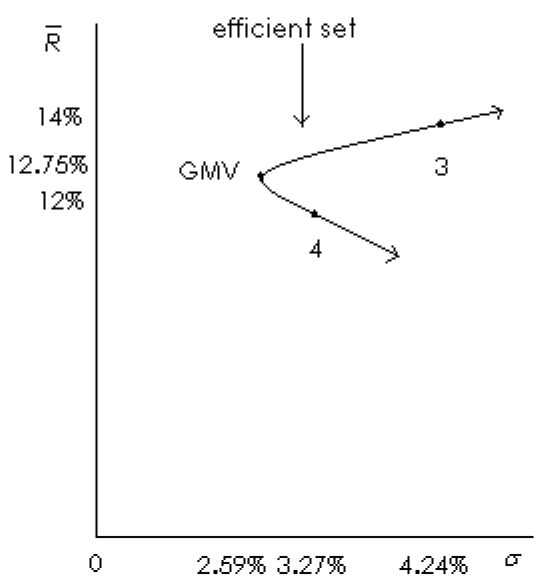
The efficient set is the positively sloped line segment through security 3 and out toward infinity.

Pair E



The efficient set is the positively sloped part of the curve, starting at the GMV portfolio and extending past security 4 toward infinity.

Pair F



The efficient set is the positively sloped part of the curve, starting at the GMV portfolio and extending past security 3 toward infinity.

C.

Pair A (assets 1 and 2):
Since the GMV portfolio of assets 1 and 2 has an expected return of 8% and a risk of 0%, then, if riskless borrowing and lending at 5% existed, one would borrow an infinite amount of money at 5% and place it in the GMV portfolio. This would be pure arbitrage (zero risk, zero net investment and positive return of 3%). With an 8% riskless lending and borrowing rate, one would hold the same portfolio one would hold without riskless lending and borrowing. (The particular portfolio held would be on the efficient frontier and would depend on the investor's degree of risk aversion.)

Pair B (assets 1 and 3):
Since short sales are allowed in Part C and since we saw in Part B that when short sales are allowed the GMV portfolio of assets 1 and 3 has an expected return of 8% and a risk of 0%, the answer is the same as that above for Pair A.

Pair C (assets 1 and 4):
We have seen that, regardless of the availability of short sales, the efficient frontier for this pair of assets was a single point representing the GMV portfolio, with a return of 12%. With riskless lending and borrowing at either 5% or 8%, the new efficient frontier (efficient set) will be a straight line extending from the vertical axis at the riskless rate and through the GMV portfolio and out to infinity. The amount that is invested in the GMV portfolio and the amount that is borrowed or lent will depend on the investor's degree of risk aversion.

Pair D (assets 2 and 3):
Since assets 2 and 3 are perfectly negatively correlated and have a GMV portfolio with an expected return of 8% and a risk of 0%, the answer is identical to that above for Pair A.

Pair E (assets 2 and 4):
We arrived at the following answer graphically; the analytical solution to this problem is presented in the subsequent chapter (Chapter 6). With a riskless rate of 5%, the new efficient frontier (efficient set) will be a straight line extending from the vertical axis at the riskless rate, passing through the portfolio where the line is tangent to the upper half of the original portfolio possibilities curve, and then out to infinity. The amount that is invested in the tangent portfolio and the amount that is borrowed or lent will depend on the investor's degree of risk aversion. The tangent portfolio has an expected return of 9.4% and a standard deviation of 1.95%. With a riskless rate of 8%, the point of tangency occurs at infinity.

Pair F (assets 3 and 4):
We arrived at the following answer graphically; the analytical solution to this problem is presented in the subsequent chapter (Chapter 6). With a riskless rate of 5%, the new efficient frontier (efficient set) will be a straight line extending from the vertical axis at the riskless rate, passing through the portfolio where the line is tangent to the upper half of the original portfolio possibilities curve, and then out to infinity. The amount that is invested in the tangent portfolio and the amount that is borrowed or lent will depend on the investor's degree of risk aversion. The tangent (optimal) portfolio has an expected return of 12.87% and a standard deviation of 2.61%. With a riskless rate of 8%, the new efficient frontier will be a straight line extending from the vertical axis at the riskless rate, passing through the portfolio where the line is tangent to the upper half of the original portfolio possibilities curve, and then out to infinity. The tangent (optimal) portfolio has an expected return of 12.94% and a standard deviation of 2.64%.

Chapter 5: Problem 5

When ρ equals 1, the least risky "combination" of securities 1 and 2 is security 2 held alone (assuming no short sales). This requires $X_1 = 0$ and $X_2 = 1$, where the X's are the investment weights. The standard deviation of this "combination" is equal to the standard deviation of security 2; $\sigma_P = \sigma_2 = 2$.

When ρ equals -1, we saw in Chapter 5 that we can always find a combination of the two securities that will completely eliminate risk, and we saw that this combination can be found by solving $X_1 = \sigma_2/(\sigma_1 + \sigma_2)$. So, $X_1 = 2/(5 + 2) = 2/7$, and since the investment weights must sum to 1, $X_2 = 1 - X_1 = 1 - 2/7 = 5/7$. So a combination of 2/7 invested in security 1 and 5/7 invested in security 2 will completely eliminate risk when ρ equals -1, and σ_P will equal 0.

When ρ equals 0, we saw in Chapter 5 that the minimum-risk combination of two assets can be found by solving $X_1 = \sigma_2^2/(\sigma_1^2 + \sigma_2^2)$. So, $X_1 = 4/(25 + 4) = 4/29$, and $X_2 = 1 - X_1 = 1 - 4/29 = 25/29$. When ρ equals 0, the expression for the standard deviation of a two-asset portfolio is

$$\sigma_P = \sqrt{X_1^2\sigma_1^2 + (1 - X_1)^2\sigma_2^2}$$

Substituting 4/29 for X_1 in the above equation, we have

$$\begin{aligned}\sigma_P &= \sqrt{\left(\frac{4}{29}\right)^2 \times 25 + \left(\frac{25}{29}\right)^2 \times 4} \\ &= \sqrt{\frac{400}{841} + \frac{2500}{841}} \\ &= \sqrt{\frac{2900}{841}} \\ &= 1.86\%\end{aligned}$$

Chapter 6: Problem 1

The simultaneous equations necessary to solve this problem are:

$$5 = 16Z_1 + 20Z_2 + 40Z_3$$

$$7 = 20Z_1 + 100Z_2 + 70Z_3$$

$$13 = 40Z_1 + 70Z_2 + 196Z_3$$

The solution to the above set of equations is:

$$Z_1 = 0.292831$$

$$Z_2 = 0.009118$$

$$Z_3 = 0.003309$$

This results in the following set of weights for the optimum (tangent) portfolio:

X_1 = .95929 (95.929%)

X_2 = .02987 (2.987%)

X_3 = .01084 (1.084%

The optimum portfolio has a mean return of 10.146% and a standard deviation of 4.106%.

Chapter 6: Problem 5

Since the given portfolios, A and B, are on the efficient frontier, the GMV portfolio can be obtained by finding the minimum-risk combination of the two portfolios:

$$X_A^{GMV} = \frac{\sigma_B^2 - \sigma_{AB}}{\sigma_A^2 + \sigma_B^2 - 2\sigma_{AB}} = \frac{16 - 20}{36 + 16 - 2 \times 20} = -\frac{1}{3}$$

$$X_B^{GMV} = 1 - X_A^{GMV} = 1\frac{1}{3}$$

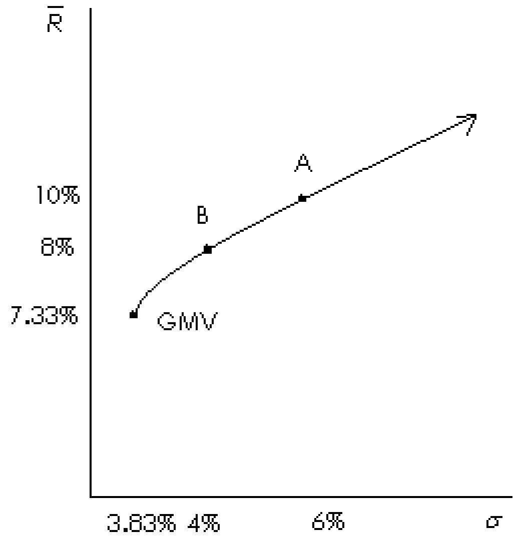
This gives $\overline{R}_{GMV} = 7.33\%$ and $\sigma_{GMV} = 3.83\%$

Also, since the two portfolios are on the efficient frontier, the entire efficient frontier can then be traced by using various combinations of the two portfolios, starting with the GMV portfolio and moving up along the efficient frontier (increasing the weight in portfolio A and decreasing the weight in portfolio B). Since $X_B = 1 - X_A$ the efficient frontier equations are:

$$\overline{R_P} = X_A \overline{R_A} + (1 - X_A)\overline{R_B} = 10X_A + 8 \times (1 - X_A)$$

$$\begin{aligned}\sigma_P &= \sqrt{X_A^2\sigma_A^2 + (1 - X_A)^2\sigma_B^2 + 2X_A(1 - X_A)\sigma_{AB}} \\ &= \sqrt{36X_A^2 + 16(1 - X_A)^2 + 40X_A(1 - X_A)}\end{aligned}$$

Since short sales are allowed, the efficient frontier will extend beyond portfolio A and out toward infinity. The efficient frontier appears as follows:



Chapter 7: Problem 5

A.
The single-index model's formula for security *i*'s mean return is

$$\overline{R_i} = \alpha_i + \beta_i \overline{R_m}$$

Since $\overline{R_m}$ equals 8%, then, e.g., for security A we have:

$$\begin{aligned}\overline{R_A} &= \alpha_A + \beta_A \overline{R_m} \\ &= 2 + 1.5 \text{ x } 8 \\ &= 2 + 12 \\ &= 14\%\end{aligned}$$

Similarly:

$$\overline{R_B} = 13.4\% \, ; \; \overline{R_C} = 7.4\% \, ; \; \overline{R_D} = 11.2\%$$

B.
The single-index model's formula for security *i*'s own variance is:

$$\sigma_i^2 = \beta_i^2 \, \sigma_m^2 + \sigma_{e_i}^2 .$$

Since $\sigma_m = 5$, then, e.g., for security A we have:

$$\begin{aligned} \sigma_A^2 &= \beta_A^2 \, \sigma_m^2 + \sigma_{e_A}^2 \\ &= (1.5)^2 \times (5)^2 + (3)^2 \\ &= 65.25 \end{aligned}$$

Similarly:

$$\sigma^2_B = 43.25; \; \sigma^2_C = 20; \; \sigma^2_D = 36.25$$

C.
The single-index model's formula for the covariance of security *i* with security *j* is

$$\sigma_{ij} = \sigma_{ji} = \beta_i \, \beta_j \, \sigma_m^2$$

Since $\sigma^2_m = 25$, then, e.g., for securities A and B we have:

$$\begin{aligned} \sigma_{AB} &= \beta_A \, \beta_B \, \sigma_m^2 \\ &= 1.5 \times 1.3 \times 25 \\ &= 48.75 \end{aligned}$$

Similarly:

$$\sigma_{AC} = 30; \; \sigma_{AD} = 33.75; \; \sigma_{BC} = 26; \; \sigma_{BD} = 29.25; \; \sigma_{CD} = 18$$

Chapter 7: Problem 6

A.
Recall that the formula for a portfolio's beta is:

$$\beta_P = \sum_{i=1}^{N} X_i \, \beta_i$$

The weight for each asset (X_i) in an equally weighted portfolio is simply $1/N$, where N is the number of assets in the portfolio.

Since there are four assets in Problem 5, N = 4 and X_i equals 1/4 for each asset in an equally weighted portfolio of those assets. So:

$$\begin{aligned}\beta_P &= \frac{1}{4}\beta_A + \frac{1}{4}\beta_B + \frac{1}{4}\beta_C + \frac{1}{4}\beta_D \\ &= \frac{1}{4}(1.5 + 1.3 + 0.8 + 0.9) \\ &= \frac{1}{4} \times 4.5 \\ &= 1.125\end{aligned}$$

B.

Recall that the definition of a portfolio's alpha is:

$$\alpha_P = \sum_{i=1}^{N} X_i \alpha_i$$

Using 1/4 as the weight for each asset, we have:

$$\begin{aligned}\alpha_P &= \frac{1}{4}\alpha_A + \frac{1}{4}\alpha_B + \frac{1}{4}\alpha_C + \frac{1}{4}\alpha_D \\ &= \frac{1}{4}(2 + 3 + 1 + 4) \\ &= \frac{1}{4} \times 10 \\ &= 2.5\end{aligned}$$

C.

Recall that a formula for a portfolio's variance using the single-index model is:

$$\sigma_P^2 = \beta_P^2 \sigma_m^2 + \sum_{i=1}^{N} X_i^2 \sigma_{e_i}^2$$

Using 1/4 as the weight for each asset, we have:

$$\begin{aligned}\sigma_P^2 &= (1.125)^2 (5)^2 + \left(\frac{1}{4}\right)^2 (3)^2 + \left(\frac{1}{4}\right)^2 (1)^2 + \left(\frac{1}{4}\right)^2 (2)^2 + \left(\frac{1}{4}\right)^2 (4)^2 \\ &= (1.125)^2 \times 25 + \frac{1}{16}(9 + 1 + 4 + 16) \\ &= 33.52\end{aligned}$$

D.

Using the single-index model's formula for a portfolio's mean return we have:

$$\begin{aligned}\overline{R_P} &= \alpha_P + \beta_P \overline{R_m} \\ &= 2.5 + 1.125 \times 8 \\ &= 11.5\%\end{aligned}$$

Chapter 8: Problem 5

The formula for a security's expected return using a general two-index model is:

$$\overline{R_i} = a_i + b_{i1} \times \overline{I_1} + b_{i2} \times \overline{I_2}$$

Using the above formula and data given in the problem, the expected return for, e.g., security A is:

$$\begin{aligned}\overline{R_A} &= a_A + b_{A1} \times \overline{I_1} + b_{A2} \times \overline{I_2} \\ &= 2 + 0.8 \times 8 + 0.9 \times 4 \\ &= 12\%\end{aligned}$$

Similarly:

$$\overline{R_B} = 17\% \; ; \; \overline{R_C} = 12.6\%$$

The two-index model's formula for a security's own variance is:

$$\sigma_i^2 = b_{i1}^2 \sigma_{I1}^2 + b_{i2}^2 \sigma_{I2}^2 + \sigma_{ci}^2$$

Using the above formula, the variance for, e.g., security A is:

$$\begin{aligned}\sigma_A^2 &= b_{A1}^2 \sigma_{I1}^2 b_{A2}^2 \sigma_{I2}^2 + \sigma_{cA}^2 \\ &= (0.8)^2 (2)^2 + (0.9)^2 (2.5)^2 + (2)^2 \\ &= 2.56 + 5.0625 + 4 = 11.6225\end{aligned}$$

Similarly, σ^2_B = 16.4025, and σ^2_C = 13.0525.

C. The two-index model's formula for the covariance of security *i* with security *j* is:

$$\sigma_{ijj} = b_{i1} b_{j1} \sigma_{I1}^2 + b_{i2} b_{j2} \sigma_{I2}^2$$

Using the above formula, the covariance of, e.g., security A with security B is:

$$\begin{aligned}\sigma_{AB} &= b_{A1} b_{B1} \sigma_{I1}^2 + b_{A2} b_{B2} \sigma_{I2}^2 \\ &= (0.8)(1.1)(2)^2 + (0.9)(1.3)(2.5)^2 \\ &= 3.52 + 7.3125 = 10.8325\end{aligned}$$

Similarly, σ_{AC} = 9.0675, and σ_{BC} = 12.8975.

Chapter 8: Problem 6

For an industry-index model, the text gives two formulas for the covariance between securities *i* and *k*. If firms *i* and *k* are both in industry *j*, the covariance between their securities' returns is given by:

$$\sigma_{ik} = b_{im} b_{km} \sigma_m^2 + b_{ij} b_{kj} \sigma_{Ij}^2$$

Otherwise, if the firms are in different industries, the covariance of their securities' returns is given by:

$$\sigma_{ik} = b_{im} b_{km} \sigma_m^2$$

If only firms A and B are in the same industry, then:

$$\begin{aligned}\sigma_{AB} &= b_{Am} b_{Bm} \sigma_m^2 + b_{A2} b_{B2} \sigma_{I2}^2 \\ &= (0.8)(1.1)(2)^2 + (0.9)(1.3)(2.5)^2 \\ &= 3.52 + 7.3125 = 10.8325\end{aligned}$$

The second formula should be used for the other pairs of firms:

$$\begin{aligned}\sigma_{AC} &= b_{Am} b_{Cm} \sigma_m^2 \\ &= (0.8)(0.9)(2)^2 = 2.88\end{aligned}$$

$$\begin{aligned}\sigma_{BC} &= b_{Bm} b_{Cm} \sigma_m^2 \\ &= (1.1)(0.9)(2)^2 = 3.96\end{aligned}$$

Chapter 13: Problem 1

The equation for the security market line is:

$$\overline{R_i} = R_F + \left(\overline{R_m} - R_F\right)\beta_i$$

Thus, from the data in the problem we have:

$$6 = R_F + \left(\overline{R_m} - R_F\right) \times 0.5 \text{ for asset 1}$$

$$12 = R_F + \left(\overline{R_m} - R_F\right) \times 1.5 \text{ for asset 2}$$

Solving the above two equations simultaneously, we find R_F = 3% and $\overline{R_m}$ = 9%. Using those values, an asset with a beta of 2 would have an expected return of:

$$3 + (9 - 3) \times 2 = 15\%$$

Chapter 13: Problem 2

Given the security market line in this problem, for the two stocks to be fairly priced their expected returns must be:

$$\overline{R_X} = 0.04 + 0.08 \times 0.5 = 0.08 \ (8\%)$$

$$\overline{R_Y} = 0.04 + 0.08 \times 2 = 0.20 \ (20\%)$$

If the expected return on either stock is higher than its return given above, the stock is a good buy.

Chapter 13: Problem 3

Given the security market line in this problem, the two funds' expected returns would be:

$$\overline{R_A} = 0.06 + 0.19 \times 0.8 = 0.212 \ (21.2\%)$$

$$\overline{R_B} = 0.06 + 0.19 \times 1.2 = 0.288 \ (28.8\%)$$

Chapter 13: Problem 7

Using the two assets in Problem 1, a portfolio with a beta of 1.2 can be constructed as follows:

$$0.5X_1 + (1.5)(1 - X_1) = 1.2$$

$$X_1 = 0.3; X_2 = 0.7$$

The return on this combination would be:

$$0.3(6\%) + 0.7(12\%) = 10.2\%$$

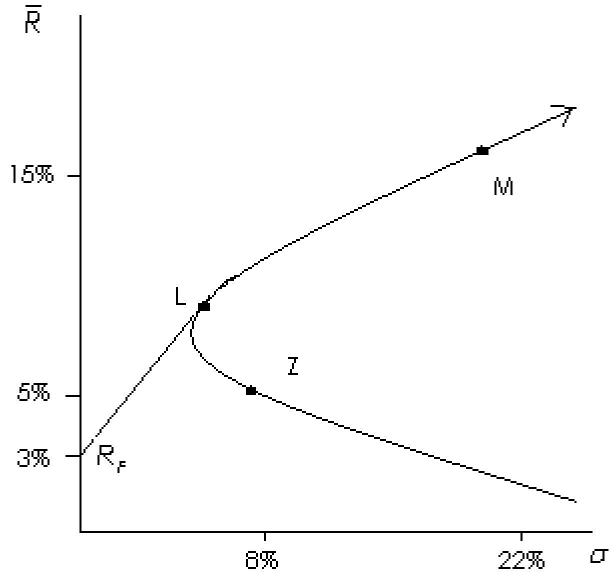
Asset 3 has a higher expected return than the portfolio of assets 1 and 2, even though asset 1 and the portfolio have the same beta. Thus, buying asset 3 and financing it by shorting the portfolio would produce a positive (arbitrage) return of 15% – 10.2% = 4.8% with zero net investment and zero beta risk.

Chapter 14: Problem 1

Given the zero-beta security market line in this problem, the return on the zero-beta portfolio equals 0.04 (4%), the intercept of the line, and the excess return of the market above the zero-beta portfolio's return (also called the "market risk premium") equals 0.10 (10%), the slope of the line. The return on the market portfolio must therefore be 0.04 + 0.10 = 0.14, or 14%.

Chapter 14: Problem 4

Since we are given $\overline{R}_Z$ and only one R_F, and since $\overline{R}_Z > R_F$, this situation is where there is riskless lending at R_F and no riskless borrowing. The efficient frontier will therefore be a ray in expected return-standard deviation space tangent to the minimum-variance curve of risky assets and intersecting the expected return axis at the riskless rate of 3% plus that part of the minimum-variance curve of risky assets to the right of the tangency point. This is depicted in the graph below, where the efficient frontier extends along the ray from R_F to the tangent portfolio *L*, then to the right of *L* along the curve through the market portfolio *M* and out toward infinity (assuming unlimited short sales). Note that, unless all investors in the economy choose to lend or invest solely in portfolio *L*, the market portfolio *M* will always be on the minimum-variance curve to the right of portfolio *L*.



Since both *M* and *Z* are on the minimum-variance curve, the entire minimum-variance curve of risky assets can be traced out by using combinations (portfolios) of *M* and *Z*. Letting *X* be the investment weight for the market portfolio, the expected return on any combination portfolio *P* of *M* and *Z* is:

$$\overline{R}_P = X\overline{R}_m + (1 - X)\overline{R}_Z \qquad (1)$$

Recognizing that M and Z are uncorrelated, the standard deviation of any combination portfolio P of M and Z is:

$$\sigma_P = \sqrt{X^2\sigma_m^2 + (1-X)^2\sigma_Z^2} \qquad (2)$$

Substituting the given values for $\bar{R}_m$ and $\bar{R}_Z$ into equation (1) gives:

$$\begin{aligned}\bar{R}_P &= 15X + 5(1-X) \\ &= 10X + 5\end{aligned} \qquad (3)$$

Substituting the given values for σ_m and σ_Z into equation (2) gives:

$$\begin{aligned}\sigma_P &= \sqrt{X^2 \times 22^2 + (1-X)^2 \times 8^2} \\ &= \sqrt{484X^2 + 64 - 128X + 64X^2} \qquad (4) \\ &= \sqrt{548X^2 - 128X + 64}\end{aligned}$$

Using equations (3) and (4) and varying X (the fraction invested in the market portfolio M) gives various coordinates for the minimum-variance curve; some of them are given below:

X	0	0.2	0.4	0.6	0.8	1.0	1.5	2.0
$\bar{R}_P$	5	7	9	11	13	15	20	25
σ_P	8	7.77	10.02	13.58	17.67	22	33.24	44.72

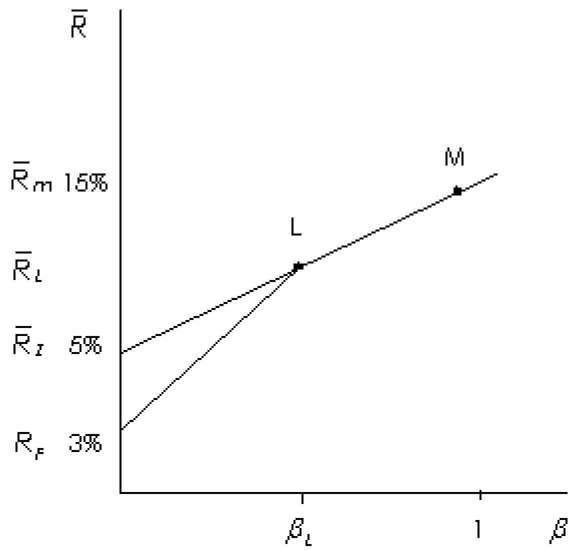
The zero-beta form of the security market line describes equilibrium beta risk and expected return relationship for all securities and portfolios (including portfolio L) except those combination portfolios composed of the riskless asset and tangent portfolio L along the ray R_F - L in the above graph:

$$\begin{aligned}\bar{R}_i &= \bar{R}_Z + (\bar{R}_m - \bar{R}_Z)\beta_i \\ &= 5 + 10\beta_i\end{aligned}$$

The equilibrium beta risk and expected return relationship for any combination portfolio C composed of the riskless asset and tangent portfolio L along the ray R_F - L in the above graph is described by the following line:

$$\bar{R}_C = R_F + \frac{(\bar{R}_L - R_F)}{\beta_L} \times \beta_C$$

Combining the two lines yields the following graph:



Comparing the above returns to the funds' actual returns, we see that both funds performed poorly, since their actual returns were below those expected given their beta risk.

Chapter 16: Problem 1

From the text we know that three points determine a plane. The APT equation for a plane is:

$$\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2}$$

Assuming that the three portfolios given in the problem are in equilibrium (on the plane), then their expected returns are determined by:

$$12 = \lambda_0 + \lambda_1 + 0.5\lambda_2 \qquad \text{(a)}$$

$$13.4 = \lambda_0 + 3\lambda_1 + 0.2\lambda_2 \qquad \text{(b)}$$

$$12 = \lambda_0 + 3\lambda_1 - 0.5\lambda_2 \qquad \text{(c)}$$

The above set of linear equations can be solved simultaneously for the three unknown values of λ_0, λ_1 and λ_2. There are many ways to solve a set of simultaneous linear equations. One method is shown below.

Subtract equation (a) from equation (b):

$$1.4 = 2\lambda_1 - 0.3\lambda_2 \qquad \text{(d)}$$

Subtract equation (a) from equation (c):

$$0 = 2\lambda_1 - \lambda_2 \qquad \text{(e)}$$

Subtract equation (e) from equation (d):

$$1.4 = 0.7\lambda_2 \text{ or } \lambda_2 = 2$$

Substitute $\lambda_2 = 2$ into equation (d):

$$1.4 = 2\lambda_1 - 0.6 \text{ or } \lambda_1 = 1$$

Substitute $\lambda_1 = 1$ and $\lambda_2 = 2$ into equation (a):

$$12 = \lambda_0 + 1 + 1 \text{ or } \lambda_0 = 10$$

Thus, the equation of the equilibrium APT plane is:

$$\bar{R}_i = 10 + b_{i1} + 2b_{i2}$$

Chapter 16: Problem 2

According to the equilibrium APT plane derived in Problem 1, any security with b_1 = 2 and b_2 = 0 should have an equilibrium expected return of 12%:

$$\bar{R}_i = 10 + b_{i1} + 2b_{i2} = 10 + 2 + 2 \times 0 = 12\%$$

Assuming the derived equilibrium APT plane holds, since portfolio D has b_{D1} = 2 and b_{D2} = 0 with an expected return of 10%, the portfolio is not in equilibrium and an arbitrage opportunity exists.

The first step is to use portfolios in equilibrium to create a replicating equilibrium investment portfolio, call it portfolio E, that has the same factor loadings (risk) as portfolio D. Using the equilibrium portfolios A, B and C in Problem 1 and recalling that an investment portfolio's weights sum to 1 and that a portfolio's factor loadings are weighted averages of the individual factor loadings we have:

$$b_{E1} = X_A b_{A1} + X_B b_{B1} + (1 - X_A - X_B) b_{C1} = 1X_A + 3X_B + 3(1 - X_A - X_B) = b_{D1} = 2$$

$$b_{E2} = X_A b_{A2} + X_B b_{B2} + (1 - X_A - X_B) b_{C2} = 0.5X_A + 0.2X_B - 0.5(1 - X_A - X_B) = b_{D2} = 0$$

Simplifying the above two equations, we have:

$$-2X_A = -1 \text{ or } X_A = \frac{1}{2}$$

$$X_A + 0.7X_B = \frac{1}{2}$$

Since $X_A = \frac{1}{2}$, $X_B = 0$ and $X_C = 1 - X_A - X_B = \frac{1}{2}$.

Since portfolio E was constructed from equilibrium portfolios, portfolio E is also on the equilibrium plane. We have seen above that any security with portfolio E's factor loadings has an equilibrium expected return of 12%, and that is the expected return of portfolio E:

$$\bar{R}_E = X_A \bar{R}_A + X_B \bar{R}_B + X_C \bar{R}_C = \frac{1}{2} \times 12 + 0 \times 13.4 + \frac{1}{2} \times 12 = 12\%$$

So now we have two portfolios with exactly the same risk: the target portfolio D and the equilibrium replicating portfolio E. Since they have the same risk (factor loadings), we can create an arbitrage portfolio, combining the two portfolios by going long in one and shorting the other. This will create a self-financing (zero net investment) portfolio with zero risk: an arbitrage portfolio.

In equilibrium, an arbitrage portfolio has an expected return of zero, but since portfolio D is not in equilibrium, neither is the arbitrage portfolio containing D and E, and an arbitrage profit may be made. We need to short sell either portfolio D or E and go long in the other. The question is: which portfolio do we short and which do we go long in? Since both portfolios have the same risk and since portfolio E has a higher expected return than portfolio D, we want to go long in E and short D; in other words, we want $X_E^{ARB} = 1$ and $X_D^{ARB} = -1$. This gives us:

$$\sum_i X_i^{ARB} = X_E^{ARB} + X_D^{ARB} = 1 - 1 = 0 \text{ (zero net investment)}$$

But since portfolio E consists of a weighted average of portfolios A, B and C, $X_E^{ARB} = 1$ is the same thing as $X_A^{ARB} = \frac{1}{2}$, $X_B^{ARB} = 0$ and $X_C^{ARB} = \frac{1}{2}$, so we have:

$$\sum_i X_i^{ARB} = X_A^{ARB} + X_B^{ARB} + X_C^{ARB} + X_D^{ARB} = \frac{1}{2} + 0 + \frac{1}{2} - 1 = 0 \text{ (zero net investment)}$$

$$\begin{aligned}
b_{ARB1} &= \sum_i X_i^{ARB} b_{i1} \\
&= X_A^{ARB} b_{A1} + X_B^{ARB} b_{B1} + X_C^{ARB} b_{C1} + X_D^{ARB} b_{D1} \quad \text{(zero factor 1 risk)} \\
&= \frac{1}{2} \times 1 + 0 \times 3 + \frac{1}{2} \times 3 - 1 \times 2 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
b_{ARB2} &= \sum_i X_i^{ARB} b_{i2} \\
&= X_A^{ARB} b_{A2} + X_B^{ARB} b_{B2} + X_C^{ARB} b_{C2} + X_D^{ARB} b_{D2} \quad \text{(zero factor 2 risk)} \\
&= \frac{1}{2} \times 0.5 + 0 \times 0.2 - \frac{1}{2} \times 0.5 - 1 \times 0 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\bar{R}_{ARB} &= \sum_i X_i^{ARB} \bar{R}_i \\
&= X_A^{ARB} \bar{R}_A + X_B^{ARB} \bar{R}_B + X_C^{ARB} \bar{R}_C + X_D^{ARB} \bar{R}_D \quad \text{(positive arbitrage return)} \\
&= \frac{1}{2} \times 12 + 0 \times 13.4 + \frac{1}{2} \times 12 - 1 \times 10 \\
&= 2\%
\end{aligned}$$

As arbitrageurs exploit the opportunity by short selling portfolio D, the price of portfolio D will drop, thereby pushing portfolio D's expected return up until it reaches its equilibrium level of 12%, at which point the expected return on the arbitrage portfolio will equal 0. There is no reason to expect any price effects on portfolios A, B and C, since the arbitrage with portfolio D can be accomplished using other assets on the equilibrium APT plane.

Chapter 16: Problem 6

A.

From the text we know that, for a 2-factor APT model to be consistent with the standard CAPM, $\lambda_j = (\bar{R}_m - R_F)\beta_{\lambda j}$. Given that $(\bar{R}_m - R_F) = 4$ and using results from Problem 1, we have:

$$1 = 4\beta_{\lambda 1} \text{ or } \beta_{\lambda 1} = 0.25;\ 2 = 4\beta_{\lambda 2} \text{ or } \beta_{\lambda 2} = 0.5.$$

B.

From the text we know that $\beta_i = b_{i1}\beta_{\lambda 1} + b_{i2}\beta_{\lambda 2}$. So we have:

$$\beta_A = 1 \times 0.25 + 0.5 \times 0.5 = 0.5$$

$$\beta_B = 3 \times 0.25 + 0.2 \times 0.5 = 0.85$$

$$\beta_C = 3 \times 0.25 - 0.5 \times 0.5 = 0.5$$

C.

Assuming all three portfolios in Problem 1 are in equilibrium, then we can use any one of them to find the risk-free rate. For example, using portfolio A gives:

$$\bar{R}_A = R_f + (\bar{R}_m - R_F)\beta_A \text{ or } R_F = \bar{R}_A - (\bar{R}_m - R_F)\beta_A$$

Given that $\bar{R}_A = 12\%$, $\beta_A = 0.5$ and $(\bar{R}_m - R_F) = 4\%$, we have:

$$R_F = 12 - 4 \times 0.5 = 10\%$$

Chapter 17: Problem 5

If a market is semi-strong-form efficient, the efficient market hypothesis says that prices should reflect all publicly available information. If publicly available information is already fully reflected in market prices, one would strongly suspect the market to be weak-form efficient as well. The only rational explanation for weak-form inefficiency is if information is incorporated into prices slowly over time, thus causing returns to be positively autocorrelated. The only exception to this might be if the market is strong-form inefficient and monopoly access to information disseminates through widening circles of investors over time.

Chapter 17: Problem 7

Recall that the zero-beta CAPM leads to lower expected returns for high-beta (above 1) stocks and higher expected returns for low-beta stocks than does the standard CAPM. If we were testing a phenomenon that tended to occur for low-beta stocks and not for high-beta stocks, then the zero-beta CAPM could show inefficiency while the standard CAPM showed efficiency.

Chapter 17: Problem 8

The betting market at roulette is in general an efficient market. Though betting on the roulette wheel has a negative expected return, there is no way that that information can be used to change the expected return. The only exception to this might be if the roulette wheel was not perfectly balanced. Since the house does not change the odds (prices) to reflect an unbalanced roulette wheel, an unbalanced wheel would make the betting market at roulette inefficient.

Chapter 25: Problem 1

Using standard deviation as the measure for variability, the reward-to-variability ratio for a fund is the fund's excess return (average return over the riskless rate) divided by the standard deviation of return, i.e., the fund's Sharpe ratio. E.g., for fund A we have:

$$\frac{\bar{R}_A - R_F}{\sigma_A} = \frac{14-3}{6} = 1.833$$

See the table in the answers to Problem 5 for the remaining funds' Sharpe ratios.

Chapter 25: Problem 2

The Treynor ratio is similar to the Sharpe ratio, except the fund's beta is used in the denominator instead of the standard deviation. E.g., for fund A we have:

$$\frac{\bar{R}_A - R_F}{\beta_A} = \frac{14-3}{1.5} = 7.833$$

See the table in the answers to Problem 5 for the remaining funds' Treynor ratios.

Chapter 25: Problem 3

A fund's differential return, using standard deviation as the measure of risk, is the fund's average return minus the return on a naïve portfolio, consisting of the market portfolio and the riskless asset, with the same standard deviation of return as the fund's. E.g., for fund A we have:

$$\bar{R}_A - \left(R_F + \frac{\bar{R}_m - R_F}{\sigma_m} \times \sigma_A \right) = 14 - \left(3 + \frac{13-3}{5} \times 6 \right) = -1\%$$

See the table in the answers to Problem 5 for the remaining funds' differential returns based on standard deviation.

Chapter 25: Problem 4

A fund's differential return, using beta as the measure of risk, is the fund's average return minus the return on a naïve portfolio, consisting of the market portfolio and the riskless asset, with the same beta as the fund's. This measure is often called "Jensen's alpha." E.g., for fund A we have:

$$\bar{R}_A - \left(R_F + \left(\bar{R}_m - R_F \right) \times \beta_A \right) = 14 - \left(3 + \left(13 - 3 \right) \times 1.5 \right) = -4\%$$

See the table in the answers to Problem 5 for the remaining funds' Jensen alphas.

Chapter 25: Problem 5

This differential return measure is the same as the one used in Problem 4, except that the riskless rate is replaced with the average return on a zero-beta asset. E.g., for fund A we have:

$$\bar{R}_A - \left(\bar{R}_Z + \left(\bar{R}_m - \bar{R}_Z \right) \times \beta_A \right) = 14 - \left(4 + \left(13 - 4 \right) \times 1.5 \right) = -3.5\%$$

The answers to Problems 1 through 5 for all five funds are as follows:

Fund	Sharpe Ratio	Treynor Ratio	Differential Return Based On Standard Deviation	Differential Return Based On Beta and R_F	Differential Return Based On Beta and $\bar{R}_Z$
A	1.833	7.333	–1%	–4%	–3.5%
B	2.250	18.000	1%	4%	3.5%
C	1.625	13.000	–3%	3%	3.0%
D	1.063	14.000	–5%	2%	1.5%
E	1.700	8.500	–3%	–3%	–2.0%