Stochastic Combinatorial Optimization Problems

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1 Undetermined stochastic combinatorial optimization problems

We are concerned with an abstract combinatorial optimization problem associated with three finite combinatorial spaces: the *input space* $\mathcal{X}$, the *output space* $\mathcal{Y}$, and a *configuration space* $\mathcal{C}$; in addition, we are given a bounded non-negative function $f(x, y, c) : \mathcal{X} \times \mathcal{Y} \times \mathcal{C} \rightarrow \mathbb{R}^+$ denoting the objective value to be maximized. Let Φ be the set of all distributions over $\mathcal{C}$. We consider *stochastic* combinatorial optimization problem in the sense that we seek to maximize f in terms of a distribution $\phi_{true} \in \Phi$ rather than a fixed configuration in $\mathcal{C}$. This is desired because the system that defines the objective values is often essentially probabilistic, which may stem from random networks or functions with random parameters. Therefore, the objective function is given by

$$F(x, y, \phi_{true}) = \int_{c \in \mathcal{C}} \phi_{true}(c) \cdot f(x, y, c)\, dc. \tag{1}$$

Given x and ϕ_{true}, we are interested in the problem

$$\max_{y \in \mathcal{Y}} F(x, y, \phi_{true}). \tag{2}$$

We may wish to compute either the optimal solution $H(x, \phi_{true}) \coloneqq \arg\max_{y \in \mathcal{Y}} F(x, y, \phi_{true})$ or its α-approximation $H_\alpha(x, \phi_{true})$ for some $\alpha \in (0, 1)$. Such problems are fundamental and also ubiquitous, ranging from the stochastic version of classic combinatorial problems to applications subject to environmental uncertainties, such as commercial recommendation, viral marketing, and behavioral decision making in autonomous systems. Taking the stochastic longest path problem as an example, the length of each edge could follow a certain distribution, and therefore, each configuration $c \in \mathcal{C}$ corresponds to a weighted graph; in such a case, $x = (u, v)$ denotes a pair of source and destination, y denotes a path from u to v, and $f(x, y, c)$ amounts to the length of y in c.

While the above formulation is natural, we are always limited by our imperfect understanding of the underlying system, which could be formalized by assuming that the distribution ϕ_{true} over the configuration is not known to us. Since the true distribution ϕ_{true} is unknown, the objective function $F(x, y, \phi_{true})$ cannot be directly optimized. Essentially, a learning process is required. In such a sense, we call these problems *undetermined* stochastic combinatorial optimization (USCO) problems.

2 Learning settings

We consider samples of input-solution pairs:

$$S_m \coloneqq \Big\{(x_i, y_i^\alpha)\Big\}_{i=1}^{m} \subseteq 2^{\mathcal{X} \times \mathcal{Y}}$$

where $y_i^\alpha \coloneqq H_\alpha(x_i, \phi_{true})$ is an α-approximation associated with the input x_i. Such samples intuitively offer the historical experience in which we successfully obtained good solutions to some inputs. From a learning perspective, we have formulated a regression task between two combinatorial spaces $\mathcal{X}$ and $\mathcal{Y}$. Some formal treatments are given as follows.

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Let us assume that the true distribution ϕ_{true} is unknown but fixed, and there is a distribution $\mathcal{D}_{\mathcal{X}}$ over $\mathcal{X}$. Our goal is to design a learning framework $A_S : \mathcal{X} \to \mathcal{Y}$ that can leverage a set S of samples to compute a prediction $A_S(x) \in \mathcal{Y}$ for each future $x \in \mathcal{X}$, with the hope that $A_S(x)$ is a good approximation to Equation (2).