Signed Permutations and Two-Rooted Graphs

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Outline

Signed permutations

A **signed permutation** of length n is an (ordered) list of n integers, $[s_1, s_2, s_3, \ldots, s_n]$, such that

$$\{|s_i| \ : \ 1 \le i \le n\} = \{1, 2, 3, \ldots, n\}.$$

Example:
$[1, 3, -4, 2]$ is a signed permutation of size four.
$[2, 5, 4, 3, 1]$ is a signed permutation of size five.

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$[1, 3, -4, 2]$ is a signed permutation of size four.
$[2, 5, 4, 3, 1]$ is a signed permutation of size five.

(**Biological motivation:** Each number represents a block of DNA, and the negative numbers represent blocks of DNA that are upside down.)

Pointers

Pointers can be assigned to entries of a signed permutation.
Example: Consider the signed permutation $[3, -5, 2, 1, -4]$.
The signed permutation with pointers is

$$[{}_{(2,3)}3_{(3,4)}, {}_{(5,6)}-5_{(4,5)}, {}_{(1,2)}2_{(2,3)}, {}_{(0,1)}1_{(1,2)}, {}_{(4,5)}4_{(3,4)}]$$

The cdr operation

cdr selects two pointers from entries with opposite signs and reverses and negates the entries between them.

Example:

$$\text{cdr}_{(3,4)} \text{ on } [3_{(3,4)}, -5, 2, 1, -4_{(3,4)}]$$
$$\text{yields } [3_{(3,4)}, {}_{(3,4)}4, -1, -2, 5]$$

More examples

Permutation : $[-2, {}_{(3,4)} - 3, -5, 1, {}_{(3,4)}4]$
After $\mathrm{cdr}_{(3,4)}$: $[-2, -1, 5, 3, 4]$.

More examples

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- A signed permutation needs a positive and negative entry in order for cdr to be applied.
- cdr always creates an **adjacency**, which is $i, i+1$ or $-(i+1), -i$.
- A signed permutation α is **cdr-sortable** if there is a sequence of cdr moves which sorts it to the identity $[1, 2, 3, \ldots, n]$.

Research questions

- Which signed permutations are cdr-sortable?

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- More generally, for signed permutations α and β, does a sequence of cdr moves exist that go from α to β?

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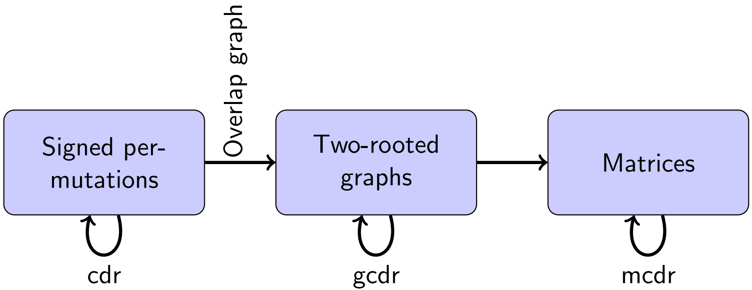
- More generally, for signed permutations α and β, does a sequence of cdr moves exist that go from α to β?
- How many signed permutations are cdr-sortable?

Big picture


Overlap graph
Signed permutations
Two-rooted graphs
Matrices
cdr
gcdr
mcdr

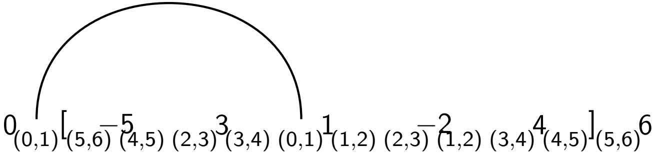
The overlap graph

$$0 \quad [\quad -5 \qquad 3 \qquad 1 \qquad -2 \qquad 4 \quad] \quad 6$$

The overlap graph

0 (0,1) [(5,6) −5 (4,5) (2,3) 3 (3,4) (0,1) 1 (1,2) (2,3) −2 (1,2) (3,4) 4 (4,5)] (5,6) 6

The overlap graph



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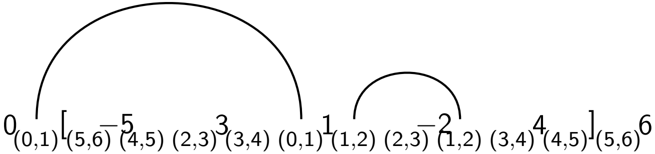

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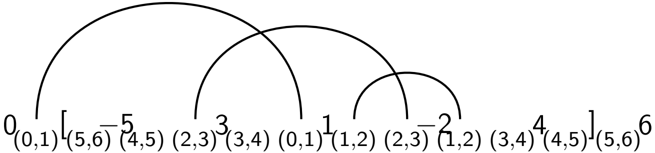

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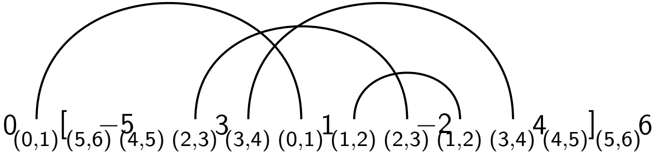

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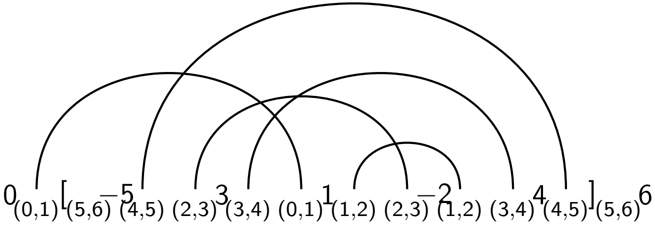

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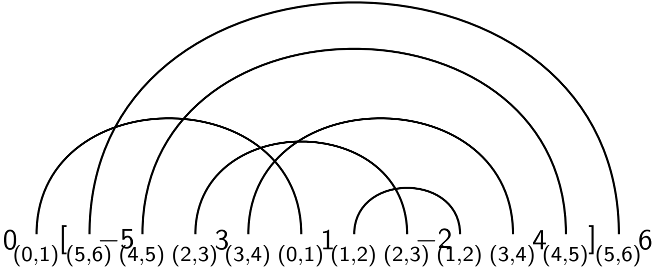

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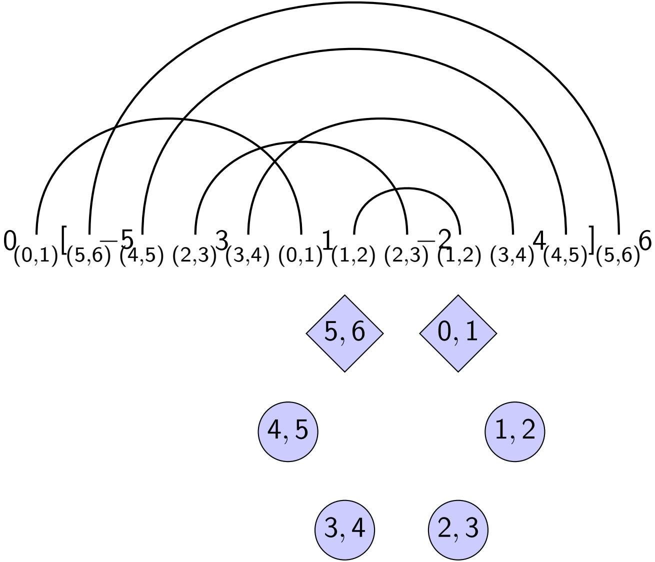

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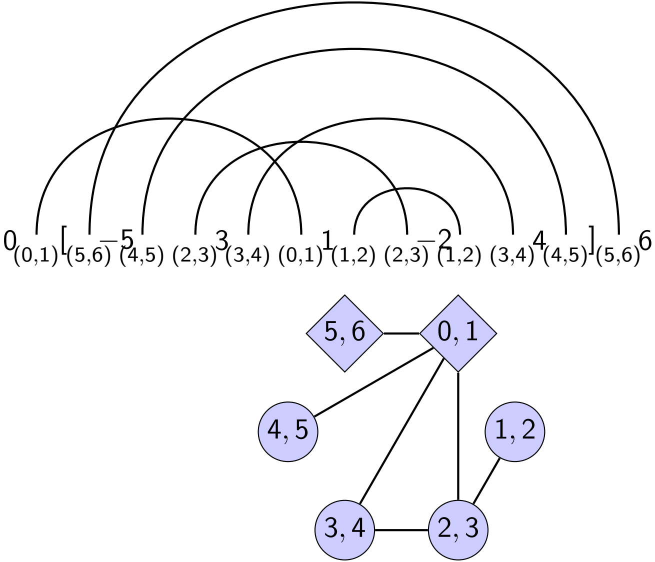
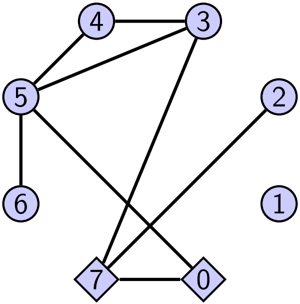


Rooted graphs

A **multi-rooted graph** $G = (V, E, V_0)$ is an undirected, simple graph with vertex set V, edge set E, and a distinguished set of vertices $V_0 \subseteq V$ called **roots**.
A **two-rooted graph** is a rooted graph in which $|V_0| = 2$. In drawing the graphs, we will denote the roots with diamonds.



The gcdr operation

We define the **gcdr** operation, an operation defined on a specific vertex of a two-rooted graph. (The operation has also been called **local complement**.)
From a two-rooted graph $G = (V, E, V_0)$, we construct $G' = (V, E', V_0) = \mathrm{gcdr}_v(G)$ by examining the closed neighborhood of v, $N[v]$, which includes v. All edges in this neighborhood are complemented.

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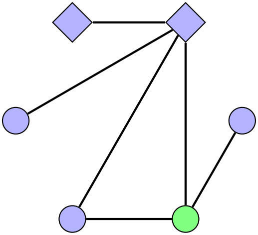
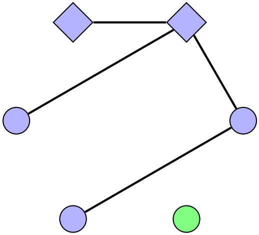
- gcdr is legal at a non-root, odd-degree vertex.
- gcdr on the vertex of an overlap graph is equivalent to cdr on the corresponding permutation and pointer.
- Applying gcdr to a vertex isolates the vertex.

gcdr

Example:



$\longmapsto$



gcdr digraph

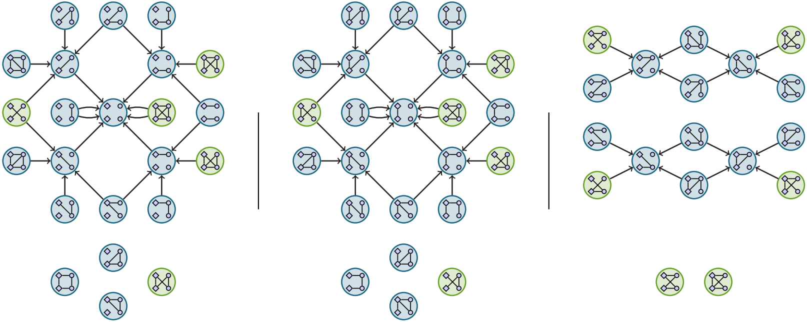
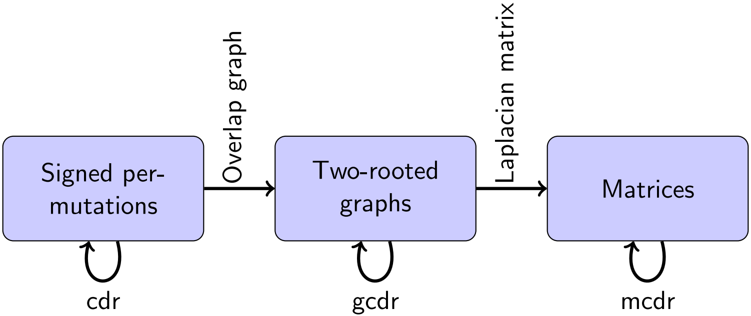


Figure : All two-rooted graphs with $n = 3$, connected by gcdr moves. Blue vertices correspond to overlap graphs of permutations.

Big picture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Laplacian matrix

For an undirected graph G with vertex set $\{1, 2, \ldots, n\}$ and edge set E, the **(mod 2) Laplacian matrix** is $A = (a_{i,j})$, where

$$a_{i,j} = \begin{cases} 1 & i \neq j, (i,j) \in E \\ 0 & i \neq j, (i,j) \notin E \\ \deg(i) & i = j \end{cases} .$$

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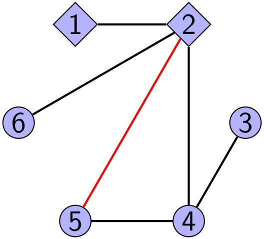
Since we are working mod 2, the diagonal entry $a_{i,i}$ is 0 if vertex i is of even degree (unoriented), and 1 if vertex i is of odd degree (oriented).

Laplacian matrix

Example:



$$\longmapsto \left[\begin{array}{cc|cccc} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ \hline 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

Laplacian matrix

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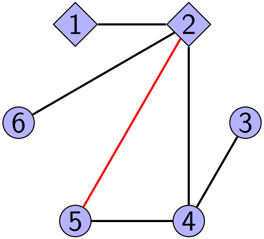

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The (mod 2) Laplacian matrix E satisfies:

- E is symmetric.
- The vector of ones is in $\ker E$.

mcdr

Let E be a matrix and let i be a non-root index. Let $\mathbb{1}_i$ be the $n \times n$ matrix which is 1 at (row i, column i), and 0 elsewhere.

mcdr

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The **mcdr** operation can be applied when $E_{i,i} = 1$, and i is not a root index. It maps

$$E \quad \overset{\text{mcdr}_i}{\longmapsto} \quad E + E\mathbb{1}_i E.$$

Trichotomy of matrices

Let $X_\diamond$ be the subset of $X = \mathbb{F}_2^n$ generated by the coordinates corresponding to root vertices. Let $P_\diamond : X \to X_\diamond$ be the projection matrix which restricts to those coordinates. The following subspace of $X_\diamond \times X_\diamond$ is invariant under mcdr:

$$\Pi(E) := \big\{(P_\diamond x, Ex) \ : \ x \in X, \ Ex \in X_\diamond\big\}.$$

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In our case, there are two roots, so $X_\diamond \cong \mathbb{F}_2^2$, and $X_\diamond \times X_\diamond \cong \mathbb{F}_2^4$. It turns out that there are only three possibilities for $\Pi(E)$:

$$\begin{aligned}
\Pi(E) &= \{(0,0,0,0),(1,0,0,0),(0,1,0,0),(1,1,0,0)\} \quad &\textbf{(a)}\\
\Pi(E) &= \{(0,0,0,0),(0,1,1,1),(1,0,1,1),(1,1,0,0)\} \quad &\textbf{(b)}\\
\Pi(E) &= \{(0,0,0,0),(1,1,0,0),(0,0,1,1),(1,1,1,1)\} \quad &\textbf{(c)}.
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\end{aligned}$$

We can translate this trichotomy back into the language of two-rooted graphs.

gcdr digraph

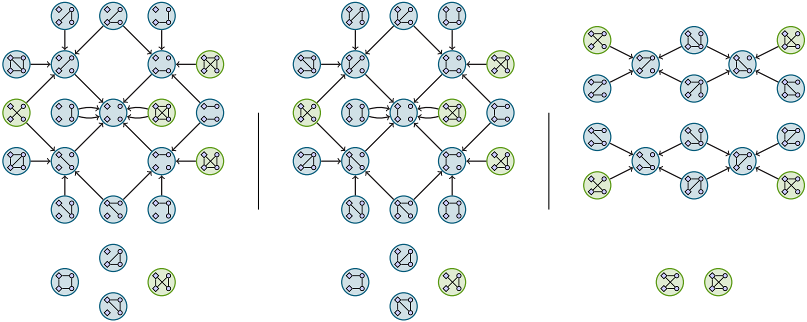
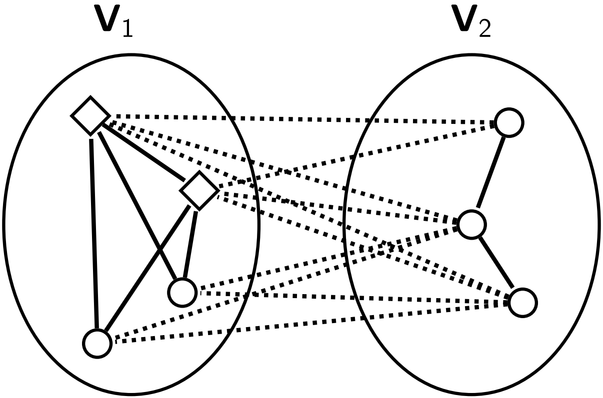


Figure : All two-rooted graphs with $n = 3$, connected by gcdr moves.

Parity cuts

A **parity cut** of a two-rooted graph $G = (V, E, (x, y))$ is a partition of V into V_1 and V_2 such that
(i) for all non-root $v \in V_1$, v is adjacent to an even number of vertices in V_2
(ii) for all non-root $w \in V_2$, w is adjacent to an even number of vertices in V_1.

Example of a Parity Cut

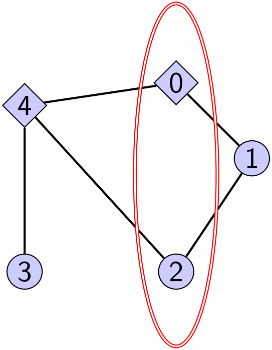


a-b-c trichotomy

A two-rooted graph G with roots x, y satisfies

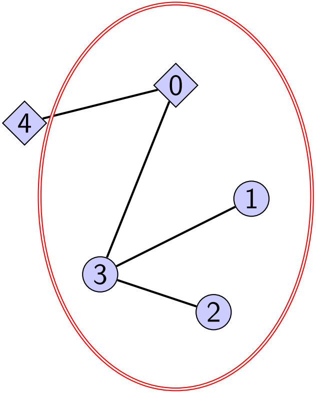
- **property (a)** if there is a parity cut $\{V_1, V_2\}$ of G such that $x \in V_1$, $y \in V_2$, x is connected to an even number of vertices in V_2, and y is connected to an even number of vertices in V_1.
- **property (b)** if there is a parity cut $\{V_1, V_2\}$ of G such that $x \in V_1$, $y \in V_2$, x is connected to an odd number of vertices in V_2, and y is connected to an odd number of vertices in V_1.
- **property (c)** if there is a parity cut $\{V_1, V_2\}$ of G such that $x, y \in V_1$, and x, y are each connected to an odd number of vertices in V_2.

Illustration



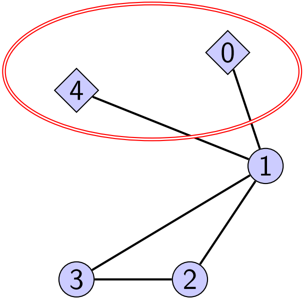
- Note $0 \in \{0, 2\}$ and $4 \in \{1, 3, 4\}$. Also, 0 connects to an even number of vertices in $\{1, 3, 4\}$.
- G satisfies property (a).

Illustration



- Note $0 \in \{0, 1, 2, 3\}$ and $4 \in \{3, 4\}$. Also, 0 connects to an odd number of vertices in $\{1, 3, 4\}$.
- G satisfies property (b).

Illustration



- Note $0, 4 \in \{0, 4\}$. 0 connects to an odd number of vertices in $\{1, 2, 3\}$.
- G satisfies property (c).

Properties of the a-b-c trichotomy

- Properties (a), (b), and (c) are invariant under (g)cdr.

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- Properties (a), (b), and (c) are invariant under (g)cdr.
- The overlap graph of the signed permutation $[1, 2, \ldots, n]$ satisfies property (a).
- The overlap graph of the signed permutation $[-n, -(n-1), \ldots, -1]$ satisfies property (b).
- If a signed permutation α is cdr-sortable, then its overlap graph satisfies property (a). But not all (a)-permutations are cdr-sortable.

gcdr digraph

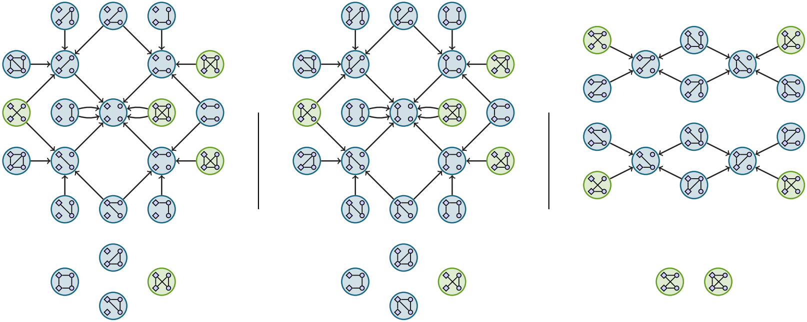


Figure : All two-rooted graphs with $n = 3$, connected by gcdr moves.

Enumeration of Graphs

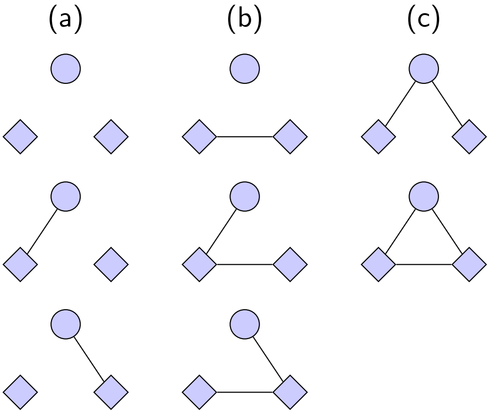
Let n be the number of non-root vertices.

n	$a(n)$	$b(n)$	$c(n)$
0	1	1	0
1	3	3	2
2	23	23	18
3	351	351	322
4	11119	11119	10530
5	703887	703887	689378

Graph counting

The total number of graphs is

$$a(n) + b(n) + c(n) = 2^{\binom{n+2}{2}}.$$

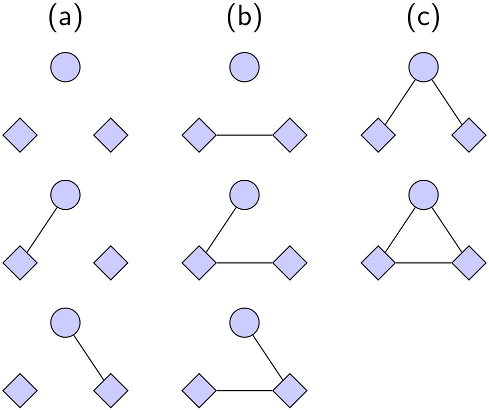


$$2^{\binom{3}{2}} = 8$$

Graph counting

Toggling the edge between $\diamond_1$ and $\diamond_2$ is a bijection from graphs of property (a) to graphs of property (b). So

$$a(n) = b(n).$$



Graph counting

A bijection shows that

$$a(n) = (2^n + 1)a(n-1) + 2^n(2^n - 2)a(n-2)\ ;\ a(0) = 1$$

from which we derive

$$a(n) + (2^n - 1)a(n-1) = \tfrac{1}{2}2^{\binom{n+2}{2}}.$$

Graph counting

We can also use

$$a(n) + (2^n - 1)a(n-1) = \frac{1}{2}2^{\binom{n+2}{2}} \; ; \; a(0) = 1$$

to get that, explicitly:

$$a(n) = \sum_{k=0}^{n} \left(\prod_{i=1}^{k} 2^{i+1} \right) \left(\prod_{i=k+1}^{n} (1 - 2^i) \right).$$

Verla Graph

Given an overlap graph, the vertex-oriented **verla graph** is constructed as follows:

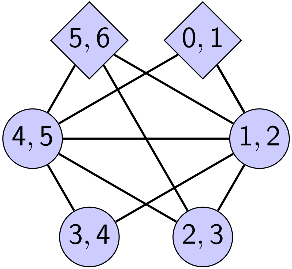
- Assign orientations to the overlap graph - vertices with odd degree are oriented and vertices with even degree are unoriented.
- Remove the $(0, 1)$ and $(n, n+1)$ vertices.

This construction is used solely for the cdr Sortability Theorem.

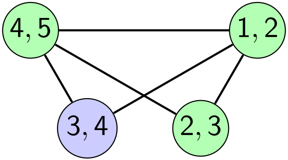
Example

$$\alpha = [-5, 3, 1, -2, 4]$$

Overlap Graph



Verla Graph



The cdr sortability theorem

Theorem

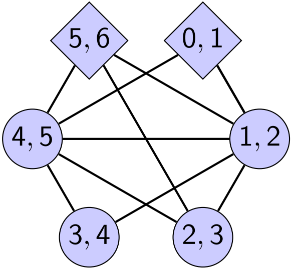
Let α be a signed permutation. The following are equivalent:

1. *α is cdr-sortable.*
2. *The overlap graph of α satisfies property (a), and every nontrivial component of the verla graph of α has an oriented vertex.*

Example

$$\alpha = [-5, 3, 1, -2, 4]$$

Overlap Graph



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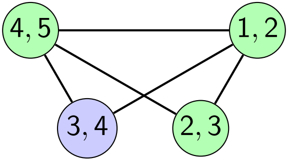

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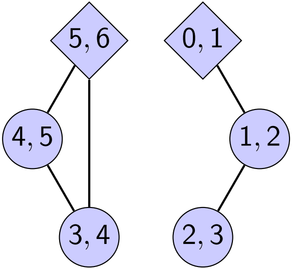

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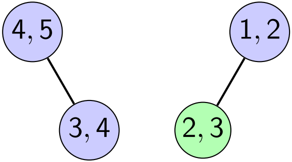

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Acknowledgements

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