Appendix A
Derivation of Linearized DAE

We present details on linearizing the DAEs in equations (1a)-(1d) and (2a)-(2d) for the transmission-level dynamics. The coefficient matrices A, B, C, and D in the linearized DAE (4) takes the forms shown in the four blocks of Table II (A: upper left, B: upper right, C: lower left, D: lower right), where all variables in the matrices represent steady-state values, but with the subscript "e" omitted for brevity. We recall that the square bracket $[x]$ denotes the diagonal matrix constructed from vector x and that x^* denotes the complex conjugate of x. The diagonals of $X_q(X_d)$ and $X'_q(X'_d)$ are denoted by vectors $x_q(x_d)$ and $x'_q(x'_d)$, respectively. In fact, all terms in Table II are functions of the steady-state power flow solution $V = [\mathcal{V}]\mathrm{e}^{j\theta}$. Specifically, we have

$$\delta = \arg(K_g(x_q)V), \tag{23}$$

and we define

$$\begin{aligned}
K_g(x) &= C_g + j[x]Y_g,\ x \in \{x_d, x_q, x'_d, x'_q\}, && (24)\\
S_g &= [C_g V]Y_g^* V^*, && (25)\\
I_g &= [\mathrm{e}^{j(\frac{\pi}{2}-\delta)}]Y_g V, && (26)\\
E_x &= [\mathrm{e}^{j(\frac{\pi}{2}-\delta)}]K_g(x)V, && (27)\\
S_l &= [C_l V]Y_l^* V^*. && (28)
\end{aligned}$$

The partial derivatives in Table II can then be compute as

$$\begin{aligned}
\frac{\partial S_g}{\partial \theta} &= [C_g V]Y_g^*[jV]^* + [Y_g^* V^*]C_g[jV], && (29)\\
\frac{\partial S_g}{\partial \mathcal{V}} &= [C_g V]Y_g^*[\mathrm{e}^{j\theta}]^* + [Y_g^* V^*]C_g[\mathrm{e}^{j\theta}], && (30)\\
\frac{\partial S_l}{\partial \theta} &= [C_l V]Y_l^*[jV]^* + [Y_l^* V^*]C_l[jV], && (31)\\
\frac{\partial S_l}{\partial \mathcal{V}} &= [C_l V]Y_l^*[\mathrm{e}^{j\theta}]^* + [Y_l^* V^*]C_g[\mathrm{e}^{j\theta}], && (32)\\
\frac{\partial I_g}{\partial \delta} &= [Y_g V][\mathrm{e}^{-j\delta}], && (33)\\
\frac{\partial I_g}{\partial \theta} &= -[\mathrm{e}^{-j\delta}]Y_g[V], && (34)\\
\frac{\partial I_g}{\partial \mathcal{V}} &= [\mathrm{e}^{j(\frac{\pi}{2}-\delta)}]Y_g[\mathrm{e}^{j\theta}], && (35)\\
\frac{\partial E_x}{\partial \delta} &= [\mathrm{e}^{j(\frac{\pi}{2}-\delta)}][K_g(x)V](-j), && (36)\\
\frac{\partial E_x}{\partial \theta} &= [\mathrm{e}^{j(\frac{\pi}{2}-\delta)}]K_g(x)[jV], && (37)\\
\frac{\partial E_x}{\partial \mathcal{V}} &= [\mathrm{e}^{j(\frac{\pi}{2}-\delta)}]K_g(x)[\mathrm{e}^{j\theta}]. && (38)
\end{aligned}$$

Appendix B
Analytical Gradient of Eigenvalues

Here we present details on calculating analytical gradients for the system's eigenvalues in equation (10). We partition the right and left eigenvectors u and v associated with the eigenvalue λ as

$$\begin{aligned}
u &= [z_1^T, z_2^T, z_3^T, z_4^T, z_5^T, z_6^T]^T, && (39)\\
v &= [w_1^T, w_2^T, w_3^T, w_4^T, w_5^T, w_6^T, w_7^T, w_8^T]^T, && (40)
\end{aligned}$$

according to the columns and rows of the matrices in Table II. Differentiating equation (23), we obtain

$$\begin{aligned}
\frac{\partial \delta}{\partial \theta} &= \mathrm{Im}\{[K_g(x_q))V]^{-1}K_g(x_q))[jV]\}, && (41)\\
\frac{\partial \delta}{\partial \mathcal{V}} &= \mathrm{Im}\{[K_g(x_q))V]^{-1}K_g(x_q))[\mathrm{e}^{j\theta}]\}. && (42)
\end{aligned}$$

Using these when taking the derivatives of the coefficients in Table II, we expand the matrix-vector product on the right hand side of equation (10) and carry out the differentiation. The resulting gradients of the eigenvalue λ with respect to θ and $\mathcal{V}$ can be expressed as

$$\begin{aligned}
\frac{\partial \lambda}{\partial \theta} = -\frac{1}{2}\big\{ & w_2^H M^{-1}\left(S_g^{\theta\theta}(z_5) + (S_g^{\theta\theta}(z_5^*))^*\right)\\
&+ w_2^H M^{-1}\left(S_g^{\mathcal{V}\theta}(z_6) + (S_g^{\mathcal{V}\theta}(z_6^*))^*\right)\\
&+ w_3^H T_{d0}'^{-1}(X_d - X'_d)\left(I_g^{\delta\theta}(z_1) + (I_g^{\delta\theta}(z_1^*))^*\right)\\
&+ w_3^H T_{d0}'^{-1}(X_d - X'_d)\left(I_g^{\theta\theta}(z_5) + (I_g^{\theta\theta}(z_5^*))^*\right)\\
&+ w_3^H T_{d0}'^{-1}\left(X_d - X'_d)(I_g^{\mathcal{V}\theta}(z_6) + (I_g^{\mathcal{V}\theta}(z_6^*))^*\right)\\
&+ jw_4^H T_{q0}'^{-1}(X_q - X'_q)\left(I_g^{\delta\theta}(z_1) - (I_g^{\delta\theta}(z_1^*))^*\right)\\
&+ jw_4^H T_{q0}'^{-1}(X_q - X'_q)\left(I_g^{\theta\theta}(z_5) - (I_g^{\theta\theta}(z_5^*))^*\right)\\
&+ jw_4^H T_{q0}'^{-1}\left(X_q - X'_q)(I_g^{\mathcal{V}\theta}(z_6) - (I_g^{\mathcal{V}\theta}(z_6^*))^*\right)\\
&+ w_5^H\left(E_{x'_q}^{\delta\theta}(z_1) + (E_{x'_q}^{\delta\theta}(z_1^*))^*\right)\\
&+ w_5^H\left(E_{x'_q}^{\theta\theta}(z_5) + (E_{x'_q}^{\theta\theta}(z_5^*))^*\right)\\
&+ w_5^H\left(E_{x'_q}^{\mathcal{V}\theta}(z_6) + (E_{x'_q}^{\mathcal{V}\theta}(z_6^*))^*\right)\\
&- jw_6^H\left(E_{x'_d}^{\delta\theta}(z_1) - (E_{x'_d}^{\delta\theta}(z_1^*))^*\right)\\
&- jw_6^H\left(E_{x'_d}^{\theta\theta}(z_5) - (E_{x'_d}^{\theta\theta}(z_5^*))^*\right)\\
&- jw_6^H\left(E_{x'_d}^{\mathcal{V}\theta}(z_6) - (E_{x'_d}^{\mathcal{V}\theta}(z_6^*))^*\right)\\
&- w_7^H\left(S_l^{\theta\theta}(z_5) + (S_l^{\theta\theta}(z_5^*))^*\right)\\
&- w_7^H\left(S_l^{\mathcal{V}\theta}(z_6) + (S_l^{\mathcal{V}\theta}(z_6^*))^*\right)\\
&+ jw_8^H\left(S_l^{\theta\theta}(z_5) - (S_l^{\theta\theta}(z_5^*))^*\right)\\
&+ jw_8^H\left(S_l^{\mathcal{V}\theta}(z_6) - (S_l^{\mathcal{V}\theta}(z_6^*))^*\right)\big\}, \qquad (43)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \lambda}{\partial \mathcal{V}} = -\frac{1}{2}\big\{ & w_2^H M^{-1}\left(S_g^{\theta\mathcal{V}}(z_5) + (S_g^{\theta\mathcal{V}}(z_5^*))^*\right)\\
&+ w_2^H M^{-1}\left(S_g^{\mathcal{V}\mathcal{V}}(z_6) + (S_g^{\mathcal{V}\mathcal{V}}(z_6^*))^*\right)\\
&+ w_3^H T_{d0}'^{-1}(X_d - X'_d)\left(I_g^{\delta\mathcal{V}}(z_1) + (I_g^{\delta\mathcal{V}}(z_1^*))^*\right)\\
&+ w_3^H T_{d0}'^{-1}(X_d - X'_d)\left(I_g^{\theta\mathcal{V}}(z_5) + (I_g^{\theta\mathcal{V}}(z_5^*))^*\right)\\
&+ w_3^H T_{d0}'^{-1}\left(X_d - X'_d)(I_g^{\mathcal{V}\mathcal{V}}(z_6) + (I_g^{\mathcal{V}\mathcal{V}}(z_6^*))^*\right)\\
&+ jw_4^H T_{q0}'^{-1}(X_q - X'_q)\left(I_g^{\delta\mathcal{V}}(z_1) - (I_g^{\delta\mathcal{V}}(z_1^*))^*\right)\\
&+ jw_4^H T_{q0}'^{-1}(X_q - X'_q)\left(I_g^{\theta\mathcal{V}}(z_5) - (I_g^{\theta\mathcal{V}}(z_5^*))^*\right)\\
&+ jw_4^H T_{q0}'^{-1}\left(X_q - X'_q)(I_g^{\mathcal{V}\mathcal{V}}(z_6) - (I_g^{\mathcal{V}\mathcal{V}}(z_6^*))^*\right)\\
&+ w_5^H\left(E_{x'_q}^{\delta\mathcal{V}}(z_1) + (E_{x'_q}^{\delta\mathcal{V}}(z_1^*))^*\right)\\
&+ w_5^H\left(E_{x'_q}^{\theta\mathcal{V}}(z_5) + (E_{x'_q}^{\theta\mathcal{V}}(z_5^*))^*\right)\\
&+ w_5^H\left(E_{x'_q}^{\mathcal{V}\mathcal{V}}(z_6) + (E_{x'_q}^{\mathcal{V}\mathcal{V}}(z_6^*))^*\right)\\
&- jw_6^H\left(E_{x'_d}^{\delta\mathcal{V}}(z_1) - (E_{x'_d}^{\delta\mathcal{V}}(z_1^*))^*\right)
\end{aligned}$$

TABLE II: Coefficient matrices of the linearized DAE model.

	δ	ω	e'_q	e'_d	θ	$\mathcal{V}$
(1a)		I				
(1b)		$-M^{-1}D$			$-M^{-1}\mathrm{Re}\{\frac{\partial S_g}{\partial \theta}\}$	$-M^{-1}\mathrm{Re}\{\frac{\partial S_g}{\partial \mathcal{V}}\}$
(1c)	$-T'^{-1}_{d0}(X_d-X'_d)\mathrm{Re}\{\frac{\partial I_g}{\partial \delta}\}$	$T'^{-1}_{d0}K_AK_S$	$-T'^{-1}_{d0}$		$-T'^{-1}_{d0}(X_d-X'_d)\mathrm{Re}\{\frac{\partial I_g}{\partial \theta}\}$	$-T'^{-1}_{d0}\left((X_d-X'_d)\mathrm{Re}\{\frac{\partial I_g}{\partial \mathcal{V}}\}+K_AC_g\right)$
(1d)	$T'^{-1}_{q0}(X_q-X'_q)\mathrm{Im}\{\frac{\partial I_g}{\partial \delta}\}$			$-T'^{-1}_{q0}$	$T'^{-1}_{q0}(X_q-X'_q)\mathrm{Im}\{\frac{\partial I_g}{\partial \theta}\}$	$T'^{-1}_{q0}(X_q-X'_q)\mathrm{Im}\{\frac{\partial I_g}{\partial \mathcal{V}}\}$
(2a)	$-\mathrm{Re}\{\frac{\partial E_{x'_q}}{\partial \delta}\}$			I	$-\mathrm{Re}\{\frac{\partial E_{x'_q}}{\partial \theta}\}$	$-\mathrm{Re}\{\frac{\partial E_{x'_q}}{\partial \mathcal{V}}\}$
(2b)	$-\mathrm{Im}\{\frac{\partial E_{x'_d}}{\partial \delta}\}$		I		$-\mathrm{Im}\{\frac{\partial E_{x'_d}}{\partial \theta}\}$	$-\mathrm{Im}\{\frac{\partial E_{x'_d}}{\partial \mathcal{V}}\}$
(2c)					$\mathrm{Re}\{\frac{\partial S_l}{\partial \theta}\}$	$\mathrm{Re}\{\frac{\partial S_l}{\partial \mathcal{V}}\}$
(2d)					$\mathrm{Im}\{\frac{\partial S_l}{\partial \theta}\}$	$\mathrm{Im}\{\frac{\partial S_l}{\partial \mathcal{V}}\}$

$$
\begin{aligned}
&- jw_6^H\left(E^{\theta\mathcal{V}}_{x'_d}(z_5)-(E^{\theta\mathcal{V}}_{x'_d}(z_5^*))^*\right)\\
&- jw_6^H\left(E^{\mathcal{V}\mathcal{V}}_{x'_d}(z_6)-(E^{\mathcal{V}\mathcal{V}}_{x'_d}(z_6^*))^*\right)\\
&- w_7^H\left(S_l^{\theta\mathcal{V}}(z_5)+(S_l^{\theta\mathcal{V}}(z_5^*))^*\right)\\
&- w_7^H\left(S_l^{\mathcal{V}\mathcal{V}}(z_6)+(S_l^{\mathcal{V}\mathcal{V}}(z_6^*))^*\right)\\
&+ jw_8^H\left(S_l^{\theta\mathcal{V}}(z_5)-(S_l^{\theta\mathcal{V}}(z_5^*))^*\right)\\
&+ jw_8^H\left(S_l^{\mathcal{V}\mathcal{V}}(z_6)-(S_l^{\mathcal{V}\mathcal{V}}(z_6^*))^*\right)\Big\},
\end{aligned} \tag{44}
$$

where we define

$$
\begin{aligned}
E_x^{\delta\theta}(z) = {}&[\mathrm{e}^{j(\frac{\pi}{2}-\delta)}][-jz]K_g(x)[jV]\\
&+[K_g(x)V][-jz][\mathrm{e}^{j(\frac{\pi}{2}-\delta)}](-j)\frac{\partial\delta}{\partial\theta},
\end{aligned} \tag{45}
$$

$$
\begin{aligned}
E_x^{\delta\mathcal{V}}(z) = {}&[\mathrm{e}^{j(\frac{\pi}{2}-\delta)}][-jz]K_g(x)[\mathrm{e}^{j\theta}]\\
&+[K_g(x)V][-jz][\mathrm{e}^{j(\frac{\pi}{2}-\delta)}](-j)\frac{\partial\delta}{\partial\mathcal{V}},
\end{aligned} \tag{46}
$$

$$
\begin{aligned}
E_x^{\theta\theta}(z) = {}&-[\mathrm{e}^{j(\frac{\pi}{2}-\delta)}]K_g(x)[V][z]\\
&+[K_g(x)[jV]z][\mathrm{e}^{j(\frac{\pi}{2}-\delta)}](-j)\frac{\partial\delta}{\partial\theta},
\end{aligned} \tag{47}
$$

$$
\begin{aligned}
E_x^{\theta\mathcal{V}}(z) = {}&[\mathrm{e}^{j(\frac{\pi}{2}-\delta)}]K_g(x)[\mathrm{e}^{j\theta}][jz]\\
&+[K_g(x)[jV]z][\mathrm{e}^{j(\frac{\pi}{2}-\delta)}](-j)\frac{\partial\delta}{\partial\mathcal{V}},
\end{aligned} \tag{48}
$$

$$
\begin{aligned}
E_x^{\mathcal{V}\theta}(z) = {}&[\mathrm{e}^{j(\frac{\pi}{2}-\delta)}]K_g(x)[\mathrm{e}^{j\theta}][jz]\\
&+[K_g(x)[\mathrm{e}^{j\theta}]z][\mathrm{e}^{j(\frac{\pi}{2}-\delta)}](-j)\frac{\partial\delta}{\partial\theta},
\end{aligned} \tag{49}
$$

$$
E_x^{\mathcal{V}\mathcal{V}}(z) = [K_g(x)[\mathrm{e}^{j\theta}]z][\mathrm{e}^{j(\frac{\pi}{2}-\delta)}](-j)\frac{\partial\delta}{\partial\mathcal{V}}, \tag{50}
$$

$$
I_g^{\delta\theta}(z) = [\mathrm{e}^{j(\frac{\pi}{2}-\delta)}][z]Y_g[V]+[Y_gV][\mathrm{e}^{-j\delta}][-jz]\frac{\partial\delta}{\partial\theta}, \tag{51}
$$

$$
I_g^{\delta\mathcal{V}}(z) = [\mathrm{e}^{-j\delta}][z]Y_g[\mathrm{e}^{j\theta}]+[Y_gV][\mathrm{e}^{-j\delta}][-jz]\frac{\partial\delta}{\partial\mathcal{V}}, \tag{52}
$$

$$
I_g^{\theta\theta}(z) = [\mathrm{e}^{j(\frac{\pi}{2}-\delta)}]Y_g[-z][V]+j[Y_g[V]z][\mathrm{e}^{-j\delta}]\frac{\partial\delta}{\partial\theta}, \tag{53}
$$

$$
I_g^{\theta\mathcal{V}}(z) = [\mathrm{e}^{j(\frac{\pi}{2}-\delta)}]Y_g[jz][\mathrm{e}^{j\theta}]+j[Y_g[V]z][\mathrm{e}^{-j\delta}]\frac{\partial\delta}{\partial\mathcal{V}}, \tag{54}
$$

$$
I_g^{\mathcal{V}\theta}(z) = [\mathrm{e}^{j(\frac{\pi}{2}-\delta)}]Y_g[jz][\mathrm{e}^{j\theta}]+[Y_g[\mathrm{e}^{j\theta}]z][\mathrm{e}^{-j\delta}]\frac{\partial\delta}{\partial\theta}, \tag{55}
$$

$$
I_g^{\mathcal{V}\mathcal{V}}(z) = [Y_g[\mathrm{e}^{j\theta}]z][\mathrm{e}^{-j\delta}]\frac{\partial\delta}{\partial\mathcal{V}}, \tag{56}
$$

$$
\begin{aligned}
S_g^{\theta\theta}(z) = {}&[C_gV]Y_g^*[z][-V^*]+[Y_g^*[jV]^*z]C_g[jV]\\
&+[Y_g^*V*]C_g[z][-V]+[C_g[jV]z]Y_g^*[jV]^*,
\end{aligned} \tag{57}
$$

$$
\begin{aligned}
S_g^{\theta\mathcal{V}}(z) = {}&[C_gV]Y_g^*[z][j\mathrm{e}^{j\theta}]^*+[Y_g^*[jV]^*z]C_g[\mathrm{e}^{j\theta}]\\
&+[Y_g^*V^*]C_g[z][j\mathrm{e}^{j\theta}]+[C_g[jV]z]Y_g^*[\mathrm{e}^{j\theta}]^*,
\end{aligned} \tag{58}
$$

$$
\begin{aligned}
S_g^{\mathcal{V}\theta}(z) = {}&[C_gV]Y_g^*[z][j\mathrm{e}^{j\theta}]^*+[Y_g^*[\mathrm{e}^{j\theta}]^*z]C_g[jV]\\
&+[Y_g^*V^*]C_g[z][j\mathrm{e}^{j\theta}]+[C_g[j\mathrm{e}^{j\theta}]z]Y_g^*[jV]^*,
\end{aligned} \tag{59}
$$

$$
S_g^{\mathcal{V}\mathcal{V}}(z) = [Y_g^*[\mathrm{e}^{j\theta}]^*z]C_g[\mathrm{e}^{j\theta}]+[C_g[j\mathrm{e}^{\theta}]z]Y_g^*[\mathrm{e}^{j\theta}]^*, \tag{60}
$$

$$
\begin{aligned}
S_l^{\theta\theta}(z) = {}&[C_lV]Y_l^*[z][-V^*]+[Y_l^*[jV]^*z]C_l[jV]\\
&+[Y_l^*V*]C_l[z][-V]+[C_l[jV]z]Y_l^*[jV]^*,
\end{aligned} \tag{61}
$$

$$
\begin{aligned}
S_l^{\theta\mathcal{V}}(z) = {}&[C_lV]Y_l^*[z][j\mathrm{e}^{j\theta}]^*+[Y_l^*[jV]^*z]C_l[\mathrm{e}^{j\theta}]\\
&+[Y_l^*V^*]C_l[z][j\mathrm{e}^{j\theta}]+[C_l[jV]z]Y_l^*[\mathrm{e}^{j\theta}]^*,
\end{aligned} \tag{62}
$$

$$
\begin{aligned}
S_l^{\mathcal{V}\theta}(z) = {}&[C_lV]Y_l^*[z][j\mathrm{e}^{j\theta}]^*+[Y_l^*[\mathrm{e}^{j\theta}]^*z]C_l[jV]\\
&+[Y_l^*V^*]C_l[z][j\mathrm{e}^{j\theta}]+[C_l[j\mathrm{e}^{j\theta}]z]Y_l^*[jV]^*,
\end{aligned} \tag{63}
$$

$$
S_l^{\mathcal{V}\mathcal{V}}(z) = [Y_l^*[\mathrm{e}^{j\theta}]^*z]C_l[\mathrm{e}^{j\theta}]+[C_l[j\mathrm{e}^{\theta}]z]Y_l^*[\mathrm{e}^{j\theta}]^*. \tag{64}
$$