Technical Notes

I. Derivation of Linearized DAE

The power system differential equation

$$\dot{\delta} = \omega - \omega_s, \tag{1a}$$
$$\dot{\omega} = M^{-1}\big(p_m - \mathrm{Re}\{[C_g V]Y_g^* V^*\} - D(w-\omega_s)\big), \tag{1b}$$
$$\dot{e}'_q = T'^{-1}_{d0}\big(e_f - e'_q - (X_d - X'_d)\mathrm{Re}\{[\mathrm{e}^{j(\frac{\pi}{2}-\delta)}]Y_g V\}\big), \tag{1c}$$
$$\dot{e}'_d = T'^{-1}_{q0}\big(-e'_d + (X_q - X'_q)\mathrm{Im}\{[\mathrm{e}^{j(\frac{\pi}{2}-\delta)}]Y_g V\}\big), \tag{1d}$$

and algebraic equation

$$e'_d = \mathrm{Re}\{[\mathrm{e}^{j(\frac{\pi}{2}-\delta)}](C_g + jX'_q Y_g)V\}, \tag{2a}$$
$$e'_q = \mathrm{Im}\{[\mathrm{e}^{j(\frac{\pi}{2}-\delta)}](C_g + jX'_d Y_g)V\}, \tag{2b}$$
$$P_l + \mathrm{Re}\{[C_l V]Y_l^* V^*\} = 0, \tag{2c}$$
$$Q_l + \mathrm{Im}\{[C_l V]Y_l^* V^*\} = 0, \tag{2d}$$

with e_f in (1c) given by

$$e_f = E_f - K_A C_g(|V| - |V_e|) + K_A K_S(\omega - \omega_s), \tag{3}$$

can be linearized into the form

$$\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \dot{s} \\ \dot{x} \end{pmatrix} = \begin{pmatrix} A(V_e) & B(V_e) \\ C(V_e) & D(V_e) \end{pmatrix}\begin{pmatrix} s \\ x \end{pmatrix}. \tag{4}$$

The coefficient matrices A, B, C, and D in the linearized DAE (4) takes the forms shown in the four blocks of Table I (A: upper left, B: upper right, C: lower left, D: lower right), where all variables in the matrices represent steady-state values, but with the subscript "e" omitted for brevity. We recall that the square bracket $[x]$ denotes the diagonal matrix constructed from vector x and that x^* denotes the complex conjugate of x. The diagonals of $X_q (X_d)$ and $X'_q (X'_d)$ are denoted by vectors $x_q (x_d)$ and $x'_q (x'_d)$, respectively. In fact, all terms in Table I are functions of the steady-state power flow solution $V = [\mathcal{V}]\mathrm{e}^{j\theta}$. Specifically, we have

$$\delta = \arg(K_g(x_q)V), \tag{5}$$

and we define

$$K_g(x) = C_g + j[x]Y_g,\ x \in \{x_d, x_q, x'_d, x'_q\}, \tag{6}$$
$$S_g = [C_g V]Y_g^* V^*, \tag{7}$$
$$I_g = [\mathrm{e}^{j(\frac{\pi}{2}-\delta)}]Y_g V, \tag{8}$$
$$E_x = [\mathrm{e}^{j(\frac{\pi}{2}-\delta)}]K_g(x)V, \tag{9}$$
$$S_l = [C_l V]Y_l^* V^*. \tag{10}$$

The partial derivatives in Table I can then be compute as

$$\frac{\partial S_g}{\partial \theta} = [C_g V]Y_g^*[jV]^* + [Y_g^* V^*]C_g[jV], \tag{11}$$
$$\frac{\partial S_g}{\partial \mathcal{V}} = [C_g V]Y_g^*[\mathrm{e}^{j\theta}]^* + [Y_g^* V^*]C_g[\mathrm{e}^{j\theta}], \tag{12}$$
$$\frac{\partial S_l}{\partial \theta} = [C_l V]Y_l^*[jV]^* + [Y_l^* V^*]C_l[jV], \tag{13}$$
$$\frac{\partial S_l}{\partial \mathcal{V}} = [C_l V]Y_l^*[\mathrm{e}^{j\theta}]^* + [Y_l^* V^*]C_g[\mathrm{e}^{j\theta}], \tag{14}$$
$$\frac{\partial I_g}{\partial \delta} = [Y_g V][\mathrm{e}^{-j\delta}], \tag{15}$$
$$\frac{\partial I_g}{\partial \theta} = -[\mathrm{e}^{-j\delta}]Y_g[V], \tag{16}$$
$$\frac{\partial I_g}{\partial \mathcal{V}} = [\mathrm{e}^{j(\frac{\pi}{2}-\delta)}]Y_g[\mathrm{e}^{j\theta}], \tag{17}$$
$$\frac{\partial E_x}{\partial \delta} = [\mathrm{e}^{j(\frac{\pi}{2}-\delta)}][K_g(x)V](-j), \tag{18}$$
$$\frac{\partial E_x}{\partial \theta} = [\mathrm{e}^{j(\frac{\pi}{2}-\delta)}]K_g(x)[jV], \tag{19}$$
$$\frac{\partial E_x}{\partial \mathcal{V}} = [\mathrm{e}^{j(\frac{\pi}{2}-\delta)}]K_g(x)[\mathrm{e}^{j\theta}]. \tag{20}$$

II. Analytical Gradients of Eigenvalues

Here we present details on calculating analytical gradients for the system's eigenvalues. We partition the right and left eigenvectors u and v associated with the eigenvalue λ as

$$u = [z_1^T, z_2^T, z_3^T, z_4^T, z_5^T, z_6^T]^T, \tag{21}$$
$$v = [w_1^T, w_2^T, w_3^T, w_4^T, w_5^T, w_6^T, w_7^T, w_8^T]^T, \tag{22}$$

according to the columns and rows of the matrices in Table I. Differentiating equation (5), we obtain

$$\frac{\partial \delta}{\partial \theta} = \mathrm{Im}\{[K_g(x_q))V]^{-1}K_g(x_q))[jV]\}, \tag{23}$$
$$\frac{\partial \delta}{\partial \mathcal{V}} = \mathrm{Im}\{[K_g(x_q))V]^{-1}K_g(x_q))[\mathrm{e}^{j\theta}]\}. \tag{24}$$

Using these when taking the derivatives of the coefficients in Table I, we can obtain the gradients of the eigenvalue λ with respect to θ and $\mathcal{V}$ can be expressed as

$$\begin{aligned}
\frac{\partial \lambda}{\partial \theta} = &-\frac{1}{2}\{w_2^H M^{-1}\left(S_g^{\theta\theta}(z_5) + (S_g^{\theta\theta}(z_5^*))^*\right) \\
&+ w_2^H M^{-1}\left(S_g^{\mathcal{V}\theta}(z_6) + (S_g^{\mathcal{V}\theta}(z_6^*))^*\right) \\
&+ w_3^H T'^{-1}_{d0}(X_d - X'_d)\left(I_g^{\delta\theta}(z_1) + (I_g^{\delta\theta}(z_1^*))^*\right) \\
&+ w_3^H T'^{-1}_{d0}(X_d - X'_d)\left(I_g^{\theta\theta}(z_5) + (I_g^{\theta\theta}(z_5^*))^*\right) \\
&+ w_3^H T'^{-1}_{d0}\left(X_d - X'_d\right)(I_g^{\mathcal{V}\theta}(z_6) + (I_g^{\mathcal{V}\theta}(z_6^*))^*) \\
&+ jw_4^H T'^{-1}_{q0}(X_q - X'_q)\left(I_g^{\delta\theta}(z_1) - (I_g^{\delta\theta}(z_1^*))^*\right) \\
&+ jw_4^H T'^{-1}_{q0}(X_q - X'_q)\left(I_g^{\theta\theta}(z_5) - (I_g^{\theta\theta}(z_5^*))^*\right) \\
&+ jw_4^H T'^{-1}_{q0}\left(X_q - X'_q\right)(I_g^{\mathcal{V}\theta}(z_6) - (I_g^{\mathcal{V}\theta}(z_6^*))^*) \\
&+ w_5^H\left(E_{x'_q}^{\delta\theta}(z_1) + (E_{x'_q}^{\delta\theta}(z_1^*))^*\right) \\
&+ w_5^H\left(E_{x'_q}^{\theta\theta}(z_5) + (E_{x'_q}^{\theta\theta}(z_5^*))^*\right) \\
&+ w_5^H\left(E_{x'_q}^{\mathcal{V}\theta}(z_6) + (E_{x'_q}^{\mathcal{V}\theta}(z_6^*))^*\right) \\
&- jw_6^H\left(E_{x'_d}^{\delta\theta}(z_1) - (E_{x'_d}^{\delta\theta}(z_1^*))^*\right) \\
&- jw_6^H\left(E_{x'_d}^{\theta\theta}(z_5) - (E_{x'_d}^{\theta\theta}(z_5^*))^*\right) \\
&- jw_6^H\left(E_{x'_d}^{\mathcal{V}\theta}(z_6) - (E_{x'_d}^{\mathcal{V}\theta}(z_6^*))^*\right) \\
&- w_7^H\left(S_l^{\theta\theta}(z_5) + (S_l^{\theta\theta}(z_5^*))^*\right) \\
&- w_7^H\left(S_l^{\mathcal{V}\theta}(z_6) + (S_l^{\mathcal{V}\theta}(z_6^*))^*\right) \\
&+ jw_8^H\left(S_l^{\theta\theta}(z_5) - (S_l^{\theta\theta}(z_5^*))^*\right) \\
&+ jw_8^H\left(S_l^{\mathcal{V}\theta}(z_6) - (S_l^{\mathcal{V}\theta}(z_6^*))^*\right)\},
\end{aligned} \tag{25}$$

TABLE I: Coefficient matrices of the linearized DAE model.

	δ	ω	e'_q	e'_d	θ	$\mathcal{V}$
(1a)		I				
(1b)		$-M^{-1}D$			$-M^{-1}\mathrm{Re}\{\frac{\partial S_g}{\partial \theta}\}$	$-M^{-1}\mathrm{Re}\{\frac{\partial S_g}{\partial \mathcal{V}}\}$
(1c)	$-T'^{-1}_{d0}(X_d-X'_d)\mathrm{Re}\{\frac{\partial I_g}{\partial \delta}\}$	$T'^{-1}_{d0}K_AK_S$	$-T'^{-1}_{d0}$		$-T'^{-1}_{d0}(X_d-X'_d)\mathrm{Re}\{\frac{\partial I_g}{\partial \theta}\}$	$-T'^{-1}_{d0}\left((X_d-X'_d)\mathrm{Re}\{\frac{\partial I_g}{\partial \mathcal{V}}\}+K_AC_g\right)$
(1d)	$T'^{-1}_{q0}(X_q-X'_q)\mathrm{Im}\{\frac{\partial I_g}{\partial \delta}\}$			$-T'^{-1}_{q0}$	$T'^{-1}_{q0}(X_q-X'_q)\mathrm{Im}\{\frac{\partial I_g}{\partial \theta}\}$	$T'^{-1}_{q0}(X_q-X'_q)\mathrm{Im}\{\frac{\partial I_g}{\partial \mathcal{V}}\}$
(2a)	$-\mathrm{Re}\{\frac{\partial E_{x'_q}}{\partial \delta}\}$			I	$-\mathrm{Re}\{\frac{\partial E_{x'_q}}{\partial \theta}\}$	$-\mathrm{Re}\{\frac{\partial E_{x'_q}}{\partial \mathcal{V}}\}$
(2b)	$-\mathrm{Im}\{\frac{\partial E_{x'_d}}{\partial \delta}\}$		I		$-\mathrm{Im}\{\frac{\partial E_{x'_d}}{\partial \theta}\}$	$-\mathrm{Im}\{\frac{\partial E_{x'_d}}{\partial \mathcal{V}}\}$
(2c)					$\mathrm{Re}\{\frac{\partial S_l}{\partial \theta}\}$	$\mathrm{Re}\{\frac{\partial S_l}{\partial \mathcal{V}}\}$
(2d)					$\mathrm{Im}\{\frac{\partial S_l}{\partial \theta}\}$	$\mathrm{Im}\{\frac{\partial S_l}{\partial \mathcal{V}}\}$

$$
\begin{aligned}
\frac{\partial \lambda}{\partial \mathcal{V}} = &-\frac{1}{2}\big\{ w_2^H M^{-1}\left(S_g^{\theta\mathcal{V}}(z_5)+(S_g^{\theta\mathcal{V}}(z_5^*))^*\right)\\
&+w_2^H M^{-1}\left(S_g^{\mathcal{V}\mathcal{V}}(z_6)+(S_g^{\mathcal{V}\mathcal{V}}(z_6^*))^*\right)\\
&+w_3^H T'^{-1}_{d0}(X_d-X'_d)\left(I_g^{\delta\mathcal{V}}(z_1)+(I_g^{\delta\mathcal{V}}(z_1^*))^*\right)\\
&+w_3^H T'^{-1}_{d0}(X_d-X'_d)\left(I_g^{\theta\mathcal{V}}(z_5)+(I_g^{\theta\mathcal{V}}(z_5^*))^*\right)\\
&+w_3^H T'^{-1}_{d0}\left(X_d-X'_d)(I_g^{\mathcal{V}\mathcal{V}}(z_6)+(I_g^{\mathcal{V}\mathcal{V}}(z_6^*))^*\right)\\
&+jw_4^H T'^{-1}_{q0}(X_q-X'_q)\left(I_g^{\delta\mathcal{V}}(z_1)-(I_g^{\delta\mathcal{V}}(z_1^*))^*\right)\\
&+jw_4^H T'^{-1}_{q0}(X_q-X'_q)\left(I_g^{\theta\mathcal{V}}(z_5)-(I_g^{\theta\mathcal{V}}(z_5^*))^*\right)\\
&+jw_4^H T'^{-1}_{q0}\left(X_q-X'_q)(I_g^{\mathcal{V}\mathcal{V}}(z_6)-(I_g^{\mathcal{V}\mathcal{V}}(z_6^*))^*\right)\\
&+w_5^H\left(E_{x'_q}^{\delta\mathcal{V}}(z_1)+(E_{x'_q}^{\delta\mathcal{V}}(z_1^*))^*\right)\\
&+w_5^H\left(E_{x'_q}^{\theta\mathcal{V}}(z_5)+(E_{x'_q}^{\theta\mathcal{V}}(z_5^*))^*\right)\\
&+w_5^H\left(E_{x'_q}^{\mathcal{V}\mathcal{V}}(z_6)+(E_{x'_q}^{\mathcal{V}\mathcal{V}}(z_6^*))^*\right)\\
&-jw_6^H\left(E_{x'_d}^{\delta\mathcal{V}}(z_1)-(E_{x'_d}^{\delta\mathcal{V}}(z_1^*))^*\right)\\
&-jw_6^H\left(E_{x'_d}^{\theta\mathcal{V}}(z_5)-(E_{x'_d}^{\theta\mathcal{V}}(z_5^*))^*\right)\\
&-jw_6^H\left(E_{x'_d}^{\mathcal{V}\mathcal{V}}(z_6)-(E_{x'_d}^{\mathcal{V}\mathcal{V}}(z_6^*))^*\right)\\
&-w_7^H\left(S_l^{\theta\mathcal{V}}(z_5)+(S_l^{\theta\mathcal{V}}(z_5^*))^*\right)\\
&-w_7^H\left(S_l^{\mathcal{V}\mathcal{V}}(z_6)+(S_l^{\mathcal{V}\mathcal{V}}(z_6^*))^*\right)\\
&+jw_8^H\left(S_l^{\theta\mathcal{V}}(z_5)-(S_l^{\theta\mathcal{V}}(z_5^*))^*\right)\\
&+jw_8^H\left(S_l^{\mathcal{V}\mathcal{V}}(z_6)-(S_l^{\mathcal{V}\mathcal{V}}(z_6^*))^*\right)\big\},
\end{aligned} \tag{26}
$$

where we define

$$
\begin{aligned}
E_x^{\delta\theta}(z) = &[\mathrm{e}^{j(\frac{\pi}{2}-\delta)}][-jz]K_g(x)[jV]\\
&+[K_g(x)V][-jz][\mathrm{e}^{j(\frac{\pi}{2}-\delta)}](-j)\frac{\partial \delta}{\partial \theta},
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
E_x^{\delta\mathcal{V}}(z) = &[\mathrm{e}^{j(\frac{\pi}{2}-\delta)}][-jz]K_g(x)[\mathrm{e}^{j\theta}]\\
&+[K_g(x)V][-jz][\mathrm{e}^{j(\frac{\pi}{2}-\delta)}](-j)\frac{\partial \delta}{\partial \mathcal{V}},
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
E_x^{\theta\theta}(z) = &-[\mathrm{e}^{j(\frac{\pi}{2}-\delta)}]K_g(x)[V][z]\\
&+[K_g(x)[jV]z][\mathrm{e}^{j(\frac{\pi}{2}-\delta)}](-j)\frac{\partial \delta}{\partial \theta},
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
E_x^{\theta\mathcal{V}}(z) = &[\mathrm{e}^{j(\frac{\pi}{2}-\delta)}]K_g(x)[\mathrm{e}^{j\theta}][jz]\\
&+[K_g(x)[jV]z][\mathrm{e}^{j(\frac{\pi}{2}-\delta)}](-j)\frac{\partial \delta}{\partial \mathcal{V}},
\end{aligned} \tag{30}
$$

$$
\begin{aligned}
E_x^{\mathcal{V}\theta}(z) = &[\mathrm{e}^{j(\frac{\pi}{2}-\delta)}]K_g(x)[\mathrm{e}^{j\theta}][jz]\\
&+[K_g(x)[\mathrm{e}^{j\theta}]z][\mathrm{e}^{j(\frac{\pi}{2}-\delta)}](-j)\frac{\partial \delta}{\partial \theta},
\end{aligned} \tag{31}
$$

$$
E_x^{\mathcal{V}\mathcal{V}}(z) = [K_g(x)[\mathrm{e}^{j\theta}]z][\mathrm{e}^{j(\frac{\pi}{2}-\delta)}](-j)\frac{\partial \delta}{\partial \mathcal{V}}, \tag{32}
$$

$$
I_g^{\delta\theta}(z) = [\mathrm{e}^{j(\frac{\pi}{2}-\delta)}][z]Y_g[V]+[Y_gV][\mathrm{e}^{-j\delta}][-jz]\frac{\partial \delta}{\partial \theta}, \tag{33}
$$

$$
I_g^{\delta\mathcal{V}}(z) = [\mathrm{e}^{-j\delta}][z]Y_g[\mathrm{e}^{j\theta}]+[Y_gV][\mathrm{e}^{-j\delta}][-jz]\frac{\partial \delta}{\partial \mathcal{V}}, \tag{34}
$$

$$
I_g^{\theta\theta}(z) = [\mathrm{e}^{j(\frac{\pi}{2}-\delta)}]Y_g[-z][V]+j[Y_g[V]z][\mathrm{e}^{-j\delta}]\frac{\partial \delta}{\partial \theta}, \tag{35}
$$

$$
I_g^{\theta\mathcal{V}}(z) = [\mathrm{e}^{j(\frac{\pi}{2}-\delta)}]Y_g[jz][\mathrm{e}^{j\theta}]+j[Y_g[V]z][\mathrm{e}^{-j\delta}]\frac{\partial \delta}{\partial \mathcal{V}}, \tag{36}
$$

$$
I_g^{\mathcal{V}\theta}(z) = [\mathrm{e}^{j(\frac{\pi}{2}-\delta)}]Y_g[jz][\mathrm{e}^{j\theta}]+[Y_g[\mathrm{e}^{j\theta}]z][\mathrm{e}^{-j\delta}]\frac{\partial \delta}{\partial \theta}, \tag{37}
$$

$$
I_g^{\mathcal{V}\mathcal{V}}(z) = [Y_g[\mathrm{e}^{j\theta}]z][\mathrm{e}^{-j\delta}]\frac{\partial \delta}{\partial \mathcal{V}}, \tag{38}
$$

$$
\begin{aligned}
S_g^{\theta\theta}(z) = &[C_gV]Y_g^*[z][-V^*]+[Y_g^*[jV]^*z]C_g[jV]\\
&+[Y_g^*V*]C_g[z][-V]+[C_g[jV]z]Y_g^*[jV]^*,
\end{aligned} \tag{39}
$$

$$
\begin{aligned}
S_g^{\theta\mathcal{V}}(z) = &[C_gV]Y_g^*[z][j\mathrm{e}^{j\theta}]^*+[Y_g^*[jV]^*z]C_g[\mathrm{e}^{j\theta}]\\
&+[Y_g^*V^*]C_g[z][j\mathrm{e}^{j\theta}]+[C_g[jV]z]Y_g^*[\mathrm{e}^{j\theta}]^*,
\end{aligned} \tag{40}
$$

$$
\begin{aligned}
S_g^{\mathcal{V}\theta}(z) = &[C_gV]Y_g^*[z][j\mathrm{e}^{j\theta}]^*+[Y_g^*[\mathrm{e}^{j\theta}]^*z]C_g[jV]\\
&+[Y_g^*V^*]C_g[z][j\mathrm{e}^{j\theta}]+[C_g[j\mathrm{e}^{j\theta}]z]Y_g^*[jV]^*,
\end{aligned} \tag{41}
$$

$$
S_g^{\mathcal{V}\mathcal{V}}(z) = [Y_g^*[\mathrm{e}^{j\theta}]^*z]C_g[\mathrm{e}^{j\theta}]+[C_g[j\mathrm{e}^{\theta}]z]Y_g^*[\mathrm{e}^{j\theta}]^*, \tag{42}
$$

$$
\begin{aligned}
S_l^{\theta\theta}(z) = &[C_lV]Y_l^*[z][-V^*]+[Y_l^*[jV]^*z]C_l[jV]\\
&+[Y_l^*V*]C_l[z][-V]+[C_l[jV]z]Y_l^*[jV]^*,
\end{aligned} \tag{43}
$$

$$
\begin{aligned}
S_l^{\theta\mathcal{V}}(z) = &[C_lV]Y_l^*[z][j\mathrm{e}^{j\theta}]^*+[Y_l^*[jV]^*z]C_l[\mathrm{e}^{j\theta}]\\
&+[Y_l^*V^*]C_l[z][j\mathrm{e}^{j\theta}]+[C_l[jV]z]Y_l^*[\mathrm{e}^{j\theta}]^*,
\end{aligned} \tag{44}
$$

$$
\begin{aligned}
S_l^{\mathcal{V}\theta}(z) = &[C_lV]Y_l^*[z][j\mathrm{e}^{j\theta}]^*+[Y_l^*[\mathrm{e}^{j\theta}]^*z]C_l[jV]\\
&+[Y_l^*V^*]C_l[z][j\mathrm{e}^{j\theta}]+[C_l[j\mathrm{e}^{j\theta}]z]Y_l^*[jV]^*,
\end{aligned} \tag{45}
$$

$$
S_l^{\mathcal{V}\mathcal{V}}(z) = [Y_l^*[\mathrm{e}^{j\theta}]^*z]C_l[\mathrm{e}^{j\theta}]+[C_l[j\mathrm{e}^{\theta}]z]Y_l^*[\mathrm{e}^{j\theta}]^*. \tag{46}
$$