

APPENDIX A DERIVATION OF LINEARIZED DAE

The power system differential equation

$$\dot{\delta} = \omega - \omega_s, \quad (1a)$$

$$\dot{\omega} = M^{-1}(p_m - \text{Re}\{[C_g V] Y_g^* V^*\} - D(\omega - \omega_s)), \quad (1b)$$

$$\dot{e}'_q = T_{d0}'^{-1}(e_f - e'_q - (X_d - X'_d) \text{Re}\{[e^{j(\frac{\pi}{2}-\delta)}] Y_g V\}), \quad (1c)$$

$$\dot{e}'_d = T_{d0}'^{-1}(-e'_d + (X_q - X'_q) \text{Im}\{[e^{j(\frac{\pi}{2}-\delta)}] Y_g V\}), \quad (1d)$$

and algebraic equation

$$e'_d = \text{Re}\{[e^{j(\frac{\pi}{2}-\delta)}](C_g + jX'_q Y_g) V\}, \quad (2a)$$

$$e'_q = \text{Im}\{[e^{j(\frac{\pi}{2}-\delta)}](C_g + jX'_d Y_g) V\}, \quad (2b)$$

$$P_l + \text{Re}\{[C_l V] Y_l^* V^*\} = 0, \quad (2c)$$

$$Q_l + \text{Im}\{[C_l V] Y_l^* V^*\} = 0. \quad (2d)$$

$$e_f = E_f - K_A C_g(|V| - |V_e|) + K_A K_S(\omega - \omega_s), \quad (3)$$

can be linearized into the form

$$\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{s} \\ \dot{x} \end{pmatrix} = \begin{pmatrix} A(V_e) & B(V_e) \\ C(V_e) & D(V_e) \end{pmatrix} \begin{pmatrix} s \\ x \end{pmatrix}. \quad (4)$$

The coefficient matrices A , B , C , and D in the linearized DAE (4) takes the forms shown in the four blocks of Table II (A : upper left, B : upper right, C : lower left, D : lower right), where all variables in the matrices represent steady-state values, but with the subscript “ e ” omitted for brevity. We recall that the square bracket $[x]$ denotes the diagonal matrix constructed from vector x and that x^* denotes the complex conjugate of x . The diagonals of $X_q(X_d)$ and $X'_q(X'_d)$ are denoted by vectors $x_q(x_d)$ and $x'_q(x'_d)$, respectively. In fact, all terms in Table II are functions of the steady-state power flow solution $V = [V]e^{j\theta}$. Specifically, we have

$$\delta = \arg(K_g(x_q)V), \quad (5)$$

and we define

$$K_g(x) = C_g + j[x]Y_g, \quad x \in \{x_d, x_q, x'_d, x'_q\}, \quad (6)$$

$$S_g = [C_g V] Y_g^* V^*, \quad (7)$$

$$I_g = [e^{j(\frac{\pi}{2}-\delta)}] Y_g V, \quad (8)$$

$$E_x = [e^{j(\frac{\pi}{2}-\delta)}] K_g(x) V, \quad (9)$$

$$S_l = [C_l V] Y_l^* V^*. \quad (10)$$

The partial derivatives in Table II can then be compute as

$$\frac{\partial S_g}{\partial \theta} = [C_g V] Y_g^* [jV]^* + [Y_g^* V^*] C_g [jV], \quad (11)$$

$$\frac{\partial S_g}{\partial V} = [C_g V] Y_g^* [e^{j\theta}]^* + [Y_g^* V^*] C_g [e^{j\theta}], \quad (12)$$

$$\frac{\partial S_l}{\partial \theta} = [C_l V] Y_l^* [jV]^* + [Y_l^* V^*] C_l [jV], \quad (13)$$

$$\frac{\partial S_l}{\partial V} = [C_l V] Y_l^* [e^{j\theta}]^* + [Y_l^* V^*] C_g [e^{j\theta}], \quad (14)$$

$$\frac{\partial I_g}{\partial \delta} = [Y_g V] [e^{-j\delta}], \quad (15)$$

$$\frac{\partial I_g}{\partial \theta} = -[e^{-j\delta}] Y_g [V], \quad (16)$$

$$\frac{\partial I_g}{\partial V} = [e^{j(\frac{\pi}{2}-\delta)}] Y_g [e^{j\theta}], \quad (17)$$

$$\frac{\partial E_x}{\partial \delta} = [e^{j(\frac{\pi}{2}-\delta)}] [K_g(x) V] (-j), \quad (18)$$

$$\frac{\partial E_x}{\partial \theta} = [e^{j(\frac{\pi}{2}-\delta)}] K_g(x) [jV], \quad (19)$$

$$\frac{\partial E_x}{\partial V} = [e^{j(\frac{\pi}{2}-\delta)}] K_g(x) [e^{j\theta}]. \quad (20)$$

APPENDIX B ANALYTICAL GRADIENT OF EIGENVALUES

Here we present details on calculating analytical gradients for the system's eigenvalues in equation (7). We partition the right and left eigenvectors u and v associated with the eigenvalue λ as

$$u = [z_1^T, z_2^T, z_3^T, z_4^T, z_5^T, z_6^T]^T, \quad (21)$$

$$v = [w_1^T, w_2^T, w_3^T, w_4^T, w_5^T, w_6^T, w_7^T, w_8^T]^T, \quad (22)$$

according to the columns and rows of the matrices in Table II. Differentiating equation (23), we obtain

$$\frac{\partial \delta}{\partial \theta} = \text{Im}\{[K_g(x_q)V]^{-1} K_g(x_q) [jV]\}, \quad (23)$$

$$\frac{\partial \delta}{\partial V} = \text{Im}\{[K_g(x_q)V]^{-1} K_g(x_q) [e^{j\theta}]\}. \quad (24)$$

Using these when taking the derivatives of the coefficients in Table II, we expand the matrix-vector product on the right hand side of equation (7) and carry out the differentiation. The resulting gradients of the eigenvalue λ with respect to θ and V can be expressed as

$$\begin{aligned} \frac{\partial \lambda}{\partial \theta} = & -\frac{1}{2} \{ w_2^H M^{-1} (S_g^{\theta\theta}(z_5) + (S_g^{\theta\theta}(z_5^*))^*) \\ & + w_2^H M^{-1} (S_g^{\nu\theta}(z_6) + (S_g^{\nu\theta}(z_6^*))^*) \\ & + w_3^H T_{d0}'^{-1} (X_d - X'_d) (I_g^{\delta\theta}(z_1) + (I_g^{\delta\theta}(z_1^*))^*) \\ & + w_3^H T_{d0}'^{-1} (X_d - X'_d) (I_g^{\theta\theta}(z_5) + (I_g^{\theta\theta}(z_5^*))^*) \\ & + w_3^H T_{d0}'^{-1} (X_d - X'_d) (I_g^{\nu\theta}(z_6) + (I_g^{\nu\theta}(z_6^*))^*) \\ & + j w_4^H T_{q0}'^{-1} (X_q - X'_q) (I_g^{\delta\theta}(z_1) - (I_g^{\delta\theta}(z_1^*))^*) \\ & + j w_4^H T_{q0}'^{-1} (X_q - X'_q) (I_g^{\theta\theta}(z_5) - (I_g^{\theta\theta}(z_5^*))^*) \\ & + j w_4^H T_{q0}'^{-1} (X_q - X'_q) (I_g^{\nu\theta}(z_6) - (I_g^{\nu\theta}(z_6^*))^*) \\ & + w_5^H (E_{x'_q}^{\delta\theta}(z_1) + (E_{x'_q}^{\delta\theta}(z_1^*))^*) \\ & + w_5^H (E_{x'_q}^{\theta\theta}(z_5) + (E_{x'_q}^{\theta\theta}(z_5^*))^*) \\ & + w_5^H (E_{x'_q}^{\nu\theta}(z_6) + (E_{x'_q}^{\nu\theta}(z_6^*))^*) \\ & - j w_6^H (E_{x'_d}^{\delta\theta}(z_1) - (E_{x'_d}^{\delta\theta}(z_1^*))^*) \\ & - j w_6^H (E_{x'_d}^{\theta\theta}(z_5) - (E_{x'_d}^{\theta\theta}(z_5^*))^*) \\ & - j w_6^H (E_{x'_d}^{\nu\theta}(z_6) - (E_{x'_d}^{\nu\theta}(z_6^*))^*) \\ & - w_7^H (S_l^{\theta\theta}(z_5) + (S_l^{\theta\theta}(z_5^*))^*) \\ & - w_7^H (S_l^{\nu\theta}(z_6) + (S_l^{\nu\theta}(z_6^*))^*) \\ & + j w_8^H (S_l^{\theta\theta}(z_5) - (S_l^{\theta\theta}(z_5^*))^*) \\ & + j w_8^H (S_l^{\nu\theta}(z_6) - (S_l^{\nu\theta}(z_6^*))^*) \}, \end{aligned} \quad (25)$$

$$\begin{aligned} \frac{\partial \lambda}{\partial V} = & -\frac{1}{2} \{ w_2^H M^{-1} (S_g^{\theta\nu}(z_5) + (S_g^{\theta\nu}(z_5^*))^*) \\ & + w_2^H M^{-1} (S_g^{\nu\nu}(z_6) + (S_g^{\nu\nu}(z_6^*))^*) \\ & + w_3^H T_{d0}'^{-1} (X_d - X'_d) (I_g^{\delta\nu}(z_1) + (I_g^{\delta\nu}(z_1^*))^*) \end{aligned}$$

TABLE II: Coefficient matrices of the linearized DAE model.

	δ	ω	e'_q	e'_d	θ	\mathcal{V}
(1a)	I					
(1b)	$-M^{-1}D$				$-M^{-1}\text{Re}\{\frac{\partial S_g}{\partial \theta}\}$	$-M^{-1}\text{Re}\{\frac{\partial S_g}{\partial \mathcal{V}}\}$
(1c)	$-T_{d0}^{\prime -1}(X_d-X_d')\text{Re}\{\frac{\partial I_g}{\partial \delta}\}$	$T_{d0}^{\prime -1}K_AK_S$	$-T_{d0}^{\prime -1}$		$-T_{d0}^{\prime -1}(X_d-X_d')\text{Re}\{\frac{\partial I_g}{\partial \theta}\}$	$-T_{d0}^{\prime -1}\left((X_d-X_d')\text{Re}\{\frac{\partial I_g}{\partial \mathcal{V}}\}+K_AC_g\right)$
(1d)	$T_{q0}^{\prime -1}(X_q-X_q')\text{Im}\{\frac{\partial I_g}{\partial \delta}\}$			$-T_{q0}^{\prime -1}$	$T_{q0}^{\prime -1}(X_q-X_q')\text{Im}\{\frac{\partial I_g}{\partial \theta}\}$	$T_{q0}^{\prime -1}(X_q-X_q')\text{Im}\{\frac{\partial I_g}{\partial \mathcal{V}}\}$
(2a)	$-\text{Re}\{\frac{\partial E_{x'_q}}{\partial \delta}\}$	I			$-\text{Re}\{\frac{\partial E_{x'_q}}{\partial \theta}\}$	$-\text{Re}\{\frac{\partial E_{x'_q}}{\partial \mathcal{V}}\}$
(2b)	$-\text{Im}\{\frac{\partial E_{x'_d}}{\partial \delta}\}$	I			$-\text{Im}\{\frac{\partial E_{x'_d}}{\partial \theta}\}$	$-\text{Im}\{\frac{\partial E_{x'_d}}{\partial \mathcal{V}}\}$
(2c)					$\text{Re}\{\frac{\partial S_l}{\partial \theta}\}$	$\text{Re}\{\frac{\partial S_l}{\partial \mathcal{V}}\}$
(2d)					$\text{Im}\{\frac{\partial S_l}{\partial \theta}\}$	$\text{Im}\{\frac{\partial S_l}{\partial \mathcal{V}}\}$

$$\begin{aligned}
& + w_3^H T_{d0}'^{-1}(X_d - X_d')(I_g^{\theta\mathcal{V}}(z_5) + (I_g^{\theta\mathcal{V}}(z_5^*))^*) \\
& + w_3^H T_{d0}'^{-1}(X_d - X_d')(I_g^{\mathcal{V}\mathcal{V}}(z_6) + (I_g^{\mathcal{V}\mathcal{V}}(z_6^*))^*) \\
& + jw_4^H T_{q0}'^{-1}(X_q - X_q')(I_g^{\delta\mathcal{V}}(z_1) - (I_g^{\delta\mathcal{V}}(z_1^*))^*) \\
& + jw_4^H T_{q0}'^{-1}(X_q - X_q')(I_g^{\theta\mathcal{V}}(z_5) - (I_g^{\theta\mathcal{V}}(z_5^*))^*) \\
& + jw_4^H T_{q0}'^{-1}(X_q - X_q')(I_g^{\mathcal{V}\mathcal{V}}(z_6) - (I_g^{\mathcal{V}\mathcal{V}}(z_6^*))^*) \\
& + w_5^H \left(E_{x'_q}^{\delta\mathcal{V}}(z_1) + (E_{x'_q}^{\delta\mathcal{V}}(z_1^*))^* \right) \\
& + w_5^H \left(E_{x'_q}^{\theta\mathcal{V}}(z_5) + (E_{x'_q}^{\theta\mathcal{V}}(z_5^*))^* \right) \\
& + w_5^H \left(E_{x'_q}^{\mathcal{V}\mathcal{V}}(z_6) + (E_{x'_q}^{\mathcal{V}\mathcal{V}}(z_6^*))^* \right) \\
& - jw_6^H \left(E_{x'_d}^{\delta\mathcal{V}}(z_1) - (E_{x'_d}^{\delta\mathcal{V}}(z_1^*))^* \right) \\
& - jw_6^H \left(E_{x'_d}^{\theta\mathcal{V}}(z_5) - (E_{x'_d}^{\theta\mathcal{V}}(z_5^*))^* \right) \\
& - jw_6^H \left(E_{x'_d}^{\mathcal{V}\mathcal{V}}(z_6) - (E_{x'_d}^{\mathcal{V}\mathcal{V}}(z_6^*))^* \right) \\
& - w_7^H (S_l^{\delta\mathcal{V}}(z_5) + (S_l^{\delta\mathcal{V}}(z_5^*))^*) \\
& - w_7^H (S_l^{\mathcal{V}\mathcal{V}}(z_6) + (S_l^{\mathcal{V}\mathcal{V}}(z_6^*))^*) \\
& + jw_8^H (S_l^{\theta\mathcal{V}}(z_5) - (S_l^{\theta\mathcal{V}}(z_5^*))^*) \\
& + jw_8^H (S_l^{\mathcal{V}\mathcal{V}}(z_6) - (S_l^{\mathcal{V}\mathcal{V}}(z_6^*))^*) \},
\end{aligned} \tag{26}$$

where we define

$$\begin{aligned}
E_x^{\delta\theta}(z) &= [\mathbf{e}^{j(\frac{\pi}{2}-\delta)}] [-jz] K_g(x) [jV] \\
&+ [K_g(x)V] [-jz] [\mathbf{e}^{j(\frac{\pi}{2}-\delta)}] (-j) \frac{\partial \delta}{\partial \theta},
\end{aligned}$$

$$\begin{aligned}
E_x^{\delta\mathcal{V}}(z) &= [\mathbf{e}^{j(\frac{\pi}{2}-\delta)}] [-jz] K_g(x) [\mathbf{e}^{j\theta}] \\
&+ [K_g(x)V] [-jz] [\mathbf{e}^{j(\frac{\pi}{2}-\delta)}] (-j) \frac{\partial \delta}{\partial \mathcal{V}},
\end{aligned}$$

$$\begin{aligned}
E_x^{\theta\theta}(z) &= -[\mathbf{e}^{j(\frac{\pi}{2}-\delta)}] K_g(x) [V] [z] \\
&+ [K_g(x)[jV]z] [\mathbf{e}^{j(\frac{\pi}{2}-\delta)}] (-j) \frac{\partial \delta}{\partial \theta},
\end{aligned} \tag{29}$$

$$\begin{aligned}
E_x^{\theta\mathcal{V}}(z) &= [\mathbf{e}^{j(\frac{\pi}{2}-\delta)}] K_g(x) [\mathbf{e}^{j\theta}] [jz] \\
&+ [K_g(x)[jV]z] [\mathbf{e}^{j(\frac{\pi}{2}-\delta)}] (-j) \frac{\partial \delta}{\partial \mathcal{V}},
\end{aligned} \tag{30}$$

$$\begin{aligned}
E_x^{\mathcal{V}\theta}(z) &= [\mathbf{e}^{j(\frac{\pi}{2}-\delta)}] K_g(x) [\mathbf{e}^{j\theta}] [jz] \\
&+ [K_g(x)[\mathbf{e}^{j\theta}]z] [\mathbf{e}^{j(\frac{\pi}{2}-\delta)}] (-j) \frac{\partial \delta}{\partial \theta},
\end{aligned} \tag{31}$$

$$E_x^{\mathcal{V}\mathcal{V}}(z) = [K_g(x) [\mathbf{e}^{j\theta}]z] [\mathbf{e}^{j(\frac{\pi}{2}-\delta)}] (-j) \frac{\partial \delta}{\partial \mathcal{V}}, \tag{32}$$

$$I_g^{\delta\theta}(z) = [\mathbf{e}^{j(\frac{\pi}{2}-\delta)}] [z] Y_g[V] + [Y_g V] [\mathbf{e}^{-j\delta}] [-jz] \frac{\partial \delta}{\partial \theta}, \tag{33}$$

$$I_g^{\delta\mathcal{V}}(z) = [\mathbf{e}^{-j\delta}] [z] Y_g[\mathbf{e}^{j\theta}] + [Y_g V] [\mathbf{e}^{-j\delta}] [-jz] \frac{\partial \delta}{\partial \mathcal{V}}, \tag{34}$$

$$I_g^{\theta\theta}(z) = [\mathbf{e}^{j(\frac{\pi}{2}-\delta)}] Y_g[-z] [V] + j[Y_g[V]z] [\mathbf{e}^{-j\delta}] \frac{\partial \delta}{\partial \theta}, \tag{35}$$

$$I_g^{\theta\mathcal{V}}(z) = [\mathbf{e}^{j(\frac{\pi}{2}-\delta)}] Y_g[jz] [\mathbf{e}^{j\theta}] + j[Y_g[V]z] [\mathbf{e}^{-j\delta}] \frac{\partial \delta}{\partial \mathcal{V}}, \tag{36}$$

$$I_g^{\mathcal{V}\theta}(z) = [\mathbf{e}^{j(\frac{\pi}{2}-\delta)}] Y_g[jz] [\mathbf{e}^{j\theta}] + [Y_g[\mathbf{e}^{j\theta}]z] [\mathbf{e}^{-j\delta}] \frac{\partial \delta}{\partial \theta}, \tag{37}$$

$$I_g^{\mathcal{V}\mathcal{V}}(z) = [Y_g[\mathbf{e}^{j\theta}]z] [\mathbf{e}^{-j\delta}] \frac{\partial \delta}{\partial \mathcal{V}}, \tag{38}$$

$$\begin{aligned}
S_g^{\theta\theta}(z) &= [C_g V] Y_g^*[z] [-V^*] + [Y_g^*[jV]^*z] C_g[jV] \\
&+ [Y_g^*V^*] C_g[z] [-V] + [C_g[jV]z] Y_g^*[jV]^*,
\end{aligned} \tag{39}$$

$$\begin{aligned}
S_g^{\theta\mathcal{V}}(z) &= [C_g V] Y_g^*[z] [j\mathbf{e}^{j\theta}]^* + [Y_g^*[jV]^*z] C_g[\mathbf{e}^{j\theta}] \\
&+ [Y_g^*V^*] C_g[z] [j\mathbf{e}^{j\theta}] + [C_g[jV]z] Y_g^*[\mathbf{e}^{j\theta}]^*,
\end{aligned} \tag{40}$$

$$\begin{aligned}
S_g^{\mathcal{V}\theta}(z) &= [C_g V] Y_g^*[z] [j\mathbf{e}^{j\theta}]^* + [Y_g^*[\mathbf{e}^{j\theta}]^*z] C_g[jV] \\
&+ [Y_g^*V^*] C_g[z] [j\mathbf{e}^{j\theta}] + [C_g[j\mathbf{e}^{j\theta}]z] Y_g^*[jV]^*,
\end{aligned} \tag{41}$$

$$S_g^{\mathcal{V}\mathcal{V}}(z) = [Y_g^*[\mathbf{e}^{j\theta}]^*z] C_g[\mathbf{e}^{j\theta}] + [C_g[j\mathbf{e}^{j\theta}]z] Y_g^*[\mathbf{e}^{j\theta}]^*, \tag{42}$$

$$\begin{aligned}
S_l^{\theta\theta}(z) &= [C_l V] Y_l^*[z] [-V^*] + [Y_l^*[jV]^*z] C_l[jV] \\
&+ [Y_l^*V^*] C_l[z] [-V] + [C_l[jV]z] Y_l^*[jV]^*,
\end{aligned} \tag{43}$$

$$\begin{aligned}
S_l^{\theta\mathcal{V}}(z) &= [C_l V] Y_l^*[z] [j\mathbf{e}^{j\theta}]^* + [Y_l^*[jV]^*z] C_l[\mathbf{e}^{j\theta}] \\
&+ [Y_l^*V^*] C_l[z] [j\mathbf{e}^{j\theta}] + [C_l[jV]z] Y_l^*[\mathbf{e}^{j\theta}]^*,
\end{aligned} \tag{44}$$

$$\begin{aligned}
S_l^{\mathcal{V}\theta}(z) &= [C_l V] Y_l^*[z] [j\mathbf{e}^{j\theta}]^* + [Y_l^*[\mathbf{e}^{j\theta}]^*z] C_l[jV] \\
&+ [Y_l^*V^*] C_l[z] [j\mathbf{e}^{j\theta}] + [C_l[j\mathbf{e}^{j\theta}]z] Y_l^*[jV]^*,
\end{aligned} \tag{45}$$

$$S_l^{\mathcal{V}\mathcal{V}}(z) = [Y_l^*[\mathbf{e}^{j\theta}]^*z] C_l[\mathbf{e}^{j\theta}] + [C_l[j\mathbf{e}^{j\theta}]z] Y_l^*[\mathbf{e}^{j\theta}]^*. \tag{46}$$