Phys 20 Lab 6 - Root Finding

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1 Part 1 - Golden Ratio Convergence of Secant Method

Within a small distance ϵ of x, the function is approximately:

$$f(x+\epsilon) = f(x) + \epsilon f'(x) + \epsilon^2 \frac{f''(x)}{2} + ...$$

Evaluating at x_1 and x_2 and plugging into the step relation for the Secant Method

$$f(x_3) = x_2 - f(x_2)\frac{x_2 - x_1}{f(x_2) - f(x_1)}$$

we get the following:

$$f(\epsilon_3) = \epsilon_2 - f(x_2)\frac{\epsilon_2 - \epsilon_1}{f(x_2) - f(x_1)}$$

When a trial solution x_i differs from the true root by ϵ we get:

$$\epsilon_3 = \epsilon_2 - f(x_2+\epsilon)\frac{\epsilon_2 - \epsilon_1}{f(x_2+\epsilon_2) - f(x_1+\epsilon)}$$

Since $f(x_i)$ is zero, we get an approximation (neglecting higher order terms):

$$f(x_i+\epsilon_i) \approx \epsilon_2 f'(x_i) + \epsilon_i^2 \frac{f''(x_i)}{2}$$

Plugging this in, we get:

$$\epsilon_3 \approx \epsilon_2 - (\epsilon_2 f'(x_2) + \epsilon_2^2 \frac{f''(x_2)}{2})\frac{\epsilon_2 - \epsilon}{\epsilon_2 f'(x_2) + \epsilon_2^2 \frac{f''(x_2)}{2} - \epsilon f'(x_1) + \epsilon^2 \frac{f''(x_1)}{2}}$$

Simplifying the denominator, we find:

$$\begin{aligned} f(x_2+\epsilon_2)-f(x_1+\epsilon) &\approx \epsilon_2 f'(x_2)+\epsilon_2^2\frac{f''(x_2)}{2}-\epsilon f'(x_1)-\epsilon^2\frac{f''(x_1)}{2} \\ &\approx f'(x_2)*(\epsilon_2-\epsilon)*(1+\frac{f''(x_2)}{2f'(x_2)}(\epsilon_2+\epsilon)) \end{aligned}$$

Which gives:

$$\begin{aligned} \epsilon_3 &= \epsilon_2-\frac{(\epsilon_2)*(\epsilon_2-\epsilon)*(f'(x_2)+\epsilon_2 f''(x_2))}{f'(x_2)*(\epsilon_2-\epsilon)*(1+\frac{f''(x_2)}{2f'(x_2)}(\epsilon_2+\epsilon))} \\ &= \epsilon_2-\frac{(\epsilon_2)*(1+\epsilon_2\frac{f''(x_2)}{f'(x_2)})}{(1+\frac{f''(x_2)}{2f'(x_2)}(\epsilon_2+\epsilon))} \\ &= \frac{\epsilon_2(1+\frac{f''(x_2)}{2f'(x_2)}(\epsilon_2+\epsilon))-(\epsilon_2)(1+\epsilon_2\frac{f''(x_2)}{f'(x_2)})}{(1+\frac{f''(x_2)}{2f'(x_2)}(\epsilon_2+\epsilon))} \\ &= \frac{\epsilon_2\epsilon\frac{f''(x_2)}{2f'(x_2)}}{(1+\frac{f''(x_2)}{2f'(x_2)}(\epsilon_2+\epsilon))} \\ &\approx \epsilon_2\epsilon\frac{f''(x_2)}{2f'(x_2)} \end{aligned}$$

Now, assume $\epsilon_{i+1}=C\epsilon_i^r$ for all i where C and r are constants independent of i. Plugging this into the recurrence, we find:

$$\begin{aligned} C\epsilon_2^r &\approx \epsilon_2\epsilon\frac{f''(x_2)}{2f'(x_2)} \\ \epsilon_2^{r-1} &\approx \epsilon\frac{f''(x_2)}{2Cf'(x_2)} \\ \epsilon_2 &\approx \epsilon^{\frac{1}{r-1}}\frac{f''(x_2)}{2Cf'(x_2)}^{\frac{1}{r-1}} \end{aligned}$$

So $C=\frac{f''(x_2)}{2Cf'(x_2)}^{\frac{1}{r-1}}$ and $r=\frac{1}{r-1}$ from the assumption. Solving for the positive r, we find the convergence rate is:

$$\begin{aligned} r &= \frac{1}{r-1} \\ r^2-r-1 &= 0 \\ r &= \frac{1+\sqrt{5}}{2} \end{aligned}$$

Which is the Golden Ratio.

2 Method Implementations

3 Code and Info

3.1 Output

The program outputs the following:

```
Running Methods on default function: f(x) = sin(x) - .76
Bisection Guess: 0.863314
Newton-Rpahson Guess: 0.863313
Secant Guess: 0.863313
Successfully Ran Script
```

3.2 Code