

Phys 20 Lab 6 - Root Finding

Chris Dudiak

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1 Part 1 - Golden Ratio Convergence of Secant Method

Within a small distance ϵ of x , the function is approximately:

$$f(x + \epsilon) = f(x) + \epsilon f'(x) + \epsilon^2 \frac{f''(x)}{2} + \dots$$

Evaluating at x_1 and x_2 and plugging into the step relation for the Secant Method

$$f(x_3) = x_2 - f(x_2) \frac{x_2 - x_1}{f(x_2) - f(x_1)}$$

we get the following:

$$f(\epsilon_3) = \epsilon_2 - f(x_2) \frac{\epsilon_2 - \epsilon_1}{f(x_2) - f(x_1)}$$

When a trial solution x_i differs from the true root by ϵ we get:

$$\epsilon_3 = \epsilon_2 - f(x_2 + \epsilon) \frac{\epsilon_2 - \epsilon_1}{f(x_2 + \epsilon_2) - f(x_1 + \epsilon)}$$

Since $f(x_i)$ is zero, we get an approximation (neglecting higher order terms):

$$f(x_i + \epsilon_i) \approx \epsilon_2 f'(x_i) + \epsilon_i^2 \frac{f''(x_i)}{2}$$

Plugging this in, we get:

$$\epsilon_3 \approx \epsilon_2 - \left(\epsilon_2 f'(x_2) + \epsilon_2^2 \frac{f''(x_2)}{2} \right) \frac{\epsilon_2 - \epsilon}{\epsilon_2 f'(x_2) + \epsilon_2^2 \frac{f''(x_2)}{2} - \epsilon f'(x_1) + \epsilon^2 \frac{f''(x_1)}{2}}$$

Simplifying the denominator, we find:

$$\begin{aligned} f(x_2 + \epsilon_2) - f(x_1 + \epsilon) &\approx \epsilon_2 f'(x_2) + \epsilon_2^2 \frac{f''(x_2)}{2} - \epsilon f'(x_1) - \epsilon^2 \frac{f''(x_1)}{2} \\ &\approx f'(x_2) * (\epsilon_2 - \epsilon) * \left(1 + \frac{f''(x_2)}{2f'(x_2)}(\epsilon_2 + \epsilon)\right) \end{aligned}$$

Which gives:

$$\begin{aligned} \epsilon_3 &= \epsilon_2 - \frac{(\epsilon_2) * (\epsilon_2 - \epsilon) * (f'(x_2) + \epsilon_2 f''(x_2))}{f'(x_2) * (\epsilon_2 - \epsilon) * \left(1 + \frac{f''(x_2)}{2f'(x_2)}(\epsilon_2 + \epsilon)\right)} \\ &= \epsilon_2 - \frac{(\epsilon_2) * \left(1 + \epsilon_2 \frac{f''(x_2)}{f'(x_2)}\right)}{\left(1 + \frac{f''(x_2)}{2f'(x_2)}(\epsilon_2 + \epsilon)\right)} \\ &= \frac{\epsilon_2 \left(1 + \frac{f''(x_2)}{2f'(x_2)}(\epsilon_2 + \epsilon)\right) - (\epsilon_2) \left(1 + \epsilon_2 \frac{f''(x_2)}{f'(x_2)}\right)}{\left(1 + \frac{f''(x_2)}{2f'(x_2)}(\epsilon_2 + \epsilon)\right)} \\ &= \frac{\epsilon_2 \epsilon \frac{f''(x_2)}{2f'(x_2)}}{\left(1 + \frac{f''(x_2)}{2f'(x_2)}(\epsilon_2 + \epsilon)\right)} \\ &\approx \epsilon_2 \epsilon \frac{f''(x_2)}{2f'(x_2)} \end{aligned}$$

Now, assume $\epsilon_{i+1} = C\epsilon_i^r$ for all i where C and r are constants independent of i. Plugging this into the recurrence, we find:

$$\begin{aligned} C\epsilon_2^r &\approx \epsilon_2 \epsilon \frac{f''(x_2)}{2f'(x_2)} \\ \epsilon_2^{r-1} &\approx \epsilon \frac{f''(x_2)}{2Cf'(x_2)} \\ \epsilon_2 &\approx \epsilon^{\frac{1}{r-1}} \frac{f''(x_2)}{2Cf'(x_2)}^{\frac{1}{r-1}} \end{aligned}$$

So $C = \frac{f''(x_2)}{2Cf'(x_2)}^{\frac{1}{r-1}}$ and $r = \frac{1}{r-1}$ from the assumption. Solving for the positive r, we find the convergence rate is:

$$\begin{aligned} r &= \frac{1}{r-1} \\ r^2 - r - 1 &= 0 \\ r &= \frac{1 + \sqrt{5}}{2} \end{aligned}$$

Which is the Golden Ratio.

2 Method Implementations

3 Code and Info

3.1 Output

The program outputs the following:

3.2 Code