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Properties of logarithms

I. $\ln(xy) = \ln(x) + \ln(y)$

II. $\ln(1) = 0$

III. $\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$

IV. $\ln(x^b) = b\ln(x)$

Differentiate $\quad \frac{d}{dx}\ln(x) = \frac{1}{x}$

$$\ln\left(\frac{x-1}{x-2}\right) = \frac{1}{\left(\frac{x-1}{x-2}\right)} \cdot \frac{d}{dx}\left(\frac{x-1}{x-2}\right)$$

requires chain rule

Use property III instead

$$\frac{d}{dx}\left[\ln(x-1) - \ln(x-2)\right]$$

$$= \frac{d}{dx}\ln(x-1) - \frac{d}{dx}\ln(x-2)$$

$$\frac{1}{x-1}\frac{d}{dx}(x-1) - \frac{1}{x-2}\frac{d}{dx}(x-2)$$

$$\frac{1}{x-1} - \frac{1}{x-2}$$

Ex: Differentiate

property I

$$\ln[x(x-1)(x-2)] = \ln x + \ln(x-1) + \ln(x-2)$$

$$\frac{d}{dx}\left(\quad " \quad \right) = \frac{d}{dx}\left[\ln x + \ln(x-1) + \ln(x-2)\right]$$

$$= \frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-2}$$

Ex: Differentiate $\ln(2x)$

$$\frac{d}{dx}\ln(2x) = \frac{1}{2x}\cdot\frac{d}{dx}(2x)$$

$$= \frac{1}{2x}\cdot 2 = \frac{1}{x}$$

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

property I

Using I, $\ln(2x) = \ln(2) + \ln(x)$

$$\frac{d}{dx}\ln(2x) = 0 + \frac{1}{x} = \frac{1}{x} \checkmark$$

We can also use log, e to solve exponential and logarithmi equations

Ex: Solve

$$e^{5x} = 2$$

$$\ln(e^{5x}) = \ln 2$$

$$5x = \ln 2$$

$$x = \frac{\ln 2}{5}$$

Ex: Solve

$$\ln\left(\frac{1}{x^2}\right) = 4$$

$$e^{\ln\left(\frac{1}{x^2}\right)} = e^4$$

$$\frac{1}{x^2} = e^4$$

$$\left(\frac{1}{x}\right)^2 = \left(e^2\right)^2$$

$$\frac{1}{x} = \pm e^2$$

$$x = \pm \frac{1}{e^2}$$

5.1 Exponential Growth and Decay

Verbal and mathematical descriptions

Verbal definition: A quantity <u>grows or decays exponentially</u> if the rate of change of the quantity is proportional ~~to the quan~~ at every instant to the value of the quantity at that instant.

mathematical definition: Such a quantity satisfies:

$$\frac{d}{dx} y = ky$$

The constant k is called the <u>growth constant</u>.

We have seen that

$$\frac{d}{dx}e^{kx} = ke^{kx}.$$

$y = e^{kx}$ grows exponentially.

If $f = e^{kx}$, then $f(0) = 1$.

Let's multiply e^{kx} by a constant C.

General form for exponential growth.

$\boxed{y = Ce^{kx}}$, then $y(0) = C$.

$$\frac{dy}{dx} = C\frac{d}{dx}e^{kx} = Cke^{kx}$$

$$= ky$$

So y grows exponentially.

- These are the only functions that represent exponential growth.

C and k are called parameters.

~~at a rate proporti~~
at a rate proportional

Ex 1 A bacterial culture grows at a rate proportional to its size. At time $t=0$, about 20,000 bacteria are present. In 5 hours 400,000 bacteria. Determine a function that represents the size of the culture after t hours.

Let $P(t)$ be the number of bacteria in the culture at time t. Since P grows exponentially

$$\frac{d}{dt}P(t) = kP(t)$$

so $P(t) = Ce^{kt}$.

To determine P, we need to determine C, k.

The facts/data give:

$$P(0) = 20{,}000 \qquad P(5) = 400{,}000$$

$P(0) = C$

$\Rightarrow C = 20{,}000$

$P(t) = 20000e^{kt}$

$P(5) = 20{,}000e^{k\cdot 5} = 400{,}000$

$\Rightarrow e^{k\cdot 5} = 20$

$5k = \ln(20)$

$k = \dfrac{\ln 20}{5}$

So $P(t) = 20{,}000e^{\frac{\ln 20}{5}t}$

What is the initial rate of growth of the bacteria population?

$P'(t) = 20{,}000\, e^{\frac{\ln 20}{5}t} \cdot \dfrac{\ln 20}{5}$

$P'(0) = k \cdot P(0) = \dfrac{\ln 20}{5} \cdot 20{,}000e$

~~C and k are called parameters.~~

Ex: A colony of fruit flies is growing exponentially and the size of the population doubles in 9 days. Determine the growth constant.

~~Since~~ Let $P(t)$ be the size of the population.

Since the colony doubles in 9 days

$$P(9) = 2 \cdot P(0).$$

$$P(t) = Ce^{kt}$$

$$P(9) = Ce^{k \cdot 9} = 2Ce^{k \cdot 0} = 2 \cdot P(0)$$

$$Ce^{9k} = 2C$$

$$e^{9k} = 2$$

$$9k = \ln 2$$

$$\boxed{k = \frac{\ln 2}{9}}$$

The growth constant does not depend on the initial value.

5.1 Exercises 1, 5, 19