4.1 Exponential functions

$$f(x) = b^x$$

exponent (variable)

base

ex: $f(x) = 2^x$

$$f(1) = 2^1 = 2$$

$$f(0) = 2^0 = 1$$

$$f(4) = 2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$$

Laws of exponents

$$b^x \cdot b^y = b^{x+y}$$

$$\frac{b^x}{b^y} = b^x b^{-y} = b^{x-y}$$

$$a^x b^x = (ab)^x$$

$$b^{-x} = \frac{1}{b^x}$$

$$(b^y)^x = b^{xy}$$

$$\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x$$

Write the following ~~s~~ in the form $\underline{2^{kx}}$ for some k.

(a) $8^{x/3} \cdot 16^{3x/4}$ (b) $\frac{10^x}{5^x}$ (c) $4^{5x/2}$

(a) $8 = 2^3$

and $16 = 2^4$ so $8^{x/3} \cdot 16^{3x/4} = (2^3)^{x/3} \cdot (2^4)^{3x/4}$

$(\log_2 8 = 3)$

$(\log_2 16 = 4)$

$= 2^x \cdot 2^{3x}$

$= 2^{x+3x}$

$= 2^{4x}$

(b) $\frac{10^x}{5^x} = \left(\frac{10}{5}\right)^x = 2^x$

(c) $4 = 2^2$ so $4^{5x/2} = (2^2)^{5x/2}$

$(\log_2 4 = 2)$

$= 2^{5x}$

We can always write any exponential function b^x

as an exponential function with another base.

Write 9^x in the form 2^{kx}.

If $9 = 2^r$

then $\log_2 9 = r$.

(In general $y = b^x \iff \log_b y = x$)

$$9 = 2^{\log_2 9}$$

$$9^x = \left(2^{\log_2 9}\right)^x$$

$$= 2^{\log_2 9 \cdot x}$$

$$= 2^{kx} \qquad k = \log_2 9$$

Solving exponential equations

ex: Let $f(x) = 3^{5x}$. Find all x such that $f(x) = 27$.

$$27 = 3^3$$

$$f(x) = 27 \Rightarrow 3^{5x} = 3^3$$

$$\boxed{\text{If } b^r = b^s \text{ then } r = s}$$

$$5x = 3$$

$$x = \frac{3}{5}$$

$$3^{5x} = 27$$

$$\log_3(3^{5x}) = \log_3(27)$$

$$5x = 3$$

$$x = \frac{3}{5}.$$

$$\log_b y = x \iff b^x = y$$

The logarithm gives the exponent (x) to which we must raise b to get y.

ex: $2^3 = 8$

$$\log_2 8 = 3$$

4.2 The expontential function

Recall: The solution the difference equation for compound interest

$$y_{n+1} = a y_n + b$$

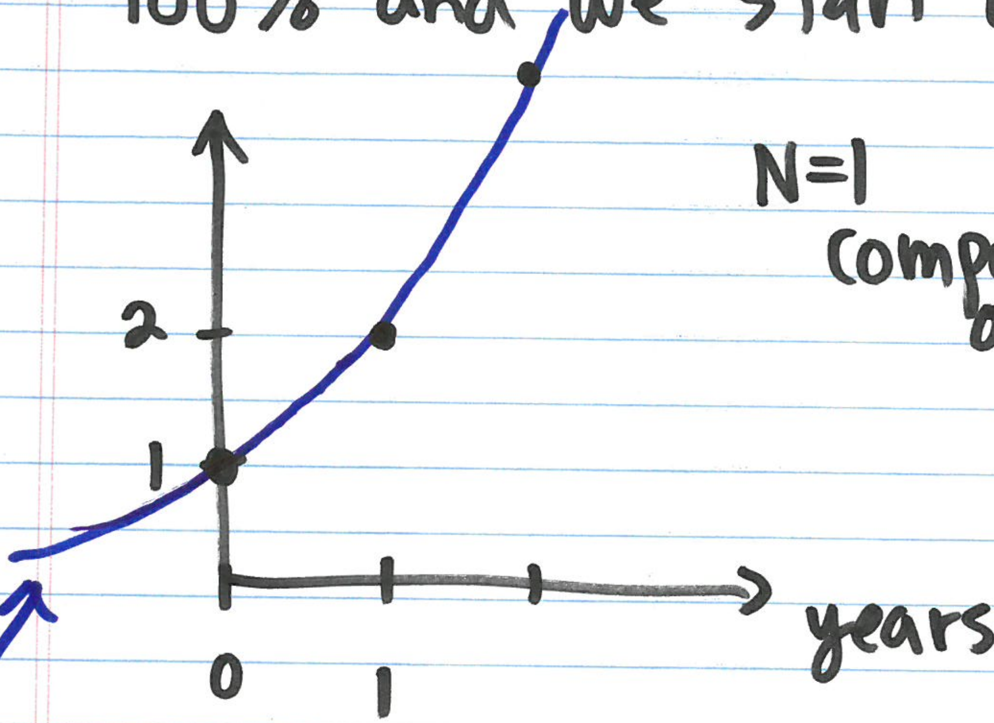
is $$y_n = \frac{b}{1-a} + \left(y_0 - \frac{b}{1-a}\right)a^n$$

If $b=0$, we have

$$y_{n+1} = a y_n$$

and $y_n = y_0 \cdot a^n$

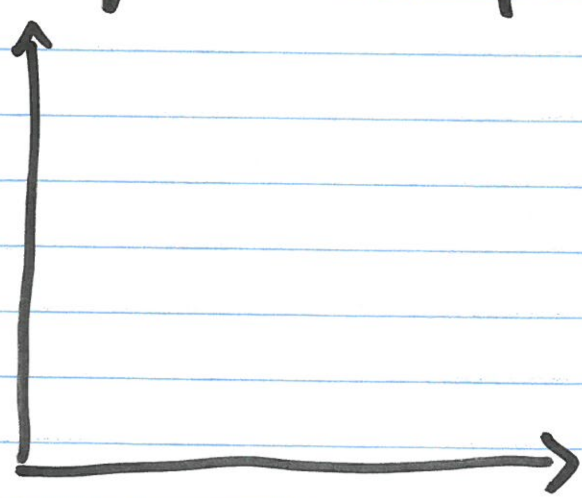
Suppose the interest rate is 100% and we start with \$1.



$N=1$
compounded annually

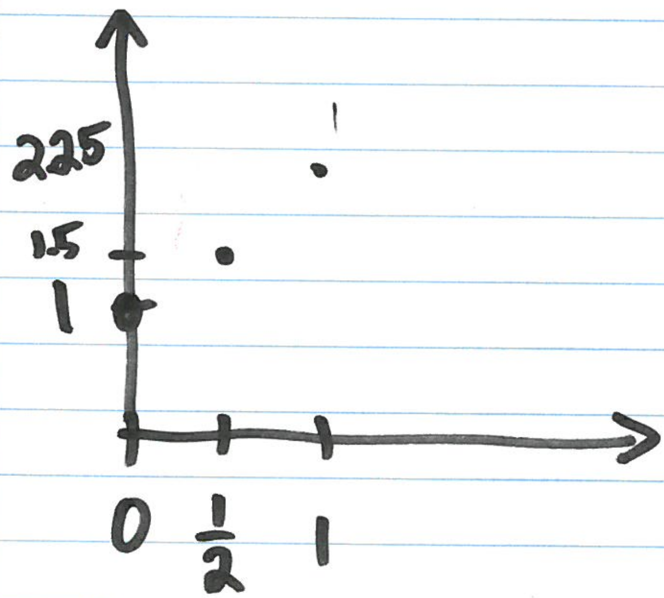
We are going to find such a function

Is there a function that passes through these points?

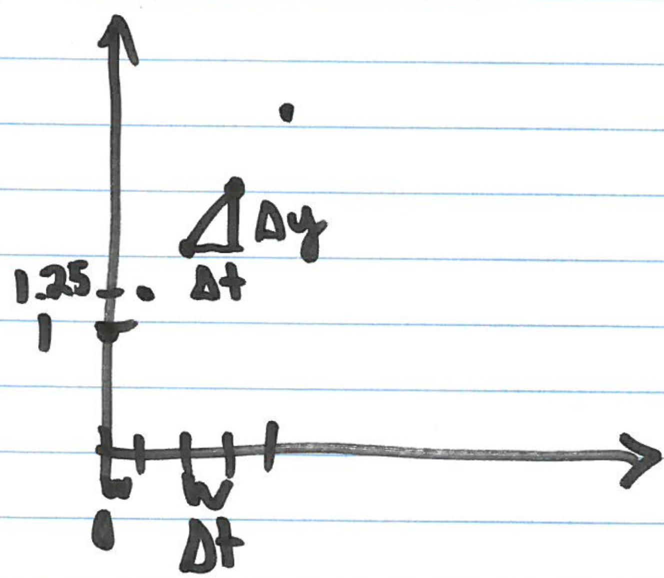


Motivation for finding such a function

Let N be the number of times interest is compounded.



$N=2$ compounded semiannually



$N=4$ compounded quarterly

$$y_{n+1}-y_n=\frac{1}{N}y_n$$

$$y_{n+1}-y_n=\Delta t\, y_n$$

$$\frac{\Delta y}{\Delta t}=y_n$$

When ~~the length of number of times~~ N gets larger Δt gets smaller and the average rate of change is computed over a smaller period.

The function $f(x) = e^x$ (the derivative equals itself

is the function defined by

$$\boxed{f'(x) = f(x)}.$$

This is called a differential equation.

The base of this exponential function is

$$e = \lim_{N\to\infty}\left(1+\frac{1}{N}\right)^N = 2.718281828\ldots$$

4.3 Derivatives of exponential functions

$$\boxed{\frac{d}{dx}e^x = e^x}$$

The derivative of the exponential function e^x equals the exponential function e^x.

Using this fact ~~we can~~ combined with the chain rule we can differentiate

$$e^{g(x)}$$

Let $f(x) = e^x$, then $e^{g(x)} = f(g(x))$

By the chain rule

$$\frac{d}{dx} e^{g(x)} = f'(g(x)) \cdot g'(x)$$

$$= e^{g(x)} \cdot g'(x)$$

(since $f'(x) = \frac{d}{dx} e^x = e^x$)

ex: Differentiate

$$y = e^{x^2+1}$$

$$g(x) = x^2+1$$

$$g'(x) = 2x$$

$$\frac{d}{dx}\left(e^{x^2+1}\right) = e^{x^2+1} \cdot 2x$$

$$= 2xe^{x^2+1}$$

Differentiate $y = e^{5x}$

$$\frac{d}{dx}e^{5x} = e^{5x}\cdot 5$$

$$= 5e^{5x}$$

More generally

$$\frac{d}{dx}e^{kx} = \cancel{k}e^{kx}\cdot\frac{d}{dx}(kx)$$

$$= e^{kx}\cdot k$$

$$= ke^{kx}$$

Exercises 4.3 #1, 5, 9, 15