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Difference equation

$$y_{n+1} - y_n = y_n$$

Solution

$$y_n = 2^n y_0$$

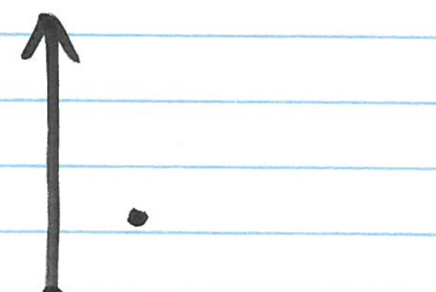
Differential eqn

$$\frac{d}{dx} y = y$$

Solution

$$y = e^x$$

$$e = 2.718281828$$



$$\boxed{\frac{d}{dx} e^x = e^x}$$

Chain rule

$$\frac{d}{dx} f(g(x)) \neq f'(g(x))$$

The derivative of the composite function $f(g(x))$

The derivative of f, evaluated at $g(x)$

Chain rule: $$\boxed{\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)}$$

Example:

Let $f(x) = e^x$.

We know $\frac{d}{dx} e^x = e^x$

$$f'(x) = f(x)$$

It does not follow that

$$\cancel{\frac{d}{dx} f(3x) = f'(3x)}$$

Find $\frac{d}{dx} e^{3x} \neq e^{3x}$

Rather, by the chain rule

$$\frac{d}{dx} e^{3x} = \frac{d}{dx} f(3x) = f'(3x) \cdot \frac{d}{dx}(3x)$$

= Derivative of the outer function evaluated at $3x$; times $\frac{d}{dx}(3x)$

$$= e^{3x} \cdot \frac{d}{dx}(3x)$$

$$= 3e^{3x}$$

Differentiate $f(x) = \sqrt{e^{x/2}+1}$

$$f(x) = (e^{x/2}+1)^{1/2}$$

$$f'(x) = \frac{1}{2}(e^{x/2}+1)^{-1/2} \cdot \frac{d}{dx}(e^{x/2}+1)$$

$$= \frac{1}{2}(e^{x/2}+1)^{-1/2}\left[e^{x/2} \cdot \frac{d}{dx}\left(\frac{x}{2}\right)\right]$$

$$= \frac{1}{2}(e^{x/2}+1)^{-1/2} \cdot \left[\frac{1}{2}e^{x/2}\right]$$

Differentiate $f(t) = t \cdot e^{-t}$

By the product rule

$$f'(t) = t \cdot \frac{d}{dt} e^{-t} + e^{-t} \cdot \frac{d}{dt} t$$

$$= t\, e^{-t} \cdot \frac{d}{dt}(-t) + e^{-t} \cdot 1$$

$$= -te^{-t} + e^{-t}$$

$$= (1-t)e^{-t}$$

Differentiate $f(x) = \dfrac{e^{2x}-1}{e^{2x}+1}$

By the quotient rule

$$f'(x) = \frac{(e^{2x}+1)\frac{d}{dx}(e^{2x}-1) - (e^{2x}-1) \cdot \frac{d}{dx}(e^{2x}+1)}{(e^{2x}+1)^2}$$

$$= \frac{(e^{2x}+1)\cdot e^{2x}\cdot 2 - (e^{2x}-1)e^{2x}\cdot 2}{(e^{2x}+1)^2}$$

$$= \frac{\cancel{2e^{4x}} + 2e^{2x} - \cancel{2e^{4x}} + 2e^{2x}}{(e^{2x}+1)^2}$$

$$= \frac{4e^{2x}}{(e^{2x}+1)^2}$$

Exercises 4.3 # 1, 5, 9, 15

Any exponential function $f(x) = b^x$ $(b>0)$ can be written as an exponential function with base e or any other base by introducing a constant R.

$$b^x = a^{kx}$$

Ex $8^x = (2^3)^x = 2^{3x}$ $k=3$

The constant k is related to the logarithm

$$\log_b(y) = x \iff b^x = y$$

$$\log_2(8) = 3$$

$$\log_e(x) = \ln x$$

$$\ln(y) = x \iff e^x = y$$

$\log_e(y)$

Another way to say this fact is to say that e^x and $\ln(x)$ are inverse functions

$$\ln(e^x) = x$$

$$e^{\ln(x)} = x$$

Ex: Represent $f(x) = 2^x$ in the form e^{kx}

$$2^x = e^{kx}$$

$$(2)^x = (e^k)^x$$

We can see that if $2 = e^{k}$ we get what we want.

$$\ln 2 = k$$

$$2^x = e^{\ln 2 \cdot x} \qquad k = \ln 2$$

4.5 - The derivative of the natural logarithm

By differentiating both sides of

$$e^{\ln(x)} = x,$$

we find

$$e^{\ln(x)} \cdot \frac{d}{dx} \ln(x) = 1$$

$$\frac{d}{dx}\ln(x) = \frac{1}{e^{\ln(x)}}$$

$$\boxed{\frac{d}{dx}\ln(x) = \frac{1}{x}}$$

Example: Differentiate

(a) $y = (\ln x)^5$ (b) $y = x \cdot \ln x$

(c) $y = \ln(x^3 + 5x^2 + 8)$

(a) $$\frac{dy}{dx} = 5(\ln x)^4 \cdot \frac{d}{dx}\ln x$$

$$= 5(\ln x)^4 \cdot \frac{1}{x}$$

(b) $$\frac{dy}{dx} = x\frac{d}{dx}\ln x + \ln x \cdot \frac{d}{dx}x$$

$$= x \cdot \frac{1}{x} + \ln x$$

$$= 1 + \ln x$$

(c) $\dfrac{dy}{dx} = \dfrac{1}{x^3+5x^2+8} \cdot \dfrac{d}{dx}(x^3+5x^2+8)$

$$= \frac{3x^2+10x}{x^3+5x^2+8}$$

$$\frac{d}{dx} \ln\left(\frac{x-1}{x-2}\right) = \frac{1}{\left(\frac{x-1}{x-2}\right)} \cdot \frac{d}{dx}\left(\frac{x-1}{x-2}\right)$$

This can be done more easily by using the fact

$$\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$$

Exercises: 4.5 #1, 3, 5, 17, 19