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Represent cost of a cylinder where the base and top cost \$2/ft^2 and the side costs \$3/ft^2.

surface area of each part:

top: πr^2

bottom: πr^2

side: $h \cdot 2\pi r$

cost: $2 \cdot \pi r^2 + 2 \cdot \pi r^2 + 3 \cdot 2\pi r h$

volume of a cylinder:

area of base $\cdot$ height

$\pi r^2 \cdot h$

3.1 Rules of Differentiation

Quotient rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

The negative sign goes with the derivative of the denominator.

$$\frac{f(x)}{g(x)} = f(x)[g(x)]^{-1}$$

$$\frac{d}{dx}[g(x)]^{-1} = \underbrace{-1[g(x)]^{-2} \cdot g'(x)}$$

<u>Ex</u>: Find $\frac{dy}{dx}$ where $y = \frac{x^3}{(x^2+1)^4}$

Let $f(x) = x^3$, $g(x) = (x^2+1)^4$

$$\frac{d}{dx}\left[\frac{x^3}{(x^2+1)^4}\right] = \frac{(x^2+1)^4 \frac{d}{dx}(x^3) - x^3 \frac{d}{dx}(x^2+1)^4}{(x^2+1)^8}$$

$$= \frac{(x^2+1)^4 \cdot 3x^2 - x^3 4(x^2+1)^3 \cdot 2x}{(x^2+1)^8}$$

$$= \frac{\cancel{(x^2+1)^3}\left[(x^2+1)\cdot 3x^2 - x^3\cdot 4\cdot 2x\right]}{\cancel{(x^2+1)^3}(x^2+1)^5}$$

$$= \frac{(x^2+1)\cdot 3x^2 - 8x^4}{(x^2+1)^5}$$

$$= \frac{3x^4+3x^2-8x^4}{(x^2+1)^5}$$

$$= \frac{-5x^4+3x^2}{(x^2+1)^5}$$

Ex: Differentiate $f(x)=\sqrt{\frac{x^2+7}{x+1}}$

$$f(x)=\left(\frac{x^2+7}{x+1}\right)^{1/2}$$

(1) $$f'(x)=\frac{1}{2}\left(\frac{x^2+7}{x+1}\right)^{-1/2}\cdot\frac{d}{dx}\left(\frac{x^2+7}{x+1}\right)$$

$$\frac{d}{dx}\left(\frac{x^2+7}{x+1}\right)=\frac{(x+1)\frac{d}{dx}(x^2+7)-(x^2+7)\frac{d}{dx}(x+1)}{(x+1)^2}$$

$$= \frac{(x+1)\cdot 2x - (x^2+7)(1)}{(x+1)^2}$$

$$= \frac{2x^2+2x-x^2-7}{(x+1)^2}$$

$$= \frac{x^2+2x-7}{(x+1)^2}$$

Plugging this into (1)

$$f'(x) = \frac{1}{2}\left(\frac{x^2+7}{x+1}\right)^{-1/2} \cdot \frac{x^2+2x-7}{(x+1)^2}$$

$$\left(\frac{a}{b}\right)^{-r} = \left(\frac{b}{a}\right)^{r}$$

$$= \frac{1}{2}\left(\frac{(x+1)}{x^2+7}\right)^{1/2} \cdot \frac{x^2+2x-7}{(x+1)^2}$$

$$= \frac{1}{2}\,\frac{x^2+2x-7}{(x^2+7)\,(x+1)^{3/2}}$$

Chain rule

composition of functions

$f \circ g$ is the composition of f and g

$f(g(x))$.

example: If $f(x) = \frac{x-1}{x+1}$ and $g(x) = x^3$, what is $f(g(x))$?

$$f(g(x)) = \frac{x^3-1}{x^3+1}.$$

prerequisite for chain: write a given expression as a composition of two functions.

a) $h(x) = (x^5+9x+3)^8$

b) $k(x) = \sqrt{4x^2+1}$

a) $h(x) = f(g(x))$ where $f(x) = x^8$ and $g(x) = x^5+9x+3$

b) $k(x) = f(g(x))$ where $f(x) = \sqrt{x}$

$g(x) = 4x^2+1$

Note: It is not so obvious how to write $h(x) = \frac{x^3 - 1}{x^3 + 1}$ as
(from earlier example)
the composition of two functions.

Chain rule

$$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)$$

Ex: Find $\frac{d}{dx}\left[f(g(x))\right]$ where

$f(x) = x^8$ and $g(x) = x^5 + 9x + 5$

$f'(x) = 8x^7$

$g'(x) = 5x^4 + 9$

$$\frac{d}{dx}\left[f(g(x))\right] = 8(x^5 + 9x + 5)^7 \cdot (5x^4 + 9)$$

Note: $\frac{d}{dx}\left[(x^5 + 9x + 5)^8\right]$

Note: When $f(x) = x^r$ $f'(x) = rx^{r-1}$ and the chain rule is exactly the general power rule.

If $h(x) = f(\sqrt{x})$ find $h'(x)$ in terms of f'

$h(x) = f(g(x))$ where $g(x) = \sqrt{x} = x^{1/2}$

$h'(x) = \frac{d}{dx}\left[f(g(x))\right]$ $\quad g'(x) = \frac{1}{2}x^{-1/2}$

$= f'(g(x)) \cdot \underbrace{g'(x)}$ $\quad$ chain rule

$= f'(\sqrt{x}) \cdot \frac{1}{2}x^{-1/2}$

$= \dfrac{f'(\sqrt{x})}{2\sqrt{x}}$