1. (9 points) A student takes out a \$12000 loan that charges an annual interest rate of 8%, compounded monthly, and makes monthly payments of \$100.

   (a) Write a difference equation that describes how to compute the balance each month based on the balance of the previous month.

   Solution: The interest rate per period is $i = 0.08/12$

$$\text{change in account balance} = \text{interest earned during month} - \text{monthly payment}$$

$$y_{n+1} - y_n = \frac{0.08}{12} y_n - 100$$

   This can be written in standard form $y_{n+1} = ay_n + b$ as:

$$y_{n+1} = \left(1 + \frac{0.08}{12}\right) y_n - 100$$

   (b) How much will the student owe after 12 years?

   Solution: We have $a = 1 + 0.08/12$, $b = -100$, $\frac{b}{1-a} = \frac{-100}{-0.08/12} = 15000$, $y_0 = 12000$, so the solution to the difference equation is

$$y_n = 15000 + (12000 - 15000)(1 + 0.08/12)^n$$

   After 12 years, or 144 months, the balance will thus be

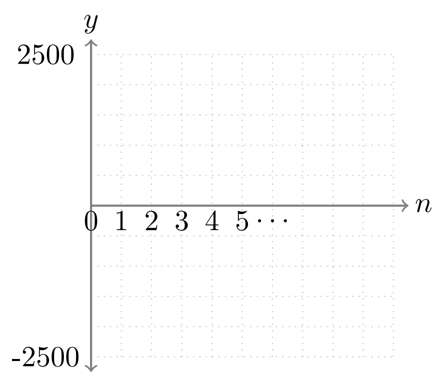
$$\begin{aligned} y_{144} &= 15000 + (12000 - 15000)(1 + 0.08/12)^{144} \\ &\approx 7189.83 \text{ dollars.} \end{aligned}$$

2. (5 points) On the coordinate grid to the right, sketch the graph of the difference equation $y_{n+1} = 1.05y_n - 100$, with initial value $y_0 = 500$. The points do not need to be exact, but your graph should correctly show the characteristics of the solution to the difference equation.

   Solution: We have $a = 1.05$, $b = -100$, $\frac{b}{1-a} = 2000$, $y_0 = 500$.

   Since $y_0 \neq \frac{b}{1-a}$ the solution is not constant. Since $|a| > 1$ the solution is unbounded. Since $a > 0$ the solution is monotonic.

   The points on the graph should be discrete, monotonic, and getting farther from the line $y = 2000$ as n increases. Since the graph starts below $y = 2000$, in order to get farther from this line, it must be monotonically decreasing.



3. (25 points) Find the first and second derivatives of the following functions:

   (a) $f(x) = e^{-3x}$

Solution:

$$f'(x) = -3e^{-3x}$$
$$f''(x) = 9e^{-3x}$$

(b) $f(x) = \frac{1}{x}$
Solution:

$$f'(x) = -\frac{1}{x^2}$$
$$f''(x) = \frac{2}{x^3}$$

(c) $f(x) = (4x+1)^{\frac{3}{2}}$
Solution:

$$f'(x) = 6(4x+1)^{1/2}$$
$$f''(x) = 12(4x+1)^{-1/2}$$

(d) $f(x) = \pi + 2x$
Solution:

$$f'(x) = 2$$
$$f''(x) = 0$$

(e) $f(x) = x \ln x$
Solution:

$$f'(x) = \ln x + 1$$
$$f''(x) = \frac{1}{x}$$

4. (20 points) Find the following integrals: (Hint: use substitution, if necessary)

(a) $\int_0^3 e^{\frac{x}{4}} dx$
Solution:

$$\int_0^3 e^{\frac{x}{4}} dx = 4e^{x/4}\Big|_0^3$$
$$= 4e^{3/4} - 4$$

(b) $\int (\frac{2}{x} + \sqrt{x} + \frac{1}{x^3}) dx$
Solution:

$$\int (\frac{2}{x} + \sqrt{x} + \frac{1}{x^3}) dx = 2 \ln|x| + \frac{2}{3} x^{3/2} - \frac{1}{2x^2} + C$$

(c) $\int_1^2 2xe^{x^2} dx$

Solution: Let $u = x^2$. Then $du = 2xdx$. So

$$\begin{aligned}\int_1^2 2xe^{x^2}dx &= \int_1^4 e^u du \\ &= e^u\Big|_1^4 \\ &= e^4 - e\end{aligned}$$

(d) $\int x^2 \frac{1}{x^3+1} dx$

Solution: Let $u = x^3 + 1$. Then $du = 3x^2 dx$. So $x^2 dx = \frac{1}{3} du$

$$\begin{aligned}\int x^2 \frac{1}{x^3+1} dx &= \int \frac{1}{3}\frac{1}{u} du \\ &= \frac{1}{3}\ln|u| + C \\ &= \frac{1}{3}\ln|x^3+1| + C\end{aligned}$$

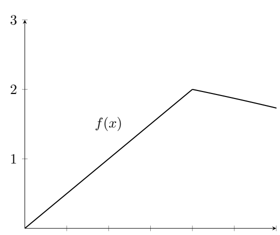
5. (5 points) For each of the following functions, find $\lim_{x\to 2} f(x)$, and say whether or not the function is continuous at $x = 2$.

(a) $f(x) = \begin{cases} \frac{2x^2-8}{x^2-2x} & x \neq 2 \\ 3 & x = 2 \end{cases}$

Solution:

$$\begin{aligned}\lim_{x\to 2} f(x) &= \lim_{x\to 2} \frac{2x^2-8}{x^2-2x} \\ &= \lim_{x\to 2} \frac{2(x-2)(x+2)}{x(x-2)} \\ &= \lim_{x\to 2} \frac{2(x+2)}{x} \\ &= 4\end{aligned}$$

$\lim_{x\to 2} f(x) \neq 3 = f(2)$, so f is not continuous at $x = 2$.

(b) 

Solution: $\lim_{x\to 2} f(x) = 2$. f is continous at $x = 2$.

6. (10 points) During a heavy downpour, a room in a building becomes flooded with water. Suppose $f(t)$ represents the height of the water line (in inches) above the floor after t hours. Suppose $f(1) = 3$ and $f'(1) = 0.5$.

(a) Estimate $f(1.2)$.

Solution (Tangent line approximation):

$$\begin{aligned} f(1.2) &\approx f(1) + f'(1) \cdot 0.2 \\ &= 3 + 0.5 \cdot 0.2 \\ &= 3.1 \end{aligned}$$

Further explanation: If we start with the formula for the average rate of change of f between $t = 1$ and $t = 1.2$, we have:

$$\begin{aligned} [\text{The average rate of change of } f \text{ between } t = 1 \text{ and } t = 1.2] &= \frac{f(1.2) - f(1)}{1.2 - 1} \\ &= \frac{f(1.2) - f(1)}{0.2} \end{aligned}$$

The average rate of change of f between $t = 1$ and $t = 1.2$, in the equation above, can be approximated by the instantaneous rate of change of f at $t = 1$ (the slope of the tangent line at $t = 1$). Using this approximation, we can replace the average rate of change above by $f'(1)$.

$$f'(1) \approx \frac{f(1.2) - f(1)}{0.2}$$

Then the approximate value in the solution above can be understood as the value we get by solving this equation for $f(1.2)$. This method of approximation is called *tangent line approximation*.

(b) Suppose that $f(1) = 3$, $f'(1) = 0.5$, and $f''(1) < 0$. Then which of the following must be true? Circle all that apply.

A. f is increasing at $t = 1$.
B. f is decreasing at $t = 1$.
C. f is concave down at $t = 1$.
D. f is concave up at $t = 1$.
E. f' is increasing $t = 1$.
F. f' is decreasing $t = 1$.

Solution: A,C,F

7. (5 points) A biochemical reaction is set up to break down x grams of starch molecules into simple sugars. The rate at which sugar is produced can be described by the function

$$v(x) = \frac{0.003x}{57 + x},$$

where x is the amount of starch, in grams. Find $v'(5)$.

Solution:

$$
\begin{aligned}
v'(x) &= \frac{(57+x)0.003 - 0.003x}{(57+x)^2} \\
v'(5) &= \frac{(62)0.003 - 0.015}{62^2} \\
&= 0.0000444
\end{aligned}
$$

8. (10 points) The population of a bacterial culture grows exponentially.

   (a) Represent the population at time t by a function using the general form for exponential growth or decay.
   Solution: $f(t) = Ce^{kt}$

   (b) If the initial population was $1,000$, and the initial rate of change was 200 per hour, what will the population be after 10 hours?
   Solution:

$$
\begin{aligned}
f(t) &= 1000e^{kt} \\
f'(t) &= 1000ke^{kt} \\
f'(0) &= 1000k = 200 \\
\Rightarrow k &= 0.2. \\
f(t) &= 1000e^{0.2t} \\
f(10) &= 1000e^2 \\
&\approx 7389
\end{aligned}
$$

   The population will be 7389 bacteria.

   (c) At what rate will the population be increasing at that time?
   Solution:

$$
\begin{aligned}
f'(t) &= 200e^{0.2t} \\
f'(10) &= 200e^2 \\
&= 0.2 \cdot 7389 \\
&\approx 1478
\end{aligned}
$$

   The population will be increasing at about 1478 bacteria per hour.

9. (10 points) Sketch the graph of $f(x) = 2x^3 + 3x^2 + 1$ in the cartesian coordinate grid below. Use information about the derivatives to find the locations of any relative minima/maxima and inflection points, and state their coordinates. You do not have to find the x-intercepts.

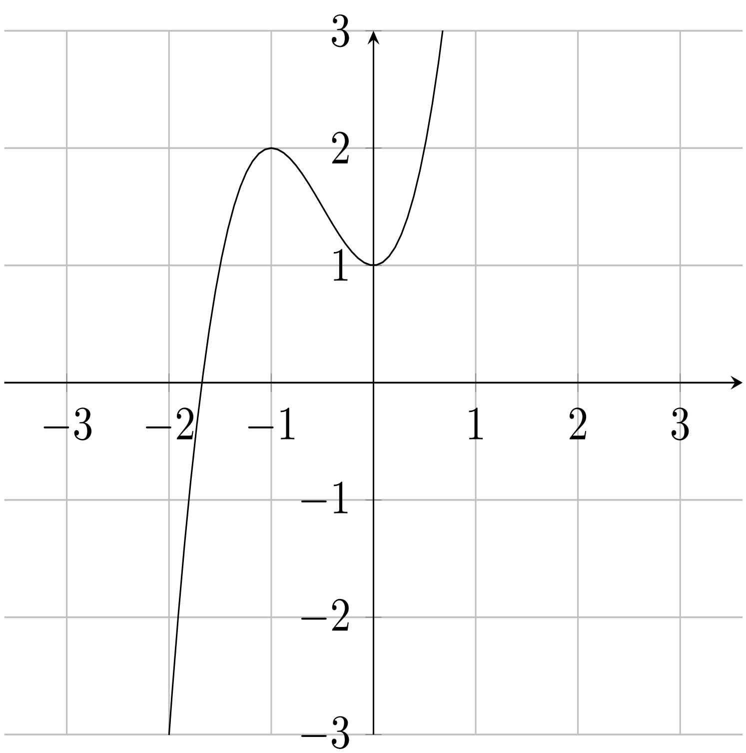
   Solution:

$$\begin{aligned}
f'(x) &= 6x^2 + 6x \\
f''(x) &= 12x + 6 \\
f'(x) &= 0 \Rightarrow 6x(x+1) = 0 \Rightarrow x = 0 \text{ or } x = -1 \\
f''(0) &= 6 > 0, \text{ so } x = 0 \text{ is a local min.} \\
f''(-1) &= -6 < 0 \text{ so } x = -1 \text{ is a local max.} \\
f(0) &= 1 \\
f(-1) &= 2
\end{aligned}$$

So $(0, 1)$ is a relative minimum point, and $(-1, 2)$ is a relative maximum point.

$$\begin{aligned}
f''(x) &= 12x + 6 = 0 \Rightarrow x = -\frac{1}{2}. \\
f''(x) &< 0 \text{ for } x < -\frac{1}{2}, \text{ and} \\
f''(x) &> 0 \text{ for } x > -\frac{1}{2}. \\
f(-\frac{1}{2}) &= -\frac{1}{4} + \frac{3}{4} + 1 = \frac{3}{2}
\end{aligned}$$

So $(-\frac{1}{2}, \frac{3}{2})$ is an inflection point.



10. (10 points) Find the area bounded by the graph of $f(x) = 3x^3 + 3x^2 - 6x$ and the x-axis, between $x = -2$ and $x = 1$.

Solution:

$$\begin{aligned}
&f(x) = 0 \\
&\quad\Rightarrow 3x^3 + 3x^2 - 6x = 3x(x^2 + x - 2) = 3x(x+2)(x-1) \\
&\quad\Rightarrow x = -2, x = 0, \text{ or } x = 1.
\end{aligned}$$

$f(x) > 0$ for $-2 < x < 0$, and $f(x) < 0$ for $0 < x < 1$. So the area is

$$\begin{aligned}
A &= \int_{-2}^{0} f(x)dx + \int_{0}^{1} -f(x)dx \\
&= \int_{-2}^{0} 3x^3 + 3x^2 - 6x - \int_{0}^{1} 3x^3 + 3x^2 - 6x \\
&= \frac{3}{4}x^4 + x^3 - 3x^2 \Big|_{-2}^{0} - \left(\frac{3}{4}x^4 + x^3 - 3x^2 \Big|_{0}^{1}\right) \\
&= 0 - [12 - 8 - 12] - ([\frac{3}{4} + 1 - 3] - 0) \\
&= 8 + 2 - \frac{3}{4} \\
&= 9.25
\end{aligned}$$

11. (10 points) You would like to build a wooden crate with 4 sides, a square base, and no top, with the minimum amount of wood possible. The volume of the box is to be 4 cubic feet. What dimensions will minimize the amount of wood you need? (Hint: surface area)

Solution: Let w be the width of the base, and let h be the height of the box. Then

$$\begin{aligned}
V &= w^2 h \\
A &= w^2 + 4hw
\end{aligned}$$

Since the volume must be 4 cubic feet,

$$w^2 h = 4 \Rightarrow h = \frac{4}{w^2}.$$

So

$$\begin{aligned}
A = w^2 + \frac{16}{w} \frac{dA}{dw} &= 2w - \frac{16}{w^2} = 0 \\
&\Rightarrow w^3 = 8 \\
&\Rightarrow w = 2.
\end{aligned}$$

Thus, $h = \frac{4}{w^2} = 1$. So the width is 2 feet, and the height is 1 foot.

12. (10 points) Find the volume of the solid of revolution obtained by rotating the region under the graph of $f(x) = 2 + x$ about the x-axis from $x = 0$ to $x = 2$.

Solution: The volume is

$$\begin{aligned} V &= \int_0^2 \pi[f(x)]^2 dx \\ &= \int_0^2 \pi(2+x)^2 dx \end{aligned}$$

Let $u = 2 + x$. Then $du = dx$. So

$$\begin{aligned} V &= \int_0^2 \pi(2+x)^2 dx \\ &= \int_2^4 \pi u^2 du \\ &= \pi \frac{u^3}{3}\bigg|_2^4 \\ &= \pi(\frac{4^3}{3} - \frac{2^3}{3}) \\ &= \frac{56\pi}{3}. \end{aligned}$$

13. (10 points) Use the midpoint or trapezoidal rule with $n = 4$ to approximate the value of the following integral:

$$\int_0^2 xe^{-x} dx.$$

Solution using trapezoidal rule: With $n = 4$ the width of each interval is $\Delta x = \frac{2-0}{4} = \frac{1}{2}$, so the endpoints of the 4 subintervals are $a_0 = 0, a_1 = 0.5, a_2 = 1, a_3 = 1.5$, and $a_4 = 2$. By the trapezoidal rule, the integral can be approximated by the sum:

$$\begin{aligned} T &= \Big(f(a_0) + 2f(a_1) + 2f(a_2) + 2f(a_3) + f(a_4)\Big)\frac{\Delta x}{2} \\ &= \big(0 + 2 \cdot 0.5e^{-0.5} + 2 \cdot 1e^{-1} + 2 \cdot 1.5e^{-1.5} + 2e^{-2}\big) \cdot \frac{1}{4} \\ &= \frac{1}{4}(e^{-0.5} + 2e^{-1} + 3e^{-1.5} + 2e^{-2}) \\ &\approx 0.5705876 \end{aligned}$$

Solution using midpoint rule: With $n = 4$ the width of each interval is $\Delta x = \frac{2-0}{4} = \frac{1}{2}$, so the midpoints of the 4 subintervals are $a_1 = 0.25, a_2 = 0.75, a_3 = 1.25$, and $a_4 = 1.75$. By the midpoint rule, the integral can be approximated by the sum:

$$\begin{aligned} M &= \Big(f(x_1) + f(x_2) + f(x_3) + f(x_4)\Big)\Delta x \\ &= \big(0.25e^{-0.25} + 0.75e^{-0.75} + 1.25e^{-1.25} + 1.75e^{-1.75}\big) \cdot \frac{1}{2} \\ &\approx 0.6056053 \end{aligned}$$