3/21 Review

Chain rule

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$

Example:

$$\frac{d}{dx} e^{kx} = e^{kx} \cdot k$$

$$\boxed{\frac{d}{dx} e^{x} = e^{x}}$$

outer function: $e^x = f(x)$

$$f'(x) = \frac{d}{dx} e^x = e^x$$

inner function: $kx = g(x)$

$$g'(x) = k$$

Example:

$$\frac{d}{dx} e^{x} = e^{x} \cdot \frac{d}{dx} x$$

$g(x) = x$
$g'(x) = 1$

$$= e^{x} \cdot 1$$

The chain rule always applies, but the factor $g'(x)$ is 1 if $g(x) = x$.

Differentiate

$$\ln(e^{2x}+1)$$

$$\boxed{\begin{aligned} \frac{d}{dx}\ln x &= \frac{1}{x} \\ \frac{d}{dx}e^x &= e^x \end{aligned}}$$

$$\frac{d}{dx}\ln(e^{2x}+1) = \frac{1}{e^{2x}+1}\cdot\frac{d}{dx}(e^{2x}+1)$$

$$= \frac{1}{e^{2x}+1}\,e^{2x}\cdot 2$$

$$\frac{d}{dx}\ln(x^2\cdot e^{2x}) = \frac{1}{x^2\cdot e^{2x}}\frac{d}{dx}(x^2\cdot e^{2x})$$

$$\cancel{= \frac{1}{x^2\cdot e^{2x}}\, x^2\cdot\frac{d}{dx}e^{2x}}$$

$$= \frac{1}{x^2\cdot e^{2x}}\cdot\left(x^2\cdot\frac{d}{dx}e^{2x} + e^{2x}\cdot\frac{d}{dx}x^2\right)$$

$$= \frac{x^2\cdot 2e^{2x} + e^{2x}\cdot 2x}{x^2\cdot e^{2x}}$$

$$= \frac{\cancel{x^2} \cdot 2\cancel{e^{2x}}}{\cancel{x^2 e^{2x}}} + \frac{\cancel{e^{2x}} \cdot 2x}{x^2 \cdot \cancel{e^{2x}}}$$

$$= 2 + \frac{2}{x}$$

We can use laws of logarithms to simplify this problem:

$$\boxed{\begin{aligned} &\ln(x^2 e^{2x}) \\ &= \ln(x^2) \\ &\quad + \ln(e^{2x}) \end{aligned}}$$

$$\frac{d}{dx}\left[\ln(x^2) + \ln(e^{2x})\right]$$

$$\frac{d}{dx}\left[2\ln x + 2x\right]$$

$$\frac{2}{x} + 2$$

Chain rule word problem

A coral grows to cover a circular region of the ocean floor. If the radius of the region is 2 meters and grows at 0.02 meters per year, how fast does the area covered increases?

We have a quantity, area A, which depends on the radius r.

$$A = \pi r^2, \quad \frac{dA}{dr} = A'(r) = 2\pi r$$

We also have that r depends on time

$$r(t_0) = 2 \quad r'(t_0) = 0.02$$

where t_0 is the time in the problem.

We want to know

$$\left.\frac{d}{dt} A(r(t))\right|_{t=t_0}$$

$$\frac{d}{dt} A(r(t)) = A'(r(t)) \cdot r'(t)$$

$$\left.\frac{d}{dt} A(r(t))\right|_{t=t_0} = \left.A'(r(t)) \cdot r'(t)\right|_{t=t_0}$$

$$= A'(r(t_0)) \cdot r'(t_0)$$

$$= 2\pi \cdot 2 \cdot 0.02$$

$$= 0.08\pi \ m^2/\text{year}$$

An insect population grows exponentially. In 10 days the population grows from 100 insects to 900. How large will the pop. be after 15 ~~years~~ days. What will the rate of growth be at that time.

$$P(t) = Ce^{kt}$$

$$P(0) = 100 = C$$

$$P(t) = 100e^{kt}$$

$$P(10) = 900 = 100e^{10k}$$

$$9 = e^{10k}$$

$$\ln 9 = 10k$$

$$k = \frac{\ln 9}{10}$$

$$P(15) = 100e^{\frac{\ln 9}{10}\cdot 15} \approx$$

$$P'(t) = k\cdot 100e^{kt}$$

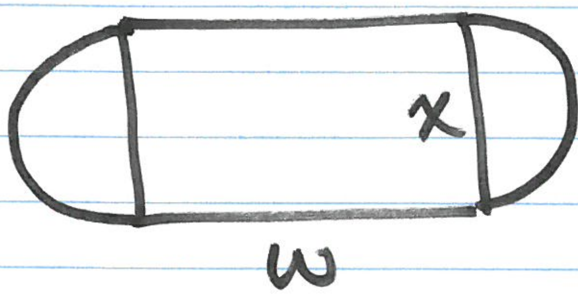
$$= \frac{\ln 9}{10}\cdot 100e^{\frac{\ln 9}{10}t}$$

$$P'(15) = 10\ln 9\, e^{\frac{\ln(9)\cdot 15}{10}} \approx$$

Optimization Practice

2.6 #11, 13, 19

An athletic field consists of a rectangular region at each end. The perimeter will be used for a 440-yard track. (See the diagram below). Find the value of x that maximizes the area of the rectangular region.



Let $A = x \cdot w$ be the area of the rectangular region, and let P be the perimeter of the track.

Objective function: $A = x \cdot w$

Constraint: $P = 440$
$$= 2w + 2\pi\left(\frac{x}{2}\right)$$

Solving the constraint equation for w gives

$$2w + \pi x = 440$$

$$w = \frac{440 - \pi x}{2}$$

So $A(x) = x\left(\frac{440 - \pi x}{2}\right)$

$$= -\frac{\pi x^2}{2} + 220x$$

Critical values:

$$A'(x) = -\pi x + 220 = 0$$

$$\Rightarrow \quad x = \frac{220}{\pi}$$

$$A''(x) = -\pi$$

So A is concave down, so

$x = \frac{220}{\pi}$ is a local max.

Since A is concave down for all x, $x = \frac{220}{\pi}$ must be a global max.

The dimensions that maximize the area are:

$$\boxed{x = \frac{220}{\pi} \text{ yards}}$$

$$\cancel{w = -11x + 220}$$

and $w = \dfrac{440 - \pi\left(\frac{220}{\pi}\right)}{2}$

$$= \frac{220}{2}$$

$$\boxed{w = 110 \text{ yards.}}$$