1 Introduction to the course

Calculus is useful for describing how one quantity changes as/when another quantity changes. The central concept studied in calculus is the function, which represents mathematically a relationship between two quantities. One of the quantities is often called *the independent variable* and the other *the dependent variable*. The independent variable will often be time. In this case, we will use calculus to describe how something changes as time goes by.

1.1 Brief overview of applications of calculus

1. Motion of an object, represented as a relationship between position and time
2. Growth or decline of populations, represented as a relationship between population size and time
3. Relationship between production and capital or labor
4. Relationship between income and production of a good

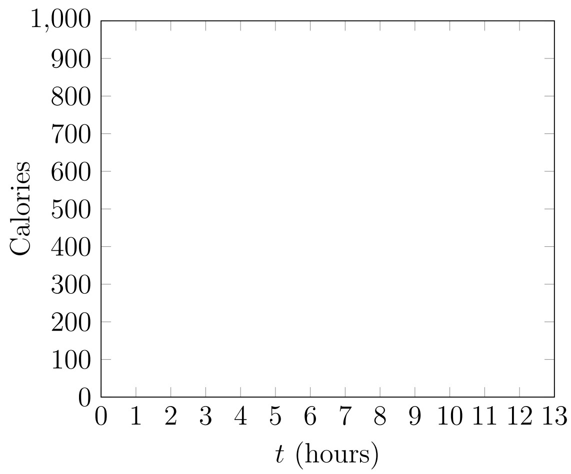
Sometimes the outcome of a process is unpredictable or uncertain but can be understood partially by assigning probabilities to outcomes. Calculus is also useful for calculating these probabilities. Thus, it can be applied to make predictions based on probabilities, and to quantify the uncertainty associated with the predictions. The part of calculus that is used to compute such probabilities is integral calculus, which we will study in the last part of this course.

1.2 Functions

The change of a quantity over time can be represented by a **function** that describes how the position of the quantity depends on time. Recall that functions are rules which describe how one value depends on another value. The set of values that the independent variable can have is called the **domain**, and the set of values that the dependent variable can have is called the **range**. To indicate the domain and the range concisely, we can say that the function *maps* from a given domain to a given range.

Example 1.1 *Consider the net number of calories that you have consumed since midnight. We will model this by a function that maps from the positive real numbers (the domain) to the positive real numbers (the range).*

Question: What do the numbers in the range represent?
Question: What do the numbers in the domain represent?



Calculus will allow us to quantify the fact that this function is increasing, to calculate how fast it is increasing at different times, and it will give us a formula for determining when the number of calories is not changing.

Another application of calculus is to describe a pattern in the way that a quantity changes. Before we introduce the concepts of calculus, we will study **difference equations**, which represent patterns of change that occur in discrete time.

2 Difference equations

Whereas the central objects of study in calculus are functions, the central objects that we use to study change in discrete time are sequences.

2.1 Sequences

A **sequence** is simply a function whose domain is the whole numbers $\{0, 1, 2, \ldots\}$. A **term** of a sequence, is the value of the sequence for a particular whole number n. If the sequence is denoted by a, we denote this value by $a(n)$ or a_n.

Example 2.1 *Imagine that every* 30 *minutes you record the net number of calories that you have consumed since midnight.*

Time	12:00	12:30	1:00	...	8:30	9:00	9:30	10:00	10:30	11:00	11:30	12:00	12:30
Calories	0	0	0	...	0	0	0	150	300	300	300	300	500

We define a sequence based on this data by letting a_n be the number of calories consumed after n 30-minute periods.

$$a_0 = 0, a_1 = 0, a_2 = 0, a_3 = 0, \ldots, a_{17} = 0, a_{18} = 0, a_{19} = 0, a_{20} = 150, a_{21} = 300, a_{22} = 300, \ldots$$

Exercise 1 *What value of n corresponds to the time 12:30?*

The above example of a sequence does not follow a simple pattern. Next we will look at sequences that have relatively simple patterns of change, and we will study how to represent the patterns of change using difference equations.

2.2 Difference equations

We will consider a particular situation in which a sequence has a relatively simple pattern of change: A bank account with regular withdrawals/deposits.

Example 2.2 *Suppose you set up a checking account with an initial balance of* \$5000*, and you withdraw* \$150 *every month. Find the first* 4 *terms in the sequence of monthly balances.*

Let y_n stand for the balance after n months. The sequence of monthly balances is:

$$y_0 = 5000, y_1 = 4850, y_2 = 4700, y_3 = 4550, \ldots$$

Given such a sequence, we form the expression $y_{n+1} - y_n$, called **the difference of consecutive terms**. A **difference equation** is an equation that gives the formula for the difference of the consecutive terms of a sequence.

Example 2.3 *Represent the sequence of monthly balances of example 2.2 by giving its initial value and its difference equation.*

The pattern in the above sequence can be expressed by that fact that the difference between consecutive terms is -150 (the negative sign occurs because of the convention that expression for the difference of consecutive terms is written with the y_{n+1} term first). So we can represent the sequence from example 2.2 by the difference equation

$$y_{n+1} - y_n = -150, \text{ with the initial value } y_0 = 5000.$$

The difference equation here is about as simple as possible: The difference of consecutive terms is just a constant number.

Note 1 *When the difference of consecutive terms is constant, the sequence increases or decreases uniformly.*

In the next example, the right hand side of the difference equation will not be constant.

Example 2.4 *Suppose the capital invested in a company is held in an account with an initial balance of* \$1000, *and the capital in the account doubles every year. Find the first* 5 *terms in the sequence of yearly balances.*

The sequence of yearly balances is

$$y_0 = 1000, y_1 = 2000, y_2 = 4000, y_3 = 8000, y_4 = 16000, \ldots$$

Example 2.5 *Represent the sequence of monthly balances of example 2.4 by giving its initial value and its difference equation.*

$$\begin{aligned} y_1 - y_0 &= 1000 \\ y_2 - y_1 &= 2000 \\ y_3 - y_2 &= 4000 \\ y_4 - y_3 &= 8000 \end{aligned}$$

Note that the values of these differences are the values y_0, y_1, y_2, y_3. The general pattern is

$$y_{n+1} - y_n = y_n, \tag{1}$$

with initial value $y_0 = 1000$.

Note 2 *The difference equation (equation 1) can be rewritten in the form:*

$$y_{n+1} = y_n + y_n = 2y_n. \tag{2}$$

Equations 1 and 2 represent the same pattern of change. Even though equation 2 does not give a formula for the difference of consecutive terms explicitly, we will still call it a difference equation, since it could be written in the form of equation 1.

Exercise 2 *Pacific salmon normally reproduce once during their lives (they are semelparous). Suppose that each female chinook salmon which survives and reproduces gives rise to* 3 *females, each of whom spawn* 6 *years later, giving rise to the next generation. If there are initially* 10 *spawning salmon, find the first* 4 *terms of the sequence representing the number of salmon which spawn in each generation.*