Data Generation and Estimation for Axially Symmetric Processes on the Sphere

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Abstract

Review and update later

Global-scale processes and phenomena are of utmost importance in the geophysical sciences. Data from global networks and satellite sensors have been used to monitor a wide array of processes and variables, such as temperature, precipitation, etc. In this dissertation, we are planning to achieve explicitly the following objectives,

1. Develop both non-parametric and parametric approaches to model global data dependency.

2. Generate global data based on given covariance structure.

3. Develop kriging methods for global prediction.

4. Explore one or more of the popularly discussed global data sets in literature such as MSU (Microwave Sounding Units) data, the tropospheric temperature data from National Oceanic and Atmospheric and TOMS (Total Ozone Mapping Spectrometer) data, total column ozone from the Laboratory for Atmospheres at NASA's Goddard Space Flight Center Administration satellite-based Microwave Sounding Unit.

Global scale data have been widely studied in literature. A common assumption on describing global dependency is the second order stationarity. However, with the scale of the Earth, this assumption is in fact unrealistic. In recent years, researchers have focused on studying the so-called axially symmetric processes on the sphere, whose spatial dependency often exhibit homogeneity on each latitude, but not across the latitudes due to the geophysical nature of the Earth. In this research, we have obtained some results on the method of non-parametric estimation procedure, in particular, the method of moments, in the estimation of spatial dependency. Our initial result shows that the spatial dependency of axially symmetric processes exhibits both anti-symmetric and symmetric characteristics across latitudes. We will also discuss detailed methods on generating global data and finally we will outline our methodologies on kriging techniques to make global prediction.

Chapter 1

Introduction

In this chapter we have given a brief introduction to some of the basic concepts in spatial statistics which are necessary to follow other chapters in this dissertation. Moreover, we have discuss about stationarity, isotropy, intrinsic stationarity, covarince function and it properties, variogram, continuity and differentiability, spectral representations, Bochner's theorem, spectral densities, circulant matrices and it's properties with special cases.

1.1 Spatial random field

A real-valued spatial process in d dimensions or a spatial random field can be denoted as $\{Z(x) : x \in D \subset \mathbb{R}^d\}$ where x is the location of process $Z(x)$ and x varies over the set D which is fixed and discrete. The distribution of the random vector $Z(\underset{\sim}{X}) = (Z(x_1), \dots, Z(x_n))$ is given by the associated finite-dimensional joint distributions

$$F\{Z(x_1), \dots, Z(x_n)\} = P\{Z(x_1) \leq h_1, \dots, Z(x_n) \leq h_n.\} \tag{1.1.1}$$

A random process is a collection of random variables $Z \in \{Z(s) : s \in D\}$, defined in a common probability space. In general, if

- $s \in N$: $Z(s)$ is a random sequence which is used in time series.
- $s \in R^1$: $Z(s)$ is a random process which is also referred as a stochastic process.
- $s \in R^d$: $Z(s)$ is a random filed or a spatial process if $d > 1$
- $s \in S^2$: $Z(s)$ is a random process on the sphere.
- $s \in R^d \times R$: $Z(s)$ is a spatio-temporal process which involves location and time.

? comment : more words

1.1.1 Stationarity and Isotropy

A spatial random field is strict stationarity, for all finite $n,\ x_1, \ldots, x_n \in \mathbb{R}^d,\ h_1, \ldots, h_n \in \mathbb{R}$ and $x \in \mathbb{R}^d$, if the random field is invariant under translation. that is,

$$P\{Z(x_1 + x) \leq h_1, \ldots, Z(x_n + x) \leq h_n\} = P\{Z(x_1) \leq h_1, \ldots, Z(x_n) \leq h_n\} \tag{1.1.2}$$

Strict stationarity is a too strong condition as it involves the distribution of the random field but many spatial methods are based moments. Therefore, it is sufficient to use weak assumptions and we could say a random process $Z(x)$ is weakly stationary if,

$$\begin{aligned} E(Z(x)) &= \mu \\ E^2(Z(x) &< \infty \\ C(h) &= Cov(Z(x), Z(x+h)) \end{aligned} \tag{1.1.3}$$

In other words a random process $Z(x)$ is weakly stationary if the second moment is finite with constant mean and $C(h)$ the covarince (also referred as auto-covariance) is a function that depends only on the spatial distance. Further strictly stationary random fields with finite second moment is also weakly stationary, but weak stationarity does not imply strict stationarity. Any random process defined in finite dimension and all its distributions are multivariate Gaussian then the random process is said to Gaussian. In the case of Gaussian random fields that weakly stationary are also strict stationarity because the first two moments (μ, σ) will explain the distribution.

Suppose $Z(x)$ is weakly stationary on $\mathbb{R}^d$ with autocavariance function $C(h)$ then it has the following properties,

(i) $C(0) \geq 0$

(ii) $C(h) = C(-h)$

(iii) $|C(h)| \leq C(0)$

(iv) If C_1, C_2, C_n are valid covariance functions then following $C(x)$ functions are also valid covarince functions

  (a) $C(x) = a_1C_1 + a_2C_2,\ \forall a_1, a_2 \geq 0$
  (b) $C(x) = C_1(x)C_2(x)$
  (c) $\lim\limits_{n\to\infty} C_n(x) = C(x),\ \forall x \in \mathbb{R}^d$

A covariance function $C(\cdot)$ on $\mathbb{R}^d$ is non-negative definite if and only if

$$\sum_{i,j=1}^{N} a_i a_j C(x_i - x_j) \geq 0, \tag{1.1.4}$$

for any integer N, any constants $a_1, a_2, \ldots, a_N$, and any locations $x_1, x_2, \ldots, x_N \in \mathbb{R}^d$. Positive definiteness is a necessary and sufficient condition to have a valid covariance function.

Theorem 1.1.1 (Mercer's theorem (simplified version))

A kernal $K : [a, b] \times [a, b] \to \mathbb{R}$ be a symmetric continuous function that is non-negative definite,

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j K(s,t) \geq 0 \quad \textit{and} \quad K(s,t) = K(t,s)$$

for all $(s, t) \in [a, b]$ and $a_i > 0$. Let $T_K : L_2 \to L_2$ be an intergral operator defined by

$$[T_K f](\cdot) = \int_a^b K(\cdot, s) f(s) ds$$

is positive, for all $f \in L_2$

$$\int_a^b K(s,t) f(s) f(t) ds dt \geq 0.$$

The corresponding orthonormal eigen functions $\psi_i \in L_2$ and non negative eigen values $\lambda_i \geq 0$ of the operator T_K is defined as

$$T_K(\psi_i(\cdot)) = \int K(\cdot, s)\psi(s) ds = \lambda_i \psi_i(\cdot), \qquad \int \psi_i(\cdot)\psi_j(\cdot) = \delta_{ij}$$

then the kernal $K(\cdot)$ is a uniformly convergent series in terms of eigen functions and associated eigen values of T_K as follows,

$$K(s,t) = \sum_{j=1}^{\infty} \lambda_i \psi_i(s) \psi_i(t)$$

A weakly stationary process with a covarince function $C(||h||)$ which is free from direction is called isotropy. The random field, $Z(x)$, on $\mathbb{R}^d$ is strictly isotropy if the joint distributions are invariant under all rigid motions. *i.e.*, for any orthogonal $d \times d$ matrix H and any $x \in \mathbb{R}^d$

$$P\{Z(Hx_1 + x) \leq h_1, \ldots, Z(Hx_n + x) \leq h_n\} = P\{Z(x_1) \leq h_1, \ldots, Z(x_n) \leq h_n\} \tag{1.1.5}$$

Isotropy assumes that it is not required to distinguish one direction from another for the random field $Z(x)$.

If the variance between two locations solely depends on the distance between the two locations then the process is said to be intrinsically stationary. Semivariogram is an alternative to the covariance function proposed by Matheron. For an intrinsically stationary random field $Z(x)$,

$$\begin{aligned} E[Z(s)] &= \mu, \\ \gamma(h) &= \frac{1}{2} Var(Z(s+h) - Z(x)), \end{aligned} \tag{1.1.6}$$

Where γ is the semivariogram and $\gamma(h) = C(0) - C(h)$ for a weakly stationary process with covariance function $C(h)$. Intrinsic stationary is defined in terms of variogram and it is more general than weak stationary which is defined in terms of covariance. Clearly, when $C(h)$ is known we can get $\gamma(h)$ but not $C(h)$ when $\gamma(h)$ is known. For example consider a linear semi variogram function,

$$\gamma(h) = \begin{cases} a^2 + \sigma^2 h & h > 0 \\ 0 & otherwise \end{cases}$$

when $\underset{h\to\infty}{lim}\gamma(h) \to \infty$ thus this is not weak stationary and $C(h)$ does not exist.

Similar to covariance the variogram is conditionally negative definite if only if

$$\sum_{i,j=1}^{N} a_i a_j 2\gamma(x_i - x_j) \leq 0, \tag{1.1.7}$$

for any integer N, any constants $a_1, a_2, \ldots, a_N$ with $\sum a_i = 0$, and any locations $x_1, x_2, \ldots, x_N \in \mathbb{R}^d$.

spectral representation of variogram

1.1.2 Mean square continuity & differentiability

There is no simple relationship between $C(h)$ and the smoothness of $Z(x)$. For a sequence of random variables $X_1, X_2, \ldots$ and a random variable X defined on a common probability space. Define, $X_n \overset{L^2}{\to} X$ if, $E(X^2) < \infty$ and $E(X_n - X)^2 \to 0$ as $n \to \infty$. We can say, $\{X_n\}$ converges in L^2 if there exists such a X.

Suppose $Z(x)$ is a random field on $\mathbb{R}^d$, Then $Z(x)$ is mean square continuous at x if,

$$\lim_{h\to 0} E(Z(x+h) - Z(x))^2 = 0$$

If $Z(x)$ is weak stationary and $C(\cdot)$ is the covariance function then $E(Z(x+h) - Z(x))^2 = 2(C(0) - C(h))$. Therefore $Z(x)$ is mean square continuous iff $C(\cdot)$ is continuous at the origin.

1.1.3 Circularly-symmetry Gaussian random vectors

Sometimes it is convenient to use complex valued random functions, rather than real valued random functions.

We say, $Z(x) = U(x) + iV(x)$ is a complex random field if $U(x), V(x)$ are real random fields. If $U(x), V(x)$ are weakly stationary so does $Z(x)$.The covariance function can be defined as,

$$C(h) = cov(Z(x+h), \overline{Z(x)}), \quad C(-x) = \overline{C(x)},$$

for any complex constants $c_1, \ldots, c_n$, and any locations $x_1, x_2, \ldots, x_n$,

$$\sum_{i,j=1}^{n} c_i \bar{c}_j C(x_i - x_j) \geq 0 \tag{1.1.8}$$

In general, a normal family has two parameters, location parameter μ and scale parameter Σ. But when we are dealing with complex normal family there is one additional parameter, the relation matrix also referred as pseudo-covariance matrix. In the case of real normal family the pseudo-covariance matrix is equivalent to covariance matrix.

According to Gallager (2008), a complex random variable $Z = Z^{Re} + iZ^{Im}$ is Gaussian, if Z^{Re}, Z^{Im} both are real and jointly Gaussian. Then Z is circularly-symmetric if both Z and $e^{i\phi}Z$ has the same probability distribution for all real ϕ. Since $E[e^{i\phi}Z] = e^{i\phi}E[Z]$, any circularly-symmetric complex random vector must have $E[Z] = 0$, in other words its mean must be zero.

Let the covariance matrix K_Z and the pseudo-covariance matrix M_Z of a zero mean $2n$ complex random vector $\underset{\sim}{Z} = (Z_1, Z_2, \ldots, Z_n)^T$, where $Z_j = (Z_j^{Re}, Z_j^{Im})^T$ and $j = 1, 2, \ldots, n$ can be defined as follows,

$$K_{\underset{\sim}{Z}} = E[\underset{\sim}{Z}\underset{\sim}{Z}^*] \tag{1.1.9}$$

$$M_{\underset{\sim}{Z}} = E[\underset{\sim}{Z}\underset{\sim}{Z}^T] \tag{1.1.10}$$

where $\underset{\sim}{Z}^*$ is the conjugate transpose of $\underset{\sim}{Z}$.

For example, consider a vector $\underset{\sim}{Z} = (Z_1, Z_2)^T$ where $Z_1 = Z_1^{Re} + iZ_1^{Im}$ and $Z_2 = Z_1^*$ ($Z_2^{Re} = Z_1^{Re}, Z_2^{Im} = -Z_1^{Im}$). The four real and imaginary parts of $\underset{\sim}{Z}$ are jointly Gaussian (each $N(0, 1/2)$), so $\underset{\sim}{Z}$ is complex Gaussian.

Now, the covariance and pseudo-covariance matrices are different defined by 1.1.9 and 1.1.10 respectively given by

$M_Z = E\begin{bmatrix} Z_1^2 & Z_1Z_1^* \\ Z_1Z_1^* & Z_1^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

similarly, $K_Z = E\begin{bmatrix} Z_1Z_1^* & Z_1^2 \\ Z_1^2 & Z_1Z_1^* \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

It is easy to notice that $E[Z_1^2] = E[z_1^{Re}z_1^{Re} - z_1^{Im}z_1^{Im}] = 1/2 - 1/2 = 0$ and if Z_1 is real (obviously also Z_2) then covarince and pseudo-covariance matrices are the same *i.e.* $M_Z \equiv K_Z$

The covariance matrix of real $2n$ random vector $\underset{\sim}{Z} = (\underset{\sim}{Z}^{Re}, \underset{\sim}{Z}^{Im})^T$ is determined by both $K_{\underset{\sim}{Z}}$ and $M_{\underset{\sim}{Z}}$ as follows,

$$\begin{aligned} E[\underset{\sim}{Z}^{Re}\underset{\sim}{Z}^{Re}] &= \frac{1}{2}Re(K_{\underset{\sim}{Z}} + M_{\underset{\sim}{Z}}), \\ E[\underset{\sim}{Z}^{Im}\underset{\sim}{Z}^{Im}] &= \frac{1}{2}Re(K_{\underset{\sim}{Z}} - M_{\underset{\sim}{Z}}), \\ E[\underset{\sim}{Z}^{Re}\underset{\sim}{Z}^{Im}] &= \frac{1}{2}Im(-K_{\underset{\sim}{Z}} + M_{\underset{\sim}{Z}}), \\ E[\underset{\sim}{Z}^{Im}\underset{\sim}{Z}^{Re}] &= \frac{1}{2}Im(K_{\underset{\sim}{Z}} + M_{\underset{\sim}{Z}}) \end{aligned} \tag{1.1.11}$$

We can get the covariance of $\underset{\sim}{Z} = (\underset{\sim}{Z}^{Re}, \underset{\sim}{Z}^{Im})^T$ as follows,

$$\begin{aligned} Cov(\underset{\sim}{Z}) &= E(\underset{\sim}{Z}\underset{\sim}{Z}^T) \\ &= \begin{pmatrix} E[\underset{\sim}{Z}^{Re}\underset{\sim}{Z}^{Re}] & E[\underset{\sim}{Z}^{Re}\underset{\sim}{Z}^{Im}] \\ E[\underset{\sim}{Z}^{Im}\underset{\sim}{Z}^{Re}] & E[\underset{\sim}{Z}^{Im}\underset{\sim}{Z}^{Im}] \end{pmatrix} \end{aligned}$$

Theorem 1.1.2 (Gallager, 2008)
Let $\underset{\sim}{Z}$ be a zero mean Gaussian random vector then $M_{\underset{\sim}{Z}} = 0$ if and only if $\underset{\sim}{Z}$ is circularly-symmetric.

1.1.4 Spectral representation of a random field

Suppose $\omega_1, \ldots, \omega_n \in \mathbb{R}^d$ and let $Z_1, \ldots, Z_n$ be mean zero complex random variables with $E(Z_i\bar{Z}_j) = 0, i \neq j$ *and* $E|Z_i|^2 = f_i$. Then the random sum

$$Z(x) = \sum_{k=1}^{n} Z_k e^{i\omega_k^T x}. \tag{1.1.12}$$

Then $Z(x)$ given above is a weakly stationary complex random field in $\mathbb{R}^d$ with covariance function $C(x) = \sum_{k=1}^{n} f_k e^{i\omega_k^T x}$

Further, if we think about the integral as a limit in L^2 of the above random sum, then the covariance function can be represented as,

$$C(x) = \int_{\mathbb{R}^d} e^{i\omega^T x} F(d\omega) \tag{1.1.13}$$

where F is the so-called spectral distribution. There is a more general result from Bochner.

Theorem 1.1.3 (Bochner's Theorem)

A complex valued covariance function $C(\cdot)$ on $\mathbb{R}$ for a weakly stationary mean square continuous complex-valued random field on $\mathbb{R}^d$ iff it can be represented as above, where F is a positive measure.

If F has a density with respect to Lebesgue measure (spectral density) denoted by f, (*i.e.* if such f exists) we can use the inversion formula to obtain f

$$f(\omega) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\omega^T x} C(x) dx \tag{1.1.14}$$

1.1.5 Spectral densities

(i) Rational Functions that are even, non-negative and integrable the corresponding covariance functions can be expressed in terms of elementary functions. For example if $f(\omega) = \phi(\alpha^2 + \omega^2)^{-1}$, then $C(h) = \pi\phi\alpha^{-1}e^{-\alpha|h|}$ (obtained by contour integration).

(ii) Gaussian are the most commonly used covariance function for a smooth process on $\mathbb{R}$ where the covariance function is given by $C(h) = ce^{-\alpha h^2}$ and the corresponding spectral density is $f(\omega) = \frac{1}{2\sqrt{\pi\alpha}} ce^{\frac{-\omega^2}{4\alpha}}$.

(iii) *Matérn* class has more practical use and more frequently used in spatial statistics. The spectral density of the form $f(\omega) = \frac{1}{\phi(\alpha^2+\omega^2)^{\nu+1/2}}$ where $\phi, \nu, \alpha > 0$ and the corresponding covariance function given by,

$$C(h) = \frac{\pi^{1/2}\phi}{2^{\nu-1}\Gamma(\nu + 1/2)\alpha^{2\nu}} (\alpha|h|)^{\nu} Y_{\nu}(\alpha|t|) \tag{1.1.15}$$

where Y_ν is the modified Bessel function, the larger the ν smoother the Y. Further, Y will be m times square differentiable iff $\nu > m$. When ν is in the form of $m + 1/2$ with m a non negative integer. The spectral density is rational and the covariance function is in the form of $e^{-\alpha|h|}\cdot$ polynomial($|h|$) for example, when $\nu = \frac{1}{2}$ $C(h)$ corresponds to exponential model and $\nu = \frac{3}{2}$ is transformation of exponential family of order 2.

$$\begin{aligned}
\nu = 1/2 \quad &: \quad C(h) = \pi\phi\alpha^{-1}e^{-\alpha|h|} \\
\nu = 3/2 \quad &: \quad C(h) = \frac{1}{2}\pi\phi\alpha^{-3}e^{-\alpha|h|}(1 + \alpha|h|)
\end{aligned}$$

1.2 Circulant matrix

A square matrix $A_{n\times n}$ is a circulant matrix if the elements of each row (except first row) has the previous row shifted by one place to the right.

$$A = circ[a_o, a_1, \cdots, a_{n-1}] \quad = \quad \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{bmatrix}. \tag{1.2.1}$$

The eigenvalues of A are given by

$$\begin{aligned} \lambda_l &= \sum_{k=0}^{n-1} a_k e^{i2lk\pi/n} \\ &= \sum_{k=0}^{n-1} a_k \rho_l^k, \qquad l = 0, 1, 2, \cdots, n-1, \end{aligned}$$

where $\rho_l = e^{i2\pi l/n}$ represents the lth root of 1, and the corresponding (unitary) eigenvector is given by

$$\psi_l = \frac{1}{\sqrt{n}}(1, \rho_l, \rho_l^2, \cdots, \rho_l^{n-1})^T.$$

If matrix A is real symmetric then its eigen values are real; for even $n = 2h$ the eigen values $\lambda_j = \lambda_{n-j}$ or there are either two eigen values or none with odd multiplicity, for odd $n = 2h - 1$ the eigen value λ_0 equal to any λ_j for $1 \leq j \leq h - 1$ or λ_0 occurs with odd multiplicity. A square matrix B is Hermitian, if and only if $B^* = B$ where B^* is the complex conjugate. If B is real then $B^* = B^T$. According to Tee (2005) Hermitian matrices has a full set of orthogonal eigen vectors with corresponding real eigen values.

1.2.1 Block circulant matrices

The idea of a block circulant matrix was first proposed by Muir (1920). A matrix $B_{np\times np}$ is a block-circulant matrix if it has the following form,

$$B = bcirc[a_o, a_1, \cdots, a_{n-1}] = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{bmatrix}. \tag{1.2.2}$$

where a_j are $(p \times p)$ sub matrices of complex or real valued elements. De Mazancourt and Gerlic (1983) proposed some methodologies to find the inverse of B. Let M be a block-permutation matrix

$$M = \begin{bmatrix} 0 & I_p & 0 & \cdots & 0 \\ 0 & 0 & I_p & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_p \\ I_p & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

where I_p is $p \times p$ identity matrix and B can be defined as follows,

$$B = \sum_{k=0}^{n-1} a_k M^k.$$

Define M^0 as $(np \times np)$ identity matrix and the eigen values of M given by ρ_l, the eigen matrix of M can be given by $Q_{np\times np} = \{\utilde{\psi_0}, \utilde{\psi_1}, \ldots, \utilde{\psi_{n-1}}\}$. From Trapp (1973) it can be shown that $Q^{-1} = Q^*/n$ where Q^* is the conjugate transpose of Q now we can write,

$$M = QDQ^{-1} = \frac{QDQ^*}{n}$$

where D is a diagonal matrix and the diagonal elements $d_i \quad i = 0, 1, n-1$ are the discrete Fourier transform of the blocks a_j,

$$d_i = \sum_{k=0}^{n-1} a_k e^{i2lk\pi/n}$$

That is the inverse of matrix B takes the following form,

$$B^{-1} = Q \cdot \begin{pmatrix} d_0^{-1} & 0 & \cdots & 0 \\ 0 & d_1^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n{-1} \end{pmatrix} \cdot Q^{-1}.$$

The eigen matrix Q is solely depending on the dimension of B and the eigen values of B (ρ_l's) or in other words B is not depending on the blocks (a_j's) *i.e.* for any block diagonal matrix $D_{np\times np}$, QDQ^{-1} is a block circulant matrix and immediately follows that the inverse of the matrix B is also a block circulant matrix.

When $a_{j_{1\times 1}}$, $B = A$, $d_i^{-1} = \lambda^{-1}$, and the eigen matrix has a dimension of $n \times n$ then

$$A^{-1} = Q\Lambda^{-1}Q^T \quad \text{where } \Lambda = \{\lambda_0, \ldots, \lambda_{n-1}\}$$

When A is real symmetric Q is real also symmetric and $Q^{-1} = Q^T$.

should we add the following special cases?

Case 1 : When a_j's are symmetric (*by Tee, add citation*)
Case 2 : When a_j's are circulant

Chapter 2

Literature Review

2.1 Spatial Data

What does it mean by spatial data? In general, spatial data or in other words geospatial data is information about a physical object or a measurement that can be represented by numerical values in a geographic coordinate system. As Cressie (1993) pointed out spatial data appeared to be in the form of maps in 1686 and spatial modeling paper in 1907. There are many questions that geoscientists and engineers are interested about spatial data. Many questions naturally arise such as how to model a spatial process and then use the model to make predictions about unobserved locations. There are many challenges when modeling spatial data; every point (location observed) is a random variable and only one observation/measurement is available. However, the number of unknowns to estimate are quite large compared to the available data, which is definitely a high-dimensional problem. As an example, if data were observed at 10 locations, one is estimating the variance-covariance matrix to characterize the spatial dependency for future predictions. Then there will have 55 unknown entities in the variance-covariance matrix to be estimated.

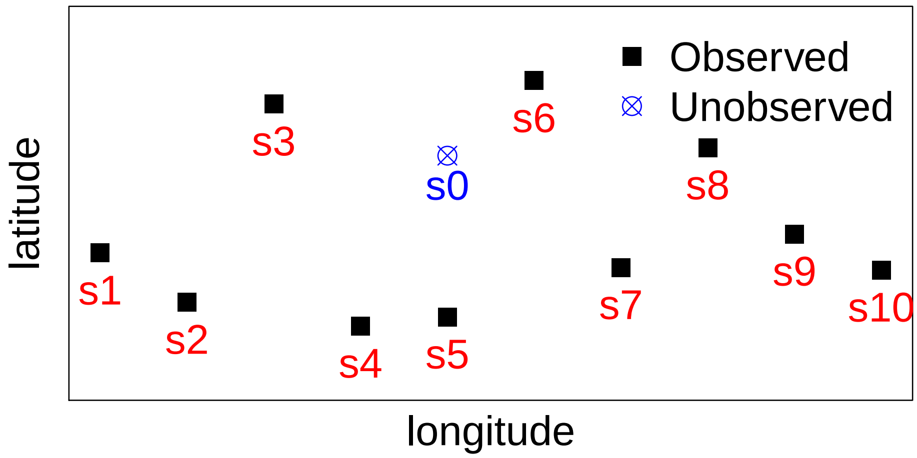

Figure 2.1: Some arbitary saptial data at 10 random locations

2.1.1 MSU data

Since 1978 Microwave Sounding Units (MSU) measure radiation emitted by the earth's atmosphere from NOAA polar orbiting satellites. The different channels of the MSU measure different frequencies of radiation proportional to the temperature of broad vertical layers of the atmosphere. Tropospheric and lower stratospheric temperature data are collected by NOAA's TIROS-N polar-orbiting satellites and adjusted for time-dependent biases by the Global Hydrology and Climate Center at the University of Alabama in Huntsville (UAH)[1]. More information about how the data is been processed can be found in Christy et al. (2000). Satellites do not measure temperature directly but measure radiances in various wavelength bands and then mathematically inverted to obtain the actual temperature.

[1] https://www.ncdc.noaa.gov/temp-and-precip/msu/overview

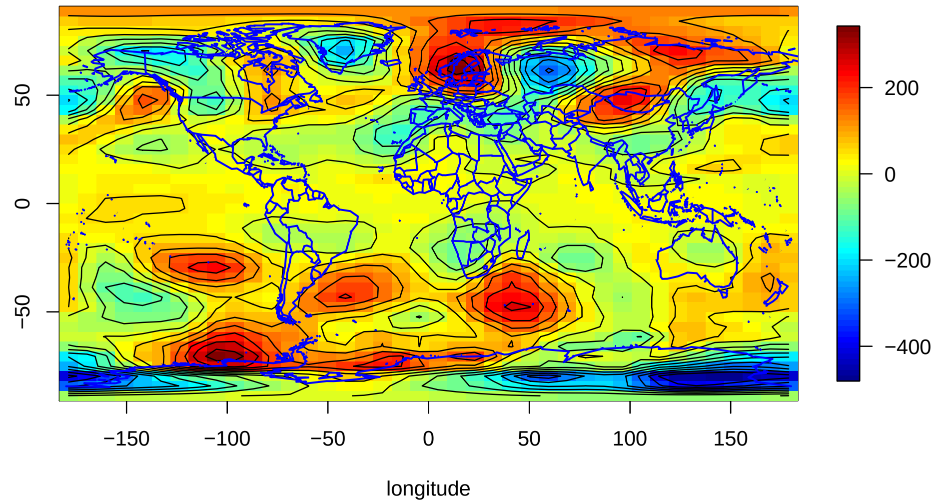

Figure 2.2: MSU data observed (without removing any spatial trends) in August 2002 : resolution 2.5° latitude × 2.5° longitude. It is easy to observe that variation is higher towards the north and south poles.

The MSU data were observed at 2.5°latitude × 2.5°longitude with total number of data observations of size 10368.

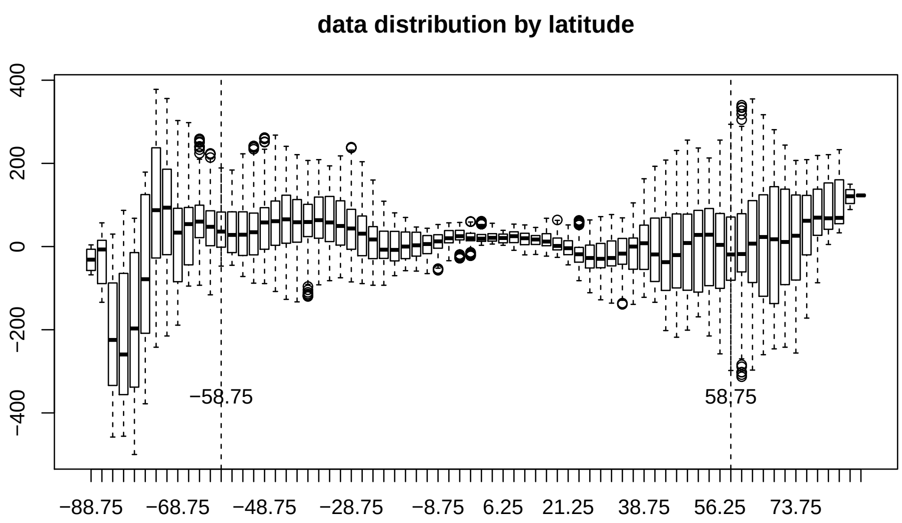

Figure 2.3: MSU data (August 2002), data distribution at each latitude (the spread is very high near north and south poles)

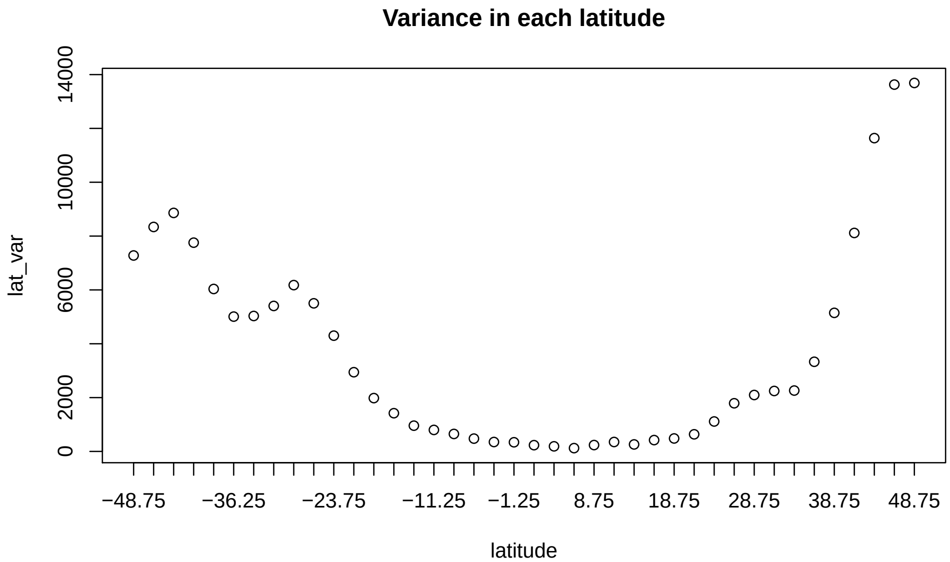

Figure 2.4: MSU data (August 2002) between $60°S$ and $60°N$, the variance at each latitude (variance is almost zero near the equator)

2.1.2 TOMS data

Extracted from Stein (2007) "The Nimbus-7 satellite carried a Total Ozone Mapping Spectrometer (TOMS) instrument that measured total column ozone daily from November 1, 1978 to May 6, 1993. This satellite followed a Sun-synchronous polar orbit with an orbital frequency of 13.825 orbits a day (cycle time about 104 minutes). As the satellite orbited, a scanning mirror repeatedly scanned across a track about 3000 km wide, each track yielding 35 total column ozone measurements. This version of the data is known as Level 2 and is publicly available [2]. "

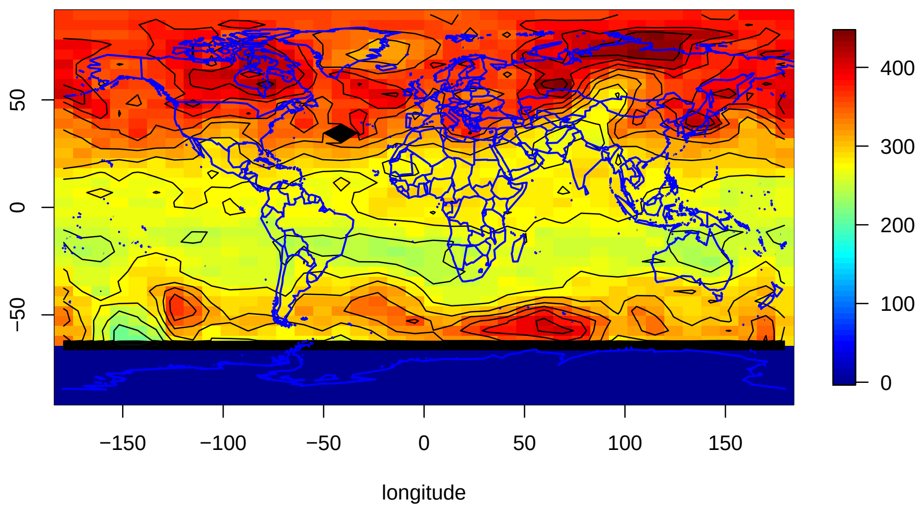

Figure 2.5: TOMS data: resolution 1°latitude × 1.25° longitude in May, 1-6 1990. The instrument used backscattered sunlight, therefore measurements were not available south of $73°S$ during this week.

There were some missing values in this data set. Stein (2007) used the average of 8 neighboring locations to replace the missing values. Further, he used spherical harmonics with associated Legendre polynomials of up to 78 covariates to remove the spatial trends to study axial symmetry of the data.

[2] `http://disc.sci.gsfc.nasa.gov/data/datapool/TOMS/Level2/`

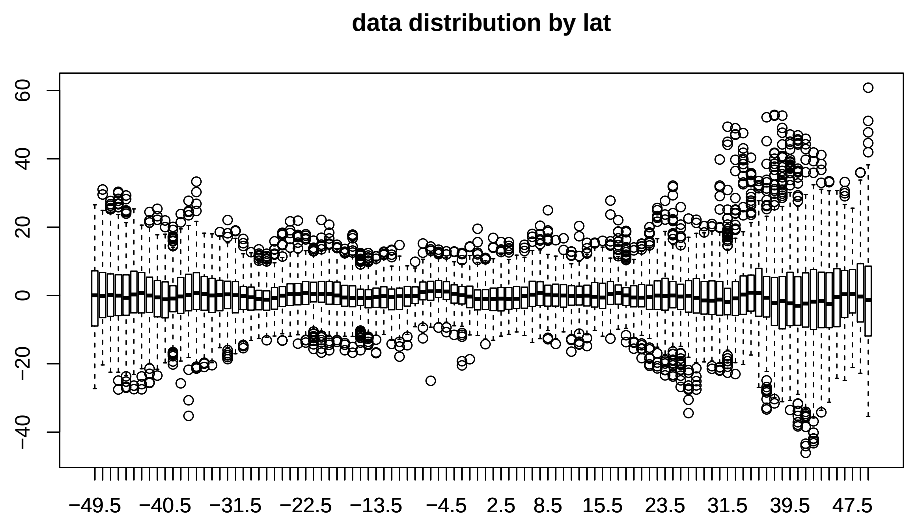

Figure 2.6: TOMS data: data distribution at each latitude (data between $50°S$ and $50°N$ were considered)

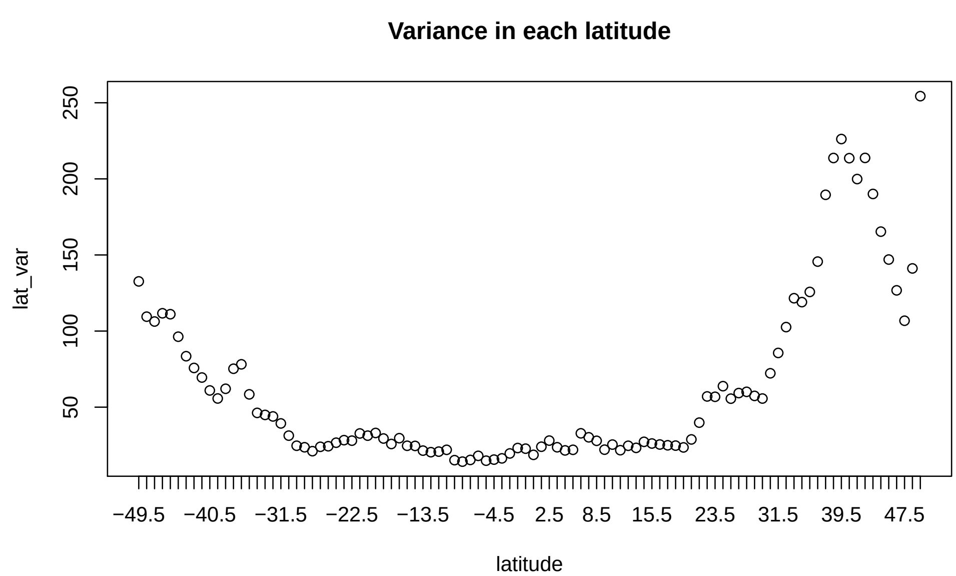

Figure 2.7: TOMS data: between $50°S$ and $50°N$, the variance at each latitude (variance is almost zero near the equator)

2.1.3 Challenges

1. There have been extensive statistical research on methodologies and techniques developed under the Euclidean space R^d. These approaches that are valid in R^d have been

applied to analyze global-scale data in recent years, due to global networks and satellite sensors that have been used to monitor a wide array of global-scale processes and variables. However, this can have unforeseen impacts, such as making use of models that are valid in R^d but in fact might not be valid under spherical coordinate systems. Huang et al. (2011) have investigated some of commonly used covariance models that are valid in R^d, and they pointed out that many of those are actually invalid on the sphere.

2. Note that, as we will see later, the spectral representation of the process on the sphere is a summation of Legendre polynomials, which is distinct from its planar counterpart as represented by an integration of Bessel functions. This distinction can be understood through group representation theory, which is the basis for an exciting new line of research we are currently pursuing.

Organize!!!

1. Axially symmetry, which means that a process is invariant to rotations about the Earth's axis. The idea was first proposed by Jones (1963), where the covariance function depends on the longitudes only through their difference.

2. In the study of a random process on a sphere, homogeneity (covariance depends solely on distance between locations) was assumed. However, this assumption may not be reasonable for actual data. Stein (2007) argued that Total Ozone Mapping Spectrometer (TOMS) data varies strongly with latitudes and homogeneous models are not suitable. Further, Cressie and Johannesson (2008), Jun and Stein (2008), Bolin and Lindgren (2011) pointed out that homogeneity assumption is not reasonable.

3. There are no methods to test axially symmetry in real data. However, this assumption is more plausible and reasonable when modeling spatial data. For example, temperature, moisture, etc. most likely symmetric on longitudes rather than latitudes. Stein (2007) propose a method to model axially symmetric process on a sphere (the fitted model is not the best, but this was a good start).

4. There are no practically useful parametric models available, for our knowledge only models available so far, Stein (1999) with 170 parameters to estimate and Cressie and Johannesson (2008) more than 396 parameters to estimate.

5. When modeling spatial data stationary models are less useful;Jun and Stein (2008) has proposed flexible class of parametric covariance models to capture the non-stationarity of global data. They used Discrete Fourier Transform (DFT) to the data on regular grids and calculated the exact likelihood for large data sets. Furthermore, they used Legendre polynomials to remove the spatial trends when fitting models to global data.

6. Lindgren et al. (2011) analyzed global temperature data with a non-stationary model defined on a sphere using Gaussian Markov Random Fields (GMRF) and Stochastic Partial Differential Equations (SPDE)

7. Monte Carlo Markov Chain (MCMC) is another approach to model non-stationary covariance models on a sphere. Bolin and Lindgren (2011) (continuation of the work proposed in Lindgren et al. (2011)) constructed a class of stochastic field models using Stochastic Partial Differential Equations (SPDEs). Non stationary covariance models were obtained by spatially varying the parameters in the SPDEs, they argue that this method is more efficient than standard MCMC procedures. There are many articles followed this techniques but we will not discuss more details about these methods.

8. Spatio-temporal mixed-effects model for dimension reduction was proposed by Katzfuss and Cressie (2011). They used MOM parameter estimation method (similar approach in FRS). This work is also based on Cressie and Johannesson (2008) spatial only Fixed Rank model. These methods are eventually focused on Bayesian approach and are less interested about topic.

9. The previous studies have argued that many processes on a sphere are not homogeneous, especially in latitude direction. Huang et al. (2012) proposed a class of statistical processes that are axially symmetric and covariance functions that depend on longitudinal differences. Moreover, they have proposed longitudinally reversible processes and some motivations to construct axially symmetric processes. The covariance models implemented in this dissertation are modified versions of the covariance models proposed by Huang et al. (2012).

10. Hitczenko and Stein (2012) discuss about the properties of an existing class of models for axially symmetric Gaussian processes on the sphere. They applied first-order differential operators to an isotropic process. draw conclusions about the local properties of the processes. Under some restrictions they derived explicit forms for the spherical harmonic representation of these processes covariance functions, and make conclusions about the local properties of the processes.

11. The issues associated when modeling axially symmetric spatial random fields on a sphere was discussed by Li (2013). They proposed convolution methods to generate random fields with a class of $Matérn$-type kernel functions by allowing the parameters in the kernel function to vary with location. Moreover, they were able to generate flexible class of covariance functions and capture the non-stationary properties on a sphere. Used FFT to get the determinant and the inverse efficiently. Further, semi-parametric variogram estimation method using spectral representation was proposed for intrinsically stationary random fields on S^2.

12. $Matérn$ covariance models are widely used when modeling spatial data, but when the smoothness parameter (ν) is greater than 0.5 it is not valid for the homogeneous processes on the Earth surface with great circle distance. Jeong and Jun (2015) proposed $Matérn$-like covariance functions for smooth processes on the earth surface that are valid with great circle distance (models were tested on sea levels pressure data).

Family	C(h)	Parameters	Validity
Matérn	$\frac{\sigma^2}{2^{\nu-1}\Gamma(\nu)}(\frac{h}{\phi})^\nu Y_\nu(\frac{h}{\phi})$	ν, σ^2, ϕ	R^3, S^2 when $\nu \leq 0.5$
Spherical	$\sigma^2(1 - \frac{3h}{2\phi} + \frac{1}{2}(\frac{h}{\phi})^3)I_{0\leq h\leq\phi}$	ϕ, σ^2	R^3, S^3
Exponential	$\sigma^2 exp\{-(h/\phi)\}$	ϕ, σ^2	R^3
Gaussian	$\sigma^2 exp\{-(h/\phi)^2\}$	ϕ, σ^2	R^3
Power	$\sigma^2(C_0 - (h/\phi)^\alpha$	ϕ, σ^2	$R^3 \alpha \in [0, 2], S^2 \alpha \in (1, 2]$

Table 2.1: Commonly used covariance and variogram models

Chapter 3

Covariance and Variogram Estimation on the Cricle

3.1 Stationary process on a circle

In this chapter we consider a real valued process $\{X(P) : P \in S\}$ on the unit circle S, with finite second moment and continuity in quadratic mean. According to Dufour and Roy (1976) the process $\{X(P)\}$ can be represented in a Fourier series which is convergent in quadratic mean,

$$X(P) = A_0 + \sum_{n=1}^{\infty}(A_n \cos(nP) + B_n \sin(nP). \tag{3.1.1}$$

$$\begin{aligned} \text{where} \quad A_0 &= \frac{1}{2\pi}\int_S X(P)dP, \\ A_n &= \frac{1}{\pi}\int_S \cos(nP)dP \\ B_n &= \frac{1}{\pi}\int_S \sin(nP)dP \end{aligned} \tag{3.1.2}$$

Let P, Q be any two points on the circle, the covariance $R(P,Q)$ between the two points can be defined as,

$$R(P,Q) = E(X(P)X(Q)) = cov(X(P), X(Q))$$

The process $X(P)$ is stationary if $E(X(P))$ is a constant and $R(P,Q)$ is function of the angular distance θ_{PQ} between P and Q. If the process $X(P)$ is stationary on the circle,

$$cov(A_n, A_m) = a_n\delta(n,m) = cov(B_n, B_m), \quad \text{for } n, m \geq 0. \tag{3.1.3}$$

Let $\{X(t_k)\}$ be a collection of gridded observations on a circle, with $t_k = (k-1)*2\pi/n, k = 1, 2, \cdots, n$. Let $C(\theta), \theta \in [0,\pi]$ denote a stationary covariance function on the circle, the underlying process is stationary if it's covariance function solely depends on the distance θ and given as follows,

$$C(\theta) = cov(X(t_k + \theta), X(t_k)), \qquad \theta \in [0, \pi]. \tag{3.1.4}$$

The above covariance function can we can be written as a Fourier summmation (Roy (1972))

$$C(\theta) = \sum_{n=0}^{\infty} a_n \cos(n\theta), \tag{3.1.5}$$

with $\sum_{n=0}^{\infty} a_n < \infty$, and $a_n \geq 0$. Note that

$$a_n = \frac{2}{\pi} \int_0^{\pi} C(\theta) \cos(n\theta) d\theta \quad \text{and} \quad a_0 = \frac{1}{\pi} \int_0^{\pi} C(\theta) d\theta.$$

3.2 Estimation

Assuming the covariance function $C(\theta)$ on a cricle is a continuous function on $[0, \pi])$ and the gridded points $\{X(t_k)\}$ on a circle can be represented as $\underset{\sim}{X} = (X_1, X_2, \cdots, X_n)^T$. The variance-covariance matrix of the sample vector $\underset{\sim}{X}$ is given by Σ. Lets assume $E(X(t)) = \mu$ is unknown, and estimated by $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X(t_i)$. Further, we can denote $\bar{X}$ in the following form,

$$\bar{X} = \frac{1}{n}\mathbf{1}_n^T \underset{\sim}{X}$$

$$\begin{aligned} \text{then,} \quad var(\bar{X}) &= cov(\frac{1}{n}\mathbf{1}_n^T \underset{\sim}{X}, \frac{1}{n}\mathbf{1}_n^T \underset{\sim}{X}) \\ &= \frac{1}{n^2}\mathbf{1}_n^T \Sigma \mathbf{1}_n \\ &= \frac{1}{n}\left(C(0) + C(\pi) + 2\sum_{m=1}^{N-1} C(m2\pi/n) \right) \end{aligned}$$

$$\begin{aligned} \text{When } n \to \infty, \frac{1}{n}\left(2\sum_{m=0}^{N} C(m2\pi/n) \right) &= \frac{1}{\pi}\frac{\pi}{N}\left(\sum_{m=0}^{N} C(m2\pi/n) \right) \\ &\to \frac{1}{\pi}\int_0^{\pi} C(\theta) d\theta = a_0. \end{aligned}$$

$$\begin{aligned} \text{Hence} \quad var(\bar{X}) &= \frac{1}{n}\left(2\sum_{m=0}^{N} C(m2\pi/n) \right) - \frac{1}{n}(C(0) + C(\pi)) \\ &\to a_0, \qquad \text{as } n \to \infty. \end{aligned}$$

$$\text{We can conclude that } var(\bar{X}) \to \frac{1}{\pi}\int_0^\pi C(\theta)d\theta \quad \text{as } n \to \infty.$$

Proposition 3.2.1 *Let $\bar{X}$ be an unbiased estimator for μ then $\bar{X}$ is not a consistent estimator for μ, mean on a circle.*

proof: *If $\bar{X}$ is consistant we get the following,*

$$P(|\bar{X} - \mu| > \varepsilon) \to 0.$$

If $var(\bar{X}) \to 0$ and from Chebyshev's inequality we have

$$P(|\bar{X} - \mu| > \varepsilon) \leq \frac{var(\bar{X})}{\varepsilon^2} \to 0, \quad \textit{for any } \varepsilon > 0.$$

Therefore, $var(\bar{X}) \to 0$ is a sufficient condition for consistency, but it is not necessary. However, if we assume $\underset{\sim}{X}$ is multivariate normally distributed, then $\bar{X}$ follows normal distribution with mean μ and approximate variance σ_0. Then ($Z \sim N(0,1)$)

$$\begin{aligned} P(|\bar{X} - \mu| > \varepsilon) &= P\left(\frac{|\bar{X} - \mu|}{\sqrt{a_0}} > \frac{\varepsilon}{\sqrt{a_0}}\right) \\ &= P\left(|Z| > \frac{\varepsilon}{\sqrt{\sigma_0}}\right) \not\to 0 \end{aligned}$$

since $\frac{\varepsilon}{\sqrt{\sigma_0}}$ is a fixed constant for each fixed $\varepsilon > 0$.

3.2.1 Estimation of covariance on a circle

We used method of moments (MOM) to estimate the covariance $C(\theta)$ on a circle, the estimator can be given by

$$\hat{C}(\Delta\lambda) = \frac{1}{n}\sum_{i=1}^{n}(X(t_i + \Delta\lambda) - \bar{X})(X(t_i) - \bar{X}), \tag{3.2.1}$$

where $\Delta\lambda = 0, 2\pi/n, 4\pi/n, \cdots, 2(N-1)\pi/n$.

We can show that the above estimator is a biased estimator for $C(\theta)$.

$$
\begin{aligned}
E(\hat{C}(\Delta\lambda)) &= \frac{1}{n}\sum_{i=1}^{n} E((X(t_i+\Delta\lambda)-\bar{X})(X(t_i)-\bar{X})) \\
&= \frac{1}{n}\sum_{i=1}^{n} E((X(t_i+\Delta\lambda)-\mu-(\bar{X}-\mu))(X(t_i)-\mu-(\bar{X})-\mu)) \\
&= \frac{1}{n}\sum_{i=1}^{n} cov(X(t_i+\Delta\lambda),X(t_i)) - \frac{1}{n}\sum_{i=1}^{n} E((X(t_i+\Delta\lambda)-\mu)(\bar{X}-\mu)) \\
&\quad -\frac{1}{n}\sum_{i=1}^{n} E((X(t_i)-\mu)(\bar{X}-\mu)) + \frac{1}{n}\sum_{i=1}^{n} E((\bar{X}-\mu)(\bar{X}-\mu)) \\
&= C(\Delta\lambda) - E((\bar{X}-\mu)(\bar{X}-\mu)) - E((\bar{X}-\mu)(\bar{X}-\mu)) \\
&\quad +E((\bar{X}-\mu)(\bar{X}-\mu)) \\
&= C(\Delta\lambda) - var(\bar{X}).
\end{aligned}
$$

Remark 1 The MOM estimator $\hat{C}(\Delta\lambda)$ of the covariance function is actually a biased estimator with the shift amount of a_0. Therefore, if $a_0 = 0$ for a covariance function, we have the unbiased estimator $\hat{C}(\Delta\lambda)$.

Remark 2 If the gridded points were on a line, for example in time series, $E(\bar{X}-\mu)^2 \to 0$ as $n \to \infty$ under the assumption that the covariance function $C(\theta) \to 0$ when $\theta \to \infty$ (which is practically feasible), that is, $\bar{X}$ is consistent in the case of points on a line. In the case of circle, we might not have $C(\theta)$ close to 0 since θ is within a bounded region ($(0, \pi)$ for the circle) and we normally assume $C(\theta)$ is continuous for θ.

3.2.2 Estimation of variogram on a circle

The theoretical variogram function is given by,

$$\gamma(\theta) = C(0) - C(\theta). \tag{3.2.2}$$

and the MOM estimator for the variogram is given by,

$$\hat{\gamma}(\Delta\lambda) = \frac{1}{2n}\sum_{i=1}^{n}(X(t_i+\Delta\lambda)-X(t_i))^2. \tag{3.2.3}$$

We can show that variogram estimator through MOM is an unbiased estimator,

$$
\begin{aligned}
E(\hat{\gamma}(\Delta\lambda)) &= \frac{1}{2n}\sum_{i=1}^{n} E(X(t_i+\Delta\lambda)-X(t_i))^2 \\
&= \frac{1}{2n}\sum_{i=1}^{n} E((X(t_i+\Delta\lambda)-\mu)-(X(t_i)-\mu))^2 \\
&= \frac{1}{2n}\sum_{i=1}^{n} cov(X(t_i+\Delta\lambda)-X(t_i), X(t_i+\Delta\lambda)-X(t_i)) \\
&= \frac{1}{2n}\sum_{i=1}^{n} \{cov(X(t_i+\Delta\lambda), X(t_i+\Delta\lambda)) + cov(X(t_i), X(t_i)) \\
&\quad -2cov(X(t_i+\Delta\lambda), X(t_i))\} \\
&= \frac{1}{2n}\sum_{i=1}^{n} (C(0)+C(0)-2C(\Delta\lambda)) \\
&= C(0)-C(\Delta\lambda) = \gamma(\Delta\lambda).
\end{aligned}
$$

need to prove consistency

3.3 Data generation on a circle

First, we will discuss how to generate correlated data at n girdded points on a circle when the covarince function is defined and compare above covariance and variogram estimators. Since the observed data are correlated, the covariance function can be written as a function of distance (angle). For the data generation process we will use exponential family and power family covariance function as given below,

$$C(\theta) = C_1 e^{-a|\theta|} \quad a > 0, C_1 > 0 \tag{3.3.1}$$

$$C(\theta) = c_0 - (|\theta|/a)^{\alpha} \quad a > 0, \alpha \in (0,2] \text{ and } c_0 \geq \int_0^{\pi} (\theta/a)^{\alpha} \sin\theta d\theta \tag{3.3.2}$$

where $\theta = i * \Delta\lambda = \pm i * 2\pi/n, i = 1, 2, \cdots, \lfloor n/2 \rfloor$

Clearly, each location is correlated with other $n-1$ locations and $C(\theta) = C(-\theta)$ the variance-covariance matrix Σ is circulant and will be in the following form,

$$
\begin{aligned}
\Sigma &= circ(C(0), C(\delta), C(2\delta), \cdots, C((N-1)\delta), C(\pi), C((N-1)\delta), \cdots, C(\delta)) \\
&= \begin{pmatrix}
C(0) & \cdots & C((N-1)\delta) & C(\pi) & C((N-1)\delta) & \cdots & C(\delta) \\
C(\delta) & \cdots & C((N-2)\delta) & C((N-1)\delta) & C(\pi) & \cdots & C(2\delta) \\
C(2\delta) & \cdots & C((N-3)\delta) & C((N-2)\delta) & C((N-1)\delta) & \cdots & C(3\delta) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
C(\delta) & \cdots & C(\pi) & C((N-1)\delta) & C((N-2)\delta) & \cdots & C(0)
\end{pmatrix} \\
&= Q\Lambda Q^T,
\end{aligned}
$$

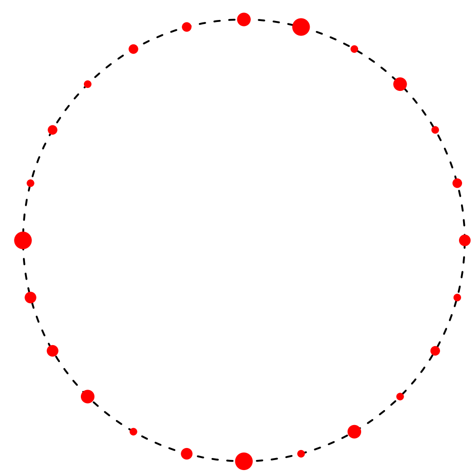

Figure 3.1: Random process on a circle at 24 points ($\Delta\lambda = 15^0$), the red dots represent the observed values at a given time and each point is associated with a random process of it's own.

where $\delta = 2\pi/n$, $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ and $Q = \{\psi_1, \psi_2, \cdots, \psi_n\}$ are the respective eigen values and eigen vectors of the above circulant matrix. Now using singular value decomposition (SVD) we can obtain the correlated data $\underset{\sim}{X}$ on a circle as follows,

$$\underset{\sim}{X} = \Sigma^{1/2} * Z = Q\Lambda^{1/2}Q^T * Z$$

where $Z \sim N(\underset{\sim}{0}, 1_n)$.

3.3.1 Compare covariance estimator

In section 3.2.1 we proved that, in general the covarince estimator (3.2.1) on a circle is biased, with a bias of $var(\bar{X})$. In order to make things simple we set $C_1, a = 1$ when $\alpha = 0.5$ $c_0 \geq \int_0^\pi (\theta)^{0.5} \sin\theta d\theta$, from Fresnel intergal it can be shown that $c_0 \geq 2.4353$. Now we compare the covariance estimator to it's theoretical covarince given by equations 3.3.1 and 3.3.2. We computed the MOM estimator $\hat{C}(\theta)$ with 48 gridded observations on the circle from 500 simulations.

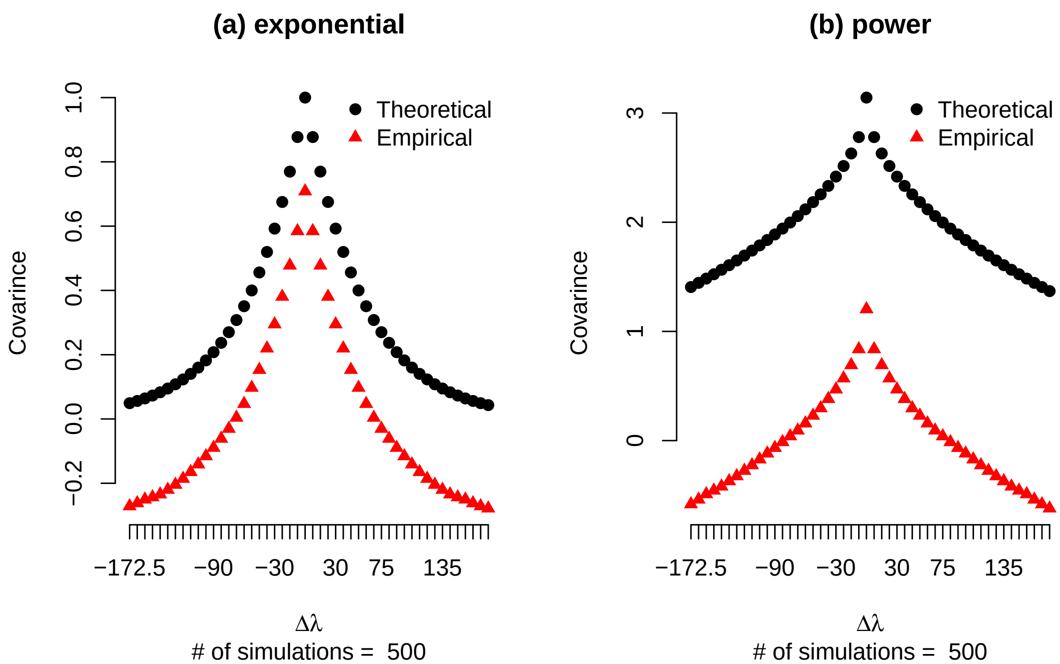

Figure 3.2: Theoretical and empirical covariance (with bias) comparison on a circle

Remark 1: We noticed that the shift between theoretical and empirical values were approximately equal to a_0 and we can obtain a_0 for the above covarince as follows,

$$\text{exponential}: \; a_0 = \frac{C_1}{a\pi}(1 - e^{-a\pi})$$

$$\text{power}: \; a_0 = c_0 - \left(\frac{\pi}{a}\right)^{\alpha} \frac{1}{\alpha + 1}$$

Now we consider the following covariance function, after subtracting a_0 from $C(\theta)$.

$$D(\theta) = C(\theta) - a_0.$$

If the new covariance function $D(\theta)$ was used to generate the data on a circle then the theoretical and empirical values will match perfectly.

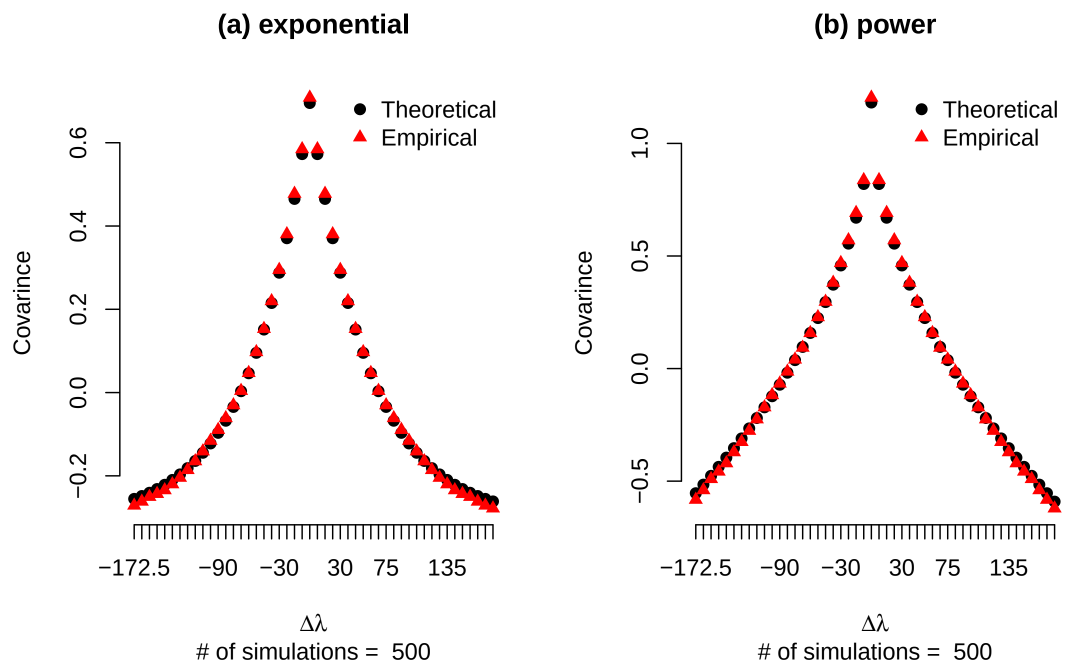

Figure 3.3: Theoretical and empirical covariance comparison on a circle

Remark 2: If the process is a zero mean process the covariance estimator will be unbiased $(i.e.Var(\bar{X}) = 0)$ hence we will get a perfect match between theoretical and empirical values.

Remark 3: We have shown that covariance estiamtor is biased and the biasness will approach to a_0. When covariance function is unknown the biaseness a_0 is also known and the biasness cannnot to estimated (cannot find the variance of $\bar{X}$) from one circle, however in both exponential and power covariance models it can be estimated if multiple copies of data (on a circle) were generated $i.e.$ $\hat{a} = var(\bar{X})$.

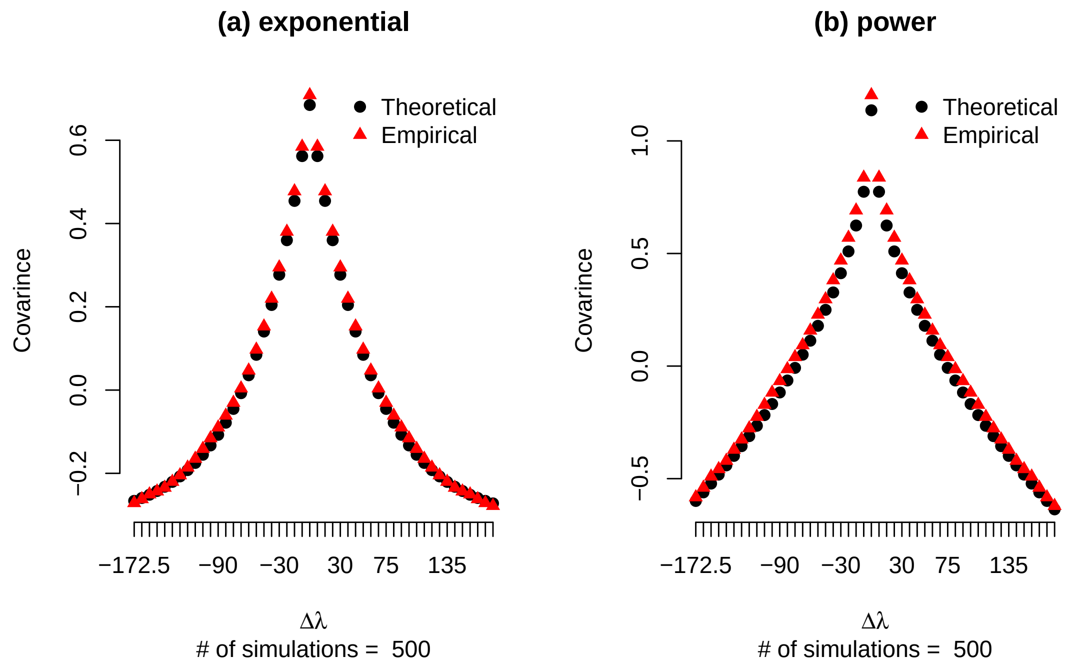

Figure 3.4: Theoretical and empirical covariance comparison on a circle, after removing biasness

3.3.2 Compare variogram estimator

We proved that in general the variogram estimator is unbiased *and consistent*. Since the semi variogram $\gamma(\theta) = C(0) - C(\theta)$, the theoretical variogram based on exponential and power covariance functions can be given in the following form,

$$\text{exponential}: \ \gamma(\theta) = C(0) - C(\theta) = C_1(1 - e^{-a|\theta|})$$

$$\text{power}: \ \gamma(\theta) = C(0) - C(\theta) = C_1(1 - e^{-a|\theta|})$$

We computed the variogram estimator $\hat{\gamma}(\theta)$ with 48 gridded observations on the circle from 500 simulations and there is a better fit between theoretical and empirical values compared to covariance models.

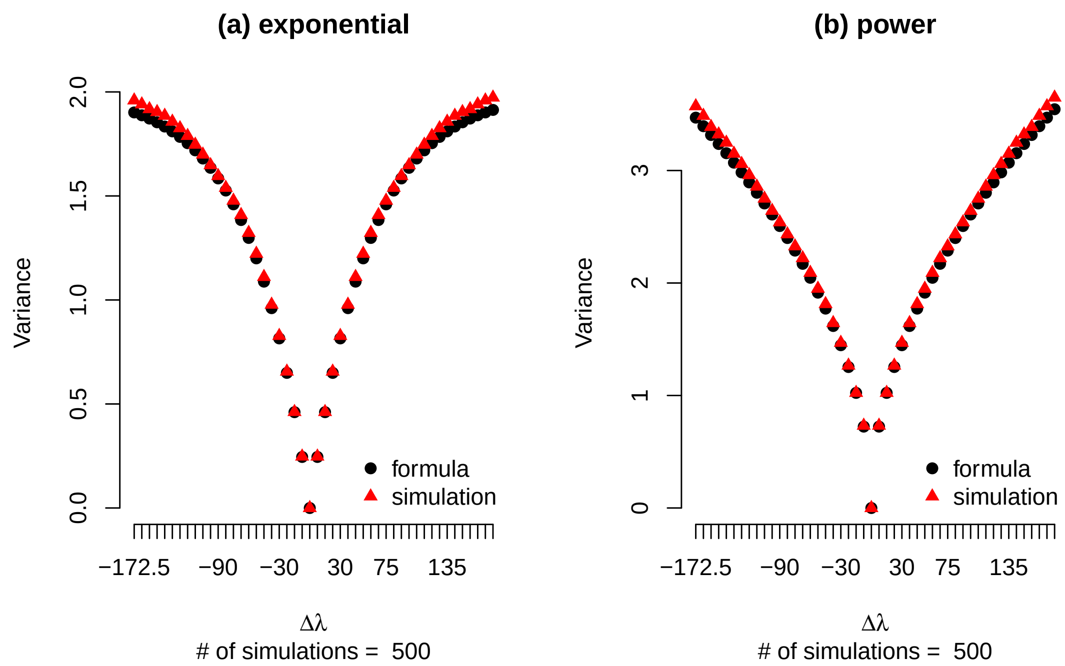

Figure 3.5: Theoretical and empirical comparison for variogram on a circle

Remark 4 *should we talk about how n_L is related with number of simulations*

Chapter 4

Random Process on a Sphere

Suppose $X \in \{X(s) : s \in D\}$, defined in a common probability space $s \in S^2$: $X(s)$ is a random process or a spatial processes on the sphere S^2 with radius r and s represents any location on sphere by latitude L and longitude l, where $0 \geq L \geq \pi$ and $-\pi \geq l \geq \pi$.

4.1 Covariance on sphere

The random process $X(\cdot)$ is said to be homogeneous (or isotropy) on S^2 if second moment is finite and invariant under the rotations on the sphere with constant mean. Similarly, we can define an isoropic random process on a sphere as,

$$\begin{aligned} E(X(s) &= \mu \quad \text{for any } s \in S^2 \\ Cov(X(s_1), X(s_2)) &= C(\theta_{s_1 s_2}) \end{aligned}$$

where $\theta_{s_1 s_2}$ is the spherical angle between two locations s_1, s_2. For a unit sphere, the distance between the two locations can be defined as great circle distance ($\text{gcd}_{s_1 s_2}$) or chordal distance ($ch_{s_1 s_2}$) as follows,

$$\theta_{s_1 s_2} = \arccos\left(\sin(L_1)\sin(L_2) + \cos(L_1)\cos(L_2)\cos(l_1 - l_2)\right)$$

In the case of $\mathbb{R}^d$, non-negative definite is a necessary and sufficient condition for a valid covariance function defined on $\mathbb{R}^d$ (1.1.4). Similarly, a real continuous function $C(\cdot)$ defined on the sphere is a valid covariance function if and only if it is non-negative definite,

$$\sum_{i,j=1}^{N} a_i a_j C(\theta_{s_1 s_2}) \geq 0, \tag{4.1.1}$$

for any integer N, any constants $a_1, a_2, \ldots, a_N$, and any locations $s_1, s_2, \ldots, s_N \in S^2$.

Let $P_k^{\nu}(\cos\theta)$ be the ultraspherical polynomials defined by the following infinite summation,

$$\frac{1}{(1-2c\cos\theta+c^2)^\nu}=\sum_{k=0}^{\infty}c^k P_k^\nu(\cos\theta) \quad \nu>0 \tag{4.1.2}$$

When $\nu=0$, $P_k^0(\cos\theta)=\cos(k\theta)$

Accoriding to Schoenberg (1942), a real continous function $C(\theta)$ is a valid covariance function on S^d, where $d=1,2,\ldots$, if and only if it can be written in the following form

$$C(\theta)=\sum_{k=0}^{\infty}c_k P_k^{(\nu)}(\cos\theta), \quad \nu=\frac{1}{2}(d-1) \tag{4.1.3}$$

where $\forall c_k \geq 0$ and $\sum c_k < \infty$.

Suppose $C(\cdot)$ is a covariance functions that is valid in S^d then it is valid on S^m where $d \leq m$. In general we have the following property,

$$S^1 \subset S^2 \subset \cdots S^d \subset \cdots S^\infty$$

$$C(S^1) \supset C(S^2) \supset \cdots C(S^d) \supset \cdots C(S^\infty)$$

and covariance functions that are valid on S^m may not be valid on S^d where $m > d$ *example?*.

In chapter 3 we discussed about random processes on a circle and the covariance on a circle ($d=1$) was expressed as follows,

$$C(\theta)=\sum_{k=0}^{\infty}c_k\cos(k\theta) \tag{4.1.4}$$

Since $\cos\theta \in S^1$ clearly $\cos\theta \in S^2$ and from the properties of covariance discussed in chapter 1, $P_k\cos\theta \in S^2$ where P_k is a Legender ploynomial. Therefore, when $d=2$ the covarince on a sphere (S^2) can be expressed as follows,

$$C(\theta)=\sum_{k=0}^{\infty}c_k P_k(\cos\theta) \tag{4.1.5}$$

Since the Legendre polynomials are orthogonal we have

$$\int_{-1}^{1}P_n(x)P_m(x)dx=\frac{2}{2n+1}\delta_{nm}$$

and on a sphere the coefficients c_k can be obtained by

$$c_\nu=\frac{2\nu+1}{2}\int_0^\pi C(\theta)P_\nu(\cos\theta)d\theta. \quad \nu=0,1,2,\ldots \tag{4.1.6}$$

One can directly use the above integral to evalute the validity of a covariance function on the sphere by checking if c_k is non-negative and $\sum c_k < \infty$.

All covariance models that are valid on $\mathbb{R}^d$ are not valid on the sphere (S^2), Huang et al. (2011) evaluated the validity of commonly used covariance on a sphere that are valid on $\mathbb{R}^d$, they showed that some models are not valid on the sphere and some models are valid only for certain parameter values.

Model	Covariance function	Validity S^2
Spherical	$\left(1 - \frac{3\theta}{2a} + \frac{1}{2}\frac{\theta^3}{a^3}\right)\mathbf{1}_{(\theta\leq a)}$	Yes
Stable	$\exp\left\{-\left(\frac{\theta}{a}\right)^{\alpha}\right\}$	Yes for $\alpha \in (0,1]$ No for $\alpha \in (1,2]$
Exponential	$\exp\{-\left(\frac{\theta}{a}\right)\}$	Yes
Gaussian	$\exp\left\{-\left(\frac{\theta}{a}\right)^2\right\}$	No
Power*	$c_0 - (\theta/a)^{\alpha}$	Yes for $\alpha \in (0,1]$ No for $\alpha \in (1,2]$
Radon transform of order 2	$e^{-\theta/a}(1+\theta/a)$	No
Radon transform of order 4	$e^{-\theta/a}(1+\theta/a+\theta^2/3a^2)$	No
Cauchy	$(1+\theta^2/a^2)^{-1}$	No
Hole - effect	$\sin a\theta/\theta$	No

Table 4.1: Validity of covariance functions on the sphere, $a > 0, \theta \in [0,\pi]$. *When $\alpha \in (0,1]$, power model is valid on the sphere for some $c_0 \geq \int_0^{\pi}(\theta/a)^{\alpha}\sin\theta d\theta$.

Furthermore, Gneiting (2013) argued that Matérn covariance function is valid on the sphere when the smoothness parameter $\nu \in (0, 1/2)$ and it is not valid if $\nu > 1/2$. Yadrenko (1983) showed that if $K(\cdot)$ is valid isotropic covariance function on $\mathbb{R}^3$ then

$$C(\theta) = K(2\sin(\theta/2))$$

is a valid isotropic covariance function on the unit sphere (where θ is *gcd*).

4.2 Variogram on a sphere

A random process $X(\cdot)$ on a sphere, Huang et al. (2011) defined, if $E(X(s)) = \mu$ for all $s \in S^2$ and $Var(X(s_1) - X(s_2)) = 2\gamma(\theta_{s_1 s_2})$ and for all $s_1, s_2 \in S^2$ then $X(\cdot)$ is intrinsically statioinary on S^2 where $2\gamma(\cdot)$ is the variogram. The variogram is conditionally negative definite

$$\sum_{i,j=1}^{N} a_i a_j 2\gamma(\theta_{s_1 s_2}) \leq 0, \tag{4.2.1}$$

for any integer N, any constants $a_1, a_2, \ldots, a_N$ with $\sum a_i = 0$, and any locations $s_1, s_2, \ldots, s_N \in S^2$. Immeditely from 4.2.2 for a continuous $2\gamma(\cdot)$ with $\gamma(0) = 0$ the variogram is negative definite if and only if

$$\gamma(\theta) = \sum_{k=0}^{\infty} c_k(1 - P_k(\cos\theta)) \tag{4.2.2}$$

where $P_k(\cdot)$ are Legendre polynomials with $\forall c_k \geq 0$ and $\sum c_k < \infty$.

In the introduction we pointed out in $\mathbb{R}^d$ one can always construct the variogram if the covariance function is given but not the converse. However, in S^2 Yaglom (1961) argued that for a valid $\gamma(\theta) \quad \theta \in [0, \pi]$ one can always construct covariance function $C(\theta) = c_0 - \gamma(\theta)$ for some $c_0 \geq \int_0^{\pi} \gamma(\theta)\sin(\theta)d\theta$.

4.3 Random process on a sphere

Jones (1963) showed that a random process on a sphere, can be written as a summation of spherical harmonics $Y_\nu^m(P)$.

A random process $X(P)$ on a unit sphere S^2, where $P = (\lambda, \phi) \in S^2$ where $P = (\lambda, \phi) \in S^2$ with longitude $\lambda \in [-\pi, \pi)$ and latitude $\phi \in [0, \pi]$.. Suppose the porcess is isotropy and continuous in quadratic mean with respect to the location P then the process can be represented by spherical harmonics, $P_\nu^m(\cdot)$ normalized associated Legendre polynomials, with the sum converges in mean square (Li and North (1997); Huang et al. (2012)).

$$X(P) = \sum_{\nu=0}^{\infty} \sum_{m=-\nu}^{\nu} Z_{\nu,m} e^{im\lambda} P_\nu^m(\cos\phi), \tag{4.3.1}$$

Since $\cos(\phi) \in [-1, 1]$ we have $\int_{-1}^{1} [P_\nu^m(cos(\phi))]^2 dcos(\phi) = 1$, and $Z_{\nu,m}$ are complex-valued coefficients satisfying.

$$Z_{\nu,m} = \int_{S^2} X(P) e^{-im\lambda} P_\nu^m(\cos\phi) dP. \tag{4.3.2}$$

Suppose the process $X(P)$ is isotropy with 0 mean (without loss of generality) which implies $E(Z_{\nu,m}) = 0$. Let $P = (\lambda_P, \phi_P)$ and $Q = (\lambda_Q, \phi_Q)$ be two arbitary locations on the sphere, if the covariance function $R(P, Q)$ on S^2 solely depends on the spherical distance between (P, Q) and from 4.2.2, 4.1.6 we can derive the covariance function as follows,

$$
\begin{aligned}
R(P,Q) &= \mathrm{E}(X(P)\overline{X(Q)}) \\
&= \sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty}\sum_{m=-\nu}^{\nu}\sum_{n=-\mu}^{\mu} \mathrm{E}(Z_{\nu,m}\overline{Z}_{\mu,n})e^{im\lambda_P}P_\nu^m(\cos\phi_P)e^{-in\lambda_Q}P_\mu^n(\cos\phi_Q) \\
&= \sum_{\nu=0}^{\infty}\frac{(2\nu+1)f_\nu}{2}P_\nu(\cos\theta_{PQ}). \qquad (4.3.3)
\end{aligned}
$$

where $\bar{Z}$ denotes the complex conjugate of Z, θ_{PQ} is the spherical distance, $P_\nu(\cdot)$ is the Legendre polynomial, and $\sum_{\nu=0}^{\infty}(2\nu+1)f_\nu < \infty$. Here, the random variable $Z_{\nu,m}$ satisfies

$$\mathrm{E}(Z_{\nu,m}\overline{Z}_{\mu,n}) = \delta_{\nu,\mu}\delta_{n,m}f_\nu,$$

where $\delta_{a,b} = 1$ if $a = b$, and 0 otherwise.

4.4 Axially symmetry

In the previous sections we discssed why it is necessary to use S^2 instead of R^3 when studying about random processes on Earth and isotropy is often assumed (Yadrenko (1983); Yaglom (1987)). However, many studies have pointed out this assumption is not resonable (Stein (2007); Jun and Stein (2008); Bolin and Lindgren (2011)) for random processes on the sphere primarly on Earth. Stein (2007) argued that Total Ozone Mapping Spectrometer (TOMS) data varies strongly with latitudes and homogeneous models are not suitable. Moreover, aerosol depth (AOD) from Multi-angle Imaging Spectrometer (MISR), Sea Surface Temprature (SST) from RRMM MIcrowave Imager (TMI) are some other example for anisotropy global data on a sphere (on Earth). In order to study non homogenous processes on the sphere Jones (1963) introduces the concept of axially symmetry, where the covarince between two spatial points depend on the longitudes only through their difference between two points.

A rondom process $X(P) : P \in S^2$ on the sphere and let $R(P,Q)$ be a valid covarince finction on the sphere where $P = (L_P, l_P), Q(L_Q, l_Q)$ then $X(P)$ is axially symmetric if and only if

$$R(L_P, L_Q, l_P, l_Q) = R(L_P, L_Q, l_P - l_Q).$$

Currently, to our knowledge there are no methods to test axially symmetry in real data. However, this assumption is more plausible and reasonable when modeling spatial data. For example, temperature, moisture, etc. most likely symmetric on longitudes rather than latitudes. Stein (2007) propose a method to model axially symmetric process on a sphere (the fitted model is not the best, but this was a good start). When modeling spatial data stationary models are less useful; but using the concept of axially symmetry Jun and Stein (2008) proposed a flexible class of parametric covariance models to capture the non-stationarity of global data. Hitczenko and Stein (2012) discussed about the properties of an existing class of models for axially symmetric Gaussian processes on the sphere. They applied first-order differential operators to an isotropic process. Huang et al. (2012) developed a new representation of axially symmetric process on the sphere and futher introduced some parametric covariance models that are valid on S^2.

if the process is axially symmetric $E(Z_{\nu,m}\overline{Z}_{\mu,n})$ can be expressed as,

$$\mathrm{E}(Z_{\nu,m}\overline{Z}_{\mu,n}) = \delta_{n,m} f_{\nu,\mu,m}.$$

Hence, for an axially symmetric process the covariance function 4.3.3 will be the following form (Huang et al. (2012))

$$\begin{aligned} R(P,Q) &= R(\phi_P, \phi_Q, \lambda_P - \lambda_Q) \\ &= \sum_{m=-\infty}^{\infty} \sum_{\nu=|m|}^{\infty} \sum_{\mu=|m|}^{\infty} f_{\nu,\mu,m} e^{im(\lambda_P - \lambda_Q)} P_\nu^m(\cos\phi_P) P_\mu^m(\cos\phi_Q). \end{aligned} \tag{4.4.1}$$

Inorder to have a valid covariance function, $f_{\nu,\mu,m} = \overline{f}_{\mu,\nu,m}$ and for each fixed integer m, the matrix $F_m(N) = \{f_{\nu,\mu,m}\}_{\nu,\mu=|m|,|m|+1,\ldots,N}$ must be positive definite for all $N \geq |m|$.

$$R(P,Q) = R(\phi_P, \phi_Q, \Delta\lambda) = \sum_{m=-\infty}^{\infty} e^{im\Delta\lambda} C_m(\phi_P, \phi_Q) \quad m = 0, \pm1, \pm2, ... \tag{4.4.2}$$

where $\Delta\lambda \in [-\pi, \pi]$ and $\phi_P, \phi_Q \in [0, \pi]$

4.4.1 Properties of $C_m(\phi_P, \phi_Q)$

The covariance function $R(P,Q)$ based on the concept of axially symmetry is clearly defined by both latitudes and longitudes (difference). The following conditions for $C_m(\phi_P, \phi_Q)$ are very important to have a valid covaraince function defined by 4.4.2.

- Hermitian and positive definite.
- $\sum_{m=-\infty}^{\infty} |C_m(\phi_P, \phi_Q)| < \infty$ for $m = 0, \pm1, \pm2$, ...
- Is a continuous function.

One can use inverse Fourier transformation to derive C_m based on an axially symmetric covariance function $R(P,Q)$ defined on a sphere, as we have

$$C_m(\phi_P, \phi_Q) = \frac{1}{2\pi} \int_{-\pi}^{\pi} R(\phi_P, \phi_Q) e^{-im\Delta\lambda} d\Delta\lambda$$

Since $C_m(\phi_P, \phi_Q)$ is continuous and both Hermitian and positive definite, Mercer's theorem (1.1.1) can be directly applied to $C_m(\phi_P, \phi_Q)$ such that there exists an orthonormal basis $\{\psi_{m,\nu}, \nu = 0, 1, \ldots\}$ in L^2 (Huang et al. (2012)) and $C_m(\phi_P, \phi_Q)$ can be given by,

$$C_m(\phi_P, \phi_Q) = \sum_{\nu=0}^{\infty} \eta_{m,\nu}\psi_{m,\nu}(\phi_P)\overline{\psi_{m,\nu}(\phi_Q)},$$

Now the covariance function on a sphere is given by,

$$R(P, Q) = \sum_{m=-\infty}^{\infty}\sum_{\nu=0}^{\infty} \eta_{m,\nu}e^{im\Delta\lambda}\psi_{m,\nu}(\phi_P)\overline{\psi_{m,\nu}(\phi_Q)}, \tag{4.4.3}$$

Where $\Delta\lambda \in [0, \pi]$, $\eta_{m,\nu} \geq 0$ and $\psi_{m,\nu}(\cdot)$ are the eigen values eigen functions of $C_m(\phi_P, \phi_Q)$ respectively.

In general for covariance function defined on a sphere (Stein (2007)) requires triple summation and required to estimate $O(n^3)$ parameters. In contrast, the covariance function 4.4.2 defined by Huang et al. (2012) requires to estimate $O(n^2)$ parameters which is a huge reduction of comoputational compelextity and we will continue to use this covariance model in our approach on global data generation which is discussed in the next chapter.

Let's consider a real-valued process with a complex valued $C_m(\phi_P, \phi_Q)$ as given below,

$$\begin{aligned} C_m(\phi_P, \phi_Q) &= c_m f(\phi_P, \phi_Q)e^{i\omega_m(\phi_P - \phi_Q)}, \quad c_m \geq 0, \omega_m \in R \\ &= c_m C_m^R(\phi_P, \phi_Q) + ic_m C_m^I(\phi_P, \phi_Q). \end{aligned}$$

Huang et al. (2012) states that if a process is real-valued then the corresponding covariance function $R(P, Q)$ is also real-valued and $C_m(\phi_P, \phi_Q) = \overline{C_{-m}(\phi_P, \phi_Q)}$, The covarince function $R(P, Q)$ on the sphere given by 4.4.2 can be simplified to the following form,

$$\begin{aligned} R(P, Q) &= C_0(\phi_P, \phi_Q) + \sum_{m=1}^{\infty} e^{-im\Delta\lambda}C_{-m}(\phi_P, \phi_Q) + \sum_{m=1}^{\infty} e^{im\Delta\lambda}C_m(\phi_P, \phi_Q) \\ &= C_0(\phi_P, \phi_Q) + \sum_{m=1}^{\infty} c_m e^{-im\Delta\lambda}(C_m^R(\phi_P, \phi_Q) - iC_m^I(\phi_P, \phi_Q)) \\ &\quad + \sum_{m=1}^{\infty} c_m e^{im\Delta\lambda}(C_m^R(\phi_P, \phi_Q) + iC_m^I(\phi_P, \phi_Q)) \\ &= c_o C_0^R(\phi_P, \phi_Q) + 2\sum_{m=1}^{\infty} c_m[\cos(m\Delta\lambda)C_m^R(\phi_P, \phi_Q) - \sin(m\Delta\lambda)C_m^I(\phi_P, \phi_Q)]. \end{aligned}$$

There are several covariance models, $R(P, Q)$, valid on a sphere were suggested by Huang et al. (2012) by carefully choosing values for c_m.

Model	c_m	paramters
model 1	: $c_m = Cp^m$	$m = 0, \pm1, \pm2, ...$ $\quad p \in (0, 1)$
model 2	: $c_m = \frac{Cp^m}{m}$ and $c_0 = 0$	$m = \pm1, \pm2, ...$ $\quad p \in (0, 1)$
model 3	: $c_m = \frac{C}{m^4}$ and $c_0 = 0$	$m = \pm1, \pm2, ...$

Table 4.2: some proposed c_m models

4.5 Longitudinally reversibile process

The idea was first introduced by Stein (2007). Suppose $K(\cdot)$ is a valid covariance function defined on a sphere where,

$$K(L_1, L_2, l_1 - l_2 = K(L_1, L_2, l_2 - l_1) \tag{4.5.1}$$

then underline process is said to be logitudinally reversible. For example the covariance model proposed by Huang et al. (2012) clearly yileds a longitudinally reveresible process as $R(\phi_P, \phi_Q, \Delta\lambda) = R(\phi_P, \phi_Q, -\Delta\lambda)$ and the reversibility holds when $C_{-m}(\phi_P, \phi_Q) = C_m(\phi_P, \phi_Q)$. Now the covariance function reduces to the following,

$$R(P, Q) = \sum_{m=0}^{\infty} C_m(\phi_P, \phi_Q) \cos(m\Delta\lambda)$$

If a random process on the sphere is real valued and longitudinally reversible so is the covariance function, $R(P, Q)$, is real valued then $C_m(\phi_P, \phi_Q)$ is real since $C_{-m}(\phi_P, \phi_Q) = C_m(\phi_P, \phi_Q)$ and $C_m(\phi_P, \phi_Q) = \overline{C_{-m}(\phi_P, \phi_Q)}$.

Chapter 5

Global Data Generation on the Sphere

5.1 Theoretical development

The global data generation process on a sphere discussed on this dissertation is primarly based on the axially symemrtic covariance structure introduced by Jones (1963) and as continuation of axially symmetric process on a sphere developed by Huang et al. (2012). Let $X(P)$ be a complex-valued random process defined on a unit sphere S^2, where $P = (\lambda, \phi) \in S^2$ with longitude $\lambda \in [-\pi, \pi)$ and latitude $\phi \in [0, \pi]$.In chapter 4 we discussed how to formualte a valid covariance function for continuous axially symmetric processes on a sphere and was given by 4.4.2:

$$R(P, Q) = R(\phi_P, \phi_Q, \Delta\lambda) = \sum_{m=-\infty}^{\infty} e^{im\Delta\lambda} C_m(\phi_P, \phi_Q)$$

where $\Delta\lambda \in [-\pi, \pi]$, and $C_m(\phi_P, \phi_Q)$ is Hermitian and $p.d.$ with $\sum_{-\infty}^{\infty} |C_m(\phi_P, \phi_Q)| < \infty$.

Since $C_m(\phi_P, \phi_Q)$ is continuous, Hermitian and positive definite Mercer's theorem (1.1.1) was appllied for $C_m(\phi_P, \phi_Q)$ hence the covariance function defined by 4.4.2 was given by the following 4.4.3:

$$R(P, Q) = \sum_{m=-\infty}^{\infty} \sum_{\nu=0}^{\infty} \eta_{m,\nu} e^{im\Delta\lambda} \psi_{m,\nu}(\phi_P) \overline{\psi_{m,\nu}(\phi_Q)},$$

Now, in the light of above two equations a continuous axially symmetric process, $X(P)$ on a unit sphere (Huang et al. (2012)[remark 2.5]), is given as:

$$X(P) = X(\phi, \lambda) = \sum_{m=-\infty}^{\infty} W_{m,\nu}(\phi) e^{im\lambda} \psi_{m,\nu}(\phi), \tag{5.1.1}$$

where λ is the longitude, ϕ is the latitude and $\psi_{m,\nu}(\cdot)$ is a orthonormal basis for $C_m(\phi_P, \phi_Q)$ and using inverse Fourier transformation we get

$$W_m(\phi) = \frac{1}{2\pi} \int_{S^2} X(P) e^{-im\lambda} \overline{\psi_{m,\nu}(\phi)} dP,$$

with $\mathrm{E}(W_{m,\nu} \overline{W_{n,\nu}}) = \delta_{m,n} \delta_{m,\mu} \eta_{m,\nu}$.

Remark 1 If the random process on a sphere is real-Gaussian, then the weights given by $W_{m,\nu}$ will be independent normal random variables. Furthermore, if ν is fixed the process defined by $X(P)$ will be equivalent to a homogeneous random process on a circle with angular distance $\Delta\lambda$. In other words a random process on a sphere at a given latitude (for fix ϕ) can be studies as a random process on the circle. In Chaper 3 a random process on a circle was given by a infinite Fourier summation (Roy (1972),Dufour and Roy (1976)) and in the case of a process on a circle, $X(P)$ (5.1.1) can be given by

$$X(\phi, \lambda) = \sum_{m=-\infty}^{\infty} W_m(\phi) e^{im\lambda}, \tag{5.1.2}$$

$$\text{where } W_m(\phi) = \frac{1}{2\pi} \int_0^{2\pi} X(\phi, \lambda) e^{-im\lambda} d\lambda,$$

with $\mathrm{E}(W_m(\phi_P) \overline{W_n(\phi_Q)}) = \delta_{m,n} C_m(\phi_P, \phi_Q)$.

5.2 Generalization of parametric models

How to discuss/link about other parametic models suggested by Jeong and Jun (2015), Jun and Stein (2008), etc... and why are we using Huang et al. (2012) and what are the advantages?

The covariance function on sphere, $R(P, Q)$, given in equation 4.4.2, is clearly a function of both longitude and latitude.

$$R(P, Q) = f(\Delta\lambda, \phi_P, \phi_Q)$$

In order to make things easier one could assume that $C_m(\phi_P, \phi_Q) = \tilde{C}_m(\phi_P - \phi_Q)$ only depends on the difference of ϕ_P and ϕ_Q, Huang et al. (2011) proposed a simple separable covariance function when both covariance components are exponential

$$R(P, Q) = c_0 e^{-a|\Delta\lambda|} e^{-b|\phi_P - \phi_Q|},$$

Where a and b are defined as decay parameters in longitude and latitude respectively.

Since a real-valued random process on sphere has a real-valued covariance fucnction and in order to have a non sperable covariance model Huang et al. (2012) considered a complex-valued $C_m(\phi_P, \phi_Q)$ which is clearly both Hermitian and positive as given below,

$$
\begin{aligned}
C_m(\phi_P, \phi_Q) &= c_m e^{-a_m|\phi_P - \phi_Q|} e^{\omega_m(\phi_P - \phi_Q)}, \quad c_m \geq 0, a_m \geq 0, \omega_m \in R. \\
C_m(\phi_P, \phi_Q) &= C_{m,R}(\phi_P, \phi_Q) + iC_{m.I}(\phi_P, \phi_Q), \quad C_m(\phi_P, \phi_Q) = \overline{C_{-m}(\phi_P, \phi_Q)}
\end{aligned}
$$

We define the following,

$$
\begin{aligned}
C_{0,R}(\phi_P, \phi_Q) &= c_0 e^{-a_0|\phi_P - \phi_Q|} \cos \omega_0(\phi_P - \phi_Q), \\
C_{m,R}(\phi_P, \phi_Q) &= c_m e^{-a_m|\phi_P - \phi_Q|} \cos \omega_m(\phi_P - \phi_Q), \\
C_{m,I}(\phi_P, \phi_Q) &= c_m e^{-a_m|\phi_P - \phi_Q|} \sin \omega_m(\phi_P - \phi_Q).
\end{aligned}
$$

Clearly, $C_m(\phi_P, \phi_Q)$ is both Hermitian and positive definite then $R(P, Q)$ can be given as follows,

$$
\begin{aligned}
R(P, Q) &= c_0 e^{-a_0|\phi_P - \phi_Q|} \cos \omega_0(\phi_P - \phi_Q) + \sum_{m=-\infty}^{-1} e^{im\Delta\lambda} C_m(\phi_P, \phi_Q) + \sum_{m=1}^{\infty} e^{im\Delta\lambda} C_m(\phi_P, \phi_Q) \\
&= c_0 e^{-a_0|\phi_P - \phi_Q|} \cos \omega_0(\phi_P - \phi_Q) + \sum_{m=1}^{\infty} e^{-im\Delta\lambda} C_{-m}(\phi_P, \phi_Q) + \sum_{m=1}^{\infty} e^{im\Delta\lambda} C_m(\phi_P, \phi_Q) \\
&= c_{0,R}(\phi_P, \phi_Q) + 2\sum_{m=1}^{\infty} cos(m\Delta\lambda) C_{m,R}(\phi_P, \phi_Q) - sin(m\Delta\lambda) C_{m,I}(\phi_P, \phi_Q) \\
&= c_0 e^{-a_0|\phi_P - \phi_Q|} \cos \omega_0(\phi_P - \phi_Q) \\
&\quad + 2\sum_{m=1}^{\infty} \cos(m\Delta\lambda)[c_m e^{-a_m|\phi_P - \phi_Q|} \cos(\omega_m(\phi_P - \phi_Q))] \\
&\quad - \sin(m\Delta\lambda)[c_m e^{-a_m|\phi_P - \phi_Q|} \sin(\omega_m(\phi_P - \phi_Q))] \\
&= c_0 e^{-a_0|\phi_P - \phi_Q|} cos \omega_m(\phi_P - \phi_Q) + 2\sum_{m=1}^{\infty} c_m e^{-a_m|\phi_P - \phi_Q|}[\cos(m\Delta\lambda)\cos(\omega_m(\phi_P - \phi_Q)) \\
&\quad - \sin(m\Delta\lambda)\sin(\omega_m(\phi_P - \phi_Q))] \\
&= c_0 e^{-a_0|\phi_P - \phi_Q|} cos \omega_m(\phi_P - \phi_Q) + 2\sum_{m=1}^{\infty} c_m e^{-a_m|\phi_P - \phi_Q|} \cos[m\Delta\lambda + \omega_m(\phi_P - \phi_Q)].
\end{aligned}
$$

Set $a_m = a$, $\omega_m = mu$ and we can get the following form for $R(P, Q)$,

$$
R(P, Q) = c_0 e^{-a|\phi_P - \phi_Q|} + 2e^{-a|\phi_P - \phi_Q|} \sum_{m=1}^{\infty} c_m \cos[m\theta(P, Q, u)],
$$

where $\theta(P, Q, u) = \Delta\lambda + u(\phi_P - \phi_Q) - 2k\pi$, and k is chosen such that $\theta(P, Q, u) \in [0, 2\pi]$.

Moreover, by carefully choosing functions for $C_m(\phi_P, \phi_Q)$ Huang et al. (2012) proposed some nonseparable covariance models ($R(P,Q)$) models valid on the sphere,

$$R(P,Q) = Ce^{-a|\phi_P-\phi_Q|}\frac{1-p^2}{1-2p\cos\theta(P,Q,u)+p^2} \tag{5.2.1}$$

$$R(P,Q) = Ce^{-a|\phi_P-\phi_Q|}\log\frac{1}{(1-2p\cos\theta(P,Q,u)+p^2)} \tag{5.2.2}$$

$$R(P,Q) = 2Ce^{-a|\phi_P-\phi_Q|}\left(\frac{\pi^4}{90}-\frac{\pi^2\theta^2(P,Q,u)}{12}+\frac{\pi\theta^3(P,Q,u)}{12}-\frac{\theta^4(P,Q,u)}{48}\right) \tag{5.2.3}$$

Modifying the covariance models to include non-stationarity

In this section we will discuss one big disadvantages of the covariance models proposed by Huang et al. (2012) and some soluntions to over come the disadvantages. The biggest disadvantage for all of them are that it is assumed not only stationarity on longitudes, but stationarity on latitudes as well.

1. We have noticed that when $\phi_P = \phi_Q$, the model 5.2.1 reduces to

$$R(P,P) = C\frac{1-p^2}{1-2p\cos(\Delta\lambda)+p^2}$$

and if we set $\Delta\lambda = 0$, the variance of latitude ϕ_P over all latitudes can be given by,

$$Var(P) = C\frac{1+p}{1-p}$$

is not a function of the latitude (a function of the parameter p) and it implies that variance is constant over latitudes. This is not supposed to be the case, since both MSU data and TOMS data in figures 2.4 and 2.7 shows that variance is highly depending on the latitude.

2. We propose a modification to the above approach by replacing the $C_m(\phi_P, \phi_Q)$ function,

$$C(\phi_P, \phi_Q) = Ce^{-a|\phi_P-\phi_Q|}$$

by a non-stationary covariance function, which depends on the latitudes, even when $\phi_P = \phi_Q$. Consider the below two functions for $C_m(\phi_P, \phi_Q)$.

$$\tilde{C}(\phi_P, \phi_Q) = C_1(C_2 - e^{-a|\phi_P|} - e^{-a|\phi_Q|} + e^{-a|\phi_P-\phi_Q|}) \tag{5.2.4}$$

$$\tilde{C}(\phi_P, \phi_Q) = C_1\left(C_2 - \frac{1}{\sqrt{a^2+\phi_P^2}} - \frac{1}{\sqrt{a^2+\phi_Q^2}} + \frac{1}{\sqrt{a^2+(\phi_P-\phi_Q)^2}}\right) \tag{5.2.5}$$

Here $C_1, a > 0$, and $C_2 \geq 1$ to ensure the positive definiteness of the above function. When $\phi_P = \phi_Q$, both functions are actually a function of ϕ_P.

$$\begin{aligned}\tilde{C}(\phi_P, \phi_P) &= C_1(C_2 - 2e^{-a|\phi_P|} + 1), \\ \tilde{C}(\phi_P, \phi_P) &= C_1\left(C_2 - \frac{2}{\sqrt{a^2 + \phi_P^2}} + \frac{1}{a}\right).\end{aligned}$$

Proposition 5.2.1 *A more general non stationary covariance function is given as following. If $C(\cdot) = C(x - y)$ is the stationary covariance function and $f(\omega) \geq 0$ is the corresponding spectral density, then*

$$\tilde{C}(x, y) = C_2 - C(x) - C(y) + C(x - y),$$

with

$$C_2 \geq \int_{-\infty}^{\infty} dF(\omega) = \int_{-\infty}^{\infty} f(\omega) d\omega > 0$$

is the non stationary covariance function. Note that the covariance function $C(\cdot)$ implies that, by Bochner's theorem, there exists a bounded measure F such that

$$C(x) = \int_{-\infty}^{\infty} e^{-ix\omega} dF(\omega).$$

When $F(\cdot)$ is absolutely continuous, there exists a spectral density $f(\cdot) \geq 0$ such that

$$C(x) = \int_{-\infty}^{\infty} e^{-ix\omega} f(\omega) d\omega.$$

Now we choose a sequence of complex numbers $a_i, i = 1, 2, \cdots, n$, and any sequence of real numbers $t_i, i = 1, 2, \cdots, n$, taking $C_2 = \int_{-\infty}^{\infty} f(\omega) d\omega$,

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n} a_i \overline{a}_j \tilde{C}(t_i, t_j) &= \sum_i \sum_j a_i \overline{a}_j (C_2 - C(t_i) - C(-t_j) + C(t_i - t_j)) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i \overline{a}_j \int_{-\infty}^{\infty} (1 - e^{-it_i\omega} - e^{it_j\omega} + e^{-i(t_i - t_j)\omega}) f(\omega) d\omega \\ &= \int_{-\infty}^{\infty} f(\omega) d\omega \left| \sum_{i=1}^{n} a_i (e^{-it_i\omega} - 1) \right|^2 \geq 0.\end{aligned}$$

So we propose six five-parameter models which are combinations of both $\tilde{C}(\phi_P, \phi_Q)$, defined by a exponential family 5.2.4 and a power family 5.2.5, and models (5.2.1, 5.2.2, 5.2.3) proposed by Huang et al. (2012) for the covariance on a sphere defined as follows,

$$R(P, Q) = \tilde{C}(\phi_P, \phi_Q) C(\theta(P, Q, u)),$$

where $\theta(P, Q, u) = \Delta\lambda + u(\phi_P - \phi_Q) \in [0, 2\pi]$, $C_1 > 0, C_2 > 0, a > 0, u \in \mathbb{R}, p \in (0, 1)$.

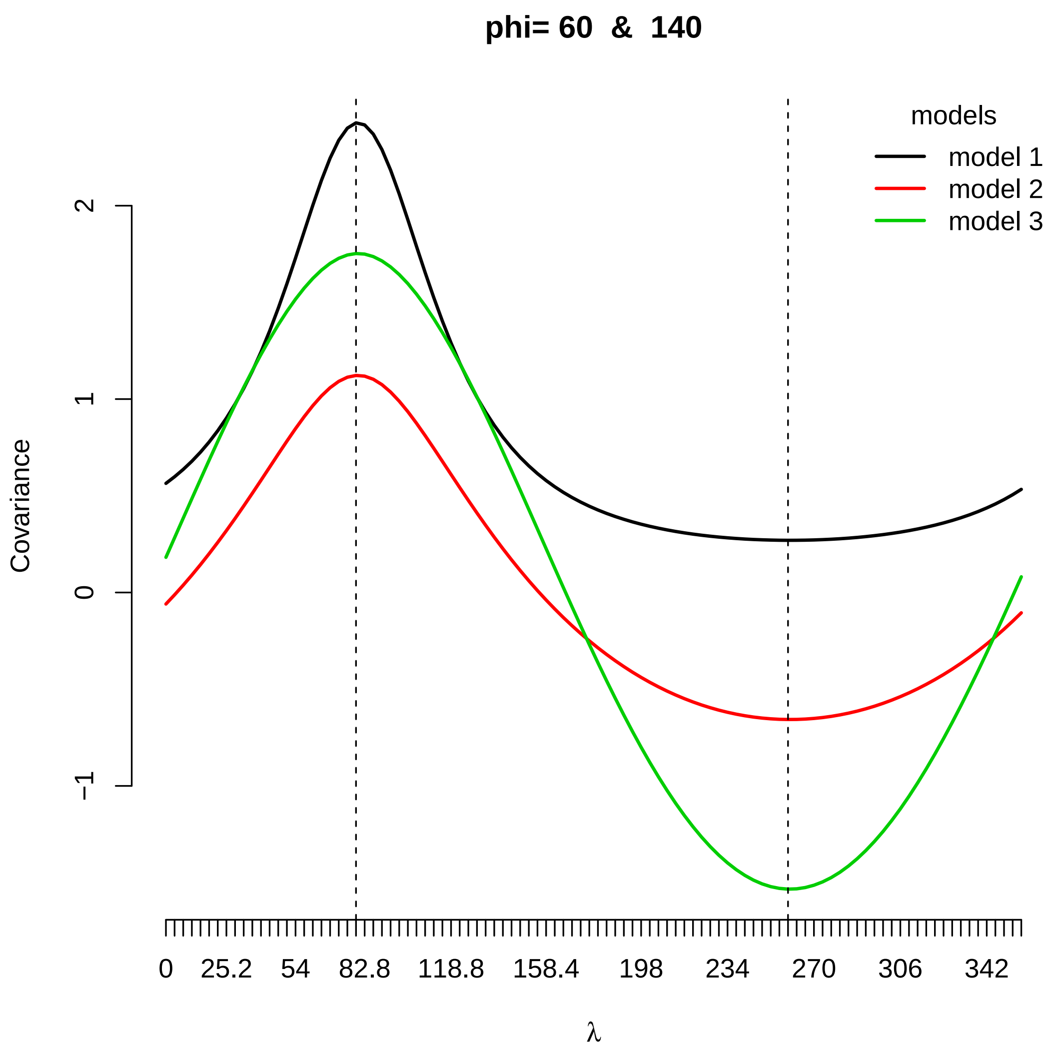

Figure 5.1: The covariance between $30^0 S$ and $50^0 N$ (latitude 60^0 and 140^0) of three covariance models with exponential family $i.e.\tilde{C}(\phi_P, \phi_Q)$ given by 5.2.4 over 100 longitudes for simplicity we set all parameters to be one.

Remark 1 The paramters C_1, C_2, a, p are scalling paramters of the covariance functions and u is a location parameter. All covariance models have a similar pattern and share one property, when there is no location shift ($u = 1$) the maximum of $R(P, Q)$ occurs at $\lambda_{max} = |\phi_P - \phi_Q|$ and the minimum of $R(P, Q)$ occurs at $\lambda_{min} = \pi + \lambda_{max}$.

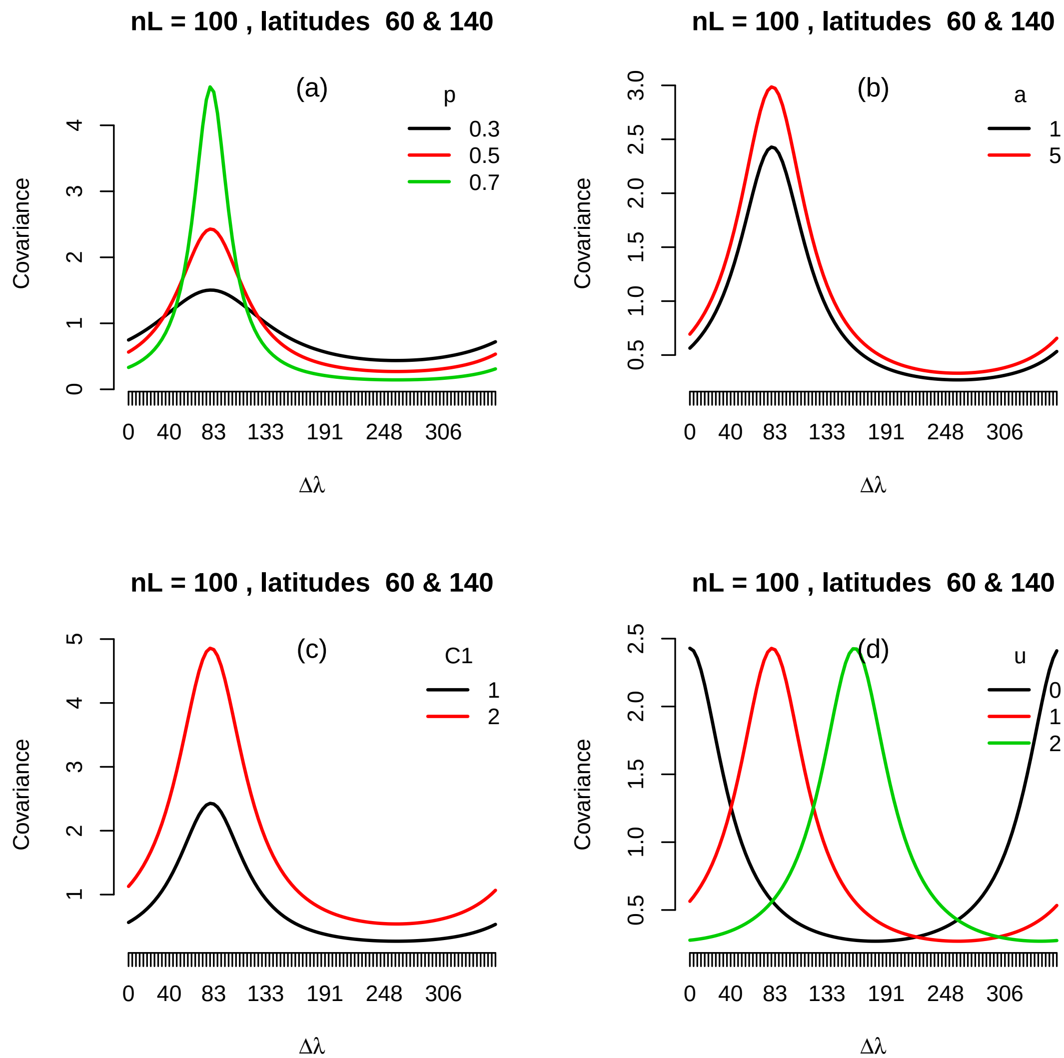

Figure 5.2: Covariance distribution for different paramter using model1: (a)-parameter p, (b)-parameter a, (c)-parameter $C1$ (similar pattern for parameter $C2$), (d)-parameter u

5.3 Method development

We can construct normal independent (complex) random variate $W_m(\phi)$ associated with the variance-covariance matrix $C_m(\phi_P, \phi_Q)$ to construct an axially symmetric process for a given latitude ϕ. Then finite summation can be used to approximate above (5.1.2) infinite summation as given below,

$$X(P) = X(\phi, \lambda) = \sum_{m=-N}^{N} W_m(\phi) e^{im\lambda} \tag{5.3.1}$$

where this would provide the gridded data. Since W_m's are independent for $m = 1, 2, \cdots$, we have

$$\begin{aligned}
Cov(X(P), X(Q)) &= Cov\left(\sum_{m=-N}^{N} W_m(\phi_P)e^{im\lambda_P}, \sum_{j=-N}^{N} W_j(\phi_Q)e^{ij\lambda_Q}\right) \\
&= \sum_{m,j} e^{im\lambda_P}e^{-ij\lambda_Q}Cov(W_m(\phi_P), W_j(\phi_Q)) \\
&= \sum_{m} e^{im(\lambda_P-\lambda_Q)}C_m(\phi_P, \phi_Q)
\end{aligned}$$

The above generated data will be complex random variates. Therefore to have the real-valued data observations or to obtain a real process, we need to have

$$C_{-m}(\phi_P, \phi_Q) = \overline{C_m(\phi_P, \phi_Q)}, \quad \text{for } m = 1, 2, \cdots, N \tag{5.3.2}$$

Lets write $W_m(\phi) = W_m^r(\phi) + iW_m^i(\phi)$ in terms of a real component and an imaginary component. We also write $C_m(\phi_P, \phi_Q) = C_m^r(\phi_P, \phi_Q) + iC_m^i(\phi_P, \phi_Q)$ and with the relationship 5.3.2 above, we have

$$C_{-m}^r(\phi_P, \phi_Q) = C_{-m}^r(\phi_P, \phi_Q), \quad C_{-m}^i(\phi_P, \phi_Q) = -C_{-m}^i(\phi_P, \phi_Q).$$

Now,

$$\begin{aligned}
Cov(W_m(\phi_P), W_m(\phi_Q)) &= Cov(W_m^r(\phi_P) + iW_m^i(\phi_P), W_m^r(\phi_Q) + iW_m^i(\phi_Q)) \\
&= \left[Cov(W_m^r(\phi_P), W_m^r(\phi_Q)) + Cov(W_m^i(\phi_P), W_m^i(\phi_Q))\right] \\
&\quad +i\left[-Cov(W_m^r(\phi_P), W_m^i(\phi_Q)) + Cov(W_m^i(\phi_P), W_m^r(\phi_Q))\right] \\
&= C_m^r(\phi_P, \phi_Q) + iC_m^i(\phi_P, \phi_Q).
\end{aligned}$$

If we let $W_{-m}(\phi) = \overline{W_m(\phi)}$, then the covariance function would satisfy the above relationship 5.3.2. In addition, we will set the following,

$$Cov(W_m^r(\phi_P), W_m^r(\phi_Q)) = Cov(W_m^i(\phi_P), W_m^i(\phi_Q)) = \frac{1}{2}C_m^r(\phi_P, \phi_Q), \tag{5.3.3}$$

$$Cov(W_m^i(\phi_P), W_m^r(\phi_Q)) = -Cov(W_m^r(\phi_P), W_m^i(\phi_Q)) = \frac{1}{2}C_m^i(\phi_P, \phi_Q). \tag{5.3.4}$$

Therefore, if we denote $\underset{\sim}{W}_m(\phi) = (W_m^r(\phi), W_m^i(\phi))^T$, then the variance-covariance matrix for $\underset{\sim}{W}_m(\phi)$ is given by

$$\frac{1}{2}\begin{pmatrix} C_m^r(\phi_P, \phi_Q) & -C_m^i(\phi_P, \phi_Q) \\ C_m^i(\phi_P, \phi_Q) & C_m^r(\phi_P, \phi_Q) \end{pmatrix}.$$

However, we cannot have a vector of random variables $\underset{\sim}{W}_m(\phi)$ with a non-symmetric variance-covariance matrix unless $C_m^i(\phi_P, \phi_Q) = 0$. In the next section we will demonstrate how to generate $\underset{\sim}{W}_m(\phi)$ with a symmetric variance-covariance

The process given by (5.1.2) is now simplified as the following (real) process,

$$\begin{aligned}
X(P) &= \sum_{m=-N}^{N} W_m(\phi)e^{im\lambda} = W_0(\phi) + \sum_{m=1}^{N} W_m(\phi)e^{im\lambda} + \sum_{m=-1}^{-N} W_m(\phi)e^{im\lambda} \\
&= W_0(\phi) + \sum_{m=1}^{N} W_m(\phi)e^{im\lambda} + \sum_{m=1}^{N} \overline{W_m(\phi)}e^{-im\lambda} \\
&= W_0(\phi) + \sum_{m=1}^{N} \big[(W_m^r(\phi) + iW_m^i(\phi))(\cos(m\lambda) + i\sin(m\lambda)) \\
&\quad + (W_m^r(\phi) - iW_m^i(\phi))(\cos(m\lambda) - i\sin(m\lambda))\big] \\
&= W_0(\phi) + 2\sum_{m=1}^{N} \left[W_m^r(\phi)\cos(m\lambda) - W_m^i(\phi)\sin(m\lambda)\right]. \qquad (5.3.5)
\end{aligned}$$

5.3.1 Data generation

Now for each fixed $m = 0, 1, 2, \cdots, N$, we consider $W_m(\phi) = W_m^r(\phi) + iW_m^i(\phi)$ then $W_m^*(\phi) = W_m^r(\phi) - iW_m^i(\phi)$ (where $W_m^*(\phi)$ is the complex conjugate of $W_m(\phi)$). We may assume that $W_m^r(\phi)$ and $W_m^i(\phi)$ are independent, each following a (Gaussian) distribution with mean zero and the same variance $\sigma_m^2(\phi) = \frac{1}{2}C_m^r(\phi,\phi)$, ($C_m^i(\phi,\phi) = 0$ implies $W_m^r(\phi)$ and $W_m^i(\phi)$ are uncorrelated, or independent for Gaussian). In chapter 1 we introduced the concept of circularly-symmetry, thus according to Gallager (2008) a complex random variable is circularly-symmetric if and only if it pseudo covariance is zero (1.1.2). In this section we will show that the Gaussian random variable $W_m(\phi)$ is a circularly-symmetric complex random variable.

Now for a set of distinct latitudes $\Phi = \{\phi_1, \phi_2, \cdots, \phi_{n_l}\}$, we consider a sequence of complex random variables $\{W_m(\phi) : \phi \in \Phi\}$, which forms a multivariate complex random vector $\underset{\sim}{W}_m = (W_m(\phi_1), W_m(\phi_2), \cdots, W_m(\phi_n))^T$ where $W_m(\phi_i) = W_m^r(\phi_i) + iW_m^r(\phi_i)$ with associated $2 \times n_l$-dimensional real random vector

$$\underset{\sim}{V}_m = (W_m^r(\phi_1), W_m^i(\phi_1), W_m^r(\phi_2), W_m^i(\phi_2), \cdots, W_m^r(\phi_{n_l}), W_m^i(\phi_{n_l}))^T.$$

Now we calculate the covariance matrix $K_W = E(\underset{\sim}{W}_m \underset{\sim}{W}_m^*)$ (where $\underset{\sim}{W}_m^*$ is the conjugated transpose) and pseudo-covariance $M_W = E(\underset{\sim}{W}_m \underset{\sim}{W}_m^T)$. Further, from 1.1.2 a complex random vector is circularly-symmetric if and only if M_W is zero.

$$\begin{aligned}
M_W &= \begin{pmatrix}
E[W_m(\phi_1)W_m(\phi_1)] & E[W_m(\phi_1)W_m(\phi_2)] & \cdots & E[W_m(\phi_1)W_m(\phi_{n_l})] \\
E[W_m(\phi_2)W_m(\phi_1)] & E[W_m(\phi_2)W_m(\phi_2)] & \cdots & E[W_m(\phi_2)W_m(\phi_{n_l})] \\
\vdots & \vdots & \ddots & \vdots \\
E[W_m(\phi_{n_l})W_m(\phi_1)] & E[W_m(\phi_{n_l})W_m(\phi_2)] & \cdots & E[W_m(\phi_{n_l})W_m(\phi_{n_l})]
\end{pmatrix} \\
&= \mathbf{0}
\end{aligned}$$

We can show the above result for $\forall\ i, j$,

$$
\begin{aligned}
& E[W_m(\phi_i)W_m(\phi_j)] \\
= \ & E[(W^r_m(\phi_i) + iW^i_m(\phi_i))(W^r_m(\phi_j) + iW^i_m(\phi_j))] \\
= \ & E(W^r_m(\phi_i)W^r_m(\phi_j)) - E(W^i_m(\phi_i)W^i_m(\phi_j)) + i[E(W^r_m(\phi_i)W^i_m(\phi_j)) + E(W^i_m(\phi_i)W^r_m(\phi_j))] \\
& \text{for } i \neq j \\
= \ & \frac{1}{2}(C^r_m(\phi_i,\phi_j) - C^r_m(\phi_i,\phi_j)) + i[-\frac{1}{2}C^i_m(\phi_i,\phi_j) + \frac{1}{2}C^i_m(\phi_i,\phi_j)] = 0 \\
& \text{for } i = j \\
= \ & \frac{1}{2}(C^r_m(\phi_i,\phi_i) - C^r_m(\phi_i,\phi_i)) + i[0+0] = 0 \quad ; W^r_m(\phi_i), W^i_m(\phi_i) \text{ are independent}
\end{aligned}
$$

Therefore, $\underset{\sim}{W}_m$ is circularly-symmetric. In addition,

$$
\begin{aligned}
K_W &= E(\underset{\sim}{W}_m \underset{\sim}{W}^*_m) \\
&= \begin{pmatrix}
E[W_m(\phi_1)W^*_m(\phi_1)] & E[W_m(\phi_1)W^*_m(\phi_2)] & \cdots & E[W_m(\phi_1)W^*_m(\phi_{n_l})] \\
E[W_m(\phi_2)W^*_m(\phi_1)] & E[W_m(\phi_2)W^*_m(\phi_2)] & \cdots & E[W_m(\phi_2)W^*_m(\phi_{n_l})] \\
\vdots & \vdots & \ddots & \vdots \\
E[W_m(\phi_{n_l})W^*_m(\phi_1)] & E[W_m(\phi_{n_l})W^*_m(\phi_2)] & \cdots & E[W_m(\phi_{n_l})W^*_m(\phi_{n_l})]
\end{pmatrix} \\
&= \begin{pmatrix}
C^r_m(\phi_1,\phi_1) & C^r_m(\phi_1,\phi_2) + iC^i_m(\phi_1,\phi_2) & \cdots & C^r_m(\phi_1,\phi_{n_l}) + iC^i_m(\phi_1,\phi_{n_l}) \\
C^r_m(\phi_2,\phi_1) - iC^i_m(\phi_2,\phi_1) & C^r_m(\phi_2,\phi_2) & \cdots & C^r_m(\phi_2,\phi_{n_l}) + iC^r_m(\phi_2,\phi_{n_l}) \\
\vdots & \vdots & \ddots & \vdots \\
C^r_m(\phi_{n_l},\phi_1) - iC^i_m(\phi_{n_l},\phi_1) & C^r_m(\phi_{n_l},\phi_2) - iC^r_m(\phi_{n_l},\phi_2) & \cdots & C^r_m(\phi_{n_l},\phi_{n_l})
\end{pmatrix} \\
&= \begin{pmatrix}
C^r_m(\phi_1,\phi_1) & C^r_m(\phi_1,\phi_2) & \cdots & C^r_m(\phi_1,\phi_{n_l}) \\
C^r_m(\phi_2,\phi_1) & C^r_m(\phi_2,\phi_2) & \cdots & C^r_m(\phi_2,\phi_{n_l}) \\
\vdots & \vdots & \ddots & \vdots \\
C^r_m(\phi_{n_l},\phi_1) & C^r_m(\phi_{n_l},\phi_2) & \cdots & C^r_m(\phi_{n_l},\phi_{n_l})
\end{pmatrix} \\
&\quad +i \begin{pmatrix}
0 & C^i_m(\phi_1,\phi_2) & \cdots & C^i_m(\phi_1,\phi_{n_l}) \\
-C^i_m(\phi_2,\phi_1) & 0 & \cdots & C^i_m(\phi_2,\phi_{n_l}) \\
\vdots & \vdots & \ddots & \vdots \\
-C^i_m(\phi_{n_l},\phi_1) & -C^i_m(\phi_{n_l},\phi_2) & \cdots & 0
\end{pmatrix} \\
&= Re(K_W) + iIm(K_W)
\end{aligned}
$$

Now,

$$K_V = E(\underset{\sim}{V}_m \underset{\sim}{V}^*_m) = E(\underset{\sim}{V}_m \underset{\sim}{V}^T_m)$$

In order to generate K_V for n_l-tuple case, we reorganize the vector $\underset{\sim}{V}_m$ into the following form.

$$\begin{aligned} \underset{\sim}{V}_m &= (W_m^r(\phi_1), \cdots, W_m^r(\phi_{n_l}), W_m^i(\phi_1), \cdots, W_m^r(\phi_{n_l}))^T \\ &= (Re(\underset{\sim}{W}_m), Im(\underset{\sim}{W}_m))^T \end{aligned}$$

that is, we grouped all real components and imaginary components together. Hence,

$$\begin{aligned} K_V &= E(\underset{\sim}{V}_m \underset{\sim}{V}_m^T) \\ &= \begin{pmatrix} E[Re(\underset{\sim}{W}_m)Re(\underset{\sim}{W}_m)^T] & E[Re(\underset{\sim}{W}_m)Im(\underset{\sim}{W}_m)^T] \\ E[Im(\underset{\sim}{W}_m)Re(\underset{\sim}{W}_m)^T] & E[Im(\underset{\sim}{W}_m)Im(\underset{\sim}{W}_m)^T] \end{pmatrix}_{2n_l \times 2n_l} \end{aligned}$$

Since $\underset{\sim}{W}_m$ is circularly-symmetric from 1.1.11 we can get the following results,

$$E[Re(\underset{\sim}{W}_m)Re(\underset{\sim}{W}_m)^T] = E[Im(\underset{\sim}{W}_m)Im(\underset{\sim}{W}_m)^T] = \frac{1}{2}(Re(K_W))_{n_l \times n_l}$$

$$E[Re(\underset{\sim}{W}_m)Im(\underset{\sim}{W}_m)^T] = -E[Im(\underset{\sim}{W}_m)Re(\underset{\sim}{W}_m)^T] = \frac{1}{2}(Im(K_W))_{n_l \times n_l}$$

$$K_V = \frac{1}{2}\begin{pmatrix} Re(K_W) & Im(K_W)^T \\ Im(K_W) & Re(K_W) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} Re(K_W) & -Im(K_W) \\ Im(K_W) & Re(K_W) \end{pmatrix}$$

Since K_V is a non-negative definite and matrix, it can be represented as follows,

$$K_V = Q\Lambda Q^T,$$

where Λ is a diagonal matrix with eigen values (real-positive) of K_V and Q are the corresponding orthonormal eigenvectors. We can choose $A = Q\Lambda^{1/2}Q^T$ to obtain,

$$\underset{\sim}{V}_m = A_{2n_l \times 2n_l} Z_{2n_l \times 1},$$

where $Z = \{z_1, z_2, \ldots, z_{n_l}, z_1^*, z_2^*, \ldots, z_{n_l}^*\}$ and each $z_i \sim N(0,1)$ hence we can get $\underset{\sim}{W}_m$. Now for each latitude $\phi_l, l = 1, 2, \cdots, n_l$ and $\lambda_k, k = 1, 2, \cdots, n_L$ ($N = n_L/2$), we denote the axially symmetric data (real) as $X(\phi_l, \lambda_k)$. These random variates can be obtained from the equation (5.3.5), let's rewrite the equation as follows,

$$X(\phi_l, \lambda_k) = W_0(\phi_l) + 2\sum_{m=1}^{N}\left[W_m^r(\phi_l)\cos(m\lambda_k) - W_m^i(\phi_l)\sin(m\lambda_k)\right] \tag{5.3.6}$$

5.3.2 Pseudo-code

- Choose a cross covariance function, $R(P, Q)$
- Initialize the parameters (C_1, C_2, a, u, p) and choose a resolution $\phi_1, \ldots, \phi_{n_l}, \lambda_1, \ldots, \lambda_{n_L}$ (or $n_l \times n_L$),

- Derive $C_m(\phi_P, \phi_Q)$ based on $R(P, Q)$ where $m = 0, 1, \ldots, n_L/2$,
    1. for each m get $Re(K_W)$ and $Im(K_W)$ hence obtain K_V
    2. use SVD to get $\underset{\sim}{V}_m$ ($n_l - tuples$)
    3. get $\underset{\sim}{W}_m$'s from $\underset{\sim}{V}_m$
- apply the equation (5.3.6) to generate grid data.

5.4 Property of MOM

Since mom, covariance esimator is biased and the variogram estimator is inconsistent which one should we use. I have added the comparison by both estimators)

The cross covariance can capture the covarince between two locations and any finite paris of locations seperated by a fixed distance (angle in the case of a sphere). In other words cross covariance can be used to to capture the covariance between points at two latitudes sperated by $\Delta\lambda \in (0, 2\pi)$.When a random process on a sphere is second-order stationary, cross covariance is a function of logitidual difference ($\Delta\lambda$). According to Wackernagel (2013) the cross covariance function is not an even function and it is easy to observe that the proposed $R(P, Q)$ functions are valid on a sphere and they are cross covarince fucntions ($R(P, Q, \Delta\lambda) \neq R(P, Q, -\Delta\lambda)$). In contrast the cross variogram is an even fucntion when the process is intrinsically stationary. We compute the empirical cross covariance and cross variogram using method of moments estimators and compared to its theoretical value.

Cross Covariance

The empirical cross covariance for axially symmetric processes on the sphere. For any two latitudes ϕ_P and ϕ_Q with $\{\lambda_i, i = 1, 2, \ldots, n\}$ representing the gridded longitudes on each circle, then $\hat{R}(\phi_P, \phi_Q, \Delta\lambda)$ is given by

$$\hat{R}(\phi_P, \phi_Q, \Delta\lambda) = \frac{1}{n}\sum_{i=1}^{n}(X(\phi_P, \lambda_i + \Delta\lambda) - \bar{X}_P)(X(\phi_Q, \lambda_i) - \bar{X}_Q), \tag{5.4.1}$$

where $\Delta\lambda = 0, 2\pi/n, 4\pi/n, \cdots, 2(N-1)\pi/n$ and $\bar{X}_P = \frac{1}{n}\sum_{i=1}^{n} X(\phi_P, \lambda_i)$ and similar for $\bar{X}_Q$. Now we will show that cross covarince estiamtor is biased,

$$
\begin{aligned}
E(\hat{R}(\phi_P,\phi_Q,\Delta\lambda)) &= \frac{1}{n}\sum_{i=1}^{n} E((X(\phi_P,\lambda_i+\Delta\lambda)-\bar{X}_P)(X(\phi_Q,\lambda_i)-\bar{X}_Q)) \\
&= \frac{1}{n}\sum_{i=1}^{n} cov(X(\phi_P,\lambda_i+\Delta\lambda),X(\phi_Q,\lambda_i)) \\
&\quad -\frac{1}{n}\sum_{i=1}^{n} E((X(\phi_P,\lambda_i+\Delta\lambda)-\mu_P)(\bar{X}_Q-\mu_Q)) \\
&\quad -\frac{1}{n}\sum_{i=1}^{n} E((X(\phi_Q,\lambda_i)-\mu_Q)(\bar{X}_P-\mu_P)) \\
&\quad +\frac{1}{n}\sum_{i=1}^{n} E((\bar{X}_P-\mu_P)(\bar{X}_Q-\mu_Q)) \\
&= R(\phi_P,\phi_Q,\Delta\lambda)-E((\bar{X}_Q-\mu_Q)(\bar{X}_P-\mu_P))-E((\bar{X}_P-\mu_P)(\bar{X}_Q-\mu_Q)) \\
&\quad +E((\bar{X}_P-\mu_P)(\bar{X}_Q-\mu_Q)) \\
&= R(\phi_P,\phi_Q,\Delta\lambda)-cov(\bar{X}_P,\bar{X}_Q).
\end{aligned}
$$

Note that,

$$
\begin{aligned}
cov(\bar{X}_P,\bar{X}_Q) &= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} cov(X(\phi_P,\lambda_i),X(\phi_Q,\lambda_j)) \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} R(\phi_P,\phi_Q,(i-j)*2\pi/n) \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} (C_0(\phi_P,\phi_Q) \\
&\quad 2\sum_{m=1}^{\infty}(C_{m,R}(\phi_P,\phi_Q)\cos(m*(i-j)*2\pi/n) \\
&\quad -C_{m,I}(\phi_P,\phi_Q)\sin(m*(i-j)*2\pi/n)) \\
&= C_0(\phi_P,\phi_Q)+2\sum_{m=1}^{\infty}C_{m,R}(\phi_P,\phi_Q)\left(\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\cos(m(i-j)*2\pi/n)\right) \\
&\quad -2\sum_{m=1}^{\infty}C_{m,I}(\phi_P,\phi_Q)\left(\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sin(m(i-j)*2\pi/n)\right) \\
&= C_0(\phi_P,\phi_Q),
\end{aligned}
$$

since

$$
\begin{aligned}
& \sum_{i=1}^{n}\sum_{j=1}^{n}\cos(m*(i-j)*2\pi/n) \\
= & \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\cos(m*i*2\pi/n)\cos(m*j*2\pi/n)-\sin(m*i*2\pi/n)\sin(m*j*2\pi/n)\right) \\
= & \left(\sum_{i=1}^{n}\cos(m*i*2\pi/n)\right)^2-\left(\sum_{i=1}^{n}\sin(m*i*2\pi/n)\right)^2=0
\end{aligned}
$$

and

$$
\begin{aligned}
& \sum_{i=1}^{n}\sum_{j=1}^{n}\sin(m*(i-j)*2\pi/n) \\
= & \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\sin(m*i*2\pi/n)\cos(m*j*2\pi/n)-\cos(m*i*2\pi/n)\sin(m*j*2\pi/n)\right) \\
= & \left(\sum_{i=1}^{n}\cos(m*i*2\pi/n)\right)*\left(\sum_{i=1}^{n}\sin(m*i*2\pi/n)\right) \\
& -\left(\sum_{i=1}^{n}\cos(m*i*2\pi/n)\right)*\left(\sum_{i=1}^{n}\sin(m*i*2\pi/n)\right)=0
\end{aligned}
$$

since for any integer m, we have

$$\sum_{k=1}^{n}\cos(mk*2\pi/n)=\begin{cases}0, & \text{for any integer } m\neq 0, \\ n, & \text{for } m=0\end{cases} \quad \text{and} \quad \sum_{k=1}^{n}\sin(mk*2\pi/n)=0.$$

Hence,

$$cov(\bar{X}_P,\bar{X}_Q)=C_0(\phi_P,\phi_Q).$$

Therefore,

$$E(\hat{R}(\phi_P,\phi_Q,\Delta\lambda))=R(\phi_P,\phi_Q,\Delta\lambda)-C_0(\phi_P,\phi_Q).$$

The cross covariance estiamatro is biased and when $\phi_P=\phi_Q$ this reduces to the same results we obtained for a random process on the circle.

Cross variogram

In general when the covariance is known once can get the variogram ($2\gamma=C(0)-C(h)$), since the cross covariance is not an even function and the variogram is defined by taking the average of $R(P,Q,\Delta\lambda)$ and $R(P,Q,-\Delta\lambda)$ and we can derive the cross covariogram as follows,

$$\begin{aligned}
\gamma(\phi_p, \phi_Q, \Delta\lambda) &= \frac{1}{2}E\left((X(\phi_P, \lambda+\Delta\lambda) - X(\phi_P, \lambda))(X(\phi_Q, \lambda+\Delta\lambda) - X(\phi_Q, \lambda))\right) \\
&= \frac{1}{2}E\left(((X(\phi_P, \lambda+\Delta\lambda) - \mu_P) - (X(\phi_P, \lambda) - \mu_P))\right. \\
&\quad ((X(\phi_Q, \lambda+\Delta\lambda) - \mu_Q) - (X(\phi_Q, \lambda) - \mu_Q)) \\
&= \frac{1}{2}\left(cov(X(\phi_P, \lambda+\Delta\lambda), X(\phi_Q, \lambda+\Delta\lambda)) - cov(X(\phi_P, \lambda+\Delta\lambda), X(\phi_Q, \lambda))\right. \\
&\quad \left. -cov(X(\phi_P, \lambda), X(\phi_Q, \lambda+\Delta\lambda)) + cov(X(\phi_P, \lambda), X(\phi_Q, \lambda))\right) \\
&= \frac{1}{2}\left(R(\phi_P, \phi_Q, 0) - R(\phi_P, \phi_Q, \Delta\lambda) - R(\phi_P, \phi_Q, -\Delta\lambda) + R(\phi_P, \phi_Q, 0)\right) \\
&= R(\phi_P, \phi_Q, 0) - \frac{1}{2}(R(\phi_P, \phi_Q, \Delta\lambda) + R(\phi_P, \phi_Q, -\Delta\lambda)).
\end{aligned}$$

$$\gamma(\phi_p, \phi_Q, \Delta\lambda) = R(\phi_p, \phi_Q, 0) - \frac{1}{2}(R(\phi_P, \phi_Q, \Delta\lambda) + R(\phi_P, \phi_Q, -\Delta\lambda)). \tag{5.4.2}$$

Regarding the cross-variogram estimation for axially symmetric processes on the sphere, we also perform the MOM estimation as following.

$$\hat{\gamma}(\phi_p, \phi_Q, \Delta\lambda) = \frac{1}{2n}\sum_{i=1}^{n}\left(X(\phi_P, \lambda_i+\Delta\lambda) - X(\phi_P, \lambda_i))(X(\phi_Q, \lambda_i+\Delta\lambda) - X(\phi_Q, \lambda_i))\right), \tag{5.4.3}$$

and we have

$$\begin{aligned}
E(\hat{\gamma}_{PQ}(\Delta\lambda)) &= \frac{1}{2n}\sum_{i=1}^{n}E\left(X(\phi_P, \lambda_i+\Delta\lambda) - X(\phi_P, \lambda_i))(X(\phi_Q, \lambda_i+\Delta\lambda) - X(\phi_Q, \lambda_i))\right) \\
&= \frac{1}{2n}\sum_{i=1}^{n}\left(2\gamma(\phi_p, \phi_Q, \Delta\lambda)\right) = \gamma(\phi_p, \phi_Q, \Delta\lambda),
\end{aligned}$$

which is unbiased.

Remark The $R(P, Q)$ function can be given re arrange such that,

$$R(\phi_P, \phi_Q, \Delta\lambda) = \frac{1}{2}(R(\phi_P, \phi_Q, \Delta\lambda) + R(\phi_P, \phi_Q, -\Delta\lambda)) + \frac{1}{2}(R(\phi_P, \phi_Q, \Delta\lambda) - R(\phi_P, \phi_Q, -\Delta\lambda))$$

the cross-covariance function $R(\phi_P, \phi_Q, \Delta\lambda)$ is decomposed into two components: the even component (the first average) and the odd component (the second average). The cross-variogram is only related to the even component of the cross-covariance function, which is different from the case on the circle (the covaraince is an even function). Therefore, using the cross-variogram function is not good enough to characterize the cross-covariance function for the data from the sphere.

Wackernagel (2013) says that cross variogram is not sufficient when there is a delayed affect example gas inout and CO_2 output
How about consistency??

5.4.1 Results

Simulated data sample:

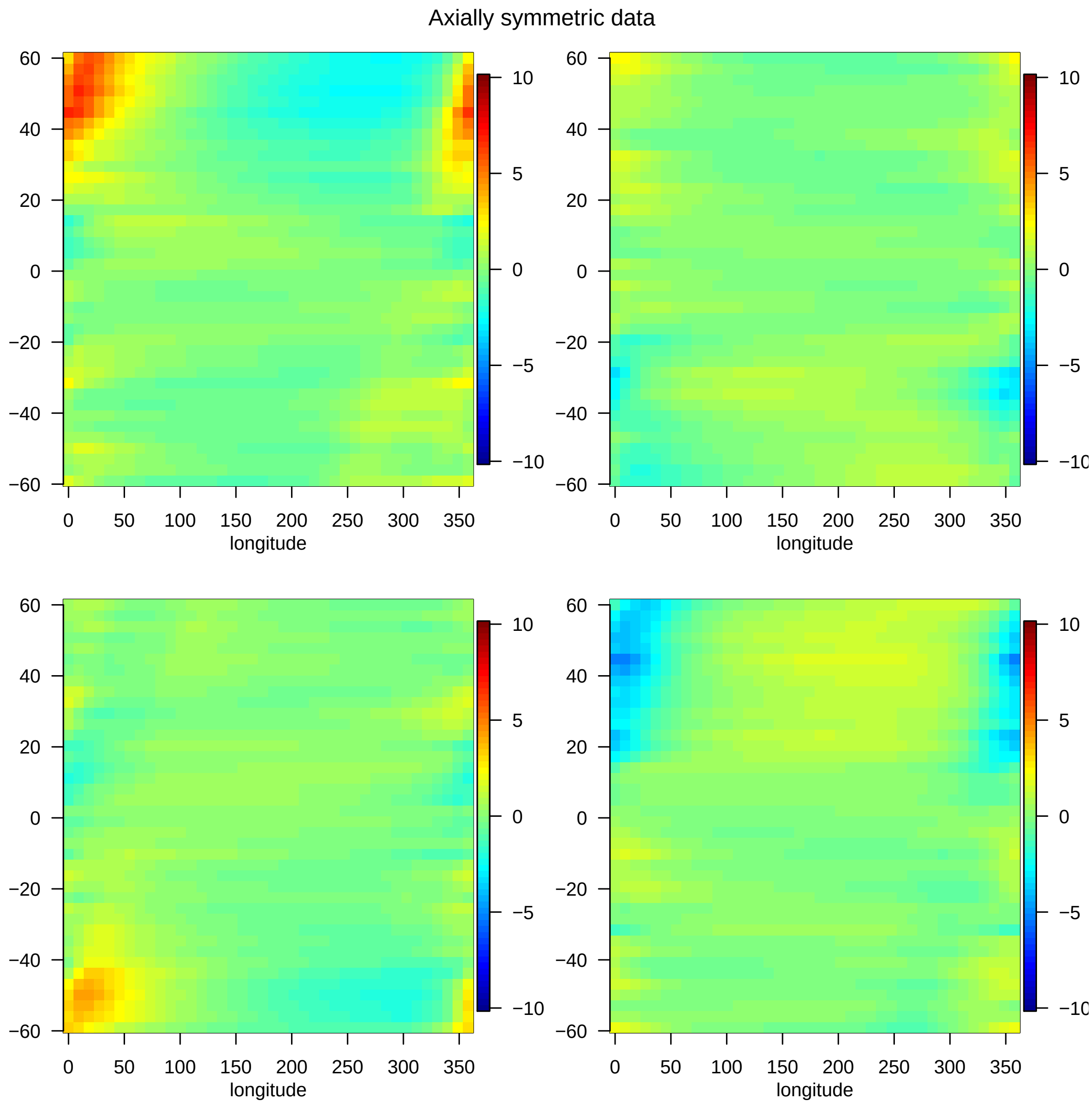

Figure 5.3: Four consecutive axially symmetric data snapshots based on model 2, grid resolution $2^0 \times 1^0$ (data scale -10 and 10).

Comprison of the proposed models with MOM estimates:

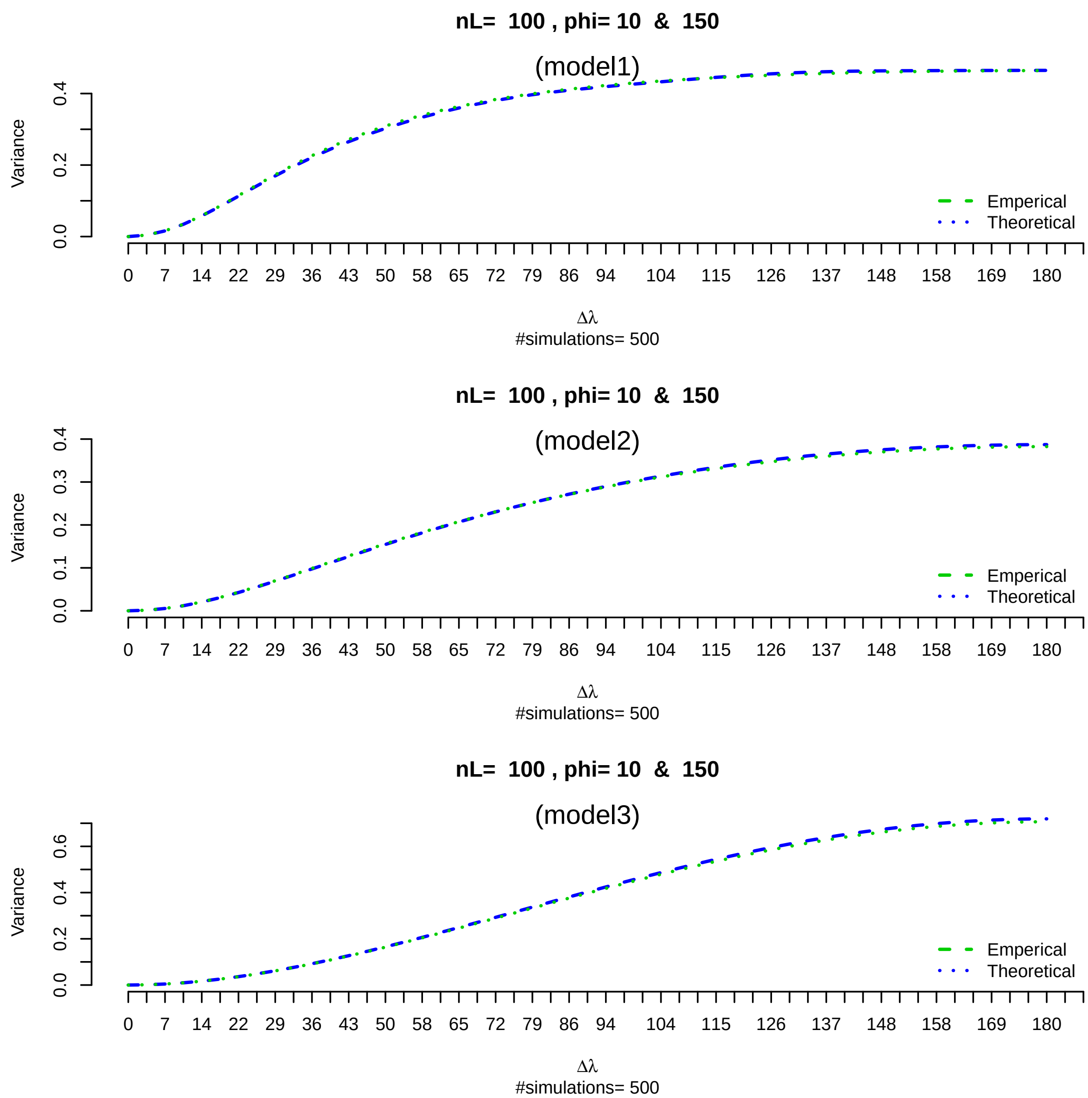

Figure 5.4: Variogram comparison when $u = 0$

- Model 1

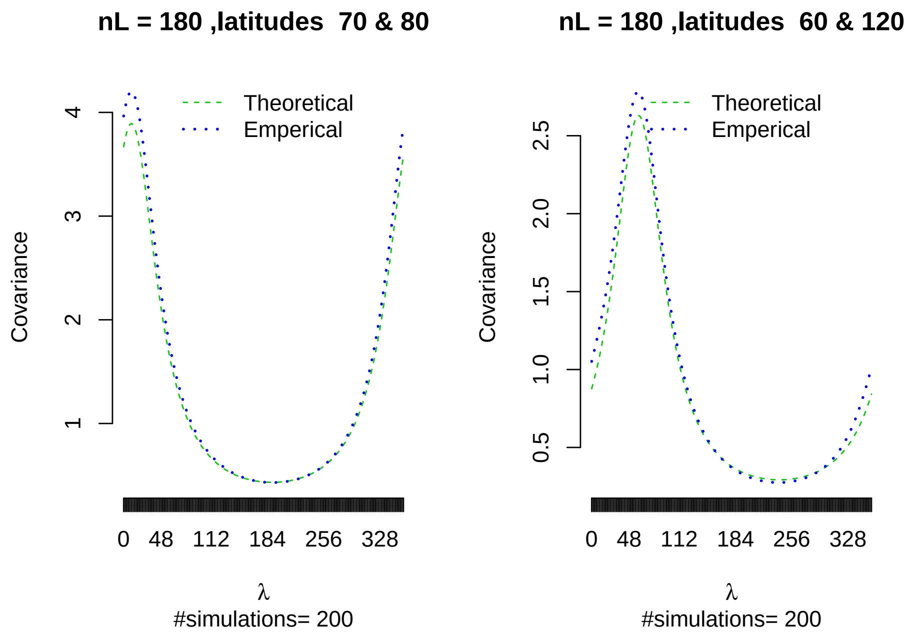

Figure 5.5: Cross covariance comparison of model1

- Model 2

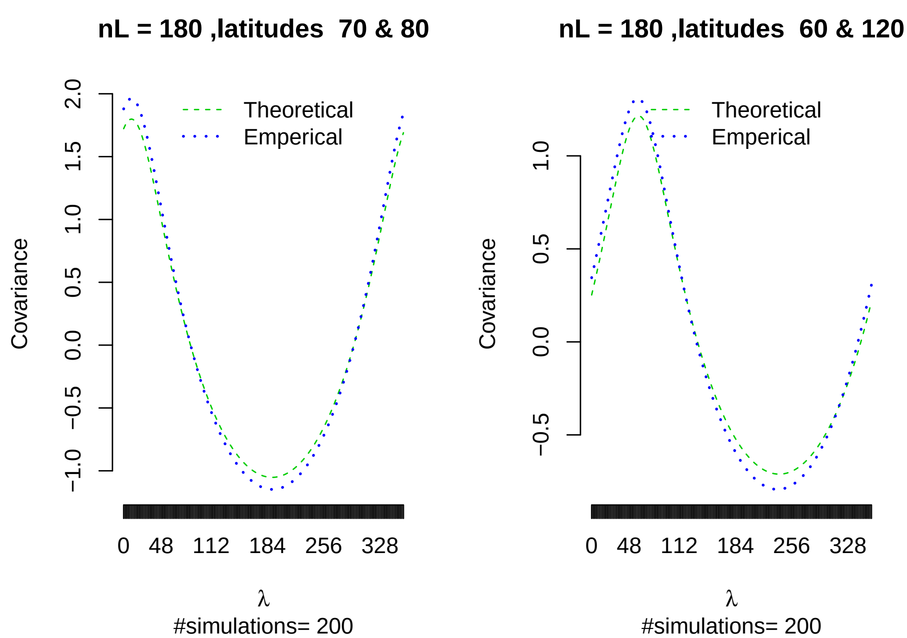

Figure 5.6: Cross covariance comparison of model2

- Model 3

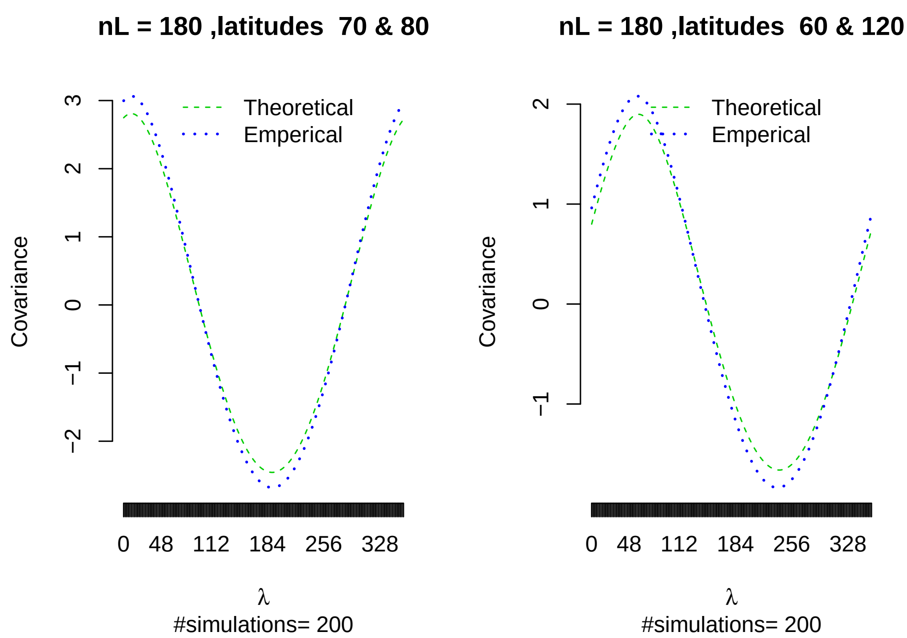

Figure 5.7: Cross covariance comparison of model3

Chapter 6

Future Research (due August 28)

Future research work !!

Bibliography

Bolin, D. and Lindgren, F. (2011). Spatial models generated by nested stochastic partial differential equations, with an application to global ozone mapping. *The Annals of Applied Statistics*, pages 523–550.

Christy, J. R., Spencer, R. W., and Braswell, W. D. (2000). Msu tropospheric temperatures: Dataset construction and radiosonde comparisons. *Journal of Atmospheric and Oceanic Technology*, 17(9):1153–1170.

Cressie, N. (1993). *Statistics for spatial data*. John Wiley & Sons.

Cressie, N. and Johannesson, G. (2008). Fixed rank kriging for very large spatial data sets. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 70(1):209–226.

De Mazancourt, T. and Gerlic, D. (1983). The inverse of a block-circulant matrix. *IEEE transactions on antennas and propagation*, 31(5):808–810.

Dufour, J.-M. and Roy, R. (1976). On spectral estimation for a homogeneous random process on the circle. *Stochastic Processes and their Applications*, 4(2):107 – 120.

Gallager, R. G. (2008). Circularly-symmetric gaussian random vectors. *preprint*, pages 1–9.

Gneiting, T. (2013). Strictly and non-strictly positive definite functions on spheres. *Bernoulli*, 19(4):1327–1349.

Hitczenko, M. and Stein, M. L. (2012). Some theory for anisotropic processes on the sphere. *Statistical Methodology*, 9(1):211–227.

Huang, C., Zhang, H., and Robeson, S. M. (2011). On the validity of commonly used covariance and variogram functions on the sphere. *Mathematical Geosciences*, 43(6):721–733.

Huang, C., Zhang, H., and Robeson, S. M. (2012). A simplified representation of the covariance structure of axially symmetric processes on the sphere. *Statistics & Probability Letters*, 82(7):1346–1351.

Jeong, J. and Jun, M. (2015). A class of matérn-like covariance functions for smooth processes on a sphere. *Spatial Statistics*, 11:1–18.

Jones, R. H. (1963). Stochastic processes on a sphere. *The Annals of mathematical statistics*, 34(1):213–218.

Jun, M. and Stein, M. L. (2008). Nonstationary covariance models for global data. *The Annals of Applied Statistics*, pages 1271–1289.

Katzfuss, M. and Cressie, N. (2011). Spatio-temporal smoothing and em estimation for massive remote-sensing data sets. *Journal of Time Series Analysis*, 32(4):430–446.

Li, T.-H. and North, G. R. (1997). Aliasing effects and sampling theorems of spherical random fields when sampled on a finite grid. *Annals of the Institute of Statistical Mathematics*, 49(2):341–354.

Li, Y. (2013). Non-parametric and semi-parametric estimation of spatial covariance function.

Lindgren, F., Rue, H., and Lindström, J. (2011). An explicit link between gaussian fields and gaussian markov random fields: the stochastic partial differential equation approach. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 73(4):423–498.

Muir, S. T. (1920). *The theory of determinants.*

Roy, R. (1972). Spectral analysis for a random process on the circle. *Journal of Applied Probability*, pages 745–757.

Schoenberg, I. J. (1942). Positive definite functions on spheres. *Duke Math. J.*, 9(1):96–108.

Stein, M. L. (1999). *Interpolation of spatial data: some theory for kriging.* Springer Science & Business Media.

Stein, M. L. (2007). Spatial variation of total column ozone on a global scale. *The Annals of Applied Statistics*, pages 191–210.

Tee, G. J. (2005). Eigenvectors of block circulant and alternating circulant matrices.

Trapp, G. E. (1973). Inverses of circulant matrices and block circulant matrices. *Kyungpook Mathematical Journal*, 13(1):11–20.

Wackernagel, H. (2013). *Multivariate geostatistics: an introduction with applications.* Springer Science & Business Media.

Yadrenko, M. I. (1983). *Spectral theory of random fields.* Optimization Software.

Yaglom, A. (1987). *Correlation Theory of Stationary and Related Random Functions: Vol.: 1: Basic Results.* Springer-Verlag.

Yaglom, A. M. (1961). Second-order homogeneous random fields. In *Proc. 4th Berkeley Sympos. Math. Statist. and Prob*, volume 2, pages 593–622.