1 Lattice Boltzmann Equation

$$\frac{\partial f}{\partial t} + \boldsymbol{c} \cdot \boldsymbol{\nabla} f + \boldsymbol{F} \cdot \boldsymbol{\nabla}_{\boldsymbol{c}} f = \Omega(f, f^{eq}) \tag{1}$$

where $f(\boldsymbol{x}, \boldsymbol{c}, t)$ is the distribution function of particles which is a function of position $\boldsymbol{x}$, microscopic speeds $\boldsymbol{c}$ and time t. $\boldsymbol{F}$ is the external force and $\Omega(f, f^{eq})$ is the collision operator that relaxes the distribution function f toward an equilibrium f^{eq}. For the classical Batnaghar-Gross-Krook (**BGK**) approximation, that collision operator simply writes:

$$\Omega(f, f^{eq}) \sim -\frac{1}{\lambda}\left[f - f^{eq}\right]$$

where λ is the collision rate.

Eq. (1), computes the evolution of f in space and time. The macrosopic quantities such as the density ρ, impulsion $\rho\boldsymbol{u}$ and energy $\rho\varepsilon$ can be derived from that distribution function by integration over the velocity space:

$$\begin{aligned} \rho &= \int f d\boldsymbol{c} \\ \rho\boldsymbol{u} &= \int f \boldsymbol{c} d\boldsymbol{c} \\ \rho\varepsilon &= \frac{1}{2}\int (\boldsymbol{c} - \boldsymbol{u})^2 f d\boldsymbol{c} \end{aligned}$$

Those macroscopic quantities are called moments of the distribution function $f(\boldsymbol{x}, \boldsymbol{c}, t)$.

After discretization of $\boldsymbol{x}$, $\boldsymbol{c}$ and t

$$f_i(\boldsymbol{x} + \boldsymbol{c}_i \delta t, t + \delta t) = f_i(\boldsymbol{x}, t) - \frac{1}{\tau}\left[f_i(\boldsymbol{x}, t) - f_i^{eq}(\boldsymbol{x}, t)\right]$$

where δt is the time step and τ is the collision rate which is related to the collision time by

$$\tau = \frac{\lambda}{\delta t}$$

The right-hand side represents the collision stage and often noted

$$f_i^{\star}(\boldsymbol{x}, t) = f_i(\boldsymbol{x}, t) - \frac{1}{\tau}\left[f_i(\boldsymbol{x}, t) - f_i^{eq}(\boldsymbol{x}, t)\right]$$

1.1 Feq NS

$$f_i^{eq}(\boldsymbol{x}, t) = w_i \rho \left[1 + \frac{\boldsymbol{c}_i \cdot \boldsymbol{u}}{c_s^2} + \frac{(\boldsymbol{c}_i \cdot \boldsymbol{u})^2}{2c_s^4} - \frac{\boldsymbol{u} \cdot \boldsymbol{u}}{2c_s^2}\right]$$

where the coefficient c_s is defined by

$$c_s = \frac{1}{\sqrt{3}} \frac{\delta x}{\delta t}$$

and w_i are weights which depend on the lattice considered.

1.2 Lattice D2Q9

$$\boldsymbol{e}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \boldsymbol{e}_3 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad \boldsymbol{e}_4 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\boldsymbol{e}_5 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{e}_6 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \boldsymbol{e}_7 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \quad \boldsymbol{e}_8 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

1.3 Explicit algorithm

1. Collision: $f_i(\boldsymbol{x}, t) \to f_i^\star(\boldsymbol{x}, t)$ (right-hand side of Eq.
2. Streaming: $f_i^\star(\boldsymbol{x}, t) \to f_i(\boldsymbol{x} + \boldsymbol{c}_i \delta t, t + \delta t)$ (left-hand side)
3. Updating the density and impulsion

$$\rho(\boldsymbol{x}, t) = \sum_i f_i(\boldsymbol{x}, t)$$

$$\rho(\boldsymbol{x}, t)\boldsymbol{u}(\boldsymbol{x}, t) = \sum_i f_i(\boldsymbol{x}, t)\boldsymbol{c}_i$$

2 Collision operators

2.1 BGK

The BGK approximation is the simplest collision operator

$$\Omega_i^{BGK}(f_i, f_i^{eq}) = -\frac{1}{\tau}\left[f_i - f_i^{eq}\right]$$

Its main advantage is its simplicity but

2.2 TRT

opposite directions

$$\boldsymbol{c}_{\bar{i}} = -\boldsymbol{c}_i$$

example on the standard D2Q9 lattice $\boldsymbol{c}_{\bar{1}} = -\boldsymbol{c}_1 = \boldsymbol{c}_3$
With that notation, the symmetric parts of f_i and f_i^{eq} are defined by

$$f_i^+ = \frac{f_i + f_{\bar{i}}}{2} \qquad \text{and} \qquad f_i^{eq+} = \frac{f_i^{eq} + f_{\bar{i}}^{eq}}{2}$$

and the anti-symmetric part by

$$f_i^- = \frac{f_i - f_{\bar{i}}}{2} \qquad \text{and} \qquad f_i^{eq-} = \frac{f_i^{eq} - f_{\bar{i}}^{eq}}{2}$$

The Two-Relaxation-Times collision operator considers the collision stage with two relaxation parameters τ^+ and τ^- acting on respectively the symmetric part and the anti-symmetric part. The LBE writes:

$$f_i(\boldsymbol{x} + \boldsymbol{c}_i\delta t, t + \delta t) = f_i - \frac{1}{\tau^+}\left[f_i^+ - f_i^{eq+}\right] - \frac{1}{\tau^-}\left[f_i^- - f_i^{eq-}\right]$$

$$\Omega_i^{TRT} = -\frac{1}{\tau^+}\left[f_i^+ - f_i^{eq+}\right] - \frac{1}{\tau^-}\left[f_i^- - f_i^{eq-}\right]$$

When the equilibrium distribution function is defined such as the Navier-Stokes equations are recovered, the kinematic viscosity is related to the parameter τ^- by:

$$\nu = c_s^2\left(\tau^- - \frac{1}{2}\right)\delta t$$

and the parameter τ^+ is a free paramater to tune to improve accuracy and stability. In practice, the parameter Λ is often for that purpose:

$$\Lambda = \left(\tau^+ - \frac{1}{2}\right)\left(\tau^- - \frac{1}{2}\right)$$

2.3 MRT

$$\begin{array}{cccccccccc} \boldsymbol{e}_0 & \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 & \boldsymbol{e}_4 & \boldsymbol{e}_5 & \boldsymbol{e}_6 & \boldsymbol{e}_7 & \boldsymbol{e}_8 & \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \begin{pmatrix} -1 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ -1 \end{pmatrix} & \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \begin{pmatrix} -1 \\ 1 \end{pmatrix} & \begin{pmatrix} -1 \\ -1 \end{pmatrix} & \begin{pmatrix} 1 \\ -1 \end{pmatrix} & \begin{matrix} \leftarrow \boldsymbol{v}_{j_x} \\ \leftarrow \boldsymbol{v}_{j_y} \end{matrix} \end{array}$$

LBE

$$\boldsymbol{f}(\boldsymbol{x} + \boldsymbol{c}_i\delta t, t + \delta t) = \boldsymbol{f}(\boldsymbol{x}, t) - \boldsymbol{M}^{-1}\boldsymbol{S}(\boldsymbol{x})\boldsymbol{M}\left[\boldsymbol{f}(\boldsymbol{x}, t) - \boldsymbol{f}^{eq}(\boldsymbol{x}, t)\right] + \boldsymbol{\mathcal{F}}\delta t$$

MRT coll ope

$$\boldsymbol{\Omega}^{MRT} = -\boldsymbol{M}^{-1}\boldsymbol{S}(\boldsymbol{x})\boldsymbol{M}\left[\boldsymbol{f}(\boldsymbol{x}, t) - \boldsymbol{f}^{eq}(\boldsymbol{x}, t)\right]$$

- $\boldsymbol{M}$ and $\boldsymbol{S}$ are two invertible matrices of dim $(N_{pop} + 1) \times (N_{pop} + 1)$
  - $\boldsymbol{M}$ represents a change of basis: "space of distribution functions" $\rightarrow$ "space of moments"
  - $\boldsymbol{S}$ contains the relaxation coefficients

$$\boldsymbol{f}(\boldsymbol{x},t) = \begin{pmatrix} f_0(\boldsymbol{x},t) \\ f_1(\boldsymbol{x},t) \\ \vdots \\ f_{N_{pop}}(\boldsymbol{x},t) \end{pmatrix}, \quad \boldsymbol{f}^{eq}(\boldsymbol{x},t) = \begin{pmatrix} f_0^{eq}(\boldsymbol{x},t) \\ f_1^{eq}(\boldsymbol{x},t) \\ \vdots \\ f_{N_{pop}}^{eq}(\boldsymbol{x},t) \end{pmatrix}, \quad \boldsymbol{\mathcal{F}}(\boldsymbol{x},t) = \begin{pmatrix} \mathcal{F}_0(\boldsymbol{x},t) \\ \mathcal{F}_1(\boldsymbol{x},t) \\ \vdots \\ \mathcal{F}_{N_{pop}}(\boldsymbol{x},t) \end{pmatrix}$$

$$\rho = \sum_i f_i = \boldsymbol{v}_\rho \cdot \boldsymbol{f}$$

$$\rho u_x = \sum_i f_i c_{ix} = \boldsymbol{v}_{j_x} \cdot \boldsymbol{f}$$

$$\rho u_y = \sum_i f_i c_{iy} = \boldsymbol{v}_{j_y} \cdot \boldsymbol{f}$$

where
$\boldsymbol{v}_\rho = (1,1,1,1,1,1,1,1,1)$
$\boldsymbol{v}_{j_x} = (0,1,0,-1,0,1,-1,-1,1)$
$\boldsymbol{v}_{j_y} = (0,0,1,0,-1,1,1,-1,-1)$
Moments

$$\boldsymbol{m} = \boldsymbol{M}\boldsymbol{f} \qquad \text{with } \boldsymbol{m} = (\rho, m_1, m_2, \rho u_x, m_4, \rho u_y, m_7, m_8)$$

where the matrix $\boldsymbol{M}$ is built with orthogonal vectors $\boldsymbol{v}_\rho$, $\boldsymbol{v}_{j_x}$, $\boldsymbol{v}_{j_y}$, …, $\boldsymbol{v}_{N_{pop}}$

$$\boldsymbol{M} = \begin{pmatrix} \boldsymbol{v}_\rho \\ \boldsymbol{v}_e \\ \boldsymbol{v}_\epsilon \\ \boldsymbol{v}_{j_x} \\ \boldsymbol{v}_{q_x} \\ \boldsymbol{v}_{j_y} \\ \boldsymbol{v}_{q_y} \\ \boldsymbol{v}_{p_{xx}} \\ \boldsymbol{v}_{p_{xy}} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -4 & -1 & -1 & -1 & -1 & 2 & 2 & 2 & 2 \\ 4 & -2 & -2 & -2 & -2 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 & -1 & -1 & 1 \\ 0 & -2 & 0 & 2 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 1 & -1 & -1 \\ 0 & 0 & -2 & 0 & 2 & 1 & 1 & -1 & -1 \\ 0 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \end{pmatrix}$$

$$\boldsymbol{S} = \mathrm{diag}(0, \omega_e, \omega_\epsilon, 0, \omega_q, 0, \omega_q, \omega_\nu, \omega_\nu)$$

$$\boldsymbol{M}^{-1} = \begin{pmatrix} \frac{1}{9} & -\frac{1}{9} & \frac{1}{9} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{9} & -\frac{1}{36} & -\frac{1}{18} & \frac{1}{6} & -\frac{1}{6} & 0 & 0 & \frac{1}{4} & 0 \\ \frac{1}{9} & -\frac{1}{36} & -\frac{1}{18} & 0 & 0 & \frac{1}{6} & -\frac{1}{6} & -\frac{1}{4} & 0 \\ \frac{1}{9} & -\frac{1}{36} & -\frac{1}{18} & -\frac{1}{6} & \frac{1}{6} & 0 & 0 & \frac{1}{4} & 0 \\ \frac{1}{9} & -\frac{1}{36} & -\frac{1}{18} & 0 & 0 & -\frac{1}{6} & \frac{1}{6} & -\frac{1}{4} & 0 \\ \frac{1}{9} & \frac{1}{18} & \frac{1}{36} & \frac{1}{6} & \frac{1}{12} & \frac{1}{6} & \frac{1}{12} & 0 & \frac{1}{4} \\ \frac{1}{9} & \frac{1}{18} & \frac{1}{36} & -\frac{1}{6} & -\frac{1}{12} & \frac{1}{6} & \frac{1}{12} & 0 & -\frac{1}{4} \\ \frac{1}{9} & \frac{1}{18} & \frac{1}{36} & -\frac{1}{6} & -\frac{1}{12} & -\frac{1}{6} & -\frac{1}{12} & 0 & \frac{1}{4} \\ \frac{1}{9} & \frac{1}{18} & \frac{1}{36} & \frac{1}{6} & \frac{1}{12} & -\frac{1}{6} & -\frac{1}{12} & 0 & -\frac{1}{4} \end{pmatrix}$$

3 Equilibrium distribution functions

$$f_i(\boldsymbol{x}+\boldsymbol{c}_i\delta t, t+\delta t) = f_i(\boldsymbol{x},t) - \frac{1}{\tau}\left[f_i(\boldsymbol{x},t) - f_i^{eq}(\boldsymbol{x},t)\right]$$

3.1 Incompressible Navier-Stokes

$$f_i^{eq}(\boldsymbol{x},t) = w_i\left[p_h + \rho_0 c_s^2\left(\frac{\boldsymbol{c}_i\cdot\boldsymbol{u}}{c_s^2} + \frac{(\boldsymbol{c}_i\cdot\boldsymbol{u})^2}{2c_s^4} - \frac{\boldsymbol{u}\cdot\boldsymbol{u}}{2c_s^2}\right)\right]$$

$$p_h(\boldsymbol{x},t) = \sum_i f_i(\boldsymbol{x},t)$$

$$\rho_0\boldsymbol{u}(\boldsymbol{x},t) = \frac{1}{c_s^2}\sum_i f_i(\boldsymbol{x},t)\boldsymbol{c}_i$$

3.2 Incompressible Navier-Stokes for two-phase flows

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$$

$$\varrho(\phi)\left[\frac{\partial\boldsymbol{u}}{\partial t} + \boldsymbol{\nabla}\cdot(\boldsymbol{u}\boldsymbol{u})\right] = -\boldsymbol{\nabla}p_h + \boldsymbol{\nabla}\cdot\eta(\boldsymbol{x},t)\left[\boldsymbol{\nabla}\boldsymbol{u} + (\boldsymbol{\nabla}\boldsymbol{u})^T\right] + \boldsymbol{F}_s + \boldsymbol{F}_g$$

$$\boldsymbol{F}_s = \mu_\phi\boldsymbol{\nabla}\phi$$

$$\boldsymbol{F}_g = \varrho(\phi)\boldsymbol{g}$$

3.2.1 Version 1 for variable density

$$f_i^{eq}(\boldsymbol{x},t) = w_i\left[p_h + \varrho(\phi) c_s^2\left(\frac{\boldsymbol{c}_i\cdot\boldsymbol{u}}{c_s^2} + \frac{(\boldsymbol{c}_i\cdot\boldsymbol{u})^2}{2c_s^4} - \frac{\boldsymbol{u}\cdot\boldsymbol{u}}{2c_s^2}\right)\right]$$

$$p_h(\boldsymbol{x},t) = \sum_i f_i(\boldsymbol{x},t)$$

$$\varrho(\phi)\boldsymbol{u}(\boldsymbol{x},t) = \frac{1}{c_s^2}\sum_i f_i(\boldsymbol{x},t)\boldsymbol{c}_i$$

- Equivalent macroscopic equations

$$\frac{\partial p_h}{\partial t} + \boldsymbol{\nabla}\cdot\left[\varrho(\phi)c_s^2\boldsymbol{u}\right] = 0$$

$$\varrho(\phi)\cancel{c_s^2}\left[\frac{\partial\boldsymbol{u}}{\partial t} + \boldsymbol{\nabla}\cdot(\boldsymbol{u}\boldsymbol{u})\right] = -\boldsymbol{\nabla}p_h\cancel{c_s^2} + \boldsymbol{\nabla}\cdot\left[\varrho(\phi)\cancel{c_s^2}\nu(\boldsymbol{\nabla}\boldsymbol{u} + (\boldsymbol{\nabla}\boldsymbol{u})^T)\right] + \mathcal{O}(\rho u^3)$$

$$\nu = \frac{1}{3}\left(\tau - \frac{1}{2}\right)\frac{\delta x^2}{\delta t}$$

3.2.2 Version 2 for variable density

$$f_i^{eq}(\boldsymbol{x},t) = w_i\left[p^\star + \left(\frac{\boldsymbol{c}_i\cdot\boldsymbol{u}}{c_s^2} + \frac{(\boldsymbol{c}_i\cdot\boldsymbol{u})^2}{2c_s^4} - \frac{\boldsymbol{u}\cdot\boldsymbol{u}}{2c_s^2}\right)\right]$$

where

$$p^\star = \frac{p_h}{\varrho(\phi)c_s^2}$$

Moments

$$p^\star(\boldsymbol{x},t) = \sum_i f_i(\boldsymbol{x},t)$$

$$\boldsymbol{u}(\boldsymbol{x},t) = \sum_i f_i(\boldsymbol{x},t)\boldsymbol{c}_i$$

$$\frac{\partial p^\star}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\nabla}\cdot(\boldsymbol{u}\boldsymbol{u}) = -\boldsymbol{\nabla}(p^\star c_s^2) + \boldsymbol{\nabla}\cdot\left[\nu(\boldsymbol{\nabla}\boldsymbol{u} + (\boldsymbol{\nabla}\boldsymbol{u})^T)\right]$$

$$\nu = \frac{1}{3}\left(\tau - \frac{1}{2}\right)\frac{\delta x^2}{\delta t}$$

Force terms to add

$$\boldsymbol{F}_p = -\frac{p_h}{\varrho}\boldsymbol{\nabla}\varrho$$

$$\boldsymbol{F}_v = \nu\left[\boldsymbol{\nabla}\boldsymbol{u} + (\boldsymbol{\nabla}\boldsymbol{u})^T\right]\cdot\boldsymbol{\nabla}\varrho(\phi)$$

4 Additional gradients

4.1 Gradients

Directional derivative method:

$$\boldsymbol{e}_i\cdot\boldsymbol{\nabla}\phi\big|_{\boldsymbol{x}} = \frac{1}{2\delta x}\left[\phi(\boldsymbol{x}+\boldsymbol{e}_i\delta x) - \phi(\boldsymbol{x}-\boldsymbol{e}_i\delta x)\right]$$

$$\boldsymbol{\nabla}\phi = \frac{1}{e^2}\sum_{i=0}^{N_{pop}} w_i\boldsymbol{e}_i\left(\boldsymbol{e}_i\cdot\boldsymbol{\nabla}\phi\big|_{\boldsymbol{x}}\right)$$

4.2 Laplacian

$$(\boldsymbol{e}_i \cdot \boldsymbol{\nabla})^2 \phi\big|_{\boldsymbol{x}} = \frac{1}{\delta x^2}\left[\phi(\boldsymbol{x} + \boldsymbol{e}_i \delta x) - 2\phi(\boldsymbol{x}) + \phi(\boldsymbol{x} - \boldsymbol{e}_i \delta x)\right]$$

$$\boldsymbol{\nabla}^2 \phi\big|_{\boldsymbol{x}} = 3 \sum_{i=0}^{N_{pop}} w_i (\boldsymbol{e}_i \cdot \boldsymbol{\nabla})^2 \phi\big|_{\boldsymbol{x}}$$