On cohomology, Gibbs properties and regularity of some nonadditive families of potentials

Carllos Eduardo Holanda

Department of Mathematics, Shantou University, Shantou, 515063, Guangdong, China

E-mail address: `c.eduarddo@gmail.com`

Abstract

We study some notions of cohomology and obtain a Livšic-like theorem for nonadditive families of potentials. Together with the existence of Gibbs states, we use this result to classify equilibrium measures for almost additive families with respect to hyperbolic flows. Moreover, building on recent examples for discrete-time dynamics, we address some Hölder and Bowen regularity problems for the physical equivalence relations between additive and asymptotically additive families with respect to hyperbolic symbolic flows and related setups.

1 Introduction

This note is a natural continuation to [Hol24], and is mainly a contribution to the study of relations between the additive classical world and the nonadditive world of families of potentials, which started in [Cun20]. In particular, we are interested in the asymptotically and almost additive cases. In this work, when considering families, the term *potential* is used interchangeably with *function*.

A family $\mathcal{A} = (a_t)_{t\geq 0}$ of functions $a_t\colon X \to \mathbb{R}$ is said to be *asymptotically additive* with respect to a flow $\Phi = (\phi_t)_{t\in\mathbb{R}}$ on a topological space X if for each $\varepsilon > 0$ there exists a function $b_\varepsilon\colon X \to \mathbb{R}$ such that

$$\limsup_{t\to\infty} \frac{1}{t}\left\| a_t - \int_0^t (b_\varepsilon \circ \phi_s)ds \right\|_\infty \leq \varepsilon,$$

where $\|\cdot\|_\infty$ is the supremum norm on X. Moreover, $\mathcal{A}$ is said to be *almost additive* with respect to Φ on X if there exists a constant $C > 0$ such that

$$-C + a_t + a_s \circ \phi_t \leq a_{t+s} \leq a_t + a_s \circ \phi_t + C$$

for every $t, s \geq 0$. It is well known that every almost additive family is asymptotically additive [FH10]. For each function $b\colon X \to \mathbb{R}$, the *additive* family $(S_t b)_{t\geq 0}$ generated by b (with respect to Φ) is denoted by $S_t b := \int_0^t (b\circ\phi_s)ds$.

It was showed in [Hol24] that, with respect to suspension flows Φ, asymptotically additive families are physically equivalent to additive families of continuous functions. That is, given an asymptotically additive family of continuous functions $\mathcal{A} = (a_t)_{t\geq 0}$ (with respect to Φ) there exists a real-valued continuous function b on the suspension manifold such that

$$\lim_{t\to\infty} \frac{1}{t} \left\| a_t - S_t b \right\|_\infty = 0. \tag{1}$$

In this case, we say that $\mathcal{A}$ is *physically equivalent* to $(S_t b)_{t\geq 0}$ (and vice-versa). Motivated by this relation, one can naturally consider the problem of studying the different levels of regularity that the physical equivalence (1) can sustain. In our framework, the most relevant types of regularity are the ones involving Bowen and Hölder functions together with families having the bounded variation condition (see Sections 2.1 and 2.2 for the definitions). In the context of hyperbolic suspension flows and related hyperbolic setups, the space of Hölder continuous functions is contained in the space of Bowen continuous functions. Furthermore, by definition, an additive family generated by a Bowen function has bounded variation with respect to any flow in general. Based on this, we are interested in three types of regularity problems:

- **Bowen regularity**. *Given any almost additive family of continuous functions $\mathcal{A} = (a_t)_{t\geq 0}$ with respect to an hyperbolic suspension flow and having bounded variation, is there a Bowen continuous function b such that $(S_t b)_{t\geq 0}$ is physically equivalent to $\mathcal{A}$?*

- **Uniform bound**. *Given any almost additive family of continuous functions $\mathcal{A} = (a_t)_{t\geq 0}$ with respect to an hyperbolic suspension flow and having bounded variation, is there a continuous function b such that*

$$\sup_{t\geq 0} \left\| a_t - S_t b \right\|_\infty < \infty \text{ ?}$$

- **Hölder regularity**. *Given any almost additive family of Hölder continuous functions $\mathcal{A} = (a_t)_{t\geq 0}$ with respect to an hyperbolic suspension flow and having bounded variation, is there a Hölder continuous function b such that $(S_t b)_{t\geq 0}$ is physically equivalent to $\mathcal{A}$?*

Here hyperbolic symbolic flows are suspensions over the two-sided full shift. These questions also can be posted with respect to hyperbolic flows or, more generally, suspension flows over subshifts of finite type (see Section 4.4). The uniform bound immediately implies Bowen regularity, and a positive answer to the Hölder regularity question in this context also gives an affirmative answer to the Bowen regularity one. These regularity issues are also pertinent in the more general case of asymptotically additive families.

Besides the intrinsic interest, there are some significant consequences regarding these regularity relations. In fact, an affirmative answer to the Bowen regularity question would imply that one can obtain the uniqueness of equilibrium measures for almost additive families with bounded variation directly from the uniqueness result for a single potential (see [Fra77, BH21a]). On the other hand, a positive answer to the Hölder regularity equivalence problem would either simplify or automatically extend some relevant results in ergodic optimization for almost and asymptotically additive families (see [BHVZ21, HLMXZ19, MSV20]).

In addition to that, the Hölder regularity problem is also connected to the existence of a relevant class of asymptotically and almost additive families admitting an analytic topological pressure function, which also could allow us to obtain higher regularities of the entropy and dimension spectrum considering many different types of level sets with respect to dynamical systems exhibiting some hyperbolic behavior (see [Rue78, BD04, BS00, PS01, BH21b, BH22c]).

In this work we obtain a characterization result for almost additive families, which is intimately connected to the three aforementioned regularity questions:

(Theorem 3). *Let $\Phi = (\phi_t)_{t\geq 0}$ be a topologically transitive continuous flow on a compact metric space X and satisfying the Closing Lemma. Let $\mathcal{B} = (b_t)_{t\geq 0}$ be an almost additive family of continuous functions with respect to Φ and satisfying the bounded variation condition. Then the following properties are equivalent:*

1. $\lim_{t\to\infty} \|b_t\|_\infty / t = 0$.
2. $\sup_{t\geq 0} \|b_t\|_\infty < \infty$.
3. there exists $K > 0$ such that $|b_t(p)| \leq K$ for all $p \in X$ and $t \geq 0$ with $\phi_t(p) = p$.

This theorem is inspired by the discrete-time counterpart obtained in [HS24] but, notwithstanding, it is proved here directly in the realm of flows, and without using any of the physical equivalence results for discrete and continuous-time dynamical systems in [Cun20] and [Hol24], respectively. The characterization of Theorem 3 gives a setup for which the uniform bound and the Bowen regularity problems are actually equivalent. In addition, this holds in particular for some types of suspensions and hyperbolic flows, showing a deeper layer in the physical equivalence relation for these types of nonadditive families (see Corollary 4). Theorem 3 is also connected to some deep results for linear cocycles, and also can be used to classify equilibrium states for almost additive families based on their cohomology classes. This later result strongly indicates that it also can be viewed as some type of nonadditive version of the classical Livšic theorem for flows ([Liv71, Liv72]). Interestingly, Theorem 3 has an optimal setup in the sense that it is no longer valid for asymptotically and subadditive families in general.

Building on some examples in [HS24], we show how to construct almost (and asymptotically) additive families of Hölder continuous potentials satisfying the bounded variation condition with respect to some symbolic flows and which are not physically equivalent to any additive family generated by a Hölder (Bowen) continuous potential. These examples show that almost and asymptotically additive families with bounded variation do not always have the same good properties of Hölder continuous functions in hyperbolic and related scenarios. Even though we are relying on examples for maps, the constructions of the counter-examples for the case of flows are rather involved and technically challenging, requiring non-trivial modifications and ideas.

The paper is organized as follows. We start studying some different notions of cohomology for almost and asymptotically additive families of functions. In the next section we restate and prove our nonadditive Livšic-type theorem for flows (Theorem 3), and show how it can be applied to the context of linear cocycles over flows. We proceed to study and compare different notions of nonadditive

Gibbs and weak Gibbs states with respect to flows, and give some examples of nonadditive families derived by volume measures and measures satisfying the Gibbs property. As another application of Theorem 3, we demonstrate how to classify almost additive families based on cohomology relations and equilibrium states. After this, we give a simple example for which the uniform bound question can always be positively answered, and another one for which the equivalences in Theorem 3 do not hold. In the next section, using the structure of suspension flows, we show how to build examples of almost additive families of Hölder continuous potentials satisfying the bounded variation condition but not physically equivalent to any additive family generated by a Hölder continuous potential, giving a negative answer to the Hölder regularity problem. Finally, we propose a way of categorizing almost additive families based on the different types of physical equivalence relations with the additive setup. We also show a construction of asymptotically additive families satisfying the bounded variation condition, with a unique equilibrium measure, but not physically equivalent to any additive family with bounded variation, giving a negative answer to the Bowen regularity problem for the asymptotically additive case. We finish the paper by quickly discussing some relevant technical matters, open problems and further explorations.

2 On Cohomology

In this section we introduce some notions of cohomology, and obtain a characterization of almost additive families of potentials. This allow us to classify equilibrium states and study regularity equivalence issues for asymptotically and almost additive families with respect to suspension flows and, in particular, hyperbolic flows (see Sections 3 and 4).

2.1 Notions for asymptotically additive families

Here we introduce some notions of cohomology for asymptotically additive families of continuous functions with respect to flows. We start recalling some basic concepts and tools in the additive (classical) setup.

A function $\psi\colon X \to \mathbb{R}$ or the additive sequence $(S_t\psi)_{t\geq 0}$ is said to satisfy the *Walters property* or is a *Walters function* (with respect to a flow Φ) if for each $\kappa > 0$ there exists $\varepsilon > 0$ such that for $x, y \in X$ and $t \geq 0$, we have that $d(\phi_s(x), \phi_s(y)) < \varepsilon$ for every $s \in [0, t]$ implies $|S_t\psi(x) - S_t\psi(y)| < \kappa$. Similarly, we say that a function $\xi\colon X \to \mathbb{R}$ satisfies the *Bowen property* or is a *Bowen function* (with respect to a flow Φ) if there exist $L > 0$ and $\varepsilon > 0$ such that for $x, y \in X$ and $t \geq 0$, we have that $d(\phi_s(x), \phi_s(y)) < \varepsilon$ for every $s \in [0, t]$ implies $|S_t\xi(x) - S_t\xi(y)| \leq L$. It is clear from the definitions that every function satisfying the Walters property also satisfies the Bowen property. Moreover, in the hyperbolic setup, the Hölder continuous functions are always Walters and, consequently, Bowen (see Proposition 7.3.1 in [FH20]).

The following property is based on the so-called *Closing Lemma* for flows, which is a classical tool in hyperbolic dynamics (see for example Theorems 5.3.11 and 6.2.4 in [FH20]).

Definition 1. Let $\Phi = (\phi_t)_{t\in\mathbb{R}}$ be a continuous flow on a compact metric space X. Φ is said to satisfy the *Closing Lemma* if for every $\varepsilon > 0$ there exists $\delta > 0$

such that if $x \in X$ and $t \geq 0$ satisfying $d(\phi_t(x), x) < \delta$, then there exists a periodic orbit $\{\phi_s(y) : 0 \leq s \leq T\}$ with $|T - t| < \varepsilon$ such that $d(\phi_s(x), \phi_s(y)) < \varepsilon$ for all $0 \leq s \leq t$.

Let us recall the notion of cohomology for functions. A continuous function $a\colon X \to \mathbb{R}$ is said to be Φ-*cohomologous to zero* if there exists a continuous function $q\colon X \to \mathbb{R}$ such that

$$a(x) = \lim_{t \to 0} \frac{q(\phi_t(x)) - q(x)}{t} \quad \text{for every } x \in X.$$

We say that a point $x \in X$ has a *forward dense orbit* if $\overline{\{\phi_s(x) : s \geq 0\}} = X$. When $\overline{\{\phi_s(x) : s \in \mathbb{R}\}} = X$, we say that $x \in X$ has a *dense orbit*. We say that a flow is *topologically transitive* if there exists at least one point with a forward dense orbit.

The following proposition is a more general version of the celebrated *Livšic theorem* originally obtained in [Liv72].

Theorem 1. *Let $\Phi = (\phi_t)_{t \in \mathbb{R}}$ be a topologically transitive continuous flow satisfying the Closing Lemma, and $a\colon X \to \mathbb{R}$ a continuous function satisfying the Walters property. Then a is cohomologous to zero if and only if for every periodic point $x = \phi_T(x)$ we have $S_T a(x) = 0$.*

Proof. See the proof of Theorem 5.3.23 in [FH20]. □

We also can obtain a characterization of additive families generated by coboundary functions. Let $\mathcal{M}(\Phi)$ be the set of Φ-invariant probability measures on X.

Proposition 2. *Under the conditions of Theorem 1, a function $a\colon X \to \mathbb{R}$ is Φ-cohomologous to zero if and only if*

$$\lim_{t \to \infty} \frac{1}{t} \left\| S_t a \right\|_\infty = 0.$$

In particular,

$$\lim_{t \to \infty} \frac{1}{t} \left\| S_t a \right\|_\infty = 0 \textit{ if and only if } \sup_{t \geq 0} \left\| S_t a \right\|_\infty < \infty.$$

Proof. Suppose a is Φ-cohomologous to zero. This implies the existence of a continuous function $q\colon X \to \mathbb{R}$ such that $S_t a = q \circ \phi_t - q$ for all $t \geq 0$. Consequently, one has $\|S_t a\|_\infty \leq 2\|q\|_\infty < \infty$ for every $t \geq 0$. Then, $\lim_{t \to \infty} \frac{1}{t} \|S_t a\|_\infty = 0$.

Conversely, let $\lim_{n \to \infty} \frac{1}{t} \|S_t a\|_\infty = 0$. The Lebesgue's dominated convergence theorem together with the Birkhoff's ergodic theorem for flows gives that

$$0 = \int_X \lim_{n \to \infty} \frac{1}{t} S_t a \, d\mu = \int_X a \, d\mu \quad \text{for all } \mu \in \mathcal{M}(\Phi). \tag{2}$$

For all $x \in X$ with $x = \phi_T(x)$, the measure $(\int_0^T \delta_{\phi_s(x)} ds)/T$ is Φ-invariant. In particular, identity (2) gives that $S_T a(x) = 0$ for all $x \in X$ with $x = \phi_T(x)$. Hence, by Theorem 1 we conclude that a is Φ-cohomologous to zero, as desired. □

Based on Proposition 2, we give a definition of cohomology for asymptotically additive families of continuous potentials.

Definition 2. We say that an asymptotically (or almost) additive family of continuous functions $\mathcal{A} = (a_t)_{t\geq 0}$ is Φ-cohomologous to a constant if there exists a continuous function $a\colon X \to \mathbb{R}$ which is Φ-cohomologous to a constant and such that

$$\lim_{t\to\infty} \frac{1}{t}\left\|a_t - S_t a\right\|_\infty = 0.$$

One can easily check that a family $\mathcal{A} = (a_t)_{t\geq 0}$ is Φ-cohomologous to a constant if and only if the sequence $(a_n/n)_{n\in\mathbb{N}}$ is uniformly convergent to a constant. In particular, $\mathcal{A}$ is Φ-cohomologous to zero if and only if $\lim_{t\to\infty} \frac{1}{t}\|a_t\|_\infty = 0$.

It is important emphasizing that the classical concept of cohomology for a function is much stronger than the one introduced for nonadditive families in Definition 2. On the other hand, Proposition 2 motivates a new definition for the nonadditive case, which is still weaker than the classical one but stronger than Definition 2.

Definition 3. An asymptotically (or almost) additive family of continuous functions $\mathcal{A} = (a_t)_{t\geq 0}$ is Φ-cohomologous to a constant if there exists a continuous function $a\colon X \to \mathbb{R}$ which is Φ-cohomologous to a constant and such that

$$\sup_{n\in\mathbb{N}} \left\|a_t - S_t a\right\|_\infty < \infty.$$

In this case, $\mathcal{A}$ is Φ-cohomologous to zero if and only if $\mathcal{A}$ is uniformly bounded.

In the next section, our main result gives a setup where the definitions 2 and 3 are, in fact, equivalent for almost additive families (see Theorem 3).

2.2 A nonadditive Livšic-type theorem for flows

We say that a family of functions $\mathcal{A} = (a_t)_{t\geq 0}$ has *bounded variation* if there exists $\varepsilon > 0$ such that

$$\sup_{t\geq 0} \sup\left\{|a_t(x) - a_t(y)| : d_t(x,y) < \varepsilon\right\} < \infty,$$

where $d_t(x,y) = \max\{d(\phi_s(x), \phi_s(y)) : s \in [0,t]\}$. Moreover, we say that $\mathcal{A}$ has *tempered variation* if

$$\limsup_{\varepsilon\to 0} \lim_{t\to\infty} \frac{\gamma_t(\varepsilon)}{t} = 0,$$

where $\gamma_t(\varepsilon) := \sup\{|a_t(x) - a_t(y)| : d_t(x,y) < \varepsilon\}$. We note that if a function ϕ satisfies the Bowen property then the additive sequence $(S_t\phi)_{t\geq 0}$ has bounded variation. The Walters property for functions (additive families) also can be extended naturally to the nonadditive case. A family of functions $\mathcal{A} = (a_t)_{t\geq 0}$ satisfies the *Walters property* if for each $\kappa > 0$ there exists $\varepsilon > 0$ such that for $x, y \in X$ and $t \geq 0$, we have that $d(\phi_s(x), \phi_s(y)) < \varepsilon$ for every $s \in [0,t]$ implies $|a_t(x) - a_t(y)| < \kappa$. It is clear from the definitions that a family satisfying the Walters property also satisfies the Bowen property.

The next result is our main theorem in this section.

Theorem 3. *Let $\Phi = (\phi_t)_{t\geq 0}$ be a topologically transitive continuous flow on a compact metric space X and satisfying the Closing Lemma. Let $\mathcal{B} = (b_t)_{t\geq 0}$ be an almost additive family of continuous functions with respect to Φ and satisfying the bounded variation property. Then, the following properties are equivalent:*

1. $\lim_{t\to\infty} \|b_t\|_\infty / t = 0$.
2. $\sup_{t\geq 0} \|b_t\|_\infty < \infty$.
3. *There exists $K > 0$ such that $|b_t(p)| \leq K$ for all $p \in X$ and $t \geq 0$ with $\phi_t(p) = p$.*

As a direct consequence, we have:

Corollary 4. *Under the hypotheses of Theorem 3, let $\mathcal{A} = (a_t)_{t\geq 0}$ be an almost additive family of continuous functions with bounded variation. Then, for a continuous function $a\colon X \to \mathbb{R}$ such that $(S_t a)_{t\geq 0}$ has bounded variation, we have*

$$\lim_{t\to\infty} \frac{1}{t} \left\| a_t - S_t a \right\|_\infty = 0 \quad \text{if and only if} \quad \sup_{t\geq 0} \left\| a_t - S_t a \right\|_\infty < \infty.$$

In particular, if $(S_t a)_{t\geq 0}$ does not have bounded variation we have

$$\sup_{t\geq 0} \left\| a_t - S_t a \right\|_\infty = \infty.$$

Corollary 4 readily implies that the Bowen regularity problem is equivalent to the uniform bound problem for topologically transitive flows satisfying the Closing Lemma. We also note that Theorem 3 is an extension of Proposition 2 to the case of almost additive families of functions.

Remark 1. Observe that Proposition 2 asks for the sequence of potentials to have the Walters property, which is stronger than the bounded variation condition. This is because the classical cohomology result obtained for a single potential is also stronger than the uniformly bounded one obtained in Theorem 3. In addition to that, as we shall see in Section 2.3, Theorem 3 is also particularly related to Theorem 1.2 in [Kal11], where control over the periodic data implies control over the full data.

To prove Theorem 3, we first need a more general auxiliary result.

Lemma 1. *Let $\Phi = (\phi_t)_{t\in\mathbb{R}}$ be a continuous flow on a compact metric space X and let $\mathcal{C} = (c_t)_{t\geq 0}$ be an almost additive family of continuous functions with uniform constant $C > 0$ and such that $\lim_{t\to\infty} \|c_t\|_\infty / t = 0$. Then:*

1. *For every τ-periodic point $x_0 \in X$, we have $\sup_{q\in\mathbb{N}} |c_{q\tau}(x_0)| \leq C$.*
2. *For every τ-periodic point x_0, there exists a constant $L := L(\tau) \geq 0$ (only depending on the period of x_0) such that $\sup_{t\geq 0} |c_t(x_0)| \leq L$.*
3. *We have*
$$\sup_{\mu\in\mathcal{M}(\Phi)} \left| \int_X c_t d\mu \right| \leq C \quad \text{for all } t \geq 0.$$

Proof. Since the family $\mathcal{C}$ is almost additive with uniform constant $C > 0$, one can see that

$$\sum_{k=0}^{p-1} c_t \circ \phi_{kt} - (p-1)C \leq c_{pt} \leq \sum_{k=0}^{p-1} c_t \circ \phi_{kt} + (p-1)C \tag{3}$$

for all $t \geq 0$ and $p \in \mathbb{N}$. Now suppose x_0 is a τ-periodic point, that is, $\phi_\tau(x_0) = x_0$. If $t = q\tau$ for some $q \in \mathbb{N}$, then

$$\phi_{kt}(x_0) = \phi_{kq\tau}(x_0) = \underbrace{(\phi_\tau \circ \phi_\tau \circ \cdots \circ \phi_\tau)}_{kq \text{ times}}(x_0) = x_0$$

for all $k \in \mathbb{N}$. In particular, this implies that

$$\lim_{p\to\infty} \frac{1}{p} \sum_{k=0}^{p-1} c_t(\phi_{kt}(x_0)) = c_t(x_0). \tag{4}$$

Since $\lim_{t\to\infty} \|c_t\|_\infty / t = 0$, it follows from (3) and (4) that

$$-C \leq c_t(x_0) = c_{q\tau}(x_0) \leq C. \tag{5}$$

By the arbitrariness of $q \in \mathbb{N}$, item 1 follows.

Let's prove item 2. Let x_0 be a τ-periodic point, consider $t = q\tau + r$ with $r \in (0, \tau)$ and fix the numbers

$$A(\tau) := \inf\left\{ \inf_{x\in X} c_s(x) : s \in [0,\tau]\right\} \quad \text{and} \quad B(\tau) := \sup\left\{ \sup_{x\in X} c_s(x) : s \in [0,\tau]\right\}. \tag{6}$$

Almost additivity together with (5) and (6) gives that

$$-2C + A(\tau) \leq -C + c_{qt}(x_0) + c_r(\phi_{q\tau}(x_0)) \leq c_t(x_0)$$

and

$$c_t(x_0) \leq c_{qt}(x_0) + c_r(\phi_{q\tau}(x_0)) + C \leq 2C + B(\tau).$$

Hence,

$$L_1(\tau) := \min\big\{A(\tau) - 2C, -C\big\} \leq c_t(x_0) \leq \max\big\{B(\tau) + 2C, C\big\} := L_2(\tau)$$

for all $t \geq 0$. Taking $L = L(\tau) := \max\{|L_1(\tau)|, |L_2(\tau)|\}$, the item 2 is proved.

Now let us prove item 3. Suppose μ is a Φ-invariant measure. Then, in particular, μ is also ϕ_t-invariant for every $t \geq 0$. By applying that $\lim_{t\to\infty} \|c_t\|_\infty / t = 0$ in the inequalities (3) together with the Birkhoff's ergodic theorem applied to the map ϕ_t, we obtain that

$$-C \leq \int_X \lim_{p\to\infty} \frac{1}{p} \sum_{k=0}^{p-1} c_t(\phi_{kt}(x)) d\mu(x) = \int_X c_t d\mu \leq C$$

for all $t \geq 0$. Since the measure $\mu \in \mathcal{M}(\Phi)$ is arbitrary, item 3 is proved. $\square$

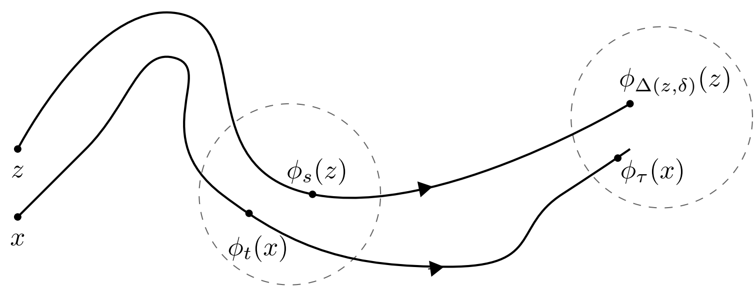

Figure 1: Approximating the orbit of any point by a finite piece of a dense orbit.

Proof of Theorem 3. The core idea of the proof is that any point in the space can always be well approximated by a point contained in a fixed finite piece of orbit.

Since $\mathcal{B}$ has bounded variation, there exists $\varepsilon > 0$ such that

$$Q := \sup_{t\geq 0} \sup \left\{ \left|b_t(x) - b_t(y)\right| : d_t(x,y) < \varepsilon \right\} < \infty. \tag{7}$$

We first show that 3 implies 2. Suppose that there exists a uniform constant $K > 0$ such that $|b_t(p)| \leq K$ for all $p \in X$ and $t \geq 0$ with $\phi_t(p) = p$. Since Φ is topologically transitive there exists a point $z \in X$ with a dense forward orbit. Now let $\delta > 0$ be the number given by the Closing Lemma. By the density of the forward orbit, there exists a number $\Delta(z,\delta) \in \mathbb{R}$ such that for each $x \in X$ and $t \in \mathbb{R}$ there exists some $s \in [0, \Delta(z,\delta)]$ with $d(\phi_t(x), \phi_s(z)) < \delta$ (see Figure 1). For $t > \Delta(z,\delta)$, in particular, there exists $s' \in [0, \Delta(z,\delta)]$ such that $d(\phi_t(z), \phi_{s'}(z)) < \delta$, which is the same as $d(\phi_{t-s'}(\phi_{s'}(z)), \phi_{s'}(z)) < \delta$. By the Closing Lemma, there exists a point $p \in X$ with $\phi_T(p) = p$ with $|T - t + s'| < \varepsilon$ and such that $d_{t-s'}(\phi_{s'}(z), p) < \varepsilon$. From almost additivity, there exists a uniform constant $L = L(\varepsilon) > 0$ such that $\|b_T - b_{t-s'}\|_\infty \leq L$. Applying the bounded variation condition (7) we have that

$$\left|b_{t-s'}(\phi_{s'}(z)) - b_{t-s'}(p)\right| \leq Q,$$

which gives that $|b_{t-s'}(\phi_{s'}(z)| \leq Q + |b_{t-s'}(p)| \leq Q + |b_T(p)| + L \leq Q + K + L$. By using almost additivity again, we get

$$\begin{aligned} \left|b_t(z)\right| = \left|b_{(t-s')+s'}(z)\right| &\leq \left|b_{s'}(z)\right| + \left|b_{t-s'}(\phi_{s'}(z))\right| + C \\ &\leq \sup_{s\in[0,\Delta(z,\delta)]} \left|b_s(z)\right| + Q + K + L + C =: \widetilde{K}. \end{aligned}$$

Since the time $t > \Delta(z,\delta)$ was arbitrary, we conclude that $|b_t(z)| \leq \widetilde{K}$ for all $t \geq 0$. Notice that the constant $\widetilde{K} > 0$ only depend on z, $\delta > 0$ and $\varepsilon > 0$. Applying the almost additivity property one more time, we have

$$\left|b_t(\phi_s(z)\right| \leq \left|b_s(z)\right| + \left|b_{t+s}(z)\right| + C \leq 2\widetilde{K} + C \quad \text{for all } t, s \geq 0.$$

Now consider any point $x \in X$. Since $\overline{\{\phi_t(z) : t \geq 0\}} = X$, there exists a sequence of points $(z_q)_{q\geq 1} \subset \{\phi_t(z) : t \geq 0\}$ such that $\lim_{q\to\infty} z_q = x$. Since every function $b_t \colon X \to \mathbb{R}$ is continuous, we obtain that

$$\left|b_t(x)\right| = \lim_{q\to\infty} \left|b_t(z_q)\right| \leq 2\widetilde{K} + C.$$

Hence, by the arbitrariness of x, we conclude that $\sup_{t\geq 0}\|b_t\|_\infty \leq 2\widetilde{K}+C<\infty$, as desired. It is obvious that 2 implies 1. Moreover, by Lemma 1, we have that 1 implies 3 and the theorem is proved. $\square$

Remark 2. Theorem 3 does not hold for asymptotically additive nor subadditive families in general. In fact, let Φ be any continuous flow on a compact metric space X and consider the family $\mathcal{A}=(a_t)_{t\geq 0}$ given by $a_t(x)=\sqrt{t}$ for all $t\geq 0$ and $x\in X$. The family $\mathcal{A}$ has bounded variation and is asymptotically additive and also subadditive with respect to Φ. Moreover, it is clear that $\lim_{t\to\infty}\|a_t\|_\infty/t=0$ but $\sup_{t\geq 0}\|a_t\|_\infty=\infty$. Proceeding as in [HS24], one can show that

$$\sup_{t\geq 0}\left\|a_t-\int_0^t (a\circ\phi_s)ds\right\|_\infty=\infty \quad \text{for every continuous function } a\colon X\to\mathbb{R}.$$

This simple example shows that Theorem 3 has the optimal nonadditive setup in the sense that it cannot be extended to more general classes of families. Moreover, it also indicates that definitions 2 and 3 are not equivalent for asymptotically additive families in general.

2.3 A connection to linear cocycles

In this section we follow closely some definitions and notions of [BH21b].

Let $\Phi=(\phi_t)_{t\in\mathbb{R}}$ be a continuous flow on a compact metric space M. Moreover, let $GL(d,\mathbb{R})$ be the set of all invertible $d\times d$ matrices. A continuous map $\mathsf{A}\colon\mathbb{R}\times M\to GL(d,\mathbb{R})$ is called a *linear cocycle* over Φ if for all $t,s\in\mathbb{R}$ and $x\in M$ we have:

1. $\mathsf{A}(0,x)=\mathrm{Id}$.
2. $\mathsf{A}(t+s,x)=\mathsf{A}(s,\phi_t(x))\mathsf{A}(t,x)$.

We shall always assume that all entries $a_{ij}(t,x)$ of $\mathsf{A}(t,x)$ are positive for every $(t,x)\in\mathbb{R}\times M$. Moreover, we consider the norm on $GL(d,\mathbb{R})$ defined by $\|B\|=\sum_{i,j=1}^d|b_{ij}|$, where b_{ij} are the entries of the matrix B.

Now we consider the family of continuous functions $\mathcal{A}_\mathrm{c}=(a_t)_{t\geq 0}$ given by

$$a_t(x)=\log\|\mathsf{A}(t,x)\| \quad \text{for all } t\geq 0 \text{ and } x\in M.$$

It follows from Proposition 12 in [BH21b] that the family $\mathcal{A}_\mathrm{c}$ is almost additive with respect to the flow Φ. We note that for a general linear cocycle, the family $\mathcal{A}_\mathrm{c}$ is only subadditive.

We say that the cocycle A has *tempered distortion* if

$$\limsup_{t\to\infty}\frac{1}{t}\log\sup\left\{\|\mathsf{A}(t,x)\mathsf{A}(t,y)^{-1}\| : z\in M \text{ and } x,y\in B_t(z,\varepsilon)\right\}=0$$

for some $\varepsilon>0$. Moreover, we say that A has *bounded distortion* if

$$\sup\left\{\|\mathsf{A}(t,x)\mathsf{A}(t,y)^{-1}\| : z\in M \text{ and } x,y\in B_t(z,\varepsilon)\right\}<\infty$$

for some $\varepsilon>0$. Clearly, bounded distortion implies tempered distortion.

Now observe that

$$\big\|\mathsf{A}(t,x)\mathsf{A}(t,x)^{-1}\big\| = \big\|\mathrm{Id}\big\| = d$$

for every $(t,x) \in \mathbb{R} \times M$, which implies that

$$\big\|\mathsf{A}(t,x)^{-1}\big\| \geq d\big\|\mathsf{A}(t,x)\big\|^{-1}.$$

Then,

$$\big\|\mathsf{A}(t,x)\mathsf{A}(t,y)^{-1}\big\| \geq \frac{K}{d}\big\|\mathsf{A}(t,x)\big\| \cdot \big\|\mathsf{A}(t,y)^{-1}\big\| \geq K\big\|\mathsf{A}(t,x)\big\| \cdot \big\|\mathsf{A}(t,y)\big\|^{-1}$$

and so

$$\big|\log\big\|\mathsf{A}(t,x)\big\| - \log\big\|\mathsf{A}(t,y)\big\|\big| \leq -\log K + \log\big\|\mathsf{A}(t,x)\mathsf{A}(t,y)^{-1}\big\|.$$

In particular, for $z \in M$ and $\varepsilon > 0$ we have

$$\sup_{x,y\in B_t(z,\varepsilon)} \big|a_t(x) - a_t(y)\big| \leq -\log K + \log \sup_{x,y\in B_t(z,\varepsilon)} \big\|\mathsf{A}(t,x)\mathsf{A}(t,y)^{-1}\big\|.$$

Hence, if A has tempered distortion, then the family $\mathcal{A}_{\mathrm{c}}$ has tempered variation, and if A has bounded distortion, then $\mathcal{A}_{\mathrm{c}}$ has bounded variation.

For a concrete example, one can consider a C^1 flow Φ on a compact set $M \subset \mathbb{R}^d$ such that for every $t \in \mathbb{R}$ and $x \in M$ the matrix $d_x\phi_t$ has only positive entries. Then $\mathsf{A}(t,x) = d_x\phi_t$ is a linear cocycle over Φ and the family $\mathcal{A}_{\mathrm{d}} = (a_t)_{t\geq 0}$ given by $a_t(x) = \log\|d_x\phi_t\|$ is an almost additive family of continuous functions with respect to Φ.

Let $GL^+(d,\mathbb{R}) \subset GL(d,\mathbb{R})$ be the set of all matrices with strictly positive entries. We have the following application of Theorem 3 to the case of continuous-time cocycles.

Theorem 5. *Let $\Phi = (\phi_t)_{t\in\mathbb{R}}$ be a topologically transitive continuous flow on a compact metric space M satisfying the Closing Lemma, and let $\mathsf{A}\colon \mathbb{R} \times M \to GL^+(d,\mathbb{R})$ be a linear cocycle over Φ with bounded distortion. Suppose there exists a compact set $\Omega \subset GL^+(d,\mathbb{R})$ such that $\mathsf{A}(t,p) \subset \Omega$ for all $t \geq 0$ and $p \in M$ with $\phi_t(p) = p$. Then there exists a compact set $\widetilde{\Omega}$ such that $\mathsf{A}(t,x) \subset \widetilde{\Omega}$ for all $t \geq 0$ and $x \in M$.*

Proof. By the hypotheses, the family of continuous functions $\mathcal{A}_{\mathrm{c}} = (a_t)_{t\geq 0}$ given by $a_t(x) = \log\|\mathsf{A}(t,x)\|$ is almost additive with respect to Φ. Moreover, since the cocycle A has bounded distortion, $\mathcal{A}_{\mathrm{c}}$ has bounded variation. Now suppose there exists a compact $\Omega \subset GL^+(d,\mathbb{R})$ where $\mathsf{A}(t,p) \subset \Omega$ for all $t \geq 0$ and $p \in M$ with $\phi_t(p) = p$. Since the map $\mathsf{A}(t,p) \mapsto \log\|\mathsf{A}(t,p)\|$ is continuous, there exists $K > 0$ such that $|a_t(p)| \leq K$ for all $t \geq 0$ and all $p \in M$ with $\phi_t(p) = p$. By Theorem 3, there exists a constant $\widetilde{K} > 0$ such that $\sup_{t\geq 0}\|a_t\|_\infty \leq \widetilde{K}$. In particular, this implies that $e^{-\widetilde{K}} \leq \|\mathsf{A}(t,x)\| \leq e^{\widetilde{K}}$ for all $t \geq 0$ and all $x \in M$. Hence, we conclude that

$$\big\|\mathsf{A}(t,x) - \mathrm{Id}\big\| \leq \big\|\mathsf{A}(t,x)\big\| + \big\|\mathrm{Id}\big\| \leq e^{\widetilde{K}} + d \quad \text{for all } t \geq 0 \text{ and } x \in M,$$

as desired. □

Remark 3. Theorem 5 is a particular continuous-time counterpart of Theorem 1.2 in [Kal11], where a uniform bound on the periodic data guarantees a uniform bound on the entire phase space.

3 Nonadditive notions of (weak) Gibbs states

In this section we compare and reconcile some notions of Gibbs states for nonadditive families of potentials and obtain a classification of equilibrium measures with respect to hyperbolic flows. We also consider families of functions derived from measures and related to Gibbs properties, which play a relevant role in our framework of Bowen regularity problems arising from physical equivalence relations (Section 4.3).

We first recall some useful concepts of the nonadditive thermodynamic formalism for maps. Let $T\colon X \to X$ be a continuous map of a compact metric space X, and let $\mathcal{F} = (f_n)_{n\in\mathbb{N}}$ be an almost additive sequence of continuous functions with tempered variation. Letting $\mathcal{M}(T)$ be the set of T-invariant probability measures, we have the following variational principle (see [Bar06, Mum06]):

$$P_T(\mathcal{F}) = \sup_{\mu\in\mathcal{M}(T)} \left(h_\mu(T) + \lim_{n\to\infty} \frac{1}{n} \int_X f_n \, d\mu \right),$$

where $P_T(\mathcal{F})$ is the *nonadditive topological pressure of* $\mathcal{F}$ *with respect to* T, and $h_\mu(T)$ is the *Kolmogorov-Sinai entropy*. A measure $\nu \in \mathcal{M}(T)$ is said to be an *equilibrium measure* or *equilibrium state* for $\mathcal{F}$ (with respect to T) if the supremum is attained at ν, that is,

$$P_T(\mathcal{F}) = h_\nu(T) + \lim_{n\to\infty} \frac{1}{n} \int_X f_n \, d\nu.$$

When $\mathcal{F}$ is an additive sequence generated by a continuous function $f\colon X \to \mathbb{R}$, one can check that

$$P_T(\mathcal{F}) = P_T(f) \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n} \int_X f_n \, d\mu = \int_X f \, d\mu \quad \text{for every } \mu \in \mathcal{M}(T),$$

where $P_T(f)$ is the classical topological pressure of f with respect to T.

Definition 4. We say that a measure μ on X (not necessarily T-invariant) is a *Gibbs measure* or a *Gibbs state* for an asymptotically additive (almost additive) sequence $\mathcal{F} = (f_n)_{n\in\mathbb{N}}$ if for any sufficiently small $\varepsilon > 0$ there exists a constant $K(\varepsilon) \geq 1$ such that

$$K(\varepsilon)^{-1} \leq \frac{\mu(B_n(x,\varepsilon))}{\exp\big[-nP_T(\mathcal{F}) + f_n(x) \big]} \leq K(\varepsilon)$$

for all $x \in X$ and $n \in \mathbb{N}$, where $B_n(x,\varepsilon)$ is the *Bowen ball* given by

$$B_n(x,\varepsilon) = \big\{ y \in X : d(T^k(x), T^k(y)) < \varepsilon \text{ for all } 0 \leq k \leq n-1 \big\}.$$

It was introduced in [Bar06] a definition of Gibbs measures for almost additive sequences using Markov partitions. In this case, one can show that this notion of Gibbs measures and the one in Definition 4 are equivalent when the dynamical system admits Markov partitions with arbitrarily small diameter (as in the case of locally maximal hyperbolic sets or repellers for C^1 diffeomorphisms, see [Bow75a]).

For the full shift map $\sigma\colon \Sigma^{\mathbb{N}} \to \Sigma^{\mathbb{N}}$, the definition is simpler. A measure μ on $\Sigma^{\mathbb{N}}$ is said to be *Gibbs* with respect to an asymptotically additive (almost additive) sequence $\mathcal{F} = (f_n)_{n\in\mathbb{N}}$ when there exists a constant $K \geq 1$ such that

$$K^{-1} \leq \frac{\mu_n(C_{i_1\ldots i_n})}{\exp\big[-nP_\sigma(\mathcal{F}) + f_n(x)\big]} \leq K$$

for all $x \in C_{i_1\ldots i_n}$ and $n \in \mathbb{N}$, where $C_{i_1\ldots i_n}$ is the *cylinder* set

$$C_{i_1\ldots i_n} = \big\{(j_1 j_2 \cdots) \in \Sigma^{\mathbb{N}} : j_1 = i_1, \ldots, j_n = i_n\big\}.$$

In the same way, we also have the following

Definition 5. We say that a measure μ on X (not necessarily T-invariant) is a *weak Gibbs measure* or a *weak Gibbs state* for an asymptotically additive (almost additive) sequence $\mathcal{F} = (f_n)_{n\in\mathbb{N}}$ if for any sufficiently small $\varepsilon > 0$ there exists a sequence $(K_n(\varepsilon))_{n\in\mathbb{N}} \subset [1, \infty)$ with $\lim_{n\to\infty} \log K_n(\varepsilon)/n = 0$ such that

$$K_n(\varepsilon)^{-1} \leq \frac{\mu(B_n(x,\varepsilon))}{\exp\big[-nP_T(\mathcal{F}) + f_n(x)\big]} \leq K_n(\varepsilon)$$

for all $x \in X$ and $n \in \mathbb{N}$.

Let X be a compact metric space. A map $S\colon X \to X$ is said to have *bounded distortion* if for each Hölder continuous function $\xi\colon X \to \mathbb{R}$ there exists a constant $D > 0$ such that if $x, y \in X$, $n \in \mathbb{N}$ and $d(T^k(x), T^k(y)) < \varepsilon$ for all $k \in \{0, \ldots, n-1\}$, then

$$\left|\sum_{k=0}^{n-1} \xi(S^k(x)) - \sum_{k=0}^{n-1} \xi(S^k(y))\right| < D\varepsilon.$$

The full shift, subshifts of finite type, uniformly expanding and hyperbolic maps all have bounded distortion (see [Wal78, Bou02]).

Let us recall some ingredients of the nonadditive thermodynamic formalism for flows. Given an almost additive family of functions $\mathcal{A} = (a_t)_{t\geq 0}$ (satisfying mild assumptions) with respect to a continuous flow Φ on a compact metric space X, we have the variational principle (see [BH21a])

$$P_\Phi(\mathcal{A}) = \sup_{\mu\in\mathcal{M}(\Phi)} \left\{h_\mu(\Phi) + \lim_{t\to\infty} \frac{1}{t} \int_X a_t \, d\mu\right\}, \tag{8}$$

where $P_\Phi(\mathcal{A})$ is the *nonadditive topological pressure of* $\mathcal{A}$ *with respect to* Φ introduced in [BH20]. Moreover, a measure $\nu \in \mathcal{M}(\Phi)$ is an *equilibrium measure* or an *equilibrium state* for $\mathcal{A}$ (with respect to Φ) if

$$P_\Phi(\mathcal{A}) = h_\nu(\Phi) + \lim_{t\to+\infty} \frac{1}{t} \int_X a_t \, d\nu.$$

Now we briefly recall the notions of suspension and hyperbolic flows, together with some useful properties. Let $T\colon X \to X$ be a homeomorphism of a compact metric space X and let $\tau\colon X \to \mathbb{R}$ be a strictly positive continuous function. Consider the space

$$W = \big\{(x, s) \in X \times \mathbb{R} : 0 \leq s \leq \tau(x)\big\},$$

and let Y be the set obtained from W identifying $(x, \tau(x))$ with $(T(x), 0)$ for each $x \in X$. Then a certain distance introduced by Bowen and Walters in [BW72] makes Y a compact metric space. The *suspension flow* over T with *height function* τ is the flow $\Phi = (\phi_t)_{t\in\mathbb{R}}$ on Y with the maps $\phi_t \colon Y \to Y$ defined by $\phi_t(x, s) = (x, s + t)$. When T is not invertible, we say that Φ is a *suspension semi-flow* on Y.

Let μ be a T-invariant probability measure on X. One can show that μ induces a Φ-invariant probability measure ν on Y such that

$$\int_Y g\, d\nu = \frac{\int_X I_g\, d\mu}{\int_X \tau\, d\mu} \tag{9}$$

for any continuous function $g \colon Y \to \mathbb{R}$, where $I_g(x) = \int_0^{\tau(x)} (g \circ \phi_s)(x)\, ds$. Conversely, any Φ-invariant probability measure ν on Y is of this form for some T-invariant probability measure μ on X. Abramov's entropy formula says that

$$h_\nu(\Phi) = \frac{h_\mu(T)}{\int_X \tau\, d\mu}. \tag{10}$$

By (9) and (10) we obtain

$$h_\nu(\Phi) + \int_Y g\, d\nu = \frac{h_\mu(T) + \int_X I_g\, d\mu}{\int_X \tau\, d\mu}. \tag{11}$$

Since $\tau > 0$, it follows from (11) that

$$P_\Phi(g) = 0 \quad \text{if and only if} \quad P_T(I_g) = 0,$$

where $P_\Phi(g)$ is the classical topological pressure of g with respect to Φ on Y and $P_T(I_g)$ is the classical topological pressure of I_g with respect to T on X. When $P_\Phi(g) = 0$, ν is an equilibrium measure for g if and only if μ is an equilibrium measure for I_g.

Let $\Phi = (\phi_t)_{t\in\mathbb{R}}$ be a C^1 flow on a smooth manifold M. A compact Φ-invariant set $\Lambda \subset M$ is called a *hyperbolic set for* Φ if there exists a splitting

$$T_\Lambda M = E^s \oplus E^u \oplus E^0$$

and constants $c > 0$ and $\lambda \in (0, 1)$ such that for each $x \in \Lambda$:

1. the vector $(d/dt)\phi_t(x)|_{t=0}$ generates $E^0(x)$.
2. for each $t \in \mathbb{R}$, we have
$$d_x\phi_t E^s(x) = E^s(\phi_t(x)) \quad \text{and} \quad d_x\phi_t E^u(x) = E^u(\phi_t(x)).$$
3. $\|d_x\phi_t v\| \le c\lambda^t \|v\|$ for $v \in E^s(x)$ and $t > 0$.
4. $\|d_x\phi_{-t} v\| \le c\lambda^t \|v\|$ for $v \in E^u(x)$ and $t > 0$.

Given a hyperbolic set Λ for a flow Φ, for each $x \in \Lambda$ and any sufficiently small $\varepsilon > 0$ we define

$$A^s(x) = \big\{ y \in B(x, \varepsilon) : d(\phi_t(y), \phi_t(x)) \searrow 0 \text{ when } t \to +\infty \big\}$$

and

$$A^u(x) = \{y \in B(x,\varepsilon) : d(\phi_t(y), \phi_t(x)) \searrow 0 \text{ when } t \to -\infty\}.$$

Moreover, let $V^s(x) \subset A^s(x)$ and $V^u(x) \subset A^u(x)$ be the largest connected components containing x. These are smooth manifolds, called respectively *local stable and unstable manifolds* of size ε at the point x, satisfying:

1. $T_xV^s(x) = E^s(x)$ and $T_xV^u(x) = E^u(x)$;
2. for each $t > 0$ we have
$$\phi_t(V^s(x)) \subset V^s(\phi_t(x)) \quad \text{and} \quad \phi_{-t}(V^u(x)) \subset V^u(\phi_{-t}(x));$$
3. there exist $\kappa > 0$ and $\mu \in (0,1)$ such that for each $t > 0$ we have
$$d(\phi_t(y), \phi_t(x)) \le \kappa\mu^t d(y,x) \quad \text{for } y \in V^s(x)$$
and
$$d(\phi_{-t}(y), \phi_{-t}(x)) \le \kappa\mu^t d(y,x) \quad \text{for } y \in V^u(x).$$

We recall that a set Λ is said to be *locally maximal* (with respect to a flow Φ) if there exists an open neighborhood U of Λ such that

$$\Lambda = \bigcap_{t\in\mathbb{R}} \phi_t(U).$$

Given a locally maximal hyperbolic set Λ and a sufficiently small $\varepsilon > 0$, there exists $\delta > 0$ such that if $x, y \in \Lambda$ satisfy $d(x,y) \le \delta$, then there exists a unique $t = t(x,y) \in [-\varepsilon, \varepsilon]$ such that

$$[x,y] := V^s(\phi_t(x)) \cap V^u(x)$$

is a single point in Λ.

By work of Bowen [Bow73] and Ratner [Rat73], any locally maximal hyperbolic set Λ has *Markov partitions* of arbitrarily small diameter. Based on this, one can see that these hyperbolic flows inherit the same good structure of a suspension flow over a symbolic map and with a Hölder continuous height function.

The variational principle (8) and the notion of equilibrium states also hold for asymptotically additive families with respect to suspension flows, including locally maximal hyperbolic sets for C^1 flows (see Section 3 in [Hol24]).

Now consider $\Phi = (\phi_t)_{t\in\mathbb{R}}$ the suspension semi-flow over a continuous map $T\colon X \to X$ satisfying the bounded distortion property and with Hölder continuous height function τ. Proposition 19 in [BS00] guarantees that for each sufficiently small $\varepsilon > 0$ there exists a constant $\kappa > 0$ such that

$$B^Y_{\tau_m(x)}(\phi_s(x), \varepsilon) \subset B^X_m(x, \kappa\varepsilon) \times (s - \kappa\varepsilon, s + \kappa\varepsilon), \tag{12}$$

$$B^X_m(x, \varepsilon/\kappa) \times (s - \varepsilon/\kappa, s + \varepsilon/\kappa) \subset B^Y_{\tau_m(x)}(\phi_s(x), \varepsilon) \tag{13}$$

for every $x \in X$, $0 < s < \tau_m(x)$ and $m \in \mathbb{N}$, where $B^Y_t(y,\delta)$ and $B^X_n(x,\delta)$ denote, respectively, the Bowen ball with respect to the flow Φ on Y and the Bowen ball with respect to the map T on X, and

$$\tau_n(x) = \sum_{k=0}^{n-1} \tau(T^k(x)) \quad \text{for all } x \in X.$$

Let $\mathcal{A} = (a_t)_{t\geq 0}$ be a family of almost additive continuous functions with respect to Φ. Following as in the proof of Lemma 3.1 in [BH21a], the sequence $\mathcal{C} = (c_n)_{n\in\mathbb{N}}$ given by $c_n(x) = a_{\tau_n(x)}(x)$ is almost additive with respect to T. Now consider μ a Gibbs measure for the sequence $\mathcal{C}$ on X and let ν be the measure on Y induced by μ (see identity (9)). In particular, $\nu = (\mu \times \lambda)/(\int_X \tau\, d\mu)$, where λ is the Lebesgue measure on $\mathbb{R}$. By the Gibbs property of μ, for any sufficiently small $\varepsilon > 0$ there exist $K_1(\varepsilon) > 0$ and $K_2(\varepsilon) > 0$ such that

$$\begin{aligned} K_1(\varepsilon)^{-1} \exp\big[-mP_T(\mathcal{C}) + c_m(x)\big] &\leq \mu(B_m^X(x, \kappa\varepsilon)) \\ &\leq K_1(\varepsilon) \exp\big[-mP_T(\mathcal{C}) + c_m(x)\big], \end{aligned} \tag{14}$$

$$\begin{aligned} K_2(\varepsilon)^{-1} \exp\big[-mP_T(\mathcal{C}) + c_m(x)\big] &\leq \mu(B_m^X(x, \varepsilon/\kappa)) \\ &\leq K_2(\varepsilon) \exp\big[-mP_T(\mathcal{C}) + c_m(x)\big] \end{aligned} \tag{15}$$

for all $x \in X$ and $m \in \mathbb{N}$. By identity (11) and the definition of τ_m, we get

$$\left(\frac{1}{\sup\tau}\right) P_T(\mathcal{C}) \leq P_\Phi(\mathcal{A}) \leq \left(\frac{1}{\inf\tau}\right) P_T(\mathcal{C}) \quad \text{and} \quad m\inf\tau \leq \tau_m(x) \leq m\sup\tau.$$

Moreover, one can check that for all $t > 0$ there exists $m \in \mathbb{N}$ such that $\tau_m(x) \leq t \leq \tau_{m+1}(x)$ with $t - \tau_m(x) \in [0, \sup\tau]$, which clearly gives that

$$\big|a_t(x) - a_{\tau_m(x)}(x)\big| \leq \sup_{s\in[0,\sup\tau]} \|a_s\|_\infty =: q.$$

It follows from (12) and (14) that

$$\begin{aligned} \nu(B_t^Y(\phi_s(x), \varepsilon)) &\leq \nu(B_{\tau_m(x)}^Y(\phi_s(x), \varepsilon)) \\ &\leq \frac{2\kappa\varepsilon K_1(\varepsilon)}{\inf\tau} \exp\big[-\tau_m(x)P_\Phi(\mathcal{A}) + a_{\tau_m(x)}(x)\big] \\ &\leq \frac{2\kappa\varepsilon K_1(\varepsilon)}{\inf\tau} \underbrace{\exp\big[(\sup\tau)P_\Phi(\mathcal{A}) + q\big]}_{=:\,L_1} \exp\big[-tP_\Phi(\mathcal{A}) + a_t(x)\big] \\ &= \frac{2\kappa\varepsilon L_1 K_1(\varepsilon)}{\inf\tau} \exp\big[-tP_\Phi(\mathcal{A}) + a_t(x)\big] \end{aligned}$$

for all $x \in X$ and $s \in [0, \tau(x)]$. The almost additivity of the family $\mathcal{A}$ readilly implies that

$$|a_t(x) - a_t(\phi_s(x))| \leq 2 \sup_{s\in[0,\sup\tau]} \|a_s\|_\infty := \widetilde{q}.$$

Since for each $y \in Y$ there exist $x \in X$ and $s \in [0, \sup\tau]$ such that $y = \phi_s(x)$, we finally obtain

$$\nu(B_t^Y(y, \varepsilon)) \leq \frac{2\kappa\varepsilon K_1(\varepsilon)e^{\widetilde{q}}}{\inf\tau} \exp\big[-tP_\Phi(\mathcal{A}) + a_t(y)\big] = \widetilde{K_1}(\varepsilon) \exp\big[-tP_\Phi(\mathcal{A}) + a_t(y)\big]$$

for all $y \in Y$ and $t > 0$, where $\widetilde{K_1}(\varepsilon) := (2\varepsilon L_1 K_1(\varepsilon)e^{\widetilde{q}})/\inf\tau$ only depends on $\varepsilon > 0$ and the function $\tau > 0$. Similarly, the identities (13) and (15) guarantee the existence of a constant $\widetilde{K_2}(\varepsilon) > 0$ such that

$$\nu(B_t^Y(y, \varepsilon)) \geq \widetilde{K_2}(\varepsilon) \exp\big[-tP_\Phi(\mathcal{A}) + a_t(y)\big] \quad \text{for all } y \in Y \text{ and } t > 0.$$

Therefore, we conclude that Gibbs measures for almost additive sequences on the base space induce the Gibbs property for almost additive families with respect to the suspension semi-flow. Analogously, one can show that weak Gibbs measures for the asymptotically additive sequence on the base induce measures that satisfy the weak Gibbs property for the asymptotically additive family with respect to the suspension semi-flow. These relations involving (weak) Gibbs properties between the map on base space and the flow also hold for suspension flows over maps having the bounded distortion[1]. In particular, by the existence of Markov partitions (see [Bow73, Rat73]), this framework includes locally maximal hyperbolic sets for topologically mixing C^1 flows.

Motivated by this, we have the following definitions.

Definition 6. Let Φ be a continuous flow on a compact metric space X. We say that a measure μ on X (not necessarily Φ-invariant) is a *Gibbs measure* or a *Gibbs state* for an asymptotically additive family of continuous functions $\mathcal{A} = (a_t)_{t\geq 0}$ (with respect to Φ) if for any sufficiently small $\varepsilon > 0$ there exists a constant $K(\varepsilon) \geq 1$ such that

$$K(\varepsilon)^{-1} \leq \frac{\mu(B_t(x,\varepsilon))}{\exp\big[-tP_\Phi(\mathcal{A}) + a_t(x)\big]} \leq K(\varepsilon)$$

for all $x \in X$ and $t > 0$.

Definition 7. Let Φ be a continuous flow on a compact metric space X. We say that a measure μ on X (not necessarily Φ-invariant) is a *weak Gibbs measure* or a *weak Gibbs state* for an asymptotically additive family of continuous functions $\mathcal{A} = (a_t)_{t\geq 0}$ (with respect to Φ) if for any sufficiently small $\varepsilon > 0$ there exists a sequence $(K_t(\varepsilon))_{t>0} \subset [1,\infty)$ with $\lim_{t\to\infty} \log K_t(\varepsilon)/t = 0$ such that

$$K_t(\varepsilon)^{-1} \leq \frac{\mu(B_t(x,\varepsilon))}{\exp\big[-tP_\Phi(\mathcal{A}) + a_t(x)\big]} \leq K_t(\varepsilon)$$

for all $x \in X$ and $t > 0$.

Remark 4. For hyperbolic flows and suspension flows over subshifts of finite type, uniformly expanding or hyperbolic maps in general, Definition 6 is a generalization of the classical notion of Gibbs measures to the nonadditive setup (see for example Definition 4.3.25 in [FH20]).

The following result guarantees the existence of Gibbs measures for almost additive families of functions with respect to hyperbolic flows.

Proposition 6 ([BH21a, Theorem 3.5])**.** *Let Λ be a locally maximal hyperbolic set for a topologically mixing C^1 flow Φ and let $\mathcal{A}$ be an almost additive family of continuous functions on Λ with bounded variation. Then:*

1. *There exists a unique equilibrium measure for $\mathcal{A}$.*
2. *There exists a unique Φ-invariant Gibbs measures for $\mathcal{A}$.*
3. *The two measures are equal and are ergodic.*

[1] In [BH21a], the authors considered hyperbolic flows and, via Markov partitions, defined the Gibbs property only on the base space. Here, we showed that the definition on the base space implies the, a priori, more general definition directly for the flow space (Definition 6).

Remark 5. The Gibbs state in Proposition 6 was obtained using the definition on the base space (see Section 3.3 in [BH21a]). As we saw above, this implies that the induced Φ-invariant measure satisfies the Gibbs property as in Definition 6. Proposition 6 also holds for appropriate versions of suspension flows over subshifts of finite type, uniformly expanding or hyperbolic maps with Hölder continuous height functions. On the other hand, for asymptotically additive families under the hypotheses of Proposition 6, we cannot guarantee uniqueness of equilibrium measures. In these cases, the measures lifted from the base space are only guaranteed to be weak Gibbs (in the sense of Definition 7).

3.1 Some families derived from Gibbs states and other measures

Many natural examples of nonadditive families were given in [Hol24]. Here, we also bring other relevant sources of almost and asymptotically additive families of functions.

If a measure η on a compact metric space X is Gibbs for some almost additive family of continuous functions with respect to a flow $\Phi = (\phi_t)_{t\in\mathbb{R}}$ on X, then for any sufficiently small $\delta > 0$ there exists a constant $K(\delta) > 1$ such that

$$\frac{1}{K(\delta)} \le \frac{\eta(B_{t+s}(x,\delta))}{\eta(B_t(x,\delta))\eta(B_s(\phi_t(x),\delta))} \le K(\delta) \quad \text{for all } x \in X \text{ and } t, s > 0.$$

In particular, for each $\delta > 0$, the family of functions $\mathcal{A}^\delta = (a_t^\delta)_{t\ge 0}$ given by $a_t^\delta(x) = \log \eta(B_t(x,\delta))$ is almost additive (defining $a_0^\delta \equiv 0$). Since every family admitting a Gibbs measure has bounded variation, it clearly follows that $\mathcal{A}^\delta$ also satisfies the bounded variation condition. We observe that the functions $a_t^\delta : X \to \mathbb{R}$ are not necessarily continuous. In fact, using the Gibbs property of η, one can only guarantee the existence of constants $K_1(\delta) \ge K_2(\delta) > 0$ such that

$$K_2(\delta) + \limsup_{x\to x_0} a_t^\delta(x) \le a_t^\delta(x_0) \le K_1(\delta) + \liminf_{x\to x_0} a_t^\delta(x)$$

for all $x_0 \in X$ and all $t \ge 0$. In particular, the functions $x \mapsto a_t^\delta(x)$ are upper semicontinuous.

Proposition 7. *Let Φ be a continuous flow on a compact metric space X, and let ν be a measure on X. Then:*

1. *If for some $\delta > 0$ there exist constants $A(\delta) \ge B(\delta) > 0$ and an almost additive family of continuous functions $\mathcal{G} = (g_t)_{t\ge 0}$ such that*

   $$B(\delta)e^{g_t(x)} \le \nu(B_t(x,\delta)) \le A(\delta)e^{g_t(x)} \quad \text{for all } x \in X \text{ and } t \ge 0,$$

   then there exists an almost additive family of Hölder continuous functions $\mathcal{H} = (h_t)_{t\ge 0}$ satisfying

   $$\sup_{t\ge 0}\sup_{x\in X} \left|\log \nu(B_t(x,\delta)) - h_t(x)\right| < \infty.$$

2. *If for some $\delta > 0$ there exist sequences of numbers $(C_t(\delta))_{t\ge 0}, (D_t(\delta))_{t\ge 0} \subset [1,\infty)$ such that $\log C_t(\delta)/t \to 0$, $\log D_t(\delta)/t \to 0$ and an asymptotically additive family of continuous functions $\mathcal{F} = (f_t)_{t\ge 0}$ such that*

   $$D_t(\delta)e^{f_t(x)} \le \nu(B_t(x,\delta)) \le C_t(\delta)e^{f_t(x)} \quad \text{for all } x \in X \text{ and } t \ge 0,$$

then there exists an asymptotically additive family of Hölder continuous functions $\mathcal{J} = (j_t)_{t\geq 0}$ *satisfying*

$$\lim_{t\to\infty} \frac{1}{t} \sup_{x\in X} \big| \log \nu(B_t(x,\delta)) - j_t(x)\big| = 0.$$

Proof. Since the space of Hölder continuous functions is dense in the space of continuous functions on compact spaces, we can guarantee the existence of families $\mathcal{H} = (h_t)_{t\geq 0}$ and $\mathcal{J} = (j_t)_{t\geq 0}$ of Hölder continuous functions such that $\sup_{x\in X} |g_t(x) - h_t(x)| \leq 1$ and $\sup_{x\in X} |f_t(x) - j_t(x)| \leq 1$ for all $t \geq 0$. Clearly $\mathcal{H}$ is almost additive and $\mathcal{J}$ is asymptotically additive with respect to Φ. $\square$

Example 1. Gibbs measures satisfy the conditions of item 1, and weak Gibbs measures satisfy the conditions of item 2 in Proposition 7. This means that, modulo physical equivalences, the families generated by them are almost and asymptotically additive families of Hölder continuous functions, respectively.

Now let M be a compact Riemannian manifold and $\Lambda \subset M$ a hyperbolic set for a topologically mixing C^1 flow Φ. For each $t > 0$, consider the continuous function $J_t : \Lambda \to \mathbb{R}$ given by $J_t(x) = -\log \|d_x\phi_t|_{E^u(x)}\|$, where $E^u(x)$ is the unstable vector space at x. Since $x \mapsto E^u(x)$ is Hölder continuous, we also have that each function $x \mapsto J_t(x)$ is Hölder. Now let λ be the Lebesgue measure on M. Assuming that Φ is C^2, the *Volume Lemma* ([FH20, Proposition 7.4.3]) says that for any sufficiently small $\delta > 0$ there exist constants $C_\delta, D_\delta > 0$ such that

$$D_\delta J_t(x) \leq \lambda(B_t(x,\delta)) \leq C_\delta J_t(x) \quad \text{for all } x \in \Lambda \text{ and } t \geq 0.$$

Moreover, one can check that the family $(J_t)_{t\geq 0}$ is additive with respect Φ. In this case, the measure λ satisfies the conditions of the first item in Proposition 7. Hence, the family $Leb^\delta = (Leb^\delta_t)_{t\geq 0}$ given by $Leb^\delta_t(x) = \log \lambda(B_t(x,\delta))$ is almost additive with bounded variation and physically equivalent to an almost additive family of Hölder continuous functions. Actually, in this particular case, Leb^δ is physically equivalent to the additive family $(J_t)_{t\geq 0}$.

3.2 Classification of nonadditive equilibrium states

In this section we apply Theorem 3 to see how we can compare families with the same equilibrium measures, only based on the information provided by the periodic data of the system.

Theorem 8. *Let Λ be a locally maximal hyperbolic set for a topologically mixing C^1 flow $\Phi = (\phi_t)_{t\geq 0}$, and let $\mathcal{A} = (a_t)_{t\geq 0}$ and $\mathcal{B} = (b_t)_{t\geq 0}$ be two almost additive families of continuous functions with bounded variation. Then $\mathcal{A}$ and $\mathcal{B}$ have the same equilibrium measure if and only if there exists a constant $K > 0$ such that*

$$\big|a_t(p) - b_t(p) - t(P_\Phi(\mathcal{A}) - P_\Phi(\mathcal{B}))\big| \leq K$$

for all $p \in \Lambda$ and $t \geq 0$ with $\phi_t(p) = p$.

Proof. Suppose that $|a_t(p) - b_t(p) - t(P_\Phi(\mathcal{A}) - P_\Phi(\mathcal{B}))| \leq K$ for all $p \in \Lambda$ and $t \geq 0$ with $\phi_t(p) = p$. It follows from Theorem 3 that

$$\sup_{t\geq 0} \sup_{x\in\Lambda} \big|a_t(x) - b_t(x) - t(P_\Phi(\mathcal{A}) - P_\Phi(\mathcal{B}))\big| < \infty. \tag{16}$$

Now consider the almost additive family $\mathcal{D} = (d_t)_{t\geq 0}$ given by $d_t := b_t + t(P_\Phi(\mathcal{A}) - P_\Phi(\mathcal{B}))$. By (16), the family $(a_t - d_t)/t$ converges uniformly to zero on Λ. Togehter with the definition of nonadditive topological pressure, we have

$$P_\Phi(\mathcal{A}) = P_\Phi(\mathcal{D}) \quad \text{and} \quad \lim_{t\to\infty} \frac{1}{t}\int_\Lambda a_t\, d\nu = \lim_{t\to\infty} \frac{1}{t}\int_\Lambda d_t\, d\nu \quad \text{for all } \nu \in \mathcal{M}(\Phi),$$

showing that $\mathcal{A}$ and $\mathcal{D}$ have the same equilibrium measures. Moreover, since $P_\Phi(\mathcal{D}) = P_\Phi(\mathcal{B}) + (P_\Phi(\mathcal{A}) - P_\Phi(\mathcal{B}))$ and

$$\begin{aligned}\sup_{\mu\in\mathcal{M}(\Phi)} \left(h_\mu(\Phi) + \lim_{t\to\infty} \frac{1}{t}\int_\Lambda d_t\, d\mu \right) = \sup_{\mu\in\mathcal{M}(\Phi)} \left(h_\mu(\Phi) + \lim_{t\to\infty} \frac{1}{t}\int_\Lambda b_t\, d\mu \right)& \\ + P_\Phi(\mathcal{A}) - P_\Phi(\mathcal{B}),&\end{aligned}$$

$\mathcal{D}$ and $\mathcal{B}$ also share the same equilibrium measures. Hence, we conclude that $\mathcal{A}$ and $\mathcal{B}$ also have the same equilibrium measures.

Let's prove the converse. By Proposition 6, $\mathcal{A}$ and $\mathcal{B}$ have unique equilibrium measures, each one of them satisfying the Gibbs property with respect to Φ. Now, by assumption, suppose these equilibrium measures are the same unique measure $\eta \in \mathcal{M}(\Phi)$. By the Gibbs property, for each sufficiently small $\varepsilon > 0$ there exist constants $K_1(\varepsilon) \geq 1$ and $K_2(\varepsilon) \geq 1$ such that

$$K_1(\varepsilon)^{-1} \leq \frac{\eta(B_t(x,\varepsilon))}{\exp\big[-tP_\Phi(\mathcal{A}) + a_t(x)\big]} \leq K_1(\varepsilon),$$

$$K_2(\varepsilon)^{-1} \leq \frac{\eta(B_t(x,\varepsilon))}{\exp\big[-tP_\Phi(\mathcal{B}) + b_t(x)\big]} \leq K_2(\varepsilon)$$

for all $x \in \Lambda$ and $t \geq 0$. This readily gives that

$$K_1(\varepsilon)^{-1}K_2(\varepsilon)^{-1} \leq \exp\big[a_t(x) - b_t(x) - t(P_\Phi(\mathcal{A}) - P_\Phi(\mathcal{B}))\big] \leq K_1(\varepsilon)K_2(\varepsilon)$$

for all $x \in \Lambda$ and $t \geq 0$, which implies

$$\sup_{x\in\Lambda} \big|a_t(x) - b_t(x) - t(P_\Phi(\mathcal{A}) - P_\Phi(\mathcal{B}))\big| \leq \log(K_1(\varepsilon)K_2(\varepsilon)) \quad \text{for all } t \geq 0.$$

In particular, we get $|a_t(p) - b_t(p) - t(P_\Phi(\mathcal{A}) - P_\Phi(\mathcal{B}))| \leq \log(K_1(\varepsilon)K_2(\varepsilon))$ for all $p \in \Lambda$ and $t \geq 0$ with $\phi_t(p) = p$, as desired. $\square$

Together with Theorem 3, Theorem 8 shows that two almost additive families with bounded variation share the same unique equilibrium state if and only if they are cohomologous to each other (modulo a uniform constant) in the sense of definitions 2 and 3. In this context, Theorem 8 is the nonadditive counterpart of the classical classification theorem for hyperbolic flows (see Theorem 7.3.24 in [FH20]).

4 On Regularity

Let us now consider the regularity problems involving the physical equivalence relations of asymptotically and almost additive families. We start investigating some natural simple examples in the non hyperbolic context. We also address the regularity issues for setups related to systems with hyperbolic behavior.

4.1 Non hyperbolic setups

One of the main ingredients in the proof of Theorem 3 is the simultaneous existence of periodic and transitive points. A reasonable point is to ask what would happen in a system with no periodic points or no transitive data. In this regard, the most natural examples seem to be linear flows on compact spaces.

Let us start with an example of a setup where the periodic data is everywhere and with the same period.

Example 2. Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ be the n-torus and consider $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ to be linear dependent, that is, there exist (not all zero) $k_j \in \mathbb{Z}$ such that $\sum_{j=1}^{n} k_j\alpha_j = 0$. The linear flow $\Phi^\alpha = (\phi_t)_{t\in\mathbb{R}}$ on $\mathbb{T}^n$ in the direction α is defined by $\phi_t(x) = x + t\alpha \pmod 1$. Letting $\mathcal{A} = (a_t)_{t\geq 0}$ be an almost additive family of continuous functions with respect to Φ^α, Theorem 1 and Example 1 in [Hol24] guarantee the existence of a continuous function $b\colon \mathbb{T}^n \to \mathbb{R}$ such that $\lim_{t\to\infty} \|a_t - S_t b\|_\infty / t = 0$, where $\|\cdot\|_\infty$ is the supremum norm on $\mathbb{T}^n$. Since Φ^α is a periodic flow, by Lemma 1 there exists a constant $L > 0$ (depending only on the period) such that

$$\sup_{t\geq 0} \left\| a_t - S_t b \right\|_\infty \leq L.$$

In this case, the uniform bound exists even if the family $\mathcal{A}$ does not have bounded variation. Moreover, it is clear that the additive family $(S_t b)_{t\geq 0}$ has bounded variation if and only if $\mathcal{A}$ also has it.

We now check what happens in the opposite extreme: transitive systems without periodic points.

Example 3. Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ be the n-torus and let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ be linear independent. In this case, the linear flow $\Phi^\alpha = (\phi_t)_{t\in\mathbb{R}}$ on $\mathbb{T}^n$ in the direction α given by $\phi_t(x) = x + t\alpha \pmod 1$ is *minimal*, that is, every orbit is dense in $\mathbb{T}^n$. Now let $\mathcal{A} = (a_t)_{t\geq 0}$ be any almost (or asymptotically) additive family of continuous functions. In particular, letting ν be the Lebesgue measure on $\mathbb{T}^n$ and $b\colon \mathbb{T}^n \to \mathbb{R}$ the continuous function given by the physical equivalence relation ([Hol24, Theorem 1]), we have

$$\lim_{t\to\infty} \frac{1}{t} \left\| a_t - t \int_{\mathbb{T}^n} b \, d\nu \right\|_\infty \leq \lim_{t\to\infty} \frac{1}{t} \left\| a_t - S_t b \right\|_\infty + \lim_{t\to\infty} \frac{1}{t} \left\| S_t b - t \int_{\mathbb{T}^n} b \, d\nu \right\|_\infty = 0.$$

This means that the function b can by replaced by the constant $\int_{\mathbb{T}^n} b \, d\nu$, which always satisfies the bounded variation property. On the other hand, the classical Gottshalk and Hedlund theorem for flows (see for example Theorem C in [McC99]), guarantees that $\sup_{t\geq 0} \|S_t g - t \int_{\mathbb{T}^n} g d\nu\|_\infty = \infty$ for every continuous function $g\colon \mathbb{T}^n \to \mathbb{R}$ not Φ^α-cohomologous to a constant. Therefore, for these types of linear flows, Theorem 3 fails even in the additive case assuming functions with any strong regularity.

Remark 6. Example 2 does not satisfy the hypotheses of Theorem 3. However, all the equivalences there are satisfied, even without asking for the bounded variation property of the families. In the opposite direction, Example 3 also does not satisfy the hypotheses of Theorem 3 but the uniform bound cannot be obtained, even asking for any type of regularity on the families of potentials.

In the next sections we consider richer setups for investigating Hölder regularity and the bounded variation condition.

4.2 Hölder regularity

We show how to construct almost and asymptotically additive families of Hölder continuous functions satisfying the bounded variation property but not physically equivalent to any additive family generated by a Hölder continuous function. Our approach is based on the following result for symbolic systems, which extends the examples in [HS24].

Here the set of symbols Σ is assumed to be finite.

Theorem 9. *Let $\sigma\colon \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ be the two-sided shift map. Then:*

1. *There exists an almost additive sequence of continuous functions with respect to σ satisfying the bounded variation condition and which is not physically equivalent to any additive sequence generated by a Hölder continuous function.*
2. *There exist almost additive sequences of Hölder continuous functions with respect to σ satisfying the bounded variation condition and which are not physically equivalent to any additive sequence generated by a Hölder continuous function.*

Proof. Let $\sigma_L\colon \Sigma^{\mathbb{N}} \to \Sigma^{\mathbb{N}}$ be the left-sided full shift. Fix $\beta > 1$ and define $s = s(\omega, \widetilde{\omega})$ on $\Sigma^{\mathbb{N}} \times \Sigma^{\mathbb{N}}$ as the smallest positive number s such that $\omega_s \neq \widetilde{\omega_s}$. In this case, we consider the distance on $\Sigma^{\mathbb{N}}$ to be $d_L(\omega, \widetilde{\omega}) = \beta^{-s(\omega,\widetilde{\omega})}$ if $\omega \neq \widetilde{\omega}$ and $d_L(\omega, \widetilde{\omega}) = 0$ if $\omega = \widetilde{\omega}$. Similarly, we define $q = q(\omega, \omega')$ on $\Sigma^{\mathbb{Z}} \times \Sigma^{\mathbb{Z}}$ as the smallest positive number q such that $\omega_{-q} \neq \omega'_{-q}$ or $\omega_q \neq \omega'_q$. From this, we consider the distance on $\Sigma^{\mathbb{Z}}$ as $d(\omega, \omega') = \beta^{-q(\omega,\omega')}$ if $\omega \neq \omega'$ and $d(\omega, \omega') = 0$ if $\omega = \omega'$.

Now let $\mathcal{F} = (f_n)_{n\in\mathbb{N}}$ be any almost additive sequence of continuous functions on $\Sigma^{\mathbb{N}}$ with respect to σ_L, satisfying the bounded variation condition and not physically equivalent to any additive sequence generated by a Hölder continuous function (for example, the sequence generated by the quasi-Bernoulli measure in Theorem 11 in [HS24]). Consider the canonical projection $\pi\colon \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{N}}$ given by

$$\omega = (\cdots \omega_{-2}\omega_{-1}\omega_0, \omega_1\omega_2 \cdots) \longmapsto \pi(\omega) = (\omega_1\omega_2\omega_3 \cdots),$$

and let $\mathcal{G} = (g_n)_{n\in}$ be the sequence on $\Sigma^{\mathbb{Z}}$ given by $g_n = f_n \circ \pi$. Since we have $d_L(\pi(\omega), \pi(\omega')) \leq d(\omega, \omega')$ for all $\omega, \omega' \in \Sigma^{\mathbb{Z}}$, g_n is continuous for each $n \in \mathbb{N}$. By the relation $(\sigma_L \circ \pi)(\omega) = (\pi \circ \sigma)(\omega)$ for all $\omega \in \Sigma^{\mathbb{Z}}$, one can easily see that $\mathcal{G}$ is almost additive with respect to σ. Moreover, since $\mathcal{F}$ has bounded variation, we get

$$\begin{aligned} &\sup_{n\in\mathbb{N}} \left\{ |g_n(\omega) - g_n(\widetilde{\omega})| : \omega, \widetilde{\omega} \in C_{i_1\ldots i_n} \right\} \\ &\leq \sup_{n\in\mathbb{N}} \left\{ |f_n(\pi(\omega)) - f_n(\pi(\widetilde{\omega}))| : \pi(\omega), \pi(\widetilde{\omega}) \in C_{i_1\ldots i_n} \cap \Sigma^{\mathbb{N}} \right\} < \infty. \end{aligned}$$

That is, $\mathcal{G}$ also satisfies the bounded variation condition. Now suppose that $\phi\colon \Sigma^{\mathbb{Z}} \to \mathbb{R}$ is a Hölder continuous function such that $\mathcal{G}$ is phisically equivalent to $(S_n\phi)_{n\in\mathbb{N}}$ with respect to σ. Lemma 1.6 in [Bow75a] (see also Section 3 in [Sin72]) guarantees the existence of a Hölder continuous function $\psi\colon \Sigma^{\mathbb{Z}} \to \mathbb{R}$ cohomologous to ϕ and such that $\psi(\cdots \omega_{-2}\omega_{-1}\omega_0\omega_1\omega_2 \cdots) = \psi(\omega_1\omega_2 \cdots)$. That is, $\psi \circ \pi = \psi$ and there exists a continuous function $v\colon \Sigma^{\mathbb{Z}} \to \mathbb{R}$ satisfying $\phi - \psi = v \circ \sigma - v$. Thus, for every $\omega \in \Sigma^{\mathbb{Z}}$, we obtain

$$\begin{aligned}\left| g_n(\omega) - \sum_{k=0}^{n-1} (\phi \circ \sigma^k)(\omega) \right| &= \left| f_n(\pi(\omega)) - \sum_{k=0}^{n-1} (\psi \circ \pi \circ \sigma^k)(\omega) + v - v \circ \sigma^n \right| \\ &\geq \left| f_n(\pi(\omega)) - \sum_{k=0}^{n-1} (\psi \circ \sigma_L^k)(\pi(\omega)) \right| - 2\|v\|_\infty.\end{aligned}$$

Since $\pi(\Sigma^{\mathbb{Z}}) = \Sigma^{\mathbb{N}}$ and $\mathcal{G}$ is physically equivalent to $(S_n\phi)_{n\in\mathbb{N}}$, we have

$$\begin{aligned}&\lim_{n\to\infty} \frac{1}{n} \sup_{\omega' \in \Sigma^{\mathbb{N}}} \left| f_n(\omega') - \sum_{k=0}^{n-1} (\psi \circ \sigma_L^k)(\omega') \right| \\ &\leq \lim_{n\to\infty} \frac{1}{n} \sup_{\omega \in \Sigma^{\mathbb{Z}}} \left| g_n(\omega) - \sum_{k=0}^{n-1} (\phi \circ \sigma^k)(\omega) \right| = 0,\end{aligned}$$

contradicting Theorem 11 in [HS24]. Therefore, $\mathcal{G}$ is not physically equivalent to any additive sequence generated by a Hölder continuous function, and item 1 is proved.

Let us prove the second item. Consider the same sequence $\mathcal{G}$ and take any real number $\gamma > 0$. By the density of Hölder functions on the space of continuous functions on $\Sigma^{\mathbb{Z}}$, for each $n \in \mathbb{N}$ there exists a Hölder continuous function $h_n^\gamma\colon \Sigma^{\mathbb{Z}} \to \mathbb{R}$ such that $\|g_n - h_n^\gamma\|_\infty \leq \gamma$. Since $\mathcal{G}$ has bounded variation, the sequence $\mathcal{H}^\gamma = (h_n^\gamma)_{n\in\mathbb{N}}$ also satisfies the bounded variation condition. Furthermore, $\mathcal{H}^\gamma$ is also almost additive with respect to σ ([Hol24, Lemma 6]). Since, in particular, $\mathcal{G}$ and $\mathcal{H}^\gamma$ are physically equivalent, $\mathcal{H}^\gamma$ is not physically equivalent to any additive sequence generated by a Hölder function. The result follows now by the arbitrariness of $\gamma \in \mathbb{R}^+$. □

Now we show how we can pass some relevant information from discrete to continuous time dynamical systems. First, an auxiliary result.

Lemma 2. *Let X be a compact metric space. Every almost additive sequence of continuous functions $\mathcal{Q} = (q_n)_{n\in\mathbb{N}}$ with respect to a continuous map $T\colon X \to X$ satisfies*

$$\sup_{n\in\mathbb{N}} \big\|q_n \circ T - q_n\big\|_\infty < \infty.$$

Proof. Since $\mathcal{Q}$ is almost additive, there exists a constant $K > 0$ such that

$$-K + q_1(x) + q_{n-1}(T(x)) \leq q_n(x) \leq q_{n-1}(T(x)) + q_1(x) + K$$

for every $x \in X$ and every $n \in \mathbb{N}$. From this, we get

$$\big|q_n(x) - q_{n-1}(T(x))\big| \leq K + \|q_1\|_\infty =: K_1 < \infty. \tag{17}$$

On the other hand, we also have

$$-K + q_1(T^{n-1}x) + q_{n-1}(x) \leq q_n(x) \leq q_{n-1}(x) + q_1(T^{n-1}x) + K,$$

which gives

$$\big|q_n(x) - q_{n-1}(x)\big| \leq K_1 \tag{18}$$

for every $x \in X$ and every $n \in \mathbb{N}$. It follows from (17) and (18) that

$$\big|q_n(T(x)) - q_n(x)\big| \leq \big|q_n(f(x)) - q_{n+1}(x)\big| + \big|q_{n+1}(x) - q_n(x)\big| \leq 2K_1$$

for every $x \in X$ and every $n \in \mathbb{N}$, as desired. □

Lemma 3. *Let $\Phi = (\phi_t)_{t\in\mathbb{R}}$ be a suspension flow on Y over a continuous invertible map $T\colon X \to X$ with continuous height function $\tau\colon X \to (0,\infty)$. Let $\mathcal{C} = (c_n)_{n\in\mathbb{N}}$ be an almost additive sequence of continuous functions with respect to T on X and satisfying the bounded variation condition. Then, there exists an almost additive family of continuous functions $\mathcal{A} = (a_t)_{t\geq 0}$ with respect to Φ on Y, satisfying the bounded variation condition and such that $a_n(x) = c_n(x)$ for all $x \in X$ and $n \in \mathbb{N}$. The same result holds for the asymptotically additive case.*

Proof. Consider the function $\lfloor\cdot\rfloor\colon \mathbb{R} \to \mathbb{Z}$ given by $\lfloor x\rfloor = \max\{m \in \mathbb{Z} : m \leq x\}$. Now for each $t > 0$ define the function $a_t : Y \to \mathbb{R}$ as

$$a_t(y) = a_t(\phi_s(x)) = a_{\lfloor t\rfloor}(\phi_s(x)) = c_{\lfloor t\rfloor}(x) \quad \text{and} \quad a_0 = c_0 := 0. \tag{19}$$

For the sake of simplicity, let us consider a constant height function $\tau = 1$. Notice that by construction, $\phi_1 = T$ on X. In addition, the sequence $(a_n)_{n\in\mathbb{N}}$ is almost additive with respect to ϕ_1 on Y. In fact, by (19) and the almost additivity of $\mathcal{C}$ on X, for all $y \in Y$ and $m, n \in \mathbb{N}$ we have

$$\begin{aligned}
a_{m+n}(y) = a_{m+n}(\phi_s(x)) &= c_{m+n}(x) \leq c_m(x) + c_n(T^m(x)) + C \\
&= a_m(\phi_s(x)) + a_n(\phi_1^m(x)) + C \\
&= a_m(y) + (a_n \circ \phi_s \circ \phi_1^m)(x) + C \\
&= a_m(y) + a_n(\phi_1^m(\phi_s(x))) + C \\
&= a_m(y) + a_n(\phi_1^m(y)) + C.
\end{aligned}$$

Proceeding in the same manner, we also have $a_{m+n}(y) \geq a_m(y) + a_n(\phi_1^m(y)) - C$ for all $y \in Y$ and $m, n \in \mathbb{N}$, and $(a_n)_{n\in\mathbb{N}}$ is indeed almost additive with respect to ϕ_1 on Y.

Let us now show that the family $\mathcal{A}$ is almost additive with respect to the flow Φ on Y. By the almost additivity of $(a_n)_{n\in\mathbb{N}}$ with respect to ϕ_1 on Y, for each $y \in Y$, $m \leq t < m+1$ and $n \leq s < n+1$, we obtain

$$\begin{aligned}
a_{t+s}(y) = a_{m+n}(y) &\leq a_m(y) + a_n(\phi_m(y)) + C = a_t(y) + a_s(\phi_m(y)) + C \\
&= a_t(y) + a_s(\phi_t(y)) + \big[a_n(\phi_m(y)) - a_n(\phi_t(y))\big] + C.
\end{aligned} \tag{20}$$

On the other hand, letting $y = \phi_s(x)$ for some $x \in X$ and $r \in [0,1)$ and $m = t+u$ with $u \in [0,1)$, we also have

$$\begin{aligned}
\big|a_n(\phi_m)(y) - a_n(\phi_t(y))\big| &= \big|a_n(\phi_m(\phi_r(x))) - a_n(\phi_t(\phi_r(x)))\big| \\
&= \big|a_n(\phi_r(\phi_m(x))) - a_n(\phi_{u+r}(\phi_m(x)))\big| \\
&\leq \big|c_n(\phi_m(x)) - c_n(\phi_1(\phi_m(x)))\big|.
\end{aligned} \tag{21}$$

Since $\mathcal{C}$ is almost additive with respect to ϕ_1 on X, Lemma 2 guarantees the existence of a uniform constant $K > 0$ such that

$$\sup_{n\in\mathbb{N}} \sup_{x\in X} \big|c_n(\phi_m(x)) - c_n(\phi_1(\phi_m(x)))\big| \leq K.$$

Together with (21), this implies that $|a_n(\phi_m)(y) - a_n(\phi_t(y))| \leq K$ for all $m, n \in \mathbb{N}$, $t > 0$ and $y \in Y$. Hence, it follows now from (20) that

$$a_{t+s}(y) \leq a_t(y) + a_s(\phi_t(y)) + K + C \quad \text{for all } y \in Y \text{ and } t, s \geq 0.$$

The other inequality can be obtained in a similar way, and we conclude that $\mathcal{A}$ is almost additive with respect to Φ on Y.

Now let d_X be any metric on the base space X and consider the *Bowen-Walters distance* d_Y on Y ([BW72]). By the continuity of each $c_n\colon X \to \mathbb{R}$, it is clear that the function $a_t\colon Y \to \mathbb{R}$ is continuous for each $t \geq 0$.

Let's show that $\mathcal{A}$ also has bounded variation with respect to Φ on Y. Take two arbitrary points $y, z \in Y$ such that $d_Y(\phi_\tau(y), \phi_\tau(z)) < \varepsilon$ for $\tau \in [0, t]$ with $m \leq t < m+1$. Writing $y = \phi_u(x)$ and $z = \phi_r(x')$ for $x, x' \in X$ and $u, r \in [0, 1)$, we have

$$\big|a_t(y) - a_t(z)\big| = \big|a_t(\phi_u(x)) - a_t(\phi_r(x'))\big| = \big|c_m(x) - c_m(x')\big| \tag{22}$$

In particular, we get

$$\begin{aligned}
&d_X(x, x') \leq d_Y(y, z) < \varepsilon, \\
&d_X(\phi_1(x), \phi_1(x')) \leq d_Y(\phi_u(\phi_1(x)), \phi_r(\phi_1(x'))) = d_Y(\phi_1(y), \phi_1(z)) < \varepsilon, \\
&\vdots \\
&d_X(\phi_{m-1}(x), \phi_{m-1}(x')) \leq d_Y(\phi_u(\phi_{m-1}(x)), \phi_r(\phi_{m-1}(x'))) \\
&\ = d_Y(\phi_{m-1}(y), \phi_{m-1}(z)) < \varepsilon.
\end{aligned} \tag{23}$$

Since the sequence $\mathcal{C}$ has bounded variation with respect to $T = \phi_1$ on X, there exists a constant $L = L(\varepsilon) > 0$ such that $|c_m(x) - c_m(x')| \leq L$. Thus, it follows from (22) that the family $\mathcal{A}$ on Y also satisfies the bounded variation condition with respect to Φ and with the same parameters ε, L as the sequence $\mathcal{C}$ on X.

For the general case where τ is any positive continuous function, we have $T(x) = \phi_{\tau(x)}(x)$ and $T^m(x) = \phi_{\tau_m(x)}(x)$ for all $m \in \mathbb{N}$ and $x \in X$, with $\tau_m = \sum_{k=0}^{m-1} \tau \circ T$. In this case, for each $t \geq 0$, we define $a_t\colon Y \to \mathbb{R}$ as

$$a_t(y) = a_t(\phi_s(x)) := a_{\tau_n(x)}(\phi_s(x)) := c_n(x) \quad \text{and} \quad a_0 = c_0 := 0. \tag{24}$$

Making the necessary modifications and proceeding as in the case with $\tau = 1$, one can see that $\mathcal{A}$ is almost additive with respect to Φ on Y. The continuity of each $c_m\colon X \to \mathbb{R}$ together with definition (24), directly implies that $a_t\colon Y \to \mathbb{R}$ is continuous for each $t \geq 0$. Moreover, since $\mathcal{C}$ has bounded variation with respect to T on X, the same relation between the distance on X and the Bowen-Walters distance on Y as in (23), guarantees the bounded variation condition for $\mathcal{A}$ with respect to Φ on Y.

Now suppose that $\mathcal{D} = (d_n)_{n\in\mathbb{N}}$ is asymptotically additive with respect to T, and consider again the family $\mathcal{A} = (a_t)_{t\geq 0}$ defined in (24) now with $c_n = d_n$ for all $n \in \mathbb{N}$. By the asymptotic additivity of $\mathcal{D}$, given any $\varepsilon > 0$ there exists a continuous function $h_\varepsilon\colon X \to \mathbb{R}$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \sup_{x\in X} \left| d_n(x) - \sum_{k=0}^{n-1} (h_\varepsilon \circ T)(x) \right| < \varepsilon. \tag{25}$$

By Lemma 2 in [Hol24], there exists a continuous function $g_\varepsilon\colon Y\to\mathbb{R}$ such that $I_{g_\varepsilon}|_X = h_\varepsilon$. By the definition of $\mathcal{A}$, for each $y=\phi_u(x)$ with $u\in[0,\sup\tau)$ and $\tau_n(x)\le t<\tau_{n+1}(x)$, we have

$$\begin{aligned}\left|a_t(y)-\int_0^t (g_\varepsilon\circ\phi_s)(y)ds\right| &= \left|a_t(\phi_u(x))-\int_u^{t+u}(g_\varepsilon\circ\phi_s)(x)ds\right|\\ &\le \left|d_n(x)-\sum_{k=0}^{n-1}(h_\varepsilon\circ T^k)(x)\right|+\sup\tau\sup g_\varepsilon+\sup h_\varepsilon.\end{aligned}$$

Since $n\to\infty$ implies $t\to\infty$, we conclude from (25) that $\mathcal{A}$ is asymptotically additive with respect to Φ on Y. The continuity and the bounded variation condition of $\mathcal{A}$ follow from the same arguments presented in the almost additive case. □

Suspension flows over two-sided subshifts of finite type with Hölder continuous height functions are also called *hyperbolic symbolic flows* (see [FH20]). One can check that additive families generated by Hölder continuous functions satisfy the bounded variation condition with respect to hyperbolic symbolic flows, and Proposition 6 also holds for these types of flows. On the other hand, it is not hard to find asymptotically additive families having bounded variation with respect to an hyperbolic symbolic flow, but admitting more than one equilibrium measure.

The following result is a continuous-time counterpart of Theorem 9:

Theorem 10. *Let $\Phi=(\phi_t)_{t\in\mathbb{R}}$ be a suspension flow over the two-sided shift map $\sigma\colon\Sigma^{\mathbb{Z}}\to\Sigma^{\mathbb{Z}}$ and with a Hölder continuous height function $\tau\colon\Sigma^{\mathbb{Z}}\to(0,\infty)$. Then:*

1. *There exist almost additive families of Hölder continuous functions with respect to Φ, satisfying the bounded variation condition and not physically equivalent to any additive family generated by a Hölder continuous function.*
2. *There exist asymptotically additive families of Hölder continuous functions with respect to Φ, satisfying the bounded variation condition, admitting a unique equilibrium measure but not physically equivalent to any additive family generated by a Hölder continuous function.*

Proof. Let $\mathcal{C}=(c_n)_{n\in\mathbb{N}}$ be any sequence given by Theorem 9, that is, $\mathcal{C}$ is an almost additive with respect to σ, satisfies the bounded variation condition and is not physically equivalent to any additive sequence generated by a Hölder continuous function. By Lemma 3 there exists an almost additive family of continuous functions $\mathcal{A}=(a_t)_{t\ge0}$ with respect to Φ on Y, with bounded variation and such that $a_n(x)=c_n(x)$ for all $x\in\Sigma^{\mathbb{Z}}$ and $n\in\mathbb{N}$. Suppose that

$$\lim_{t\to\infty}\frac{1}{t}\sup_{y\in Y}\left|a_t(y)-\int_0^t(b\circ\phi_s)(y)ds\right|=0,\quad \text{where}\quad b\colon Y\to\mathbb{R}\text{ is Hölder.}$$

In particular, we have

$$\lim_{t\to\infty}\frac{1}{t}\sup_{x\in\Sigma^{\mathbb{Z}}}\left|a_t(x)-\int_0^t(b\circ\phi_s)(x)ds\right|=0. \tag{26}$$

By the proof of Lemma 15 in [BH21b], for each $t > 0$ there exists a unique $n \in \mathbb{N}$ with $t = \tau_n(x) + \kappa$ for some $\kappa \in [0, \sup \tau]$ such that

$$\Big| \int_0^t (b \circ \phi_s)(x) ds - \sum_{k=0}^{n-1} (I_b \circ \sigma^k)(x) \Big| \le \sup b \sup \tau,$$

where $I_b(x) = \int_0^{\tau(x)} (b \circ \phi_s)(x) ds$. Thus, it follows from (26) that

$$\lim_{n\to\infty} \frac{1}{n} \sup_{x\in\Sigma^{\mathbb{Z}}} \Big| c_n(x) - \sum_{k=0}^{n-1} (I_b \circ \sigma^k)(x) \Big| = \lim_{n\to\infty} \frac{1}{n} \sup_{x\in\Sigma^{\mathbb{Z}}} \Big| a_n(x) - \int_0^n (b \circ \phi_s)(x) ds \Big| = 0.$$

Since $b\colon Y \to \mathbb{R}$ is Hölder, the function $I_b : X \to \mathbb{R}$ is also Hölder ([BS00, Proposition 18]). Hence, $\mathcal{C}$ is physically equivalent to the additive sequence generated by I_b, which is a contradiction.

Now fix a number $\gamma > 0$. By the density of Hölder functions on the space of continuous functions, for each $t \ge 0$ there exists a Hölder continuous function $b_t^\gamma\colon Y \to \mathbb{R}$ such that $\sup_{y\in Y} |b_t^\gamma(y) - a_t(y)| \le \gamma$. It is clear that the family $\mathcal{B}^\gamma := (b_t^\gamma)_{t\ge 0}$ is almost additive and satisfy the bounded variation condition with respect to the flow Φ on Y. Moreover, since in particular $\mathcal{B}^\gamma$ is physically equivalent to $\mathcal{A}$, it is obvious that the family $\mathcal{B}^\gamma$ cannot be physically equivalent to any additive family generated by a Hölder continuous function, as desired.

Now let us prove the second item. It was showed in [HS24] the existence of asymptotically additive sequences of continuous functions $\mathcal{D} = (d_n)_{n\in\mathbb{N}}$ with respect to the left-sided shift map $\sigma_L\colon \Sigma^{\mathbb{N}} \to \Sigma^{\mathbb{N}}$, satisfying the bounded variation condition, with a unique equilibrium measure, but not physically equivalent to any additive sequence generated by a Hölder continuous function. Proceeding as in the proof of Theorem 9, one also can assume that $\mathcal{D}$ is asymptotically additive with respect to the two-sided shift map $\sigma\colon \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$. By Lemma 3, following as in the proof of the last item and also using the density of Hölder functions, we can guarantee the existence of an asymptotically additive family of Hölder continuous functions $\mathcal{A}$ with respect to Φ on Y, satisfying the bounded variation condition but not physically equivalent to any additive family generated by a Hölder function. Moreover, it is clear from the identity (11) that $\mathcal{A}$ also admits a unique equilibrium measure on Y, which is induced by the unique equilibrium measure for $\mathcal{D}$ on the base space $\Sigma^{\mathbb{Z}}$. $\square$

Remark 7. By Theorem 11 in [HS24], one can obtain the same class of examples in Theorem 10 for the case of suspension semi-flows over the left-sided full shift map σ_L. These counter-examples show that the physical equivalence relation [Hol24, Theorem 1] does not always allow us to reduce the study of asymptotically additive families with bounded variation to the case of single potentials with Hölder regularity. Since the thermodynamic and multifractal formalisms for Hölder continuous potentials are suitable and well understood for uniformly hyperbolic setups and related ones, Theorem 10 actually indicates a significant barrier regarding the exchange of information between the additive, almost additive and asymptotically additive worlds with respect to continuous-time dynamical systems.

4.3 Bowen regularity

In this section, taking into consideration Theorem 3, we address more general regularity aspects of almost and asymptotically additive families. Equilibrium states satisfying the Gibbs property play a relevant role in our approach.

Recall that a function $\xi\colon X \to \mathbb{R}$ is Bowen (with respect to a flow Φ on a topological space X) if there exist $\kappa > 0$ and $\varepsilon > 0$ such that for $x, y \in X$ and $t \geq 0$, we have that $d(\phi_s(x), \phi_s(y)) < \varepsilon$ for every $s \in [0, t]$ implies $|S_t\xi(x) - S_t\xi(y)| \leq \kappa$. This also means that the additive family $(S_t\xi)_{t\geq 0}$ has bounded variation with respect to Φ on X.

Theorem 11. *Let $\Phi = (\phi_t)_{t\in\mathbb{R}}$ be a suspension flow over the two-sided shift map $\sigma\colon \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ and with a Hölder continuous height function $\tau\colon \Sigma^{\mathbb{Z}} \to (0, \infty)$. Let $\mathcal{A} = (a_t)_{t\geq 0}$ be an almost additive family of continuous functions with respect to Φ on Y, and having bounded variation. Then the following properties are equivalent:*

1. *The equilibrium measure for $\mathcal{A}$ satisfies the Gibbs property for a continuous Bowen function.*

2. *There exists a continuous Bowen function $b\colon Y \to \mathbb{R}$ such that*

$$\lim_{t\to\infty} \frac{1}{t} \sup_{y\in Y} \bigl|a_t(y) - S_t b(y)\bigr| = 0.$$

3. *There exists a continuous Bowen function $b\colon Y \to \mathbb{R}$ such that*

$$\sup_{t\geq 0} \sup_{y\in Y} \bigl|a_t(y) - S_t b(y)\bigr| < \infty.$$

Proof. Let us start proving that 1 implies 3. By an appropriate version of Proposition 6 for hyperbolic symbolic flows, $\mathcal{A}$ has a unique equilibrium measure ν, which satisfies the Gibbs property (now in the sense of Definition 6). By hypothesis, ν is also Gibbs for some continuous Bowen function $b\colon Y \to \mathbb{R}$. Then, for some sufficiently small $\delta > 0$, there exist constants $K_1 = K_1(\delta) \geq 1$ and $K_2 = K_2(\delta) \geq 1$ such that

$$K_1^{-1} \leq \frac{\nu(B_t(y,\delta))}{\exp\bigl[-tP_\Phi(\mathcal{A}) + a_t(y)\bigr]} \leq K_1, \tag{27}$$

$$K_2^{-1} \leq \frac{\nu(B_t(y,\delta))}{\exp\bigl[-tP_\Phi(b) + S_t b(y)\bigr]} \leq K_2 \tag{28}$$

for all $y \in Y$ and $t \geq 0$, where $P_\Phi(b)$ is the classical topological pressure of b with respect to Φ. Taking the new function $\widetilde{b} := b + P_\Phi(b) - P_\Phi(\mathcal{A})$, by (27) and (28) we clearly have $|a_t(y) - S_t\widetilde{b}(y)| \leq \log K_1 K_2$ for all $y \in Y$ and $t \geq 0$. Since $\widetilde{b}$ is also a continuous Bowen function, item 3 is proved.

Now suppose 3 holds, that is, there exist a uniform constant $K_3 > 0$ and a continuous Bowen function b such that $|a_t(y) - S_t b(y)| \leq K_3$ for all $y \in Y$ and $t \geq 0$. Then, by the Gibbs property for $\mathcal{A}$ in (27) and the fact that $P_\Phi(\mathcal{A}) = P_\Phi(b)$ in this case, for a sufficiently small $\delta > 0$ we obtain

$$(K_1 e^{K_3})^{-1} = K_1^{-1} e^{-K_3} \leq \frac{\nu(B_t(y,\delta))}{\exp\bigl[-tP_\Phi(b) + S_t b(y)\bigr]} \leq K_1 e^{K_3}$$

for all $y \in Y$ and $t \geq 0$, which is item 1.

Finally, since every hyperbolic symbolic flow is topologically transitive ([FH20, Proposition 1.6.30]) and satisfy the hypotheses of the Closing Lemma ([KH12, Corollary 18.1.8]), Corollary 4 immediately gives that items 2 and 3 are equivalent, and the result is proved. □

For the case of hyperbolic symbolic flows or locally maximal hyperbolic sets for C^1 topologically mixing flows, it is not hard to see that an almost additive family satisfies the bounded variation condition if and only if it admits a Gibbs measure. In Theorem 11, the equivalence between items 1 and 2 indicates a possible way of classifying almost additive families with bounded variation with respect to hyperbolic symbolic flows or locally maximal hyperbolic sets for C^1 topologically mixing flows. Also motivated by [HS24], we propose the following classification of families with respect to hyperbolic symbolic flows:

- **Type 1:** Almost additive families with bounded variation and admitting a Gibbs state for a Bowen continuous function.
- **Type 2:** Almost additive families with bounded variation but not admitting Gibbs states for Bowen continuous functions.
- **Type 3:** Almost additive families without the bounded variation condition but having a unique equilibrium state.
- **Type 4:** Almost additive families having more than one equilibrium state.

Remark 8. Following the same lines as in [HS24], one can construct families of types 1, 3 and 4. On the other hand, examples of type 2 seem to be much more complicated to produce or they actually don't exist. In the discrete-time setup, the existence of sequences of type 2 is connected with the problem of relating quasi-Bernoulli and Gibbs measures with respect to the full shift map (see also [Cun20]).

Asymptotically additive families. Here we show how to treat the Bowen regularity problem for asymptotically additive families. Let $\mathcal{G} = (g_n)_{n\in\mathbb{N}}$ be an asymptotically additive sequence of continuous functions with respect to $\sigma\colon \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$, having bounded variation, with a unique equilibrium measure, but not physically equivalent to any additive sequence generated by a Bowen function (such an example was given in [HS24]). By Lemma 3, there exists an asymptotically additive family $\mathcal{A} = (a_t)_{t\geq 0}$ with respect to the hyperbolic symbolic flow Φ on Y (with height function τ) and such that $a_n(x) = g_n(x)$ for all $x \in \Sigma^{\mathbb{Z}}$ and $n \in \mathbb{N}$. Now suppose the existence of a continuous Bowen function $b\colon Y \to \mathbb{R}$ such that $\mathcal{A}$ is physically equivalent to $(S_t b)_{t\geq 0}$. By the appropriate versions of Lemmas 3.1 and 3.3 in [BH21a] for hyperbolic symbolic flows, the sequence $\mathcal{C} = (c_n)_{n\in\mathbb{N}}$ given by $c_n(x) = \int_0^{\tau_n(x)} (b \circ \phi_s)(x)ds$ is additive and satisfy the bounded variation condition with respect to σ. By the physical equivalence relation between $\mathcal{A}$ and $(S_t b)_{t\geq 0}$, we have in particular that

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \sup_{x\in\Sigma^{\mathbb{Z}}} \left| g_n(x) - c_n(x) \right| &= \lim_{n\to\infty} \frac{1}{n} \sup_{x\in\Sigma^{\mathbb{Z}}} \left| a_n(x) - c_n(x) \right| \\ &\leq \lim_{t\to\infty} \frac{1}{t} \sup_{y\in Y} \left| a_t(y) - S_t b(y) \right| = 0.\end{aligned} \tag{29}$$

Since, by the proof of Lemma 15 in [BH21b], $c_n(x) = \sum_{k=0}^{n-1} I_b \circ \sigma^k(x) =: S_n I_b(x)$ for all $x \in \Sigma^{\mathbb{Z}}$ and all $n \in \mathbb{N}$, the sequence $(S_n I_b)_{n\in\mathbb{N}}$ has the bounded variation condition. Hence, it follows from (29) that $\mathcal{G}$ is physically equivalent to the additive sequence $(S_n I_b)_{n\in\mathbb{N}}$ generated by the Bowen function $I_b \colon \Sigma^{\mathbb{Z}} \to \mathbb{R}$. This is a contradiction. Therefore, by construction, $\mathcal{A}$ satisfies the bounded variation condition and has a unique equilibrium measure (with respect to Φ on Y), but cannot be physically equivalent to any additive family generated by a Bowen continuous function. □

4.4 Concluding remarks

Observe that all the results in the regularity sections are developed for hyperbolic symbolic flows. There is a deeper reason for that, which comes all the way from [BKM20]. In this last work, studying almost additivity in the context of planar matrix cocycles, the authors showed an example of a quasi-Bernoulli measure that is not Gibbs for any Hölder continuous function with respect to the left-sided full shift map ([BKM20, Example 2.10 (2)]). In view of the non-additive versions of the Livšic theorem for maps and flows (Theorem 5 in [HS24] and Theorem 3, respectively), this particular example plays a fundamental role in the production of the counter-examples in [HS24] for the full shift map and, consequently, the ones in Theorem 10 for symbolic flows.

Based on this, morally speaking, all the counter-examples and results discussed here in the regularity section can be adapted to the case of hyperbolic flows and, more generally, to suspension flows over topologically mixing subshifts of finite type. To achieve this, one needs to obtain appropriate versions of Theorems 2.8 and 2.9 in [BKM20] for topologically mixing Markov chains using the classical thermodynamic machinery developed in [Bow75a].

Finally, let us mention the still open problem of the existence of sequences and families of type 2. A reasonable starting point to attack this question is to understand how the aforementioned theorems in [BKM20] could accommodate Bowen continuous functions, going beyond the Hölder regularity previously considered by them. A positive answer in this direction would finally reveal the existence of quasi-Bernoulli measures that do not satisfy the Gibbs property, giving as well a type 2 sequence.

Acknowledgments: The author was partially supported by NSF of China, grant no. 12222110.

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