On cohomology, Gibbs properties and regularity of some nonadditive families of functions

Carllos Eduardo Holanda

Department of Mathematics, Shantou University, Shantou, 515063, Guangdong, China

E-mail address: c.eduarddo@gmail.com

Abstract

We explore notions of cohomology and obtain a kind of Livšic theorem for nonadditive families of functions. Together with the existence of Gibbs states, we use this result to classify equilibrium measures for almost additive families with respect to hyperbolic systems, improving the nonadditive thermodynamic theory for flows. Moreover, building on recent examples for discrete-time dynamics, we address some Hölder and Bowen regularity problems for the physical equivalence relations between additive and asymptotically additive families with respect to hyperbolic symbolic flows and related dynamical systems.

1 Introduction

This note is a natural continuation to [Hol24], and is mainly a contribution to the study of relations between the additive classical world and the nonadditive world of families of potentials, which started in [Cun20]. In particular, we are interested in the asymptotically and almost additive cases. In this work, when considering families or sequences, the term *potential* is used interchangeably with *function*.

A family $\mathcal{A} = (a_t)_{t\geq 0}$ of functions $a_t \colon X \to \mathbb{R}$ is said to be *asymptotically additive* with respect to a flow $\Phi = (\phi_t)_{t\in\mathbb{R}}$ on a topological space X if for each $\varepsilon > 0$ there exists a function $b_\varepsilon \colon X \to \mathbb{R}$ such that

$$\limsup_{t\to\infty} \frac{1}{t}\left\| a_t - \int_0^t (b_\varepsilon \circ \phi_s)ds \right\|_\infty \leq \varepsilon,$$

where $\|\cdot\|_\infty$ is the supremum norm on X. Moreover, $\mathcal{A}$ is said to be *almost additive* with respect to Φ on X if there exists a constant $C > 0$ such that

$$-C + a_t + a_s \circ \phi_t \leq a_{t+s} \leq a_t + a_s \circ \phi_t + C$$

for every $t, s \geq 0$. It is well known that every almost additive family is asymptotically additive [FH10]. For each function $b \colon X \to \mathbb{R}$, the *additive* family $(S_t b)_{t\geq 0}$ generated by b (with respect to Φ) is denoted by $S_t b := \int_0^t (b \circ \phi_s)ds$.

It was showed in [Hol24] that, with respect to suspension flows Φ, asymptotically additive families are physically equivalent to additive families of continuous functions. That is, given an asymptotically additive family of continuous functions $\mathcal{A} = (a_t)_{t\geq 0}$ (with respect to Φ) there exists a real-valued continuous function b on the suspension manifold such that

$$\lim_{t\to\infty} \frac{1}{t} \left\| a_t - S_t b \right\|_\infty = 0. \tag{1}$$

In this case, we say that $\mathcal{A}$ is *physically equivalent* to $(S_t b)_{t\geq 0}$ (and vice-versa). Motivated by this equivalence relation, one can naturally consider the problem of studying the different levels of regularity that the physical equivalence (1) can sustain. In our framework, the most relevant types of regularity are the ones involving Bowen and Hölder functions together with families having the bounded variation property (see Sections 2.1 and 2.2 for the definitions). In the context of hyperbolic suspension flows and related hyperbolic setups, the space of Hölder continuous functions is contained in the space of Bowen continuous functions. Furthermore, by definition, an additive family generated by a Bowen function has bounded variation with respect to any flow in general. Based on this, we are interested in three types of regularity problems:

- **Bowen regularity**. *Given any almost additive family of continuous functions $\mathcal{A} = (a_t)_{t\geq 0}$ with respect to an hyperbolic suspension flow and having bounded variation, is there a Bowen continuous function b such that $(S_t b)_{t\geq 0}$ is physically equivalent to $\mathcal{A}$?*

- **Uniform bound**. *Given any almost additive family of continuous functions $\mathcal{A} = (a_t)_{t\geq 0}$ with respect to an hyperbolic suspension flow and having bounded variation, is there a continuous function b such that*

$$\sup_{t\geq 0} \left\| a_t - S_t b \right\|_\infty < \infty \text{ ?}$$

- **Hölder regularity**. *Given any almost additive family of Hölder continuous functions $\mathcal{A} = (a_t)_{t\geq 0}$ with respect to an hyperbolic suspension flow and having bounded variation, is there a Hölder continuous function b such that $(S_t b)_{t\geq 0}$ is physically equivalent to $\mathcal{A}$?*

Here hyperbolic symbolic flows are suspensions over the two-sided full shift. These questions also can be posted with respect to hyperbolic flows or, more generally, suspension flows over subshifts of finite type (see Section 4.4). The uniform bound immediately implies Bowen regularity, and a positive answer to the Hölder regularity question in this context also gives an affirmative answer to the Bowen regularity one. All these regularity issues are also pertinent in the more general case of asymptotically additive families. Moreover, definitive answers to these questions can close the final gap in the comparison between additive and nonadditive families taking into consideration uniqueness of equilibrium states (see [Fra77, BH21a]), ergodic optimization (see for example [BHVZ21, HLMXZ19, MSV20]), and regularity of topological pressure together with multifractal analysis (see [Rue78, BD04, BS00, PS01, BH21b, BH22c]).

It is important emphasizing that, considering the physical equivalence relations and associated problems, the passage of information from discrete-time to continuous-time is not direct, and can actually be quite nuanced and tricky. For instance, we recall here that the full physical equivalence problem for continuous flows in general is still open (see [Hol24]), despite already existing a complete answer for continuous maps given by Cuneo in [Cun20].

In the first part of this work, we obtain a characterization result for almost additive families with bounded variation (Theorem 3), which is intimately connected to the three aforementioned regularity questions. This result is inspired by the discrete-time counterpart obtained in [HS24] but, notwithstanding, it is proved here directly in the realm of flows, and without using any of the physical equivalence results for discrete and continuous-time dynamical systems in [Cun20] and [Hol24], respectively. Moreover, the characterization gives a setup for which the uniform bound and the Bowen regularity problems are actually equivalent (holding in particular for some types of suspensions and hyperbolic flows), is linked to some deep results for linear cocycles, and also can be applied to classify equilibrium states for almost additive families based on their cohomology classes, working as a nonadditive version of the classical Livšic theorem for flows ([Liv71, Liv72]), also complementing the nonadditive formalism developed in [BH20, BH21a].

Building on some examples in [HS24], we show how to construct almost (and asymptotically) additive families of Hölder continuous potentials satisfying the bounded variation property with respect to some symbolic flows and which are not physically equivalent to any additive family generated by a Hölder (Bowen) continuous potential. These examples show that almost and asymptotically additive families with bounded variation do not always have the same good properties of Hölder continuous functions in hyperbolic and related frameworks. Even though we are relying on examples for maps, the constructions of the counter-examples for the case of flows are rather involved and somewhat delicate, requiring new tools and non-trivial modifications.

The paper is organized in three parts. In the first one, we study some different notions of cohomology for almost and asymptotically additive families, we establish our nonadditive Livšic theorem for flows (Theorem 3), and show how it can be applied to the context of linear cocycles over flows. In the second part, we study and compare different notions of nonadditive Gibbs and weak Gibbs states with respect to flows, give some examples of nonadditive families derived by volume measures and measures satisfying the Gibbs property and, as another application of Theorem 3, we demonstrate how to classify almost additive families based on cohomology relations and equilibrium states. In the last part, dealing with the regularity issues, we start giving a simple example for which the uniform bound question can always be positively answered, and another one for which the equivalences in Theorem 3 do not hold. In the following, using the structure of suspension flows, we show how to build the aforementioned counter-examples of almost additive families of potentials, giving a negative answer to the Hölder regularity problem. After proposing a way of categorizing almost additive families with respect to hyperbolic and symbolic flows, we show a construction demonstrating that the Bowen regularity problem cannot be positively answered in the asymptotically additive case. We finish the paper by quickly discussing some relevant technical matters, open problems and further explorations.

2 On Cohomology

In this section we introduce some notions of cohomology, and obtain a characterization of almost additive families of potentials. This allow us to classify equilibrium states and study regularity equivalence issues for asymptotically and almost additive families with respect to suspension flows and, in particular, hyperbolic flows (see Sections 3 and 4).

2.1 Exploring notions for asymptotically additive families

Here, based on the physical equivalence in [Hol24] and inspired by concepts recently studied in [HS24] for the discrete-time case, we introduce some, a priori, distinct cohomology notions for asymptotically additive families with respect to flows. We start recalling some concepts and tools in the additive framework.

We say that a function $\psi\colon X \to \mathbb{R}$ is *Walters* (with respect to a flow Φ) if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that for $x, y \in X$ and $t \geq 0$, we have that

$$d(\phi_s(x), \phi_s(y)) < \delta \text{ for every } s \in [0, t] \implies |S_t\psi(x) - S_t\psi(y)| < \varepsilon.$$

In this case, we also say that the additive family $(S_t\psi)_{t\geq 0}$ satisfies the *Walters property*. Moreover, we say that a function $\xi\colon X \to \mathbb{R}$ is *Bowen* (with respect to Φ) if there exist $L > 0$ and $\delta > 0$ such that for $x, y \in X$ and $t \geq 0$, we have that

$$d(\phi_s(x), \phi_s(y)) < \delta \text{ for every } s \in [0, t] \implies |S_t\xi(x) - S_t\xi(y)| \leq L.$$

Clearly every Walters function is also Bowen. In the hyperbolic framework, the Hölder continuous functions are always Walters and, consequently, Bowen (see Proposition 7.3.1 in [FH20]).

A continuous flow Φ on a compact metric space X is said to satisfy the *Closing Lemma* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in X$ and $t \geq 0$ satisfying $d(\phi_t(x), x) < \delta$, then there exists a periodic orbit $\{\phi_s(y) : 0 \leq s \leq T\}$ with $|T - t| < \varepsilon$ such that $d(\phi_s(x), \phi_s(y)) < \varepsilon$ for all $0 \leq s \leq t$ (see for example Theorems 5.3.11 and 6.2.4 in [FH20]).

Let us recall the notion of cohomology for functions with respect to flows. A continuous function $a\colon X \to \mathbb{R}$ is said to be Φ-*cohomologous to zero* if there exists a continuous function $q\colon X \to \mathbb{R}$ such that

$$a(x) = \lim_{t\to 0} \frac{q(\phi_t(x)) - q(x)}{t} \quad \text{for every } x \in X.$$

We say that a point $x \in X$ has a *forward dense orbit* if $\overline{\{\phi_s(x) : s \geq 0\}} = X$. When $\overline{\{\phi_s(x) : s \in \mathbb{R}\}} = X$, we say that $x \in X$ has a *dense orbit*. We say that a flow is *topologically transitive* if there exists at least one point with a forward dense orbit.

The next result is a slightly more general version of the celebrated *Livšic theorem* for flows ([Liv72]).

Theorem 1. *Let $\Phi = (\phi_t)_{t\in\mathbb{R}}$ be a topologically transitive continuous flow satisfying the Closing Lemma, and $a\colon X \to \mathbb{R}$ a continuous function satisfying the Walters property. Then a is cohomologous to zero if and only if for every periodic point $x = \phi_T(x)$ we have $S_T a(x) = 0$.*

Proof. See, for example, the proof of Theorem 5.3.23 in [FH20]. $\square$

Based on Theorem 1, we also obtain a simple characterization of additive families generated by *coboundaries*. Let $\mathcal{M}(\Phi)$ be the set of Φ-invariant probability measures on X.

Proposition 2. *Under the conditions of Theorem 1, a function $a\colon X \to \mathbb{R}$ is Φ-cohomologous to zero if and only if*

$$\lim_{t\to\infty} \frac{1}{t}\left\|S_t a\right\|_\infty = 0.$$

In particular,

$$\lim_{t\to\infty} \frac{1}{t}\left\|S_t a\right\|_\infty = 0 \textit{ if and only if } \sup_{t\geq 0}\left\|S_t a\right\|_\infty < \infty.$$

Proof. Suppose a is Φ-cohomologous to zero. This implies the existence of a continuous function $q\colon X \to \mathbb{R}$ such that $S_t a = q\circ\phi_t - q$ for all $t \geq 0$. Consequently, one has $\|S_t a\|_\infty \leq 2\|q\|_\infty < \infty$ for every $t \geq 0$. Then, $\lim_{t\to\infty} \frac{1}{t}\|S_t a\|_\infty = 0$.

Conversely, let $\lim_{n\to\infty} \frac{1}{t}\|S_t a\|_\infty = 0$. The Lebesgue's dominated convergence theorem gives that

$$0 = \int_X \lim_{n\to\infty} \frac{1}{t} S_t a \; d\mu = \int_X a \; d\mu \quad \text{for all } \mu \in \mathcal{M}(\Phi). \tag{2}$$

For all $x \in X$ with $x = \phi_T(x)$, the measure $(\int_0^T \delta_{\phi_s(x)} ds)/T$ is Φ-invariant. In particular, identity (2) gives that $S_T a(x) = 0$ for all $x \in X$ with $x = \phi_T(x)$. Hence, by Theorem 1 we conclude that a is Φ-cohomologous to zero. $\square$

Now based on Theorem 1 in [Hol24] and Proposition 2, we propose our first definition of cohomology for asymptotically additive families.

Definition 1. We say that an asymptotically (or almost) additive family of continuous functions $\mathcal{A} = (a_t)_{t\geq 0}$ is Φ-cohomologous to a constant if there exists a continuous function $a\colon X \to \mathbb{R}$ which is Φ-cohomologous to some constant and satisfies

$$\lim_{t\to\infty} \frac{1}{t}\left\|a_t - S_t a\right\|_\infty = 0.$$

One can easily check that a family $\mathcal{A} = (a_t)_{t\geq 0}$ is Φ-cohomologous to a constant if and only if the sequence $(a_n/n)_{n\in\mathbb{N}}$ is uniformly convergent to a constant. In particular, $\mathcal{A}$ is Φ-cohomologous to zero if and only if $\lim_{t\to\infty} \frac{1}{t}\|a_t\|_\infty = 0$.

Remark 1. Observe that the classical concept of cohomology for a function is much stronger than the one introduced for nonadditive families in Definition 1.

Proposition 2 also motivates a new definition for the nonadditive case, which is still weaker than the classical one but stronger than Definition 1.

Definition 2. An asymptotically (or almost) additive family of continuous functions $\mathcal{A} = (a_t)_{t\geq 0}$ is Φ-cohomologous to a constant if there exists a continuous function $a\colon X \to \mathbb{R}$ which is Φ-cohomologous to a constant and satisfies

$$\sup_{n\in\mathbb{N}}\left\|a_t - S_t a\right\|_\infty < \infty.$$

In this case, observe that $\mathcal{A}$ is uniformly bounded if and only if $\mathcal{A}$ is Φ-cohomologous to zero.

2.2 An almost additive Livšic theorem for flows

We say that a family of functions $\mathcal{A} = (a_t)_{t\geq 0}$ has *bounded variation* or satisfies the *bounded variation property* (with respect to a flow $\Phi = (\phi_t)_{t\in\mathbb{R}}$) if there exists $\varepsilon > 0$ such that

$$\sup_{t\geq 0} \sup \left\{ |a_t(x) - a_t(y)| : d_t(x,y) < \varepsilon \right\} < \infty,$$

where $d_t(x,y) = \max\{d(\phi_s(x), \phi_s(y)) : s \in [0,t]\}$. It is clear that if a function ϕ is Bowen, then the additive family $(S_t\phi)_{t\geq 0}$ has bounded variation.

Following closely the property in the additive setup, we say that a family of functions $\mathcal{A} = (a_t)_{t\geq 0}$ satisfies the *Walters property* if for each $\kappa > 0$ there exists $\varepsilon > 0$ such that for $x, y \in X$ and $t \geq 0$, we have that

$$d(\phi_s(x), \phi_s(y)) < \varepsilon \text{ for every } s \in [0,t] \implies |a_t(x) - a_t(y)| < \kappa.$$

Clearly every family satisfying the Walters property also has bounded variation.

The next result is our main theorem in this section.

Theorem 3. *Let $\Phi = (\phi_t)_{t\geq 0}$ be a topologically transitive continuous flow on a compact metric space X and satisfying the Closing Lemma. Let $\mathcal{B} = (b_t)_{t\geq 0}$ be an almost additive family of continuous functions with respect to Φ and satisfying the bounded variation property. Then, the following properties are equivalent:*

1. $\lim_{t\to\infty} \|b_t\|_\infty / t = 0$.
2. $\sup_{t\geq 0} \|b_t\|_\infty < \infty$.
3. *There exists $K > 0$ such that $|b_t(p)| \leq K$ for all $p \in X$ and $t \geq 0$ with $\phi_t(p) = p$.*

Remark 2. Comparing Theorem 3 with the classical Livšic result (Theorem 1), we notice that the Walters property is required in the last but not in the former. This happens because, as it turns out, the nonadditive notions of cohomology (Definitions 1 and 2) are weaker than the notion of cohomolgy for functions (see also Theorem 8 for the connections with equilibrium states and periodic points).

As a direct consequence, we have:

Corollary 4. *Under the hypotheses of Theorem 3, let $\mathcal{A} = (a_t)_{t\geq 0}$ be an almost additive family of continuous functions with bounded variation. Then, for a continuous function $a\colon X \to \mathbb{R}$ such that $(S_t a)_{t\geq 0}$ has bounded variation, we have*

$$\lim_{t\to\infty} \frac{1}{t} \big\|a_t - S_t a\big\|_\infty = 0 \quad \textit{if and only if} \quad \sup_{t\geq 0} \big\|a_t - S_t a\big\|_\infty < \infty.$$

In particular, if $(S_t a)_{t\geq 0}$ does not have bounded variation we have

$$\sup_{t\geq 0} \big\|a_t - S_t a\big\|_\infty = \infty.$$

Corollary 4 readily implies that the Bowen regularity problem is equivalent to the uniform bound problem for topologically transitive flows satisfying the Closing Lemma. We also note that Theorem 3 is an extension of Proposition 2 to the case of almost additive families of functions.

Proof of Theorem 3. We start with a key auxiliary lemma.

Lemma 1. *Let $\Phi = (\phi_t)_{t\in\mathbb{R}}$ be a continuous flow on a compact metric space X and let $\mathcal{C} = (c_t)_{t\geq 0}$ be an almost additive family of continuous functions with uniform constant $C > 0$ and such that $\lim_{t\to\infty} \|c_t\|_\infty/t = 0$. Then:*

1. *For every τ-periodic point $x_0 \in X$, we have $\sup_{q\in\mathbb{N}} |c_{q\tau}(x_0)| \leq C$.*
2. *For each periodic point $x_0 \in X$, there exists a constant $L := L(\tau) \geq 0$ (only depending on the period of x_0) such that $\sup_{t\geq 0} |c_t(x_0)| \leq L$.*
3. *We have*
$$\sup_{\mu\in\mathcal{M}(\Phi)} \left| \int_X c_t d\mu \right| \leq C \quad \textit{for all } t \geq 0.$$

Proof of the lemma. Since the family $\mathcal{C}$ is almost additive with uniform constant $C > 0$, one can see that

$$\sum_{k=0}^{p-1} c_t \circ \phi_{kt} - (p-1)C \leq c_{pt} \leq \sum_{k=0}^{p-1} c_t \circ \phi_{kt} + (p-1)C \tag{3}$$

for all $t \geq 0$ and $p \in \mathbb{N}$. Now suppose x_0 is a τ-periodic point, that is, $\phi_\tau(x_0) = x_0$. If $t = q\tau$ for some $q \in \mathbb{N}$, then

$$\phi_{kt}(x_0) = \phi_{kq\tau}(x_0) = \underbrace{(\phi_\tau \circ \phi_\tau \circ \cdots \circ \phi_\tau)}_{kq \text{ times}}(x_0) = x_0$$

for all $k \in \mathbb{N}$. In particular, this implies that

$$\lim_{p\to\infty} \frac{1}{p} \sum_{k=0}^{p-1} c_t(\phi_{kt}(x_0)) = c_t(x_0). \tag{4}$$

Since $\lim_{t\to\infty} \|c_t\|_\infty/t = 0$, it follows from (3) and (4) that

$$-C \leq c_t(x_0) = c_{q\tau}(x_0) \leq C. \tag{5}$$

By the arbitrariness of $q \in \mathbb{N}$, item 1 follows.

Let's prove item 2. Let x_0 be a τ-periodic point, consider $t = q\tau + r$ with $r \in (0, \tau)$ and fix the numbers

$$A(\tau) := \inf \left\{ \inf_{x\in X} c_s(x) : s \in [0,\tau] \right\} \quad \text{and} \quad B(\tau) := \sup \left\{ \sup_{x\in X} c_s(x) : s \in [0,\tau] \right\}. \tag{6}$$

Almost additivity together with (5) and (6) gives that

$$-2C + A(\tau) \leq -C + c_{qt}(x_0) + c_r(\phi_{q\tau}(x_0)) \leq c_t(x_0)$$

and

$$c_t(x_0) \leq c_{qt}(x_0) + c_r(\phi_{q\tau}(x_0)) + C \leq 2C + B(\tau).$$

Hence,

$$L_1(\tau) := \min \big\{ A(\tau) - 2C, -C \big\} \leq c_t(x_0) \leq \max \big\{ B(\tau) + 2C, C \big\} := L_2(\tau)$$

for all $t \geq 0$. Taking $L = L(\tau) := \max\{|L_1(\tau)|, |L_2(\tau)|\}$, the item 2 is proved.

Now we prove item 3. Suppose μ is a Φ-invariant measure. Then, in particular, μ is also ϕ_t-invariant for every $t \geq 0$. By applying that $\lim_{t\to\infty} \|c_t\|_\infty/t = 0$ in the inequalities (3), we get

$$-C \leq \int_X \lim_{p\to\infty} \frac{1}{p} \sum_{k=0}^{p-1} c_t(\phi_{kt}(x))d\mu(x) = \int_X c_t d\mu \leq C$$

for all $t \geq 0$. Since the measure $\mu \in \mathcal{M}(\Phi)$ is arbitrary, item 3 is proved. $\square$

Let's proceed with the proof of the theorem. Since $\mathcal{B}$ satisfies the bounded variation property, there exists $\varepsilon > 0$ such that

$$Q := \sup_{t\geq 0} \sup \left\{ \left|b_t(x) - b_t(y)\right| : d_t(x,y) < \varepsilon \right\} < \infty. \tag{7}$$

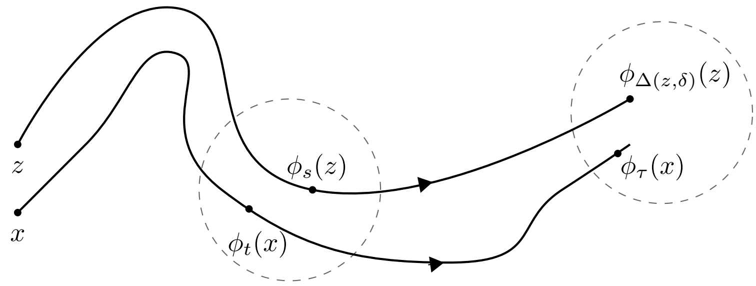

Figure 1: Approximating the orbit of any point by a finite piece of a dense orbit.

We first show that 3 implies 2. Suppose that there exists a uniform constant $K > 0$ such that $|b_t(p)| \leq K$ for all $p \in X$ and $t \geq 0$ with $\phi_t(p) = p$. Since Φ is topologically transitive, there exists a point $z \in X$ with a dense forward orbit. Now let $\delta > 0$ be the number given by the Closing Lemma. By the density of the forward orbit, there exists a number $\Delta(z,\delta) \in \mathbb{R}$ such that for each $x \in X$ and $t \in \mathbb{R}$ there exists some $s \in [0, \Delta(z,\delta)]$ with $d(\phi_t(x), \phi_s(z)) < \delta$ (see Figure 1). For $t > \Delta(z,\delta)$, in particular, there exists $s' \in [0, \Delta(z,\delta)]$ such that $d(\phi_t(z), \phi_{s'}(z)) < \delta$, which is the same as $d(\phi_{t-s'}(\phi_{s'}(z)), \phi_{s'}(z)) < \delta$. By the Closing Lemma, there exists $p \in X$ satisfying $\phi_T(p) = p$, with $|T - t + s'| < \varepsilon$ and such that $d_{t-s'}(\phi_{s'}(z), p) < \varepsilon$. From almost additivity, there exists a uniform constant $L = L(\varepsilon) > 0$ such that $\|b_T - b_{t-s'}\|_\infty \leq L$. By the bounded variation property (7), we have

$$\left|b_{t-s'}(\phi_{s'}(z)) - b_{t-s'}(p)\right| \leq Q,$$

which implies $|b_{t-s'}(\phi_{s'}(z)| \leq Q + |b_{t-s'}(p)| \leq Q + |b_T(p)| + L \leq Q + K + L$. Applying almost additivity again, we get

$$\begin{aligned}\left|b_t(z)\right| = \left|b_{(t-s')+s'}(z)\right| &\leq \left|b_{s'}(z)\right| + \left|b_{t-s'}(\phi_{s'}(z))\right| + C \\ &\leq \sup_{s\in[0,\Delta(z,\delta)]} \left|b_s(z)\right| + Q + K + L + C =: \widetilde{K}.\end{aligned}$$

Since the time $t > \Delta(z,\delta)$ was arbitrarily chosen, we conclude that $|b_t(z)| \leq \widetilde{K}$ for all $t \geq 0$. Notice that the constant $\widetilde{K} > 0$ only depends on z, $\delta > 0$ and

$\varepsilon > 0$. By using the almost additivity property one more time, we have

$$\left|b_t(\phi_s(z)\right| \le \left|b_s(z)\right| + \left|b_{t+s}(z)\right| + C \le 2\widetilde{K} + C \quad \text{for all } t, s \ge 0.$$

Now consider any point $x \in X$. Since $\overline{\{\phi_t(z) : t \ge 0\}} = X$, there exists a sequence of points $(z_q)_{q\ge1} \subset \{\phi_t(z) : t \ge 0\}$ such that $\lim_{q\to\infty} z_q = x$. Since each function $b_t \colon X \to \mathbb{R}$ is continuous, we finally obtain

$$\left|b_t(x)\right| = \lim_{q\to\infty} \left|b_t(z_q)\right| \le 2\widetilde{K} + C.$$

Hence, by the arbitrariness of x, we have $\sup_{t\ge0} \|b_t\|_\infty \le 2\widetilde{K} + C < \infty$, which is item 2. Obviously item 2 implies item 1. Moreover, it follows from Lemma 1 that item 1 implies item 3, and the theorem follows. □

Remark 3. Theorem 3 is no longer valid in the asymptotically additive nor the subadditive framework in general. In fact, let Φ be any continuous flow on a compact metric space X and consider the family $\mathcal{A} = (a_t)_{t\ge0}$ given by $a_t(x) = \sqrt{t}$ for all $t \ge 0$ and $x \in X$. Clearly $\mathcal{A}$ is asymptotically additive, subadditive and satisfies the bounded variation property with respect to Φ. Moreover, one can check that $\lim_{t\to\infty} \|a_t\|_\infty/t = 0$ but $\sup_{t\ge0} \|a_t\|_\infty = \infty$, confirming that Theorem 3 has the optimal nonadditive framework, in the sense that it does not work for other bigger classes of families. Furthermore, the example above also indicates that Definitions 1 and 2 are not equivalent for asymptotically additive families in general.

2.3 A connection to linear cocycles

In this section, for the sake of clarity, we give some definitions and notions following closely [BH21b]. Let $\Phi = (\phi_t)_{t\in\mathbb{R}}$ be a continuous flow on a compact metric space M. Moreover, let $GL(d,\mathbb{R})$ be the set of all invertible $d\times d$ matrices. A continuous map $\mathsf{A} \colon \mathbb{R} \times M \to GL(d,\mathbb{R})$ is called a *linear cocycle* over Φ if for all $t, s \in \mathbb{R}$ and $x \in M$ we have:

1. $\mathsf{A}(0, x) = \mathrm{Id}$.
2. $\mathsf{A}(t + s, x) = \mathsf{A}(s, \phi_t(x))\mathsf{A}(t, x)$.

We shall always assume that all entries $a_{ij}(t, x)$ of $\mathsf{A}(t, x)$ are positive for every $(t, x) \in \mathbb{R} \times M$. Moreover, we consider the norm on $GL(d,\mathbb{R})$ defined by $\|B\| = \sum_{i,j=1}^{d} |b_{ij}|$, where b_{ij} are the entries of the matrix B.

Now we consider the family of continuous functions $\mathcal{A}_\mathrm{c} = (a_t)_{t\ge0}$ given by

$$a_t(x) = \log \left\|\mathsf{A}(t, x)\right\| \quad \text{for all } t \ge 0 \text{ and } x \in M.$$

By Proposition 12 in [BH21b], $\mathcal{A}_\mathrm{c}$ is almost additive with respect to Φ. We note that for a general linear cocycle, the family $\mathcal{A}_\mathrm{c}$ is only subadditive.

We say that a cocycle A has *bounded distortion* if

$$\sup\left\{\left\|\mathsf{A}(t, x)\mathsf{A}(t, y)^{-1}\right\| : z \in M \text{ and } x, y \in B_t(z, \varepsilon)\right\} < \infty$$

for some $\varepsilon > 0$.

Notice that

$$\left\|\mathsf{A}(t, x)\mathsf{A}(t, x)^{-1}\right\| = \left\|\mathrm{Id}\right\| = d$$

for every $(t,x) \in \mathbb{R} \times M$, which implies that

$$\|\mathsf{A}(t,x)^{-1}\| \geq d\|\mathsf{A}(t,x)\|^{-1}.$$

Thus,

$$\|\mathsf{A}(t,x)\mathsf{A}(t,y)^{-1}\| \geq \frac{K}{d}\|\mathsf{A}(t,x)\| \cdot \|\mathsf{A}(t,y)^{-1}\| \geq K\|\mathsf{A}(t,x)\| \cdot \|\mathsf{A}(t,y)\|^{-1}$$

for some uniform constant $K > 0$, and so

$$\big|\log\|\mathsf{A}(t,x)\| - \log\|\mathsf{A}(t,y)\|\big| \leq -\log K + \log\|\mathsf{A}(t,x)\mathsf{A}(t,y)^{-1}\|.$$

In particular, we have

$$\sup_{x,y\in B_t(z,\varepsilon)} \big|a_t(x) - a_t(y)\big| \leq -\log K + \log \sup_{x,y\in B_t(z,\varepsilon)} \|\mathsf{A}(t,x)\mathsf{A}(t,y)^{-1}\|$$

for $z \in M$ and $\varepsilon > 0$. Hence, if A has bounded distortion, then $\mathcal{A}_{\mathrm{c}}$ has bounded variation.

For a concrete example, one can consider a C^1 flow Φ on a compact set $M \subset \mathbb{R}^d$ such that for every $t \in \mathbb{R}$ and $x \in M$ the matrix $d_x\phi_t$ has only positive entries. Thus $\mathsf{A}(t,x) = d_x\phi_t$ is a linear cocycle over Φ and the family $\mathcal{A}_{\mathrm{d}} = (a_t)_{t\geq 0}$ given by $a_t(x) = \log\|d_x\phi_t\|$ is an almost additive family of continuous functions with respect to Φ.

Let $GL^+(d,\mathbb{R}) \subset GL(d,\mathbb{R})$ be the set of all matrices with strictly positive entries. We have the following application of Theorem 3 to the case of continuous-time cocycles.

Theorem 5. *Let $\Phi = (\phi_t)_{t\in\mathbb{R}}$ be a topologically transitive continuous flow on a compact metric space M satisfying the Closing Lemma, and let $\mathsf{A}\colon \mathbb{R} \times M \to GL^+(d,\mathbb{R})$ be a linear cocycle over Φ with bounded distortion. Suppose there exists a compact set $\Omega \subset GL^+(d,\mathbb{R})$ such that $\mathsf{A}(t,p) \subset \Omega$ for all $t \geq 0$ and $p \in M$ with $\phi_t(p) = p$. Then there exists a compact set $\widetilde{\Omega}$ such that $\mathsf{A}(t,x) \subset \widetilde{\Omega}$ for all $t \geq 0$ and $x \in M$.*

Proof. By the hypotheses, the family of continuous functions $\mathcal{A}_{\mathrm{c}} = (a_t)_{t\geq 0}$ given by $a_t(x) = \log\|\mathsf{A}(t,x)\|$ is almost additive with respect to Φ. Moreover, since the cocycle A has bounded distortion, $\mathcal{A}_{\mathrm{c}}$ has bounded variation. Now suppose that there is a compact $\Omega \subset GL^+(d,\mathbb{R})$ such that $\mathsf{A}(t,p) \subset \Omega$ for all $t \geq 0$ and $p \in M$ with $\phi_t(p) = p$. Since the map $\mathsf{A}(t,p) \mapsto \log\|\mathsf{A}(t,p)\|$ is continuous, there exists a constant $K > 0$ such that $|a_t(p)| \leq K$ for all $t \geq 0$ and all $p \in M$ with $\phi_t(p) = p$. By Theorem 3, there exists $\widetilde{K} > 0$ such that $\sup_{t\geq 0}\|a_t\|_\infty \leq \widetilde{K}$. In particular, we get $e^{-\widetilde{K}} \leq \|\mathsf{A}(t,x)\| \leq e^{\widetilde{K}}$ for all $t \geq 0$ and all $x \in M$. Hence, we finally obtain

$$\|\mathsf{A}(t,x) - \mathrm{Id}\| \leq \|\mathsf{A}(t,x)\| + \|\mathrm{Id}\| \leq e^{\widetilde{K}} + d \quad \text{for all } t \geq 0 \text{ and } x \in M,$$

concluding the proof. □

Remark 4. Theorem 5 is a particular continuous-time counterpart of a deep result by Kalinin ([Kal11, Theorem 1.2]), where a uniform bound on the periodic data guarantees a uniform bound on the entire phase space (see also related results in [Wal99, LZ22]).

3 Nonadditive notions of (weak) Gibbs states

In this section we compare and reconcile some notions of Gibbs states for nonadditive families of functions and obtain a classification of equilibrium measures with respect to hyperbolic flows. We also consider families of functions derived from measures and related to Gibbs properties, which play a relevant role in our framework of Bowen regularity problems arising from physical equivalence relations (Section 4.3).

Let us recall some ingredients of the nonadditive thermodynamic formalism for flows. Given an almost additive family of functions $\mathcal{A} = (a_t)_{t\geq 0}$ (satisfying mild assumptions) with respect to a continuous flow Φ on a compact metric space X, we have the variational principle (see [BH21a])

$$P_\Phi(\mathcal{A}) = \sup_{\mu\in\mathcal{M}(\Phi)} \left\{ h_\mu(\Phi) + \lim_{t\to\infty} \frac{1}{t} \int_X a_t \, d\mu \right\}, \tag{8}$$

where $P_\Phi(\mathcal{A})$ is the *nonadditive topological pressure of* $\mathcal{A}$ *with respect to* Φ introduced in [BH20]. Moreover, a measure $\nu \in \mathcal{M}(\Phi)$ is an *equilibrium measure* or an *equilibrium state* for $\mathcal{A}$ (with respect to Φ) if

$$P_\Phi(\mathcal{A}) = h_\nu(\Phi) + \lim_{t\to+\infty} \frac{1}{t} \int_X a_t \, d\nu.$$

Now we briefly recall the notions of suspension and hyperbolic flows, together with some useful properties. Let $T\colon X \to X$ be a homeomorphism of a compact metric space X and let $\tau\colon X \to \mathbb{R}$ be a strictly positive continuous function. Consider the space

$$W = \big\{ (x,s) \in X \times \mathbb{R} : 0 \leq s \leq \tau(x) \big\},$$

and let Y be the set obtained from W identifying $(x,\tau(x))$ with $(T(x),0)$ for each $x \in X$. Then a certain distance introduced by Bowen and Walters in [BW72] makes Y a compact metric space. The *suspension flow* over T with *height function* τ is the flow $\Phi = (\phi_t)_{t\in\mathbb{R}}$ on Y with the maps $\phi_t\colon Y \to Y$ defined by $\phi_t(x,s) = (x, s+t)$. When T is not invertible, we say that Φ is a *suspension semi-flow* on Y.

Let μ be a T-invariant probability measure on X. One can show that μ induces a Φ-invariant probability measure ν on Y such that

$$\int_Y g \, d\nu = \frac{\int_X I_g \, d\mu}{\int_X \tau \, d\mu} \tag{9}$$

for any continuous function $g\colon Y \to \mathbb{R}$, where $I_g(x) = \int_0^{\tau(x)} (g\circ\phi_s)(x)\, ds$. Conversely, any Φ-invariant probability measure ν on Y is of this form for some T-invariant probability measure μ on X. Abramov's entropy formula says that

$$h_\nu(\Phi) = \frac{h_\mu(T)}{\int_X \tau \, d\mu}. \tag{10}$$

By (9) and (10) we obtain

$$h_\nu(\Phi) + \int_Y g \, d\nu = \frac{h_\mu(T) + \int_X I_g \, d\mu}{\int_X \tau \, d\mu}. \tag{11}$$

Since $\tau > 0$, it follows from (11) that

$$P_\Phi(g) = 0 \quad \text{if and only if} \quad P_T(I_g) = 0,$$

where $P_\Phi(g)$ is the classical topological pressure of g with respect to Φ on Y and $P_T(I_g)$ is the classical topological pressure of I_g with respect to T on X. When $P_\Phi(g) = 0$, ν is an equilibrium measure for g if and only if μ is an equilibrium measure for I_g.

By the seminal works of Bowen [Bow73] and Ratner [Rat73], any locally maximal hyperbolic set for a C^1 flow on a Riemmanian manifold has *Markov partitions* of arbitrarily small diameter. Based on this, one can see that these systems inherit the same good structure of a suspension flow over a symbolic map and with a Hölder continuous height function.

Let X be a compact metric space. A map $S\colon X \to X$ is said to have *bounded distortion* if for each Hölder continuous function $\xi\colon X \to \mathbb{R}$ there exists a constant $D > 0$ such that if $x, y \in X$, $n \in \mathbb{N}$ and $d(T^k(x), T^k(y)) < \varepsilon$ for all $k \in \{0, \ldots, n-1\}$, then

$$\left| \sum_{k=0}^{n-1} \xi(S^k(x)) - \sum_{k=0}^{n-1} \xi(S^k(y)) \right| < D\varepsilon.$$

The full shift, subshifts of finite type, uniformly expanding and hyperbolic maps all have bounded distortion (see [Wal78, Bou02]).

The variational principle (8) and the notion of equilibrium states also hold for asymptotically additive families with respect to suspension flows, including locally maximal hyperbolic sets for C^1 flows (see Section 3 in [Hol24]).

Now consider $\Phi = (\phi_t)_{t\in\mathbb{R}}$ the suspension semi-flow over a continuous map $T\colon X \to X$ satisfying the bounded distortion property and with Hölder continuous height function τ. Proposition 19 in [BS00] guarantees that for each sufficiently small $\varepsilon > 0$ there exists a constant $\kappa > 0$ such that

$$B^Y_{\tau_m(x)}(\phi_s(x), \varepsilon) \subset B^X_m(x, \kappa\varepsilon) \times (s - \kappa\varepsilon, s + \kappa\varepsilon), \tag{12}$$

$$B^X_m(x, \varepsilon/\kappa) \times (s - \varepsilon/\kappa, s + \varepsilon/\kappa) \subset B^Y_{\tau_m(x)}(\phi_s(x), \varepsilon) \tag{13}$$

for every $x \in X$, $0 < s < \tau_m(x)$ and $m \in \mathbb{N}$, where $B^Y_t(y, \delta)$ and $B^X_n(x, \delta)$ denote, respectively, the Bowen ball with respect to the flow Φ on Y and the Bowen ball with respect to the map T on X, and

$$\tau_n(x) = \sum_{k=0}^{n-1} \tau(T^k(x)) \quad \text{for all } x \in X.$$

Let $\mathcal{A} = (a_t)_{t\geq 0}$ be a family of almost additive continuous functions with respect to Φ. Following as in the proof of Lemma 3.1 in [BH21a], the sequence $\mathcal{C} = (c_n)_{n\in\mathbb{N}}$ given by $c_n(x) = a_{\tau_n(x)}(x)$ is almost additive with respect to T. Now consider μ a Gibbs measure for the sequence $\mathcal{C}$ on X (for the proper definitions of Gibbs and weak-Gibbs measures for sequences with respect to maps, see [Bar06, Mum06]) and let ν be the measure on Y induced by μ (see identity (9)). In particular, $\nu = (\mu \times \lambda)/(\int_X \tau \, d\mu)$, where λ is the Lebesgue measure on $\mathbb{R}$. By the Gibbs property of μ, for any sufficiently small $\varepsilon > 0$ there exist $K_1(\varepsilon) > 0$ and $K_2(\varepsilon) > 0$ such that

$$\begin{aligned} K_1(\varepsilon)^{-1} \exp\big[-mP_T(\mathcal{C}) + c_m(x)\big] &\leq \mu(B_m^X(x, \kappa\varepsilon)) \\ &\leq K_1(\varepsilon) \exp\big[-mP_T(\mathcal{C}) + c_m(x)\big], \end{aligned} \tag{14}$$

$$\begin{aligned} K_2(\varepsilon)^{-1} \exp\big[-mP_T(\mathcal{C}) + c_m(x)\big] &\leq \mu(B_m^X(x, \varepsilon/\kappa)) \\ &\leq K_2(\varepsilon) \exp\big[-mP_T(\mathcal{C}) + c_m(x)\big] \end{aligned} \tag{15}$$

for all $x \in X$ and $m \in \mathbb{N}$. By identity (11) and the definition of τ_m, we get

$$\left(\frac{1}{\sup \tau}\right) P_T(\mathcal{C}) \leq P_\Phi(\mathcal{A}) \leq \left(\frac{1}{\inf \tau}\right) P_T(\mathcal{C}) \quad \text{and} \quad m \inf \tau \leq \tau_m(x) \leq m \sup \tau.$$

Moreover, one can check that for all $t > 0$ there exists $m \in \mathbb{N}$ such that $\tau_m(x) \leq t \leq \tau_{m+1}(x)$ with $t - \tau_m(x) \in [0, \sup \tau]$, which clearly gives that

$$\big|a_t(x) - a_{\tau_m(x)}(x)\big| \leq \sup_{s \in [0, \sup \tau]} \|a_s\|_\infty =: q.$$

It follows from (12) and (14) that

$$\begin{aligned} \nu(B_t^Y(\phi_s(x), \varepsilon)) &\leq \nu(B_{\tau_m(x)}^Y(\phi_s(x), \varepsilon)) \\ &\leq \frac{2\kappa\varepsilon K_1(\varepsilon)}{\inf \tau} \exp\big[-\tau_m(x) P_\Phi(\mathcal{A}) + a_{\tau_m(x)}(x)\big] \\ &\leq \frac{2\kappa\varepsilon K_1(\varepsilon)}{\inf \tau} \underbrace{\exp\big[(\sup \tau) P_\Phi(\mathcal{A}) + q\big]}_{=: L_1} \exp\big[-tP_\Phi(\mathcal{A}) + a_t(x)\big] \\ &= \frac{2\kappa\varepsilon L_1 K_1(\varepsilon)}{\inf \tau} \exp\big[-tP_\Phi(\mathcal{A}) + a_t(x)\big] \end{aligned}$$

for all $x \in X$ and $s \in [0, \tau(x)]$. The almost additivity of the family $\mathcal{A}$ readilly implies that

$$|a_t(x) - a_t(\phi_s(x))| \leq 2 \sup_{s \in [0, \sup \tau]} \|a_s\|_\infty := \widetilde{q}.$$

Since for each $y \in Y$ there exist $x \in X$ and $s \in [0, \sup \tau]$ such that $y = \phi_s(x)$, we finally obtain

$$\nu(B_t^Y(y, \varepsilon)) \leq \frac{2\kappa\varepsilon K_1(\varepsilon) e^{\widetilde{q}}}{\inf \tau} \exp\big[-tP_\Phi(\mathcal{A}) + a_t(y)\big] = \widetilde{K_1}(\varepsilon) \exp\big[-tP_\Phi(\mathcal{A}) + a_t(y)\big]$$

for all $y \in Y$ and $t > 0$, where $\widetilde{K_1}(\varepsilon) := (2\varepsilon L_1 K_1(\varepsilon) e^{\widetilde{q}})/\inf \tau$ only depends on $\varepsilon > 0$ and the function $\tau > 0$. Similarly, the identities (13) and (15) guarantee the existence of a constant $\widetilde{K_2}(\varepsilon) > 0$ such that

$$\nu(B_t^Y(y, \varepsilon)) \geq \widetilde{K_2}(\varepsilon) \exp\big[-tP_\Phi(\mathcal{A}) + a_t(y)\big] \quad \text{for all } y \in Y \text{ and } t > 0.$$

Therefore, we conclude that Gibbs measures for almost additive sequences on the base space induce the Gibbs property for almost additive families with respect to the suspension semi-flow. Analogously, one can show that weak Gibbs measures for the asymptotically additive sequence on the base induce measures that satisfy the weak Gibbs property for the asymptotically additive family with respect to the suspension semi-flow. These relations involving (weak) Gibbs properties between the map on base space and the flow also hold for suspension flows over maps having the bounded distortion[1]. In particular, by the existence of Markov partitions (see [Bow73, Rat73]), this framework includes locally maximal hyperbolic sets for topologically mixing C^1 flows.

Motivated by this, we have the following definitions.

Definition 3. Let Φ be a continuous flow on a compact metric space X. We say that a measure μ on X (not necessarily Φ-invariant) is a *Gibbs measure* or a *Gibbs state* for an asymptotically additive family of continuous functions $\mathcal{A} = (a_t)_{t\geq 0}$ (with respect to Φ) if for any sufficiently small $\varepsilon > 0$ there exists a constant $K(\varepsilon) \geq 1$ such that

$$K(\varepsilon)^{-1} \leq \frac{\mu(B_t(x,\varepsilon))}{\exp\left[-tP_\Phi(\mathcal{A}) + a_t(x)\right]} \leq K(\varepsilon)$$

for all $x \in X$ and $t > 0$.

Definition 4. Let Φ be a continuous flow on a compact metric space X. We say that a measure μ on X (not necessarily Φ-invariant) is a *weak Gibbs measure* or a *weak Gibbs state* for an asymptotically additive family of continuous functions $\mathcal{A} = (a_t)_{t\geq 0}$ (with respect to Φ) if for any sufficiently small $\varepsilon > 0$ there exists a sequence of numbers $(K_t(\varepsilon))_{t>0} \subset [1,\infty)$ with $\lim_{t\to\infty} \log K_t(\varepsilon)/t = 0$ such that

$$K_t(\varepsilon)^{-1} \leq \frac{\mu(B_t(x,\varepsilon))}{\exp\left[-tP_\Phi(\mathcal{A}) + a_t(x)\right]} \leq K_t(\varepsilon)$$

for all $x \in X$ and $t > 0$.

Remark 5. For hyperbolic flows and suspension flows over subshifts of finite type, uniformly expanding or hyperbolic maps in general, Definition 3 is a generalization of the classical notion of Gibbs measures to the nonadditive setup (see for example Definition 4.3.25 in [FH20]).

The following result guarantees the existence of Gibbs states for almost additive families of functions with respect to hyperbolic flows.

Proposition 6 ([BH21a, Theorem 3.5]). *Let Λ be a locally maximal hyperbolic set for a topologically mixing C^1 flow Φ and let $\mathcal{A}$ be an almost additive family of continuous functions on Λ with bounded variation. Then:*

1. *There exists a unique equilibrium measure for $\mathcal{A}$.*
2. *There exists a unique Φ-invariant Gibbs measures for $\mathcal{A}$.*
3. *The two measures are equal and are ergodic.*

Remark 6. The Gibbs state in Proposition 6 was obtained using the definition on the base space (see Section 3.3 in [BH21a]). As we saw above, this implies that the induced Φ-invariant measure satisfies the Gibbs property as in Definition 3. Proposition 6 also holds for appropriate versions of suspension flows over

[1] In [BH21a], the authors considered hyperbolic flows and, via Markov partitions, defined the Gibbs property only on the base space. Here, we showed that the definition on the base space implies the, a priori, more general definition directly for the flow space (Definition 3).

subshifts of finite type, uniformly expanding or hyperbolic maps with Hölder continuous height functions. On the other hand, for asymptotically additive families under the hypotheses of Proposition 6, we cannot guarantee uniqueness of equilibrium states. In these cases, the measures lifted from the base space are only guaranteed to be weak Gibbs (in the sense of Definition 4).

3.1 Some nonadditive families derived from (weak) Gibbs states and other measures

Many natural examples of nonadditive families were given in [Hol24]. Here, we also bring other relevant sources of almost and asymptotically additive families of functions.

If a measure η on a compact metric space X is Gibbs for some almost additive family of continuous functions with respect to a flow $\Phi = (\phi_t)_{t\in\mathbb{R}}$ on X, then for any sufficiently small $\delta > 0$ there exists a constant $K(\delta) > 1$ such that

$$\frac{1}{K(\delta)} \leq \frac{\eta(B_{t+s}(x,\delta))}{\eta(B_t(x,\delta))\eta(B_s(\phi_t(x),\delta))} \leq K(\delta) \quad \text{for all } x \in X \text{ and } t, s > 0.$$

In particular, for each $\delta > 0$, the family of functions $\mathcal{A}^\delta = (a_t^\delta)_{t\geq 0}$ given by $a_t^\delta(x) = \log \eta(B_t(x,\delta))$ is almost additive (defining $a_0^\delta \equiv 0$). Since every family admitting a Gibbs measure has bounded variation, it clearly follows that $\mathcal{A}^\delta$ also satisfies the bounded variation property. We observe that the functions $a_t^\delta \colon X \to \mathbb{R}$ are not necessarily continuous. In fact, using the Gibbs property of η, one can only guarantee the existence of constants $K_1(\delta) \geq K_2(\delta) > 0$ such that

$$K_2(\delta) + \limsup_{x\to x_0} a_t^\delta(x) \leq a_t^\delta(x_0) \leq K_1(\delta) + \liminf_{x\to x_0} a_t^\delta(x)$$

for all $x_0 \in X$ and all $t \geq 0$. In particular, the functions $x \mapsto a_t^\delta(x)$ are upper semicontinuous.

Proposition 7. *Let Φ be a continuous flow on a compact metric space X, and let ν be a measure on X. Then:*

1. *If for some $\delta > 0$ there exist constants $A(\delta) \geq B(\delta) > 0$ and an almost additive family of continuous functions $\mathcal{G} = (g_t)_{t\geq 0}$ such that*

$$B(\delta)e^{g_t(x)} \leq \nu(B_t(x,\delta)) \leq A(\delta)e^{g_t(x)} \quad \text{for all } x \in X \text{ and } t \geq 0,$$

   then there exists an almost additive family of Hölder continuous functions $\mathcal{H} = (h_t)_{t\geq 0}$ satisfying

$$\sup_{t\geq 0}\sup_{x\in X} \big|\log \nu(B_t(x,\delta)) - h_t(x)\big| < \infty.$$

2. *If for some $\delta > 0$ there exist sequences of numbers $(C_t(\delta))_{t\geq 0}, (D_t(\delta))_{t\geq 0} \subset [1,\infty)$ such that $\log C_t(\delta)/t \to 0$, $\log D_t(\delta)/t \to 0$ and an asymptotically additive family of continuous functions $\mathcal{F} = (f_t)_{t\geq 0}$ such that*

$$D_t(\delta)e^{f_t(x)} \leq \nu(B_t(x,\delta)) \leq C_t(\delta)e^{f_t(x)} \quad \text{for all } x \in X \text{ and } t \geq 0,$$

   then there exists an asymptotically additive family of Hölder continuous functions $\mathcal{J} = (j_t)_{t\geq 0}$ satisfying

$$\lim_{t\to\infty} \frac{1}{t} \sup_{x\in X} \big|\log \nu(B_t(x,\delta)) - j_t(x)\big| = 0.$$

Proof. Since the space of Hölder continuous functions is dense in the space of continuous functions on compact spaces, we can guarantee the existence of families $\mathcal{H} = (h_t)_{t\geq 0}$ and $\mathcal{J} = (j_t)_{t\geq 0}$ of Hölder continuous functions such that $\sup_{x\in X} |g_t(x) - h_t(x)| \leq 1$ and $\sup_{x\in X} |f_t(x) - j_t(x)| \leq 1$ for all $t \geq 0$. Clearly $\mathcal{H}$ is almost additive and $\mathcal{J}$ is asymptotically additive with respect to Φ. $\square$

Example 1. Gibbs measures satisfy the conditions of item 1, and weak Gibbs measures satisfy the conditions of item 2 in Proposition 7. This means that, modulo physical equivalences, the families generated by them are almost and asymptotically additive families of Hölder continuous functions, respectively.

Now let M be a compact Riemannian manifold and $\Lambda \subset M$ a hyperbolic set for a topologically mixing C^1 flow Φ. For each $t > 0$, consider the continuous function $J_t : \Lambda \to \mathbb{R}$ given by $J_t(x) = -\log \|d_x\phi_t|_{E^u(x)}\|$, where $E^u(x)$ is the unstable vector space at x. Since $x \mapsto E^u(x)$ is Hölder continuous, we also have that each function $x \mapsto J_t(x)$ is Hölder. Now let λ be the Lebesgue measure on M. Assuming that Φ is C^2, the *Volume Lemma* ([FH20, Proposition 7.4.3]) says that for any sufficiently small $\delta > 0$ there exist constants $C_\delta, D_\delta > 0$ such that

$$D_\delta J_t(x) \leq \lambda(B_t(x,\delta)) \leq C_\delta J_t(x) \quad \text{for all } x \in \Lambda \text{ and } t \geq 0.$$

Moreover, one can check that the family $(J_t)_{t\geq 0}$ is additive with respect Φ. In this case, the measure λ satisfies the conditions of the first item in Proposition 7. Hence, the family $Leb^\delta = (Leb^\delta_t)_{t\geq 0}$ given by $Leb^\delta_t(x) = \log \lambda(B_t(x,\delta))$ is almost additive with bounded variation and physically equivalent to an almost additive family of Hölder continuous functions. Actually, in this particular case, Leb^δ is physically equivalent to the additive family $(J_t)_{t\geq 0}$.

3.2 Classification of nonadditive equilibrium states

In this section we apply Theorem 3 to see how we can compare families with the same equilibrium measures, only based on the information provided by the periodic data of the system.

Theorem 8. *Let Λ be a locally maximal hyperbolic set for a topologically mixing C^1 flow $\Phi = (\phi_t)_{t\geq 0}$, and let $\mathcal{A} = (a_t)_{t\geq 0}$ and $\mathcal{B} = (b_t)_{t\geq 0}$ be two almost additive families of continuous functions with bounded variation. Then $\mathcal{A}$ and $\mathcal{B}$ have the same equilibrium measure if and only if there exists a constant $K > 0$ such that*

$$\left|a_t(p) - b_t(p) - t(P_\Phi(\mathcal{A}) - P_\Phi(\mathcal{B}))\right| \leq K$$

for all $p \in \Lambda$ and $t \geq 0$ with $\phi_t(p) = p$.

Proof. Suppose that $|a_t(p) - b_t(p) - t(P_\Phi(\mathcal{A}) - P_\Phi(\mathcal{B}))| \leq K$ for all $p \in \Lambda$ and $t \geq 0$ with $\phi_t(p) = p$. It follows from Theorem 3 that

$$\sup_{t\geq 0}\sup_{x\in\Lambda} \left|a_t(x) - b_t(x) - t(P_\Phi(\mathcal{A}) - P_\Phi(\mathcal{B}))\right| < \infty. \tag{16}$$

Now consider the almost additive family $\mathcal{D} = (d_t)_{t\geq 0}$ given by $d_t := b_t + t(P_\Phi(\mathcal{A}) - P_\Phi(\mathcal{B}))$. By (16), the family $(a_t - d_t)/t$ converges uniformly to zero on Λ. Togehter with the definition of nonadditive topological pressure, we have

$$P_\Phi(\mathcal{A}) = P_\Phi(\mathcal{D}) \quad \text{and} \quad \lim_{t\to\infty} \frac{1}{t}\int_\Lambda a_t\, d\nu = \lim_{t\to\infty} \frac{1}{t}\int_\Lambda d_t\, d\nu \quad \text{for all } \nu \in \mathcal{M}(\Phi),$$

showing that $\mathcal{A}$ and $\mathcal{D}$ have the same equilibrium measures. Moreover, since $P_\Phi(\mathcal{D}) = P_\Phi(\mathcal{B}) + (P_\Phi(\mathcal{A}) - P_\Phi(\mathcal{B}))$ and

$$\sup_{\mu\in\mathcal{M}(\Phi)} \left(h_\mu(\Phi) + \lim_{t\to\infty} \frac{1}{t} \int_\Lambda d_t \, d\mu \right) = \sup_{\mu\in\mathcal{M}(\Phi)} \left(h_\mu(\Phi) + \lim_{t\to\infty} \frac{1}{t} \int_\Lambda b_t \, d\mu \right) + P_\Phi(\mathcal{A}) - P_\Phi(\mathcal{B}),$$

$\mathcal{D}$ and $\mathcal{B}$ also share the same equilibrium measures. Hence, we conclude that $\mathcal{A}$ and $\mathcal{B}$ also have the same equilibrium measures.

Let's prove the converse. By Proposition 6, $\mathcal{A}$ and $\mathcal{B}$ admit unique equilibrium measures, each one of them satisfying the Gibbs property with respect to Φ. Now, by hypothesis, suppose these equilibrium states are the same unique measure $\eta \in \mathcal{M}(\Phi)$. The Gibbs property says that, for each sufficiently small $\varepsilon > 0$, there exist constants $K_1(\varepsilon) \geq 1$ and $K_2(\varepsilon) \geq 1$ such that

$$K_1(\varepsilon)^{-1} \leq \frac{\eta(B_t(x,\varepsilon))}{\exp\big[- tP_\Phi(\mathcal{A}) + a_t(x) \big]} \leq K_1(\varepsilon),$$

$$K_2(\varepsilon)^{-1} \leq \frac{\eta(B_t(x,\varepsilon))}{\exp\big[- tP_\Phi(\mathcal{B}) + b_t(x) \big]} \leq K_2(\varepsilon)$$

for all $x \in \Lambda$ and $t \geq 0$. This readily gives

$$K_1(\varepsilon)^{-1} K_2(\varepsilon)^{-1} \leq \exp\big[a_t(x) - b_t(x) - t(P_\Phi(\mathcal{A}) - P_\Phi(\mathcal{B})) \big] \leq K_1(\varepsilon) K_2(\varepsilon)$$

for all $x \in \Lambda$ and $t \geq 0$, which implies

$$\sup_{x\in\Lambda} \big| a_t(x) - b_t(x) - t(P_\Phi(\mathcal{A}) - P_\Phi(\mathcal{B})) \big| \leq \log(K_1(\varepsilon)K_2(\varepsilon)) \quad \text{for all } t \geq 0.$$

In particular, we obtain $|a_t(p) - b_t(p) - t(P_\Phi(\mathcal{A}) - P_\Phi(\mathcal{B}))| \leq \log(K_1(\varepsilon)K_2(\varepsilon))$ for all $p \in \Lambda$ and $t \geq 0$ with $\phi_t(p) = p$, and the result follows. □

Together with Theorem 3, Theorem 8 shows that two almost additive families with bounded variation share the same unique equilibrium state if and only if they are cohomologous to each other (modulo a uniform constant) in the sense of Definitions 1 and 2. In this scenario, Theorem 8 is the nonadditive version of the classical classification theorem for hyperbolic flows (see for example Theorem 7.3.24 in [FH20]).

4 On Regularity

Let us now consider the regularity problems involving the physical equivalence relations of asymptotically and almost additive families. We start investigating some natural simple examples in non hyperbolic setups. We also address the regularity issues for frameworks related to systems with hyperbolic behavior.

4.1 Linear flows on the flat torus

One of the main ingredients in the proof of Theorem 3 is the simultaneous existence of periodic and transitive points. A reasonable point then, is to ask what would happen in a system with no periodic points or no transitive data.

In this regard, the most natural examples seem to be linear flows on compact spaces. In this direction, let us start with an example of a setup where the periodic data is everywhere and with the same period.

Example 2. Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ be the n-torus and consider $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ to be linear dependent, that is, there exist (not all zero) $k_j \in \mathbb{Z}$ such that $\sum_{j=1}^n k_j\alpha_j = 0$. The linear flow $\Phi^\alpha = (\phi_t)_{t\in\mathbb{R}}$ on $\mathbb{T}^n$ in the direction α is defined by $\phi_t(x) = x + t\alpha \pmod 1$. Letting $\mathcal{A} = (a_t)_{t\geq 0}$ be an almost additive family of continuous functions with respect to Φ^α, Theorem 1 and Example 1 in [Hol24] guarantee the existence of a continuous function $b\colon \mathbb{T}^n \to \mathbb{R}$ such that $\lim_{t\to\infty} \|a_t - S_t b\|_\infty / t = 0$, where $\|\cdot\|_\infty$ is the supremum norm on $\mathbb{T}^n$. Since Φ^α is a periodic flow, by Lemma 1 there exists a constant $L > 0$ (depending only on the period) such that

$$\sup_{t\geq 0} \big\|a_t - S_t b\big\|_\infty \leq L.$$

In this case, the uniform bound exists even if the family $\mathcal{A}$ does not satisfy the bounded variation property. Moreover, it is clear that the additive family $(S_t b)_{t\geq 0}$ has bounded variation if and only if $\mathcal{A}$ also has it.

We now check what happens in the opposite extreme: transitive systems without periodic points.

Example 3. Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ be the n-torus and let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ be linear independent. In this case, the linear flow $\Phi^\alpha = (\phi_t)_{t\in\mathbb{R}}$ on $\mathbb{T}^n$ in the direction α given by $\phi_t(x) = x + t\alpha \pmod 1$ is *minimal*, that is, every orbit is dense in $\mathbb{T}^n$. Now let $\mathcal{A} = (a_t)_{t\geq 0}$ be any almost (or asymptotically) additive family of continuous functions. In particular, letting ν be the Lebesgue measure on $\mathbb{T}^n$ and $b\colon \mathbb{T}^n \to \mathbb{R}$ the continuous function given by the physical equivalence relation ([Hol24, Theorem 1]), we have

$$\lim_{t\to\infty} \frac{1}{t}\left\|a_t - t\int_{\mathbb{T}^n} b\,d\nu\right\|_\infty \leq \lim_{t\to\infty} \frac{1}{t}\big\|a_t - S_t b\big\|_\infty + \lim_{t\to\infty} \frac{1}{t}\left\|S_t b - t\int_{\mathbb{T}^n} b\,d\nu\right\|_\infty = 0.$$

This means that the function b can by replaced by the constant $\int_{\mathbb{T}^n} b\,d\nu$, which always satisfies the bounded variation property. On the other hand, the classical Gottshalk and Hedlund theorem for flows (see for example Theorem C in [McC99]), guarantees that $\sup_{t\geq 0} \|S_t g - t\int_{\mathbb{T}^n} g d\nu\|_\infty = \infty$ for every continuous function $g\colon \mathbb{T}^n \to \mathbb{R}$ not Φ^α-cohomologous to a constant. Therefore, for these types of linear flows, Theorem 3 fails even in the additive case assuming functions with any strong regularity.

Remark 7. Example 2 does not satisfy the hypotheses of Theorem 3. However, all the equivalences there are satisfied, even without asking for the bounded variation property of the families. In the opposite direction, Example 3 also does not satisfy the hypotheses of Theorem 3 but the uniform bound cannot be obtained, even asking for any type of regularity on the families of potentials.

In the next couple of sections, we treat the regularity issues in setups with hyperbolic behavior. These are natural and richer scenarios for investigating Hölder regularity and the bounded variation property (Bowen regularity).

4.2 Hölder regularity

We show how to construct almost and asymptotically additive families of Hölder continuous functions satisfying the bounded variation property but not physically equivalent to any additive family generated by a Hölder continuous function. Our approach is based on the following result for symbolic systems, which extends the examples in [HS24].

Here the set of symbols Σ is assumed to be finite.

Theorem 9. *Let $\sigma\colon \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ be the two-sided shift map. Then:*

1. *There exists an almost additive sequence of continuous functions with respect to σ satisfying the bounded variation property and which is not physically equivalent to any additive sequence generated by a Hölder continuous function.*
2. *There exist almost additive sequences of Hölder continuous functions with respect to σ satisfying the bounded variation property and which are not physically equivalent to any additive sequence generated by a Hölder continuous function.*

Proof. Let $\sigma_L\colon \Sigma^{\mathbb{N}} \to \Sigma^{\mathbb{N}}$ be the left-sided full shift. Fix $\beta > 1$ and define $s = s(\omega, \widetilde{\omega})$ on $\Sigma^{\mathbb{N}} \times \Sigma^{\mathbb{N}}$ as the smallest positive number s such that $\omega_s \neq \widetilde{\omega_s}$. In this case, we consider the distance on $\Sigma^{\mathbb{N}}$ to be $d_L(\omega, \widetilde{\omega}) = \beta^{-s(\omega,\widetilde{\omega})}$ if $\omega \neq \widetilde{\omega}$ and $d_L(\omega, \widetilde{\omega}) = 0$ if $\omega = \widetilde{\omega}$. Similarly, we define $q = q(\omega, \omega')$ on $\Sigma^{\mathbb{Z}} \times \Sigma^{\mathbb{Z}}$ as the smallest positive number q such that $\omega_{-q} \neq \omega'_{-q}$ or $\omega_q \neq \omega'_q$. From this, we consider the distance on $\Sigma^{\mathbb{Z}}$ as $d(\omega, \omega') = \beta^{-q(\omega,\omega')}$ if $\omega \neq \omega'$ and $d(\omega, \omega') = 0$ if $\omega = \omega'$.

Now let $\mathcal{F} = (f_n)_{n\in\mathbb{N}}$ be any almost additive sequence of continuous functions on $\Sigma^{\mathbb{N}}$ with respect to σ_L, satisfying the bounded variation property and not physically equivalent to any additive sequence generated by a Hölder continuous function (for example, the sequence generated by the quasi-Bernoulli measure in Theorem 11 in [HS24]). Consider the canonical projection $\pi\colon \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{N}}$ given by

$$\omega = (\cdots \omega_{-2}\omega_{-1}\omega_0, \omega_1\omega_2 \cdots) \longmapsto \pi(\omega) = (\omega_1\omega_2\omega_3 \cdots),$$

and let $\mathcal{G} = (g_n)_{n\in}$ be the sequence on $\Sigma^{\mathbb{Z}}$ given by $g_n = f_n \circ \pi$. Since we have $d_L(\pi(\omega), \pi(\omega')) \leq d(\omega, \omega')$ for all $\omega, \omega' \in \Sigma^{\mathbb{Z}}$, g_n is continuous for each $n \in \mathbb{N}$. By the relation $(\sigma_L \circ \pi)(\omega) = (\pi \circ \sigma)(\omega)$ for all $\omega \in \Sigma^{\mathbb{Z}}$, one can easily see that $\mathcal{G}$ is almost additive with respect to σ. Moreover, since $\mathcal{F}$ has bounded variation, we get

$$\begin{aligned}
&\sup_{n\in\mathbb{N}} \left\{ |g_n(\omega) - g_n(\widetilde{\omega})| : \omega, \widetilde{\omega} \in C_{i_1\dots i_n} \right\} \\
&\leq \sup_{n\in\mathbb{N}} \left\{ |f_n(\pi(\omega)) - f_n(\pi(\widetilde{\omega}))| : \pi(\omega), \pi(\widetilde{\omega}) \in C_{i_1\dots i_n} \cap \Sigma^{\mathbb{N}} \right\} < \infty,
\end{aligned}$$

where $C_{i_1\dots i_n}$ is the *cylinder* set

$$C_{i_1\dots i_n} = \left\{ (j_1 j_2 \cdots) \in \Sigma^{\mathbb{N}} : j_1 = i_1, \dots, j_n = i_n \right\}.$$

That is, $\mathcal{G}$ also satisfies the bounded variation property. Now suppose that $\phi\colon \Sigma^{\mathbb{Z}} \to \mathbb{R}$ is a Hölder continuous function such that $\mathcal{G}$ is phisically equivalent to $(S_n\phi)_{n\in\mathbb{N}}$ with respect to σ. Lemma 1.6 in [Bow75a] (see also Section 3 in [Sin72]) guarantees the existence of a Hölder continuous function $\psi\colon \Sigma^{\mathbb{Z}} \to \mathbb{R}$ cohomologous to ϕ and such that $\psi(\cdots\omega_{-2}\omega_{-1}\omega_0\omega_1\omega_2\cdots) = \psi(\omega_1\omega_2\cdots)$. That is, $\psi\circ\pi = \psi$ and there exists a continuous function $v\colon \Sigma^{\mathbb{Z}} \to \mathbb{R}$ satisfying $\phi - \psi = v\circ\sigma - v$. Thus, for every $\omega \in \Sigma^{\mathbb{Z}}$, we obtain

$$\begin{aligned}\left|g_n(\omega) - \sum_{k=0}^{n-1}(\phi\circ\sigma^k)(\omega)\right| &= \left|f_n(\pi(\omega)) - \sum_{k=0}^{n-1}(\psi\circ\pi\circ\sigma^k)(\omega) + v - v\circ\sigma^n\right| \\ &\geq \left|f_n(\pi(\omega)) - \sum_{k=0}^{n-1}(\psi\circ\sigma_L^k)(\pi(\omega))\right| - 2\|v\|_\infty.\end{aligned}$$

Since $\pi(\Sigma^{\mathbb{Z}}) = \Sigma^{\mathbb{N}}$ and $\mathcal{G}$ is physically equivalent to $(S_n\phi)_{n\in\mathbb{N}}$, we have

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{n}\sup_{\omega'\in\Sigma^{\mathbb{N}}}\left|f_n(\omega') - \sum_{k=0}^{n-1}(\psi\circ\sigma_L^k)(\omega')\right| \\ &\leq \lim_{n\to\infty}\frac{1}{n}\sup_{\omega\in\Sigma^{\mathbb{Z}}}\left|g_n(\omega) - \sum_{k=0}^{n-1}(\phi\circ\sigma^k)(\omega)\right| = 0,\end{aligned}$$

which is a contradiction. Therefore, $\mathcal{G}$ is not physically equivalent to any additive sequence generated by a Hölder continuous function, and item 1 is proved.

Let us prove the second item. Consider the same sequence $\mathcal{G}$ and take any real number $\gamma > 0$. By the density of Hölder functions on the space of continuous functions on $\Sigma^{\mathbb{Z}}$, for each $n \in \mathbb{N}$ there exists a Hölder continuous function $h_n^\gamma\colon \Sigma^{\mathbb{Z}} \to \mathbb{R}$ such that $\|g_n - h_n^\gamma\|_\infty \leq \gamma$. Since $\mathcal{G}$ has bounded variation, the sequence $\mathcal{H}^\gamma = (h_n^\gamma)_{n\in\mathbb{N}}$ also satisfies the bounded variation property. Furthermore, it is clear that $\mathcal{H}^\gamma$ is also almost additive with respect to σ. Since, in particular, $\mathcal{G}$ and $\mathcal{H}^\gamma$ are physically equivalent, $\mathcal{H}^\gamma$ is not physically equivalent to any additive sequence generated by a Hölder function. The result follows now by the arbitrariness of $\gamma \in \mathbb{R}^+$. $\square$

Now we show how we can pass some relevant information from discrete to continuous-time systems. First, a simple auxiliary result.

Lemma 2. *Let X be a compact metric space. Every almost additive sequence of continuous functions $\mathcal{Q} = (q_n)_{n\in\mathbb{N}}$ with respect to a continuous map $T\colon X \to X$ satisfies*

$$\sup_{n\in\mathbb{N}}\left\|q_n\circ T - q_n\right\|_\infty < \infty.$$

Proof. Since $\mathcal{Q}$ is almost additive, there exists a constant $K > 0$ such that

$$-K + q_1(x) + q_{n-1}(T(x)) \leq q_n(x) \leq q_{n-1}(T(x)) + q_1(x) + K$$

for every $x \in X$ and every $n \in \mathbb{N}$. From this, we get

$$\left|q_n(x) - q_{n-1}(T(x))\right| \leq K + \|q_1\|_\infty =: K_1 < \infty. \tag{17}$$

On the other hand, we also have

$$-K + q_1(T^{n-1}x) + q_{n-1}(x) \leq q_n(x) \leq q_{n-1}(x) + q_1(T^{n-1}x) + K,$$

which gives

$$\big|q_n(x) - q_{n-1}(x)\big| \leq K_1 \tag{18}$$

for every $x \in X$ and every $n \in \mathbb{N}$. It follows from (17) and (18) that

$$\big|q_n(T(x)) - q_n(x)\big| \leq \big|q_n(f(x)) - q_{n+1}(x)\big| + \big|q_{n+1}(x) - q_n(x)\big| \leq 2K_1$$

for every $x \in X$ and every $n \in \mathbb{N}$, as desired. $\square$

Lemma 3. *Let $\Phi = (\phi_t)_{t\in\mathbb{R}}$ be a suspension flow on Y over a continuous invertible map $T\colon X \to X$ with continuous height function $\tau\colon X \to (0,\infty)$. Let $\mathcal{C} = (c_n)_{n\in\mathbb{N}}$ be an almost additive sequence of continuous functions with respect to T on X and satisfying the bounded variation property. Then, there exists an almost additive family of continuous functions $\mathcal{A} = (a_t)_{t\geq 0}$ with respect to Φ on Y, satisfying the bounded variation property and such that $a_n(x) = c_n(x)$ for all $x \in X$ and $n \in \mathbb{N}$. The same result holds for the asymptotically additive case.*

Proof. Consider the function $\lfloor\cdot\rfloor\colon \mathbb{R} \to \mathbb{Z}$ given by $\lfloor x\rfloor = \max\{m \in \mathbb{Z} : m \leq x\}$. Now for each $t > 0$ define the function $a_t : Y \to \mathbb{R}$ as

$$a_t(y) = a_t(\phi_s(x)) = a_{\lfloor t\rfloor}(\phi_s(x)) = c_{\lfloor t\rfloor}(x) \quad \text{and} \quad a_0 = c_0 := 0. \tag{19}$$

For the sake of simplicity, let us consider a constant height function $\tau = 1$. Notice that by construction, $\phi_1 = T$ on X. In addition, the sequence $(a_n)_{n\in\mathbb{N}}$ is almost additive with respect to ϕ_1 on Y. In fact, by (19) and the almost additivity of $\mathcal{C}$ on X, for all $y \in Y$ and $m, n \in \mathbb{N}$ we have

$$\begin{aligned}
a_{m+n}(y) = a_{m+n}(\phi_s(x)) = c_{m+n}(x) &\leq c_m(x) + c_n(T^m(x)) + C\\
&= a_m(\phi_s(x)) + a_n(\phi_1^m(x)) + C\\
&= a_m(y) + (a_n \circ \phi_s \circ \phi_1^m)(x) + C\\
&= a_m(y) + a_n(\phi_1^m(\phi_s(x))) + C\\
&= a_m(y) + a_n(\phi_1^m(y)) + C.
\end{aligned}$$

Proceeding in the same manner, we also have $a_{m+n}(y) \geq a_m(y) + a_n(\phi_1^m(y)) - C$ for all $y \in Y$ and $m, n \in \mathbb{N}$, and $(a_n)_{n\in\mathbb{N}}$ is indeed almost additive with respect to ϕ_1 on Y.

Let us now show that the family $\mathcal{A}$ is almost additive with respect to the flow Φ on Y. By the almost additivity of $(a_n)_{n\in\mathbb{N}}$ with respect to ϕ_1 on Y, for each $y \in Y$, $m \leq t < m+1$ and $n \leq s < n+1$, we obtain

$$\begin{aligned}
a_{t+s}(y) = a_{m+n}(y) \leq a_m(y) + a_n(\phi_m(y)) + C &= a_t(y) + a_s(\phi_m(y)) + C\\
&= a_t(y) + a_s(\phi_t(y)) + \big[a_n(\phi_m(y)) - a_n(\phi_t(y))\big] + C.
\end{aligned} \tag{20}$$

On the other hand, letting $y = \phi_s(x)$ for some $x \in X$ and $r \in [0,1)$ and $m = t+u$ with $u \in [0,1)$, we also have

$$\begin{aligned}
\big|a_n(\phi_m)(y) - a_n(\phi_t(y))\big| &= \big|a_n(\phi_m(\phi_r(x))) - a_n(\phi_t(\phi_r(x)))\big|\\
&= \big|a_n(\phi_r(\phi_m(x))) - a_n(\phi_{u+r}(\phi_m(x)))\big|\\
&\leq \big|c_n(\phi_m(x)) - c_n(\phi_1(\phi_m(x)))\big|.
\end{aligned} \tag{21}$$

Since $\mathcal{C}$ is almost additive with respect to ϕ_1 on X, Lemma 2 guarantees the existence of a uniform constant $K > 0$ such that

$$\sup_{n\in\mathbb{N}} \sup_{x\in X} \left|c_n(\phi_m(x)) - c_n(\phi_1(\phi_m(x)))\right| \leq K.$$

Together with (21), this implies that $|a_n(\phi_m)(y) - a_n(\phi_t(y))| \leq K$ for all $m, n \in \mathbb{N}$, $t > 0$ and $y \in Y$. Hence, it follows now from (20) that

$$a_{t+s}(y) \leq a_t(y) + a_s(\phi_t(y)) + K + C \quad \text{for all } y \in Y \text{ and } t, s \geq 0.$$

The other inequality can be obtained in a similar way, and we conclude that $\mathcal{A}$ is almost additive with respect to Φ on Y.

Now let d_X be any metric on the base space X and consider the *Bowen-Walters distance* d_Y on Y ([BW72]). By the continuity of each $c_n\colon X \to \mathbb{R}$, it is clear that the function $a_t\colon Y \to \mathbb{R}$ is continuous for each $t \geq 0$.

Let's show that $\mathcal{A}$ also has bounded variation with respect to Φ on Y. Take two arbitrary points $y, z \in Y$ such that $d_Y(\phi_\tau(y), \phi_\tau(z)) < \varepsilon$ for $\tau \in [0, t]$ with $m \leq t < m+1$. Writing $y = \phi_u(x)$ and $z = \phi_r(x')$ for $x, x' \in X$ and $u, r \in [0, 1)$, we have

$$\left|a_t(y) - a_t(z)\right| = \left|a_t(\phi_u(x)) - a_t(\phi_r(x'))\right| = \left|c_m(x) - c_m(x')\right| \tag{22}$$

In particular, we get

$$\begin{aligned}
&d_X(x, x') \leq d_Y(y, z) < \varepsilon,\\
&d_X(\phi_1(x), \phi_1(x')) \leq d_Y(\phi_u(\phi_1(x)), \phi_r(\phi_1(x'))) = d_Y(\phi_1(y), \phi_1(z)) < \varepsilon,\\
&\vdots\\
&d_X(\phi_{m-1}(x), \phi_{m-1}(x')) \leq d_Y(\phi_u(\phi_{m-1}(x)), \phi_r(\phi_{m-1}(x')))\\
&\quad = d_Y(\phi_{m-1}(y), \phi_{m-1}(z)) < \varepsilon.
\end{aligned} \tag{23}$$

Since the sequence $\mathcal{C}$ has bounded variation with respect to $T = \phi_1$ on X, there exists a constant $L = L(\varepsilon) > 0$ such that $|c_m(x) - c_m(x')| \leq L$. Thus, it follows from (22) that the family $\mathcal{A}$ on Y also have bounded variation with respect to Φ, and with the same parameters ε, L as the sequence $\mathcal{C}$ on X.

For the general case where τ is any positive continuous function, we have $T(x) = \phi_{\tau(x)}(x)$ and $T^m(x) = \phi_{\tau_m(x)}(x)$ for all $m \in \mathbb{N}$ and $x \in X$, with $\tau_m = \sum_{k=0}^{m-1} \tau \circ T$. In this case, for each $t \geq 0$, we define $a_t\colon Y \to \mathbb{R}$ as

$$a_t(y) = a_t(\phi_s(x)) := a_{\tau_n(x)}(\phi_s(x)) := c_n(x) \quad \text{and} \quad a_0 = c_0 := 0. \tag{24}$$

Making the necessary modifications and proceeding as in the case with $\tau = 1$, one can see that $\mathcal{A}$ is almost additive with respect to Φ on Y. The continuity of each $c_m\colon X \to \mathbb{R}$ together with definition (24), directly implies that $a_t\colon Y \to \mathbb{R}$ is continuous for each $t \geq 0$. Moreover, since $\mathcal{C}$ has bounded variation with respect to T on X, the same relation between the distance on X and the Bowen-Walters distance on Y as in (23), guarantees the bounded variation property for $\mathcal{A}$ with respect to Φ on Y.

Now suppose that $\mathcal{D} = (d_n)_{n\in\mathbb{N}}$ is asymptotically additive with respect to T, and consider again the family $\mathcal{A} = (a_t)_{t\geq 0}$ defined in (24) now with $c_n = d_n$

for all $n \in \mathbb{N}$. By the asymptotic additivity of $\mathcal{D}$, given any $\varepsilon > 0$ there exists a continuous function $h_\varepsilon\colon X \to \mathbb{R}$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \sup_{x\in X} \left| d_n(x) - \sum_{k=0}^{n-1} (h_\varepsilon \circ T)(x) \right| < \varepsilon. \tag{25}$$

By Lemma 2 in [Hol24], there exists a continuous function $g_\varepsilon\colon Y \to \mathbb{R}$ such that $I_{g_\varepsilon}|_X = h_\varepsilon$. By the definition of $\mathcal{A}$, for each $y = \phi_u(x)$ with $u \in [0, \sup \tau)$ and $\tau_n(x) \le t < \tau_{n+1}(x)$, we have

$$\begin{aligned} \left| a_t(y) - \int_0^t (g_\varepsilon \circ \phi_s)(y) ds \right| &= \left| a_t(\phi_u(x)) - \int_u^{t+u} (g_\varepsilon \circ \phi_s)(x) ds \right| \\ &\le \left| d_n(x) - \sum_{k=0}^{n-1} (h_\varepsilon \circ T^k)(x) \right| + \sup \tau \sup g_\varepsilon + \sup h_\varepsilon. \end{aligned}$$

Since $n \to \infty$ implies $t \to \infty$, we conclude from (25) that $\mathcal{A}$ is asymptotically additive with respect to Φ on Y. The continuity and the bounded variation property of $\mathcal{A}$ follow from the same arguments presented in the almost additive case. $\square$

Suspension flows over two-sided subshifts of finite type with Hölder continuous height functions are also called *hyperbolic symbolic flows* (see [FH20]). One can check that additive families generated by Hölder continuous functions satisfy the bounded variation property with respect to hyperbolic symbolic flows, and Proposition 6 also holds for these types of flows. On the other hand, it is not hard to find asymptotically additive families having bounded variation with respect to an hyperbolic symbolic flow, but admitting more than one equilibrium state.

The following result is a continuous-time counterpart of Theorem 9, and gives a negative answer to the Hölder regularity problem.

Theorem 10. *Let $\Phi = (\phi_t)_{t\in\mathbb{R}}$ be a suspension flow over the two-sided shift map $\sigma\colon \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ and with a Hölder continuous height function $\tau\colon \Sigma^{\mathbb{Z}} \to (0, \infty)$. Then:*

1. *There exist almost additive families of Hölder continuous functions with respect to Φ, satisfying the bounded variation property and not physically equivalent to any additive family generated by a Hölder continuous function.*
2. *There exist asymptotically additive families of Hölder continuous functions with respect to Φ, satisfying the bounded variation property, admitting a unique equilibrium state but not physically equivalent to any additive family generated by a Hölder continuous function.*

Proof. Let $\mathcal{C} = (c_n)_{n\in\mathbb{N}}$ be any sequence given by Theorem 9, that is, $\mathcal{C}$ is almost additive with respect to σ, has bounded variation and is not physically equivalent to any additive sequence generated by a Hölder continuous function. By Lemma 3 there exists an almost additive family of continuous functions

$\mathcal{A} = (a_t)_{t\geq 0}$ with respect to Φ on Y, with bounded variation and such that $a_n(x) = c_n(x)$ for all $x \in \Sigma^{\mathbb{Z}}$ and $n \in \mathbb{N}$. Suppose that

$$\lim_{t\to\infty} \frac{1}{t} \sup_{y\in Y} \left| a_t(y) - \int_0^t (b \circ \phi_s)(y) ds \right| = 0, \quad \text{where} \quad b\colon Y \to \mathbb{R} \text{ is Hölder.}$$

In particular, we have

$$\lim_{t\to\infty} \frac{1}{t} \sup_{x\in\Sigma^{\mathbb{Z}}} \left| a_t(x) - \int_0^t (b \circ \phi_s)(x) ds \right| = 0. \tag{26}$$

By the proof of Lemma 15 in [BH21b], for each $t > 0$ there exists a unique $n \in \mathbb{N}$ with $t = \tau_n(x) + \kappa$ for some $\kappa \in [0, \sup \tau]$ such that

$$\left| \int_0^t (b \circ \phi_s)(x) ds - \sum_{k=0}^{n-1} (I_b \circ \sigma^k)(x) \right| \leq \sup b \sup \tau,$$

where $I_b(x) = \int_0^{\tau(x)} (b \circ \phi_s)(x) ds$. Thus, it follows from (26) that

$$\lim_{n\to\infty} \frac{1}{n} \sup_{x\in\Sigma^{\mathbb{Z}}} \left| c_n(x) - \sum_{k=0}^{n-1} (I_b \circ \sigma^k)(x) \right| = \lim_{n\to\infty} \frac{1}{n} \sup_{x\in\Sigma^{\mathbb{Z}}} \left| a_n(x) - \int_0^n (b \circ \phi_s)(x) ds \right| = 0.$$

Since $b\colon Y \to \mathbb{R}$ is Hölder, the function $I_b : X \to \mathbb{R}$ is also Hölder ([BS00, Proposition 18]). Hence, $\mathcal{C}$ is physically equivalent to the additive sequence generated by I_b, which is a contradiction.

Now fix a number $\gamma > 0$. By the density of Hölder functions on the space of continuous functions, for each $t \geq 0$ there exists a Hölder continuous function $b_t^\gamma\colon Y \to \mathbb{R}$ such that $\sup_{y\in Y} |b_t^\gamma(y) - a_t(y)| \leq \gamma$. It is clear that the family $\mathcal{B}^\gamma := (b_t^\gamma)_{t\geq 0}$ is almost additive and satisfy the bounded variation property with respect to the flow Φ on Y. Moreover, since in particular $\mathcal{B}^\gamma$ is physically equivalent to $\mathcal{A}$, it is obvious that the family $\mathcal{B}^\gamma$ cannot be physically equivalent to any additive family generated by a Hölder continuous function, as desired.

Now let us prove the second item. It was showed in [HS24] the existence of asymptotically additive sequences of continuous functions $\mathcal{D} = (d_n)_{n\in\mathbb{N}}$ with respect to the left-sided shift map $\sigma_L\colon \Sigma^{\mathbb{N}} \to \Sigma^{\mathbb{N}}$, satisfying the bounded variation property, with a unique equilibrium measure, but not physically equivalent to any additive sequence generated by a Hölder continuous function. Proceeding as in the proof of Theorem 9, one also can assume that $\mathcal{D}$ is asymptotically additive with respect to the two-sided shift map $\sigma\colon \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$. By Lemma 3, following as in the proof of the last item, we can guarantee the existence of an asymptotically additive family of Hölder continuous functions $\mathcal{A}$ with respect to Φ on Y, having bounded variation but not physically equivalent to any additive family generated by a Hölder function. Moreover, it is clear from the identity (11) that $\mathcal{A}$ also admits a unique equilibrium measure on Y, which is induced by the unique equilibrium measure for $\mathcal{D}$ on the base space $\Sigma^{\mathbb{Z}}$. □

Remark 8. These counter-examples show that the physical equivalence [Hol24, Theorem 1] does not always allow us to reduce the study of asymptotically additive families with bounded variation to the case of single functions with Hölder regularity. Since the thermodynamic and multifractal formalisms are well understood for the case of Hölder continuous potentials in hyperbolic setups, Theorem 10 also shows, as in the case of maps, a significant barrier regarding the exchange of information between the additive, almost additive and asymptotically additive worlds with respect to continuous-time systems.

4.3 Bowen regularity and a proposed classification of almost additive families

In this section, taking into consideration Theorem 3, we address the issues related to bounded variation and Bowen regularity of additive, almost and asymptotically additive families. In our approach here, equilibrium states satisfying the Gibbs property play a crucial role.

Theorem 11. *Let $\Phi = (\phi_t)_{t\in\mathbb{R}}$ be a suspension flow over the two-sided shift map $\sigma\colon \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ and with a Hölder continuous height function $\tau\colon \Sigma^{\mathbb{Z}} \to (0,\infty)$. Let $\mathcal{A} = (a_t)_{t\geq 0}$ be an almost additive family of continuous functions with respect to Φ on Y, and having bounded variation. Then the following properties are equivalent:*

1. *The equilibrium measure for $\mathcal{A}$ satisfies the Gibbs property for a continuous Bowen function.*
2. *There exists a continuous Bowen function $b\colon Y \to \mathbb{R}$ such that*
$$\lim_{t\to\infty} \frac{1}{t} \sup_{y\in Y} \big|a_t(y) - S_t b(y)\big| = 0.$$
3. *There exists a continuous Bowen function $b\colon Y \to \mathbb{R}$ such that*
$$\sup_{t\geq 0}\sup_{y\in Y} \big|a_t(y) - S_t b(y)\big| < \infty.$$

Proof. We start showing that 1 implies 3. By an appropriate version of Proposition 6 for hyperbolic symbolic flows, $\mathcal{A}$ has a unique equilibrium state ν, which satisfies the Gibbs property (now in the sense of Definition 3). By hypothesis, ν is also Gibbs for some continuous Bowen function $b\colon Y \to \mathbb{R}$. Then, for some sufficiently small $\delta > 0$, there exist constants $K_1 = K_1(\delta) \geq 1$ and $K_2 = K_2(\delta) \geq 1$ such that

$$K_1^{-1} \leq \frac{\nu(B_t(y,\delta))}{\exp\big[-tP_\Phi(\mathcal{A}) + a_t(y)\big]} \leq K_1, \tag{27}$$

$$K_2^{-1} \leq \frac{\nu(B_t(y,\delta))}{\exp\big[-tP_\Phi(b) + S_t b(y)\big]} \leq K_2 \tag{28}$$

for all $y \in Y$ and $t \geq 0$, where $P_\Phi(b)$ is the classical topological pressure of b with respect to Φ. Taking the new function $\widetilde{b} := b + P_\Phi(b) - P_\Phi(\mathcal{A})$, by (27) and (28) we clearly have $|a_t(y) - S_t\widetilde{b}(y)| \leq \log K_1K_2$ for all $y \in Y$ and $t \geq 0$. Since $\widetilde{b}$ is also a continuous Bowen function, item 3 is proved.

Now suppose 3 holds, that is, there exist a uniform constant $K_3 > 0$ and a continuous Bowen function b such that $|a_t(y) - S_t b(y)| \leq K_3$ for all $y \in Y$ and

$t \geq 0$. In this case, $P_\Phi(\mathcal{A}) = P_\Phi(b)$. Moreover, by the Gibbs property for $\mathcal{A}$ in (27), we obtain

$$(K_1 e^{K_3})^{-1} = K_1^{-1} e^{-K_3} \leq \frac{\nu(B_t(y,\delta))}{\exp\big[-tP_\Phi(b) + S_t b(y)\big]} \leq K_1 e^{K_3}$$

for all $y \in Y$ and $t \geq 0$, which is item 1.

Finally, since every hyperbolic symbolic flow is topologically transitive ([FH20, Proposition 1.6.30]) and satisfy the hypotheses of the Closing Lemma ([KH12, Corollary 18.1.8]), it follows from Corollary 4 that items 2 and 3 are equivalent, and the theorem is proved. □

For the case of hyperbolic symbolic flows or locally maximal hyperbolic sets for C^1 topologically mixing flows, it is not hard to see that an almost additive family satisfies the bounded variation property if and only if it admits a Gibbs state. In Theorem 11, the equivalence between items 1 and 2 indicates a possible way of classifying almost additive families with bounded variation with respect to hyperbolic symbolic flows or locally maximal hyperbolic sets for C^1 topologically mixing flows. Then, we propose the following classification of almost additive families:

- **Type 1:** Almost additive families with bounded variation and admitting a Gibbs state for a Bowen continuous function.
- **Type 2:** Almost additive families with bounded variation but not admitting Gibbs states for Bowen continuous functions.
- **Type 3:** Almost additive families without bounded variation but having a unique equilibrium state.
- **Type 4:** Almost additive families having more than one equilibrium state.

Remark 9. One can construct families of types 1, 3 and 4 ([HS24]). On the other hand, examples of type 2 seem to be much more complicated to produce or they actually don't exist. In the discrete-time framework, the existence of sequences of type 2 is connected with the problem of showing that every quasi-Bernoulli measure is a Gibbs state for some Bowen function (see also [Cun20]).

Asymptotically additive families. Now we show how to treat the Bowen regularity problem for asymptotically additive families. Let $\mathcal{G} = (g_n)_{n\in\mathbb{N}}$ be an asymptotically additive sequence of continuous functions with respect to $\sigma\colon \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$, having bounded variation, with a unique equilibrium measure, but not physically equivalent to any additive sequence generated by a Bowen function. By Lemma 3, there exists an asymptotically additive family $\mathcal{A} = (a_t)_{t\geq 0}$ with respect to the hyperbolic symbolic flow Φ on Y (with height function τ) and such that $a_n(x) = g_n(x)$ for all $x \in \Sigma^{\mathbb{Z}}$ and $n \in \mathbb{N}$. Now suppose the existence of a continuous Bowen function $b\colon Y \to \mathbb{R}$ such that $\mathcal{A}$ is physically equivalent to $(S_t b)_{t\geq 0}$. By the appropriate versions of Lemmas 3.1 and 3.3 in [BH21a] for hyperbolic symbolic flows, the sequence $\mathcal{C} = (c_n)_{n\in\mathbb{N}}$ given by $c_n(x) = \int_0^{\tau_n(x)} (b \circ \phi_s)(x) ds$ is additive and satisfy the bounded variation property with respect to σ. By the physical equivalence relation between $\mathcal{A}$ and

$(S_t b)_{t\geq 0}$, we have in particular that

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\sup_{x\in\Sigma^{\mathbb{Z}}}\Big|g_n(x)-c_n(x)\Big| &= \lim_{n\to\infty}\frac{1}{n}\sup_{x\in\Sigma^{\mathbb{Z}}}\Big|a_n(x)-c_n(x)\Big| \\ &\leq \lim_{t\to\infty}\frac{1}{t}\sup_{y\in Y}\big|a_t(y)-S_t b(y)\big| = 0.\end{aligned} \tag{29}$$

Since, by the proof of Lemma 15 in [BH21b], $c_n(x) = \sum_{k=0}^{n-1} I_b\circ\sigma^k(x) =: S_n I_b(x)$ for all $x\in\Sigma^{\mathbb{Z}}$ and all $n\in\mathbb{N}$, the sequence $(S_n I_b)_{n\in\mathbb{N}}$ has bounded variation. Hence, it follows from (29) that $\mathcal{G}$ is physically equivalent to the additive sequence $(S_n I_b)_{n\in\mathbb{N}}$ generated by the Bowen function $I_b\colon \Sigma^{\mathbb{Z}}\to\mathbb{R}$. This is a contradiction. Therefore, by construction, $\mathcal{A}$ has bounded variation and admits a unique equilibrium state (with respect to Φ on Y), but cannot be physically equivalent to any additive family generated by a Bowen continuous function. □

4.4 Concluding remarks

Observe that all the results in the regularity sections are developed for hyperbolic symbolic flows. There is a deeper reason for that, which comes all the way from [BKM20]. In this last work, studying almost additivity in the context of planar matrix cocycles, the authors showed an example of a quasi-Bernoulli measure that is not Gibbs for any Hölder continuous function with respect to the left-sided full shift map ([BKM20, Example 2.10 (2)]). In view of the non-additive versions of the Livšic theorem for maps and flows (Theorem 5 in [HS24] and Theorem 3, respectively), this particular example plays a fundamental role in the production of the counter-examples in [HS24] for the left-sided full shift map and, consequently, the ones in Theorem 10 for symbolic flows.

Based on this, morally speaking, all the counter-examples and results discussed here in the regularity section can be adapted to the case of hyperbolic flows and, more generally, to suspension flows over topologically mixing subshifts of finite type. To achieve this, one needs to obtain appropriate versions of Theorems 2.8 and 2.9 in [BKM20] for topologically mixing Markov chains using the classical thermodynamic machinery developed in [Bow75a].

Finally, let us mention the still open problem of the existence of sequences and families of type 2. A reasonable starting point to attack this question is to understand how the aforementioned theorems in [BKM20] could accommodate Bowen continuous functions, going beyond the Hölder regularity previously considered by them. A positive answer in this direction would finally reveal the existence of quasi-Bernoulli measures that do not satisfy the Gibbs property for any continuous function, consequently giving examples of sequences of type 2 with respect to the full shift of finite type. Based on this, by our constructions in this note, we could as well give examples of families of type 2 with respect to hyperbolic symbolic flows and hyperbolic flows (via Markov partitions).

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