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Dynamic analysis of unstable Hopfield networks

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Abstract In this paper, the dynamic behaviors of unstable Hopfield neural networks (HNNs) with asymmetric connections are studied. It is found that the solution of the HNN is bounded and the HNN is a dissipative system. In addition, sufficient conditions for the instability of the equilibrium point and the existence of stable limit cycles are proposed. Some numerical simulations are given to illustrate the effectiveness of the proposed results. It is shown that some HNNs exhibit two independent limit cycles or chaotic attractors which are symmetric to each other with respect to the origin.

Keywords Hopfield neural network · Dissipative system · Asymmetric connection · Instability · Limit cycle · Chaotic attractor

1 Introduction

The Hopfield neural network (HNN) is a form of recurrent artificial neural network, which appears to be a powerful tool for memory retrieval. The dynamic behaviors of HNNs have been intensively studied in [1–14]. However, most of these researches focus on the local or global stability analysis of the equilibrium point, which only make up a small fraction of its dynamic behaviors. Generally, HNNs are complex nonlinear systems which inherently display rich dynamic behaviors. Some instability results of the HNNs have been introduced in [6, 8].

Studies of biological neural networks show that the brain exhibits some chaotic dynamics, and it is believed that chaos plays an important role for information creation and storage in biological neural networks [15–17]. Many artificial neural networks have been proposed to simulate chaotic behavior of the brain. Recently, studies of periodic and chaotic motions in HNNs have been an active area of research [18–26]. Researches on limit cycles and chaotic attractors existing in unstable HNNs can narrow the gap between the artificial neural networks and natural neuronal systems. Dynamic analysis of unstable HNNs can give us a comprehensive understanding to its dynamic behavior and help us finding more chaotic neural networks.

From this point of view, in this paper, the dynamic behaviors of unstable HNNs with asymmetric connections are studied. It is found that the solution of the HNN is bounded and the HNN is a dissipative system. In addition, sufficient conditions for the instability of the equilibrium points and the existence of stable limit cycles are proposed. Moreover, the symmetry properties of HNNs are discussed briefly. In simulations, limit cycles and chaotic attractors existing in HNNs are proposed. It is shown that some HNNs display two independent limit cycles or chaotic attractors

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which are symmetric to each other with respect to the origin.

2 Dynamic behavior analysis

Consider the HNN of the form

$$\dot{U} = -AU + Wg(U) + \Theta, \tag{1}$$

where $U = [u_1, u_2, \ldots, u_n]^T \in \mathbb{R}^n$ denotes the neuron state vector, $\Theta = [\theta_1, \theta_2, \ldots, \theta_n]^T \in \mathbb{R}^n$ is a real constant vector, $A = \mathrm{diag}[a_1, a_2, \ldots, a_n] \in \mathbb{R}^{n\times n}$ is a positive definite diagonal matrix, $W = [w_{ij}] \in \mathbb{R}^{n\times n}$ is a synaptic weights matrix, g: $\mathbb{R}^n \to \mathbb{R}^n$ denotes the neuron activation function with $g(U) = [g_1(u_1), g_2(u_2), \ldots, g_n(u_n)]^T$. $g_i(\cdot)$ $(i = 1, 2, \ldots, n)$ are nonlinear differentiable functions satisfying the following conditions

$$|g_i(x)| \le M,$$
$$\text{for some } M \in \mathbb{R}^+,\ \forall x \in \mathbb{R},\ i = 1, 2, \ldots, n, \tag{2}$$

$$\dot{g}_i(x) > 0, \quad \forall x \in \mathbb{R},\ i = 1, 2, \ldots, n. \tag{3}$$

Note that Conditions (2) and (3) coincide with the traditional HNNs (e.g., [2–6]). In addition, the synaptic weights matrix W can be symmetrically or asymmetrically defined. Unless otherwise stated, the following results hold for both symmetrically and asymmetrically connected HNNs.

Proposition 1 *The solution of Network (1) is bounded, and Network (1) is a dissipative system.*

Proof Let $U^* = [u_1^*, u_2^*, \ldots, u_n^*]^T$ be an equilibrium point of Network (1). Define a continuously differentiable energy function of Network (1) as

$$V(Y) = \sum_{i=1}^{n} \int_0^{y_i} \varphi_i(s)\,\mathrm{d}s, \tag{4}$$

where $Y = U - U^* = [y_1, y_2, \ldots, y_n]^T$, and

$$\varphi_i(y_i) = g_i(y_i + u_i^*) - g_i(u_i^*), \quad i = 1, 2, \ldots, n. \tag{5}$$

Hence

$$\dot{Y} = -AY + W\varphi(Y), \tag{6}$$

where $\varphi(Y) = [\varphi_1(y_1), \varphi_2(y_2), \ldots, \varphi_n(y_n)]^T$.

Denote $Y^* = [y_1^*, y_2^*, \ldots, y_n^*]^T = 0$. It follows from (3) and (5) that $\varphi_i(\cdot)$ $(i = 1, 2, \ldots, n)$ are monotonically increasing functions, and $\varphi_i(y) = 0$ if and only if $y = 0$. Then

$$\int_0^{y_i} \varphi_i(s)\,\mathrm{d}s \ge 0, \quad i = 1, 2, \ldots, n.$$

Thus, $V(Y)$ is lower bounded with $V(Y) \ge 0$, and $V(Y) = 0$ if and only if $Y = Y^* = 0$. Therefore, Y^* is the global minimum of $V(Y)$. Furthermore, if $Y = Y^* = 0$ then $U = U^*$, i.e., Y^* corresponds to the equilibrium point U^* of Network (1).

Let $H = \frac{1}{2}(W + W^T)$. Then H is a symmetric matrix, and for any $Y \in \mathbb{R}^n$, it holds

$$Y^T WY + \left(Y^T WY\right)^T = 2Y^T WY,$$
$$Y^T WY + Y^T W^T Y = 2Y^T HY,$$

then

$$Y^T HY = Y^T WY. \tag{7}$$

It follows from (4), (6) and (7) that the derivative of $V(Y)$ along the trajectory of the network can be described as

$$\begin{aligned}\dot{V}(Y) &= \varphi^T(Y)\dot{Y} \\ &= \varphi^T(Y)[-AY + W\varphi(Y)] \\ &= -\varphi^T(Y)AY + \varphi^T(Y)W\varphi(Y) \\ &= -\varphi^T(Y)AY + \varphi^T(Y)H\varphi(Y). \end{aligned} \tag{8}$$

It follows from (2) and (5) that

$$|\varphi_i(y)| \le |g_i(y + u_i^*)| + |g_i(u_i^*)| \le 2M,$$
$$\forall y \in \mathbb{R},\ i = 1, 2, \ldots, n,$$

and

$$\varphi_i(y)y \ge 0, \quad \forall y \in \mathbb{R},\ i = 1, 2, \ldots, n.$$

Define $B_\delta(Y) = \{Y : \|Y\| < \delta\}$, and $B_\delta^c(Y) = \mathbb{R}^n \backslash B_\delta(Y)$. There always exists a δ which is large enough, such that for any $Y \in B_\delta^c(Y)$

$$|y_i| > 2M\lambda_{\max}(H)/a_i, \quad i = 1, 2, \ldots, n.$$

Thus

$$a_i\varphi_i(y_i)y_i > \lambda_{\max}(H)\varphi_i(y_i)\varphi_i(y_i), \quad i = 1, 2, \ldots, n.$$

It follows that

$$\begin{aligned}
&\varphi^T(Y)AY \\
&\quad = \sum_{i=1}^{n} a_i\varphi_i(y_i)y_i > \lambda_{\max}(H)\sum_{i=1}^{n}\varphi_i(y_i)\varphi_i(y_i) \\
&\quad \geq \varphi^T(Y)H\varphi(Y).
\end{aligned} \tag{9}$$

It follows from (8) and (9) that $\dot{V}(Y) < 0$ for any $Y \in B_\delta^c(Y)$. Moreover, if δ is large enough, then there is no equilibrium point in $B_\delta^c(Y)$ (see the Appendix). Therefore, given an initial state $Y(0) \in B_\delta^c(Y)$, $V(Y)$ dissipates monotonically, and the phase space volume contracts along the trajectory of Network (1). The solution of the network trends to converge into $B_\delta(Y)$ rather than stays in $B_\delta^c(Y)$. Hence, the solution of Network (1) is bounded, and Network (1) is a dissipative system. □

Limit cycles and chaotic attractors usually emerge in HNNs which have no stable equilibrium point. In the following, some sufficient conditions for the instability of the equilibrium points are proposed in Proposition 2.

Proposition 2 *If* $H - AD(U^*) > 0$, *where* $D(U) = \mathrm{diag}[1/\dot{g}_1(u_1), 1/\dot{g}_2(u_2), \ldots, 1/\dot{g}_n(u_n)]$ *and* $H = \frac{1}{2}(W + W^T)$, *then* U^* *is an unstable equilibrium point of Network* (1).

Proof It follows from (5) that

$$\dot{\varphi}_i(y) = \dot{g}_i(y + u_i^*), \quad \forall y \in \mathbb{R}, i = 1, 2, \ldots, n.$$

Denote $Y^* = [y_1^*, y_2^*, \ldots, y_n^*]^T = 0$. Hence

$$\dot{\varphi}_i(y_i^*) = \dot{\varphi}_i(0) = \dot{g}_i(u_i^*), \quad i = 1, 2, \ldots, n.$$

Then, the linearization of $\varphi(Y)$ at Y^* is

$$Y = D(U^*)\varphi(Y). \tag{10}$$

It follows from (8) and (10) that

$$\begin{aligned}
\dot{V}(Y) &= -\varphi^T(Y)AY + \varphi^T(Y)H\varphi(Y) \\
&= \varphi^T(Y)(H - AD(U^*))\varphi(Y).
\end{aligned} \tag{11}$$

In a certain neighborhood of Y^*, if $H - AD(U^*) > 0$ then $\dot{V}(Y) > 0$. Given an initial state which is sufficiently close to Y^*, $V(Y)$ increases monotonically, which keeps the network trajectory away from Y^*. In addition, Y^* corresponds to the equilibrium point U^* of Network (1). According to Lyapunov stability criterion, U^* is an unstable equilibrium point. □

Note that Proposition 2 gives sufficient conditions for identifying the unstable equilibrium point. Moreover, it is worth mentioning that some instability results of the asymmetric HNNs have been introduced in [8]. It can be verified that Proposition 2 is less restrictive than the results in [8] (see details in Example 3.1 in Sect. 3). In addition, some instability results of the symmetric HNNs have been proposed in [6]. In the following corollary, a more concise result, which follows directly from the Proposition 2, is proposed.

Corollary 1 *If* W *is a symmetric matrix and* $W - AD(U^*) > 0$, *where* $D(U) = \mathrm{diag}[1/\dot{g}_1(u_1), 1/\dot{g}_2(u_2), \ldots, 1/\dot{g}_n(u_n)]$, *then* U^* *is an unstable equilibrium point*.

Remark 1 As illustrated in Proposition 1, Network (1) is a dissipative system. Hence, if Network (1) has no stable equilibrium point, then it may display periodic, quasi-periodic or chaotic motions.

In the following, by virtue of Propositions 1, sufficient conditions for the existence of stable limit cycles in HNNs are proposed.

Proposition 3 *If* $U \in \mathbb{R}^2$ *and Network* (1) *has no stable equilibrium point. Then Network* (1) *at least has one stable limit cycle.*

Proof As illustrated in Proposition 1, there always exists a region $B \subset \mathbb{R}^2$ out of which all trajectories of Network (1) passing through the boundary of the region point inwards. Moreover, Network (1) has no stable equilibrium point. Therefore, according to the Poincare–Bendixson Theorem, there exists at least one stable limit cycle. □

The HNNs proposed in [22–26] have a common property, namely, $\Theta = 0$. Network (1) appears to be a symmetric system when $\Theta = 0$. In the following remark, the symmetry properties of HNNs are discussed briefly.

Remark 2 If $\Theta = 0$, then Network (1) has a natural symmetry under the coordinate transform $U = -U$, i.e., under a reflection in the origin. Hence, if $\Theta = 0$ and Network (1) exhibits a limit cycle (or chaotic attractor), then the limit cycle (or chaotic attractor) is symmetric with respect to the origin or there exists another independent limit cycle (or chaotic attractor) under a reflection in the origin.

It should be mentioned that some limit cycles and chaotic attractors of the HNN have been introduced in [26], which are not symmetric about the origin (see Fig. 3 in [26]). Then by virtue of Remark 2, we can reflect the limit cycle (or chaotic attractor) through the origin, and find another one (see details in Example 3.3 in Sect. 3).

3 Numerical simulations

3.1 Example 1

We illustrate the effectiveness of Propositions 2 and 3 in this example.

Consider Network (1) with $n = 2$, $a_1 = a_2 = 1$, $\theta_1 = -1.12$, $\theta_2 = 0.02$, $g_1(\cdot) = g_2(\cdot) = \tanh(\cdot)$, and

$$W = \begin{bmatrix} 2 & 2 \\ -2 & 3 \end{bmatrix}.$$

The evidence of the uniqueness of the equilibrium point is given as follows. Consider the system of nonlinear equations

$$-u_1 + 2\tanh(u_1) + 2\tanh(u_2) - 1.12 = 0, \tag{12}$$

$$-u_2 - 2\tanh(u_1) + 3\tanh(u_2) + 0.02 = 0. \tag{13}$$

Add the two equations (12) and (13). We get

$$u_1 = -u_2 + 5\tanh(u_2) - 1.1. \tag{14}$$

Substituting u_1 by (14) in (13) yields

$$-u_2 - 2\tanh(-u_2 + 5\tanh(u_2) - 1.1) + 3\tanh(u_2) + 0.02 = 0.$$

Let $f(u_2) = -u_2 - 2\tanh(-u_2 + 5\tanh(u_2) - 1.1) + 3\tanh(u_2) + 0.02$. Figure 1 illustrates the graph of $f(u_2)$. It can be verified that $U^* = [0.4, 0.4]^T$ is an equilibrium point. Moreover, as shown in Fig. 1, $f(u_2) = 0$ if and only if $u_2 = 0.4$. Hence, $U^* = [0.4, 0.4]^T$ is the unique equilibrium point of the defined network.

It is calculated that $D(U^*) = \mathrm{diag}[1.17, 1.17]$, then it is shown that $H - AD(U^*) > 0$. Hence, by Proposition 2, U^* is unstable. As illustrated in Fig. 2, given an initial state $U_1(0) = [0.5, 0.5]^T$ which is very close to U^*, but as the network dynamic evolves with time, the solution keeps away from U^*. This confirms that U^* is unstable. However, the defined network does not satisfy the instability conditions of the Theorem 3 in [8]. Thus, Proposition 2 is less restrictive than the results in [8]. This illustrates the effectiveness of Proposition 2.

It follows from Proposition 3 that the defined network has at least one stable limit cycle. Figure 2 shows the limit cycle of the defined network. Given another two initial states $U_2(0) = [-3, 4]^T$ and $U_3(0) = [3, -4]^T$, as illustrated in Fig. 2, the trajectories of the network converge to the limit cycle as the network dynamic evolves with time, which implies the limit cycle is stable. This illustrates the effectiveness of the Proposition 3.

3.2 Example 2

We illustrate two independent stable limit cycles coexisting in a 3-neuron HNN in this example.

Consider Network (1) with $n = 3$, $a_1 = a_2 = a_3 = 1$, $\theta_1 = \theta_2 = \theta_3 = 0$, $g_1(\cdot) = g_2(\cdot) = g_3(\cdot) = \arctan(\cdot)$, and

$$W = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 2 & 0 \\ -2 & 0 & 2.1 \end{bmatrix}.$$

It is obvious that $U^* = [0, 0, 0]^T$ is an equilibrium point of the defined network, and $D(U^*) = \mathrm{diag}[1, 1, 1]$, then $\lambda_{\min}(H - AD(U^*)) = 0.03 > 0$, which implies $H - AD(U^*) > 0$. Hence, by Proposition 2, U^* is unstable. Given an initial state $U_1(0) = [0.1, -0.1, 0.1]^T$, as shown in Fig. 3, $U_1(0)$ is very close to U^*, but as the network dynamic evolves with time, the state trajectory retreats from U^*, and it finally converges to a stable limit cycle (blue). This illustrates the effectiveness of Proposition 2.

It is obvious that, the blue limit cycle is not symmetric with respect to the origin. Then, according to Remark 2, there exists another limit cycle under a reflection in the origin. Given another initial state $U_2(0) = [-0.1, 0.1, -0.1]^T$, as shown in Fig. 3, the

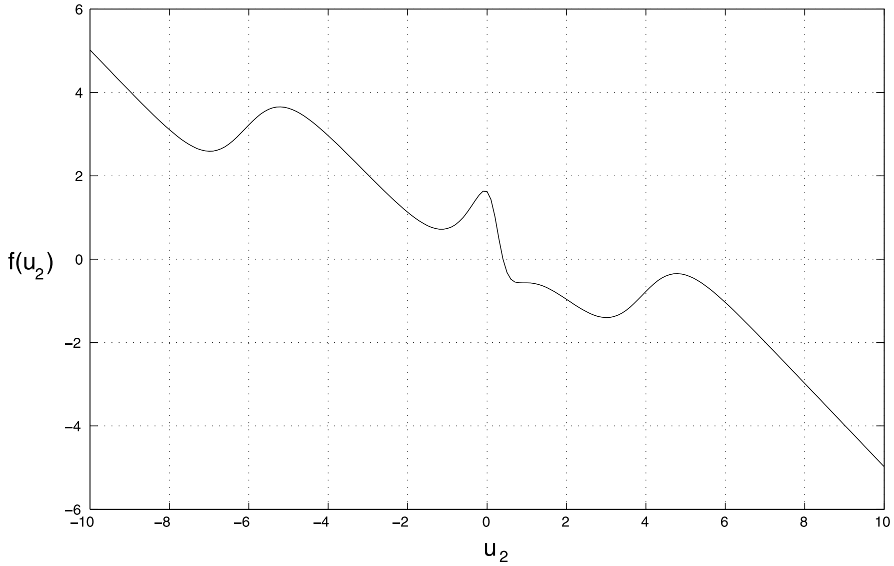


Fig. 1 The graph of $f(u_2)$

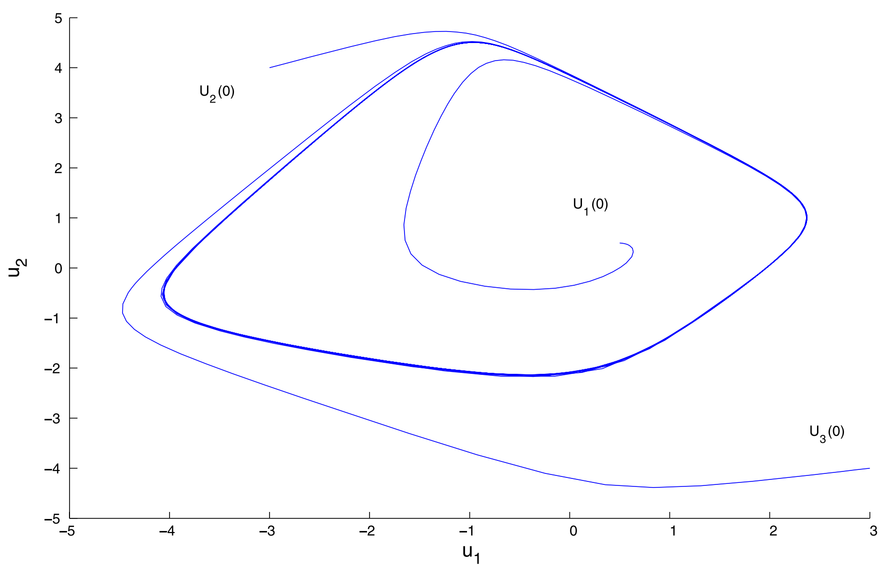


Fig. 2 Stable limit cycle of the defined network in Example 1

network finally converges to another independent stable limit cycle (red), and two limit cycles are symmetric to each other with respect to the origin. This illustrates the effectiveness of the analysis in Remark 2.

3.3 Example 3

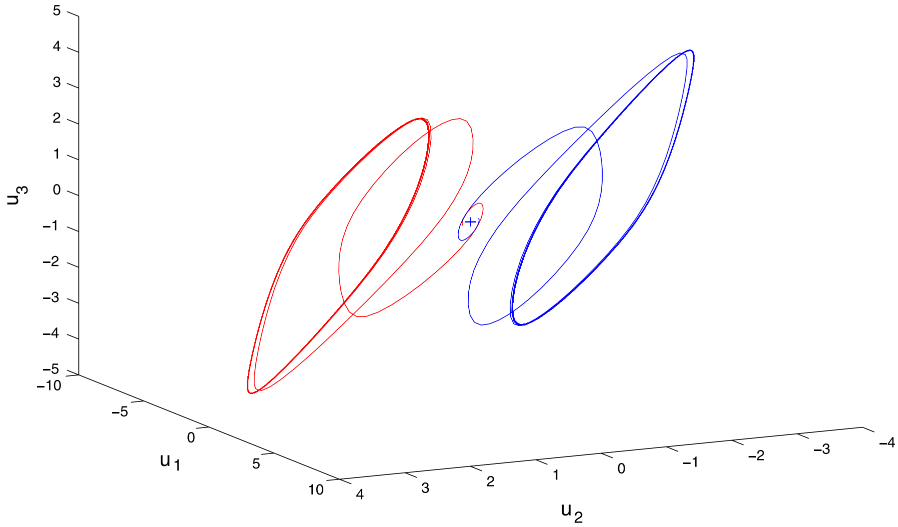
We illustrate the two independent chaotic attractors of a 3-neuron HNN in this example.

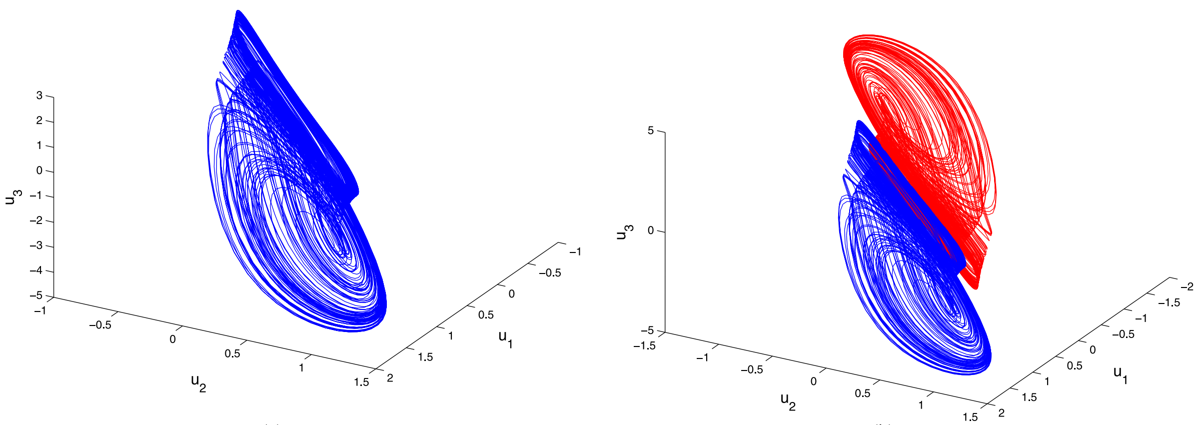
Consider Network (1) with $n = 3$, $a_1 = a_2 = a_3 = 1$, $\theta_1 = \theta_2 = \theta_3 = 0$, $g_1(\cdot) = g_2(\cdot) = g_3(\cdot) = \tanh(\cdot)$, and

$$W = \begin{bmatrix} 3.8 & -1.9 & 0.7 \\ 2.5 & 0.06 & 1 \\ -6.6 & 1.3 & 0.07 \end{bmatrix}.$$

Given an initial state $U_1(0) = [0.1, -0.1, 0.1]^T$ (blue), then the network dynamic evolves with time. Figure 4(a) shows the state trajectory which appears to be a chaotic attractor. We can obtain the Lyapunov exponents as 0.09, 0 and −0.52, respectively, which

Fig. 3 Limit cycles of the defined network in Example 2 (‘+’ denotes the origin)





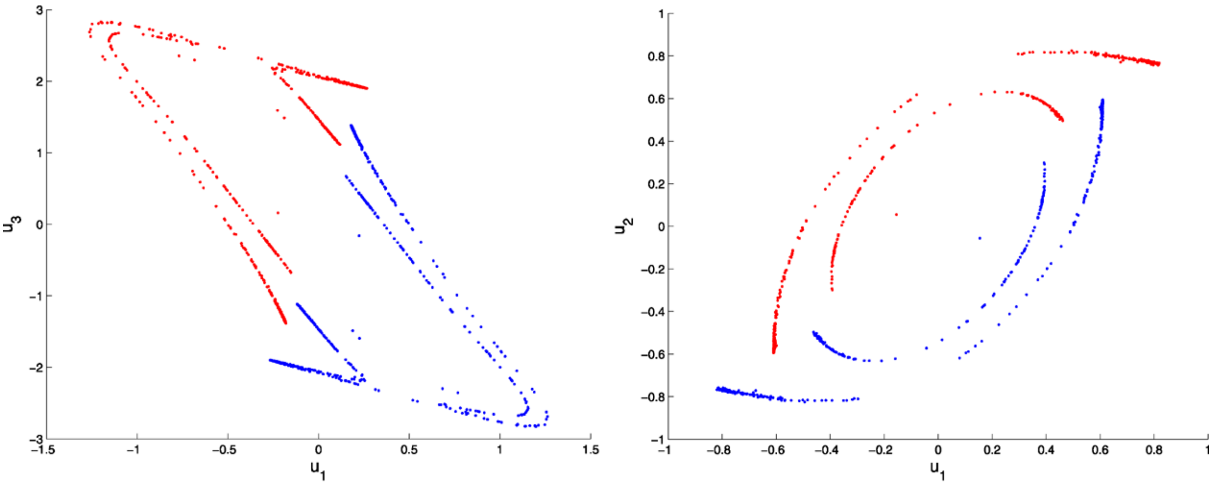
(a) (b)

Fig. 4 The chaotic attractors in Example 3

confirm that the motion displayed in Fig. 4(a) is indeed a chaotic attractor. It is obvious that the blue chaotic attractor is not symmetric about the origin. Thus, according to Remark 2, there exists another independent chaotic attractor under a reflection in the origin. Given the network another initial state $U_2(0) = [-0.1, 0.1, -0.1]^T$, as illustrated in Fig. 4(b), the network finally converges to the red chaotic attractor. Figure 5 displays the Poincaré sections of the chaotic attractors shown in Fig. 4(b). As shown in Fig. 5, the two chaotic attractors are independent and symmetric to each other with respect to the origin. Moreover, as shown in Examples 3.1, 3.2 and 3.3, all the solutions are bounded as the network dynamic evolves with time, which illustrates the effectiveness of Proposition 1.

4 Conclusion

In this paper, we focus on the dynamic behaviors of HNNs which have no stable equilibrium points, and several fundamental results have been obtained. Some limit cycles and chaotic attractors existing in HNNs are proposed. In numerical simulations, some HNNs exhibit two independent limit cycles or chaotic attractors which are symmetric to each other with respect to the origin.



(a) Cross-section $u_2 = 0$ (b) Cross-section $u_3 = 0$

Fig. 5 The Poincaré sections of the chaotic attractors shown in Fig. 4(b)

Appendix

Define $B_k(U) = \{U : \|U\| \leq k\}$, $B_k^c(U) = \mathbb{R}^n \backslash B_k(U)$. If k is large enough, then Network (1) has no equilibrium point in $B_k^c(U)$.

Proof It is obvious that

$$Wg(U^*) + \Theta = AU^*. \tag{15}$$

It follows from (2) that

$$\sum_{j=1}^{n} w_{ij} g_j(u_j) + \theta_i \leq M \sum_{j=1}^{n} |w_{ij}| + |\theta_i|, \quad i = 1, 2, \ldots, n. \tag{16}$$

Let

$$\rho_i = M \sum_{j=1}^{n} |w_{ij}|/a_i + |\theta_i|/a_i, \; i = 1, 2, \ldots, n.$$

If $k \geq \rho_i$ (for some $i \in \{1, 2, \ldots, n\}$), then for any $U \in B_k^c(U)$

$$M \sum_{j=1}^{n} |w_{ij}| + |\theta_i| < a_i |u_i|.$$

Hence, there is no $U \in B_k^c(U)$ satisfying Condition (15). Thus Network (1) has no equilibrium point in $B_k^c(U)$. $\square$

References

1. Hopfield, J.J.: Neural networks and physical systems with emergent collective computational abilities. Proc. Natl. Acad. Sci. USA **79**, 2554–2558 (1982)
2. Hopfield, J.J.: Neurons with graded response have collective computational properties like those of two-state neurons. Proc. Natl. Acad. Sci. USA **81**, 3088–3092 (1984)
3. Kelly, D.G.: Stability in contractive nonlinear neural networks. IEEE Trans. Biomed. Eng. **37**(3), 231–242 (1990)
4. Matsuoka, K.: Stability conditions for nonlinear continuous neural networks with asymmetric connection weights. Neural Netw. **5**, 495–500 (1992)
5. Forti, M., Manetti, S., Marini, M.: Necessary and sufficient condition for absolute stability of neural networks. IEEE Trans. Circuits Syst. **41**(7), 491–494 (1994)
6. Yang, H., Dillon, T.S.: Exponential stability and oscillation of Hopfield graded response neural network. IEEE Trans. Neural Netw. **5**, 719–729 (1994)
7. Zhang, Y., Heng, P.A., Fu, A.W.C.: Estimate of exponential convergence rate and exponential stability for neural networks. IEEE Trans. Neural Netw. **10**(6), 1487–1493 (1999)
8. Guan, Z., Chen, G., Qin, Y.: On equilibria stability, and instability of Hopfield neural networks. IEEE Trans. Neural Netw. **11**(2), 534–540 (2000)
9. Liang, X.B., Si, J.: Global exponential stability of neural networks with globally Lipschitz continuous activations and its application to linear variational inequality problem. IEEE Trans. Neural Netw. **12**(2), 349–359 (2001)
10. Qiao, H., Peng, J., Xu, Z.B.: Nonlinear measures: a new approach to exponential stability analysis for Hopfield-type neural networks. IEEE Trans. Neural Netw. **12**(2), 360–370 (2001)
11. Chen, T.P., Amari, S.I.: Stability of asymmetric Hopfield networks. IEEE Trans. Neural Netw. **12**, 159–163 (2001)
12. Liu, X., Chen, T.P.: A new result on the global convergence of Hopfield neural networks. IEEE Trans. Circuits Syst. **49**(10), 1514–1516 (2002)

13. Qiao, H., Peng, J., Xu, Z.B., Zhang, B.: A reference model approach to stability analysis of neural networks. IEEE Trans. Syst. Man Cybern. **33**(6), 925–936 (2003)
14. Peng, J., Xu, Z.B., Qiao, H., Zhang, B.: A critical analysis on global convergence of Hopfield-type neural networks. IEEE Trans. Circuits Syst. **52**(4), 804–814 (2005)
15. Guevara, M.R., Glass, L., Mackey, M.C., Shrier, A.: Chaos in neurobiology. IEEE Trans. Syst. Man Cybern. **13**, 790–798 (1983)
16. Babloyantz, A., Lourenco, C.: Brain chaos and computation. Int. J. Neural Syst. **7**, 461–471 (1996)
17. Freeman, W.J.: The physiology of perception. Sci. Am. **264**, 78–85 (1991)
18. Zou, F., Nossek, J.A.: Bifurcation and chaos in cellular neural networks. IEEE Trans. Circuits Syst. **40**, 166–173 (1993)
19. Bersini, H.: The frustrated and compositional nature of chaos in small Hopfield networks. Neural Netw. **11**, 1017–1025 (1998)
20. Ruiz, A., Owens, D.H., Townley, S.: Existence, learning, and replication of periodic motions in recurrent neural networks. IEEE Trans. Neural Netw. **9**(4), 651–661 (1998)
21. Bersini, H., Sener, P.: The connections between the frustrated chaos and the intermittency chaos in small Hopfield networks. Neural Netw. **15**, 1197–1204 (2002)
22. Li, Q.D., Yang, X.S., Yang, F.Y.: Hyperchaos in Hopfield-type neural networks. Neurocomputing **67**, 275–280 (2005)
23. Yang, X.S., Yuan, Q.: Chaos and transient chaos in simple Hopfield neural networks. Neurocomputing **69**, 232–241 (2005)
24. Huang, Y., Yang, X.S.: Hyperchaos and bifurcation in a new class of four-dimensional Hopfield neural networks. Neurocomputing **69**, 1787–1795 (2006)
25. Yang, X.S., Huang, Y.: Complex dynamics in simple Hopfield neural networks. Chaos **16**, 033114 (2006)
26. Huang, W.Z., Huang, Y.: Chaos of a new class of Hopfield neural networks. Appl. Math. C **206**, 1–11 (2008)