HOMEWORK #4: Unilateral Forward Laplace Transforms (SOLUTION)

1. Unilateral Forward Laplace Transforms

(a) Compute by hand the unilateral Laplace Transform of each of the following time functions. Verify each of your answers using MATLAB and the Symbolic Toolbox.

i. $f(t) = [5 + 2t^5 + 6e^{-3t}]u(t)$

$$\boldsymbol{F}(\boldsymbol{s}) = \mathcal{L}\big[[5 + 2t^5 + 6e^{-3t}]u(t)\big] = 5\mathcal{L}[u(t)] + 2\mathcal{L}[t^5 u(t)] + 6\mathcal{L}[e^{-3t}u(t)]$$

$$\boldsymbol{F}(\boldsymbol{s}) = 5\frac{1}{\boldsymbol{s}} + 2\frac{5!}{\boldsymbol{s}^{5+1}} + 6\frac{1}{\boldsymbol{s}+3} = \frac{5}{\boldsymbol{s}} + \frac{240}{\boldsymbol{s}^6} + \frac{6}{\boldsymbol{s}+3} = \frac{5(\boldsymbol{s}^5)(\boldsymbol{s}+3) + 240(\boldsymbol{s}+3) + 6(\boldsymbol{s}^6)}{\boldsymbol{s}^6(\boldsymbol{s}+3)}$$

$$\boxed{\boldsymbol{F}(\boldsymbol{s}) = \frac{11\boldsymbol{s}^6 + 15\boldsymbol{s}^5 + 240\boldsymbol{s} + 720}{\boldsymbol{s}^7 + 3\boldsymbol{s}^6}}$$

ii. $f(t) = [4\cos(t) + 7\sin(\sqrt{5}t)]u(t)$

$$\boldsymbol{F}(\boldsymbol{s}) = \mathcal{L}\left[[4\cos(t) + 7\sin(\sqrt{5}t)]u(t)\right] = 4\mathcal{L}[\cos(t)\,u(t)] + 7\mathcal{L}[\sin(\sqrt{5}t)\,u(t)]$$

$$\boldsymbol{F}(\boldsymbol{s}) = 4\frac{\boldsymbol{s}}{\boldsymbol{s}^2 + (1)^2} + 7\frac{\sqrt{5}}{\boldsymbol{s}^2 + (\sqrt{5})^2} = \frac{4\boldsymbol{s}}{\boldsymbol{s}^2+1} + \frac{7\sqrt{5}}{\boldsymbol{s}^2+5} = \frac{4\boldsymbol{s}(\boldsymbol{s}^2+5) + 7\sqrt{5}(\boldsymbol{s}^2+1)}{(\boldsymbol{s}^2+1)(\boldsymbol{s}^2+5)}$$

$$\boxed{\boldsymbol{F}(\boldsymbol{s}) = \frac{4\boldsymbol{s}^3 + 7\sqrt{5}\boldsymbol{s}^2 + 20\boldsymbol{s} + 7\sqrt{5}}{s^4 + 6\boldsymbol{s}^2 + 5}}$$

iii. $f(t) = \sqrt{t-1}\,r(t-3)u(t-2)\delta(t-5)$

Using the sampling property of the Dirac Delta/Impulse, one can simplify the above as follows

$$f(t) = \sqrt{(5)-1}\,r((5)-3)u((5)-2)\delta(t-5)$$

$$f(t) = \sqrt{4}\,r(2)u(3)\delta(t-5) = (2)(2)(1)\delta(t-5) = 4\delta(t-5)$$

$$\boldsymbol{F}(\boldsymbol{s}) = \mathcal{L}[4\delta(t-5)] = 4\mathcal{L}[\delta(t-5)]$$

Using the time shift property,

$$f(t) = 4e^{-5s}\mathcal{L}[\delta(t)] = 4e^{-5s}(1)$$

$$\boxed{\boldsymbol{F}(\boldsymbol{s}) = 4e^{-5s}}$$

iv. $f(t) = 3t^4 e^{-10t}u(t)$

$\boldsymbol{F}(\boldsymbol{s}) = \mathcal{L}[3t^4 e^{-10t}u(t)] = 3\mathcal{L}[t^4 e^{-10t}u(t)]$. Using the freq. shift/multiply-by-exp() property, $\boldsymbol{F}(\boldsymbol{s}) = 3\mathcal{L}[t^4 e^{-10t}u(t)] = 3\mathcal{L}[t^4 u(t)]|_{\boldsymbol{s}=\boldsymbol{s}+10} = 3\left(\frac{4!}{\boldsymbol{s}^{4+1}}\right)|_{\boldsymbol{s}=\boldsymbol{s}+10} = \left(\frac{72}{\boldsymbol{s}^5}\right)|_{\boldsymbol{s}=\boldsymbol{s}+10}$

$$\boxed{\boldsymbol{F}(\boldsymbol{s}) = \frac{72}{(\boldsymbol{s}+10)^5} = \frac{72}{\boldsymbol{s}^5 + 50\boldsymbol{s}^4 + 1000\boldsymbol{s}^3 + 10000\boldsymbol{s}^2 + 50{,}000\boldsymbol{s} + 100000}}$$

v. $f(t) = e^{-2t}\, cos(3t)\, u(t)$

$\boldsymbol{F(s)} = \mathcal{L}\big[[e^{-2t}\, cos(3t)]u(t)\big]$. Using the frequency-shift/multiply-by-exponential property,

$$\boldsymbol{F(s)} = \mathcal{L}[\cos(3t)\, u(t)]|_{s=s+2} = \left(\frac{\boldsymbol{s}}{\boldsymbol{s}^2 + (3)^2}\right)|_{s=s+2} = \left(\frac{\boldsymbol{s}}{\boldsymbol{s}^2 + (3)^2}\right)|_{s=s+2} = \frac{\boldsymbol{s}+2}{(\boldsymbol{s}+2)^2 + 9}$$

$$\boxed{\boldsymbol{F(s)} = \frac{\boldsymbol{s}+2}{\boldsymbol{s}^2 + 4\boldsymbol{s} + 13}}$$

vi. $f(t) = 2e^{-2(t-3)}u(t)$

$f(t) = 2[e^{-2t+6}]u(t) = 2e^6 e^{-2t}u(t)$

$$\boldsymbol{F(s)} = \mathcal{L}[2e^6 e^{-2t}u(t)] = 2e^6\mathcal{L}[e^{-2t}u(t)] = 2e^6\frac{1}{\boldsymbol{s}+2} \rightarrow \boxed{\boldsymbol{F(s)} = \frac{2e^6}{\boldsymbol{s}+2}}$$

vii. $f(t) = 2tu(t-4)$

Express $f(t)$ in a form where all functions of t are shifted (time delayed) by 4 since the step function is shifted by 4. Doing so will allow for the proper application of the time shift property.

$f(t) = 2(t-4+4)u(t-4) = 2(t-4)u(t-4) + 8u(t-4)$

$\boldsymbol{F(s)} = \mathcal{L}[2(t-4)u(t-4) + 8u(t-4)] = 2\mathcal{L}[(t-4)u(t-4)] + 8\mathcal{L}[u(t-4)]$

Now apply the time shifting property

$\boldsymbol{F(s)} = 2\mathcal{L}[tu(t)]e^{-4s} + 8\mathcal{L}[u(t)]e^{-4s} = e^{-4\boldsymbol{s}}(2\mathcal{L}[tu(t)] + 8\mathcal{L}[u(t)])$

$$\boldsymbol{F(s)} = e^{-4s}\left[2\frac{1}{\boldsymbol{s}^2} + 8\frac{1}{\boldsymbol{s}}\right] = \frac{2e^{-4s}}{\boldsymbol{s}^2} + \frac{8e^{-4s}}{\boldsymbol{s}} = \frac{2(4\boldsymbol{s}+1)}{\boldsymbol{s}^2}e^{-4s} \rightarrow \boxed{\boldsymbol{F(s)} = \frac{8\boldsymbol{s}+2}{\boldsymbol{s}^2}e^{-4s}}$$

viii. $f(t) = 7(t-4)u(t-2)$

Express $f(t)$ in a form where all functions of t are shifted (time delayed) by 2 since the step function is shifted by 2. Doing so will allow for the proper application of the time shift property.

$f(t) = 7(t-4)u(t-2) = 7(t-4+2-2)u(t-2) = 7(t-2)u(t-2) - 14u(t-2)$

$\boldsymbol{F(s)} = \mathcal{L}[7(t-2)u(t-2) - 14u(t-2)] = 7\mathcal{L}[(t-2)u(t-2)] - 14\mathcal{L}[u(t-2)]$

Now apply the time shifting property.

$\boldsymbol{F(s)} = 7\mathcal{L}[tu(t)]e^{-2s} - 14\mathcal{L}[u(t)]e^{-2s} = e^{-2\boldsymbol{s}}(7\mathcal{L}[tu(t)] - 14\mathcal{L}[u(t)])$

$$\boldsymbol{F(s)} = e^{-2s}\left[7\frac{1}{\boldsymbol{s}^2} - 14\frac{1}{\boldsymbol{s}}\right] = \frac{7e^{-2s}}{\boldsymbol{s}^2} - \frac{14e^{-2s}}{\boldsymbol{s}} = \frac{7(1-2\boldsymbol{s})}{\boldsymbol{s}^2}e^{-2s} \rightarrow \boxed{\boldsymbol{F(s)} = \frac{7-14\boldsymbol{s}}{\boldsymbol{s}^2}e^{-2s}}$$

ix. $f(t) = 2e^{-4t}u(t-1)$

Express $f(t)$ in a form where all functions of t are shifted (time delayed) by 1 since the step function is shifted by 1. Doing so will allow for the proper application of the time shift property.

$f(t) = 2e^{-4(t-1+1)}u(t-1) = 2e^{-4(t-1)-4}u(t-1) = 2e^{-4}e^{-4(t-1)}u(t-1)$

$\boldsymbol{F(s)} = \mathcal{L}\big[2e^{-4}e^{-4(t-1)}u(t-1)\big] = 2e^{-4}\mathcal{L}\big[e^{-4(t-1)}u(t-1)\big]$

Now apply the time shifting property,

$$\boldsymbol{F(s)} = 2e^{-4}\mathcal{L}\left[e^{-4(t-1)}u(t-1)\right] = 2e^{-4}\mathcal{L}[e^{-4t}u(t)]e^{-s} = 2e^{-(s+4)}\mathcal{L}[e^{-4t}u(t)]$$

$$\boldsymbol{F(s)} = 2e^{-(s+4)}\mathcal{L}[e^{-4t}u(t)] = 2e^{-(s+4)}\frac{1}{s+4} \rightarrow \boxed{\boldsymbol{F(s)} = \frac{2e^{-(s+4)}}{\boldsymbol{s}+4}}$$

x. $f(t) = e^{-2(t+1)}(t+2)u(t-3)$

Express $f(t)$ in a form where all functions of t are shifted (time delayed) by 3 since the step function is shifted by 3. Doing so will allow for the proper application of the time shift property.

$$f(t) = e^{-2(t+1)}(t+2)u(t-3) = e^{-2(t+1-4+4)}(t+2-5+5)u(t-3)$$

$$f(t) = e^{-2(t-3)-8}(t-3+5)u(t-3) = e^{-8}e^{-2(t-3)}(t-3+5)u(t-3)$$

$$f(t) = e^{-8}e^{-2(t-3)}(t-3)u(t-3) + 5e^{-8}e^{-2(t-3)}u(t-3)$$

$$\boldsymbol{F(s)} = \mathcal{L}\left[e^{-8}e^{-2(t-3)}(t-3)u(t-3) + 5e^{-8}e^{-2(t-3)}u(t-3)\right]$$

$$\boldsymbol{F(s)} = e^{-8}\mathcal{L}\left[e^{-2(t-3)}(t-3)u(t-3)\right] + 5e^{-8}\mathcal{L}\left[e^{-2(t-3)}u(t-3)\right]$$

Now apply the time-shift/phase-shift property,

$$\boldsymbol{F(s)} = e^{-8}\mathcal{L}[e^{-2t}tu(t)]e^{-3s} + 5e^{-8}\mathcal{L}[e^{-2t}u(t)]e^{-3s}$$

$$\boldsymbol{F(s)} = e^{-(3s+8)}\mathcal{L}[e^{-2t}tu(t)] + 5e^{-(3s+8)}\mathcal{L}[e^{-2t}u(t)]$$

$$\boldsymbol{F(s)} = e^{-(3s+8)}(\mathcal{L}[e^{-2t}tu(t)] + 5\mathcal{L}[e^{-2t}u(t)])$$

Now apply the frequency-shift/multiply-by-exponential property,

$$\boldsymbol{F(s)} = e^{-(3s+8)}(\mathcal{L}[tu(t)]|_{s=s+2} + 5\mathcal{L}[e^{-2t}u(t)])$$

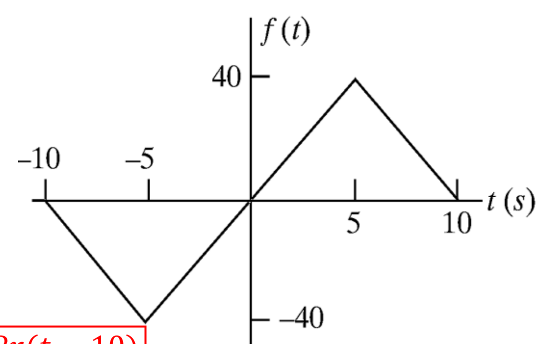
$$\boldsymbol{F(s)} = e^{-(3s+8)}\left(\frac{1}{s^2}|_{s=s+2} + 5\frac{1}{\boldsymbol{s}+2}\right) = e^{-(3s+8)}\left(\frac{1}{(\boldsymbol{s}+2)^2} + \frac{5}{\boldsymbol{s}+2}\right)$$

$$\boldsymbol{F(s)} = e^{-(3s+8)}\left(\frac{1+5(\boldsymbol{s}+2)}{(\boldsymbol{s}+2)^2}\right) = e^{-(3s+8)}\left[\frac{5\boldsymbol{s}+11}{(\boldsymbol{s}+2)^2}\right] \rightarrow \boxed{\boldsymbol{F(s)} = \frac{(5\boldsymbol{s}+11)e^{-(3s+8)}}{\boldsymbol{s}^2+4\boldsymbol{s}+4}}$$

(b) Consider the plot of function $f(t)$ shown below.

i. Express $f(t)$ as a linear combination of singularity functions. Ensure each term of is expressed in such a way that the Laplace Transform time-shift property may be properly applied.

$$\boxed{\begin{aligned} f(t) &= (-8t-80)[u(t+10)-u(t+5)] \\ &\quad +8t[u(t+5)-u(t-5)] \\ &\quad +(-8t+80)[u(t-5)-u(t-10)] \\ f(t) &= -8(t+10)u(t+10)+16(t+5)u(t+5) \\ &\quad -16(t-5)u(t-5)+8(t-10)u(t-10) \end{aligned}}$$



$$\boxed{f(t) = -8r(t+10) + 16r(t+5) - 16r(t-5) + 8r(t-10)}$$

ii. Compute the <u>unilateral Laplace transform</u> of $f(t)$.

Since the unilateral Laplace transform is valid for time $t = [0^-, \infty)$, we need to determine $g(t) = f(t), t > 0^-$. With similar manipulations performed for part (i), one can obtain:

$$\boxed{g(t) = f(t), t > 0^- = 8r(t) - 16r(t-5) + 8r(t-10), t > 0^-}$$

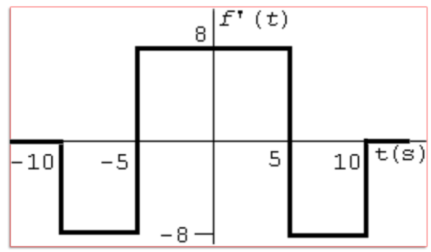
$$\boldsymbol{G(s)} = 8\mathcal{L}[r(t)] - 16\mathcal{L}[r(t-5)] + 8\mathcal{L}[r(t-10)]$$

$$\boldsymbol{G(s)} = 8\mathcal{L}[r(t)] - 16\mathcal{L}[r(t)]e^{-5s} + 8\mathcal{L}[r(t)]e^{-10s}$$

$$\boldsymbol{G(s)} = \frac{8}{s^2} - 16\frac{1}{s^2}e^{-5s} + 8\frac{1}{s^2}e^{-10s} \rightarrow \boxed{\boldsymbol{G(s)} = \frac{8}{s^2}(1 - 2e^{-5s} + e^{-10s})}$$

iii. Compute and sketch a plot of $f'(t)$.

$$\boxed{\begin{aligned} f'(t) &= -8[u(t+10) - u(t+5)] + 8[u(t+5) - u(t-5)] \\ &\quad +(-8)[u(t-5) - u(t-10)] \\ f'(t) &= -8u(t+10) + 16u(t+5) - 16u(t-5) + 8u(t-10) \end{aligned}}$$



iv. Compute the unilateral Laplace transform of $f'(t)$.

Since the unilateral Laplace transform is valid for time $t = [0^-, \infty)$, we need to find $h(t) = f'^{(t)}, t > 0^-$. With similar manipulations performed for part (iii), one can obtain:

$$\boxed{h(t) = f'(t), t > 0^- = 8u(t) - 16u(t-5) + 8u(t-10), t > 0^-}$$

$$\mathcal{L}[h(t)] = \mathcal{L}[8u(t) - 16u(t-5) + 8u(t-10)]$$

$$\mathcal{L}[h(t)] = 8\mathcal{L}[u(t)] - 16\mathcal{L}[u(t-5)] + 8\mathcal{L}[u(t-10)]$$

$$\boldsymbol{H(s)} = \frac{8}{s} - 16\frac{1}{s}e^{-5s} + 8\frac{1}{s}e^{-10s} \rightarrow \boxed{\boldsymbol{H(s)} = \frac{8}{s}(1 - 2e^{-5s} + e^{-10s})}$$

You could have instead used the 1st derivative property to obtain the above solution as follows.

$$\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0^-)$$

$$\mathcal{L}[f'(t)] = s\left[\frac{8}{s^2}(1 - 2e^{-5s} + e^{-10s})\right] - [8r(0^-) - 16r(0^- - 5) + 8r(0^- - 10)]$$

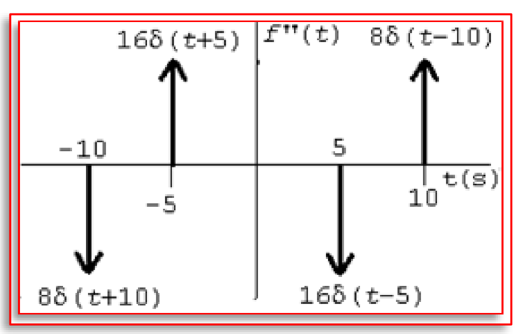
$$\mathcal{L}[f'(t)] = \frac{8}{s}(1 - 2e^{-5s} + e^{-10s}) - [8r(0^-) - 16r(-5) + 8r(-10)]$$

$$\mathcal{L}[f'(t)] = \frac{8}{s}(1 - 2e^{-5s} + e^{-10s}) - [8(0) - 16(0) + 8(0)]$$

$$\boxed{\mathcal{L}[f'(t)] = \frac{8}{s}(1 - 2e^{-5s} + e^{-10s})}$$

v. Compute and sketch a plot of $f''(t)$.

$$\boxed{\begin{aligned} f''(t) = &-8\delta(t+10) + 16\delta(t+5) \\ &- 16\delta(t-5) + 8\delta(t-10) \end{aligned}}$$



vi. Compute the unilateral Laplace transform of $f''(t)$.

Since the unilateral Laplace transform is valid for time $t = [0^-, \infty)$, we need to find $f''(t)$ for $t > 0^-$. Therefore:

$$\boxed{f''(t) = -16\delta(t-5) + 8\delta u(t-10), t > 0^-}$$

$$\mathcal{L}[f''(t)] = \mathcal{L}[-16\delta(t-5) + 8\delta u(t-10)] = -16\mathcal{L}[\delta(t)]e^{-5s} + 8\mathcal{L}[\delta(t)]e^{-10s}$$

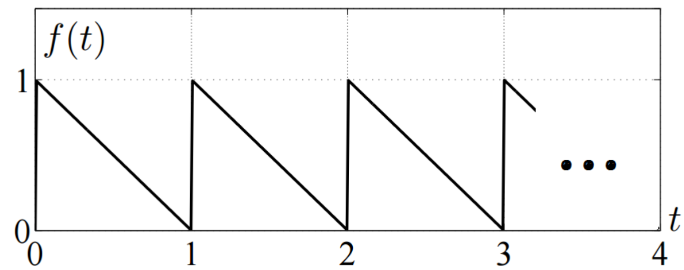
$$\mathcal{L}[f''(t)] = -16(1)e^{-5s} + 8(1)e^{-10s}$$

$$\boxed{\mathcal{L}[f''(t)] = 8(-2e^{-5s} + e^{-10s})}$$

The solution could have also used the derivative property similar to the steps in (iv).

(c) Consider the periodic right-sided time function $f(t)$ shown below.

i. Express $f(t)$ as an infinite sum of singularity functions.



$f(t)$ can be written as follows:

$$f(t) = f_1(t)[u(t) - u(t-T_0)] + f_1(t-T_0)[u(t-T_0) - u(t-2T_0)] + \cdots$$

$$f(t) = f_1(t)u(t) + [f_1(t-T_0) - f_1(t)]u(t-T_0) + [f_1(t-2T_0) - f_1(t-T_0)]u(t-2T_0) + [f_1(t-3T_0) - f_1(t-2T_0)]u(t-3T_0) + \cdots$$

$$\boxed{f(t) = f_1(t)u(t) + \sum_{k=1}^{\infty}[f_1(t-kT_0) - f_1(t-(k-1)T_0)]u(t-kT_0)}$$

where $f_1(t)$ is the periodic function expressed over one period. In this problem, the periodic function has a period of $T_0 = 1$ and $f_1(t)$ can be described as follows:

$$f_1(t) = 1 - \frac{1}{T_0}t \to f_1(t) = 1 - \frac{t}{1} \to \boxed{f_1(t) = 1 - t}$$

Next, compute the $f_1(t-kT_0)$ and $f_1(t-(k-1)T_0)$

$$f_1(t-kT_0) = 1 - (t-kT_0) = 1 - t + kT_0 \to \boxed{f_1(t-kT_0) = 1 + k - t}$$

$$f_1(t-(k-1)T_0) = 1 - (t-(k-1)T_0) \to \boxed{f_1(t-(k-1)T_0) = -t + k}$$

Next, compute the difference $f_1(t-kT_0) - f_1(t-(k-1)T_0)$

$$f_1(t-kT_0) - f_1(t-(k-1)T_0) = 1 + k - t - (-t + k)]$$

$$\boxed{f_1(t-kT_0) - f_1(t-(k-1)T_0) = 1}$$

Therefore, plugging in the above results yields:

$$f(t) = (1-t)u(t) + \sum_{k=1}^{\infty} 1u(t-(1)k) \to \boxed{f(t) = (1-t)u(t) + \sum_{k=1}^{\infty} u(t-k)}$$

ii. Compute the unilateral Laplace transform of $f(t)$. You may find the following formula very useful.

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}, -1 < r < 1$$

$$\boldsymbol{F(s)} = \mathcal{L}[(1-t)u(t)] + \mathcal{L}\left[\sum_{k=1}^{\infty} u(t-k)\right] = \mathcal{L}[u(t)] - \mathcal{L}[tu(t)] + \mathcal{L}\left[\sum_{k=1}^{\infty} u(t-k)\right]$$

$$\boldsymbol{F(s)} = \frac{1}{\boldsymbol{s}} - \frac{1}{\boldsymbol{s}^2} + \sum_{k=1}^{\infty} \mathcal{L}[u(t-k)] = \frac{1}{\boldsymbol{s}} - \frac{1}{\boldsymbol{s}^2} + \sum_{k=1}^{\infty} \frac{1}{\boldsymbol{s}} e^{-k\boldsymbol{s}} = \frac{1}{\boldsymbol{s}} - \frac{1}{\boldsymbol{s}^2} + \frac{1}{\boldsymbol{s}} \sum_{k=1}^{\infty} (e^{-\boldsymbol{s}})^k$$

$$\boldsymbol{F(s)} = \frac{1}{\boldsymbol{s}} - \frac{1}{\boldsymbol{s}^2} + \frac{1}{\boldsymbol{s}}\left[\sum_{k=0}^{\infty} (e^{-\boldsymbol{s}})^k - 1\right] = \frac{1}{\boldsymbol{s}} - \frac{1}{\boldsymbol{s}^2} + \frac{1}{\boldsymbol{s}}\left[\frac{1}{1-e^{-\boldsymbol{s}}} - 1\right] = -\frac{1}{\boldsymbol{s}^2} + \frac{1}{\boldsymbol{s}}\left[1 + \frac{1}{1-e^{-\boldsymbol{s}}} - 1\right]$$

$$\boldsymbol{F(s)} = -\frac{1}{\boldsymbol{s}^2} + \frac{1}{\boldsymbol{s}}\left[\frac{1}{1-e^{-\boldsymbol{s}}}\right] \rightarrow \boxed{\boldsymbol{F(s)} = \frac{-(1-e^{-\boldsymbol{s}}) + \boldsymbol{s}}{\boldsymbol{s}^2(1-e^{-\boldsymbol{s}})} = \frac{\boldsymbol{s} - 1 + e^{-\boldsymbol{s}}}{\boldsymbol{s}^2(1-e^{-\boldsymbol{s}})}}$$

2. Laplace Transforms and Integro-Differential Equations

(a) Compute $\boldsymbol{V_o(s)} = \mathcal{L}\{v_o(t)u(t)\}$ for the following integro-differential equation. Identify the characteristic polynomial, the zero state transform component $\boldsymbol{V_{o,ZS}(s)}$ of $\boldsymbol{V_o(s)}$, and the zero input transform component $\boldsymbol{V_{o,ZI}(s)}$ of $\boldsymbol{V_o(s)}$.

$$v_o''(t) + 5v_o'(t) + 6v_o(t) + 2\int_{0^-}^{t} v_o(\lambda)d\lambda = 4\delta(t), \qquad v_o(0^-) = 2\text{V}, v_o'(0^-) = 4\text{ V/s}$$

$$\mathcal{L}\left[v_o''(t) + 5v_o'(t) + 6v_o(t) + 2\int_{0^-}^{t} v_o(\lambda)d\lambda = 4\delta(t)\right]$$

$$\mathcal{L}[v_o''(t)] + 5\mathcal{L}[v_o'(t)] + 6\mathcal{L}[v_o(t)] + 2\mathcal{L}\left[\int_{0^-}^{t} v_o(\lambda)d\lambda\right] = 4\mathcal{L}[\delta(t)]$$

$$\boldsymbol{s}^2\boldsymbol{V_o(s)} - \boldsymbol{s}v_o(0^-) - v_o'(0^-) + 5[\boldsymbol{sV_o(s)} - v_o(0^-)] + 6\boldsymbol{V_o(s)} + \frac{2\boldsymbol{V_o(s)}}{\boldsymbol{s}} = 4\boldsymbol{\Delta(s)}$$

$$\boldsymbol{s}^3\boldsymbol{V_o(s)} - \boldsymbol{s}^2v_o(0^-) - \boldsymbol{s}v_o'(0^-) + 5[\boldsymbol{s}^2\boldsymbol{V_o(s)} - \boldsymbol{s}v_o(0^-)] + 6\boldsymbol{sV_o(s)} + 2\boldsymbol{V_o(s)} = 4\boldsymbol{s\Delta(s)}$$

$$\boldsymbol{V_o(s)}[\boldsymbol{s}^3 + 5\boldsymbol{s}^2 + 6\boldsymbol{s} + 2] = \boldsymbol{s}^2v_o(0^-) + \boldsymbol{s}v_o'(0^-) + 5\boldsymbol{s}v_o(0^-) + 4\boldsymbol{s\Delta(s)}$$

$$\boldsymbol{V_o(s)} = \boldsymbol{V_{o,ZI}(s)} + \boldsymbol{V_{o,ZS}(s)} = \frac{\boldsymbol{s}^2v_o(0^-) + \boldsymbol{s}[v_o'(0^-) + 5v_o(0^-)]}{\boldsymbol{s}^3 + 5\boldsymbol{s}^2 + 6\boldsymbol{s} + 2} + \frac{4\boldsymbol{s}}{\boldsymbol{s}^3 + 5\boldsymbol{s}^2 + 6\boldsymbol{s} + 2}\boldsymbol{\Delta(s)}$$

$$\boldsymbol{V_o(s)} = \boldsymbol{V_{o,ZI}(s)} + \boldsymbol{V_{o,ZS}(s)} = \frac{\boldsymbol{s}^2(2) + \boldsymbol{s}[4 + 5(2)]}{\boldsymbol{s}^3 + 5\boldsymbol{s}^2 + 6\boldsymbol{s} + 2} + \frac{4\boldsymbol{s}}{\boldsymbol{s}^3 + 5\boldsymbol{s}^2 + 6\boldsymbol{s} + 2}(1)$$

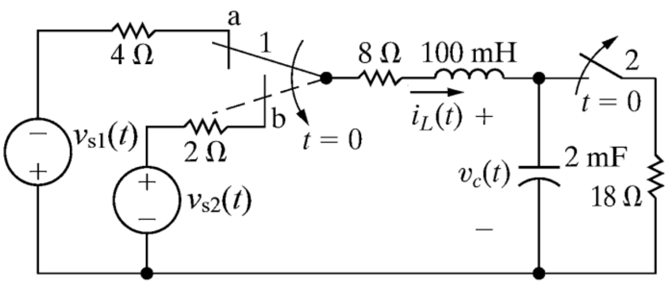
$$\boldsymbol{V_o(s)} = \boldsymbol{V_{o,ZI}(s)} + \boldsymbol{V_{o,ZS}(s)} = \frac{2\boldsymbol{s}(\boldsymbol{s} + 7)}{\boldsymbol{s}^3 + 5\boldsymbol{s}^2 + 6\boldsymbol{s} + 2} + \frac{4\boldsymbol{s}}{\boldsymbol{s}^3 + 5\boldsymbol{s}^2 + 6\boldsymbol{s} + 2}$$

$$\boxed{\boldsymbol{V_{o,ZI}(s)} = \frac{2\boldsymbol{s}(\boldsymbol{s} + 7)}{\boldsymbol{s}^3 + 5\boldsymbol{s}^2 + 6\boldsymbol{s} + 2}} \qquad \boxed{\boldsymbol{V_{o,ZS}(s)} = \frac{4\boldsymbol{s}}{\boldsymbol{s}^3 + 5\boldsymbol{s}^2 + 6\boldsymbol{s} + 2}}$$

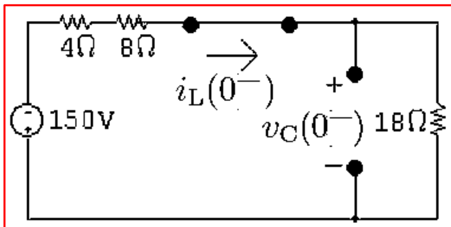
$$\boldsymbol{V_o(s)} = \frac{2\boldsymbol{s}(\boldsymbol{s} + 7) + 4\boldsymbol{s}}{\boldsymbol{s}^3 + 5\boldsymbol{s}^2 + 6\boldsymbol{s} + 2} = \frac{2\boldsymbol{s}^2 + 18\boldsymbol{s}}{\boldsymbol{s}^3 + 5\boldsymbol{s}^2 + 6\boldsymbol{s} + 2} \rightarrow \boxed{\boldsymbol{V_o(s)} = \frac{2\boldsymbol{s}(\boldsymbol{s} + 9)}{\boldsymbol{s}^3 + 5\boldsymbol{s}^2 + 6\boldsymbol{s} + 2}}$$

Characteristic polynomial is the denominator of $\boldsymbol{V_{o,ZI}(s)}$. Therefore, the characteristic polynomial is $\boxed{\boldsymbol{s}^3 + 5\boldsymbol{s}^2 + 6\boldsymbol{s} + 2}$. This means the integro-differential equation models the behavior of a third order system.

(b) The second order network shown below has $v_{s1}(t) = 150\text{V}$ and $v_{s2}(t) = 60\text{V}$. At $t = 0$, switch 1 moves to position (b) after being at position (a) for a long time. Simultaneously, switch 2 opens after being closed for a long time.

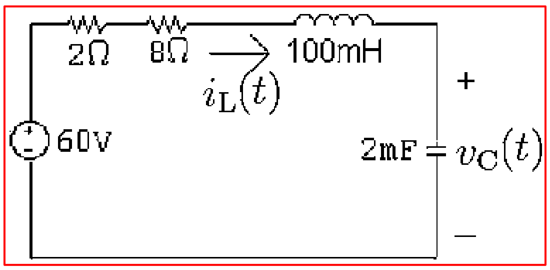


i. Analyze the network at time $t = 0^-$ to compute the state variable values $v_C(0^-)$ and $i_L(0^-)$, respectively.



$$i_L(0^-) = \frac{-150}{30} = -5\,\text{A}$$
$$v_C(0^-) = 18\,i_L(0^-) = -90\,\text{V}$$

ii. Analyze the network for $t > 0^-$ using <u>mesh analysis</u> to obtain a single linear second order constant coefficient integro-differential equation that describes the inductor current $i_L(t)u(t)$ for $t > 0^-$.



Mesh Analysis:

$$-v_{S_2}(t) + v_{R_{eq}}(t) + v_L(t) + v_C(t) = 0\text{V}$$

$$R_{eq} i_L(t) + L\frac{di_L(t)}{dt} + v_C(0^-) + \frac{1}{C}\int_{0^-}^{t} i_L(x)dx = v_{S_2}(t)$$

$$10\Omega i_L(t) + 0.1\text{H}\frac{di_L(t)}{dt} + v_C(0^-) + \frac{1}{2\text{mF}}\int_{0^-}^{t} i_L(x)dx = v_{S_2}(t)$$

$$\boxed{100 i_L(t) + \frac{di_L(t)}{dt} + 10 v_C(0^-) + 5\text{k}\int_{0^-}^{t} i_L(x)dx = 10 v_{S_2}(t)}$$

$$\boxed{\frac{di_L(t)}{dt} + 100i_L(t) + 5\text{k}\int_{0^-}^{t} i_L(x)dx = 10\left[v_{S_2}(t) - v_C(0^-)\right] = 1.5\text{kV}}$$

iii. Take the Laplace Transform of the equation found in (ii) and compute the complete response transform $\boldsymbol{I_L(s)} = \mathcal{L}\{i_L(t)u(t)\}$. identify the characteristic polynomial of the network, the zero state response transform component $\boldsymbol{I_{L,ZS}(s)}$ of $\boldsymbol{I_L(s)}$, and the zero input response transform component $\boldsymbol{I_{L,ZI}(s)}$ of $\boldsymbol{I_L(s)}$.

$$\mathcal{L}\left[100i_L(t) + \frac{di_L(t)}{dt} + 10v_C(0^-) + 5\text{k}\int_{0^-}^{t} i_L(x)dx = 10v_{S_2}(t)\right]$$

$$100\mathcal{L}[i_L(t)] + \mathcal{L}\left[\frac{di_L(t)}{dt}\right] + 10v_C(0^-)\mathcal{L}[1] + 5\text{k}\mathcal{L}\left[\int_{0^-}^{t} i_L(x)dx\right] = 10\mathcal{L}\left[v_{S_2}(t)\right]$$

$$100\boldsymbol{I_L(s)} + [\boldsymbol{sI_L(s)} - i_L(0^-)] + \frac{10v_C(0^-)}{\boldsymbol{s}} + 5\text{k}\left[\frac{\boldsymbol{I_L(s)}}{\boldsymbol{s}}\right] = 10\boldsymbol{V_{S_2}(s)}$$

$$\boldsymbol{I_L(s)}\left(100 + \boldsymbol{s} + \frac{5\text{k}}{\boldsymbol{s}}\right) = 10\boldsymbol{V_{S_2}(s)} + i_L(0^-) - \frac{10v_C(0^-)}{\boldsymbol{s}}$$

$$\boldsymbol{I_L(s)}(\boldsymbol{s}^2 + 100\boldsymbol{s} + 5\text{k}) = 10\boldsymbol{sV_{S_2}(s)} + \boldsymbol{s}i_L(0^-) - 10v_C(0^-)$$

$$\boldsymbol{I_L(s)} = \boldsymbol{I_{L,ZS}(s)} + \boldsymbol{I_{L,ZI}(s)} = \frac{10\boldsymbol{s}}{\boldsymbol{s}^2 + 100\boldsymbol{s} + 5\text{k}}\boldsymbol{V_{S_2}(s)} + \frac{\boldsymbol{s}i_L(0^-) - 10v_C(0^-)}{\boldsymbol{s}^2 + 100\boldsymbol{s} + 5\text{k}}$$

$$\boldsymbol{I_L(s)} = \boldsymbol{I_{L,ZS}(s)} + \boldsymbol{I_{L,ZI}(s)} = \frac{10\boldsymbol{s}}{\boldsymbol{s}^2 + 100\boldsymbol{s} + 5\text{k}}\left(\frac{60\text{V}}{\boldsymbol{s}}\right) + \frac{\boldsymbol{s}(-5\text{A}) - 10(-90\text{V})}{\boldsymbol{s}^2 + 100\boldsymbol{s} + 5\text{k}}$$

$$\boldsymbol{I_L(s)} = \boldsymbol{I_{L,ZS}(s)} + \boldsymbol{I_{L,ZI}(s)} = \frac{600}{\boldsymbol{s}^2 + 100\boldsymbol{s} + 5\text{k}} + \frac{900 - 5\boldsymbol{s}}{\boldsymbol{s}^2 + 100\boldsymbol{s} + 5\text{k}}$$

$$\boxed{\boldsymbol{I_{L,ZS}(s)} = \frac{600}{\boldsymbol{s}^2 + 100\boldsymbol{s} + 5\text{k}}} \qquad \boxed{\boldsymbol{I_{L,ZI}(s)} = \frac{900 - 5\boldsymbol{s}}{\boldsymbol{s}^2 + 100\boldsymbol{s} + 5\text{k}} = 5\frac{180 - \boldsymbol{s}}{\boldsymbol{s}^2 + 100\boldsymbol{s} + 5\text{k}}}$$

$$\boldsymbol{I_L(s)} = \frac{600 + 900 - 5\boldsymbol{s}}{\boldsymbol{s}^2 + 100\boldsymbol{s} + 5\text{k}} \to \boxed{\boldsymbol{I_L(s)} = \frac{1.5\text{k} - 5\boldsymbol{s}}{\boldsymbol{s}^2 + 100\boldsymbol{s} + 5\text{k}} = 5\frac{300 - \boldsymbol{s}}{\boldsymbol{s}^2 + 100\boldsymbol{s} + 5\text{k}}}$$

Characteristic polynomial is the denominator of $\boldsymbol{I_{L,ZI}(s)}$. Therefore, the characteristic polynomial is $\boxed{\boldsymbol{s}^2 + 100\boldsymbol{s} + 5\text{k}}$. This means the integro-differential equation models the behavior of a second order system, which is what the network is because it contains one independent capacitor and one independent inductor.

iv. Write a time-domain expression that relates the capacitor voltage $v_C(t)u(t)$ to the inductor current $i_L(t)u(t)$. Then use the expression compute $\boldsymbol{V_C(s)} = \mathcal{L}\{v_C(t)u(t)\}$.

$$v_C(t) = v_C(0^-) + \frac{1}{C}\int_{0^-}^{t} i_L(x)dx = -90\text{V} + \frac{1}{2\text{mF}}\int_{0^-}^{t} i_L(x)dx$$

$$v_C(t) = -90 + 500\int_{0^-}^{t} i_L(x)dx \to \mathcal{L}\left[v_C(t) = -90 + 500\int_{0^-}^{t} i_L(x)dx\right]$$

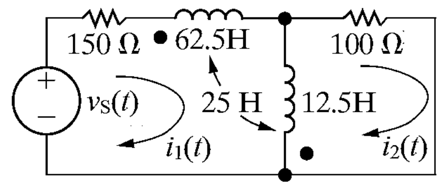
$$\mathcal{L}[v_C(t)] = -90\mathcal{L}[1] + 500\mathcal{L}\left[\int_{0^-}^{t} i_L(x)dx\right]$$

$$\boldsymbol{V_C(s)} = \frac{-90}{\boldsymbol{s}} + 500\left[\frac{\boldsymbol{I_L(s)}}{\boldsymbol{s}}\right] = \frac{1}{\boldsymbol{s}}[-90 + 500\boldsymbol{I_L(s)}]$$

$$\boldsymbol{V_C(s)} = \frac{1}{\boldsymbol{s}}\left[-90 + 500\frac{1.5\text{k} - 5\boldsymbol{s}}{\boldsymbol{s}^2 + 100\boldsymbol{s} + 5\text{k}}\right] = \frac{1}{\boldsymbol{s}}\left(-90 + \frac{750\text{k} - 2.5\text{k}\boldsymbol{s}}{\boldsymbol{s}^2 + 100\boldsymbol{s} + 5\text{k}}\right)$$

$$\boldsymbol{V_C(s)} = \frac{-90(\boldsymbol{s}^2 + 100\boldsymbol{s} + 5\text{k}) + 750\text{k} - 2.5\text{k}\boldsymbol{s}}{\boldsymbol{s}(\boldsymbol{s}^2 + 100\boldsymbol{s} + 5\text{k})} \rightarrow \boxed{\boldsymbol{V_C(s)} = \frac{-90\boldsymbol{s}^2 - 11.5\text{k}\boldsymbol{s} + 300\text{k}}{\boldsymbol{s}(\boldsymbol{s}^2 + 100\boldsymbol{s} + 5\text{k})}}$$

(c) Consider the magnetically coupled network shown below with $v_s(t) = 625u(t)$V. Assume the network is in the zero state (i.e. the state variable for each energy storage element is initially zero).



i. Apply mesh analysis to derive two ordinary linear constant coefficient first order differential equations that govern the behavior of the network.

$$\boxed{\begin{aligned}
&625 = 150i_1 + 62.5\frac{di_1}{dt} + 25\frac{d}{dt}(i_2 - i_1) + 12.5\frac{d}{dt}(i_1 - i_2) - 25\frac{di_1}{dt}\\
&0 = 12.5\frac{d}{dt}(i_2 - i_1) + 25\frac{di_1}{dt} + 100i_2\\
&\text{Simplifying the above equations gives:}\\
&625 = 150i_1 + 25\frac{di_1}{dt} + 12.5\frac{di_2}{dt}\\
&0 = 100i_2 + 12.5\frac{di_1}{dt} + 12.5\frac{di_2}{dt}
\end{aligned}}$$

ii. Take the Laplace Transform of the both equations developed in (i).

$$\boxed{\begin{aligned}
&\frac{625}{s} = (25s + 150)I_1(s) + 12.5sI_2(s)\\
&0 = 12.5sI_1(s) + (12.5s + 100)I_2(s)
\end{aligned}}$$

iii. Solve for the Laplace Transform of the two mesh currents, i.e., solve for $\boldsymbol{I_1(s)} = \mathcal{L}\{i_1(t)u(t)\}$ and $\boldsymbol{I_2(s)} = \mathcal{L}\{i_2(t)u(t)\}$.

Solve the second equation for $\boldsymbol{I_1(s)}$:

$$\boldsymbol{I_1(s)} = -\frac{\boldsymbol{I_2(s)}[12.5\boldsymbol{s} + 100]}{12.5\boldsymbol{s}}$$

Substitute above into the first equation and solve for $\boldsymbol{I_2(s)}$

$$\frac{625}{\boldsymbol{s}} = (25\boldsymbol{s} + 150)\left[-\frac{[12.5\boldsymbol{s} + 100]}{12.5\boldsymbol{s}}\right]\boldsymbol{I_2(s)} + 12.5\boldsymbol{s}\boldsymbol{I_2(s)}$$

$$\frac{625}{\boldsymbol{s}} = \left[-\frac{(25\boldsymbol{s} + 150)[12.5\boldsymbol{s} + 100]}{12.5\boldsymbol{s}} + 12.5\boldsymbol{s}\right]\boldsymbol{I_2(s)}$$

$$\frac{625}{s} = \left[\frac{-(25s+150)[12.5s+100]+(12.5s)^2}{12.5s}\right] I_2(s)$$

$$I_2(s) = \frac{625}{s}\left[\frac{12.5s}{-(25s+150)[12.5s+100]+(12.5s)^2}\right]$$

$$I_2(s) = \left[\frac{7812.5}{-(156.25s^2+4375s+15\text{k})}\right] = \left[\frac{7812.5}{-156.25\left(s^2+\frac{4375}{156.25}s+\frac{15\text{k}}{156.25}\right)}\right]$$

$$\boxed{I_2(s) = \frac{50}{-(s^2+28s+96)} = -\frac{50}{(s+4)(s+24)}}$$

Substitute $I_2(s)$ into the equation for $I_1(s)$ and solve for $I_1(s)$

$$I_1(s) = -I_2(s)\frac{[12.5s+100]}{12.5s} = -\left[\frac{50}{-(s^2+28s+96)}\right]\frac{[12.5s+100]}{12.5s}$$

$$I_1(s) = \left[\frac{50[12.5s+100]}{12.5s(s^2+28s+96)}\right] = \left[\frac{50(12.5)[s+100/12.5]}{12.5s(s^2+28s+96)}\right] = \frac{50(s+8)}{s(s^2+28s+96)}$$

$$\boxed{I_1(s) = \frac{50(s+8)}{s(s^2+28s+96)} = \frac{50(s+8)}{s(s+4)(s+24)}}$$