Lecture #3(b): Basic Signal Waveforms
Examples

ECE 20200: Linear Circuit Analysis II
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Example #1

- Consider the piecewise continuous function defined below.

$$f(t) = \begin{cases} 3 & t < 0 \\ -2 & 0 < t < 2 \\ 2t - 4 & t > 2 \end{cases}$$

  - Express $f(t)$ as a linear combination (i.e. a sum) of singularity function.
  - Write a MATLAB script to plot $f(t)$ for the time interval $-4 < t < 4$

Example #1 (SOLUTION)

- Express $f(t)$ as a linear combination of singularity function.
  - $f(t)$ can be expressed as a sum of three non-zero functions that are "turned on" during specific intervals of time.

$$f(t) = \begin{cases} f_1(t) = 3 & t < 0 \\ f_2(t) = -2 & 0 < t < 2 \\ f_3(t) = 2t - 4 & t > 2 \end{cases}$$

  - Using the window/gate function, $f(t)$ can be expressed as

$$f(t) = f_1(t)[1 - u(t)] + f_2(t)[u(t) - u(t-2)] + f_3(t)u(t-2)$$

  - Simplify $f(t)$ by collecting "terms"

$$f(t) = f_1(t) + [f_2(t) - f_1(t)]u(t) + [f_3(t) - f_2(t)]u(t-2)$$

  - Substitute $f_1(t)$ through $f_3(t)$ into $f(t)$ and simplify

$$f(t) = 3 - 5u(t) + 2(t-1)u(t-2)$$

Example #1 (SOLUTION cont'd)

- Express $f(t)$ as a linear combination of singularity functions.
  - $f(t)$ can be expressed as a sum of three non-zero functions that are "turned on" during specific intervals of time.

$$f(t) = \begin{cases} f_1(t) = 3 & t < 0 \\ f_2(t) = -2 & 0 < t < 2 \\ f_3(t) = 2t - 4 & t > 2 \end{cases}$$

  - Manipulate $f(t)$ so that it may be written in terms of as many ramp functions as possible

$$f(t) = 3 - 5u(t) + 2(t - 1 - 1 + 1)u(t - 2)$$

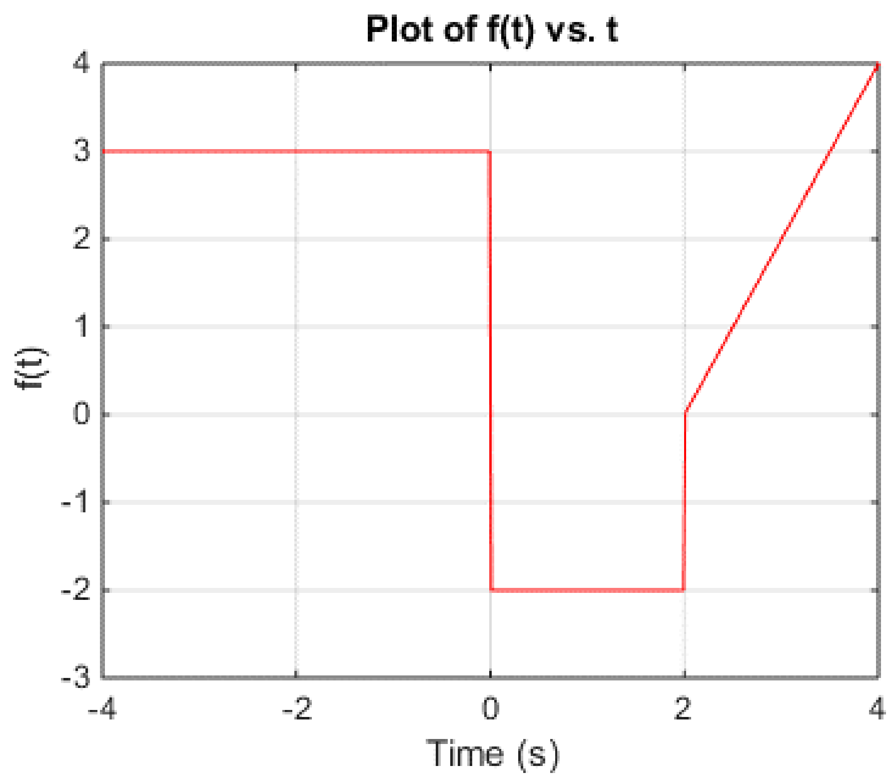
$$f(t) = 3 - 5u(t) + 2(t - 2)u(t - 2) + 2u(t - 2)$$

$$f(t) = 3 - 5u(t) + 2r(t - 2) + 2u(t - 2)$$

Example #1 (SOLUTION cont'd)

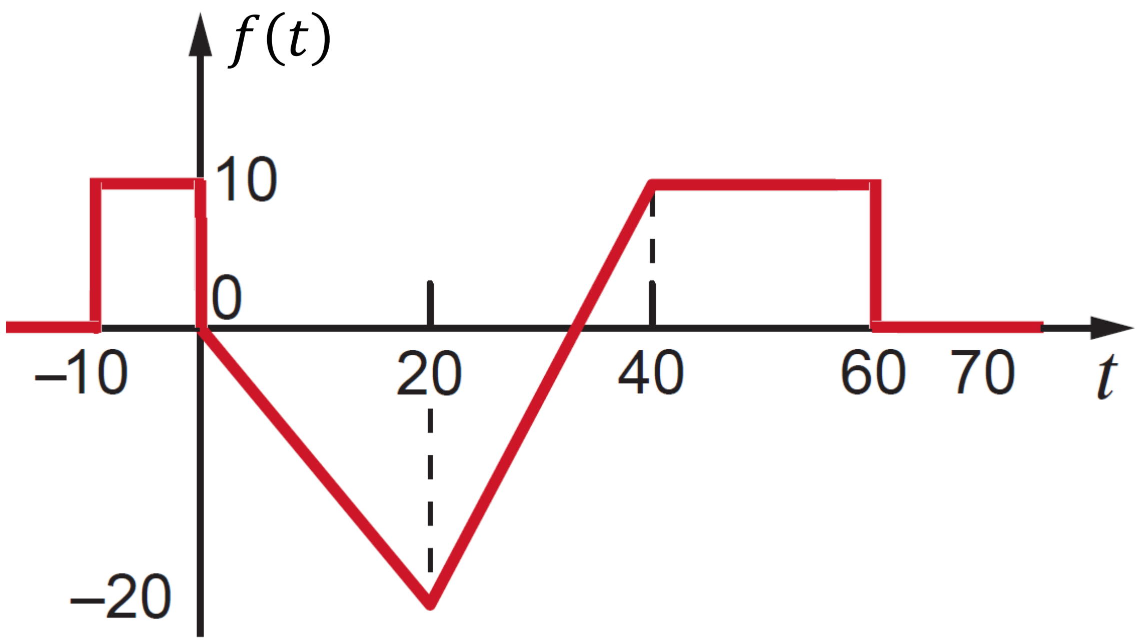
- Write a MATLAB script to plot $f(t)$ for the interval $-4 < t < 4$

```
% clear everything
clc;  clear all;  close all;
%----------------------------
% define the time vector
t = -4:0.01:4;
%----------------------------
% define each piece of f(t)
f1 = 3; f2 = -2; f3 = 2*t - 4;
%----------------------------
% define f(t) using pieces and
% built-in Heaviside() function
f = f1 + (f2-f1).*heaviside(t) ...
    + (f3-f2).*heaviside(t-2);
%----------------------------
% plot f(t)
plot(t, f, 'r'); grid on;
xlabel('Time (s)'); ylabel('f(t)');
title('Plot of f(t) vs. t');
%----------------------------
% set range of vertical axis
ylim([-3,4]);
```



Example #2

- Consider the piecewise continuous function shown below.
  - Express $f(t)$ as a linear combination (i.e. a sum of) singularity function.
  - Compute $f'(t)$ and sketch $f'(t)$



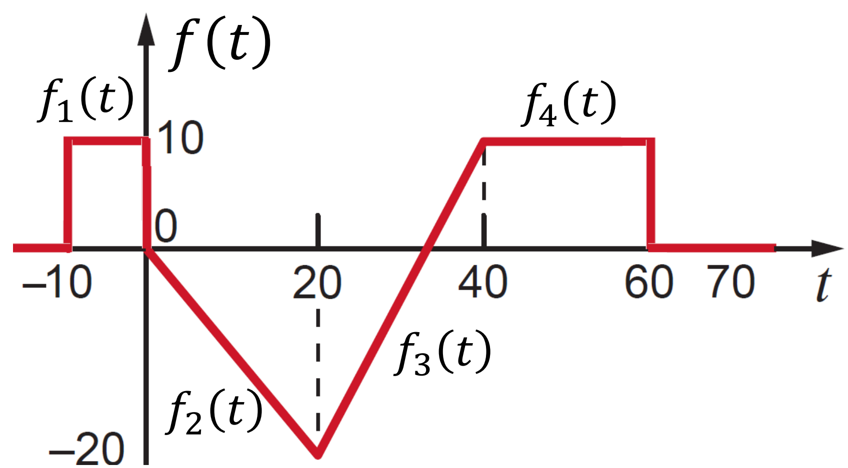
Example #2 (SOLUTION cont'd)

- Express $f(t)$ as a combination of singularity functions.
  - $f(t)$ can be expressed as a sum of four non-zero functions that are "turned on" during specific intervals of time. Therefore,

$$f(t) = f_1(t)[u(t+10) - u(t)] + f_2(t)[u(t) - u(t-20)] + f_3(t)[u(t-20) - u(t-40)] + f_4(t)[u(t-40) - u(t-60)]$$

  - Before computing $f_1(t)$ through $f_4(t)$, you may want to simplify $f(t)$ by collecting "terms" for each unit-step function

$$\begin{aligned} f(t) = f_1(t)u(t+10) & \\ +[f_2(t) - f_1(t)]u(t) & \\ +[f_3(t) - f_2(t)]u(t-20) & \\ +[f_4(t) - f_3(t)]u(t-40) & \\ -f_4(t)u(t-60) & \end{aligned}$$



Example #2 (SOLUTION cont'd)

- Express $f(t)$ as a combination of singularity functions.
  - Compute $f_1(t)$ and $f_4(t)$

$$\boxed{f_1(t) = f_4(t) = +10}$$

  - Compute $f_2(t)$ – Use point-slope formula of a line

$$f_2(t) = f_2(t_2) + m_2(t - t_2) = f_2(20) + (-1)(t - 20)$$

$$f_2(t) = -20 + (-1)(t - 20) \rightarrow \boxed{f_2(t) = -t}$$

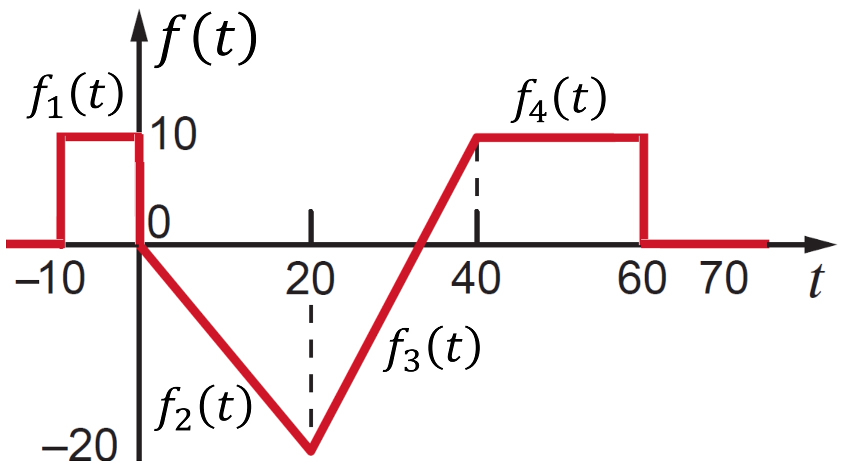
  - Compute $f_3(t)$ – Use point-slope formula of a line

$$f_3(t) = f_3(t_3) + m_3(t - t_3)$$

$$f_3(t) = f_3(40) + (1.5)(t - 40)$$

$$f_3(t) = 10 + (1.5)(t - 40)$$

$$\boxed{f_3(t) = 1.5t - 50}$$



Example #2 (SOLUTION cont'd)

- Express $f(t)$ as a combination of singularity functions.
  - Recall the simplified form of $f(t)$ and each function f1 to f4.

$$f(t) = f_1(t)u(t+10) + [f_2(t) - f_1(t)]u(t) + [f_3(t) - f_2(t)]u(t-20)$$
$$+[f_4(t) - f_3(t)]u(t-40) - f_4(t)u(t-60)$$

$\boxed{f_1(t) = f_4(t) = +10}$ $\boxed{f_2(t) = -t}$ $\boxed{f_3(t) = 1.5t - 50}$

  - Compute each term difference

$$f_2(t) - f_1(t) = -t - 10 = -(t+10)$$
$$f_3(t) - f_2(t) = 1.5t - 50 - (-t) = 2.5t - 50 = 2.5(t-20)$$
$$f_4(t) - f_3(t) = 10 - (1.5t - 50) = 60 - 1.5t = -1.5(t-40)$$

  - Substitute above into $f(t)$.

$$f(t) = 10u(t+10) - (t+10)u(t) + 2.5(t-20)u(t-20)$$
$$-1.5(t-40)u(t-40) - 10u(t-60)$$

Example #2 (SOLUTION cont'd)

- Express $f(t)$ as a combination of singularity functions.
  - The expression for $f(t)$ is now

$$f(t) = 10u(t+10) - (t+10)u(t) + 2.5(t-20)u(t-20)$$
$$-1.5(t-40)u(t-40) - 10u(t-60)$$

  - Finally, try to find as many ramp functions as possible in $f(t)$ and re-express $f(t)$ with those ramp functions

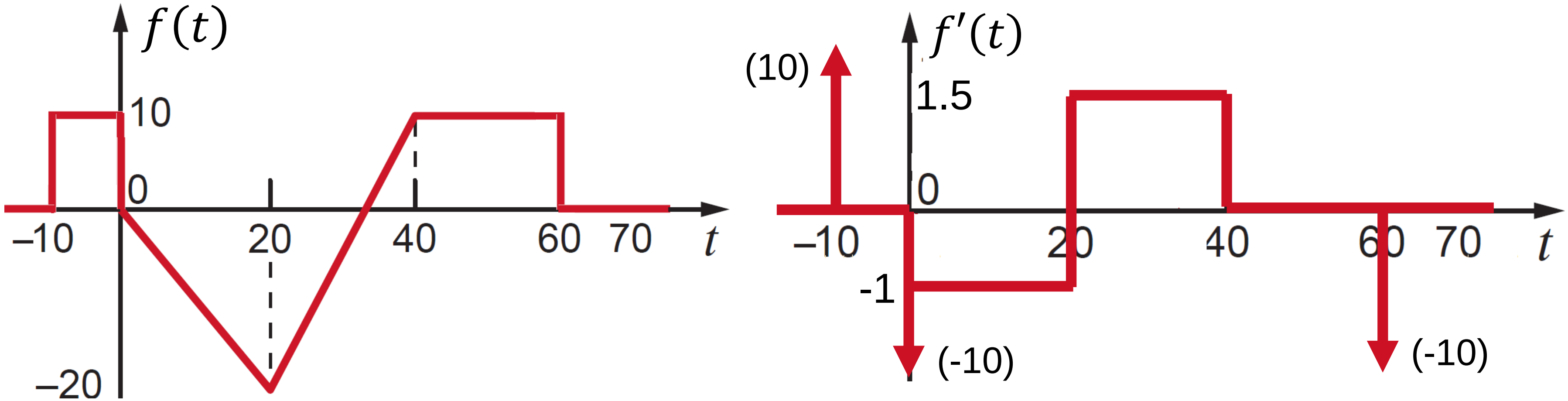
$$f(t) = 10u(t+10) - r(t) - 10u(t) + 2.5r(t-20)$$
$$-1.5r(t-40) - 10u(t-60)$$

- Compute and sketch $f'(t)$
  - Compute $f'(t)$ term-wise

$$f'(t) = 10\delta(t+10) - u(t) - 10\delta(t) + 2.5u(t-20)$$
$$-1.5u(t-40) - 10\delta(t-60)$$

Example #2 (SOLUTION cont'd)

- Compute and Sketch $f'(t)$
  - Recall, $f'(t)$ is $f'(t) = 10\delta(t+10) - u(t) - 10\delta(t) + 2.5u(t-20)$
    $$-1.5u(t-40) - 10\delta(t-60)$$
  - The sketch of both $f(t)$ and $f'(t)$ is shown below



Example #3

- Evaluate the following expressions involving the Dirac Delta/Impulse
  - $f(t) = \sqrt{t}[\delta(t-4) + \delta(t-9) + \delta(t-16)]$
  - $f(t) = u(t-1)r(t-2)\delta(t-4)$
  - $\int_{-2}^{2} u(t+2)\delta(t-1)dt$
  - $\int_{-\infty}^{2^-} r(t+1)\delta(t-2)dt$
  - $\int_{-3^-}^{3^+} \cos\left(\frac{\pi}{3}t - \frac{\pi}{6}\right) e^{-t^2}[\delta(t+3) + \delta(t-3)]dt$

Example #3 (SOLUTION)

- $f(t) = \sqrt{t}[\delta(t-4) + \delta(t-9) + \delta(t-16)]$
  - Using the sampling theorem of the Dirac Delta/Impulse, we have

$$f(t) = \sqrt{t}\delta(t-4) + \sqrt{t}\delta(t-9) + \sqrt{t}\delta(t-16)$$
$$f(t) = \sqrt{+4}\delta(t-4) + \sqrt{+9}\delta(t-9) + \sqrt{+16}\delta(t-16)$$
$$\boxed{f(t) = 2\delta(t-4) + 3\delta(t-9) + 4\delta(t-16)}$$

- $f(t) = u(t-1)r(t-2)\delta(t-4)$
  - Using the sampling theorem of the Dirac Delta/Impulse, we have

$$f(t) = u(t-1)r(t-2)\delta(t-4)$$
$$f(t) = u(4-1)r(4-2)\delta(t-4)$$
$$f(t) = u(3)r(2)\delta(t-4)$$
$$f(t) = (1)(2)\delta(t-4)$$
$$\boxed{f(t) = 2\delta(t-4)}$$

Example #3 (SOLUTION cont'd)

- $\int_{-2}^{2} u(t+2)\delta(t-1)dt$
  - Note that the impulse occurs, or is concentrated, at time $t = 1$
  - The limits of integration **do envelop** the time at which the impulse occurs. Therefore, the integral may have a non-zero value!
  - Based on the **sifting theorem** of the Dirac Delta/Impulse, we have

$$\int_{-2}^{2} u(t+2)\delta(t-1)dt = \int_{1^-}^{1^+} u(t+2)\delta(t-1)dt$$

$$\int_{-2}^{2} u(t+2)\delta(t-1)dt = \int_{1^-}^{1^+} u(1+2)\delta(t-1)dt$$

$$\int_{-2}^{2} u(t+2)\delta(t-1)dt = u(3)\int_{1^-}^{1^+} \delta(t-1)dt$$

$$\int_{-2}^{2} u(t+2)\delta(t-1)dt = u(3)(1) = \boxed{1}$$

Example #3 (SOLUTION cont'd)

- $\int_{-\infty}^{2^-} r(t+1)\delta(t-2)dt$
  - Note that the impulse occurs, or is concentrated, at time $t = +2$
  - The limits of integration **do not envelop** the time at which the impulse occurs. Therefore, the integral **always** evaluates to zero!
  - Based on the **sifting theorem** of the Dirac Delta/Impulse, we have

$$\int_{-\infty}^{2^-} r(t+1)\delta(t-2)dt = \int_{2^-}^{2^-} r(t+1)\delta(t-2)dt$$

$$\int_{-2}^{2^-} r(t+1)\delta(t-2)dt = \boxed{0}$$

Example #3 (SOLUTION cont'd)

- $\int_{-3^-}^{3^+} \cos\left(\frac{\pi}{3}t - \frac{\pi}{6}\right) e^{-t^2}[\delta(t+3) + \delta(t-3)]dt$
  - Note that the impulses occur, or are concentrated, at time $t = \pm 3$
  - The limits of integration envelop both impulse locations. Therefore, the entire integral may evaluate to a non-zero value!
  - Based on the sifting theorem of the Dirac Delta/Impulse, we have

$$\int_{-3^-}^{3^+} \cos\left(\frac{\pi}{3}t - \frac{\pi}{6}\right) e^{-t^2}\delta(t+3)dt + \int_{-3^-}^{3^+} \cos\left(\frac{\pi}{3}t - \frac{\pi}{6}\right) e^{-t^2}\delta(t-3)dt$$

$$\int_{-3^-}^{3^+} \cos\left(\frac{\pi}{3}(-3) - \frac{\pi}{6}\right) e^{-(-3)^2}\delta(t+3)dt + \int_{-3^-}^{3^+} \cos\left(\frac{\pi}{3}(3) - \frac{\pi}{6}\right) e^{-(3)^2}\delta(t-3)dt$$

$$\cos\left(-\frac{7\pi}{6}\right) e^{-9} \int_{-3^-}^{3^+} \delta(t+3)dt + \cos\left(\frac{5\pi}{6}\right) e^{-9} \int_{-3^-}^{3^+} \delta(t-3)dt$$

$$\left[\cos\left(-\frac{7\pi}{6}\right) + \cos\left(\frac{5\pi}{6}\right)\right] e^{-9} = \boxed{-\sqrt{3}e^{-9}}$$

Example #4

- Evaluate the following expressions
  - $f(t) = \frac{d}{dt}[r(t-1)u(t-4)]$
  - $g(t) = \frac{d}{dt}[e^{-2t}u(t-3)]$
  - $h(t) = \frac{d}{dt}[cos(2t)u(t)]$
- Solution
  - $f(t) = \frac{d}{dt}[r(t-1)u(t-4)]$
  - Use the product rule for a derivative of a product of two functions

$$f(t) = r'(t-1)u(t-4) + r(t-1)u'(t-4)$$
$$f(t) = u(t-1)u(t-4) + r(t-1)\delta(t-4)$$
$$f(t) = u(t-4) + r(3)\delta(t-4)$$
$$\boxed{f(t) = u(t-4) + 3\delta(t-4)}$$

Example #4 (SOLUTION cont'd)

- $g(t) = \frac{d}{dt}[e^{-2t}u(t-3)]$
  - Use the product rule for a derivative of a product of two functions

    $$g(t) = (e^{-2t})'u(t-3) + e^{-2t}u'(t-3)$$
    $$g(t) = -2e^{-2t}u(t-3) + e^{-2t}\delta(t-3)$$
    $$\boxed{g(t) = -2e^{-2t}u(t-3) + e^{-6}\delta(t-3)}$$

- $h(t) = \frac{d}{dt}[cos(2t)u(t)]$
  - Use the product rule for a derivative of a product of two functions

    $$h(t) = [cos(2t)]'u(t) + cos(2t)u'(t)$$
    $$h(t) = -2sin(2t)u(t) + cos(2t)\delta(t)$$
    $$h(t) = -2sin(2t)u(t) + cos(2(0))\delta(t)$$
    $$\boxed{h(t) = -2sin(2t)u(t) + \delta(t)}$$

Example #5

- Consider the following functions which can be considered sinusoids in the most general notion therefore.

$f_1(t) = 10$ $\quad$ $f_4(t) = 8\cos(5t)$

$f_2(t) = 3e^{-2t}$ $\quad$ $f_5(t) = 14e^{-2t}\cos(5t)$

$f_3(t) = 6e^{4t}$ $\quad$ $f_6(t) = 20e^{5t}\cos(6t + 30°)$

  - Express each function as a sum of complex exponentials.
  - What complex frequency or complex frequencies constitute each function?

Example #5 (SOLUTION)

- $f_1(t) = 10$
  - Express $f_1(t)$ as a sum of complex exponentials

    $$f_1(t) = 10 = (10e^{j0})e^{(0+j0)t} = (|\boldsymbol{K}|e^{j\phi})e^{(\sigma_0+j\omega_0)t} = \boldsymbol{K}e^{\boldsymbol{s_0}t}$$

    $$\boldsymbol{s_0} = \sigma_0 + j\omega_0 = 0 + j0$$

- $f_2(t) = 3e^{-2t}$
  - Express $f_2(t)$ as a sum of complex exponentials

    $$f_2(t) = 3e^{-2t} = (3e^{j0})e^{(-2+j0)t} = \left(|\boldsymbol{K}|e^{j\phi}\right)e^{(\sigma_0+j\omega_0)t} = \boldsymbol{K}e^{\boldsymbol{s_0}t}$$

    $$\boldsymbol{s_0} = \sigma_0 + j\omega_0 = -2$$

- $f_3(t) = 6e^{4t}$
  - Express $f_3(t)$ as a sum of complex exponentials

    $$f_3(t) = 6e^{4t} = (6e^{j0})e^{(4+j0)t} = \left(|\boldsymbol{K}|e^{j\phi}\right)e^{(\sigma_0+j\omega_0)t} = \boldsymbol{K}e^{\boldsymbol{s_0}t}$$

    $$\boldsymbol{s_0} = \sigma_0 + j\omega_0 = 4$$

Example #5 (SOLUTION cont'd)

- $f_4(t) = 8\cos(2t)$
  - Express $f_4(t)$ as a sum of complex exponentials

$$f_4(t) = 8cos(2t) = \frac{8e^{j0°}}{2}\left[e^{j2t} + e^{-j2t}\right] = \frac{8e^{j0°}}{2}e^{j2t} + \frac{8e^{j0°}}{2}e^{-j2t}$$

$$f_4(t) = \frac{8e^{j0°}}{2}e^{(0+j2)t} + \frac{8e^{j0°}}{2}e^{(0-j2)t}$$

$$f_4(t) = \frac{\left(|\boldsymbol{K}|e^{j\phi}\right)}{2}e^{(\sigma_0+j\omega_0)t} + \frac{\left(|\boldsymbol{K}|e^{-j\phi}\right)}{2}e^{(\sigma_0-j\omega_0)t}$$

$$f_4(t) = \frac{\boldsymbol{K}}{2}e^{\boldsymbol{s_0}t} + \frac{\boldsymbol{K}^*}{2}e^{\boldsymbol{s_0^*}t}$$

$$\boldsymbol{s_{1,2}} = \boldsymbol{s_0}, \boldsymbol{s_0^*} = \sigma_0 \pm j\omega_0 = \pm j2$$

Example #5 (SOLUTION cont'd)

- $f_5(t) = 14e^{-2t}\cos(5t)$
  - Express $f_5(t)$ as a sum of complex exponentials

$$f_5(t) = 14e^{-2t}cos(5t) = 14e^{j0}e^{-2t}\frac{1}{2}\left[e^{j5t} + e^{-j5t}\right]$$

$$f_5(t) = \frac{14e^{j0°}}{2}e^{-2t}e^{j5t} + \frac{14e^{j0°}}{2}e^{-2t}e^{-j5t}$$

$$f_5(t) = \frac{14e^{j0°}}{2}e^{(-2+j5)t} + \frac{14e^{j0°}}{2}e^{(-2-j5)t}$$

$$f_5(t) = \frac{\left(|\boldsymbol{K}|e^{j\phi}\right)}{2}e^{(\sigma_0+j\omega_0)t} + \frac{\left(|\boldsymbol{K}|e^{-j\phi}\right)}{2}e^{(\sigma_0-j\omega_0)t}$$

$$f_5(t) = \frac{\boldsymbol{K}}{2}e^{\boldsymbol{s_0}t} + \frac{\boldsymbol{K}^*}{2}e^{\boldsymbol{s_0^*}t}$$

$$\boldsymbol{s_{1,2}} = \boldsymbol{s_0}, \boldsymbol{s_0^*} = \sigma_0 \pm j\omega_0 = -2 \pm j5$$

Example #5 (SOLUTION cont'd)

- $f_6(t) = 20e^{5t}\cos(6t + 30°)$
  - Express $f_6(t)$ as a sum of complex exponentials

$$f_6(t) = 20e^{5t}cos(6t + 30°) = 20e^{j0}e^{5t}\frac{1}{2}\left[e^{j(6t+30°)} + e^{-j(6t+30°)}\right]$$

$$f_5(t) = \frac{20e^{j0°}}{2}e^{5t}e^{j6t}e^{j30°} + \frac{20e^{j0°}}{2}e^{5t}e^{-j6t}e^{-j30°}$$

$$f_4(t) = \frac{20e^{j30°}}{2}e^{(5+j6)t} + \frac{20e^{-j30°}}{2}e^{(5-j6)t}$$

$$f_4(t) = \frac{\left(|\boldsymbol{K}|e^{j\phi}\right)}{2}e^{(\sigma_0+j\omega_0)t} + \frac{\left(|\boldsymbol{K}|e^{-j\phi}\right)}{2}e^{(\sigma_0-j\omega_0)t}$$

$$f_4(t) = \frac{\boldsymbol{K}}{2}e^{\boldsymbol{s_0}t} + \frac{\boldsymbol{K}^*}{2}e^{\boldsymbol{s_0^*}t}$$

$$\boldsymbol{s_{1,2}} = \boldsymbol{s_0}, \boldsymbol{s_0^*} = \sigma_0 \pm j\omega_0 = 5 \pm j6$$