HOMEWORK #3: Basic Signal Waveforms (SOLUTIONS)

1. Expressing Functions in Terms of Singularity Functions

(a) Express the following functions of time using a linear combination of singularity functions.

i. $v_1(t) = \begin{cases} 3 & t < 1 \\ -2 & 1 < t < 2 \\ 0 & elsewhere \end{cases}$

$v_1(t) = 3[1 - u(t-1)] + (-2)[u(t-1) - u(t-2)] = 3 + [-3-2]u(t-1) + 2u(t-2)$

$\boxed{v_1(t) = 3 - 5u(t-1) + 2u(t-2)}$. Alternatively, one can express $v_1(t)$ as follows

$\boxed{v_1(t) = 5u(1-t) - 2u(2-t)}$

ii. $x(t) = \begin{cases} t-1 & 1 < t < 2 \\ 1 & 2 < t < 3 \\ -t+4 & 3 < t < 4 \\ 0 & elsewhere \end{cases}$

$x(t) = (t-1)[u(t-1) - u(t-2)] + (1)[u(t-2) - u(t-3)] + (-t+4)[u(t-3) - u(t-4)]$

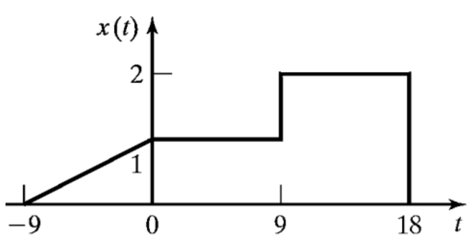
$x(t) = (t-1)u(t-1) + [1-(t-1)]u(t-2) + [-t+4-(1)]u(t-3) + [t-4]u(t-4)$

$x(t) = (t-1)u(t-1) + [2-t]u(t-2) + [-t+3]u(t-3) + [t-4]u(t-4)$

$x(t) = (t-1)u(t-1) - [t-2]u(t-2) - [t-3]u(t-3) + [t-4]u(t-4)$

$\boxed{x(t) = r(t-1) - r(t-2) - r(t-3) + r(t-4)}$

(b) Consider the plot of each of the following functions of time shown on the right. Express each as a linear combination of singularity functions. Simplify each expression as much as possible.



$x(t) = f_1(t)[u(t+9) - u(t)] + (1)[u(t) - u(t-9)] + 2[u(t-9) - u(t-18)]$

$x(t) = f_1(t)u(t+9) + [1 - f_1(t)]u(t) + [2-1]u(t-9) - 2u(t-18)$

$$f_1(t) = f_1(t_1) + m_1(t - t_1) = f(0) + \left(\frac{1}{9}\right)(t-0) \rightarrow \boxed{f_1(t) = 1 + \frac{1}{9}t}$$

$x(t) = \left[1 + \frac{1}{9}t\right]u(t+9) + \left[1 - 1 - \frac{1}{9}t\right]u(t) + u(t-9) - 2u(t-18)$

$x(t) = \left[1 + \frac{1}{9}t\right]u(t+9) - \left[\frac{1}{9}t\right]u(t) + u(t-9) - 2u(t-18)$

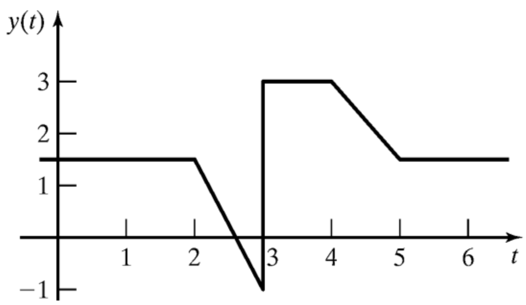
$$x(t)=u(t+9)+\frac{1}{9}tu(t+9)-\left[\frac{1}{9}t\right]u(t)+u(t-9)-2u(t-18)$$

$$x(t)=u(t+9)+\frac{1}{9}(t+9-9)u(t+9)-\frac{1}{9}tu(t)+u(t-9)-2u(t-18)$$

$$x(t)=u(t+9)+\frac{1}{9}(t+9)u(t+9)-u(t+9)-\frac{1}{9}tu(t)+u(t-9)-2u(t-18)$$

$$x(t)=\frac{1}{9}(t+9)u(t+9)-\frac{1}{9}tu(t)+u(t-9)-2u(t-18)$$

$$\boxed{x(t)=\frac{1}{9}r(t+9)-\frac{1}{9}r(t)+u(t-9)-2u(t-18)}$$



$$y(t)=1.5[1-u(t-2)]+f_1(t)[u(t-2)-u(t-3)]+3[u(t-3)-u(t-4)]+f_2(t)[u(t-4)-u(t-5)]+1.5u(t-5)$$

$$y(t)=1.5+[f_1(t)-1.5]u(t-2)+[3-f_1(t)]u(t-3)+[f_2(t)-3]u(t-4)+[1.5-f_2(t)]u(t-5)$$

$$f_1(t)=f_1(t_1)+m_1(t-t_1)=f_1(3)+(-2.5)(t-3)=-1-2.5t+7.5=6.5-2.5t$$

$$f_2(t)=f_2(t_2)+m_2(t-t_2)=f_2(4)+(-1.5)(t-4)=3-1.5t+6=9-1.5t$$

$$y(t)=1.5+[6.5-2.5t-1.5]u(t-2)+[3-6.5+2.5t]u(t-3)+[9-1.5t-3]u(t-4)+[1.5-9+1.5t]u(t-5)$$

$$y(t)=1.5+[5-2.5t]u(t-2)+[-3.5+2.5t]u(t-3)+[6-1.5t]u(t-4)+[-7.5+1.5t]u(t-5)$$

$$y(t)=1.5-[2.5t-5]u(t-2)+[2.5t-3.5]u(t-3)-[1.5t-6]u(t-4)+[1.5t-7.5]u(t-5)$$

$$y(t)=1.5-[2.5(t-\mathbf{2}+\mathbf{2})-5]u(t-2)+[2.5(t-\mathbf{3}+\mathbf{3})-3.5]u(t-3)-[1.5(t-\mathbf{4}+\mathbf{4})-6]u(t-4)+[1.5(t-\mathbf{5}+\mathbf{5})-7.5]u(t-5)$$

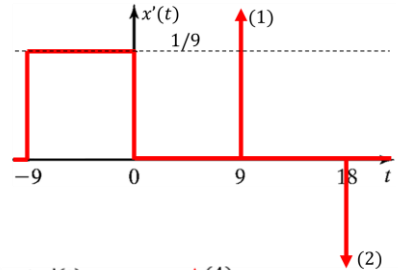
$$y(t)=1.5-[2.5(t-\mathbf{2})+\mathbf{5}-5]u(t-2)+[2.5(t-\mathbf{3})+\mathbf{7.5}-3.5]u(t-3)-[1.5(t-\mathbf{4})+\mathbf{6}-6]u(t-4)+[1.5(t-\mathbf{5})+\mathbf{7.5}-7.5]u(t-5)$$

$$y(t)=1.5-[2.5(t-\mathbf{2})]u(t-2)+[2.5(t-\mathbf{3})+\mathbf{4}]u(t-3)-[1.5(t-\mathbf{4})]u(t-4)+[1.5(t-\mathbf{5})]u(t-5)$$

$$\boxed{y(t)=1.5-2.5r(t-2)+2.5r(t-3)+4u(t-3)-1.5r(t-4)+1.5r(t-5)}$$

2. Sketching Waveforms Involving Singularity Functions

(a) Sketch each of the following functions by hand. Clearly label each sketches. Use MATLAB to plot $f_1(t)$ through $f_4(t)$ and use MATLAB's output to verify your hand sketches.



i. $\frac{d}{dt}x(t)$ (Q1(b)).

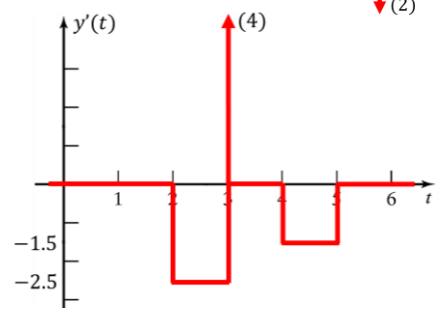
$$x'(t) = (1/9)r'(t+9) - (1/9)r'(t) + u'(t-9) - 2u'(t-18)$$

$$x'(t) = (1/9)u(t+9) - (1/9)u(t) + \delta(t-9) - 2\delta(t-18)$$

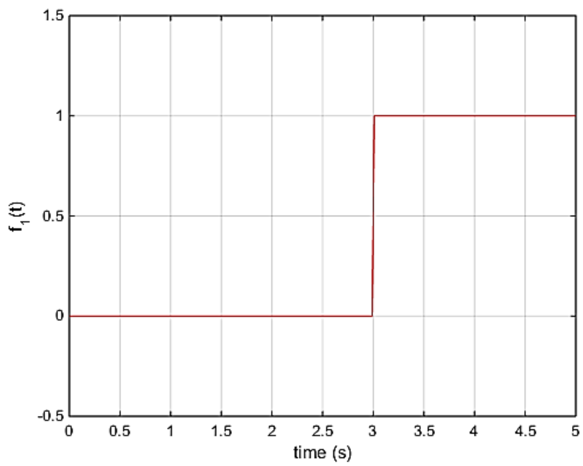
$\frac{d}{dt}y(t)$ (Q1(b))

$$y'(t) = (1.5)' - 2.5r'(t-2) + 2.5r'(t-3) + 4u'(t-3) - 1.5r'(t-4) + 1.5r'(t-5)$$

$$y'(t) = -2.5u(t-2) + 2.5u(t-3) + 4\delta(t-3) - 1.5u(t-4) + 1.5u(t-5)$$

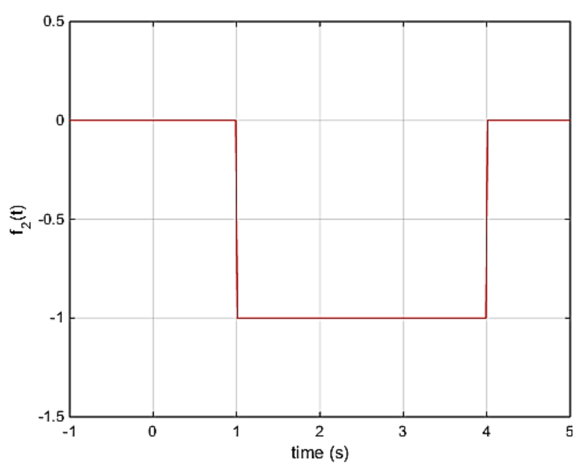


ii. $f_1(t) = u(2t-6)$ – Transition occurs at $2t_0 - 6 = 0 \rightarrow t_0 = 3s$. Note, $u(2t-6) = u(t-3)$



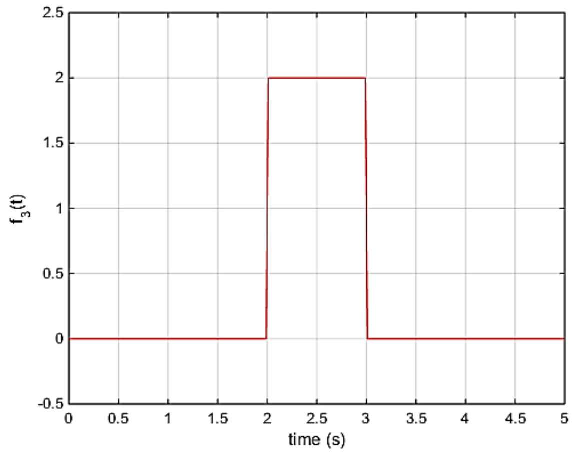

```
clear all;
clc;
close all;
t = -2:0.01:5;
f1 = heaviside(2*t-6);
plot(t,f1,'r');
axis([0, 5, -0.5, 1.5]);
xlabel('time (s)');
ylabel('f_1(t)');
grid on;
```

iii. $f_2(t) = u(t-4) - u(t-1)$ – Note, $f_2(t)$ can be expressed as $f_2(t) = -[u(t-1) - u(t-4)]$



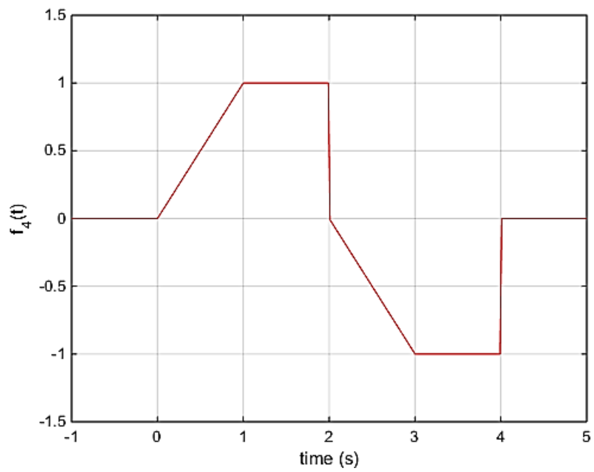

```
clear all;
clc;
close all;
t = -1:0.01:5;
f2 = heaviside(t-4)-heaviside(t-1);
plot(t,f2,'r');
axis([-1, 5, -1.5, 0.5]);
xlabel('time (s)');
ylabel('f_2(t)');
grid on;
```

iv. $f_3(t) = 2u(t-2)u(3-t)$ – Note, $u(t-2)$ transitions at $t_0 = 2$ and is "on" for $t > t_0$ while $u(3-t)$ transitions at $t_0 = 3s$ and is "on" for $t < t_0$. The scaled product $2u(t-2)u(3-t)$ is therefore equivalent to $2[u(t-2) - u(t-3)]$.



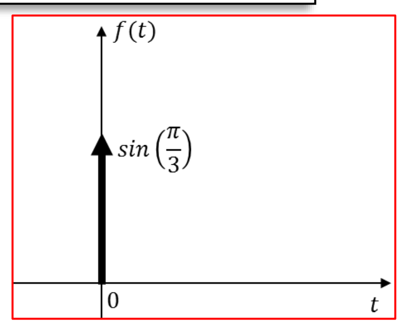

```
clear all;
clc;
close all;
t = 0:0.01:5;
f3 = 2*heaviside(t-2).*heaviside(3-t);
plot(t,f3,'r');
axis([0, 5, -0.5, 2.5]);
xlabel('time (s)');
ylabel('f_3(t)');
grid on;
```

v. $f_4(t) = r(t) - r(t-1) - u(t-2) - r(t-2) + r(t-3) + u(t-4)$




```
clear all;
clc;
close all;
t = -1:0.01:5;
fa = t.*heaviside(t);
fb = (t-1).*heaviside(t-1);
fc = heaviside(t-2);
fd = (t-2).*heaviside(t-2);
fe = (t-3).*heaviside(t-3);
ff = heaviside(t-4);
f4 = fa - fb - fc - fd + fe + ff;
plot(t,f4,'r');
xlabel('time (s)');
ylabel('f_4(t)');
axis([-1, 5, -1.5, 1.5]);
grid on;
```

vi. $f_5(t) = \frac{sin\left(t+\frac{\pi}{3}\right)}{t^2+1}\delta(t)$ - $f_5(t) = \frac{sin\left(0+\frac{\pi}{3}\right)}{(0)^2+1}\delta(t) = sin\left(\frac{\pi}{3}\right)\delta(t)$. So $f_5(t) = 0$ everywhere except at $t = 0$ where there exists an impulse of weight/strength/area of $sin\left(\frac{\pi}{3}\right)$.



vii. $f_6(t) = [e^{2t^2} - e^{-2t^2}][\delta(t+1) - \delta(t-1)]$

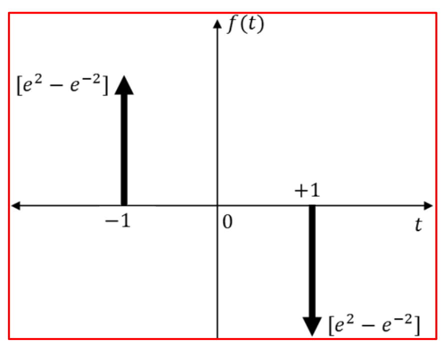
$$f_6(t) = \left[e^{2t^2} - e^{-2t^2}\right][\delta(t+1) - \delta(t-1)]$$

$$f_6(t) = \left[e^{2t^2} - e^{-2t^2}\right]\delta(t+1) + \left[e^{-2t^2} - e^{2t^2}\right]\delta(t-1)$$

$$f_6(t) = \left[e^{2(-1)^2} - e^{-2(-1)^2}\right]\delta(t+1) + \left[e^{-2(1)^2} - e^{2(1)^2}\right]\delta(t-1)$$

$$f_6(t) = [e^2 - e^{-2}]\delta(t+1) + [e^{-2} - e^2]\delta(t-1)$$

So $f_6(t) = 0$ everywhere except at $t = +1$ and $t = -1$ where there exists an impulse. The impulse at $t = -1$ has weight/strength/area of $[e^2 - e^{-2}]$ while the impulse at $t = +1$ has weight/strength/area of $-[e^2 - e^{-2}]$.



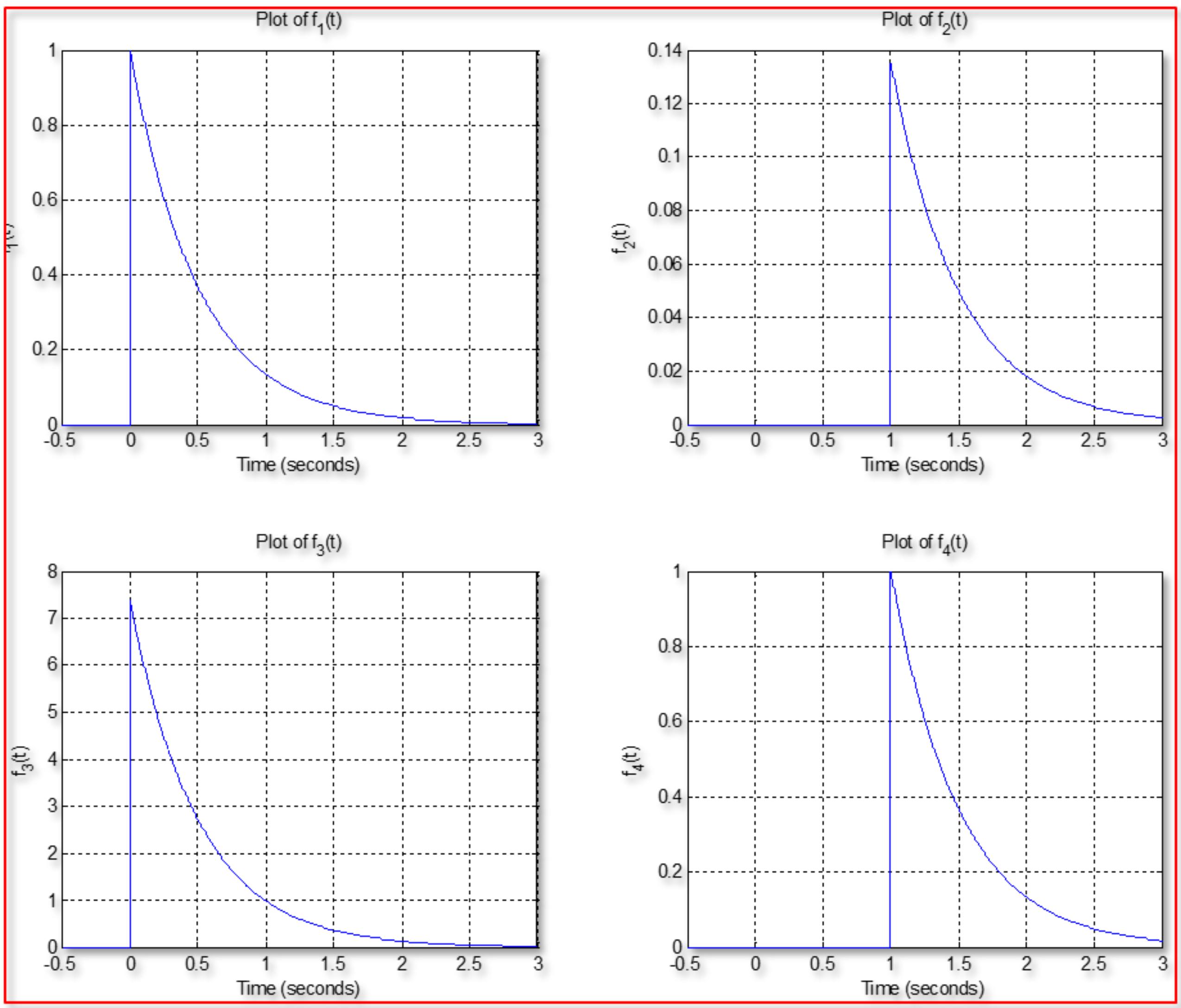
(b) Sketch each of the following "similar" time functions. Clearly label each sketch. Note the similarities and differences among the four waveforms.

i. $f_1(t) = e^{-2t}u(t)$

ii. $f_2(t) = e^{-2t}u(t-1)$

iii. $f_3(t) = e^{-2(t-1)}u(t)$

iv. $f_4(t) = e^{-2(t-1)}u(t-1)$



3. Evaluating Derivative and Integral Expressions Involving Singularity Functions

(a) Apply the sifting property of the Dirac Delta/Impulse to evaluate each of the following integrals.

i. $\int_{-3}^{2} \cos(t)\,\delta(t)dt$

$$\int_{-3}^{2} \cos(t)\,\delta(t)dt = \int_{0^-}^{0^+} \cos(0)\,\delta(t)dt = \cos(0)\int_{0^-}^{0^+} \delta(t)dt = \cos(0)\,1 = \boxed{1}$$

ii. $\int_{-3}^{2} t^2\,\delta(t-1)dt$

$$\int_{-3}^{2} t^2\,\delta(t-1)dt = \int_{1^-}^{1^+} (1)^2\delta(t-1)dt = (1)^2\int_{1^-}^{1^+}\delta(t-1)dt = (1)^2 1 = \boxed{1}$$

iii. $\int_{-3}^{1} ln(t)\,\delta(t-2)dt$

$\boxed{\int_{-3}^{1}\ln(t)\,\delta(t-2)dt = 0}$ since the limits do not include the location of the impulse at $t = 2$.

iv. $\int_{-3}^{5^+} sin(t)\,\delta(t-5)dt$

$$\int_{-3}^{5^+} sin(t)\,\delta(t-5)dt = \int_{5^-}^{5^+}\sin(5)\,\delta(t-5)dt = \sin(5)\int_{5^-}^{5^+}\delta(t-5)dt = \sin(5)(1) = \boxed{sin(5)}$$

v. $\int_{-3^+}^{4^-} e^{-5t}\,\delta(t-4)dt$

$\boxed{\int_{-3^+}^{4^-} e^{-5t}\,\delta(t-4)dt = 0}$ since the limits do not include the location of the impulse at $t = 4$.

vi. $\int_{-3^-}^{3^+} e^{t^2}\,\delta(t+3)dt$

$$\int_{-3^-}^{3^+} e^{t^2}\,\delta(t+3)dt = \int_{-3^-}^{-3^+} e^{(-3)^2}\,\delta(t+3)dt = e^9\int_{-3^-}^{-3^+}\delta(t+3)dt = (e^9)(1) = \boxed{e^9}$$

(b) Evaluate the following expressions involving time derivatives and singularity functions.

i. $f_1(t) = [u(t+1)u(t-1)]'$

$$\boxed{\begin{aligned}&\frac{d}{dt}[u(t-1)\,u(t+1)] = \delta(t-1)u(t+1) + \\ &u(t-1)\delta(t+1) = \delta(t-1)1 + 0\delta(t+1) = \underline{\delta(t-1)}\end{aligned}}$$

ii. $f_2(t) = [r(t-6)u(t-2)]'$

$$\boxed{\begin{aligned}&\frac{d}{dt}[r(t-6)\,u(t-2)] = u(t-6)u(t-2) + \\ &r(t-6)\delta(t-2) = u(t-6)1 + 0\delta(t-2) = \underline{u(t-6)}\end{aligned}}$$

iii. $f_3(t) = \left[\sin(4t)\,u\left(t-\frac{\pi}{8}\right)\right]'$

$$f_3(t) = \sin'(4t)\,u\left(t-\frac{\pi}{8}\right) + \sin(4t)\,u'\left(t-\frac{\pi}{8}\right)$$

$$f_3(t) = 4\cos(4t)\,u\left(t-\frac{\pi}{8}\right) + \sin(4t)\,\delta\left(t-\frac{\pi}{8}\right)$$

$$f_3(t) = 4\cos(4t)\,u\left(t-\frac{\pi}{8}\right) + \sin\left(4\left(\frac{\pi}{8}\right)\right)\delta\left(t-\frac{\pi}{8}\right)$$

$$\boxed{f_3(t) = 4\cos(4t)\,u\left(t-\frac{\pi}{8}\right) + \delta\left(t-\frac{\pi}{8}\right)}$$