Purdue University Northwest

Lecture #6(a): Frequency Response I:
Frequency Response, SSS, and Bode Diagrams: Theory

ECE 20200: Linear Circuit Analysis II
Steve Naumov (Instructor)

Lecture Overview

- This set of notes presents
  - Frequency Response, Network Functions, and Sinusoidal Steady State Response
  - Review of Logs, The Bel and Decibel Scales
  - Bode Diagrams
  - Sketching of Bode Diagrams

Lecture #6(a): Frequency Response I: *Frequency Response, SSS, and Bode Diagrams: Theory*

Frequency Response, Network Functions, and SSS

System Functions & Responses: Frequency Response

- **Frequency Response** $x_{o,freq}(t)$ is the <u>response</u> of a <u>stable linear system</u> to a sinusoidal excitation $x_i(t) = A_m cos(\omega t + \theta)u(t) = \boldsymbol{Re}[A_m e^{j(\omega t+\theta)}]u(t)$.
  - When considering the frequency response of a stable linear system, we are typically interested in the <u>steady state</u> portion of the response
- Time-Domain Block Diagram Representation

$$x_i(t) = \boldsymbol{Re}[A_m e^{j(\omega t+\theta)}]\,u(t) \longrightarrow \boxed{\begin{array}{c}\text{Stable LTI}\\ \text{System}\\ g(t)\end{array}} \longrightarrow \boldsymbol{Re}\left[x_{o,freq}(t)\right] = \boldsymbol{Re}[x_{o,sss}(t) + x_{o,tr}(t)]$$

- s-Domain Block Diagram Representation

$$\boldsymbol{X_i(s)} = \boldsymbol{Re}\left[\frac{A_m e^{j\theta}}{\boldsymbol{s} - j\omega}\right] \longrightarrow \boxed{\begin{array}{c}\text{Stable LTI}\\ \text{System}\\ \boldsymbol{G(s)}\end{array}} \longrightarrow \begin{array}{l}\boldsymbol{Re}\left[\boldsymbol{X_{o,freq}(s)}\right] = \boldsymbol{Re}[\boldsymbol{X_{o,sss}(s)} + \boldsymbol{X_{o,tr}(s)}]\\ \boldsymbol{Re}\left[\boldsymbol{X_{o,freq}(s)}\right] = \boldsymbol{Re}\left[\boldsymbol{G(s)}\,\frac{A_m e^{j\theta}}{\boldsymbol{s}-j\omega}\right]\end{array}$$

- We will find $\boldsymbol{X_{o,sss}(s)}$ and $x_{o,sss}(t)$ by assuming $\boldsymbol{X_i(s)} = \frac{A_m e^{j\theta}}{\boldsymbol{s}-j\omega}$ and then taking the real part of $\boldsymbol{X_{o,sss}(s)}$ and $x_{o,sss}(t)$

System Functions & Responses: Frequency Response

- *s*-Domain Block Diagram Representation

$$\boldsymbol{X_i(s)} = \boldsymbol{Re}\left[\frac{A_m e^{j\theta}}{\boldsymbol{s} - j\omega}\right] \longrightarrow \boxed{\begin{array}{c}\text{Stable LTI}\\ \text{System}\\ \boldsymbol{G(s)}\end{array}} \longrightarrow \begin{array}{l}\boldsymbol{Re}\left[\boldsymbol{X_{o,freq}(s)}\right] = \boldsymbol{Re}\left[\boldsymbol{X_{o,sss}(s)} + \boldsymbol{X_{o,tr}(s)}\right]\\ \boldsymbol{Re}\left[\boldsymbol{X_{o,freq}(s)}\right] = \boldsymbol{Re}\left[\boldsymbol{G(s)}\frac{A_m e^{j\theta}}{\boldsymbol{s}-j\omega}\right]\end{array}$$

- Find $\boldsymbol{X_{o,sss}(s)}$ using partial fraction expansion

$$\boldsymbol{X_{o,freq}(s)} = \boldsymbol{X_{o,sss}(s)} + \boldsymbol{X_{o,tr}(s)} = \frac{A_m e^{j\theta}}{\boldsymbol{s} - j\omega}\boldsymbol{G(s)} = \frac{\boldsymbol{r_0}}{\boldsymbol{s} - j\omega} + \frac{r_1}{\boldsymbol{s} + p_1} + \cdots + \frac{r_n}{\boldsymbol{s} + p_n}$$

$$A_m e^{j\theta}\boldsymbol{G(s)} = r_0 + \frac{r_1}{\boldsymbol{s} + p_1}(\boldsymbol{s} - j\omega) + \cdots + \frac{r_n}{\boldsymbol{s} + p_n}(\boldsymbol{s} - j\omega)$$

$$\left[A_m e^{j\theta}\boldsymbol{G(s)}\right]\Big|_{\boldsymbol{s}=j\omega} = \left[r_0 + \frac{r_1}{\boldsymbol{s} + p_1}(\boldsymbol{s} - j\omega) + \cdots + \frac{r_n}{\boldsymbol{s} + p_n}\left(\boldsymbol{s} - j\omega\right)\right]\Big|_{\boldsymbol{s}=j\omega}$$

$$\boxed{r_0 = A_m e^{j\theta}\boldsymbol{G}(j\omega) = A_m e^{j\theta}|\boldsymbol{G}(j\omega)|e^{j\angle\boldsymbol{G}(j\omega)} = A_m|\boldsymbol{G}(j\omega)|e^{j[\theta + \angle\boldsymbol{G}(j\omega)]}}$$

$$\boxed{\boldsymbol{X_{o,sss}(s)} = \frac{A_m|\boldsymbol{G}(j\omega)|e^{j[\theta+\angle\boldsymbol{G}(j\omega)]}}{\boldsymbol{s} - j\omega}}$$

System Functions & Responses: Frequency Response

- s-Domain Block Diagram Representation

$$\boldsymbol{X_i(s)} = \boldsymbol{Re}\left[\frac{A_m e^{j\theta}}{\boldsymbol{s} - j\omega}\right] \longrightarrow \boxed{\begin{array}{c}\text{Stable LTI}\\ \text{System}\\ \boldsymbol{G(s)}\end{array}} \longrightarrow \begin{array}{l}\boldsymbol{Re}[\boldsymbol{X_{o,freq}(s)}] = \boldsymbol{Re}[\boldsymbol{X_{o,sss}(s)} + \boldsymbol{X_{o,tr}(s)}]\\ \boldsymbol{Re}[\boldsymbol{X_{o,freq}(s)}] = \boldsymbol{Re}\left[\boldsymbol{G(s)}\frac{A_m e^{j\theta}}{\boldsymbol{s}-j\omega}\right]\end{array}$$

$$\boxed{\boldsymbol{X_{o,sss}(s)} = \frac{A_m|\boldsymbol{G}(j\omega)|e^{j[\theta+\angle G(j\omega)]}}{\boldsymbol{s} - j\omega}}$$

- Find $x_{o,sss}(t) = \boldsymbol{Re}\left(\mathcal{L}^{-1}\left[\boldsymbol{X_{o,sss}(s)}\right]\right)$ using Inverse Transform

$$x_{o,sss}(t) = \boldsymbol{Re}\left(\mathcal{L}^{-1}\left[\frac{A_m|\boldsymbol{G}(j\omega)|e^{j[\theta+\angle \boldsymbol{G}(j\omega)]}}{\boldsymbol{s} - j\omega}\right]\right)u(t)$$

$$x_{o,sss}(t) = \boldsymbol{Re}\left[A_m|\boldsymbol{G}(j\omega)|e^{j[\theta+\angle \boldsymbol{G}(j\omega)]}e^{j\omega t}\right]u(t)$$

$$x_{o,sss}(t) = \boldsymbol{Re}\left[A_m|\boldsymbol{G}(j\omega)|e^{j[\omega t+\theta+\angle \boldsymbol{G}(j\omega)]}\right]u(t)$$

$$\boxed{x_{o,sss}(t) = A_m|\boldsymbol{G}(j\omega)|\cos\left(\omega t + \theta + \angle \boldsymbol{G}(j\omega)\right)u(t)}$$

System Functions & Responses: Frequency Response

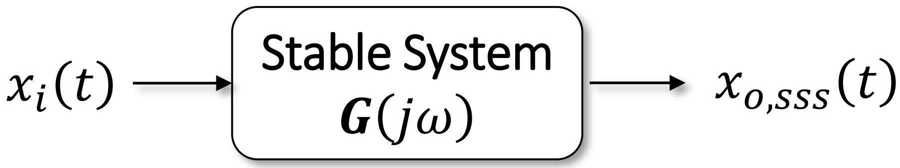
$$x_i(t) \longrightarrow \boxed{\begin{array}{c}\text{Stable LTI}\\ \text{System } \boldsymbol{G}(j\omega)\end{array}} \longrightarrow x_{o,sss}(t)$$

$$x_i(t) = A_m \cos(\omega t + \theta)\, u(t)$$

$$x_{o,sss}(t) = A_m |\boldsymbol{G}(j\omega)| \cos\big(\omega t + \theta + \angle \boldsymbol{G}(j\omega)\big)\, u(t)$$

- If a <u>stable linear system</u> is excited by a sinusoid, the <u>steady state response</u> $x_{o,sss}(t)$ of the system is also a sinusoid.
- In the <u>steady state</u>, the <u>stable linear system</u>
  - operates at the same frequency as the input sinusoid: ω
  - scales the amplitude A_m of the input sinusoid by $|\boldsymbol{G}(j\omega)|$: $A_m|\boldsymbol{G}(j\omega)|$
  - offsets the phase angle θ of the input sinusoid by $\angle\boldsymbol{G}(j\omega)$: $\theta + \angle\boldsymbol{G}(j\omega)$
- If $\boldsymbol{G}(j\omega)$ is known and we know how $|\boldsymbol{G}(j\omega)|$ and$\angle\boldsymbol{G}(j\omega)$ varies as a function of ω, $x_{o,sss}(t)$ of any <u>stable linear system</u> can be found

System Functions & Responses: Frequency Response



$$x_i(t) = A_m \cos(\omega t + \theta)\, u(t)$$

$$x_{o,sss}(t) = A_m |G(j\omega)| \cos\big(\omega t + \theta + \angle G(j\omega)\big)\, u(t)$$

- How to find $x_{o,sss}(t)$ using the system function of a stable linear system
  - Find the frequency response function $\boldsymbol{G}(j\omega) = \boldsymbol{X_o}(j\omega)/\boldsymbol{X_i}(j\omega)$

$$\boldsymbol{G}(j\omega) = \boldsymbol{G}(\boldsymbol{s})\Big|_{\boldsymbol{s}=j\omega} = \frac{\boldsymbol{X_o}(\boldsymbol{s})}{\boldsymbol{X_i}(\boldsymbol{s})}\Big|_{\boldsymbol{s}=j\omega} = \frac{\boldsymbol{X_o}(j\omega)}{\boldsymbol{X_i}(j\omega)}$$

  - Represent $\boldsymbol{G}(j\omega)$ in complex exponential form $\boldsymbol{G}(j\omega) = |\boldsymbol{G}(j\omega)| e^{j\angle \boldsymbol{G}(j\omega)}$

$$|\boldsymbol{G}(j\omega)| = \left|\frac{\boldsymbol{X_o}(j\omega)}{\boldsymbol{X_i}(j\omega)}\right| = \frac{|\boldsymbol{X_o}(j\omega)|}{|\boldsymbol{X_i}(j\omega)|} \qquad \angle \boldsymbol{G}(j\omega) = \angle\left(\frac{\boldsymbol{X_o}(j\omega)}{\boldsymbol{X_i}(j\omega)}\right) = \angle \boldsymbol{X_o}(j\omega) - \angle \boldsymbol{X_i}(j\omega)$$

  - Express $x_{o,sss}(t)$

$$\boxed{x_{o,sss}(t) = A_m |\boldsymbol{G}(j\omega)| \cos\big(\omega t + \theta + \angle \boldsymbol{G}(j\omega)\big)\, u(t)}$$

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Review of Logs, The Bel and Decibel Scales

The Bel, the Decibel, and Logarithm Properties

- **Review**: Properties of Logarithms

$$\log(xy) = \log(x) + \log(y) \qquad \log(x/y) = \log(x) - \log(y)$$

$$log(x^n) = n\log(x) \qquad log(1) = 0$$

$$20\log(\sqrt{2}) \approx 3 \qquad 20\log(1/\sqrt{2}) \approx -3$$

- **Bel**: Unit of measure that expresses the base-10 logarithm of a ratio of two physical quantities usually of the same type

$$\boxed{\#\ of\ Bels = \log_{10}(A_2/A_1)}$$

- **Decibel**: Unit of measure that is a factor of 10 <u>smaller</u> than a Bel

$$\boxed{\#\ of\ Decibels = 10 \times \#\ of\ Bels = 10 \times \log_{10}(A_2/A_1)}$$

$$\boxed{\frac{\#\ of\ Decibels}{\#\ of\ Bels} = \frac{10 \times \log_{10}(A_2/A_1)}{\log_{10}(A_2/A_1)} = 10}$$

The Decibel Scale: Power, Voltage, and Current

- For ECE, the decibel scale usually is applied to:
  - Power: P_{out}/P_{in} Voltage: V_{out}/V_{in} Current: I_{out}/I_{in}
- **Power Gains in Decibels (dBs)** $\boxed{H_{P,dB} = 10\log_{10}(P_{out}/P_{in})}$
- **Voltage Gains in Decibels (dBs)** $\boxed{H_{V,dB} = 20\log_{10}(V_{out}/V_{in})}$

$$H_{P,dB} = 10\log_{10}\left(\frac{V_{out}^2}{R_{out}} / \frac{V_{in}^2}{R_{in}}\right) = 10\log_{10}\left(\frac{V_{out}^2}{V_{in}^2}\frac{R_{in}}{R_{out}}\right)$$

$$H_{P,dB} = \boxed{20\log_{10}\left(\frac{V_{out}}{V_{in}}\right)} + \boxed{10\log_{10}\left(\frac{R_{in}}{R_{out}}\right)} = \boldsymbol{H_{V,dB}} + constant$$

- **Current Gains in Decibels (dB's)** $\boxed{H_{I,dB} = 20\log_{10}(I_{out}/I_{in})}$

$$H_{P,dB} = \boxed{20\log_{10}\left(\frac{I_{out}}{I_{in}}\right)} + \boxed{10\log_{10}\left(\frac{R_{out}}{R_{in}}\right)} = \boldsymbol{H_{I,dB}} + constant$$

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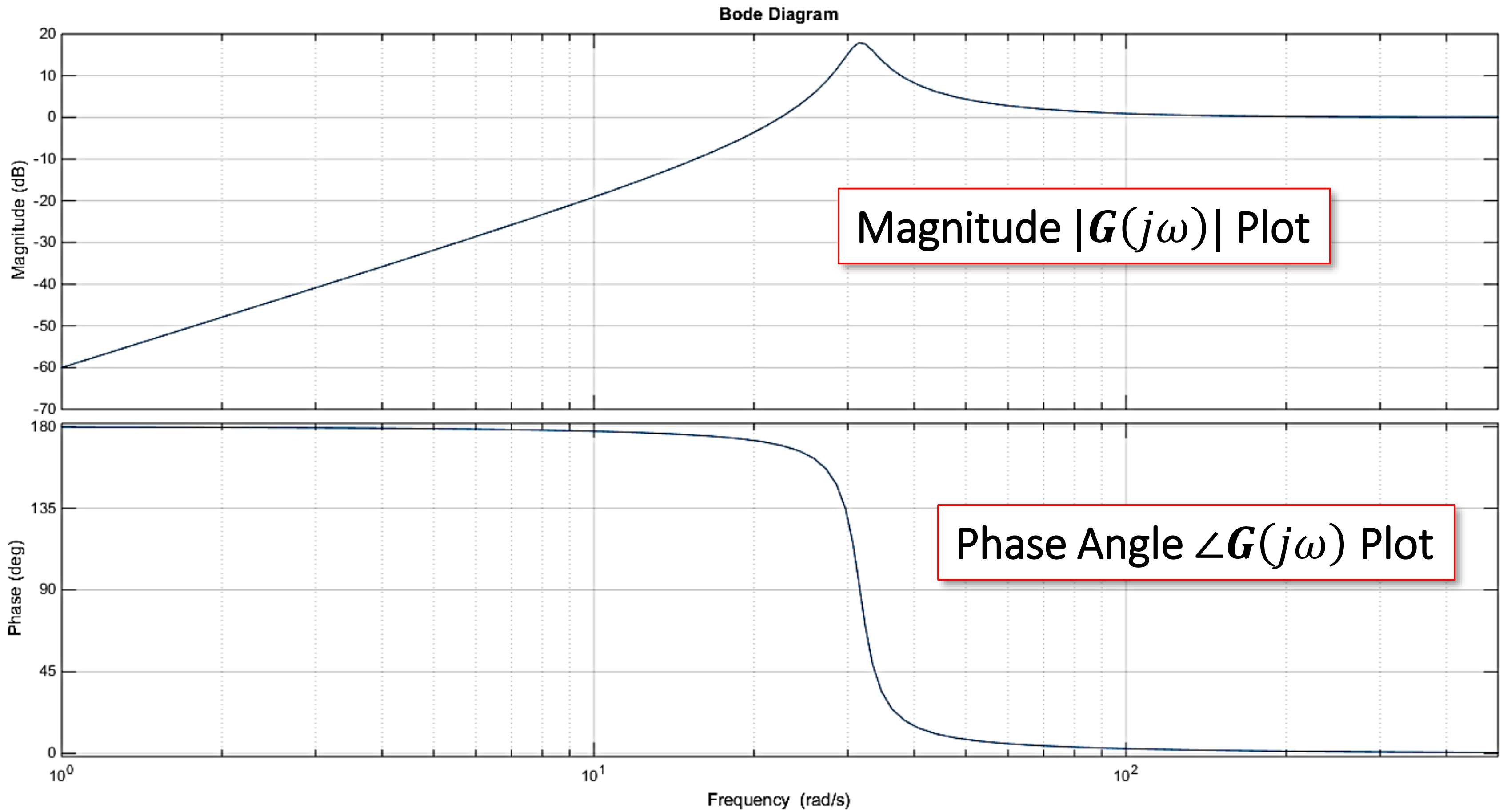
Bode Diagrams

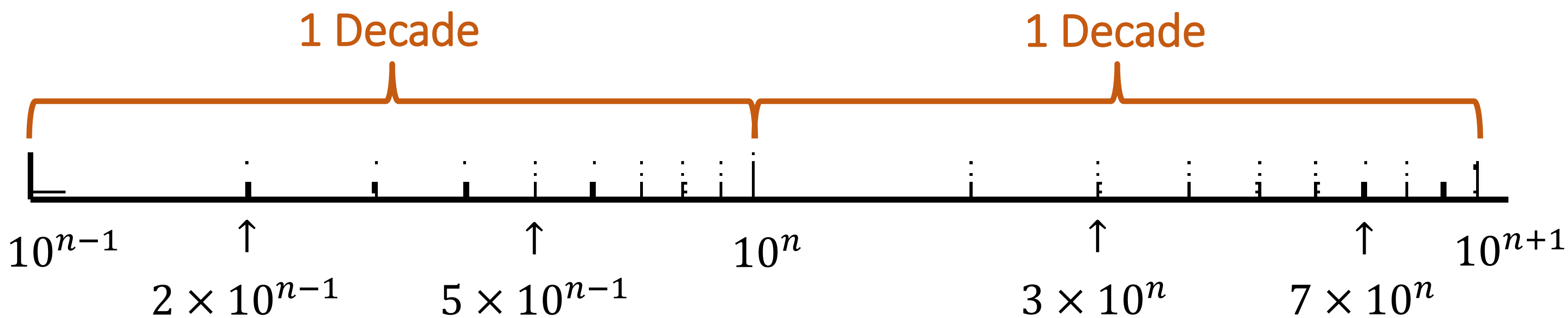
Bode Plots/Diagram: Overview

- **Bode Plots**: <u>Logarithmic</u> plots of the **<u>magnitude</u>** $|\boldsymbol{G}(j\omega)|$ and **<u>phase angle</u>** $\angle\boldsymbol{G}(j\omega)$ of a frequency response function $\boldsymbol{G}(j\omega)$ w.r.t. frequency



Magnitude Plot of a Bode Diagram

- The **magnitude plot** of a Bode Diagram is a semi-log-x plot of the magnitude of the gain $|\boldsymbol{G}|$ of as a function of frequency
  - **Horizontal Axis**: Represents ω in rad/s using a base-10 log scale

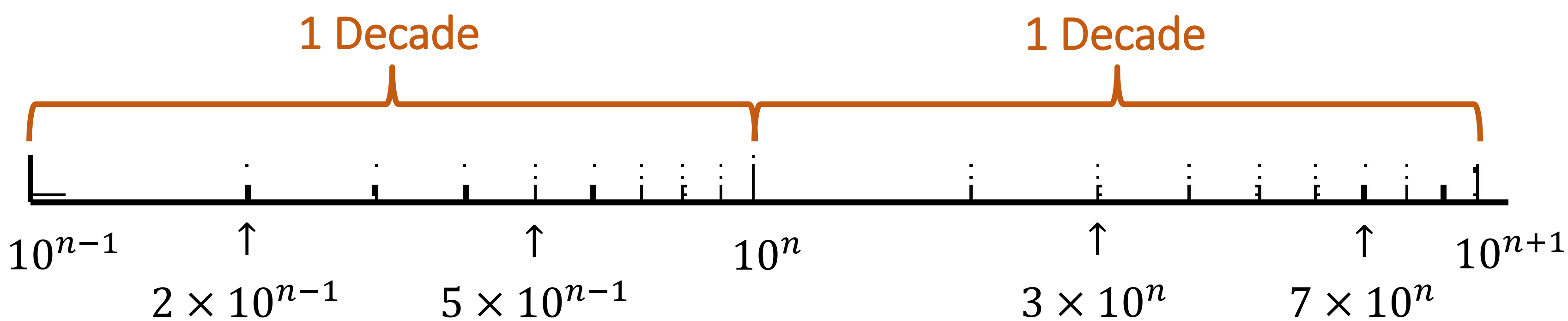


  - **Vertical Axis**: Represents $|\boldsymbol{G}|$ in decibels (dB's) on a linear scale (i.e. vertical axis represents the quantity $|\boldsymbol{G}|_{dB} = 20\log_{10}(|\boldsymbol{G}|)$)

$\lvert\boldsymbol{G}\rvert$	10^{-3}	10^{-2}	10^{-1}	2^{-1}	$\sqrt{2}^{\,-1}$	10^{0}	$\sqrt{2}$	2	10^{1}	10^{2}	10^{3}
$\lvert\boldsymbol{G}\rvert_{dB}$	-60	-40	-20	-6	-3	0	3	6	20	40	60

Phase Angle Plot of a Bode Diagram

- The **phase angle plot** of a Bode Diagram is a semi-log-x plot of the gain's phase angle $\angle \boldsymbol{G}$ as a function of frequency
  - **Horizontal Axis**: Represents ω in rad/s using a base-10 log scale



  - **Vertical Axis**: Represents $\angle \boldsymbol{G}$ in degrees or radians on a linear scale

Approximately Plotting Bode Diagrams: Overview

- Until the late 1980's, engineers hand-drew approximations of Bode plots
  - Many rules of rules-of-thumb, tables, and templates to help!
- Now, engineer's use MATLAB (or equivalent) to draw exact Bode plots
- Why teach hand-drawing approximations to Bode plots?
  - It gives the engineer insight as to how poles, zeros, and the scale factor of a system function influence the Bode plot
  - These ideas are used for transfer function synthesis, analog circuit design, control system design, etc.
- We will discuss simple methods of hand-drawing approximate Bode plots based on the asymptotic behavior of $\boldsymbol{G}(j\omega)$

Approximately Plotting Bode Diagrams

- **Step #1**: Represent the $\boldsymbol{G(s)}$ in <u>standard (normalized) Bode form</u>

$$\boldsymbol{G}(\boldsymbol{s}) = K\frac{(\boldsymbol{s}+z_1)\times\cdots\times(\boldsymbol{s}+z_m)}{(\boldsymbol{s}+p_1)\times\cdots\times(\boldsymbol{s}+p_n)}$$

$$\boldsymbol{G}(\boldsymbol{s}) = \left(K\frac{z_1 \ldots z_m}{p_1 \ldots p_n}\right)\frac{(\boldsymbol{s}/z_1+1)\times\cdots\times(\boldsymbol{s}/z_m+1)}{(\boldsymbol{s}/p_1+1)\times\cdots\times(\boldsymbol{s}/p_n+1)}$$

$$\boxed{\boldsymbol{G}(\boldsymbol{s}) = K_0\frac{(\boldsymbol{s}/z_1+1)\times\cdots\times(\boldsymbol{s}/z_m+1)}{(\boldsymbol{s}/p_1+1)\times\cdots\times(\boldsymbol{s}/p_n+1)}}$$

- **Step #2**: Find $\boldsymbol{G}(j\omega)$ of $\boldsymbol{G}(\boldsymbol{s})$ represented in <u>standard Bode form</u>.

$$\boldsymbol{G}(j\omega) = K_o\frac{(j\omega/z_1+1)\times\cdots\times(j\omega/z_m+1)}{(j\omega/p_1+1)\times\cdots\times(j\omega/p_n+1)} = K_o\frac{\prod_{k=1}^{m}(j\omega/z_k+1)}{\prod_{k=1}^{n}(j\omega/p_k+1)}$$

$$\boxed{\boldsymbol{G}(j\omega) = K_o\frac{\boldsymbol{N}_1(j\omega)\times\cdots\times\boldsymbol{N}_m(j\omega)}{\boldsymbol{D}_1(j\omega)\times\cdots\times\boldsymbol{D}_n(j\omega)} = K_o\frac{\prod_{k=1}^{m}\boldsymbol{N}_k(j\omega)}{\prod_{k=1}^{n}\boldsymbol{D}_k(j\omega)}}$$

Approximately Plotting Bode Diagrams

- **Step #3(a)**: Find the magnitude $|\boldsymbol{G}(j\omega)|$ of $\boldsymbol{G}(j\omega)$

$$|\boldsymbol{G}(j\omega)| = \left| K_o \frac{(j\omega/z_1 + 1) \times \cdots \times (j\omega/z_m + 1)}{(j\omega/p_1 + 1) \times \cdots \times (j\omega/p_n + 1)} \right|$$

$$|\boldsymbol{G}(j\omega)| = \frac{|K_o(j\omega/z_1 + 1) \times \cdots \times (j\omega/z_m + 1)|}{|(j\omega/p_1 + 1) \times \cdots \times (j\omega/p_n + 1)|}$$

$$|\boldsymbol{G}(j\omega)| = |K_0| \frac{|j\omega/z_1 + 1| \times \cdots \times |j\omega/z_m + 1|}{|j\omega/p_1 + 1| \times \cdots \times |j\omega/p_n + 1|}$$

$$|\boldsymbol{G}(j\omega)| = |K_0| \frac{|\boldsymbol{N}_1(j\omega)| \times \cdots \times |\boldsymbol{N}_m(j\omega)|}{|\boldsymbol{D}_1(j\omega)| \times \cdots \times |\boldsymbol{D}_n(j\omega)|}$$

Approximately Plotting Bode Diagrams

- **Step #3(b)**: Find the Decibel magnitude $|\boldsymbol{G}(j\omega)|_{dB}$ of $\boldsymbol{G}(j\omega)$

$$|\boldsymbol{G}(j\omega)|_{dB} = 20\log_{10}\left(|K_0|\frac{|j\omega/z_1+1|\times\cdots\times|j\omega/z_m+1|}{|j\omega/p_1+1|\times\cdots\times|j\omega/p_n+1|}\right)$$

$$|\boldsymbol{G}(j\omega)|_{dB} = 20\log_{10}(|K_0||j\omega/z_1+1|\times\cdots\times|j\omega/z_m+1|)$$
$$-20\log_{10}(|j\omega/p_1+1|\times\cdots\times|j\omega/p_n+1|)$$

$$|\boldsymbol{G}(j\omega)|_{dB} = 20\log_{10}(|K_0|) + \sum_{k=1}^{m} 20\log_{10}(|j\omega/z_k+1|)$$
$$-\sum_{k=1}^{n} 20\log_{10}(|j\omega/p_k+1|)$$

$$|\boldsymbol{G}(j\omega)|_{dB} = |K_0|_{dB} + \sum_{k=1}^{m}|j\omega/z_k+1|_{dB} - \sum_{k=1}^{n}|j\omega/p_k+1|_{dB}$$

$$\boxed{|\boldsymbol{G}(j\omega)|_{dB} = |K_0|_{dB} + \sum_{k=1}^{m}|\boldsymbol{N}_k(j\omega)|_{dB} - \sum_{k=1}^{n}|\boldsymbol{D}_k(j\omega)|_{dB}}$$

  - Approximate plot of $|\boldsymbol{G}(j\omega)|_{dB}$ may be obtained by superimposing the approximate plot of the Decibel magnitude of each term!

Approximately Plotting Bode Diagrams

- **Step #4**: Find the phase angle $\angle\boldsymbol{G}(j\omega)$ of $\boldsymbol{G}(j\omega)$

$$\boldsymbol{G}(j\omega) = |\boldsymbol{G}(j\omega)|e^{j\angle\boldsymbol{G}(j\omega)} = |K_0|e^{j\theta_0}\frac{|j\omega/z_1+1|e^{j\theta_1}\times\cdots\times|j\omega/z_m+1|e^{j\theta_m}}{|j\omega/p_1+1|e^{j\phi_1}\times\cdots\times|j\omega/p_n+1|e^{j\phi_n}}$$

$$\boldsymbol{G}(j\omega) = |K_0|\frac{|j\omega/z_1+1|\times\cdots\times|j\omega/z_m+1|e^{j\sum_{k=0}^{m}\theta_k}}{|j\omega/p_1+1|\times\cdots\times|j\omega/p_n+1|e^{j\sum_{k=1}^{n}\phi_k}}$$

$$\boldsymbol{G}(j\omega) = |K_0|\frac{|j\omega/z_1+1|\times\cdots\times|j\omega/z_m+1|}{|j\omega/p_1+1|\times\cdots\times|j\omega/p_n+1|}e^{j\left(\sum_{k=0}^{m}\theta_k-\sum_{k=1}^{n}\phi_k\right)}$$

$$\angle\boldsymbol{G}(j\omega) = \angle K_0 + \sum_{k=1}^{m}\angle(j\omega/z_k+1) - \sum_{k=1}^{n}\angle(j\omega/p_k+1)$$

$$\boxed{\angle\boldsymbol{G}(j\omega) = \theta_0 + \sum_{k=1}^{m}\theta_k - \sum_{k=1}^{n}\phi_k}$$

  - Approximate plot of $\angle\boldsymbol{G}(j\omega)$ may be obtained by superimposing the approximate plot of the phase angle of each term!

Approximately Plotting Bode Diagrams: Summary

- **Step #1**: Represent the $\boldsymbol{G}(\boldsymbol{s})$ in <u>standard (normalized) Bode form</u>

$$\boxed{\boldsymbol{G}(\boldsymbol{s}) = K_0 \frac{(\boldsymbol{s}/z_1 + 1) \times \cdots \times (\boldsymbol{s}/z_m + 1)}{(\boldsymbol{s}/p_1 + 1) \times \cdots \times (\boldsymbol{s}/p_n + 1)}}$$

- **Step #2**: Find $\boldsymbol{G}(j\omega)$ of $\boldsymbol{G}(\boldsymbol{s})$ represented in <u>standard Bode form</u>.

$$\boxed{\boldsymbol{G}(j\omega) = K_o \frac{(j\omega/z_1 + 1) \dots (j\omega/z_m + 1)}{(j\omega/p_1 + 1) \dots (j\omega/p_n + 1)} = K_o \frac{\prod_{k=1}^{m}(j\omega/z_k + 1)}{\prod_{k=1}^{n}(j\omega/p_k + 1)}}$$

- **Step #3(b)**: Find the Decibel magnitude $|\boldsymbol{G}(j\omega)|_{dB}$ of $\boldsymbol{G}(j\omega)$

$$\boxed{|\boldsymbol{G}(j\omega)|_{dB} = |K_0|_{dB} + \sum_{k=1}^{m} |j\omega/z_k + 1|_{dB} - \sum_{k=1}^{n} |j\omega/p_k + 1|_{dB}}$$

- **Step #4**: Find the phase angle $\angle\boldsymbol{G}(j\omega)$ of $\boldsymbol{G}(j\omega)$

$$\boxed{\angle\boldsymbol{G}(j\omega) = \angle K_0 + \sum_{k=1}^{m} \angle(j\omega/z_k + 1) - \sum_{k=1}^{n} \angle(j\omega/p_k + 1)}$$

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Sketching Bode Diagrams: Magnitude of Real Zeros/Poles

Plotting dB Magnitude of Constant Terms $G(j\omega) = K_0$

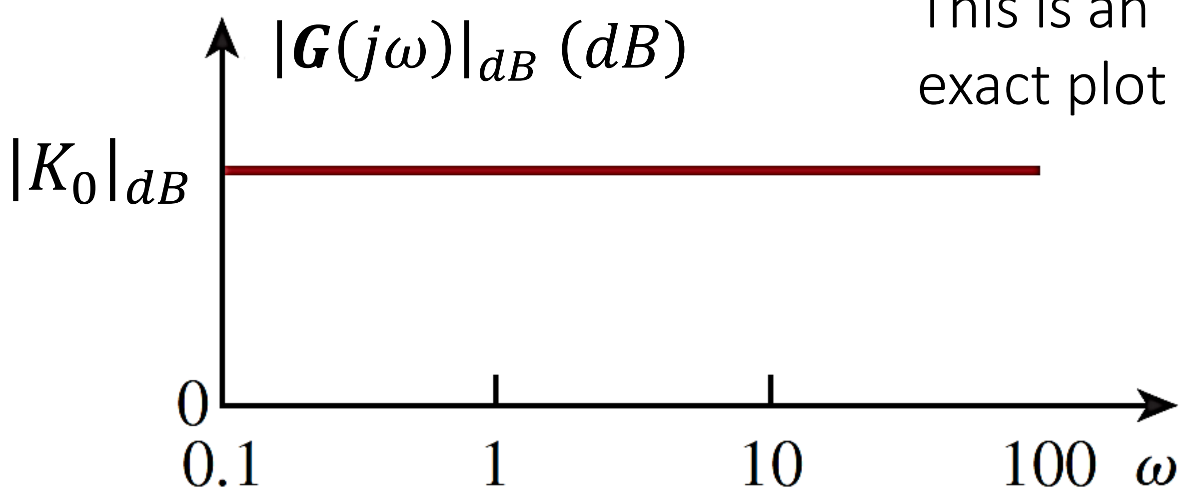
- Magnitude $|G(j\omega)|$ of $G(j\omega) = K_0$

$$\boxed{|G(j\omega)| = |K_0|}$$

- Decibel Magnitude $|G(j\omega)|_{dB}$ of $G(j\omega) = K_0$

$$|G(j\omega)|_{dB} = 20\log_{10}(|G(j\omega)|)$$

$$\boxed{|G(j\omega)|_{dB} = 20\log_{10}(|K_0|)}$$



This is an exact plot

Plotting dB Magnitude of Zeros Located at the Origin $\boldsymbol{G}(j\omega) = (j\omega)^N, N > 0$

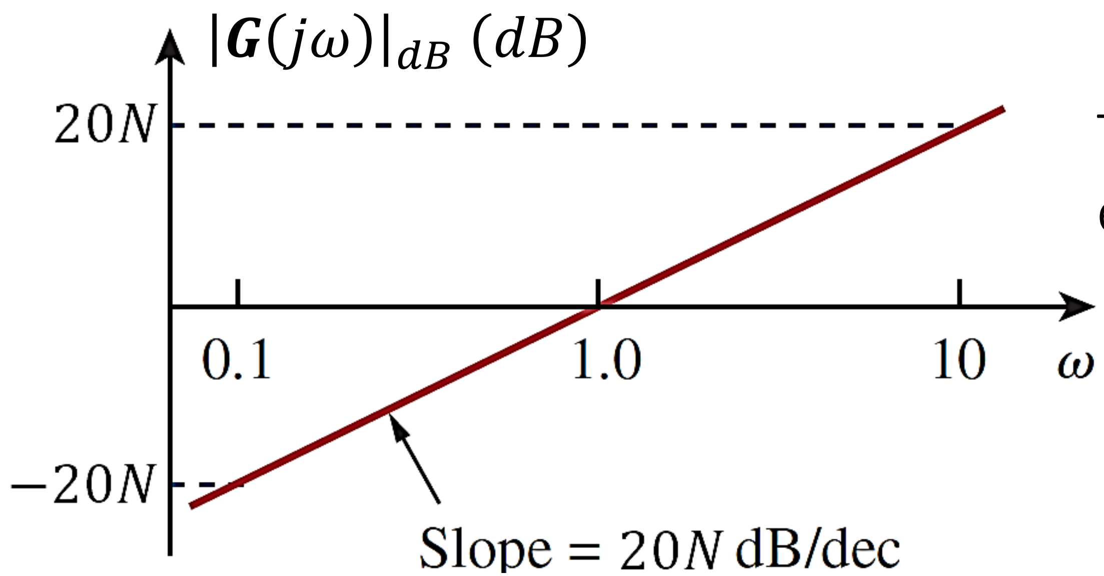
- Magnitude $|\boldsymbol{G}(j\omega)|$ of $\boldsymbol{G}(j\omega) = (j\omega)^N$

$$|\boldsymbol{G}(j\omega)| = \left|(j\omega)^N\right| = |(j\omega)|^N \Rightarrow \boxed{|\boldsymbol{G}(j\omega)| = \omega^N}$$

- Decibel Magnitude $|\boldsymbol{G}(j\omega)|_{dB}$ of $\boldsymbol{G}(j\omega) = (j\omega)^N$

$$|\boldsymbol{G}(j\omega)|_{dB} = 20\log_{10}(|\boldsymbol{G}(j\omega)|) = 20\log_{10}\left(\omega^N\right)$$

$$\boxed{|\boldsymbol{G}(j\omega)|_{dB} = 20N\log_{10}(\omega)}$$



This is an exact plot

Plotting dB Magnitude of Poles Located at the Origin $\boldsymbol{G}(j\omega) = 1/(j\omega)^N, N > 0$

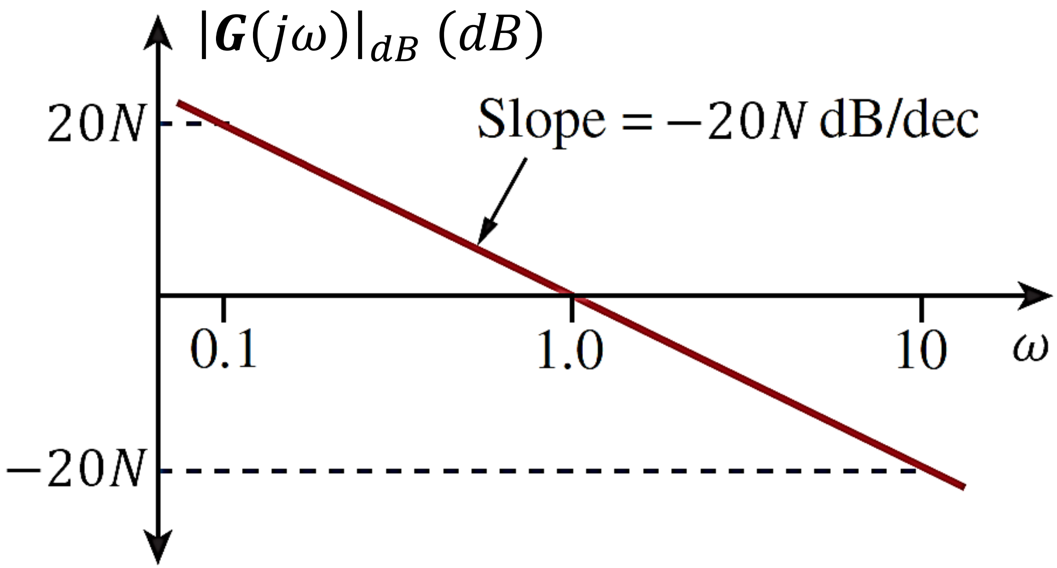
- Magnitude $|\boldsymbol{G}(j\omega)|$ of $\boldsymbol{G}(j\omega) = 1/(j\omega)^N$

$$|\boldsymbol{G}(j\omega)| = \left|1/(j\omega)^N\right| = |1/(j\omega)|^N = (1/\omega)^N \Rightarrow \boxed{|\boldsymbol{G}(j\omega)| = \omega^{-N}}$$

- Decibel Magnitude $|\boldsymbol{G}(j\omega)|_{dB}$ of $\boldsymbol{G}(j\omega) = 1/(j\omega)^N$

$$|\boldsymbol{G}(j\omega)|_{dB} = 20\log_{10}(|\boldsymbol{G}(j\omega)|) = 20\log_{10}\left(\omega^{-N}\right)$$

$$\boxed{|\boldsymbol{G}(j\omega)|_{dB} = -20N\log_{10}(\omega)}$$



This is an exact plot

Plotting dB Magnitude of Real Zeros at $s = |z_k|$ $\boldsymbol{G}(j\omega) = (j\omega/|z_k| + 1)^N, N > 0$

- Magnitude $|\boldsymbol{G}(j\omega)|$ of $\boldsymbol{G}(j\omega) = (j\omega/|z_k| + 1)^N$

  $$|\boldsymbol{G}(j\omega)| = \left|(j\omega/|z_k| + 1)^N\right| = |(j\omega/|z_k| + 1)|^N = \sqrt{1 + (\omega/|z_k|)^2}^N$$

  - Region #1: $\omega/|z_k| \ll 1 \Rightarrow \omega \ll |z_k|$

    $$|\boldsymbol{G}(j\omega)| = \sqrt{1 + (\omega/|z_k|)^2}^N = \left|\sqrt{1 + (small)^2}\right|^N$$

    $$\boxed{|\boldsymbol{G}(j\omega)| \approx 1}$$

  - Region #2: $\omega/|z_k| = 1 \Rightarrow \omega = |z_k|$

    $$|\boldsymbol{G}(j|z_k|)| = \sqrt{1 + (|z_k|/|z_k|)^2}^N$$

    $$\boxed{|\boldsymbol{G}(j|z_k|)| = \left(\sqrt{2}\right)^N}$$

  - Region #3: $\omega/|z_k| \gg 1 \Rightarrow \omega \gg |z_k|$

    $$|\boldsymbol{G}(j\omega)| = \sqrt{1 + (\omega/|z_k|)^2}^N \approx \sqrt{(\omega/|z_k|)^2}^N$$

    $$\boxed{|\boldsymbol{G}(j\omega)| \approx (\omega/|z_k|)^N}$$

Plotting dB Magnitude of Real Zeros at $s = |z_k|$ (cont'd) $\boldsymbol{G}(j\omega) = (j\omega/|z_k| + 1)^N, N > 0$

- Decibel Magnitude $|\boldsymbol{G}(j\omega)|$ of $\boldsymbol{G}(j\omega) = (j\omega/|z_k| + 1)^N$

$$|\boldsymbol{G}(j\omega)|_{dB} = 20\log_{10}(|\boldsymbol{G}(j\omega)|) = 20\log_{10}\left(\sqrt{1 + (\omega/|z_k|)^2}^N\right)$$

  - **Region #1**: $\omega/|z_k| \ll 1 \Rightarrow \omega \ll |z_k|$

$$|\boldsymbol{G}(j\omega)|_{dB} \approx 20\log_{10}(1)$$
$$\boxed{|\boldsymbol{G}(j\omega)|_{dB} \approx 0\ dB}$$

  - **Region #2**: $\omega/|z_k| = 1 \Rightarrow \omega = |z_k|$

$$|\boldsymbol{G}(j|z_k|)|_{dB} = 20\log_{10}\left(\sqrt{2}^N\right) = 20N\log_{10}\left(\sqrt{2}\right)$$
$$\boxed{|\boldsymbol{G}(j|z_k|)|_{dB} \approx 3N\ dB}$$

  - **Region #3**: $\omega/|z_k| \gg 1 \Rightarrow \omega \gg |z_k|$

$$|\boldsymbol{G}(j\omega)|_{db} \approx 20\log((\omega/|z_k|)^N) \approx 20N\log_{10}(\omega/|z_k|)$$
$$\boxed{|\boldsymbol{G}(j\omega)|_{db} \approx 20N\log_{10}(\omega) - 20N\log_{10}(|z_k|)}$$

Plotting dB Magnitude of Real Zeros at $s = |z_k|$ (cont'd) $\boldsymbol{G}(j\omega) = (j\omega/|z_k| + 1)^N, N > 0$

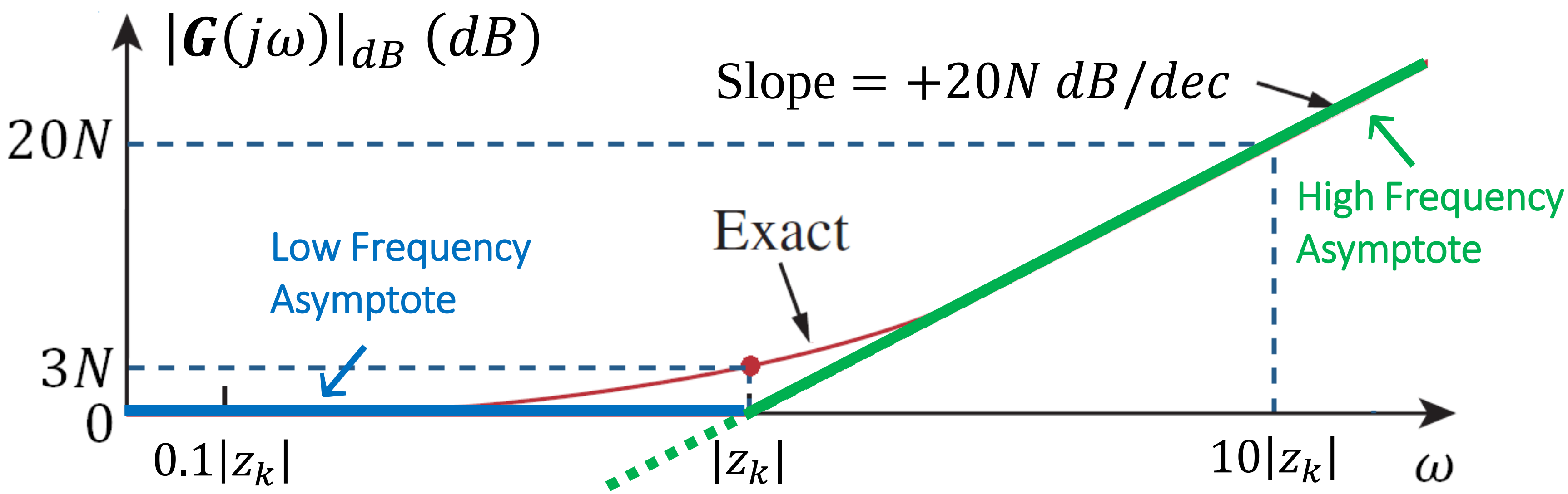
- Decibel Magnitude $|\boldsymbol{G}(j\omega)|$ of $\boldsymbol{G}(j\omega) = (j\omega/|z_k| + 1)^N$
  - Region #1: $\omega \ll |z_k|$

    $\boxed{|\boldsymbol{G}(j\omega)|_{dB} \approx 0\ dB}$

    Region #2: $\omega = |z_k|$

    $\boxed{|\boldsymbol{G}(j|z_k|)|_{dB} \approx 3N\ dB}$
  - Region #3: $\omega \gg |z_k|$

    $\boxed{|\boldsymbol{G}(j\omega)|_{dB} \approx 20N\log_{10}(\omega) - 20N\log_{10}(|z_k|)}$



Plotting dB Magnitude of Real Poles at $s = |p_k|$ $\boldsymbol{G}(j\omega) = 1/(j\omega/|p_k| + 1)^N, N > 0$

- Magnitude $|\boldsymbol{G}(j\omega)|$ of $\boldsymbol{G}(j\omega) = 1/(j\omega/|p_k| + 1)^N$

$$|\boldsymbol{G}(j\omega)| = \left|1/(j\omega/|p_k| + 1)^N\right| = |(j\omega/|p_k| + 1)|^{-N} = \sqrt{1 + (\omega/|z_k|)^2}^{\,-N}$$

  - **Region #1**: $\omega/|p_k| \ll 1 \Rightarrow \omega \ll |p_k|$

$$|\boldsymbol{G}(j\omega)| = \sqrt{1 + (\omega/|p_k|)^2}^{\,-N} = \sqrt{1 + (small)^2}^{\,-N}$$

$$\boxed{|\boldsymbol{G}(j\omega)| \approx 1}$$

  - **Region #2**: $\omega/|p_k| = 1 \Rightarrow \omega = |p_k|$

$$|\boldsymbol{G}(j|p_k|)| = \sqrt{1 + (|p_k|/|p_k|)^2}^{\,-N} = \left|\sqrt{2}\right|^{-N}$$

$$\boxed{|\boldsymbol{G}(j|p_k|)| = \left(\sqrt{2}\right)^{-N}}$$

  - **Region #3**: $\omega/|p_k| \gg 1 \Rightarrow \omega \gg |p_k|$

$$|\boldsymbol{G}(j\omega)| = \sqrt{1 + (\omega/|p_k|)^2}^{\,-N} \approx \sqrt{(\omega/|p_k|)^2}^{\,-N}$$

$$\boxed{|\boldsymbol{G}(j\omega)| \approx (\omega/|p_k|)^{-N}}$$

Plotting dB Magnitude of Real Poles at $s = |p_k|$ (cont'd) $\boldsymbol{G}(j\omega) = 1/(j\omega/|p_k| + 1)^N, N > 0$

- Decibel Magnitude $|\boldsymbol{G}(j\omega)|$ of $\boldsymbol{G}(j\omega) = 1/(j\omega/|p_k| + 1)^{-N}$

$$|\boldsymbol{G}(j\omega)|_{dB} = 20\log_{10}(|\boldsymbol{G}(j\omega)|) = 20\log_{10}\left(\sqrt{1 + (\omega/|p_k|)^2}^{-N}\right)$$

  - **Region #1**: $\omega/|p_k| \ll 1 \Rightarrow \omega \ll |p_k|$

$$|\boldsymbol{G}(j\omega)|_{dB} \approx 20\log_{10}(1)$$
$$\boxed{|\boldsymbol{G}(j\omega)|_{dB} \approx 0\ dB}$$

  - **Region #2**: $\omega/|p_k| = 1 \Rightarrow \omega = |p_k|$

$$|\boldsymbol{G}(j|p_k|)|_{dB} = 20\log_{10}\left(\sqrt{2}^{-N}\right) = -20N\log_{10}\left(\sqrt{2}\right)$$
$$\boxed{|\boldsymbol{G}(j|p_k|)|_{dB} \approx -3N\ dB}$$

  - **Region #3**: $\omega/|p_k| \gg 1 \Rightarrow \omega \gg |p_k|$

$$|\boldsymbol{G}(j\omega)|_{dB} \approx 20\log((\omega/|p_k|)^{-N}) \approx -20N\log_{10}(\omega/|p_k|)$$
$$\boxed{|\boldsymbol{G}(j\omega)|_{dB} \approx -20N\log_{10}(\omega) + 20N\log_{10}(|p_k|)}$$

Plotting dB Magnitude of Real Poles at $s = |p_k|$ (cont'd) $\boldsymbol{G}(j\omega) = 1/(j\omega/|p_k| + 1)^N, N > 0$

- Decibel Magnitude $|\boldsymbol{G}(j\omega)|$ of $\boldsymbol{G}(j\omega) = 1/(j\omega/|p_k| + 1)^{-N}$
  - Region #1: $\omega \ll |p_k|$

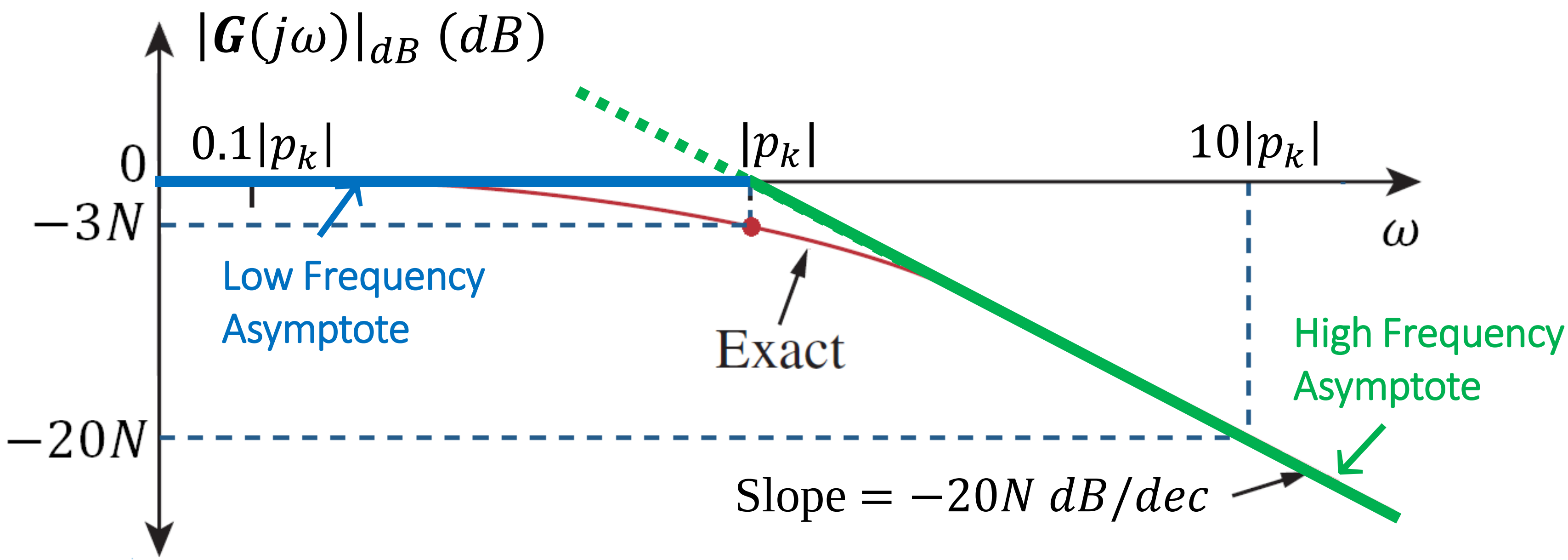
    $\boxed{|\boldsymbol{G}(j\omega)|_{dB} \approx 0\ dB}$

    Region #2: $\omega = |p_k|$

    $\boxed{|\boldsymbol{G}(j|z_k|)|_{dB} \approx -3N\ dB}$

  - Region #3: $\omega \gg |p_k|$

    $\boxed{|\boldsymbol{G}(j\omega)|_{dB} \approx -20N\log_{10}(\omega) + 20N\log_{10}(|p_k|)}$



Plotting dB Magnitude of Real Poles/Zeros: Summary

$$\boxed{|\boldsymbol{G}(j\omega)|_{dB} = |K_0|_{dB} + \sum_{k=1}^{m} |j\omega/z_k + 1|_{dB} - \sum_{k=1}^{n} |j\omega/p_k + 1|_{dB}}$$

- Real zeros/poles are **"corner"/"break"** frequencies in plot of $|\boldsymbol{G}(j\omega)|_{dB}$
  - Plot exhibits a drastic **"break"** in behavior at every zero/pole frequency
  - Each real, finite, non-zero zero causes the plot of $|\boldsymbol{G}(j\omega)|_{dB}$ to **increase** linearly by $20\mathrm{N}$ dBs/decade for $\omega > |z_k|$
  - Each real, finite, non-zero pole causes the plot of $|\boldsymbol{G}(j\omega)|_{dB}$ to **decrease** linearly by $20\mathrm{N}$ dBs/decade for $\omega > |p_k|$
- The constant term K_0 just vertically shifts the plot of $|\boldsymbol{G}(j\omega)|_{dB}$ by $|K_0|_{dB}$
- Each real, finite, pole/zero at the origin linearly offsets the plot of $|\boldsymbol{G}(j\omega)|_{dB}$ with a slope of $20\mathrm{N}$ dBs/decade

Lecture #6(a): Frequency Response I:
Frequency Response, SSS, and Bode Diagrams: Theory

Sketching Bode Diagrams: Phase Angle of Real Zeros/Poles

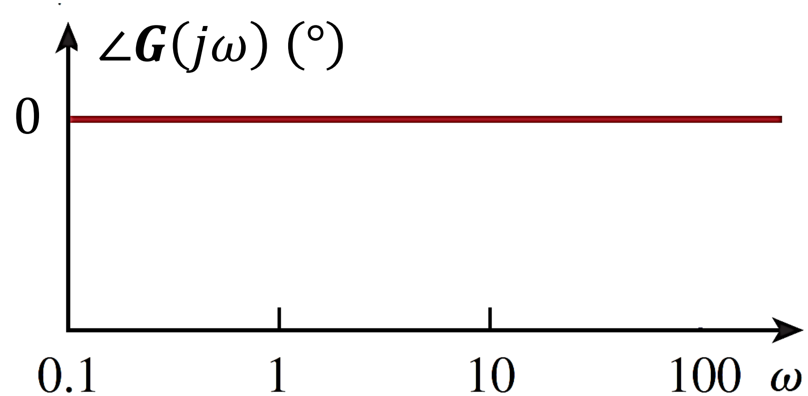
Plotting Phase Angle of Constant Term $G(j\omega) = K_0$

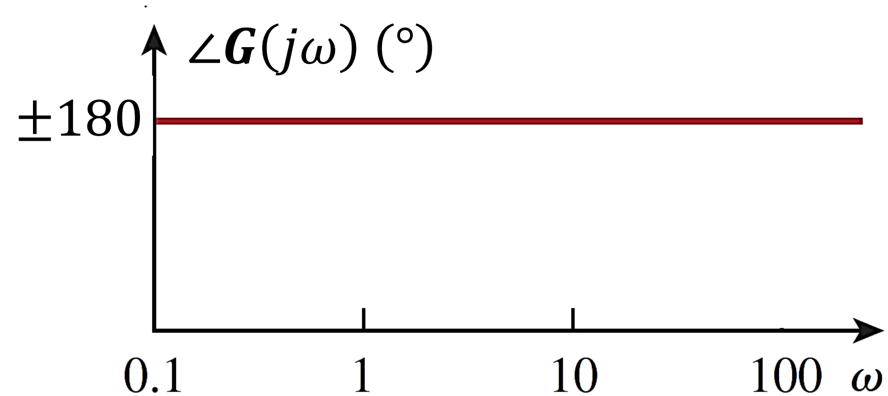
- Phase Angle $\angle G(j\omega)$ of $G(j\omega) = K_0$

$$\boxed{\angle G(j\omega) = \begin{cases} 0°, & K_0 > 0 \\ \pm 180°, & K_0 < 0 \end{cases}}$$

- Sketches of Phase Angle $\angle G(j\omega)$ of $G(j\omega) = K_0$

$K_0 > 0$ $\qquad\qquad$ $K_0 < 0$





These are exact plots

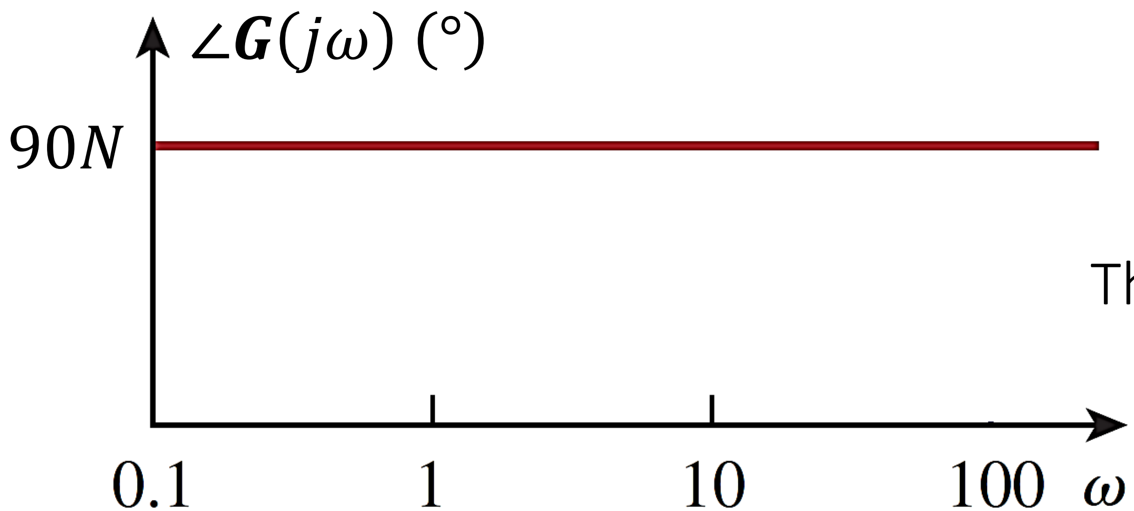
Plotting Phase Angle of Zeros Located at the Origin $\boldsymbol{G}(j\omega) = (j\omega)^N, N > 0$

- Phase Angle $\angle\boldsymbol{G}(j\omega)$ of $\boldsymbol{G}(j\omega) = (j\omega)^N$
  - To clearly see the phase angle, express $G(j\omega)$ in exponential form

$$\boldsymbol{G}(j\omega) = (j\omega)^N = \left(\omega e^{+j90^\circ}\right)^N = \omega^N e^{+jN90^\circ}$$

$$\boxed{\angle\boldsymbol{G}(j\omega) = 90N^\circ}$$

- Sketch of Phase Angle $\angle\boldsymbol{G}(j\omega)$ of $\boldsymbol{G}(j\omega) = (j\omega)^N$



This is an exact plot

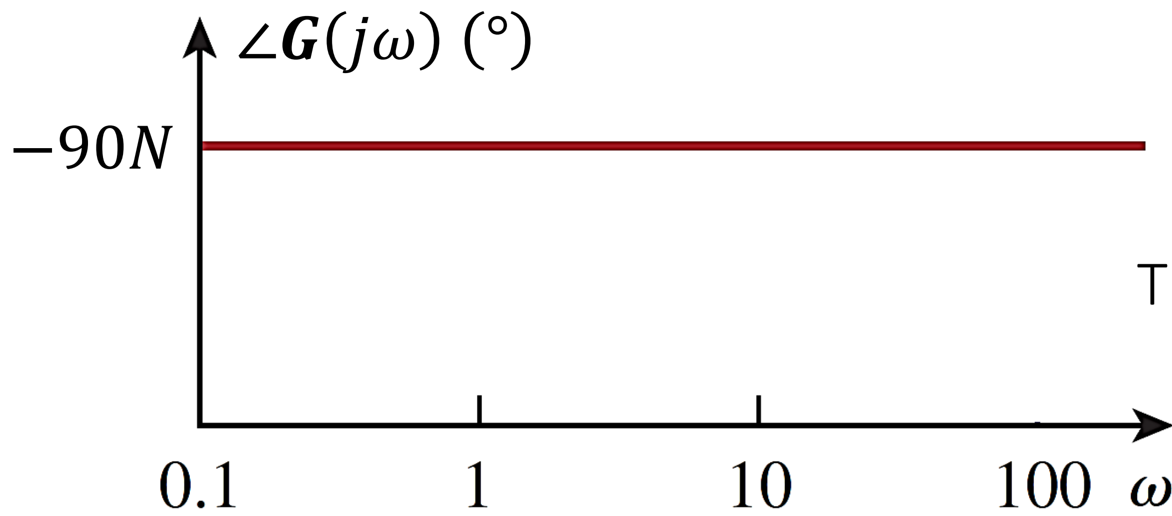
Plotting Phase Angle of Poles Located at the Origin $\boldsymbol{G}(j\omega) = 1/(j\omega)^N, N > 0$

- Phase Angle $\angle\boldsymbol{G}(j\omega)$ of $\boldsymbol{G}(j\omega) = 1/(j\omega)^N$
  - To clearly see the phase angle, express $G(j\omega)$ in exponential form

$$\boldsymbol{G}(j\omega) = 1/(j\omega)^N = \left(\omega e^{+j90°}\right)^{-N} = \omega^{-N} e^{-jN90°}$$

$$\boxed{\angle\boldsymbol{G}(j\omega) = -90N°}$$

- Sketch of Phase Angle $\angle\boldsymbol{G}(j\omega)$ of $\boldsymbol{G}(j\omega) = 1/(j\omega)^N$



This is an exact plot

Plotting Phase Angle of Real LHP Zeros at $s = |z_k|$ $\boldsymbol{G}(j\omega) = (j\omega/|z_k| + 1)^N, N > 0$

- Phase Angle $\angle\boldsymbol{G}(j\omega)$ of $\boldsymbol{G}(j\omega) = (j\omega/|z_k| + 1)^N$
  - To clearly see the phase angle, express $G(j\omega)$ in exponential form

$$\boldsymbol{G}(j\omega) = [\angle(j\omega/|z_k| + 1)]^N = \sqrt[2]{[1 + (\omega/|z_k|)^2]^N}\, e^{jNtan^{-1}(\omega/|z_k|)}$$

$$\boxed{\angle\boldsymbol{G}(j\omega) = Ntan^{-1}(\omega/|z_k|)}$$

  - **Region #1**: $\omega/|z_k| \ll 1 \Rightarrow \omega \ll |z_k|$

$$\angle\boldsymbol{G}(j\omega) = Ntan^{-1}(\omega/|z_k|) = Ntan^{-1}(small)$$

$$\boxed{\angle\boldsymbol{G}(j\omega) \approx 0°}$$

  - **Region #2**: $\omega/|z_k| = 1 \Rightarrow \omega = |z_k|$

$$\angle\boldsymbol{G}(j|z_k|) = Ntan^{-1}(|z_k|/|z_k|) = Ntan^{-1}(1)$$

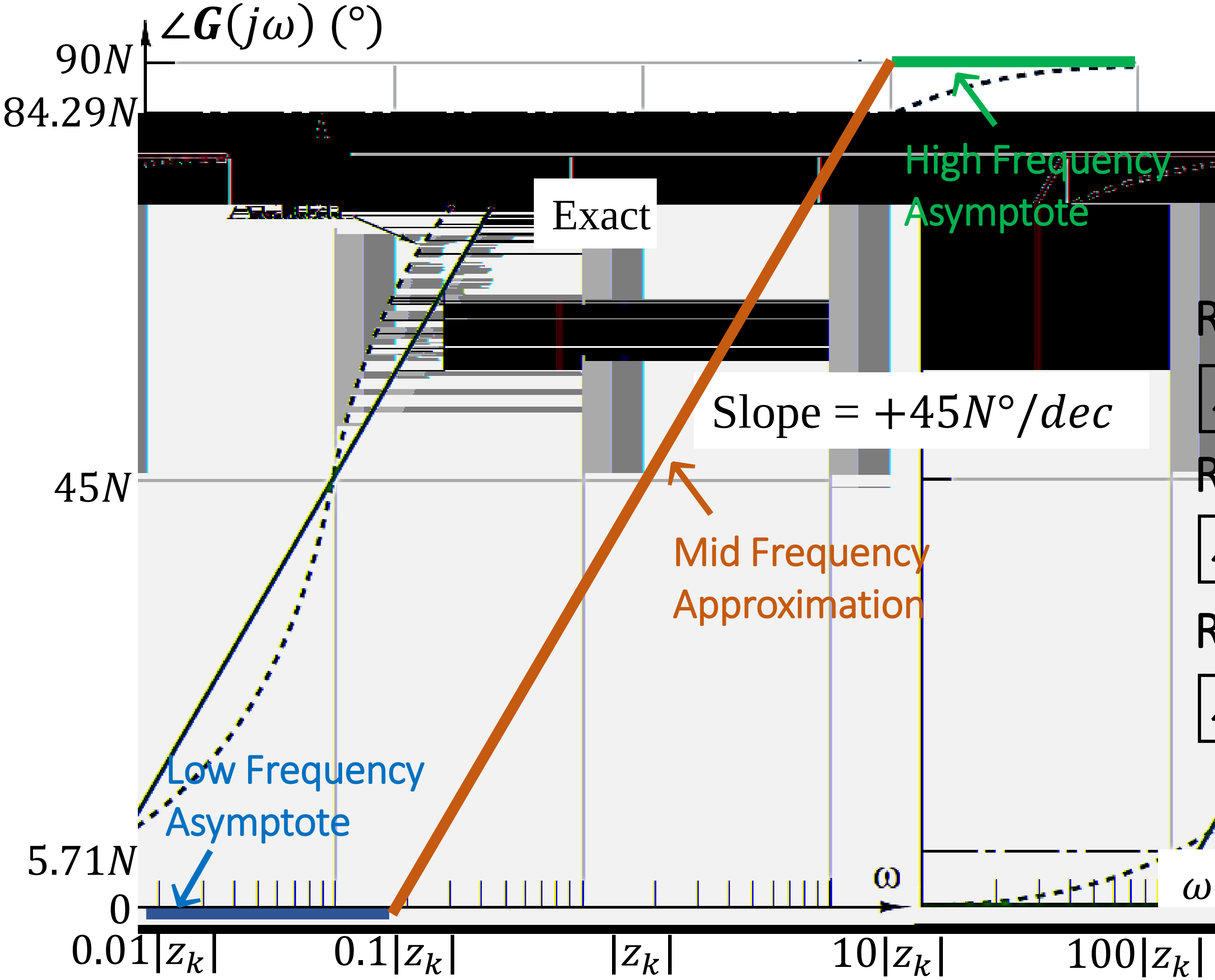
$$\boxed{\angle\boldsymbol{G}(j|z_k|) = +45N°}$$

  - **Region #3**: $\omega/|z_k| \gg 1 \Rightarrow \omega \gg |z_k|$

$$\angle\boldsymbol{G}(j\omega) = Ntan^{-1}(\omega/|z_k|) = Ntan^{-1}(big)$$

$$\boxed{\angle\boldsymbol{G}(j\omega) \rightarrow +90N°}$$

Plotting Phase Angle of Real LHP Zeros at $s = |z_k|$
$\boldsymbol{G}(j\omega) = (j\omega/|z_k| + 1)^N, N > 0$



Region #1: $\omega \ll |z_k|$

$\boxed{\angle \boldsymbol{G}(j\omega) \approx 0°}$

Region #2: $\omega = |z_k|$

$\boxed{\angle \boldsymbol{G}(j|z_k|) = +45N°}$

Region #3: $\omega \gg |z_k|$

$\boxed{\angle \boldsymbol{G}(j\omega) \rightarrow +90N°}$

Plotting Phase of Real RHP Zeros at $s = |z_k|$ w/ Form $\boldsymbol{G}(j\omega) = (-j\omega/|z_k| + 1)^N, N > 0$

- Phase Angle $\angle\boldsymbol{G}(j\omega)$ of $\boldsymbol{G}(j\omega) = (-j\omega/|z_k| + 1)^N$
  - To clearly see the phase angle, express $G(j\omega)$ in exponential form

    $$\boldsymbol{G}(j\omega) = [\angle(-j\omega/|z_k| + 1)]^N = \sqrt[2]{[1 + (-\omega/|z_k|)^2]^N} e^{-jNtan^{-1}(\omega/|z_k|)}$$

    $$\boxed{\angle\boldsymbol{G}(j\omega) = -Ntan^{-1}(\omega/|z_k|)}$$

  - **Region #1**: $\omega/|z_k| \ll 1 \Rightarrow \omega \ll |z_k|$

    $$\angle\boldsymbol{G}(j\omega) = -Ntan^{-1}(\omega/|z_k|) = -Ntan^{-1}(small)$$

    $$\boxed{\angle\boldsymbol{G}(j\omega) \approx 0°}$$

  - **Region #2**: $\omega/|z_k| = 1 \Rightarrow \omega = |z_k|$

    $$\angle\boldsymbol{G}(j|z_k|) = -Ntan^{-1}(|z_k|/|z_k|) = -Ntan^{-1}(1)$$

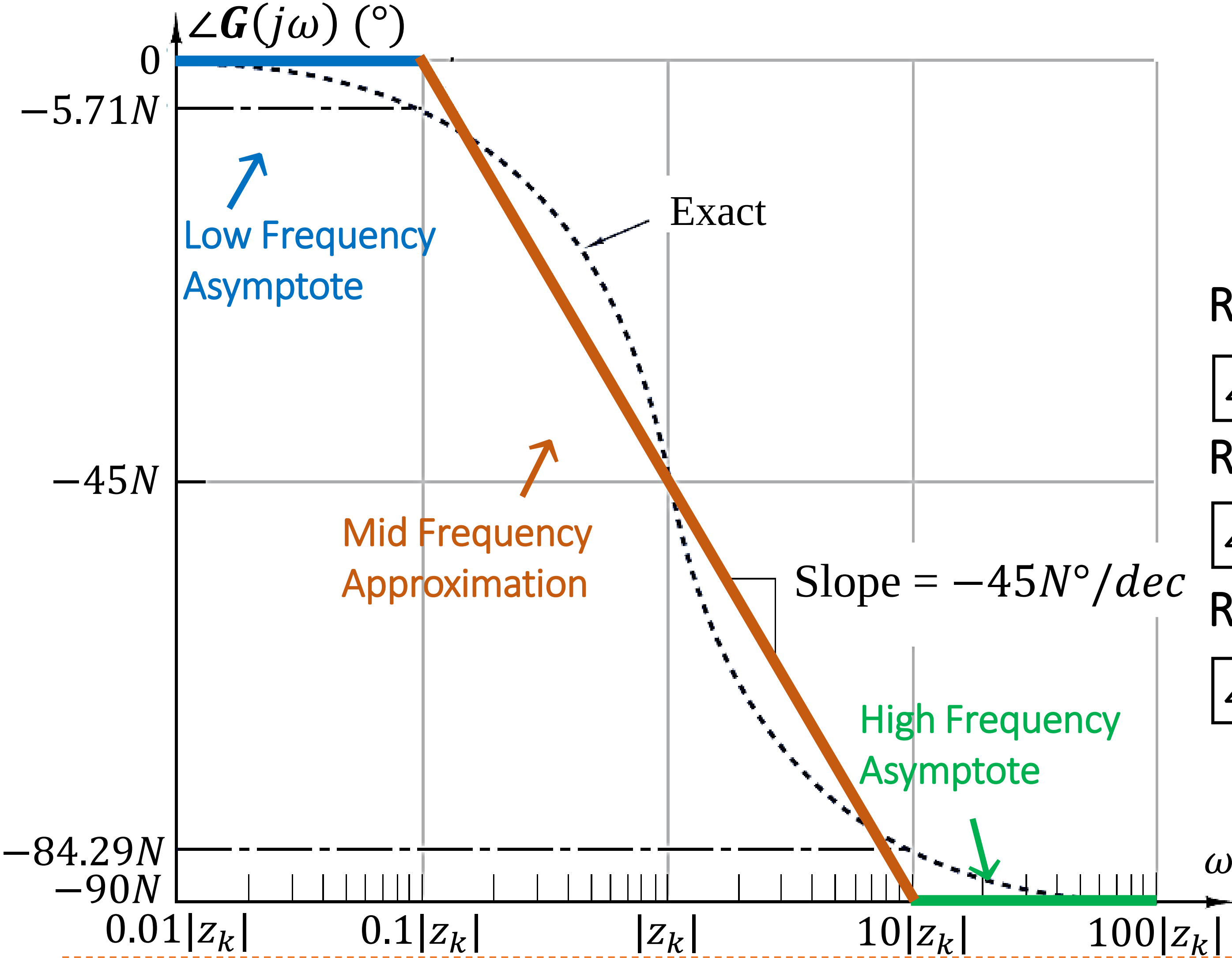
    $$\boxed{\angle\boldsymbol{G}(j|z_k|) = -45N°}$$

  - **Region #3**: $\omega/|z_k| \gg 1 \Rightarrow \omega \gg |z_k|$

    $$\angle\boldsymbol{G}(j\omega) = -Ntan^{-1}(\omega/|z_k|) = -Ntan^{-1}(big)$$

    $$\boxed{\angle\boldsymbol{G}(j\omega) \rightarrow -90N°}$$

Plotting Phase of Real RHP Zeros at $s = |z_k|$ w/ Form $G(j\omega) = (-j\omega/|z_k| + 1)^N, N > 0$



Region #1: $\omega \ll |z_k|$

$\boxed{\angle G(j\omega) \approx 0°}$

Region #2: $\omega = |z_k|$

$\boxed{\angle G(j|z_k|) = -45N°}$

Region #3: $\omega \gg |z_k|$

$\boxed{\angle G(j\omega) \to -90N°}$

Plotting Phase of Real RHP Zeros at $s = |z_k|$ w/ Form $\boldsymbol{G}(j\omega) = (j\omega/|z_k| - 1)^N, N > 0$

- Phase Angle $\angle\boldsymbol{G}(j\omega)$ of $\boldsymbol{G}(j\omega) = (j\omega/|z_k| - 1)^N$
  - To clearly see the phase angle, express $G(j\omega)$ in exponential form

    $\boldsymbol{G}(j\omega) = [\angle(j\omega/|z_k| - 1)]^N = \sqrt[2]{[1 + (\omega/|z_k|)^2]^N}\, e^{jN(180 - tan^{-1}(\omega/|z_k|))}$

    $\boxed{\angle\boldsymbol{G}(j\omega) = 180N° - Ntan^{-1}(\omega/|z_k|)}$
  - **Region #1**: $\omega/|z_k| \ll 1 \Rightarrow \omega \ll |z_k|$

    $\angle\boldsymbol{G}(j\omega) = 180N° - Ntan^{-1}(\omega/|z_k|) = 180N° - Ntan^{-1}(small)$

    $\boxed{\angle\boldsymbol{G}(j\omega) \approx 180N°}$
  - **Region #2**: $\omega/|z_k| = 1 \Rightarrow \omega = |z_k|$

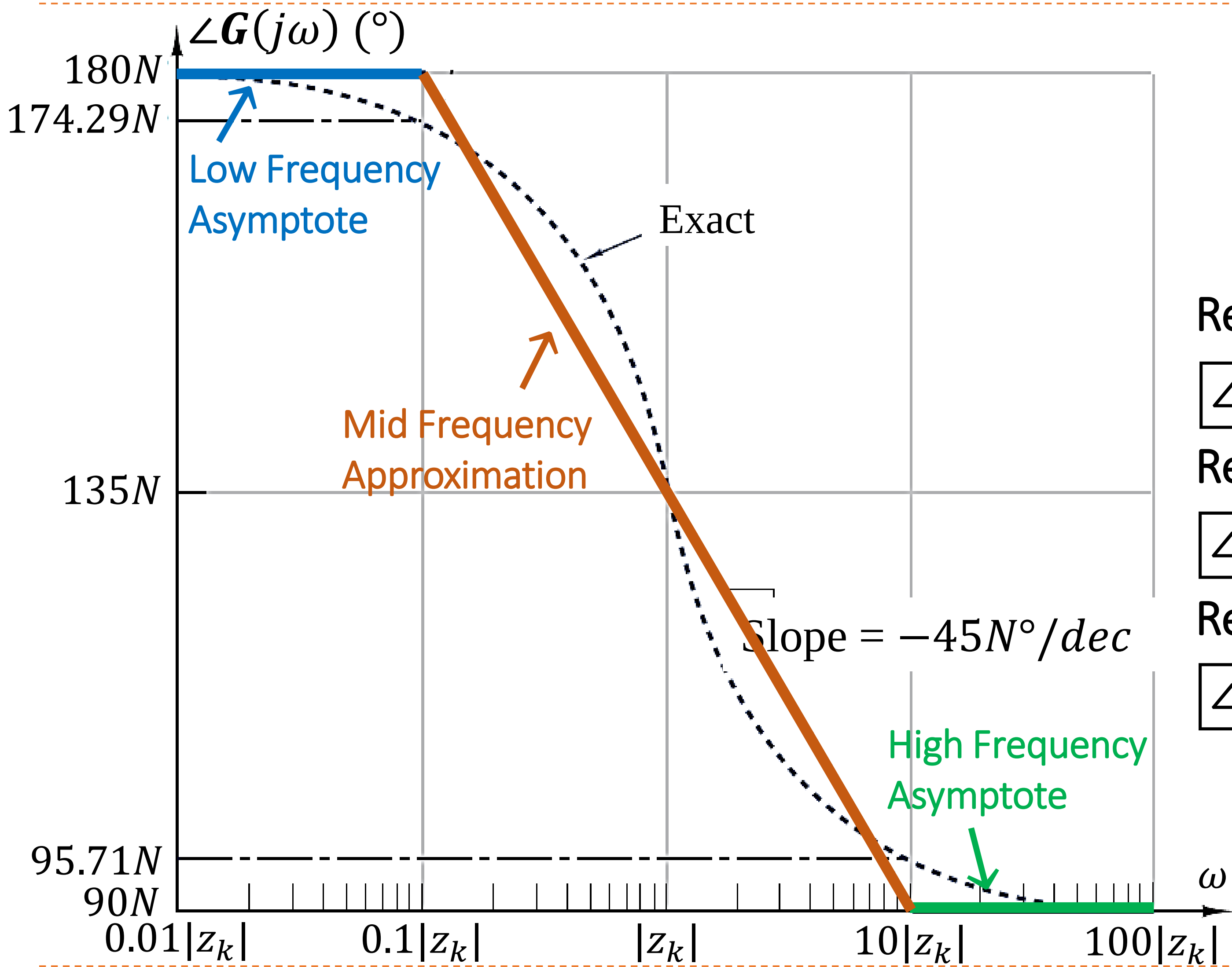
    $\angle\boldsymbol{G}(j|z_k|) = 180N° - Ntan^{-1}(|z_k|/|z_k|) = 180N° - Ntan^{-1}(1)$

    $\boxed{\angle\boldsymbol{G}(j|z_k|) = 135N°}$
  - **Region #3**: $\omega/|z_k| \gg 1 \Rightarrow \omega \gg |z_k|$

    $\angle\boldsymbol{G}(j\omega) = 180N° - Ntan^{-1}(\omega/|z_k|) = 180N° - Ntan^{-1}(big)$

    $\boxed{\angle\boldsymbol{G}(j\omega) \rightarrow 90N°}$

Plotting Phase of Real RHP Zeros at $s = |z_k|$ w/ Form $\boldsymbol{G}(j\omega) = (j\omega/|z_k| - 1)^N, N > 0$



Region #1: $\omega \ll |z_k|$

$\boxed{\angle \boldsymbol{G}(j\omega) \approx 180N\,°}$

Region #2: $\omega = |z_k|$

$\boxed{\angle \boldsymbol{G}(j|z_k|) = 135N°}$

Region #3: $\omega \gg |z_k|$

$\boxed{\angle \boldsymbol{G}(j\omega) \rightarrow 90N°}$

Plotting Phase Angle of Real LHP Poles at $s = |p_k|$ $\boldsymbol{G}(j\omega) = 1/(j\omega/|p_k| + 1)^N, N > 0$

- Phase Angle $\angle\boldsymbol{G}(j\omega)$ of $\boldsymbol{G}(j\omega) = 1/(j\omega/|p_k| + 1)^N$
  - To clearly see the phase angle, express $G(j\omega)$ in exponential form

    $$\boldsymbol{G}(j\omega) = [\angle(j\omega/|p_k| + 1)]^{-N} = \sqrt[2]{[1 + (\omega/|z_k|)^2]^{-N}}e^{-jNtan^{-1}(\omega/|p_k|)}$$

    $$\boxed{\angle\boldsymbol{G}(j\omega) = -Ntan^{-1}(\omega/|p_k|)}$$

  - **Region #1**: $\omega/|p_k| \ll 1 \Rightarrow \omega \ll |p_k|$

    $$\angle\boldsymbol{G}(j\omega) = -Ntan^{-1}(\omega/|p_k|) = -Ntan^{-1}(small)$$

    $$\boxed{\angle\boldsymbol{G}(j\omega) \approx 0°}$$

  - **Region #2**: $\omega/|p_k| = 1 \Rightarrow \omega = |p_k|$

    $$\angle\boldsymbol{G}(j|p_k|) = -Ntan^{-1}(|p_k|/|p_k|) = -Ntan^{-1}(1)$$

    $$\boxed{\angle\boldsymbol{G}(j|p_k|) = -45N°}$$

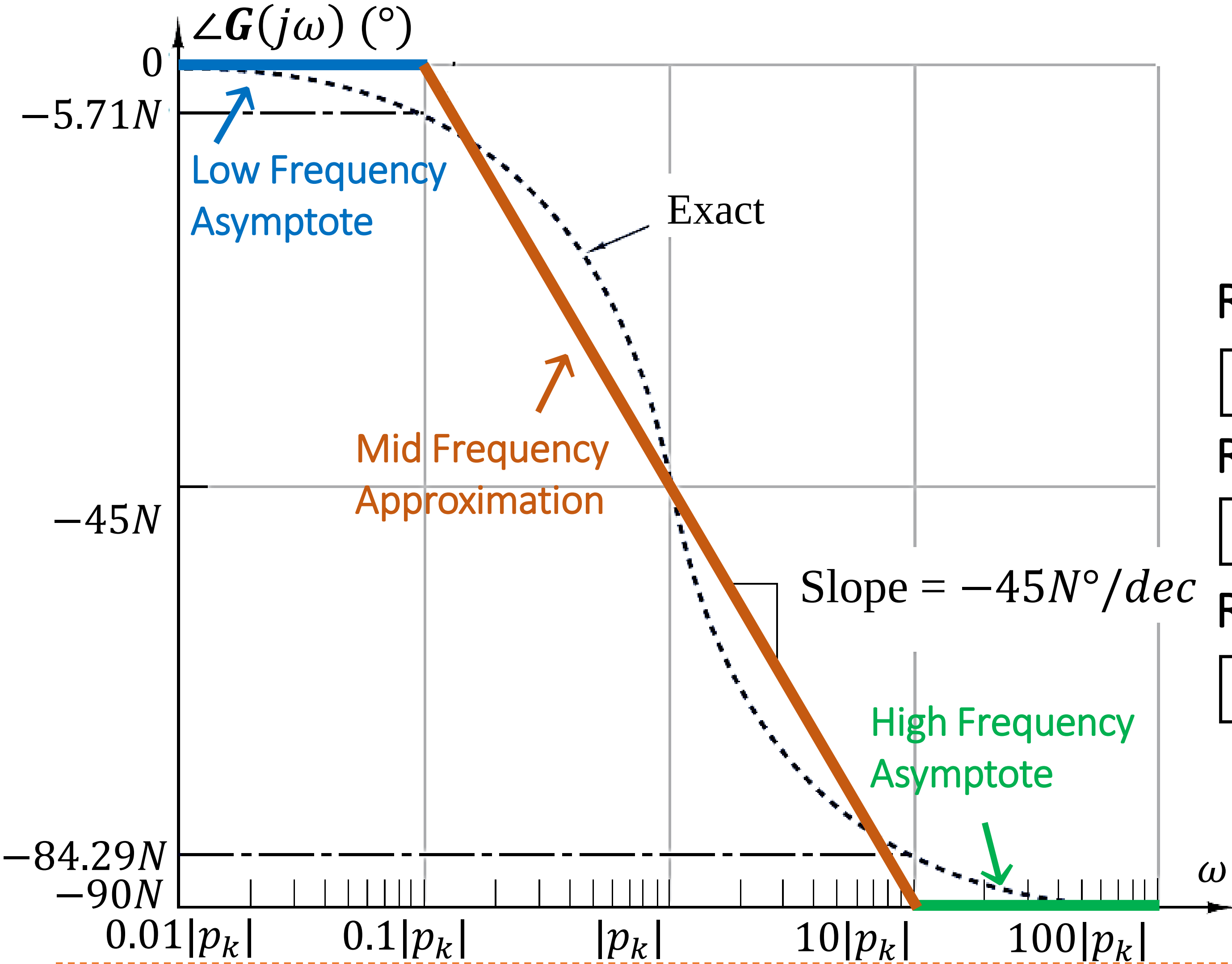
  - **Region #3**: $\omega/|p_k| \gg 1 \Rightarrow \omega \gg |p_k|$

    $$\angle\boldsymbol{G}(j\omega) = -Ntan^{-1}(\omega/|p_k|) = -Ntan^{-1}(big)$$

    $$\boxed{\angle\boldsymbol{G}(j\omega) \to -90N°}$$

Plotting Phase Angle of Real LHP Poles at $s = |p_k|$

$\boldsymbol{G}(j\omega) = 1/(j\omega/|p_k| + 1)^N, N > 0$



Region #1: $\omega \ll |p_k|$

$\boxed{\angle \boldsymbol{G}(j\omega) \approx 0°}$

Region #2: $\omega = |p_k|$

$\boxed{\angle \boldsymbol{G}(j|p_k|) = -45N°}$

Region #3: $\omega \gg |p_k|$

$\boxed{\angle \boldsymbol{G}(j\omega) \to -90N°}$

Plotting Angle of Real Poles/Zeros: Summary

$$\boxed{\angle \boldsymbol{G}(j\omega) = \angle K_0 + \sum_{k=1}^{m} \angle(j\omega/z_k + 1) - \sum_{k=1}^{n} \angle(j\omega/p_k + 1)}$$

- Finite, real, non-zero, zeros cause phase shifts that tend to either $\pm 90N°$ or $+180N°$
  - Effect spans from one decade before to one decade after $\omega = |z_k|$
    - LHP zero with form $(j\omega/|z_k| + 1)^N$ tends to $+90N°$ from $0°$
    - RHP zero with form $(-j\omega/|z_k| + 1)^N$ tends to $-90N°$ from $0°$
    - RHP zero with form $(j\omega/|z_k| - 1)^N$ tends to $+90N°$ from $+180N°$
- Finite, real, non-zero, LHP poles cause phase shifts tending to $-90N°$
  - Effect spans from one decade before to one decade after $\omega = |p_k|$
- Constant term K_0 causes a phase shift of $0°$ $(K_0 > 0)$ or $\pm 180°$ $(K_0 < 0)$
- Each real pole/zero at the origin causes a phase shift of $90°$ for all ω

Lecture #6(a): Frequency Response I: *Frequency Response, SSS, and Bode Diagrams: Theory*

Sketching Bode Diagrams:
Magnitude/Phase Complex LHP Zeros/Poles

Un(der)damped 2nd Order Systems: $0 \leq \zeta < 1$

- We want to approximately sketch Bode diagrams of frequency responses stemming from system functions having complex poles
  - This means we are interested in 2nd order systems with $0 \leq \zeta < 1$

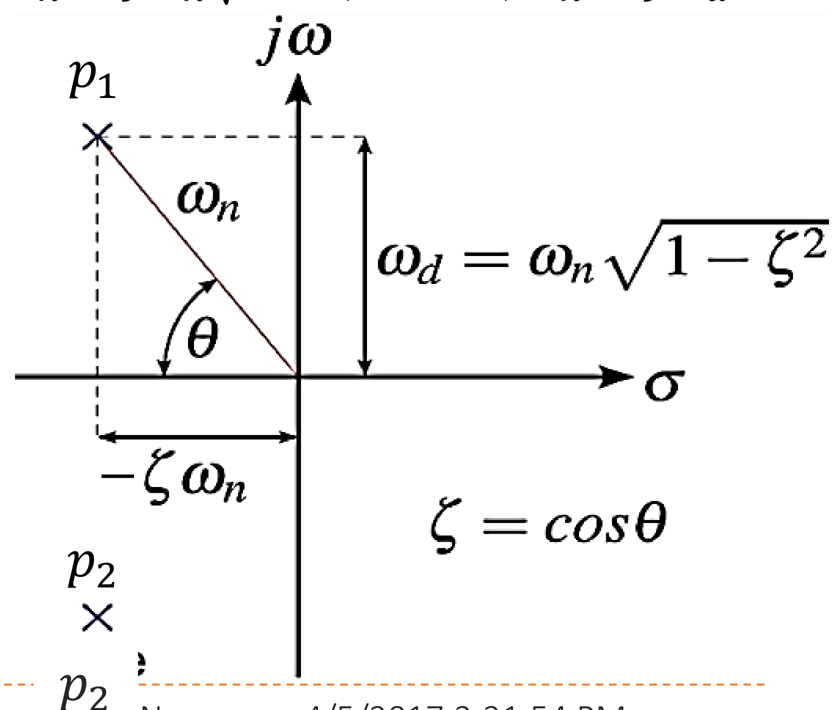
Undamped ($\zeta = 0$): Imaginary poles: $p_{1,2} = \pm j\omega_n$

Under-damped ($0 < \zeta < 1$): $p_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -\zeta\omega_n \pm j\omega_d$

ω_d is the **damped natural frequency**

$$G(s) = \frac{N(s)}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$0 \leq \zeta < 1$$



Plotting dB Magnitude of Complex Poles $G(j\omega) = [(j\omega/\omega_n)^2 + 2\zeta(j\omega/\omega_n) + 1]^{-N}, N > 0, 0 \leq \zeta < 1$

- Magnitude $|G(j\omega)|$ of $G(j\omega) = [(j\omega/\omega_n)^2 + 2\zeta(j\omega/\omega_n) + 1]^{-N}$

  $$|G(j\omega)| = \left|[(j\omega/\omega_n)^2 + 2\zeta(j\omega/\omega_n) + 1]^{-N}\right|$$

  $$|G(j\omega)| = \sqrt{[1 - (\omega/\omega_n)^2]^2 + (2\zeta)^2(\omega/\omega_n)^2}^{\,-N}$$

  - **Region #1**: $\omega/\omega_n \ll 1 \Rightarrow \omega \ll \omega_n$

    $$|G(j\omega)| = \sqrt{[1 - (small)^2]^2 + (2\zeta)^2(small)^2}^{\,-N} \Rightarrow \boxed{|G(j\omega)| \approx 1}$$

  - **Region #2**: $\omega/\omega_n = 1 \Rightarrow \omega = \omega_n$

    $$|G(j\omega_n)| = \sqrt{[1 - (\omega_n/\omega_n)^2]^2 + (2\zeta)^2(\omega_n/\omega_n)^2}^{\,-N}$$

    $$\boxed{|G(j\omega_n)| = (2\zeta)^{-N}}$$

  - **Region #3**: $\omega/\omega_n \gg 1 \Rightarrow \omega \gg \omega_n$

    $$|G(j\omega)| = \sqrt{[1 - (\omega/\omega_n)^2]^2 + (2\zeta)^2(\omega/\omega_n)^2}^{\,-N}$$

    $$|G(j\omega)| \approx \sqrt{(\omega/\omega_n)^4 + (2\zeta)^2(\omega/\omega_n)^2}^{\,-N} \approx [(\omega/\omega_n)^2]^{-N}$$

    $$\boxed{|G(j\omega)| \approx (\omega/\omega_n)^{-2N}}$$

Plotting dB Magnitude of Complex Poles (cont'd)
$\boldsymbol{G}(j\omega) = [(j\omega/\omega_n)^2 + 2\zeta(j\omega/\omega_n) + 1]^{-N}, N > 0, 0 \leq \zeta < 1$

- Decibel Magnitude $|\boldsymbol{G}(j\omega)|_{dB}$ of $\boldsymbol{G}(j\omega)$

$$|\boldsymbol{G}(j\omega)|_{dB} = 20\log_{10}\left(\sqrt{[1 - (\omega/\omega_n)^2]^2 + (2\zeta)^2(\omega/\omega_n)^2}^{\,-N}\right)$$

  - **Region #1**: $\omega/\omega_n \ll 1 \Rightarrow \omega \ll \omega_n$

$$|\boldsymbol{G}(j\omega)|_{dB} \approx 20\log_{10}(1)$$
$$\boxed{|\boldsymbol{G}(j\omega)|_{dB} \approx 0\ dB}$$

  - **Region #2**: $\omega/\omega_n = 1 \Rightarrow \omega = \omega_n$

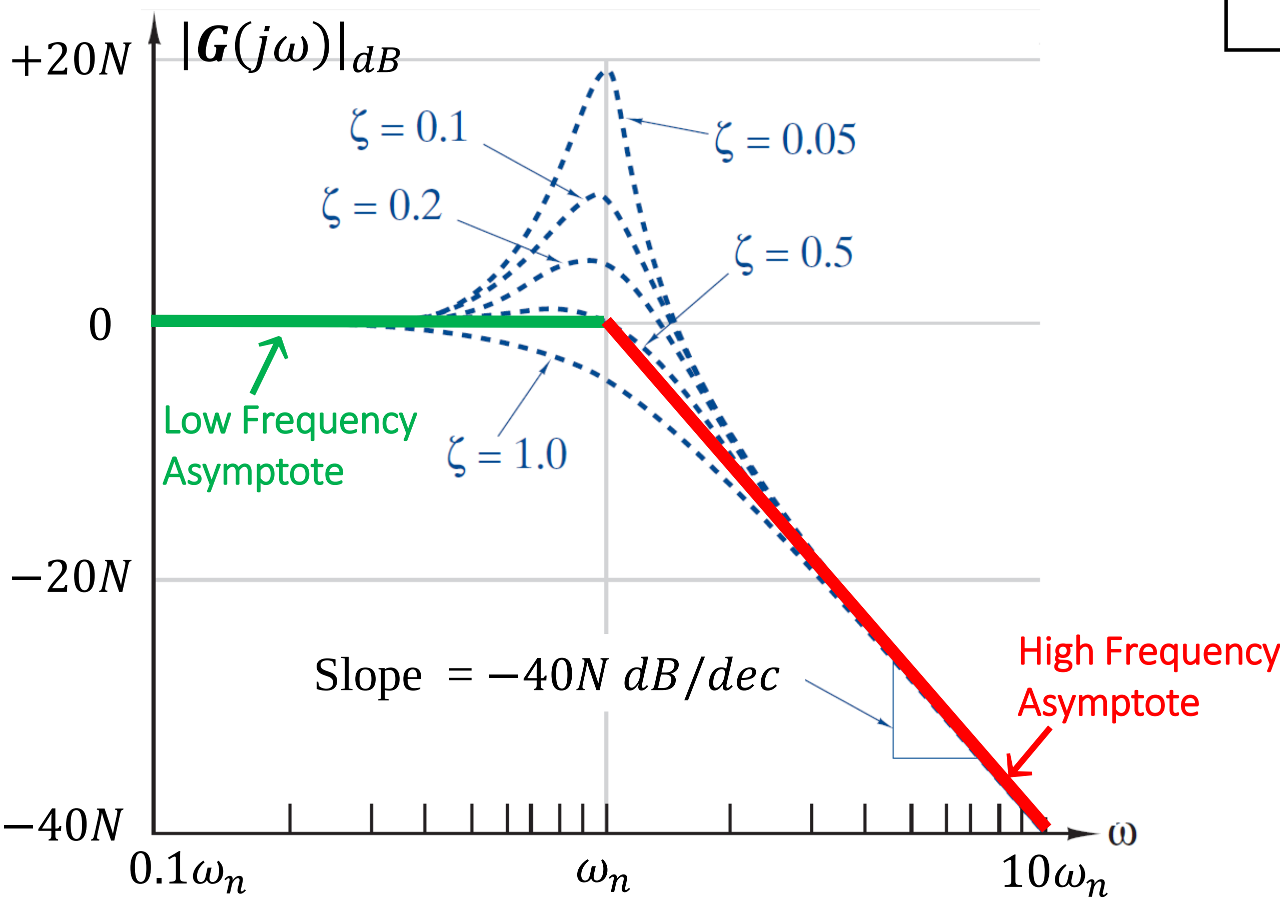
$$|\boldsymbol{G}(j\omega_n)|_{dB} = 20\log_{10}\left((2\zeta)^{-N}\right)$$
$$\boxed{|\boldsymbol{G}(j\omega_n)|_{dB} = -20N\log_{10}(2\zeta)}$$

  - **Region #3**: $\omega/\omega_n \gg 1 \Rightarrow \omega \gg \omega_n$

$$|\boldsymbol{G}(j\omega)|_{db} \approx 20\log((\omega/\omega_n)^{-2N}) \approx -40N\log_{10}(\omega/\omega_n)$$
$$\boxed{|\boldsymbol{G}(j\omega)|_{db} \approx -40N\log_{10}(\omega) + 40N\log_{10}(\omega_n)}$$

Plotting dB Magnitude of Complex Poles (cont'd)
$\boldsymbol{G}(j\omega) = [(j\omega/\omega_n)^2 + 2\zeta(j\omega/\omega_n) + 1]^{-N}, N > 0, 0 \le \zeta < 1$

- **Region #1**: $\omega \ll \omega_n$ $\boxed{|\boldsymbol{G}|_{dB} \approx 0\ dB}$ **Region #2**: $\omega = \omega_n$ $\boxed{|\boldsymbol{G}|_{dB} = -20N\log_{10}(2\zeta)}$ **Region #3**: $\omega \gg \omega_n$ $\boxed{|\boldsymbol{G}|_{dB} \approx -40N\log_{10}(\omega) + 40N\log_{10}(\omega_n)}$



Maximum of dB Magnitude of Complex Poles

- What conditions and at what frequency will $|\boldsymbol{G}(j\omega)|$ be maximized?

$$|\boldsymbol{G}(j\omega)| = \left(\sqrt{[1-(\omega/\omega_n)^{\,2}]^2 + (2\zeta)^2(\omega/\omega_n)^2}\right)^{-1}$$

  - For $0 \le \zeta \le 1/\sqrt{2}$, $|\boldsymbol{G}(j\omega)|$has a maximum value $|\boldsymbol{G}(j\omega)|_{max} \ge 1$
  - The frequency that causes $|\boldsymbol{G}(j\omega)|_{max}$ to occur is known as the **resonant frequency $\boldsymbol{\omega_r}$**. An expression for ω_r is given below:

$$\boxed{\omega_r = \omega_n\sqrt{1-2\zeta^2}}$$

$$\boxed{\text{As } \zeta \to 0,\ \omega_r \to \omega_n}$$

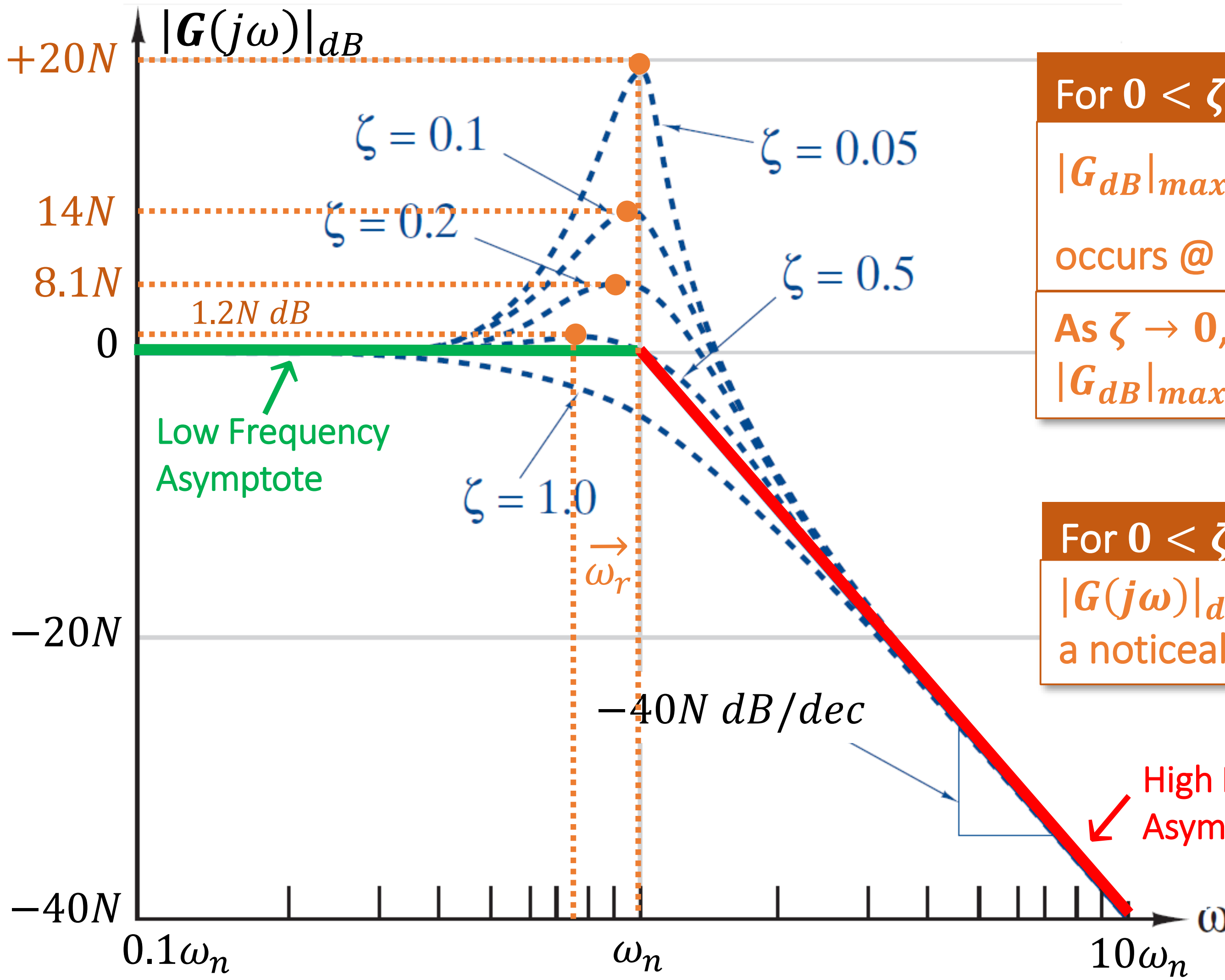
  - The value of $|\boldsymbol{G}(j\omega)|_{max}$ at $\omega = \omega_r = \omega_n\sqrt{1-2\zeta^2}$ is

$$\boxed{|\boldsymbol{G}(j\omega)|_{max} = |\boldsymbol{G}(j\omega_r)| = \left(2\zeta\sqrt{1-\zeta^2}\right)^{-1}}$$

$$\boxed{\text{As } \zeta \to 0,\ |\boldsymbol{G}(j\omega)|_{max} \to (2\zeta)^{-1}}$$

  - When $\zeta = 1/\sqrt{2}$, $|\boldsymbol{G}(j\omega)| = 1$ and $|\boldsymbol{G}(j\omega)|$ is called **maximally flat**
  - When $0 < \zeta < 1/2$, $|\boldsymbol{G}(j\omega)|$ exhibits a noticeable **peak**

Maximum of dB Magnitude of Complex Poles (cont'd)



For $0 < \zeta \le 1/\sqrt{2}$

$|G_{dB}|_{max} = -N\left|2\zeta\sqrt{1-\zeta^2}\right|_{dB}$

occurs @ $\omega_r = \omega_n\sqrt{1-2\zeta^2}$

As $\zeta \to 0$, $\omega_r \to \omega_n$ and $|G_{dB}|_{max} \to -N|2\zeta|_{dB}$

For $0 < \zeta < 0.5$

$|G(j\omega)|_{dB}$ exhibits a noticeable Peak

Plotting Phase Angle of Complex LHP Poles
$\boldsymbol{G}(j\omega) = [(j\omega/\omega_n)^2 + 2\zeta(j\omega/\omega_n) + 1]^{-N}, N > 0, 0 \le \zeta < 1$

- Phase Angle $\angle\boldsymbol{G}(j\omega)$ of $\boldsymbol{G}(j\omega) = 1/[(j\omega/\omega_n)^2 + 2\zeta(j\omega/\omega_n) + 1]^N$

$$\boldsymbol{G}(j\omega) = [(j\omega/\omega_n)^2 + 2\zeta(j\omega/\omega_n) + 1]^{-N}$$
$$\boldsymbol{G}(j\omega) = \sqrt[-N]{[1-(\omega/\omega_n)^{\,2}]^2 + (2\zeta)^2(\omega/\omega_n)^2}\, e^{-jNtan^{-1}\left(\frac{2\zeta[\omega/\omega_n]}{1-(\omega/\omega_n)^2}\right)}$$

$$\boxed{\angle\boldsymbol{G}(j\omega) = -Ntan^{-1}\left([(2\zeta)(\omega/\omega_n)]/[1-(\omega/\omega_n)^{\,2}]\right)}$$

- **Region #1**: $\omega/\omega_n \ll 1 \Rightarrow \omega \ll \omega_n$

$$\angle\boldsymbol{G}(j\omega) = -Ntan^{-1}\left([(2\zeta)(small)]/[1-(small)^{\,2}]\right) \rightarrow \boxed{\angle\boldsymbol{G}(j\omega) \approx 0°}$$

- **Region #2**: $\omega/\omega_n = 1 \Rightarrow \omega = \omega_n$

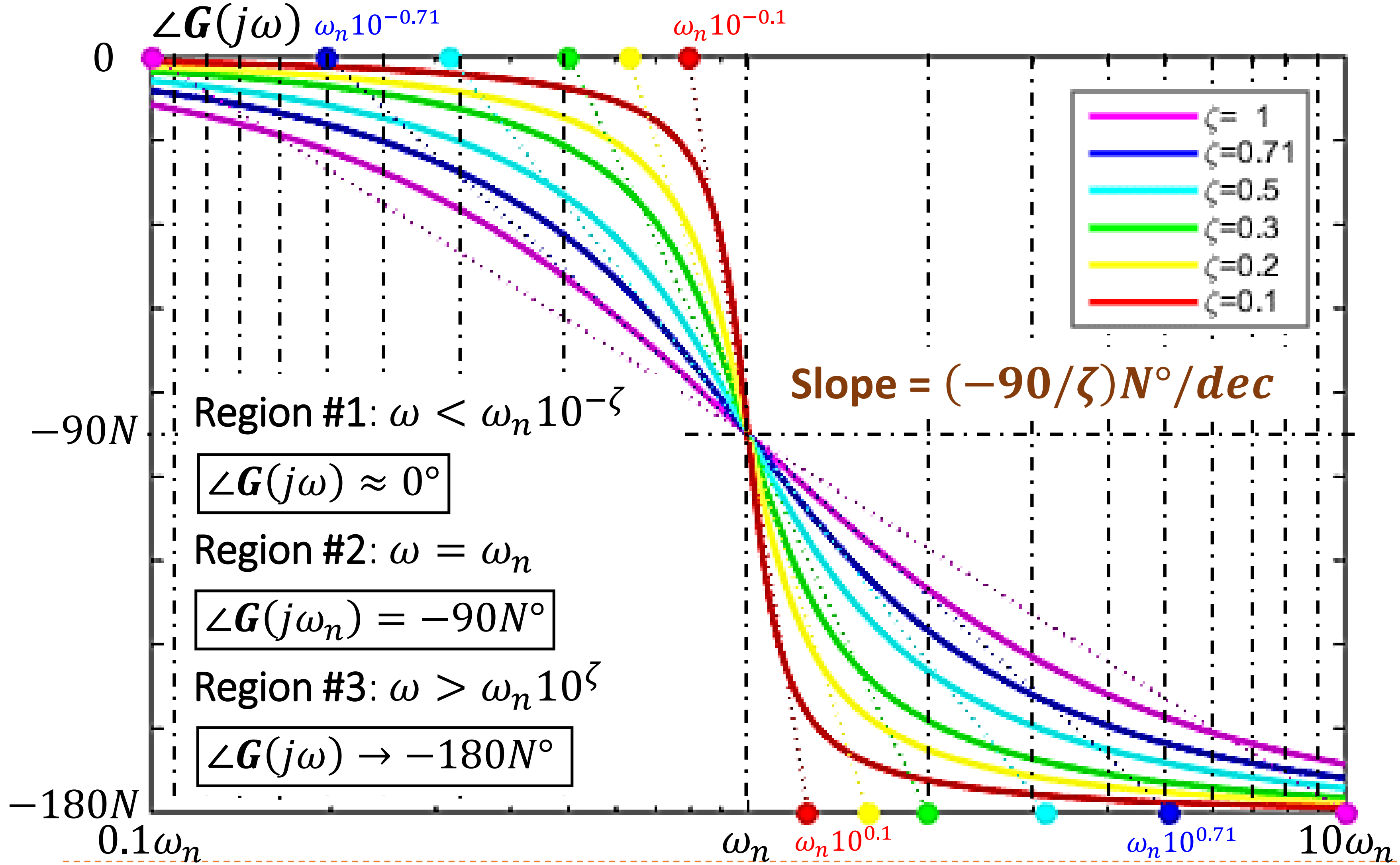
$$\angle\boldsymbol{G}(j\omega_n) = -Ntan^{-1}\left([(2\zeta)(\omega_n/\omega_n)]/[1-(\omega_n/\omega_n)^2]\right) = -Ntan^{-1}(2\zeta/0)$$
$$\boxed{\angle\boldsymbol{G}(j\omega_n) = -90N°}$$

- **Region #3**: $\omega/\omega_n \gg 1 \Rightarrow \omega \gg \omega_n$

$$\angle\boldsymbol{G}(j\omega) = -Ntan^{-1}\left([(2\zeta)(\omega/\omega_n)]/[1-(\omega/\omega_n)^{\,2}]\right)$$
$$\angle\boldsymbol{G}(j\omega) \approx -Ntan^{-1}((2\zeta)/[-(\omega/\omega_n)]) \approx -Ntan^{-1}([(small)]/[-(big)])$$
$$\boxed{\angle\boldsymbol{G}(j\omega) \rightarrow -180N°}$$

Plotting Phase Angle of Complex LHP Poles

$\boldsymbol{G}(j\omega) = [(j\omega/\omega_n)^2 + 2\zeta(j\omega/\omega_n) + 1]^{-N}, N > 0, 0 \leq \zeta < 1$



Plotting dB Magnitude of Complex Zeros
$\boldsymbol{G}(j\omega) = [(j\omega/\omega_n)^2 + 2\zeta(j\omega/\omega_n) + 1]^N, N > 0, 0 \le \zeta < 1$

- Magnitude $|\boldsymbol{G}(j\omega)|$ of $\boldsymbol{G}(j\omega) = [(j\omega/\omega_n)^2 + 2\zeta(j\omega/\omega_n) + 1]^N$

  $$|\boldsymbol{G}(j\omega)| = \left|[(j\omega/\omega_n)^2 + 2\zeta(j\omega/\omega_n) + 1]^N\right|$$

  $$|\boldsymbol{G}(j\omega)| = \sqrt{[1 - (\omega/\omega_n)^2]^2 + (2\zeta)^2(\omega/\omega_n)^2}^{\,N}$$

  - **Region #1**: $\omega/\omega_n \ll 1 \Rightarrow \omega \ll \omega_n$

    $$|\boldsymbol{G}(j\omega)| = \sqrt{[1 - (small)^2]^2 + (2\zeta)^2(small)^2}^{\,N} \Rightarrow \boxed{|\boldsymbol{G}(j\omega)| \approx 1}$$

  - **Region #2**: $\omega/\omega_n = 1 \Rightarrow \omega = \omega_n$

    $$|\boldsymbol{G}(j\omega_n)| = \sqrt{[1 - (\omega_n/\omega_n)^2]^2 + (2\zeta)^2(\omega_n/\omega_n)^2}^{\,N}$$

    $$\boxed{|\boldsymbol{G}(j\omega_n)| = (2\zeta)^N}$$

  - **Region #3**: $\omega/\omega_n \gg 1 \Rightarrow \omega \gg \omega_n$

    $$|\boldsymbol{G}(j\omega)| = \sqrt{[1 - (\omega/\omega_n)^2]^2 + (2\zeta)^2(\omega/\omega_n)^2}^{\,N}$$

    $$|\boldsymbol{G}(j\omega)| \approx \sqrt{(\omega/\omega_n)^4 + (2\zeta)^2(\omega/\omega_n)^2}^{\,N} \approx [(\omega/\omega_n)^2]^N$$

    $$\boxed{|\boldsymbol{G}(j\omega)| \approx (\omega/\omega_n)^{2N}}$$

Plotting dB Magnitude of Complex Zeros (cont'd)
$\boldsymbol{G}(j\omega) = [(j\omega/\omega_n)^2 + 2\zeta(j\omega/\omega_n) + 1]^N, N > 0, 0 \leq \zeta < 1$

- Decibel Magnitude $|\boldsymbol{G}(j\omega)|_{dB}$ of $\boldsymbol{G}(j\omega)$

$$|\boldsymbol{G}(j\omega)|_{dB} = 20\log_{10}\left(\sqrt{[1-(\omega/\omega_n)^{\,2}]^2 + (2\zeta)^2(\omega/\omega_n)^2}^{\;N}\right)$$

  - **Region #1**: $\omega/\omega_n \ll 1 \Rightarrow \omega \ll \omega_n$

$$|\boldsymbol{G}(j\omega)|_{dB} \approx 20\log_{10}(1)$$
$$\boxed{|\boldsymbol{G}(j\omega)|_{dB} \approx 0\ dB}$$

  - **Region #2**: $\omega/\omega_n = 1 \Rightarrow \omega = \omega_n$

$$|\boldsymbol{G}(j\omega_n)|_{dB} = 20\log_{10}\left((2\zeta)^{\,N}\right)$$
$$\boxed{|\boldsymbol{G}(j\omega_n)|_{dB} = 20N\log_{10}(2\zeta)}$$

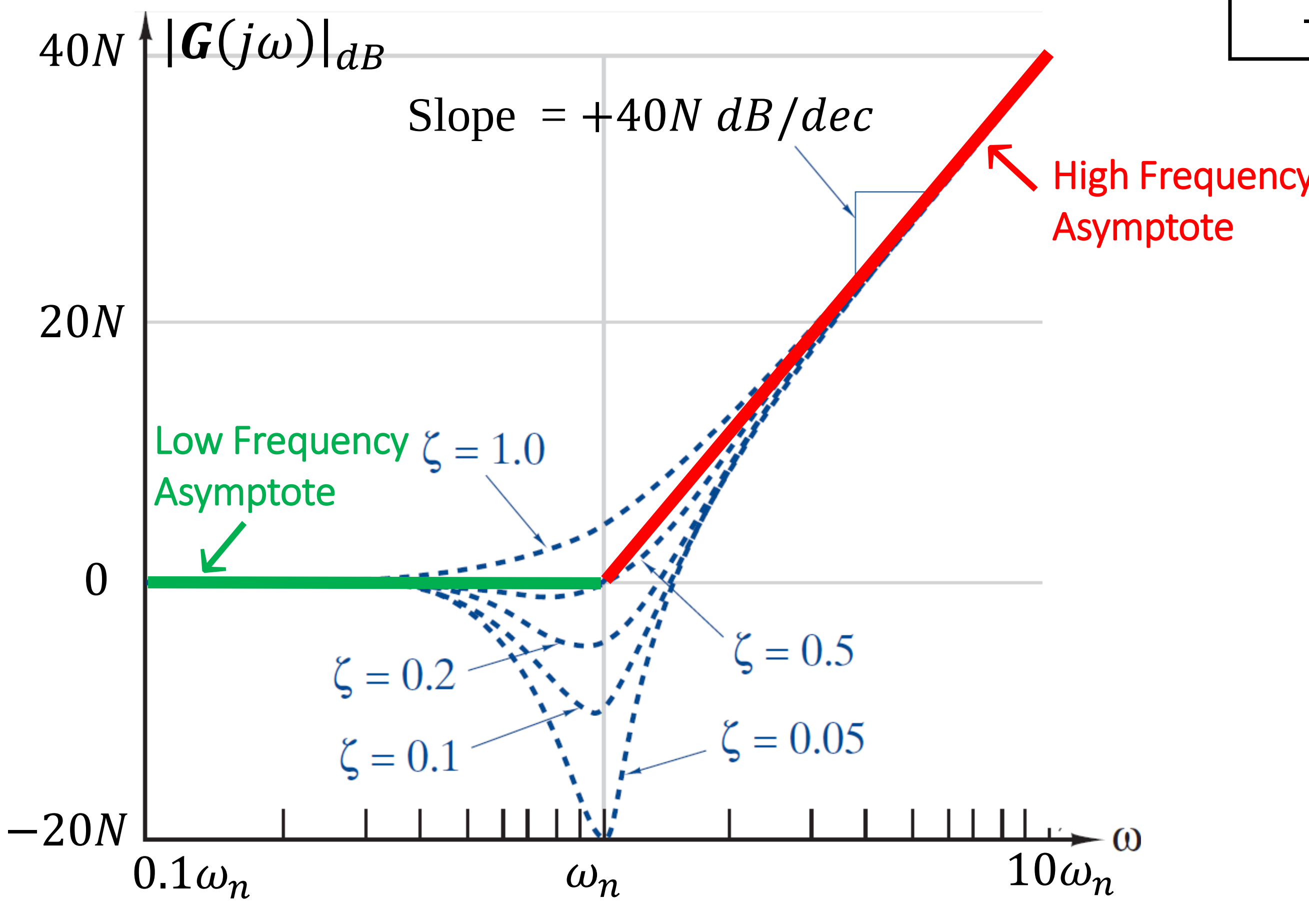
  - **Region #3**: $\omega/\omega_n \gg 1 \Rightarrow \omega \gg \omega_n$

$$|\boldsymbol{G}(j\omega)|_{db} \approx 20\log((\omega/\omega_n)^{2N}) \approx 40N\log_{10}(\omega/\omega_n)$$
$$\boxed{|\boldsymbol{G}(j\omega)|_{db} \approx 40N\log_{10}(\omega) - 40N\log_{10}(\omega_n)}$$

Plotting dB Magnitude of Complex Zeros (cont'd)

$$\boldsymbol{G}(j\omega) = [(j\omega/\omega_n)^2 + 2\zeta(j\omega/\omega_n) + 1]^N, N > 0, 0 \le \zeta < 1$$

- Region #1: $\omega \ll \omega_n$ — $\boxed{|\boldsymbol{G}|_{dB} \approx 0\ dB}$
- Region #2: $\omega = \omega_n$ — $\boxed{|\boldsymbol{G}|_{dB} = 20N\log_{10}(2\zeta)}$
- Region #3: $\omega \gg \omega_n$ — $\boxed{|\boldsymbol{G}|_{dB} \approx 40N\log_{10}(\omega) - 40N\log_{10}(\omega_n)}$



Minimum of dB Magnitude of Complex Zeros

- What conditions and at what frequency will $|\boldsymbol{G}(j\omega)|$ be minimized?

$$|\boldsymbol{G}(j\omega)| = \sqrt{[1-(\omega/\omega_n)^{\,2}]^2 + (2\zeta)^2(\omega/\omega_n)^2}$$

  - For $0 \le \zeta \le 1/\sqrt{2}$, $|\boldsymbol{G}(j\omega)|$has a minimum value $|\boldsymbol{G}(j\omega)|_{min} \le 1$
  - The frequency that causes $|\boldsymbol{G}(j\omega)|_{min}$ to occur is also the **resonant frequency $\boldsymbol{\omega_r}$**. An expression for ω_r is:

$$\boxed{\omega_r = \omega_n\sqrt{1-2\zeta^2}}$$

$$\boxed{\text{As } \zeta \to 0,\ \omega_r \to \omega_n}$$

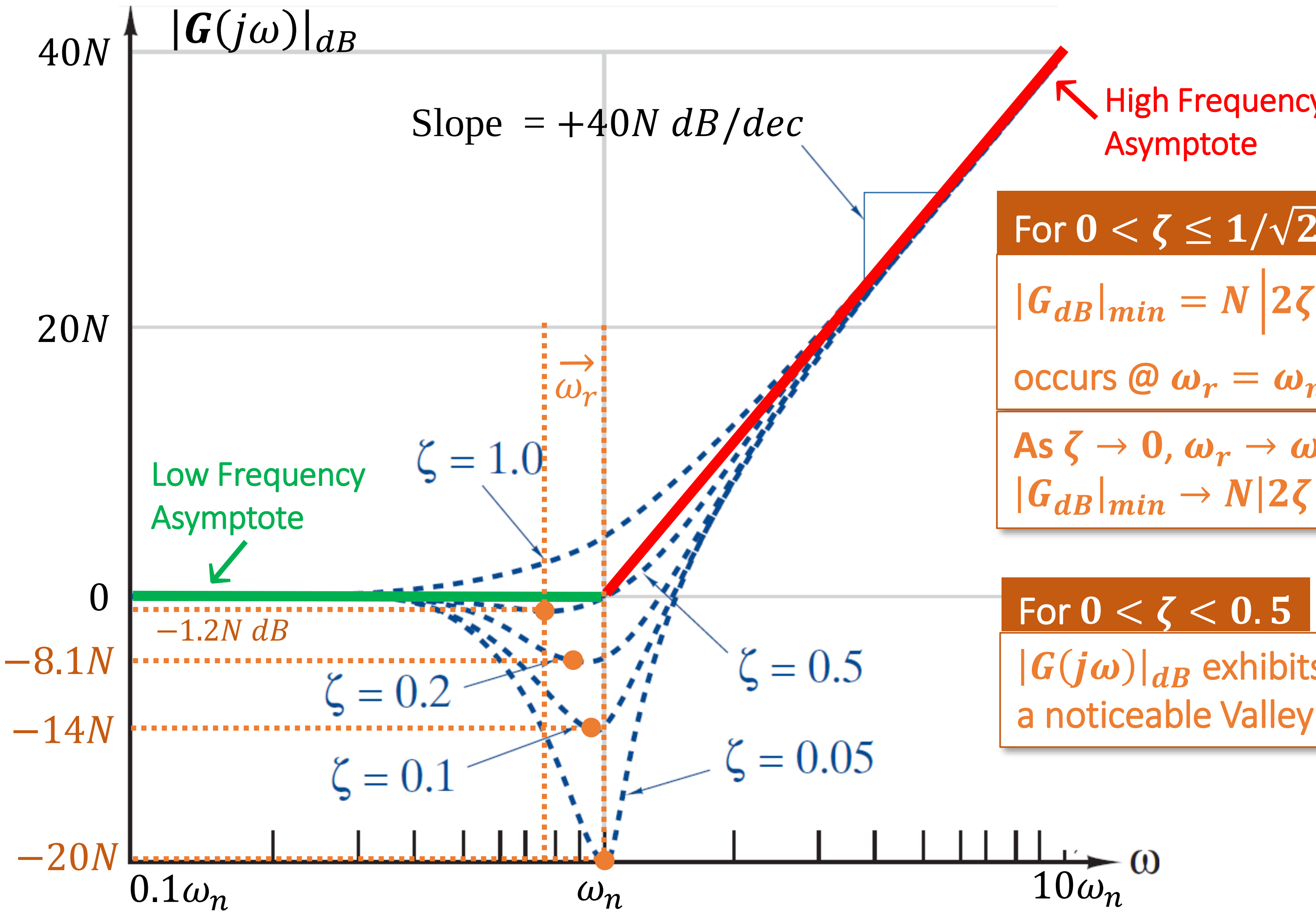
  - The value of $|\boldsymbol{G}(j\omega)|_{min}$ at $\omega = \omega_r = \omega_n\sqrt{1-2\zeta^2}$ is

$$\boxed{|\boldsymbol{G}(j\omega)|_{min} = |\boldsymbol{G}(j\omega_r)| = 2\zeta\sqrt{1-\zeta^2}}$$

$$\boxed{\text{As } \zeta \to 0,\ |\boldsymbol{G}(j\omega)|_{min} \to 2\zeta}$$

  - When $\zeta = 1/\sqrt{2}$, $|\boldsymbol{G}(j\omega)| = 1$ and $|\boldsymbol{G}(j\omega)|$ is called **maximally flat**
  - When $0 < \zeta < 1/2$, $|\boldsymbol{G}(j\omega)|$ exhibits a noticeable **valley**

Minimum of dB Magnitude of Complex Zeros (cont'd)



For $0 < \zeta \le 1/\sqrt{2}$

$|G_{dB}|_{min} = N\left|2\zeta\sqrt{1-\zeta^2}\right|_{dB}$

occurs @ $\omega_r = \omega_n\sqrt{1-2\zeta^2}$

As $\zeta \to 0$, $\omega_r \to \omega_n$ and $|G_{dB}|_{min} \to N|2\zeta|_{dB}$

For $0 < \zeta < 0.5$

$|G(j\omega)|_{dB}$ exhibits a noticeable Valley

Plotting Phase Angle of Complex LHP Zeros
$\boldsymbol{G}(j\omega) = [(j\omega/\omega_n)^2 + 2\zeta(j\omega/\omega_n) + 1]^N, N > 0, 0 \le \zeta < 1$

- Phase Angle $\angle\boldsymbol{G}(j\omega)$ of $\boldsymbol{G}(j\omega) = [(j\omega/\omega_n)^2 + 2\zeta(j\omega/\omega_n) + 1]^N$

$$\boldsymbol{G}(j\omega) = [\angle(j\omega/\omega_n)^2 + 2\zeta(j\omega/\omega_n) + 1]^N$$

$$\boldsymbol{G}(j\omega) = \sqrt{[1-(\omega/\omega_n)^2]^2 + (2\zeta)^2(\omega/\omega_n)^2}^{\,N} e^{jNtan^{-1}\left(\frac{2\zeta[\omega/\omega_n]}{1-(\omega/\omega_n)^2}\right)}$$

$$\boxed{\angle\boldsymbol{G}(j\omega) = Ntan^{-1}\left([(2\zeta)(\omega/\omega_n)]/[1-(\omega/\omega_n)^2]\right)}$$

- **Region #1**: $\omega/\omega_n \ll 1 \Rightarrow \omega \ll \omega_n$

$$\angle\boldsymbol{G}(j\omega) = Ntan^{-1}\left([(2\zeta)(small)]/[1-(small)^2]\right) \rightarrow \boxed{\angle\boldsymbol{G}(j\omega) \approx 0°}$$

- **Region #2**: $\omega/\omega_n = 1 \Rightarrow \omega = \omega_n$

$$\angle\boldsymbol{G}(j\omega_n) = Ntan^{-1}\left([(2\zeta)(\omega_n/\omega_n)]/[1-(\omega_n/\omega_n)^2]\right) = Ntan^{-1}(2\zeta/0)$$

$$\boxed{\angle\boldsymbol{G}(j\omega_n) = +90N°}$$

- **Region #3**: $\omega/\omega_n \gg 1 \Rightarrow \omega \gg \omega_n$

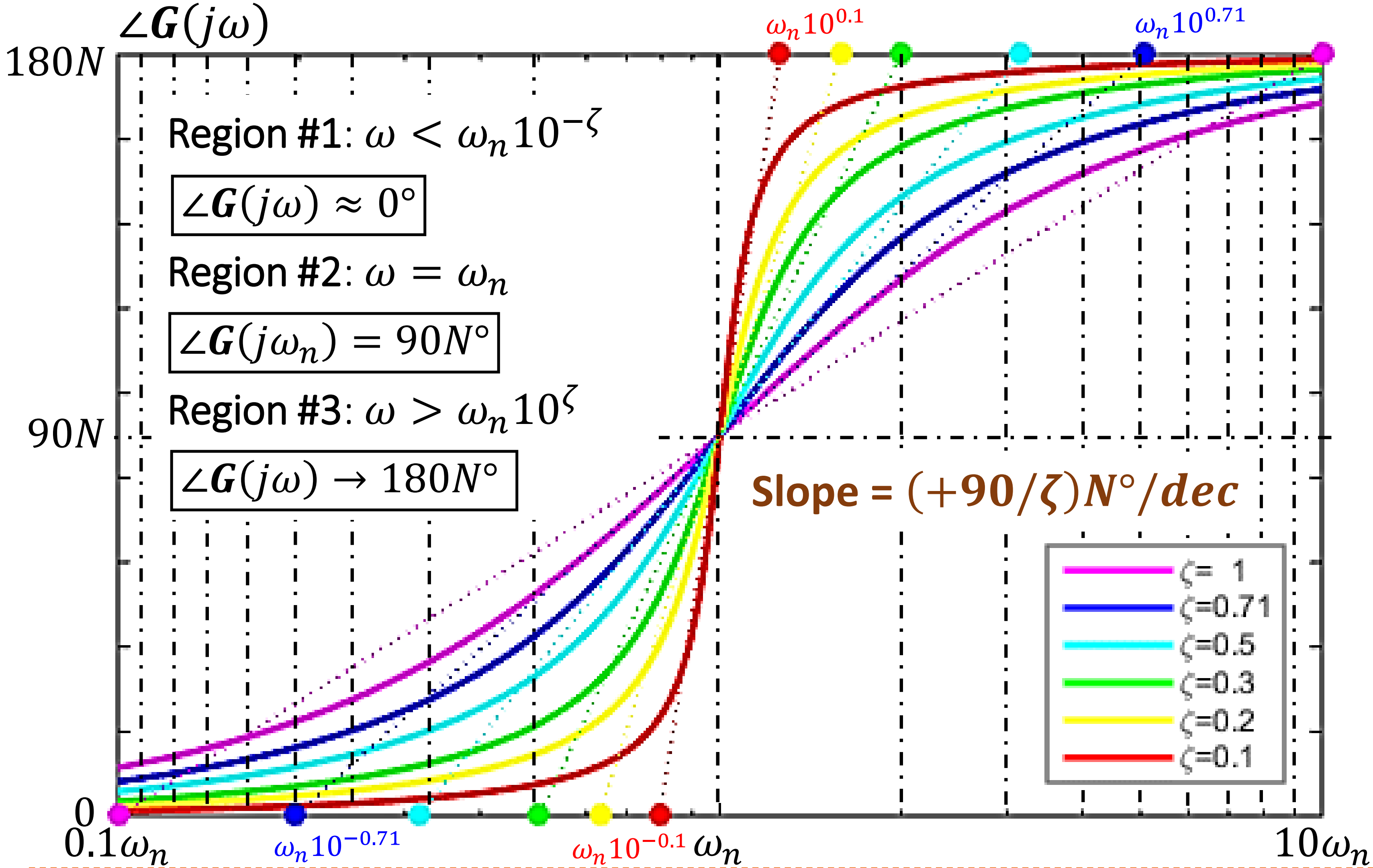
$$\angle\boldsymbol{G}(j\omega) = Ntan^{-1}\left([(2\zeta)(\omega/\omega_n)]/[1-(\omega/\omega_n)^2]\right)$$

$$\angle\boldsymbol{G}(j\omega) \approx Ntan^{-1}([(2\zeta)]/[-(\omega/\omega_n)]) \approx Ntan^{-1}([(small)]/[-(big)])$$

$$\boxed{\angle\boldsymbol{G}(j\omega) \rightarrow +180N°}$$

Plotting Phase Angle of Complex LHP Zeros

$$\boldsymbol{G}(j\omega) = [(j\omega/\omega_n)^2 + 2\zeta(j\omega/\omega_n) + 1]^N, N > 0, 0 \le \zeta < 1$$



Lecture Summary

- This set of notes presented
  - Frequency Response, Network Functions, and Sinusoidal Steady State Response
  - Review of Logs, The Bel and Decibel Scales
  - Bode Diagrams
  - Sketching of Bode Diagrams