Lecture #3(a): Basic Signal Waveforms
Theory

ECE 20200: Linear Circuit Analysis II
Steve Naumov (Instructor)

Lecture Overview

- This set of slides presents the following
  - Basis Functions for Engineering
    - Singularity Functions
      - Heaviside Step Function
      - Gate/Window Function
      - Ramp Function
      - Dirac Delta/Impulse “Function”
      - Relating Singularity Functions
      - Synthesizing Functions Using Singularity Functions
    - The Generalized Sinusoid and Complex Frequency Plane

Basis Functions

- **Basis Functions**: A set of elementary mathematical functions that may be combined in some manner to **model/approximate** many real world signals/waveforms.
- **Typical set of basis functions**
  - Heaviside Step Function
  - Gate Function
  - Ramp Function
  - Dirac Delta/Impulse "Function"
  - Complex Exponential Function
- **Singularity Functions**: A function that is either discontinuous or has at least one discontinuous derivative. Examples include
  - Heaviside Step
  - Dirac Delta/Impulse
  - Gate
  - Ramp

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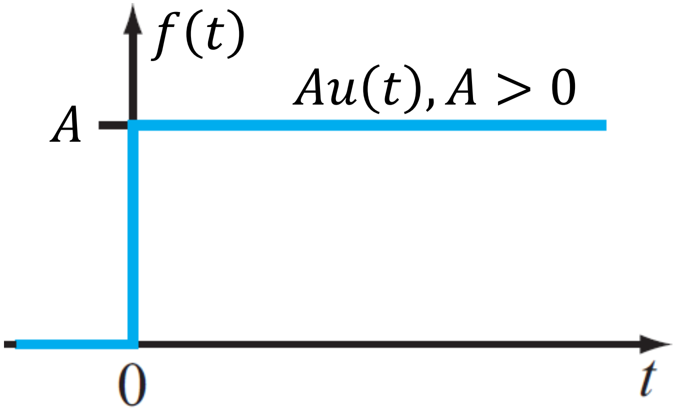
Heaviside Step Function

Heaviside (Unit) Step Function

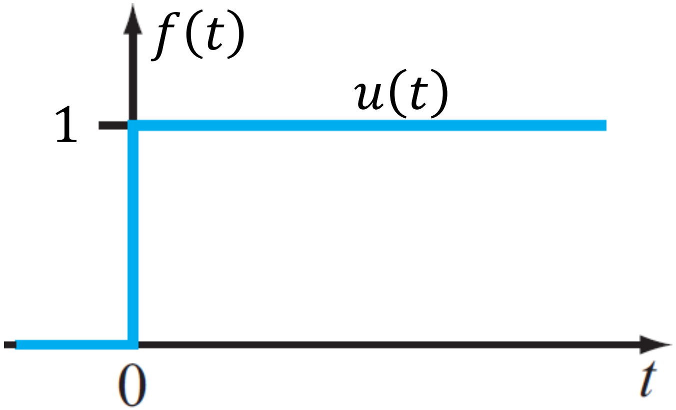
- Heaviside Step Function



$$f(t) = Au(t) = \begin{cases} 0, & t < 0 \\ A, & t > 0 \end{cases}$$

  - Value is zero **before** $t = 0$ and A **after** $t = 0$
  - Discontinuous at $t = 0 \rightarrow f(0) = Au(0)$ is undefined!

- Heaviside Unit Step Function



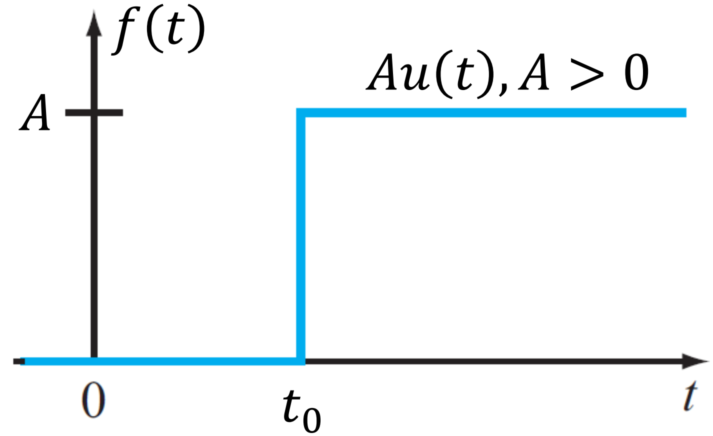
$$f(t) = u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

  - Value is zero **before** $t = 0$ and 1 **after** $t = 0$
  - Discontinuous at $t = 0 \rightarrow f(0) = u(0)$ is undefined!

Time-Shifted Heaviside (Unit) Step Functions

- **Delayed** Heaviside **Step** Function

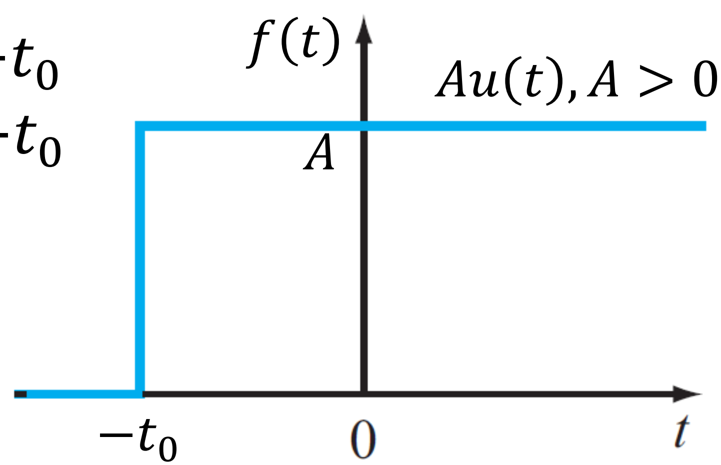
$$f(t) = Au(t - t_0) = \begin{cases} 0, & t < t_0 \\ A, & t > t_0 \end{cases}$$



  - Value is zero **before** $t = t_0 > 0$ and and A **after** $t = t_0 > 0$
  - $f(t_0) = Au(t_0)$ is undefined!

- **Advanced** Heaviside **Step** Function

$$f(t) = Au(t + t_0) = \begin{cases} 0, & t < -t_0 \\ A, & t > -t_0 \end{cases}$$

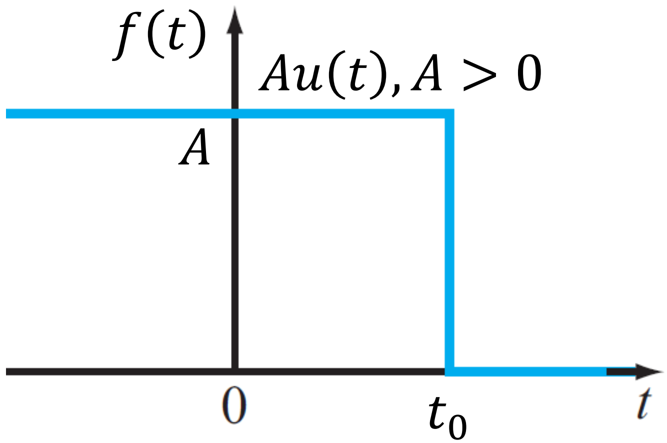


  - Value is zero **before** $t = t_0 < 0$ and A **after** $t = t_0 < 0$
  - $f(-t_0) = u(-t_0)$ is undefined!

Time-Reversed Heaviside Step Function

- **Time-Reversed Delayed** Heaviside **Step** Function

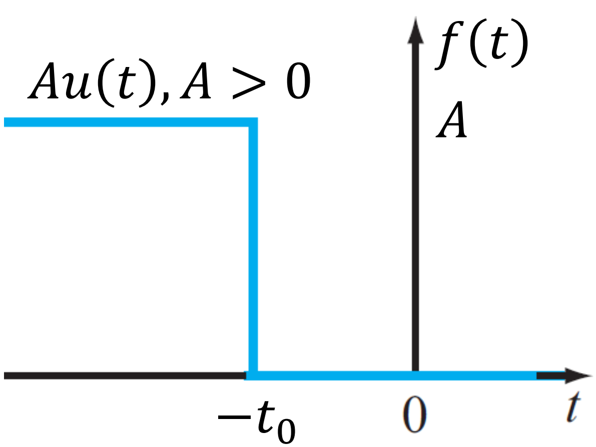
$$f(t) = Au(t_0 - t) = \begin{cases} A, & t < t_0 \\ 0, & t > t_0 \end{cases}$$



  - Value is A before $t = t_0 > 0$ and and 0 after $t = t_0 > 0$
  - $f(t_0) = Au(t_0)$ is undefined!

- **Time-Reversed Advanced** Heaviside **Step** Function

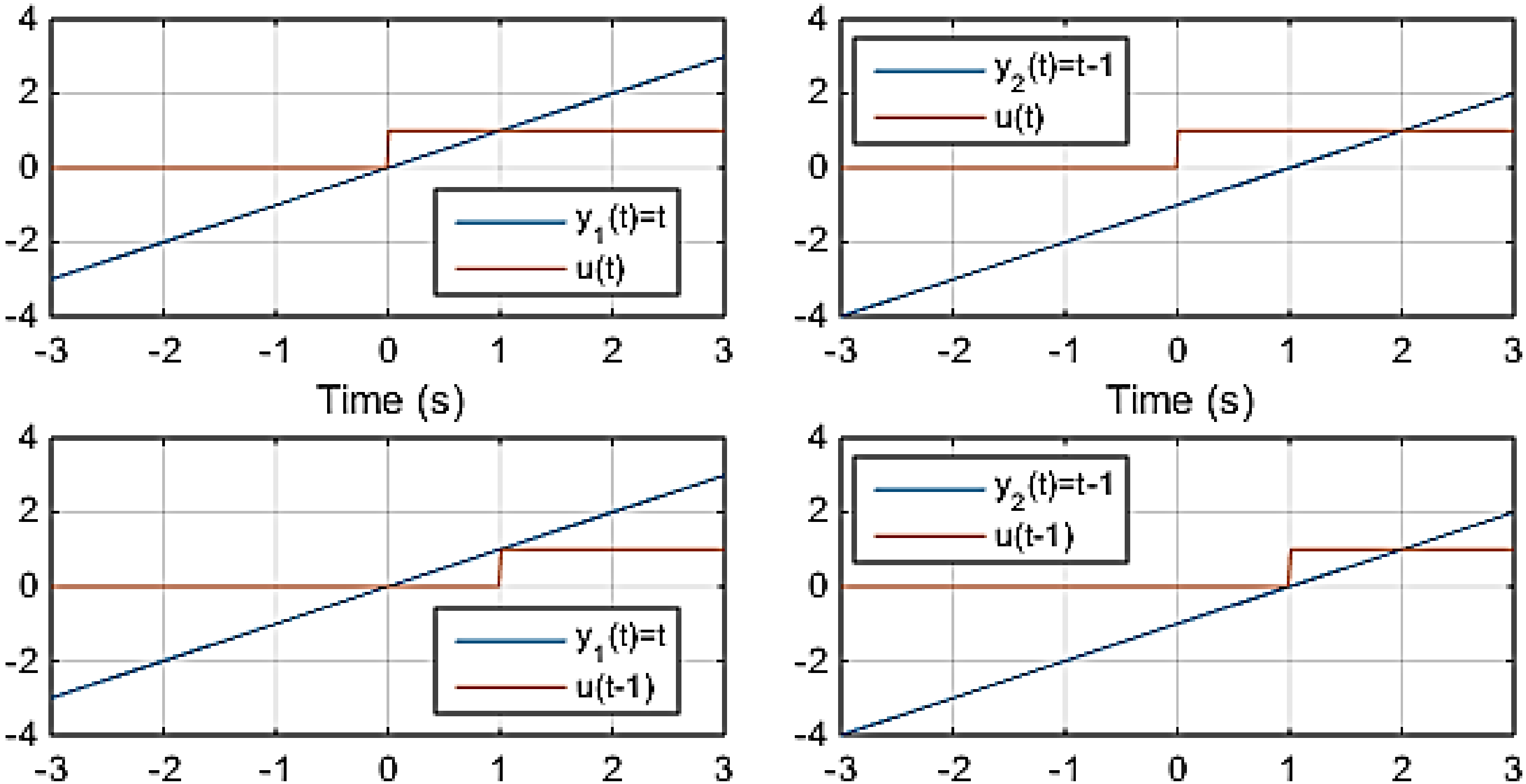
$$f(t) = Au(-t_0 - t) = \begin{cases} A, & t < -t_0 \\ 0, & t > -t_0 \end{cases}$$



  - Value is A before $t = t_0 < 0$ and and 0 after $t = t_0 < 0$
  - $f(-t_0) = Au(-t_0)$ is undefined!

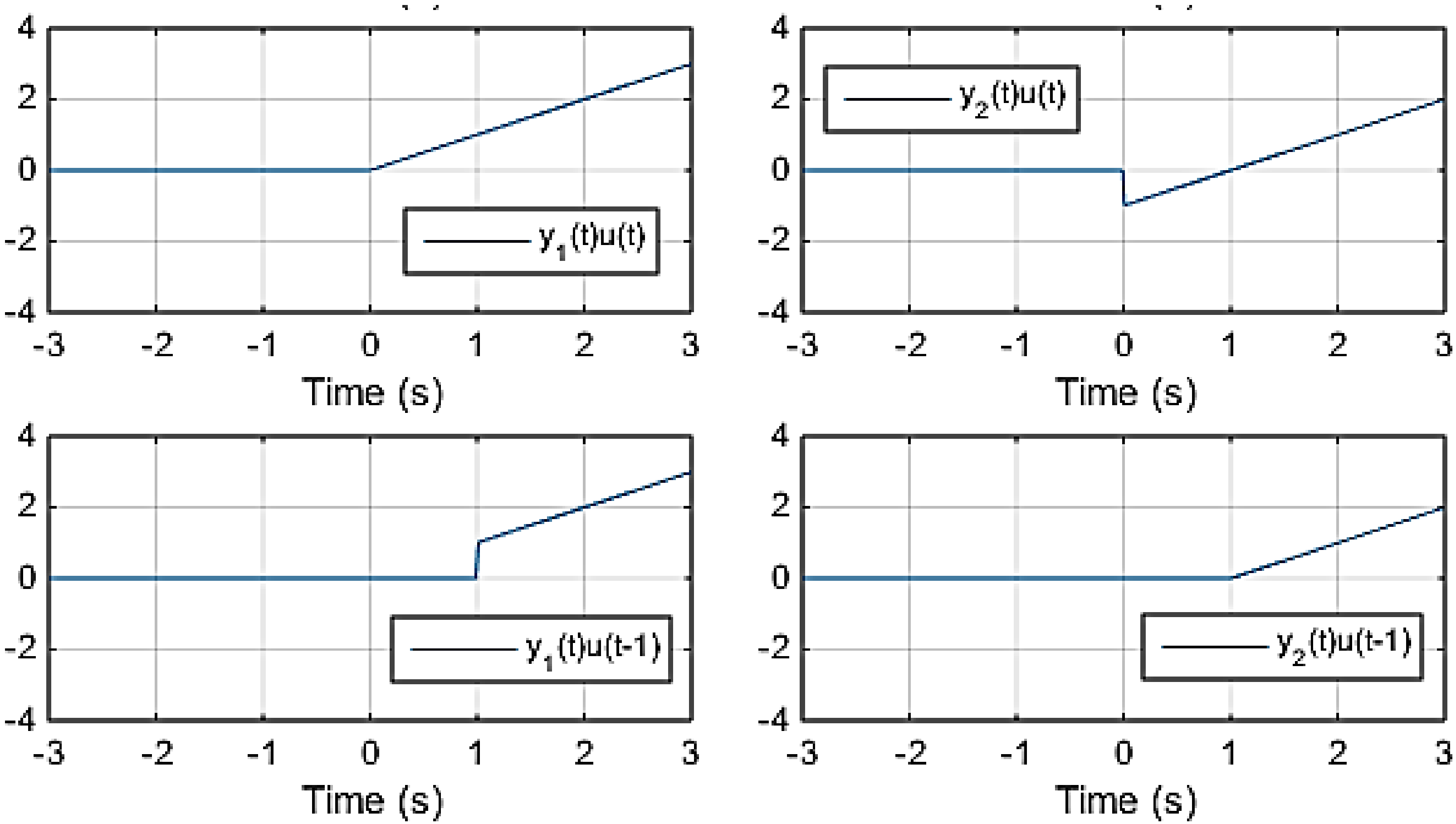
Heaviside Step: Turning On/Off Functions

- What is the result of multiplying the two functions in each graph shown below?



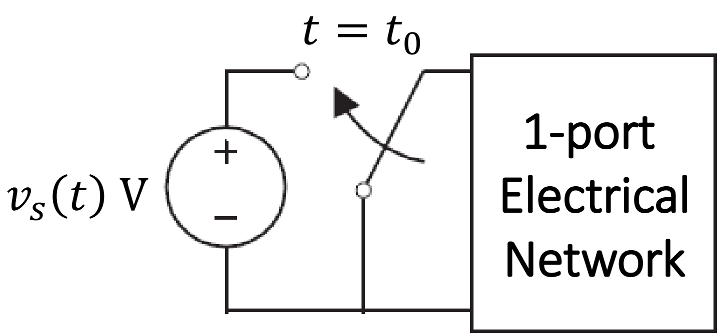
Heaviside Step: Turning On/Off Functions (cont’d)

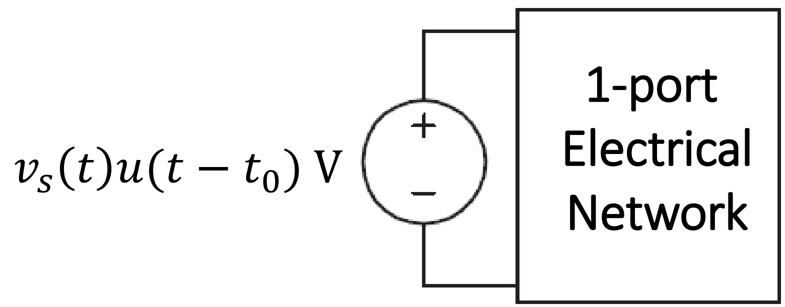
- The result of multiplying the two functions in each graph are shown below.



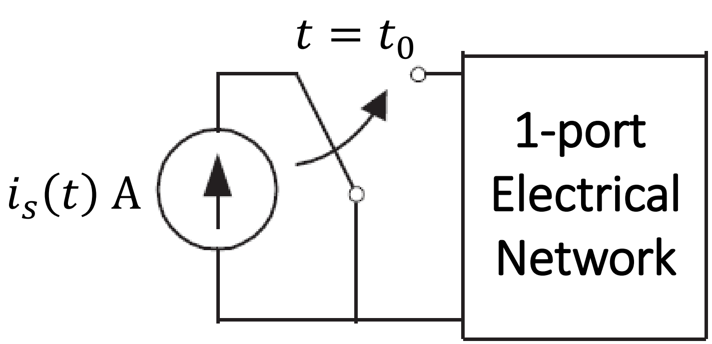
Heaviside Step: Modeling Switched Sources

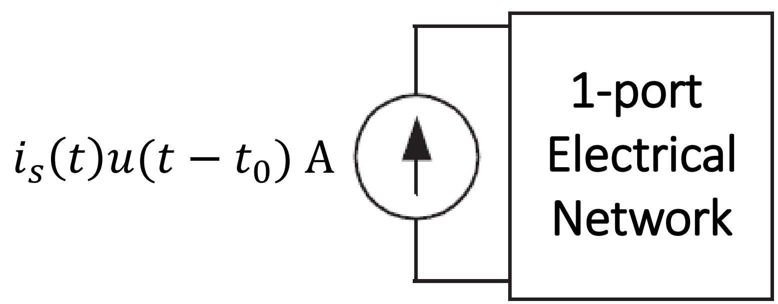
- Modeling Switched Voltage Sources
  - The two 1-port networks below are equivalent for all time





- Modeling Switched Current Sources
  - The two 1-port networks below are equivalent for all time





Lecture #3(a): Basic Signal Waveforms
Theory

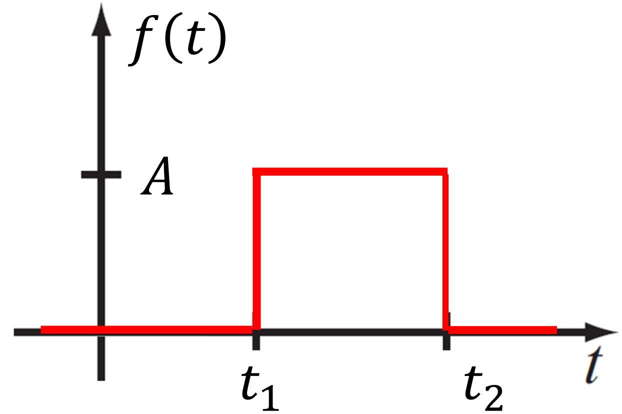
Gate/Window Function

Gate/Window Function

- Gate/Window Function

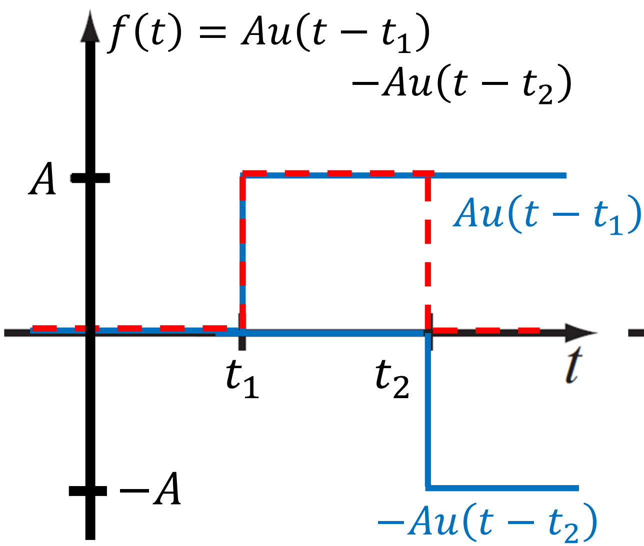
$$f(t) = \begin{cases} A, & t_1 < t < t_2 \\ 0, & elsewhere \end{cases}$$

  - Assumption: $t_2 > t_1$

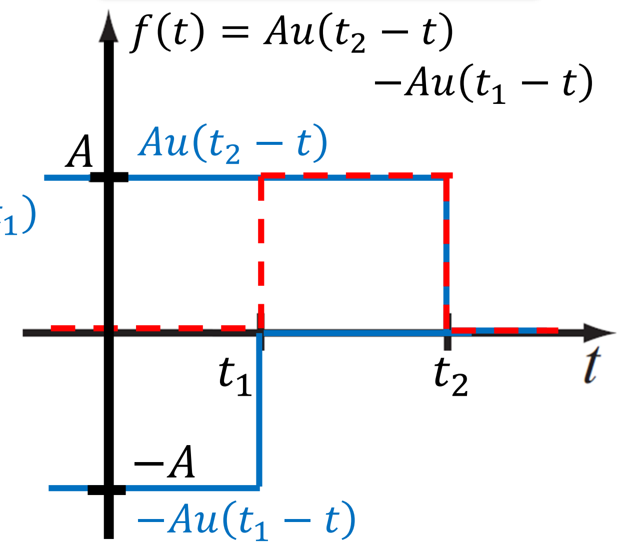


- Synthesizing the Gate Function $(t_2 > t_1)$

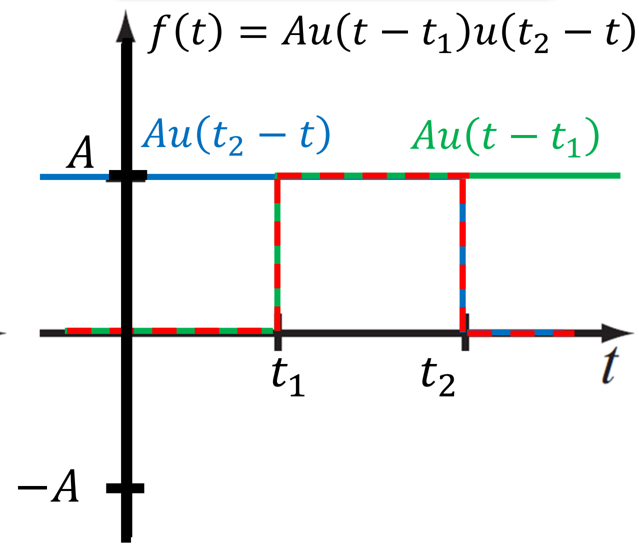
Synthesis Option #1



Synthesis Option #2



Synthesis Option #3



Lecture #3(a): Basic Signal Waveforms
Theory

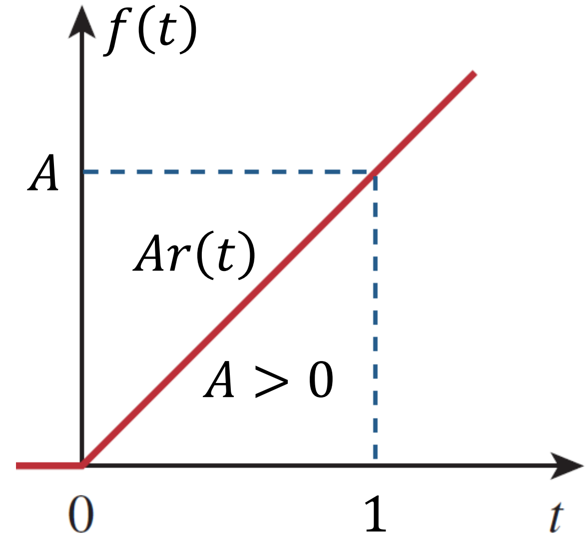
Ramp Function

Ramp Function

- **Ramp Function**

$$f(t) = Ar(t) = Atu(t) = \begin{cases} 0, & t \leq 0 \\ At, & t \geq 0 \end{cases}$$

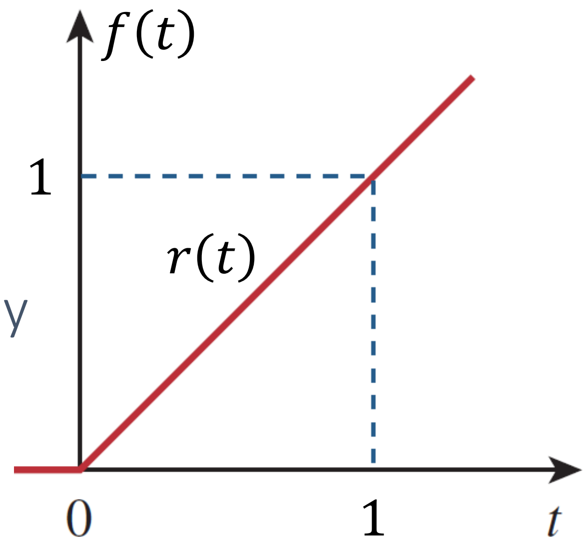
  - Value is zero **before** $t = 0$ and changes linearly with slope A **after** $t = 0$
  - Slope A has units s^{-1} (Hz)



- **Unit Ramp Function**

$$f(t) = r(t) = tu(t) = \begin{cases} 0, & t \leq 0 \\ t, & t \geq 0 \end{cases}$$

  - Value is zero **before** $t = 0$ and changes linearly with unity slope **after** $t = 0$
  - Slope has units s^{-1} (Hz)

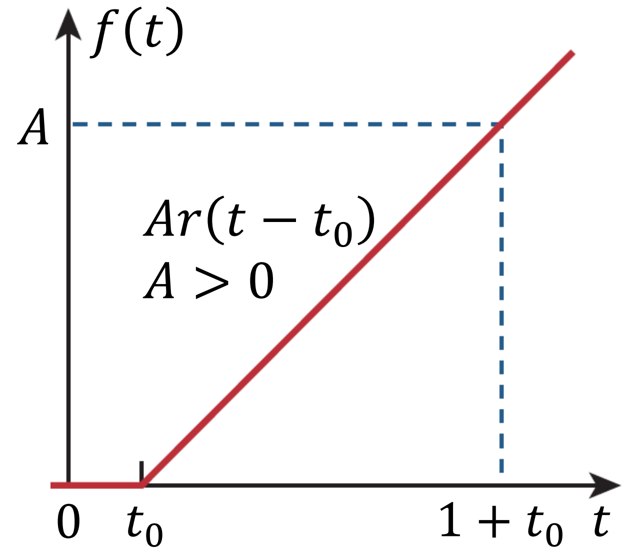


Time-Shifted Ramp Function

- **Delayed Ramp Function**

$$f(t) = Ar(t - t_0) = \begin{cases} 0, & t \leq t_0 \\ A(t - t_0), & t \geq t_0 \end{cases}$$

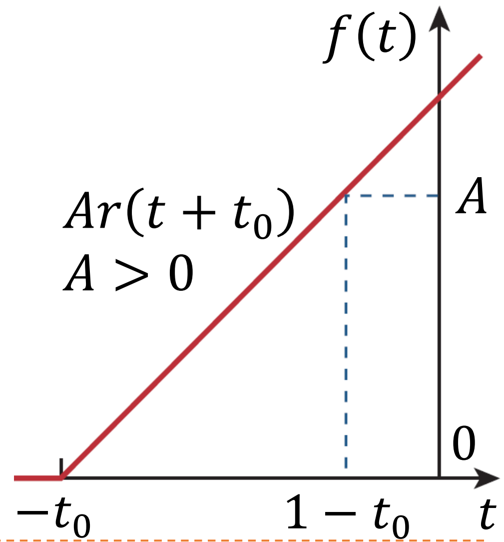
  - Value is zero **before** $t = t_0 > 0$ and changes linearly with slope A **after** $t = t_0 > 0$
  - Slope A has units s^{-1} (Hz)



- **Advanced Ramp Function**

$$f(t) = Ar(t + t_0) = \begin{cases} 0, & t \leq -t_0 \\ A(t + t_0), & t \geq -t_0 \end{cases}$$

  - Value is zero **before** $t = t_0 < 0$ and changes linearly with slope A **after** $t = t_0 < 0$
  - Slope A has units s^{-1} (Hz)



Lecture #3(a): Basic Signal Waveforms

Theory

Dirac Delta/Impulse Function

Intuition Behind the Dirac Delta/Impulse "Function"

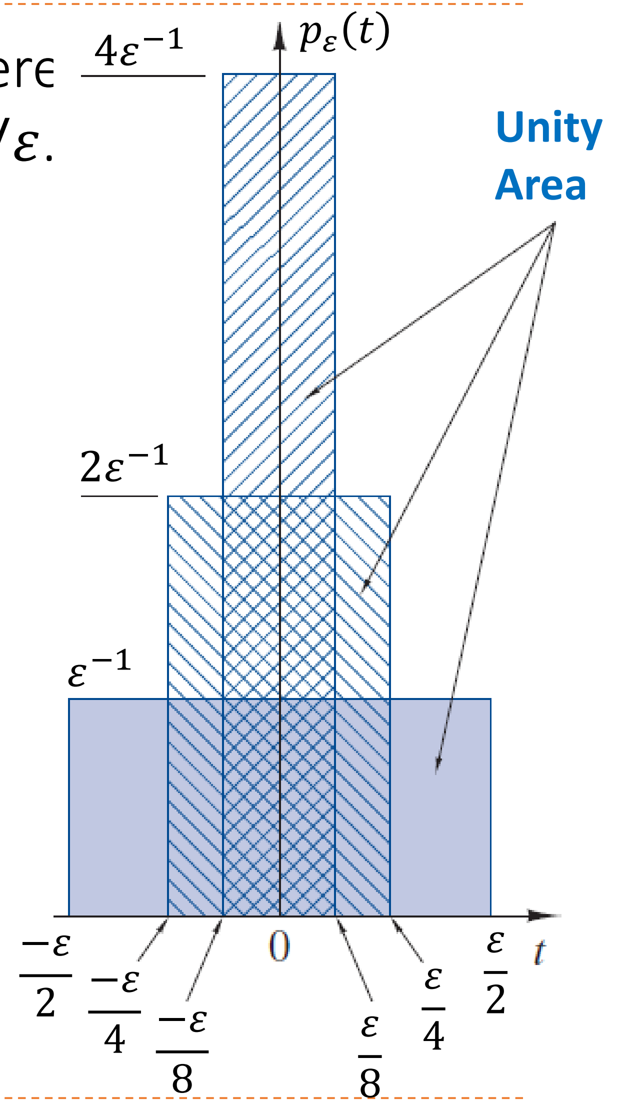
- Consider the **unit-area** rectangular pulse centere about $t = 0$ with duration ε and amplitude $1/\varepsilon$.
- The pulse $p_\varepsilon(t)$ can be described as a gate function using the following equation

$$p_\varepsilon(t) = \varepsilon^{-1}[u(t + 0.5\varepsilon) - u(t - 0.5\varepsilon)]$$

- As $\varepsilon \to 0$, **duration decreases** while **amplitude increases** thereby **preserving unity-area**
- The "function" obtained as $\varepsilon \to 0$ is known as the **Unit Dirac Delta/Impulse.**

$$\delta(t) = \lim_{\varepsilon \to 0} p_\varepsilon(t)$$

- The area is known as the **strength or intensity** and has units of **seconds s**



Dirac Delta/Impulse “Function”

- **Unit Dirac Delta/Impulse**

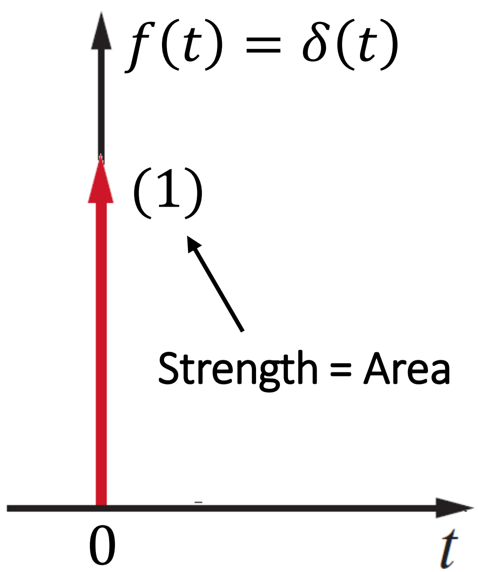
$$f(t) = \delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$$

$$\int_{t_1}^{t_2} \delta(t)dt = \begin{cases} 1, & t_1 < 0 < t_2 \\ 0, & elsewhere \end{cases}$$

Dirac Delta/Impulse

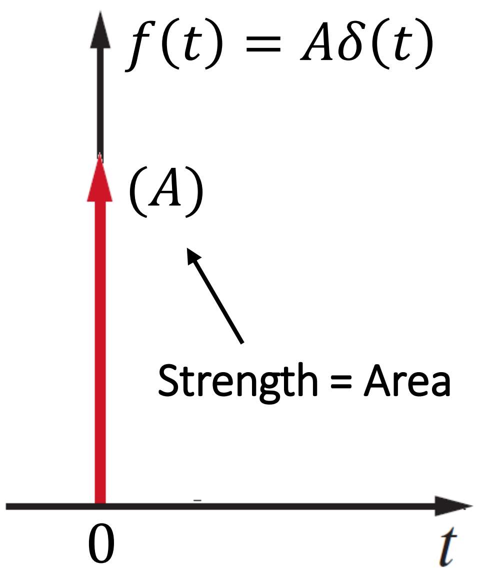
$$f(t) = A\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$$

$$\int_{t_1}^{t_2} A\delta(t)dt = \begin{cases} A, & t_1 < 0 < t_2 \\ 0, & elsewhere \end{cases}$$



<u>Assumption</u>

$t_2 > t_1$



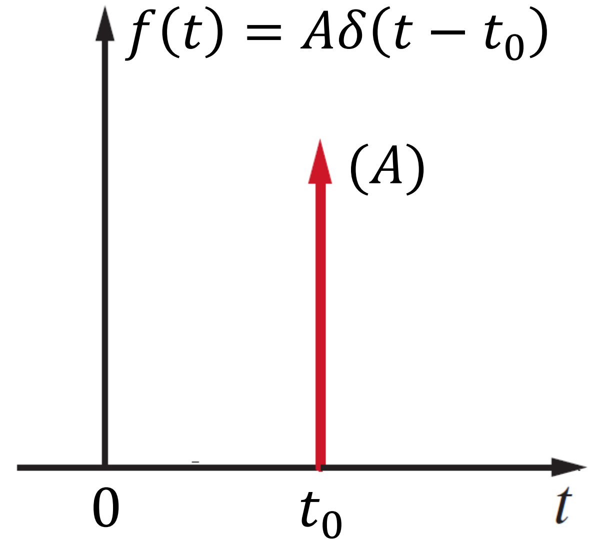
Time-Shifted Dirac Delta/Impulse "Function"

- Delayed Dirac Delta/Impulse Function

$$f(t) = A\delta(t - t_0) = \begin{cases} \infty, & t = t_0 \\ 0, & t \neq t_0 \end{cases}$$

$$\int_{t_1}^{t_2} A\delta(t - t_0)dt = \begin{cases} A, & t_1 < t_0 < t_2 \\ 0, & elsewhere \end{cases}$$

  - Assumption: $t_2 > t_1$

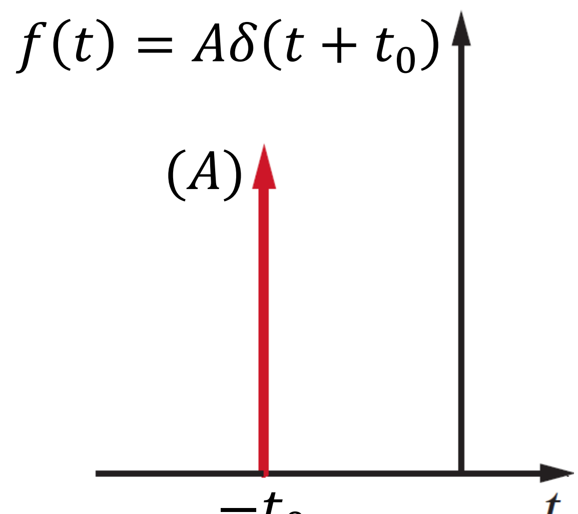


- Advanced Dirac Delta/Impulse Function

$$f(t) = A\delta(t + t_0) = \begin{cases} \infty, & t = -t_0 \\ 0, & t \neq -t_0 \end{cases}$$

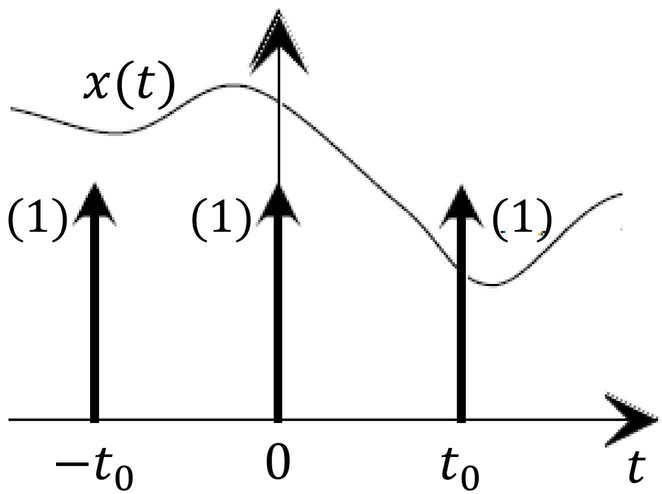
$$\int_{t_1}^{t_2} A\delta(t - t_0)dt = \begin{cases} A, & t_1 < -t_0 < t_2 \\ 0, & elsewhere \end{cases}$$

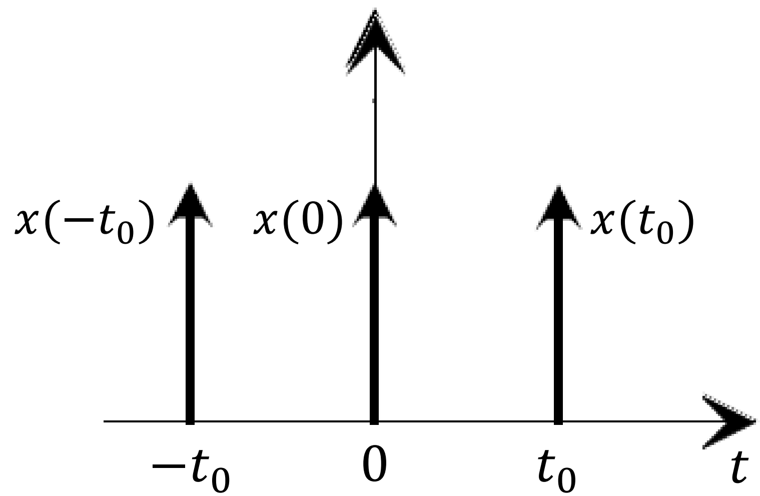
  - Assumption: $t_2 > t_1$



Sampling Property of the Dirac Delta/Impulse

- If a function of time $x(t)$ is continuous at time $t = t_0 > 0$, then $x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)$
- If a function of time $x(t)$ is continuous at time $t = -t_0 < 0$, then $x(t)\delta(t + t_0) = x(-t_0)\delta(t + t_0)$
- **Explanation**: Product of $x(t)$ with a unit-strength impulse occurring at $t = \pm t_0 > 0$ is an impulse with strength $x(\pm t_0)$ occurring at $t = \pm\, t_0 > 0$





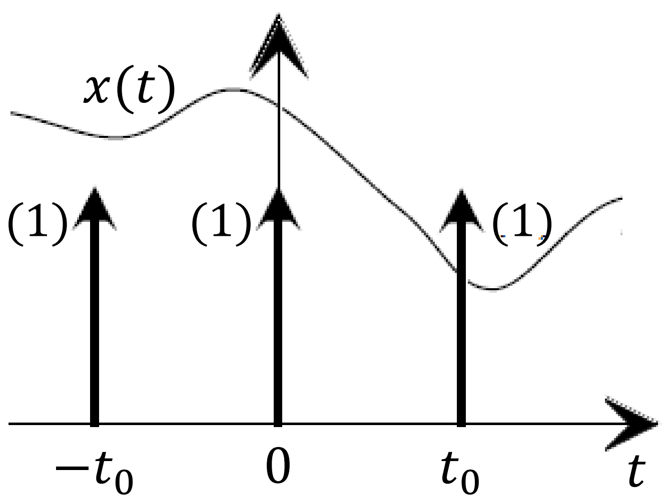
Sifting Property of the Dirac Delta/Impulse

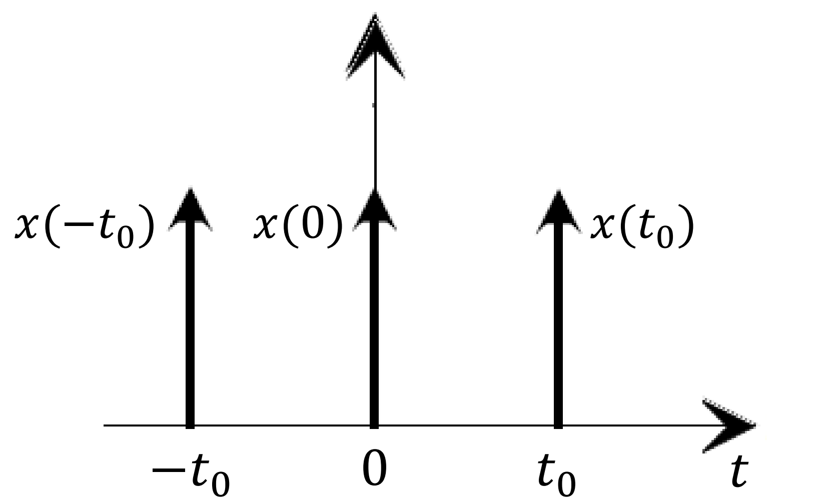
- If a function of time $x(t)$ is continuous at time $t = t_0 > 0$, then

$$\int_{t_1}^{t_2} x(t)\delta(t - t_0)dt = \int_{t_1}^{t_2} \overset{\text{Sampling Property}}{\boxed{x(t_0)\delta(t - t_0)}}dt = \begin{cases} x(t_0), & t_1 < t_0 < t_2 \\ 0, & elsewhere \end{cases}$$

- If a function of time $x(t)$ is continuous at time $t = -t_0 < 0$, then

$$\int_{t_1}^{t_2} x(t)\delta(t + t_0)dt = \int_{t_1}^{t_2} \overset{\text{Sampling Property}}{\boxed{x(-t_0)\delta(t + t_0)}}dt = \begin{cases} x(-t_0), & t_1 < -t_0 < t_2 \\ 0, & elsewhere \end{cases}$$





Lecture #3(a): Basic Signal Waveforms
Theory

Relating Singularity Functions

Relating Singularity Functions (Differentiation)

- Discontinuity at $t = t_0 > 0$

$$\frac{d}{dt}r(t - t_0) = \begin{cases} 0, & t < t_0 \\ 1, & t > t_0 \end{cases} = u(t - t_0)$$

$$\frac{d}{dt}u(t - t_0) = \begin{cases} 0, & t \neq t_0 \\ \infty, & t = t_0 \end{cases} = \delta(t - t_0)$$

- Discontinuity at $t = -t_0 < 0$

$$\frac{d}{dt}r(t + t_0) = \begin{cases} 0, & t < -t_0 \\ 1, & t > -t_0 \end{cases} = u(t + t_0)$$

$$\frac{d}{dt}u(t + t_0) = \begin{cases} 0, & t \neq -t_0 \\ \infty, & t = -t_0 \end{cases} = \delta(t + t_0)$$

Relating Singularity Functions (Integration)

- Discontinuity at $t = t_0 > 0$

$$\int_{\tau \to -\infty}^{\tau = t} \delta(\tau - t_0) d\tau = \begin{cases} 0, & t < t_0 \\ 1, & t > t_0 \end{cases} = u(t - t_0)$$

$$\int_{\tau \to -\infty}^{\tau = t} u(\tau - t_0) d\tau = \begin{cases} 0, & t \leq t_0 \\ t, & t \geq t_0 \end{cases} = r(t - t_0)$$

- Discontinuity at $t = -t_0 < 0$

$$\int_{\tau \to -\infty}^{\tau = t} \delta(\tau + t_0) d\tau = \begin{cases} 0, & t < -t_0 \\ 1, & t > -t_0 \end{cases} = u(t + t_0)$$

$$\int_{\tau \to -\infty}^{\tau = t} u(\tau + t_0) d\tau = \begin{cases} 0, & t \leq -t_0 \\ t, & t \geq -t_0 \end{cases} = r(t + t_0)$$

Lecture #3(a): Basic Signal Waveforms
Theory

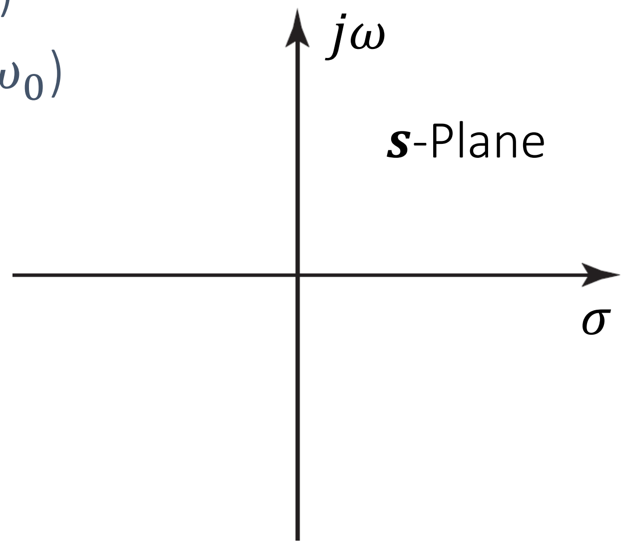
Generalized Sinusoidal Function

Generalized Sinusoidal Function

- The most general form of a sinusoidal function is given as
  $f(t) = \boldsymbol{Re}[\boldsymbol{K}e^{s_0 t}] \ = |\boldsymbol{K}|e^{\sigma_0 t}cos(\omega_0 t + \phi)$
  - **Complex Frequency** Variable: $\boldsymbol{s_0} = \sigma_0 \pm j\omega_0$
    - **Radian Frequency** $\boldsymbol{Im}[\boldsymbol{s_0}] = \pm\omega_0 = \pm 2\pi f_0 = \pm 2\pi/T_0$: Indicates how quickly the sinusoid oscillates (measured in rad/s).
    - **Neper Frequency** $\boldsymbol{Re}[\boldsymbol{s}_0] = \sigma_0 = 1/\tau$: Indicates how quickly the sinusoid converges or diverges w.r.t. time (measured in nepers/s).
  - $\boldsymbol{K} = |\boldsymbol{K}|e^{j\phi}$: Amplitude of complex exponential $\boldsymbol{e^{st}}$
- Going from $\boldsymbol{Re}[\boldsymbol{K}e^{s_0 t}]$ to $|\boldsymbol{K}|e^{\sigma_0 t}cos(\omega_0 t + \phi)$
  $f(t) = \boldsymbol{Re}[\boldsymbol{K}e^{s_0 t}] = \boldsymbol{Re}\left[|\boldsymbol{K}|e^{j\phi}e^{(\sigma_0 + j\omega_0)t}\right] \rightarrow$
  $f(t) = \boldsymbol{Re}\left[|\boldsymbol{K}|e^{j\phi + (\sigma_0 + j\omega_0)t}\right] = \boldsymbol{Re}\left[|\boldsymbol{K}|e^{\sigma_0 t}e^{j(\omega_0 t + \phi)}\right] \rightarrow$
  $f(t) = |\boldsymbol{K}|e^{\sigma_0 t}\boldsymbol{Re}[cos(\omega_0 t + \phi) + jsin(\omega_0 t + \phi)] \rightarrow$
  $f(t) = |\boldsymbol{K}|e^{\sigma_0 t}cos(\omega_0 t + \phi)$

Generalized Sinusoidal Function (cont'd)

- Function $f(t) = \boldsymbol{Re}[\boldsymbol{K}e^{s_0 t}]$ is a general case of several specific functions of which you have some familiarity.
- Depending on the value of complex frequency variable $\boldsymbol{s_0}$, special cases of $f(t) = \boldsymbol{Re}[\boldsymbol{K}e^{s_0 t}]$ arise. Those cases include
  - Constant/DC Value: $(\boldsymbol{s_0} = 0 \rightarrow \sigma_0 = \pm\omega_0 = 0)$
  - Real Monotonic Exponential: $(\boldsymbol{s_0} = \sigma_0 \rightarrow \sigma_0 \neq 0, \pm\omega_0 = 0)$
  - Un-damped Sinusoid: $(\boldsymbol{s_0} = \pm j\omega_0 \rightarrow \sigma_0 = 0)$
  - Exponentially Varying Sinusoid: $(\boldsymbol{s_0} = \sigma_0 \pm j\omega_0)$
- Complex Frequecy Plane ($\boldsymbol{s}$-Plane)
  - An Cartesian plane used to graphically visualize the form of $f(t)$ as a function of complex frequency variable $\boldsymbol{s}$



Generalized Sinusoid: Constant & Real Exponential

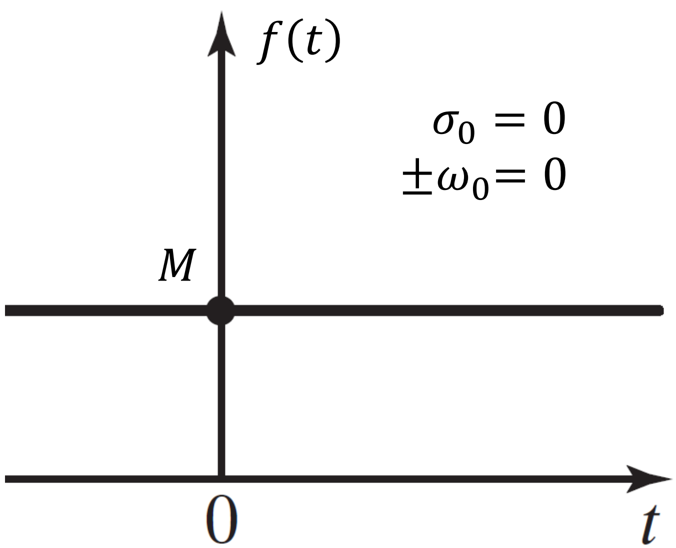
- Constant (DC) ($\sigma_0 = 0, \pm\omega_0 = 0$)

$$f(t) = \boldsymbol{Re}[\boldsymbol{K}e^{s_0 t}] = |\boldsymbol{K}|e^{\sigma_0 t}cos(\omega_0 t + \phi)$$

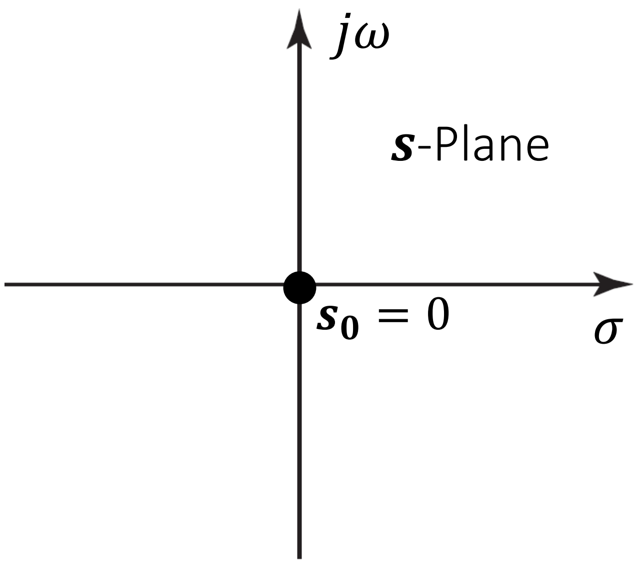
$$f(t) = |\boldsymbol{K}|e^{(0)t}cos\big((0)t + \phi\big) = |\boldsymbol{K}|cos(\phi)$$

$$\boxed{f(t) = M}$$

Time Domain



Complex Frequency Domain



Generalized Sinusoid: Real Monotonic Exponential

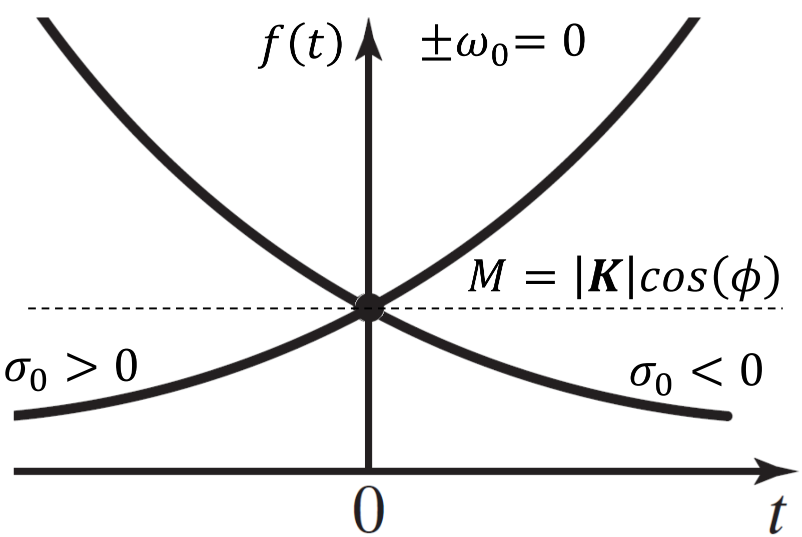
- Real Exponential ($\sigma_0 \neq 0, \pm\omega_0 = 0$)

$$f(t) = \boldsymbol{Re}[\boldsymbol{K}e^{s_0 t}] = |\boldsymbol{K}|e^{\sigma_0 t}cos(\omega_0 t + \phi)$$

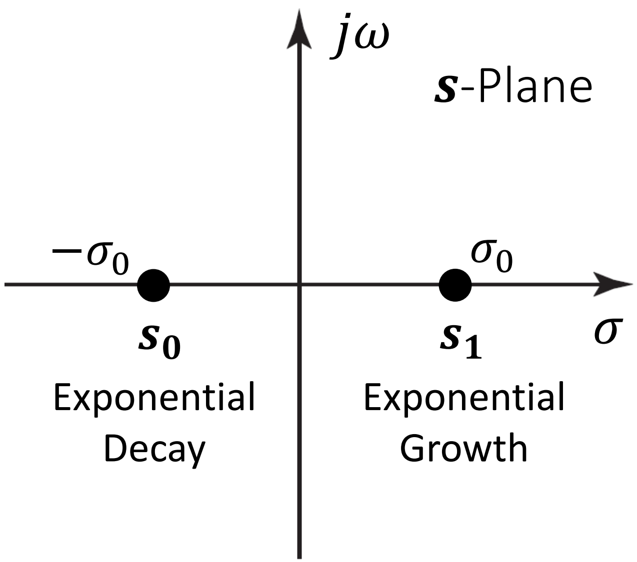
$$f(t) = |\boldsymbol{K}|e^{\sigma_0 t}cos\big((0)t + \phi\big) = |\boldsymbol{K}|cos(\phi)e^{\sigma_0 t}$$

$$\boxed{f(t) = Me^{\sigma_0 t}}$$

Time Domain



Complex Frequency Domain



Generalized Sinusoid: Un-dapmed Sinusoid

- Un-damped Sinusoid ($\sigma_0 = 0, \pm\omega_0 \neq 0$)

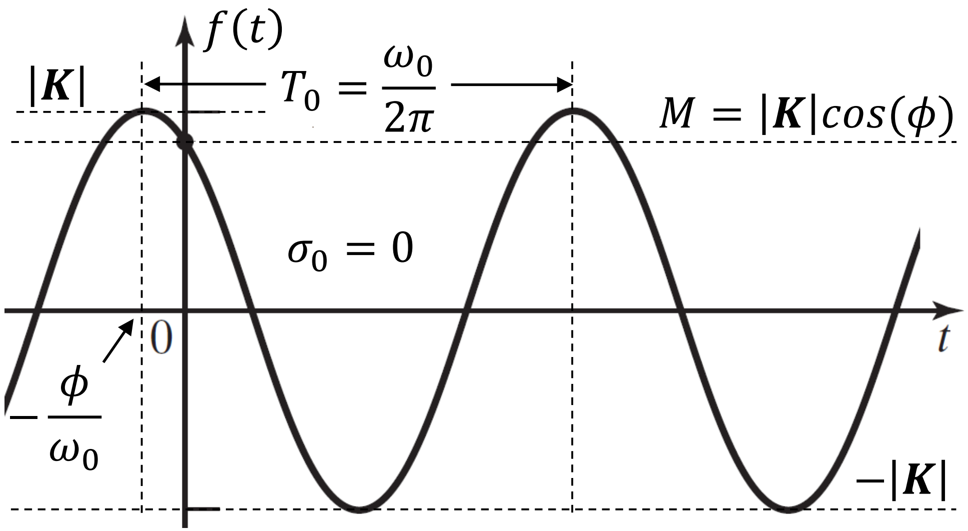
$$f(t) = \boldsymbol{Re}[\boldsymbol{K}e^{\boldsymbol{s}_0 t}] = |\boldsymbol{K}|e^{\sigma_0 t}cos(\omega_0 t + \phi) = |\boldsymbol{K}|e^{(0)t}cos(\omega_0 t + \phi)$$

$$\boxed{f(t) = |\boldsymbol{K}|cos(\omega_0 t + \phi)}$$

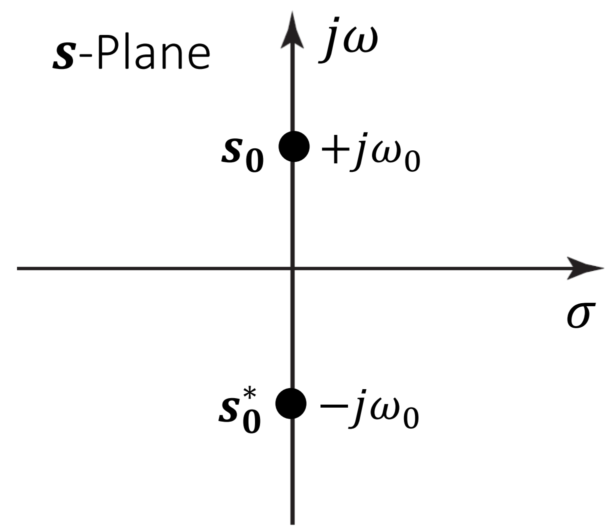
$$f(t) = 0.5|\boldsymbol{K}|\left[e^{j(\omega_0 t+\phi)} + e^{-j(\omega_0 t+\phi)}\right] = 0.5\left[|\boldsymbol{K}|e^{j\phi}e^{j\omega_0 t} + |\boldsymbol{K}|e^{-j\phi}e^{-j\omega_0 t}\right]$$

$$f(t) = 0.5\boldsymbol{K}e^{j\omega_0 t} + 0.5\boldsymbol{K}^*e^{-j\omega_0 t} \rightarrow \boxed{f(t) = 0.5\boldsymbol{K}e^{\boldsymbol{s}_0 t} + 0.5\boldsymbol{K}^*e^{\boldsymbol{s}_0^* t}}$$

Time Domain



Complex Frequency Domain



Generalized Sinusoid: Exponentially Growing Sinusoid

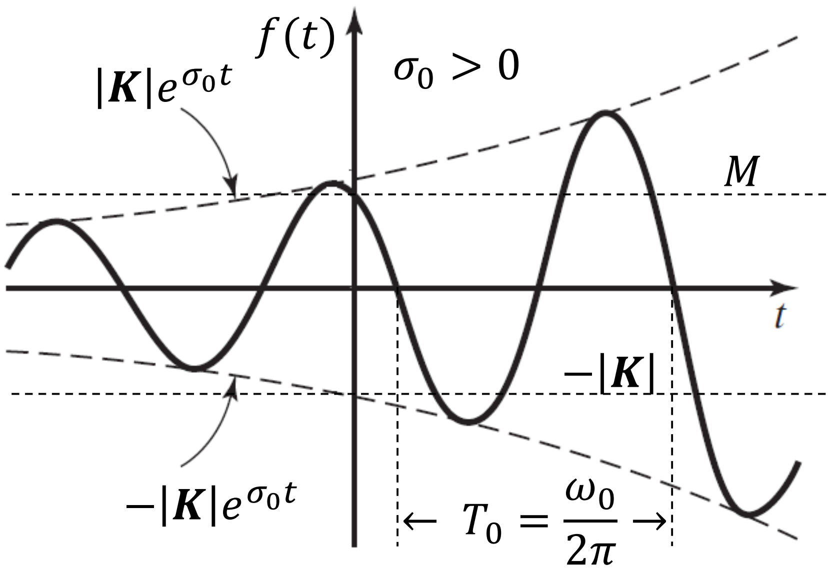
- Exponentially Growing Sinusoid $(\sigma_0 > 0, \omega_0 \neq 0)$

$$\boxed{f(t) = |\boldsymbol{K}|e^{\sigma_0 t} cos(\omega_0 t + \phi)} \rightarrow f(t) = 0.5|\boldsymbol{K}|e^{\sigma_0 t}\left[e^{j(\omega_0 t + \phi)} + e^{-j(\omega_0 t + \phi)}\right]$$

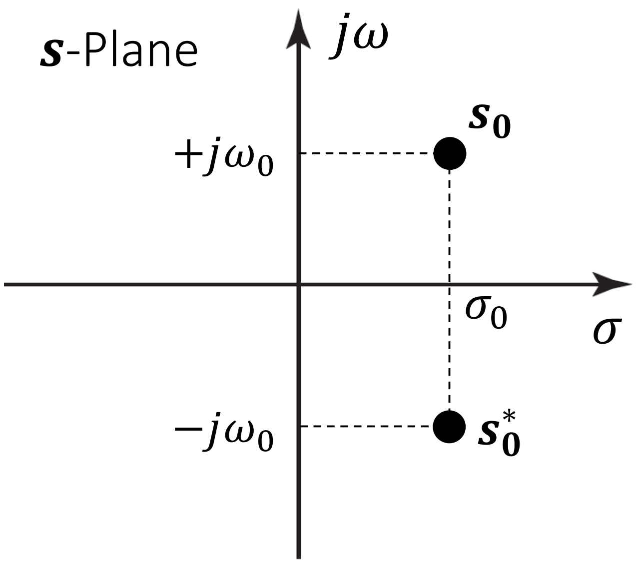
$$f(t) = 0.5\left[|\boldsymbol{K}|e^{j\phi}e^{(\sigma_0 + j\omega_0)t} + |\boldsymbol{K}|e^{-j\phi}e^{(\sigma_0 - j\omega_0)t}\right]$$

$$\boxed{f(t) = 0.5\boldsymbol{K}e^{\boldsymbol{s}_0 t} + 0.5\boldsymbol{K}^* e^{\boldsymbol{s}_0^* t}}$$

Time Domain



Complex Frequency Domain



Generalized Sinusoid: Exponentially Decaying Sinusoid

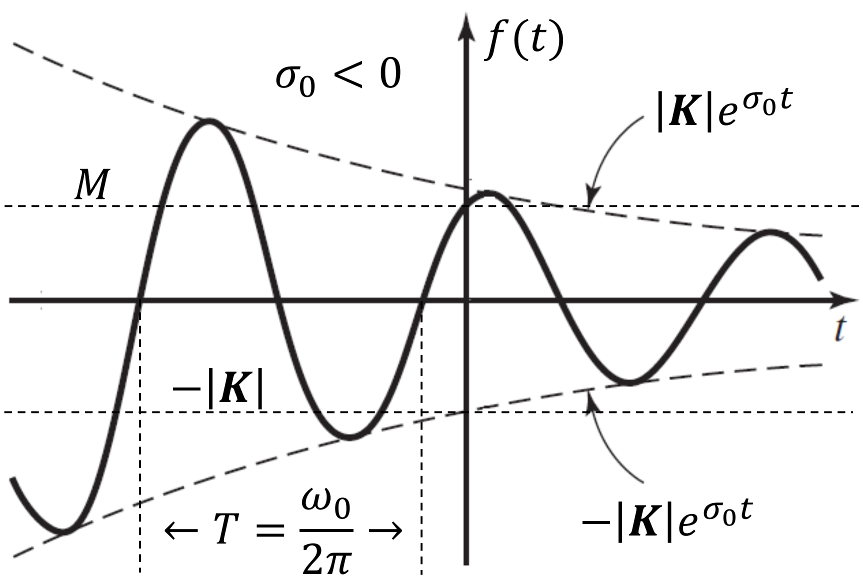
- Exponentially Decaying Sinusoid ($\sigma_0 < 0, \omega_0 \neq 0$)

$$\boxed{f(t) = |\boldsymbol{K}|e^{\sigma_0 t}cos(\omega_0 t + \phi)} \rightarrow f(t) = 0.5|\boldsymbol{K}|e^{\sigma_0 t}\left[e^{j(\omega_0 t+\phi)} + e^{-j(\omega_0 t+\phi)}\right]$$

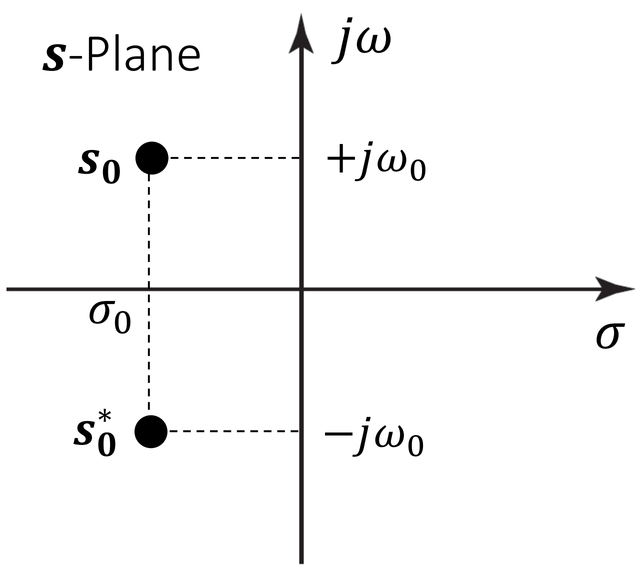
$$f(t) = 0.5\left[|\boldsymbol{K}|e^{j\phi}e^{(\sigma_0+j\omega_0)t} + |\boldsymbol{K}|e^{-j\phi}e^{(\sigma_0-j\omega_0)t}\right]$$

$$\boxed{f(t) = 0.5\boldsymbol{K}e^{\boldsymbol{s}_0 t} + 0.5\boldsymbol{K}^* e^{\boldsymbol{s}_0^* t}}$$

Time Domain



Complex Frequency Domain



Lecture Summary

- This set of slides presented the following
  - Basis Functions for Engineering
    - Singularity Functions
      - Heaviside Step Function
      - Gate/Window Function
      - Ramp Function
      - Dirac Delta/Impulse “Function”
      - Relating Singularity Functions
      - Synthesizing Functions Using Singularity Functions
    - The Generalized Sinusoid and Complex Frequency Plane