HOMEWORK #5: Inverse Laplace Transforms, Poles/Zeros, and {I,F}VT (Selected Answers)

1. "Simple" Inverse Laplace Transforms

Compute the inverse Laplace Transform of each of the following rational functions of a complex frequency. Completing the square may be required, but partial fraction expansion is unnecessary.

(a) $\boldsymbol{F}(\boldsymbol{s}) = \frac{3}{(2\boldsymbol{s}-5)^5}$ $\boxed{\mathcal{L}^{-1}\{\boldsymbol{F}(\boldsymbol{s})\} = f(t) = \frac{1}{256}e^{2.5t}t^4u(t)}$

(b) $\boldsymbol{F}(\boldsymbol{s}) = \frac{3\boldsymbol{s}+1}{\boldsymbol{s}+4}$ $\boxed{\mathcal{L}^{-1}\{\boldsymbol{F}(\boldsymbol{s})\} = f(t) = 3\delta(t) - 11e^{-4t}u(t)}$

(c) $\boldsymbol{F}(\boldsymbol{s}) = \frac{s-5}{s^2+4s+5}$ $\boxed{\mathcal{L}^{-1}\{\boldsymbol{F}(\boldsymbol{s})\} = e^{-2t}[\cos(t) - 7\sin(t)]u(t) \approx 7.07e^{-2t}\cos(t + 81.87°)u(t)}$

(d) $\boldsymbol{F}(\boldsymbol{s}) = \frac{2s^4+3s^3-s^2+8s+4}{s^3}$ $\boxed{\mathcal{L}^{-1}\{\boldsymbol{F}(\boldsymbol{s})\} = f(t) = 2\delta'(t) + 3\delta(t) + [2t^2 + 8t - 1]u(t)}$

(e) $\boldsymbol{F}(\boldsymbol{s}) = \frac{\boldsymbol{s}(1+e^{-\pi\boldsymbol{s}})}{\boldsymbol{s}^2+4\boldsymbol{s}+5}$

$$\boxed{\mathcal{L}^{-1}\{\boldsymbol{F}(\boldsymbol{s})\} = f(t) = e^{-2t}[\cos(t) - 2\sin(t)]u(t) + e^{-2(t-\pi)}[\cos(t-\pi) - 2\sin(t-\pi)]u(t-\pi)}$$

$$\boxed{\mathcal{L}^{-1}\{\boldsymbol{F}(\boldsymbol{s})\} = f(t) = e^{-2t}[\cos(t) - 2\sin(t)][u(t) - e^{2\pi}u(t-\pi)]}$$

2. Inverse Laplace Transforms via Partial Fraction Expansion

Compute the right sided time functions corresponding to each of the following rational functions of a complex frequency. <u>Verify all partial fraction expansion results with MATLAB</u>.

(a) <u>Strictly Proper, Distinct Real Poles</u>

i. $\boldsymbol{F}(\boldsymbol{s}) = \frac{2\boldsymbol{s}^3+33\boldsymbol{s}^2+93\boldsymbol{s}+54}{\boldsymbol{s}(\boldsymbol{s}+1)(\boldsymbol{s}^2+5\boldsymbol{s}+6)}$

$$\boxed{f(t) = [9 + 4e^{-t} - 8e^{-2t} - 3e^{-3t}]u(t)}$$

(b) <u>Strictly Proper, Repeated Real Poles</u>

i. $\boldsymbol{F}(\boldsymbol{s}) = \frac{2\boldsymbol{s}^2+4\boldsymbol{s}+1}{(\boldsymbol{s}+1)(\boldsymbol{s}+2)^3}$ $\boxed{f(t) = [-e^{-t} + e^{-2t}(1 + 3t - 0.5t^2)]u(t)}$

(c) <u>Strictly Proper, Distinct Complex Poles (**Complex Number Method**)</u>

i. $\boldsymbol{F}(\boldsymbol{s}) = \frac{-\boldsymbol{s}^2+52\boldsymbol{s}+445}{\boldsymbol{s}(\boldsymbol{s}^2+10\boldsymbol{s}+89)}$

$$\boxed{f(t) = [5 + 7.2e^{-5t}\cos(8t - 146.31°)]u(t)}$$

ii. $\boldsymbol{F}(\boldsymbol{s}) = \frac{14\boldsymbol{s}^2+56\boldsymbol{s}+152}{(\boldsymbol{s}+6)(\boldsymbol{s}^2+4\boldsymbol{s}+20)}$

$$\boxed{f(t) = [10e^{-6t} + 5.66e^{-2t}\cos(4t + 45°)]u(t)}$$

(d) <u>Strictly Proper, Distinct Complex Poles (**Real Number Method**)</u>

i. $\boldsymbol{F}(\boldsymbol{s}) = \frac{20\boldsymbol{s}+40}{\boldsymbol{s}(\boldsymbol{s}^2+6\boldsymbol{s}+25)}$ $\boxed{f(t) = \frac{1}{5}\left[8 + e^{-3t}[19\sin(4t) - 8\cos(4t)]\right]u(t)}$

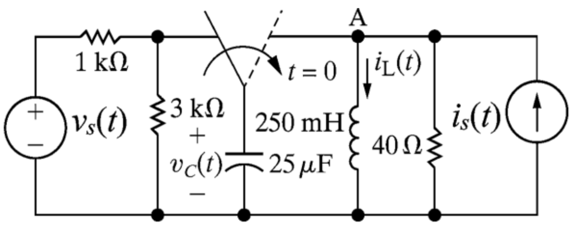
ii. $\boldsymbol{F}(\boldsymbol{s}) = \frac{s+1}{(s+2)(s^2+2s+5)}$ $\boxed{f(t) = \frac{1}{5}\left[-e^{-2t} + e^{-t}[\cos(2t) + 2\sin(2t)]\right]u(t)}$

(e) Proper/Improper

i. $\boldsymbol{F}(\boldsymbol{s}) = \frac{5s^3+20s^2-49s-108}{s^2+7s+10}$ $\boxed{f(t) = 5\delta'(t) - 15\delta(t) + [10e^{-2t} - 4e^{-5t}]u(t)}$

3. Inverse Laplace Transforms, Integro-Differential Equations, and Network Analysis

(a) Consider the second order network shown with $v_s(t) = 100\text{V}$ and $i_s(t) = 100\text{ mA}$. The switch moves to the "right" position after being in the "left" position for a long time.



i. Analyze the network at time $t = 0^-$ to compute the state variable values $v_C(0^-)$ and $i_L(0^-)$.

$\boxed{v_C(0^-) = 75\text{V}}$ $\boxed{i_L(0^-) = 100\text{mA}}$

ii. Analyze the network for $t > 0^-$ using nodal analysis at node A to obtain an integro-differential equation that describes the voltage $v_C(t)u(t)$ for $t > 0^-$.

$$\boxed{\frac{dv_C(t)}{dt} + \frac{v_C(t)}{RC} + \frac{i_L(0^-)}{C} + \frac{1}{LC}\int_{0^-}^{t} v_C(x)dx = \frac{i_S(t)}{C}}$$

$$\boxed{1\text{k}v_C(t) + \frac{dv_C(t)}{dt} + (40\text{k})i_L(0^-) + 160\text{k}\int_{0^-}^{t} v_C(x)dx = (40\text{k})i_S(t)}$$

iii. Take the Laplace Transform of the equation found in (ii) and compute the complete capacitor voltage response transform $\boldsymbol{V_C}(\boldsymbol{s}) = \mathcal{L}\{v_C(t)u(t)\}$. As part of your computation, identify the characteristic polynomial of $\boldsymbol{V_C}(\boldsymbol{s})$, the zero state component $\boldsymbol{V_{C,ZS}}(\boldsymbol{s})$ of $\boldsymbol{V_C}(\boldsymbol{s})$, and the zero input component $\boldsymbol{V_{C,ZI}}(\boldsymbol{s})$ of $\boldsymbol{V_C}(\boldsymbol{s})$.

$$\boxed{\boldsymbol{V_{C,ZS}}(\boldsymbol{s}) = \frac{4\text{k}}{\boldsymbol{s}^2 + 1\text{k}\boldsymbol{s} + 160\text{k}}} \quad \boxed{\boldsymbol{V_{C,ZI}}(\boldsymbol{s}) = \frac{75\boldsymbol{s} - 4\text{k}}{\boldsymbol{s}^2 + 1\text{k}\boldsymbol{s} + 160\text{k}}} \quad \boxed{\boldsymbol{V_C}(\boldsymbol{s}) = \frac{75\boldsymbol{s}}{\boldsymbol{s}^2 + 1\text{k}\boldsymbol{s} + 160\text{k}}}$$

Characteristic Polynomial: $\boxed{\boldsymbol{s}^2 + 1\text{k}\boldsymbol{s} + 160\text{k}}$.

iv. Compute the complete capacitor voltage response $v_C(t)u(t)$ by taking the inverse Laplace Transform of the complete capacitor voltage response transform $\boldsymbol{V_C}(\boldsymbol{s})$.

$\boxed{v_C(t) = \mathcal{L}^{-1}\{\boldsymbol{V_C}(\boldsymbol{s})\} = 25[4e^{-800t} - e^{-200t}]u(t)}$

v. Write an expression that relates the complete inductor current response $i_L(t)u(t)$ to the complete capacitor voltage response $v_C(t)u(t)$. Then use the expression for $i_L(t)u(t)$ to compute complete inductor current response transform $\boldsymbol{I_L}(\boldsymbol{s}) = \mathcal{L}\{i_L(t)u(t)\}$.

$$\boxed{\boldsymbol{I_L(s)} = \frac{1}{10}\left[\frac{\boldsymbol{s}^2 + 4\text{k}\boldsymbol{s} + 160\text{k}}{\boldsymbol{s}(\boldsymbol{s}^2 + 1\text{k}\boldsymbol{s} + 160\text{k})}\right]}$$

vi. Compute the complete inductor current response $i_L(t)u(t)$ by taking the inverse Laplace Transform of the complete inductor current response transform $\boldsymbol{I_L(s)}$.

$$\boxed{i_L(t) = \mathcal{L}^{-1}\{\boldsymbol{I_L(s)}\} = [0.1 - 0.5e^{-800t} + 0.5e^{-200t}]u(t)}$$

4. Pole-Zero Representation of Rational Functions and Pole-Zero Diagrams

(a) Consider the rational function $\boldsymbol{F(s)} = \boldsymbol{N(s)}/\boldsymbol{D(s)}$ of a complex frequency.

$$\boldsymbol{F(s)} = \frac{(8\boldsymbol{s} + 40)(4\boldsymbol{s}^2 + 8\boldsymbol{s} + 36)}{(2\boldsymbol{s} + 14)(\boldsymbol{s} + 3)(\boldsymbol{s}^2 + 5\boldsymbol{s} + 6)}$$

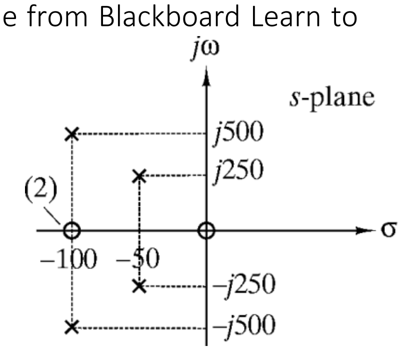
i. Compute the scale factor K. $\boxed{K = 16}$

ii. Compute the poles (finite, infinite) of $\boldsymbol{F(s)}$. $\boxed{p_1 = -7}$, $\boxed{p_{2,3} = -3}$, and $\boxed{p_4 = -2}$.

iii. Compute the zeros (finite, infinite) of $\boldsymbol{F(s)}$. $\boxed{z_1 = -5}$, $\boxed{z_{2,3} = -1 \pm j2\sqrt{2}}$, and $\boxed{z_4 \to \infty}$

iv. Sketch the pole-zero diagram for $\boldsymbol{F(s)}$. Include any infinite poles and zeros in your sketch. Then, use MATLAB and the **`pzplot2()`** user-defined function file from Blackboard Learn to create a pole-zero diagram of $\boldsymbol{F(s)}$.

(b) Consider the pole-zero diagram of $\boldsymbol{F(s)} = \boldsymbol{N(s)}/\boldsymbol{D(s)}$ shown. Compute the expression for $\boldsymbol{F(s)}$ if $\boldsymbol{F}(150) = \frac{400}{41}$.



$$\boxed{\boldsymbol{F(s)} = (1/3)100\times10^3 \frac{\boldsymbol{s}(\boldsymbol{s} + 100)^2}{(\boldsymbol{s}^2 + 100\boldsymbol{s} + 65000)(\boldsymbol{s}^2 + 200\boldsymbol{s} + 260000)}}$$

5. Initial and Final Value Theorems

Compute, if possible, $f_k(0^+)$ and $f_k(\infty)$ of the right-sided time function corresponding to each of the following rational functions of a complex frequency. If it is not possible, briefly explain why.

(a) $\boldsymbol{F_1(s)} = \frac{s+3}{s^2+s}$ $\boxed{f_1(0^+) = 1}$ $\boxed{f_1(\infty) = 3}$

(b) $\boldsymbol{F_2(s)} = \frac{5}{(s+1)(s^2+9)}$ $\boxed{f_2(0^+) = 0}$ Cannot apply the FVT

(c) $\boldsymbol{F_3(s)} = \frac{3s^3+6s^2+12s+3}{s(s+3)^2}$ $\boxed{f_3(0^+) = -12}$ $\boxed{f_3(\infty) = 1/3}$