ALE method for gravity waves

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Abstract

We investigate the capabilities of the ALE method for the numerical simulation of gravity waves.

1 A 2D Finite-Element scheme for free-surface Stokes flows

1.1 IBVP

$$\mathcal{D}(t) = \{z \in (b(x), \eta(t,x)) \; x \in (0,1)\} .$$

$$\partial_t \boldsymbol{u} + \boldsymbol{\nabla} p - \operatorname{div}(2\eta \boldsymbol{D}(\boldsymbol{u})) = \boldsymbol{f} \quad \operatorname{div} \boldsymbol{u} = 0 , \tag{1.1}$$

$$u|_{z=b(x,y)} \partial_x b - w|_{z=b(x,y)} = 0 , \tag{1.2}$$

$$\partial_t \eta + u|_{z=\eta(t,x,y)} \partial_x \eta - w|_{z=\eta(t,x,y)} = 0 . \tag{1.3}$$

$\boldsymbol{u} = \boldsymbol{\nabla} \phi$ $(\boldsymbol{\nabla} \times \boldsymbol{u} = 0)$

$$\operatorname{div} \phi = 0 \quad \partial_t \phi + \frac{|\boldsymbol{\nabla} \phi|^2}{2} + p - \boldsymbol{f} \cdot \boldsymbol{x} = 0 \tag{1.4}$$

(éventuellement des conditions aux parois latérales du domaine) et une condition initiale pour (ϕ, η), le système dit “des vagues”

$$\operatorname{div} \phi = 0 \quad \partial_t \phi|_{z=\eta(t,x,y)} + \frac{|\boldsymbol{\nabla} \phi|^2|_{z=\eta(t,x,y)}}{2} + p_0 - \boldsymbol{f} \cdot \boldsymbol{x}|_{z=\eta(t,x,y)} = 0 \tag{1.5}$$

$$(\partial_t + \boldsymbol{u} \cdot \boldsymbol{\nabla}) \omega - \omega \cdot \boldsymbol{\nabla} \boldsymbol{u} = \nu \Delta \omega , \tag{1.6}$$

1.2 Scheme

$([\mathbb{P}^k]^d, \mathbb{P}^{k-1})$

$$\int_\Omega (\partial_t \boldsymbol{u} \cdot \boldsymbol{v} - p \operatorname{div} \boldsymbol{v} + q \operatorname{div} \boldsymbol{u} + 2\eta \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{v}) - \boldsymbol{f} \cdot \boldsymbol{v}) = 0 \quad \forall (\boldsymbol{v}, q) \in ([\mathbb{P}^k]^d, \mathbb{P}^{k-1})$$

$$\frac{d}{dt}\int_\Omega \boldsymbol{u}\cdot\boldsymbol{v}+\int_\Omega\left([(-\boldsymbol{w}\cdot\boldsymbol{\nabla})\boldsymbol{u}]\cdot\boldsymbol{v}-\frac{1}{2}(\operatorname{div}\boldsymbol{w})\boldsymbol{u}\cdot\boldsymbol{v}-p\operatorname{div}\boldsymbol{v}+q\operatorname{div}\boldsymbol{u}+2\eta\boldsymbol{D}(\boldsymbol{u}):\boldsymbol{D}(\boldsymbol{v})-\boldsymbol{f}\cdot\boldsymbol{v}\right)=0\,.$$

$$\begin{aligned}&\int_{\Omega^{n+1}}\boldsymbol{u}^{n+1}\cdot\boldsymbol{v}-\int_{\Omega^n}\boldsymbol{u}^n\cdot\boldsymbol{v}+\Delta t\int_{\Omega^{n+1}}\left([(-\boldsymbol{w}^n\circ(\boldsymbol{I}+\Delta t\boldsymbol{w}^n)\cdot\boldsymbol{\nabla})\boldsymbol{u}^{n+1}]\cdot\boldsymbol{v}-\frac{1}{2}(\operatorname{div}\boldsymbol{w}^n)\boldsymbol{u}^{n+1}\cdot\boldsymbol{v}\right)\\ &+\Delta t\int_{\Omega^{n+1}}\left(-p^{n+1}\operatorname{div}\boldsymbol{v}+q\operatorname{div}\boldsymbol{u}^{n+1}+2\eta\boldsymbol{D}(\boldsymbol{u})^{n+1}:\boldsymbol{D}(\boldsymbol{v})-\boldsymbol{f}\cdot\boldsymbol{v}\right)=0\quad\forall(\boldsymbol{v},q)\in([\mathbb{P}^k]^d,\mathbb{P}^{k-1})\end{aligned}\tag{1.7}$$

$\Omega^n\to\Omega^{n+1}:\boldsymbol{x}\to\boldsymbol{x}+\Delta t\boldsymbol{w}^n(\boldsymbol{x})$

$$\begin{aligned}&\int_{\Omega^{n+1}}\left(\boldsymbol{u}^{n+1}\cdot\boldsymbol{v}-[\boldsymbol{u}^n\circ(\boldsymbol{I}+\Delta t\boldsymbol{w}^n)]\cdot\boldsymbol{v}+\Delta t[((\boldsymbol{u}^n-\boldsymbol{w}^n)\circ(\boldsymbol{I}+\Delta t\boldsymbol{w}^n)\cdot\boldsymbol{\nabla})\boldsymbol{u}^{n+1}]\cdot\boldsymbol{v}\right)\\ &+\Delta t\int_{\Omega^{n+1}}\left(-p^{n+1}\operatorname{div}\boldsymbol{v}+q\operatorname{div}\boldsymbol{u}^{n+1}+2\eta\boldsymbol{D}(\boldsymbol{u})^{n+1}:\boldsymbol{D}(\boldsymbol{v})-\boldsymbol{f}\cdot\boldsymbol{v}\right)=0\quad\forall(\boldsymbol{v},q)\in([\mathbb{P}^k]^d,\mathbb{P}^{k-1})\end{aligned}\tag{1.8}$$

References