Chapter 1

What will we do ?

We are interested in the Euclidean space $\mathbb{R}^n$.

1. In the first major part of the course, we discuss about the n-dimensional volume of a subset $E \subset \mathbb{R}^n$. The first objective is to construct the Lebesgue measure $\mathcal{L}^n$.

2. In the second major part of the course, we update our tool of Integral, namely from the Riemann Integral to the Lebesgue Integral. This is based on the measure theory developed by abstraction of the Lebesgue measure in the first part.

Chapter 2

Measuring n-dimensional volume of $E \subset \mathbb{R}^n$

1. In the Euclidean space $\mathbb{R}^n$, we are able to measure the distance between two points $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$ of $\mathbb{R}^n$,

$$d(x, y) = \Big((x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots (x_n - y_n)^2 \Big)^{\frac{1}{2}}.$$

2. This gives rise to the n-dimensional volume formula for a few classes of subsets in $\mathbb{R}^n$. For example in $\mathbb{R}^3$, we take the formula:

   (a) If E is the cube $[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$, we take the value

   $$(b_1 - a_1)(b_2 - a_2)(b_3 - a_3)$$

   as its 3-dimensional volume.

   (b) If we consider a tetrahedron with base area A and the height h, we take the value

   $$\frac{1}{3} Ah$$

   as its 3-dimensional volume.

   (c) other examples...

Knowing the n-dimensional volumes of such a class of elementary sets,

1. We may extend our knowledge base on calculating n-volume: For a set made by assembling a few such elementary sets, the n-volume would be the sum of n-volumes of elementary sets.

2. That we wrote right above is the theory we want to develop. It is a difficult task: To make this consistent mathematically, any such theory should provide a proof that the n-volume assigned on a certain set $E \subset \mathbb{R}^n$ would be calculated independently of ways of cutting the set.

3. For example, for a given set $E \subset \mathbb{R}^n$, there are two persons. The first person cuts E into G_1, G_2, G_3, and the second person cuts E into H_1 and H_2. More specifically, G_1, G_2, G_3 are pairwise disjoint and $E = G_1 \cup G_2 \cup G_3$, and H_1, H_2 are pairwise disjoint and $E = H_1 \cup H_2$. n-volumes of G_i, and H_j are known. Theory should be certain about the equality
$$|G_1| + |G_2| + |G_3| = |H_1| + |H_2|$$
where $|S|$ denotes n-volume of the set S, if known. This consistency should be the case for all different ways of cutting the set E.

4. We consider the following humble goal: Let $\mathcal{R}$ be the collection of all cubes in $\mathbb{R}^n$ (whose n-volumes are as we know). Let R be any cube in $\mathcal{R}$ with the n-volume $|R|$. Let $(R_j)_{j=1}^{\infty}$ be pairwise disjoint partition of R, and $(Q_k)_{k=1}^{\infty}$ be another pairwise disjoint partition of R. The goal is that the n-volume of a cube was actually *correct*:
$$|R| = \sum_{j=1}^{\infty} |R_j| = \sum_{k=1}^{\infty} |Q_k|.$$
This is the task of our courses for a while, and this is a difficult problem.

Consistent family with n-volume

The word 'family', or 'collection' are synonyms of set. We use family or collection to avoid confusion.

Definition 1. *A pair* $(\mathcal{G}, \lambda)$ *of* $\mathcal{G}$*, a nonempty collection of subsets of* $\mathbb{R}^n$ *containing* $\emptyset$*,* X*, and* $\lambda : \mathcal{G} \to [0, \infty]$*, is said to be consistent if*

1. $\lambda(\emptyset) = 0$,
2. *If* G *is a set in* $\mathcal{G}$*,* $v = \lambda(G)$ *and* $(G_j)_{j=1}^{\infty}$ *is any sequence of sets in* $\mathcal{G}$ *that are pairwise disjoint and* $G = \bigcup_{j=1}^{\infty} G_j$*, then*

$$\sum_{j=1}^{\infty} \lambda(G_j) = v.$$

Let $n = 2$ and consider $\mathbb{R}^2$.

By half-open intervals we mean the intervals of one of the following forms

$$\emptyset, \quad [a, b), \quad [a, \infty), \quad (-\infty, b), \quad (-\infty, \infty),$$

where $a, b \in \mathbb{R}$ and are assumed to be $a < b$.

Definition 2. *The collection* $\mathcal{R}$ *is the collection of all cartesian products of two half-open intervals. The member of* $\mathcal{R}$ *is called a rectangle.*

1. *If* R *is an unbounded rectangle, we define* $|R| = \infty$.
2. *If* R *is a nonempty bounded rectangle* $[a_1, b_1) \times [a_2, b_2)$*,* $|R| = (b_1 - a_1)(b_2 - a_2)$.
3. $|\emptyset| = 0$.

We prove that $(\mathcal{R}, | \cdot |)$ is consistent from now on.

Towards a consistent family

At this moment, we prove a proposition stating that, out of somewhat arbitrary volume function ρ and a collection, one may extract its refined version of volume function λ.

Proposition 3. *Let X be a nonempty set, $\mathcal{G} \subset \mathcal{P}(X)$, and $\rho : \mathcal{G} \to [0, \infty]$ be such that $\emptyset \in \mathcal{G}$, $X \in \mathcal{G}$, and $\rho(\emptyset) = 0$. For any set $S \subset X$, define*

$$\lambda(S) := \inf_{(G_j) \text{ of } \mathcal{G} \text{ that covers } S} \Big\{ \sum_{j=1}^{\infty} \rho(G_j) \Big\}.$$

Then, λ, defined on $\mathcal{P}(X)$, satisfies the followings:

1. *$\lambda(\emptyset) = 0$.*
2. *If (S_j) of $\mathcal{P}(X)$ covers S, i.e., $\bigcup_j S_j \supset S$, then*

$$\lambda(S) \leq \sum_{j=1}^{\infty} \lambda(S_j). \tag{2.0.1}$$

Remark 2.1. We will define on $\mathcal{P}(\mathbb{R}^2)$

$$\lambda(S) := \inf_{(R_j) \text{ of rectangles that covers } S} \Big\{ \sum_{j=1}^{\infty} |R_j| \Big\}.$$

Examples:

proof of Proposition 3. Let S be any subset of X.

1. The set of coverings of S by sets in $\mathcal{G}$ is nonempty because $X \in \mathcal{G}$ and $(X, \emptyset, \emptyset, \cdots)$ covers S. Obviously, the set

$$\left\{ \sum_{j=1}^{\infty} \rho(G_j) : (G_j) \text{ of } \mathcal{G} \text{ covers } S \right\} \subset [0, \infty]$$

is nonempty and bounded below by 0. Therefore, $\lambda(S)$, the infimum over the set, is well-defined.

2. Since $(\emptyset, \emptyset, \cdots)$ covers $\emptyset$, $\rho(\emptyset) = 0$, and $\sum_j 0 = 0$, $\lambda(\emptyset)$ must be 0.

3. Now, let (S_j) be any sequence of subsets of X that covers S. Suppose any of $\lambda(S_j) = \infty$. Then the inequality (2.0.1) is trivially true. Now, we assume $\lambda(S_j) < \infty$ for every j.

4. Let $\epsilon > 0$. By the definition of infimum, for each j, there exists a covering $(G^j_\alpha)_{\alpha=1}^{\infty}$ of S_j by sets in $\mathcal{G}$ such that

$$\sum_{\alpha=1}^{\infty} \rho(G^j_\alpha) \leq \lambda(S_j) + \frac{\epsilon}{2^j}.$$

Obviously, $\bigcup_\alpha \bigcup_j G^j_\alpha \supset S$ and thus $(G^j_\alpha)_{j,\alpha=1}^{\infty}$ is a countable covering of S by sets in $\mathcal{G}$. Thus,

$$\lambda(S) \leq \sum_{j=1}^{\infty} \sum_{\alpha=1}^{\infty} \rho(G^j_\alpha) \leq \sum_{j=1}^{\infty} \left[\lambda(S_j) + \frac{\epsilon}{2^j} \right] = \sum_{j=1}^{\infty} \lambda(S_j) + \epsilon.$$

Since this inequality holds for every $\epsilon > 0$, we conclude that

$$\lambda(S) \leq \sum_{j=1}^{\infty} \lambda(S_j).$$

$\square$

1. Out of mere formula $\big|[a_1,b_1)\times[a_2,b_2)\big| = (b_1-a_1)(b_2-a_2)$, we suddenly have a definition of λ for all the subsets of $\mathbb{R}^2$.
2. However, we restrict ourselves the use of λ only on rectangles for a while to complete the proof of that $(\mathcal{R}, |\cdot|)$ is consistent.
3. Now, we aim to prove that for a rectangle R, $|R| = \lambda(R)$, namely the area formula $|\cdot|$ was already good to some extent.
4. Since, $\lambda(R) \le |R|$, we only need to prove $\lambda(R) \ge |R|$.

We prove a few lemmas.

Lemma 4. *If R and R' are rectangels and $R \subset R'$, then $|R| \le |R'|$.*

Proof. Omitted. □

Lemma 5. *Let R be a nonempty bounded rectangle $[a,b)\times[c,d)$, and consider*

$$a = t_1 < t_2 < \cdots < t_N = b, \quad c = s_1 < s_2 < \cdots < s_K = d,$$

and consider rectangles $R_{i,j} = [t_i, t_{i+1}) \times [s_j, s_{j+1})$ for $i = 1,2,\cdots,N-1$ and $j = 1,2,\cdots,K-1$. Then,

$$|R| = \sum_{i=1}^{N-1}\sum_{j=1}^{K-1} |R_{i,j}| \quad \textit{and} \quad R = \bigcup_{i=1}^{N-1}\bigcup_{j=1}^{K-1} R_{i,j} \quad \textit{of disjoint union.}$$

Proof.

$$\begin{aligned}|R| = (b-a)(c-d) &= \Big(\sum_{i=1}^{N-1}(t_{i+1}-t_i)\Big)\Big(\sum_{j=1}^{K-1}(s_{j+1}-s_j)\Big)\\ &= \sum_{i=1}^{N-1}\sum_{j=1}^{K-1}(t_{i+1}-t_i)(s_{j+1}-s_j) = \sum_{i=1}^{N-1}\sum_{j=1}^{K-1}|R_{i,j}|.\end{aligned}$$

By definition,

$$\begin{aligned}R_{i,j} &= \{(x,y) \mid t_i \le x < t_{i+1} \text{ and } s_j \le y < s_{j+1}\}\\ R_{i',j'} &= \{(x,y) \mid t'_i \le x < t_{i'+1} \text{ and } s'_j \le y < s_{j'+1}\}\end{aligned}$$

and if $(i,j) \ne (i',j')$, then $i \ne i'$ or $j \ne j'$, and they must be disjoint. Again by definition

$$\begin{aligned}R = \ & \{(x,y) \mid a \le x < b \quad \text{and} \quad c \le y < d\}\\ = \ & \Big\{(x,y) \ \Big| \ \Big[t_1 \le x < t_2 \text{ or } t_2 \le x < t_3 \text{ or } \cdots \text{ or } t_{N-1} \le x < t_N\Big]\\ & \quad \text{and} \quad \Big[s_1 \le y < s_2 \text{ or } s_2 \le y < s_3 \text{ or } \cdots \text{ or } s_{K-1} \le y < y_N\Big]\Big\}\\ = \ & \bigcup_{i=1}^{N-1}\bigcup_{j=1}^{K-1} R_{i,j}.\end{aligned}$$

□

Lemma 6. *Suppose R be a nonempty bounded rectangle. If $(R_k)_{k=1}^M$ is a covering of R by sets in $\mathcal{R}$, then*

$$|R| \le \sum_{k=1}^{M} |R_k|.$$

Proof.

1. If any of R_k is unbounded, then $|R_k| = \infty$ and the inequality trivially holds. Now we assume R_k is a bounded rectangle for every k.

2. If we can prove the same inequality on any subcover of (R_k), then the inequality still stands with the cover itself. Thus we consider a subcover of (R_k) by discarding every R_k that is the empty set, and prove the inequality with this subcover: Below, we assume R_k is nonempty for every k.

3. Let us write for each $R_k = [a_k, b_k) \times [c_k, d_k)$.
   Let $t_1 < t_2 < \cdots < t_N$ be an enumeration of the finite set
   $$\{a, a_1, a_2, \cdots, a_M, b, b_1, b_2, \cdots, b_M\}$$
   in ascending order.
   Let $s_1 < s_2 < \cdots < s_K$ be an enumeration of the finite set
   $$\{c, c_1, c_2, \cdots, c_K, d, d_1, d_2, \cdots, d_K\}$$
   in ascending order. We consider the rectangles $Q_{i,j} = [t_i, t_{i+1}) \times [s_j, s_{j+1})$, pairwise disjoint.

4. Note that for each $R_k = [a_k, b_k) \times [c_k, d_k)$, there exist indices $i_{begin}(k)$ and $i_{end}(k)$ such that $t_{i_{begin}(k)} = a_k$ and $t_{i_{end}(k)} = b_k$. Similarly $j_{begin}(k)$ and $j_{end}(k)$ exist. By the previous lemma,
   $$R_k = \bigcup_{i=i_{begin}(k)}^{i_{end}(k)-1} \bigcup_{j=j_{begin}(k)}^{j_{end}(k)-1} Q_{i,j} \quad \text{of disjoint union.}$$
   Because of this equality and that $(Q_{i,j})$ are pairwise disjoint, the following is true:
   $$\text{For every } k \text{ and every } (i,j), \text{ either } Q_{i,j} \subset R_k \text{ or } Q_{i,j} \cap R_k = \emptyset.$$

5. The similar is true for R.

6. We define
   $$\Gamma = \{(i,j) \mid Q_{i,j} \subset R\}, \quad \Gamma_k = \{(i,j) \mid Q_{i,j} \subset R_k\}.$$
   By the previous Lemma,
   $$|R| = \sum_{(i,j)\in\Gamma} |Q_{i,j}|, \quad |R_k| = \sum_{(i,j)\in\Gamma_k} |Q_{i,j}|.$$

7. That $R \subset \bigcup_k R_k$ implies that $(i,j) \in \Gamma$ implies that $Q_{i,j}$ intersects some R_k. Otherwise, (R_k) is not a covering of R.

8. This R_k-intersecting $Q_{i,j}$ in fact must be a subset of R_k. But $Q_{i,j} \subset R_k$ iff $(i,j) \in \Gamma_k$. We thus conclude: $\Gamma \subset \bigcup_k \Gamma_k$.

9. Finally,
   $$|R| = \sum_{(i,j)\in\Gamma} |Q_{i,j}| \le \sum_{(i,j)\in\bigcup_k \Gamma_k} |Q_{i,j}| \le \sum_k \sum_{(i,j)\in\Gamma_k} |Q_{i,j}| = \sum_k |R_k|.$$

$\square$

Proposition 7. *For a rectangle R, $\lambda(R) = |R|$.*

Proof.

1. If $R = \emptyset$, $\lambda(\emptyset) = 0 = |\emptyset|$.

2. Now, assume first that R is a bounded rectangle. We prove that $\lambda(R) \geq |R|$ below. Note we know that $\lambda(R) \leq |R| < \infty$.

3. By definition of $\lambda(R)$, for any $\epsilon > 0$ there exists a (Q_k) of $\mathcal{R}$ that covers R such that
$$\lambda(R) + \epsilon \geq \sum_k |Q_k|.$$

4. Now, it is possible to enlarge each rectangle Q_k a little to form an open rectangle $\tilde{Q}_k \supset Q_k$ but satisfying
$$|Q_k| \geq |\tilde{Q}_k| - \frac{\epsilon}{2^k}.$$

5. $(\tilde{Q}_k)$ forms an open covering of the closure of R that is compact. Hence, there is a finite subcover of the closure of R. (that is a finite cover of R too.) We have
$$\begin{aligned}
\sum_k |Q_k| &\geq \sum_k \left(|\tilde{Q}_k| - \frac{\epsilon}{2^k} \right) \\
&\geq \sum_k |\tilde{Q}_k| - \epsilon \\
&\geq \sum_{k \in subcover} |\tilde{Q}_k| - \epsilon \\
&\geq |R| - \epsilon,
\end{aligned}$$
where in the last inequality, we used the Lemma 6. In conclusion,
$$\lambda(R) + 2\epsilon \geq |R|$$
for every $\epsilon > 0$, and we conclude $\lambda(R) \geq |R|$.

6. Finally, let R be an unbounded rectangle. If so, we can consider $R_1 \subset R_2 \subset \cdots$ of subsets of R with $|R_j| < \infty$ and $|R_j| \to \infty$ as $j \to \infty$. Then for every j,
$$\lambda(R) \geq \lambda(R_j) = |R_j|,$$
which implies that $\lambda(R) = \infty$. The equality $\lambda(R) = |R| = \infty$ holds.

$\square$

For later purpose, we also prove the following equality.

Lemma 8. *Let R be a nonempty bounded rectangle. If $R = \bigcup_{k=1}^{M} R_k$ of disjoint union of rectangles R_1, R_2, $\cdots$, R_M, then*

$$|R| = \sum_{k=1}^{M} |R_k|.$$

Proof. Exercise. $\square$

Justify first that $(i, j) \in \Gamma$ iff $(i, j) \cup_k \Gamma_k$, and second that $\cup_k \Gamma_k$ is a disjoint union.

Chapter 3

Arguments repeatedly used

[Argument with the infimum]

Let $A \subset \mathbb{R}$ lower bounded. Then $m := \inf A$ is well-defined. For any positive $\epsilon > 0$, $m + \epsilon$ is not a lower bound of A, and thus there must be $a \in A$ such that $a \leq m + \epsilon$.

[Inequality holding for all $\epsilon > 0$]

Let $a, b \in \mathbb{R}$. If $a \leq b + \epsilon$ for every $\epsilon > 0$, then $a \leq b$.

[Countable sum of nonnegative numbers]

Let (c_j) be a sequence of nonnegative numbers. Then, the summation of the series is independent of changing orders, such as $c_{\sigma(j)}$ with $\sigma : \mathbb{N} \to \mathbb{N}$ a bijection. One of the following two is the case.

(i) $\displaystyle\sum_{j=1}^{\infty} c_j = \lim_{N\to\infty} \sum_{j=1}^{N} c_j = s_* < \infty.$

The series absolutely converges, and the limit s_* is independent of changing orders of c_j

(ii) $\displaystyle\sum_{j=1}^{\infty} c_j = \lim_{N\to\infty} \sum_{j=1}^{N} c_j = s_* = \infty.$

The limit $+\infty$ is independent of changing orders of c_j.

[From (E_j) of sequence of sets to $(\hat{E}_j)$ of pairwise disjoint sets]

Lemma 9. *Let (E_j) be a sequence of sets. Define $(\hat{E}_j)$ recursive by*

$$\hat{E}_1 = E_1$$
$$\hat{E}_j = E_j \setminus \Big(\bigcup_{i=1}^{j-1} E_i \Big)$$

Then, for any N,

$$(i) \quad \bigcup_{j=1}^{N} \hat{E}_j = \bigcup_{j=1}^{N} E_j,$$

(ii) $(\hat{E}_j)_{j=1}^{N}$ *is a sequence of pairwise disjoint sets.*

Proof. The two assertions are obviously true for $N = 1$. If the assertion is true for $1, 2, \cdots, N-1$,

$$\hat{E}_N = E_N \setminus \Big(\bigcup_{j=1}^{N-1} E_j \Big) = E_N \setminus \Big(\bigcup_{j=1}^{N-1} \hat{E}_j \Big).$$

Obviously, $\hat{E}_N$ is disjoint from $\bigcup_{j=1}^{N-1} \hat{E}_j$. Therefore, $(\hat{E}_j)_{j=1}^{N}$ is pairwise disjoint. Also,

$$\begin{aligned}
\bigcup_{j=1}^{N} \hat{E}_j = \hat{E}_N \cup \Big(\bigcup_{j=1}^{N-1} \hat{E}_j \Big) &= \hat{E}_N \cup \Big(\bigcup_{j=1}^{N-1} E_j \Big) \\
&= \Big[E_N \cap \Big(\bigcup_{j=1}^{N-1} E_j \Big)^c \Big] \cup \Big(\bigcup_{j=1}^{N-1} E_j \Big) \\
&= E_N \cup \Big(\bigcup_{j=1}^{N-1} E_j \Big) = \bigcup_{j=1}^{N} E_j
\end{aligned}$$

Remark 3.1. Since the assertion in Lemma 9 is true for any N, it also holds that

$$(i) \quad \bigcup_{j=1}^{\infty} \hat{E}_j = \bigcup_{j=1}^{\infty} E_j,$$

(ii) $(\hat{E}_j)_{j=1}^{\infty}$ is a sequence of pairwise disjoint sets.

because

$$x \in \bigcup_{j=1}^{\infty} \hat{E}_j \implies x \in \hat{E}_{j_0} \text{ for some } j_0 \implies x \in \bigcup_{j=1}^{j_0} \hat{E}_j = \bigcup_{j=1}^{j_0} E_j \implies x \in \bigcup_{j=1}^{\infty} E_j,$$

$$x \in \bigcup_{j=1}^{\infty} E_j \implies x \in E_{j_1} \text{ for some } j_1 \implies x \in \bigcup_{j=1}^{j_1} E_j = \bigcup_{j=1}^{j_1} \hat{E}_j \implies x \in \bigcup_{j=1}^{\infty} \hat{E}_j,$$

and for any $\hat{E}_{i_0}$ and $\hat{E}_{i_1}$, we let $N = \max\{i_0, i_1\}$ and we know $(\hat{E}_j)_{j=1}^{N}$ is pairwise disjoint.

Remark 3.2. [(For any N)-assertion by induction] & [(limit)-assertion proven in addition] style of proof will appear repeatedly.

$\square$

Chapter 4

Measure Theoretic Separation

We would like to have that if a set $S \subset \mathbb{R}^2$ is made by assembling two disjoint sets S_1 and S_2,

$$\lambda(S_1 \cup S_2) = \lambda(S_1) + \lambda(S_2).$$

We then would like to have its countable version.

Since the inequality $\lambda(S_1 \cup S_2) \leq \lambda(S_1) + \lambda(S_2)$ already is established, worry is in whether there is a case

$$\lambda(S_1 \cup S_2) < \lambda(S_1) + \lambda(S_2)$$

Over-estimation by Truely 2-dimensional covering

Look at the definition of $\lambda(S)$,

$$\lambda(S) := \inf_{(R_j) \text{ of rectangles that covers } S} \left\{ \sum_{j=1}^{\infty} |R_j| \right\}.$$

The importance of the rectangle in our theory lies in that it is a Truely 2-dimensional lump.

1. The set $\displaystyle\bigcup_{j=1}^{\infty} R_j \supset S$ is thus a Truely 2-dimensional lump replacement of S.
2. We estimate its 2-dimensional area by $\displaystyle\sum_{j=1}^{\infty} |R_j|$, that is certainly an over-estimation.
3. This over-estimation is minimized as much as possible, over all the coverings.

How does this 2-dim-over-estimation $\rightarrow$ minimization properly works? For example consider the singletone set $\{x_0\}$. Intuitively, 0 has to be its 2-dimsnional area.

1. We see that one square R_ℓ with side length $\ell > 0$ whose center is x_0 is a Truely 2-dimensional replacement of $\{x_0\}$. $(R_\ell, \emptyset, \emptyset, \cdots)$ covers $\{x_0\}$.
2. Its over-estimation is thus, $\ell^2 > 0$.
3. By minimization of over-estimation by letting $\ell \rightarrow 0$, we conclude that the infimum $\lambda(\{x_0\}) = 0$.

Thus, it makes sense to take the area of one point set is 0.

Question: Can the over-estimation be not resolved by the minimization process?

One speculative example about the question of resolving over-estimation is the following in 1 dimension. The role of rectangles is taken by intervals. Let

$$A = [0,1] \cap \mathbb{Q}, \quad B = [0,1] \cap \mathbb{Q}^c$$

1. If (R_j) is a Truely 1-dimensional covering of A by intervals, and (Q_k) is a Truely 1-dimensional covering of B by intervals, let us write this replacement

$$A' = \bigcup_j R_j, \quad B' = \bigcup_k Q_k.$$

2. Because of density of rationals and irrationals, the invasion of A' into the portion of B', and the invasion of B' into the portion of A' must have occured. In other words,

$$\sum_{j=1}^{\infty} |R_j| + \sum_{k=1}^{\infty} |Q_k| > 1.$$

3. Is it for certain thing that by the followed minimization step, this is to be resolved properly ? In other words, are we sure

$$\lambda(A) + \lambda(B) = 1 \quad ?$$

This is why we ask a question if there can be a case of two disjoint set S_1 and S_2 with

$$\lambda(S_1 \cup S_2) < \lambda(S_1) + \lambda(S_2)$$

measure-theoretic separation

Since we are very speculative about this over-estimation-resolving procedure, we adopt a stronger notion of separation over the notion of being disjoint.

Definition 1. *We say a set $E \subset \mathbb{R}^2$ separates S_1 and S_2 if*

$$\Big(S_1 \subset E \quad \textit{and} \quad S_2 \subset E^c\Big) \quad \textit{or} \quad \Big(S_2 \subset E \quad \textit{and} \quad S_1 \subset E^c\Big)$$

Remark 4.1. If there exists a set E that separates S_1 and S_2, then S_1 and S_2 must be disjoint.

Example: Let E be an open ball of radius $r > 0$ and S_1 and S_2 be two compact sets.

Example: Let E be the upper half plane $x_2 \geq 0$ and S_1 and S_2 be two sets one of which is in the half plane, and the other is outside of the half plane.

Definition 2. *We say $E \subset \mathbb{R}^2$ is λ-separating if the following is true.*

$$E \text{ separates } S_1 \text{ and } S_2 \quad \Longrightarrow \quad \lambda(S_1 \cup S_2) = \lambda(S_1) + \lambda(S_2).$$

Question 1: What kind of sets can have such a separating property?

We answer to the following question first, before the Q1.

Question 2: What are the consequences of being such a set?

Theorem 3. *Let E_1, E_2, E_3, $\cdots$ be pairwise disjoint λ-separating sets and $S_1, S_2, \cdots$ be any sequence in $\mathcal{P}(\mathbb{R}^2)$ such that $S_j \subset E_j$ for every j. Then,*

$$(i) \text{ for any } N \quad \lambda\Big(\bigcup_{j=1}^{N} S_j\Big) = \sum_{j=1}^{N} \lambda(S_j), \quad \text{and} \quad (ii)\ \lambda\Big(\bigcup_{j=1}^{\infty} S_j\Big) = \sum_{j=1}^{\infty} \lambda(S_j).$$

Proof. 1. We prove the first assertion.

Certainly $\lambda\Big(\bigcup_{j=1}^{1} S_j\Big) = \sum_{j=1}^{1} \lambda(S_j)$. Now, if equality holds for

$$\lambda\Big(\bigcup_{j=1}^{k-1} S_j\Big) = \sum_{j=1}^{k-1} \lambda(S_j)$$

we assert that

$$\lambda\Big(\bigcup_{j=1}^{k} S_j\Big) = \lambda\Big(\bigcup_{j=1}^{k-1} S_j \cup S_k\Big).$$

Since E_k separates S_k and $\Big(\bigcup_{j=1}^{k-1} S_j\Big)$, the (RHS) equals to

$$\lambda\Big(\bigcup_{j=1}^{k-1} S_j\Big) + \lambda(S_k) = \sum_{j=1}^{k-1} \lambda(S_j) + \lambda(S_k) = \sum_{j=1}^{k} \lambda(S_k).$$

2. For the second assertion,

$$\lambda\Big(\bigcup_{j=1}^{\infty} S_j\Big) \le \sum_{j=1}^{\infty} \lambda(S_j) = \lim_{N\to\infty} \sum_{j=1}^{N} \lambda(S_j) = \lim_{N\to\infty} \lambda\Big(\bigcup_{j=1}^{N} S_j\Big) \le \lim_{N\to\infty} \lambda\Big(\bigcup_{j=1}^{\infty} S_j\Big) = \lambda\Big(\bigcup_{j=1}^{\infty} S_j\Big)$$

Hence, every quantity appeared equals to each other.

$\square$

Remark 4.2. One important example is the case where $S_j = E_j$ itself for every j, that are pairwise disjoint and λ-separating. They always satisfies

$$\lambda\Big(\bigcup_{j=1}^{\infty} E_j\Big) = \sum_{j=1}^{\infty} \lambda(E_j).$$

Remark 4.3. To get back to our first objective, to show $(\mathcal{R}, |\cdot|)$ is consistent, (that is to show $(\mathcal{R}, \lambda)$ is consistent since $\lambda(R) = |R|$ for any rectangle $R \in \mathcal{R}$), we will be done once we prove that any rectangle is λ-separating.

Proposition 4. *For any $R \in \mathcal{R}$, the following is true.*

$$R \text{ separates } S_1 \text{ and } S_2 \quad \Longrightarrow \quad \lambda(S_1 \cup S_2) = \lambda(S_1) + \lambda(S_2).$$

Proof. We prove that

$$R \text{ separates } S_1 \text{ and } S_2 \quad \Longrightarrow \quad \lambda(S_1 \cup S_2) \geq \lambda(S_1) + \lambda(S_2).$$

1. If $\lambda(S_1 \cup S_2) = \infty$, then the inequality trivially holds.
2. From now on, we assume $\lambda(S_1 \cup S_2) < \infty$. It also follows that $\lambda(S_1) < \infty$ and $\lambda(S_2) < \infty$. Without loss, we consider the case $S_1 \subset R$.
3. For any $\epsilon > 0$, there exists a (R_j) of $\mathcal{R}$ that covers $S_1 \cup S_2$ such that

   $$\lambda(S_1 \cup S_2) + \epsilon \geq \sum_{j=1}^{\infty} \lambda(R_j).$$

   Note that every R_j must be a bounded rectangle and the series in (RHS) absolutely converges, since (LHS) is finite.
4. Now, we notice that R^c can always be written as a disjoint union of four rectangles Q_1, Q_2, Q_3, and Q_4.
5. Let $R = Q_0$. We can write for every j

   $$R_j = Q_j^0 \cup Q_j^1 \cup Q_j^2 \cup Q_j^3 \cup Q_j^4, \quad Q_j^\alpha = R_j \cap Q_\alpha, \quad \alpha = 0, 1, 2, 3, 4$$

   Each intersection is again a rectangle, and this is a disjoint union of five rectangles.
6. Now, $(Q_j^0)_{j=1}^{\infty}$ covers S_1, and $(Q_j^\alpha)_{j=1,\alpha=1}^{j=\infty,\alpha=4}$ covers S_2.
7. Therefore,

   $$\begin{aligned}\lambda(S_1 \cup S_2) + \epsilon \geq \sum_{j=1}^{\infty} \lambda(R_j) &= \sum_{j=1}^{\infty}\sum_{\alpha=0}^{4} \lambda(Q_j^\alpha) \\ &= \sum_{j=1}^{\infty} \lambda(Q_j^0) + \sum_{j=1}^{\infty}\sum_{\alpha=1}^{4} \lambda(Q_j^\alpha) \\ &\geq \lambda(S_1) + \lambda(S_2).\end{aligned}$$

8. Since the inequality holds for every $\epsilon > 0$, $\lambda(S_1 \cup S_2) \geq \lambda(S_1) + \lambda(S_2)$.

□

Theorem 5. $(\mathcal{R}, |\cdot|)$ *is consistent.*

Proof. This is by Proposition 4. □

Seen from the proof of Proposition 4, it is not hard to prove that for two rectangles R and R', the union $A = R \cup R'$, which is not a rectangle in general, is also λ-separating.

Proposition 6. *For any $R, R' \in \mathcal{R}$, $R \cup R'$ is λ-separating.*

Proof. From the proof of Proposition 4, the only modifications we need to make are the followings.

1. $R \cup R' = (R \cap R'^c) \cup (R \cap R') \cup (R' \cap R^c) = \bigcup_{\alpha=1}^{m} Q_\alpha$ of disjoint union of finite numbers of rectangles.

2. Similarly, $(R \cup R')^c = \bigcup_{\alpha=m+1}^{m+m'} Q_\alpha$ of disjoint union of finite numbers of rectangles.

3. If (R_j) covers $S_1 \cup S_2$, then

$$R_j = \bigcup_{\alpha=1}^{m+m'} Q_j^\alpha \text{of disjoint union of rectangles, where} \quad Q_j^\alpha = Q_\alpha \cap R_j.$$

4. $(Q_j^\alpha)_{j=1,\alpha=1}^{j=\infty,\alpha=m}$ covers S_1, and $(Q_j^\alpha)_{j=1,\alpha=m+1}^{j=\infty,\alpha=m+m'}$ covers S_2

$\square$

1. We have established that each member of $\mathcal{R}$ is λ-separating.
2. Instead of giving a proof that a certain set of interest is λ-separating individually, we use the induction below. This way of development of the theory is due to Caratheodory.

Theorem 7 (Caratheodory)**.** *Suppose $E_1, E_2, E_3, \cdots$ are λ-separating. Then,*

$$(i) \quad \textit{For any } N, \bigcup_{j=1}^{N} E_j \textit{ is } \lambda\textit{-separating.}$$

$$(ii) \quad \bigcup_{j=1}^{\infty} E_j \textit{ is } \lambda\textit{-separating.}$$

Proof. 1. We let $(\hat{E}_j)$ be the pairwise disjoint sequence obtained from (E_j) by Proposition before.

2. We prove the stronger assertion over (i):

   $(i)'$ For any N, the following is true.

$$S_1 \subset \bigcup_{j=1}^{N} E_j, \quad S_2 \subset \Big(\bigcup_{j=1}^{N} E_j\Big)^c \quad \Longrightarrow \quad \lambda(S_1 \cup S_2) = \sum_{j=1}^{N} \lambda(S_1 \cap \hat{E}_j) + \lambda(S_2).$$

3. Indeed,

$$\sum_{j=1}^{N} \lambda(S_1 \cap \hat{E}_j) \geq \lambda\Big(S_1 \cap \bigcup_{j=1}^{N} \hat{E}_j\Big) = \lambda\Big(S_1 \cap \bigcup_{j=1}^{N} E_j\Big) = \lambda(S_1),$$

   which implies the assertion (i) in the statement.

4. The stronger assertion $(i)'$ holds for $N = 1$, because $\bigcup_{j=1}^{1} E_j = E_1 = \hat{E}_1$, which is λ-separating. Suppose that the assertion $(i)'$ is true for $1, 2, \cdots, N-1$. Now, let $S_1 \subset \bigcup_{j=1}^{N} E_j$ and $S_2 \subset \Big(\bigcup_{j=1}^{N} E_j\Big)^c$. Then,

$$\begin{aligned}\lambda(S_1 \cup S_2) &= \lambda\Big(\Big(S_1 \cap \bigcup_{j=1}^{N} E_j\Big) \cup S_2\Big) = \lambda\Big(\Big(S_1 \cap \bigcup_{j=1}^{N} \hat{E}_j\Big) \cup S_2\Big) \\ &= \lambda\Big(\Big(S_1 \cap \bigcup_{j=1}^{N-1} \hat{E}_j\Big) \cup (S_1 \cap \hat{E}_N) \cup S_2\Big)\end{aligned}$$

   Because the set $\Big(S_1 \cap \bigcup_{j=1}^{N-1} \hat{E}_j\Big) \subset \bigcup_{j=1}^{N-1} E_j$ and the set $\Big((S_1 \cap \hat{E}_N) \cup S_2\Big) \subset \Big(\bigcup_{j=1}^{N-1} E_j\Big)^c$,

$$= \lambda\Big(S_1 \cap \bigcup_{j=1}^{N-1} \hat{E}_j\Big) + \lambda\Big((S_1 \cap \hat{E}_N) \cup S_2\Big)$$

Because the set $S_1 \cap \hat{E}_N \subset E_N$ and the set $S_2 \subset E_N^c$

$$= \lambda\Big(S_1 \cap \bigcup_{j=1}^{N-1} \hat{E}_j\Big) + \lambda\big(S_1 \cap \hat{E}_N\big) + \lambda(S_2)$$

$$\text{(by } (i)' \text{ on } N-1) \quad = \sum_{j=1}^{N-1} \lambda(S_1 \cap \hat{E}_j) + \lambda\big(S_1 \cap \hat{E}_N\big) + \lambda(S_2)$$

$$= \sum_{j=1}^{N} \lambda(S_1 \cap \hat{E}_j) + \lambda(S_2).$$

5. Now, we prove the second assertion stronger in the similar sense.
   Let $S_1 \subset \bigcup_{j=1}^{\infty} \hat{E}_j$ and $S_2 \subset \Big(\bigcup_{j=1}^{\infty} \hat{E}_j\Big)^c$.

$$\begin{aligned}
\lambda(S_1 \cup S_2) &= \lambda\bigg(\Big(S_1 \cap \bigcup_{j=1}^{\infty} E_j\Big) \cup S_2\bigg) \\
&\geq \lambda\bigg(\Big(S_1 \cap \bigcup_{j=1}^{N} E_j\Big) \cup S_2\bigg) \quad \text{(here, we took } S_1' = S_1 \cap \bigcup_{j=1}^{N} E_j, \quad S_2' = S_2) \\
&= \sum_{j=1}^{N} \lambda(S_1 \cap \hat{E}_j) + \lambda(S_2)
\end{aligned}$$

   for any N. Taking the limit $N \to \infty$,

$$\begin{aligned}
\lambda(S_1 \cup S_2) &\geq \sum_{j=1}^{\infty} \lambda(S_1 \cap \hat{E}_j) + \lambda(S_2) \\
&\geq \lambda\Big(S_1 \cap \bigcup_{j=1}^{\infty} \hat{E}_j\Big) + \lambda(S_2) = \lambda(S_1) + \lambda(S_2) \geq \lambda(S_1 \cup S_2).
\end{aligned}$$

   Hence, every quantity appeared equals to each other.

□

Remark 4.4. Thanks to the Caratheodory Theorem, out of $\mathcal{R}$, we grow the collection of sets by adding sets assembled by countable union and complement. Consistency is kept by the transitive λ-separating property.

Once we have enlarged collection, say $\mathcal{G}$, then we grow it again by using the countable union and complement. We repeat this over and over again. This procedure will be detailed in the next chapter.

Chapter 5

The mathematics of "one after another" and consistent family

1. Let the collection $\mathcal{G}_0 = \mathcal{R}$ of rectangles. $(\mathcal{R}, \lambda)$ is consistent.
2. One should note, to enlarge a family while keeping consistency is not at all trivial.

   Example: Assume that we knew that $(\mathcal{R}, \lambda)$ and $(\mathcal{T}, \lambda)$ are consistent individually, where $\mathcal{T}$ is a suitable collection of triangles. How many new consistency checkings are needed for the new collection $\mathcal{G} = \mathcal{R} \cup \mathcal{T}$?
3. Given that, if we define the new collection denoted by $(\mathcal{G}_0)+$ out of $\mathcal{G}_0$

   $$(\mathcal{G}_0)+ = \left\{ G = \bigcup_{j=1}^{\infty} P_j \mid \text{for every } j \quad P_j \in \mathcal{G}_0 \ \text{ or } \ P_j^c \in \mathcal{G}_0 \right\} =: \mathcal{G}_1,$$

   then every member of $\mathcal{G}_1$ is λ-separating.
4. $(\mathcal{G}_1, \lambda)$ is consistent, i.e.,

   $$\mathcal{G}_1 \ni G = \bigcup_{j=1}^{\infty} G_j \ \text{ disjoint union of sets in } \mathcal{G}_1 \qquad \Longrightarrow \qquad \lambda(G) = \sum_{j=1}^{\infty} \lambda(G_j).$$
5. In the similar fashion, $(\mathcal{G}_2, \lambda)$, $(\mathcal{G}_3, \lambda)$, $\cdots$ will be consistent. More precisely, for any N, $(\mathcal{G}_N, \lambda)$ is consistent. (This will be proven by induction.)
6. The limit statement: $\Big(\mathcal{G}_\infty, \lambda\Big)$ with $\mathcal{G}_\infty = \cup_{N=1}^{\infty} \mathcal{G}_N$ is consistent.

   This is because, if

   $$\mathcal{G}_\infty \ni G = \bigcup_{j=1}^{\infty} G_j \ \text{ disjoint union of sets in } \mathcal{G}_\infty,$$

   then for every j, $G_j \in \mathcal{G}_{N(j)}$ for some $N(j)$. In other words, G_j has been included at $\mathcal{G}_{N(j)}$ as a λ-separating set. Thus, $\lambda(G) = \displaystyle\sum_{j=1}^{\infty} \lambda(G_j)$.
7. Since we can, we enlarge $\mathcal{G}_\infty$ again to obtain $(\mathcal{G}_{\infty,1}, \lambda)$ consistent.
8. We do this over and over again.

We can pose a few questions on families

$$\mathcal{G}_0 \subset \mathcal{G}_1 \subset \mathcal{G}_2 \subset \cdots$$

Certainly, growing cannot go beyond the power collection $\mathcal{P}(\mathbb{R}^2)$. Considering this, we examine a few possibilities.

Possibility (0-0). The collection neither stop growing nor reaching $\mathcal{P}(\mathbb{R}^2)$.

Possibility (0-1). The collection keeps strictly growing to becomes $\mathcal{P}(\mathbb{R}^2)$.

Possibility (1). The collection from the initial family $\mathcal{G}_0$ might stop growing if no new sets are added by the expansion $(\ \cdot\ )+$, i.e., at the moment

$$\mathcal{G} = (\mathcal{G})+ = \Big\{ H = \bigcup_{j=1}^{\infty} P_j \mid \text{for every } j \quad P_j \in \mathcal{G} \ \text{ or } \ P_j^c \in \mathcal{G} \Big\}.$$

We have a definite answer to that question. To do this, we need the family

$$(\mathcal{G}_\alpha)_{\alpha\in A}$$

where A is a set other than $\mathbb{N}$.

Indexing

1. In most of our experience, we use index $j \in \mathbb{N}$ to denote a member of sequence a_1, a_2, $\cdots$.
2. This notion of "indexing by $\mathbb{N}$" has been certainly useful. This usefulness is abstracted mathematically and used elsewhere. We have a few examples.

   Example: Let $\mathbb{N}^+ = \mathbb{N} \cup \{\infty\}$.

   (a) If (a_j) is a convergent sequence $a_j \to a_*$ as $j \to \infty$. We may use indexing by $\mathbb{N}^+$ including the limit.

   (b) We have seen many examples where a statement is parametrized by $(\text{statement})_N$. We gave a proof in the style that we prove (i) $(\text{statement})_N$ for any N, and (ii) $(\text{statement})_\infty$. This is to give a proof for statement indexed by $\mathbb{N}^+$.

   Example: Consider $\mathbb{N}^+ \times \mathbb{N}^+$.

Definition 1 (order, linear order, well order on X)**.** *Let X be a nonempty set.*

1. *A subset $P \subset X \times X$ is called a partial order on X if*

   (a) If $(a,b),(b,c) \in P$ then $(a,c) \in P$.

   (b) If $(a,b),(b,a) \in P$ then $a = b$.

   (c) For every $a \in X$, $(a,a) \in P$.

2. *A partial order P on X is called a linear order on X if in addition*

   (d) For every pair $a, b \in X$, either $(a,b) \in P$ or $(b,a) \in P$.

3. *A linear order on X is called a well order on X if in addition*

   (e) For every nonempty subset $A \subset X$, the least element $a \in A$.

Remark 5.1. .

1. $\leq$ is a well-order on $\mathbb{N}$.
2. $\leq$ on $\mathbb{R}$ is a linear order but is not a well-order. This is because the condition (e) is not true in general, for example $A = (0, 1)$.
3. We will also use the symbol $<$

Definition 2. *For a nonempty set X with well order, denoted by $\leq$, we define*

$$a < b \iff a \leq b \quad \text{and} \quad a \neq b.$$

According to the set theory, the following statement is true.

Theorem 3. *There exists an uncountable set with well-order.*

In our course, we do not intend to proceed with a set theory, giving a proof of this. We only consider a family $(\mathcal{G}_\alpha)$ indexed by such a set.

We fix X that is uncountable and with well-order, denoted by $\leq$.

Proposition 4. *There exists a subset $A \subset X$ such that*

(i) for any $\alpha \in A$, $I_\alpha = \{\beta \in X \mid \beta < \alpha\}$ is countable $\quad(ii)$ A is uncountable.

Proof. Define $S = \{\alpha \in X \mid I_\alpha$ is uncountable$\}$. In case S is empty, we define $A = X$. If not, there exists the least element $\omega_1 \in S$ and define $A = I_{\omega_1}$. □

Remark 5.2. We omit the discussion but well-ordered sets with properties in Proposition 4 are order isomorphic to each other. For the role of index, use of any such a set leads to the equivalent result in our class.

Definition 5. *(1) Define $\mathcal{G}_0 = \mathcal{R}$, where 0 refers to the least element of A.*

(2) For a given $\alpha \in A$, if $\mathcal{G}_\beta$ is defined for every $\beta \in A$ with $\beta < \alpha$, define

$$\mathcal{G}_\alpha := \bigcup_{\beta<\alpha} (\mathcal{G}_\beta) + .$$

Proposition 6. *$\mathcal{G}_\alpha$ is defined for every $\alpha \in A$, thus defining expanding families $(\mathcal{G}_\alpha)_{\alpha \in A}$.*

Proof. This is the induction we use:
Let $S = \{\alpha \in A \mid \mathcal{G}_\alpha$ is not defined.$\}$. If S is nonempty, then there exists the least element $\omega \in S$. Then $\mathcal{G}_\beta$ with $\beta < \omega$ must have been defined. In turn, $\mathcal{G}_\omega$ has a definition by Definition 5, contradiction. Therefore S is the empty set. □

Definition 7. *Define the collection*

$$\mathcal{B} = \bigcup_{\alpha \in A} \mathcal{G}_\alpha.$$

Theorem 8.

$$(\mathcal{B})+ = \mathcal{B}.$$

Proof.

1. We prove that

   (i) $\mathcal{B}$ is closed under complement operation.

   (ii) $\mathcal{B}$ is closed under countable union operation.

2. Suppose $E \in \mathcal{B}$. By definition, $E \in \mathcal{G}_{\alpha_0}$ for some $\alpha_0 \in A$.
3. Let $S = \{\beta \in A \mid \alpha_0 < \beta\}$. S cannot be the empty set: If S is empty, then for any $\alpha \in A$, $\alpha_0 = \alpha$ or $\alpha < \alpha_0$. In other words, $I_{\alpha_0} \cup \{\alpha_0\} \supset A$. This contradicts to that (LHS) is countable while (RHS) is uncountable.
4. There exists $\beta \in A$ such that $\alpha_0 < \beta$, and E^c must have been included in $\mathcal{G}_\beta$.
5. Now, $E_1, E_2, E_3, \cdots \in \mathcal{B}$ with $E_j \in \mathcal{G}_{\alpha_j}$ for some $\alpha_j \in A$.
6. Let $S' = \{\beta \in A \mid \alpha_j < \beta \quad \text{for every } j\}$. S' cannot be the empty set: If S' is empty, then for any $\alpha \in A$, there exists some j such that $\alpha_j = \alpha$ or $\alpha < \alpha_j$. In other words, $\bigcup_{j=1}^{\infty} I_{\alpha_j} \cup \{\alpha_j\} \supset A$, which is contradiction.
7. There exists $\beta \in A$ such that $\alpha_j < \beta$ for every j. Then $\displaystyle\bigcup_{j=1}^{\infty} E_j$ must have been included in $\mathcal{G}_\beta$.

□

Definition 9. *A nonempty collection $\mathcal{E} \subset \mathcal{P}(\mathbb{R}^2)$ containing $\emptyset$ is called a σ-algebra if*

$$(i) \text{ If } E \in \mathcal{E} \text{ then } E^c \in \mathcal{E}.$$

$$(ii) \text{ If } E_1, E_2, \cdots \in \mathcal{E} \text{ then } \bigcup_{j=1}^{\infty} E_j \in \mathcal{E}.$$

We will come to the definition of σ-algebra again in the next class.

Remark 5.3. .

1. We are done with defining the area, the 2 dimensional Lebesgue measure, on every set $E \in \mathcal{B}$.
2. The collection $\mathcal{B} = \mathcal{B}(\mathbb{R}^2)$ is called the σ-algebra of all *borel sets*.

Remark 5.4. .

1. The expanding families $(\mathcal{G}_\alpha)$ certainly depends on the initial family $\mathcal{G}_0$, which was $\mathcal{R}$ in our case.
2. More precisely, for any given $\mathcal{G}_0$ containing $\emptyset$, $\bigcup_{\alpha \in A} \mathcal{G}_\alpha$ is <u>the smallest σ-algebra containing $\mathcal{G}_0$</u>.
3. Regardless of the initial family, we can certainly define

$$\mathcal{E}^\lambda(\mathbb{R}^2) := \{E \subset \mathbb{R}^2 \mid E \text{ is } \lambda\text{-separating.}\}$$

   We will call $\mathcal{E}^\lambda(\mathbb{R}^2)$ the σ-algebra of all λ-*measurable sets* or the σ-algebra of all *Lebesgue measurable sets.*

   (Instead of calling it the collection of all λ-separating sets.)

Remark 5.5. .

1. We have not yet answered to the question if $\mathcal{B}(\mathbb{R}^2) = \mathcal{P}(\mathbb{R}^2)$ or not.

   We will verify
$$\mathcal{B}(\mathbb{R}^2) \subsetneq \mathcal{E}^\lambda(\mathbb{R}^2) \subsetneq \mathcal{P}(\mathbb{R}^2).$$
2. Before that, we have one important result to know.

   We show that every open set $U \subset \mathbb{R}^2$ is in $\mathcal{B}(\mathbb{R}^2)$. More precisely, $U \in \mathcal{G}_1$.

Theorem 10. *Any open set $U \subset \mathbb{R}^2$ is a countable disjoint union of rectangles in $\mathcal{R}$.*

Proof. 1. For $m = 0, 1, 2, \cdots$, we consider the depth m grid lines of $\mathbb{R}^2$: At each m, the grid lines are drawn by the grid points and the grid points are those points whose x-coordinate and y-coordinate are in the form

$$\text{integer} + \sum_{j=1}^{m} \frac{b_j}{2^j}, \quad b_j \in \{0, 1\}.$$

$\mathbb{R}^2$ is a countable union of those pairwise disjoint depth m rectangles partitioned by grid lines. The collection of depth m rectangles is denoted by $\mathcal{R}_m$.

2. Now, we inductively define $\mathcal{Q}_{m,0}$ and $\mathcal{Q}_{m,1}$ of depth m rectangles so that

$$\Big(\bigcup_{j=0}^{m} \mathcal{Q}_{j,0}\Big) \cup \mathcal{Q}_{m,1} \quad \text{covers } U. \tag{C}$$

At $m = 0$, define

$$\mathcal{Q}_{0,0} = \{Q \in \mathcal{R}_0 \mid Q \subset U\}, \quad \mathcal{Q}_{0,1} = \{Q \in \mathcal{R}_0 \mid Q \cap U \neq \emptyset \text{ and } Q \not\subset U\}.$$

Certainly, $\mathcal{Q}_{0,0} \cup \mathcal{Q}_{0,1}$ covers U.

3. Now, suppose $(\mathcal{Q}_{j,0}, \mathcal{Q}_{j,1})$ are defined up to $j = 0, 1, \cdots, m-1$, satisfying (C). Depth $m-1$ rectangles in $\mathcal{Q}_{m-1,1}$ are pairwise disjoint and each of them is a disjoint union of four depth m rectangles. We define $\mathcal{R}'_m$ be the collection of pairwise disjoint depth m rectangles obtained from $\mathcal{Q}_{m-1,1}$. Now,

$$\mathcal{Q}_{m,0} = \{Q \in \mathcal{R}'_m \mid Q \subset U\}, \quad \mathcal{Q}_{m,1} = \{Q \in \mathcal{R}'_m \mid Q \cap U \neq \emptyset \text{ and } Q \not\subset U\}.$$

Certainly, $U \cap \bigcup_{Q' \in \mathcal{Q}_{m-1,1}} Q'$ is covered by $\mathcal{Q}_{m,0} \cup \mathcal{Q}_{m,1}$. Hence, $\Big(\bigcup_{j=0}^{m} \mathcal{Q}_{j,0}\Big) \cup \mathcal{Q}_{m,1}$ covers U.

4. Let $\mathcal{Q} := \bigcup_{m=0}^{\infty} \mathcal{Q}_{m,0}$ and define the set G as the union over the collection $\mathcal{Q}$.

   By definition, $G \subset U$.

5. We show that $G \supset U$.

   If $x \in U$, then there exists an open square of side length $\ell > 0$ containing x that is a subset of U. Inside of this open square, there exists a half open square $\hat{Q}$ containing x with smaller side length that are aligned along with the grid lines of some depth $\hat{m}$.

6. That $\hat{Q} \subset U$ implies

   (i) $\Big(\bigcup_{j=0}^{\hat{m}} \mathcal{Q}_{j,0}\Big) \cup \mathcal{Q}_{\hat{m},1}$ covers $\hat{Q}$

   (ii) $\hat{Q}$ is disjoint from every rectangles in $\mathcal{Q}_{\hat{m},1}$.

   Hence, $\Big(\bigcup_{j=0}^{\hat{m}} \mathcal{Q}_{j,0}\Big)$ covers $\hat{Q}$, or $G \supset \hat{Q}$. Thus, $G \ni x$.

$\square$

Chapter 6

Abstraction of the Lebesgue measure

σ-algebra

Let X be a nonempty set.

If (P) is any property on subsets of X such that

(i) $\emptyset$ has the property.

(ii) The property is transitive for taking complement and countable union.

then, certainly $\exists \mathcal{Q}$ a seed family containing $\emptyset$ with members having (P).

You always end up with two σ-algebras:

1. By considering $\mathcal{G}_0 = \mathcal{Q}$, $\mathcal{G}_1 = (\mathcal{G}_0)+$, $\mathcal{G}_2 = (\mathcal{G}_1)+$, $\cdots$, to define

$$\underline{\mathcal{E}}(\mathcal{Q}) = \bigcup_{\alpha \in A} \mathcal{G}_\alpha.$$

2. $\mathcal{E}^P = \{E \subset X \mid E \text{ has the property } (P).\}$

The former is called the smallest σ-algebra containing $\mathcal{Q}$. The latter is the σ-algebra of sets having (P).

We recall the definition of σ-algebra of subsets of X.

Definition 1. *Let X be a nonempty set. A collection $\mathcal{E} \subset \mathcal{P}(X)$ containing $\emptyset$ is called a σ-algebra of subsets of X if*

$$(i)\ \textit{If } E \in \mathcal{E} \textit{ then } E^c \in \mathcal{E}.$$

$$(ii)\ \textit{If } E_1, E_2, \cdots \in \mathcal{E} \textit{ then } \bigcup_{j=1}^{\infty} E_j \in \mathcal{E}.$$

Proposition 2. *Let X be a nonempty set, and $\mathcal{E}$ be a σ-algebra of subsets of X. Then it holds that*

$$(iii)\quad \textit{If } E_1, E_2, \cdots \in \mathcal{E} \textit{ then } \bigcap_{j=1}^{\infty} E_j \in \mathcal{E}.$$

Proof. This is because $\bigcup_{j=1}^{\infty} E_j^c \in \mathcal{E}$ and $\left(\bigcup_{j=1}^{\infty} E_j^c\right)^c = \bigcap_{j=1}^{\infty} E_j$. □

The term "smallest" is from the following observations.

1. If $\mathcal{Q}$ is any seed collection containing $\emptyset$, the set

$$\Sigma := \{\mathcal{E} \subset \mathcal{P}(X) \mid \mathcal{E} \text{ is a } \sigma\text{-algebra and } \mathcal{E} \supset \mathcal{Q}\}$$

   is nonempty because $\mathcal{P}(X) \in \Sigma$.

2. Let $\underline{\mathcal{E}}$ be the intersection of all the members of Σ, i.e.,

$$\underline{\mathcal{E}} = \{E \subset X \mid E \text{ is member of } \mathcal{E} \text{ for every } \mathcal{E} \in \Sigma\}.$$

   It easily follows that $\underline{\mathcal{E}}$ is again a σ-algebra since

   (a) $\emptyset \in \mathcal{E}$ for every $\mathcal{E} \in \Sigma$.

   (b) If $\underline{E}_1$, $\underline{E}_2$, $\cdots$ are members of $\mathcal{E}$ for every $\mathcal{E} \in \Sigma$, then so is $\bigcup_{j=1}^{\infty} \underline{E}_j$.

3. Lastly, we show $\bigcup_{\alpha \in A} \mathcal{G}_\alpha \subset \underline{\mathcal{E}}$ with $\mathcal{G}_0 = \mathcal{Q}$ below.

Proposition 3. *With $\mathcal{G}_0 = \mathcal{Q}$, $\bigcup_{\alpha \in A} \mathcal{G}_\alpha \subset \underline{\mathcal{E}}$*

Proof. This is because

$$(i)\ \text{Certainly, } \mathcal{Q} = \mathcal{G}_0 \text{ is contained in } \underline{\mathcal{E}}.$$

$$(ii)\ \text{If } \mathcal{G}_\beta \subset \underline{\mathcal{E}} \text{ for every } \beta < \alpha, \text{ then so is } \mathcal{G}_\alpha = \bigcup_{\beta < \alpha} (\mathcal{G}_\beta)+.$$

If we take $S = \{\alpha \in A \mid \mathcal{G}_\alpha \not\subset \underline{\mathcal{E}}\}$, then S is empty set, otherwise, there exists the least element $\omega \in S$, but this contradicts to (ii) above. □

The one of the role of the smallest σ-algebra, (or of a few first families in $(\mathcal{G}_\alpha)$) is played for the pair $(\mathcal{B}, \lambda)$ in the following manner.

Theorem 4. *For any set $S \subset \mathbb{R}^2$, there exists a borel set $E \supset S$ with $\lambda(E) = \lambda(S)$.*

Proof.

1. If $\lambda(S) = \infty$, we take $E = \mathbb{R}^2$ and we are done. Now we assume $\lambda(S) < \infty$.
2. For every $\alpha = 1, 2, 3 \cdots$, there exists (R_j^α) of rectangles that cover S with

$$\lambda(S) + \frac{1}{\alpha} \geq \sum_{j=1}^{\infty} \lambda(R_j^\alpha)$$

3. We define

$$E^\alpha := \bigcup_{j=1}^{\infty} R_j^\alpha, \quad E := \bigcap_{\alpha=1}^{\infty} E^\alpha$$

   that are borel sets. Every E^α contains S as a subset, and so is the E.

4. Now, for every α,

$$\lambda(S) + \frac{1}{\alpha} \geq \sum_{j=1}^{\infty} \lambda(R_j^\alpha) \geq \lambda(E^\alpha) \geq \lambda(E) \geq \lambda(S).$$

   Taking the limit $\alpha \to \infty$, we obtain $\lambda(S) = \lambda(E)$.

$\square$

Measure

Let X be a nonempty set.

Definition 5. *Let $\mathcal{E}$ be a σ-algebra of subsets of X. A set function $\mu_0 : \mathcal{E} \to [0, \infty]$ is called a measure on $\mathcal{E}$ if*

(*i*) $\mu_0(\emptyset) = 0$,

(*ii*) *If* $E = \bigcup_{j=1}^{\infty} E_j$, *where* (E_j) *is pairwise disjoint sets in* $\mathcal{E}$ *then* $\mu_0(E) = \sum_{j=1}^{\infty} \mu_0(E_j)$.

Definition 6. *Let X be a nonempty set and $\mathcal{E}$ be a σ-algebra of subsets of X.*

1. *The pair $(X, \mathcal{E})$ is called a measurable space.*
2. *A member of $\mathcal{E}$ is called a $\mathcal{E}$-measurable set.*

Definition 7. *Let $(X, \mathcal{E})$ be a measurable space and μ be a measure on $\mathcal{E}$. The triple $(X, \mathcal{E}, \mu)$ is called a measure space.*

Outer measure and regularity

Let X be a nonempty set.

Definition 8. *A set function $\mu : \mathcal{P}(X) \to [0, \infty]$ is called an (outer) measure on X if*

$$(i)\ \mu(\emptyset) = 0,$$
$$(ii)\ \textit{If } S \subset \bigcup_{j=1}^{\infty} S_j \textit{ then } \quad \mu(S) \leq \sum_{j=1}^{\infty} \mu(S_j).$$

Exercise 9. *Re-do the parts Definition 1, 2, Theorem 3, Theorem 7 in Chapter 4, not for $\mathbb{R}^2$ but for X.*

Definition 10. *Let μ be an outer measure on X. The collection*

$$\mathcal{E}^{\mu} := \Big\{ E \subset X \mid E \textit{ is } \mu\textit{-separating} \Big\}$$

is called the σ-algebra of $\mathcal{E}^{\mu}$-measurable sets, or shortly of μ-measurable sets.

Definition 11. *An outer measure μ on X is a regular outer measure if*

for every $S \subset X$, there exists a μ-measurable set $E \supset S$ with $\mu(E) = \mu(S)$.

Let $X = \mathbb{R}^n$.

Definition 12. *Let*

$$\mathcal{B}(\mathbb{R}^n) = \underline{\mathcal{E}}(\mathcal{Q}) \quad \textit{the smallest } \sigma\textit{-algebra containing } \mathcal{Q} \textit{ of half open } n\textit{-cubes.}$$

We say $\mathcal{B}$ is the σ-algebra of borel sets.

Definition 13. *An outer measure μ on $\mathbb{R}^n$ is called a borel outer measure if every borel set is a μ-measurable set.*

Definition 14. *A borel outer measure μ on $\mathbb{R}^n$ is a borel regular outer measure if*

for every $S \subset \mathbb{R}^n$, there exists a borel set $E \supset S$ with $\mu(E) = \mu(S)$.

Exercise 15. *Let* $\big(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \mu_0\big)$ *be a measure space, i.e.,* μ_0 *is a borel measure on* $\mathbb{R}^n$*. Then the extension* μ *on* $\mathcal{P}(\mathbb{R}^n)$ *of* μ_0 *by*

$$\mu(S) = \inf_{(E_j) \text{ of } \mathcal{B}(\mathbb{R}^n) \text{ that covers } S} \sum_{j=1}^{\infty} \mu_0(E_j)$$

is well-defined, and μ *is a borel regular outer measure.*

Exercise 16. *Let* $\big(X, \mathcal{E}, \mu_0\big)$ *be a measure space. Then the extension* μ *on* $\mathcal{P}(X)$ *of* μ_0 *by*

$$\mu(S) = \inf_{(E_j) \text{ of } \mathcal{E} \text{ that covers } S} \sum_{j=1}^{\infty} \mu_0(E_j)$$

is well-defined, and μ *is a regular outer measure.*

Examples of measurable spaces and measure spaces

Consequences of countable additivity

Proposition 17. *Let $(X, \mathcal{E}, \mu)$ be a measure space. Let (E_j) be a sequence of $\mathcal{E}$-measurable sets such that $E_1 \subset E_2 \subset E_3 \subset \cdots$. Then*

$$(i) \text{ For any } N,\ \mu\Big(\bigcup_{j=1}^{N} E_j\Big) = \mu(E_N), \quad (ii)\ \mu\Big(\bigcup_{j=1}^{\infty} E_j\Big) = \lim_{N\to\infty} \mu(E_N).$$

Proof. 1. In fact, the Proposition is to prove (ii).

2. We use the pairwise disjoint sequence $(\hat{E}_j)$ obtained from (E_j). At this point, we know that $\hat{E}_j$ are all $\mathcal{E}$-measurable sets.

3. Thanks to the countable additivity,

$$\lim_{N\to\infty} \mu(E_N) = \lim_{N\to\infty} \mu\Big(\bigcup_{j=1}^{N} E_j\Big) = \lim_{N\to\infty} \mu\Big(\bigcup_{j=1}^{N} \hat{E}_j\Big) = \sum_{j=1}^{\infty} \mu(\hat{E}_j) = \mu\Big(\bigcup_{j=1}^{\infty} \hat{E}_j\Big) = \mu\Big(\bigcup_{j=1}^{\infty} E_j\Big).$$

□

Proposition 18. *Let $(X, \mathcal{E}, \mu)$ be a measure space. Let (E_j) be a sequence of $\mathcal{E}$-measurable sets such that $\mu(E_1) < \infty$ and $E_1 \supset E_2 \supset E_3 \supset \cdots$. Then*

$$(i) \text{ For any } N,\ \mu\Big(\bigcap_{j=1}^{N} E_j\Big) = \mu(E_N), \quad (ii)\ \mu\Big(\bigcap_{j=1}^{\infty} E_j\Big) = \lim_{N\to\infty} \mu(E_N).$$

Proof. 1. In fact, the proposition is to prove (ii).

2. Let $F_j = E_1 \setminus E_j$ so that $F_1 \subset F_2 \subset F_3 \subset \cdots$.

   All of them are $\mathcal{E}$-measurable subsets of E_1 with finite measure, and we have

$$\mu(F_N) + \mu(E_N) = \mu(E_1) \quad \Longleftrightarrow \quad \mu(E_N) = \mu(E_1) - \mu(F_N).$$

3. (RHS) has the limit,

$$\mu(E_1) - \lim_{N\to\infty} \mu(F_N) = \mu(E_1) - \mu\Big(\bigcup_{j=1}^{\infty} F_j\Big).$$

4. On the other hand, $\bigcup_{j=1}^{\infty} F_j = E_1 \setminus \bigcap_{j=1}^{\infty} E_j$. This implies that

$$\mu(E_1) - \mu\Big(\bigcup_{j=1}^{\infty} F_j\Big) = \mu\Big(\bigcap_{j=1}^{\infty} E_j\Big).$$

□

summary

1. On $(X, \mathcal{E}, \mu)$, we now define the Integral.
2. Further questions on the set and measure, in particular on subsets of $\mathbb{R}^n$ and the Lebesgue measure, are left for the later study:
   (a) The existence of a set $S \notin \mathcal{E}^\lambda$, a non-Lebesgue-measurable set.
   (b) Many interesting examples of sets: The Cantor set, The Fat Cantor set, $\cdots$ will be examined too.

Chapter 7

When we need (multiplicity, addition), not (set, union)

In our class,

$$\text{“measurable multiplicity”} \quad = \quad \text{“a measurable function valued in } [0,\infty]\text{”}$$

1. The “Area” is such a notion that total area of certain regions does not count overlaping region doubly,

   i.e., even if E and E' have an overlaping region $E \cap E' \neq \emptyset$, the total area is

   $$\lambda(E \cup E').$$

2. We may want to count doubly for the region $E \cap E'$. Total multiplicity

   $$m\bigl(\{E, E'\}\bigr) = \lambda(E \setminus E') + 2\lambda(E \cap E') + \lambda(E' \setminus E)$$

   Example: suppose we measure “Brightness”.

How is a multiplicity θ on X defined?

Let $(X, \mathcal{E})$ be a measurable space.

(try to imagine "Brightness" decided by bulbs.)

1. Consider a data (c_1, E_1), (c_2, E_2), (c_3, E_3), $\cdots$ where $(c_j, E_j) \in [0, \infty] \times \mathcal{E}$.
2. We let the sequence $L = (c_j, E_j)_{j=1}^{\infty}$. Because of nonnegativity of c_j, the way we enumerate is irrelevant in what we will do here.
3. The data L induces a function $\theta : X \mapsto [0, \infty]$. For each x, we count

$$x \mapsto \sum_{E_j \ni x} c_j.$$

1. Now, let $\mathcal{L}^+ = \mathcal{L}^+(X, \mathcal{E})$ be the set of all sequences in $[0, \infty] \times \mathcal{E}$.
2. Then we define
$$\Big\{ x \mapsto \sum_{E_j \ni x} c_j \mid (c_j, E_j)_{j=1}^{\infty} \in \mathcal{L}^+ \Big\}.$$
3. This is the set of all measurable multiplicities on X.

Now, let $(X, \mathcal{E})$ be equipped with a measure μ on $\mathcal{E}$.

1. For each measurable multiplicy θ, we wish to assign a number for instance

   If θ is $x \mapsto \displaystyle\sum_{E_j \ni x} c_j$ for some $(c_j, E_j) \in \mathcal{L}^+$, we wish to assign $I[\theta] = \displaystyle\sum_{j=1}^{\infty} c_j \mu(E_j)$

2. At the moment, we can't. Because many different data can induce the same multiplicity θ.

   Example:

3. Now, we resolve this problem of well-definedness. This is the theory of Integral.

Chapter 8

Integral of a measurable multiplicity

Let $(X, \mathcal{E}, \mu)$ be a measure space.

We first consider a simpler kind of multiplicities.

1. We consider the set of finite sequences of a form $(c_j, E_j)_{j=1}^m$ in $[0,\infty] \times \mathcal{E}$.
2. If we want, this can be considered as a member of $\mathcal{L}^+$ with $E_j = \emptyset$ for $j > m$.
3. The multiplicity θ induced by a finite sequence is defined in the same way.
4. We impose further restriction. We let

$$\mathcal{L}_0^+ = \left\{ (c_j, E_j)_{j=1}^m \;\middle|\; m \in \mathbb{N}, \quad (c_j, E_j) \in [0,\infty) \times \mathcal{E}. \right\}$$

Remark 8.1. That is to say, m is finite and also $c_j < \infty$.

Definition 1. *A measurable multiplicity θ is simple and nonnegative if θ is induced from a finite sequence with further assumption $(c_j, E_j)_{j=1}^m \in \mathcal{L}_0^+$.*

More common notation for the multiplicity $x \mapsto \sum_{E_j \ni x} c_j$ is to use the characteristic function. For $S \subset X$, the characteristic function of the set S is

$$\chi_S(x) = \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{if } x \notin S. \end{cases}$$

One can write

$$x \mapsto \sum_{E_j \ni x} c_j = \sum_{j=1}^{\infty} c_j \chi_{E_j}(x).$$

For a nonnegative simple function induced by $(c_j, E_j)_{j=1}^m$ is thus

$$x \mapsto \sum_{j=1}^{m} c_j \chi_{E_j}(x).$$

Definition 2. *An element $(c_j, E_j)_{j=1}^m \in \mathcal{L}_0^+$ is canonical if*

$$(i)\ c_1, c_2, \cdots, c_m \text{ are all distinct and nonzero}$$
$$(ii)\ (E_j) \text{ is pairwise disjoint.}$$

Expression $\sum_{j=1}^m c_j \chi_{E_j}(x)$ is said to be in a canonical form if $(c_j, E_j)_{j=1}^m$ is canonical.

Theorem 3. *Any nonnegative simple function is induced from a canonical data.*

Proof.

1. Consider a nonnegative simple function represented by

$$\theta(x) = \sum_{j=1}^m c_j \chi_{E_j}(x)$$

We now define a canonical data that induces the same function.

2. Let Γ be a set of finite binary sequence $\beta = (\beta_1, \beta_2, \cdots, \beta_m)$, $\beta_j \in \{0, 1\}$.
3. For each $\beta \in \Gamma$, we define the $\mathcal{E}$-measurable set in the following manner:

$$E_\beta = H_1 \cap H_2 \cap \cdots \cap H_m, \quad H_j = \begin{cases} E_j & \text{if } \beta_j = 1 \\ E_j^c & \text{if } \beta_j = 0 \end{cases}$$

4. We note that $X = \bigcup_{\beta \in \Gamma} E_\beta$ a disjoint union.
5. For each $\beta \in \Gamma$, define

$$c_\beta = \sum_{j=1}^m c_j \beta_j = \sum_{j,\ \beta_j \neq 0} c_j.$$

Then, because for every $x \in X$, x belongs to unique $E_{\bar{\beta}}$ for some $\bar{\beta}$,

$$\theta(x) = \sum_{j=1}^m c_j \chi_{E_j}(x) = \sum_{j,\ \bar{\beta}_j \neq 0} c_j = c_{\bar{\beta}} = \sum_{\beta \in \Gamma} c_\beta \chi_{E_\beta}(x).$$

6. Now, we enumerate the set $\{c_\beta \mid \beta \in \Gamma\} \setminus \{0\}$, that is $a_1, a_2, \cdots, a_{m'}$.
7. Define $\Gamma_k = \{\beta \in \Gamma \mid c_\beta = a_k\}$ for $k = 1, 2, \cdots, m'$. We have that

$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_k \cup \Gamma_0 \quad \text{of disjoint union, where } \Gamma_0 = \{\beta \in \Gamma \mid c_\beta = 0\}.$$

Define $F_k = \bigcup_{\beta \in \Gamma_k} E_\beta$, which is $\mathcal{E}$-measurable.

8. Finally

$$\begin{aligned} \theta(x) &= \sum_{j=1}^m c_j \chi_{E_j(x)} = \sum_{\beta \in \Gamma} c_\beta \chi_{E_\beta}(x) = \sum_{\beta \in \Gamma \setminus \Gamma_0} c_\beta \chi_{E_\beta}(x) \\ &= \sum_{k=1}^{m'} \sum_{\beta \in \Gamma_k} c_\beta \chi_{E_\beta}(x) = \sum_{k=1}^{m'} \sum_{\beta \in \Gamma_k} a_k \chi_{E_\beta}(x) = \sum_{k=1}^{m'} a_k \sum_{\beta \in \Gamma_k} \chi_{E_\beta}(x) = \sum_{k=1}^{m'} a_k \chi_{F_k}(x). \end{aligned}$$

9. Note that $(a_k, F_k)_{k=1}^{m'}$ is canonical.

$\square$

Remark 8.2. If $\theta(x) = \sum_{j=1}^{m} c_j \chi_{E_j}(x)$ is in a canonical form, the range set of θ is precisely

$$\{c_1, c_2, \cdots, c_m\} \cup \{0\} \quad \text{of } m+1 \text{ elements}$$

and E_j is precisely the inverse image $\theta^{-1}(c_j)$. Canonical data of a given nonnegative simple function θ is unique up to the enumeration of the data.

Definition 4. *Let θ be a nonnegative simple function. We define*

$$\int \theta \, d\mu = \sum_{j=1}^{m} c_j \mu(E_j), \quad (c_j, E_j)_{j=1}^{m} \textit{ is a canonical data for } \theta.$$

Remark 8.3. The quantity is well-defined because canonical data exists and is unique up to the enumeration of the data.

Theorem 5 (finite representation independence)**.** *Let $(c_j, E_j) \in \mathcal{L}_0^+$ induces a nonnegative simple function $\theta(x) = \sum_{j=1}^{m} c_j \chi_{E_j}(x)$ that is not necessarily in a canonical form. Then, the equality*

$$\int \theta \, d\mu = \sum_{j=1}^{m} c_j \mu(E_j) \quad \textit{holds.}$$

Proof.

1. We use the same notations used in the proof of Theorem 3.
2. Define for each j and β
$$r_{j,\beta} = \begin{cases} c_j \mu(E_\beta) & \text{if } \beta_j \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$
3. Then,
$$\begin{aligned} \sum_{j=1}^{m} c_j \mu(E_j) &= \sum_{j=1}^{m} c_j \sum_{\beta,\ \beta_j \neq 0} \mu(E_\beta) = \sum_{j=1}^{m} \sum_{\beta,\ \beta_j \neq 0} c_j \mu(E_\beta) = \sum_{j=1}^{m} \sum_{\beta \in \Gamma} r_{j,\beta} \\ &= \sum_{\beta \in \Gamma} \sum_{j=1}^{m} r_{j,\beta} = \sum_{\beta \in \Gamma} \sum_{j,\ \beta_j \neq 0} c_j \mu(E_\beta) = \sum_{\beta \in \Gamma} \mu(E_\beta) \sum_{j,\ \beta_j \neq 0} c_j = \sum_{\beta \in \Gamma} c_\beta \mu(E_\beta). \end{aligned}$$
4. Now,
$$\begin{aligned} \sum_{\beta \in \Gamma} c_\beta \mu(E_\beta) &= \sum_{\beta \in \Gamma \setminus \Gamma_0} c_\beta \mu(E_\beta) = \sum_{k=1}^{m'} \sum_{\beta \in \Gamma_k} c_\beta \mu(E_\beta) = \sum_{k=1}^{m'} \sum_{\beta \in \Gamma_k} a_k \mu(E_\beta) \\ &= \sum_{k=1}^{m'} a_k \sum_{\beta \in \Gamma_k} \mu(E_\beta) = \sum_{k=1}^{m'} a_k \mu(F_k) = \int \theta \, d\mu. \end{aligned}$$

$\square$

Let us define

$$\Lambda_0^+ = \{x \mapsto \sum_{j=1}^m c_j \chi_{E_j}(x) \mid (c_j, E_j)_{j=1}^m \in \mathcal{L}_0^+\}$$

the set of all nonnegative simple functions.

Λ_0^+ is a highly structured set in the following senses.

1. Λ_0^+ is closed under the nonnegative scalar multiplication, i.e.,

$$\text{if } \theta = \sum_{j=1}^m c_j \chi_{E_j} \in \Lambda_0^+ \text{ and } c \in [0,\infty) \text{ then } c\theta = \sum_{j=1}^m cc_j \chi_{E_j} \in \Lambda_0^+.$$

2. Λ_0^+ is closed under the finite summation, i.e.,

$$\begin{aligned} &\text{if} \quad \theta_1 = \sum_{j=1}^{m_1} c_{1,j}\chi_{E_{1,j}},\ \theta_2 = \sum_{j=1}^{m_2} c_{2,j}\chi_{E_{2,j}}, \cdots,\ \theta_N = \sum_{j=1}^{m_N} c_{N,j}\chi_{E_{N,j}} \in \Lambda_0^+ \\ &\text{then} \quad \theta_1 + \theta_2 + \cdots + \theta_N = \sum_{\alpha=1}^N \sum_{j=1}^{m_\alpha} c_{\alpha,j}\chi_{E_{\alpha,j}} \in \Lambda_0^+. \end{aligned}$$

A fancy way to say this: Λ_0^+ is a convex cone.

Also, in the sense that we can check if $\theta_1(x) \le \theta_2(x)$ for every $x \in X$, Λ_0^+ is partially ordered.

In fact, Λ_0^+ is closed under finite products too, but we only consider the following case: For $E \in \mathcal{E}$,

$$\text{if} \quad \theta(x) = \sum_{j=1}^m c_j \chi_{E_j}(x), \quad \text{then} \quad \theta(x)\chi_E(x) = \sum_{j=1}^m c_j \chi_{E_j}(x)\chi_E(x) = \sum_{j=1}^m c_j \chi_{E_j \cap E}(x) \in \Lambda_0^+.$$

The integral goes well along with these structures.

Theorem 6. *Let $\theta, \tilde\theta \in \Lambda_0^+$, $c \in [0,\infty)$. Then,*

1. $\displaystyle\int c\theta\, d\mu = c\int \theta\, d\mu.$
2. $\displaystyle\int \theta + \tilde\theta\, d\mu = \int \theta\, d\mu + \int \tilde\theta\, d\mu.$
3. *If $\theta \le \tilde\theta$, then* $\displaystyle\int \theta\, d\mu \le \int \tilde\theta\, d\mu.$
4. *For $E \in \mathcal{E}$, denote $\int \theta\chi_E\, d\mu =: \int_E \theta$. The map $\rho : \mathcal{E} \to [0,\infty]$ defined by*

$$E \mapsto \int_E \theta\, d\mu$$

is a measure on $(X, \mathcal{E})$.

Proof. 1. Let $\theta(x) = \sum_{j=1}^m c_j \chi_{E_j}(x)$, and $c\theta = \sum_{j=1}^m cc_j \chi_{E_j}$. By finite representation independence,

$$\int c\theta \, d\mu = \sum_{j=1}^m cc_j \mu(E_j) = c \sum_{j=1}^m c_j \mu(E_j) = c \int \theta \, d\mu.$$

2. Let $\theta(x) = \sum_{j=1}^{m_1} c_{1,j} \chi_{E_{1,j}}(x)$, $\tilde{\theta}(x) = \sum_{j=1}^{m_2} c_{2,j} \chi_{E_{2,j}}(x)$, and write

$$\theta(x) + \tilde{\theta}(x) = \sum_{\alpha=1}^2 \sum_{j=1}^{m_\alpha} c_{\alpha,j} \chi_{E_{\alpha,j}}(x).$$

Again by finite representation independence,

$$\begin{aligned} &\int \theta + \tilde{\theta} \, d\mu \\ = \; &\sum_{\alpha=1}^2 \sum_{j=1}^{m_\alpha} c_{\alpha,j} \mu(E_{\alpha,j}) = \sum_{j=1}^{m_1} c_{1,j} \mu(E_{1,j}) + \sum_{j=1}^{m_2} c_{2,j} \mu(E_{2,j}) = \int \theta \, d\mu + \int \tilde{\theta} \, d\mu. \end{aligned}$$

3. Let us represent $\theta(x) = \sum_{j=1}^m c_j \chi_{E_j}(x)$, $\tilde{\theta}(x) = \sum_{k=1}^{m'} d_k \chi_{F_k}(x)$. We let

$$(G_1, G_2, \cdots, G_m, G_{m+1}, G_{m+2}, \cdots, G_{m+m'}) = (E_1, E_2, \cdots, E_m, F_1, F_2, \cdots, F_{m'})$$

and write

$$\begin{aligned} \theta(x) &= \sum_{\ell=1}^{m+m'} c_\ell \chi_{G_\ell}(x), \quad \text{where } c_\ell = 0 \text{ if } \ell > m, \\ \tilde{\theta}(x) &= \sum_{\ell=1}^{m+m'} d_\ell \chi_{G_\ell}(x), \quad \text{where } d_\ell = 0 \text{ if } \ell \le m. \end{aligned}$$

Now, let us consider the decompositions of $(G_\ell)_{\ell=1}^{m+m'}$ similarly done in the proof of Theorem 3 to write

$$\theta(x) = \sum_{\beta \in \Gamma} c_\beta \chi_{H_\beta}(x), \quad \tilde{\theta}(x) = \sum_{\beta \in \Gamma} d_\beta \chi_{H_\beta}(x)$$

Since $\theta(x) \le \tilde{\theta}(x)$ for every $x \in X$ and (H_β) is pairwise disjoint, we have $c_\beta \le d_\beta$ for every $\beta \in \Gamma$. Hence,

$$\int \theta \, d\mu = \sum_{\beta \in \Gamma} c_\beta \mu(H_\beta) \le \sum_{\beta \in \Gamma} d_\beta \mu(H_\beta) = \int \tilde{\theta} \, d\mu.$$

4. Let $\theta(x) = \sum_{j=1}^m c_j \chi_{E_j}(x)$. To show ρ is a measure, first,

$$\rho(\emptyset) = \int \sum_{j=1}^m c_j \chi_{E_j \cap \emptyset} \, d\mu = \int \sum_{j=1}^m c_j \chi_\emptyset \, d\mu = 0.$$

If $F_1, F_2, F_3, \cdots$ are pairwise disjoint $\mathcal{E}$-measurable sets,

$$\begin{aligned} \rho\Big(\bigcup_{k=1}^\infty F_k\Big) &= \sum_{j=1}^m c_j \mu\Big(E_j \cap \bigcup_{k=1}^\infty F_k\Big) = \sum_{j=1}^m c_j \mu\Big(\bigcup_{k=1}^\infty (E_j \cap F_k)\Big) = \sum_{j=1}^m c_j \sum_{k=1}^\infty \mu(E_j \cap F_k) \\ &= \sum_{k=1}^\infty \sum_{j=1}^m c_j \mu(E_j \cap F_k) = \sum_{k=1}^\infty \int \sum_{j=1}^m c_j \chi_{E_j \cap F_k} \, d\mu = \sum_{k=1}^\infty \rho(F_k). \end{aligned}$$

We used that series for a sequence of members in $[0, \infty]$ is independent of reordering.

$\square$

We recall the set of all measurable multiplicities

$$\Lambda^+ = \{x \mapsto \sum_{j=1}^{\infty} c_j \chi_{E_j}(x) \mid (c_j, E_j)_{j=1}^{\infty} \in \mathcal{L}^+\}.$$

Now we define the integral of $\theta \in \Lambda^+$.

Definition 7. *For $\theta \in \Lambda^+$, we define*

$$\int \theta d\mu = \sup\Big\{ \int \varphi d\mu \mid \varphi \textit{ is nonnegative simple and } 0 \le \varphi(x) \le \theta(x) \textit{ for every } x \in X \Big\}$$

Remark 8.4. The integral is well-defined:

1. The zero function is nonnegative simple and thus the set above is always nonempty.
2. For any nonempty subset $A \subset [0, \infty]$, if A is bounded above by a real number, $\sup A$ is as we know. If A is not bounded above, $\sup A = \infty$.

Remark 8.5. If θ is nonnegative simple, then the new definition coincides with the old one, because for the set

$$A = \Big\{ \int \varphi \, d\mu \mid \varphi \text{ is nonnegative simple and } 0 \le \varphi(x) \le \theta(x) \text{ for every } x \in X \Big\}$$

$A \ni \displaystyle\int \theta \, d\mu$ (in old definition) itself. This number is an upper bound of A by monotonicity proven in Theorem 6.

Remark 8.6. .

1. Note that we are maximizing the under-estimations.
2. Now, every measurable multiplicity has the integral definition with respect to μ.
3. In fact, we can give the definition for any function $f : X \to [0, \infty]$. But, we will not discuss further in this direction.

We first check the monotonicity still stands in $\Lambda^+ \supset \Lambda_0^+$.

Proposition 8. *.*

1. *If $\theta_1, \theta_2 \in \Lambda^+$ and $\theta_1 \le \theta_2$ then* $\displaystyle\int \theta_1 \, d\mu \le \int \theta_2 \, d\mu$.
2. *If $\theta \in \Lambda^+$ and $c \in [0, \infty)$ then* $\displaystyle\int c\theta \, d\mu = c \int \theta \, d\mu$.

Proof. We write for $\theta \in \Lambda^+$

$$C_\theta = \Big\{\varphi \in \mathcal{L}_0^+ \mid 0 \le \varphi \le \theta\Big\}, \quad A_\theta = \Big\{\int \varphi \, d\mu \mid \varphi \in C_\theta\Big\}.$$

1. Because $\theta_1(x) \le \theta_2(x)$ for every $x \in X$,

$$\varphi \in C_{\theta_1} \quad \Longrightarrow \quad \varphi \in C_{\theta_2} \quad \text{hence} \quad C_{\theta_1} \subset C_{\theta_2}, \quad A_{\theta_1} \subset A_{\theta_2}.$$

   Taking sup on A_{θ_1}, A_{θ_2} results in that $\displaystyle\int \theta_1 \, d\mu \le \int \theta_2 \, d\mu$.

2. If $c = 0$, the $(LHS) = (RHS) = 0$ (arithmetics always assumes $0 \times \infty = 0$.) Let $c > 0$. For $c\theta \in \Lambda_0^+$,

$$C_{c\theta} = \Big\{\varphi \in \mathcal{L}_0^+ \mid 0 \le \varphi \le c\theta\Big\} = \Big\{\varphi \in \mathcal{L}_0^+ \mid 0 \le \frac{\varphi}{c} \le \theta\Big\} = \Big\{c\phi \in \mathcal{L}_0^+ \mid 0 \le \phi \le \theta\Big\}.$$

   By Theorem 6,

$$A_{c\theta} = \{ca \mid a \in A_\theta\}.$$

   Taking sup on $A_{c\theta}$, $\displaystyle\int c\theta \, d\mu = c\int \theta \, d\mu$.

$\square$

1. The set $\Lambda^+ \supset \Lambda_0^+$ is also a convex cone, partially ordered, and $\theta\chi_E \in \Lambda^+$ if $\theta \in \Lambda^+$ and E is $\mathcal{E}$-measurable.

2. Importantly, Λ^+ is closed under countable series

$$\sum_{\alpha=1}^{\infty} \theta_\alpha = \sum_{\alpha=1}^{\infty}\sum_{j=1}^{\infty} c_{\alpha,j}\chi_{E_{\alpha,j}} \in \Lambda^+.$$

Example

Remark 8.7. In next chapter, we will also prove that Λ^+ is closed under other limit procedures of sup, inf, lim inf, lim sup, lim (if exists), for a sequence $(\theta_\alpha)_{\alpha=1}^{\infty}$. This is one of the contrasted features of new integral over the Riemann Integral.

Our primary goal was to establish the equaltiy

$$\theta = \sum_{j=1}^{\infty} c_j \chi_{E_j} \quad \Longrightarrow \quad \int \theta \, d\mu = \sum_{j=1}^{\infty} c_j \mu(E_j) \quad (c_j, E_j) \in \mathcal{L}^+.$$

Our ultimate goal is to establish that

$$\theta_1, \theta_2, \dots \in \Lambda^+ \quad \text{and} \quad \theta = \sum_{\alpha=1}^{\infty} \theta_\alpha \Longrightarrow \quad \int \theta \, d\mu = \sum_{\alpha=1}^{\infty} \int \theta_\alpha \, d\mu.$$

Theorem 9.

$$\textit{If } \theta = \sum_{j=1}^{\infty} c_j \chi_{E_j} \in \Lambda^+ \textit{ then } \quad \int \theta \, d\mu = \sum_{j=1}^{\infty} c_j \mu(E_j).$$

Proof. 1. Let $\sum_{j=1}^{\infty} c_j \chi_{E_j}(x)$ represents θ. We may assume the followings.

(a) We may discard all (c_j, E_j) where $E_j = \emptyset$. This is from that arithmetics in $[0, \infty]$ is such that $0 \times \infty = 0$.

2. Admitting this, if any of c_j is infinite, say the c_{j_0} is infinite, then

$$\Big\{ \int \varphi \, d\mu \mid \varphi \in \Lambda_0^+, \quad 0 \le \varphi \le \theta \Big\} \subset [0, \infty]$$

is unbounded, because $M\chi_{E_{j_0}} \le \theta$ for arbitrary $M > 0$. Thus

$$\int \theta \, d\mu = \sum_{j=1}^{\infty} c_j \mu(E_j) = \infty.$$

From now on, we also assume $c_j \ne \infty$ for every j.

3. Let $\theta_N(x) = \sum_{j=1}^{N} c_j \chi_{E_j}(x)$ a nonnegative simple. Since $\theta \ge \theta_N$ for any N,

$$\int \theta \, d\mu \ge \lim_{N\to\infty} \int \theta_N \, d\mu = \lim_{N\to\infty} \sum_{j=1}^{N} c_j \mu(E_j) = \sum_{j=1}^{\infty} c_j \mu(E_j).$$

4. Now we show that $\int \theta \, d\mu \le \sum_{j=1}^{\infty} c_j \mu(E_j)$.

Let φ be nonnegative and simple such that $0 \le \varphi \le \theta$.

5. Fix $0 < r < 1$ and consider $r\varphi$, a nonnegative simple function having a property:

$$\text{if } x \in \big\{ x \mid \theta(x) > 0 \big\} \quad \text{then} \quad \theta(x) > r\varphi(x). \tag{8.0.1}$$

6. For each N, two nonnegative simple functions $r\varphi$ and θ_N can be represented by common pairwise disjoint $\mathcal{E}$-measurable sets E_β with $X = \bigcup_\beta E_\beta$ as before, for instance
$$r\varphi(x) = \sum_\beta rc_\beta \chi_{E_\beta}(x), \quad \theta_N(x) = \sum_\beta d_\beta \chi_{E_\beta}(x).$$
Then
$$F_{N,r} = \bigcup_{\beta,\ rc_\beta \le d_\beta} E_\beta = \big\{x \in X \mid r\varphi(x) \le \theta_N(x)\big\},$$
and $F_{N,r}$ is $\mathcal{E}$-measurable.

7. We have that
$$\{x \in X \mid \theta(x) = 0\} \subset F_{1,r} \subset F_{2,r} \subset F_{3,r} \subset \cdots \quad .$$
Furthermore, (8.0.1) implies that
$$\bigcup_{N=1}^\infty F_{N,r} = X.$$
Indeed, if $\theta(x) = 0$, then $x \in F_{1,r}$ and if $\theta(x) > 0$ there exists some N_1 so that $\theta_{N_1}(x) > r\varphi(x)$.

(If r wasn't multiplied, (or $r = 1$) this might not be true: a possible situation is that $\varphi(x_0) = \theta(x_0) = 1 > 1 - \frac{1}{N} = \theta_N(x_0)$ for all N and this x_0 would be excluded in all of $F_{N,r}$ with $r = 1$.)

8. Hence, we have for every $0 < r < 1$ and N
$$\int_{F_{N,r}} r\varphi \, d\mu \le \int_{F_{N,r}} \theta_N \, d\mu \le \int \theta_N \, d\mu.$$

9. Taking the limit $N \to \infty$ first, we see that on (LHS)
$$\lim_{N\to\infty} \int_{F_{N,r}} r\varphi \, d\mu = \lim_{N\to\infty} \rho(F_{N,r}) = \rho\Big(\bigcup_{N=1}^\infty F_{N,r}\Big) = \rho(X) = \int r\varphi \, d\mu,$$
where ρ is the measure $E \mapsto \int_E r\varphi \, d\mu$. Hence,
$$r \int \varphi \, d\mu \le \sum_{j=1}^\infty c_j \mu(E_j).$$

10. Taking the limit $r \to 1$, we have
$$\int \varphi \, d\mu \le \sum_{j=1}^\infty c_j \mu(E_j).$$

11. Finally, take the sup over all $\varphi \in \Lambda_0^+$ with $0 \le \varphi \le \theta$ to have
$$\int \theta \, d\mu \le \sum_{j=1}^\infty c_j \mu(E_j).$$

$\square$

Theorem 10. .

1. *For* $\theta_1, \theta_2, \cdots \in \Lambda^+$, $\displaystyle\int \sum_{\alpha=1}^{\infty} \theta_\alpha \, d\mu = \sum_{\alpha=1}^{\infty} \int \theta_\alpha \, d\mu.$

2. *For* $\theta \in \Lambda^+$, *the map* $\rho : \mathcal{E} \to [0, \infty]$ *given by*

$$\rho(E) = \int_E \theta \, d\mu = \int \theta \chi_E \, d\mu \quad \textit{is a measure on } (X, \mathcal{E}).$$

Proof. 1. Let θ_α be represented by $\sum_{j=1}^{\infty} c_{\alpha,j} \chi_{E_{\alpha,j}}(x)$. Then

$$\int \sum_{\alpha=1}^{\infty} d\mu = \int \sum_{\alpha=1}^{\infty} \sum_{j=1}^{\infty} c_{\alpha,j} \chi_{E_{\alpha,j}} \, d\mu = \sum_{\alpha=1}^{\infty} \sum_{j=1}^{\infty} c_{\alpha,j} \mu(E_{\alpha,j}) = \sum_{\alpha=1}^{\infty} \int \theta \, d\mu.$$

2. Let $\theta(x) = \sum_{j=1}^{\infty} c_j \chi_{E_j}(x)$. To show ρ is a measure, first,

$$\rho(\emptyset) = \int \sum_{j=1}^{\infty} c_j \chi_{E_j \cap \emptyset} \, d\mu = \int \sum_{j=1}^{\infty} c_j \chi_\emptyset \, d\mu = 0.$$

If $F_1, F_2, F_3, \cdots$ are pairwise disjoint $\mathcal{E}$-measurable sets,

$$\begin{aligned}
\rho\Big(\bigcup_{k=1}^{\infty} F_k\Big) &= \int_{\bigcup_{k=1}^{\infty} F_k} \theta \, d\mu = \sum_{j=1}^{\infty} c_j \mu\Big(E_j \cap \bigcup_{k=1}^{\infty} F_k\Big) = \sum_{j=1}^{\infty} c_j \mu\Big(\bigcup_{k=1}^{\infty} (E_j \cap F_k)\Big) \\
&= \sum_{j=1}^{\infty} c_j \sum_{k=1}^{\infty} \mu(E_j \cap F_k) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} c_j \mu(E_j \cap F_k) = \sum_{k=1}^{\infty} \int_{F_k} \theta \, d\mu = \sum_{k=1}^{\infty} \rho(F_k).
\end{aligned}$$

□

We have completed the theory of (multiplicity, addition, integral), using the theory of (set, union, measure).

Chapter 9

$[-\infty, \infty]$-valued multiplicities

$\theta_+(x)$ and $\theta_-(x)$

Let $f : X \to [-\infty, \infty]$. We define

$$f_+(x) := \max\{0, f(x)\}$$
$$f_-(x) := -\min\{0, f(x)\}.$$

Then, for every $x \in X$,

1. One of $f_+(x)$ or $f_-(x)$ is always zero.
2. f_+, f_- are both $[0, \infty]$-valued.
3. $f(x) = f_+(x) - f_-(x)$.

Remark 9.1. .

1. Analogous definitions appeared in the previous section can be given for $[-\infty, \infty]$-valued, or $\mathbb{C}$-valued, or $\mathbb{R}^k$-valued cases, provided that the infinity values are treated correctly.
2. Importantly, in the above equality $f(x) = f_+(x) - f_-(x)$, it never is the case $\infty - \infty$, since one of the $f_+(x)$ and $f_-(x)$ is always 0.
3. Note that for $[0, \infty]$-valued functions f_1, f_2, this is not true in general.

   if $f_1(x) = f_2(x) = \infty$, $f_1(x) - f_2(x)$ is not defined.

   Similarly, the positive part of $f_1 - f_2$ is not f_1 in general.

We begin with showing the following proposition for a nonnegative valued function.

Proposition 1. *Let $\theta : X \to [0,\infty]$ such that*

$$\textit{for every } c \in [0,\infty), \quad \textit{the inverse image} \quad \{x \in X \mid \theta(x) \geq c\} \in \mathcal{E}.$$

Then, there exists $(c_j, E_j)_{j=1}^{\infty}$ of sequence in $[0,\infty] \times \mathcal{E}$ such that

$$\theta(x) = \sum_{j=1}^{\infty} c_j \chi_{E_j}(x).$$

Proof. 1. We consider the partition of the whole $[0,\infty]$

$$[0,\infty] = \{\infty\} \cup [0,1) \cup [1,2) \cup [2,3) \cup \cdots .$$

2. Define

$$\begin{aligned} \text{for } \alpha = 1,2,\cdots \quad F_\alpha &= \theta^{-1}\big([\alpha-1,\alpha)\big) \\ &= \big\{x \in X \mid \alpha - 1 \leq \theta(x)\big\} \quad \cap \quad \big\{x \in X \mid \alpha \leq \theta(x)\big\}^c \in \mathcal{E}, \\ F_0 &= \theta^{-1}\big(\{\infty\}\big) = \bigcap_{c\in\mathbb{N}} \big\{x \in X \mid \theta(x) \geq c\big\} \in \mathcal{E}. \end{aligned}$$

Because $X = \displaystyle\bigcup_{\alpha=0}^{\infty} F_\alpha$, a disjoint union, we can write

$$\theta(x) = \sum_{\alpha=0}^{\infty} \theta(x)\chi_{F_\alpha}(x) = \sum_{\alpha=0}^{\infty} \theta_\alpha(x).$$

It suffices to show that for each α, $\theta_\alpha \in \Lambda^+$.

3. Certainly, $\theta_0(x) = c_0\chi_{F_0(x)} \in \Lambda^+$, with $c_0 = \infty$.

4. Now, consider the case $\alpha = 1$, where θ_1 is valued in $[0,1)$.

5. At each depth $m = 1,2,3,\cdots$, we consider the division

$$[0,1) = I_m \cup I'_m \quad \text{of disjoint union}$$

in the following manner.

6. For $m = 1,2,3,\cdots$, we define

Γ_m to be the set of bindary m-tuples $\beta = (\beta_1,\beta_2,\cdots,\beta_m)$, $\beta_k \in \{0,1\}$.

For each $\beta \in \Gamma_m$, we let the interval $I_{m,\beta} = \Big[\displaystyle\sum_{k=1}^{m} \big(\frac{1}{2}\big)^k \beta_k,\quad \sum_{k=1}^{m} \big(\frac{1}{2}\big)^k \beta_k + \big(\frac{1}{2}\big)^m\Big)$

and this is to partition $[0,1) = \displaystyle\bigcup_{\beta\in\Gamma^m} I_{m,\beta}$, a disjoint union of length $\left(\frac{1}{2}\right)^m$. Now,

$$I_m := \bigcup_{\beta\in\Gamma_m,\quad \beta_m=1} I_{m,\beta}, \qquad I'_m := \bigcup_{\beta\in\Gamma_m,\quad \beta_m=0} I_{m,\beta}$$

For instance,

$$
\begin{aligned}
&\text{if } m=1 \quad [0,1) = \left[0, \frac{1}{2}\right) \quad \cup \quad \left[\frac{1}{2}, \frac{2}{2}\right), \\
&\text{if } m=2 \quad [0,1) = \left(\left[0, \frac{1}{4}\right) \cup \left[\frac{2}{4}, \frac{3}{4}\right)\right) \quad \cup \quad \left(\left[\frac{1}{4}, \frac{2}{4}\right) \cup \left[\frac{3}{4}, \frac{4}{4}\right)\right), \\
&\vdots
\end{aligned}
$$

This is to conduct the following: every real number in $[0,1)$ is represented by the unique right binary representation $(\beta_m)_{m=1}^{\infty}$. Numbers whose binary representation has $\beta_m = 1$ at m-th digit comprise I_m and remaining numbers comprise I'_m.

7. Let $E_m = \theta^{-1}(I_m) = \displaystyle\bigcup_{\beta \in \Gamma_m,\ \beta_m = 1} \theta^{-1}\big(I_{m,\beta}\big)$, which is $\mathcal{E}$-measurable by assumption, and that each $I_{m,\beta}$ is of the form $[a,b)$.
8. We assert that $\displaystyle\sum_{m=1}^{\infty} \left(\frac{1}{2}\right)^m \chi_{E_m(x)} = \theta_1(x)$.
9. Fix any $\bar{x}$, let $\bar{c} = \theta_1(\bar{x}) \in [0,1)$. The number c has the unique right binary representation $(\bar{\beta}_m)_{m=1}^{\infty}$,
$$\bar{c} = \sum_{m=1}^{\infty} \left(\frac{1}{2}\right)^m \bar{\beta}_m.$$
Then we observe that for every m,
$$\begin{cases} \bar{c} \in I_m & \text{iff } \bar{\beta}_m = 1 \\ \bar{c} \in I'_m & \text{iff } \bar{\beta}_m = 0 \end{cases} \quad \Longrightarrow \quad \begin{cases} \bar{x} \in E_m & \text{iff } \bar{\beta}_m = 1 \\ \bar{x} \notin E_m & \text{iff } \bar{\beta}_m = 0 \end{cases}$$
10. This implies that
$$\sum_{m=1}^{\infty} \left(\frac{1}{2}\right)^m \chi_{E_m(\bar{x})} = \sum_{m=1}^{\infty} \left(\frac{1}{2}\right)^m \bar{\beta}_m = \bar{c} = \theta_1(\bar{x}).$$
11. We can do the same for $\alpha = 2, 3, \cdots$.

□

Remark 9.2. The key part is that, the set of all real numbers in $[0,1)$ whose right binary representation's m-th digit is 1, is a countable union of intervals.

Theorem 2. *Let $\theta : X \to [-\infty,\infty]$. Then the followings are equivalent.*

(1) $\theta_+, \theta_- \in \Lambda^+$.

(2) For every $c \in (-\infty,\infty)$, $\{x \in X \mid \theta(x) > c\} \in \mathcal{E}$.

(3) For every $c \in (-\infty,\infty)$, $\{x \in X \mid \theta(x) \le c\} \in \mathcal{E}$.

(4) For every $c \in (-\infty,\infty)$, $\{x \in X \mid \theta(x) \ge c\} \in \mathcal{E}$.

(5) For every $c \in (-\infty,\infty)$, $\{x \in X \mid \theta(x) < c\} \in \mathcal{E}$.

Proof. 1. Assume (2). Then (3) is obviously true. Also,

$$\{x \in X \mid \theta(x) < c\} = \bigcup_{k=1}^{\infty} \{x \in X \mid \theta(x) \le c - \frac{1}{k}\}$$

and thus (5) is true. Then (4) is obviously true.

2. Assume (4). Then

$$\{x \in X \mid \theta(x) > c\} = \bigcup_{k=1}^{\infty} \{x \in X \mid \theta(x) \ge c + \frac{1}{k}\}$$

and thus (2) is true. Thus (2),(3),(4),(5) are equivalent.

3. Now, assume (2),(3),(4),(5). We note that for each $c \in [0,\infty)$,

$$\{x \in X \mid \theta_+(x) \ge c\} = \begin{cases} X & \text{if } c = 0 \\ \{x \in X \mid \theta(x) \ge c\} & \text{if } c > 0 \end{cases} \in \mathcal{E},$$

$$\{x \in X \mid \theta_-(x) \ge c\} = \begin{cases} X & \text{if } c = 0 \\ \{x \in X \mid \theta(x) \le -c\} & \text{if } c > 0 \end{cases} \in \mathcal{E}.$$

By Propsition 1, $\theta_+, \theta_- \in \Lambda^+$.

4. Finally, assume (1). We prove the following first. Let $\varphi \in \Lambda^+$ be represented by

$$\varphi(x) = \sum_{j=0}^{\infty} c_j \chi_{E_j(x)}.$$

We may assume that

(a) $c_0 = \infty$, and $c_j < \infty$ for all $j \ge 1$.
(b) $E_j \ne \emptyset$ for all $j \ge 1$.

in the representation. We write

$$\varphi(x) = c_0 \chi_{E_0}(x) + \sum_{j=1}^{\infty} c_j \chi_{E_j(x)},$$

$$\varphi_N(x) = \sum_{j=1}^{N} c_j \chi_{E_j(x)} \quad \text{a nonnegative simple function.}$$

5. For each N, we represent φ_N by

$$\varphi_N(x) = \sum_{\beta} c_\beta H_\beta, \quad \bigcup_{\beta} H_\beta = X \quad \text{disjoint union.}$$

6. Fix any $d \in [0,\infty)$. The set

$$F_N = \{x \mid \varphi_N(x) > d\} = \bigcup_{\beta,\ c_\beta > d} H_\beta \quad \text{is } \mathcal{E}\text{-measurable.}$$

Then,

$$\{x \mid \varphi(x) > d\} = E_0 \cup \bigcup_{N=1}^{\infty} F_N \quad \text{is } \mathcal{E}\text{-measurable.}$$

More specifically, if $x \in (LHS)$ and $x \notin E_0$, then $\lim_{N\to\infty} \varphi_N(x) = \varphi(x)$, and for some N_1, $\varphi_N(x)$ exceeds d for every $N \geq N_1$. Converse inclusion $(LHS) \supset (RHS)$ is straightforward.

7. Having established above, we show (2).

Fix $c \in (-\infty,\infty)$. We first observe that

$$\big\{x \in X \mid \theta(x) > c\big\} = \begin{cases} \big\{x \in X \mid \theta(x)_+ > c\big\} & \text{if } c > 0 \text{ or } c = 0, \\ \big\{x \in X \mid \theta(x)_- < |c|\big\} & \text{if } c < 0. \end{cases}$$

If $c \geq 0$, $\big\{x \in X \mid \theta_+(x) > c\big\}$ is $\mathcal{E}$-measurable by above argument. If $c < 0$, in similar fashion above,

$$\begin{aligned}\big\{x \in X \mid \theta_-(x) < |c|\big\} &= \bigcup_{k=1}^{\infty} \{x \in X \mid \theta_-(x) \leq |c| - \frac{1}{k}\} \\ &= \bigcup_{k=1}^{\infty} \Big(\{x \in X \mid \theta_-(x) > |c| - \frac{1}{k}\}\Big)^c \in \mathcal{E}.\end{aligned}$$

□

Definition 3. *We say $\theta : X \to [-\infty,\infty]$ is $\mathcal{E}$-measurable if*

$$\textit{for every } c \in (-\infty,\infty), \quad \{x \in X \mid \theta(x) > c\} \in \mathcal{E}.$$

We also define Λ to be the set of all $\mathcal{E}$-measurable multiplicities. Of course $\Lambda \supset \Lambda^+$.

1. We recall Λ^+ was a convex cone, closed under the series.
2. The set Λ, as it is, cannot be a vector space, since addition and subtraction may not be defined: The only allowed infinity addtion or subtraction are $\infty + \infty = \infty$, and $-\infty - \infty = -\infty$.
3. We can consider $\hat{\Lambda} \subset \Lambda$ that are $(-\infty,\infty)$-valued measurable multiplicities.
4. We show below that $\hat{\Lambda}$ is a vector space, closed under sup, inf, lim sup, and lim inf.
5. We will, however, proceed with Λ as much as possible.

Proposition 4. *If $\theta \in \Lambda$ and $c \in (-\infty,\infty)$, then $c\theta \in \Lambda$.*

Proof. If $c = 0$, $c\theta$ is a zero function in Λ. Assume $c \neq 0$. By assumption, $\theta_+, \theta_- \in \Lambda^+$. We know that $|c|\theta_+, |c|\theta_- \in \Lambda^+$.

$$\left.\begin{array}{ll} (c\theta)_\pm = |c|\theta_\pm & \text{if } c > 0 \\ (c\theta)_\pm = |c|\theta_\mp & \text{if } c < 0 \end{array}\right. \quad \in \Lambda^+.$$

□

Proposition 5. *Suppose $\theta_1, \theta_2 \in \Lambda$, and suppose further that*

$$\theta_1(x) + \theta_2(x) \textit{ is defined for every } x \in X.$$

Then $\theta_1 + \theta_2 \in \Lambda$.

Proof. 1. Fix $c \in (-\infty,\infty)$. Then $\theta_1, c - \theta_2 \in \Lambda$.

$$\{x \in X \mid \theta_1(x) > c - \theta_2(x)\} = \{x \in X \mid \theta_1(x) + \theta_2(x) \text{ is defined and } \quad \theta_1(x) + \theta_2(x) > c\}.$$

Indeed,

x is s.t.	$\theta_1(x) = \infty$	$\theta_1(x) = -\infty$	$\theta_1(x) \in (-\infty,\infty)$
$\theta_2(x) = \infty$	(1)	$\times$	(2)
$\theta_2(x) = -\infty$	$\times$	(3)	(4)
$\theta_2(x) \in (-\infty,\infty)$	(5)	(6)	(7)

The membership of x to (LHS) and to (RHS) is identical for cases (1) $\simeq$ (7).

2. Now,

$$\{x \in X \mid \theta_1(x) > c - \theta_2(x)\} = \bigcup_{q \in \mathbb{Q}} \{x \in X \mid \theta_1(x) > q\} \cap \{x \in X \mid q \geq c - \theta_2(x)\},$$

The set (RHS) is certainly $\mathcal{E}$-measurable. The set equality is because: if $x \in (LHS)$, let $q \in \mathbb{Q}$ be such that $\theta_1(x) > q \geq c - \theta_2(x)$. Converse inclusion is straightforward.

□

Proposition 6. *If $\theta, \varphi \in \Lambda$, then $\theta\varphi \in \Lambda$.*

Proof. Let $\omega = \theta\varphi$. We prove the statement by showing that

$$\omega_+ = \theta_+\varphi_+ + \theta_-\varphi_- \in \Lambda^+, \quad \omega_- = \theta_-\varphi_+ + \theta_+\varphi_- \in \Lambda^+.$$

1. Suppose $\omega(x) > 0$. Then $\omega_+(x) = \theta(x)\varphi(x)$ and $\omega_-(x) = 0$.

$$\begin{array}{lll} \omega(x) > 0 & \Longrightarrow & \text{either } [\theta(x) > 0 \text{ and } \varphi(x) > 0] \quad \text{or } [\theta(x) < 0 \text{ and } \varphi(x) < 0] \\ & \Longleftrightarrow & \text{either } [\theta_+(x) = \theta(x), \quad \theta_-(x) = 0, \quad \varphi(x)_+ = \varphi(x), \quad \text{and } \varphi_-(x) = 0] \\ & & \text{or } [\theta_-(x) = -\theta(x), \quad \theta_+(x) = 0, \quad \varphi_-(x) = -\varphi(x), \quad \text{and } \varphi_+(x) = 0] \\ & \Longrightarrow & \theta_+(x)\varphi_+(x) + \theta_-(x)\varphi_-(x) = \theta(x)\varphi(x) = w_+(x) \\ & & \theta_-(x)\varphi_+(x) + \theta_+(x)\varphi_-(x) = 0 = w_-(x). \end{array}$$

2. Suppose $\omega(x) = 0$. Then $\omega_+(x) = \omega_-(x) = 0$.

$$\begin{aligned}
\omega(x) = 0 \quad &\Longrightarrow \quad [\theta(x) = 0] \text{ or } [\varphi(x) = 0] \\
&\Longleftrightarrow \quad [\theta_+(x) = \theta_-(x) = 0] \text{ or } [\varphi_+(x) = \varphi_-(x) = 0] \\
&\Longrightarrow \quad \theta_+(x)\varphi_+(x) + \theta_-(x)\varphi_-(x) = 0 = w_+(x) = 0. \\
&\qquad\quad \theta_-(x)\varphi_+(x) + \theta_+(x)\varphi_-(x) = 0 = w_-(x).
\end{aligned}$$

3. Suppose $\omega(x) < 0$. Then $\omega_-(x) = -\theta(x)\varphi(x)$ and $\omega_+(x) = 0$.

$$\begin{aligned}
\omega(x) < 0 \quad &\Longrightarrow \quad \text{either } [\theta(x) > 0 \text{ and } \varphi(x) < 0] \quad \text{or } [\theta(x) < 0 \text{ and } \varphi(x) > 0] \\
&\Longleftrightarrow \quad \text{either } [\theta(x)_+ = \theta(x), \quad \theta_-(x) = 0, \quad \varphi_-(x) = -\varphi(x), \quad \text{and } \varphi_+(x) = 0] \\
&\qquad\quad \text{or } [\theta(x)_- = -\theta(x), \quad \theta_+(x) = 0, \quad \varphi(x)_+ = \varphi(x), \quad \text{and } \varphi_-(x) = 0] \\
&\Longrightarrow \quad \theta_+(x)\varphi_+(x) + \theta_-(x)\varphi_-(x) = 0 = w_+(x) \\
&\qquad\quad \theta_-(x)\varphi_+(x) + \theta_+(x)\varphi_-(x) = -\theta(x)\varphi(x) = w_-(x).
\end{aligned}$$

$\square$

Remark 9.3. In particular, for $E \in \mathcal{E}$ and $\theta \in \Lambda$, $\theta(x)\chi_E(x) \in \Lambda$.

closedness of Λ under pointwise limit

We recall sup and inf respectively for upper bounded and lower bounded nonempty subset of $\mathbb{R}$.

For a subset $A \subset [-\infty,\infty]$ we define

$$\sup A = \begin{cases} \sup A \;\; \text{as we know} & \text{if } A \text{ is nonempy and bounded above by a real number.} \\ -\infty & \text{if } A = \emptyset \\ +\infty & \text{otherwise.} \end{cases}$$

$$\inf A = \begin{cases} \inf A \;\; \text{as we know} & \text{if } A \text{ is nonempy and bounded below by a real number.} \\ +\infty & \text{if } A = \emptyset \\ -\infty & \text{otherwise.} \end{cases}$$

Proposition 7. *Let $\theta_1, \theta_2, \cdots \in \Lambda$. Then $\sup_\alpha \theta_\alpha$, $\inf_\alpha \theta_\alpha \in \Lambda$, where*

$$\sup_\alpha \theta(x) = \sup_\alpha \theta_\alpha(x), \quad \inf_\alpha \theta(x) = \inf_\alpha \theta_\alpha(x) \quad \textit{for each } x \in X$$

Proof. Fix $c \in (-\infty,\infty)$.

1. $\big(\sup_\alpha \theta\big)^{-1}\big([-\infty,c]\big) = \bigcap_\alpha \theta_\alpha^{-1}\big([-\infty,c]\big)$ is $\mathcal{E}$-measurable.
2. $\big(\inf_\alpha \theta\big)^{-1}\big([c,\infty]\big) = \bigcap_\alpha \theta_\alpha^{-1}\big([c,\infty]\big)$ is $\mathcal{E}$-measurable.

□

$\limsup$ and $\liminf$ of a sequence

Let (a_n) be a sequence in $[-\infty,\infty]$.

1. Define a new sequence
$$b_n = \sup_{k\geq n} a_k.$$
Then $b_1 \geq b_2 \geq b_3 \geq \cdots$
Since (b_n) is monotone, its limit exists in $[-\infty,\infty]$. The limit
$$\lim_{n\to\infty} b_n = \limsup a_n$$
Of course $\lim_{n\to\infty} b_n$ is attained as $\inf_n b_n$.
2. Define a new sequence
$$c_n = \inf_{k\geq n} a_k.$$
Then $c_1 \leq c_2 \leq c_3 \leq \cdots$
Since (c_n) is monotone, its limit exists in $[-\infty,\infty]$. The limit
$$\lim_{n\to\infty} c_n = \liminf a_n$$
Of course $\lim_{n\to\infty} c_n$ is attained as $\sup_n c_n$.

The lim sup and lim inf of a sequence in $[-\infty,\infty]$ are convenient because the two limits always exist. The limit of a sequence may not exists, on the other hand. The limit exists if and only if $\limsup_n a_n = \liminf_n a_n$.

For a given sequence (f_α) of functions valued in $[-\infty, \infty]$, we define new functions by defining

$$\text{For each } x \in X, \qquad \begin{cases} g_\alpha(x) = \sup_{k \geq \alpha} f_k(x), \\ h_\alpha(x) = \inf_{k \geq \alpha} f_k(x), \\ \limsup_\alpha f(x) = \inf_\alpha g_\alpha(x), \\ \liminf_\alpha f(x) = \sup_\alpha h_\alpha(x). \end{cases}$$

Having established the Proposition 7, and having seen above definitions, we see that if $\theta_\alpha \in \Lambda$ for $\alpha = 1, 2, \cdots$, then $\limsup_\alpha \theta_\alpha$, $\liminf_\alpha \theta_\alpha \in \Lambda$, defined pointwisely as in the above discussion.

Proposition 8. *If* $\theta_1, \theta_2, \cdots \in \Lambda$, $\limsup_\alpha \theta_\alpha, \liminf_\alpha \theta_\alpha \in \Lambda$. *If the pointwise limit* $\lim_{\alpha\to\infty} \theta_\alpha$ *exists, then* $\lim_{\alpha\to\infty} \theta_\alpha \in \Lambda$.

Proof. Done in the discussion. $\square$

Chapter 10

Integral with respect to μ

Now, we fix a measure μ.

Definition 1. *For each $\theta \in \Lambda$, we assign two numbers*

$$\Big(\int \theta_+ \, d\mu, \quad \int \theta_- \, d\mu\Big).$$

In case $\displaystyle\int \theta_+ \, d\mu - \int \theta_- \, d\mu$ is defined, i.e., it is not the case $\infty - \infty$, we define

$$\int \theta \, d\mu = \int \theta_+ \, d\mu - \int \theta_- \, d\mu.$$

For instance, for $\theta(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$, the integral is not defined.

We have been pursuing theory that allows use of infinities.

Now, let us restrict ourselves to $(-\infty, \infty)$-valued measurable multiplicities $\hat{\Lambda}$.

Exercise 2. *Repeat the same closedness in previous section for Λ^+ of $[0, \infty]$-valued multiplicities. This shows that Λ^+ is a convex cone closed under pointwise* $\sup$, $\inf$, $\limsup$, *and* $\liminf$, *in addition to the monotone limit in series.*

Exercise 3. *Repeat the same closedness in previous section for $\hat{\Lambda}$ of $(-\infty, \infty)$-valued multiplicities. This shows that $\hat{\Lambda}$ is a vector space closed under pointwise* $\sup$, $\inf$, $\limsup$, *and* $\liminf$.

The restriction to the $(-\infty, +\infty)$-valued measurable multiplicities is justified by the observations in the last part of this chapter.

Functions of finite integral.

Definition 4. *We define the set*

$$\begin{aligned} L^1(\mu) &= \{\theta \in \hat{\Lambda} \mid \int \theta \, d\mu \in (-\infty, \infty)\} \\ &= \{\theta \in \hat{\Lambda} \mid \int \theta_+ \, d\mu < \infty \quad \textit{and} \quad \int \theta_- \, d\mu < \infty\} \\ &= \{\theta \in \hat{\Lambda} \mid \int \theta_+ + \theta_- \, d\mu < \infty\} \\ &= \{\theta \in \hat{\Lambda} \mid |\theta| \in \Lambda^+ \quad \textit{with finite integral.}\} \end{aligned}$$

The function $\theta_+ + \theta_-$ is denoted by $|\theta| \in \Lambda^+$.

Having defined the $L^1(\mu)$, the set of $(-\infty, +\infty)$-valued multiplicities of finite integral, we check that $L^1(\mu)$ is structured set in the following sense:

Theorem 5. *.*

1. *If $\theta \in L^1(\mu)$ and $c \in (-\infty, \infty)$, then $c\theta \in L^1(\mu)$.*
2. *If $\theta_1, \theta_2 \in L^1(\mu)$, then $\theta_1 + \theta_2 \in L^1(\mu)$.*
3. *If $\theta \in L^1(\mu)$ and $E \in \mathcal{E}$, then $\theta\chi_E \in L^1(\mu)$.*

Proof. In item 1,2,3, the membership to $\hat{\Lambda}$ of each resultant function is checked in Exercises.

1. $|c\theta| \leq |c| \, |\theta| \in \Lambda^+$ with finite integral.
2. $|\theta_1 + \theta_2| \leq |\theta_1| + \theta_2| \in \Lambda^+$ with finite integral.
3. $|\theta\chi_E| \leq |\theta| \in \Lambda^+$ with finite integral.

□

Theorem 6. *.*

1. *If $\theta \in L^1(\mu)$ and $c \in (-\infty, \infty)$, then $\int c\theta \, d\mu = c \int \theta \, d\mu$.*
2. *If $\theta_1, \theta_2 \in L^1(\mu)$, then $\int \theta_1 + \theta_2 \, d\mu = \int \theta_1 \, d\mu + \int \theta_2 \, d\mu$.*
3. *If $\theta_1, \theta_2 \in L^1(\mu)$ with $\theta_1(x) \leq \theta_2(x)$ for every $x \in X$, then $\int \theta_1 \, d\mu \leq \int \theta_2 \, d\mu$.*
4. *If $\theta \in L^1(\mu)$ and $E \in \mathcal{E}$, then*

$$E \mapsto \int \theta\chi_E \, d\mu = \int_E \theta \, d\mu \quad \textit{is countably additive.}$$

Proof. 1. If $c = 0$, then $(LHS) = (RHS) = 0$. If $c \neq 0$,

$$\text{if } c > 0 \quad \int (c\theta)_+ \, d\mu - \int (c\theta_-) \, d\mu = \int c\theta_+ \, d\mu - \int c\theta_- \, d\mu = c\Big(\int \theta_+ \, d\mu - \int \theta_- \, d\mu \Big)$$

$$\begin{aligned} \text{if } c < 0 \quad \int (c\theta)_+ \, d\mu - \int (c\theta_-) \, d\mu &= \int |c|\theta_- \, d\mu - \int |c|\theta_+ \, d\mu = |c|\Big(\int \theta_- \, d\mu - \int \theta_+ \, d\mu \Big) \\ &= c\Big(\int \theta_+ \, d\mu - \int \theta_- \, d\mu \Big) \end{aligned}$$

2. Let $\varphi = \theta_1 + \theta_2$. Then for each $x \in X$,

$$\varphi_+(x) - \varphi_-(x) = \varphi(x) = \theta_1(x) + \theta_2(x) = \theta_{1+}(x) - \theta_{1-}(x) + \theta_{2+}(x) - \theta_{2-}(x),$$

where the equality is such that every term is finite real number. Hence,

$$\begin{aligned} &\text{for every } x \in X \quad \varphi_+(x) + \theta_{1-}(x) + \theta_{2-}(x) = \varphi_-(x) + \theta_{1+}(x) + \theta_{2+}(x) \\ &\implies \int \varphi_+ + \theta_{1-} + \theta_{2-} \, d\mu = \int \varphi_- + \theta_{1+} + \theta_{2+} \, d\mu \\ &\iff \int \varphi_+ \, d\mu + \int \theta_{1-} \, d\mu + \int \theta_{2-} \, d\mu = \int \varphi_- \, d\mu + \int \theta_{1+} \, d\mu + \int \theta_{2+} \, d\mu. \end{aligned}$$

In the last equality, every term is finite real number. Hence,

$$\int \varphi_+ \, d\mu - \int \varphi_- \, d\mu = \int \theta_{1+} \, d\mu - \int \theta_{1-} \, d\mu + \int \theta_{2+} \, d\mu - \int \theta_{1-} \, d\mu.$$

3. We have inequality

$$\text{for every } x \in X \quad \theta_{1+}(x) - \theta_{1-}(x) \leq \theta_{2+}(x) - \theta_{2-}(x),$$

where in the inequality every term is finite real number. Similarly as in item 2,

$$\begin{aligned} &\int \theta_{1+} \, d\mu + \int \theta_{2-} \, d\mu \leq \int \theta_{2+} \, d\mu + \int \theta_{1-} \, d\mu \\ \iff &\int \theta_{1+} \, d\mu - \int \theta_{1-} \, d\mu \leq \int \theta_{2+} \, d\mu - \int \theta_{2-} \, d\mu. \end{aligned}$$

4. We have that

$$\begin{aligned} \rho(E) = \int \theta\chi_E \, d\mu &= \int \theta_+\chi_E \, d\mu - \int \theta_-\chi_E \, d\mu \\ &= \rho_+(E) - \rho_-(E), \quad \rho_\pm \text{ is a measure.} \end{aligned}$$

Thus, if $E_1, E_2, \cdots$ are pairwise disjoint $\mathcal{E}$-measurable sets,

$$\rho\Big(\bigcup_{j=1}^{\infty} E_j\Big) = \int \theta\chi_{\bigcup_j E_j} \, d\mu = \rho_+\Big(\bigcup_{j=1}^{\infty} E_j\Big) - \rho_-\Big(\bigcup_{j=1}^{\infty} E_j\Big),$$

where the RHS is a difference of two finite numbers. The two numbers are limits in the sense

$$\rho_+\Big(\bigcup_{j=1}^{\infty} E_j\Big) = \sum_{j=1}^{\infty} \rho_+(E_j), \qquad \rho_-\Big(\bigcup_{j=1}^{\infty} E_j\Big) = \sum_{j=1}^{\infty} \rho_-(E_j).$$

Since $\sum_{j=1}^{N} \rho_+(E_j)$ and $\sum_{j=1}^{N} \rho_-(E_j)$ both are convergent sequences with finite limits, the limit of difference

$$\sum_{j=1}^{N} \rho_+(E_j) - \sum_{j=1}^{N} \rho_-(E_j) = \sum_{j=1}^{N} \rho_+(E_j) - \rho_-(E_j) = \sum_{j=1}^{N} \rho(E_j)$$

exists and equals to the difference of limits. In other words,

$$\rho\Big(\bigcup_{j=1}^{\infty} E_j\Big) = \sum_{j=1}^{\infty} \rho(E_j).$$

□

For a given measure μ, we develop the following notions.

Definition 7. .

1. *We say that $S \subset X$ is μ-negligible if S is a subset of some $E \in \mathcal{E}$ and $\mu(E) = 0$.*
2. *If $P(x)$ is a condition on element of X, we say $P(x)$ holds μ-almost every $x \in X$*

$$\text{if} \quad \{x \mid P(x) \text{ is false}\} \text{ is } \mu\text{-negiligible.}$$

Remark 10.1. If S is not μ-negligible and $E \supset S$ is an $\mathcal{E}$-measurable set, then $\mu(E) > 0$. In case S is $\mathcal{E}$-measurable, then $\mu(S) > 0$.

Proposition 8. *Let $\theta \in \Lambda$. Suppose $\int \theta \, d\mu$ is defined and also $\int \theta \, d\mu$ is finite. Then $\theta^{-1}\big(\{-\infty, +\infty\}\big)$ is μ-negligible.*

Proof. .

1. We recall $\int \theta \, d\mu$ is finite iff $\int \theta_+ \, d\mu < \infty$ and $\int \theta_- \, d\mu < \infty$.
2. Suppose
$$\theta^{-1}\big(\{-\infty, +\infty\}\big) = \theta^{-1}\big(\{-\infty\}\big) \,\cup\, \theta^{-1}\big(\{\infty\}\big)$$
is not μ-negligible. Then one of $\theta^{-1}\big(\{-\infty\}\big)$ or $\theta^{-1}\big(\{+\infty\}\big)$ is not μ-negligible. Let us consider the case $\theta^{-1}\big(\{+\infty\}\big)$ is not μ-negligible.
3. The set $\theta^{-1}\big(\{+\infty\}\big) \in \mathcal{E}$ then has a strictly posivive measure, and $\int \theta_+ \, d\mu = \infty$, contradiction.

□

Proposition 9. *Suppose that $\theta, \tilde{\theta} \in \Lambda$ such that both $\int \theta \, d\mu$ and $\int \tilde{\theta} \, d\mu$ are defined and both are finite.*

$$\text{If} \quad \int_E \theta \, d\mu = \int_E \tilde{\theta} \, d\mu \quad \text{for every } E \in \mathcal{E},$$

then $\theta(x) = \tilde{\theta}(x)$ for μ-almost every $x \in X$.

Proof. .

1. Suppose that
$$\{x \in X \mid \theta(x) \neq \tilde{\theta}(x)\} = \{x \in X \mid \theta(x) > \tilde{\theta}(x)\} \cup \{x \in X \mid \theta(x) < \tilde{\theta}(x)\} \in \mathcal{E}$$
is not μ-negligible. We consider the case $E_+ = \big\{x \in X \mid \theta(x) > \tilde{\theta}(x)\big\}$ is not μ-negligible.

2. We write

$$E_{+} = \bigcup_{k=1}^{\infty} \Big\{ x \in X \mid \theta(x) \geq \tilde{\theta}(x) + \frac{1}{k} \Big\}, \quad E_{k+} = \Big\{ x \in X \mid \theta(x) \geq \tilde{\theta}(x) + \frac{1}{k} \Big\},$$

all $\mathcal{E}$-measurable. At least one of E_{k+} must not be μ-negligible, say at k_0.

3. Since $F = \theta^{-1}\big(\{-\infty, +\infty\}\big) \cup \tilde{\theta}^{-1}\big(\{-\infty, +\infty\}\big) \in \mathcal{E}$ is μ-negligible, $E_{k_0+} \setminus F \in \mathcal{E}$ is not μ-negligible.

Take an $\mathcal{E}$-measurable subset $E \subset E_{k_0+} \setminus F$ with $0 < \mu(E) < \infty$. Now,

$$\begin{aligned}
\text{for every } x \in E, \quad & \theta_{+}(x) - \theta_{-}(x) = \theta(x) \geq \tilde{\theta}(x) + \frac{1}{k_0} = \tilde{\theta}_{+}(x) - \tilde{\theta}_{-}(x) + \frac{1}{k_0} \\
\iff \quad & \theta_{+}(x) + \tilde{\theta}_{-}(x) \geq \theta_{-}(x) + \tilde{\theta}_{+}(x) + \frac{1}{k_0}, \\
\implies \quad & \int_E \theta_{+}\, d\mu - \int_E \theta_{-}\, d\mu \geq \int_E \tilde{\theta}_{+}\, d\mu - \int_E \tilde{\theta}_{-}\, d\mu + \frac{1}{k_0}\mu(E) \\
\implies \quad & \int_E \theta_{+}\, d\mu - \int_E \theta_{-}\, d\mu > \int_E \tilde{\theta}_{+}\, d\mu - \int_E \tilde{\theta}_{-}\, d\mu,
\end{aligned}$$

contradicting to the assumption.

□

Remark 10.2. .

1. Proposition 9 says that $\theta, \tilde{\theta} \in \Lambda$ with finite integrals that coincide at μ-almost every $x \in X$, are not distinguishable, in terms of integral over $\mathcal{E}$-measurable sets with the measure μ.

2. Proposition 8 says that any $\tilde{\theta} \in \Lambda$ whose integral is defined and finite, it equals to some $\theta \in \hat{\Lambda}$ for μ-almost every $x \in X$.

Sometimes, we carelessly speak of integrals with the measure μ, abused in some sense, as follows.

Let us consider the set

$$\begin{aligned}
\tilde{L}^1(\mu) = \Big\{ \tilde{\theta} : & X \setminus S \to [-\infty, \infty] \mid \text{for some } \theta \in L^1(\mu) \\
& \{x \mid \tilde{\theta}(x) \text{ is not defined}\} \cup \{x \mid \theta(x) \neq \tilde{\theta}(x)\} \quad \text{is } \mu\text{-negiligible} \Big\}
\end{aligned}$$

We will carelessly speak of the integral of $\tilde{\theta} \in \tilde{L}^1(\mu)$, which is in fact to speak of integral of some $\theta \in L^1(\mu)$ such that

$$\{x \mid \tilde{\theta}(x) \text{ is not defined}\} \cup \{x \mid \theta(x) \neq \tilde{\theta}(x)\}$$

is μ-negiligible. Even if there are two such members in $L^1(\mu)$, the integrals are same.

Example

$$\tilde{\theta}(x) = \frac{1}{x^2}, \quad \text{with respect to the Lebesgue measure.}$$

Chapter 11

Towards an example of $\lambda(S_1 \cup S_2) < \lambda(S_1) + \lambda(S_2)$

Now, we answer to a few questions we left before on subsets of $\mathbb{R}^n$.

We recall the definition of outer measure λ

$$\lambda(S) = \inf_{(R_j) \text{ of } n\text{-cubes that covers } S} \Big\{ \sum_{j=1}^{\infty} |R_j| \Big\},$$

which we take as two step definition: (i) replacing S by a truely n-dimensional cover, to have an over-estimation; (ii) minimization of over-estimation.

We consider questions in $\mathbb{R}$. We have worry, for instance for the case:

1. The union $S_1 \cup S_2 = [0, 1)$ itself is a very good set,
2. while its partition into two disjoint sets S_1 and S_2 is so entangled that
3. a n-dimensional covering of S_1 and that of S_2 cannot be effectively separated and have to invade each other's territory. To put this in the other way around, if we count the cubes of covering for the union $[0, 1)$, which is for S_1, and which is for S_2, many cubes could be doubly counted, possibly resulting in

$$\lambda([0, 1)) < \lambda(S_1) + \lambda(S_2)$$

We examine a few such disjoing but entangled S_1 and S_2:

1. $S_1 = [0, 1) \cap \mathbb{Q}$ and $S_2 = [0, 1) \cap \mathbb{Q}^c$.
2. $S_1 = C \subset [0, 1)$ the Cantor set (half-open version) and $S_2 = [0, 1) \setminus S_1$.
3. $S_1 = D \subset [0, 1)$ the Fat Cantor set (half-open version) and $S_2 = [0, 1) \setminus S_2$
4. $S_1 = V \subset [0, 1)$ the Vitali set and S_2 its translation.

Remark 11.1. We proceed with the results known: Among those example, the only case $\lambda(S_1 \cup S_2) < \lambda(S_1) + \lambda(S_2)$ occurs is the case where S_1 is the Vitali set. All other sets turn out to be a Borel sets.

$S_1 = [0,1) \cap \mathbb{Q}$, $S_2 = [0,1) \cap \mathbb{Q}^c$

Although, we can show that S_1 and S_2 are borel sets, and thus

$$\lambda([0,1)) < \lambda(S_1) + \lambda(S_2)$$

will never be true, we compute the outer measures from the definition.

Proposition 1. $\lambda(S_1) = 0$ *and* $\lambda(S_2) = 1$.

Proof. We can enumerate rationals in S_1 by

$$q_1, q_2, q_3, \cdots .$$

Fix any $\epsilon > 0$. Then with $R_j = \left[q_j, q_j + \frac{\epsilon}{2^j}\right)$, (R_j) covers S_1. We compute

$$\sum_{j=1}^{\infty} |R_j| = \epsilon \sum_{j=1}^{\infty} \frac{1}{2^j} = \epsilon.$$

Therefore

$$\left\{ \sum_{j=1}^{\infty} |R_j| \;\middle|\; (R_j) \text{ of half-open intervals that covers } S \right\} \supset \{\epsilon \mid \epsilon > 0\},$$

and the infimum must be 0.

We can also cover S_2 by $[0,1)$ alone, which proves that $\lambda(S_2) \leq 1$.

Since $\lambda(S_1) + \lambda(S_2) \geq 1$, $\lambda(S_1) = 0$ and $\lambda(S_2) = 1$. □

So, we failed to construct an example where

$$\lambda([0,1)) < \lambda(S_1) + \lambda(S_2) \quad \text{occurs.}$$

We could find the very effective coverings for S_1 that are not far from being separated from S_2:

Even though there is unavoidable invasion of coverings to S_2 region, the overestimation of this part is diminished in the limit.

S_1 is the Cantor set, $S_2 = [0,1) \setminus S_1$

Definition of the Cantor set (half-open version)

The cantor set
$$C = \bigcap_{n=0}^{\infty} C_n,$$
and we define C_n for $n = 0, 1, 2, \cdots$ in the following manner.

1. At $n = 0$: $C_0 = [0,1)$. C_0 consists of $1 = 2^0$ interval of length $1 = \left(\frac{1}{3}\right)^0$.
2. At $n = 1$: Out of C_0, we subtract the mid-third interval,
$$C_1 = \left[0, \frac{1}{3}\right) \cup \left[\frac{2}{3}, \frac{3}{3}\right)$$
It is written as above as disjoint union of $2 = 2^1$ intervals of length $\left(\frac{1}{3}\right)^1$.
3. At $n = 2$: For each of intervals of C_1, we subtract the mid-third interval,
$$C_1 = \left[0, \frac{1}{9}\right) \cup \left[\frac{2}{9}, \frac{3}{9}\right) \cup \left[\frac{6}{9}, \frac{7}{9}\right) \cup \left[\frac{8}{9}, \frac{9}{9}\right)$$
It is written as above as disjoint union of $4 = 2^2$ intervals of length $\left(\frac{1}{3}\right)^2$.
4. If C_{n-1} consists of 2^{n-1} disjoint intervals of length $\left(\frac{1}{3}\right)^{n-1}$, for each of the intervals, we subtract the mid-third interval, to define C_n, consists of 2^n disjoint intervals of length $\left(\frac{1}{3}\right)^n$.

The definitions of C_n can be understood by the unique right ternary representation of $r \in [0,1)$: $r = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n \gamma_n, \quad \gamma_n \in \{0,1,2\}$.

1. $C_0 = [0,1)$.
2. C_1 consists of those numbers of C_0, whose right ternary representation's first digit is not 1.
3. C_2 consists of those numbers of C_1, whose right ternary representation's second digit is not 1.
4. C_n, consists of those numbers of C_{n-1}, whose right ternary representation's n-th digit is not 1.

We define the Cantor set as the intersection of every C_n.

Likewise $[0,1) \cap \mathbb{Q}$, the Cantor set $C \subset [0,1)$ and its complement in $[0,1)$ is *quite entangled*: for every interval (a,b) intersecting C must have element not in C.

Again, by the abstract theory of measures, we can show that C is a borel set and it is measure 0 set.

But let us directly compute $\lambda(S_1)$ from the definition.

Proposition 2. *$\lambda(S_1) = 0$ and $\lambda(S_2) = 1$.*

Proof. In fact, intervals comprising C_n covers S_1. Therefore

$$\left\{ \sum_{j=1}^{\infty} |R_j| \mid (R_j) \text{ of half-open intervals that covers } S \right\} \supset \left\{ \left(\frac{1}{3}\right)^n \times 2^n \mid n \in \mathbb{N} \right\},$$

and thus the infimum must be 0. Similarly as before, $\lambda(S_2) = 0$. □

So, we failed again to construct an example where

$$\lambda([0,1)) < \lambda(S_1) + \lambda(S_2) \quad \text{occurs.}$$

We could find the very effective coverings for S_1 that are not far from being separated from S_2: The covering by intervals comprising C_n themselves.

Even though there is unavoidable invasion of coverings to S_2 region, the overestimation of this part is diminished in the limit.

One may think we need to take an example where the outer measure of S_1 is non-zero.

S_1 is the Fat Cantor set, $S_2 = [0,1) \setminus S_1$

Definition of the Fat Cantor set (half-open version)

We can construct a likewise set with a positive measure, that is the Fat Cantor set.

$$D = \bigcap_{n=0}^{\infty} D_n,$$

and we define D_n for $n = 0, 1, 2, \cdots$ in the following manner.

1. At $n = 0$: $D_0 = [0, 1)$. D_0 is of $1 = 2^0$ interval of length 1.
2. At $n = 1$: Out of D_0, we subtract the middle $\left(\frac{1}{4}\right)^1$ interval, which is a disjoint union of $2 = 2^1$ intervals.
3. At $n = 2$: For each of intervals of D_1, we subtract the middle $\left(\frac{1}{4}\right)^2$ interval, which is a disjoint union of $2 = 2^2$ intervals.
4. If D_{n-1} consists of 2^{n-1} disjoint intervals, for each of the intervals, we subtract the middle $\left(\frac{1}{4}\right)^n$ interval, to define D_n, consists of 2^n disjoint intervals.

One notices that we can also find very effective coverings of S_1 too, as in the previous case. We check this:

Proposition 3. $\lambda(S_1) = \frac{1}{2}$ *and* $\lambda(S_2) = \frac{1}{2}$.

Proof. .

1. We again take the covering of S_1, those intervals comprising D_n themselves. We notice that at n-th level, the length of the interval is computed as

$$\ell_0 = 1$$
$$\ell_1 = \frac{\ell_0 - \frac{1}{4}}{2}$$
$$\ell_n = \frac{\ell_{n-1} - \left(\frac{1}{4}\right)^n}{2}$$

   Hence, the total sum $\sigma_n = \sum_{j=1}^{2^n} |R_j|$ is computed by

$$\sigma_0 = \ell_0 2^0 = 1,$$
$$\sigma_n = \ell_n 2^n = \frac{\ell_{n-1} - \left(\frac{1}{4}\right)^n}{2} 2^n = \ell_{n-1} 2^{n-1} - \frac{1}{2^{n+1}} = \sigma_{n-1} - \frac{1}{2^{n+1}}.$$

   Therefore

$$\left\{ \sum_{j=1}^{\infty} |R_j| \mid (R_j) \text{ of half-open intervals that covers } S \right\} \supset \left\{ \sigma_n \mid n \in \mathbb{N} \right\},$$

   and thus the infimum $\lambda(S_1) \leq \frac{1}{2}$.

2. Now we compute $\lambda(S_2)$. We write

$$S_2 = \bigcup_{n=1}^{\infty} D'_n,$$

where D'_n is the what is newly removed at n-th stage. Here, the coverings obtained from intervals of D'_n for all n itself is a countable covering of S_2. The total sum

$$\sum_{n=1}^{\infty} \sum_{j=1}^{2^{n-1}} |R_j| = \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n \times 2^{n-1} = \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} = \frac{1}{2}.$$

Hence, we again conclude that the infimum $\lambda(S_2) \le \frac{1}{2}$.

3. Since $\lambda(S_1) + \lambda(S_2) \ge 1$, we must have

$$\lambda(S_1) = \lambda(S_2) = \frac{1}{2}.$$

$\square$

So, we failed again to construct an example where

$$\lambda([0,1)) < \lambda(S_1) + \lambda(S_2) \quad \text{occurs.}$$

We could find the very effective coverings for S_1 that are not far from being separated from S_2: The covering by intervals of D_n themselves. We could find the very effective coverings of S_2 that are separated from S_1:

Even though there is unavoidable invasion of coverings to S_2 region, the overestimation of this part is diminished in the limit.

Now, we come to the Vitali set.

S_1 is the Vitali set

Definition of the Vitali set

We begin with partitioning $\mathbb{R}$ into uncountably many disjoint copies of $\mathbb{Q}$.

1. For $r, t \in \mathbb{R}$, we define

$$r \sim t \quad \text{if } r = t + q \text{ for some } q \in \mathbb{Q}.$$

   This defines an equivalence relation, namely

$$\begin{aligned} r \sim r \qquad & \text{for every } r \in \mathbb{R} \\ r \sim t \quad \Longrightarrow \quad & t \sim r \\ r \sim s, \quad s \sim t \quad \Longrightarrow \quad & r \sim t. \end{aligned}$$

   and $\mathbb{R}$ is a disjoint union of all (uncountably many) equivalence classes.

2. Examples of equivalance classes:
   (a) For any $q \in \mathbb{Q}$, $[q] = \mathbb{Q}$.
   (b) We can think of $[\pi]$, $\left[\sqrt{2}\right]$ for instances.
3. Obviously, every class intersects $[0, 1]$.
4. We make exactly one choice for every equivalance classes in $[0, 1]$, which shows the existence of the set $V \subset [0, 1]$.

Now, we parition $\mathbb{R}$ into countably many disjoint copies of V. Define

$$V_q = V + q = \{v + q \mid v \in V\}, \quad q \in \mathbb{Q}.$$

claim: $(V_q)_{q\in\mathbb{Q}}$ is pairwise disjoint

Suppose $p, q \in \mathbb{Q}$ and $V_p \cap V_q \neq \emptyset$. Let $x \in V_p \cap V_q$. If so,

$$x - q \in V \quad \text{and} \quad x - p \in V.$$

The element $x - q$ and $x - p$ must be same since $x - q \sim x - p$, $(x - q = x - p + (p - q))$ and V has only one element of each class. Hence $p = q$.

claim: $\bigcup_{q\in\mathbb{Q}} V_q = \mathbb{R}$

Let $x \in \mathbb{R}$. Then there exist $\hat{x} \in V$ with $\hat{x} = x + q$ for some $q \in \mathbb{Q}$.

Proposition 4. $\lambda(V) > 0$

Proof. Suppose not, i.e., $\lambda(V) = 0$. Then $\lambda(V) = \lambda(V_q) = 0$ for every $q \in \mathbb{Q}$.

$$\mathbb{R} = \bigcup_{q\in\mathbb{Q}} V_q \quad \Longrightarrow \quad \lambda(\mathbb{R}) \leq \sum_{q\in\mathbb{Q}} \lambda(V_q) = 0 \quad \text{contradiction.}$$

□

Proposition 5. *There exists $N \in \mathbb{N}$ such that for any distinct $q_1, q_2, \cdots, q_N \in [0,1]\cap\mathbb{Q}$*

$$\lambda\Big(\bigcup_{j=1}^{N} V_{q_j}\Big) \ < \ \sum_{j=1}^{N} \lambda(V_{q_j}).$$

Proof. Since $\lambda(V) > 0$, we can choose N so that $N \times \lambda(V) > 3$. Take any distinct $q_1, q_2, \cdots, q_N \in [0,1] \cap \mathbb{Q}$. Note that

$$\bigcup_{j=1}^{N} V_{q_j} \subset [0,2] \quad \text{and hence} \quad \lambda\Big(\bigcup_{j=1}^{N} V_{q_j}\Big) \leq \lambda([0,2]) \leq 2.$$

On the other hand,

$$\sum_{j=1}^{N} \lambda(V_{q_j}) = N\lambda(V) > 3.$$

□

1. The set V clearly cannot be λ-separating, and thus this gives the example showing that
$$\mathcal{E}^{\lambda}(\mathbb{R}) \subsetneq \mathcal{P}(\mathbb{R}).$$

2. The Cantor set C gives rise to the following conclusion
$$\mathcal{B}(\mathbb{R}) \subsetneq \mathcal{E}^{\lambda}(\mathbb{R}).$$

   This is justified by checking the followings. Because its exposition needs the notion of cardinality, we briefly mention the arguments:
$$\mathsf{card}(\mathcal{B}(\mathbb{R})) = \mathsf{card}(\mathbb{R}) = \mathsf{card}([0,1)) = \mathsf{card}(C) < \mathsf{card}(\mathcal{P}(C)) \leq \mathsf{card}(\mathcal{E}^{\lambda}(\mathbb{R})).$$

   (a) One can show that the cardinality of $\mathcal{B}(\mathbb{R})$ equals to the cardinality of $\mathbb{R}$, by the induction argument.

   (b) One can show that C is an uncountable set with the Lebesgue measure zero.

   The argument is the following. We characterize C as the set of numbers in $[0,1)$ whose right ternary representation have only values in $\{0,2\}$. On the other hand, every numbers in $[0,1)$ has the unique right binary representation, and thus each binary representation corresponds to a ternary representation where 1 is replaced by 2. This defines an injective map from $[0,1)$ to C, and thus C is uncountable. More specifically, the cardinality of C equals to the cardinality of $\mathbb{R}$.

   (c) Thus, the collection of all subsets of C, $\mathcal{P}(C)$ has cardinality strictly greater than that of $\mathbb{R}$. All of them have the outer measure zero.

   (d) Since $\mathcal{E}^{\lambda}(\mathbb{R})$ contains every set S with the outer measure $\lambda(S) = 0$, we conclude that $\mathcal{E}^{\lambda}(\mathbb{R})$ has the cardinality strictly greater than that of $\mathbb{R}$.

Chapter 12

A few earlier families of $\mathcal{G}_0 \subset \mathcal{G}_1 \subset \cdots \subset \mathcal{B}(\mathbb{R}^2)$

Although $\mathcal{E}^\lambda(\mathbb{R}^2)$, and $\mathcal{B}(\mathbb{R})$ as well, contains lots of subsets of $\mathbb{R}^2$, due to the construction of the Lebesgue measure out of rectangles, sets in the family $\mathcal{G}_0 \subset \mathcal{G}_1 \subset \mathcal{G}_2 \subset \mathcal{G}_3$ would do the most jobs in the following sense.

Theorem 1. *Suppose $E \subset \mathbb{R}^2$ is Lebesgue measurable. For each $\epsilon > 0$, there exists a covering (R_k) such that $\lambda\Big(\bigcup_{k=1}^{\infty} R_k \setminus E\Big) < \epsilon$.*

Proof. .

1. Let $\mathbb{R}^2 = \bigcup_{\alpha=1}^{\infty} H_\alpha$ a disjoint countable union of integer translations of $[0,1) \times [0,1)$.

2. Write $E = \bigcup_{\alpha=1}^{\infty} H_\alpha \cap E$ a disjoint union and let $E_\alpha = H_\alpha \cap E$. $\lambda(E_\alpha) < \infty$.

3. For the given $\epsilon > 0$, there exists a covering $(Q_{\alpha,j})$ of E_α such that

$$\lambda(E_\alpha) \le \lambda\Big(\bigcup_{j=1}^{\infty} Q_{\alpha,j}\Big) \le \sum_{j=1}^{\infty} \lambda(Q_{\alpha,j}) \le \lambda(E_\alpha) + \frac{\epsilon}{2^\alpha}.$$

   We may assume $Q_{\alpha,j} \subset H_\alpha$, if not, we replace it by $Q_{\alpha,j} \cap H_\alpha$.

4. Because E_α is Lebesgue measurable and $\lambda(E_\alpha) < \infty$,

$$\lambda\Big(\bigcup_{j=1}^{\infty} Q_{\alpha,j}\Big) = \lambda(E_\alpha) + \lambda\Big(\bigcup_{j=1}^{\infty} Q_{\alpha,j} \ \setminus \ E_\alpha\Big)$$
$$\implies \lambda\Big(\bigcup_{j=1}^{\infty} Q_{\alpha,j} \ \setminus \ E_\alpha\Big) = \lambda\Big(\bigcup_{j=1}^{\infty} Q_{\alpha,j}\Big) - \lambda(E_\alpha) \le \frac{\epsilon}{2^\alpha}.$$

5. Therefore,

$$\lambda\Big(\bigcup_{\alpha=1}^{\infty}\bigcup_{j=1}^{\infty} Q_{\alpha,j} \ \setminus \ E\Big) = \lambda\Big(\bigcup_{\alpha=1}^{\infty}\Big(\bigcup_{j=1}^{\infty} Q_{\alpha,j} \ \setminus \ E_\alpha\Big)\Big)$$
$$\leq \sum_{\alpha=1}^{\infty}\lambda\Big(\bigcup_{j=1}^{\infty} Q_{\alpha,j} \ \setminus \ E_\alpha\Big) \leq \epsilon.$$

□

Theorem 2. *Suppose $E \subset \mathbb{R}^2$ is Lebesgue measurable and $\lambda(E) < \infty$. For every $\epsilon > 0$, there exists a finite union of rectangles $A = \bigcup_{k=1}^{N} R_k$ such that*

$$\lambda(A \setminus E) + \lambda(E \setminus A) \leq \epsilon.$$

Proof. .

1. For the given $\epsilon > 0$, by Theorem 1, there exists (R_k) that covers E such that

$$\lambda\Big(\bigcup_{k=1}^{\infty} R_k\Big) \leq \sum_{k=1}^{\infty}\lambda(R_k) \leq \lambda(E) + \frac{\epsilon}{2}, \quad \lambda\Big(\bigcup_{k=1}^{\infty} R_k \setminus E\Big) \leq \frac{\epsilon}{2}.$$

Hence,
$$\lambda\Big(\bigcup_{k=1}^{N} R_k \setminus E\Big) \leq \Big(\bigcup_{k=1}^{\infty} R_k \setminus E\Big) \leq \frac{\epsilon}{2} \quad \text{for any } N.$$
We choose N later.

2. By continuity of the measure on the increasing sequence of $\mathcal{E}$-measurable sets,

$$\lambda\Big(\bigcup_{k=1}^{\infty} R_k\Big) = \lim_{N\to\infty}\lambda\Big(\bigcup_{k=1}^{N} R_k\Big).$$

Since the limit exists as a finite number, there exists N_1 such that

$$\lambda\Big(\bigcup_{k=1}^{\infty} R_k\Big) - \lambda\Big(\bigcup_{k'=1}^{N_1} R_{k'}\Big) \leq \frac{\epsilon}{2}$$

and this implies
$$\lambda\Big(\big(\bigcup_{k=1}^{\infty} R_k\big) \setminus \big(\bigcup_{k'=1}^{N_1} R_{k'}\big)\Big) \leq \frac{\epsilon}{2}.$$

Hence,
$$\lambda\Big(E \setminus \big(\bigcup_{k'=1}^{N_1} R_{k'}\big)\Big) \leq \lambda\Big(\big(\bigcup_{k=1}^{\infty} R_k\big) \setminus \big(\bigcup_{k'=1}^{N_1} R_{k'}\big)\Big) \leq \frac{\epsilon}{2}.$$

□

Theorem 3. *Suppose $E \subset \mathbb{R}^2$ is Lebesgue measurable. For every $\epsilon > 0$, there exists a set $K \subset E$, a members of $\mathcal{G}_2$ such that $\lambda(E \setminus K) < \epsilon$.*

Proof. .

1. E^c is Lebesgue measurable. By Theorem 1, there exists a set $G \supset E^c$, a member of $\mathcal{G}_1$ such that
$$\lambda\Big(G \setminus E^c\Big) < \epsilon.$$
2. Note that $G \setminus E^c = G \cap E = E \setminus G^c$ and $G^c \in \mathcal{G}_2$.
3. Also $G^c \subset E$.

□

Remark 12.1. By modifying the proof of Theorem 1, we can take $\bigcup_{k=1}^{\infty} R_k = U$ to be the union of open rectangles, and U to be open. (enlarge slightly the rectangle $Q_{\alpha,j}$ to an open rectangle $\tilde{Q}_{\alpha,j}$). Consequently, in Theorem 3 we can take K to be a closed set.

Remark 12.2. The membership checking for $x \in \mathbb{R}^2$ with respect to the set $A = \bigcup_{k=1}^{N} R_k$ can be done in finite procedure.

Exercise 4. *Prove the statement: Let $S \subset X$. Suppose that for each $\epsilon > 0$, there exists a covering (R_k) such that $\lambda\Big(\bigcup_{k=1}^{\infty} R_k \setminus S\Big) < \epsilon$. Then S is Lebesgue measurable.*

Remark 12.3. That is to say, Lebesgue measurable sets are precisely those sets that invasion of $\bigcup_{k=1}^{\infty} R_k$ into S^c can be made arbitrarily small.

Exercise 5. *Prove the statement: Suppose $E \subset \mathbb{R}^2$ is Lebesgue measurable. Then there exists $G \supset E$ and $K \subset E$, members of $\mathcal{G}_3$ such that*

$$\lambda(E) = \lambda(G) = \lambda(K) \quad \textit{and} \quad \lambda(G \setminus E) = \lambda(E \setminus K) = 0.$$

Chapter 13

Take home Exam

In $\mathbb{R}$, we define the Stieltjes measure μ.

1. Let $F : \mathbb{R} \mapsto \mathbb{R}$ be a monotone increasing function.
2. Let $\mathcal{J}$ be the collection of half-open intervals, as in our class.
3. Let $J \in \mathcal{J}$. We define

$$\begin{array}{rll} (i) & |J| = \infty & \text{if } J \text{ is unbounded} \\ (ii) & |J| = F(b-) - F(a-) & \text{if } J \text{ is bounded, nonempty, and } J = [a, b) \\ \Big(\text{or} \quad (ii)' & |J| = F(b) - F(a) & \text{and assume } F \text{ is left continuous.}\Big) \\ (iii) & |J| = 0 & \text{if } J = \emptyset. \end{array}$$

P 1 (20 pts). Let $J \in \mathcal{J}$ and $J_k \in \mathcal{J}$ for $k = 1, 2, \cdots, m$. Show that if $J \subset \bigcup_{k=1}^{m} J_k$ then

$$|J| \leq \sum_{k=1} |J_k|.$$

P 2 (20 pts). Let $J \in \mathcal{J}$ and $J_k \in \mathcal{J}$ for $k = 1, 2, \cdots, m$. Show that if $J = \bigcup_{k=1}^{m} J_k$ of disjoint union, then

$$|J| = \sum_{k=1} |J_k|.$$

P 3 (30 pts). For any $S \subset \mathbb{R}$, define

$$\mu(S) = \inf_{(J_k) \text{ of } \mathcal{J} \text{ that covers } S} \Big\{ \sum_{k=1}^{\infty} |J_k| \Big\}.$$

Show that $\mu(S)$ is well-defined for every $S \subset \mathbb{R}$ and μ is an outer measure on $\mathbb{R}$.

P 4 (40 pts). Show that for $J \in \mathcal{J}$, $\mu(S) = |S|$.

P 5 (40 pts). Show that for $J \in \mathcal{J}$, J is μ-separating.

P 6 (20 pts). Let $E_1, E_2, \cdots$ are μ-separating, pairwise disjoint, and $S_k \subset E_k$ for each k. Show that

$$\mu\Big(\bigcup_{k=1}^{\infty} S_k\Big) = \sum_{k=1}^{\infty} \mu(S_k).$$

---

1. $(X, \mathcal{E}, \mu)$ is a measure space.

P 7 (30 pts). Let $\theta(x) = \sum_{j=1}^{m} c_j \chi_{E_j(x)}$ and $\varphi(x) = \sum_{k=1}^{m} d_k \chi_{F_k(x)}$ be two nonnegative and simple functions. Show that

$$\{x \in X \mid \theta(x) \leq \varphi(x)\}$$

is $\mathcal{E}$-measurable.

P 8 (30 pts). Using Theorem 10 in p.58 and Proposition 5 in p.64, prove the Monotone Convergence Theorem:

Let $\theta_1, \theta_2, \cdots \in \Lambda^+$ such that for every $x \in X$

$$\theta_1(x) \leq \theta_2(x) \leq \cdots$$

Then

$$\int \lim_{\alpha\to\infty} \theta_\alpha \, d\mu = \lim_{\alpha\to\infty} \int \theta_\alpha \, d\mu.$$

P 9 (20 pts). Let C be the Cantor set. Show that there exists no open set contained in C.

P 10 (30 pts). Let D be the Fat Cantor set. Show that $\chi_D(x)$ is not Riemann Integrable.

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Note link :
https://github.com/cebumactan/ming-lee/blob/master/materials/real_analysis_2025.pdf
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