Chapter 1

$\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$

The set $\mathbb{R}^n$

- $\mathbb{R}$ is the set of all real numbers.
- Let n be a positive integer.
- We can write $(x_1, x_2, \cdots, x_n)$ of real numbers, which is an ordered n-tuple of real numbers.
- $\mathbb{R}^n$ is the set of all ordered n-tuples of real numbers.
- An element $x \in \mathbb{R}^n$ can be written in

  $$\text{row} \quad x = (x_1, x_2, x_3, \cdots, x_n) \quad \text{or in column} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix}, \quad \text{or any other form.}$$

  As long as the order of listed n real numbers are seen, there would not be a problem.

Addition and Scalar multiplication

Addition

1. If $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ and $y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$, we can add the two to obtain an element in $\mathbb{R}^n$:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \\ \vdots \\ x_n + y_n \end{pmatrix},$$

which is denoted by $x + y \in \mathbb{R}^n$.

Scalar multiplication

1. In this course, the scalar is a synonym of real number.

2. If λ is a real number and $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, then we can scale x to obtain λx:

$$\lambda \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \\ \vdots \\ \lambda x_n \end{pmatrix} \in \mathbb{R}^n.$$

The set $\mathbb{R}^{m\times n}$

- Let $\mathbb{R}$ be again the set of all real numbers.
- Let m and n be two positive integers.
- The set of all $(A_{ij})_{i=1,j=1}^{i=m,j=n}$ of real numbers indexed by $(i,j) \in \{1,2,\cdots,m\} \times \{1,2,\cdots,n\}$ is denoted by $\mathbb{R}^{m\times n}$.
- The notation for an element $A \in \mathbb{R}^{m\times n}$ in this time is more specific. We distinguish two indices $i = 1,2,\cdots,m$ and $j = 1,2,\cdots,n$. We list mn real numbers in a box so that i is a row index, and j is a column index:

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1n} \\ A_{21} & A_{22} & A_{23} & \cdots & A_{2n} \\ A_{31} & A_{32} & A_{33} & \cdots & A_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{m1} & A_{m2} & A_{m3} & \cdots & A_{mn} \end{pmatrix}.$$

- We call an element of $\mathbb{R}^{m\times n}$ an $(m \times n)$ matrix, which reads as "m by n matrix".
- Why ..?

Addition and Scalar multiplication

Addition

1. If $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{m\times n}$, i.e., if the matrix A and B are in same shape, we can add the two to obtain an element in $\mathbb{R}^{m\times n}$:

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1n} \\ A_{21} & A_{22} & A_{23} & \cdots & A_{2n} \\ A_{31} & A_{32} & A_{33} & \cdots & A_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{m1} & A_{m2} & A_{m3} & \cdots & A_{mn} \end{pmatrix} + \begin{pmatrix} B_{11} & B_{12} & B_{13} & \cdots & B_{1n} \\ B_{21} & B_{22} & B_{23} & \cdots & B_{2n} \\ B_{31} & B_{32} & B_{33} & \cdots & B_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ B_{m1} & B_{m2} & B_{m3} & \cdots & B_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} A_{11}+B_{11} & A_{12}+B_{12} & A_{13}+B_{13} & \cdots & A_{1n}+B_{1n} \\ A_{21}+B_{21} & A_{22}+B_{22} & A_{23}+B_{23} & \cdots & A_{2n}+B_{2n} \\ A_{31}+B_{31} & A_{32}+B_{32} & A_{33}+B_{33} & \cdots & A_{3n}+B_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{m1}+B_{m1} & A_{m2}+B_{m2} & A_{m3}+B_{m3} & \cdots & A_{mn}+B_{mn} \end{pmatrix},$$

which is denoted by $A + B \in \mathbb{R}^{m\times n}$.

Scalar multiplication

1. If λ is a real number and $A \in \mathbb{R}^{m\times n}$, then we can scale A to obtain λA:

$$\lambda \begin{pmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1n} \\ A_{21} & A_{22} & A_{23} & \cdots & A_{2n} \\ A_{31} & A_{32} & A_{33} & \cdots & A_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{m1} & A_{m2} & A_{m3} & \cdots & A_{mn} \end{pmatrix} = \begin{pmatrix} \lambda A_{11} & \lambda A_{12} & \lambda A_{13} & \cdots & \lambda A_{1n} \\ \lambda A_{21} & \lambda A_{22} & \lambda A_{23} & \cdots & \lambda A_{2n} \\ \lambda A_{31} & \lambda A_{32} & \lambda A_{33} & \cdots & \lambda A_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda A_{m1} & \lambda A_{m2} & \lambda A_{m3} & \cdots & \lambda A_{mn} \end{pmatrix}.$$

In summary,

1. We have the set $\mathbb{R}^n$, equipped with the addition and the scalar multiplication, that we denote by $\big(\mathbb{R}^n, +, s\big)$.

$$
\begin{aligned}
+ &: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \\
s &: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n.
\end{aligned}
$$

2. We havethe set $\mathbb{R}^{m\times n}$, equipped with the addition and the scalar multiplication, that we denote by $\big(\mathbb{R}^{m\times n}, +, s\big)$.

$$
\begin{aligned}
+ &: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \\
s &: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n.
\end{aligned}
$$

1. That we work with $\big(\mathbb{R}^n, +, s\big)$, i.e., that $\mathbb{R}^n$ are equipped with the addition and the scalar multiplication may, more importantly, mean that we do not perform other operations.

   These are illegal expressions:

   (a) $\mathbb{R}^n + \mathbb{R}$, $n \geq 2$:

   (b) $\mathbb{R}^n + \mathbb{R}^m$, $n \neq m$.

   (c) product in general. $\times : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, though in some dimensions we can define meaningful product.

   (d) comparison in general.

2. Likewise, for the case of $\mathbb{R}^{m\times n}$, we do not perform other operations, unless otherwise defined later of this course.

- We have so far
$$(\mathbb{R}^n, +, s) \quad \text{and} \quad (\mathbb{R}^{m\times n}, +, s),$$
respectively a set equipped with the addition and scalar multiplication.

- Expression that makes use of the addition and scalar multiplication, and use only of them such as
$$\lambda_1 x + \lambda_2 y + \lambda_3 z + \lambda_4 w, \quad x, y, z, w \in \mathbb{R}^n$$
comes as an important expression to us.

- The expression is called a "Linear Combination" of x, y, z, w; Linear, meaning that the scalar multiplication $x \mapsto \lambda x$ is to make something proportional, and Combination, meaning here we are adding things.

The final comment on $(\mathbb{R}^n, +, s)$ and $(\mathbb{R}^{m\times n}, +, x)$:

As stated earlier, in some dimensions we may additionally define product, for example in $\mathbb{R}^2$ we can define
$$(x_1, x_2) \times (y_1, y_2) = (v_1, v_2), \quad \text{where} \quad v_1 + iv_2 = (x_1 + ix_2)(y_1 + iy_2).$$

However, all of such operations are treated in this course to be exceptional and come as addendum, emphasizing that there are only (i) the addition and (ii) the scalar multiplication we are allowed to operate so far.

Chapter 2

Matrix Matrix multiplication

- Emphasizing that there are no other general product operations, now we define the unique product operation between matrices.
- For given matrix $A \in \mathbb{R}^{\ell \times m}$ and matrix $B \in \mathbb{R}^{m \times n}$, we define the matrix $C \in \mathbb{R}^{\ell \times n}$ to be the product denoted by AB so that

$$C_{ij} = \sum_{\alpha=1}^{m} A_{i\alpha} B_{\alpha j} \quad \text{for } i = 1, 2, \cdots, \ell \text{ and } j = 1, 2, \cdots, n.$$

- Importantly, the product is defined only for the case where the second component of shape of A and the first component of shape of B are the same, and the product is not defined for all remaining cases.
- Why .. ?

The defined product can be regarded in a few different ways.

1. The product can be regarded as to build new n linear combinations of columns of A, as designated by numbers in B:

2. The product can be regarded as to build new ℓ linear combinations of rows of B, as designated by numbers in A:

3. The product can be regarded as to build new ℓn real numbers, out of rows of A and columns of B.

The product is useful in many places.

Perfect for the chain rule in multivariable calculus

Suppose that $X = \mathbb{R}^\ell$, $Y = \mathbb{R}^m$, and $Z = \mathbb{R}^n$, and

$$f : X \to Y, \quad g : Y \to Z.$$

Multivariable Calculus:

1. Also assume f and g are many times diffentiable functions.
2. For a given point $\bar{x}$, we collect numbers
$$B_{\alpha j} = \frac{\partial f^\alpha}{\partial x^j}(\bar{x}) = \lim_{h\to 0} \frac{f^\alpha(\bar{x} + he_j) - f^\alpha(\bar{x})}{h}.$$
3. Similarly, for a given poit $\bar{y}$, we collect numbers
$$A_{i\alpha} = \frac{\partial g^i}{\partial y^\alpha}(\bar{y}) = \lim_{h\to 0} \frac{g^i(\bar{y} + he_\alpha) - g^i(\bar{y})}{h}.$$
4. Now we consider the composition
$$g \circ f : X \to Z, \quad g \circ f(x) = g(f(x))$$
and collect numbers
$$C_{ij} = \frac{\partial (g\circ f)^i}{\partial x^j}(\bar{x}) = \lim_{h\to 0} \frac{(g\circ f)^i(\bar{x} + he_j) - (g\circ f)^i(\bar{x})}{h}.$$
5. Then C turns out to equal to AB, for $\bar{y} = f(\bar{x})$.

Writing a system of linear equations

If we are given

$$\begin{cases} 3x - 7y + 4z = -2, \\ 9x - 2y - 6z = 0, \\ -5x + 3y - 11z = -8 \end{cases}$$

It is nice that we can write also

$$\begin{pmatrix} 3 & -7 & 4 \\ 9 & -2 & -6 \\ -5 & 3 & -11 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ -8 \end{pmatrix}$$

Using $\ell = 3$, $m = 3$, and $n = 1$ in the left-hand-side.

The question of why..?

1. Why does it have to be in that way?
2. Why not we define for $A, B \in \mathbb{R}^{m\times n}$ a product with the same shape in $\mathbb{R}^{m\times n}$?

$$\begin{pmatrix} A_{11}B_{11} & A_{12}B_{12} & A_{13}B_{13} & \cdots & A_{1n}B_{1n} \\ A_{21}B_{21} & A_{22}B_{22} & A_{23}B_{23} & \cdots & A_{2n}B_{2n} \\ A_{31}B_{31} & A_{32}B_{32} & A_{33}B_{33} & \cdots & A_{3n}B_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{m1}B_{m1} & A_{m2}B_{m2} & A_{m3}B_{m3} & \cdots & A_{mn}B_{mn} \end{pmatrix}$$

   (a) Actually you can. In most of computer language, this entry-wise product in the same shape is provided by the broadcasting.

3. All why-questions come to the following one,

   Would other people be interested in the new definition ?

A few important observations

1. Important: You don't mess up with the order of multiplication here.
2. Unlike multiplication we know for two real numbers, for the Matrix-Matrix multiplication for $A \in \mathbb{R}^{\ell \times m}$ and $B \in \mathbb{R}^{m \times n}$, the product BA is not even defined in general. This is simply because in general $n \neq \ell$.
3. In the special case of that $\ell = m = n$, BA is defined. However even in such a case

$$AB \neq BA \quad \text{in general.}$$

Summary

1. We have $\left(\mathbb{R}^{m \times n}, +, s\right)$ for m and n positive integers.
2. Linear combinations of members of $\mathbb{R}^{m \times n}$ are expressions using $+$ and s such as

$$\lambda_1 A_1 + \lambda_2 A_2 + \cdots \lambda_j A_j \in \mathbb{R}^{m \times n}.$$

3. We have a definition of multiplication for $A \in \mathbb{R}^{\ell \times m}$ and matrix $B \in \mathbb{R}^{m \times n}$ resulting in $AB \in \mathbb{R}^{\ell \times n}$.

Knowledge so far will be applied to mathematics, science, and engineering, and will be extremely important and powerful.

Chapter 3

Linear Independence and Basis

Let us consider elements in $\mathbb{R}^2$,

$$x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \quad z = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

It is easy to notice that if we are with x and y in our hands, z can be reproduced by

$$z = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix},$$

a linear combination of x and y.

Linear Combinations and Span

For a given finite number of elements $a_1, a_2, \cdots, a_k \in \mathbb{R}^n$, the set of all linear combinations of them is denoted by

$$\mathsf{span}\left\langle a_1, a_2, \cdots, a_k \right\rangle = \left\{ x \in \mathbb{R}^n \mid x \text{ is a linear combination of } a_1, a_2, \cdots, a_k \right\}.$$

Example

Let

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \quad \cdots \quad e_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

Then, $e_1, e_2, \cdots, e_n$ span all elements in $\mathbb{R}^n$, i.e., $\mathsf{span}\left\langle e_1, e_2, \cdots, e_n \right\rangle = \mathbb{R}^n$.

Linear Independence

As far as reproducibility is concerned, we notice that we keep only two elements

$$x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 \\ 2 \end{pmatrix},$$

and every element in $\mathbb{R}^2$ can be reproduced by linear combination of x and y.

This necessiates the notion of linear independece of a few elements in $\mathbb{R}^n$.

Example 1 z above is linearly dependent with respect to x and y.

Example 2 On the other hand, in $\mathbb{R}^3$,

$$x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \quad z = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

Then, x, y, z are linearly independent. By no chance, z is a linear combination of x and y.

Definition 1. *We say $a_1, a_2, \cdots, a_k \in \mathbb{R}^n$ are linearly independent if*

$$\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_k a_k = 0 \quad \Longrightarrow \quad \lambda_1 = \lambda_2 = \cdots = \lambda_k = 0.$$

$a_1, a_2, \cdots, a_k \in \mathbb{R}^n$ are said to be linearly dependent if they are not linearly independent.

Basis

Definition 2. *We say* $a_1, a_2 \cdots, a_k$ *of* $\mathbb{R}^n$ *form a basis of* $\mathbb{R}^n$ *if*

1. $a_1, a_2 \cdots, a_k$ *span* $\mathbb{R}^n$*,*
2. $a_1, a_2 \cdots, a_k$ *are linearly independent.*

Examples

$e_1, e_2, \cdots, e_n$ form a basis.

We will prove in later section the following seemingly natural facts:

1. Elements in $\mathbb{R}^n$ fewer than n cannot span <u>all</u> of $\mathbb{R}^n$.
2. Elements in $\mathbb{R}^n$ exceeding n must linearly <u>dependent</u>.
3. n elements in $\mathbb{R}^n$ are linearly independent if and only if they <u>do span all</u> of $\mathbb{R}^n$.

Chapter 4

Power and Importance of $\mathbb{R}^{m\times n}$

We saw that $\left(\mathbb{R}^{m\times n}, +, s\right)$ is so simply defined. The power of $\mathbb{R}^{m\times n}$ comes from that, (i) there are so many instances in mathematics, science, engineering, and etc., where entities of interests are such simple that we can add and scale them and that (ii) we will learn nice tools for $\mathbb{R}^{m\times n}$.

Example 1 In fact, this is an example of $\mathbb{C}^{m\times n}$, with $\mathbb{C}$ of complex numbers.
Wolfgang Pauli, a physicist, introduced *Pauli matrices* in developing theory of quantum mechanics in early 20c.

We consider a multiplication table for a 4 elements set $\{1, \sigma_x, \sigma_y, \sigma_z\}$:

Table 4.1: A multiplicative table

	1	σ_x	σ_y	σ_z
1	1	σ_x	σ_y	σ_z
σ_x	σ_x	-1	σ_z	$-\sigma_y$
σ_y	σ_y	$-\sigma_z$	-1	σ_x
σ_z	σ_z	σ_y	$-\sigma_x$	-1

Note that $\sigma_x\sigma_y \neq \sigma_y\sigma_x$, and this set cannot be represented by elements in $\mathbb{R}$.

One of the way we achieve the table:

We consider
$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right\}$$
for $1, \sigma_x, \sigma_y, \sigma_z$ from the left.

The set $\mathsf{span}\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right\rangle$ contains all the elements that can be generated.

note: The role played by $\mathbb{R}$, the scalar factor, can be taken by the mathematical object of *field*. But we will stick to $\mathbb{R}$ in this course.

Example 2 Colors (Light)

Although there are so many colors such as yellow, green, blue, white, black, $\cdots$, it turns out that the only three of them, red, green, and blue are independent. All others are linear combination of the three.

We choose $[0,1]^3 \subset \mathbb{R}^3$ represents the colors, with

1. $(1,0,0) \simeq$ red,
2. $(0,1,0) \simeq$ green,
3. $(0,0,1) \simeq$ blue,
4. $(0,0,0) \simeq$ black,
5. $(1,1,1) \simeq$ white.

The value ranged in $[0,1]$ represents the intensity.

Example 3 Bacteria with manipulatable genes.

In Bio-Engineering, ceartain bacateria is called a *vector*. This reflects that a certain kind of bacteria may or may not contain a few genes of our interests, and can be manipulated in the laboratory in the way of addition.

By a reproduction, a bacteria with gene A and a bacteria with gene B can result in a bacteria with both genes A and B. The number of such genes that can be maniplulated in lab. and bacteria can be represented by $\mathbb{R}^n$.

Example 4 Sauce

Sauce can be perfectly represented by $\mathbb{R}^n$. They can be added and scaled:

We linear combinate the sugar, salt, vinegar, soy sauce, water, what you want, in grams.

So a recipe you have to remember for a certain sauce is an element in $\mathbb{R}^n$, and if you want to make sauce a lot, you scale the element.

1. We saw many examples of *entities* that can be added and scaled, and sometimes we employ multiplication.

2. They are well-represented by $\mathbb{R}^{m\times n}$ or $\mathbb{R}^n$.

3. In all of examples, we see that there are two different roles:

$$\text{Respresentation} \quad \text{vs.} \quad \text{Essential Entities}$$

4. The theory has been developed in Linear Algebra so that the two roles are explicitly detached from each other. The latter, of essential entities, is defined in mathematics as *vector space* and *algebra*.

5. In our course, we use the word *vector* to denote the essential entity, while we simply use the word element of $\mathbb{R}^n$, or the matrix in $\mathbb{R}^{m\times n}$ for the representational use.

6. The connection from the set of essential entities to for instance $\mathbb{R}^n$, is done by fixing the basis. Let us be clear on this below.

- The definition of vector space X over the scalar $\mathbb{R}$ can be found in math textbooks and webpages. For completeness, we also contained the mathematical definition in the last page of this chapter.
- If you are not familiar with mathematical definitions, consider a vector space X as a set of symbols that can be added and scaled.
- k-dimensional vector space for k a positive integer.

  Definition 3. *A vector space X that admits a basis consists of k elements is called a k-dimensional vector space.*

  Example The space of colors admits a basis of three elements. Hence the space is 3-dimensional vector space.
- Connection from the *essential entities* to *representing elements* is done once we fix what are the basis elements in X.

  Example The space of sauces consists of $a = sugar$, $b = salt$, $c = vinegar$, and $d = water$.

  Once we fix a, b, c, and d to be the basis, we will understand an element

$$\begin{pmatrix} 5 \\ 5 \\ 3 \\ 100 \end{pmatrix} \in \mathbb{R}^4$$

  to represent a sauce consisting of 5-sugar, 5-salt, 3-vinegar, and 100 water, in grams.

  We may want to take the diluted ingredients $a + 100d$, $b + 100d$, $c + 100d$, and d to be the basis. Then

$$\begin{pmatrix} 5 \\ 5 \\ 3 \\ 100 \end{pmatrix} \in \mathbb{R}^4$$

  to represent a sauce consisting of 5-sugar, 5-salt, 3-vinegar, and 1400 water, in grams.
- You can see the detached roles of *representation* and *essential entities*.
- And how the basis choice connects the two.

A vector space X over the scalar $\mathbb{R}$ is mathematically defined as follows.

1. X is a nonempty set.
2. $+ : X \times X \to X$ is well-defined with following properties.
   (a) For any $x, y, z \in X$, $x + (y + z) = (x + y) + z$.
   (b) For any $x, y \in X$, $x + y = y + x$.
   (c) $\exists 0 \in X$ such that for any $x \in X$ $x + 0 = x$.
   (d) For any $x \in X$, $\exists(-x) \in X$ such that $x + (-x) = 0$.
3. $s : \mathbb{R} \times X \to X$ is well-defined with follwoing properties.
   (a) For any $\lambda_1, \lambda_2 \in \mathbb{R}$ and $x \in X$, $\lambda_1(\lambda_2 x) = (\lambda_1\lambda_2)x$
   (b) For any $x \in X$, $1x = x$.
4. $+$ and s work in compatible ways satisfying the following.
   (a) For any $\lambda \in \mathbb{R}$ and $x, y \in X$, $\lambda(x + y) = \lambda x + \lambda y$.
   (b) For any $\lambda_1, \lambda_2 \in \mathbb{R}$ and $x \in X$, $(\lambda_1 + \lambda_2)x = \lambda_1 x + \lambda_2 x$.