Chapter 1

$\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$

The set $\mathbb{R}^n$

- $\mathbb{R}$ is the set of all real numbers.
- Let n be a positive integer.
- We can write $(x_1, x_2, \cdots, x_n)$ of real numbers, which is an ordered n-tuple of real numbers.
- $\mathbb{R}^n$ is the set of all ordered n-tuples of real numbers.
- An element $x \in \mathbb{R}^n$ can be written in

  $$\text{row} \quad x = (x_1, x_2, x_3, \cdots, x_n) \quad \text{or in column} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix}, \quad \text{or any other form.}$$

  As long as the order of listed n real numbers are seen, there would not be a problem.

Addition and Scalar multiplication

Addition

1. If $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ and $y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$, we can add the two to obtain an element in $\mathbb{R}^n$:
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \\ \vdots \\ x_n + y_n \end{pmatrix},$$
which is denoted by $x + y \in \mathbb{R}^n$.

Scalar multiplication

1. In this course, the scalar is a synonym of real number.
2. If λ is a real number and $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, then we can scale x to obtain λx:
$$\lambda \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \\ \vdots \\ \lambda x_n \end{pmatrix} \in \mathbb{R}^n.$$

The set $\mathbb{R}^{m\times n}$

- Let $\mathbb{R}$ be again the set of all real numbers.
- Let m and n be two positive integers.
- The set of all $(A_{ij})_{i=1,j=1}^{i=m,j=n}$ of real numbers indexed by $(i,j) \in \{1,2,\cdots,m\} \times \{1,2,\cdots,n\}$ is denoted by $\mathbb{R}^{m\times n}$.
- The notation for an element $A \in \mathbb{R}^{m\times n}$ in this time is more specific. We distinguish two indices $i = 1,2,\cdots,m$ and $j = 1,2,\cdots,n$. We list mn real numbers in a box so that i is a row index, and j is a column index:

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1n} \\ A_{21} & A_{22} & A_{23} & \cdots & A_{2n} \\ A_{31} & A_{32} & A_{33} & \cdots & A_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{m1} & A_{m2} & A_{m3} & \cdots & A_{mn} \end{pmatrix}.$$

- We call an element of $\mathbb{R}^{m\times n}$ an $(m \times n)$ matrix, which reads as "m by n matrix".
- Why ..?

Addition and Scalar multiplication

Addition

1. If $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{m\times n}$, i.e., if the matrix A and B are in same shape, we can add the two to obtain an element in $\mathbb{R}^{m\times n}$:

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1n} \\ A_{21} & A_{22} & A_{23} & \cdots & A_{2n} \\ A_{31} & A_{32} & A_{33} & \cdots & A_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{m1} & A_{m2} & A_{m3} & \cdots & A_{mn} \end{pmatrix} + \begin{pmatrix} B_{11} & B_{12} & B_{13} & \cdots & B_{1n} \\ B_{21} & B_{22} & B_{23} & \cdots & B_{2n} \\ B_{31} & B_{32} & B_{33} & \cdots & B_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ B_{m1} & B_{m2} & B_{m3} & \cdots & B_{mn} \end{pmatrix}$$
$$= \begin{pmatrix} A_{11}+B_{11} & A_{12}+B_{12} & A_{13}+B_{13} & \cdots & A_{1n}+B_{1n} \\ A_{21}+B_{21} & A_{22}+B_{22} & A_{23}+B_{23} & \cdots & A_{2n}+B_{2n} \\ A_{31}+B_{31} & A_{32}+B_{32} & A_{33}+B_{33} & \cdots & A_{3n}+B_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{m1}+B_{m1} & A_{m2}+B_{m2} & A_{m3}+B_{m3} & \cdots & A_{mn}+B_{mn} \end{pmatrix},$$

which is denoted by $A + B \in \mathbb{R}^{m\times n}$.

Scalar multiplication

1. If λ is a real number and $A \in \mathbb{R}^{m\times n}$, then we can scale A to obtain λA:

$$\lambda \begin{pmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1n} \\ A_{21} & A_{22} & A_{23} & \cdots & A_{2n} \\ A_{31} & A_{32} & A_{33} & \cdots & A_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{m1} & A_{m2} & A_{m3} & \cdots & A_{mn} \end{pmatrix} = \begin{pmatrix} \lambda A_{11} & \lambda A_{12} & \lambda A_{13} & \cdots & \lambda A_{1n} \\ \lambda A_{21} & \lambda A_{22} & \lambda A_{23} & \cdots & \lambda A_{2n} \\ \lambda A_{31} & \lambda A_{32} & \lambda A_{33} & \cdots & \lambda A_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda A_{m1} & \lambda A_{m2} & \lambda A_{m3} & \cdots & \lambda A_{mn} \end{pmatrix}.$$

In summary,

1. We have the set $\mathbb{R}^n$, equipped with the addition and the scalar multiplication, that we denote by $\left(\mathbb{R}^n, +, s\right)$.

$$+ : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n,$$
$$s : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n.$$

2. We havethe set $\mathbb{R}^{m\times n}$, equipped with the addition and the scalar multiplication, that we denote by $\left(\mathbb{R}^{m\times n}, +, s\right)$.

$$+ : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n,$$
$$s : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n.$$

1. That we work with $(\mathbb{R}^n, +, s)$, i.e., that $\mathbb{R}^n$ are equipped with the addition and the scalar multiplication may, more importantly, mean that we do not perform other operations.

   These are illegal expressions:

   (a) $\mathbb{R}^n + \mathbb{R}$, $n \geq 2$:

   (b) $\mathbb{R}^n + \mathbb{R}^m$, $n \neq m$.

   (c) product in general. $\times : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, though in some dimensions we can define meaningful product.

   (d) comparison in general.

2. Likewise, for the case of $\mathbb{R}^{m\times n}$, we do not perform other operations, unless otherwise defined later of this course.