Chapter 1

What will we do ?

We are interested in the Euclidean space $\mathbb{R}^n$.

1. In the first major part of the course, we discuss about the n-dimensional volume of a subset $E \subset \mathbb{R}^n$. The first objective is to construct the Lebesgue measure $\mathcal{L}^n$.

2. In the second major part of the course, we update our tool of Integral, namely from the Riemann Integral to the Lebesgue Integral. This is based on the measure theory developed by abstraction of the Lebesgue measure in the first part.

Chapter 2

Measuring n-dimensional volume of $E \subset \mathbb{R}^n$

1. In the Euclidean space $\mathbb{R}^n$, we are able to measure the distance between two points $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$ of $\mathbb{R}^n$,

$$d(x, y) = \Big((x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots (x_n - y_n)^2 \Big)^{\frac{1}{2}}.$$

2. This gives rise to the n-dimensional volume formula for a few classes of subsets in $\mathbb{R}^n$. For example in $\mathbb{R}^3$, we take the formula:
   (a) If E is the cube $[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$, we take the value

   $$(b_1 - a_1)(b_2 - a_2)(b_3 - a_3)$$

   as its 3-dimensional volume.

   (b) If we consider a tetrahedron with base area A and the height h, we take the value

   $$\frac{1}{3} Ah$$

   as its 3-dimensional volume.

   (c) other examples...

Knowing the n-dimensional volumes of such a class of elementary sets,

1. We may extend our knowledge base on calculating n-volume: For a set made by assembling a few such elementary sets, the n-volume would be the sum of n-volumes of elementary sets.

2. That we wrote right above is the theory we want to develop. It is a difficult task: To make this consistent mathematically, any such theory should provide a proof that the n-volume assigned on a certain set $E \subset \mathbb{R}^n$ would be calculated independently of ways of cutting the set.

3. For example, for a given set $E \subset \mathbb{R}^n$, there are two persons. The first person cuts E into G_1, G_2, G_3, and the second person cuts E into H_1 and H_2. More specifically, G_1, G_2, G_3 are pairwise disjoint and $E = G_1 \cup G_2 \cup G_3$, and H_1, H_2 are pairwise disjoint and $E = H_1 \cup H_2$. n-volumes of G_i, and H_j are known. Theory should be certain about the equality

$$|G_1| + |G_2| + |G_3| = |H_1| + |H_2|$$

where $|S|$ denotes n-volume of the set S, if known. This consistency should be the case for all different ways of cutting the set E.

4. We consider the following humble goal: Let $\mathcal{R}$ be the collection of all cubes in $\mathbb{R}^n$ (whose n-volumes are as we know). Let R be any cube in $\mathcal{R}$ with the n-volume $|R|$. Let $(R_j)_{j=1}^{\infty}$ be pairwise disjoint partition of R, and $(Q_k)_{k=1}^{\infty}$ be another pairwise disjoint partition of R. The goal is that the n-volume of a cube was actually *correct*:

$$|R| = \sum_{j=1}^{\infty} |R_j| = \sum_{k=1}^{\infty} |Q_k|.$$

This is the task of our courses for a while, and this is a difficult problem.

Consistent family with n-volume

The word 'family', or 'collection' are synonyms of set. We use family or collection to avoid confusion.

Definition 1. *A pair $(\mathcal{G}, \lambda)$ of $\mathcal{G}$, a nonempty collection of subsets of $\mathbb{R}^n$ containing $\emptyset$, X, and $\lambda : \mathcal{G} \to [0, \infty]$, is said to be consistent if*

1. $\lambda(\emptyset) = 0$,
2. *If G is a set in $\mathcal{G}$, $v = \lambda(G)$ and $(G_j)_{j=1}^{\infty}$ is any sequence of sets in $\mathcal{G}$ that are pairwise disjoint and $G = \bigcup_{j=1}^{\infty} G_j$, then*

$$\sum_{j=1}^{\infty} \lambda(G_j) = v.$$

Let $n = 2$ and consider $\mathbb{R}^2$.

By half-open intervals we mean the intervals of one of the following forms

$$\emptyset, \quad [a, b), \quad [a, \infty), \quad (-\infty, b), \quad (-\infty, \infty),$$

where $a, b \in \mathbb{R}$ and are assumed to be $a < b$.

Definition 2. *The collection $\mathcal{R}$ is the collection of all cartesian products of two half-open intervals. The member of $\mathcal{R}$ is called a rectangle.*

1. *If R is an unbounded rectangle, we define $|R| = \infty$.*
2. *If R is a nonempty bounded rectangle $[a_1, b_1) \times [a_2, b_2)$, $|R| = (b_1 - a_1)(b_2 - a_2)$.*
3. $|\emptyset| = 0$.

We prove that $(\mathcal{R}, |\cdot|)$ is consistent from now on.

Towards a consistent family

At this moment, we prove a proposition stating that, out of somewhat arbitrary volume function ρ and a collection, one may extract its refined version of volume function λ.

Proposition 3. *Let X be a nonempty set, $\mathcal{G} \subset \mathcal{P}(X)$, and $\rho : \mathcal{G} \to [0, \infty]$ be such that $\emptyset \in \mathcal{G}$, $X \in \mathcal{G}$, and $\rho(\emptyset) = 0$. For any set $S \subset X$, define*

$$\lambda(S) := \inf_{(G_j) \text{ of } \mathcal{G} \text{ that covers } S} \Big\{ \sum_{j=1}^{\infty} \rho(G_j) \Big\}.$$

Then, λ, defined on $\mathcal{P}(X)$, satisfies the followings:

1. *$\lambda(\emptyset) = 0$.*
2. *If (S_j) of $\mathcal{P}(X)$ covers S, i.e., $\bigcup_j S_j \supset S$, then*

$$\lambda(S) \leq \sum_{j=1}^{\infty} \lambda(S_j). \tag{2.0.1}$$

Remark 2.1. We will define on $\mathcal{P}(\mathbb{R}^2)$

$$\lambda(S) := \inf_{(R_j) \text{ of rectangles that covers } S} \Big\{ \sum_{j=1}^{\infty} |R_j| \Big\}.$$

Examples:

proof of Proposition 3. Let S be any subset of X.

1. The set of coverings of S by sets in $\mathcal{G}$ is nonempty because $X \in \mathcal{G}$ and $(X, \emptyset, \emptyset, \cdots)$ covers S. Obviously, the set

$$\left\{ \sum_{j=1}^{\infty} \rho(G_j) : (G_j) \text{ of } \mathcal{G} \text{ covers } S \right\} \subset [0, \infty]$$

is nonempty and bounded below by 0. Therefore, $\lambda(S)$, the infimum over the set, is well-defined.

2. Since $(\emptyset, \emptyset, \cdots)$ covers $\emptyset$, $\rho(\emptyset) = 0$, and $\sum_j 0 = 0$, $\lambda(\emptyset)$ must be 0.

3. Now, let (S_j) be any sequence of subsets of X that covers S. Suppose any of $\lambda(S_j) = \infty$. Then the inequality (2.0.1) is trivially true. Now, we assume $\lambda(S_j) < \infty$ for every j.

4. Let $\epsilon > 0$. By the definition of infimum, for each j, there exists a covering $(G_\alpha^j)_{\alpha=1}^{\infty}$ of S_j by sets in $\mathcal{G}$ such that

$$\sum_{\alpha=1}^{\infty} \rho(G_\alpha^j) \leq \lambda(S_j) + \frac{\epsilon}{2^j}.$$

Obviously, $\bigcup_{\alpha} \bigcup_{j} G_\alpha^j \supset S$ and thus $(G_\alpha^j)_{j,\alpha=1}^{\infty}$ is a countable covering of S by sets in $\mathcal{G}$. Thus,

$$\lambda(S) \leq \sum_{j=1}^{\infty} \sum_{\alpha=1}^{\infty} \rho(G_\alpha^j) \leq \sum_{j=1}^{\infty} \left[\lambda(S_j) + \frac{\epsilon}{2^j} \right] = \sum_{j=1}^{\infty} \lambda(S_j) + \epsilon.$$

Since this inequality holds for every $\epsilon > 0$, we conclude that

$$\lambda(S) \leq \sum_{j=1}^{\infty} \lambda(S_j).$$

$\square$

1. Out of mere formula $\big|[a_1, b_1) \times [a_2, b_2)\big| = (b_1 - a_1)(b_2 - a_2)$, we suddenly have a definition of λ for all the subsets of $\mathbb{R}^2$.
2. However, we restrict ourselves the use of λ only on rectangles for a while to complete the proof of that $(\mathcal{R}, |\cdot|)$ is consistent.
3. Now, we aim to prove that for a rectangle R, $|R| = \lambda(R)$, namely the area formula $|\cdot|$ was already good to some extent.
4. Since, $\lambda(R) \leq |R|$, we only need to prove $\lambda(R) \geq |R|$.

We prove a few lemmas.

Lemma 4. *If R and R' are rectangels and $R \subset R'$, then $|R| \leq |R'|$.*

Proof. Omitted. □

Lemma 5. *Let R be a nonempty bounded rectangle $[a, b) \times [c, d)$, and consider*

$$a = t_1 < t_2 < \cdots < t_N = b, \quad c = s_1 < s_2 < \cdots < s_K = d,$$

and consider rectangles $R_{i,j} = [t_i, t_{i+1}) \times [s_j, s_{j+1})$ for $i = 1, 2, \cdots, N-1$ and $j = 1, 2, \cdots, K-1$. Then,

$$|R| = \sum_{i=1}^{N-1} \sum_{j=1}^{K-1} |R_{i,j}| \quad \textit{and} \quad R = \bigcup_{i=1}^{N-1} \bigcup_{j=1}^{K-1} R_{i,j} \quad \textit{of disjoint union.}$$

Proof.

$$\begin{aligned} |R| = (b-a)(c-d) &= \Big(\sum_{i=1}^{N-1} (t_{i+1} - t_i) \Big) \Big(\sum_{j=1}^{K-1} (s_{j+1} - s_j) \Big) \\ &= \sum_{i=1}^{N-1} \sum_{j=1}^{K-1} (t_{i+1} - t_i)(s_{j+1} - s_j) = \sum_{i=1}^{N-1} \sum_{j=1}^{K-1} |R_{i,j}|. \end{aligned}$$

By definition,

$$\begin{aligned} R_{i,j} &= \{(x, y) \mid t_i \leq x < t_{i+1} \text{ and } s_j \leq y < s_{j+1}\} \\ R_{i',j'} &= \{(x, y) \mid t'_i \leq x < t_{i'+1} \text{ and } s'_j \leq y < s_{j'+1}\} \end{aligned}$$

and if $(i, j) \neq (i', j')$, then $i \neq i'$ or $j \neq j'$, and they must be disjoint. Again by definition

$$\begin{aligned} R = &\ \{(x, y) \mid a \leq x < b \quad \text{and} \quad c \leq y < d\} \\ = &\ \Big\{(x, y) \ \Big| \ \Big[t_1 \leq x < t_2 \text{ or } t_2 \leq x < t_3 \text{ or } \cdots \text{ or } t_{N-1} \leq x < t_N\Big] \\ &\quad \text{and} \quad \Big[s_1 \leq y < s_2 \text{ or } s_2 \leq y < s_3 \text{ or } \cdots \text{ or } s_{K-1} \leq y < y_N\Big]\Big\} \\ = &\ \bigcup_{i=1}^{N-1} \bigcup_{j=1}^{K-1} R_{i,j}. \end{aligned}$$

□

Lemma 6. *Suppose R be a nonempty bounded rectangle. If $(R_k)_{k=1}^M$ is a covering of R by sets in $\mathcal{R}$, then*

$$|R| \le \sum_{k=1}^{M} |R_k|.$$

Proof.

1. If any of R_k is unbounded, then $|R_k| = \infty$ and the inequality trivially holds. Now we assume R_k is a bounded rectangle for every k.
2. If we can prove the same inequality on any subcover of (R_k), then the inequality still stands with the cover itself. Thus we consider a subcover of (R_k) by discarding every R_k that is the empty set, and prove the inequality with this subcover: Below, we assume R_k is nonempty for every k.
3. Let us write for each $R_k = [a_k, b_k) \times [c_k, d_k)$.
   Let $t_1 < t_2 < \cdots < t_N$ be an enumeration of the finite set
   $$\{a, a_1, a_2, \cdots, a_M, b, b_1, b_2, \cdots, b_M\}$$
   in ascending order.
   Let $s_1 < s_2 < \cdots < s_K$ be an enumeration of the finite set
   $$\{c, c_1, c_2, \cdots, c_K, d, d_1, d_2, \cdots, d_K\}$$
   in ascending order. We consider the rectangles $Q_{i,j} = [t_i, t_{i+1}) \times [s_j, s_{j+1})$, pairwise disjoint.
4. Note that for each $R_k = [a_k, b_k) \times [c_k, d_k)$, there exist indices $i_{begin}(k)$ and $i_{end}(k)$ such that $t_{i_{begin}(k)} = a_k$ and $t_{i_{end}(k)} = b_k$. Similarly $j_{begin}(k)$ and $j_{end}(k)$ exist. By the previous lemma,
   $$R_k = \bigcup_{i=i_{begin}(k)}^{i_{end}(k)-1} \bigcup_{j=j_{begin}(k)}^{j_{end}(k)-1} Q_{i,j} \quad \text{of disjoint union.}$$
   Because of this equality and that $(Q_{i,j})$ are pairwise disjoint, the following is true:
   $$\text{For every } k \text{ and every } (i,j), \text{ either } Q_{i,j} \subset R_k \text{ or } Q_{i,j} \cap R_k = \emptyset.$$
5. The similar is true for R.
6. We define
   $$\Gamma = \{(i,j) \mid Q_{i,j} \subset R\}, \quad \Gamma_k = \{(i,j) \mid Q_{i,j} \subset R_k\}.$$
   By the previous Lemma,
   $$|R| = \sum_{(i,j)\in\Gamma} |Q_{i,j}|, \quad |R_k| = \sum_{(i,j)\in\Gamma_k} |Q_{i,j}|.$$
7. That $R \subset \bigcup_k R_k$ implies that $(i,j) \in \Gamma$ implies that $Q_{i,j}$ intersects some R_k. Otherwise, (R_k) is not a covering of R.
8. This R_k-intersecting $Q_{i,j}$ in fact must be a subset of R_k. But $Q_{i,j} \subset R_k$ iff $(i,j) \in \Gamma_k$. We thus conclude: $\Gamma \subset \bigcup_k \Gamma_k$.
9. Finally,
   $$|R| = \sum_{(i,j)\in\Gamma} |Q_{i,j}| \le \sum_{(i,j)\in\bigcup_k \Gamma_k} |Q_{i,j}| \le \sum_k \sum_{(i,j)\in\Gamma_k} |Q_{i,j}| = \sum_k |R_k|.$$

$\square$

Proposition 7. *For a rectangle R, $\lambda(R) = |R|$.*

Proof.

1. If $R = \emptyset$, $\lambda(\emptyset) = 0 = |\emptyset|$.
2. Now, assume first that R is a bounded rectangle. We prove that $\lambda(R) \geq |R|$ below. Note we know that $\lambda(R) \leq |R| < \infty$.
3. By definition of $\lambda(R)$, for any $\epsilon > 0$ there exists a (Q_k) of $\mathcal{R}$ that covers R such that
$$\lambda(R) + \epsilon \geq \sum_k |Q_k|.$$
4. Now, it is possible to enlarge each rectangle Q_k a little to form an open rectangle $\tilde{Q}_k \supset Q_k$ but satisfying
$$|Q_k| \geq |\tilde{Q}_k| - \frac{\epsilon}{2^k}.$$
5. $(\tilde{Q}_k)$ forms an open covering of the closure of R that is compact. Hence, there is a finite subcover of the closure of R. (that is a finite cover of R too.) We have
$$\begin{aligned}\sum_k |Q_k| &\geq \sum_k \left(|\tilde{Q}_k| - \frac{\epsilon}{2^k}\right)\\ &\geq \sum_k |\tilde{Q}_k| - \epsilon\\ &\geq \sum_{k \in subcover} |\tilde{Q}_k| - \epsilon\\ &\geq |R| - \epsilon,\end{aligned}$$
where in the last inequality, we used the Lemma 6. In conclusion,
$$\lambda(R) + 2\epsilon \geq |R|$$
for every $\epsilon > 0$, and we conclude $\lambda(R) \geq |R|$.
6. Finally, let R be an unbounded rectangle. If so, we can consider $R_1 \subset R_2 \subset \cdots$ of subsets of R with $|R_j| < \infty$ and $|R_j| \to \infty$ as $j \to \infty$. Then for every j,
$$\lambda(R) \geq \lambda(R_j) = |R_j|,$$
which implies that $\lambda(R) = \infty$. The equality $\lambda(R) = |R| = \infty$ holds.

□

For later purpose, we also prove the following equality.

Lemma 8. *Let R be a nonempty bounded rectangle. If $R = \bigcup_{k=1}^{M} R_k$ of disjoint union of rectangles R_1, R_2, $\cdots$, R_M, then*

$$|R| = \sum_{k=1}^{M} |R_k|.$$

Proof. Exercise. $\square$

Justify first that $(i, j) \in \Gamma$ iff $(i, j) \cup_k \Gamma_k$, and second that $\cup_k \Gamma_k$ is a disjoint union.

Chapter 3

Arguments repeatedly used

[Argument with the infimum]

Let $A \subset \mathbb{R}$ lower bounded. Then $m := \inf A$ is well-defined. For any positive $\epsilon > 0$, $m + \epsilon$ is not a lower bound of A, and thus there must be $a \in A$ such that $a \leq m + \epsilon$.

[Inequality holding for all $\epsilon > 0$]

Let $a, b \in \mathbb{R}$. If $a \leq b + \epsilon$ for every $\epsilon > 0$, then $a \leq b$.

[Countable sum of nonnegative numbers]

Let (c_j) be a sequence of nonnegative numbers. Then, the summation of the series is independent of changing orders, such as $c_{\sigma(j)}$ with $\sigma : \mathbb{N} \to \mathbb{N}$ a bijection. One of the following two is the case.

(i) $$\sum_{j=1}^{\infty} c_j = \lim_{N\to\infty} \sum_{j=1}^{N} c_j = s_* < \infty.$$

The series absolutely converges, and the limit s_* is independent of changing orders of c_j

(ii) $$\sum_{j=1}^{\infty} c_j = \lim_{N\to\infty} \sum_{j=1}^{N} c_j = s_* = \infty.$$

The limit $+\infty$ is independent of changing orders of c_j.

[From (E_j) of sequence of sets to $(\hat{E}_j)$ of pairwise disjoint sets]

Lemma 9. *Let (E_j) be a sequence of sets. Define $(\hat{E}_j)$ recursive by*

$$\hat{E}_1 = E_1$$
$$\hat{E}_j = E_j \setminus \Big(\bigcup_{i=1}^{j-1} E_i \Big)$$

Then, for any N,

$$(i) \quad \bigcup_{j=1}^{N} \hat{E}_j = \bigcup_{j=1}^{N} E_j,$$

(ii) $(\hat{E}_j)_{j=1}^{N}$ is a sequence of pairwise disjoint sets.

Proof. The two assertions are obviously true for $N = 1$. If the assertion is true for $1, 2, \cdots, N-1$,

$$\hat{E}_N = E_N \setminus \Big(\bigcup_{j=1}^{N-1} E_j \Big) = E_N \setminus \Big(\bigcup_{j=1}^{N-1} \hat{E}_j \Big).$$

Obviously, $\hat{E}_N$ is disjoint from $\bigcup_{j=1}^{N-1} \hat{E}_j$. Therefore, $(\hat{E}_j)_{j=1}^{N}$ is pairwise disjoint. Also,

$$\begin{aligned}
\bigcup_{j=1}^{N} \hat{E}_j = \hat{E}_N \cup \Big(\bigcup_{j=1}^{N-1} \hat{E}_j \Big) &= \hat{E}_N \cup \Big(\bigcup_{j=1}^{N-1} E_j \Big) \\
&= \Big[E_N \cap \Big(\bigcup_{j=1}^{N-1} E_j \Big)^c \Big] \cup \Big(\bigcup_{j=1}^{N-1} E_j \Big) \\
&= E_N \cup \Big(\bigcup_{j=1}^{N-1} E_j \Big) = \bigcup_{j=1}^{N} E_j
\end{aligned}$$

Remark 3.1. Since the assertion in Lemma 9 is true for any N, it also holds that

$$(i) \quad \bigcup_{j=1}^{\infty} \hat{E}_j = \bigcup_{j=1}^{\infty} E_j,$$

(ii) $(\hat{E}_j)_{j=1}^{\infty}$ is a sequence of pairwise disjoint sets.

because

$$x \in \bigcup_{j=1}^{\infty} \hat{E}_j \implies x \in \hat{E}_{j_0} \text{ for some } j_0 \implies x \in \bigcup_{j=1}^{j_0} \hat{E}_j = \bigcup_{j=1}^{j_0} E_j \implies x \in \bigcup_{j=1}^{\infty} E_j,$$

$$x \in \bigcup_{j=1}^{\infty} E_j \implies x \in E_{j_1} \text{ for some } j_1 \implies x \in \bigcup_{j=1}^{j_1} E_j = \bigcup_{j=1}^{j_1} \hat{E}_j \implies x \in \bigcup_{j=1}^{\infty} \hat{E}_j,$$

and for any $\hat{E}_{i_0}$ and $\hat{E}_{i_1}$, we let $N = \max\{i_0, i_1\}$ and we know $(\hat{E}_j)_{j=1}^{N}$ is pairwise disjoint.

Remark 3.2. [(For any N)-assertion by induction] & [(limit)-assertion proven in addition] style of proof will appear repeatedly.

$\square$

Chapter 4

Measure Theoretic Separation

We would like to have that if a set $S \subset \mathbb{R}^2$ is made by assembling two disjoint sets S_1 and S_2,

$$\lambda(S_1 \cup S_2) = \lambda(S_1) + \lambda(S_2).$$

We then would like to have its countable version.

Since the inequality $\lambda(S_1 \cup S_2) \leq \lambda(S_1) + \lambda(S_2)$ already is established, worry is in whether there is a case

$$\lambda(S_1 \cup S_2) < \lambda(S_1) + \lambda(S_2)$$

Over-estimation by Truely 2-dimensional covering

Look at the definition of $\lambda(S)$,

$$\lambda(S) := \inf_{(R_j) \text{ of rectangles that covers } S} \left\{ \sum_{j=1}^{\infty} |R_j| \right\}.$$

The importance of the rectangle in our theory lies in that it is a Truely 2-dimensional lump.

1. The set $\displaystyle\bigcup_{j=1}^{\infty} R_j \supset S$ is thus a Truely 2-dimensional lump replacement of S.
2. We estimate its 2-dimensional area by $\displaystyle\sum_{j=1}^{\infty} |R_j|$, that is certainly an over-estimation.
3. This over-estimation is minimized as much as possible, over all the coverings.

How does this 2-dim-over-estimation $\to$ minimization properly works? For example consider the singletone set $\{x_0\}$. Intuitively, 0 has to be its 2-dimsnional area.

1. We see that one square R_ℓ with side length $\ell > 0$ whose center is x_0 is a Truely 2-dimensional replacement of $\{x_0\}$. $(R_\ell, \emptyset, \emptyset, \cdots)$ covers $\{x_0\}$.
2. Its over-estimation is thus, $\ell^2 > 0$.
3. By minimization of over-estimation by letting $\ell \to 0$, we conclude that the infimum $\lambda(\{x_0\}) = 0$.

Thus, it makes sense to take the area of one point set is 0.

Question: Can the over-estimation be not resolved by the minimization process?

One speculative example about the question of resolving over-estimation is the following in 1 dimension. The role of rectangles is taken by intervals. Let

$$A = [0,1] \cap \mathbb{Q}, \quad B = [0,1] \cap \mathbb{Q}^c$$

1. If (R_j) is a Truely 1-dimensional covering of A by intervals, and (Q_k) is a Truely 1-dimensional covering of B by intervals, let us write this replacement

$$A' = \bigcup_j R_j, \quad B' = \bigcup_k Q_k.$$

2. Because of density of rationals and irrationals, the invasion of A' into the portion of B', and the invasion of B' into the portion of A' must have occured. In other words,

$$\sum_{j=1}^{\infty} |R_j| + \sum_{k=1}^{\infty} |Q_k| > 1.$$

3. Is it for certain thing that by the followed minimization step, this is to be resolved properly ? In other words, are we sure

$$\lambda(A) + \lambda(B) = 1 \quad ?$$

This is why we ask a question if there can be a case of two <u>disjoint</u> set S_1 and S_2 with

$$\lambda(S_1 \cup S_2) < \lambda(S_1) + \lambda(S_2)$$

measure-theoretic separation

Since we are very speculative about this over-estimation-resolving procedure, we adopt a stronger notion of separation over the notion of being disjoint.

Definition 1. *We say a set $E \subset \mathbb{R}^n$ separates S_1 and S_2 if*

$$\Big(S_1 \subset E \quad \textit{and} \quad S_2 \subset E^c\Big) \quad \textit{or} \quad \Big(S_2 \subset E \quad \textit{and} \quad S_1 \subset E^c\Big)$$

Remark 4.1. If there exists a set E that separates S_1 and S_2, then S_1 and S_2 must be disjoint.

Example: Let E be an open ball of radius $r > 0$ and S_1 and S_2 be two compact sets.

Example: Let E be a half space $x_n \geq 0$ and S_1 and S_2 be two sets one of which is in the half space, and the other is outside of the half space.

Definition 2. *We say $E \subset \mathbb{R}^n$ is λ-separating if the following is true.*

$$E \text{ separates } S_1 \text{ and } S_2 \quad \Longrightarrow \quad \lambda(S_1 \cup S_2) = \lambda(S_1) + \lambda(S_2).$$

Question 1: What kind of sets can have such a separating property?

We answer to the following question first, before the Q1.

Question 2: What are the consequences of being such a set?

Theorem 3. *Let E_1, E_2, E_3, $\cdots$ be pairwise disjoint λ-separating sets and $S_1, S_2, \cdots$ be any sequence in $\mathcal{P}(\mathbb{R}^n)$ such that $S_j \subset E_j$ for every j. Then,*

$$(i) \text{ for any } N \quad \lambda\Big(\bigcup_{j=1}^{N} S_j\Big) = \sum_{j=1}^{N} \lambda(S_j), \quad \text{and} \quad (ii)\ \lambda\Big(\bigcup_{j=1}^{\infty} S_j\Big) = \sum_{j=1}^{\infty} \lambda(S_j).$$

Proof. 1. We prove the first assertion.

Certainly $\lambda\Big(\bigcup_{j=1}^{1} S_j\Big) = \sum_{j=1}^{1} \lambda(S_j)$. Now, if equality holds for

$$\lambda\Big(\bigcup_{j=1}^{k-1} S_j\Big) = \sum_{j=1}^{k-1} \lambda(S_j)$$

we assert that

$$\lambda\Big(\bigcup_{j=1}^{k} S_j\Big) = \lambda\Big(\bigcup_{j=1}^{k-1} S_j \cup S_k\Big).$$

Since E_k separates S_k and $\Big(\bigcup_{j=1}^{k-1} S_j\Big)$, the (RHS) equals to

$$\lambda\Big(\bigcup_{j=1}^{k-1} S_j\Big) + \lambda(S_k) = \sum_{j=1}^{k-1} \lambda(S_j) + \lambda(S_k) = \sum_{j=1}^{k} \lambda(S_k).$$

2. For the second assertion,

$$\lambda\Big(\bigcup_{j=1}^{\infty} S_j\Big) \leq \sum_{j=1}^{\infty} \lambda(S_j) = \lim_{N\to\infty} \sum_{j=1}^{N} \lambda(S_j) = \lim_{N\to\infty} \lambda\Big(\bigcup_{j=1}^{N} S_j\Big) \leq \lim_{N\to\infty} \lambda\Big(\bigcup_{j=1}^{\infty} S_j\Big) = \lambda\Big(\bigcup_{j=1}^{\infty} S_j\Big)$$

Hence, every quantity appeared equals to each other.

$\square$

Remark 4.2. One important example is the case where $S_j = E_j$ itself for every j, that are pairwise disjoint and λ-separating. They always satisfies

$$\lambda\Big(\bigcup_{j=1}^{\infty} E_j\Big) = \sum_{j=1}^{\infty} \lambda(E_j).$$

Remark 4.3. To get back to our first objective, to show $(\mathcal{R}, |\cdot|)$ is consistent, (that is to show $(\mathcal{R}, \lambda)$ is consistent since $\lambda(R) = |R|$ for any rectangle $R \in \mathcal{R}$), we will be done once we prove that any rectangle is λ-separating.

Proposition 4. *For any $R \in \mathcal{R}$, the following is true.*

$$R \textit{ separates } S_1 \textit{ and } S_2 \quad \Longrightarrow \quad \lambda(S_1 \cup S_2) = \lambda(S_1) + \lambda(S_2).$$

Proof. We prove that

$$R \text{ separates } S_1 \text{ and } S_2 \quad \Longrightarrow \quad \lambda(S_1 \cup S_2) \geq \lambda(S_1) + \lambda(S_2).$$

1. If $\lambda(S_1 \cup S_2) = \infty$, then the inequality trivially holds.
2. From now on, we assume $\lambda(S_1 \cup S_2) < \infty$. It also follows that $\lambda(S_1) < \infty$ and $\lambda(S_2) < \infty$. Without loss, we consider the case $S_1 \subset R$.
3. For any $\epsilon > 0$, there exists a (R_j) of $\mathcal{R}$ that covers $S_1 \cup S_2$ such that
$$\lambda(S_1 \cup S_2) + \epsilon \geq \sum_{j=1}^{\infty} \lambda(R_j).$$
Note that every R_j must be a bounded rectangle and the series in (RHS) absolutely converges, since (LHS) is finite.
4. Now, we notice that R^c can always be written as a disjoint union of four rectangles Q_1, Q_2, Q_3, and Q_4.
5. Let $R = Q_0$. We can write for every j
$$R_j = Q_j^0 \cup Q_j^1 \cup Q_j^2 \cup Q_j^3 \cup Q_j^4, \quad Q_j^\alpha = R_j \cap Q_\alpha, \quad \alpha = 0, 1, 2, 3, 4$$
Each intersection is again a rectangle, and this is a disjoint union of five rectangles.
6. Now, $(Q_j^0)_{j=1}^{\infty}$ covers S_1, and $(Q_j^\alpha)_{j=1,\alpha=1}^{j=\infty,\alpha=4}$ covers S_2.
7. Therefore,
$$\begin{aligned}\lambda(S_1 \cup S_2) + \epsilon \geq \sum_{j=1}^{\infty} \lambda(R_j) &= \sum_{j=1}^{\infty}\sum_{\alpha=0}^{4} \lambda(Q_j^\alpha) \\ &= \sum_{j=1}^{\infty} \lambda(Q_j^0) + \sum_{j=1}^{\infty}\sum_{\alpha=1}^{4} \lambda(Q_j^\alpha) \\ &\geq \lambda(S_1) + \lambda(S_2).\end{aligned}$$
8. Since the inequality holds for every $\epsilon > 0$, $\lambda(S_1 \cup S_2) \geq \lambda(S_1) + \lambda(S_2)$.

□

Theorem 5. $(\mathcal{R}, |\cdot|)$ *is consistent.*

Proof. This is by Proposition 4. □

Seen from the proof of Proposition 4, it is not hard to prove that for two rectangles R_1 and R_2, the union $A = R_1 \cup R_2$, which is not a rectangle in general, is also λ-separating.

Proposition 6. *For any $R, R' \in \mathcal{R}$, $R \cup R'$ is λ-separating.*

Proof. From the proof of Proposition 4, the only modifications we need to make are the followings.

1. $R \cup R' = (R \cap R'^c) \cup (R \cap R') \cup (R' \cap R^c) = \bigcup_{\alpha=1}^{m} Q_\alpha$ of disjoint union of finite numbers of rectangles.
2. Similarly, $(R \cup R')^c = \bigcup_{\alpha=m+1}^{m+m'} Q_\alpha$ of disjoint union of finite numbers of rectangles.
3. If (R_j) covers $S_1 \cup S_2$, then

$$R_j = \bigcup_{\alpha=1}^{m+m'} Q_j^\alpha \text{of disjoint union of rectangles, where} \quad Q_j^\alpha = Q_\alpha \cap R_j.$$

4. $(Q_j^\alpha)_{j=1,\alpha=1}^{j=\infty,\alpha=m}$ covers S_1, and $(Q_j^\alpha)_{j=1,\alpha=m+1}^{j=\infty,\alpha=m+m'}$ covers S_2

□