

# Chapter 1

## $\mathbb{R}^n$ and $\mathbb{R}^{m \times n}$

### The set $\mathbb{R}^n$

- $\mathbb{R}$  is the set of all real numbers.
- Let  $n$  be a positive integer.
- We can write  $(x_1, x_2, \dots, x_n)$  of real numbers, which is an ordered  $n$ -tuple of real numbers.
- $\mathbb{R}^n$  is the set of all ordered  $n$ -tuples of real numbers.
- An element  $x \in \mathbb{R}^n$  can be written in

$$\text{row } x = (x_1, x_2, x_3, \dots, x_n) \quad \text{or in column } x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix}, \quad \text{or any other form.}$$

As long as the order of listed  $n$  real numbers are seen, there would not be a problem.

## Addition and Scalar multiplication

### Addition

- If  $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$  and  $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$ , we can add the two to obtain an element in  $\mathbb{R}^n$ :

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \\ \vdots \\ x_n + y_n \end{pmatrix},$$

which is denoted by  $x + y \in \mathbb{R}^n$ .

### Scalar multiplication

- In this course, the scalar is a synonym of real number.
- If  $\lambda$  is a real number and  $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ , then we can scale  $x$  to obtain  $\lambda x$ :

$$\lambda \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \\ \vdots \\ \lambda x_n \end{pmatrix} \in \mathbb{R}^n.$$

## The set $\mathbb{R}^{m \times n}$

- Let  $\mathbb{R}$  be again the set of all real numbers.
- Let  $m$  and  $n$  be two positive integers.
- The set of all  $(A_{ij})_{i=1,j=1}^{i=m,j=n}$  of real numbers indexed by  $(i,j) \in \{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$  is denoted by  $\mathbb{R}^{m \times n}$ .
- The notation for an element  $A \in \mathbb{R}^{m \times n}$  in this time is more specific. We distinguish two indices  $i = 1, 2, \dots, m$  and  $j = 1, 2, \dots, n$ . We list  $mn$  real numbers in a box so that  $i$  is a row index, and  $j$  is a column index:

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1n} \\ A_{21} & A_{22} & A_{23} & \cdots & A_{2n} \\ A_{31} & A_{32} & A_{33} & \cdots & A_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{m1} & A_{m2} & A_{m3} & \cdots & A_{mn} \end{pmatrix}.$$

- We call an element of  $\mathbb{R}^{m \times n}$  an  $(m \times n)$  matrix, which reads as “ $m$  by  $n$  matrix”.
- Why ..?

## Addition and Scalar multiplication

### Addition

1. If  $A \in \mathbb{R}^{m \times n}$  and  $B \in \mathbb{R}^{m \times n}$ , i.e., if the matrix  $A$  and  $B$  are in same shape, we can add the two to obtain an element in  $\mathbb{R}^{m \times n}$ :

$$\begin{aligned} & \begin{pmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1n} \\ A_{21} & A_{22} & A_{23} & \cdots & A_{2n} \\ A_{31} & A_{32} & A_{33} & \cdots & A_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & A_{m3} & \cdots & A_{mn} \end{pmatrix} + \begin{pmatrix} B_{11} & B_{12} & B_{13} & \cdots & B_{1n} \\ B_{21} & B_{22} & B_{23} & \cdots & B_{2n} \\ B_{31} & B_{32} & B_{33} & \cdots & B_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_{m1} & B_{m2} & B_{m3} & \cdots & B_{mn} \end{pmatrix} \\ &= \begin{pmatrix} A_{11} + B_{11} & A_{12} + B_{12} & A_{13} + B_{13} & \cdots & A_{1n} + B_{1n} \\ A_{21} + B_{21} & A_{22} + B_{22} & A_{23} + B_{23} & \cdots & A_{2n} + B_{2n} \\ A_{31} + B_{31} & A_{32} + B_{32} & A_{33} + B_{33} & \cdots & A_{3n} + B_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{m1} + B_{m1} & A_{m2} + B_{m2} & A_{m3} + B_{m3} & \cdots & A_{mn} + B_{mn} \end{pmatrix}, \end{aligned}$$

which is denoted by  $A + B \in \mathbb{R}^{m \times n}$ .

### Scalar multiplication

1. If  $\lambda$  is a real number and  $A \in \mathbb{R}^{m \times n}$ , then we can scale  $A$  to obtain  $\lambda A$ :

$$\lambda \begin{pmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1n} \\ A_{21} & A_{22} & A_{23} & \cdots & A_{2n} \\ A_{31} & A_{32} & A_{33} & \cdots & A_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & A_{m3} & \cdots & A_{mn} \end{pmatrix} = \begin{pmatrix} \lambda A_{11} & \lambda A_{12} & \lambda A_{13} & \cdots & \lambda A_{1n} \\ \lambda A_{21} & \lambda A_{22} & \lambda A_{23} & \cdots & \lambda A_{2n} \\ \lambda A_{31} & \lambda A_{32} & \lambda A_{33} & \cdots & \lambda A_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda A_{m1} & \lambda A_{m2} & \lambda A_{m3} & \cdots & \lambda A_{mn} \end{pmatrix}.$$

In summary,

1. We have the set  $\mathbb{R}^n$ , equipped with the addition and the scalar multiplication, that we denote by  $(\mathbb{R}^n, +, s)$ .

$$\begin{aligned} + : \mathbb{R}^n \times \mathbb{R}^n &\rightarrow \mathbb{R}^n, \\ s : \mathbb{R} \times \mathbb{R}^n &\rightarrow \mathbb{R}^n. \end{aligned}$$

2. We have the set  $\mathbb{R}^{m \times n}$ , equipped with the addition and the scalar multiplication, that we denote by  $(\mathbb{R}^{m \times n}, +, s)$ .

$$\begin{aligned} + : \mathbb{R}^n \times \mathbb{R}^n &\rightarrow \mathbb{R}^n, \\ s : \mathbb{R} \times \mathbb{R}^n &\rightarrow \mathbb{R}^n. \end{aligned}$$

- That we work with  $(\mathbb{R}^n, +, s)$ , i.e., that  $\mathbb{R}^n$  are equipped with the addition and the scalar multiplication may, more importantly, mean that we do not perform other operations.

**These are illegal expressions:**

- (a)  $\mathbb{R}^n + \mathbb{R}$ ,  $n \geq 2$ :
- (b)  $\mathbb{R}^n + \mathbb{R}^m$ ,  $n \neq m$ .
- (c) product in general.  $\times : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ , though in some dimensions we can define meaningful product.
- (d) comparison in general.

- Likewise, for the case of  $\mathbb{R}^{m \times n}$ , we do not perform other operations, unless otherwise defined later of this course.

- We have so far

$$(\mathbb{R}^n, +, s) \quad \text{and} \quad (\mathbb{R}^{m \times n}, +, s),$$

respectively a set equipped with the addition and scalar multiplication.

- Expression that makes use of the addition and scalar multiplication, and use only of them such as

$$\lambda_1 x + \lambda_2 y + \lambda_3 z + \lambda_4 w, \quad x, y, z, w \in \mathbb{R}^n$$

comes as an important expression to us.

- The expression is called a “Linear Combination” of  $x, y, z, w$ ; Linear, meaning that the scalar multiplication  $x \mapsto \lambda x$  is to make something proportional, and Combination, meaning here we are adding things.

The final comment on  $(\mathbb{R}^n, +, s)$  and  $(\mathbb{R}^{m \times n}, +, s)$ :

As stated earlier, in some dimensions we may additionally define product, for example in  $\mathbb{R}^2$  we can define

$$(x_1, x_2) \times (y_1, y_2) = (v_1, v_2), \quad \text{where } v_1 + iv_2 = (x_1 + ix_2)(y_1 + iy_2).$$

However, all of such operations are treated in this course to be exceptional and come as addendum, emphasizing that there are only (i) the addition and (ii) the scalar multiplication we are allowed to operate so far.

## Chapter 2

# Matrix Matrix multiplication

- Emphasizing that there are no other general product operations, now we define the unique product operation between matrices.
- For given matrix  $A \in \mathbb{R}^{\ell \times m}$  and matrix  $B \in \mathbb{R}^{m \times n}$ , we define the matrix  $C \in \mathbb{R}^{\ell \times n}$  to be the product denoted by  $AB$  so that

$$C_{ij} = \sum_{\alpha=1}^m A_{i\alpha} B_{\alpha j} \quad \text{for } i = 1, 2, \dots, \ell \text{ and } j = 1, 2, \dots, n.$$

- Importantly, the product is defined only for the case where the second component of shape of  $A$  and the first component of shape of  $B$  are the same, and the product is not defined for all remaining cases.
- Why .. ?

The defined product can be regarded in a few different ways.

1. The product can be regarded as to build new  $n$  linear combinations of columns of  $A$ , as designated by numbers in  $B$ :

2. The product can be regarded as to build new  $\ell$  linear combinations of rows of  $B$ , as designated by numbers in  $A$ :
  3. The product can be regarded as to build new  $\ell n$  real numbers, out of rows of  $A$  and columns of  $B$ .

## The product is useful in many places.

### Perfect for the chain rule in multivariable calculus

Suppose that  $X = \mathbb{R}^\ell$ ,  $Y = \mathbb{R}^m$ , and  $Z = \mathbb{R}^n$ , and

$$f : X \rightarrow Y, \quad g : Y \rightarrow Z.$$

Multivariable Calculus:

1. Also assume  $f$  and  $g$  are many times differentiable functions.
2. For a given point  $\bar{x}$ , we collect numbers

$$B_{\alpha j} = \frac{\partial f^\alpha}{\partial x^j}(\bar{x}) = \lim_{h \rightarrow 0} \frac{f^\alpha(\bar{x} + he_j) - f^\alpha(\bar{x})}{h}.$$

3. Similarly, for a given point  $\bar{y}$ , we collect numbers

$$A_{i\alpha} = \frac{\partial g^i}{\partial y^\alpha}(\bar{y}) = \lim_{h \rightarrow 0} \frac{g^i(\bar{y} + he_\alpha) - g^i(\bar{y})}{h}.$$

4. Now we consider the composition

$$g \circ f : X \rightarrow Z, \quad g \circ f(x) = g(f(x))$$

and collect numbers

$$C_{ij} = \frac{\partial(g \circ f)^i}{\partial x^j}(\bar{x}) = \lim_{h \rightarrow 0} \frac{(g \circ f)^i(\bar{x} + he_j) - (g \circ f)^i(\bar{x})}{h}.$$

5. Then  $C$  turns out to equal to  $AB$ , for  $\bar{y} = f(\bar{x})$ .

### Writing a system of linear equations

If we are given

$$\begin{cases} 3x - 7y + 4z = -2, \\ 9x - 2y - 6z = 0, \\ -5x + 3y - 11z = -8 \end{cases}$$

It is nice that we can write also

$$\begin{pmatrix} 3 & -7 & 4 \\ 9 & -2 & -6 \\ -5 & 3 & -11 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ -8 \end{pmatrix}$$

Using  $\ell = 3$ ,  $m = 3$ , and  $n = 1$  in the left-hand-side.

## The question of why..?

1. Why does it have to be in that way?
2. Why not we define for  $A, B \in \mathbb{R}^{m \times n}$  a product with the same shape in  $\mathbb{R}^{m \times n}$  ?

$$\begin{pmatrix} A_{11}B_{11} & A_{12}B_{12} & A_{13}B_{13} & \cdots & A_{1n}B_{1n} \\ A_{21}B_{21} & A_{22}B_{22} & A_{23}B_{23} & \cdots & A_{2n}B_{2n} \\ A_{31}B_{31} & A_{32}B_{32} & A_{33}B_{33} & \cdots & A_{3n}B_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{m1}B_{m1} & A_{m2}B_{m2} & A_{m3}B_{m3} & \cdots & A_{mn}B_{mn} \end{pmatrix}$$

- (a) Actually you can. In most of computer language, this entry-wise product in the same shape is provided by the broadcasting.
3. All why-questions come to the following one,  
Would other people be interested in the new definition ?

## A few important observations

1. Important: You don't mess up with the order of multiplication here.
2. Unlike multiplication we know for two real numbers, for the Matrix-Matrix multiplication for  $A \in \mathbb{R}^{\ell \times m}$  and  $B \in \mathbb{R}^{m \times n}$ , the product  $BA$  is not even defined in general. This is simply because in general  $n \neq \ell$ .
3. In the special case of that  $\ell = m = n$ ,  $BA$  is defined. However even in such a case

$$AB \neq BA \quad \text{in general.}$$

## Summary

1. We have  $(\mathbb{R}^{m \times n}, +, s)$  for  $m$  and  $n$  positive integers.
2. Linear combinations of members of  $\mathbb{R}^{m \times n}$  are expressions using  $+$  and  $s$  such as

$$\lambda_1 A_1 + \lambda_2 A_2 + \cdots + \lambda_j A_j \in \mathbb{R}^{m \times n}.$$

3. We have a definition of multiplication for  $A \in \mathbb{R}^{\ell \times m}$  and matrix  $B \in \mathbb{R}^{m \times n}$  resulting in  $AB \in \mathbb{R}^{\ell \times n}$ .

Knowledge so far will be applied to mathematics, science, and engineering, and will be extremely important and powerful.



## Chapter 3

# Linear Independence and Basis

Let us consider elements in  $\mathbb{R}^2$ ,

$$x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \quad z = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

It is easy to notice that if we are with  $x$  and  $y$  in our hands,  $z$  can be reproduced by

$$z = 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix},$$

a linear combination of  $x$  and  $y$ .

## Linear Combinations and Span

For a given finite number of elements  $a_1, a_2, \dots, a_k \in \mathbb{R}^n$ , the set of all linear combinations of them is denoted by

$$\text{span} \langle a_1, a_2, \dots, a_k \rangle = \{x \in \mathbb{R}^n \mid x \text{ is a linear combination of } a_1, a_2, \dots, a_k\}.$$

### Example

Let

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \quad \dots \quad e_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

Then,  $e_1, e_2, \dots, e_n$  span all elements in  $\mathbb{R}^n$ , i.e.,  $\text{span} \langle e_1, e_2, \dots, e_n \rangle = \mathbb{R}^n$ .

## Linear Independence

As far as reproducibility is concerned, we notice that we keep only two elements

$$x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 \\ 2 \end{pmatrix},$$

and every element in  $\mathbb{R}^2$  can be reproduced by linear combination of  $x$  and  $y$ .

This necessitates the notion of linear independence of a few elements in  $\mathbb{R}^n$ .

**Example 1**  $z$  above is linearly dependent with respect to  $x$  and  $y$ .

**Example 2** On the other hand, in  $\mathbb{R}^3$ ,

$$x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \quad z = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

Then,  $x, y, z$  are linearly independent. By no chance,  $z$  is a linear combination of  $x$  and  $y$ .

**Definition 1.** We say  $a_1, a_2, \dots, a_k \in \mathbb{R}^n$  are linearly independent if

$$\lambda_1 a_1 + \lambda_2 a_2 + \dots + \lambda_k a_k = 0 \implies \lambda_1 = \lambda_2 = \dots = \lambda_k = 0.$$

$a_1, a_2, \dots, a_k \in \mathbb{R}^n$  are said to be linearly dependent if they are not linearly independent.

## Basis

**Definition 2.** We say  $a_1, a_2 \dots, a_k$  of  $\mathbb{R}^n$  form a basis of  $\mathbb{R}^n$  if

1.  $a_1, a_2 \dots, a_k$  span  $\mathbb{R}^n$ ,
2.  $a_1, a_2 \dots, a_k$  are linearly independent.

### Examples

$e_1, e_2, \dots, e_n$  form a basis.

We will prove in later section the following seemingly natural facts:

1. Elements in  $\mathbb{R}^n$  fewer than  $n$  cannot span all of  $\mathbb{R}^n$ .
2. Elements in  $\mathbb{R}^n$  exceeding  $n$  must linearly dependent.
3.  $n$  elements in  $\mathbb{R}^n$  are linearly independent if and only if they do span all of  $\mathbb{R}^n$ .



## Chapter 4

# Power and Importance of $\mathbb{R}^{m \times n}$

We saw that  $(\mathbb{R}^{m \times n}, +, s)$  is so simply defined. The power of  $\mathbb{R}^{m \times n}$  comes from that, (i) there are so many instances in mathematics, science, engineering, and etc., where entities of interests are such simple that we can add and scale them and that (ii) we will learn nice tools for  $\mathbb{R}^{m \times n}$ .

**Example 1** In fact, this is an example of  $\mathbb{C}^{m \times n}$ , with  $\mathbb{C}$  of complex numbers. Wolfgang Pauli, a physicist, introduced *Pauli matrices* in developing theory of quantum mechanics in early 20c.

We consider a multiplication table for a 4 elements set  $\{1, \sigma_x, \sigma_y, \sigma_z\}$ :

Table 4.1: A multiplicative table

	1	$\sigma_x$	$\sigma_y$	$\sigma_z$
1	1	$\sigma_x$	$\sigma_y$	$\sigma_z$
$\sigma_x$	$\sigma_x$	-1	$\sigma_z$	$-\sigma_y$
$\sigma_y$	$\sigma_y$	$-\sigma_z$	-1	$\sigma_x$
$\sigma_z$	$\sigma_z$	$\sigma_y$	$-\sigma_x$	-1

Note that  $\sigma_x \sigma_y \neq \sigma_y \sigma_x$ , and this set cannot be represented by elements in  $\mathbb{R}$ .

One of the way we achieve the table:

We consider

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right\}$$

for  $1, \sigma_x, \sigma_y, \sigma_z$  from the left.

The set  $\text{span} \langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \rangle$  contains all the elements that can be generated.

**note:** The role played by  $\mathbb{R}$ , the scalar factor, can be taken by the mathematical object of *field*. But we will stick to  $\mathbb{R}$  in this course.

**Example 2** Colors (Light)

Although there are so many colors such as yellow, green, blue, white, black,  $\dots$ , it turns out that the only three of them, red, green, and blue are independent. All others are linear combination of the three.

We choose  $[0, 1]^3 \subset \mathbb{R}^3$  represents the colors, with

1.  $(1, 0, 0) \simeq$  red,
2.  $(0, 1, 0) \simeq$  green,
3.  $(0, 0, 1) \simeq$  blue,
4.  $(0, 0, 0) \simeq$  black,
5.  $(1, 1, 1) \simeq$  white.

The value ranged in  $[0, 1]$  represents the intensity.

**Example 3** Bacteria with manipulatable genes.

In Bio-Engineering, certain bacteria is called a *vector*. This reflects that a certain kind of bacteria may or may not contain a few genes of our interests, and can be manipulated in the laboratory in the way of addition.

By a reproduction, a bacteria with gene  $A$  and a bacteria with gene  $B$  can result in a bacteria with both genes  $A$  and  $B$ . The number of such genes that can be manipulated in lab. and bacteria can be represented by  $\mathbb{R}^n$ .

**Example 4** Sauce

Sauce can be perfectly represented by  $\mathbb{R}^n$ . They can be added and scaled:

We linearly combine the sugar, salt, vinegar, soy sauce, water, what you want, in grams.

So a recipe you have to remember for a certain sauce is an element in  $\mathbb{R}^n$ , and if you want to make sauce a lot, you scale the element.

1. We saw many examples of *entities* that can be added and scaled, and sometimes we employ multiplication.
2. They are well-represented by  $\mathbb{R}^{m \times n}$  or  $\mathbb{R}^n$ .
3. In all of examples, we see that there are two different roles:

Representation vs. Essential Entities

4. The theory has been developed in Linear Algebra so that the two roles are explicitly detached from each other. The latter, of essential entities, is defined in mathematics as *vector space* and *algebra*.
5. In our course, we use the word *vector* to denote the essential entity, while we simply use the word element of  $\mathbb{R}^n$ , or the matrix in  $\mathbb{R}^{m \times n}$  for the representational use.
6. The connection from the set of essential entities to for instance  $\mathbb{R}^n$ , is done by fixing the basis. Let us be clear on this below.

- The definition of vector space  $X$  over the scalar  $\mathbb{R}$  can be found in math textbooks and webpages. For completeness, we also contained the mathematical definition in the last page of this chapter.
- If you are not familiar with mathematical definitions, consider a vector space  $X$  as a set of symbols that can be added and scaled.
- $k$ -dimensional vector space for  $k$  a positive integer.

**Definition 3.** A vector space  $X$  that admits a basis consists of  $k$  elements is called a  $k$ -dimensional vector space.

**Example** The space of colors admits a basis of three elements. Hence the space is 3-dimensional vector space.

- Connection from the *essential entities to representing elements* is done once we fix what are the basis elements in  $X$ .

**Example** The space of sauces consists of  $a = \text{sugar}$ ,  $b = \text{salt}$ ,  $c = \text{vinegar}$ , and  $d = \text{water}$ .

Once we fix  $a$ ,  $b$ ,  $c$ , and  $d$  to be the basis, we will understand an element

$$\begin{pmatrix} 5 \\ 5 \\ 3 \\ 100 \end{pmatrix} \in \mathbb{R}^4$$

to represent a sauce consisting of 5-sugar, 5-salt, 3-vinegar, and 100 water, in grams.

We may want to take the diluted ingredients  $a + 100d$ ,  $b + 100d$ ,  $c + 100d$ , and  $d$  to be the basis. Then

$$\begin{pmatrix} 5 \\ 5 \\ 3 \\ 100 \end{pmatrix} \in \mathbb{R}^4$$

to represent a sauce consisting of 5-sugar, 5-salt, 3-vinegar, and 1400 water, in grams.

- You can see the detached roles of *representation* and *essential entities*.
- And how the basis choice connects the two.

A vector space  $X$  over the scalar  $\mathbb{R}$  is mathematically defined as follows.

1.  $X$  is a nonempty set.
2.  $+ : X \times X \rightarrow X$  is well-defined with following properties.
  - (a) For any  $x, y, z \in X$ ,  $x + (y + z) = (x + y) + z$ .
  - (b) For any  $x, y \in X$ ,  $x + y = y + x$ .
  - (c)  $\exists 0 \in X$  such that for any  $x \in X$   $x + 0 = x$ .
  - (d) For any  $x \in X$ ,  $\exists (-x) \in X$  such that  $x + (-x) = 0$ .
3.  $s : \mathbb{R} \times X \rightarrow X$  is well-defined with following properties.
  - (a) For any  $\lambda_1, \lambda_2 \in \mathbb{R}$  and  $x \in X$ ,  $\lambda_1(\lambda_2 x) = (\lambda_1 \lambda_2)x$
  - (b) For any  $x \in X$ ,  $1x = x$ .
4.  $+$  and  $s$  work in compatible ways satisfying the following.
  - (a) For any  $\lambda \in \mathbb{R}$  and  $x, y \in X$ ,  $\lambda(x + y) = \lambda x + \lambda y$ .
  - (b) For any  $\lambda_1, \lambda_2 \in \mathbb{R}$  and  $x \in X$ ,  $(\lambda_1 + \lambda_2)x = \lambda_1 x + \lambda_2 x$ .

## Summary

1. For an  $m$ -dimensional vector space  $(X, +, s, \{b_1, b_2, \dots, b_m\})$  with  $m$  basis vectors  $b_1, b_2, \dots, b_m$ , we consider the representation of its element for example

$$\begin{pmatrix} 3 \\ -5 \\ 0 \\ \vdots \\ 0.8 \end{pmatrix} \in \mathbb{R}^{m \times 1}$$

to represent  $3b_1 - 5b_2 + 0b_3 + \dots + 0.8b_m$ .

2. Be careful when  $X$  itself is  $\mathbb{R}^m$ , which is the very possible example. One way we will follow in this course is that, for the representational use, we make use of

$$\mathbb{R}^{m \times 1} \text{ of one column matrices.}$$

and try best to distinguish the essential entity  $X$  and its representation.

## **Chapter 5**

# **Transformation of Solid shapes**

We will spend long times studying transformations of geometrical shapes.

### **Examples**

Cube, Airplane, Right Hand.

#### **1. Rotation**

- How do we make an airplane heading to a certain direction ?

## 2. Dilation

- Dilation in all direction.
- Dilation differently for semi-axes.
- Dilation differently for a few non-orthogonal directions.

## 3. Reflection with respect to a hyperplane.

- Right hand vs. Left hand.
- Composition of many reflections.

4. Projection

- Same projection twice ?
- Composition of many projections.

5. Shearing

For those solid shapes and transformations, we do the following.

For solid shapes:

- We let the set

$$X = \{\text{Locations}\} \quad \text{with respect to a fixed origin.}$$

- Location with respect to an origin can be scaled by a real number.

We let locations be added using the parallelogram law.

You can check  $X$  of locations to a fixed origin is a vector space.

More precisely,  $X$  is equipped with the basis locations  $e_1, e_2, e_3$  for the case of 3-dimensions. In general, up to  $e_n$  for  $n$ -dimensional locations.

- A solid shape is then a subset occupying a region of locations.

$$G \subset X$$

A transformation of a shape can be then a mapping from Location space to itself.

- A transformation is then a mapping

$$T : X \rightarrow X.$$

- We let the set of all linear transformations from Location space  $X$  to itself:

$$L_{X,X} = \{T : X \rightarrow X \mid T \text{ is linear}\}.$$

- Of course, in general we can think of

$X$ , the space of  $n$ -dimensional locations to some fixed origin,

$Y$ , the space of  $m$ -dimensional locations to some fixed origin,

$L_{X,Y}$ , the space of linear mappings  $T : X \rightarrow Y$ .

**Representation:**

- We will use

$$\mathbb{R}^{n \times 1}, \quad \mathbb{R}^{n \times n} \quad \text{for } X \text{ and } L_{X,X}.$$

- Indeed,  $n \times n$  matrix  $A$  and  $n \times 1$  column  $x$  can be multiplied to produce another  $n \times 1$  column  $Ax$ , representing a new location.
- By techniques in  $\mathbb{R}^{n \times n}$  (or in general  $\mathbb{R}^{m \times n}$ ), you will learn the surprising fact

that every linear transformation is a few compositions of rotation, dialation, reflection, shearing, and projections. No others.

The composition of linear transformations will be stacking matrix multiplications

$$T = A_k A_{k-1} A_{k-2} \cdots A_2 A_1$$

of  $n \times n$  matrices.

**Home Work**

Suppose  $A \in \mathbb{R}^{k \times \ell}$ ,  $B \in \mathbb{R}^{\ell \times m}$ , and  $C \in \mathbb{R}^{m \times n}$  so that we can multiply  $A$  and  $B$ , and multiply  $B$  and  $C$ . Try to check that

$$(AB)C = A(BC).$$

A map  $T : X \rightarrow Y$  is linear if

for a linear combination  $\lambda_1x_1 + \lambda_2x_2 + \cdots + \lambda_kx_k$

$T(\lambda_1x_1 + \lambda_2x_2 + \cdots + \lambda_kx_k)$  must equal to  $\lambda_1T(x_1) + \lambda_2T(x_2) + \cdots + \lambda_kT(x_k)$ .

- We will write and analyze matrices for rotations, reflections, dilations, shearings, and projections one by one from next lecture.
- This is to study orthogonal matrices, diagonal matrices, symmetric matrices, triangular matrices, and so on, which we will introduce.

We have  $x \in \mathbb{R}^{n \times 1}$  and  $A \in \mathbb{R}^{n \times n}$ , by which, the new location

$$y = Ax \in \mathbb{R}^{n \times 1}$$

We proceed with  $n = 3$ .

Today, we introduce a few kinds of matrices. We will need only one more technique to deal with axes that are not coordinate axes. The point is that other than ones we introduce today + the technique, you do not need to know any further kinds of matrices.

## Identity Matrix

1. We consider the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{R}^{3 \times 3}$$

2. It is a special element in  $\mathbb{R}^{3 \times 3}$ , and denoted by  $I$ .

3. Note that for any  $x = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ ,

$$y = Ax = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

4. Be careful that  $I$  is not the one below !

$$I \neq \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

- Diagonal entries are numbers in diagonal of the square.
- Off diagonal entries are entries not in diagonal.
- All of diagonal entries of  $I$  are 1
- All of off diagonal entries of  $I$  are 0.

## Diagonal Matrices

1. Let  $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}^3$ . We consider a matrix

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \in \mathbb{R}^{3 \times 3}.$$

2. First, let us consider the special cases  $\lambda_1 > 0, \lambda_2 > 0, \lambda_3 > 0$

(a) Example 1 :  $x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$      $Ax =$

(b) Example 2 :  $x = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$      $Ax =$

(c) Example 3 :  $x = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$      $Ax =$

(d) Example 4 :  $x = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$      $Ax =$

3.  $\lambda_i = 0$ : One component is shrinked to 0.

4.  $\lambda_i < 0$ : Dilate with sign flip.

5. More matrices for the dilations below will be introduced later.

(a) Principal axes are not coordinate axes.

(b) Dilation with directions not orthogonal to each other.

## Elementary Rotation Matrices

1. Let  $\theta \in \mathbb{R}$ . We consider a matrix

$$\begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{R}^{3 \times 3}.$$

(a) Example 1 :  $x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad Ax =$

(b) Example 2 :  $x = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad Ax =$

(c) Example 3 :  $x = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad Ax =$

(d) Example 4 :  $x = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad Ax =$

2. We consider a matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix} \in \mathbb{R}^{3 \times 3}.$$

- (a) Example 1 :  $x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$      $Ax =$
- (b) Example 2 :  $x = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$      $Ax =$
- (c) Example 3 :  $x = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$      $Ax =$
- (d) Example 4 :  $x = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$      $Ax =$

3. We consider a matrix

$$\begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix} \in \mathbb{R}^{3 \times 3}.$$

(a) Example 1 :  $x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$      $Ax =$

(b) Example 2 :  $x = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$      $Ax =$

(c) Example 3 :  $x = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$      $Ax =$

(d) Example 4 :  $x = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$      $Ax =$

We will introduce more rotation matrices later.

## Elementary Orthogonal Projection Matrices

1. We consider a matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{3 \times 3}.$$

(a) Example 1 :  $x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$      $Ax =$

(b) Example 2 :  $x = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$      $Ax =$

(c) Example 3 :  $x = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$      $Ax =$

(d) Example 4 :  $x = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$      $Ax =$

2. We will introduce more orthogonal projection matrices onto the projecting plane that is not aligned with coordinate.
3. There are non-orthogonal projections, but we will not use them much. We will introduce them later.

## Elementary Reflection Matrices

1. We consider a matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in \mathbb{R}^{3 \times 3}.$$

(a) Example 1 :  $x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$      $Ax =$

(b) Example 2 :  $x = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$      $Ax =$

(c) Example 3 :  $x = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$      $Ax =$

(d) Example 4 :  $x = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$      $Ax =$

2. We will introduce more reflection matrices whose symmetry hyperplane is not aligned with coordinate.

## Elementary Shearing Matrices

1. We consider a matrix

$$\begin{pmatrix} \lambda_1 & \alpha & \beta \\ 0 & \lambda_2 & \gamma \\ 0 & 0 & \lambda_3 \end{pmatrix} \in \mathbb{R}^{3 \times 3}.$$

2. This kind of matrices are called upper triangular matrices.

(a) Example 1 :  $x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$      $Ax =$

(b) Example 2 :  $x = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$      $Ax =$

(c) Example 3 :  $x = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$      $Ax =$

(d) Example 4 :  $x = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$      $Ax =$

1. As soon we are able to freely go back and forth from one representation to another representation w.r.t choice of basis vectors, we will know in full generality about matrices of above kinds.
2. Again, you recall that you do not need to know more kinds other than those.

1. Now, we learn the technique to deal with axes oblique to coordinate axes.
2. To be concrete, let us be given an objective to come up with dilation but with respect to three axes, specified by three location points

$$\begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

with dilation factors  $\lambda_1 = \frac{1}{2}$ ,  $\lambda_2 = 2$ , and  $\lambda_3 = 1$ .

3. We will use the three as our basis and seek a *new representations* of locations.
4. Typically, as seen in the example, we would know old representations of your preferred temporal basis.

## I know $B$ . The Old representation of your new basis

Let us list those old representations of your temporal basis in columns of a  $n \times n$  matrix  $B$ :

$$B = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} | & | & | \\ b_1 & b_2 & b_3 \\ | & | & | \end{pmatrix}.$$

All you need to remember is the following equality:

If your new representation of a certain location  $x$  is  $\begin{pmatrix} a' \\ b' \\ c' \end{pmatrix}$ ,

$$\begin{pmatrix} | & | & | \\ b_1 & b_2 & b_3 \\ | & | & | \end{pmatrix} \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

is nothing but its old representation. To remember:

$$B_{ij'} a_{j'} = a_i.$$

Because

$$\begin{pmatrix} | & | & | \\ b_1 & b_2 & b_3 \\ | & | & | \end{pmatrix} \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix},$$

(recall that  $B$  is the Old representations of you new basis)

if we want to know what are  $\begin{pmatrix} a' \\ b' \\ c' \end{pmatrix}$ , the temporal representation, we get it by having

$$B^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

We will re-visit the inverse matrix later to study it properly, but here we just admit that  $B^{-1}$  of our matrix  $B$  exists with the property we want.

**Definition 4.** For a given  $A \in \mathbb{R}^{3 \times 3}$ , if there exists a matrix  $L \in \mathbb{R}^{3 \times 3}$  such that

$$AL = LA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which must be then unique,  $L$  is said to be the inverse matrix of  $A$ , and denoted by  $A^{-1}$ .

If we multiply  $B^{-1}$  both sides,

- (RHS) =  $B^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ ,
- (LHS) =  $B^{-1}B \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} = \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} = (\text{RHS})$ .

Now, we are able to achieve our objective of non-elementary dialation:

- Get the new representation, do the elementary dilation in new representation, and don't forget to return the result, not in your representation, but in old representation.

This is to conduct:

$$\begin{aligned}
 x &= \begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \\
 &\mapsto \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \\
 &\mapsto \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = y
 \end{aligned}$$

Because I know that the inverse matrix

$$\begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

I can compute one by one.

## Example 2: non-elementary reflection

- We want to compose a matrix that does the reflection with respect to a hyperplane we want.
- That I know what the hypline is: I know representations of normal vector and tangent vectors of the hyperplane.
- What we have to do:
  1. Get the new repretation of  $x$  in temporal basis of normal and tangents.
  2. Flip the sign of normal. (This is an elementary reflection).
  3. Return back to the old representation.

- For example, let the hyperplane is normal to  $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ .

1. I know one choice of basis:

$$b_1 = \begin{pmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{pmatrix}, \quad b_2 = \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}, \quad b_3 = \begin{pmatrix} -\frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{6} \\ \frac{2\sqrt{6}}{6} \end{pmatrix}.$$

2. The matrix  $B$  of old representation of new temporal basis is

$$B = \begin{pmatrix} \frac{\sqrt{3}}{3} & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} \\ \frac{\sqrt{3}}{3} & \frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} \\ \frac{\sqrt{3}}{3} & 0 & \frac{2\sqrt{6}}{6} \end{pmatrix}$$

3. We then conduct

$$\begin{aligned} x = \begin{pmatrix} a \\ b \\ c \end{pmatrix} &\mapsto \begin{pmatrix} \frac{\sqrt{3}}{3} & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} \\ \frac{\sqrt{3}}{3} & \frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} \\ \frac{\sqrt{3}}{3} & 0 & \frac{2\sqrt{6}}{6} \end{pmatrix}^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \\ &\mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{3} & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} \\ \frac{\sqrt{3}}{3} & \frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} \\ \frac{\sqrt{3}}{3} & 0 & \frac{2\sqrt{6}}{6} \end{pmatrix}^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \\ &\mapsto \begin{pmatrix} \frac{\sqrt{3}}{3} & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} \\ \frac{\sqrt{3}}{3} & \frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} \\ \frac{\sqrt{3}}{3} & 0 & \frac{2\sqrt{6}}{6} \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{3} & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} \\ \frac{\sqrt{3}}{3} & \frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} \\ \frac{\sqrt{3}}{3} & 0 & \frac{2\sqrt{6}}{6} \end{pmatrix}^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = y \end{aligned}$$

### Example 3: non-elementary, non-orthogonal projection

- Goal: We want to compose a matrix that does the projection of a point onto the two-dimensional, or one-dimensional plane, in the manner described below.
- We let

$$b_1 \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}, \quad b_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

1. Get the new representation of  $x$  in temporal basis.
2. Project on to the axes. (This is an elementary projection).
3. Return back to the old representation.

- In Matrix Multiplications:

1. The matrix  $B$  of old representation of new temporal basis is

$$B = \begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2. Let us admit that

$$B^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

3. We then conduct for example for  $x = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}$

$$\begin{aligned} x = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} &\mapsto \begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \\ &\mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ &\mapsto \begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

4. We also compute this non-elementary, non-orthogonal projection matrix

$$A = \begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

You do the examples on Rotation, and Shearing that takes places with respect to axes you choose.

### **Home Work 2.**

Consider

$$b_1 = \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}, \quad b_2 = \begin{pmatrix} -\frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{6} \\ \frac{2\sqrt{6}}{6} \end{pmatrix}, \quad b_3 = \begin{pmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{pmatrix}.$$

Write the  $3 \times 3$  rotation matrix, that makes a rotation  $\theta$  counter-clock-wisely in the plane specified by  $b_1$  and  $b_2$ , and that keeps the remaining component unchanged.

## Optional Facts.

**Question.** How do I recognize which  $n \times n$  matrix does which ?

- There are a few recognizable patterns. We introduce FACTS without proofs:

1. Dilations with respect to orthogonal axes.

$$\mathbb{S} = \{A \in \mathbb{R}^{n \times n} \mid A \text{ is symmetric}\} \subset \mathbb{R}^{n \times n}.$$

This includes, of course, diagonal and identity matrix.

We saw in the lecture the example of  $A = \begin{pmatrix} \frac{5}{4} & -\frac{3}{4} & 0 \\ -\frac{3}{4} & \frac{5}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ .

2. Projections are those: transform twice does not count.  $PP = P$ .

- (a) Projecting twice does not count:

$$\mathbb{P} = \{A \in \mathbb{R}^{n \times n} \mid AA = A\} \subset \mathbb{R}^{n \times n}.$$

(b) Indeed, If  $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ , you can see  $PP = P$ .

- (c) If  $P$  is projection, then complementary matrix  $I - P$  automatically has the same property, namely,

$$(I - P)(I - P) = I I - P - P + PP = I - 2P + P = I - P.$$

- (d) Indeed, we check  $A = BPB^{-1}$ , where  $P$  is an elementary projection.

$$AA = BPB^{-1}BPB^{-1} = BPIP B^{-1} = BPPB^{-1} = BPB^{-1}.$$

3. Rotations are those:

$$\text{SO} = \{A \in \mathbb{R}^{n \times n} \mid (\text{i}) \text{ columns of } A \text{ are orthonormal. (ii) columns are right-handed.}\}.$$

We saw the example of an elementary rotation  $A = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

4. Reflections are those:

$$\text{H} = \{A \in \mathbb{R}^{n \times n} \mid (\text{i}) \text{ columns of } A \text{ are orthonormal. (ii) columns are left-handed.}\}.$$

We saw the example of an elementary reflection  $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

5. We know that upper triangular matrix does the shearing.

$$\text{T} = \{A \in \mathbb{R}^{n \times n} \mid A \text{ is upper triangular.}\}$$

As a matter of fact, every  $n \times n$  matrix is a rotation of an upper triangular matrix.

- More importantly and as a matter of fact, every  $n \times n$  matrix  $A$  can be written in the form

$$A = B E_k E_{k-1} E_{k-2} \cdots E_2 E_1 B^{-1},$$

where  $E_k E_{k-1} E_{k-2} \cdots E_2 E_1$  is the finite consecutive applications of elementary transforms.

- General answer to the question is thus: We run the decomposition algorithm to obtain  $A = B E_k E_{k-1} E_{k-2} \cdots E_2 E_1 B^{-1}$ . Then you know what it does.

## Final Remark

You should be able to tell clearly the differences of the following two:

If you can, then you understood very well what is going on by manipulating matrices.

### 1. Transformation 1

- Let the  $3 \times 3$  matrix  $A_1 = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ,
- Let the  $3 \times 3$  matrix  $A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ,
- Let the  $3 \times 3$  matrix  $A_3 = \begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ .
- We suppose  $A_1$ ,  $A_2$  and  $A_3$  are the three representations of some linear transformations.
- Then for a location  $x$  represented by  $\begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}$ , we consider the three consecutive linear transformations:

$$\begin{aligned} x = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} &\mapsto \begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \\ &\mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ &\mapsto \begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

- But, what is being essentially done here is to obtain:

$$\begin{aligned} \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} &\mapsto \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = y_1 \quad \text{the 1st new location in the same representation} \\ \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} &\mapsto \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = y_2 \quad \text{the 2nd new location in the same representation} \\ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} &\mapsto \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = y_3 \quad \text{the 3rd new location in the same representation.} \end{aligned}$$

### 2. Transformation 2 : the projection we did today.

- Here, what is being essentially done is different things:

Get the new representation of  $x \mapsto$  Get the new location  $y \mapsto$  Get  $y$ 's old representation

$\Rightarrow$  What are being done essentially are different. Their representational matrix multiplications are the same. But you understand what's going on here now.

## Chapter 6

# Linear mappings from $X$ to $Y$

- We will re-visit  $\mathbb{R}^{n \times n}$  of square matrices, to discuss the determinant of a square matrix, the inverse if exists, how we can calculate them, and so on.
  - Continuing discussions on the question: What does the matrix  $A$  do ?
  - On the matrix  $A \in \mathbb{R}^{m \times n}$  that is not square.
  - A matrix in  $\mathbb{R}^{n \times n}$  is called a square matrix. A matrix in  $\mathbb{R}^{m \times n}$  is called a rectangular matrix. Of course, a square matrix is a rectangular matrix.
- 
- Because we have developed the same on : “what does a square matrix  $A$  do?”,
  - We can also easily understand “what does a rectangular matrix do?” too.
  - This is done through the notion of the rank of matrix.

First of all, in this chapter,

- $\mathbb{R}^{n \times 1}$  represent a vector space  $X$  of dimension  $n$ ,
- $\mathbb{R}^{m \times 1}$  represent another vector space  $Y$  of dimension  $m$ ,
- $\mathbb{R}^{m \times n}$  represent the space of linear mappings from  $X$  to  $Y$ .

## Rank of a matrix $A$

Let  $A \in \mathbb{R}^{m \times n}$ , i.e., it has  $m$  rows and  $n$  columns, in the way

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1n} \\ A_{21} & A_{22} & A_{23} & \cdots & A_{2n} \\ A_{31} & A_{32} & A_{33} & \cdots & A_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{m1} & A_{m2} & A_{m3} & \cdots & A_{mn} \end{pmatrix}.$$

**Motivation and observation: when the number “one” is relevant**

Consider a special matrix  $A \in \mathbb{R}^{3 \times 5}$ ,

$$A = \begin{pmatrix} 1 & 2 & 10 & 200 & 3 \\ 2 & 4 & 20 & 400 & 6 \\ 3 & 6 & 30 & 600 & 9 \end{pmatrix}$$

so that every columns are constant multiple of the first column.

But, it is also such that every rows are constant multiple of the first row.

Try differently. Is this always so ?

**Motivation and observation: when the number “two” is relevant**

Consider a special matrix  $A \in \mathbb{R}^{2 \times 3}$ ,

$$A = \begin{pmatrix} 3 & 0 & -9 \\ 1 & 2 & 1 \end{pmatrix}$$

Certainly,  $(3, 0, -9)$  and  $(1, 2, 1)$  are the two independent rows.

Are the three columns independent ?

No. Certainly,  $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$  and  $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$  are linearly independent.

However,  $\begin{pmatrix} -9 \\ 1 \end{pmatrix} = -3 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ , i.e.,  $\begin{pmatrix} -9 \\ 1 \end{pmatrix} \in \text{span} \langle \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \rangle$ .

Thus, the number of linearly independent columns coincide the number “two”.

It is not obvious to see why, but the number of linearly independent rows and the number of linearly independent columns seem to be the same.

To be able to check the statement in mathematical way, we do the followings.

1. We collect

$$\text{span} \langle n \text{ columns of } A \rangle \subset \mathbb{R}^{m \times 1}.$$

2. And we also collect

$$\text{span} \langle m \text{ rows of } A \rangle \subset \mathbb{R}^{1 \times n}.$$

The question is how many number of linearly independent elements in them ?

Before we proceed, we introduce two notations.

- For  $A \in \mathbb{R}^{m \times n}$ ,  $A^T$ , the transpose of  $A$ , is the matrix in  $\mathbb{R}^{n \times m}$  where the rows and columns are interchanged, i.e.,

$$(A^T)_{ij} = A_{ji}, \quad j = 1, 2, \dots, m, \quad i = 1, 2, \dots, n.$$

- For two columns  $x, y \in \mathbb{R}^{\ell \times 1}$ , we say  $x$  and  $y$  are orthogonal to each other if

$$x^T y = 0.$$

We now give a name for the above two we collected.

1. We call the first one, as

$$\text{Ran}(A), \quad \text{the range of } A$$

2. We consider the *transpose* the second one,

$$\text{Trow}(A) = \text{span} \langle m \text{ transpose of rows of } A \rangle \subset \mathbb{R}^{n \times 1}.$$

The range  $\text{Ran}(A)$  is literally the range of  $A$ .

- Check: if  $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ , then according to the Matrix Multiplication rule,

$Ax = x_1c_1 + x_2c_2 + \cdots + x_nc_n$ , a linear combination of columns of  $A$ .

- Here,  $c_1, c_2, \dots, c_n$  are columns of  $A$ .

A subset  $E \subset X$  is called a vector subspace of  $X$  if  $E$  is also a vector space.

$\text{Ran}(A) \subset \mathbb{R}^{m \times 1}$  is a *vector subspace* of  $\mathbb{R}^{m \times 1}$ , and  $\text{Trow}(A) \in \mathbb{R}^{n \times 1}$  is a *vector subspace* of  $\mathbb{R}^{n \times 1}$ .

- Indeed, any linear combination of elements in the set, is again in the set, i.e.,
 

if  $\mathbb{R}^{m \times 1}$  columns  $a_1, a_2, \dots, a_k \in \text{Ran}(A)$   $\implies \lambda_1a_1 + \lambda_2a_2 + \cdots + \lambda_ka_k \in \text{Ran}(A)$ .  
 if  $\mathbb{R}^{n \times 1}$  columns  $b_1, b_2, \dots, b_\ell \in \text{Trow}(A)$   $\implies \lambda_1b_1 + \lambda_2b_2 + \cdots + \lambda_\ell b_\ell \in \text{Trow}(A)$ .
- We also check other conditions to be a vector space.

The objective of the next lecture is to reveal the following:

1.  $\text{Trow}(A)$  and  $\text{Ran}(A)$  are both  $r$ -dimensional, and
2. “The question we posed” can be understood by looking at the square matrix of  $\mathbb{R}^{r \times r}$   
 $\text{Trow}(A) \rightarrow \text{Ran}(A)$ .  
 between the two.
3. What does the square matrix do? We completely know.

This is known as the Fundamental Theorem of Linear Algebra. We will go through this in the next lecture in detail.

## Chapter 7

# Fundamental Theorem of Linear Algebra

Here, we verify the picture:

through the fundamental theorem of linear algebra. We divide it into two parts.

We recall the orthogonality.

**Theorem 1.** Let  $x_0 \in \mathbb{R}^{n \times 1}$ .

$$x_0 \perp \text{Trow}(A) \iff Ax_0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

*Proof.*

$$\begin{aligned} x_0 \perp \text{Trow}(A) &\iff x_0^T(\lambda_1 b_1 + \lambda_2 b_2 + \cdots + \lambda_m b_m) = 0, \quad b_1, b_2, \dots, b_m \text{ are transpose of rows of } A \\ &\iff x_0^T b_i = 0 \quad \text{for every } i = 1, 2, \dots, m \\ &\iff i\text{-th component of } Ax_0 = 0 \text{ for every } i = 1, 2, \dots, m, \end{aligned}$$

according to the Matrix Multiplication Rule.  $\square$

The theorem tells the following.

1. In  $\mathbb{R}^{n \times 1}$ , we choose orthonormal basis elements of  $\mathbb{R}^{n \times 1}$   $v_1, v_2, \dots, v_r, v_{r+1}, \dots, v_n$  such that
2. The first  $r$  basis elements  $v_1, v_2, \dots, v_r$  span  $\text{Trow}(A)$ .
3. We let

$$V = \begin{pmatrix} | & | & \cdots & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & \cdots & | \end{pmatrix}.$$

4. Then we can think of the  $n \times n$  projection matrix  $P$ :

$$P = V \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} V^{-1}.$$

This gives, for any  $x \in \mathbb{R}^{m \times 1}$ , the projections

$$x_0 = (I - P)x, \quad x' = Px, \quad x = x_0 + x'.$$

$x'$  is an element of  $\text{Trow}(A)$ .

5.  $x_0$  part of  $x$  does not do anything under the multiplication of  $A$ :

$$Ax = A(x_0 + x') = 0 + Ax' = Ax'.$$

**Theorem 2.** For any matrix  $A \in \mathbb{R}^{m \times n}$ ,

the number of lin. indep. elements in  $\text{Trow}(A)$  = the number of lin. indep. elements in  $\text{Ran}(A)$ .

- In fact we only need to check that for every matrix  $A \in \mathbb{R}^{m \times n}$ ,  
 the number of lin. indep. elements in  $\text{Trow}(A)$   $\leq$  the number of lin. indep. elements in  $\text{Ran}(A)$ .
- This is because, if that is true, then do the same on another matrix  $B = A^T$ ,  
 the number of lin. indep. elements in  $\text{Trow}(A^T)$   $\leq$  the number of lin. indep. elements in  $\text{Ran}(A^T)$ .

*Proof.* We only check the inequality:

1. To do that: Suppose  $b_1, b_2, \dots, b_r \in \text{Trow}(A)$  are linearly independent. We show that  $a_1 = Ab_1, a_2 = Ab_2, \dots, a_r = Ab_r$  are linearly independent.
2. Method by proving contrapositive statement.
3. Suppose that  $a_1, a_2, \dots, a_r$  are linearly dependent. i.e., we can find coefficients  $\lambda_1, \lambda_2, \dots, \lambda_r$ , not all of them are zero, such that

$$\lambda_1 a_1 + \lambda_2 a_2 + \dots + \lambda_r a_r = 0.$$

4. In other words, by linearity,

$$A(\lambda_1 b_1 + \lambda_2 b_2 + \dots + \lambda_r b_r) = 0.$$

5. But Theorem 1 says that then

$$x_0 = \lambda_1 b_1 + \lambda_2 b_2 + \dots + \lambda_r b_r \in \text{Trow}(A) \text{ is orthogonal to all of } \text{Trow}(A).$$

6. The only element in  $\text{Trow}(A)$  that is orthogonal to all of  $\text{Trow}(A)$  is only

$$x_0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

7. This concludes that  $b_1, b_2, \dots, b_r \in \text{Trow}(A)$  are linearly dependent.

□

The theorem tells the following.

1. In  $\mathbb{R}^{n \times 1}$ , we choose orthonormal basis elements of  $\mathbb{R}^{n \times 1} v_1, v_2, \dots, v_r, v_{r+1}, \dots, v_n$  such that
2. The first  $r$  basis elements  $v_1, v_2, \dots, v_r$  span  $\text{Trow}(A)$ .
3. In  $\mathbb{R}^{m \times 1}$ , we do the similar, choose orthonormal basis elements of  $\mathbb{R}^{m \times 1} w_1, w_2, \dots, w_r, w_{r+1}, \dots, w_m$  such that
4. The first  $r$  basis elements  $w_1, w_2, \dots, w_r$  span  $\text{Ran}(A)$ .
5. Importantly the number  $r$  is shared !
6. The mapping goes in the three steps: Let  $x \in \mathbb{R}^{n \times 1}$ .

step 1  $x$  first undergoes the reduced projection:

$$x \mapsto \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix} V^{-1} x = x''.$$

$x''$  is in temporal representation with new basis, but for convenience, we let the components from  $(r+1)$ -th is removed. Thus,  $x'' \in \mathbb{R}^{r \times 1}$ .

step 2 The next step is the mapping from  $\text{Trow}(A)$  to  $\text{Ran}(A)$ . We divide this into two steps, one is to take *internal transformation*

$$x'' \mapsto T x'', \quad T : \mathbb{R}^{r \times 1} \rightarrow \mathbb{R}^{r \times 1}.$$

step 3 The last step is the image copy. In the last stage, the  $\mathbb{R}^{r \times 1}$  column  $T x'' = x'''$  is in temporal representation with  $v_1, v_2 \dots, v_r$ .

$$\begin{pmatrix} x_1''' \\ x_2''' \\ \vdots \\ x_r''' \end{pmatrix}.$$

This is mapped to

$$x_1'''w_1 + x_2'''w_2 + \cdots + x_r'''w_r = \begin{pmatrix} | & | & \cdots & | \\ w_1 & w_2 & \cdots & w_r \\ | & | & \cdots & | \end{pmatrix} \begin{pmatrix} x_1''' \\ x_2''' \\ \vdots \\ x_r''' \end{pmatrix}.$$

This ends the whole picture. A rectangular matrix  $A \in \mathbb{R}^{m \times n}$  can be understood by the  $r \times r$  matrix  $T$  in the following decomposition

$$A = W T P V^{-1}.$$

$V^{-1}$	: $n \times n$ matrix	basis selection w.r.t $\text{Trow}(A)$ .
$P$	: $r \times n$ matrix	projection onto $\text{Trow}(A)$ , reduced
$T$	: $r \times r$ matrix	some $\mathbb{R}^{r \times r}$ transformation of $\text{Trow}(A)$
$W$	: $m \times r$ matrix	the final image mapping to $\text{Ran}(A)$

## Chapter 8

# Computing by Hands: The Picture of Fundamental Theorem of Linear Algebra

In this chapter, we discuss about following questions. Suppose a matrix  $A \in \mathbb{R}^{m \times n}$  is given.

1. How to pick up basis elements of  $\text{Ran}(A)$  ?
2. How to pick up basis elements of  $\text{Trow}(A)$  ?
3. How to pick up basis elements of  $\text{Trow}(A)^\perp$ ?
4. How to pick up basis elements of  $\text{Ran}(A)^\perp$ ?
5. How to calculate the rank  $r$ ?

In case the shape of the matrix  $A$  is of very big size, in general, those questions are subjected to running some algorithm using computer.

However, if the size of the matrix  $A$  is not too large, you would want to do it by your hands with pen and paper. Today, we introduce one technique starting with two Good News.

1. In fact, you already know this.
2. The way it achieves is so nice that you would extract good enough information from a certain amount of work.

## Row operation: Replacement of $A$ by $A'$

- Here, we consider  $A'$  of same shape, instead of  $A$ , anyone satisfying  $\text{Trow}(A') = \text{Trow}(A)$ , i.e., the **span** of rows of  $A'$  and the **span** of rows of  $A$  are the same.
- You are free to change the order of rows.
- One typical example is to pick up one of the row of  $A$ , and replace it by the row itself + a linear combination of remaining rows of  $A$ .
- We could equivalently do the Column operation. Traditionally, this technique has been done in the way of Row Operation.

Here comes the first good news, that you already know this:

### Example

$$\begin{cases} 2x - 3y = 5 \\ x - 2y = 2 \end{cases} \quad \text{Let } A = \begin{pmatrix} 2 & -3 & 5 \\ 1 & -2 & 2 \end{pmatrix}$$

1. According to the Matrix Multiplication Rule, Row operation is implemented by multiplying  $m \times m$  matrix from the left.
2. For our purposes, it is matter of choice whether you involves those  $m \times m$  matrices or you omit them. Do what are convenient for you.

## The purpose of the replacement, and why it is easy

1. We are here, replacing the matrix  $A$  sequentially by

$$A \rightarrow A' \rightarrow A'' \rightarrow A''' \rightarrow \dots$$

while keeping  $\text{Trow}(A) = \text{Trow}(A') = \text{Trow}(A'') = \text{Trow}(A''') = \dots$ .

2. Of course, those matrices appear later have to be simpler than ones appeared before, in what so ever senses. For example, there are more 0 in the entries. Or we can easily extract some basis elements of  $\text{Trow}(A)$  out of it.
3. By contrast, the information on  $\text{Ran}(A)$  are completely forgotten. In general,

$$\text{Ran}(A) \neq \text{Ran}(A'), \quad \text{Ran}(A') \neq \text{Ran}(A''), \quad \dots$$

and in this stage we simply do not care whatever happens in range.

$\implies$  The consequence then will be:

- We will see much simpler picture of  $\text{Trow}(A)$ ,
- We will still keep the same  $\text{rank}(A)$ , thanks to the Fundamental Theorem.
- We will completely lose the picture of  $\text{Ran}(A)$ .
- But once the Row operation is finished, column picture will be so nicely recovered.  
This is the second good news.
- We do this one by one.
- The goal of this is known as the Column-Row (CR)-decomposition, that can be seen in the Lecture of G. Strang.

What do we do ?:

1. Introduce 0 as many as possible.
2. Turn it to the row echelon form.
3. Turn it to the reduced row echelon form.
4. Check if calculations went correct: Complete the Column-Row decomposition.

**Example 2**

$$A = \begin{pmatrix} 1 & 3 & 10 & 200 \\ 2 & 6 & 20 & 400 \\ 3 & 9 & 30 & 600 \end{pmatrix}$$

**Example 3**

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 2 & 3 & 7 \\ 1 & 4 & 6 \end{pmatrix}$$

**Example 4**

$$A = \begin{pmatrix} 0 & -7 & -4 & 2 \\ 2 & 4 & 6 & 12 \\ 3 & 1 & -1 & -2 \end{pmatrix}$$

**Example 5**

$$A = \begin{pmatrix} 0 & 1 & 2 & 1 & 2 \\ 1 & 0 & 0 & 0 & 1 \\ 2 & 1 & 2 & 1 & 5 \\ 1 & 2 & 4 & 3 & 6 \end{pmatrix}$$

## More on computation by hands

**Example:** Complete the CR decomposition of

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 2 & 3 & 7 \\ 1 & 4 & 6 \end{pmatrix}$$

Summarize: For example, we arrived at

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 2 & 3 & 7 \\ 1 & 4 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}.$$

1.  $\text{rank}(A)$ : (RHS) is a product of  $(3 \times 2)$  and  $(2 \times 3)$  matrices.  $\text{rank}(A) = 2$ .
2.  $\text{Trow}(A)$ :  $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}$  is in much simpler form. Big improvement.
3.  $\text{Ran}(A)$ :  $\begin{pmatrix} 1 & 1 \\ 2 & 3 \\ 1 & 4 \end{pmatrix}$  at least reveals which columns are independent. Small improvement

1. We already know the  $\text{rank}(A)$ .
2. We select basis elements of the domain  $\mathbb{R}^{n \times 1}$ , adapted to  $\text{Trow}(A)$ .
3. We select basis elements of the codomain  $\mathbb{R}^{m \times 1}$ , adapted to  $\text{Ran}(A)$ .

**Selecting basis elements of the domain  $\mathbb{R}^{n \times 1}$ , adapted to  $\text{Trow}(A)$**

1. We select basis of  $\text{Trow}(A)$ .

Out of the last form

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}$$

We select two elements

$$\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

that span  $\text{Trow}(A)$ .

2. Now, we select basis of  $\text{Trow}(A)^\perp$ .

(a) This is to find element  $x_0 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$  such that

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

(b) This equation is super easy to solve.

(c) Recalling that the 1st and 2nd columns are marked independent columns, we

put  $c = -1$ , and let  $\begin{pmatrix} a \\ b \end{pmatrix}$  be the precisely  $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ . This gives rise to

$$\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}.$$

We have selected all the basis for the domain:

$$\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}.$$

The first two span  $\text{Trow}(A)$ , and the last span  $\text{Trow}(A)^\perp$ . They are not orthogonal to each other though.

### Selecting basis elements of the codomain $\mathbb{R}^{m \times 1}$ , adapted to $\text{Ran}(A)$

One possible way:

1. We select basis of  $\text{Ran}(A)$ .

Out of the last form

$$\begin{pmatrix} 1 & 1 \\ 2 & 3 \\ 1 & 4 \end{pmatrix}$$

We may select two elements

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}.$$

that span  $\text{Ran}(A)$ .

2. To select basis of  $\text{Ran}(A)^\perp$ , we solve

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

One another way I would do:

- Swapping the role of row and column, I consider  $A^T$ , and turn  $A^T$  into the Reduce Row Echelon Form.
- In fact, it suffices to turn  $C^T$  into the Reduced Row Echelon Form.
- Let us do that. Out of  $C^T$ ,

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 4 \end{pmatrix}$$

- We turn it into

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \end{pmatrix}$$

1. Now, we select basis of  $\text{Ran}(A)$ .

$$\begin{pmatrix} 1 \\ 0 \\ -5 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}.$$

2. Select basis of  $\text{Ran}(A)^\perp$ :

$$\begin{pmatrix} -5 \\ 3 \\ -1 \end{pmatrix}.$$

We have selected all the basis for the domain:

$$\begin{pmatrix} 1 \\ 0 \\ -5 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} -5 \\ 3 \\ -1 \end{pmatrix}.$$

The first two span  $\text{Ran}(A)$ , and the last span  $\text{Ran}(A)^\perp$ . They are not orthogonal to each other though.

### (Optional Material)

If you also have completed the decomposition of  $C^T$ ,

$$C^T = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \end{pmatrix},$$

You may write

$$A = CR = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -5 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}.$$

### Home Work

Do each of following steps for the matrices written below.

1. Turn the matrix  $A$  into the Reduced Row Echelon Form.
2. Complete the  $A = CR$  decomposition, checking if the calculation was correct.
3. Turn the matrix  $C^T$  into the Reduced Row Echelon Form.
4. Compute the rank of  $A$ .
5. Select basis elements of domain  $\mathbb{R}^{n \times 1}$ :
  - (a) basis elements of  $\text{Trow}(A)$ ,
  - (b) basis elements of  $\text{Trow}(A)^\perp$ .
6. Select basis elements of codomain  $\mathbb{R}^{m \times 1}$ :
  - (a) basis elements of  $\text{Ran}(A)$ ,
  - (b) basis elements of  $\text{Ran}(A)^\perp$ .

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 5 \\ 2 & 4 & 6 & 9 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$



## Chapter 9

# Invertible Square Matrices

We get back to the special cases of  $n \times n$  square matrices. In this chapter, unless otherwise specified, all matrices are square  $n \times n$  matrices.

We begin with observing the following nice feature on  $n \times n$  matrices.

1. If  $A$  and  $B$  are  $n \times n$  matrices, we can perform multiplications both  $AB$  and  $BA$ , although in general  $AB \neq BA$ .
2.  $AB$  and  $BA$  are again  $n \times n$  matrices. Hence, we can for instance perform  $ABCDEF$  of many  $n \times n$  matrices.
3. There exists unique  $n \times n$  matrix  $I \in \mathbb{R}^{n \times n}$  such that

$$\text{for every } A \in \mathbb{R}^{n \times n}, \quad AI = IA = A.$$

Hence, one naturally asks a question that for a given  $A$

1. Is there a  $G$  such that  $GA = I$  ?
2. Is there a  $H$  such that  $AH = I$  ?
3. If so, how can I compute  $G$  or  $H$  ?

We do the following in this chapter:

- We give the definition of invertibility of a matrix  $A \in \mathbb{R}^{n \times n}$ .
- Calculation of  $A^{-1}$  in case  $A$  is invertible, by the Row Reduction.

Let us state two important theorems about the invertibility of  $A$ .

**Theorem 1.** *Let  $A \in \mathbb{R}^{n \times n}$ . Then followings are all equivalent.*

1. *There exists  $G$  such that  $AG = I$ .*
2. *There exists  $H$  such that  $HA = I$ .*
3. *There exists  $E$  such that  $AE = EA = I$ .*
4.  *$A$  has full rank, that is  $n$ .*

**Theorem 2** (uniqueness of inverse). *Suppose that  $EA = AE = I$  and  $FA = AF = I$ . Then  $E = F$ .*

*Proof.*

$$EA = I \implies EAF = F \implies E = F.$$

□

Based on the two theorems, we give the following definition.

**Definition 3.** *For a given  $A \in \mathbb{R}^{n \times n}$ , we say  $A$  is invertible if there exists a matrix  $E \in \mathbb{R}^{n \times n}$  such that*

$$AE = EA = I,$$

*which must then be unique.  $E$  is said to be the inverse matrix of  $A$ , and denoted by  $A^{-1}$ .*

Because this course is more about practical aspects of Linear Algebra, we give proof of the Theorem 1 after we learn

- how to know if  $A$  is invertible or not,
- if invertible, how to compute  $A^{-1}$ .

## Second look : Turning $A$ to RREF

- Although we perform each of replacement turning  $A$  to RREF in the high speed and easy-going manner,
- All we did consists of the following two:
  1. Swapping two rows of  $A$ .
  2. Replace a row of  $A$  by

The row itself  $\times a$  ( $a \neq 0$ ) + Linear Combination of the remaining rows.

- The each of above two is implemented by multiplying a square matrix  $E$  from left: Indeed, for example if  $A \in \mathbb{R}^{4,n}$ ,

1. Swapping 1st and 3rd row in  $\mathbb{R}^{4 \times 4}$ :

$$E = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{to } A$$

2. Replace the 2nd row in  $\mathbb{R}^{4 \times 4}$ :

$$E' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \lambda_1 & a & \lambda_3 & \lambda_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{to } A \quad a \neq 0$$

implements the operations.

- Hence, RREF after a finite number of such implementations to  $A$  are described by

$$E_k E_{k-1} E_{k-2} \cdots E_2 E_1 A = R, \quad R \text{ is the RREF of } A.$$

Associativity law gives that we perform all of the implementation first,

$$R = (E_k E_{k-1} E_{k-2} \cdots E_2 E_1) A = EA, \quad E = E_k E_{k-1} E_{k-2} \cdots E_2 E_1.$$

- Now, let us think about what will be the RREF of  $A$ .

1. Suppose  $A$  is invertible. What is its RREF?

It must be the identity matrix, no other choice.

$$EA = I.$$

If we record  $E$  of the accumulation of all implementations, then it is the inverse.

2. On the other hands, if  $A$  is not invertible, its RREF must not be the identity matrix.

## How to record RREF implementations?

- We do the exactly same operations parallelly to both  $A$  and the identity matrix  $I$ .
- Then all the implementations will turn  $A$  into RREF (identity matrix if  $A$  is invertible),
- and will turn  $I$  into all its accumulation  $E$ , that must be  $A^{-1}$ .

### Example 1

$$A = \begin{pmatrix} 1 & 3 & 200 \\ 2 & 6 & 400 \\ 3 & 9 & 600 \end{pmatrix}$$

### Example 2

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 2 & 3 & 7 \\ 1 & 4 & 6 \end{pmatrix}$$

### Example 3

$$A = \begin{pmatrix} 0 & -7 & -4 \\ 2 & 4 & 6 \\ 3 & 1 & -1 \end{pmatrix}$$

### Example 4

$$A = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 4 & 3 \end{pmatrix}$$

### Example 5

$$A = \begin{pmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ 2 & 1 & 1 & 5 \\ 1 & 2 & 3 & 6 \end{pmatrix}$$

### Proof of Theorem 1

**Theorem 1.** Let  $A \in \mathbb{R}^{n \times n}$ . Then followings are all equivalent.

1. There exists  $G$  such that  $AG = I$ .
2. There exists  $H$  such that  $HA = I$ .
3. There exists  $E$  such that  $AE = EA = I$ .
4.  $A$  has full rank, that is  $n$ .

*Proof.* We will prove

1. (1)  $\implies$  (4)  $\implies$  (3)  $\implies$  (1),
2. (2)  $\implies$  (4)  $\implies$  (3)  $\implies$  (2).

Then (1),(2),(3),(4) are all equivalent.

- (1)  $\implies$  (4).

We show that  $\text{Ran}(A) = \mathbb{R}^{n \times 1}$ .

For every  $y \in \mathbb{R}^{n \times 1}$ , let  $x = Gy$ . Then  $Ax = AGy = Iy = y$ .

- (4)  $\implies$  (3).

- Because  $A$  is  $n \times n$  square matrix with full rank, its Reduced Row Echelon Form must be precisely the identity matrix. In other words, there exists  $HA = I$ .

- Because  $A^T$  is  $n \times n$  square matrix with full rank, its Reduced Row Echelon Form must be precisely the identity matrix. In other words, there exists  $G^T A^T = I$ .

- If so,

$$HA = I \implies HAG = G \implies H = G.$$

Hence, by letting  $E = H = G$ , (3) is proven.

- (3)  $\implies$  (1)

This is obvious.

- (2)  $\implies$  (4)

We show that  $\text{Ran}(A^T) = \text{Trow}(A) = \mathbb{R}^{n \times 1}$ . We know that  $A^T H^T = I$ .

For every  $y \in \mathbb{R}^{n \times 1}$ , let  $x = H^T y$ . Then  $A^T x = A^T H^T y = Iy = y$ .

- (4)  $\implies$  (3).

This has been done.

- (3)  $\implies$  (2)

This is obvious.

□



## Chapter 10

# Determinant of a square Matrix

Let  $A \in \mathbb{R}^{m \times n}$ .

- $\text{rank}(A) \in \mathbb{Z}$ ,
- $\text{Trow}(A) \subset \mathbb{R}^{n \times 1}$ ,
- $\text{Ran}(A) \subset \mathbb{R}^{m \times 1}$

then provide clearer picture on the roles of  $A$  played between  $\mathbb{R}^{n \times 1}$  and  $\mathbb{R}^{m \times 1}$ .

We now introduce a real number, out of a matrix  $A$  when  $A$  is a square  $n \times n$  matrix.

$$\det(A) \in \mathbb{R}, \quad \text{for } A \text{ square matrix.}$$

It is called the determinant of  $A$ .

We will do:

1. What is the determinant ?
2. Calculation method
3. What it is used for
4. Properties.

From now on in this chapter, every matrix is square.

## What is the determinant of $A$ ?

- We use notation  $\det(A)$ , and in addition, sometimes it is useful to use notation

$\det(a_1, a_2, \dots, a_n)$ ,  $a_1, a_2, \dots, a_n$  are rows of  $A$  from top to bottom.

- This is to interpret  $\det$  as a function of  $n$  rows of  $A$ .
- One can also view  $\det$  as a function of  $n$  columns of  $A$

$\det(b_1, b_2, \dots, b_n)$   $b_1, b_2, \dots, b_n$  are columns of  $A$  from left to right.

which gives the same real number.

- The use of rows is just for traditional reason.

The real-valued function  $\det : (a_1, a_2, \dots, a_n) \mapsto \mathbb{R}$  we are interested in is, for some reason, a function such that

1. Linear in a certain sense:

$$\begin{aligned} \det\left(\left(\sum_{k=1}^m \lambda_k x_k\right), a_2, a_3, \dots, a_n\right) &= \sum_{k=1}^m \lambda_k \det(x_k, a_2, a_3, \dots, a_n) \\ \det\left(a_1, \left(\sum_{k=1}^m \lambda_k x_k\right), a_2, a_3, \dots, a_n\right) &= \sum_{k=1}^m \lambda_k \det(a_1, x_k, a_3, a_4, \dots, a_n) \\ &\vdots \\ \det\left(a_1, a_2, a_3, \dots, a_{n-1}, \left(\sum_{k=1}^m \lambda_k x_k\right)\right) &= \sum_{k=1}^m \lambda_k \det(a_1, a_2, a_3, \dots, a_{n-1}, x_k) \end{aligned}$$

(This is not to say  $\det(A + B) = \det(A) + \det(B)$ !, in general,  $\det(A + B) \neq \det(A) + \det(B)$ !)

2. Anti-symmetric about consecutive row swap, i.e.,

for any  $i$ ,  $\det(\dots, a_i, a_{i+1}, \dots) = -\det(\dots, a_{i+1}, a_i, \dots)$ .

For the moment, we will not discuss why.

Let us take an example calculation what this real number should be for a given matrix.

Let  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ . Using the item 1,

$$\begin{aligned}\det(A) &= a \det \begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix} + b \det \begin{pmatrix} 0 & 1 \\ c & d \end{pmatrix} \\ &= ac \det \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + ad \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + bc \det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + bd \det \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.\end{aligned}$$

Note that the item 2 implies that if two consecutive rows are same, its det is zero:

$$\det(\dots, a, a, \dots) = -\det(\dots, a, a, \dots) \quad \text{can happen only if the number is zero.}$$

Therefore, using item 2,

$$\begin{aligned}\det(A) &= ad \det \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + bc \det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= (ad - bc) \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.\end{aligned}$$

What number for the det of the identity matrix would you want to assign ?

Thus, in addition to the item 1 and 2,

3. We want the det function to be such that

$$\det(I) = 1.$$

Let us also compute the determinant of  $3 \times 3$  matrix

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

From the calculation, we see that we only need to know the determinant of the very special form of matrices such as

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \dots$$

These are permutations that swaps the rows if multiplied from the left.

How many permutations exists from row index  $i \in \{1, 2, \dots, n\}$  to the new row index  $i' \in \{1, 2, \dots, n\}$  ?  $n!$

In fact, we only need to know, how many consecutive swaps are needed to turn a permutation matrix to the identity matrix.

**Proposition 1.** *For a permutation matrix, the number of consecutive swaps to turn to the identity matrix is either even or odd, and the evenness or the oddness is independent of the way we turn.*

**Definition 2.** *Let  $A \in \mathbb{R}^{n \times n}$ . We define a real number*

$$\begin{aligned} \det(A) &= \sum_{\text{among every permutation } \sigma} \left( A_{1,\sigma(1)} A_{2,\sigma(2)} A_{3,\sigma(3)} \cdots A_{n,\sigma(n)} \right) \det(P_\sigma) \\ &= \sum_{\text{among every permutation } \sigma} \left( A_{1,\sigma(1)} A_{2,\sigma(2)} A_{3,\sigma(3)} \cdots A_{n,\sigma(n)} \right) \operatorname{sgn}(P_\sigma). \end{aligned}$$

We will show that the real number  $\det(A)$  is the number:

1. Its sign tells that rows of  $A$  from the top to bottom is right-handed or left-handed, together with fact that
2.  $\det(A) = 0$  if and only if rows of  $A$  is not independent.
3. Its magnitude is precisely the volume of the parallelepiped the  $n$  rows of  $A$  with origin makes.



## Chapter 11

# Determinant and RREF

Row operations and RREF again plays important role in proving many properties about the det.

Let us consider the two elementary row operations, the matrix  $E$  multiplied from the left on  $A$ , say in turnig  $A$  into RREF:

1. Swapping rows, i.e.,  $E$  is a permutation. For example,

$$E = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

2. Replacing  $i$ -th row with  $a \times \text{row}_i + \text{linear combination of the remaining row}$ , with  $a \neq 0$ . For example,

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \lambda_3 \neq 0.$$

The effect of muliplying  $E$  on  $A$  from left, in terms of det change from  $\det(A)$  to  $\det(EA)$ , is in fact very simple.

Recall that If any of two rows of  $A$  are same, then the det is 0. It is easy to see then

**Proposition 1.**

1. If rows of  $A$  is dependent, then  $\det(A) = 0$ .
2. If there is 0 row in  $A$ , then  $\det(A) = 0$ .

*Proof.* If  $i$ -th row can be written by linear combination of remaning rows,

$$a_i = \sum_{j \neq i} \lambda_j a_j \quad \text{then} \quad \det(\dots, a_{i-1}, \sum_{j \neq i} \lambda_j a_j, \dots) = \sum_{j \neq i} \lambda_j \det(\dots, a_{i-1}, a_j, \dots)$$

but all the terms in (RHS) has same rows ( $i$ -th and  $j$ -th), and thus sum must be 0.  $\square$

Having observe those, we can now see:

1. If  $E$  is a permutation, then

$$\det(EA) = \text{sign}(E) \det(A).$$

2. If  $E$  is to replace  $i$ -row by  $\sum_{j=1}^n \lambda_j a_j$ ,  $\lambda_i \neq 0$ , then

$$\begin{aligned} \det(EA) &= \det(\dots, a_{i-1}, \sum_{j=1}^n \lambda_j a_j, \dots) \\ &= \det(\dots, \lambda_i a_i, \dots) \\ &= \lambda_i \det(\dots, a_i, \dots) = \lambda_i \det(A). \end{aligned}$$

Moreover,

1. If  $E$  is a permutation, by definition

$$\text{sign}(E) = \det(E).$$

2. If  $E$  is to replace  $\sum_{j=1}^n \lambda_j a_j$ ,  $\lambda_i \neq 0$ , then

$$\begin{aligned} \det(E) &= \det(e_1, e_2, \dots, e_{i-1}, \sum_{j=1}^n \lambda_j e_j, e_{i+1}, \dots, e_n) \\ &= \det(e_1, e_2, \dots, e_{i-1}, \lambda_i e_i, e_{i+1}, \dots, e_n) \\ &= \lambda_i \det(e_1, e_2, \dots, e_{i-1}, e_i, e_{i+1}, \dots, e_n) = \lambda_i \det(I) = \lambda_i. \end{aligned}$$

We proved that (i):  $\det(E)$  of above two types never is 0. Also we proved that (ii):

**Proposition 2.**

$$\det(EA) = \det(E) \det(A).$$

## Determinant of $A$ and determinant of its RREF

Now, let

$$E_k E_{k-1} E_{k-2} \cdots E_1 A = R,$$

where  $R$  is the RREF of  $A$ .

Using the Proposition 2, we have

$$\begin{aligned} \det(R) &= \det(E_k E_{k-1} E_{k-2} \cdots E_1 A) \\ &= \det(E_k) \det(E_{k-1} E_{k-2} \cdots E_1 A) \\ &= \cdots \\ &= \det(E_k) \det(E_{k-1}) \cdots \det(E_1) \det(A). \end{aligned}$$

Hence we proved that:

**Proposition 3.**

$$\det(A) = \frac{\det(R)}{\det(E_k) \det(E_{k-1}) \cdots \det(E_1)},$$

where we used that  $\det(E_k) \det(E_{k-1}) \cdots \det(E_1) \neq 0$ .

**Proposition 4.**  $\det(A) = 0$  if and only if rows of  $A$  is dependent.

*Proof.* .

- We already checked  $\det(A) = 0$  if rows of  $A$  is dependent, begining this chapter.
- We need to show that rows of  $A$  is dependent if  $\det(A) = 0$ . This is obvious from the formula in Proposition 3.

□

Continuing investigation of  $\det$ ,

**Proposition 5.**

$$\det(AB) = \det(A)\det(B).$$

*Proof.* .

1. Suppose  $\det(A) = 0$ .

Then  $\text{rank}(A) < n$ . In other words,  $\text{Ran}(A) \neq \mathbb{R}^{n \times 1}$ . Because any linear combination of columns of  $AB$  is a linear combination of columns  $A$ ,  $\text{Ran}(AB) \neq \mathbb{R}^{n \times 1}$  too. Hence  $\det(AB) = 0$ .

2. Suppose  $\det(B) = 0$ .

Then  $\text{rank}(B) < n$ . In other words,  $\text{Trow}(B) \neq \mathbb{R}^{n \times 1}$ , and there is nontrivial  $x_0$  such that  $Bx_0 = 0$ . Because  $ABx_0 = 0$ .  $\text{Trow}(AB) \neq \mathbb{R}^{n \times 1}$  too. Hence  $\det(AB) = 0$ .

3. Suppose  $\det(A) \neq 0$  and  $\det(B) \neq 0$ .

$A$  and  $B$  are both invertible. Let  $C = AB$  and write  $B = A^{-1}C$ .

Recall that there exist row operations so that

$$A^{-1} = E_k E_{k-1} E_{k-2} \cdots E_1.$$

Therefore,

$$\begin{aligned} \det(B) &= \det(E_k) \det(E_{k-1}) \cdots \det(E_1) \det(C) \\ \iff \det(B) &= \frac{\det(\text{RREF}(A))}{\det(A)} \det(C). \end{aligned}$$

But, since  $A$  is invertible, its RREF is identity matrix. Hence we derived

$$\det(B) = \frac{\det(C)}{\det(A)}.$$

□

Using Proposition 5, if  $A$  is invertible,  $\det(A^{-1}) = \frac{1}{\det(A)}$ .

Next, we verify that  $\det(A)$  can be computed column perspective too:

**Proposition 6.**  $\det(A^T) = \det(A)$ .

*Proof.* .

1. From the definition of  $\det(A)$ ,

$$\det(A) = \sum_{\substack{\text{among every permutation} \\ \sigma: \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}}} (A_{1,\sigma(1)} A_{2,\sigma(2)} A_{3,\sigma(3)} \cdots A_{n,\sigma(n)}) \operatorname{sgn}(P_\sigma).$$

2. We note one observation: For each iteration, the product of  $n$  real numbers  $A_{1,\sigma(1)} A_{2,\sigma(2)} A_{3,\sigma(3)} \cdots A_{n,\sigma(n)}$  is same as the product of the following  $n$  real numbers

$$A_{\sigma^{-1}(1),1} A_{\sigma^{-1}(2),2} A_{\sigma^{-1}(3),3} \cdots A_{\sigma^{-1}(n),n},$$

3. Hence,

$$\det(A) = \sum_{\substack{\text{among every permutation} \\ \sigma: \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}}} (A_{\sigma^{-1}(1),1} A_{\sigma^{-1}(2),2} A_{\sigma^{-1}(3),3} \cdots A_{\sigma^{-1}(n),n}) \operatorname{sgn}(P_\sigma)$$

(writing  $s = \sigma^{-1}$ )

$$\begin{aligned} &= \sum_{\substack{\text{among every permutation} \\ s: \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}}} (A_{s(1),1} A_{s(2),2} A_{s(3),3} \cdots A_{s(n),n}) \operatorname{sgn}(P_{s^{-1}}) \\ &= \sum_{\substack{\text{among every permutation} \\ s: \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}}} (A_{s(1),1} A_{s(2),2} A_{s(3),3} \cdots A_{s(n),n}) \operatorname{sgn}(P_s) \\ &= \sum_{\substack{\text{among every permutation} \\ s: \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}}} (A_{1,s(1)}^T A_{2,s(2)}^T A_{3,s(3)}^T \cdots A_{n,s(n)}^T) \operatorname{sgn}(P_s) \\ &= \det(A^T), \end{aligned}$$

4. where we admited the fact that  $\operatorname{sign}(P_s) = \operatorname{sign}(P_{s^{-1}})$ .

□

$|\det(A)|$  is the  $n$ -dimensional volume.

We do this for the simple  $2 \times 2$  case, and generalize to  $n \times n$  case.

### Notations

We do this from column perspective.

- Let  $A$  be a  $2 \times 2$  matrix with columns  $x_1$  and  $x_2$ .

$$A = \begin{pmatrix} | & | \\ x_1 & x_2 \\ | & | \end{pmatrix}.$$

- The locations in  $\mathbb{R}^2$  from  $x_1$  and  $x_2$ , decides a parallelogram, together with the origin 0. We denote the area of this parallelogram by

$$\text{vol}(0, x_1, x_2).$$

We define  $\text{vol}(0, x_1, x_2) = 0$  if  $A$  has dependent columns.

### Upper triangular $2 \times 2$ matrix

- Consider an invertible  $2 \times 2$  upper triangular matrix,

$$U = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \quad a \neq 0, d \neq 0.$$

- Let  $x_1$  and  $x_2$  to be its two columns from left. Then, the parallelogram we see is such that

$$\text{base length} = |a| \quad \text{height length} = |d|,$$

$$\text{vol}(0, x_1, x_2) = |ad|.$$

- On the other hand, the determinant of the triangular matrix  $U$  is simply

$$\det(U) = ad.$$

- Hence, we conclude that

If  $U$  is  $2 \times 2$  upper triangular matrix, then  $\text{vol}(0, x_1, x_2) = |\det(U)|$ .

This is true even when  $U$  is not invertible.

## Rotation

- Now, let  $A$  be a general  $2 \times 2$  matrix written as

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

- In this  $2 \times 2$  case, it is obvious that there is a rotation matrix  $Q$  so that the rotated location

$$x'_1 = Qx_1 \quad \text{is aligned in the 1st-axis.}$$

In other words,

$$QA = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix} = U.$$

- By a rotation, the area of parallelogram must not change:

$$\text{vol}(0, x_1, x_2) = \text{vol}(0, x'_1, x'_2).$$

- Therefore,

$$\text{vol}(0, x_1, x_2) = \text{vol}(0, x'_1, x'_2) = |\det(U)| = |\det Q \det A| = |\det A|$$

and we are done.

$A$  is  $n \times n$  matrix

- Let

$$A = \begin{pmatrix} | & | & \cdots & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & \cdots & | \end{pmatrix}$$

- The locations in  $\mathbb{R}^n$  from  $x_1, x_2, \dots, x_n$  decides a  $n$ -dimensional parallelepiped, together with the origin 0. We consider similarly

$$\text{vol}(0, x_1, x_2, \dots, x_n).$$

- Now, we consider  $n \times n$  upper triangular matrix  $U$ :

$$U = \begin{pmatrix} U_{11} & U_{12} & U_{13} & \cdots & U_{1n} \\ 0 & U_{22} & U_{23} & \cdots & U_{2n} \\ 0 & 0 & U_{33} & \cdots & U_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & U_{nn} \end{pmatrix}$$

**Proposition 7.** *The determinant of upper triangular or lower triangular matrix is the product of diagonal components.*

*Proof.* This is obvious from the definition of

$$\det(A) = \sum_{\substack{\text{among every permutation} \\ \sigma: \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}}} (A_{1,\sigma(1)} A_{2,\sigma(2)} A_{3,\sigma(3)} \cdots A_{n,\sigma(n)}) \text{sgn}(P_\sigma).$$

□

We have that for the upper triangular matrix case

$$\text{vol}(0, x_1, x_2, \dots, x_n) = |U_{11} U_{22} U_{33} \cdots U_{nn}|.$$

- Hence, we conclude that

If  $U$  is  $n \times n$  upper triangular matrix, then  $\text{vol}(0, x_1, x_2, \dots, x_n) = |\det(U)|$ .

- We admit the following fact without proof:

**Theorem.** *For any  $n \times n$  matrix  $A$ , there exists an  $n \times n$  rotation matrix  $Q$  with  $\det(Q) = 1$  such that  $QA$  is an upper triangular matrix.*

- Finally, since the volume must not change under the rotation,

$$\text{vol}(0, x_1, x_2, \dots, x_n) = \text{vol}(0, x'_1, x'_2, \dots, x'_n) = |\det(U)| = |\det(Q) \det(A)| = |\det(A)|.$$

## Chapter 12

# Collecting $n!$ numbers

Here, we re-visit the calculation of  $\det$ , which is to collect  $n!$  numbers.

We recall the computation of  $\det(A)$  of  $3 \times 3$  matrix

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

1. We first expand  $\det(A)$  into  $n$  terms

$$\det(A) = a \det \begin{pmatrix} 1 & 0 & 0 \\ d & e & f \\ g & h & i \end{pmatrix} + b \det \begin{pmatrix} 0 & 1 & 0 \\ d & e & f \\ g & h & i \end{pmatrix} + c \det \begin{pmatrix} 0 & 0 & 1 \\ d & e & f \\ g & h & i \end{pmatrix}$$

2. Each of  $n$  terms, one more depth expansion only involves  $n - 1$  terms, since two same rows implies zero  $\det$ : From the first term above,

$$a \det \begin{pmatrix} 1 & 0 & 0 \\ d & e & f \\ g & h & i \end{pmatrix} = ae \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ g & h & i \end{pmatrix} + af \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ g & h & i \end{pmatrix}.$$

3. Recursively, each of  $n - 1$  terms, one more depth expansion only involves  $n - 2$  terms: From the first term above,

$$ae \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ g & h & i \end{pmatrix} = ae \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

Collecting exactly  $n!$  numbers for the  $\det$ .

We do the  $\det(A)$  of  $4 \times 4$  matrix

$$A = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & \ell \\ m & n & o & p \end{pmatrix}$$

1. We first expand  $\det(A)$  into 4 terms

$$\det(A) = a \det \begin{pmatrix} 1 & 0 & 0 & 0 \\ e & f & g & h \\ i & j & k & \ell \\ m & n & o & p \end{pmatrix} + b \det \begin{pmatrix} 0 & 1 & 0 & 0 \\ e & f & g & h \\ i & j & k & \ell \\ m & n & o & p \end{pmatrix} + c \det \begin{pmatrix} 0 & 0 & 1 & 0 \\ e & f & g & h \\ i & j & k & \ell \\ m & n & o & p \end{pmatrix} + d \det \begin{pmatrix} 0 & 0 & 0 & 1 \\ e & f & g & h \\ i & j & k & \ell \\ m & n & o & p \end{pmatrix}$$

2. Each of 4 terms, one more depth expansion only involves 3 terms, since two same rows implies zero det: From the first term above,

$$a \det \begin{pmatrix} 1 & 0 & 0 & 0 \\ e & f & g & h \\ i & j & k & \ell \\ m & n & o & p \end{pmatrix} = af \det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ i & j & k & \ell \\ m & n & o & p \end{pmatrix} + ag \det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ i & j & k & \ell \\ m & n & o & p \end{pmatrix} + ah \det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ i & j & k & \ell \\ m & n & o & p \end{pmatrix}.$$

3. Recursively, for each of 3 terms, one more depth expansion only involves 2 terms: From the first term above,

$$af \det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ i & j & k & \ell \\ m & n & o & p \end{pmatrix} = afk \det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ m & n & o & p \end{pmatrix} + af\ell \det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ m & n & o & p \end{pmatrix}.$$

4. Finally, for each of 2 terms, one more depth expansion only involves 1 term: From the first term above,

$$afk \det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ m & n & o & p \end{pmatrix} = afkp \det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Collecting exactly  $n!$  numbers for the det.

We recall,

**Swapping two consecutive rows, or swapping two consecutive columns makes sign flipped.**

We do the collection, taking care of the sign factor at the same time.

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

1. We first expand  $\det(A)$  into  $n$  terms

$$\begin{aligned} \det(A) &= a \det \begin{pmatrix} 1 & 0 & 0 \\ d & e & f \\ g & h & i \end{pmatrix} + b \det \begin{pmatrix} 0 & 1 & 0 \\ d & e & f \\ g & h & i \end{pmatrix} + c \det \begin{pmatrix} 0 & 0 & 1 \\ d & e & f \\ g & h & i \end{pmatrix} \\ &= a \det \begin{pmatrix} 1 & 0 & 0 \\ d & e & f \\ g & h & i \end{pmatrix} - b \det \begin{pmatrix} 1 & 0 & 0 \\ e & d & f \\ h & g & i \end{pmatrix} + c \det \begin{pmatrix} 1 & 0 & 0 \\ f & e & d \\ i & h & g \end{pmatrix} \end{aligned}$$

2. Each of  $n$  terms, one more depth expansion only involves  $n - 1$  terms: From the second term above,

$$\begin{aligned} -b \det \begin{pmatrix} 1 & 0 & 0 \\ e & d & f \\ h & g & i \end{pmatrix} &= -bd \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ h & g & i \end{pmatrix} - bf \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ h & g & i \end{pmatrix} \\ &= -bd \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ h & g & i \end{pmatrix} + bf \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ h & i & g \end{pmatrix} \end{aligned}$$

3. Recursively, each of  $n - 1$  terms, one more depth expansion only involves  $n - 2$  terms: From the second term above,

$$bf \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ h & i & g \end{pmatrix} = bfg \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = bfg.$$

We keep the parts we have already taken care of, as the smaller size identity block, and pass to the next depth.

In practice, we can write

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

1. We first expand  $\det(A)$  into  $n$  terms

$$\det(A) = a \det \begin{pmatrix} e & f \\ h & i \end{pmatrix} - b \det \begin{pmatrix} d & f \\ g & i \end{pmatrix} + c \det \begin{pmatrix} e & d \\ h & g \end{pmatrix}$$

2. Each of  $n$  terms, one more depth expansion only involves  $n - 1$  terms: From the second term above,

$$-b \det \begin{pmatrix} d & f \\ g & i \end{pmatrix} = -bd \det(i) + bf \det(g)$$

3. Recursively, each of  $n - 1$  terms, one more depth expansion only involves  $n - 2$  terms: From the second term above,

$$bf \det(g) = bfg.$$

### Examples of calculation

$$\begin{pmatrix} 3 & 4 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 3 \\ 6 & 8 & -4 & -1 \end{pmatrix} \quad \begin{pmatrix} -2 & 6 & 0 & 9 \\ 3 & 4 & 8 & 2 \\ 4 & 3 & 0 & 1 \\ 3 & 1 & 2 & -1 \end{pmatrix} \quad \begin{pmatrix} 2 & 5 & 4 & 1 \\ 4 & 7 & 6 & 2 \\ 6 & -2 & -4 & 0 \\ -6 & 7 & 7 & 0 \end{pmatrix} \quad \begin{pmatrix} 4 & 3 & 2 & 1 \\ 5 & 4 & -3 & 0 \\ 9 & -8 & -7 & 0 \\ 4 & 6 & 2 & 1 \end{pmatrix}$$

- As much as you can recognize, and as much as you can keep sign factors correct, **Do the row swap and row replacement towards a triangular matrix**, introducing as many as zeroes in the resulting matrix.
- Then do the  $n!$  expansion.

## $\text{cof}(A)$ , $A^{-1}$ formula, and Cramer's formula.

Let us go back to the first depth expansion

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

1. We first expand  $\det(A)$  into  $n$  terms

$$\det(A) = a \det \begin{pmatrix} 1 & 0 & 0 \\ d & e & f \\ g & h & i \end{pmatrix} + b \det \begin{pmatrix} 0 & 1 & 0 \\ d & e & f \\ g & h & i \end{pmatrix} + c \det \begin{pmatrix} 0 & 0 & 1 \\ d & e & f \\ g & h & i \end{pmatrix}$$

2. Assuming the  $\det(A)$  calculation has already met end, in other words, assuming we have calculated the three numbers

$$\det \begin{pmatrix} 1 & 0 & 0 \\ d & e & f \\ g & h & i \end{pmatrix}, \quad \det \begin{pmatrix} 0 & 1 & 0 \\ d & e & f \\ g & h & i \end{pmatrix}, \quad \det \begin{pmatrix} 0 & 0 & 1 \\ d & e & f \\ g & h & i \end{pmatrix}$$

Let us take this three numbers to be yet another row,

$$c_1 = \left( \det \begin{pmatrix} 1 & 0 & 0 \\ d & e & f \\ g & h & i \end{pmatrix}, \det \begin{pmatrix} 0 & 1 & 0 \\ d & e & f \\ g & h & i \end{pmatrix}, \det \begin{pmatrix} 0 & 0 & 1 \\ d & e & f \\ g & h & i \end{pmatrix} \right)$$

We can write then

$$a_1 \cdot c_1 = \det(A)$$

In fact, we may do the following too.

1. We first expand  $\det(A)$  into  $n$  terms

$$\det(A) = d \det \begin{pmatrix} a & b & c \\ 1 & 0 & 0 \\ g & h & i \end{pmatrix} + e \det \begin{pmatrix} a & b & c \\ 0 & 1 & 0 \\ g & h & i \end{pmatrix} + f \det \begin{pmatrix} a & b & c \\ 0 & 0 & 1 \\ g & h & i \end{pmatrix}$$

2. Assuming we have calculated the three numbers

$$\det \begin{pmatrix} a & b & c \\ 1 & 0 & 0 \\ g & h & i \end{pmatrix}, \quad \det \begin{pmatrix} a & b & c \\ 0 & 1 & 0 \\ g & h & i \end{pmatrix}, \quad \det \begin{pmatrix} a & b & c \\ 0 & 0 & 1 \\ g & h & i \end{pmatrix}$$

Let us take this three numbers to be yet another row,

$$c_2 = \left( \det \begin{pmatrix} a & b & c \\ 1 & 0 & 0 \\ g & h & i \end{pmatrix}, \det \begin{pmatrix} a & b & c \\ 0 & 1 & 0 \\ g & h & i \end{pmatrix}, \det \begin{pmatrix} a & b & c \\ 0 & 0 & 1 \\ g & h & i \end{pmatrix} \right)$$

We can write then

$$a_2 \cdot c_2 = \det(A)$$

3. Importantly, we note

$$a_1 \cdot c_2 = 0$$

This is simply because

$$a_1 \cdot c_2 = a \det \begin{pmatrix} a & b & c \\ 1 & 0 & 0 \\ g & h & i \end{pmatrix} + b \det \begin{pmatrix} a & b & c \\ 0 & 1 & 0 \\ g & h & i \end{pmatrix} + c \det \begin{pmatrix} a & b & c \\ 0 & 0 & 1 \\ g & h & i \end{pmatrix} = \det \begin{pmatrix} a & b & c \\ a & b & c \\ g & h & i \end{pmatrix} = 0.$$

We can collect  $n$  new rows  $c_i$  that has the property:

$$a_i \cdot c_j = \begin{cases} \det(A) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Provided we have calculated all those  $c_i$ ,  $i = 1, 2, \dots, n$ , and also provided  $\det(A) \neq 0$ , we then have equality

$$\frac{1}{\det(A)} C^T = A^{-1},$$

where  $C$  is the matrix with  $n$  rows  $c_i$  from the top to bottom. It is called the cofactor matrix of  $A$  and denoted by  $\text{cof}(A)$ .

If we want to compute  $A^{-1}$  we have two means:

1. Do the RREF row operation on  $(A \mid I)$ .
2. Do compute  $\det(A)$  and collect rows  $c_i$ .

Actually, computing  $\det(A)$  and  $\text{cof}(A)$  in general does not reduce computations.

## Home Work

1. Compute the det for the above four examples.

$$\begin{pmatrix} 3 & 4 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 3 \\ 6 & 8 & -4 & -1 \end{pmatrix} \quad \begin{pmatrix} -2 & 6 & 0 & 9 \\ 3 & 4 & 8 & 2 \\ 4 & 3 & 0 & 1 \\ 3 & 1 & 2 & -1 \end{pmatrix} \quad \begin{pmatrix} 2 & 5 & 4 & 1 \\ 4 & 7 & 6 & 2 \\ 6 & -2 & -4 & 0 \\ -6 & 7 & 7 & 0 \end{pmatrix} \quad \begin{pmatrix} 4 & 3 & 2 & 1 \\ 5 & 4 & -3 & 0 \\ 9 & -8 & -7 & 0 \\ 4 & 6 & 2 & 1 \end{pmatrix}$$

2. Let

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} | & | & | \\ b_1 & b_2 & b_3 \\ | & | & | \end{pmatrix} = \begin{pmatrix} - & a_1 & - \\ - & a_2 & - \\ - & a_3 & - \end{pmatrix}$$

consider the columnwise expansion, for example

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a \det \begin{pmatrix} 1 & b & c \\ 0 & e & f \\ 0 & h & i \end{pmatrix} + d \det \begin{pmatrix} 0 & b & c \\ 1 & e & f \\ 0 & h & i \end{pmatrix} + g \det \begin{pmatrix} 0 & b & c \\ 0 & e & f \\ 1 & h & i \end{pmatrix},$$

in other words, we collect this times  $d_1$

$$d_1 = \left( \det \begin{pmatrix} 1 & b & c \\ 0 & e & f \\ 0 & h & i \end{pmatrix}, \begin{pmatrix} 0 & b & c \\ 1 & e & f \\ 0 & h & i \end{pmatrix}, \begin{pmatrix} 0 & b & c \\ 0 & e & f \\ 1 & h & i \end{pmatrix} \right).$$

and also similarly  $d_j$ ,  $j = 1, 2, \dots, n$ .

Show that  $d_{ji}$ , the  $i$ -th component of  $d_j$ , and  $c_{ij}$ , the  $j$ -th component of  $c_i$  are the same number.

3. Using the Home Work 2, prove the theorem on Cramer's formula:

**Theorem 1.** Let  $A$  be an  $n \times n$  invertible matrix whose columns are denoted by  $b_1, b_2, \dots, b_n$  from the left. Suppose  $y \in \mathbb{R}^{n \times 1}$  is given and  $x \in \mathbb{R}^{n \times 1}$  solves the equation  $Ax = y$ .

Then the  $i$ -th component of the solution  $x$

$$x_i = \frac{\det(\dots, b_{i-1}, y, b_{i+1}, \dots)}{\det A}.$$



## Chapter 13

# System of $m$ equations and $n$ unknowns

We consider a system of linear equations:

$$\begin{aligned} A_{1,1}x_1 + A_{1,2}x_2 + A_{1,3}x_3 + \cdots + A_{1,n}x_n &= b_1 \\ A_{2,1}x_1 + A_{2,2}x_2 + A_{2,3}x_3 + \cdots + A_{2,n}x_n &= b_2 \\ &\vdots \\ A_{m,1}x_1 + A_{m,2}x_2 + A_{m,3}x_3 + \cdots + A_{m,n}x_n &= b_m \end{aligned}$$

We write the system

$$Ax = b,$$

where

$$A \in \mathbb{R}^{m \times n}, \quad x \in \mathbb{R}^{n \times 1} = X, \quad b \in \mathbb{R}^{m \times 1} = Y.$$

One might think that a meaningful solution map

$$R : Y \rightarrow X, \quad b \mapsto x = Rb$$

is only possible when  $m = n$  and actually  $m = n = r$ , with  $r = \text{rank}(A)$ .

However, this is not true. You know everything to say that in fact, you can always single out one element  $\tilde{x} \in X$ , defining a meaningful map

$$\tilde{R} : Y \rightarrow X, \quad b \mapsto \tilde{x} = \tilde{R}b,$$

using the fact that the map  $x \mapsto Ax$  is a bijection between  $\text{Trow}(A)$  and  $\text{Ran}(A)$ .

Today, we outline the program how we study the problem of solving linear system:

1. Overview
2. Solving skill by hand: REF and RREF
3. Other methods such as Cholesky methods.
4. If possible, iterative methods.

## Two Obvious Facts

1. If  $b \notin \text{Ran}(A)$ , then there can be no way to have any solution.

2. Suppose  $b \in \text{Ran}(A)$ .

If  $\text{Trow}(A) \neq \mathbb{R}^{n \times 1}$ , i.e.,  $\text{Ker}(A) = \text{Trow}(A)^\perp$  is not trivial, then there are infinitely many solutions.

This is because, that  $b \in \text{Ran}(A)$  implies there exists  $x \in \mathbb{R}^{n \times 1}$  with  $Ax = b$ . Now, for infinitely many choices of  $x_0 \in \text{Ker}(A)$ ,  $x + x_0$  is again a solution.

## What do we want?

1. We make a choice, in  $Y$ , a projection onto  $\text{Ran}(A)$  for any  $b \in Y$ .
2. Not surprisingly,  $b \mapsto b' \in \text{Ran}(A)$  that is by the orthogonal projection is the one used mostly.
3. Now, for given the projection  $b'$ , we completely know that:
  - (a) There exists unique element  $x' \in \text{Trow}(A)$  such that  $Ax' = b'$ .
  - (b) The complete set of *solutions* are  $x' + x_0$  for any  $x_0 \in \text{Ker}(A)$ .
4. In most cases, the map

$$\tilde{R} : b \mapsto b' \mapsto x' \in \text{Trow}(A)$$

will be the meaningful map, singling out unique element.

5. The  $\tilde{x}$  it returns is the one where

$$A\tilde{x} = b', \quad b' \text{ is the element in } \text{Ran}(A) \text{ such that } |b - b'|_2 \text{ is minimized.}$$

## Solutions of $Ax = 0$ , by REF or RREF

Let

$$A = \begin{pmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{pmatrix}$$

- We already know how to obtain basis of  $\text{Trow}(A)$  and  $\text{Ker}(A) = \text{Trow}(A)^\perp$ .
- Let us repeat the procedure in detail again though.

We remark one thing in row operations:

Suppose we make a row operation into

$$A' = \begin{pmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 3 & 6 & 8 & 10 \end{pmatrix}$$

This is to do

$$EA = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 3 & 6 & 8 & 10 \end{pmatrix} = A'$$

and we recall that

1. We multiply  $E$  such that

$$\text{Trow}(A) = \text{Trow}(EA), \quad \text{and consequently} \quad \text{Trow}(A)^\perp = \text{Trow}(EA)^\perp.$$

2. However, because all the linear combinations of columns of  $A$  is passed to  $E$  to produce linear combinations of columns of  $E$

$$\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

we have that

$$\text{Ran}(EA) = \{b \in Y \mid b = Ew \text{ for some } w \in \text{Ran}(A)\}$$

and this is keep changing by multiplying  $E$ .

Having this in minds, we completely solve the equation  $Ax = 0$ .



## Chapter 14

# Solutions of $Ax = b'$ when $b' \in \text{Ran}(A)$

- In this chapter, we calculate solutions of the equation, when we do know  $b' \in \text{Ran}(A)$ .
- Or, we may think that we already know sufficiently about  $A$  and  $b'$  given above is the one after we passed the (r-h-s) to the projection onto  $\text{Ran}(A)$ .

We keep working with

$$A = \begin{pmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{pmatrix}$$

and take

$$b' = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

to solve  $Ax = b'$ .

- If  $x$  solves the equation  $EAx = Eb'$ , and  $E$  is a row operation, then  $x$  solves

$$Ax = b',$$

because  $E$  is always invertible.

- Therefore, in searching for any solution  $x$ , the same operation by  $E$  has to be applied on the (r-h-s) too, keeping

$$(A, b) \rightarrow (E_1 A, E_1 b') \rightarrow (E_2 E_1 A, E_2 E_1 b') \rightarrow \dots$$

- Or, as same as we did for the inverse calculation, we append the  $b'$  column to  $A$  to have

$$\left( \begin{array}{cccc|c} 1 & 2 & 2 & 2 & 1 \\ 2 & 4 & 6 & 8 & 1 \\ 3 & 6 & 8 & 10 & 2 \end{array} \right)$$

- In preceding chapter, we considered  $Ax = 0$ , and there we might have also appended 0 column, which has no effect.

## REF or RREF of the augmented matrix

1. Let us first turn the matrix into REF:

$$\left( \begin{array}{cccc|c} 1 & 2 & 2 & 2 & 1 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

2. Observations:

- (a) We observe the 1st and 3rd columns are the pivot column, and in writing

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix},$$

we called the  $x_1$  and  $x_3$  the primary or pivot variables, and the remaining  $x_2$  and  $x_4$  the free variables.

- (b) Also, we see the last row is 0 row, including the (r-h-s) part. Otherwise,  $b' \notin \text{ran}(A)$ .

3. Backsubstitution.

- (a) We first put all free variables 0.

$$\text{We look for a particular solution } x_p = \begin{pmatrix} x_1 \\ 0 \\ x_3 \\ 0 \end{pmatrix}.$$

(b) If so, we have

$$2x_3 = -1 \iff x_3 = -\frac{1}{2}$$

(c) Now,

$$x_1 + 2x_3 = 1, \quad x_3 = -\frac{1}{2} \implies x_1 = 2.$$

A particular solution  $x_p = \begin{pmatrix} 2 \\ 0 \\ -\frac{1}{2} \\ 0 \end{pmatrix}.$

4. We recall that

$$\text{Ker}(A) = \text{span} \left\langle \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -2 \\ 1 \end{pmatrix} \right\rangle,$$

in other words

$$c_1 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 0 \\ -2 \\ 1 \end{pmatrix}$$

span all solutions of  $Ax = 0$ .

5. Thus,

$$x = \begin{pmatrix} 2 \\ 0 \\ -\frac{1}{2} \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 0 \\ -2 \\ 1 \end{pmatrix}$$

comprises the complete set of solutions for  $Ax = b'$ .

6. The particular solution we found is not a member of  $\text{Trow}(A) = \text{Ker}(A)^\perp$  though.

1. Suppose we further turn REF to RREF,

$$\left( \begin{array}{cccc|c} 1 & 2 & 0 & -2 & 2 \\ 0 & 0 & 1 & 2 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

2. Reading off a particular solution

- (a) We first put all free variables 0.

We look for a particular solution  $x_p = \begin{pmatrix} x_1 \\ 0 \\ x_3 \\ 0 \end{pmatrix}$ .

- (b) If so,

A particular solution  $x_p = \begin{pmatrix} 2 \\ 0 \\ -\frac{1}{2} \\ 0 \end{pmatrix}$ .

## Algorithm summary

- Turning  $(A|b')$  into REF, and solving by back substitution can be put into the algorithm:  
known as the row-pivoting (row swap allowed) Gauss elimination.
- In the algorithm, we turn

$$E_n E_{n-1} \cdots E_1 A = U,$$

where each of  $E_j$  is either row swap or row replacement.

- Once REF  $U$  is obtained, we do the backsubstitution.
- Backsubstituting a rectangular REF  $U$  means, we suitably treat the pivot and free variables to solve

$$Ax_0 = 0, \quad \text{and} \quad Ax_p = b'.$$

- Turning  $(A|b')$  into RREF is known as the row-pivoting (row swap allowed) Gauss-Jordan elimination.

## PLU deomposition of $A$ and REF:

In the algorithm we did not need to remember what were  $E_k$  in the procedure.

Although not necessary, if they were recorded, we did the PLU decomposition of  $A$ .

### The reason why LU:

1. Suppose  $A$  was so special that, we did not need any row swap in turning it into REF,
2. or more practically in computer implementation, the numbers in the next pivot position is always not too close to 0 value.
3. Then, what we only need to do is to make entries below the pivot into 0.
4. That is always done by multiplying  $E$ , for example,

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

that is a lower triangular matrix.

5. Thus, what we did can be written as

$$L_n L_{n-1} \cdots L_1 A = U$$

6. A lower triangular matrix  $\times$  a lower triangular matrix is again lower triangular.  
The inverse of a lower triangular matrix is again lower triangular.

$$LA = U \quad \text{or} \quad A = \hat{L}U.$$

### The reason why P

1. Now, suppose we did need row swap in turning it into REF.
2. Hence, what we did can be written as, for example,

$$L_n P_{n-1} P_{n-2} L_{n-3} L_{n-4} L_{n-5} P_{n-6} \cdots P_2 L_1 A = U.$$

3. We need one more knowledge:

- (a) If  $L$  is any lower triangular matrix and  $P$  is any permutation, one can always find another lower triangular  $L'$  and another permutation  $P'$  so that

$$LP = P'L'.$$

- (b) Hence, we did:

$$LPA = U \quad \text{or} \quad A = \hat{P}\hat{L}U \quad \text{or} \quad \tilde{P}A = \tilde{L}U.$$

## Chapter 15

# Orthogonal Projection of $b$ onto $b' \in \text{Ran}(A)$

- Now, suppose we are given  $b \in \mathbb{R}^{m \times 1}$ , but we do not know if  $b \in \text{Ran}(A)$  or not.
  - Here, we look for  $b' \in \text{Ran}(A) \subset \mathbb{R}^{m \times 1}$  that is the orthogonal projection of  $b$  onto  $\text{Ran}(A)$ .
  - Once  $b'$  is obtained, we solve complete set of solutions for  $Ax = b'$  by turning  $(A \mid b')$  into REF, and using backsubstitution.
- 
- This leads to solving the least square problem
    1. Of course, the distance  $|b - b'|$  is minimum distance among all distances  $|b - c|$ ,  $c \in \text{Ran}(A)$ .
    2. The Euclidean length

$$|b - b'| = \sqrt{(b_1 - b'_1)^2 + \cdots + (b_m - b'_m)^2}.$$

## Orthogonal Projection on $\text{Ran}(A)$

- To compose the orthogonal projection, what we need is the basis  $w_1, w_2, \dots, w_r$  that span  $\text{Ran}(A)$  and basis  $w_{r+1}, w_{r+2}, \dots, w_m$  that span  $\text{Ran}(A)^\perp$ .
- We at least know one way to calculate  $w_1, w_2, \dots, w_r$ , and also  $w_{r+1}, w_{r+2}, \dots, w_m$ , using RREF.
- We also know how to compose the projection.
- So, we at least know one way to compute  $b'$ .

- In other words: we are given

1. Linearly independent  $\mathbb{R}^{n \times 1}$  columns  $w_1, w_2, \dots, w_m$
2. such that

$$\begin{aligned} \text{span} \langle w_1, w_2, \dots, w_r \rangle &= \text{Ran}(A), \quad \text{span} \langle w_{r+1}, w_{r+2}, \dots, w_m \rangle = \text{Ran}(A)^\perp, \\ (\text{By definition of } \text{Ran}(A)^\perp, \text{span} \langle w_1, w_2, \dots, w_r \rangle \perp \text{span} \langle w_{r+1}, w_{r+2}, \dots, w_m \rangle). \end{aligned}$$

- If  $w_1, w_2, \dots, w_m$  are so,

1. Let

$$W = \begin{pmatrix} | & | & \cdots & | \\ w_1 & w_2 & \cdots & w_m \\ | & | & \cdots & | \end{pmatrix}$$

2. Let the  $m \times m$  elementary projection

$$\hat{P} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

3. The projection

$$P = W \hat{P} W^{-1}$$

will give  $b \mapsto b' = Pb$ .

## A few comments before entering calculation

1. We will do this today using what we have learned so far.
  2. While to obtain the basis out of RREF, and CR decomposition of  $A$  is good because we can achieve this by hand,
  3. CR decomposition is usually for pedagogical purposes, not for real problems.
  4. Motivated from this, we will introduce QR decomposition and SVD for the purpose of obtaining (orthonormal) basis, for real world problems.
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5. Another remark is the following: One might have seen, in solving the least square problem, the system

$$A^T A x = A^T b.$$

- (a) We will clarify how this is a projection-involved least square problem.
- (b) As a matter of fact, this works in the special case where the matrix is thin and full rank:

$$m \geq n = r.$$

We work again with

$$\begin{pmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{pmatrix}$$

with

$$b = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}.$$

**Home Work:** Obtain all the least square solutions for

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 10 \\ 3 & 6 & 9 & 12 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}.$$

This consists of six steps:

1. Choose basis for  $\text{Ran}(A)$ .
2. Choose basis for  $\text{Ran}(A)^\perp$ .
3. Compute the projection matrix  $P$  and compute  $b' = Pb$ .
4. Do the REF or RREF.
5. Compute all the solutions  $Ax = 0$ .
6. Compute a particular solution of  $Ax_p = b'$

## Chapter 16

# General Idea of Basis Revealing Decomposition

We saw examples of writing

$$A = YU,$$

where  $Y$  and  $U$  are either

- $Y$  is full-rank invertible  $m \times m$  matrix,
- $U$  is  $m \times n$  matrix.

or

- $Y$  is a full-rank  $m \times r$  matrix,
- $U$  is a full-rank  $r \times n$  matrix.

**REF by row operations:**  $E_N E_{N-1} \cdots E_1 A = U$

1. If we recorded row operations in turning  $A$  to REF  $U$ ,

- This will be the case  $\tilde{L}\tilde{P}A = U$ , or  $A = PLU$ , where
  - $Y = PL$ ,
  - $U$  is REF of  $A$ .

2. If there are zero rows in  $U$  in its bottom part, one can reduce in the form

$$A = \hat{Y}\hat{U},$$

- $\hat{U}$ , with zero rows of  $U$  deleted,
- $\hat{Y}$ , consisting of only first  $r$  columns of  $Y$ .

**RREF by row operations:**  $E_N E_{N-1} \cdots E_1 A = U$

1. If we turned  $A$  to RREF, with zero row deleted,

- This will be the case  $A = CR$ , where
  - $Y = C$ ,
  - $U$  is RREF of  $A$ , with zero row deleted.

## Basis of $\text{Ran}(A)$

- Because  $\text{Ran}(U)$  always span  $\mathbb{R}^r \times \{(0, 0, \dots, 0)\}$  ( $m - r$  times)

$$\text{Ran}(Y) = \text{Ran}(A).$$

- Thus, in seeking basis of  $\text{Ran}(A)$ , we rather worked with  $\text{Ran}(Y)$ .

In fact, one can arrive one of the REF of  $A$  by multiplying orthogonal matrices,

$$O_N O_{N-1} O_{N-2} \cdots O_1 A = U, \quad U \text{ is REF of } A,$$

- Before, we turned  $A$  to REF by row operations. This can be done by hands.
- Orthogonal matrix  $\times$  orthogonal matrix is again orthogonal matrix. The inverse of an orthogonal matrix is again orthogonal matrix.

$$\tilde{O}A = U \iff A = OU,$$

- Arriving at REF by orthogonal matrices is done by row-pivoting Householder method.

Thus, there is another  $A = YU$

1.  $Y = O$  is an  $m \times m$  orthogonal matrix, (A matrix consisting of orthonormal columns),
2.  $U$  is an REF of  $A$ .
3. Again, they can be reduced to  $\hat{Y}(m \times r)$  and  $\hat{U}(r \times n)$ .

*Remark 16.1.* Typically, algorithm in library does not do the pivoting or does the column-pivoting. Resultant information on  $\text{Ran}(A)$  is same. In this case  $U$  part may not be a standard REF.

## Summary

- So far, we know

1.  $A = YU,$

$Y = C,$   $U$  is the RREF of  $A.$

2.  $A = YU,$

$Y = PL,$   $U$  is an REF of  $A.$  They can be reduced to  $\hat{Y}$  and  $\hat{U}.$

3.  $A = YU,$

$Y = O,$   $U$  is an REF of  $A.$  They can be reduced to  $\hat{Y}$  and  $\hat{U}.$

### Basis reading from $Y$

1. If those decompositions are available, the best choice is the last one, where we read the orthonormal basis for  $\text{Ran}(A).$
2. In PLU,  $PL$  may contain some zeroes.
3. In CR,  $C$  consists of columns of  $A$  as they are.

### Least Square and QR decomposition

1. Let  $A = YU$  be the QR decomposition as in the above.
2. We write  $A = \hat{Y}\hat{U}$  by deleting zero rows of  $U$  and collecting first  $r$  columns of  $Y$ .
3. The columns of  $\hat{Y}$ , denoted by  $o_1, o_2, \dots, o_r$  span  $\text{Ran}(A)$ .
4. The remaining columns of  $Y$ , denoted by  $o_{r+1}, o_{r+2}, \dots, o_m$  span  $\text{Ran}(A)^\perp$ .
5. Then, write

$$Y = \begin{pmatrix} | & | & \cdots & | \\ o_1 & o_2 & \cdots & o_m \\ | & | & \cdots & | \end{pmatrix}$$

consisting of orthonormal columns.

6. The projection matrix

$$P = Y\hat{P}Y^{-1} = Y\hat{P}Y^T = \begin{pmatrix} | & | & \cdots & | \\ o_1 & o_2 & \cdots & o_r \\ | & | & \cdots & | \end{pmatrix} \begin{pmatrix} - & o_1 & - \\ - & o_2 & - \\ \vdots & \vdots & \vdots \\ - & o_r & - \end{pmatrix} = \hat{Y}\hat{Y}^T.$$

7. In fact, we do not return to the old representation. In the temporal representation, we directly do the backsubstitution step:

$$\hat{U}x = \hat{Y}^T b, \quad \hat{U} : r \times n.$$

8. More precisely,

$$Ax = Pb \iff YUx = \hat{Y}\hat{Y}^T b \iff Ux = Y^T \hat{Y}\hat{Y}^T b,$$

$$U = \begin{pmatrix} \hat{U} \\ 0 \end{pmatrix},$$

$$\begin{aligned} Y^T \hat{Y}\hat{Y}^T &= \begin{pmatrix} - & o_1 & - \\ - & o_2 & - \\ \vdots & & \vdots \\ - & o_m & - \end{pmatrix} \begin{pmatrix} | & | & \cdots & | \\ o_1 & o_2 & \cdots & o_r \\ | & | & \cdots & | \end{pmatrix} \begin{pmatrix} - & o_1 & - \\ - & o_2 & - \\ \vdots & \vdots & \vdots \\ - & o_r & - \end{pmatrix} \\ &= \begin{pmatrix} \hat{Y}^T \\ 0 \end{pmatrix} \end{aligned}$$

### Conclusion:

If you run the reduced QR decomposition  $A = \hat{O}\hat{U}$ , then do the backsubstitution on

$$\hat{U}x = \hat{Y}^T b.$$

## Normal equation $A^T A x = A^T b$ when $m \geq n = r$

1. We first show that  $A^T A$  is  $n \times n$  invertible matrix.

**Proposition 1.** Suppose  $A$  is an  $m \times n$  matrix with  $\text{rank}(A) = n$  and  $m \geq n$ . Then  $A^T A$  is  $n \times n$  matrix with full rank.

*Proof.* .

- (a) Since  $\text{rank}(A) = n$ , there exist linearly independent  $x_1, x_2, \dots, x_n$  in  $\text{Trow}(A)$ .
- (b) To prove the Proposition, we show that  $\text{Ran}(A^T A)$  has dimensions  $n$ :

$$A^T A x_1, A^T A x_2, \dots, A^T A x_n \text{ are linearly independent.}$$

- (c) Suppose not, i.e.,  $\exists(\lambda_1, \lambda_2, \dots, \lambda_n)$ , not all of them are zero, such that

$$\begin{aligned} 0 &= \lambda_1 A^T A x_1 + \lambda_2 A^T A x_2 + \dots + \lambda_n A^T A x_n \\ &= A^T A(\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n) = A^T A x_\lambda. \end{aligned}$$

This implies,

$$x_\lambda^T A^T A x_\lambda = |A x_\lambda|^2 = 0 \implies A x_\lambda = 0 \implies x_\lambda \in \text{Trow}(A) \cap \text{Ker}(A) \implies x_\lambda = 0.$$

This contradicts to that  $x_1, \dots, x_n$  are linearly independent.

□

2. Thus, if  $m \geq n = r$ , there exists the unique least square solution  $x = (A^T A)^{-1} A^T b$ .

3. Now, we check this  $x$  is precisely same solution by our previous method:

Let  $A = \hat{Y} \hat{U}$ ,  $\hat{Y}$  is  $m \times n$  matrix with orthonormal columns,  $\hat{U}$  is  $n \times n$  REF of  $A$  with full rank.

$$\begin{aligned} (A^T A)^{-1} A^T b &= (\hat{U}^T \hat{Y}^T \hat{Y} \hat{U})^{-1} \hat{U}^T \hat{Y}^T b \\ &= (\hat{U}^T \hat{U})^{-1} \hat{U}^T \hat{Y}^T b = \hat{U}^{-1} (\hat{U}^T)^{-1} \hat{U}^T \hat{Y}^T b = \hat{U}^{-1} \hat{Y}^T b. \end{aligned}$$

The normal equation appears from the following calculation. We consider the function

$$\begin{aligned} e(x) &= |Ax - b|^2 = (Ax - b)^T(Ax - b) = (x^T A^T - b^T)(Ax - b) \\ &= x^T A^T Ax - x^T A^T b - b^T Ax + b^T b \\ &= x^T A^T Ax - 2x^T A^T b + |b|^2. \end{aligned}$$

I know how the graph of the function  $e(x)$  looks like: it is quadratic, and has a global minimum and global minimizer.

To find the global minimizer, we solve

$$\nabla e(x) = 0, \quad \nabla e(x) = 2A^T Ax - 2A^T b.$$

## Chapter 17

# Summary of course and Perspective of Linear Algebra beyond the course

### The $A = CR$ decomposition

1. We turn  $A$  into RREF  $R$ ,

$$E_N E_{N-1} \cdots E_1 A = R.$$

2. We may write the same equation as

$$E_N E_{N-1} \cdots E_1 = E, \quad A = E^{-1} R.$$

3. Super nice thing about  $A = CR$  is that even though we do not record  $E$ , we are forced to know what  $C$  is, it must be the pivot columns of  $A$ .
4. We can extract information on  $\text{Trow}(A)$  and  $\text{Ker}(A)$  out of RREF  $R$ .
5. We can extract information on  $\text{Ran}(A)$  and  $\text{Ran}(A)^\perp$  out of  $C$ .
6. If we are to solve least square problem with general  $b$ , we can build a projection matrix based on the information on  $\text{Ran}(A)$  side.
7. It is nice that we can perform  $A = CR$  by hands.

## The $A = PLU$ decomposition

1. We turn  $A$  into REF  $U$ ,

$$E_N E_{N-1} \cdots E_1 A = U.$$

2. We may write the same equation as

$$E_N E_{N-1} \cdots E_1 = E, \quad A = E^{-1}U.$$

3. Because  $E_k$  were row operations, if we recorded them,

$$E^{-1} = PL, \quad \Leftrightarrow \quad E = L^{-1}P^T.$$

4. We can extract information on  $\text{Trow}(A)$  and  $\text{Ker}(A)$  out of REF  $U$ .

5. We can extract information on  $\text{Ran}(A)$  and  $\text{Ran}(A)^\perp$  out of  $PL$ .

6. Suppose  $b' \in \text{Ran}(A)$ . We are to solve the equation

$$Ux = Eb'$$

7. We can do:

$$(A \mid b') \quad \text{into} \quad (U \mid \tilde{b}') = (U \mid Eb')$$

and solve equation  $Ux = \tilde{b}'$ .

8. Now, suppose we are given  $A = PLU$  (by computer). Note that  $PL = E^{-1}$ .

- (a) In that case, to solve  $PLUx = b'$ , we first solve

$$L\tilde{b}' = P^T b.$$

and we secondly solve

$$Ux = \tilde{b}'.$$

The both systems are triangular, and solution  $\tilde{b}'$  is obtained by substitution from top to bottom and from bottom to top respectively.

- (b) Note that  $\tilde{b}' = Eb' = L^{-1}P^T b'$ .

9. If we have many  $b'$ 's to solve, we may try to obtain  $LU$  by computer and obtain  $\tilde{b}'$  by solving  $L\tilde{b}' = P^T b'$ , instead of turning  $(A \mid b')$  to REF every time.

10. If we are to solve least square problem with general  $b$ , we can build a projection matrix based on the information on  $\text{Ran}(A)$  side.

## The $A = QR$ decomposition

1. Although it is known as  $QR$  decomposition, let us write  $A = OU$ . We turn  $A$  into

$$O_N O_{N-1} \cdots O_1 A = U.$$

2. We may write the same equation as

$$O_N O_{N-1} \cdots O_1 = O^T, \quad A = OU.$$

3. We can extract information on  $\text{Trow}(A)$  and  $\text{Ker}(A)$  out of  $U$ .

4. We can extract information on  $\text{Ran}(A)$  and  $\text{Ran}(A)^\perp$  out of  $O$ .

In particular, this  $\text{Ran}(A)$  side information is done very nicely thanks to that  $O$  is orthogonal.

5. This also makes the projection of  $b$  onto  $b' \in \text{Ran}(A)$  very easy.

6. Suppose  $b' \in \text{Ran}(A)$ . We are to solve the equation

$$Ux = O^T b'$$

.

7. For a general  $b$ , projection can be simply built-in in the reduction of  $O$  and  $U$  to  $\hat{O}$  and  $\hat{U}$ :

$$\hat{U}x = \hat{O}^T b.$$

## The $A = O U Q^T$ decomposition

1. We only introduce the statement for the Singular Value Decomposition

$$A = O U Q^T$$

- 2.  $O$  is  $m \times m$  orthogonal matrix,
- 3.  $Q$  is  $n \times n$  orthogonal matrix,
- 4.  $U$  is rank  $r$  diagonal matrix whose diagonals are all nonnegative.

$$A = \begin{pmatrix} & & & \\ | & | & \cdots & | \\ o_1 & o_2 & \cdots & o_m \\ | & | & \cdots & | \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \sigma_2 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \sigma_3 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \sigma_r & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} - & q_1 & - \\ - & q_2 & - \\ \vdots & \vdots & \vdots \\ - & q_n & - \end{pmatrix}$$

5. Once this decomposition is made, we pretty much know all about  $A$ :

$$\begin{aligned} \text{Trow}(A) &= \text{span} \langle q_1, q_2, \dots, q_r \rangle, \\ \text{Ker}(A) &= \text{span} \langle q_{r+1}, q_{r+2}, \dots, q_n \rangle, \\ \text{Ran}(A) &= \text{span} \langle o_1, o_2, \dots, o_r \rangle, \\ \text{Ran}(A)^\perp &= \text{span} \langle o_{r+1}, o_{r+2}, \dots, o_m \rangle. \end{aligned}$$

6. The least square problem is then solved by the following. Let  $A = \hat{O} \hat{U} \hat{Q}^T$  be the reduced decomposition.

- (a) Solve

$$\hat{U}x' = \hat{O}^T b,$$

Since  $\hat{U}$  is  $r \times r$  diagonal, this solution  $x'$  is just read off.

- (b) The particular solution  $x_p = \hat{Q}x'$ .  
(c) The complete set of solutions consists of

$$x_p + c_{r+1}q_{r+1} + c_{r+2}q_{r+2} + \cdots + c_nq_n.$$

## Perspective beyond this course

- For every  $m \times n$  matrix  $A$ , four decompositions
  1.  $A = CR$ ,
  2.  $A = PLU$ ,
  3.  $A = OU$ ,
  4.  $A = OUQ^T$

can be computed by hand or computer. Other than this four, you might not need to know further in most cases.

- We do not introduce the algorithm for the latter two, QR and Singular Value Decomposition. So, you may want to study what are the methods in the future.

- For every  $n \times n$  square matrix  $A$ , of course, all above four can be applied.
- For a square matrix  $A$ , one sometimes look for a decomposition in the form

$$A = XUX^{-1}$$

in the sense that basis change effects in the same way for the domain and codomain.

- There is thus 5-th decomposition for a square matrix,
- 5.  $A = QUQ^T$ , where  $U$  is upper triangular and  $Q$  is orthogonal.

$\left( \begin{array}{l} \text{More precisely, if we keep every matrix in real numbers, } U \text{ is block upper} \\ \text{triangular (Schur Form.)} \end{array} \right)$

- This is known as Schur Decomposition. You may want to study what is the method for the Schur Decomposition.



## Chapter 18

# Solving a symmetric and positive definite system

- An  $n \times n$  square matrix that is symmetric and positive definite is one of the nicest matrices.
- When a linear system is given by a spd matrix, there are methods work better than rest of cases, two major methods known are
  1. Cholesky Decomposition
  2. Conjugate Gradient.

**Definition 1** (A symmetric positive definite matrix). .

- An  $n \times n$  square matrix  $A$  is called symmetric if  $A^T = A$ .
- A symmetric matrix  $A$  is said to be positive definite if

for all  $x \neq 0$ ,  $x^T Ax$  is strictly positive.

1.  $\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix}$  is symmetric, but not positive definite.

2.  $\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix}$  is symmetric, but not positive definite.

3. If  $A$  is  $m \times n$  matrix with  $m \geq n = r$ , then  $A^T A$  is symmetric positive definite.
4. If  $A$  is spd, then  $x \neq 0$  implies  $Ax \neq 0$ . In other words, it is always full rank.

## Solving by Cholesky Decomposition

From now on,  $A$  is (full rank) symmetric and positive definite  $n \times n$  matrix.

### Simplest Example

$$A = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix}$$

Although it is totally unnecessary, if we want, we can write  $A$  in the form

$$A = \begin{pmatrix} \sqrt{3} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{1} & 0 \\ 0 & 0 & 0 & \sqrt{7} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{1} & 0 \\ 0 & 0 & 0 & \sqrt{7} \end{pmatrix}$$

The cholesky decomposition is not a new decomposition. It is an  $LU$  decomposition

$$A = LU, \quad U \text{ part is adjusted so that } U = L^T.$$

(Since  $A$  is full rank, it is  $LU$ , not  $PLU$ . No need for row swap.)

We just jump to the example.

We consider a full rank  $L$  matrix to compose  $A = LL^T$  and do the  $LU$  decomposition of  $A$  to see if REF becomes  $L^T$ .

Do the LU decomposition, but to distribute factors into  $L$  and  $L^T$  equally, do the square root the pivots, and divide the row by that.

$$L = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ -2 & 2 & 1 & 0 \\ 5 & -1 & 3 & -2 \end{pmatrix}, \quad L^T = \begin{pmatrix} 3 & 0 & -2 & 5 \\ 0 & -4 & 2 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -2 \end{pmatrix}, \quad A = LL^T = \begin{pmatrix} 9 & 0 & -6 & 15 \\ 0 & 16 & -8 & 4 \\ -6 & -8 & 9 & -9 \\ 15 & 4 & -9 & 39 \end{pmatrix},$$