## Chapter 1

# What will we do ?

We are interested in the Euclidean space $\mathbb{R}^n$.

1. In the first major part of the course, we discuss about the $n$-dimensional volume of a subset $E \subset \mathbb{R}^n$. The first objective is to construct the Lebesgue measure $\mathcal{L}^n$.

2. In the second major part of the course, we update our tool of Integral, namely from the Riemann Integral to the Lebesgue Integral. This is based on the measure theory developed by abstraction of the Lebesgue measure in the first part.

# Chapter 2

# Measuring $n$-dimensional volume of $E \subset \mathbb{R}^n$

1. In the Euclidean space $\mathbb{R}^n$, we are able to measure the distance between two points $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$ of $\mathbb{R}^n$,

$$d(x, y) = \Big( (x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots (x_n - y_n)^2 \Big)^{\frac{1}{2}}.$$

2. This gives rise to the $n$-dimensional volume formula for a few classes of subsets in $\mathbb{R}^n$. For example in $\mathbb{R}^3$, we take the formula:

   (a) If $E$ is the cube $[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$, we take the value

   $$(b_1 - a_1)(b_2 - a_2)(b_3 - a_3)$$

   as its 3-dimensional volume.

   (b) If we consider a tetrahedron with base area $A$ and the height $h$, we take the value

   $$\frac{1}{3} Ah$$

   as its 3-dimensional volume.

   (c) other examples...

Knowing the $n$-dimensional volumes of such a class of elementary sets,

1. We may extend our knowledge base on calculating $n$-volume: For a set made by assembling a few such elementary sets, the $n$-volume would be the sum of $n$-volumes of elementary sets.

2. That we wrote right above is the theory we want to develop. It is a difficult task: To make this consistent mathematically, any such theory should provide a proof that the $n$-volume assigned on a certain set $E \subset \mathbb{R}^n$ would be calculated independently of ways of cutting the set.

3. For example, for a given set $E \subset \mathbb{R}^n$, there are two persons. The first person cuts $E$ into $G_1, G_2, G_3$, and the second person cuts $E$ into $H_1$ and $H_2$. More specifically, $G_1, G_2, G_3$ are pairwise disjoint and $E = G_1 \cup G_2 \cup G_3$, and $H_1, H_2$ are pairwise disjoint and $E = H_1 \cup H_2$. $n$-volumes of $G_i$, and $H_j$ are known. Theory should be certain about the equality

$$|G_1| + |G_2| + |G_3| = |H_1| + |H_2|$$

where $|S|$ denotes $n$-volume of the set $S$, if known. This consistency should be the case for all different ways of cutting the set $E$.

4. We consider the following humble goal: Let $\mathcal{R}$ be the collection of all cubes in $\mathbb{R}^n$ (whose $n$-volumes are as we know). Let $R$ be any cube in $\mathcal{R}$ with the $n$-volume $|R|$. Let $(R_j)_{j=1}^{\infty}$ be pairwise disjoint partition of $R$, and $(Q_k)_{k=1}^{\infty}$ be another pairwise disjoint partition of $R$. The goal is that the $n$-volume of a cube was actually *correct*:

$$|R| = \sum_{j=1}^{\infty} |R_j| = \sum_{k=1}^{\infty} |Q_k|.$$

This is the task of our courses for a while, and this is a difficult problem.

# Consistent family with $n$-volume

The word 'family', or 'collection' are synonyms of set. We use family or collection to avoid confusion.

**Definition 1.** *A pair* $(\mathcal{G}, \lambda)$ *of* $\mathcal{G}$*, a nonempty collection of subsets of* $\mathbb{R}^n$ *containing* $\emptyset$*,* $X$*, and* $\lambda : \mathcal{G} \to [0, \infty]$*, is said to be consistent if*

1. $\lambda(\emptyset) = 0$,
2. *If* $G$ *is a set in* $\mathcal{G}$*,* $v = \lambda(G)$ *and* $(G_j)_{j=1}^{\infty}$ *is any sequence of sets in* $\mathcal{G}$ *that are pairwise disjoint and* $G = \bigcup_{j=1}^{\infty} G_j$*, then*

$$\sum_{j=1}^{\infty} \lambda(G_j) = v.$$

Let $n = 2$ and consider $\mathbb{R}^2$.

By half-open intervals we mean the intervals of one of the following forms

$$\emptyset, \quad [a, b), \quad [a, \infty), \quad (-\infty, b), \quad (-\infty, \infty),$$

where $a, b \in \mathbb{R}$ and are assumed to be $a < b$.

**Definition 2.** *The collection* $\mathcal{R}$ *is the collection of all cartesian products of two half-open intervals. The member of* $\mathcal{R}$ *is called a rectangle.*

1. *If* $R$ *is an unbounded rectangle, we define* $|R| = \infty$.
2. *If* $R$ *is a nonempty bounded rectangle* $[a_1, b_1) \times [a_2, b_2)$*,* $|R| = (b_1 - a_1)(b_2 - a_2)$.
3. $|\emptyset| = 0$.

We prove that $(\mathcal{R}, |\cdot|)$ is consistent from now on.

## Towards a consistent family

At this moment, we prove a proposition stating that, out of somewhat arbitrary volume function $\rho$ and a collection, one may extract its refined version of volume function $\lambda$.

**Proposition 3.** *Let $X$ be a nonempty set, $\mathcal{G} \subset \mathcal{P}(X)$, and $\rho : \mathcal{G} \to [0, \infty]$ be such that $\emptyset \in \mathcal{G}$, $X \in \mathcal{G}$, and $\rho(\emptyset) = 0$. For any set $S \subset X$, define*

$$\lambda(S) := \inf_{(G_j) \text{ of } \mathcal{G} \text{ that covers } S} \Big\{ \sum_{j=1}^{\infty} \rho(G_j) \Big\}.$$

*Then, $\lambda$, defined on $\mathcal{P}(X)$, satisfies the followings:*

1. $\lambda(\emptyset) = 0$.
2. *If $(S_j)$ of $\mathcal{P}(X)$ covers $S$, i.e., $\bigcup_j S_j \supset S$, then*

$$\lambda(S) \leq \sum_{j=1}^{\infty} \lambda(S_j). \tag{2.0.1}$$

*Remark* 2.1. We will define on $\mathcal{P}(\mathbb{R}^2)$

$$\lambda(S) := \inf_{(R_j) \text{ of rectangles that covers } S} \Big\{ \sum_{j=1}^{\infty} |R_j| \Big\}.$$

Examples:

*proof of Proposition 3.* Let $S$ be any subset of $X$.

1. The set of coverings of $S$ by sets in $\mathcal{G}$ is nonempty because $X \in \mathcal{G}$ and $(X, \emptyset, \emptyset, \cdots)$ covers $S$. Obviously, the set

$$\left\{ \sum_{j=1}^{\infty} \rho(G_j) : (G_j) \text{ of } \mathcal{G} \text{ covers } S \right\} \subset [0, \infty]$$

is nonempty and bounded below by 0. Therefore, $\lambda(S)$, the infimum over the set, is well-defined.

2. Since $(\emptyset, \emptyset, \cdots)$ covers $\emptyset$, $\rho(\emptyset) = 0$, and $\sum_j 0 = 0$, $\lambda(\emptyset)$ must be 0.

3. Now, let $(S_j)$ be any sequence of subsets of $X$ that covers $S$. Suppose any of $\lambda(S_j) = \infty$. Then the inequality (2.0.1) is trivially true. Now, we assume $\lambda(S_j) < \infty$ for every $j$.

4. Let $\epsilon > 0$. By the definition of infimum, for each $j$, there exists a covering $(G_\alpha^j)_{\alpha=1}^{\infty}$ of $S_j$ by sets in $\mathcal{G}$ such that

$$\sum_{\alpha=1}^{\infty} \rho(G_\alpha^j) \leq \lambda(S_j) + \frac{\epsilon}{2^j}.$$

Obviously, $\bigcup_\alpha \bigcup_j G_\alpha^j \supset S$ and thus $(G_\alpha^j)_{j,\alpha=1}^{\infty}$ is a countable covering of $S$ by sets in $\mathcal{G}$. Thus,

$$\lambda(S) \leq \sum_{j=1}^{\infty} \sum_{\alpha=1}^{\infty} \rho(G_\alpha^j) \leq \sum_{j=1}^{\infty} \left[ \lambda(S_j) + \frac{\epsilon}{2^j} \right] = \sum_{j=1}^{\infty} \lambda(S_j) + \epsilon.$$

Since this inequality holds for every $\epsilon > 0$, we conclude that

$$\lambda(S) \leq \sum_{j=1}^{\infty} \lambda(S_j).$$

□

1. Out of mere formula $\big|[a_1, b_1) \times [a_2, b_2)\big| = (b_1 - a_1)(b_2 - a_2)$, we suddenly have a definition of $\lambda$ for all the subsets of $\mathbb{R}^2$.
2. However, we restrict ourselves the use of $\lambda$ only on rectangles for a while to complete the proof of that $(\mathcal{R}, |\cdot|)$ is consistent.
3. Now, we aim to prove that for a rectangle $R$, $|R| = \lambda(R)$, namely the area formula $|\cdot|$ was already good to some extent.
4. Since, $\lambda(R) \leq |R|$, we only need to prove $\lambda(R) \geq |R|$.

We prove a few lemmas.

**Lemma 4.** *If $R$ and $R'$ are rectangels and $R \subset R'$, then $|R| \leq |R'|$.*

*Proof.* Omitted. □

**Lemma 5.** *Let $R$ be a nonempty bounded rectangle $[a, b) \times [c, d)$, and consider*

$$a = t_1 < t_2 < \cdots < t_N = b, \quad c = s_1 < s_2 < \cdots < s_K = d,$$

*and consider rectangles $R_{i,j} = [t_i, t_{i+1}) \times [s_j, s_{j+1})$ for $i = 1, 2, \cdots, N-1$ and $j = 1, 2, \cdots, K-1$. Then,*

$$|R| = \sum_{i=1}^{N-1} \sum_{j=1}^{K-1} |R_{i,j}| \quad \text{and} \quad R = \bigcup_{i=1}^{N-1} \bigcup_{j=1}^{K-1} R_{i,j} \quad \text{of disjoint union.}$$

*Proof.*

$$\begin{aligned}
|R| = (b-a)(c-d) &= \Big( \sum_{i=1}^{N-1} (t_{i+1} - t_i) \Big) \Big( \sum_{j=1}^{K-1} (s_{j+1} - s_j) \Big) \\
&= \sum_{i=1}^{N-1} \sum_{j=1}^{K-1} (t_{i+1} - t_i)(s_{j+1} - s_j) = \sum_{i=1}^{N-1} \sum_{j=1}^{K-1} |R_{i,j}|.
\end{aligned}$$

By definition,

$$\begin{aligned}
R_{i,j} &= \{(x, y) \mid t_i \leq x < t_{i+1} \text{ and } s_j \leq y < s_{j+1}\} \\
R_{i',j'} &= \{(x, y) \mid t_i' \leq x < t_{i'+1} \text{ and } s_j' \leq y < s_{j'+1}\}
\end{aligned}$$

and if $(i, j) \neq (i', j')$, then $i \neq i'$ or $j \neq j'$, and they must be disjoint. Again by definition

$$\begin{aligned}
R = \ & \{(x, y) \mid a \leq x < b \quad \text{and} \quad c \leq y < d\} \\
= \ & \Big\{ (x, y) \ \Big| \ \Big[ t_1 \leq x < t_2 \text{ or } t_2 \leq x < t_3 \text{ or } \cdots \text{ or } t_{N-1} \leq x < t_N \Big] \\
& \quad \text{and} \quad \Big[ s_1 \leq y < s_2 \text{ or } s_2 \leq y < s_3 \text{ or } \cdots \text{ or } s_{K-1} \leq y < y_N \Big] \Big\} \\
= \ & \bigcup_{i=1}^{N-1} \bigcup_{j=1}^{K-1} R_{i,j}.
\end{aligned}$$

□

**Lemma 6.** *Suppose $R$ be a nonempty bounded rectangle. If $(R_k)_{k=1}^M$ is a covering of $R$ by sets in $\mathcal{R}$, then*

$$|R| \le \sum_{k=1}^{M} |R_k|.$$

*Proof.*

1. If any of $R_k$ is unbounded, then $|R_k| = \infty$ and the inequality trivially holds. Now we assume $R_k$ is a bounded rectangle for every $k$.
2. If we can prove the same inequality on any subcover of $(R_k)$, then the inequality still stands with the cover itself. Thus we consider a subcover of $(R_k)$ by discarding every $R_k$ that is the empty set, and prove the inequality with this subcover: Below, we assume $R_k$ is nonempty for every $k$.
3. Let us write for each $R_k = [a_k, b_k) \times [c_k, d_k)$.
   Let $t_1 < t_2 < \cdots < t_N$ be an enumeration of the finite set
   $$\{a, a_1, a_2, \cdots, a_M, b, b_1, b_2, \cdots, b_M\}$$
   in ascending order.
   Let $s_1 < s_2 < \cdots < s_K$ be an enumeration of the finite set
   $$\{c, c_1, c_2, \cdots, c_K, d, d_1, d_2, \cdots, d_K\}$$
   in ascending order. We consider the rectangles $Q_{i,j} = [t_i, t_{i+1}) \times [s_j, s_{j+1})$, pairwise disjoint.
4. Note that for each $R_k = [a_k, b_k) \times [c_k, d_k)$, there exist indices $i_{begin}(k)$ and $i_{end}(k)$ such that $t_{i_{begin}(k)} = a_k$ and $t_{i_{end}(k)} = b_k$. Similarly $j_{begin}(k)$ and $j_{end}(k)$ exist. By the previous lemma,
   $$R_k = \bigcup_{i=i_{begin}(k)}^{i_{end}(k)-1} \bigcup_{j=j_{begin}(k)}^{j_{end}(k)-1} Q_{i,j} \quad \text{of disjoint union.}$$
   Because of this equality and that $(Q_{i,j})$ are pairwise disjoint, the following is true:
   $$\text{For every } k \text{ and every } (i,j), \text{ either } Q_{i,j} \subset R_k \text{ or } Q_{i,j} \cap R_k = \emptyset.$$
5. The similar is true for $R$.
6. We define
   $$\Gamma = \{(i,j) \mid Q_{i,j} \subset R\}, \quad \Gamma_k = \{(i,j) \mid Q_{i,j} \subset R_k\}.$$
   By the previous Lemma,
   $$|R| = \sum_{(i,j)\in\Gamma} |Q_{i,j}|, \quad |R_k| = \sum_{(i,j)\in\Gamma_k} |Q_{i,j}|.$$
7. That $R \subset \bigcup_k R_k$ implies that $(i,j) \in \Gamma$ implies that $Q_{i,j}$ intersects some $R_k$. Otherwise, $(R_k)$ is not a covering of $R$.
8. This $R_k$-intersecting $Q_{i,j}$ in fact must be a subset of $R_k$. But $Q_{i,j} \subset R_k$ iff $(i,j) \in \Gamma_k$. We thus conclude: $\Gamma \subset \bigcup_k \Gamma_k$.
9. Finally,
   $$|R| = \sum_{(i,j)\in\Gamma} |Q_{i,j}| \le \sum_{(i,j)\in\bigcup_k \Gamma_k} |Q_{i,j}| \le \sum_k \sum_{(i,j)\in\Gamma_k} |Q_{i,j}| = \sum_k |R_k|.$$

$\square$

**Proposition 7.** *For a rectangle $R$, $\lambda(R) = |R|$.*

*Proof.*

1. If $R = \emptyset$, $\lambda(\emptyset) = 0 = |\emptyset|$.
2. Now, assume first that $R$ is a bounded rectangle. We prove that $\lambda(R) \geq |R|$ below. Note we know that $\lambda(R) \leq |R| < \infty$.
3. By definition of $\lambda(R)$, for any $\epsilon > 0$ there exists a $(Q_k)$ of $\mathcal{R}$ that covers $R$ such that
$$\lambda(R) + \epsilon \geq \sum_k |Q_k|.$$
4. Now, it is possible to enlarge each rectangle $Q_k$ a little to form an open rectangle $\tilde{Q}_k \supset Q_k$ but satisfying
$$|Q_k| \geq |\tilde{Q}_k| - \frac{\epsilon}{2^k}.$$
5. $(\tilde{Q}_k)$ forms an open covering of the closure of $R$ that is compact. Hence, there is a finite subcover of the closure of $R$. (that is a finite cover of $R$ too.) We have
$$\begin{aligned}\sum_k |Q_k| &\geq \sum_k \left(|\tilde{Q}_k| - \frac{\epsilon}{2^k}\right) \\ &\geq \sum_k |\tilde{Q}_k| - \epsilon \\ &\geq \sum_{k \in subcover} |\tilde{Q}_k| - \epsilon \\ &\geq |R| - \epsilon,\end{aligned}$$
where in the last inequality, we used the Lemma 6. In conclusion,
$$\lambda(R) + 2\epsilon \geq |R|$$
for every $\epsilon > 0$, and we conclude $\lambda(R) \geq |R|$.
6. Finally, let $R$ be an unbounded rectangle. If so, we can consider $R_1 \subset R_2 \subset \cdots$ of subsets of $R$ with $|R_j| < \infty$ and $|R_j| \to \infty$ as $j \to \infty$. Then for every $j$,
$$\lambda(R) \geq \lambda(R_j) = |R_j|,$$
which implies that $\lambda(R) = \infty$. The equality $\lambda(R) = |R| = \infty$ holds.

$\square$

For later purpose, we also prove the following equality.

**Lemma 8.** *Let $R$ be a nonempty bounded rectangle. If $R = \bigcup_{k=1}^{M} R_k$ of disjoint union of rectangles $R_1$, $R_2$, $\cdots$, $R_M$, then*

$$|R| = \sum_{k=1}^{M} |R_k|.$$

*Proof.* Exercise. $\square$

Justify first that $(i,j) \in \Gamma$ iff $(i,j) \cup_k \Gamma_k$, and second that $\cup_k \Gamma_k$ is a disjoint union.

# Chapter 3

# Arguments repeatedly used

[Argument with the infimum]

Let $A \subset \mathbb{R}$ lower bounded. Then $m := \inf A$ is well-defined. For any positive $\epsilon > 0$, $m + \epsilon$ is not a lower bound of $A$, and thus there must be $a \in A$ such that $a \leq m + \epsilon$.

[Inequality holding for all $\epsilon > 0$]

Let $a, b \in \mathbb{R}$. If $a \leq b + \epsilon$ for every $\epsilon > 0$, then $a \leq b$.

[Countable sum of nonnegative numbers]

Let $(c_j)$ be a sequence of nonnegative numbers. Then, the summation of the series is independent of changing orders, such as $c_{\sigma(j)}$ with $\sigma : \mathbb{N} \to \mathbb{N}$ a bijection. One of the following two is the case.

$(i)$ $\displaystyle\sum_{j=1}^{\infty} c_j = \lim_{N\to\infty} \sum_{j=1}^{N} c_j = s_* < \infty.$

The series absolutely converges, and the limit $s_*$ is independent of changing orders of $c_j$

$(ii)$ $\displaystyle\sum_{j=1}^{\infty} c_j = \lim_{N\to\infty} \sum_{j=1}^{N} c_j = s_* = \infty.$

The limit $+\infty$ is independent of changing orders of $c_j$.

[From $(E_j)$ of sequence of sets to $(\hat{E}_j)$ of pairwise disjoint sets]

**Lemma 9.** *Let $(E_j)$ be a sequence of sets. Define $(\hat{E}_j)$ recursive by*

$$\hat{E}_1 = E_1$$
$$\hat{E}_j = E_j \setminus \Big( \bigcup_{i=1}^{j-1} E_i \Big)$$

*Then, for any $N$,*

$$(i) \quad \bigcup_{j=1}^{N} \hat{E}_j = \bigcup_{j=1}^{N} E_j,$$

*(ii)* $(\hat{E}_j)_{j=1}^{N}$ *is a sequence of pairwise disjoint sets.*

*Proof.* The two assertions are obviously true for $N = 1$. If the assertion is true for $1, 2, \cdots, N-1$,

$$\hat{E}_N = E_N \setminus \Big(\bigcup_{j=1}^{N-1} E_j\Big) = E_N \setminus \Big(\bigcup_{j=1}^{N-1} \hat{E}_j\Big).$$

Obviously, $\hat{E}_N$ is disjoint from $\bigcup_{j=1}^{N-1} \hat{E}_j$. Therefore, $(\hat{E}_j)_{j=1}^{N}$ is pairwise disjoint. Also,

$$\begin{aligned}\bigcup_{j=1}^{N} \hat{E}_j = \hat{E}_N \cup \Big(\bigcup_{j=1}^{N-1} \hat{E}_j\Big) &= \hat{E}_N \cup \Big(\bigcup_{j=1}^{N-1} E_j\Big) \\ &= \Big[E_N \cap \Big(\bigcup_{j=1}^{N-1} E_j\Big)^c\Big] \cup \Big(\bigcup_{j=1}^{N-1} E_j\Big) \\ &= E_N \cup \Big(\bigcup_{j=1}^{N-1} E_j\Big) = \bigcup_{j=1}^{N} E_j\end{aligned}$$

*Remark* 3.1. Since the assertion in Lemma 9 is true for any $N$, it also holds that

$$(i) \quad \bigcup_{j=1}^{\infty} \hat{E}_j = \bigcup_{j=1}^{\infty} E_j,$$

*(ii)* $(\hat{E}_j)_{j=1}^{\infty}$ is a sequence of pairwise disjoint sets.

because

$$x \in \bigcup_{j=1}^{\infty} \hat{E}_j \implies x \in \hat{E}_{j_0} \text{ for some } j_0 \implies x \in \bigcup_{j=1}^{j_0} \hat{E}_j = \bigcup_{j=1}^{j_0} E_j \implies x \in \bigcup_{j=1}^{\infty} E_j,$$

$$x \in \bigcup_{j=1}^{\infty} E_j \implies x \in E_{j_1} \text{ for some } j_1 \implies x \in \bigcup_{j=1}^{j_1} E_j = \bigcup_{j=1}^{j_1} \hat{E}_j \implies x \in \bigcup_{j=1}^{\infty} \hat{E}_j,$$

and for any $\hat{E}_{i_0}$ and $\hat{E}_{i_1}$, we let $N = \max\{i_0, i_1\}$ and we know $(\hat{E}_j)_{j=1}^{N}$ is pairwise disjoint.

*Remark* 3.2. [(For any $N$)-assertion by induction] & [(limit)-assertion proven in addition] style of proof will appear repeatedly.

□

# Chapter 4

# Measure Theoretic Separation

We would like to have that if a set $S \subset \mathbb{R}^2$ is made by assembling two disjoint sets $S_1$ and $S_2$,

$$\lambda(S_1 \cup S_2) = \lambda(S_1) + \lambda(S_2).$$

We then would like to have its countable version.

Since the inequality $\lambda(S_1 \cup S_2) \leq \lambda(S_1) + \lambda(S_2)$ already is established, worry is in whether there is a case

$$\lambda(S_1 \cup S_2) < \lambda(S_1) + \lambda(S_2)$$

## Over-estimation by Truely 2-dimensional covering

Look at the definition of $\lambda(S)$,

$$\lambda(S) := \inf_{(R_j) \text{ of rectangles that covers } S} \left\{ \sum_{j=1}^{\infty} |R_j| \right\}.$$

The importance of the rectangle in our theory lies in that it is a Truely 2-dimensional lump.

1. The set $\displaystyle\bigcup_{j=1}^{\infty} R_j \supset S$ is thus a Truely 2-dimensional lump replacement of $S$.

2. We estimate its 2-dimensional area by $\displaystyle\sum_{j=1}^{\infty} |R_j|$, that is certainly an over-estimation.

3. This over-estimation is minimized as much as possible, over all the coverings.

How does this 2-dim-over-estimation $\to$ minimization properly works? For example consider the singletone set $\{x_0\}$. Intuitively, 0 has to be its 2-dimsnional area.

1. We see that one square $R_\ell$ with side length $\ell > 0$ whose center is $x_0$ is a Truely 2-dimensional replacement of $\{x_0\}$. $(R_\ell, \emptyset, \emptyset, \cdots)$ covers $\{x_0\}$.

2. Its over-estimation is thus, $\ell^2 > 0$.

3. By minimization of over-estimation by letting $\ell \to 0$, we conclude that the infimum $\lambda(\{x_0\}) = 0$.

Thus, it makes sense to take the area of one point set is 0.

Question: Can the over-estimation be not resolved by the minimization process?

One speculative example about the question of resolving over-estimation is the following in 1 dimension. The role of rectangles is taken by intervals. Let

$$A = [0, 1] \cap \mathbb{Q}, \quad B = [0, 1] \cap \mathbb{Q}^c$$

1. If $(R_j)$ is a Truely 1-dimensional covering of $A$ by intervals, and $(Q_k)$ is a Truely 1-dimensional covering of $B$ by intervals, let us write this replacement

$$A' = \bigcup_j R_j, \quad B' = \bigcup_k Q_k.$$

2. Because of density of rationals and irrationals, the invasion of $A'$ into the portion of $B'$, and the invasion of $B'$ into the portion of $A'$ must have occured. In other words,

$$\sum_{j=1}^{\infty} |R_j| + \sum_{k=1}^{\infty} |Q_k| > 1.$$

3. Is it for certain thing that by the followed minimization step, this is to be resolved properly ? In other words, are we sure

$$\lambda(A) + \lambda(B) = 1 \quad ?$$

This is why we ask a question if there can be a case of two disjoint set $S_1$ and $S_2$ with

$$\lambda(S_1 \cup S_2) < \lambda(S_1) + \lambda(S_2)$$

## measure-theoretic separation

Since we are very speculative about this over-estimation-resolving procedure, we adopt a stronger notion of separation over the notion of being disjoint.

**Definition 1.** *We say a set $E \subset \mathbb{R}^2$ separates $S_1$ and $S_2$ if*

$$\Big(S_1 \subset E \quad \textit{and} \quad S_2 \subset E^c\Big) \quad \textit{or} \quad \Big(S_2 \subset E \quad \textit{and} \quad S_1 \subset E^c\Big)$$

*Remark* 4.1. If there exists a set $E$ that separates $S_1$ and $S_2$, then $S_1$ and $S_2$ must be disjoint.

**Example**: Let $E$ be an open ball of radius $r > 0$ and $S_1$ and $S_2$ be two compact sets.

**Example**: Let $E$ be the upper half plane $x_2 \geq 0$ and $S_1$ and $S_2$ be two sets one of which is in the half plane, and the other is outside of the half plane.

**Definition 2.** *We say $E \subset \mathbb{R}^2$ is $\lambda$-separating if the following is true.*

$$E \text{ separates } S_1 \text{ and } S_2 \quad \Longrightarrow \quad \lambda(S_1 \cup S_2) = \lambda(S_1) + \lambda(S_2).$$

**Question 1**: What kind of sets can have such a separating property?

We answer to the following question first, before the Q1.

**Question 2**: What are the consequences of being such a set?

**Theorem 3.** *Let $E_1$, $E_2$, $E_3$, $\cdots$ be pairwise disjoint $\lambda$-separating sets and $S_1, S_2, \cdots$ be any sequence in $\mathcal{P}(\mathbb{R}^2)$ such that $S_j \subset E_j$ for every $j$. Then,*

$$(i) \text{ for any } N \quad \lambda\Big(\bigcup_{j=1}^{N} S_j\Big) = \sum_{j=1}^{N} \lambda(S_j), \quad \text{and} \quad (ii)\ \lambda\Big(\bigcup_{j=1}^{\infty} S_j\Big) = \sum_{j=1}^{\infty} \lambda(S_j).$$

*Proof.* 1. We prove the first assertion.

Certainly $\lambda\Big(\bigcup_{j=1}^{1} S_j\Big) = \sum_{j=1}^{1} \lambda(S_j)$. Now, if equality holds for

$$\lambda\Big(\bigcup_{j=1}^{k-1} S_j\Big) = \sum_{j=1}^{k-1} \lambda(S_j)$$

we assert that

$$\lambda\Big(\bigcup_{j=1}^{k} S_j\Big) = \lambda\Big(\bigcup_{j=1}^{k-1} S_j \cup S_k\Big).$$

Since $E_k$ separates $S_k$ and $\Big(\bigcup_{j=1}^{k-1} S_j\Big)$, the (RHS) equals to

$$\lambda\Big(\bigcup_{j=1}^{k-1} S_j\Big) + \lambda(S_k) = \sum_{j=1}^{k-1} \lambda(S_j) + \lambda(S_k) = \sum_{j=1}^{k} \lambda(S_k).$$

2. For the second assertion,

$$\lambda\Big(\bigcup_{j=1}^{\infty} S_j\Big) \leq \sum_{j=1}^{\infty} \lambda(S_j) = \lim_{N\to\infty} \sum_{j=1}^{N} \lambda(S_j) = \lim_{N\to\infty} \lambda\Big(\bigcup_{j=1}^{N} S_j\Big) \leq \lim_{N\to\infty} \lambda\Big(\bigcup_{j=1}^{\infty} S_j\Big) = \lambda\Big(\bigcup_{j=1}^{\infty} S_j\Big)$$

Hence, every quantity appeared equals to each other.

$\square$

*Remark* 4.2. One important example is the case where $S_j = E_j$ itself for every $j$, that are pairwise disjoint and $\lambda$-separating. They always satisfies

$$\lambda\Big(\bigcup_{j=1}^{\infty} E_j\Big) = \sum_{j=1}^{\infty} \lambda(E_j).$$

*Remark* 4.3. To get back to our first objective, to show $(\mathcal{R}, |\cdot|)$ is consistent, (that is to show $(\mathcal{R}, \lambda)$ is consistent since $\lambda(R) = |R|$ for any rectangle $R \in \mathcal{R}$), we will be done once we prove that any rectangle is $\lambda$-separating.

**Proposition 4.** *For any $R \in \mathcal{R}$, the following is true.*

$$R \text{ separates } S_1 \text{ and } S_2 \quad \Longrightarrow \quad \lambda(S_1 \cup S_2) = \lambda(S_1) + \lambda(S_2).$$

*Proof.* We prove that

$$R \text{ separates } S_1 \text{ and } S_2 \quad \Longrightarrow \quad \lambda(S_1 \cup S_2) \geq \lambda(S_1) + \lambda(S_2).$$

1. If $\lambda(S_1 \cup S_2) = \infty$, then the inequality trivially holds.

2. From now on, we assume $\lambda(S_1 \cup S_2) < \infty$. It also follows that $\lambda(S_1) < \infty$ and $\lambda(S_2) < \infty$. Without loss, we consider the case $S_1 \subset R$.

3. For any $\epsilon > 0$, there exists a $(R_j)$ of $\mathcal{R}$ that covers $S_1 \cup S_2$ such that

   $$\lambda(S_1 \cup S_2) + \epsilon \geq \sum_{j=1}^{\infty} \lambda(R_j).$$

   Note that every $R_j$ must be a bounded rectangle and the series in (RHS) absolutely converges, since (LHS) is finite.

4. Now, we notice that $R^c$ can always be written as a disjoint union of four rectangles $Q_1$, $Q_2$, $Q_3$, and $Q_4$.

5. Let $R = Q_0$. We can write for every $j$

   $$R_j = Q_j^0 \cup Q_j^1 \cup Q_j^2 \cup Q_j^3 \cup Q_j^4, \quad Q_j^\alpha = R_j \cap Q_\alpha, \quad \alpha = 0, 1, 2, 3, 4$$

   Each intersection is again a rectangle, and this is a disjoint union of five rectangles.

6. Now, $(Q_j^0)_{j=1}^{\infty}$ covers $S_1$, and $(Q_j^\alpha)_{j=1,\alpha=1}^{j=\infty,\alpha=4}$ covers $S_2$.

7. Therefore,

   $$\begin{aligned}\lambda(S_1 \cup S_2) + \epsilon \geq \sum_{j=1}^{\infty} \lambda(R_j) &= \sum_{j=1}^{\infty}\sum_{\alpha=0}^{4} \lambda(Q_j^\alpha) \\ &= \sum_{j=1}^{\infty} \lambda(Q_j^0) + \sum_{j=1}^{\infty}\sum_{\alpha=1}^{4} \lambda(Q_j^\alpha) \\ &\geq \lambda(S_1) + \lambda(S_2).\end{aligned}$$

8. Since the inequality holds for every $\epsilon > 0$, $\lambda(S_1 \cup S_2) \geq \lambda(S_1) + \lambda(S_2)$.

$\square$

**Theorem 5.** $(\mathcal{R}, |\cdot|)$ *is consistent.*

*Proof.* This is by Proposition 4. $\square$

Seen from the proof of Proposition 4, it is not hard to prove that for two rectangles $R$ and $R'$, the union $A = R \cup R'$, which is not a rectangle in general, is also $\lambda$-separating.

**Proposition 6.** *For any $R, R' \in \mathcal{R}$, $R \cup R'$ is $\lambda$-separating.*

*Proof.* From the proof of Proposition 4, the only modifications we need to make are the followings.

1. $R \cup R' = (R \cap R'^c) \cup (R \cap R') \cup (R' \cap R^c) = \bigcup_{\alpha=1}^{m} Q_\alpha$ of disjoint union of finite numbers of rectangles.

2. Similarly, $(R \cup R')^c = \bigcup_{\alpha=m+1}^{m+m'} Q_\alpha$ of disjoint union of finite numbers of rectangles.

3. If $(R_j)$ covers $S_1 \cup S_2$, then

$$R_j = \bigcup_{\alpha=1}^{m+m'} Q_j^\alpha \text{of disjoint union of rectangles, where} \quad Q_j^\alpha = Q_\alpha \cap R_j.$$

4. $(Q_j^\alpha)_{j=1,\alpha=1}^{j=\infty,\alpha=m}$ covers $S_1$, and $(Q_j^\alpha)_{j=1,\alpha=m+1}^{j=\infty,\alpha=m+m'}$ covers $S_2$

$\square$

1. We have established that each member of $\mathcal{R}$ is $\lambda$-separating.
2. Instead of giving a proof that a certain set of interest is $\lambda$-separating individually, we use the induction below. This way of development of the theory is due to Caratheodory.

**Theorem 7** (Caratheodory)**.** *Suppose $E_1, E_2, E_3, \cdots$ are $\lambda$-separating. Then,*

$$(i) \quad \textit{For any } N, \bigcup_{j=1}^{N} E_j \textit{ is } \lambda\textit{-separating.}$$

$$(ii) \quad \bigcup_{j=1}^{\infty} E_j \textit{ is } \lambda\textit{-separating.}$$

*Proof.* 1. We let $(\hat{E}_j)$ be the pairwise disjoint sequence obtained from $(E_j)$ by Proposition before.

2. We prove the stronger assertion over $(i)$:

   $(i)'$ For any $N$, the following is true.

$$S_1 \subset \bigcup_{j=1}^{N} E_j, \quad S_2 \subset \Big(\bigcup_{j=1}^{N} E_j\Big)^c \quad \Longrightarrow \quad \lambda(S_1 \cup S_2) = \sum_{j=1}^{N} \lambda(S_1 \cap \hat{E}_j) + \lambda(S_2).$$

3. Indeed,

$$\sum_{j=1}^{N} \lambda(S_1 \cap \hat{E}_j) \geq \lambda\Big(S_1 \cap \bigcup_{j=1}^{N} \hat{E}_j\Big) = \lambda\Big(S_1 \cap \bigcup_{j=1}^{N} E_j\Big) = \lambda(S_1),$$

   which implies the assertion $(i)$ in the statement.

4. The stronger assertion $(i)'$ holds for $N = 1$, because $\bigcup_{j=1}^{1} E_j = E_1 = \hat{E}_1$, which is $\lambda$-separating. Suppose that the assertion $(i)'$ is true for $1, 2, \cdots, N-1$. Now, let $S_1 \subset \bigcup_{j=1}^{N} E_j$ and $S_2 \subset \Big(\bigcup_{j=1}^{N} E_j\Big)^c$. Then,

$$\lambda(S_1 \cup S_2) = \lambda\Big(\Big(S_1 \cap \bigcup_{j=1}^{N} E_j\Big) \cup S_2\Big) = \lambda\Big(\Big(S_1 \cap \bigcup_{j=1}^{N} \hat{E}_j\Big) \cup S_2\Big)$$
$$= \lambda\Big(\Big(S_1 \cap \bigcup_{j=1}^{N-1} \hat{E}_j\Big) \cup (S_1 \cap \hat{E}_N) \cup S_2\Big)$$

   Because the set $\Big(S_1 \cap \bigcup_{j=1}^{N-1} \hat{E}_j\Big) \subset \bigcup_{j=1}^{N-1} E_j$ and the set $\Big((S_1 \cap \hat{E}_N) \cup S_2\Big) \subset \Big(\bigcup_{j=1}^{N-1} E_j\Big)^c$,

$$= \lambda\Big(S_1 \cap \bigcup_{j=1}^{N-1} \hat{E}_j\Big) + \lambda\Big((S_1 \cap \hat{E}_N) \cup S_2\Big)$$

Because the set $S_1 \cap \hat{E}_N \subset E_N$ and the set $S_2 \subset E_N^c$

$$
\begin{aligned}
&= \lambda\Big(S_1 \cap \bigcup_{j=1}^{N-1} \hat{E}_j\Big) + \lambda\big(S_1 \cap \hat{E}_N\big) + \lambda(S_2) \\
\text{(by } (i)' \text{ on } N-1) \quad &= \sum_{j=1}^{N-1} \lambda(S_1 \cap \hat{E}_j) + \lambda\big(S_1 \cap \hat{E}_N\big) + \lambda(S_2) \\
&= \sum_{j=1}^{N} \lambda(S_1 \cap \hat{E}_j) + \lambda(S_2).
\end{aligned}
$$

5. Now, we prove the second assertion stronger in the similar sense.

   Let $S_1 \subset \bigcup_{j=1}^{\infty} \hat{E}_j$ and $S_2 \subset \Big(\bigcup_{j=1}^{\infty} \hat{E}_j\Big)^c$.

$$
\begin{aligned}
\lambda(S_1 \cup S_2) &= \lambda\bigg(\Big(S_1 \cap \bigcup_{j=1}^{\infty} E_j\Big) \cup S_2\bigg) \\
&\geq \lambda\bigg(\Big(S_1 \cap \bigcup_{j=1}^{N} E_j\Big) \cup S_2\bigg) \quad \text{(here, we took } S_1' = S_1 \cap \bigcup_{j=1}^{N} E_j, \quad S_2' = S_2) \\
&= \sum_{j=1}^{N} \lambda(S_1 \cap \hat{E}_j) + \lambda(S_2)
\end{aligned}
$$

   for any $N$. Taking the limit $N \to \infty$,

$$
\begin{aligned}
\lambda(S_1 \cup S_2) &\geq \sum_{j=1}^{\infty} \lambda(S_1 \cap \hat{E}_j) + \lambda(S_2) \\
&\geq \lambda\bigg(S_1 \cap \bigcup_{j=1}^{\infty} \hat{E}_j\bigg) + \lambda(S_2) = \lambda(S_1) + \lambda(S_2) \geq \lambda(S_1 \cup S_2).
\end{aligned}
$$

   Hence, every quantity appeared equals to each other.

□

*Remark* 4.4. Thanks to the Caratheodory Theorem, out of $\mathcal{R}$, we grow the collection of sets by adding sets assembled by countable union and complement. Consistency is kept by the transitive $\lambda$-separating property.

Once we have enlarged collection, say $\mathcal{G}$, then we grow it again by using the countable union and complement. We repeat this over and over again. This procedure will be detailed in the next chapter.

# Chapter 5

# The mathematics of "one after another" and consistent family

1. Let the collection $\mathcal{G}_0 = \mathcal{R}$ of rectangles. $(\mathcal{R}, \lambda)$ is consistent.
2. One should note, to enlarge a family while keeping consistency is not at all trivial.

   Example: Assume that we knew that $(\mathcal{R}, \lambda)$ and $(\mathcal{T}, \lambda)$ are consistent individually, where $\mathcal{T}$ is a suitable collection of triangles. How many new consistency checkings are needed for the new collection $\mathcal{G} = \mathcal{R} \cup \mathcal{T}$ ?
3. Given that, if we define the new collection denoted by $(\mathcal{G}_0)+$ out of $\mathcal{G}_0$

   $$(\mathcal{G}_0)+ = \left\{ G = \bigcup_{j=1}^{\infty} P_j \mid \text{for every } j \quad P_j \in \mathcal{G}_0 \quad \text{or} \quad P_j^c \in \mathcal{G}_0 \right\} =: \mathcal{G}_1,$$

   then every member of $\mathcal{G}_1$ is $\lambda$-separating.
4. $(\mathcal{G}_1, \lambda)$ is consistent, i.e.,

   $$\mathcal{G}_1 \ni G = \bigcup_{j=1}^{\infty} G_j \;\; \text{disjoint union of sets in } \mathcal{G}_1 \qquad \Longrightarrow \qquad \lambda(G) = \sum_{j=1}^{\infty} \lambda(G_j).$$
5. In the similar fashion, $(\mathcal{G}_2, \lambda)$, $(\mathcal{G}_3, \lambda)$, $\cdots$ will be consistent. More precisely, for any $N$, $(\mathcal{G}_N, \lambda)$ is consistent. (This will be proven by induction.)
6. The limit statement: $\Big(\mathcal{G}_\infty, \lambda\Big)$ with $\mathcal{G}_\infty = \cup_{N=1}^{\infty} \mathcal{G}_N$ is consistent.

   This is because, if

   $$\mathcal{G}_\infty \ni G = \bigcup_{j=1}^{\infty} G_j \;\; \text{disjoint union of sets in } \mathcal{G}_\infty,$$

   then for every $j$, $G_j \in \mathcal{G}_{N(j)}$ for some $N(j)$. In other words, $G_j$ has been included at $\mathcal{G}_{N(j)}$ as a $\lambda$-separating set. Thus, $\lambda(G) = \displaystyle\sum_{j=1}^{\infty} \lambda(G_j)$.
7. Since we can, we enlarge $\mathcal{G}_\infty$ again to obtain $(\mathcal{G}_{\infty,1}, \lambda)$ consistent.
8. We do this over and over again.

We can pose a few questions on families

$$\mathcal{G}_0 \subset \mathcal{G}_1 \subset \mathcal{G}_2 \subset \cdots$$

Certainly, growing cannot go beyond the power collection $\mathcal{P}(\mathbb{R}^2)$. Considering this, we examine a few possibilities.

**Possibility (0-0).** The collection neither stop growing nor reaching $\mathcal{P}(\mathbb{R}^2)$.

**Possibility (0-1).** The collection keeps strictly growing to becomes $\mathcal{P}(\mathbb{R}^2)$.

**Possibility (1).** The collection from the initial family $\mathcal{G}_0$ might stop growing if no new sets are added by the expansion $(\ \cdot\ )+$, i.e., at the moment

$$\mathcal{G} = (\mathcal{G})+ = \Big\{ H = \bigcup_{j=1}^{\infty} P_j \mid \text{for every } j \quad P_j \in \mathcal{G} \ \text{ or } \ P_j^c \in \mathcal{G} \Big\}.$$

We have a definite answer to that question. To do this, we need the family

$$(\mathcal{G}_\alpha)_{\alpha \in A}$$

where $A$ is a set other than $\mathbb{N}$.

## Indexing

1. In most of our experience, we use index $j \in \mathbb{N}$ to denote a member of sequence $a_1$, $a_2$, $\cdots$.
2. This notion of "indexing by $\mathbb{N}$" has been certainly useful. This usefulness is abstracted mathematically and used elsewhere. We have a few examples.

   **Example**: Let $\mathbb{N}^+ = \mathbb{N} \cup \{\infty\}$.

   (a) If $(a_j)$ is a convergent sequence $a_j \to a_*$ as $j \to \infty$. We may use indexing by $\mathbb{N}^+$ including the limit.

   (b) We have seen many examples where a statement is parametrized by (statement)$_N$. We gave a proof in the style that we prove $(i)$ (statement)$_N$ for any $N$, and $(ii)$ (statement)$_\infty$. This is to give a proof for statement indexed by $\mathbb{N}^+$.

   **Example**: Consider $\mathbb{N}^+ \times \mathbb{N}^+$.

**Definition 1** (order, linear order, well order on $X$). *Let $X$ be a nonempty set.*

1. *A subset $P \subset X \times X$ is called a partial order on $X$ if*

   *(a) If $(a, b), (b, c) \in P$ then $(a, c) \in P$.*

   *(b) If $(a, b), (b, a) \in P$ then $a = b$.*

   *(c) For every $a \in X$, $(a, a) \in P$.*

2. *A partial order $P$ on $X$ is called a linear order on $X$ if in addition*

   *(d) For every pair $a, b \in X$, either $(a, b) \in P$ or $(b, a) \in P$.*

3. *A linear order on $X$ is called a well order on $X$ if in addition*

   *(e) For every nonempty subset $A \subset X$, the least element $a \in A$.*

*Remark* 5.1. .

1. $\leq$ is a well-order on $\mathbb{N}$.
2. $\leq$ on $\mathbb{R}$ is a linear order but is not a well-order. This is because the condition (e) is not true in general, for example $A = (0, 1)$.
3. We will also use the symbol $<$

**Definition 2.** *For a nonempty set $X$ with well order, denoted by $\leq$, we define*

$$a < b \quad \Longleftrightarrow \quad a \leq b \quad \text{and} \quad a \neq b.$$

According to the set theory, the following statement is true.

**Theorem 3.** *There exists an uncountable set with well-order.*

In our course, we do not intend to proceed with a set theory, giving a proof of this. We only consider a family $(\mathcal{G}_\alpha)$ indexed by such a set.

We fix $X$ that is uncountable and with well-order, denoted by $\leq$.

**Proposition 4.** *There exists a subset $A \subset X$ such that*

*$(i)$ for any $\alpha \in A$, $I_\alpha = \{\beta \in X \mid \beta < \alpha\}$ is countable $(ii)$ $A$ is uncountable.*

*Proof.* Define $S = \{\alpha \in X \mid I_\alpha \text{ is uncountable}\}$. In case $S$ is empty, we define $A = X$. If not, there exists the least element $\omega_1 \in S$ and define $A = I_{\omega_1}$. □

*Remark* 5.2. We omit the discussion but well-ordered sets with properties in Proposition 4 are order isomorphic to each other. For the role of index, use of any such a set leads to the equivalent result in our class.

**Definition 5.** *(1) Define $\mathcal{G}_0 = \mathcal{R}$, where 0 refers to the least element of $A$.*

*(2) For a given $\alpha \in A$, if $\mathcal{G}_\beta$ is defined for every $\beta \in A$ with $\beta < \alpha$, define*

$$\mathcal{G}_\alpha := \bigcup_{\beta < \alpha} (\mathcal{G}_\beta) + .$$

**Proposition 6.** *$\mathcal{G}_\alpha$ is defined for every $\alpha \in A$, thus defining expanding families $(\mathcal{G}_\alpha)_{\alpha \in A}$.*

*Proof.* This is the induction we use:
Let $S = \{\alpha \in A \mid \mathcal{G}_\alpha \text{ is not defined.}\}$. If $S$ is nonempty, then there exists the least element $\omega \in S$. Then $\mathcal{G}_\beta$ with $\beta < \omega$ must have been defined. In turn, $\mathcal{G}_\omega$ has a definition by Definition 5, contradiction. Therefore $S$ is the empty set. □

**Definition 7.** *Define the collection*

$$\mathcal{B} = \bigcup_{\alpha \in A} \mathcal{G}_\alpha.$$

**Theorem 8.**

$$(\mathcal{B}) + = \mathcal{B}.$$

*Proof.*

1. We prove that

   $(i)$ $\mathcal{B}$ is closed under complement operation.

   $(ii)$ $\mathcal{B}$ is closed under countable union operation.

2. Suppose $E \in \mathcal{B}$. By definition, $E \in \mathcal{G}_{\alpha_0}$ for some $\alpha_0 \in A$.
3. Let $S = \{\beta \in A \mid \alpha_0 < \beta\}$. $S$ cannot be the empty set: If $S$ is empty, then for any $\alpha \in A$, $\alpha_0 = \alpha$ or $\alpha < \alpha_0$. In other words, $I_{\alpha_0} \cup \{\alpha_0\} \supset A$. This contradicts to that $(LHS)$ is countable while $(RHS)$ is uncountable.
4. There exists $\beta \in A$ such that $\alpha_0 < \beta$, and $E^c$ must have been included in $\mathcal{G}_\beta$.
5. Now, $E_1, E_2, E_3, \cdots \in \mathcal{B}$ with $E_j \in \mathcal{G}_{\alpha_j}$ for some $\alpha_j \in A$.
6. Let $S' = \{\beta \in A \mid \alpha_j < \beta \quad \text{for every } j\}$. $S'$ cannot be the empty set: If $S'$ is empty, then for any $\alpha \in A$, there exists some $j$ such that $\alpha_j = \alpha$ or $\alpha < \alpha_j$. In other words, $\bigcup_{j=1}^{\infty} I_{\alpha_j} \cup \{\alpha_j\} \supset A$, which is contradiction.
7. There exists $\beta \in A$ such that $\alpha_j < \beta$ for every $j$. Then $\displaystyle\bigcup_{j=1}^{\infty} E_j$ must have been included in $\mathcal{G}_\beta$.

□

**Definition 9.** *A nonempty collection $\mathcal{E} \subset \mathcal{P}(\mathbb{R}^2)$ containing $\emptyset$ is called a $\sigma$-algebra if*

$$(i) \text{ If } E \in \mathcal{E} \text{ then } E^c \in \mathcal{E}.$$

$$(ii) \text{ If } E_1, E_2, \cdots \in \mathcal{E} \text{ then } \bigcup_{j=1}^{\infty} E_j \in \mathcal{E}.$$

We will come to the definition of $\sigma$-algebra again in the next class.

*Remark* 5.3. .

1. We are done with defining the area, the 2 dimensional Lebesgue measure, on every set $E \in \mathcal{B}$.
2. The collection $\mathcal{B} = \mathcal{B}(\mathbb{R}^2)$ is called the $\sigma$-algebra of all *borel sets*.

*Remark* 5.4. .

1. The expanding families $(\mathcal{G}_\alpha)$ certainly depends on the initial family $\mathcal{G}_0$, which was $\mathcal{R}$ in our case.
2. More precisely, for any given $\mathcal{G}_0$ containing $\emptyset$, $\bigcup_{\alpha \in A} \mathcal{G}_\alpha$ is the smallest $\sigma$-algebra containing $\mathcal{G}_0$.
3. Regardless of the initial family, we can certainly define

   $$\mathcal{E}^\lambda(\mathbb{R}^2) := \{E \subset \mathbb{R}^2 \mid E \text{ is } \lambda\text{-separating.}\}$$

   We will call $\mathcal{E}^\lambda(\mathbb{R}^2)$ the $\sigma$-algebra of all $\lambda$-*measurable sets* or the $\sigma$-algebra of all *Lebesgue measurable sets.*

   (Instead of calling it the collection of all $\lambda$-separating sets.)

*Remark* 5.5. .

1. We have not yet answered to the question if $\mathcal{B}(\mathbb{R}^2) = \mathcal{P}(\mathbb{R}^2)$ or not.

   We will verify

   $$\mathcal{B}(\mathbb{R}^2) \subsetneq \mathcal{E}^\lambda(\mathbb{R}^2) \subsetneq \mathcal{P}(\mathbb{R}^2).$$

2. Before that, we have one important result to know.

   We show that every open set $U \subset \mathbb{R}^2$ is in $\mathcal{B}(\mathbb{R}^2)$. More precisely, $U \in \mathcal{G}_1$.

**Theorem 10.** *Any open set $U \subset \mathbb{R}^2$ is a countable disjoint union of rectangles in $\mathcal{R}$.*

*Proof.* 1. For $m = 0, 1, 2, \cdots$, we consider the depth $m$ grid lines of $\mathbb{R}^2$: At each $m$, the grid lines are drawn by the grid points and the grid points are those points whose $x$-coordinate and $y$-coordinate are in the form

$$\text{integer} + \sum_{j=1}^{m} \frac{b_j}{2^j}, \quad b_j \in \{0, 1\}.$$

$\mathbb{R}^2$ is a countable union of those pairwise disjoint depth $m$ rectangles partitioned by grid lines. The collection of depth $m$ rectangles is denoted by $\mathcal{R}_m$.

2. Now, we inductively define $\mathcal{Q}_{m,0}$ and $\mathcal{Q}_{m,1}$ of depth $m$ rectangles so that

$$\Big(\bigcup_{j=0}^{m} \mathcal{Q}_{j,0}\Big) \cup \mathcal{Q}_{m,1} \quad \text{covers } U. \tag{C}$$

At $m = 0$, define

$$\mathcal{Q}_{0,0} = \{Q \in \mathcal{R}_0 \mid Q \subset U\}, \quad \mathcal{Q}_{0,1} = \{Q \in \mathcal{R}_0 \mid Q \cap U \neq \emptyset \text{ and } Q \not\subset U\}.$$

Certainly, $\mathcal{Q}_{0,0} \cup \mathcal{Q}_{0,1}$ covers $U$.

3. Now, suppose $\big(\mathcal{Q}_{j,0}, \mathcal{Q}_{j,1}\big)$ are defined up to $j = 0, 1, \cdots, m-1$, satisfying (C). Depth $m-1$ rectangles in $\mathcal{Q}_{m-1,1}$ are pairwise disjoint and each of them is a disjoint union of four depth $m$ rectangles. We define $\mathcal{R}'_m$ be the collection of pairwise disjoint depth $m$ rectangles obtained from $\mathcal{Q}_{m-1,1}$. Now,

$$\mathcal{Q}_{m,0} = \{Q \in \mathcal{R}'_m \mid Q \subset U\}, \quad \mathcal{Q}_{m,1} = \{Q \in \mathcal{R}'_m \mid Q \cap U \neq \emptyset \text{ and } Q \not\subset U\}.$$

Certainly, $U \cap \bigcup_{Q' \in \mathcal{Q}_{m-1,1}} Q'$ is covered by $\mathcal{Q}_{m,0} \cup \mathcal{Q}_{m,1}$. Hence, $\Big(\bigcup_{j=0}^{m} \mathcal{Q}_{j,0}\Big) \cup \mathcal{Q}_{m,1}$ covers $U$.

4. Let $\mathcal{Q} := \bigcup_{m=0}^{\infty} \mathcal{Q}_{m,0}$ and define the set $G$ as the union over the collection $\mathcal{Q}$.

   By definition, $G \subset U$.

5. We show that $G \supset U$.

   If $x \in U$, then there exists an open square of side length $\ell > 0$ containing $x$ that is a subset of $U$. Inside of this open square, there exists a half open square $\hat{Q}$ containing $x$ with smaller side length that are aligned along with the grid lines of some depth $\hat{m}$.

6. That $\hat{Q} \subset U$ implies

$$(i) \ \Big(\bigcup_{j=0}^{\hat{m}} \mathcal{Q}_{j,0}\Big) \cup \mathcal{Q}_{\hat{m},1} \quad \text{covers } \hat{Q}$$

$(ii)$ $\hat{Q}$ is disjoint from every rectangles in $\mathcal{Q}_{\hat{m},1}$.

Hence, $\Big(\bigcup_{j=0}^{\hat{m}} \mathcal{Q}_{j,0}\Big)$ covers $\hat{Q}$, or $G \supset \hat{Q}$. Thus, $G \ni x$.

$\square$

# Chapter 6

# Abstraction of the Lebesgue measure

## $\sigma$-algebra

Let $X$ be a nonempty set.

If $(P)$ is any property on subsets of $X$ such that

$(i)$ $\emptyset$ has the property.
$(ii)$ The property is transitive for taking complement and countable union.

then, certainly $\exists \mathcal{Q}$ a seed family containing $\emptyset$ with members having $(P)$.

You always end up with two $\sigma$-algebras:

1. By considering $\mathcal{G}_0 = \mathcal{Q}$, $\mathcal{G}_1 = (\mathcal{G}_0)+$, $\mathcal{G}_2 = (\mathcal{G}_1)+$, $\cdots$, to define
$$\underline{\mathcal{E}}(\mathcal{Q}) = \bigcup_{\alpha \in A} \mathcal{G}_\alpha.$$
2. $\mathcal{E}^P = \{E \subset X \mid E \text{ has the property } (P).\}$

The former is called the smallest $\sigma$-algebra containing $\mathcal{Q}$. The latter is the $\sigma$-algebra of sets having $(P)$.

We recall the definition of $\sigma$-algebra of subsets of $X$.

**Definition 1.** *Let $X$ be a nonempty set. A collection $\mathcal{E} \subset \mathcal{P}(X)$ containing $\emptyset$ is called a $\sigma$-algebra of subsets of $X$ if*

$$(i) \text{ If } E \in \mathcal{E} \text{ then } E^c \in \mathcal{E}.$$

$$(ii) \text{ If } E_1, E_2, \cdots \in \mathcal{E} \text{ then } \bigcup_{j=1}^{\infty} E_j \in \mathcal{E}.$$

**Proposition 2.** *Let $X$ be a nonempty set, and $\mathcal{E}$ be a $\sigma$-algebra of subsets of $X$. Then it holds that*

$$(iii) \quad \text{If } E_1, E_2, \cdots \in \mathcal{E} \text{ then } \bigcap_{j=1}^{\infty} E_j \in \mathcal{E}.$$

*Proof.* This is because $\displaystyle\bigcup_{j=1}^{\infty} E_j^c \in \mathcal{E}$ and $\displaystyle\Big(\bigcup_{j=1}^{\infty} E_j^c\Big)^c = \bigcap_{j=1}^{\infty} E_j$. □

The term "smallest" is from the following observations.

1. If $\mathcal{Q}$ is any seed collection containing $\emptyset$, the set

$$\Sigma := \{\mathcal{E} \subset \mathcal{P}(X) \mid \mathcal{E} \text{ is a } \sigma\text{-algebra and } \mathcal{E} \supset \mathcal{Q}\}$$

   is nonempty because $\mathcal{P}(X) \in \Sigma$.

2. Let $\underline{\mathcal{E}}$ be the intersection of all the members of $\Sigma$, i.e.,

$$\underline{\mathcal{E}} = \{E \subset X \mid E \text{ is member of } \mathcal{E} \text{ for every } \mathcal{E} \in \Sigma\}.$$

   It easily follows that $\underline{\mathcal{E}}$ is again a $\sigma$-algebra since

   (a) $\emptyset \in \mathcal{E}$ for every $\mathcal{E} \in \Sigma$.

   (b) If $\underline{E}_1$, $\underline{E}_2$, $\cdots$ are members of $\mathcal{E}$ for every $\mathcal{E} \in \Sigma$, then so is $\displaystyle\bigcup_{j=1}^{\infty} \underline{E}_j$.

3. Lastly, we show $\displaystyle\bigcup_{\alpha \in A} \mathcal{G}_\alpha \subset \underline{\mathcal{E}}$ with $\mathcal{G}_0 = \mathcal{Q}$ below.

**Proposition 3.** *With $\mathcal{G}_0 = \mathcal{Q}$, $\displaystyle\bigcup_{\alpha \in A} \mathcal{G}_\alpha \subset \underline{\mathcal{E}}$*

*Proof.* This is because

$$(i) \text{ Certainly, } \mathcal{Q} = \mathcal{G}_0 \text{ is contained in } \underline{\mathcal{E}}.$$

$$(ii) \text{ If } \mathcal{G}_\beta \subset \underline{\mathcal{E}} \text{ for every } \beta < \alpha, \text{ then so is } \mathcal{G}_\alpha = \bigcup_{\beta < \alpha} (\mathcal{G}_\beta)+.$$

If we take $S = \{\alpha \in A \mid \mathcal{G}_\alpha \not\subset \underline{\mathcal{E}}\}$, then $S$ is empty set, otherwise, there exists the least element $\omega \in S$, but this contradicts to $(ii)$ above. □

The one of the role of the smallest $\sigma$-algebra, (or of a few first families in $(\mathcal{G}_\alpha)$) is played for the pair $(\mathcal{B}, \lambda)$ in the following manner.

**Theorem 4.** *For any set $S \subset \mathbb{R}^2$, there exists a borel set $E \supset S$ with $\lambda(E) = \lambda(S)$.*

*Proof.*

1. If $\lambda(S) = \infty$, we take $E = \mathbb{R}^2$ and we are done. Now we assume $\lambda(S) < \infty$.

2. For every $\alpha = 1, 2, 3 \cdots$, there exists $(R_j^\alpha)$ of rectangles that cover $S$ with

$$\lambda(S) + \frac{1}{\alpha} \geq \sum_{j=1}^{\infty} \lambda(R_j^\alpha)$$

3. We define

$$E^\alpha := \bigcup_{j=1}^{\infty} R_j^\alpha, \quad E := \bigcap_{\alpha=1}^{\infty} E^\alpha$$

that are borel sets. Every $E^\alpha$ contains $S$ as a subset, and so is the $E$.

4. Now, for every $\alpha$,

$$\lambda(S) + \frac{1}{\alpha} \geq \sum_{j=1}^{\infty} \lambda(R_j^\alpha) \geq \lambda(E^\alpha) \geq \lambda(E) \geq \lambda(S).$$

Taking the limit $\alpha \to \infty$, we obtain $\lambda(S) = \lambda(E)$.

□

## Measure

Let $X$ be a nonempty set.

**Definition 5.** *Let $\mathcal{E}$ be a $\sigma$-algebra of subsets of $X$. A set function $\mu_0 : \mathcal{E} \to [0, \infty]$ is called a measure on $\mathcal{E}$ if*

(*i*) $\mu_0(\emptyset) = 0$,

(*ii*) *If* $E = \bigcup_{j=1}^{\infty} E_j$, *where* $(E_j)$ *is pairwise disjoint sets in* $\mathcal{E}$ *then* $\mu_0(E) = \sum_{j=1}^{\infty} \mu_0(E_j)$.

**Definition 6.** *Let $X$ be a nonempty set and $\mathcal{E}$ be a $\sigma$-algebra of subsets of $X$.*

1. *The pair $(X, \mathcal{E})$ is called a measurable space.*

2. *A member of $\mathcal{E}$ is called a $\mathcal{E}$-measurable set.*

**Definition 7.** *Let $(X, \mathcal{E})$ be a measurable space and $\mu$ be a measure on $\mathcal{E}$. The triple $(X, \mathcal{E}, \mu)$ is called a measure space.*

# Outer measure and regularity

Let $X$ be a nonempty set.

**Definition 8.** *A set function $\mu : \mathcal{P}(X) \to [0,\infty]$ is called an (outer) measure on $X$ if*

$$(i)\ \mu(\emptyset) = 0,$$

$$(ii)\ \text{If } S \subset \bigcup_{j=1}^{\infty} S_j \text{ then } \quad \mu(S) \leq \sum_{j=1}^{\infty} \mu(S_j).$$

**Exercise 9.** *Re-do the parts Definition 1, 2, Theorem 3, Theorem 7 in Chapter 4, not for $\mathbb{R}^2$ but for $X$.*

**Definition 10.** *Let $\mu$ be an outer measure on $X$. The collection*

$$\mathcal{E}^{\mu} := \Big\{ E \subset X \mid E \text{ is } \mu\text{-separating} \Big\}$$

*is called the $\sigma$-algebra of $\mathcal{E}^{\mu}$-measurable sets, or shortly of $\mu$-measurable sets.*

**Definition 11.** *An outer measure $\mu$ on $X$ is a regular outer measure if*

*for every $S \subset X$, there exists a $\mu$-measurable set $E \supset S$ with $\mu(E) = \mu(S)$.*

Let $X = \mathbb{R}^n$.

**Definition 12.** *Let*

$$\mathcal{B}(\mathbb{R}^n) = \underline{\mathcal{E}}(\mathcal{Q}) \quad \text{the smallest } \sigma\text{-algebra containing } \mathcal{Q} \text{ of half open } n\text{-cubes.}$$

*We say $\mathcal{B}$ is the $\sigma$-algebra of borel sets.*

**Definition 13.** *An outer measure $\mu$ on $\mathbb{R}^n$ is called a borel outer measure if every borel set is a $\mu$-measurable set.*

**Definition 14.** *A borel outer measure $\mu$ on $\mathbb{R}^n$ is a borel regular outer measure if*

*for every $S \subset \mathbb{R}^n$, there exists a borel set $E \supset S$ with $\mu(E) = \mu(S)$.*

**Exercise 15.** *Let* $\big(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \mu_0\big)$ *be a measure space, i.e.,* $\mu_0$ *is a borel measure on* $\mathbb{R}^n$*. Then the extension* $\mu$ *on* $\mathcal{P}(\mathbb{R}^n)$ *of* $\mu_0$ *by*

$$\mu(S) = \inf_{(E_j) \textit{ of } \mathcal{B}(\mathbb{R}^n) \textit{ that covers } S} \sum_{j=1}^{\infty} \mu_0(E_j)$$

*is well-defined, and* $\mu$ *is a borel regular outer measure.*

**Exercise 16.** *Let* $\big(X, \mathcal{E}, \mu_0\big)$ *be a measure space. Then the extension* $\mu$ *on* $\mathcal{P}(X)$ *of* $\mu_0$ *by*

$$\mu(S) = \inf_{(E_j) \textit{ of } \mathcal{E} \textit{ that covers } S} \sum_{j=1}^{\infty} \mu_0(E_j)$$

*is well-defined, and* $\mu$ *is a regular outer measure.*

## Examples of measurable spaces and measure spaces

## Consequences of countable additivity

**Proposition 17.** *Let $(X, \mathcal{E}, \mu)$ be a measure space. Let $(E_j)$ be a sequence of $\mathcal{E}$-measurable sets such that $E_1 \subset E_2 \subset E_3 \subset \cdots$. Then*

$$(i) \text{ For any } N,\ \mu\Big(\bigcup_{j=1}^{N} E_j\Big) = \mu(E_N), \quad (ii)\ \mu\Big(\bigcup_{j=1}^{\infty} E_j\Big) = \lim_{N\to\infty} \mu(E_N).$$

*Proof.* 1. In fact, the Proposition is to prove $(ii)$.

2. We use the pairwise disjoint sequence $(\hat{E}_j)$ obtained from $(E_j)$. At this point, we know that $\hat{E}_j$ are all $\mathcal{E}$-measurable sets.

3. Thanks to the countable additivity,

$$\lim_{N\to\infty} \mu(E_N) = \lim_{N\to\infty} \mu\Big(\bigcup_{j=1}^{N} E_j\Big) = \lim_{N\to\infty} \mu\Big(\bigcup_{j=1}^{N} \hat{E}_j\Big) = \sum_{j=1}^{\infty} \mu(\hat{E}_j) = \mu\Big(\bigcup_{j=1}^{\infty} \hat{E}_j\Big) = \mu\Big(\bigcup_{j=1}^{\infty} E_j\Big).$$

□

**Proposition 18.** *Let $(X, \mathcal{E}, \mu)$ be a measure space. Let $(E_j)$ be a sequence of $\mathcal{E}$-measurable sets such that $\mu(E_1) < \infty$ and $E_1 \supset E_2 \supset E_3 \supset \cdots$. Then*

$$(i) \text{ For any } N,\ \mu\Big(\bigcap_{j=1}^{N} E_j\Big) = \mu(E_N), \quad (ii)\ \mu\Big(\bigcap_{j=1}^{\infty} E_j\Big) = \lim_{N\to\infty} \mu(E_N).$$

*Proof.* 1. In fact, the proposition is to prove $(ii)$.

2. Let $F_j = E_1 \setminus E_j$ so that $F_1 \subset F_2 \subset F_3 \subset \cdots$.
   All of them are $\mathcal{E}$-measurable subsets of $E_1$ with finite measure, and we have

$$\mu(F_N) + \mu(E_N) = \mu(E_1) \quad \Longleftrightarrow \quad \mu(E_N) = \mu(E_1) - \mu(F_N).$$

3. $(RHS)$ has the limit,

$$\mu(E_1) - \lim_{N\to\infty} \mu(F_N) = \mu(E_1) - \mu\Big(\bigcup_{j=1}^{\infty} F_j\Big).$$

4. On the other hand, $\bigcup_{j=1}^{\infty} F_j = E_1 \setminus \bigcap_{j=1}^{\infty} E_j$. This implies that

$$\mu(E_1) - \mu\Big(\bigcup_{j=1}^{\infty} F_j\Big) = \mu\Big(\bigcap_{j=1}^{\infty} E_j\Big).$$

□

On $(X, \mathcal{E}, \mu)$, we now define the Integral.