Chapter 1

What will we do ?

We are interested in the Euclidean space $\mathbb{R}^n$.

1. In the first major part of the course, we discuss about the n-dimensional volume of a subset $E \subset \mathbb{R}^n$. The first objective is to construct the Lebesgue measure $\mathcal{L}^n$.

2. In the second major part of the course, we update our tool of Integral, namely from the Riemann Integral to the Lebesgue Integral. This is based on the measure theory developed by abstraction of the Lebesgue measure in the first part.

Chapter 2

Measuring n-dimensional volume of $E \subset \mathbb{R}^n$

1. In the Euclidean space $\mathbb{R}^n$, we are able to measure the distance between two points $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$ of $\mathbb{R}^n$,

$$d(x, y) = \Big((x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots (x_n - y_n)^2 \Big)^{\frac{1}{2}}.$$

2. This gives rise to the n-dimensional volume formula for a few classes of subsets in $\mathbb{R}^n$. For example in $\mathbb{R}^3$, we take the formula:

   (a) If E is the cube $[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$, we take the value

   $$(b_1 - a_1)(b_2 - a_2)(b_3 - a_3)$$

   as its 3-dimensional volume.

   (b) If we consider a tetrahedron with base area A and the height h, we take the value

   $$\frac{1}{3} Ah$$

   as its 3-dimensional volume.

   (c) other examples...

Knowing the n-dimensional volumes of such a class of elementary sets,

1. We may extend our knowledge base on calculating n-volume: For a set made by assembling a few such elementary sets, the n-volume would be the sum of n-volumes of elementary sets.

2. That we wrote right above is the theory we want to develop. It is a difficult task: To make this consistent mathematically, any such theory should provide a proof that the n-volume assigned on a certain set $E \subset \mathbb{R}^n$ would be calculated independently of ways of cutting the set.

3. For example, for a given set $E \subset \mathbb{R}^n$, there are two persons. The first person cuts E into G_1, G_2, G_3, and the second person cuts E into H_1 and H_2. More specifically, G_1, G_2, G_3 are pairwise disjoint and $E = G_1 \cup G_2 \cup G_3$, and H_1, H_2 are pairwise disjoint and $E = H_1 \cup H_2$. n-volumes of G_i, and H_j are known. Theory should be certain about the equality
$$|G_1| + |G_2| + |G_3| = |H_1| + |H_2|$$
where $|S|$ denotes n-volume of the set S, if known. This consistency should be the case for all different ways of cutting the set E.

4. We consider the following humble goal: Let $\mathcal{R}$ be the collection of all cubes in $\mathbb{R}^n$ (whose n-volumes are as we know). Let R be any cube in $\mathcal{R}$ with the n-volume $|R|$. Let $(R_j)_{j=1}^{\infty}$ be pairwise disjoint partition of R, and $(Q_k)_{k=1}^{\infty}$ be another pairwise disjoint partition of R. The goal is that the n-volume of a cube was actually *correct*:
$$|R| = \sum_{j=1}^{\infty} |R_j| = \sum_{k=1}^{\infty} |Q_k|.$$
This is the task of our courses for a while, and this is a difficult problem.

Consistent family with n-volume

The word 'family', or 'collection' are synonyms of set. We use family or collection to avoid confusion.

Definition 1. *A pair $(\mathcal{G}, \lambda)$ of $\mathcal{G}$, a nonempty collection of subsets of $\mathbb{R}^n$ containing $\emptyset$, X, and $\lambda : \mathcal{G} \to [0, \infty]$, is said to be consistent if*

1. $\lambda(\emptyset) = 0$,
2. *If G is a set in $\mathcal{G}$, $v = \lambda(G)$ and $(G_j)_{j=1}^{\infty}$ is any sequence of sets in $\mathcal{G}$ that are pairwise disjoint and $G = \bigcup_{j=1}^{\infty} G_j$, then*

$$\sum_{j=1}^{\infty} \lambda(G_j) = v.$$

Let $n = 2$ and consider $\mathbb{R}^2$.

By half-open intervals we mean the intervals of one of the following forms

$$\emptyset, \quad [a, b), \quad [a, \infty), \quad (-\infty, b), \quad (-\infty, \infty),$$

where $a, b \in \mathbb{R}$ and are assumed to be $a < b$.

Definition 2. *The collection $\mathcal{R}$ is the collection of all cartesian products of two half-open intervals. The member of $\mathbb{R}$ is called a rectangle.*

1. *If R is an unbounded rectangle, we define $|R| = \infty$.*
2. *If R is a nonempty bounded rectangle $[a_1, b_1) \times [a_2, b_2)$, $|R| = (b_1 - a_1)(b_2 - a_2)$.*
3. $|\emptyset| = 0$.

We prove that $(\mathcal{R}, |\cdot|)$ is consistent from now on.

Towards a consistent family

At this moment, we prove a proposition stating that, out of somewhat arbitrary volume function ρ and a collection, one may extract its refined version of volume function λ.

Proposition 3. *Let X be a nonempty set, $\mathcal{G} \subset \mathcal{P}(X)$, and $\rho : \mathcal{G} \to [0, \infty]$ be such that $\emptyset \in \mathcal{G}$, $X \in \mathcal{G}$, and $\rho(\emptyset) = 0$. For any set $S \subset X$, define*

$$\lambda(S) := \inf_{(G_j) \textit{ of } \mathcal{G} \textit{ that covers } S} \Big\{ \sum_{j=1}^{\infty} \rho(G_j) \Big\}.$$

Then, λ, defined on $\mathcal{P}(X)$, satisfies the followings:

1. *$\lambda(\emptyset) = 0$.*
2. *If (S_j) of $\mathcal{P}(X)$ covers S, i.e., $\bigcup_j S_j \supset S$, then*

$$\lambda(S) \leq \sum_{j=1}^{\infty} \lambda(S_j). \tag{2.0.1}$$

Remark 2.1. We will define on $\mathcal{P}(\mathbb{R}^2)$

$$\lambda(S) := \inf_{(R_j) \text{ of rectangles that covers } S} \Big\{ \sum_{j=1}^{\infty} |R_j| \Big\}.$$

Examples:

proof of Proposition 3. Let S be any subset of X.

1. The set of coverings of S by sets in $\mathcal{G}$ is nonempty because $X \in G$ and $(X, \emptyset, \emptyset, \cdots)$ covers S. Obviously, the set

$$\left\{ \sum_{j=1}^{\infty} \rho(G_j) : (G_j) \text{ of } \mathcal{G} \text{ covers } S \right\} \subset \mathbb{R}$$

is nonempty and bounded below by 0. Therefore, $\lambda(S)$, the infimum over the set, is well-defined.

2. Since $(\emptyset, \emptyset, \cdots)$ covers $\emptyset$, $\rho(\emptyset) = 0$, and $\sum_j 0 = 0$, $\lambda(\emptyset)$ must be 0.

3. Now, let (S_j) be any sequence of subsets of X that covers S. Suppose any of $\lambda(S_j) = \infty$. Then the inequality (2.0.1) is trivially true. Now, we assume $\lambda(S_j) < \infty$ for every j.

4. Let $\epsilon > 0$. By the definition of infimum, for each j, there exists a covering $(G^j_\alpha)_{\alpha=1}^{\infty}$ of S_j by sets in $\mathcal{G}$ such that

$$\sum_{\alpha=1}^{\infty} \rho(G^j_\alpha) \leq \lambda(S_j) + \frac{\epsilon}{2^j}.$$

Obviously, $\bigcup_\alpha \bigcup_j G^j_\alpha \supset S$ and thus $(G^j_\alpha)_{j,\alpha=1}^{\infty}$ is a countable covering of S by sets in $\mathcal{G}$. Thus,

$$\lambda(S) \leq \sum_{j=1}^{\infty} \sum_{\alpha=1}^{\infty} \rho(G^j_\alpha) \leq \sum_{j=1}^{\infty} \left[\lambda(S_j) + \frac{\epsilon}{2^j} \right] = \sum_{j=1}^{\infty} \lambda(S_j) + \epsilon.$$

Since this inequality holds for every $\epsilon > 0$, we conclude that

$$\lambda(S) \leq \sum_{j=1}^{\infty} \lambda(S_j).$$

$\square$

1. Out of mere formula $\big|[a_1, b_1) \times [a_2, b_2)\big| = (b_1 - a_1)(b_2 - a_2)$, we suddenly have a definition of λ for all the subsets of $\mathbb{R}^2$.
2. However, we restrict ourselves the use of λ only on rectangles for a while to complete the proof of that $(\mathcal{R}, |\cdot|)$ is consistent.
3. Now, we aim to prove that for a rectangle R, $|R| = \lambda(R)$, namely the area formula $|\cdot|$ was already good to some extent.
4. Since, $\lambda(R) \leq |R|$, we only need to prove $\lambda(R) \geq |R|$.

We prove a few lemmas.

Lemma 4. *If R and R' are rectangels and $R \subset R'$, then $|R| \leq |R'|$.*

Proof. Omitted. □

Lemma 5. *Let R be a nonempty bounded rectangle $[a, b) \times [c, d)$, and consider*

$$a = t_1 < t_2 < \cdots < t_N = b, \quad c = s_1 < s_2 < \cdots < s_K = d,$$

and consider rectangles $R_{i,j} = [t_i, t_{i+1}) \times [s_j, s_{j+1})$ for $i = 1, 2, \cdots, N-1$ and $j = 1, 2, \cdots, K-1$. Then,

$$|R| = \sum_{i=1}^{N-1} \sum_{j=1}^{K-1} |R_{i,j}| \quad \textit{and} \quad R = \bigcup_{i=1}^{N-1} \bigcup_{j=1}^{K-1} R_{i,j} \quad \textit{of disjoint union.}$$

Proof.

$$\begin{aligned} |R| = (b-a)(c-d) &= \Big(\sum_{i=1}^{N-1} (t_{i+1} - t_i)\Big)\Big(\sum_{j=1}^{K-1} (s_{j+1} - s_j)\Big) \\ &= \sum_{i=1}^{N-1} \sum_{j=1}^{K-1} (t_{i+1} - t_i)(s_{j+1} - s_j) = \sum_{i=1}^{N-1} \sum_{j=1}^{K-1} |R_{i,j}|. \end{aligned}$$

By definition,

$$\begin{aligned} R_{i,j} &= \{(x, y) \mid t_i \leq x < t_{i+1} \text{ and } s_j \leq y < s_{j+1}\} \\ R_{i',j'} &= \{(x, y) \mid t'_i \leq x < t_{i'+1} \text{ and } s'_j \leq y < s_{j'+1}\} \end{aligned}$$

and if $(i, j) \neq (i', j')$, then $i \neq i'$ or $j \neq j'$, and they must be disjoint. Again by definition

$$\begin{aligned} R = \;& \{(x, y) \mid a \leq x < b \quad \text{and} \quad c \leq y < d\} \\ = \;& \Big\{(x, y) \;\Big|\; \Big[t_1 \leq x < t_2 \text{ or } t_2 \leq x < t_3 \text{ or } \cdots \text{ or } t_{N-1} \leq x < t_N\Big] \\ & \quad \text{and} \quad \Big[s_1 \leq y < s_2 \text{ or } s_2 \leq y < s_3 \text{ or } \cdots \text{ or } s_{K-1} \leq y < y_N\Big]\Big\} \\ = \;& \bigcup_{i=1}^{N-1} \bigcup_{j=1}^{K-1} R_{i,j}. \end{aligned}$$

□

Lemma 6. *Suppose R be a nonempty bounded rectangle. If $(R_k)_{k=1}^M$ is a finite covering of R by sets in $\mathcal{R}$, then*

$$|R| \leq \sum_{k=1}^{M} |R_k|.$$

Proof. 1. Let us write for each $R_k = [a_k, b_k) \times [c_k, d_k)$.

Let $t_1 < t_2 < \cdots < t_N$ be an enumeration of the finite set

$$\{a, a_1, a_2, \cdots, a_M, b, b_1, b_2, \cdots, b_M\}$$

in ascending order.

Let $s_1 < s_2 < \cdots < s_K$ be an enumeration of the finite set

$$\{c, c_1, c_2, \cdots, c_M, d, d_1, d_2, \cdots, d_M\}$$

in ascending order. We consider the rectangles $Q_{i,j} = [t_i, t_{i+1}) \times [s_j, s_{j+1})$, pairwise disjoint.

2. Note that for each $R_k = [a_k, b_k) \times [c_k, d_k)$, there exist indices $i_{begin}(k)$ and $i_{end}(k)$ such that $t_{i_{begin}(k)} = a_k$ and $t_{i_{end}(k)} = b_k$. Similarly $j_{begin}(k)$ and $j_{end}(k)$ exist. By the previous lemma,

$$R_k = \bigcup_{i=i_{begin}(k)}^{i_{end}(k)-1} \bigcup_{j=j_{begin}(k)}^{j_{end}(k)-1} Q_{i,j} \quad \text{of disjoint union.}$$

Because of this equality and that $(Q_{i,j})$ are pairwise disjoint, the following is true:

For every k and every (i, j), either $Q_{i,j} \subset R_k$ or $Q_{i,j} \cap R_k = \emptyset$.

3. The similar is true for R.

4. We define

$$\Gamma = \{(i, j) \mid Q_{i,j} \subset R\}, \quad \Gamma_k = \{(i, j) \mid Q_{i,j} \subset R_k\}.$$

By the previous Lemma,

$$|R| = \sum_{(i,j)\in\Gamma} |Q_{i,j}|, \quad |R_k| = \sum_{(i,j)\in\Gamma_k} |Q_{i,j}|.$$

5. That $R \subset \bigcup_k R_k$ implies that $(i, j) \in \Gamma$ implies that $Q_{i,j}$ intersects some R_k. Otherwise, (R_k) is not a covering of R.

6. This R_k-intersecting $Q_{i,j}$ in fact must be a subset of R_k. But $Q_{i,j} \subset R_k$ iff $(i, j) \in \Gamma_k$. We thus conclude: $\Gamma \subset \bigcup_k \Gamma_k$.

7. Finally,

$$|R| = \sum_{(i,j)\in\Gamma} |Q_{i,j}| \leq \sum_{(i,j)\in\bigcup_k \Gamma_k} |Q_{i,j}| \leq \sum_k \sum_{(i,j)\in\Gamma_k} |Q_{i,j}| = \sum_k |R_k|.$$

$\square$

Proposition 7. *For a rectangle R, $\lambda(R) = |R|$.*

Proof.

1. If $R = \emptyset$, $\lambda(\emptyset) = 0 = |\emptyset|$.
2. Now, assume first that R is a bounded rectangle. We prove that $\lambda(R) \geq |R|$ below. Note in this case $\lambda(R) \leq |R| < \infty$.
3. By definition of $\lambda(R)$, for any $\epsilon > 0$ there exists a covering (Q_k) of R such that
$$\lambda(R) + \epsilon \geq \sum_k |Q_k|.$$
4. Now, it is possible to enlarge each rectangle Q_k a little to form an open rectange $\tilde{Q}_k \supset Q_k$ but satisfying
$$|Q_k| \geq |\tilde{Q}_k| - \frac{\epsilon}{2^k}.$$
5. $(\tilde{Q}_k)$ forms an open covering of the closure of R that is compact. Hence, there is a finite subcover of the closure of R. (that is a finite cover of R too.) We have
$$\begin{aligned}\sum_k |Q_k| &\geq \sum_k \left(|\tilde{Q}_k| - \frac{\epsilon}{2^k}\right)\\ &\geq \sum_k |\tilde{Q}_k| - \epsilon\\ &\geq \sum_{k \in subcover} |\tilde{Q}_k| - \epsilon\\ &\geq |R| - \epsilon,\end{aligned}$$
where in the last inequality, we used the Lemma 6. In conclusion,
$$\lambda(R) + 2\epsilon \geq |R|$$
for every $\epsilon > 0$, and we conclude $\lambda(R) \geq |R|$.
6. Finally, let R be an unbounded rectangle. If so, we can consider $R_1 \subset R_2 \subset \cdots$ of subsets of R with $|R_j| < \infty$ and $|R_j| \to \infty$ as $j \to \infty$. Then for every j,
$$\lambda(R) \geq \lambda(R_j) = |R_j|,$$
which implies that $\lambda(R) = \infty$. The equality $\lambda(R) = |R| = \infty$ holds.

□