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Localization in adiabatic shear flow via geometric theory of singular perturbations --Manuscript Draft--

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Localization in adiabatic shear flow via geometric theory of singular perturbations

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Abstract

We study localization occurring during high speed shear deformations of metals leading to the formation of shear bands. The localization instability results from the competition among Hadamard instability (caused by softening response) and the stabilizing effects of strain-rate hardening. We consider a hyperbolic-parabolic system that expresses the above mechanism and construct self-similar solutions of localizing type that arise as the outcome of the above competition. The existence of self-similar solutions is turned, via a series of transformations, into a problem of constructing a heteroclinic orbit for an induced dynamical system. The dynamical system is four dimensional but has a fast-slow structure with respect to a small parameter capturing the strength of strain-rate hardening. Geometric singular perturbation theory is applied to construct the heteroclinic orbit as a transversal intersection of two invariant manifolds in the phase space.

1 Introduction

Shear bands are narrow zones of intensely localized shear that are formed during the high speed shear motion of metals [23, 2]. They often precede rupture and are one of the striking instances of material instability leading to failure. Our main objective is to capture this phenomenon in the simplest possible meaningful framework of one-dimensional thermo-visco-plasticity.

A specimen located in the xy -plane undergoes shear motion in the y -direction. The motion is described by the (plastic) shear strain $\gamma(x, t)$, the strain rate $u(x, t) = \gamma_t(x, t)$, the velocity $v(x, t)$ in the shear direction, the temperature $\theta(x, t)$ and the shear stress $\sigma(x, t)$ all defined in $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$. The shear motion of the thermoviscoplastic material is described by the equations

$$\begin{aligned} \gamma_t &= v_x, & (\text{kinematic compatibility}) \\ v_t &= \sigma_x, & (\text{momentum equation}) \\ \theta_t &= \sigma v_x, & (\text{adiabatic energy equation}) \\ \sigma &= \theta^{-\alpha} \gamma^m (v_x)^n, & (\text{constitutive law}) \end{aligned} \tag{1}$$

They lead to the hyperbolic-parabolic system:

$$\begin{aligned} u_t &= (\theta^{-\alpha} \gamma^m u^n)_{xx}, \\ \gamma_t &= u, \\ \theta_t &= \theta^{-\alpha} \gamma^m u^{n+1}. \end{aligned} \tag{2}$$

We refer to [10] for a derivation of the non-dimensional forms (1) and (2) and to [2, 15, 22] for further information on the mechanical aspects of the model.

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The constitutive law $\sigma = \theta^{-\alpha} \gamma^m u^n$ characterizes, in the form of an empirical power law, the nature of the material. The parameter $\alpha > 0$ measures the degree of the thermal softening, $m > 0$ measures that of strain hardening (or $m < 0$ in case of a softening plastic flow), and $n > 0$ measures the degree of strain-rate hardening and is assumed to be small, $n \ll 1$. The system (1) captures the simplest mechanism proposed for shear localization in high-speed deformations of metals [23, 2], and an (isothermal) variant appears in early studies of necking [8]. There is an extensive mechanics literature on the problem, *e.g.* [15, 2, 22], and early mathematical results appear in [19, 20] and [18] where a proof of shear band formation appears. The model with $m = 0$ corresponds to shear of a viscous temperature-dependent fluid and stability or instability results of the model appear in [1, 3, 17, 18, 9]. The model with $\alpha = 0$ corresponds to isothermal plasticity and localizing self-similar solutions were obtained for this model in [13, 11]. The above studies concern simplified cases of (2) with a reduced number of equations; by contrast, here we consider the full model.

Our objective is to construct a family of localizing solutions for (1) of self-similar form

$$\begin{aligned}\bar{\gamma}(t, x) &= (t+1)^a \Gamma((t+1)^\lambda x), & \bar{v}(t, x) &= (t+1)^b V((t+1)^\lambda x), & \bar{\theta}(t, x) &= (t+1)^c \Theta((t+1)^\lambda x), \\ \bar{\sigma}(t, x) &= (t+1)^d \Sigma((t+1)^\lambda x), & \bar{u}(t, x) &= (t+1)^{a-1} U((t+1)^\lambda x),\end{aligned}\tag{3}$$

with exponents a, b, c, d as in (8) exploiting scale invariance properties of (2). The parameter $\lambda > 0$ accounts for the rate of localization, and $\xi = (t+1)^\lambda x$ stands for the self-similar variable. The idea of self-similar localizing solutions was first introduced in [9]. Note that the exponent λ is taken of the opposite sign to that typical for parabolic problems, thus leading to localizing solutions.

The model (1) admits the class of *uniform shearing solution* independently of the exponents (α, m, n) , emerging from constant initial data γ_0 and θ_0 , and reading

$$\begin{aligned}\gamma &= t + \gamma_0, & v &= x, & u &= 1, & \theta &= \left(\frac{1+\alpha}{1+m} ((t+\gamma_0)^{1+m} - \gamma_0^{1+m}) + \theta_0^{1+\alpha} \right)^{\frac{1}{1+\alpha}}, \\ \sigma &= \left(\frac{1+\alpha}{1+m} ((t+\gamma_0)^{1+m} - \gamma_0^{1+m}) + \theta_0^{1+\alpha} \right)^{-\frac{\alpha}{1+\alpha}} (t+\gamma_0)^m.\end{aligned}\tag{4}$$

Uniform shear should be contrasted to the solutions that will be constructed here which exhibit localizing instability. In the latter case, the growth of the strain is superlinear at origin and the $x - t$ -profile localizes in a narrow zone.

The due cause for the instability of (4) is the loss of hyperbolicity occurring when $n = 0$. Note that, for $n = 0$, (1) is a first order system. As illustrated in Appendix A, along the uniform shearing solution the model is elliptic in the t -direction past a critical threshold for the parameter range $-\alpha + m < 0$. This leads to expect catastrophic growth of oscillations in the initial value problem, what has coined the term *Hadamard Instability*. However, as pointed out in [9], Hadamard Instability cannot by itself explain the formation of shear bands, as what is observed in the process is the orderly development of localization [23] rather than the arbitrary growth of oscillations. Opposed to Hadamard Instability, lies the regularizing effect of small viscosity ($0 < n \ll 1$). When n is sufficiently large, the diffusive mechanism supersedes and localization is suppressed, see [3, 17, 20]. A conjecture as to where the threshold of instability occurs is provided by the asymptotic analysis in [10], devising an effective equation that changes type along a threshold from forward to backward parabolic. It leads to expect instability when $-\alpha + m + n$ changes sign from positive to negative value. Indeed, this conjecture will be validated here and localising solutions are constructed in the regime $-\alpha + m + n < 0$.

This article is organized as follows: Sections 2 and 3 deal with the formulation of the problem. A series of techniques and transformations were devised in [9] and successfully adapted to the strain independent model ($\sigma = \varphi(\theta)u^n$) [14], and to the temperature independent model ($\sigma = \varphi(\gamma)u^n$) [13, 11]. They are further adapted to the larger sytem of equations here. Applying the *ansatz* (3) leads

to the system of singular ordinary differential equations

$$\begin{aligned}
a\Gamma(\xi) + \lambda\xi\Gamma'(\xi) &= U(\xi), \\
bV(\xi) + \lambda\xi V'(\xi) &= \Sigma'(\xi), \\
c\Theta(\xi) + \lambda\xi\Theta'(\xi) &= \Sigma(\xi)U(\xi), \\
\Sigma(\xi) &= \Theta(\xi)^{-\alpha}\Gamma(\xi)^m U(\xi)^n, \\
V'(\xi) &= U(\xi), \\
\Gamma(0) = \Gamma_0 > 0, \quad U(0) = U_0 > 0, \quad \xi \in [0, \infty),
\end{aligned} \tag{5}$$

where, we look for $(\Gamma, U, \Theta, \Sigma)$ that is even and V that is odd. We impose

$$V(0) = U'(0) = \Gamma'(0) = \Sigma'(0) = \Theta'(0) = 0. \tag{6}$$

so that the symmetric extensions are regular for the self-similar profiles.

The system (5) is singular (at $\xi = 0$) and non-autonomous and as such it does not fit under a general existence theory. Nevertheless, the singularity in (5) can be resolved and the system desingularized. Furthermore, upon introducing a series of nonlinear transformations, the construction of profiles for (5) is turned to the construction of a heteroclinic orbit for the four-dimensional dynamical system for (p, q, r, s)

$$\begin{aligned}
\dot{p} &= p \left(\frac{1}{\lambda}(r - a) + 2 - \lambda pr - q \right), \\
\dot{q} &= q \left(1 - \lambda pr - q \right) + bpr, \\
n\dot{r} &= r \left(\frac{\alpha - m - n}{\lambda(1 + \alpha)}(r - a) + \lambda pr + q + \frac{\alpha}{\lambda}r \left(s - \frac{1 + m + n}{1 + \alpha} \right) + \frac{n\alpha}{\lambda(1 + \alpha)} \right), \\
\dot{s} &= s \left(\frac{\alpha - m - n}{\lambda(1 + \alpha)}(r - a) + \lambda pr + q - \frac{1}{\lambda}r \left(s - \frac{1 + m + n}{1 + \alpha} \right) - \frac{n}{\lambda(1 + \alpha)} \right),
\end{aligned} \tag{S}$$

parametrized by (λ, α, m, n) . The initial conditions (6) are transmitted to asymptotic conditions for the heteroclinic as $\eta \rightarrow -\infty$ while the behavior as $\eta \rightarrow \infty$ captures the asymptotic behavior of the profiles.

In Sections 4, 5 and 6, we turn to the existence of the heteroclinic orbit. In Section 4 are listed the equilibria of (S) together with the linearized flow around them in phase space. Selection conditions for the heteroclinic orbit are presented in Section 5.

The heteroclinic is constructed in Section 6. The smallness of n is exploited throughout the analysis. Its implications are twofold, one is singular and the other is regular. Note that we expect that in the limit $n = 0$ only two variables (out of the original four) will play a role. First, by (1)₃,

$$\partial_t \left(\frac{\theta^{1+\alpha}}{1 + \alpha} \right) = (\gamma_t^n) \partial_t \left(\frac{\gamma^{1+m}}{1 + m} \right),$$

which states that at $n = 0$ modulo initial data one variable is linked to the other, thus reducing the number of variables by one. When $n > 0$, while we cannot remove one of the variables, their co-evolving feature is expected to persist; this feature is regularly perturbed. Second, the constitutive law $\sigma = \theta^{-\alpha}\gamma^m(v_x)^n$ is a constraint equation specifying a manifold on the state space of $(\gamma, v, \theta, \sigma)$, which by solving the differential constraint equation reduces the number of variables by one. The constraint equation is singularly perturbed: at $n = 0$ it is an algebraic equation $\sigma = \theta^{-\alpha}\gamma^m$; at $n > 0$ its role turns into a differential equation which determines the evolution after solving the singular differential system. The evolution turns out to be a relaxation process towards a slow manifold that is perturbed from the one at $n = 0$; this is the singularly perturbed part. We capture this slow manifold in our study. This procedure will be clarified in Section 6.

The above mechanism manifests itself in the (p, q, r, s) -system (S) by (i) the presence of so called *fast-slow* structure with respect to the small parameter n and (ii) the appearance of the invariant manifold that is two dimensional. *Geometric singular perturbation theory* is employed to construct the perturbed heteroclinic orbit. We detail this in Section 6. More advanced results on Geometric singular

perturbation theory can be found in [4, 5, 6, 7, 16, 21, 12]. In section 7, we describe the emergence of localization in the constructed two-parameter family of solutions via asymptotic analysis. The numerical computations of the solution, which have been described in [14], are provided again here for illustrative purposes.

The emerging self-similar solutions depend on two parameters $(U(0), \Gamma(0))$ describing the initial nonuniformity; the rate of localization λ is in turn determined by (26) and, due to the construction necessities, has to obey the bound (26). They provide an example of instability resulting to localization. To our knowledge, they are the first instance depicting such behavior for a sufficiently broad model (1) or (2) that includes the key contributing factors of thermal softening, strain hardening and strain-rate hardening, that entering in the basic mechanism of shear band formation proposed by Zener and Hollomon [23] and Clifton [2]. They complement the original proof of shear band formation [18], where shear band formation is shown but in a set-up where energy is supplied via the boundary. Some of the key predictions of stress-collapse are common in both results, but the present has the conceptual advantage to capture the emergence of localization as the combined result of Hadamard instability with small viscosity effects.

2 Self-similar solutions

The system (2) has a scale invariance property: If $(\gamma, u, v, \theta, \sigma)$ is a solution, then the rescaled function $(\gamma_\rho, u_\rho, v_\rho, \theta_\rho, \sigma_\rho)$ defined by

$$\begin{aligned}\gamma_\rho(t, x) &= \rho^a \gamma(\rho^{-1}t, \rho^\lambda x), & v_\rho(t, x) &= \rho^b v(\rho^{-1}t, \rho^\lambda x), \\ \theta_\rho(t, x) &= \rho^c \theta(\rho^{-1}t, \rho^\lambda x), & \sigma_\rho(t, x) &= \rho^d \sigma(\rho^{-1}t, \rho^\lambda x), \\ u_\rho(t, x) &= \rho^{b+\lambda} \gamma(\rho^{-1}t, \rho^\lambda x)\end{aligned}\tag{7}$$

is again a solution, provided

$$\begin{aligned}a &:= a_0 + a_1 \lambda = \frac{2 + 2\alpha - n}{D} + \frac{2(1 + \alpha)}{D} \lambda, & b &:= b_0 + b_1 \lambda = \frac{1 + m}{D} + \frac{1 + m + n}{D} \lambda, \\ c &:= c_0 + c_1 \lambda = \frac{2(1 + m)}{D} + \frac{2(1 + m + n)}{D} \lambda, & d &:= d_0 + d_1 \lambda = \frac{-2\alpha + 2m + n}{D} + \frac{2(-\alpha + m + n)}{D} \lambda,\end{aligned}\tag{8}$$

for each $\lambda \in \mathbb{R}$, with the denominator

$$D = 1 + 2\alpha - m - n.\tag{9}$$

Throughout this paper, the material parameters (α, m, n) run only through the ranges

$$\begin{aligned}\alpha &> 0 && \text{(thermal softening),} \\ m &> -1 && \text{(strain softening/hardening),} \\ n &> 0 && \text{(strain rate sensitivity),} \\ -\alpha + m + n &< 0 && \text{(net softening).}\end{aligned}\tag{10}$$

that are associated to physical properties of the problem at hand (in parentheses). In this regime we note $D > 1 + \alpha > 1$. Having λ negative is the typical scaling used in parabolic problems and capturing the effect of diffusion. Here, we are interested in the opposite behavior, of localization, and throughout this work we restrict attention to the opposite range $\lambda > 0$.

Motivated by the scale invariance property, we seek for $\lambda > 0$ solutions of the form

$$\begin{aligned}\gamma(t, x) &= t^a \Gamma(t^\lambda x), & v(t, x) &= t^b V(t^\lambda x), & \theta(t, x) &= t^c \Theta(t^\lambda x), \\ \sigma(t, x) &= t^d \Sigma(t^\lambda x), & u(t, x) &= t^{b+\lambda} U(t^\lambda x),\end{aligned}\tag{11}$$

setting $\xi = t^\lambda x$. Plugging the ansatz into the system (1) gives the system of ordinary differential and algebraic equations for $(\Gamma(\xi), V(\xi), \Theta(\xi), \Sigma(\xi), U(\xi))$:

$$\begin{aligned}
a\Gamma(\xi) + \lambda\xi\Gamma'(\xi) &= U(\xi), \\
bV(\xi) + \lambda\xi V'(\xi) &= \Sigma'(\xi), \\
c\Theta(\xi) + \lambda\xi\Theta'(\xi) &= \Sigma(\xi)U(\xi), \\
\Sigma(\xi) &= \Theta(\xi)^{-\alpha}\Gamma(\xi)^m U(\xi)^n, \\
V'(\xi) &= U(\xi).
\end{aligned} \tag{12}$$

Note that the uniform shear solution is attained as a self-similar solution at $\lambda = -\frac{1+m}{2(1+\alpha)} < 0$, with

$$\Gamma(\xi) = U(\xi) = U_0, \quad V(\xi) = U_0\xi, \quad \Theta(\xi) = \left(\frac{1+\alpha}{1+m}U_0^{1+m+n}\right)^{\frac{1}{1+\alpha}}, \quad \Sigma(\xi) = \left(\frac{1+\alpha}{1+m}\right)^{\frac{-\alpha}{1+\alpha}}U_0^{\frac{-\alpha+m+n}{1+\alpha}}.$$

The system (12) is non-autonomous and singular at $\xi = 0$. In the sequel, we de-singularize (12) and turn it into an autonomous system.

3 Reduction to the construction of a heteroclinic orbit

The goal of this section is to derive an equivalent system (S) to (12) that is autonomous and to turn the problem of constructing profiles for (12) to the construction of a heteroclinic orbit for (S). We employ techniques from [9], where a temperature dependent viscous stress with exponential law $\sigma = \mu(\theta)u^n = e^{-\alpha\theta}u^n$ is studied. However, the present analysis and especially the construction of the heteroclinic orbit is quite more complicated here.

3.1 De-singularization

Observe that if $(\Gamma(\xi), V(\xi), \Theta(\xi), \Sigma(\xi), U(\xi))$ solves (12), then $(\Gamma(-\xi), -V(-\xi), \Theta(-\xi), \Sigma(-\xi), U(-\xi))$ is also a solution. Using this symmetry, we will look for self-similar profiles such that $\Gamma(\xi), \Theta(\xi), \Sigma(\xi), U(\xi)$ are even functions of ξ , and $V(\xi)$ is an odd function of ξ . Accordingly, we impose the conditions

$$V(0) = U'(0) = \Gamma'(0) = \Sigma'(0) = \Theta'(0) = 0 \tag{13}$$

at the origin. We regard (12) as a boundary-value problem in the right half space $\xi \in [0, \infty)$ subject to the boundary conditions (13). Because the system (12) is singular, it is not clear in advance that how many conditions are needed to single out the solution. We will come to this point later in Section 5.

The system (12) is itself scale invariant: Given a solution $(\Gamma(\xi), V(\xi), \Theta(\xi), \Sigma(\xi), U(\xi))$ the rescaled function $(\Gamma_\rho(\xi), V_\rho(\xi), \Theta_\rho(\xi), \Sigma_\rho(\xi), U_\rho(\xi))$ defined by

$$\begin{aligned}
\Gamma_\rho(\xi) &= \rho^{a_1}\Gamma(\rho\xi), & V_\rho(\xi) &= \rho^{b_1}V(\rho\xi), & \Theta_\rho(\xi) &= \rho^{c_1}\Theta(\rho\xi), \\
\Sigma_\rho(\xi) &= \rho^{d_1}\Sigma(\rho\xi), & U_\rho(\xi) &= \rho^{b_1+1}U(\rho\xi) = \rho^{a_1}U(\rho\xi)
\end{aligned} \tag{14}$$

is again a solution. A self-similar solution under this scaling would be of the form

$$(\Gamma(\xi), V(\xi), \Theta(\xi), \Sigma(\xi), U(\xi)) = (A\xi^{-a_1}, B\xi^{-b_1}, C\xi^{-c_1}, D\xi^{-d_1}, E\xi^{-a_1})$$

with A, B, C, D, E appropriate constants. Such a solution is singular at $\xi = 0$ and fails with the initial conditions. Its form suggests to consider the change of variables

$$\begin{aligned}
\bar{\gamma}(\xi) &= \xi^{a_1}\Gamma(\xi), & \bar{v}(\xi) &= \xi^{b_1}V(\xi), & \bar{\theta}(\xi) &= \xi^{c_1}\Theta(\xi), \\
\bar{\sigma}(\xi) &= \xi^{d_1}\Sigma(\xi), & \bar{u}(\xi) &= \xi^{b_1+1}U(\xi),
\end{aligned} \tag{15}$$

with a_1, b_1, c_1 and d_1 as in (8), in order to de-singularize the problem. After some cumbersome, but straightforward calculation, we find that $(\bar{\gamma}, \bar{v}, \bar{\theta}, \bar{\sigma}, \bar{u})$ satisfies

$$\begin{aligned} a_0 \bar{\gamma} + \lambda \xi \bar{\gamma}' &= \bar{u}, \\ b_0 \bar{v} + \lambda \xi \bar{v}' &= -d_1 \bar{\sigma} + \xi \bar{\sigma}', \\ c_0 \bar{\theta} + \lambda \xi \bar{\theta}' &= \bar{\sigma} \bar{u}, \\ \bar{\sigma} &= \bar{\theta}^{-\alpha} \bar{\gamma}^m \bar{u}^n, \\ -b_1 \bar{v} + \xi \bar{v}' &= \bar{u}. \end{aligned} \tag{16}$$

Next, introduce a new independent variable $\eta = \log \xi$ and define $(\tilde{\gamma}, \tilde{v}, \tilde{\theta}, \tilde{\sigma}, \tilde{u})$ by

$$\begin{aligned} \tilde{\gamma}(\log \xi) &= \bar{\gamma}(\xi), \quad \tilde{v}(\log \xi) = \bar{v}(\xi), \quad \tilde{\theta}(\log \xi) = \bar{\theta}(\xi), \\ \tilde{\sigma}(\log \xi) &= \bar{\sigma}(\xi), \quad \tilde{u}(\log \xi) = \bar{u}(\xi). \end{aligned} \tag{17}$$

Noticing that $\frac{d}{d\eta} \tilde{\gamma}(\eta) = \xi \frac{d}{d\xi} \bar{\gamma}(\xi)$, we obtain an autonomous system

$$\begin{aligned} a_0 \tilde{\gamma} + \lambda \dot{\tilde{\gamma}} &= \tilde{u}, \\ b_0 \tilde{v} + \lambda \dot{\tilde{v}} &= -d_1 \tilde{\sigma} + \dot{\tilde{\sigma}}, \\ c_0 \tilde{\theta} + \lambda \dot{\tilde{\theta}} &= \tilde{\sigma} \tilde{u}, \\ \tilde{\sigma} &= \tilde{\theta}^{-\alpha} \tilde{\gamma}^m \tilde{u}^n, \\ -b_1 \tilde{v} + \dot{\tilde{v}} &= \tilde{u}, \end{aligned} \tag{18}$$

where the notation $\dot{f} = \frac{df}{d\eta}$ is used.

The system (18) is autonomous and one might attempt to consider its equilibria. However, it is easy to conclude that we cannot expect a heteroclinic that tends to equilibria of (18). Indeed, suppose $\tilde{u} \rightarrow \tilde{u}_\infty \geq 0$ as $\eta \rightarrow \infty$. Then from the last equation in (18), we conclude that $\tilde{v} \rightarrow \infty$. This suggests to enlarge the scope and consider solutions that grow as polynomials (or faster) at infinities.

3.2 The (p, q, r, s) -system derivation

Next, we attempt to come up with a new choice of variables that tend to equilibria as $\eta \rightarrow \pm\infty$ and accommodate orbits that have power behavior at infinities. We rewrite (18) in the form

$$\begin{aligned} \frac{d}{d\eta}(\ln \tilde{\gamma}) &= \frac{1}{\lambda} \left(-a_0 + \frac{\tilde{u}}{\tilde{\gamma}} \right), \\ \frac{d}{d\eta}(\ln \tilde{v}) &= -b_1 + \frac{\tilde{u}}{\tilde{v}}, \\ \frac{d}{d\eta}(\ln \tilde{\theta}) &= \frac{1}{\lambda} \left(-c_0 + \frac{\tilde{\sigma} \tilde{u}}{\tilde{\theta}} \right), \\ \frac{d}{d\eta}(\ln \tilde{\sigma}) &= d_1 + b \frac{\tilde{v}}{\tilde{\sigma}} + \lambda + \frac{\tilde{u}}{\tilde{\sigma}} \end{aligned} \tag{19}$$

and view it as describing the evolution of $(\tilde{\gamma}, \tilde{v}, \tilde{\theta}, \tilde{\sigma})$ with \tilde{u} determined by $\tilde{u} = \left(\frac{\tilde{\sigma}}{\tilde{\theta}^{-\alpha} \tilde{\gamma}^m} \right)^{\frac{1}{n}}$.

This leads us to define

$$p := \frac{\tilde{\gamma}}{\tilde{\sigma}}, \quad q := b \frac{\tilde{v}}{\tilde{\sigma}}, \quad r = \frac{\tilde{u}}{\tilde{\gamma}} = \left(\frac{\tilde{\sigma}}{\tilde{\theta}^{-\alpha} \tilde{\gamma}^{m+n}} \right)^{\frac{1}{n}}, \quad s := \frac{\tilde{\sigma} \tilde{\gamma}}{\tilde{\theta}}. \tag{20}$$

The transformation $(p, q, r, s) \leftrightarrow (\tilde{\gamma}, \tilde{v}, \tilde{\theta}, \tilde{\sigma})$ is a bijection in the first quadrant with the inverse determined by

$$\tilde{\gamma} = p^{\frac{1+\alpha}{D}} s^{\frac{\alpha}{D}} r^{\frac{n}{D}} \quad \tilde{\theta} = p^{\frac{1+m+n}{D}} s^{\frac{m+n-1}{D}} r^{\frac{2n}{D}}$$

and then

$$\sigma = \frac{1}{\tilde{\gamma}} p, \quad v = \frac{1}{b} \sigma q$$

Using (19) and (20), we write

$$\begin{aligned} \frac{\dot{p}}{p} &= \frac{\dot{\tilde{\gamma}}}{\tilde{\gamma}} - \frac{\dot{\tilde{\sigma}}}{\tilde{\sigma}} &= \left[\frac{1}{\lambda} \left(\frac{\tilde{u}}{\tilde{\gamma}} - a_0 \right) \right] &- \left[d_1 + b \frac{\tilde{v}}{\tilde{\sigma}} + \lambda \frac{\tilde{u}}{\tilde{v}} \frac{\tilde{v}}{\tilde{\sigma}} \right] \\ \frac{\dot{q}}{q} &= \frac{\dot{\tilde{v}}}{\tilde{v}} - \frac{\dot{\tilde{\sigma}}}{\tilde{\sigma}} &= \left[b_1 + \frac{\tilde{u}}{\tilde{v}} \right] &- \left[d_1 + b \frac{\tilde{v}}{\tilde{\sigma}} + \lambda \frac{\tilde{u}}{\tilde{v}} \frac{\tilde{v}}{\tilde{\sigma}} \right] \\ n \frac{\dot{r}}{r} &= -(m+n) \frac{\dot{\tilde{\gamma}}}{\tilde{\gamma}} + \frac{\dot{\tilde{\sigma}}}{\tilde{\sigma}} + \alpha \frac{\dot{\tilde{\theta}}}{\tilde{\theta}} &= \left[\frac{-(m+n)}{\lambda} \left(\frac{\tilde{u}}{\tilde{\gamma}} - a_0 \right) \right] &+ \left[d_1 + b \frac{\tilde{v}}{\tilde{\sigma}} + \lambda \frac{\tilde{u}}{\tilde{v}} \frac{\tilde{v}}{\tilde{\sigma}} \right] + \left[\frac{\alpha}{\lambda} \left(\frac{\tilde{\sigma} \tilde{u}}{\tilde{\theta}} - c_0 \right) \right] \\ \frac{\dot{s}}{s} &= \frac{\dot{\tilde{\gamma}}}{\tilde{\gamma}} + \frac{\dot{\tilde{\sigma}}}{\tilde{\sigma}} - \frac{\dot{\tilde{\theta}}}{\tilde{\theta}} &= \left[\frac{1}{\lambda} \left(\frac{\tilde{u}}{\tilde{\gamma}} - a_0 \right) \right] &+ \left[d_1 + b \frac{\tilde{v}}{\tilde{\sigma}} + \lambda \frac{\tilde{u}}{\tilde{v}} \frac{\tilde{v}}{\tilde{\sigma}} \right] - \left[\frac{1}{\lambda} \left(\frac{\tilde{\sigma} \tilde{u}}{\tilde{\theta}} - c_0 \right) \right]. \end{aligned}$$

We note that

$$\frac{\tilde{\sigma} \tilde{u}}{\tilde{\theta}} = r s, \quad \frac{\tilde{u}}{\tilde{v}} = \frac{b p r}{q}, \quad \frac{\tilde{u}}{\tilde{\sigma}} = p r,$$

and using (8) and (9), after a cumbersome but straightforward calculation, we derive the (p, q, r, s) -system:

$$\begin{aligned} \dot{p} &= p \left(\frac{1}{\lambda} (r - a) + 2 - \lambda p r - q \right), \\ \dot{q} &= q \left(1 - \lambda p r - q \right) + b p r, \\ n \dot{r} &= r \left(\frac{\alpha - m - n}{\lambda(1 + \alpha)} (r - a) + \lambda p r + q + \frac{\alpha}{\lambda} r \left(s - \frac{1 + m + n}{1 + \alpha} \right) + \frac{n \alpha}{\lambda(1 + \alpha)} \right), \\ \dot{s} &= s \left(\frac{\alpha - m - n}{\lambda(1 + \alpha)} (r - a) + \lambda p r + q - \frac{1}{\lambda} r \left(s - \frac{1 + m + n}{1 + \alpha} \right) - \frac{n}{\lambda(1 + \alpha)} \right). \end{aligned} \tag{S}$$

In the sequel, we analyze (S) as an autonomous system: We begin with sorting its equilibria and analyzing their linear stability. Most importantly, (S) possesses the *fast-slow* structure because of the small parameter n in the left-hand-side of (S)₃; the dynamics of r can be distinctively faster than those of the other variables.

4 Equilibria and their linear stability

System (S) admits several equilibria listed in the Appendix C. Our region of interest is the sector $\{(p, q, r, s) \mid p \geq 0, q \geq 0, r > 0, s > 0\}$. That $p, q \geq 0$ comes from the requirement that $\tilde{\gamma}, \tilde{v}, \tilde{\sigma} \geq 0$. The reason we reject the region $r, s \leq 0$ stems from mechanical considerations: If we transform back to the original variables, then we find

$$r(\eta)|_{\eta=t^\lambda x} = t \partial_t \log \gamma(t, x), \quad r(\eta) s(\eta)|_{\eta=t^\lambda x} = t \partial_t \log \theta(t, x).$$

Shear band initiation is related to conditions of loading where both the plastic strain and the temperature are increasing. This motivates to restrict on the region $r > 0, s > 0$.

There are two equilibria that can reside in this region:

$$\begin{aligned} M_0 &= (0, 0, r_0, s_0), & r_0 &= \frac{2 + 2\alpha - n}{D} + \frac{2 + 2\alpha}{D} \lambda, & s_0 &= \frac{1 + m + n}{1 + \alpha} - \frac{n}{(1 + \alpha) r_0}, \\ M_1 &= (0, 1, r_1, s_1), & r_1 &= r_0 - \frac{1 + \alpha}{\alpha - m - n} \lambda, & s_1 &= \frac{1 + m + n}{1 + \alpha} - \frac{n}{(1 + \alpha) r_1}. \end{aligned}$$

We find $r_0 > 0$ and $r_0 s_0 = c = \frac{2(1+m)}{D} + \frac{2(1+m+n)}{D} \lambda > 0$, or $r_0, s_0 > 0$. Thus M_0 always resides in the region. However, M_1 can be out of the region $r > 0, s > 0$ if λ is large enough. Note that $r_1, s_1 > 0$

only if $\frac{1+m+n}{1+\alpha}r_1 > \frac{n}{(1+\alpha)}$. This reads

$$\frac{1+m+n}{1+\alpha} \left(\frac{2+2\alpha-n}{D} - \frac{(1+\alpha)(1+m+n)}{D(\alpha-m-n)} \lambda \right) > \frac{n}{(1+\alpha)},$$

and thus M_1 resides in the region only under the constraint

$$0 < \lambda < \frac{2(\alpha-m-n)}{1+m+n} \left(\frac{1+m}{1+m+n} \right). \quad (21)$$

Henceforth, we restrict attention to rates that (21) holds.

We denote the four eigenvalues and four eigenvectors of M_i , $i = 0, 1$, by μ_{ij} and X_{ij} with $j = 1, 2, 3, 4$.

- M_0 is a saddle; it has three positive eigenvalues and one negative eigenvalue.

$$\mu_{01} = 2, \quad \mu_{02} = 1, \quad \mu_{03} = \mu_0^+ = \mathcal{O}\left(\frac{1}{n}\right) > 0, \quad \mu_{04} = \mu_0^- < 0, \quad (22)$$

where μ_0^\pm are respectively a positive and a negative solution of the quadratic equation

$$\left(\mu - \frac{r_0}{n} \left(\frac{1-s_0}{\lambda} - \frac{n}{\lambda r_0} \right) \right) \left(\mu + \frac{s_0 r_0}{\lambda} \right) - \frac{s_0 r_0}{n} \frac{1-s_0}{\lambda} \frac{\alpha r_0}{\lambda} = 0.$$

The leading orders of μ_0^\pm are given by

$$\mu_0^+ = \frac{\alpha-m}{n\lambda(1+\alpha)} \frac{2(1+\alpha)(1+\lambda)}{(1+2\alpha-m)} + \mathcal{O}(1), \quad \mu_0^- = -\frac{1+m}{\lambda} \frac{2(1+\alpha)(1+\lambda)}{(1+2\alpha-m)} + \mathcal{O}(n).$$

Notice that the one of the positive eigenvalue μ_{03} is $\mathcal{O}(\frac{1}{n})$, which indicates the separably fast dynamics along the direction X_{03} . We will make use of this structure later. The precise values of the eigenvector components are presented in the Appendix C, the directions of the eigenvectors are pointed out in Figure 1 for n sufficiently small.

- M_1 is a saddle; it has one positive eigenvalue and three negative eigenvalues.

$$\mu_{11} = -\frac{1+m+n}{\alpha-m-n}, \quad \mu_{12} = -1, \quad \mu_{13} = \mu_1^+ = \mathcal{O}\left(\frac{1}{n}\right) > 0, \quad \mu_{14} = \mu_1^- < 0, \quad (23)$$

where μ_1^\pm is respectively a positive and a negative solution of the quadratic equation

$$\left(\mu - \frac{r_1}{n} \left(\frac{1-s_1}{\lambda} - \frac{n}{\lambda r_1} \right) \right) \left(\mu + \frac{s_1 r_1}{\lambda} \right) - \frac{s_1 r_1}{n} \frac{1-s_1}{\lambda} \frac{\alpha r_1}{\lambda} = 0$$

The leading orders of μ_1^\pm are given by

$$\begin{aligned} \mu_1^+ &= \frac{\alpha-m}{n\lambda(1+\alpha)} \left(\frac{2(1+\alpha)(1+\lambda)}{(1+2\alpha-m)} - \frac{1+\alpha}{\alpha-m-n} \lambda \right) + \mathcal{O}(1), \\ \mu_1^- &= -\frac{1+m}{\lambda} \left(\frac{2(1+\alpha)(1+\lambda)}{(1+2\alpha-m)} - \frac{1+\alpha}{\alpha-m-n} \lambda \right) + \mathcal{O}(n). \end{aligned}$$

Notice again that the positive eigenvalue μ_{13} is $\mathcal{O}(\frac{1}{n})$.

Differently from M_0 , eigenvalues of M_1 have chances to be repeated. The exposition in the Appendix C specifies the possible combinations completely and the eigenvectors or the generalized eigenvectors are provided accordingly.

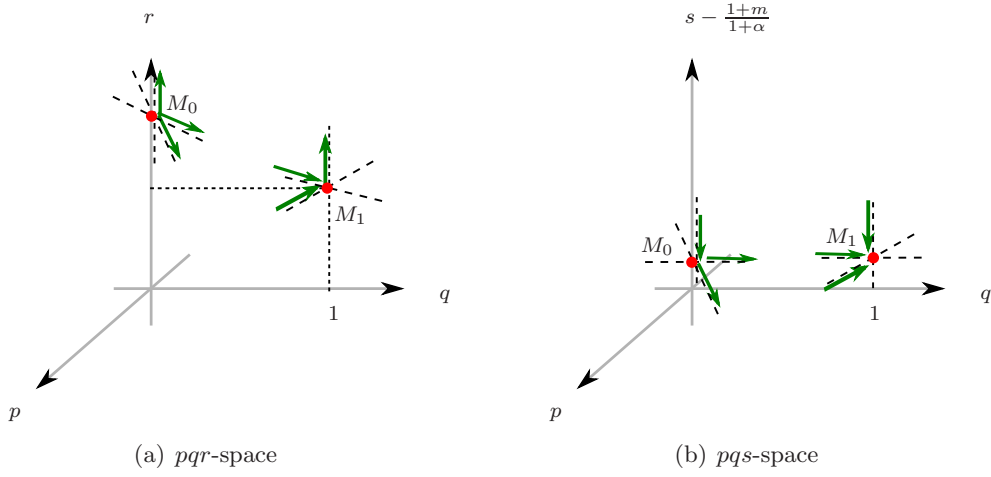


Figure 1: Eigenvectors around M_0 and M_1 in pqr -space and in pqs -space respectively ($\mu_{11} \neq -1$ and $n \ll 1$).

5 Characterization of the heteroclinic orbit

The equilibrium M_0 has three dimensions for the unstable manifold and one for the stable manifold while the equilibrium M_1 has three dimensions for the stable manifold and one for the unstable manifold. Due to the high dimensionality, it is not immediate to read off the behavior of the flow in phase space. This section aims to develop a rough picture illustrating the flow on the positive sector $p, q, r, s \geq 0$. We discuss how the local stable and unstable manifolds extend globally and form a family of heteroclinic orbits. Finally, we give a geometric characterization of the orbit of interest among those, which we will call $\chi(\eta)$ for the rest of this paper.

Recall that we determined the behavior of $\chi(\eta)$ as $\eta \rightarrow \infty$, namely $\chi(\eta) \rightarrow M_1$, from the properties of strict increase (in time) of $\theta(t, x)$ and $\gamma(t, x)$ and the fact that M_1 is the only equilibrium at this range. Now, we study the asymptotic behavior of $\chi(\eta)$ as $\eta \rightarrow -\infty$ in terms of the boundary conditions (13) at $\xi = 0$. The following proposition states how (13) are transmitted to asymptotic conditions on (p, q, r, s) .

Proposition 5.1. *Suppose $(\Gamma, V, \Theta, \Sigma, U)$ is a solution of (12), (13) that is smooth and bounded near $\xi = 0$. Then the corresponding orbit defined by transformations (15), (17), (20) $\chi(\eta) = (p(\eta), q(\eta), r(\eta), s(\eta)) \rightarrow M_0$ as $\eta \rightarrow -\infty$. Furthermore, it tends to M_0 along the direction of the first eigenvector X_{01} , i.e.,*

$$e^{-2\eta}(\chi(\eta) - M_0) \rightarrow \kappa X_{01}, \quad \text{for some constant } \kappa \neq 0 \text{ as } \eta \rightarrow -\infty. \quad (24)$$

Remark 5.1. That the orbit meets M_0 along X_{01} is nontrivial. M_0 has three dimensions of unstable subspaces and $\mu_{02}(=1) < \mu_{01}(=2) < \mu_{03}(=\mathcal{O}(\frac{1}{n}))$. Then orbits that meet M_0 in the direction of X_{01} with rate $e^{-2\eta}$ are all on the two dimensional manifold whose tangent space at M_0 is spanned by X_{01} and X_{03} .

Proof. Assuming smoothness and boundedness of $(\Gamma, V, \Theta, \Sigma, U)$ in the neighborhood of $\xi = 0$ and from the boundary conditions (13), the derivatives of $(\Gamma, V, \Theta, \Sigma, U)$ evaluated at $\xi = 0$ are obtained by differentiating the system (12) repeatedly. Re-write (12)

$$\begin{aligned} a + \lambda \xi \frac{\Gamma'}{\Gamma} &= \frac{U}{\Gamma}, & c + \lambda \xi \frac{\Theta'}{\Theta} &= \frac{\Sigma \Gamma U}{\Theta \Gamma}, \\ (b + \lambda)U + \lambda \xi U'(\xi) &= \Sigma'' = \left(\frac{\Sigma \Gamma \Theta}{\Theta \Gamma} \right)'', & \frac{\left(\frac{\Sigma \Gamma}{\Theta} \right)''}{\frac{\Sigma \Gamma}{\Theta}} &= (1 + m + n) \frac{\Gamma''}{\Gamma} - (1 + \alpha) \frac{\Theta''}{\Theta} + n \frac{\left(\frac{U}{\Gamma} \right)''}{\frac{U}{\Gamma}} \end{aligned}$$

from where after a computation we conclude

$$\begin{aligned} \frac{U}{\Gamma}(0) &= a = r_0, & \left(\frac{U}{\Gamma}\right)'(0) &= 0, & \left(\frac{U}{\Gamma}\right)''(0) &= \frac{\Gamma(0)}{\Sigma(0)} \frac{-2(b+\lambda)r_0}{\frac{1-s_0}{\lambda} - \frac{n}{r_0} \left(\frac{2}{s_0} + \frac{r_0}{\lambda}\right) \left(\frac{\frac{1}{\lambda}+2}{\frac{1+\alpha}{\lambda}r_0 + \frac{2}{s_0}}\right)}, \\ \frac{\Sigma\Gamma}{\Theta}(0) &= \frac{c}{a} = s_0, & \left(\frac{\Sigma\Gamma}{\Theta}\right)'(0) &= 0, & \left(\frac{\Sigma\Gamma}{\Theta}\right)''(0) &= \frac{n}{r_0} \left(\frac{\frac{1}{\lambda}+2}{\frac{1+\alpha}{\lambda}r_0 + \frac{2}{s_0}}\right) \left(\frac{U}{\Gamma}\right)''(0). \end{aligned}$$

Now, we consider the Taylor expansions of $p(\log \xi)$, $q(\log \xi)$, $r(\log \xi)$ and $s(\log \xi)$ at $\xi = 0$ using above and (13).

$$\begin{aligned} p(\log \xi) &= \frac{\tilde{\gamma}}{\tilde{\sigma}} = \frac{\xi^{a_1}\Gamma(\xi)}{\xi^{d_1}\Sigma(\xi)} = \xi^2 \frac{\Gamma(\xi)}{\Sigma(\xi)} = \xi^2 \frac{\Gamma(0)}{\Sigma(0)} + o(\xi^2) \\ q(\log \xi) &= b \frac{\tilde{v}}{\tilde{\sigma}} = b \frac{\xi^{b_1}V(\xi)}{\xi^{d_1}\Sigma(\xi)} = b\xi \frac{V(\xi)}{\Sigma(\xi)} = b\xi^2 \frac{U(0)}{\Sigma(0)} + o(\xi^2) = \xi^2 br_0 \frac{\Gamma(0)}{\Sigma(0)} + o(\xi^2) \\ r(\log \xi) &= \frac{\tilde{u}}{\tilde{\gamma}} = \frac{\xi^{1+b_1}U(\xi)}{\xi^{a_1}\Gamma(\xi)} = \frac{U(0)}{\Gamma(0)} + \xi \left(\frac{U}{\Gamma}\right)'(0) + \frac{1}{2}\xi^2 \left(\frac{U}{\Gamma}\right)''(0) + o(\xi^2) \\ &= \frac{U}{\Gamma}(0) + \xi^2 \frac{\Gamma(0)}{\Sigma(0)} \frac{-(b+\lambda)r_0}{\frac{1-s_0}{\lambda} - \frac{n}{r_0} \left(\frac{2}{s_0} + \frac{r_0}{\lambda}\right) \left(\frac{\frac{1}{\lambda}+2}{\frac{1+\alpha}{\lambda}r_0 + \frac{2}{s_0}}\right)}, \\ s(\log \xi) &= \frac{\tilde{\sigma}\tilde{\gamma}}{\tilde{\theta}} = \frac{\xi^{a_1+d_1}\Sigma(\xi)\Gamma(\xi)}{\xi^{c_1}\Theta(\xi)} = \frac{\Sigma\Gamma}{\Theta}(0) + \xi \left(\frac{\Sigma\Gamma}{\Theta}\right)'(0) + \frac{1}{2}\xi^2 \left(\frac{\Sigma\Gamma}{\Theta}\right)''(0) + o(\xi^2) \\ &= \frac{\Sigma\Gamma}{\Theta}(0) + \xi^2 n \left(\frac{\left(\frac{1}{\lambda}+2\right)\frac{1}{r_0}}{\frac{1+\alpha}{\lambda}r_0 + \frac{2}{s_0}}\right) \frac{\Gamma(0)}{\Sigma(0)} \frac{-(b+\lambda)r_0}{\frac{1-s_0}{\lambda} - \frac{n}{r_0} \left(\frac{2}{s_0} + \frac{r_0}{\lambda}\right) \left(\frac{\frac{1}{\lambda}+2}{\frac{1+\alpha}{\lambda}r_0 + \frac{2}{s_0}}\right)} + o(\xi^2). \end{aligned}$$

Therefore,

$$\chi(\log \xi) - M_0 = (p(\log \xi), q(\log \xi), r(\log \xi), s(\log \xi)) - M_0 = \frac{\Gamma(0)}{\Sigma(0)} \xi^2 X_{01} + o(\xi^2),$$

which is the (24) for $\eta = \log \xi$. □

Remark 5.2. For n small enough, we find signs of second derivatives are definite: $\left(\frac{U}{\Gamma}\right)''(0) < 0$, $\left(\frac{\Sigma\Gamma}{\Theta}\right)''(0) < 0$, and

$$\begin{aligned} \frac{\Gamma''(0)}{\Gamma(0)} &= \frac{1}{2\lambda} \left(\frac{U}{\Gamma}\right)''(0) < 0, & \frac{\Theta''(0)}{\Theta(0)} &= \frac{1}{2\lambda} \left(s_0 \left(\frac{U}{\Gamma}\right)''(0) + r_0 \left(\frac{\Sigma\Gamma}{\Theta}\right)''(0)\right) < 0, \\ \frac{U''(0)}{U(0)} &= \frac{\Gamma''(0)}{\Gamma(0)} + \frac{\left(\frac{U}{\Gamma}\right)''(0)}{\frac{U}{\Gamma}(0)} < 0, & \Sigma''(0) &= (b+\lambda)U(0) > 0. \end{aligned} \tag{25}$$

5.1 A two-parameter family of heteroclinic orbits

Assuming the existence of the heteroclinic $\chi(\eta)$, this section is devoted to giving an interpretation of it in terms of the given data. By data we refer to $(\Gamma(0), \Theta(0), \Sigma(0), U(0))$. Since the system (12) is singular it is not clear how many boundary conditions, for $(\Gamma(0), \Theta(0), \Sigma(0), U(0))$, are independent. Below, we clarify this issue.

The orbit satisfying (24) is hypothesized to be one dimensional and achieved for each set of parameters (λ, α, m, n) . If $\chi(\eta)$ is a heteroclinic orbit, then so is the reparametrization $\chi(\eta - \eta_0)$ for any $\eta_0 \in \mathbb{R}$. In conclusion, in order to fix one heteroclinic orbit, five parameters λ , η_0 , α , m , and n are required. The latter three are the material properties that account for thermal softening, strain hardening, and strain-rate hardening.

Other than those three, we associate the localization rate λ and the translation factor η_0 to the physical data. Due to the singular nature of (12), not all of them are independent but two out of six numbers λ , η_0 , $\Gamma(0)$, $\Theta(0)$, $\Sigma(0)$, and $U(0)$ fixes the rest. In what follows, we choose $\Gamma(0)$ and $U(0)$ as the primary parameters and specify the rest in terms of them. $\Gamma(0)$ and $U(0)$ are the tip sizes of the strain and strain rate at $\xi = 0$. To recap, we take a view that for each given triple of material parameters (α, m, n) there is a two-parameters family of heteroclinic orbits by $\Gamma(0)$ and $U(0)$.

As $r_0 = \frac{2(1+\alpha)-n}{D} + \frac{2(1+\alpha)}{D}\lambda = \frac{U(0)}{\Gamma(0)}$, we put

$$\lambda = \left(\frac{U(0)}{\Gamma(0)} - \frac{2(1+\alpha)-n}{D} \right) \frac{D}{2(1+\alpha)}. \quad (26)$$

In particular, the restriction (21) on λ reads as the ratio condition

$$\begin{aligned} \frac{2(1+\alpha)-n}{D} < \frac{U(0)}{\Gamma(0)} < \frac{2(1+\alpha)-n}{D} + \frac{4(1+\alpha)(\alpha-m-n)(1+m)}{D(1+m+n)^2} \\ = \frac{2(1+\alpha)}{1+m+n} - \frac{n}{D} \left(\frac{4(1+\alpha)(\alpha-m-n)}{(1+m+n)^2} + 1 \right). \end{aligned} \quad (27)$$

Once λ is fixed, a straightforward calculation gives

$$\Theta(0) = c^{-\frac{1}{1+\alpha}} \Gamma(0)^{\frac{m}{1+\alpha}} U(0)^{\frac{1+n}{1+\alpha}}, \quad \Sigma(0) = c^{\frac{\alpha}{1+\alpha}} \Gamma(0)^{\frac{m}{1+\alpha}} U(0)^{-\frac{\alpha-n}{1+\alpha}}.$$

η_0 is defined by the following procedure: Any orbit $\varphi(\eta)$ escaping M_0 in the direction X_{01} is asymptotically expanded by

$$\varphi(\eta) - M_0 = \kappa_1 e^{\mu_{01}\eta} + \kappa_3 e^{\mu_{03}\eta} + \text{higher-order terms as } \eta \rightarrow -\infty. \quad (28)$$

Let $(\bar{\kappa}_1, \bar{\kappa}_3)$ be fixed and let $\bar{\chi}(\eta)$ be the heteroclinic orbit whose asymptotic expansion (28) is characterized by that. We look for $\chi(\eta) = \bar{\chi}(\eta - \eta_0)$. Then (24) gives

$$\kappa_1 X_{01} = \lim_{\eta \rightarrow -\infty} (\chi(\eta) - M_0) e^{2\eta} = \lim_{\eta \rightarrow -\infty} (\bar{\chi}(\eta - \eta_0) - M_0) e^{2(\eta - \eta_0)} e^{2\eta_0} = e^{2\eta_0} \bar{\kappa}_1 X_{01}.$$

Thus $\eta_0 = \frac{1}{2} \log \frac{\kappa_1}{\bar{\kappa}_1}$ but from the proof of Proposition 5.1, we know that $\kappa_1 = \frac{\Gamma(0)}{\Sigma(0)}$, or

$$\eta_0 = \frac{1}{2} \log \left(\frac{\Gamma(0)}{\Sigma(0) \bar{\kappa}_1} \right). \quad (29)$$

6 Existence via Geometric theory of singular perturbations

This section is devoted to proving the existence of a heteroclinic orbit $\chi(\eta)$ with limiting behavior as determined in the preceding section.

Theorem 1. *Let H be a domain for the tuple $(\lambda, \alpha, m, n) \in \mathbb{R}^4$ defined by restricting (α, m, n) to take values in the range (10) and λ to satisfy the bound*

$$0 < \lambda < \frac{2(\alpha - m - n)}{1 + m + n} \left(\frac{1 + m}{1 + m + n} \right) \quad (\text{localizing rate bound}).$$

For $(\lambda, \alpha, m, 0) \in H$, there is $n_0(\lambda, \alpha, m)$ such that for $n \in [0, n_0)$ and $(\lambda, \alpha, m, n) \in H$ System (S) admits a heteroclinic orbit $\chi^{\lambda, \alpha, m, n}(\eta)$ joining equilibrium $M_0^{\lambda, \alpha, m, n}$ to equilibrium $M_1^{\lambda, \alpha, m, n}$ with the property that

$$e^{-2\eta} (\chi^{\lambda, \alpha, m, n}(\eta) - M_0^{\lambda, \alpha, m, n}) \rightarrow \kappa X_{01}^{\lambda, \alpha, m, n} \quad \text{as } \eta \rightarrow -\infty \text{ for some } \kappa \neq 0. \quad (30)$$

The heteroclinic orbit $\chi^{\lambda, \alpha, m, n}(\eta)$ is achieved via *geometric singular perturbation theory*; the presence of the small parameter $n > 0$ in the left-hand-side of (S)₃ elucidates the *fast-slow* structure of the system, having r as a fast variable and the rest as slow variables. In the interest of the reader, we present some preliminary steps in detail. Experts with Geometric theory of singular perturbations may wish to proceed directly to Sections 6.2, 6.3.

Recall that (S) accounts for a family of dynamical systems parametrized by (λ, α, m, n) ; the heteroclinic orbit will be achieved respectively for each admissible (λ, α, m, n) . To simplify notations we suppress the dependence on λ , α , and m but retain the dependence on n .

6.1 Invariant manifold theory and geometric singular perturbation theory

We quickly state some rudiments of the geometric singular perturbation theory that are in use, following definitions from [5, 6]. We will use [6, Theorem 12.2], which is wrapped up in the form of [16, Theorem 2.2], and [16, Theorem 3.1].

Let $r \geq 2$. and let X be a C^r vector field in \mathbb{R}^d . $\bar{\Lambda} = \Lambda \cup \partial\Lambda$ is a compact, connected C^{r+1} manifold in \mathbb{R}^d . $F^t : \mathbb{R}^d \mapsto \mathbb{R}^d$ denotes the time t -map associated with the vector field X and DF^t denotes its differential.

$\bar{\Lambda}$ is said to be overflowing invariant under X if for every $m \in \bar{\Lambda}$ and $t \leq 0$, $F^t(m) \in \bar{\Lambda}$ and X is pointing strictly outward on $\partial\Lambda$. $T\mathbb{R}^d|_{\Lambda}$ denotes the tangent bundle of \mathbb{R}^d along Λ and $T\Lambda$ denotes the tangent bundle of Λ . A subbundle $E \subset T\mathbb{R}^d|_{\bar{\Lambda}}$ is said to be negatively invariant if $E \subset DF^t(E)$ for all $t \leq 0$. Let $E \subset T\mathbb{R}^d|_{\bar{\Lambda}}$ be a subbundle that is negatively invariant and is containing $T\Lambda$. With a given such E , $T\mathbb{R}^d|_{\bar{\Lambda}}$ then splits into $T\mathbb{R}^d|_{\bar{\Lambda}} = E \oplus E' = T\Lambda \oplus N \oplus E'$, where $N \subset E$ is any complement of $T\Lambda$ in E and $E' \subset T\mathbb{R}^d|_{\bar{\Lambda}}$ is any complement of E in $T\mathbb{R}^d|_{\bar{\Lambda}}$.

With those introduced, we probe the separabilities of the three subbundles by their asymptotic growth rates backward in time: Let $m \in \bar{\Lambda}$, $t \leq 0$ and $v^0 \in T_m\Lambda$; $w^0 \in N_m$; $x^0 \in E'_m$; $v^t = DF^t(m)v^0$; $w^t = \pi^N DF^t(m)w^0$; $x^t = \pi^{E'} DF^t(m)x^0$, where π^N and $\pi^{E'}$ are bundle projections onto N and E' respectively. Now we define five numbers for each $m \in \bar{\Lambda}$:

$$\begin{aligned} \nu^s(m) &\triangleq \inf \left\{ \nu > 0 : \frac{1}{|x^{-t}|} = o(\nu^t) \quad \text{as } t \rightarrow \infty \quad \forall x^0 \in E'_m \right\}. \quad \text{If } \nu^s(m) < 1, \text{ define} \\ \sigma^s(m) &\triangleq \inf \left\{ \sigma > 0 : |v^{-t}| = o(|x^{-t}|^\sigma) \quad \text{as } t \rightarrow \infty \quad \forall x^0 \in E'_m, v^0 \in T_m\Lambda \right\}. \end{aligned}$$

Further, define

$$\begin{aligned} \alpha^u(m) &\triangleq \inf \left\{ \alpha > 0 : |w^{-t}| = o(\alpha^t) \quad \text{as } t \rightarrow \infty \quad \forall w^0 \in N_m \right\}. \quad \text{If } \alpha^u(m) < 1, \text{ define} \\ \rho^u(m) &\triangleq \inf \left\{ \rho > 0 : \frac{|w^{-t}|}{|v^{-t}|} = o(\rho^t) \quad \text{as } t \rightarrow \infty \quad \forall w^0 \in N_m, v^0 \in T_m\Lambda \right\}. \quad \text{If } \rho^u(m) < 1, \text{ define} \\ \tau^u(m) &\triangleq \inf \left\{ \tau > 0 : |\hat{v}^{-t}| = o\left(\left(\frac{|v^{-t}|}{|w^{-t}|}\right)^\tau\right) \quad \text{as } t \rightarrow \infty \quad \forall w^0 \in N_m, v^0 \in T_m\Lambda, \hat{v}^0 \in T_m\Lambda \right\}. \end{aligned}$$

Definition 6.1. Let $\bar{\Lambda} = \Lambda \cup \partial\Lambda$ an overflowing invariant manifold and E a subbundle over it be given as above. We say an overflowing invariant manifold $\bar{\Lambda}$ satisfies assumptions (A_r) and $(B_{r'})$, $r' \leq r - 1$, with the subbundle E if for all $m \in \bar{\Lambda}$ the rate numbers

$$\nu^s(m) < 1, \quad \sigma^s(m) < \frac{1}{r}, \tag{A_r}$$

$$\alpha^u(m) < 1, \quad \rho^u(m) < 1, \quad \tau^u(m) < \frac{1}{r'}. \tag{B_{r'}}$$

Remark 6.1. [5] considered more general rate number assumptions without (A_r) but requiring two additional rate numbers $\rho_2 < 1$ and $\tau_2 < \frac{1}{r'}$. (A_r) and $(B_{r'})$ are stronger conditions to have them valid.

Definition 6.2 (Normally Hyperbolic Invariant Manifold). Let Λ be a compact manifold without boundary, invariant under X . Let E^s and E^u be subbundles of $T\mathbb{R}^d|_{\Lambda}$ such that $E^s + E^u = T\mathbb{R}^d|_{\Lambda}$, $E^s \cap E^u = T\Lambda$, E^u is negatively invariant under X and E^s is so under $-X$. We say Λ is r -normally hyperbolic if Λ is an overflowing invariant manifold with a subbundle E^u satisfying the rate assumptions (A_r) and Λ is so with E^s under $-X$.

The two notions for a certain invariant manifold in Definition 6.1 and 6.2 are for a certain type of persistence theorem under the perturbation of the vector field. Before we introduce the persistence theorem by Fenichel, we introduce the notion of the transversal intersection of two submanifolds in a phase space \mathcal{M} .

Definition 6.3 (Transversal Intersection). ([16, Definition 3.1]) Let \mathcal{M}_1 and \mathcal{M}_2 be submanifolds of a manifold \mathcal{M} . The manifolds \mathcal{M}_1 and \mathcal{M}_2 intersect transversally at a point $m \in \mathcal{M}_1 \cap \mathcal{M}_2$ iff

$$T_m \mathcal{M} = T_m \mathcal{M}_1 + T_m \mathcal{M}_2$$

holds, where $T_m \mathcal{M}$ denotes the tangent space of the manifold \mathcal{M} and similarly for \mathcal{M}_1 and \mathcal{M}_2 .

The persistence theorem is specialized for the dynamical system that has the *fast-slow* structure,

$$\begin{cases} \dot{x} = f(x, y, \epsilon), \\ \epsilon \dot{y} = g(x, y, \epsilon), \end{cases} \quad \text{where } \epsilon \in (-\epsilon_0, \epsilon_0), \epsilon_0 > 0 \text{ small, } x \in \mathbb{R}^\ell, y \in \mathbb{R}^k, \ell + k = d. \quad (31)$$

We say x is a slow variable and y is a fast variable. We assume f and g have enough smoothness in their domains of definitions. The meanings of the term are indicated by the following two limiting problems,

$$\begin{aligned} \text{(Reduced Problem)} \quad & \begin{cases} \dot{x} = f(x, y, 0), \\ 0 = g(x, y, 0), \end{cases} & \text{(Layer Problem)} \quad & \begin{cases} x' = 0, \\ y' = g(x, y, 0), \end{cases} \quad (\cdot)' = \frac{d}{d(t/\epsilon)}. \end{aligned}$$

The zeroset \mathcal{S} of $g(x, y, 0)$ defines a manifold where the orbits of the Reduced problem are restricted. On the other hand, this manifold consists of equilibria of the Layer problem. We consider

$$\begin{aligned} \mathcal{S} &\subset \{(x, y) \mid g(x, y, 0) = 0\}, \\ \mathcal{S}_R &\subset \{(x, y) \in \mathcal{S} \mid D_y g(x, y, 0) \text{ has the full rank } k\} \quad \text{open}, \\ \mathcal{S}_H &\subset \{(x, y) \in \mathcal{S}_R \mid \text{all eigenvalues of } D_y g(x, y, 0) \text{ have nontrivial real parts}\} \quad \text{open}. \end{aligned}$$

On \mathcal{S}_R , the equation $0 = g(x, y, 0)$ is locally solvable and we speak of the reduced vector field X_R on slow variables. (See equation (7.8) in [6].)

Next, we use Fenichel's Theorems in the form of [16, Theorem 2.2]. In particular this extended version with overflowing invariant manifold needs [5, Theorem 3]. We omit the statements, but the result states the persistence properties of the compact branch $K \subset \mathcal{S}_H$ and its stable and unstable manifolds under a small perturbation. The upshot of the theorem is that any $\mathcal{N} \hookrightarrow K$ in the reduced phase space that is normally hyperbolic invariant under the reduced vector field X_R persists under the perturbation in a suitable sense. The stable and unstable manifolds of \mathcal{N} has a local lifting to the unreduced phase space and persist under the perturbation as well. In particular, as far as the perturbations of \mathcal{N} and its unstable manifold are concerned, it is enough to have \mathcal{N} overflowing invariant as in Definition 6.1 with its center-unstable bundle.

Ingredients acquired from the Fenichel's Theorem enable one to compose a variety of geometric arguments, such as a transversal intersection, to achieve solutions. The one used here belongs to one of the simplest settings [16, Theorem 3.1]: We take a simply connected branch of \mathcal{S}_H and its compact subset K . We pick \mathcal{N}_0 and \mathcal{N}_1 in K . The heteroclinic orbit is achieved by the transversal intersection of the unstable manifold of \mathcal{N}_0 and the stable manifold of \mathcal{N}_1 in K . This transversal intersection in K then lifts to that in the unreduced phase space. (See [16].)

6.2 Singular orbits for the inviscid system with $n = 0$

Let us instantiate the perturbation theory for (S). We take two normally hyperbolic manifolds \mathcal{N}_0 and \mathcal{N}_1 , which are simply the equilibrium points M_0 and M_1 . The goal of this section is to establish the transversal intersection of the \mathcal{N}_0^u , the unstable manifold of \mathcal{N}_0 , and \mathcal{N}_1^s , the stable manifold of \mathcal{N}_1 , in the *pgrs*-space. Unless otherwise explicitly mentioned, quantities without superscript n are understood as ones for $n = 0$.

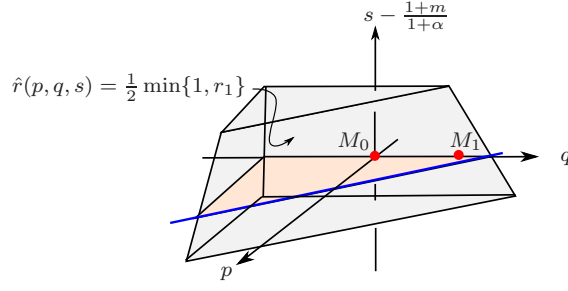


Figure 2: The trapezoid D , the domain of the graph.

We begin by writing the Reduced problem and the Layer problem for (S) :

$$\begin{aligned}
 \dot{p} &= p \left(\frac{1}{\lambda} (r - a) + 2 - \lambda pr - q \right), \\
 \dot{q} &= q \left(1 - \lambda pr - q \right) + bpr, \\
 0 &= r \left(\frac{\alpha - m}{\lambda(1 + \alpha)} (r - a) + \lambda pr + q + \frac{\alpha}{\lambda} r \left(s - \frac{1 + m}{1 + \alpha} \right) \right), \\
 \dot{s} &= s \left(\frac{\alpha - m}{\lambda(1 + \alpha)} (r - a) + \lambda pr + q - \frac{1}{\lambda} r \left(s - \frac{1 + m}{1 + \alpha} \right) \right),
 \end{aligned} \tag{R}$$

is the Reduced problem while

$$p' = 0, \quad q' = 0, \quad r' = r \left(\frac{\alpha - m}{\lambda(1 + \alpha)} (r - a) + \lambda pr + q + \frac{\alpha}{\lambda} r \left(s - \frac{1 + m}{1 + \alpha} \right) \right), \quad y' = 0, \tag{32}$$

is the Layer problem, where $(\cdot)' = \frac{d}{d\tilde{\eta}} = \frac{d}{d(\eta/n)}$ denotes differentiation with respect to the fast independent variable $\tilde{\eta}$. The zero-set

$$g(p, q, r, s) \triangleq r \left(\frac{\alpha - m}{\lambda(1 + \alpha)} (r - a) + \lambda pr + q + \frac{\alpha}{\lambda} r \left(s - \frac{1 + m}{1 + \alpha} \right) \right)$$

consists entirely of the equilibria of (32). We simply take one branch \mathcal{S}_H and $K \subset \mathcal{S}_H$ compact, which will be called a *critical manifold*. We choose K to be a graph $(D, \hat{r}(D))$, where

$$\hat{r}(p, q, s) = \frac{\frac{\alpha - m}{\lambda(1 + \alpha)} a - q}{\frac{\alpha - m}{\lambda(1 + \alpha)} + \lambda p + \frac{\alpha}{\lambda} \left(s - \frac{1 + m}{1 + \alpha} \right)}$$

or implicitly

$$\frac{\alpha - m}{\lambda(1 + \alpha)} (\hat{r} - a) + \lambda p \hat{r} + q + \frac{\alpha}{\lambda} \hat{r} \left(s - \frac{1 + m}{1 + \alpha} \right) = 0 \tag{33}$$

and D is a trapezoid in pqs -space

$$\begin{aligned}
 D \triangleq \left\{ (p, q, s) \mid p \geq -\epsilon, \quad |q| \leq 2, \quad \left| s - \frac{1 + m}{1 + \alpha} \right| \leq \frac{1}{2} \min \left\{ \frac{\alpha - m}{\alpha(1 + \alpha)}, \frac{1 + m}{(1 + \alpha)} \right\}, \right. \\
 \left. \hat{r}(p, q, s) \geq \frac{1}{2} \min \{1, r_1\} \right\}.
 \end{aligned}$$

Now $K \triangleq (D, \hat{r}(D))$. In particular, K is chosen so that M_0 and M_1 are on K ; s and $r = \hat{r}(p, q, s)$ have positive lower bound on K . See the trapezoid D in Figure 2. ϵ will be taken sufficiently small later.

We notice that the implicit formula (33) defines affine level sets of \hat{r} . In Figure 3 are the level sets $\hat{r}(p, q, s) = R$ in the pqs -space, $0 \leq R \leq a$. When $R = a$, it passes the origin, which is the equilibrium point M_0 . As R decreases the affine level sets sweep out the positive p, q sector.

Now, we verify $K \subset \mathcal{S}_H$.

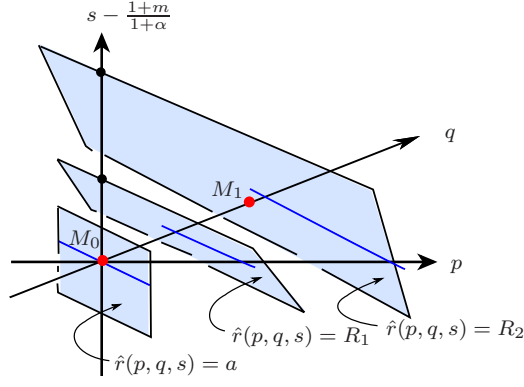


Figure 3: Affine level sets $\hat{r}(p, q, s) = R$, $0 \leq R \leq a$ in pqs -space

Proposition 6.1. $K \subset \mathcal{S}_H$, i.e., the partial jacobian $\frac{\partial g}{\partial r}(p, q, r, s)|_{r=\hat{r}(p, q, s)} > 0$ for all $(p, q, r, s) \in K$.

Proof.

$$\begin{aligned} \left. \frac{\partial g}{\partial r} \right|_K &= \left(\frac{\alpha - m}{\lambda(1 + \alpha)}(\hat{r} - a) + \lambda p \hat{r} + q + \frac{\alpha}{\lambda} \hat{r} \left(s - \frac{1 + m}{1 + \alpha} \right) \right) + \hat{r} \left(\frac{\alpha - m}{\lambda(1 + \alpha)} + \lambda p + \frac{\alpha}{\lambda} \left(s - \frac{1 + m}{1 + \alpha} \right) \right) \\ &= \hat{r} \left(\frac{\alpha - m}{\lambda(1 + \alpha)} + \lambda p + \frac{\alpha}{\lambda} \left(s - \frac{1 + m}{1 + \alpha} \right) \right) \geq \frac{1}{2} \min\{1, r_1\} \left(\frac{\alpha - m}{2\lambda(1 + \alpha)} - \lambda \epsilon \right). \end{aligned}$$

Taking $\epsilon < \frac{\alpha - m}{4\lambda^2(1 + \alpha)}$, independently of n , is enough. \square

Let us rewrite (R), for the reduced vector field X_R , in an amenable form with the identity (33):

$$\begin{aligned} \dot{p} &= p \left(\frac{D}{\lambda(1 + \alpha)}(\hat{r} - a_0) + \frac{\alpha}{\lambda} \hat{r} \left(s - \frac{1 + m}{1 + \alpha} \right) \right), \\ \dot{q} &= q \left(1 - \lambda p \hat{r} - q \right) + b p \hat{r}, \\ \dot{s} &= -\frac{1 + \alpha}{\lambda} \hat{r} s \left(s - \frac{1 + m}{1 + \alpha} \right). \end{aligned} \tag{34}$$

Having set forth the critical manifold K in \mathcal{S}_H and the reduced vector field X_R on K , the theorem of Fenichel holds in K . Let W_0^u denotes the unstable manifold of M_0 of the reduced problem in K and W_1^s denotes similarly the stable manifold of M_1 in K . As explained in [16], because M_0 and M_1 are taken from the same branch K , the task reduces now to make sure W_0^u intersects transversally W_1^s in K .

Before we discuss further, we turn to the reduced linear stability of M_0 and M_1 in the pqs -space. We find that the expression in Section 4 for the third eigenvalue μ_{03} and its eigenvector X_{03} of M_0 (μ_{13} and X_{13} respectively of M_1) does not make sense for $n = 0$; the orbits are restricted on K and the flow is genuinely three dimensional. Thus when $n = 0$, we only speak of three dimensional tangent space embedded in $pqrs$ -space. We find the formulas in Section 4 of the other three eigenvectors and eigenvalues still valid for $n = 0$. In the complementary three dimensional tangent space, we find that M_1 is a stable node and M_0 is a saddle that has two positive and one negative eigenvalues. Therefore, W_0^u will be the two dimensional unstable manifold of M_0 and W_1^s the three dimensional stable manifold of M_1 .

With those in mind, we find that for the inviscid problem, $n = 0$, all the interesting flow occurs on the two-dimensional plane $s \equiv \frac{1+m}{1+\alpha}$ that is invariant. (34) then decouples,

$$\begin{aligned} \dot{p} &= p \left(\frac{D}{\lambda(1 + \alpha)}(\hat{r} - a_0) \right), \\ \dot{q} &= q \left(1 - \lambda p \hat{r} - q \right) + b p \hat{r}, \end{aligned} \tag{35}$$

where $\hat{r} = \hat{r}(p, q, \frac{1+m}{1+\alpha})$. This two dimensional flow on the projected pq -plane around M_0 defines W_0^u that is now contained in the plane $s \equiv \frac{1+m}{1+\alpha}$. The following Lemma completely characterizes the flow on a triangle T in the first quadrant of the pq -plane.

Lemma 6.1. *Let T be the closed triangle on $s \equiv \frac{1+m}{1+\alpha}$ enclosed by $p = 0$, $q = 0$, and the level set $\hat{r}(p, q, \frac{1+m}{1+\alpha}) = \frac{1}{2} \min\{1, r_1\}$ that is intersected by $s \equiv \frac{1+m}{1+\alpha}$. Then $T \setminus M_0 \in W^s(M_1)$.*

Proof. T is a two dimensional compact positively invariant set: (1) on $p = 0$, $\nu = (1, 0)$ and $X_R \cdot \nu = \dot{p} = 0$, where X_R stands for the reduced vector field of (34); (2) on $q = 0$, $\nu = (0, 1)$ and $X_R \cdot \nu = \dot{q} = bp\hat{r} \geq 0$; lastly on the hypotenuse, let $\underline{r} = \frac{1}{2} \min\{1, r_1\}$. The inward normal pointing the origin is $\nu = (-\lambda\underline{r}, -1)$. We compute

$$\begin{aligned} X_R \cdot \nu &= -\lambda\underline{r}\dot{p} - \dot{q} = -\lambda\underline{r}p \left(1 - \lambda\underline{r}p - q + \frac{1}{\lambda}(\underline{r} - a) + 1\right) - q(1 - \lambda\underline{r}p - q) - b\underline{r}p \\ &= (1 - \lambda\underline{r}p - q)(-\lambda\underline{r}p - q) - \underline{r}p \left((\underline{r} - a) + \lambda + b\right) \\ &= \left(\frac{\alpha - m}{\lambda(1 + \alpha)}\right)^2 (\underline{r} - r_0)(\underline{r} - r_1) + \underline{r}p(1 - \underline{r}) \geq \delta > 0. \end{aligned} \quad (36)$$

Having that shown, let Ω be an ω -limit set of the orbit portrait from $x_0 \in T \setminus M_0$. It is non-empty because T is compact. It cannot contain M_0 , because M_0 does not have a stable subspace. It cannot contain a periodic orbit because if so, there would be a fixed point in the interior of T and this is not true. It cannot contain a separatrix cycle because T has only two fixed points M_0 and M_1 and again M_0 does not have a stable subspace. By Poincaré-Bendixson Theorem, the ω -limit set is M_1 . \square

In particular, let $\mathcal{F}_{M_0}^u \subset W_0^u$ be the most rapidly escaping orbit from M_0 satisfying (30) that is characterizable by the Unstable manifold theorem for the hyperbolic fixed point. By Lemma 6.1, $\mathcal{F}_{M_0}^u$ survives to end up arriving at M_1 and this gives the proof for $n = 0$ of Theorem 1.

Remark 6.2 (Neither of T nor $\chi(\eta)$ are normally hyperbolic). Under the reduced vector field, neither T nor $\chi(\eta)$ falls into the domain of Definitions 6.1 or 6.2. Checking the assumptions in Definition 6.1, we find that the rate numbers as determined at M_1 depend on the ordering of eigenvalues of M_1 . The ordering depends on the material parameters and in the range of parameters (10) the conditions (A_r) , $(B_{r'})$ are violated. Note that, if either of them were normally hyperbolic, then the heteroclinic $\chi^n(\eta)$ for $n > 0$ small would have been obtained immediately by the theorem of Fenichel. Instead of working with those manifolds that lack persistence, we deduce $\chi(\eta)$ as a transversal intersection of the persistent (under small perturbations of n) manifolds W_0^u and W_1^s .

Proposition 6.2. *Let $\mathcal{N}_0 = M_0$, $\mathcal{N}_1 = M_1$, $\mathcal{F}_{M_0}^u \subset W_0^u = W^u(M_0)$ the most rapidly escaping orbit from M_0 satisfying (30), $W_1^s = \Phi_{-t_0}(W_{loc}^s(M_1))$, the time $-t_0$ image of the local stable manifold of M_1 for large enough $t_0 < \infty$. Then $\mathcal{F}_{M_0}^u$ intersects W_1^s transversally in pqs -space.*

Proof of Proposition 6.2. For large enough $t_0 < \infty$, by Lemma 6.1 the orbit point $x \in \mathcal{F}_{M_0}^u$ must be attained in W_1^s as an interior point. Therefore the tangent space $T_x W_1^s$ is the whole of $T_x \mathbb{R}^3$. Then the intersection with $\mathcal{F}_{M_0}^u$ is trivially transversal. \square

6.3 Persistence for $n > 0$

Lemma 6.2. *Let $\mathcal{N}_0 = M_0$, $\mathcal{F}_{M_0}^u \subset W_0^u$ the most rapidly escaping orbit from M_0 satisfying (30). Then, for sufficiently small n , $\mathcal{F}_{M_0}^u$ perturbs in a C^{r-1} manner to $\mathcal{F}_{M_0^n}^u$ the most rapidly escaping orbit from M_0^n satisfying (30).*

Proof. We pick an one dimensional orbit portraits in W_0^1 that are not most rapid, the line segment $q \in [-\frac{1}{2}, \frac{1}{2}]$ on q -axis, which we will denote $\bar{\Lambda}$. We claim that $\bar{\Lambda}$ is an overflowing invariant manifold as in Definition 6.1 of the reduced problem. More precisely, it satisfies (A_r) and $(B_{r'})$ with $r' = r - 1$ and E the tangent pq -plane. If so, the most rapidly escaping orbit exists as an unstable C^{r-1} family

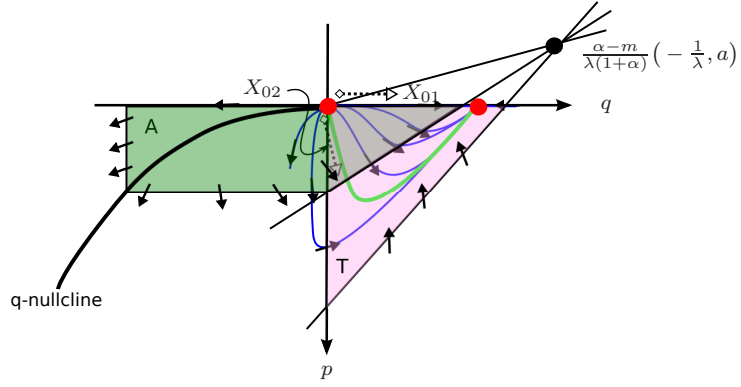


Figure 4: The schematic sketch of the flow on the invariant plane $s = \frac{1+m}{1+\alpha}$. On $s = \frac{1+m}{1+\alpha}$, M_0 is an unstable node and M_1 is a stable node. Two directions of unstable subspaces of M_0 are denoted by X_{01} and X_{02} ; the straight lines emanating from the point $\frac{\alpha-m}{\lambda(1+\alpha)}(-\frac{1}{\lambda}, a)$ are the intersections of the level sets of \hat{r} with the plane $s = \frac{1+m}{1+\alpha}$; the curve in the fourth quadrant is the nullcline of the equation $(34)_2$; the triangle T is a 2-dimensional positively invariant set; the trapezoid A is a 2-dimensional negatively invariant set; the ω -limit set of any point in T is M_1 ; the α -limit set of any point in A is M_0 ; in particular there is a heteroclinic orbit (green one) that is the most rapidly escaping orbit from M_0 through X_{01} .

of manifolds \mathcal{F}_x^u with $x = M_0$ of the reduced problem by [5, Theorem 3] and it is persistent in a basepoint-wise manner under the perturbation.

From (35), $\dot{q} = q(1-q)$ on q -axis, it is clear that $\bar{\Lambda}$ is overflowing invariant. Let E be the pq -plane along $\bar{\Lambda}$ and E' be the lines parallel to the s -axis. Then, $T\mathbb{R}^3|_{\bar{\Lambda}}$ splits into three one dimensional bundles $T\Lambda \oplus N \oplus E'$ with any N complementary to $T\Lambda$ in E such that N_{M_0} is parallel to X_{01} . Asymptotic rate numbers are determined at M_0 by the eigenvalues of M_0 . At M_0 , E'_{M_0} is the stable subspace with eigenvalue $-\mu_{04}$ and N_{M_0} and $T_{M_0}\Lambda$ are the unstable ones with $\mu_{01} = 2$ and $\mu_{02} = 1$ respectively. From these, we compute

$$\nu^s = e^{-\mu_{04}}, \quad \sigma^s = 0, \quad \alpha^u = e^{-2}, \quad \rho^u = e^{-1}, \quad \tau^u = 0.$$

□

Now Theorem 1 follows in the same way as in [16, Theorem 3.1] by the transversal intersection.

Proof of Theorem 1. By the theorem of Fenichel, n_0 can be taken sufficiently small so that for given (λ, α, m) if $n \in [0, n_0)$ then $(\lambda, \alpha, m, n) \in H$ and the system (S) admits a transversal heteroclinic orbit joining equilibrium M_0^n to equilibrium M_1^n : $\mathcal{F}_{M_0}^u$ perturbs to $\mathcal{F}_{M_0^n}^u$ by Lemma 6.2 and W_1^s perturbs to $W_1^{s,n}$ and the transversal intersection is stable under the perturbation. □

7 Emergence of localization

By transforming back with (11), (15), (17), and (20), $(\Gamma(\xi), V(\xi), \Theta(\xi), \Sigma(\xi), U(\xi))$ and $(\gamma(x, t), v(x, t), \theta(x, t), \sigma(x, t), u(x, t))$ are recovered. We replace $t \leftarrow t + 1$ in the final expressions,

$$\begin{aligned} \gamma(t, x) &= (t+1)^a \Gamma((t+1)^\lambda x), & v(t, x) &= (t+1)^b V((t+1)^\lambda x), & \theta(t, x) &= (t+1)^c \Theta((t+1)^\lambda x), \\ \sigma(t, x) &= (t+1)^d \Sigma((t+1)^\lambda x), & u(t, x) &= (t+1)^{b+\lambda} U((t+1)^\lambda x), \end{aligned}$$

so that we interpret $(\Gamma(\xi), V(\xi), \Theta(\xi), \Sigma(\xi), U(\xi)) = (\gamma(0, x), v(0, x), \theta(0, x), \sigma(0, x), u(0, x))|_{x=\xi}$, the initial states that give rise to the localization. For given material parameters (α, m, n) , additionally two degrees of freedom fully accounts for the above two-parameters family of solutions. As explained in Section 5.1, the choices of $U(0)$ and $\Gamma(0)$ fix one solution and other boundary values and the localizing

rate λ are dependently decided. The valid ranges of $U(0)$ and $\Gamma(0)$ are such that the ratio $\frac{U(0)}{\Gamma(0)}$ is not too small and not too big, i.e.,

$$\frac{2(1+\alpha)-n}{D} < \frac{U(0)}{\Gamma(0)} < \frac{2(1+\alpha)-n}{D} + \frac{4(1+\alpha)(\alpha-m-n)(1+m)}{D(1+m+n)^2}.$$

The localizing rate λ is then determined by (26) accordingly in the range

$$0 < \lambda < \frac{2(\alpha-m-n)}{1+m+n} \left(\frac{1+m}{1+m+n} \right).$$

In the following sections, we examine characteristics of these two-parameters family of initial states and discuss the emergence of localization from them as time proceeds.

7.0.1 Properties of the self-similar solutions

We state that each of the strain, strain-rate, and temperature has a small bump at the origin out of the asymptotically flat state, whose tip sizes are parameterized by $\Gamma(0)$ and $U(0)$. Accordingly, the velocity is an increasing function of x that has a slightly steeper slope at origin than at the rest of the places; the stress is then the convex increasing function of $|x|$. These initial non-uniformities can be viewed as perturbations of the uniform shearing motion at a certain time.

Proposition 7.1. *Let $(\Gamma(\xi), V(\xi), \Theta(\xi), \Sigma(\xi), U(\xi))$ be the self-similar profiles that is defined by transformations of (15), (17), and (20) upon to the heteroclinic orbit $\chi(\eta) = (p(\eta), q(\eta), r(\eta), s(\eta))$ constructed by Theorem 1 in the valid range of the parameters $\Gamma(0) = \Gamma_0$ and $U(0) = U_0$ of (27). Then it follows that:*

(i) *The self-similar profile achieves the boundary condition at $\xi = 0$,*

$$V(0) = \Gamma_\xi(0) = \Theta_\xi(0) = \Sigma_\xi(0) = U_\xi(0) = 0, \quad \Gamma(0) = \Gamma_0, \quad U(0) = U_0.$$

(ii) *Its asymptotic behavior as $\xi \rightarrow 0$ is given by*

$$\begin{aligned} \Gamma(\xi) - \Gamma_0 &= \Gamma''(0) \frac{\xi^2}{2} + o(\xi^2), \quad \Gamma''(0) < 0, \\ \Theta(\xi) - c^{-\frac{1}{1+\alpha}} \Gamma_0^{\frac{m}{1+\alpha}} U_0^{\frac{1+n}{1+\alpha}} &= \Theta''(0) \frac{\xi^2}{2} + o(\xi^2), \quad \Theta''(0) < 0, \\ \Sigma(\xi) - c^{\frac{\alpha}{1+\alpha}} \Gamma_0^{\frac{m}{1+\alpha}} U_0^{-\frac{\alpha-n}{1+\alpha}} &= \Sigma''(0) \frac{\xi^2}{2} + o(\xi^2), \quad \Sigma''(0) > 0, \\ U(\xi) - U_0 &= U''(0) \frac{\xi^2}{2} + o(\xi^2), \quad U''(0) < 0, \\ V(\xi) - U_0 \xi &= U''(0) \frac{\xi^3}{6} + o(\xi^3), \quad U''(0) < 0. \end{aligned} \tag{37}$$

(iii) *Its asymptotic behavior as $\xi \rightarrow \infty$ is given by*
if $\mu_{11} \neq -1$, or $\mu_{11} = -1$ but $b = \lambda$,

$$\begin{aligned} \Gamma(\xi) &= \mathcal{O}(\xi^{-\frac{1+\alpha}{\alpha-m-n}}), \quad V(\xi) = \mathcal{O}(1), \quad \Theta(\xi) = \mathcal{O}(\xi^{-\frac{1+m+n}{\alpha-m-n}}), \\ \Sigma(\xi) &= \mathcal{O}(\xi), \quad U(\xi) = \mathcal{O}(\xi^{-\frac{1+\alpha}{\alpha-m-n}}) \end{aligned} \tag{38}$$

otherwise

$$\begin{aligned} \Gamma(\xi) &= \mathcal{O}(\xi^{-\frac{1+\alpha}{\alpha-m-n}} (\log \xi)^{\frac{1+\alpha}{D}}), \quad V(\xi) = \mathcal{O}((\log \xi)^{-\frac{\alpha-m-n}{D}}), \\ \Theta(\xi) &= \mathcal{O}(\xi^{-\frac{1+m+n}{\alpha-m-n}} (\log \xi)^{\frac{1+m+n}{D}}), \\ \Sigma(\xi) &= \mathcal{O}(\xi (\log \xi)^{-\frac{\alpha-m-n}{D}}), \quad U(\xi) = \mathcal{O}(\xi^{-\frac{1+\alpha}{\alpha-m-n}} (\log \xi)^{\frac{1+\alpha}{D}}) \end{aligned} \tag{39}$$

Proof. The proof of the Proposition 5.1 and Remark 5.2 contains (i) and (ii) and thus we are left to prove (iii). In the similar fashion to (28), any orbit $\psi(\eta)$ in the local stable manifold of $W^s(M_1)$ is characterized by the triple $(\kappa'_1, \kappa'_2, \kappa'_3)$ in association with the asymptotic expansion

$$\begin{aligned} \psi(\eta) - M_1 &= \begin{cases} \kappa'_1 e^{\mu_{11}\eta} X_{11} + \kappa'_2 e^{\mu_{12}\eta} X_{12} + \kappa'_4 e^{\mu_{14}\eta} X_{14} + \text{high order terms} & \text{if } \mu_{11} \neq -1, \text{ or } \mu_{11} = -1 \text{ but } b = \lambda, \\ \kappa'_1 \eta e^{\mu_{11}\eta} X'_{11} + \kappa'_2 e^{\mu_{12}\eta} X_{12} + \kappa'_4 e^{\mu_{14}\eta} X_{14} + \text{high order terms} & \text{otherwise} \end{cases} \end{aligned} \quad (40)$$

as $\xi \rightarrow \infty$. As it may be associated with the generalized eigenvector, we have separated cases as above.

Now, we have $q \rightarrow 1$, $r \rightarrow r_1$, $s \rightarrow s_1$ but $p \rightarrow 0$ and the leading order of p is to be found. We look up X_{11} term in the above because p -component of the vectors X_{12} and X_{14} is 0. Because the plane $p \equiv 0$ is an invariant plane for a non-linear flow, triples of the form $(0, \kappa'_2, \kappa'_3)$ spans this invariant plane. Because our heteroclinic orbit $\chi(\eta)$ ventures out from the plane $p \equiv 0$, κ'_1 of the $\chi(\eta)$ cannot be 0. This implies that the leading order of $p(\log \xi)$ is

$$p(\log \xi) = \begin{cases} \mathcal{O}(\xi^{\mu_{11}}) & \text{if } \mu_{11} \neq -1 \text{ or } \mu_{11} = -1 \text{ but } b = \lambda, \\ \mathcal{O}(\xi^{\mu_{11}} \log \xi) & \text{otherwise} \end{cases}$$

as $\xi \rightarrow \infty$.

Asymptotics (38) and (39) are the straightforward calculations from the reconstruction formulas

$$\begin{aligned} \tilde{\gamma} &= p^{\frac{1+\alpha}{D}} r^{\frac{n}{D}} s^{\frac{\alpha}{D}}, & \tilde{v} &= \frac{1}{b} p^{-\frac{\alpha-m-n}{D}} q r^{\frac{n}{D}} s^{\frac{\alpha}{D}}, & \tilde{\theta} &= p^{\frac{1+m+n}{D}} r^{\frac{2n}{D}} s^{-\frac{1-m-n}{D}}, \\ \tilde{\sigma} &= p^{-\frac{\alpha-m-n}{D}} r^{\frac{n}{D}} s^{\frac{\alpha}{D}}, & \tilde{u} &= p^{\frac{1+\alpha}{D}} r^{\frac{n}{D}+1} s^{\frac{\alpha}{D}}, \end{aligned}$$

(15), and (17). □

7.0.2 Emergence of localization

As time proceeds, the initial states evolve out emerging the localization and this section is devoted to describing this localization: the deviation in the growth or decay rate at the origin from those at the rest of the places are contrasted, which is the emergence of the localization.

For its illustrations, we presented in the below the generic cases of $-\frac{1+m+n}{\alpha-m-n} \neq -1$; in non-generic cases we would have the logarithmic corrections according to the Proposition 7.1.

- Strain : The strain keeps increasing at a fixed x as time proceeds. However the growth at the origin is faster than that at the rest of the places,

$$\gamma(t, 0) = (1+t)^{\frac{2+2\alpha-n}{D} + \frac{2+2\alpha}{D}\lambda} \Gamma(0), \quad \gamma(t, x) \sim t^{\frac{2+2\alpha-n}{D} - \frac{(1+\alpha)(1+m+n)}{D(\alpha-m-n)}\lambda} |x|^{-\frac{1+\alpha}{\alpha-m-n}}, \quad \text{as } t \rightarrow \infty, x \neq 0.$$

Remember that the positivity of the growth rate $\frac{2+2\alpha-n}{D} - \frac{(1+\alpha)(1+m+n)}{D(\alpha-m-n)}\lambda$ was the ground for imposing the constraint (21).

- Temperature : The temperature keeps increasing at a fixed x as time proceeds. The growth at the origin is faster than that at the rest of the places,

$$\theta(t, 0) = (1+t)^{\frac{2(1+m)}{D} + \frac{2(1+m+n)}{D}\lambda} \Theta(0), \quad \theta(t, x) \sim t^{\frac{2(1+m)}{D} - \frac{(1+m+n)^2}{D(\alpha-m-n)}\lambda} |x|^{-\frac{1+m+n}{\alpha-m-n}}, \quad \text{as } t \rightarrow \infty, x \neq 0.$$

Again, the positivity of the growth rate $\frac{2(1+m)}{D} - \frac{(1+m+n)^2}{D(\alpha-m-n)}\lambda$ is from (21).

- Strain rate : The growth rates of the strain-rate is by definition less by one to those of the strain, again illustrating the localization.

$$u(t, 0) = (1+t)^{\frac{1+m}{D} + \frac{2+2\alpha}{D}\lambda} U(0), \quad u(t, x) \sim t^{\frac{1+m}{D} - \frac{(1+\alpha)(1+m+n)}{D(\alpha-m-n)}\lambda} |x|^{-\frac{1+\alpha}{\alpha-m-n}}, \quad \text{as } t \rightarrow \infty, x \neq 0.$$

- Stress : The stress keeps decreasing at a fixed x as time proceeds. However, collapsing down of the stress at the origin is severer than that at the rest of the places,

$$\sigma(t, 0) = (1+t)^{\frac{-2\alpha+2m+n}{D} + \frac{-2\alpha+2m+2n}{D}\lambda} \Sigma(0), \quad \sigma(t, x) \sim t^{\frac{-2\alpha+2m+n}{D} + \frac{1+m+n}{D}\lambda} |x|, \quad \text{as } t \rightarrow \infty, x \neq 0,$$

noticing that $\frac{-2\alpha+2m+n}{D} + \frac{1+m+n}{D}\lambda \leq -\frac{n}{1+m+n}$ in the valid range of the λ .

- Velocity : The velocity is an odd function of x . At each time t , $v(t, x)$ is an increasing function of x from $-v_\infty(t)$ to $v_\infty(t)$, as x runs from $-\infty$ to ∞ , where $v_\infty(t) \triangleq \lim_{x \rightarrow \infty} v(t, x)$. The velocity field is contrasted with the linear field of uniform shear motion. The self-similar scaling $\xi = (1+t)^\lambda x$ implies that as time goes by the most of the transition takes place around the origin making the slope at origin steeper accordingly. It eventually forms a step-function type singularity at origin. Note that the asymptotic velocity

$$v_\infty(t) = (1+t)^b V_\infty = (1+t)^{\frac{1+m}{D} + \frac{1+m+n}{D}\lambda} V_\infty, \quad V_\infty \triangleq \lim_{\xi \rightarrow \infty} V(\xi) < \infty.$$

This dictates that the particle of our solutions gets faster as it moves forward and we find the far field loading condition on the velocity different from that of uniform shearing motion, for the latter we have the constant velocity boundary condition. This deviation is a consequence of our simplifying assumption of self-similarity.

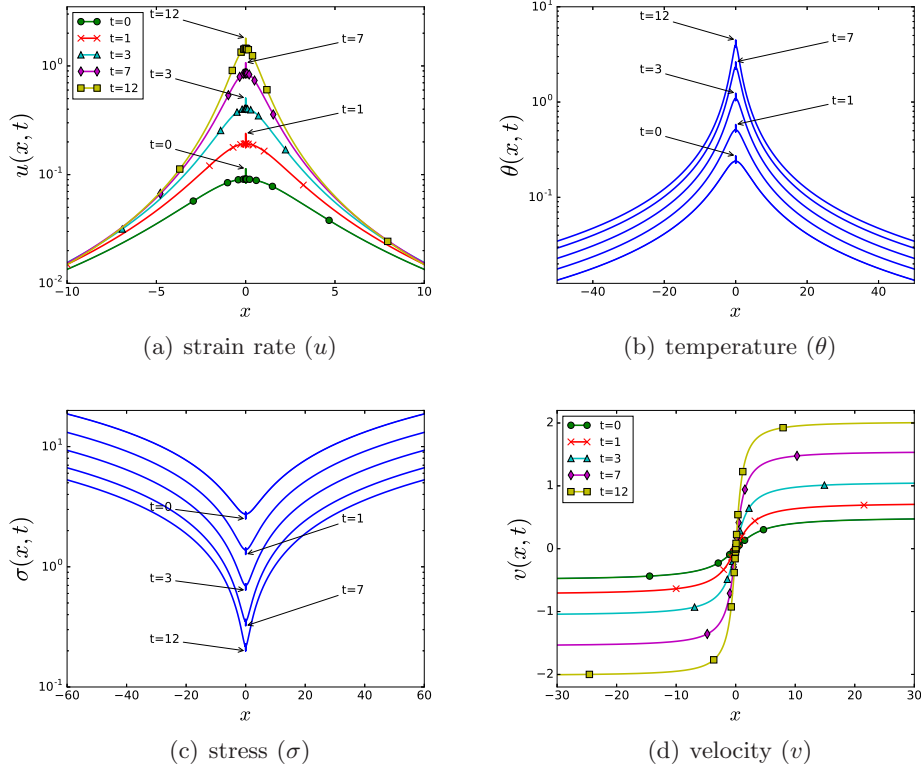


Figure 5: The localizing solution, for $\alpha = 1.2$, $m = 0$, $n = 0.3$, $\lambda = 0.5325$, and $\eta_0 = -12.94$, sketched in the original variables u , θ , σ , v . All graphs except v are in logarithmic scale.

In Figure 5 are the computational results of the original field variables ($\gamma(t, x)$, $u(t, x)$, $\theta(t, x)$, $\sigma(t, x)$, $v(t, x)$) that are computed from the heteroclinic orbit of (S) succeeded by transformations (20), (17), (15), and (11). Parameters $(\alpha, m, n, \lambda) = (1.2, 0, 0.3, 0.5325)$ was chosen for the computation. Before we examine them, we address one technical issue on the nontrivial choice of $m = 0$ for its capturing. M_0 and M_1 are hyperbolic equilibrium points and their spectrum split into those with positive and negative real parts. While the Stable Manifold Theorem gives the separation of the corresponding stable and unstable manifolds, to achieve the same computationally is challenging task

because the dynamical system is highly stiff in such a way that the presence of unstable subspace makes the computation destabilized. Since M_0 and M_1 both are saddle, computation in neither of time direction resolves this difficulty.

While (1) bears this difficulty, with $m = 0$, $(1)_2$ -(1)₄ comprises a closed system, decoupling the first equation. The dimension is reduced by one in the corresponding dynamical system either (See [14] for the precise formulation) and there, M_0 is an unstable node. By contrast to the saddle-saddle connection, in cases one of the equilibrium points is a stable or unstable *node*, solving the system in the stabilizing time direction produces numerically stable computations. We were able to compute the heteroclinic orbit backward in time, and to find Figure 5. Capturing the saddle-saddle connection will need more substantial considerations in numerical approach than that has been approached here.

Figures 5 illustrates the emergence of localization by depicting the profiles at a few instances of time. The vertical axes, except for the in Figure 5(d) for the velocity, are in logarithmic scale. In Figure 5(a), one sees the localization in strain rate; the initial profile is a small perturbation of a constant, at later time localization occurs at the origin. The same is seen in Figure 5(b) for the temperature and in Figure 5(d), which eventually looks like a step function. On the other hand, stress collapses down to zero as seen from Figure 5(c), with different rates at the origin and at the rest of the points respectively.

A The loss of hyperbolicity

Consider the system (1) for $n = 0$ when the viscoplastic effects are turned off. It is rewritten in the form of first order transport equations,

$$\begin{pmatrix} \gamma_t \\ \theta_t \\ v_t \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & \theta^{-\alpha}\gamma^m \\ m\theta^{-\alpha}\gamma^{m-1} & -\alpha\theta^{-\alpha-1}\gamma^m & 0 \end{pmatrix}}_{\triangleq B} \begin{pmatrix} \gamma_x \\ \theta_x \\ v_x \end{pmatrix}. \quad (41)$$

(41) is hyperbolic if B has three real eigenvalues and three linearly independent eigenvectors.

$$\begin{aligned} \det(B - \mu I) &= -\mu(\mu^2 + \theta^{-\alpha}\gamma^{m-1}(\alpha\theta^{-\alpha-1}\gamma^{1+m} - m)) \\ &= -\mu(\mu^2 + \theta^{-\alpha}\gamma^{m-1}\left(\frac{\alpha - m}{1 + \alpha} + \alpha\left(\frac{\gamma^{1+m}}{\theta^{1+\alpha}} - \frac{1 + m}{1 + \alpha}\right)\right)) \\ &= -\mu(\mu^2 + \theta^{-\alpha}\gamma^{m-1}\left(\frac{\alpha - m}{1 + \alpha} + \alpha\frac{\gamma_0(x)^{1+m} - \frac{1+m}{1+\alpha}\theta_0(x)^{1+\alpha}}{\theta^{1+\alpha}}\right)), \end{aligned}$$

where $\gamma_0(x)$ and $\theta_0(x)$ are initial values of γ and θ respectively. If $\alpha - m > 0$ and θ increases along the evolution then (41) loses hyperbolicity in a finite time. In particular, this happens along the uniform shearing solutions (4) as $t \rightarrow \infty$.

B Other equilibria and the reason for rejecting them

Other possible equilibria of System (S) are listed below. Underlined the reader will find the components that lie outside the region of interest and leads to rejection of them.

- (1) $p = 0, q = 0, \underline{r = 0}, s = 0,$
- (2) $p = 0, q = 0, \underline{r = 0}, s = s, r_0 = \frac{-n}{\alpha - m - n},$
- (3) $p = 0, q = 0, r = \frac{n\alpha - r_0(\alpha - m - n)}{(1 + \alpha)(m + n)}, \underline{s = 0},$
- (4) $p = 0, q = 1, \underline{r = 0}, s = 0,$
- (5) $p = 0, q = 1, \underline{r = 0}, s = s, r_1 = \frac{-n}{\alpha - m - n},$

$$(6) \quad p = 0, \quad q = 1, \quad r = \frac{n\alpha - r_1(\alpha - m - n)}{(1 + \alpha)(m + n)}, \quad s = 0,$$

$$(7) \quad p = p, \quad q = 0, \quad r = 0, \quad s = 0, \quad \frac{r_0}{\lambda} = 2,$$

$$(8) \quad p = p, \quad q = 1, \quad r = 0, \quad s = 0, \quad \frac{r_0}{\lambda} = 1,$$

$$(9) \quad p = p, \quad q = 0, \quad r = 0, \quad s = s, \quad \frac{r_0}{\lambda} = 2, \quad r_0 = \frac{-n}{\alpha - m - n},$$

$$(10) \quad p = p, \quad q = 1, \quad r = 0, \quad s = s, \quad \frac{r_0}{\lambda} = 1, \quad r_1 = \frac{-n}{\alpha - m - n},$$

$$(11) \quad p = -\frac{(\alpha - m - n)(1 + m + n)}{(1 + \alpha)(1 + m)} < 0, \quad q = \frac{2(\alpha - m - n)}{1 + m}b, \quad r = a_0, \quad s = \frac{1 + m + n}{1 + \alpha} - \frac{n}{(1 + \alpha)a_0},$$

$$(12) \quad p = \left(\frac{2\alpha(1 + m)}{D(1 - m - n)} + \frac{2(\alpha - m - n)}{D}\lambda \right) \left(\frac{2\alpha(1 + m)}{D(1 - m - n)} - \frac{1 + m + n}{D}\lambda \right) \frac{1 - m - n}{\lambda(2 - n)} \frac{1 - m - n}{\lambda(1 + m)},$$

$$q = \left(\frac{2\alpha(1 + m)}{D(1 - m - n)} + \frac{2(\alpha - m - n)}{D}\lambda \right) \left(\frac{1 + m}{D} + \frac{1 + m + n}{D}\lambda \right) \frac{1 - m - n}{\lambda(1 + m)},$$

$$r = \frac{2 - n}{1 - m - n}, \quad s = 0.$$

C Linearized problems around M_0 and M_1

The coefficient matrix for the linearized system around the equilibrium M_0 is

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ br_0 & 1 & 0 & 0 \\ \frac{r_0}{n}(\lambda r_0) & \frac{r_0}{n} & \frac{r_0}{n} \left(\frac{\alpha - m - n}{\lambda(1 + \alpha)} - \frac{n\alpha}{\lambda(1 + \alpha)r_0} \right) & \frac{r_0}{n} \left(\frac{\alpha r_0}{\lambda} \right) \\ s_0(\lambda r_0) & s_0 & s_0 \left(\frac{\alpha - m - n}{\lambda(1 + \alpha)} + \frac{n}{\lambda(1 + \alpha)r_0} \right) & s_0 \left(-\frac{r_0}{\lambda} \right) \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ br_0 & 1 & 0 & 0 \\ \frac{r_0}{n}(\lambda r_0) & \frac{r_0}{n} & \frac{r_0}{n} \frac{1}{\lambda} \left(1 - s_0 - \frac{n}{r_0} \right) & \frac{r_0}{n} \left(\frac{\alpha r_0}{\lambda} \right) \\ s_0(\lambda r_0) & s_0 & s_0 \frac{1}{\lambda} (1 - s_0) & s_0 \left(-\frac{r_0}{\lambda} \right) \end{pmatrix}$$

The corresponding eigenvectors X_{0j} are collected in the matrix S_0 as j -th column vector, $j = 1, 2, 3, 4$.

$$S_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ br_0 & 1 & 0 & 0 \\ y_1 & y_2 & 1 & y_4 \\ z_1 & z_2 & z_3 & 1 \end{pmatrix}, \quad \begin{pmatrix} y_1 \\ z_1 \end{pmatrix} = -(\lambda + b)r_0 \begin{pmatrix} \frac{1 + \alpha}{\lambda} r_0 + \frac{\mu_{01}}{s_0} \\ \frac{\Delta_1}{\frac{n}{r_0} \left(\frac{1}{\lambda} + \mu_{01} \right)} \end{pmatrix}, \quad \begin{pmatrix} y_2 \\ z_2 \end{pmatrix} = - \begin{pmatrix} \frac{1 + \alpha}{\lambda} r_0 + \frac{\mu_{02}}{s_0} \\ \frac{\Delta_2}{\frac{n}{r_0} \left(\frac{1}{\lambda} + \mu_{02} \right)} \end{pmatrix}$$

$$z_3 = n \left(\frac{1 - s_0}{\lambda} + \frac{n\mu_0}{\lambda + s_0} \right), \quad y_4 = \frac{r_0 + \frac{\mu_0}{\lambda}}{1 - s_0}, \quad (42)$$

where $\Delta_1 = \frac{1 - s_0}{\lambda} \left(\frac{1 + \alpha}{\lambda} r_0 + \frac{\mu_{01}}{s_0} \right) - \frac{n}{r_0} \left(\frac{1}{\lambda} + \mu_{01} \right) \left(\frac{r_0}{\lambda} + \frac{\mu_{01}}{s_0} \right)$ and $\Delta_2 = \frac{1 - s_0}{\lambda} \left(\frac{1 + \alpha}{\lambda} r_0 + \frac{\mu_{02}}{s_0} \right) - \frac{n}{r_0} \left(\frac{1}{\lambda} + \mu_{02} \right) \left(\frac{r_0}{\lambda} + \frac{\mu_{02}}{s_0} \right)$. We find that $y_1, y_2, y_4 < 0$; $z_1, z_2, z_3 \sim \mathcal{O}(n)$, provided n is sufficiently small.

Next, the coefficient matrix for the linearized equations around M_1 is

$$\begin{pmatrix} -\frac{1 + m + n}{\alpha - m - n} & 0 & 0 & 0 \\ (b - \lambda)r_1 & -1 & 0 & 0 \\ \frac{r_1}{n}(\lambda r_1) & \frac{r_1}{n} & \frac{r_1}{n} \left(\frac{\alpha - m - n}{\lambda(1 + \alpha)} - \frac{n\alpha}{\lambda(1 + \alpha)r_1} \right) & \frac{r_1}{n} \left(\frac{\alpha r_1}{\lambda} \right) \\ s_1(\lambda r_1) & s_1 & s_1 \left(\frac{\alpha - m - n}{\lambda(1 + \alpha)} + \frac{n}{\lambda(1 + \alpha)r_1} \right) & s_1 \left(-\frac{r_1}{\lambda} \right) \end{pmatrix} = \begin{pmatrix} -\frac{1 + m + n}{\alpha - m - n} & 0 & 0 & 0 \\ (b - \lambda)r_1 & -1 & 0 & 0 \\ \frac{r_1}{n}(\lambda r_1) & \frac{r_1}{n} & \frac{r_1}{n} \frac{1}{\lambda} \left(1 - s_1 - \frac{n}{r_1} \right) & \frac{r_1}{n} \left(\frac{\alpha r_1}{\lambda} \right) \\ s_1(\lambda r_1) & s_1 & s_1 \frac{1}{\lambda} (1 - s_1) & s_1 \left(-\frac{r_1}{\lambda} \right) \end{pmatrix}$$

The following exposition specifies all possible combinations but, except for the case $\mu_{11} = \mu_{12} = -1$, four linearly independent eigenvectors are attained. When the exceptional case occurs the repeated eigenvalue -1 has one less geometric multiplicity than the algebraic multiplicity.

As to the eigenvectors, notice first that the eigenvalues for M_1 , differently from those for M_0 , have chances to be repeated. While the exposition in Appendix C specifies quite a few possible combinations, what is explained below is that unless $\mu_{11} = \mu_{12} = -1$, the four linearly independent

eigenvectors are attained, and when the exceptional case takes place we will supplement precisely one generalized eigenvector for the repeated eigenvalue -1 .

Case 1. $-\frac{1+m+n}{\alpha-m-n} \neq -1$; or $-\frac{1+m+n}{\alpha-m-n} = -1$ but $b = \lambda$. This case yields the four linearly independent eigenvectors. The eigenvector X_{1j} is collected in the matrix S_1 as j -th column vector, $j = 1, 2, 3, 4$, and in cases eigenvalues are repeated the corresponding eigenvectors are understood as a basis for the corresponding subspaces:

$$S_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x_1 & 1 & 0 & 0 \\ y_1 & y_2 & 1 & y_4 \\ z_1 & z_2 & z_3 & 1 \end{pmatrix}, \quad x_1 = \begin{cases} \frac{(b-\lambda)r_1}{1+\mu_{11}} & \text{if } \mu_{11} \neq -1, \\ 0 & \text{otherwise,} \end{cases}$$

$$z_3 = n \left(\frac{\frac{1-s_1}{\lambda}}{\frac{nr_1}{\lambda} + \frac{n\mu_1}{s_1}} \right), \quad y_4 = \frac{\frac{r_1}{\lambda} + \frac{\mu_1}{s_1}}{\frac{1-s_1}{\lambda}},$$

$$\begin{pmatrix} y_1 \\ z_1 \end{pmatrix} = \begin{cases} -(\lambda r_1 + x_1) \begin{pmatrix} \frac{\lambda}{1-s_1} \\ 0 \end{pmatrix} & \text{if } \mu_{14} = \mu_{11}, \\ -(\lambda r_1 + x_1) \begin{pmatrix} \frac{1+\alpha}{\lambda} r_1 + \frac{\mu_{11}}{s_1} \\ \frac{\Delta_3}{r_1} \left(\frac{1}{\lambda} + \mu_{11} \right) \end{pmatrix} & \text{otherwise,} \end{cases} \quad \begin{pmatrix} y_2 \\ z_2 \end{pmatrix} = \begin{cases} - \begin{pmatrix} \frac{\lambda}{1-s_1} \\ 0 \end{pmatrix} & \text{if } \mu_{14} = \mu_{12}, \\ - \begin{pmatrix} \frac{1+\alpha}{\lambda} r_1 + \frac{\mu_{12}}{s_1} \\ \frac{\Delta_4}{r_1} \left(\frac{1}{\lambda} + \mu_{12} \right) \end{pmatrix} & \text{otherwise,} \end{cases} \quad (43)$$

where

$$\Delta_3 = \frac{1-s_1}{\lambda} \left(\frac{1+\alpha}{\lambda} r_1 + \frac{\mu_{11}}{s_1} \right) - \frac{n}{r_1} \left(\frac{1}{\lambda} + \mu_{11} \right) \left(\frac{r_1}{\lambda} + \frac{\mu_{11}}{s_1} \right)$$

$$= \frac{-n}{r_1 s_1} \det \left[\begin{pmatrix} \frac{r_1}{n} \left(\frac{1-s_1}{\lambda} - \frac{n}{\lambda r_1} \right) & \frac{r_1}{n} \frac{\alpha r_1}{\lambda} \\ s_1 \frac{1-s_1}{\lambda} & -s_1 \frac{r_1}{\lambda} \end{pmatrix} - \mu_{11} I \right] \neq 0,$$

$$\Delta_4 = \frac{1-s_1}{\lambda} \left(\frac{1+\alpha}{\lambda} r_1 + \frac{\mu_{12}}{s_1} \right) - \frac{n}{r_1} \left(\frac{1}{\lambda} + \mu_{12} \right) \left(\frac{r_1}{\lambda} + \frac{\mu_{12}}{s_1} \right)$$

$$= \frac{-n}{r_1 s_1} \det \left[\begin{pmatrix} \frac{r_1}{n} \left(\frac{1-s_1}{\lambda} - \frac{n}{\lambda r_1} \right) & \frac{r_1}{n} \frac{\alpha r_1}{\lambda} \\ s_1 \frac{1-s_1}{\lambda} & -s_1 \frac{r_1}{\lambda} \end{pmatrix} - \mu_{12} I \right] \neq 0$$

respectively for the corresponding cases.

Case 2. $-\frac{1+m+n}{\alpha-m-n} = -1$ and $b \neq \lambda$: For this case -1 has one less geometric multiplicity than the algebraic multiplicity and we replace the first column of S_1 by the generalized eigenvector $\left(\frac{1}{(b-\lambda)r_1}, 0, y'_1, z'_1 \right)^T$, where

$$\begin{pmatrix} y'_1 \\ z'_1 \end{pmatrix} = \begin{cases} \begin{pmatrix} -\frac{\lambda}{1-s_1} \left(\frac{\lambda}{b-\lambda} - \frac{n}{r_1} z_2 \right) \\ 0 \end{pmatrix} & \text{if } \mu_{14} = -1, \\ -\frac{\lambda}{b-\lambda} \begin{pmatrix} \frac{1+\alpha}{\lambda} r_1 + \frac{\mu_{11}}{s_1} \\ \frac{\Delta_3}{r_1} \left(\frac{1}{\lambda} + \mu_{11} \right) \end{pmatrix} + \frac{n}{r_1} \begin{pmatrix} \frac{y_2 \left(\frac{r_1}{\lambda} + \frac{\mu_{11}}{s_1} \right) + z_2 \frac{\alpha r_1}{\lambda}}{\Delta_3} \\ \frac{y_2 \left(\frac{1-s_1}{\lambda} \right) + z_2 \left(-\frac{1-s_1}{\lambda} + \frac{n}{r_1} \left(\frac{1}{\lambda} + \mu_{11} \right) \right)}{\Delta_3} \end{pmatrix} & \text{otherwise.} \end{cases} \quad (44)$$

$$0 = Mat_1 \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} - \mu \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (\mu_{11} - \mu)w \\ (b-\lambda)r_1 w + (\mu_{12} - \mu)x \\ \frac{r_1}{n} \left[\lambda r_1 w + x + \left(\frac{1-s_1}{\lambda} - \frac{n}{r_1} \left(\frac{1}{\lambda} + \mu \right) \right) y + \frac{\alpha r_1}{\lambda} z \right] \\ s_1 \left[\lambda r_1 w + x + \left(\frac{1-s_1}{\lambda} \right) y - \left(\frac{r_1}{\lambda} + \frac{\mu}{s_1} \right) z \right] \end{pmatrix},$$

$$A \triangleq \begin{pmatrix} \frac{1-s_1}{\lambda} - \frac{n}{r_1} \left(\frac{1}{\lambda} + \mu \right) & \frac{\alpha r_1}{\lambda} \\ \frac{1-s_1}{\lambda} & -\left(\frac{r_1}{\lambda} + \frac{\mu}{s_1} \right) \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = -(\lambda r_1 w + x) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$A^{-1} = \frac{1}{\Delta} \begin{pmatrix} \left(\frac{r_1}{\lambda} + \frac{\mu}{s_1} \right) & \frac{\alpha r_1}{\lambda} \\ \frac{1-s_1}{\lambda} & -\frac{1-s_1}{\lambda} + \frac{n}{r_1} \left(\frac{1}{\lambda} + \mu \right) \end{pmatrix}, \quad \Delta = \frac{1-s_1}{\lambda} \left(\frac{1+\alpha}{\lambda} r_1 + \frac{\mu}{s_1} \right) - \frac{n}{r_1} \left(\frac{1}{\lambda} + \mu \right) \left(\frac{r_1}{\lambda} + \frac{\mu}{s_1} \right)$$

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