Localization of Thermoviscoplastic Materials In Adiabatic Deformations

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Abstract We study a focusing instability occurring during high strain-rate deformations of metals and leading to the formation of shear bands. We consider shear motions of a thermal-viscoplastic material exhibting thermal softening and strain hardening. Under adiabatic conditions thermal softening induces a destabilizing mechanism that may lead to localization of shear strain in a narrow band. We show the existence of a family of focusing self-similar solutions, that capture the nature of this instability, in particular its emergence as a net result of the competition between Hadamard instability and viscosity. The focusing self-similar solutions we obtain can grow at a polynomial order. To construct the solutions, we introduce a non-linear transformation and to turn the existence problem into that of finding a heteroclinic orbit of an associated dynamical system. The mechanism of instability is captured by a Chapman-Enskog type asymptotic reduction. The existence of the heteroclinic orbit is obtained via the geometric singular perturbation theory.

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1 Introduction

We study a focusing type instability occurring in one dimensional adiabatic shear flows. We consider shear motions of a slab located at in the $x-y$ plane, and moving in the y-direction. We are interested in two situations: (a) the material occupies the slab between two parallel plates, $(x,y) \in (0,1) \times \mathbb{R}$, and is sheared by assigned velocities at $x = 0,1$. (b) The material occupies the entire $x-y$-space and is sheared in the y-direction. As the material keeps loading, the shear deformation and the internal energy increase in time. Materials typically is *thermally softening* and this, in some instances, can lead to the loss of hyperbolicity in the corresponding thermo-mechanical system. As the process is adiabatic, energy accumulates at each of point x possibly non-uniformly and this imbalance of the temperature distribution can lead to the localization of shear.

To understand this phenomenon, we study the following one dimensional thermo-mechanical model. $y(t,x) \in \mathbb{R}$ is the spatial coordinate in the y-direction at time t of the material initially at x on the x-axis. The motion is described by following quantities,

$$
\begin{aligned}
\gamma(t,x) &\triangleq \partial_x y(t,x) : \text{shear strain},\\
v(t,x) &\triangleq \partial_t y(t,x) : \text{vertical velocity},\\
u(t,x) &\triangleq \partial_t \gamma(t,x) : \text{shear strain rate},\\
\theta(t,x) &: \text{temperature},\\
\tau(t,x) &: \text{shear stress}.
\end{aligned} \tag{1}
$$

Here, we focus on the problem in the entire space $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$. The system of equations describing the motion is

$$
\begin{aligned}
\gamma_t &= u \quad \text{(kinematic compatibility)},\\
v_t &= \tau_x \quad \text{(momentum conservation)},\\
\theta_t &= \tau u \quad \text{(energy equation that is adiabatic)},\\
\tau &= \tau(\theta,\gamma,u) \quad \text{(constitutive law)}.
\end{aligned} \tag{A}
$$

In terms of classification, the model (A) belongs to the framework of one-dimensional thermo-visco-elasticity. It is also instructive to interprete $(\mathrm{A})_4$ as a constitutive law for thermo-visco-plastic materials viewing $\gamma = \gamma_p$ the plastic strain. (See the hierarchy of models in [5, 13].) The latter context suggests the terminology: The material exhibits thermal softening at (θ,γ,u) where $f_\theta(\theta,\gamma,u) < 0$, strain hardening where $f_\gamma(\theta,\gamma,u) > 0$, and strain softening where $f_\gamma(\theta,\gamma,u) < 0$.

To carry on the parametric study accordingly to the stiffness of these slopes, we focus on the law of form

$$
\tau = \varphi(\theta,\gamma)u^n = \theta^{-\alpha}\gamma^m u^n, \tag{2}
$$

where n is the strain-rate sensitivity and is assumed to be very small, $0 < n \ll 1$. We further introduce two subclasses of (A), where φ is independent of either θ or γ

respectively. The strain independent model (B) consists of

$$v_t = \tau_x, \quad \theta_t = \tau u, \quad \tau = \mu(\theta) u^n \tag{B}$$

and the temperature indenpendent model (C) consists of

$$\gamma_t = u, \quad v_t = \tau_x \quad \tau, = \varphi(\gamma) u^n. \tag{C}$$

The main results of this paper is the construction of the family of self-similar solutions of *focusing type* to the problems (B), and (C). The same for the problem (A) is in progress. This study on the focusing behavior is motivated by the phenomena of shear bands, which is the narrow zones shear localizes intensely. It was observed by Zener and Hollomon [14] during the high speed of deformation of metals. For further mechanical interests on the problem, we refer to [4, 11] and references therein.

In the study of the stability, one finds the solution of uniform shearing motion as the referential motion, which arises universally to (A). At the simplest normalization constants,

$$\gamma = t, \quad v = x, \quad u = 1, \quad \theta \text{ such that } \theta_t = \varphi(\theta, t)$$

constitutes a solution. More concrete form is available for (B),

$$h(\theta) = \int_{\Theta_0}^{\theta} \frac{d\theta'}{\mu(\theta')}, \quad \theta = h^{-1}(t).$$

Note that the growth rate of the strain is linear and uniform in x. In the focusing solutions we construct, the growth rate is not uniform; it grows faster at the focusing core and slower at the rest of the points.

The linear stability around the uniform shearing solution [4] and the study of Chapman-Enskog type relaxation to the effective equation [5] indicate the system becomes unstable in the regime

$$-\alpha + m + n < 0. \tag{3}$$

In the other regime $-\alpha + m + n > 0$, it is expected that the uniform shearing solution is the globally asymptotically stable solution. Non-linear stability of the problem (B) ($m = 0$) in the regime $-\alpha + n > 0$ has been studied from early stage of the research, by Dafermos and Hsiao [2] when $n = 1$, and by Tzavaras [12] when $n \neq 1$. The system (B) has an interpretation that the model describes a fluid with temperature dependent viscosity $\mu(\theta)$ in the rect-linear shear motion. Similar result for the problem (C) ($\alpha = 0$) in the regime $m + n > 0$ is obtained by Tzavaras [13]. For the problem (B), the failure of the asymptotic stability is treated by Bertsch et. al. [1] when $n = 1$ and by Katsaounis and Tzavaras [5] when $n \neq 1$. Our objective is to show the onset of localization by constructing solutions directly.

The special role of the viscosity in the model is of our central insterest. In the absence of the viscosity ($n = 0$) for the regime (3), the system loses hyperbolicity and the instability system exhibits has coined the term *Hadamard Instability*. This refers to that the oscillations will grow exponentially, i.e., the systems are catastrophically ill-posed. In the presence of the viscosity on the other hand, it alters the nature of the instability. There is a competition between the viscosity and the inelastic softening. Even when the viscosity effects merely in the way (3) holds, it is one of our main objective to state that, among the self-similar family, the growth and focusing rates are polynomial and the rate is bounded above according to the material property.

2 Main results

2.1 Self-similar structure

We investigate the scale invariance property of the system (A), and consequently that of (B) and (C) too. Suppose $(\gamma, u, v, \theta, \tau)$ is a solution of system (A). Then a rescaled version of it $(\gamma_\rho, u_\rho, v_\rho, \theta_\rho, \tau_\rho)$ given by

$$\begin{aligned} \gamma_\rho(t,x) &= \rho^a \gamma(\rho^{-1}t, \rho^\lambda x), & v_\rho(t,x) &= \rho^b v(\rho^{-1}t, \rho^\lambda x), \\ \theta_\rho(t,x) &= \rho^c \theta(\rho^{-1}t, \rho^\lambda x), & \tau_\rho(t,x) &= \rho^d \tau(\rho^{-1}t, \rho^\lambda x), \\ u_\rho(t,x) &= \rho^{b+\lambda} \gamma(\rho^{-1}t, \rho^\lambda x), \end{aligned}$$

is also a solution of (A) provided

$$\begin{aligned} a &= \frac{2+2\alpha-n}{D} + \frac{2+2\alpha}{D}\lambda =: a_0 + a_1\lambda, \\ b &= \frac{1+m}{D} + \frac{1+m+n}{D}\lambda =: b_0 + b_1\lambda, \\ c &= \frac{2(1+m)}{D} + \frac{2(1+m+n)}{D}\lambda =: c_0 + c_1\lambda, \\ d &= \frac{-2\alpha+2m+n}{D} + \frac{-2\alpha+2m+2n}{D}\lambda =: d_0 + d_1\lambda, \end{aligned}$$

for each $\lambda \in \mathbb{R}$, where $D = 1+2\alpha-m-n$. Motivated by the scale invariance property parametrized by λ, we look for the solutions of the form

$$\begin{aligned} \gamma(t,x) &= t^a \Gamma(t^\lambda x), & v(t,x) &= t^b V(t^\lambda x), & \theta(t,x) &= t^c \Theta(t^\lambda x), \\ \tau(t,x) &= t^d \Sigma(t^\lambda x), & u(t,x) &= t^{b+\lambda} U(t^\lambda x) \end{aligned}$$

and set $\xi = t^\lambda x$. In this format, $\lambda > 0$ accounts for the focusing behavior as time proceeds whereas $\lambda < 0$ accounts for the de-focusing behavior. This family includes the uniform shearing solution at $\lambda = -\frac{1+m}{2(1+\alpha)}$. Since we are interested in the focusing solutions, we consider $\lambda > 0$ in the rest of the paper.

Plugging in the ansatz to the system (A) we obtain a system of ordinary differential and algebraic equations that $\big(\Gamma(\xi), V(\xi), \Theta(\xi), \Sigma(\xi), U(\xi)\big)$ satisfies.

$$
\begin{aligned}
a\Gamma(\xi)+\lambda\xi\Gamma'(\xi) &= U(\xi),\\
bV(\xi)+\lambda\xi V'(\xi) &= \Sigma'(\xi),\\
c\Theta(\xi)+\lambda\xi\Theta'(\xi) &= \Sigma(\xi)U(\xi),\\
\Sigma(\xi) &= \Theta(\xi)^{-\alpha}\Gamma(\xi)^m U(\xi)^n,\\
V'(\xi) &= U(\xi).
\end{aligned}
\tag{4}
$$

2.2 Main theorem

We first state the existence of two parameters family of solutions for (B) where $m=0$. See [9] for the detailed discussion.

Theorem 1. *Let $\alpha, n>0$, $\alpha \neq 2n+1$ the given material parameters and fix $U_0>0$ and $\Theta_0>0$. Suppose that*

$$\frac{2}{1+2\alpha-n} < \frac{U_0^{1+n}}{\Theta_0^{1+\alpha}} < \frac{2}{1+n}, \tag{5}$$

$-\alpha+n<0$, and n is sufficiently small. Then there is a focusing self-similar solution of the form

$$
\begin{aligned}
&v(t,x)=(t+1)^b V((t+1)^\lambda x), \quad \theta(t,x)=(t+1)^c\Theta((t+1)^\lambda x),\\
&\tau(t,x)=(t+1)^d\Sigma((t+1)^\lambda x), \quad u(t,x)=(t+1)^{b+\lambda}U((t+1)^\lambda x)
\end{aligned}
$$

to the system (B), where the focusing rate is

$$\lambda=\frac{1+2\alpha-n}{2+2n}\frac{U_0^{1+n}}{\Theta_0^{1+\alpha}}-\frac{2}{2+2n}>0. \tag{6}$$

Furthermore, the self-similar profile $\big(V(\xi),\Theta(\xi),\Sigma(\xi),U(\xi)\big)$, $\xi=(t+1)^\lambda x$, has the following properties :

(i) Satisfies the boundary condition at $\xi=0$,

$$V(0)=\Theta_\xi(0)=\Sigma_\xi(0)=U_\xi(0)=0, \quad U(0)=U_0, \Theta(0)=\Theta_0.$$

(ii) Its asymptotic behavior as $\xi\to 0$ is given by

$$
\begin{aligned}
\Theta(\xi) &= \Theta(0) + \Theta''(0)\frac{\xi^2}{2} + o(\xi^2), && \Theta''(0) < 0,\\
\Sigma(\xi) &= \Theta(0)^{-\alpha}U(0)^n + \Sigma''(0)\frac{\xi^2}{2} + o(\xi^2), && \Sigma''(0) > 0,\\
U(\xi) &= U(0) + U''(0)\frac{\xi^2}{2} + o(\xi^2), && U''(0) < 0,\\
V(\xi) &= U(0)\xi + U''(0)\frac{\xi^3}{6} + o(\xi^3), && U''(0) < 0.
\end{aligned}
\tag{7}
$$

(iii) Its asymptotic behavior as $\xi \to \infty$ is given by

$$
\begin{aligned}
&V(\xi) = O(1), \quad \Theta(\xi) = O(\xi^{-\frac{1+n}{\alpha-n}}),\\
&\Sigma(\xi) = O(\xi), \quad U(\xi) = O(\xi^{-\frac{1+\alpha}{\alpha-n}}).
\end{aligned}
\tag{8}
$$

In case of system (C), where $\alpha = 0$, there exists a two parameter family of solutions, see [10], [8] for the detailed discussion.

Theorem 2. *Let $-1 \le m < 0$ and $n > 0$, $m+n \ne \frac{1}{2}$ the given material parameters and fix $U_0 > 0$ and $\Gamma_0 > 0$. Suppose that*

$$
\frac{2-n}{1-m-n} \;<\; \frac{U_0}{\Gamma_0} \;<\; \frac{2-n}{1+m+n},
$$

$m+n < 0$, and n is sufficiently small. Then there is a focusing self-similar solution of the form

$$
\begin{aligned}
\gamma(t,x) &= (t+1)^a\Gamma((t+1)^\lambda x), \quad v(t,x) = (t+1)^b V((t+1)^\lambda x),\\
\tau(t,x) &= (t+1)^d\Sigma((t+1)^\lambda x), \quad u(t,x) = (t+1)^{b+\lambda} U((t+1)^\lambda x),
\end{aligned}
$$

to the system (C), where the focusing rate is

$$
\lambda = \frac{1-m-n}{2}\Big(\frac{U_0}{\Gamma_0} - \frac{2-n}{1-m-n}\Big) > 0. \tag{9}
$$

Furthermore, the self-similar profile $\big(V(\xi),\Theta(\xi),\Sigma(\xi),U(\xi)\big)$, $\xi = (t+1)^\lambda x$, has the following properties :

(i) Satisfies the boundary condition at $\xi = 0$,

$$
V(0) = \Gamma_\xi(0) = \Sigma_\xi(0) = U_\xi(0) = 0, \quad U(0) = U_0, \Gamma(0) = \Gamma_0.
$$

(ii) Its asymptotic behavior as $\xi \to 0$ is given by

$$\begin{aligned}
\Gamma(\xi) &= \frac{1}{a}U(0) + \Gamma''(0)\frac{\xi^2}{2} + o(\xi^2), && \Gamma''(0) < 0,\\
\Sigma(\xi) &= \Gamma(0)^m U(0)^n + \Sigma''(0)\frac{\xi^2}{2} + o(\xi^2), && \Sigma''(0) > 0,\\
U(\xi) &= U(0) + U''(0)\frac{\xi^2}{2} + o(\xi^2), && U''(0) < 0,\\
V(\xi) &= U(0)\xi + U''(0)\frac{\xi^3}{6} + o(\xi^3), && U''(0) < 0.
\end{aligned} \tag{10}$$

(iii) Its asymptotic behavior as $\xi \to \infty$ is given by

$$\begin{aligned}
\Gamma(\xi) &= O(\xi^{\frac{1}{m+n}}), \quad & V(\xi) &= O(1),\\
\Sigma(\xi) &= O(\xi), & U(\xi) &= O(\xi^{\frac{1}{m+n}}).
\end{aligned} \tag{11}$$

2.3 Emergence of localization

We describe the emergence of localization of the family of solutions for system (B) constructed by Theorem 1. The corresponding localized solutions for system (C) constructed by Theorem 2 is similar, thus omitted.

In both cases we replace $t \leftarrow t+1$,

$$\begin{aligned}
v(t,x) &= (t+1)^b V((t+1)^\lambda x), \quad & \theta(t,x) &= (t+1)^c \Theta((t+1)^\lambda x),\\
\tau(t,x) &= (t+1)^d \Sigma((t+1)^\lambda x), & u(t,x) &= (t+1)^{b+\lambda} U((t+1)^\lambda x),
\end{aligned}$$

so that we interpret

$$\big(V(\xi),\Theta(\xi),\Sigma(\xi),U(\xi)\big) = \big(v(0,x),\theta(0,x),\tau(0,x),u(0,x)\big)|_{x=\xi},$$

the initial states of the self-similar solutions.

- Initial non-uniformities : The profile $\big(V(\xi),\Theta(\xi),\Sigma(\xi),U(\xi)\big)$ is the initial profile of the self-similar solution. $\Theta(\xi)$ and $U(\xi)$ have a small bump at the origin from the asymptotically flat state. The tip sizes at the origin Θ_0 and U_0 are the two parameters that fixes the solution. The velocity $V(\xi)$ is an odd function of ξ that connects $-V_\infty$ and V_∞ as ξ runs from $-\infty$ to ∞, where $V_\infty \triangleq \lim_{\xi\to\infty} V(\xi)$. The slope near the origin is slightly steeper, which reflects the initial non-uniformity in the velocity.
- Temperature : The temperature is an increasing function of t for a fixed x. The growth rate at the origin is faster than any other x, which dictates the localization near the origin.

$$\theta(t,0) = (1+t)^{\frac{2}{D}+\frac{2+2n}{D}\lambda}\Theta(0), \quad \theta(t,x) \sim t^{\frac{2}{D}-\frac{(1+n)^2}{D(\alpha-n)}\lambda}|x|^{-\frac{1+\alpha}{\alpha-n}}, \quad \text{as } t\to\infty, x\neq 0.$$

- Strain rate : The growth rate at the origin is faster than the rest of the points, which dictates the localization near the origin.

$$
\begin{aligned}
u(t,0) &= (1+t)^{\frac{1}{D}+\frac{2+2\alpha}{D}\lambda} U(0), \\
u(t,x) &\sim t^{\frac{1}{D}-\frac{(1+\alpha)(1+n)}{D(\alpha-n)}\lambda} |x|^{-\frac{1+\alpha}{\alpha-n}}, \quad \text{as } t\to\infty, x\neq 0.
\end{aligned}
$$

- Stress : The stress is a decreasing function of t for fixed x. However, the decay rate at the origin is faster than the rest of the points.

$$
\begin{aligned}
\tau(t,0) &= (1+t)^{\frac{-2\alpha+n}{D}+\frac{-2\alpha+2n}{D}\lambda} \Sigma(0), \\
\tau(t,x) &\sim t^{\frac{-2\alpha+n}{D}+\frac{1+n}{D}\lambda} |x|^{-\frac{1+\alpha}{\alpha-n}}, \quad \text{as } t\to\infty, x\neq 0.
\end{aligned}
$$

  Note that the rate of the latter is always less than $-\frac{n}{1+n}$ in the valid range of λ.
- Velocity : The velocity $v(x,t)$ is an odd function of x. It connects $-v_\infty$ to v_∞, as x runs from $-\infty$ to ∞, where $v_\infty \triangleq \lim_{x\to\infty} v(t,x)$. Because of the scaling law $\xi = (1+t)^\lambda x$, the transition from $-v_\infty$ to v_∞ localizes around the origin as time increases. The slope becomes steeper and steeper and develops a step function type singularity, see Figure 1(c). The far field velocity

$$
v_\infty(t) = (1+t)^b V_\infty = (1+t)^{\frac{1}{D}+\frac{1+n}{D}\lambda} V_\infty
$$

  itself grows at a polynomial rate. This is not in agreement with the uniform shearing motion. This deviation is a consequence of our simplifying assumption for the self-similarity.

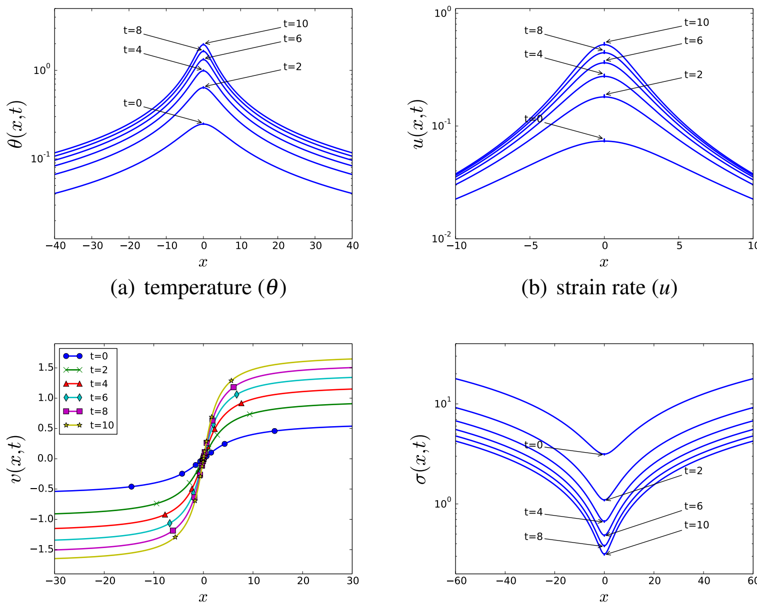
3 Existence via Geometric singular perturbation theory

Among steps to prove Theorem 1 and 2, we provide the one key step of the existence proof, for system (B). The problem boils down to show the existence of a heteroclinic orbit for the system.

$$
\begin{aligned}
\frac{\dot p}{p} &= \Big[\frac{1+\alpha}{1+n}\frac{1}{\lambda}\Big(r^{1+n}-c_0\Big)\Big] - \Big[d_1+q+\lambda pr\Big], \\
\frac{\dot q}{q} &= \Big[b_1+\frac{bpr}{q}\Big] - \Big[d_1+q+\lambda pr\Big], \\
n\frac{\dot r}{r} &= \Big[\frac{\alpha-n}{\lambda(1+n)}\Big(r^{1+n}-c_0\Big)\Big] + \Big[d_1+q+\lambda pr\Big].
\end{aligned}
\tag{P}
$$

The objective is to construct the heteroclinic orbit that connects equilibrium points

$$
M_0 = \Big(0,0,\big(\frac{2}{D}+\frac{2(1+n)}{D}\lambda\big)^{\frac{1}{1+n}}\Big), \quad M_1 = \Big(0,1,\big(\frac{2}{D}-\frac{(1+n)^2}{D(\alpha-n)}\lambda\big)^{\frac{1}{1+n}}\Big).
$$



(a) temperature (θ) (b) strain rate (u)

(c) velocity (v) (d) stress (σ)

Fig. 1 The localizing solutions for system (B), for $n = 0.3$ and $\lambda = 0.39$. All graphs except v are in logarithmic scale. See [10] for the system (C).

We describe now briefly how system (P) is derived. The technique we employ was mainly developed in Katsaounis et. al. [6]. Following [6, 8, 9], we introduce a series of non-linear transformations described by (12), (13) while the definition of (p,q,r)-variables is given in (14),

$$\begin{aligned} \bar{\gamma}(\xi) &= \xi^{a_1}\Gamma(\xi), \quad \bar{v}(\xi) = \xi^{b_1}V(\xi), \qquad \bar{\theta}(\xi) = \xi^{c_1}\Theta(\xi), \\ \bar{\tau}(\xi) &= \xi^{d_1}\Sigma(\xi), \quad \bar{u}(\xi) = \xi^{b_1+1}U(\xi). \end{aligned} \tag{12}$$

$$\begin{aligned} \tilde{\gamma}(\log\xi) &= \bar{\gamma}(\xi), \quad \tilde{v}(\log\xi) = \bar{v}(\xi), \quad \tilde{\theta}(\log\xi) = \bar{\theta}(\xi), \\ \tilde{\tau}(\log\xi) &= \bar{\tau}(\xi), \quad \tilde{u}(\log\xi) = \bar{u}(\xi), \end{aligned} \tag{13}$$

$$p \triangleq \frac{\tilde{\theta}^{\frac{1+\alpha}{1+n}}}{\tilde{\tau}}, \quad q \triangleq b\frac{\tilde{v}}{\tilde{\tau}}, \quad r \triangleq \frac{\tilde{u}}{\tilde{\theta}^{\frac{1+\alpha}{1+n}}}, \tag{14}$$

and $\eta \triangleq \log\xi$ the new independent variable $\left(\frac{df}{d\eta} = \dot{f}\right)$. The system (P) has a *fast-slow* structure due to n in front of $\dot{r}$. We conduct a Chapman-Enskog type reduction via geometric singular perturbation theory [3, 7]. The reduced problem becomes a planar dynamical system and the heteroclinic orbit is obtained by phase space analysis, [9].

3.1 Critical manifold

The surface the orbit relaxes is the zero set of the right-hand-side of $(\mathrm{P})_3$. The zero set, that is away from $r = 0$ plane, is the surface specified by

$$r = \frac{\frac{\alpha c_0}{\lambda} - d_1 - q}{\frac{\alpha}{\lambda} + \lambda p} \triangleq h(p,q;n=0), \quad \text{or} \quad q + \lambda r p + \frac{\alpha}{\lambda}\Big(r - r_0\Big) = 0.$$

We take the triangle T in the first quadrant enclosed by p-axis, q-axis and the contour line $\underline{r} = h(p,q;n=0)$ and a compact set $K \supset\supset T$. We set the critical manifold

$$G(\lambda,\alpha,n=0) \triangleq \Big\{ (p,q,r) \mid (p,q) \in K, \text{ and } r = \frac{\frac{\alpha c_0}{\lambda} - d_1 - q}{\frac{\alpha}{\lambda} + \lambda p} \Big\}. \tag{15}$$

The system in *fast scale* with the independent variable $\tilde{\eta} = \eta/n$ is

$$\begin{aligned}
p' &= np\Big(\Big[\frac{1+\alpha}{1+n}\frac{1}{\lambda}\Big(r^{1+n} - c_0\Big)\Big] - \Big[d_1 + q + \lambda pr\Big]\Big),\\
q' &= nq\Big(\Big[b_1 + \frac{bpr}{q}\Big] - \Big[d_1 + q + \lambda pr\Big]\Big),\\
r' &= r\Big(\Big[\frac{\alpha - n}{\lambda(1+n)}\Big(r^{1+n} - c_0\Big)\Big] + \Big[d_1 + q + \lambda pr\Big]\Big),
\end{aligned} \tag{$\tilde{P}$}$$

where we denoted $(\cdot)' = \dfrac{d}{d\tilde{\eta}}(\cdot)$. When $n = 0$, $(\tilde{P})|_{n=0}$ reads

$$p' = 0, \quad q' = 0, \quad r' = r\Big(\Big[\frac{\alpha}{\lambda}\Big(r - c_0\Big)\Big] + \Big[d_1 + q + \lambda pr\Big]\Big).$$

Lemma 1. *$G(\lambda,\alpha,0)$ is a normally hyperbolic invariant manifold with respect to the system $(\tilde{P})|_{n=0}$.*

3.2 Chapman-Enskog type reduction

By the theorem of geometric singular perturbation theory, [3, 7], if n is sufficiently small, there exists the locally invariant manifold $G(\lambda,\alpha,n)$ with respect to (P). Then, on this manifold, $\big(p(\eta), q(\eta)\big)$ satisfies the planar system

$$\begin{aligned}
\dot{p} &= p\Big\{\Big[\frac{1+\alpha}{1+n}\frac{1}{\lambda}\Big(h^{1+n} - c_0\Big)\Big] - \Big[d_1 + q + \lambda ph\Big]\Big\},\\
\dot{q} &= q\Big(1 - q - \lambda ph\Big) + bph,
\end{aligned} \tag{R}$$

where h stands for $h(p,q;n)$.

3.3 Confinement of the orbit

Lemma 2. *The triangle T is positively invariant for the system (R) when $n = 0$.*

We can compute the inward normal component of $(\dot{p}, \dot{q})$ on the boundary of the triangle T for $(R)|_{n=0}$:

$$\dot{p} = -\frac{D}{\alpha} p\Big(d_1 + q + \lambda ph\Big),$$
$$\dot{q} = q\Big(1 - q - \lambda ph\Big) + bph.$$

Essential calculation is on the hypotenuse and the fact that it is the contour line $\underline{r} = h(p,q,n=0)$ helps us obtain the estimate. Define $\underline{p}$ and $\underline{q}$ to be the p-intercept and q-intercept of the contour line respectively : $\underline{q} = \lambda \underline{p}\underline{r} = \frac{\alpha}{\lambda}(r_0 - \underline{r})$. With $(-\underline{q}, -\underline{p})$ being the inward normal vector, the inward normal component on the hypotenuse is

$$\begin{aligned}
(\dot{p},\dot{q})\cdot(-\underline{q},-\underline{p}) &= \frac{D}{\alpha}\underline{q}p\Big(d_1 + q + \lambda p\underline{r}\Big) - \underline{p}\Big\{q\Big(1 - q - \lambda p\underline{r}\Big) + bp\underline{r}\Big\} \\
&= \frac{D}{\alpha}\underline{q}p\Big(d_1 + \underline{q}\Big) - \underline{p}\Big\{\Big(\underline{q} - \lambda p\underline{r}\Big)\Big(1 - \underline{q}\Big) + bp\underline{r}\Big\} \\
&= -\underline{p}\underline{q}(1-\underline{q}) + \underline{q}p\Big(\frac{D}{\alpha}d_1 + \frac{D}{\alpha}\underline{q} + (1-\underline{q}) - \frac{b}{\lambda}\Big) \\
&= -\underline{p}\underline{q}(1-\underline{q}) + \underline{q}p\frac{1+\alpha}{\lambda}\Big(\frac{1}{1+\alpha} - \underline{r}\Big) \\
&\geq -\underline{p}\underline{q}(1-\underline{q}) \quad \text{for } \underline{r} < \frac{1}{1+\alpha} \\
&\geq \delta_0 > 0.
\end{aligned}$$

Lemma 3. *The triangle T is positively invariant for the system (R) provided n is sufficiently small.*

Now the hypotenuse is not anymore a contour line of the function $h(p,q;\lambda,\alpha,n)$. We arrange terms of right-hand-sides of (R) in the form

$$\begin{aligned}
\dot{p} &= p\Big\{\Big[\frac{1+\alpha}{\lambda}\Big(\underline{r} - c_0\Big)\Big] - \Big[d_1 + q + \lambda p\underline{r}\Big]\Big\} \\
&\quad + \underbrace{p\Big\{\Big[\frac{1+\alpha}{1+n}\frac{1}{\lambda}\Big(h^{1+n} - c_0\Big)\Big] - \Big[\frac{1+\alpha}{\lambda}\Big(\underline{r} - c_0\Big)\Big] - \lambda p(h - \underline{r})\Big]\Big\}}_{\triangleq g_1(p,q,n)}, \\
\dot{q} &= q\Big(1 - q - \lambda p\underline{r}\Big) + bp\underline{r} + \underbrace{(-q\lambda p + b)(h - \underline{r})}_{\triangleq g_2(p,q,n)}.
\end{aligned}$$

Since h is a smooth function of n and D is compact, provided n is sufficiently small, we have an estimate

$$|g_1(p,q,n)| + |g_2(p,q,n)| \leq C_0 n, \quad \text{where } C_0 \text{ does not depend on } p,\ q, \text{ and } n.$$

Therefore

$$(\dot{p},\dot{q}) \cdot (-\underline{q}, -\underline{p}) \geq \delta_0 + C_0' n \quad \text{for another uniform constant } C_0'.$$

Take n sufficiently small so that the last expression is positive. After having the orbit confined in the positive invariant set T, we further conduct the phase space analysis to capture the heteroclinic orbit but is omitted here.

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