

This document is intended to serve as personal notes for self-study of *Monte Carlo Methods for Statistical Physics* by M.E.J. Newman and G.T. Barkema.

# 1 Introduction

## 1.1 Statistical Mechanics

Many systems of interest are composed of very many simple constituent systems. For example, a liter of oxygen is composed of  $10^{22}$  oxygen molecules. The equations of motion for a single oxygen molecule are relatively simple but the sheer number of oxygen molecules in the larger system render the exact solutions infeasible. Instead, we approach the larger system via probabilistic methods. The goal is to express the state of the larger system as a set of probabilities of being in one state or another.

The typical systems studied in physics for which MCM are useful are systems described by a Hamiltonian  $H$  and an associated energy spectrum that is either continuous or given by a discrete set  $E_0, E_1, E_2$ , etc. We will also consider systems that interact with a thermal reservoir with which the system can exchange heat. Heat exchanges with the reservoir manifest as a small (negligible) perturbation in the Hamiltonian which pushes the system from one energy state to another.

Modeling the effect of the perturbation on the Hamiltonian is done by defining a *dynamics* for the system. The dynamics of the system may take the form of a set of transition rates. Suppose the system is in a state  $\mu$ . Then,  $R(\mu \rightarrow \nu)dt$  is defined to be the probability that the system transitions to a state  $\nu$  in time  $dt$ , where  $R(\mu \rightarrow \nu)$  is defined to be the *transition rate*. We may also define a set of weights  $w_\mu(t)$  which represent that probability that the system will be found in state  $\mu$  at time  $t$ . Using these elements, we can write down a *master equation*:

$$\frac{dw_\mu}{dt} = \sum_\nu [w_\nu(t)R(\nu \rightarrow \mu) - w_\mu(t)R(\mu \rightarrow \nu)] \quad (1)$$

The first term in the sum is the rate at which the system is transitioning into state  $\mu$ . The second term is the rate at which the system is transitioning out of state  $\mu$ .

We may define the expectation value of some observable  $Q$ , which takes the value  $Q_\mu$  in state  $\mu$ , as

$$\langle Q \rangle = \sum_\mu Q_\mu w_\mu(t)$$

The expectation value of  $Q$  can be interpreted two ways:

- It is the mean of the observed value of  $Q$  that would be obtained if we measured many identically prepared systems simultaneously.
- It is the time average of the quantity  $Q$ . That is, it might be the average observed value of  $Q$  after many measurements over time.

## 1.2 Equilibrium

An *equilibrium* state can be defined as a state in which all of the rates of change  $dw_\mu/dt$  vanish, and therefore the transition rate into and out of each state  $\mu$  must be equal. All systems governed by the master equation must come to equilibrium after some time and so we can define the equilibrium values of the weights, the *equilibrium occupation probabilities*, as

$$p_\mu = \lim_{t \rightarrow \infty} w_\mu(t)$$

For a system in equilibrium with thermal reservoir at temperature  $T$ , the Boltzmann distribution is

$$p_\mu = \frac{1}{Z} e^{-E_\mu/kT} = \frac{1}{Z} e^{-\beta E_\mu}$$

where  $Z$  is the normalization constant, called the *partition function*, and defined as

$$Z = \sum_{\mu} e^{-\beta E_\mu}$$

Plugging this into our definition of expectation values gives us

$$\langle Q \rangle = \sum_{\mu} Q_\mu p_\mu(t) = \frac{1}{Z} \sum_{\mu} Q_\mu e^{-\beta E_\mu}$$

which is the expression for the expectation value of a system in equilibrium. For example, the internal energy  $U$  of a system is given by

$$U = \langle E \rangle = \frac{1}{Z} \sum_{\mu} E_\mu e^{-\beta E_\mu} = -\frac{1}{Z} \frac{\partial Z}{\partial \beta} = -\frac{\partial \log Z}{\partial \beta}$$

It is sometimes appropriate in MCM calculations to calculate the partition function, from which many other quantities can be derived. Some examples can be found in Section 1.2 of Newman and Barkema.

## 1.3 Fluctuations, Correlations, and Responses

It is useful to assess the fluctuations in observable quantities. Doing so allows us to determine how the quantity we are measuring varies over time and therefore how much of an approximation is being made by calculating the expectation value. As an example, the root-mean-square fluctuation in internal energy is given by

$$\sqrt{\langle E^2 \rangle - \langle E \rangle^2} = \sqrt{\frac{\partial^2 \log Z}{\partial \beta^2}}$$

In the limit of a large system, called the *thermodynamic limit*, it is common for fluctuations to become negligible. For example, RMS fluctuations in internal energy scale as  $\sqrt{V}$  in a system of volume  $V$ , but the internal energy itself scales

as  $V$  so as the system becomes very large, the fluctuations grow increasingly negligible.

Every parameter that can be fixed in a system (e.g. volume, external field) has a conjugate variable (e.g. pressure, magnetization) that is given by a derivative of the free energy. For example,

$$p = -\frac{\partial F}{\partial V}$$

$$M = \frac{\partial F}{\partial B}$$

The Hamiltonian contains terms of the form  $-XY$  where  $Y$  is a "field" whose value is fixed, and  $X$  is the conjugate variable to which it couples. A common trick to calculate the thermal average of a quantity is the following:

- Make up a fictitious field that couples to the quantity of interest.
- Add an appropriate term to the Hamiltonian.
- Set the field to zero after performing the derivative on the free energy.

The *susceptibility* of  $X$  to  $Y$  is a measurement of the strength of the response of  $X$  to variation in  $Y$ . Typically, this quantity is denoted by  $\chi$  and can be defined as

$$\chi = \frac{\partial \langle X \rangle}{\partial Y}$$

The *linear response theorem* tell us that the fluctuations in a variable are proportional to the susceptibility of the variable to its conjugate field.

### 1.3.1 Correlation Functions

The *disconnected correlation function*  $G^{(2)}(i, j)$  is defined as

$$G^{(2)}(i, j) = \langle x_i x_j \rangle$$

This function gives us roughly an idea of how the variables  $x_i$  and  $x_j$  are correlated. Consider the following scenarios:

- If  $x_i$  and  $x_j$  move (roughly) together most of the time around the origin zero, then the average of their product will be a positive value.
- If  $x_i$  and  $x_j$  move (roughly) together but in opposite directions most of the time, then the average of their product will be a negative value.
- If  $x_i$  and  $x_j$  move with no discernible pattern relative to one another, then their product will be sometimes positive and sometimes negative, averaging to zero over a long time.

This definition can be problematic in the case of a *spontaneously broken symmetry state*. This occurs when a variable  $x$  develops a non-zero expectation value. If this happens,  $x_i$  and  $x_j$  may, for instance, always be large positive values leading to a large positive correlation value even if they have nothing to do with one another. To address this problem, we can use the *two-point connected correlation function*  $G_c^{(2)}(i, j)$  defined as

$$\begin{aligned} G_c^{(2)}(i, j) &= \langle x_i x_j \rangle - \langle x_i \rangle \langle x_j \rangle \\ &= \langle (x_i - \langle x_i \rangle) \times (x_j - \langle x_j \rangle) \rangle \end{aligned}$$

For reference, the higher-order connected correlation functions for three and four sites can be found in section 1.2.1.

## 1.4 Example: The Ising Model

The two-dimensional Ising model of a magnet consists of a lattice of sites, where each site contains a magnetic dipole or spin. Each spin is a two-level system denoted by the variable  $s_i$ , which can only take on the values  $\pm 1$ . Exchange interactions are modeled by including terms proportional to  $s_i s_j$  in the Hamiltonian. In the simplest case, these terms are identical for each pair of spins (indicating the same interaction strength  $J$  all over the lattice). Another simplification is to consider only nearest neighbor interactions. An external magnetic field  $B$  can also be introduced. In this case we obtain the following Hamiltonian

$$H = -J \sum_{\langle ij \rangle} s_i s_j - B \sum_i s_i$$

where  $\langle ij \rangle$  indicates a sum over nearest neighbors. The minus signs are conventional and fix the signs of the parameters  $J$  and  $B$ . In the current form:

- A positive  $J$  value (*ferromagnetic*) makes the spins want to align to minimize the Hamiltonian.
- A negative  $J$  value (*anti-ferromagnetic*) makes the spins want to anti-align.
- A positive  $B$  value makes the spins want to orient positively.
- A negative  $B$  value makes the spins want to orient negatively - that is, they want to align with the magnetic field.

Each state of the Ising system is an specific "setting" of the spins in the lattice. For  $N$  sites in the lattice, there are  $2^N$  states. Earlier we defined the partition function  $Z$  as

$$Z = \sum_{\mu} e^{-\beta E_{\mu}}$$

where  $\{\mu\}$  represents a set of states. So our Ising partition function is given by

$$Z = \sum_{s_1=\pm 1} \sum_{s_2=\pm 1} \dots \sum_{s_N=\pm 1} e^{-\beta H} = \sum_{\{s_i\}} e^{-\beta H}$$

From the partition function, we can calculate the internal energy, entropy, free energy, specific heat, and so on (see section 1.2). The mean magnetization  $\langle M \rangle$  is given by

$$\langle M \rangle = \left\langle \sum_i s_i \right\rangle$$

and the mean magnetization per spin  $\langle m \rangle$  is just

$$\langle m \rangle = \frac{1}{N} \langle M \rangle$$

The magnetic susceptibility is calculated as

$$\frac{\partial \langle M \rangle}{\partial B} = \beta (\langle M^2 \rangle - \langle M \rangle^2)$$

and the magnetic susceptibility per spin is calculated as

$$\chi = \frac{1}{N} \frac{\partial \langle M \rangle}{\partial B}$$

The specific heat is given by

$$c = \frac{k\beta^2}{N} (\langle E^2 \rangle - \langle E \rangle^2)$$

If we introduce a spatially-varying magnetic field by subscripting  $B$ , our Hamiltonian becomes

$$H = -J \sum_{\langle ij \rangle} s_i s_j - \sum_i B_i s_i$$

and the mean magnetization per site becomes

$$\langle m_i \rangle = \langle s_i \rangle = \frac{1}{\beta} \frac{\partial \log Z}{\partial B_i}$$

which allows us to write down the connected correlation function

$$G_c^{(2)}(i, j) = \frac{1}{\beta^2} \frac{\partial^2 \log Z}{\partial B_i \partial B_j}$$

## 1.5 Numerical Methods

There are several computational methods that can be used to solve for quantities of interest in a given model. The most straightforward method involves mapping the model of interest to a lattice of finite size so that the partition function is exactly calculable by computer.

### 1.5.1 Example: 2D Ising Model

We start with a two-dimensional lattice of 25 spins, arranged in a  $5 \times 5$  grid. We assume periodic boundary conditions such that spins at one end of the lattice experience interactions with spins on the other end. We also start with no external magnetic field ( $B = 0$ ).

There are  $2^{25}$  possible configurations, or states. However, we can utilize the symmetry of the system to argue that every state has a reflective partner where every spin is flipped. If  $B = 0$ , this state must have the same energy as its reflective partner. Therefore, we only need to consider one of each pair in the calculation of our partition function and then double the sum.

To calculate the mean magnetization per spin, we can use

$$\langle m \rangle = \frac{1}{N} \langle M \rangle = \frac{1}{N} \left\langle \sum_i s_i \right\rangle$$

and to calculate the specific heat per spin we use

$$C = \frac{\partial U}{\partial T} = -k\beta^2 \frac{\partial U}{\partial \beta} = k\beta^2 \frac{\partial^2 \log Z}{\partial \beta^2}$$

To do: calculate partition function, specific heat and mean magnetization and plot last two. Compare against closed form equations by Onsager. The key conclusion is that the physics is not well modeled on a finite lattice around the critical point ( $kT = 2.3J$ ), but we still get fairly accurate results away from this point. It is still desirable to study the largest finite-sized system possible to most accurately reason about the properties of the system in the thermodynamic limit. However, brute force calculations scale rapidly in time with the lattice size. We must find other ways to calculate quantities of interest on a large lattice size.

### 1.5.2 Monte Carlo Methods

To calculate the partition function of a larger lattice for the Ising model, we turn to Monte Carlo simulation. The fundamental idea is to simulate the thermal fluctuations from state to state - in accordance with a set of weights  $w_\mu(t)$  as they would be in the real system. We then take a time average of the quantity of interest, which is regarded as an expectation value of that quantity.

For this approach to work, we must find an accurate dynamics for the system that will govern its time evolution. That is, we must determine the master equation of the system. We do this by

- Choosing a set of rates  $R(\mu \rightarrow \nu)$  for the transitions.
- Forcing the equilibrium solution for the master equation to be the Boltzmann distribution.
- Use this set of rates to time-evolve the simulated system.

- Estimate the observable quantity in which we're interested.

This approach has the benefit that we do not need to recreate every state to get an accurate solution. We only need to time-evolve through a small fraction of states. However, there are statistical errors that arise from not recreating every state (these can be reduced by lengthening the simulation time). For this reason, it is not wise to calculate the partition function and use this to calculate other values. Instead, we should calculate quantities of interest directly (to reduce the noise of the result).

## 1.6 Problems

- **1.1** "If a system is in equilibrium with a thermal reservoir at temperature  $T$ , the probability of its having a total energy  $E$  varies with  $E$  in proportion to  $e^{-\beta E}$ ." True or False?

False. If a system is in equilibrium with a thermal reservoir of temperature  $T$ , the probability of its having a total energy  $E$  is given by its probability of being in any state with characteristic energy  $E$ . This can be written as

$$P_E = \frac{1}{Z} \sum_{E_\mu=E} e^{-\beta E_\mu}$$

The partition function factor  $1/Z$  is found in all  $P_E$  values for any given  $E$ . We can also note that, by definition, all  $E_\mu$  values in the sum of exponentials are simply  $E$ . Therefore, if we know the degeneracy of the energy value  $E$  (denoted  $g_E$ ), then the probability of the system having total energy  $E$  is given by

$$P_E = \frac{1}{Z} g_E e^{-\beta E} \propto g_E e^{-\beta E}$$

- **1.2** A certain simple system has only two energy states, with energies  $E_0$  and  $E_1$ , and transitions between the two states take place at rates  $R(0 \rightarrow 1) = R_0 \exp[-\beta(E_1 - E_0)]$  and  $R(1 \rightarrow 0) = R_0$ . Solve the master equation for the probabilities  $w_0$  and  $w_1$  of occupation of the two states as a functions of time with the initial conditions  $w_0 = 0$ ,  $w_1 = 1$ . Show that as  $t \rightarrow \infty$  these solutions tend to the Boltzmann probabilities.

The general form of the master equation is

$$\frac{dw_\mu}{dt} = \sum_\nu [w_\nu(t)R(\nu \rightarrow \mu) - w_\mu(t)R(\mu \rightarrow \nu)]$$

For our two state system, we obtain two coupled equations:

$$\begin{aligned} \frac{dw_0}{dt} &= w_1(t)R(1 \rightarrow 0) - w_0(t)R(0 \rightarrow 1) \\ &= w_1(t)R_0 - w_0(t)R_0 \exp[-\beta(E_1 - E_0)] \end{aligned} \tag{2}$$

and

$$\begin{aligned}\frac{dw_1}{dt} &= w_0(t)R(0 \rightarrow 1) - w_1(t)R(1 \rightarrow 0) \\ &= w_0(t)R_0 \exp[-\beta(E_1 - E_0)] - w_1(t)R_0\end{aligned}\tag{3}$$

which we can write as

$$\begin{bmatrix} \dot{w}_0 \\ \dot{w}_1 \end{bmatrix} = \begin{bmatrix} -R_0 \exp[-\beta(E_1 - E_0)] & R_0 \\ R_0 \exp[-\beta(E_1 - E_0)] & -R_0 \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix}$$

Solving for  $w_0(t)$  and  $w_1(t)$  involves solving the eigenvalue problem for the  $2 \times 2$  matrix above. The characteristic equation is

$$\begin{aligned}\begin{vmatrix} -R_0 \exp[-\beta(E_1 - E_0)] - \lambda & R_0 \\ R_0 \exp[-\beta(E_1 - E_0)] & -R_0 - \lambda \end{vmatrix} &= 0 \\ (-R_0 \exp[-\beta(E_1 - E_0)] - \lambda)(-R_0 - \lambda) - R_0^2 \exp[-\beta(E_1 - E_0)] &= 0 \\ \lambda(R_0 \exp[-\beta(E_1 - E_0)] + R_0) + \lambda^2 &= 0\end{aligned}$$

Our two eigenvalues are  $\lambda = 0$  and  $\lambda = -R_0(\exp[-\beta(E_1 - E_0)] + 1)$ . For simplicity, we will temporarily set  $\alpha = \exp[-\beta(E_1 - E_0)]$ . The associated eigenvectors are

$$C_1 \begin{bmatrix} 1 \\ \alpha \end{bmatrix}$$

and

$$C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Where  $C_1$  and  $C_2$  are arbitrary constants. Our solutions are therefore given by

$$\begin{aligned}w_0(t) &= C_1 + C_2 e^{-R_0(\alpha+1)t} \\ w_1(t) &= \alpha C_1 - C_2 e^{-R_0(\alpha+1)t}\end{aligned}$$

Using the initial condition  $w_0(0) = 0$ , we discover  $C_1 = -C_2$ . Then, using the initial condition  $w_1(0) = 1$ , we find that  $C_1 = 1/(\alpha + 1)$ . Putting all of this together, we obtain the final solutions

$$\begin{aligned}w_0(t) &= \frac{1 - e^{-R_0(\alpha+1)t}}{\alpha + 1} \\ w_1(t) &= \frac{\alpha + e^{-R_0(\alpha+1)t}}{\alpha + 1}\end{aligned}$$

In the limit  $t \rightarrow \infty$ ,

$$\begin{aligned}w_0(t) &= \frac{1}{\alpha + 1} \\ &= \frac{1}{e^{-\beta(E_1 - E_0)} + 1} \\ &= \frac{e^{-\beta E_0}}{e^{-\beta E_1} + e^{-\beta E_0}} \\ &= \frac{e^{-\beta E_0}}{Z}\end{aligned}$$



$$\begin{aligned}
w_1(t) &= \frac{\alpha}{\alpha + 1} \\
&= \frac{e^{-\beta(E_1 - E_0)}}{e^{-\beta(E_1 - E_0)} + 1} \\
&= \frac{e^{-\beta E_1}}{e^{-\beta E_1} + e^{-\beta E_0}} \\
&= \frac{e^{-\beta E_1}}{Z}
\end{aligned}$$

So, we can see that in the limit  $t \rightarrow \infty$ , we recover the Boltzmann distribution.

- **1.3** A slightly more complex system contains  $N$  distinguishable particles, each of which can be in one of two boxes. The particles in the first box have energy  $E_0 = 0$  and the particles in the second have energy  $E_1$ , and particles are allowed to move back and forward between the boxes under the influence of thermal excitations from a reservoir at temperature  $T$ . Find the partition function for this system and then use this result to calculate the internal energy.

The partition function is given by

$$Z = \sum_{\mu} e^{-\beta E_{\mu}}$$

where  $\{\mu\}$  represents the set of possible states. For a system of  $N$  particles, each of which can be in one of two possible states, we have  $2^N$  possible system configurations. Since  $E_0 = 0$ ,  $E_{\mu}$  is simply given by  $g_1 E_1$  where  $g_1$  is the number of particles in the second box. There is one way to place 0 particles ( $E_{\mu} = 0$ ) in the second box. There are  $N$  ways to place 1 particle ( $E_{\mu} = E_1$ ) in the second box. There are  $N(N-1)/2$  ways to place 2 particles ( $E_{\mu} = 2E_1$ ) in the second box. Continuing this pattern, we find that the partition function is given by

$$Z = \sum_{n=0}^N \binom{N}{n} e^{-\beta n E_1}$$

where the binomial coefficient determines the degeneracy of energy.

- **1.4** Solve the Ising model in one dimension for the case  $B = 0$  as follows. Define a new set of variables  $\sigma_i$  which take values 0 and 1 according to  $\sigma_i = \frac{1}{2}(1 - s_i s_{i+1})$  and rewrite the Hamiltonian in terms of these variables for a system of  $N$  spins with periodic boundary conditions. Show that the resulting system is equivalent to the one studied in Problem 1.3 in the limit of large  $N$  and hence calculate the internal energy as a function of temperature.