KTH Royal Institute of Technology

SF2822: Applied Nonlinear Optimization

Project 2A2 - London traffic modeling

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1 Abstract

In this report the traffic in the London area and its suburbs is modeled. The model is represented as a network consisting of nodes that each represents an area. There are links between most of the nodes to which a traveling time is assigned. As a result, the problem is a routing problem of different commuters with the aim to minimize the total traveling time. Two different scenarios are considered in the model; with and without congestion.

This problem is implemented in GAMS as a linear and nonlinear problem, respectively for the no congestion scenario and for the congestion scenario. The optimal solution for the linear problem is determined using the CPLEX solver, whereas the SNOPT solver is used for the nonlinear problem. Due to the convexity of the nonlinear optimization problem, it can be concluded that a local optimum is also a global optimum.

The optimal solution from the linear problem shows that the shortest path is used for each commuter. In the nonlinear problem paths are chosen using the same objective function, but in some cases another path is chosen due to the high level of congestion that exists on the shortest paths. Analyzing the different levels of congestion leads to the conclusion that the capacity for some paths can be increased by adding another lane.

The model could be improved by including the dynamic behavior of the system, defining the amount of cars as an integer instead of a continuous variable and for instance by using a different objective function, i.e. minimizing the maximum level of congestion.

2 Problem description and background information

In this assignment the traffic modeling of an urban region consisting of multiple areas is considered. In this particular case, the city of London with its surrounding suburbs is used as a guideline to determine realistic distances and travel times between each area. The urban region is modeled as a network with each area representing a node in the network and each road between the areas defined as an arc between two nodes. An arbitrary amount of areas surrounding the center of London is chosen in order to construct an initial linear program that minimizes the total amount of travel time between the areas. A travel time between each of the areas where a route exists is determined based on geographical data. Initially, it is assumed that there is no congestion on the roads and that the speed limits are the same, implying that the travel time between each area is equal in both directions. The different areas and corresponding existing routes between them are visible in the image below.

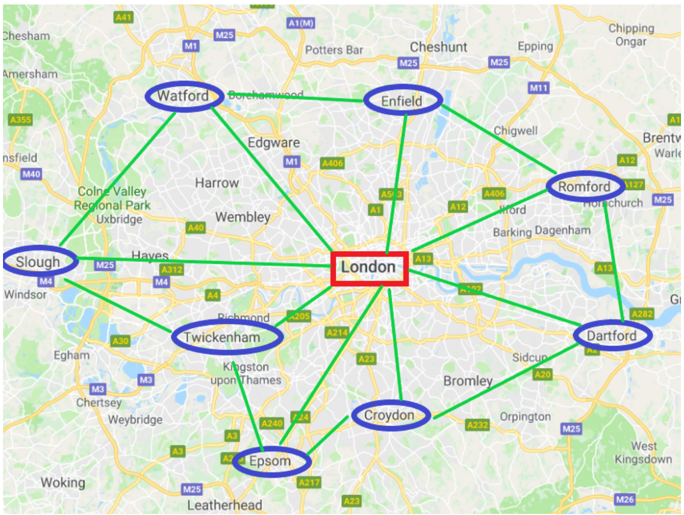

Figure 1: London area and its suburbs. In green the existing links - accessible in both directions- are represented. In red there is London- downtown and finally in blue there are all the suburbs.

The model covers links that are highways in reality, so each link is assumed to have two lanes for a given direction. A significant number of the commuters want to travel from a suburb to the center but there are also commuters that want to go to another suburb, and it is assumed that all commuters travel by car.

The following sections explain the modeling of the original problem (without congestion) as well as a more realistic model including congested traffic during peak times. Reasonable values of the capacity on each route as well as the increased travel times are defined based on real data. Afterwards, it is determined whether the solution of this problem can be considered a global optimum and alterations are implemented in order to increase the capacity of certain roads. Finally, potential model improvements are made in order to obtain a more realistic solution in the future.

3 Mathematical formulation

3.1 Original problem: no-congestion

The original problem refers to the no-congestion scenario. It results to be a commuters routing problem; there is a demand of commuters between different areas. As stated above, it is assumed that most people move from the surrounding suburbs towards London downtown.

Definitions

The sets are:

- $\mathcal{I} = \{1, \cdots 9\}$ where i corresponds to the i-th area,
- $\mathcal{J} = \{1, \cdots 9\}$ where j corresponds to the j-th area,
- $\mathcal{K} = \{1, \cdots 9\}$ where k corresponds to the k-th area.

The parameters are:

Symbol	: **Definition**
$d_{i,j}$	: the amount of cars that want to move from the area i to the area j,
$t_{i,j}$	: the time needed to travel the road (i, j).

The variables are:

Symbol	: **Definition**
$x_{i,j}$	: the amount of cars traveling on the road (i, j),
$w_{i,j,k}$	: the amount of cars traveling on the road (j, k) and that started in the area i,
z	: the sum of all the travel times.

The constraints are:

- The right amount of cars, starting from the area i, has to be launched on the road:
$$\sum_{j\in\mathcal{J}} d_{i,j} = \sum_{j\in\mathcal{J}} w_{i,i,j} \qquad \forall i \in \mathcal{I}.$$
- Flow-balance of the cars for each area:
$$d_{i,j} = \sum_{k\in\mathcal{K}} (w_{i,k,j} - w_{i,j,k}) \qquad \forall (i, j) \in (\mathcal{I}, \mathcal{J}) \text{ with } i \neq j.$$
- The amount of people using the road (i,j) is given by :
$$x_{i,j} = \sum_{k\in\mathcal{K}} w_{k,i,j} \qquad \forall (i, j) \in (\mathcal{I}, \mathcal{J}).$$

The objective function represents the sum of all travel times over the existing links:

$$z = \sum_{i\in\mathcal{I}} \sum_{i\in\mathcal{J}} t_{i,j} x_{i,j}$$

To summarize

$$\min_{w_{i,j,k}} \sum_{i\in\mathcal{I}} \sum_{i\in\mathcal{J}} t_{i,j} x_{i,j}$$

subject to:

$$\begin{aligned}
&\sum_{j\in\mathcal{J}} d_{i,j} = \sum_{j\in\mathcal{J}} w_{i,i,j} && \forall i \in \mathcal{I},\\
&d_{i,j} = \sum_{k\in\mathcal{K}} (w_{i,k,j} - w_{i,j,k}) && \forall (i,j) \in (\mathcal{I},\mathcal{J}) \text{ with } i \neq j,\\
&x_{i,j} = \sum_{k\in\mathcal{K}} w_{k,i,j} && \forall (i,j) \in (\mathcal{I},\mathcal{J}),\\
&x_{i,j} \geq 0 && \forall (i,j) \in (\mathcal{I},\mathcal{J}),\\
&w_{i,j,k} \geq 0 && \forall (i,j,k) \in (\mathcal{I},\mathcal{J},\mathcal{K}).
\end{aligned}$$

3.2 Advanced problem: with congestion

The advanced problem refers to the scenario with congestion.

Definitions

The sets are the same.

Two parameters are added:

Symbol	: **Definition**
$u_{i,j}$	: the maximum capacity that the road (i,j) can welcome,
$a_{i,j}$	: the coefficient of congestion for the road (i,j).

The variables are the same.

One constraint is added:

- Each road has a maximum capacity:

$$x_{i,j} \leq u_{i,j} \qquad \forall (i,j) \in (\mathcal{I},\mathcal{J}).$$

The objective function represents the sum of all travel times over the existing links:

$$z = \sum_{i\in\mathcal{I}} \sum_{i\in\mathcal{J}} x_{i,j} \left(t_{i,j} + \frac{a_{i,j} x_{i,j}}{u_{i,j} - x_{i,j}} \right)$$

To summarize

$$\min_{w_{i,j,k}} \sum_{i\in\mathcal{I}} \sum_{i\in\mathcal{J}} x_{i,j}\left(t_{i,j} + \frac{a_{i,j}x_{i,j}}{u_{i,j} - x_{i,j}}\right)$$

subject to:

$$
\begin{aligned}
&\sum_{j\in\mathcal{J}} d_{i,j} = \sum_{j\in\mathcal{J}} w_{i,i,j} && \forall i \in \mathcal{I},\\
&d_{i,j} = \sum_{k\in\mathcal{K}} (w_{i,k,j} - w_{i,j,k}) && \forall (i,j) \in (\mathcal{I},\mathcal{J}) \text{ with } i \neq j,\\
&x_{i,j} = \sum_{k\in\mathcal{K}} w_{k,i,j} && \forall (i,j) \in (\mathcal{I},\mathcal{J}),\\
&x_{i,j} \leq u_{i,j} && \forall (i,j) \in (\mathcal{I},\mathcal{J}),\\
&x_{i,j} \geq 0 && \forall (i,j) \in (\mathcal{I},\mathcal{J}),\\
&w_{i,j,k} \geq 0 && \forall (i,j,k) \in (\mathcal{I},\mathcal{J},\mathcal{K}).
\end{aligned}
$$

3.3 Convexity investigation

The convexity is studied for the congested scenario, since the non-congested scenario is a linear problem, hence intrinsically convex.

When the problem is nonlinear its convexity must be investigated to understand the nature of solution, specifically if the equivalence between local optimum and global optimum holds.

The problem in order to be convex, this must hold:

- objective function is convex;

- the set of constraints is convex.

In this case the constraints are almost all equalities, as consequence the set in order to be convex requires the constraints to be affine, which is the case. The only constraint which is not an equality is the upper bound for $x_{i,j}$. It results to be a linear constraint.Hence, the second bullet is valid.

The objective function is:

$$z = \sum_{i=1}^{m} \sum_{j=1}^{m} x_{i,j}\left(t_{i,j} + \frac{a_{i,j}x_{i,j}}{u_{i,j} - x_{i,j}}\right)$$

To prove its convexity, the theorem which states that the sum of convex functions is still a convex function is used. As a result, the convexity is here investigated only for a given pair of (i, j).

First of all, the objective function addend-wise is rewritten, defining y as the (i-th, j-th) addend

$$y = x_{i,j}t_{i,j} + \frac{a_{i,j}x_{i,j}^2}{u_{i,j} - x_{i,j}}$$

Taking the first derivative of y it holds:

$$\frac{dy}{dx_{i,j}} = t_{i,j} + \frac{2x_{i,j}a_{i,j}(u_{i,j} - x_{i,j}) + a_{i,j}x_{i,j}^2}{(u_{i,j} - x_{i,j})^2}$$

Rearranging it:

$$\frac{dy}{dx_{i,j}} = t_{i,j} + \frac{2a_{i,j}u_{i,j}x_{i,j} - a_{i,j}x_{i,j}^2}{(u_{i,j} - x_{i,j})^2}$$

Taking the second derivative of z it holds:

$$\frac{d^2y}{dx_{i,j}^2} = \frac{d}{dx_{i,j}}\left(\frac{dy}{dx_{i,j}}\right) = \frac{(u_{i,j} - x_{i,j})^2\,(2a_{i,j}u_{i,j} - 2a_{i,j}x_{i,j}) + 2(u_{i,j} - x_{i,j})\left(2a_{i,j}u_{i,j}x_{i,j} - a_{i,j}x_{i,j}^2\right)}{(u_{i,j} - x_{i,j})^4}$$

Grouping and simplifying $(u_{i,j} - x_{i,j})$:

$$\frac{d^2y}{dx_{i,j}^2} = \frac{\cancel{(u_{i,j} - x_{i,j})}\left[(u_{i,j} - x_{i,j})\,(2a_{i,j}u_{i,j} - 2a_{i,j}x_{i,j}) + 4a_{i,j}u_{i,j}x_{i,j} - 2a_{i,j}x_{i,j}^2\right]}{(u_{i,j} - x_{i,j})^{\cancel{4}\,3}}$$

Again, rearranging it:

$$\frac{d^2y}{dx_{i,j}^2} = \frac{2a_{i,j}u_{i,j}^2 - \cancel{2a_{i,j}u_{i,j}x_{i,j}} - \cancel{2a_{i,j}u_{i,j}x_{i,j}} + \cancel{2a_{i,j}x_{i,j}^2} + \cancel{4a_{i,j}u_{i,j}x_{i,j}} - \cancel{2a_{i,j}x_{i,j}^2}}{(u_{i,j} - x_{i,j})^3}$$

Finally, it results in:

$$\frac{d^2y}{dx_{i,j}^2} = \frac{2a_{i,j}u_{i,j}^2}{(u_{i,j} - x_{i,j})^3}$$

The numerator results to be always positive since it is the product of positive constants; the denominator is positive if $u_{i,j} > x_{i,j}$, which is the case since, by definition the number of commuters on the link (i,j) cannot exceed the upper bound $u_{i,j}$ ($x_{i,j} \leq u_{i,j}\ \forall i,j$).
Since the second derivative of y is in this problem always greater than zero it holds that also it is convex; furthermore, given the theorem declared above, it holds that the sum z of convex functions is convex too. This implies the first bullet point to be true.

Finally, the problem results to be convex. The one-to-one correspondence between local and global optimum is guaranteed.

4 Parameters and numerical values

4.1 Numerical values for parameters

Firstly, a representation of the geographical area with its links is given and modeled as a graph.

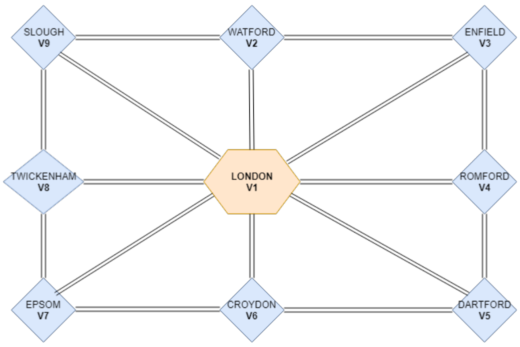

Figure 2: Areas links represented as graph

As the figure shows, each area as a node associated and the link is in both direction. Furthermore, as stated in *Problem description and problem background* each direction has two lanes. This will be taken into account in the capacity computation.

In order to define proper and realistic numerical values for all the parameters, a great amount of online research has been done. In particular, the population and economic activity of London and its suburbs are studied. For simplicity, the economic activity is classified as "high", "medium" or "low"; as a result, a suburb where the economic activity is high is expected to have more people moving towards it in the morning and more people leaving in the afternoon. An exception is done for London whose economic activity is regarded as "very high". Furthermore, the number of people moving/coming from/to a suburb or London is assumed to be proportional to the population.
These values are shown in the table below.

Area	**Area number**	**Node**	**Population**	**Economic activity**
London	i=1	v_1	8,787,892	very high
Watford	i=2	v_2	90,301	low
Enfield	i=3	v_3	331,400	low
Romford	i=4	v_4	95,894	high
Dartford	i=5	v_5	105,500	medium
Croydon	i=6	v_6	382,300	high
Epsom	i=7	v_7	41,489	low
Twickenham	i=8	v_8	52,396	low
Slough	i=9	v_9	161,055	high

Table 1: London and its suburbs: population and economic activity

As stated above, the demand between nodes is set accordingly to the population and the economic activity. Furthermore, to reduce the order of magnitude of demand, the following assumption is done: a demand of 1 from the i-th node to the j-th node - in reality - corresponds to a demand of 100. In addition to this, it is assumed there is no demand from the i-th node to itself.
The numerical values for the demand at the two peak times, 8:15 and 17:00 are given in table 2 and 3 below, respectively.

$\mathbf{d_{i,j}}$	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9
v_1	/	10			30	20			
v_2	150	/							40
v_3	30		/	90	20	30			
v_4	50			/					
v_5	92			80	/				22
v_6	50					/			
v_7	90			25		10	/		
v_8	50					10	3	/	10
v_9	40								/

Table 2: Demand between each pair of nodes at 8:15

$\mathbf{d_{i,j}}$	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9
v_1	/	150	30	50	92	50	90	50	40
v_2	10	/							
v_3			/						
v_4			90	/	80		25		
v_5	30		20		/				
v_6	20		30			/	10	10	
v_7							/	3	
v_8								/	
v_9		40			22			10	/

Table 3: Demand between each pair of nodes at 17:00

Moreover, the traveling times are defined accordingly to the online research on Google Maps when there is no congestion. These following values were calculated setting the departure time at 04:00. The matrix is assumed to be symmetric, hence when there is no congestion traveling from i to j takes the same time as traveling from j to i.

$\mathbf{t_{i,j}}$	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9
v_1	/	49	53	39	40	42	47	36	38
v_2	49	/	30						36
v_3	53	30	/	33					
v_4	39		33	/	35				
v_5	40			35	/	41			
v_6	42				41	/	25		
v_7	47					25	/	32	
v_8	36						32	/	33
v_9	38	36						33	/

Table 4: Traveling time between each pair of nodes - expressed in minutes

With respect to the capacity of each path, the upper limit $u_{i,j}$ is calculated as follows:

$$u_{i,j} = \left\lfloor \frac{2 \cdot (\text{length } i \to j)}{6.5} \right\rfloor$$

The different values for $u_{i,j}$ are determined using the distances between each of the nodes as well as the average space occupied by a vehicle. It is assumed that the average length of a vehicle is 4.5m and that a 2m safety distance is considered (1m in front and 1m at the back). It is assumed that each path consists of two lanes. The paths' lengths were found by online research on Google Maps (shown in *Appendix* A). The length from i to j is considered to be the same as the length from j to i.

$\mathbf{u_{i,j}}$	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9
v_1	/	100	74	81	96	55	92	56	118
v_2	100	/	94						119
v_3	74	94	/	117					
v_4	81		117	/	92				
v_5	96			92	/	111			
v_6	55				111	/	50		
v_7	92					50	/	50	
v_8	56						50	/	109
v_9	118	119						109	/

Table 5: Maximum capacity between each pair of nodes - expressed in number of vehicles

The travel time when the road is congested is given by:

$$t^*_{i,j} = t_{i,j} + a_{i,j} \frac{x_{i,j}}{u_{i,j} - x_{i,j}}$$

where $\frac{x_{i,j}}{u_{i,j}-x_{i,j}}$, which is ranging between 0 and $u_{i,j} - 1$, is a measure of the congestion. A way to find a suitable value of $a_{i,j}$ would be to isolate it in a particular case. The chosen case is when the road is fully congested. In that case the measure of the congestion is maximum and isolating $a_{i,j}$ gives:

$$\begin{aligned} a_{i,j} &= \frac{t_{i,j}^{\text{congested}} - t_{i,j}}{u_{i,j} - 1} \\ &= \frac{(1 + \text{increase}_{i,j}) \cdot t_{i,j} - t_{i,j}}{u_{i,j} - 1} \\ &= \text{increase}_{i,j} \frac{t_{i,j}}{u_{i,j} - 1} \end{aligned}$$

The different values for increase$_{i,j}$ are defined as the average values of the increase in travel times at different peak times: 08:15 in the morning and 17:00 in the afternoon. Since the amount of increased travel time depends highly on the direction of the traffic, the average values are divided into three categories: traffic towards London, traffic from London and traffic within the suburbs. Given in the table below are the average values for increase$_{i,j}$, based on estimated travel times using Google Maps.

Time	**Towards London**	**From London**	**Within Suburbs**
8:15	1.03	0.71	0.74
17:00	0.65	0.93	0.64

Table 6: Average increase in travel times for different peak times and directions

Implementing these values gives a coefficient of congestion $a_{i,j}$ for each connection between a pair of nodes. The calculated coefficients are presented in table 7 and 8 below for the corresponding peak times 8:15 and 17:00, respectively.

$\mathbf{a_{i,j}}$	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	$\mathbf{v_9}$
v_1	/	0.3514	0.5155	0.3461	0.2989	0.5522	0.3667	0.4647	0.2306
v_2	0.5098	/	0.2387						0.2337
v_3	0.7478	0.2387	/	0.2105					
v_4	0.5021		0.2105	/	0.2846				
v_5	0.4337			0.2846	/	0.2758			
v_6	0.8011				0.2758	/	0.3776		
v_7	0.5320					0.3776	/	0.4833	
v_8	0.6742						0.4833	/	0.2261
v_9	0.3345	0.2337						0.2261	/

Table 7: Coefficient of congestion between each pair of nodes at 8:15

$\mathbf{a_{i,j}}$	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9
v_1	/	0.4603	0.6752	0.4534	0.3916	0.7233	0.4803	0.6087	0.3021
v_2	0.3217	/	0.2065						0.2021
v_3	0.4719	0.2065	/	0.1821					
v_4	0.3169		0.1821	/	0.2462				
v_5	0.2737			0.2462	/	0.2385			
v_6	0.5056				0.2385	/	0.3265		
v_7	0.3357					0.3265	/	0.4180	
v_8	0.4255						0.4180	/	0.1956
v_9	0.2111	0.2021						0.1956	/

Table 8: Coefficient of congestion between each pair of nodes at 17:00

5 Results and analysis

5.1 Original problem: no-congestion

Implementing the original problem according to the given demand at time 8 : 15, as expected, the solution is such that for each demand the shortest path is chosen. This is due to the absence of any upper limit on the links and any further constraints; as a matter of fact, since the aim is to minimize the sum of all travel times the quickest paths are selected.

Given the numerical values shown in *Parameters* section, the total travel time is 43282 minutes, hence approximately 721 hours. The total number of moving cars is 952. Thus, each car in average takes approximately 45.5 minutes to go from its starting point to its destination.

The following table shows the number of cars on each link (i, j):

$\mathbf{x_{i,j}}$	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9
v_1	/	10		25	30	50			22
v_2	150	/							40
v_3	60		/	110					
v_4	50			/	20				
v_5	114			80	/				
v_6	50					/			
v_7	115					20	/		
v_8	50						13	/	10
v_9	40								/

Table 9: The amount of vehicles $x_{i,j}$ that use the path from i to j

Obviously when the direct link exists it is used, since in this model it happens that given two areas if the direct link exists it is shorter in time than any other paths. When there is not a direct path, as said before, the fastest path is selected. For example, the 20 cars demand between Enfield to Dartford ($3 \rightarrow 5$) goes through Romford, taking 68 minutes. If another path was chosen, such as passing through London instead, 93 were necessary.

A remark is, since no upper capacity limit on links exists, no demand flow needs to be dispatched, hence each demand flow is never split. This can be seen in the following figure:

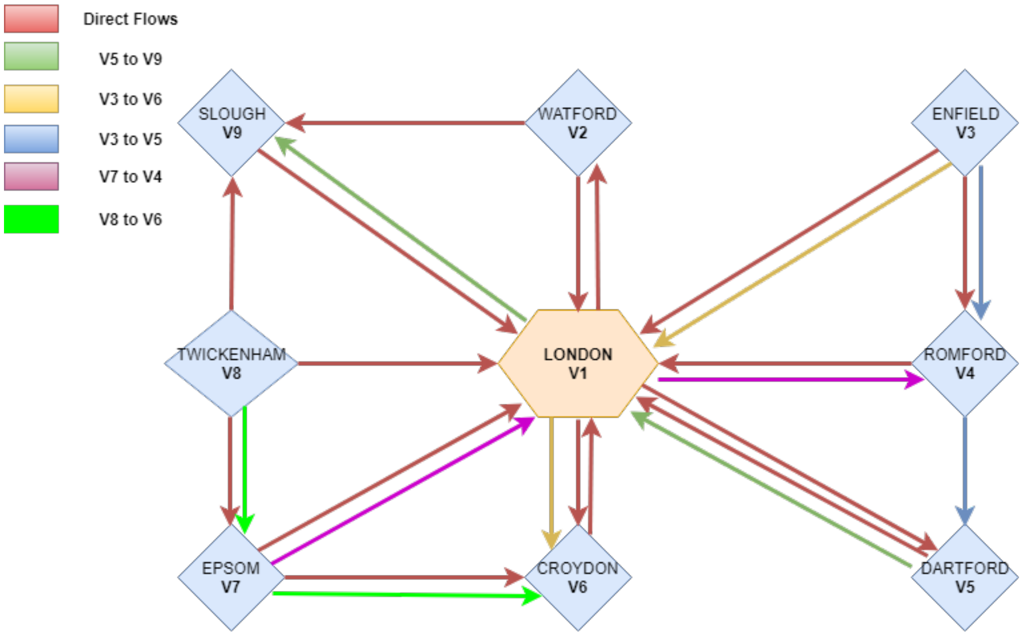

Figure 3: Demands' flows graphically illustrated.

The detailed flows can be read in the *Appendix* B.1.

5.2 Congestion scenario

5.2.1 Morning Traffic

Implementing the problem where congestion can occur and referring to the commuters' demand at 8:15, the cumulative total travel time is approximately 52417 minutes, hence approximately 873 hours. Thus, a commuter spends approximately 55 minutes in his/her car.This increase in time, if compared to the not congested scenario, is expected since congestion delays flows.

The table below shows the amount of vehicles that use each path, either to reach their final destination or to use it as a link to another destination.

$\mathbf{x_{i,j}}$	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9
v_1	/	10.0		18.4	30.0	45.6			
v_2	92.2	/	9.4						92.8
v_3	64.9	4.4	/	110.0					
v_4	50.9			/	20.0				
v_5	91.0			87.6	/	22			
v_6	50.4				6.6	/	22		
v_7	86.3					31.5	/	43.6	
v_8	51.1						17.9	/	52.4
v_9	108.8							4.4	/

Table 10: The amount of vehicles $x_{i,j}$ that use the path from i to j when morning congestion occurs

In this case, flows can now be dispatched - depending on capacity constraints and traffic condition. As a consequence of this, as can be noticed in the table, now- if compared to the not congested scenario - all the links are used to make the traffic flow as quick as possible.
Furthermore, some links happen to be overcrowded, meaning that they are close to be fully loaded. This can be observed clearly in the congestion graph reported below. The percentage of usage of each link is calculated as

$$\text{usage}_{\%}^{(i,j)} = \frac{x_{i,j}}{u_{i,j}}$$

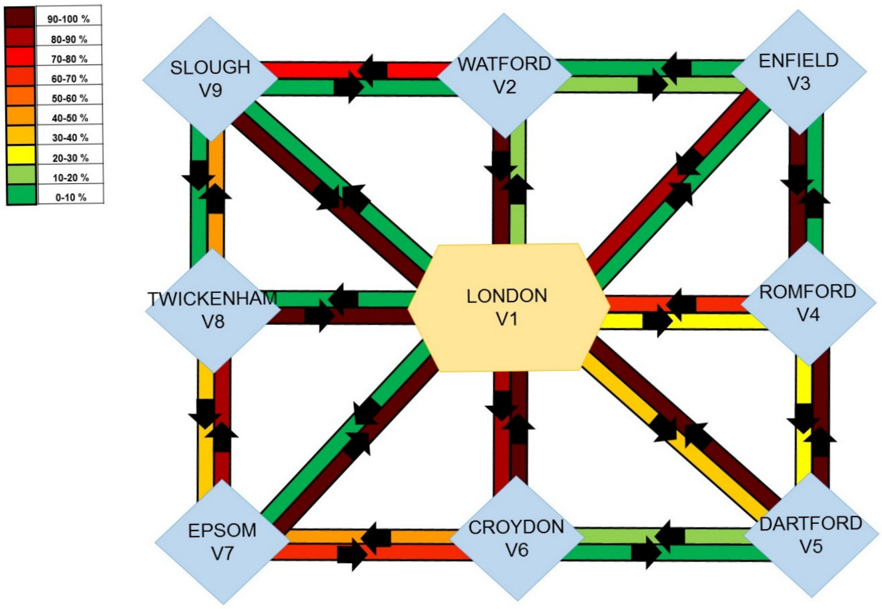

Figure 4: Level of congestion at 8:15 graphically illustrated

To better understand the graph the usage matrix is here reported:

$\mathbf{usage}_{\%}^{(\mathbf{i,j})}$	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9
v_1	/	10.00%		22.72 %	31.25 %	82.91%			
v_2	92.20 %	/	10.00%						77.98%
v_3	87.84%	4.68%	/	94.02%					
v_4	62.96%			/	21.74%				
v_5	94.79%			95.22%	/	19.82%			
v_6	91.82%				5.95%	/	44.00%		
v_7	93.80%					63.00%	/	87.20%	
v_8	91.43%						34.80 %	/	48.07%
v_9	92.20%							4.04%	/

Table 11: Usage percentage $usage_{\%}^{(i,j)}$ for each link at 8:15

As stated before, the flows' paths are different from the ones in the not congested scenario. This is clearly evident in the following graphs which depicts the flows:

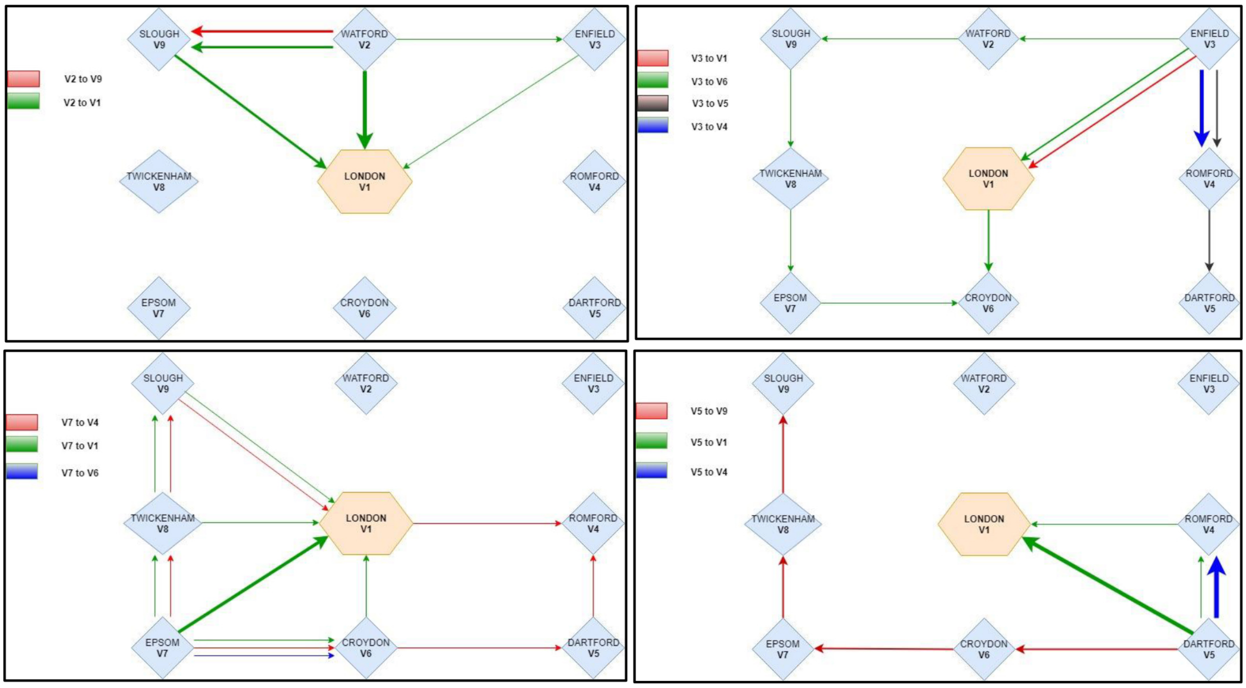

Figure 5: Flow from nodes 2, 3, 5 and 7 (clockwise starting upper left). The thicker arrows describe a more intense flow.

In the congestion case, most of the demand is still satisfied by the same link as the linear case. But for other links the congestion make it impossible or not optimal. Those modifications are depicted in the graphs above. Here are some important observations:

- The number of cars starting in V_2 that wants to join V_1 is important. The demand is so big (150 cars) that the cars starting from V_2 take three different ways: 61.5% go directly to V_1, 32.3% go there via V_9 and 6.3% go there via V_3.
- The shortest way for the 30 cars that wants to go to V_6 starting from V_3 would be to go there via V_1. But because those roads are really congested, 14.7% of the cars prefer to make a large detour. Without considering the congestion the detour takes 156 minutes instead of 95 minutes via the city-center!
- For the demand between V_5 and V_9, once again the city-center is avoided. The totality of the 22 cars prefers to take a detour. It highlights the importance of the delay due to the congestion; because if the congestion is disregarded taking the detour takes 131 minutes being 68% more than through the city-center.

The detailed flows can be read in the *Appendix* B.2.

5.2.2 Evening Traffic

Implementing the problem where congestion can occur and referring to the commuters' demand at 17:00, the cumulative travel time is approximately 53352 minutes, hence approximately 889 hours. Thus, a commuter spends approximately 56 minutes in his/her car.

The following table shows the number of cars on each link (i, j):

$\mathbf{x_{i,j}}$	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9
v_1	/	92.4	65.1	50.9	91.1	50.5	86.4	51.3	108.9
v_2	10.0	/	3.3						40.0
v_3		8.5	/						
v_4	18.2		110.1	/	87.7				
v_5	30.0			20.1	/	6.8			
v_6	46.0				22.1	/	30.6		
v_7						22.0	/	16.3	
v_8							43.2	/	3.3
v_9		92.3						51.9	/

Table 12: The amount of vehicles $x_{i,j}$ that use the path from i to j when evening congestion occurs

Similarly to what happens in the morning, flows are dispatched when congestion is too intense. Furthermore, if this new scenario is compared to the morning one, as expected, the congestion happens in the other direction. This is coherent to the model but also to reality, describing the commuters going back their hometown.

This can be observed in the following picture:

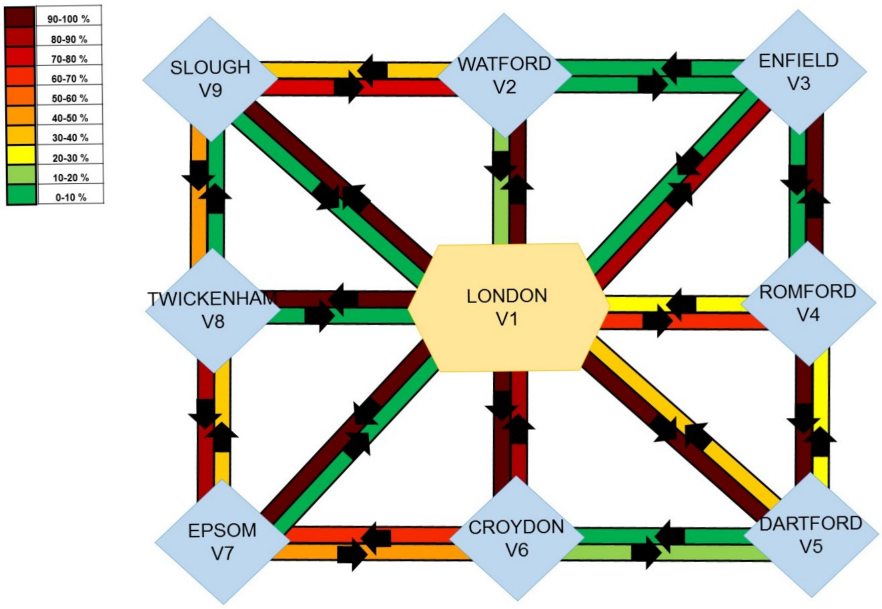

Figure 6: Level of congestion at 17:00 graphically illustrated

To better understand the graph the usage matrix is here reported:

$\mathbf{usage}_{\%}^{\mathbf{(i,j)}}$	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9
v_1	/	92.40%	97.97%	62.84%	94.90%	91.82%	93.91%	91.61%	92.37 %
v_2	10.00 %	/	3.51%						33.61%
v_3		9.04%	/						
v_4	22.47%		94.10%	/	95.33%				
v_5	31.25%			21.85%	/	6.13%			
v_6	84.73%				19.91%	/	61.20%		
v_7						44.00%	/	32.60%	
v_8							86.40%	/	3.03%
v_9		77.56%						47.61%	/

Table 13: Usage percentage $usage_{\%}^{(i,j)}$ for each link at 17:00

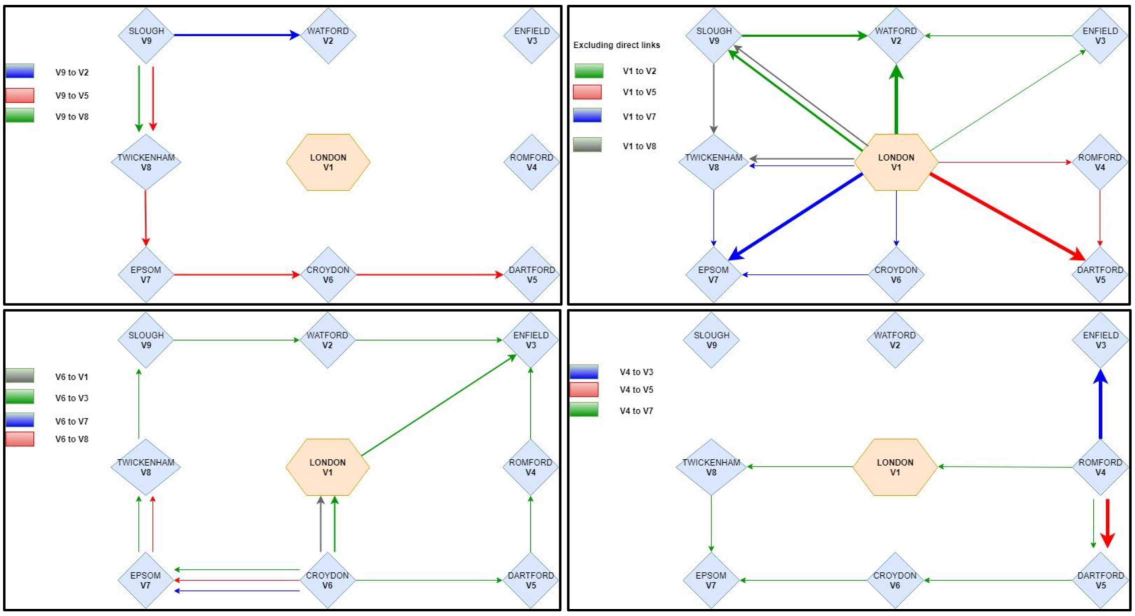

Figure 7: Flow from nodes 1, 4, 6 and 9 (clockwise starting upper left). The thicker arrows describe a more intense flow.

Most of the journeys are the same for the linear and the non-linear model, but as depicted above some are different. When the demand on one link is too high, different paths will be used for same departure-arrival. Like in the previous example, the city-center is avoided. Because of the high congestion in the city-center, it is usually better to take a long detour rather than going through the city-center.

The detailed flows can be read in the *Appendix* B.2.

5.3 What if the overcrowded links get their capacity increased?

Due to low capacity of link Watford to London ($2 \to 1$), it is assumed that a good choice would be to increase the capacity. This can be done by adding one extra lane in the direction $2 \to 1$. This would result in increasing by 50% the current total capacity $2 \to 1$. A decrease in cumulative travel time is expected since a less dispatched flow from $2 \to 1$ is expected. This occurs because more cars are allowed to travel through the direct flows.

As expected, a decrease in total time occurs. The total travel time is 50414 minutes. Furthermore, the link $x_{2,1} = 130.4$ which results in approximately 39 additional cars which are allowed to travel on that route. What must be remarked is that this relevant decrease occurs for two reasons: not only the low capacity limit but also the commuters' demand on that link exceeds the capacity.

Another relevant modification could be to increase the capacity, for instance add one extra lane, to the most loaded link: Dartford to Romford ($5 \to 4$). As before, this results in a 50% increase of the cumulative capacity of the link. Again, the expected output throughout the model is a decrease in the objective function.
As expected, the total travel time results to be 50304 minutes. Furthermore, the link $x_{5,4} = 116.9$, which means that approximately 20 cars more are allowed on the link.

The next and last modification is directly derived from the above ones: the two improvements together. Again, an important decrease is now strongly expected because two of the most crowded links' capacities are boosted.
As a matter of fact, the total travels time is 48934 minutes.

The following table summarizes these links' capacity modifications and it shows the percentage improvement in the objective function calculated with respect to the objective function of congested problem at 8:15, where the total time results in 52417 minutes.

Overcrowded link	**Original $\mathbf{u_{i,j}}$**	**Increased $\mathbf{u_{i,j}}$**	**Total travel time**	**% Improvement**
$x_{2,1}$	100	150	50414	3.82 %
$x_{5,4}$	92	138	50304	4.03 %
$x_{2,1}$ and $x_{5,4}$	100, 92	150, 138	48943	6.95 %

Table 14: Modifications summary table

5.4 Potential improvements

Firstly, the traffic modeling is strictly related to the queuing theory, as a result the traffic evolves as time passes. In this model this "time passing" is not taken into account. As a consequence, for example, the 150 commuters' demand from Watford to London is assumed to occur all at the same time, when in reality the situation can change within seconds. An example is that, at a given moment the link is congested and the queue is full, but as the first car in the queue leaves a new spot is available. Hence, a potential improvement could be take into account the different moments, given a time interval such as from 8:00 to 8:15. This would imply to keep track of the congestion at each time, knowing that some cars will leave at some time.

Another improvement would be with respect to the domain of both x and w. When it comes to moving cars, the problem should be formulated as an integer one. This is because it is not possible to route 4.2 cars in a link, it should be 4 cars at most. The problem with this variables' domain, as expected, returns an larger total travel time.It is approximately 53553 minutes, 892 hours. Thus, a car spends approximately 56.3 minutes to complete its journey. The increase is not that high and therefore the following conclusion is drawn: the non-integer problem gives a good lower bound for the integer one. Detailed values for both x and w can be found in the *Appendix* C. Note that this new model is run using MINLP solver and KNITRO option.

Another option would be to generate a model that minimizes a different objective function, e.g. the maximum level of congestion in the system. This comes from the common habit to prefer a longer but emptier route rather than being stuck in a traffic jam. By minimizing the level of congestion the overall traffic in the system will be more evenly distributed leading to a more balanced network.

The new implementation is similar to the old one. A new constraint is introduced and the objective function is altered. γ variable represents the maximum congestion.

$$\min_{w_{i,j,k}} \gamma$$

$$\begin{aligned}
&\text{subject to:} \\
&\sum_{j\in\mathcal{J}} d_{i,j} = \sum_{j\in\mathcal{J}} w_{i,i,j} && \forall i \in \mathcal{I}, \\
&d_{i,j} = \sum_{k\in\mathcal{K}} (w_{i,k,j} - w_{i,j,k}) && \forall (i,j) \in (\mathcal{I},\mathcal{J}) \text{ with } i \neq j, \\
&x_{i,j} = \sum_{k\in\mathcal{K}} w_{k,i,j} && \forall (i,j) \in (\mathcal{I},\mathcal{J}), \\
&x_{i,j} \leq u_{i,j} && \forall (i,j) \in (\mathcal{I},\mathcal{J}), \\
&\frac{x_{i,j}}{u_{i,j}} \leq \gamma && \forall (i,j) \in (\mathcal{I},\mathcal{J}) \text{ if } u_{i,j} > 0, \\
&x_{i,j} \geq 0 && \forall (i,j) \in (\mathcal{I},\mathcal{J}), \\
&w_{i,j,k} \geq 0 && \forall (i,j,k) \in (\mathcal{I},\mathcal{J},\mathcal{K}), \\
&\gamma \in \mathbb{R}
\end{aligned}$$

After implementation of this improved model, the optimal solution shows that the total traveling time has increased, as was expected. The level of congestion is still high (93.2%) due to the fact that the system is heavily loaded in general. Nevertheless, in most situations it will be a more reliable solution since less people will be dependent on the accessibility of the roads. For instance, when an accident happens on a very important road that is 95% congested, it might become closed or reduced in capacity leading to a high increase in travel time for many people, whereas less people would be affected in a more balanced network.

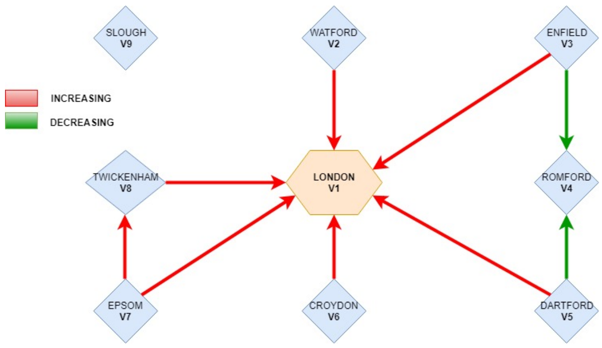

Figure 8: Graph of the most congested roads. Cars starting from V_7 in direction of V_1 will inevitably go through one of the most congested road, this is the limiting link.

Since commuters will not decide on their routes considering the total traveling time of all commuters, another final improvement with respect to the objective function would be to minimize the individual traveling time instead of the sum. This would correspond to what happens in reality when a person looks for the quickest way to reach a destination in Google Maps. His/her own traveling time is individually optimized given the current network situation. So a realistic way to think that is to simulate a real network scenario, e.g. Poisson process arrivals, and according to that, for each commuter given a demand the individual time should be minimized.

6 Summary and Conclusions

In this report the traffic in London's network is modeled and analyzed. For this model, the downtown area and some important suburbs regarding economic activity and population are taken into account. The network is a simplified representation of the real geographical network with respect to the highway connections between these areas. After defining the areas and existing links as well as traveling times for each link, two different scenarios are investigated; one with congestion and one without.

A commuters' demand, expressed in number of cars, between each area is defined based on research about areas' economic activity and population. In particular, the congestion-free scenario returns for each demand the shortest path with respect to travel time. This is expected because no upper capacity is fixed for links neither other constraints. Precisely, the quicker one is chosen. The total travel time results to be around 721 hours.

With respect to the congested scenario, two sub-scenarios are analyzed to better understand the reality: when people travel to work (at 8:15) and when they come back (at 17:00). A capacity and positive coefficient of congestion is assigned to each link. The problem is examined according to these values which depend on the sub-scenario. The morning scenario has a total travel time equal to approximately 873 hours, whereas the evening scenario equals approximately 889 hours.
Since this optimization problem is considered to be convex, it can be concluded that the local optimum is also a global optimum. In this congested scenario, demands' flows are often split depending on both congestion and links' capacity. Sometimes, it happens that a larger detour is preferred because the congestion causes a remarkable delay that makes the quicker path undesirable.

Furthermore, some capacities' modifications are implemented to reduce traffic on the overloaded links. More specifically, the links whose demand exceeds its capacity as well as the most congested link are subjected to these modifications. These modifications are implemented in the congested morning scenario leading to an individual and combined capacity increase of 50%. This generates an expected decrease in total travel time of around $3.5 - 6.9\%$.

Potential model improvements are discussed in order to obtain a more realistic representation of the real system. Firstly, a passing-time treatment is suggested, since this model assumes that each commuter is leaving exactly at the same time. The demand as well the congestion are functions of time and this could be modeled more accurately based on the queuing theory. A different intensity parameter could be set to describe different intensity of demand arrival. Secondly, the current model assumes the variables to be continuous. However, in reality routing for example 4.3 cars is ambiguous. As a result, an integer formulation is implemented.

Lastly, new potential objective functions are suggested, for instance minimizing the maximum level of congestion. This arises from two reasons: common habit to prefer an emptier road rather than being stuck in a queue and making the network less sensitive to possible incidents. Another possible objective function would be to minimize individual traveling times instead of the sum, generating a more accurate representation of the commuters' behavior.

A Links' lengths between nodes

The table below shows the length of paths between nodes expressed in kilometers. It was used to compute the capacity table.

$\mathbf{l_{i,j}}$	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9
v_1	/	32.6	24.3	26.6	31.5	18	30	18.5	38.5
v_2	32.6	/	30.6						37.5
v_3	24.3	30.6	/	38.2					
v_4	26.6		38.2	/	30				
v_5	31.5			30	/	36.3			
v_6	18				36.3	/	16.3		
v_7	30					16.3	/	16.5	
v_8	18.5						16.5	/	35.5
v_9	38.5	37.5						35.5	/

Table 15: Length of link i, j expressed in km

B Detailed demands' flows

B.1 Non-congested scenario

Demand	Departure	Arrival	Fulfillment	
10	v_1	v_2	10	$v_1 \to v_2$
30	v_1	v_5	30	$v_1 \to v_5$
20	v_1	v_6	20	$v_1 \to v_6$
150	v_2	v_1	150	$v_2 \to v_1$
40	v_2	v_9	40	$v_2 \to v_9$
30	v_3	v_1	30	$v_3 \to v_1$
90	v_3	v_4	90	$v_3 \to v_4$
20	v_3	v_5	20	$v_3 \to v_4 \to v_5$
30	v_3	v_6	30	$v_3 \to v_1 \to v_6$
50	v_4	v_1	50	$v_4 \to v_1$
92	v_5	v_1	92	$v_5 \to v_1$
80	v_5	v_4	80	$v_5 \to v_4$
22	v_5	v_9	22	$v_5 \to v_1 \to v_9$
50	v_6	v_1	50	$v_6 \to v_1$
90	v_7	v_1	90	$v_7 \to v_1$
25	v_7	v_4	25	$v_7 \to v_1 \to v_4$
10	v_7	v_6	10	$v_7 \to v_6$
50	v_8	v_1	50	$v_8 \to v_1$
10	v_8	v_6	10	$v_8 \to v_7 \to v_6$
3	v_8	v_7	3	$v_8 \to v_7$
10	v_8	v_9	10	$v_8 \to v_9$
40	v_9	v_1	40	$v_9 \to v_1$

Table 16: Demand fulfillment for non-congested problem

B.2 Congested scenarios

Here the detailed flows' dispatching for the congested scenarios are summarized.

Demand	**Departure**	**Arrival**	**Fulfillment**	
10	v_1	v_2	10	$v_1 \to v_2$
30	v_1	v_5	30	$v_1 \to v_5$
20	v_1	v_6	20	$v_1 \to v_6$
150	v_2	v_1	92.2	$v_2 \to v_1$
			48.4	$v_2 \to v_9 \to v_1$
			9.4	$v_2 \to v_9 \to v_1$
40	v_2	v_9	40	$v_2 \to v_9$
30	v_3	v_1	30	$v_3 \to v_1$
90	v_3	v_4	90	$v_3 \to v_4$
20	v_3	v_5	20	$v_3 \to v_4 \to v_5$
30	v_3	v_6	25.6	$v_3 \to v_1 \to v_6$
			4.4	$v_3 \to v_2 \to v_9 \to v_8 \to v_7 \to v_6$
50	v_4	v_1	50	$v_4 \to v_1$
92	v_5	v_1	91	$v_5 \to v_1$
	v_5	v_1	1	$v_5 \to v_4 \to v_1$
80	v_5	v_4	80	$v_5 \to v_4$
22	v_5	v_9	22	$v_5 \to v_6 \to v_7 \to v_8 \to v_9$
50	v_6	v_1	50	$v_6 \to v_1$
90	v_7	v_1	86.3	$v_7 \to v_1$
			0.5	$v_7 \to v_6 \to v_1$
			1.2	$v_7 \to v_8 \to v_1$
			2	$v_7 \to v_8 \to v_9 \to v_1$
25	v_7	v_4	18.4	$v_7 \to v_8 \to v_9 \to v_1 \to v_4$
			6.6	$v_7 \to v_6 \to v_5 \to v_4$
10	v_7	v_6	10	$v_7 \to v_6$
50	v_8	v_1	50	$v_8 \to v_1$
10	v_8	v_6	10	$v_8 \to v_7 \to v_6$
3	v_8	v_7	3	$v_8 \to v_7$
10	v_8	v_9	10	$v_8 \to v_9$
40	v_9	v_1	40	$v_9 \to v_1$

Table 17: Demand fulfillment for congested problem at 8:15

Demand	**Departure**	**Arrival**	**Fulfillment**	
150	v_1	v_2	92.4	$v_1 \to v_2$
			8.5	$v_1 \to v_3 \to v_2$
			49.1	$v_1 \to v_9 \to v_2$
30	v_1	v_3	30	$v_1 \to v_3$
50	v_1	v_4	50	$v_1 \to v_4$
92	v_1	v_5	91.1	$v_1 \to v_5$
			0.9	$v_1 \to v_4 \to v_5$
50	v_1	v_6	50	$v_1 \to v_6$
90	v_1	v_7	86.4	$v_1 \to v_7$
			0.5	$v_1 \to v_6 \to v_7$
			3.1	$v_1 \to v_8 \to v_7$
50	v_1	v_8	30.1	$v_1 \to v_8$
			19.9	$v_1 \to v_9 \to v_8$
40	v_1	v_9	40	$v_1 \to v_9$
10	v_2	v_1	10	$v_2 \to v_1$
90	v_4	v_3	90	$v_4 \to v_3$
80	v_4	v_5	80	$v_4 \to v_5$
25	v_4	v_7	18.2	$v_4 \to v_1 \to v_8 \to v_7$
			6.8	$v_4 \to v_5 \to v_6 \to v_7$
30	v_5	v_1	30	$v_5 \to v_1$
20	v_5	v_3	20	$v_5 \to v_4 \to v_3$
20	v_6	v_1	20	$v_6 \to v_1$
30	v_6	v_3	26.6	$v_6 \to v_1 \to v_3$
			0.1	$v_6 \to v_5 \to v_4 \to v_3$
			3.3	$v_6 \to v_7 \to v_8 \to v_9 \to v_2 \to v_3$
10	v_6	v_7	10	$v_6 \to v_7$
10	v_6	v_8	10	$v_6 \to v_7 \to v_8$
3	v_7	v_8	3	$v_7 \to v_8$
40	v_9	v_2	40	$v_9 \to v_2$
22	v_9	v_5	22	$v_9 \to v_8 \to v_7 \to v_6 \to v_5$
10	v_9	v_8	10	$v_9 \to v_8$

Table 18: Demand fulfillment for congested problem at 17:00

C Numerical values for integer problem

Numerical values for variables x and w for the integer congested problem at 8:15.

----   88 VARIABLE x.L

	v1	v2	v3	v4	v5	v6
v1		10.000		18.000	40.000	33.000
v2	93.000		15.000			
v3	68.000	17.000		100.000		
v4	51.000				10.000	
v5	91.000			88.000		22.000
v6	51.000				7.000	
v7	87.000					45.000
v8	52.000					
v9	100.000					

+	v7	v8	v9
v2			99.000
v6	22.000		
v7		42.000	
v8	30.000		50.000
v9		17.000	

Figure 9: x values

----   88 VARIABLE w.L

	v1	v2	v3	v4	v5	v6
v1.v1		10.000			30.000	20.000
v2.v2	93.000		15.000			
v2.v3	15.000					
v2.v9	42.000					
v3.v1					10.000	13.000
v3.v3	53.000	17.000		100.000		
v3.v4					10.000	
v3.v7						17.000
v4.v4	50.000					
v5.v4	1.000					
v5.v5	91.000			81.000		22.000
v6.v6	50.000					
v7.v1				18.000		
v7.v5				7.000		
v7.v6	1.000				7.000	
v7.v7	87.000					18.000
v7.v8	12.000					
v7.v9	8.000					
v8.v7						10.000
v8.v8	40.000					
v8.v9	10.000					
v9.v9	40.000					

+	v7	v8	v9
v2.v2			82.000
v3.v2			17.000
v3.v8	17.000		
v3.v9		17.000	
v5.v6	22.000		
v5.v7		22.000	
v5.v8			22.000
v7.v7		20.000	
v7.v8			8.000
v8.v8	13.000		20.000

Figure 10: w values