Problem 1. Using truth tables or otherwise, determine whether the following propositions are valid, satisfiable or unsatisfiable:

$$(((P \rightarrow Q) \rightarrow P) \rightarrow P)$$
$$(((P \rightarrow Q) \rightarrow P) \rightarrow Q)$$
$$((A \oplus B) \oplus B) \Leftrightarrow A$$
$$(A \rightarrow (B \wedge C)) \rightarrow ((A \wedge B) \vee (A \wedge C))$$
$$((A \vee (B \wedge C)) \rightarrow (D \wedge (E \vee F))) \Leftrightarrow (((A \vee B) \wedge (A \vee C)) \rightarrow ((D \wedge E) \vee (D \wedge F)))$$

Problem 2. Consider an arbitary function f. Formulate as predicates the following statements about f:

f is injective.

f is surjective.

Consider the predicate *loves* defined such that $loves(A, B)$ is interpreted as A *loves* B. Write the following sentences as formulae using predicate logic:

Everyone loves somebody

There is someone that everyone loves.

Problem 3. As you have seen, determining the satisfiability or validity of a proposition quickly becomes intractable when we introduce longer formulae or more literals. We would like an algorithmic way to evaluate propositions. Before we do this, however, we would like some standard form which allows us to systemically simplify or process our propositions.

The first such form to consider is **Negation Normal Form**. A formula is in negation normal form if it contains only the connectives $\vee$, $\wedge$ and $\neg$, and $\neg$ is only applied to atomic formulae (i.e. the only negated terms are literals).

To convert a formula to Negation Normal form, we:

1. Eliminate all $\rightarrow$ and $\Leftrightarrow$ connectives using the following equivalences:

$$A \Leftrightarrow B \approx (A \rightarrow B) \wedge (B \rightarrow A)$$

$$A \rightarrow B \approx \neg A \vee B$$

2. Apply negations to their arguments until only atoms are negated, using DeMorgan's Laws and the Law of the Excluded Middle:

$$\neg(A \wedge B) \approx \neg A \vee \neg B$$

$$\neg(A \vee B) \approx \neg A \wedge \neg B$$

$$\neg\neg A \approx A$$

3. Once this is done, the proposition is in Negation Normal Form.

Convert the following formulae to Negation Normal Form:

$$P \vee Q \rightarrow Q \vee R$$
$$P \wedge Q \rightarrow Q \wedge P$$
$$((P \rightarrow Q) \rightarrow P) \rightarrow P$$
$$(P \rightarrow Q) \wedge (Q \rightarrow P)$$
$$((P \wedge Q) \vee R) \wedge \neg(P \vee R)$$
$$\neg(P \vee Q \vee R) \vee ((P \wedge Q) \vee R)$$

By converting a formula to Negation Normal Form we have removed the redundant $\rightarrow$ connective and can easily convert our formula to a canonical form. For this course we shall consider the canonical form **CNF** (Conjunctive Normal Form). Formulae in CNF are of the form $C_1 \wedge C_2 \wedge ...C_m$ where each C_i is called a **clause**. A clause has the form $a_1 \vee a_2 \vee ...a_n$ where each a_j is a literal. To convert a proposition to CNF:

1. Convert it to Negation Normal Form.
2. Push disjunctions until they only apply to literals using the distributive law:
$$A \vee (B \wedge C) \approx (A \vee B) \wedge (A \vee C)$$
3. The formula is now in CNF. Observe that a clause evaluates to true if any of its literals evaluates to true, and the formula evaluates to true if all the clauses evaluate to true. As such, we can apply the following further simplifications:
   - Delete any clause containing the literals A and $\neg A$ - these clauses evaluate to true by the law of the excluded middle.
   - Delete any clause which contains every literal from another clause - these clauses are redundant by absorption.
   - Combine each pair of clauses which differ only in the following way: one contains the literal A and the other its negation $\neg A$. To do so, use the following logical equivalence:
   $$(A \vee B) \wedge (\neg A \vee B) \approx B$$

   Convert the above formulae from Negation Normal Form to CNF. Determine which are valid, satisfiable and unsatisfiable. Show that all valid formulae have the same CNF.

Problem 4. A **Horn Clause** is a clause with at most one non-negated literal. Prove that this definition is equivalent to the definition presented in your lectures.