Laurent Bourgeois, Philippe Moireau

Data completion and identification in problems governed by PDEs

February 4, 2019

Springer

Contents

Acronyms

Here is a list of symbols used in this book:

Ψ Operator to be inversed
A semigroup generator
C Observation operator

Chapter 1
Inverse and ill-posed problems

1.1 Introduction

Solving an inverse problems generally consists in identifying some hidden parameters or defects in a system with the help of accessible data. More precisely, we impose some solicitations to the system and measure the corresponding responses in order to recover those parameters or defects. The inverse problems may be linear or not, in the sense that the mapping from the parameter/defect to the data may be linear or not. Most often, those inverse problems are ill-posed in the sense that a small perturbation of the data implies a strong error on the parameter/defect to identify. More precisely, the French Mathematician Jacques Hadamard gave the following definition of a well-posed problem in mathematical physics:

- existence of a solution
- uniqueness of the solution
- continuous dependance of this solution on the data.

A problem is ill-posed if one of this properties at least is not satisfied. Let $\Psi : \mathcal{Z} \to \mathcal{Y}$ be a mapping from a normed space $\mathcal{Z}$ to a normed space $\mathcal{Y}$. The equation $\Psi(z) = y$ is called well-posed if Ψ is bijective and $\Psi^{-1} : \mathcal{Y} \to \mathcal{Z}$ is continuous. The equation is ill-posed otherwise. A particular and frequent case is when $\mathcal{Z}$ and $\mathcal{Y}$ are both Banach spaces and Ψ is linear and continuous. Then if Ψ is bijective, its inverse Ψ^{-1} is continuous, as a consequence of the open mapping theorem. In other words, the first and second properties of a well-posed problem automatically implies the third one. An important case is when Ψ is a compact operator: then the equation $\Psi z = y$ is always ill-posed unless $\mathcal{Z}$ is finite dimensional. Indeed, if we assume that Ψ^{-1} exists and is continuous, then $\Psi^{-1}\Psi = I : \mathcal{Z} \to \mathcal{Z}$ is a compact operator by composition of a continuous and a compact operator. But the identity operator is a compact operator on $\mathcal{Z}$ if and only if $\mathcal{Z}$ is finite dimensional. Let us now present several elementary examples of inverse and ill-posed problems.

1.2 A toy example: differentiation operator

We consider the operator $\Psi : L^2(0,1) \to L^2(0,1)$ defined by

$$(\Psi z)(t) = \int_0^t z(s)\,ds.$$

Clearly, the inverse of Ψ is differentiation. Let us show that the equation $\Psi z = y$ is ill-posed. For it suffices to prove that Ψ is a compact operator. If we set $y = \Psi z$, we have $y' = z \in L^2(0,1)$, so that Ψ is a continuous operator from $L^2(0,1)$ to $H^1(0,1)$. Since the embedding $H^1(0,1) \to L^2(0,1)$ is compact, then $\Psi : L^2(0,1) \to L^2(0,1)$ is compact. Note, however, that Ψ is injective. We also observe that another expression of Ax is

$$(\Psi z)(t) = \int_0^1 K(t,s)z(s)\,ds,$$

where K is defined by $K(t,s) = 1$ of $s \le t$ and 0 otherwise. We readily see that $K \in L^2((0,1)\times(0,1))$ and it is a general result that an integral operator on $L^2(a,b)$, the kernel of which is in $L^2((a,b)\times(a,b))$, is compact.

Theorem 1.1. *Let us consider an operator* $\Psi : L^2(a,b) \mapsto L^2(a,b)$ *defined by*

$$(\Psi u)(t) = \int_a^b K(s,t)u(s)\,dt, \quad t \in (a,b)$$

for $K \in L^2((a,b)\times(a,b))$ *with* $a < b$. *Then* Ψ *is compact.*

Proof. Proving that Ψ is compact amounts to prove that if a sequence (z_n) weakly converges to z in $L^2(a,b)$ then (Ψz_n) strongly converges to Ψz in $L^2(a,b)$. We have

$$(\Psi z_n)(t) = \int_0^1 K(t,s)z_n(s)\,ds.$$

For almost all $t \in (a,b)$, the function $s \mapsto K(t,s)$ belongs to $L^2(a,b)$. This is a consequence of Fubini's theorem. Hence for almost $t \in (a,b)$, we have $(Ax_n)(t) \to (Ax)(t)$. In addition, by the Cauchy-Schwarz inequality, for almost all $t \in (a,b)$,

$$\begin{aligned}|\Psi z_n(t) - \Psi z(t)|^2 &\le \int_0^1 |K(t,s)|^2\,ds \int_0^1 |z_n(s)-z(s)|^2\,ds\\ &\le C\int_0^1 |K(t,s)|^2\,ds,\end{aligned}$$

since the weakly convergent sequence (z_n) is bounded. We observe that the right-hand side function of t is integrable in (a,b) and independent of n. From the Lebesgue's theorem, we conclude that (Ψz_n) converges to Ψz in $L^2(a,b)$. $\square$

Since the operator Ψ is compact, the problem of finding z from y such that $\Psi z = y$ is ill-posed. This has a practical consequence on the instability of numerical differentiation. If one wants to differentiate a noisy function which is given at discrete values one has to proceed carefully, that is by introducing a regularization, because such procedure is strongly unstable.

1.3 The backward heat equation

Let us first consider the heat equation for $(x,t) \in (0,\pi) \times (0,T)$ with initial condition

$$\begin{cases} \partial_t u - \partial_x^2 u = 0 & \text{in } (0,\pi)\times(0,T), \\ u(0,t) = u(\pi,t) = 0 & \text{on } (0,T), \\ u(x,0) = \phi(x) & \text{on } (0,\pi). \end{cases} \tag{1.1}$$

As soon as $\phi \in L^2(0,\pi)$, such problem is well-posed in $C^0(0,T;L^2(0,\pi)) \cap L^2(0,T;H^1(0,\pi))$. To show that, it is natural to decompose the solution with respect to x in the complete basis $\sin(nx)$ of $L^2(0,\pi)$. Doing so, we obtain that

$$u(x,t) = \sum_{n=1}^{+\infty} \phi_n e^{-n^2 t} \sin(nx), \phi_n := \frac{2}{\pi}\int_0^\pi \phi(y)\sin(ny)\,dy.$$

Now, let us consider the same problem with a final condition, which is the so-called backward heat equation

$$\begin{cases} \partial_t u - \partial_x^2 u = 0 & \text{in } (0,\pi)\times(0,T), \\ u(0,t) = u(\pi,t) = 0 & \text{on } (0,T), \\ u(x,T) = f(x) & \text{on } (0,\pi). \end{cases} \tag{1.2}$$

By using the previous decomposition, we obtain

$$u(x,t) = \sum_{n=1}^{+\infty} f_n e^{n^2(T-t)} \sin(nx), \quad f_n := \frac{2}{\pi}\int_0^\pi f(y)\sin(ny)\,dy.$$

In particular,

$$\|u(x,0)\|^2_{L^2(0,\pi)} = \frac{\pi}{2}\sum_{n=1}^{+\infty} |f_n|^2 e^{2n^2 T},$$

which means that $\phi(x) = u(x,0)$ is in general undefined in $L^2(0,\pi)$ for some $f \in L^2(0,\pi)$, unless the sequence of (f_n) decreases extremely fast. To be more specific, we observe that finding ϕ from f consists in solving $\Psi\phi = f$ with $\Psi : L^2(0,\pi) \to L^2(0,\pi)$ and

$$(\Psi\phi)(x) = \int_0^\pi K(x,y)\phi(y)\,dy, \quad K(x,y) \coloneqq \frac{2}{\pi}\sum_{n=1}^{+\infty} e^{-n^2 T}\sin(nx)\sin(ny).$$

Obviously, the function K belongs to $L^2((0,\pi)\times(0,\pi))$, which implies from Theorem 1.1 that the operator Ψ is compact, and hence the problem of finding the initial condition ϕ from the final one f by solving (1.2) is ill-posed. However, it can be proved that such problem has at most one solution.

1.4 The Cauchy problem for the Laplace equation

Assume that $\Omega \subset \mathbb{R}^d$ is a bounded open and connected domain of class $C^{0,1}$, and Γ a non-empty open subpart of $\partial\Omega$. We consider the following problem: for a pair of data $(g_0, g_1) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$, find $u \in H^1(\Omega)$ such that

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = g_0 & \text{on } \Gamma, \\ \partial_\nu u = g_1 & \text{on } \Gamma, \end{cases} \tag{1.3}$$

where ν is the outward unit normal $\partial\Omega$.
Such problem is the Cauchy problem for the Laplace's equation, which arises in many practical situations : the (g_0, g_1) are measurements on the accessible part Γ on the domain and are redundant, while no data is available on the complementary part $\tilde{\Gamma}$ of $\partial\Omega$, which is not accessible. Note that, from Theorem 3.6, problem (1.3) has at most one solution.

A practical method to solve the Cauchy problem (1.3) is to view it as a data completion problem, following the ideas given in Andrieux et al. (2006); Azaïez et al. (2006); Ben Belgacem & El Fekih (2005). In particular, the proofs that follow are borrowed from Azaïez et al. (2006); Ben Belgacem & El Fekih (2005). In order to simplify the presentation, we assume that $\overline{\Gamma} \cap \overline{\tilde{\Gamma}} = \emptyset$, which implies in particular that space $H^{1/2}(\Gamma)$ coincides with space $\tilde{H}^{1/2}(\Gamma)$, which is defined as the dual space of $H^{-1/2}(\Gamma)$. The idea is to complete the boundary data on the inaccessible part of the boundary $\tilde{\Gamma}$ so that the solutions to the Dirichlet problem and the Neumann problem formed in Ω with data g_0 and g_1 on Γ coincide with each other. More precisely, for $\mu \in H^{1/2}(\tilde{\Gamma})$ we consider $u_D(\mu, g_0)$ and $u_N(\mu, g_1)$ the solutions in $H^1(\Omega)$ of well-posed problems

$$\begin{cases} \Delta u_D = 0 & \text{in } \Omega, \\ u_D = g_0 & \text{on } \Gamma, \\ u_D = \mu & \text{on } \tilde{\Gamma}, \end{cases} \tag{1.4}$$

and

$$\begin{cases} \Delta u_N = 0 & \text{in } \Omega, \\ \partial_\nu u_N = g_1 & \text{on } \Gamma, \\ u_N = \mu \text{on } \tilde{\Gamma}. \end{cases} \tag{1.5}$$

Remark 1.1. Let us remark that if we do not assume that $\overline{\Gamma} \cap \overline{\tilde{\Gamma}} = \emptyset$, problem (1.4) is undefined in general for $g_0 \in H^{1/2}(\Gamma)$ and $\mu \in H^{1/2}(\tilde{\Gamma})$.

We have the following proposition.

Proposition 1.1. *For Cauchy data $(g_0, g_1) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$, the Cauchy problem (1.3) has a solution in $H^1(\Omega)$ if and only if there exists $\mu \in H^{1/2}(\tilde{\Gamma})$ such that*

$$\partial_\nu u_D(\mu, g_0)|_{\tilde{\Gamma}} = \partial_\nu u_N(\mu, g_1)|_{\tilde{\Gamma}}. \tag{1.6}$$

Proof. First we assume that equation (1.6) is satisfied. Then the function $u_D - u_N \in H^1(\Omega)$ solves the homogeneous Cauchy problem (3.1) with $\tilde{\Gamma}$ playing the role of Γ and then $u_D = u_N$ in Ω in view of Theorem 3.6. The solution $u \coloneqq u_D = u_N$ is a solution of the Cauchy problem (1.3).
Conversely, if $u \in H^1(\Omega)$ is the solution to the Cauchy problem, we just have to set $\mu \coloneqq u|_\Gamma \in H^{1/2}(\tilde{\Gamma})$. □

Next, we establish an equivalence between equation (1.6) and a Steklov-Poincaré problem. In this view we define $u_D(\mu) \coloneqq u_D(\mu, 0)$ and $\breve{u}_D(g_0) = u_D(0, g_0)$ and we adopt similar notations for u_N. We define the bilinear form on $H^{1/2}(\tilde{\Gamma}) \times H^{1/2}(\tilde{\Gamma})$

$$s(\lambda, \mu) = \int_\Omega \nabla u_D(\lambda) \cdot \nabla u_D(\mu)\, dx - \int_\Omega \nabla u_N(\lambda) \cdot \nabla u_N(\mu)\, dx$$

and the linear form on $H^{1/2}(\tilde{\Gamma})$

$$\ell(\mu) = -\int_\Omega \nabla \breve{u}_D(g_0) \cdot \nabla u_D(\mu)\, dx + \langle g_1, u_N(\mu) \rangle_\Gamma,$$

where the bracket denotes duality pairing between $H^{-1/2}(\Gamma)$ and $\tilde{H}^{1/2}(\Gamma)$. We have the following theorem.

Theorem 1.2. *The function $\lambda \in H^{1/2}(\tilde{\Gamma})$ solves equation (1.6) if and only if it solves the weak formulation*

$$s(\lambda, \mu) = \ell(\mu), \quad \forall \mu \in H^{\frac{1}{2}}(\tilde{\Gamma}).$$

Proof. let us assume that λ in $H^{1/2}(\tilde{\Gamma})$ satisfies (1.6). Then for all μ in $H^{1/2}(\Gamma)$,

$$\int_{\tilde{\Gamma}} \partial_\nu u_D(\lambda, g_0) \mu\, ds = \int_{\tilde{\Gamma}} \partial_\nu u_N(\lambda, g_1) \mu\, ds.$$

Hence

$$\int_{\partial\Omega} \partial_\nu u_D(\lambda, g_0) u_D(\mu)\, ds = \int_{\partial\Omega} \partial_\nu u_N(\lambda, g_1) u_N(\mu)\, ds - \int_{\Gamma} g_1 u_N(\mu)\, ds.$$

Integration by parts implies

$$\int_{\Omega} \nabla u_D(\lambda, g_0) \cdot \nabla u_D(\mu)\, dx = \int_{\Omega} \nabla u_N(\lambda, g_1) \cdot \nabla u_N(\mu)\, dx + \int_{\Gamma} g_1 u_N(\mu)\, ds.$$

Remarking that

$$u_D(\lambda, g_0) = u_D(\lambda) + \breve{u}_D(g_0), \quad u_N(\lambda, g_1) = u_N(\lambda) + \breve{u}_N(g_1),$$

we obtain

$$\begin{aligned} \int_{\Omega} \nabla u_D(\lambda) \cdot \nabla u_D(\mu)\, dx &+ \int_{\Omega} \nabla \breve{u}_D(g_0) \cdot \nabla u_D(\mu)\, dx \\ &= \int_{\Omega} \nabla u_N(\lambda) \cdot \nabla u_N(\mu)\, dx + \int_{\Omega} \nabla \breve{u}_N(g_1) \cdot \nabla u_N(\mu)\, dx \\ &\qquad + \int_{\Gamma} g_1 u_N(\mu)\, ds. \end{aligned} \tag{1.7}$$

On the other hand,

$$\int_{\Omega} \nabla \breve{u}_N(g_1) \cdot \nabla u_N(\mu)\, dx = \int_{\Gamma} \breve{u}_N(g_1) \partial_\nu u_N(\mu)\, ds + \int_{\tilde{\Gamma}} \breve{u}_N(g_1) \partial u_N(\mu)\, ds = 0.$$

We hence obtain

$$s(\lambda, \mu) = \ell(\mu),$$

which completes the proof. The converse assertion follows the same lines. □

The weak formulation of Theorem 1.2 is equivalent to

$$S\lambda = L, \tag{1.8}$$

where $S : H^{1/2}(\tilde{\Gamma}) \to H^{-1/2}(\tilde{\Gamma})$ and $L \in H^{-1/2}(\tilde{\Gamma})$ are defined with the help of the bilinear form s and the linear form ℓ.

Let us now show that the operator S is compact. In order to highlight this fact, we notice that

$$s(\lambda, \mu) = \langle \partial_\nu (u_D - u_N)(\lambda), \mu \rangle_{\tilde{\Gamma}},$$

That is

$$S\lambda = \partial_\nu (u_D - u_N)(\lambda)|_{\tilde{\Gamma}}\,, \tag{1.9}$$

which justifies the fact that we call S the Steklov-Poincaré operator.

Proposition 1.2. *The operator $S : H^{1/2}(\tilde{\Gamma}) \to H^{-1/2}(\tilde{\Gamma})$ is compact.*

Proof. For the sake of simplicity, we assume that Ω is a $C^{1,1}$ domain in order to use some simple regularity results for the Laplace equation. The case when Ω is a polygonal domain ($d = 2$) or polyhedral ($d = 3$) would be treated with the help of Grisvard (1985) and the general case of Lipschitz domains for $d = 2, 3$ would be treated with the help of Jerison & Kenig (1981).
In view of the expression (1.9) we focus on the regularity of solutions u_D and u_N of problems (1.4) and (1.5). We have supposed that $\overline{\Gamma} \cap \overline{\tilde{\Gamma}} = \emptyset$. We hence may find a smooth cut-off function ϕ such that $\phi = 1$ in a neighborhood of Γ and $\phi = 0$ in a neighborhood of $\tilde{\Gamma}$. By denoting $v = \phi\, u_N(\lambda)$, the function v solves the problem

$$\begin{cases} \Delta v = f & \text{in } \Omega, \\ \partial_\nu v = 0 & \text{on } \Gamma, \\ v = 0 & \text{on } \tilde{\Gamma}, \end{cases}$$

with $f = (u_N(\lambda)\Delta\phi + 2\nabla u_N(\lambda) \cdot \nabla\phi) \in L^2(\Omega)$. From standard regularity results for the Laplace equation, $v \in H^2(\Omega)$ and

$$\|v\|_{H^2(\Omega)} \leq C\,\|f\|_{L^2(\Omega)} \leq C\,\|u_N(\lambda)\|_{H^1(\Omega)} \leq C\,\|\lambda\|_{H^{\frac{1}{2}}(\tilde{\Gamma})},$$

and lastly

$$\|u_N(\lambda)\|_{H^{\frac{3}{2}}(\Gamma)} \leq C\,\|\lambda\|_{H^{\frac{1}{2}}(\tilde{\Gamma})}.$$

In view of (1.9) we now consider $w = u_D(\lambda) - u_N(\lambda)$, which solves the problem

$$\begin{cases} \Delta w = 0 & \text{in } \Omega, \\ w = -u_N(\lambda) & \text{on } \Gamma, \\ w = 0 & \text{on } \tilde{\Gamma}. \end{cases}$$

We conclude that

$$\|w\|_{H^2(\Omega)} \leq C\,\|u_N(\lambda)\|_{H^{\frac{3}{2}}(\Gamma)} \leq C\,\|\lambda\|_{H^{\frac{1}{2}}(\tilde{\Gamma})}.$$

Lastly

$$\|S\lambda\|_{H^{\frac{1}{2}}(\tilde{\Gamma})} \leq C\,\|\lambda\|_{H^{\frac{1}{2}}(\tilde{\Gamma})}.$$

We complete the proof by recalling that the embedding $H^{1/2}(\tilde{\Gamma}) \to H^{-1/2}(\tilde{\Gamma})$ is compact. □

From Proposition 1.1 and Theorem 1.2, we see that solving the Cauchy problem for the Laplace equation (1.3) is equivalent to invert the operator S, which from Proposition 1.2 happens to be compact. This means that problem (1.3) is ill-posed. We will see in the sequel how such problem can be regularized in some sense.

1.5 The wave equation with interior measurements

Let us now consider the wave equation for $(x,t) \in \Omega \times (0,T)$ where Ω is a regular bounded domain of boundary $\Gamma = \partial\Omega$. We denote (u_0, v_0) the initial condition and consider no external solicitation so that the wave system reads

$$\begin{cases} \partial_t^2 u - \Delta u = 0 & \text{in } \Omega \times (0,T) \\ u = 0 & \text{on } \Gamma \times (0,T) \\ u = u_0 & \text{in } \Omega \\ v = v_0 & \text{in } \Omega \end{cases} \tag{1.10}$$

We denote by

$$H^0(\Omega, \Delta) = \Big\{ u \in H_0^1(\Omega) \,|\, \Delta u \in L^2(\Omega) \Big\},$$

the space allowing to consider strong solutions of (5.1). Indeed from Appendix A, two types of solutions can be expected. If

$$(u_0, v_0) \in H^0(\Omega, \Delta) \times H_0^1(\Omega),$$

then the problem (5.1) admits a strong solution

$$u \in C^2([0,T]; L^2(\Omega)) \cap C^1([0,T]; H_0^1(\Omega)) \cap C^0([0,T]; H^0(\Omega, \Delta)).$$

However, if we consider an initial condition in the energy space

$$(u_0, v_0) \in H_0^1(\Omega) \times L^2(\Omega),$$

then the problem (5.1) admits a weak solution

$$u \in C^1([0,T]; L^2(\Omega)) \cap C^0([0,T]; H_0^1(\Omega)).$$

Let us now consider that we have at our disposal some measurements of a target solution of (5.1) initialized from an unknown initial condition (u_0, v_0). Typically, we assume to measure $u \in \omega \times (0,T)$ where ω is a subdomain of Ω. Our inverse problem is then to assess if we can reconstruct the unknown (u_0, v_0) from the measurement y of u denoted in $\omega \times (0,T)$. Keeping in mind Hadamard's definition, two questions arise: (1) do we have enough information in y to reconstruct (u_0, v_0) and (2) is this reconstruction stable with respect to the data at hand? The first question has been studied by numerous authors since the pioneer work of Lions (1988). In essence, there is enough information in the measurements if T is large enough so that the all the information contained in the initial solution propagates up to the subdomain ω. The most precise results were first introduced by Bardos et al. (1988) clarifying what we have meant by "information", and constentely refined since then – see for instance some very recent results in Gagnon (2017); Burq & Gérard (2017).

Definition 1.1. (Strong Geometric Control Condition) Let $\omega \subset \Omega$ and $T_0 > 0$. We say that the couple (ω, T_0) satisfies the SGCC if every generalized geodesic $s \mapsto x_\xi(s)$ (i.e. ray of geometric optics) from a point ξ and traveling at speed one in Ω meets ω in a time $t < T_0$, in the sense

$$\text{(SGCC)}: \quad \forall \xi \in \Omega, \exists s \in (0,T), \exists \delta > 0, B(x_\xi(s), \delta) \subset \omega.$$

We say that ω satisfies the SGCC if there exists $T_0 > 0$ such that (ω, T_0) satisfies the SGCC

Let us then introduce the linear operator

$$\Psi_T : \left| \begin{array}{rl} \mathcal{Z} & \to \mathcal{Y}_T \\ z_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} & \mapsto y = u_\omega \end{array} \right.$$

and $\mathcal{Y}_T = H^1([0,T], L^2(\omega))$ or $\mathcal{Y}_T = L^2([0,T], H^1(\omega))$ – namely $\mathcal{Y}_T = H^k([0,T], H^{1-k}(\omega))$ for $k = 0, 1$. Note that assuming $\mathcal{Y}_T = H^1([0,T], L^2(\omega))$ leads to equivalently consider the derivative in the sense of distribution $\Psi_T z_0 = \partial_t u|_\omega \in L^2([0,T], L^2(\omega))$. Then, we have the following coercivity result.

Claim. Let $k = 0, 1$ and assuming that ω satisfies the SGCC, then Ψ_T is coercive from $H_0^1(\Omega) \times L^2(\Omega)$ to $H^k([0,T], H^{1-k}(\omega))$ for large time, namely there exists (T, α) such that for every $(u_0, v_0) \in H_0^1(\Omega) \times L^2(\Omega)$

$$\int_0^T \int_\omega |\partial_t u|^2 \geq \alpha \Big(\|u_0\|^2_{H_0^1(\Omega)} + \|v_0\|^2_{L^2(\Omega)} \Big),$$

or

$$\int_0^T \int_\omega |\nabla u|^2 \geq \alpha \Big(\|u_0\|^2_{H_0^1(\Omega)} + \|v_0\|^2_{L^2(\Omega)} \Big).$$

Therefore, establishing the injectivity and/or surjectivity of Ψ_T depends of our choices of spaces $\mathcal{Z}$ and $\mathcal{Y}_T$. Namely assuming $\mathcal{Z} = H_0^1(\Omega) \times L^2(\Omega)$, if $\mathcal{Y}_T = H^k([0,T], H^{1-k}(\omega))$, the problem is injective as soon as (ω, T_0) satisfies the GCC and $T > T_0$. However, Ψ is not injective for $T < T_0$. However the question of its surjectivity is not obvious and it is certainly not satisfied when considering the – rather natural– space $\mathcal{Y}_T = L^2([0,T], L^2(\omega))$.

Remark 1.2. Note finally that other kinds of measurements can be envisioned, in particular boundary measurements. Considering the model (5.1), we could consider boundary forces measurements

$$y = \partial_\nu u, \quad \text{on } \partial\Gamma \times (0,T).$$

Another possibility could be to replace partially the Dirichlet boundary conditions in (5.1) by Neumann boundary conditions

$$\begin{cases} \partial_t^2 u - \Delta u = 0 & \text{in } \Omega \times (0,T) \\ \partial_\nu u = 0 & \text{on } \Gamma_N \times (0,T) \\ u = 0 & \text{on } \Gamma_D \times (0,T) \\ v = v_0 & \text{in } \Omega \end{cases}$$

Then the measurements could be of the Dirichlet type

$$y = u, \quad \text{in } \Gamma_N \times (0,T).$$

1.6 The inverse Robin problem

Up to now we have stuck to linear inverse problems. Let us now address a non linear one, that is the inverse Robin problem. We hence consider a bounded, connected and open domain $\Omega \in \mathbb{R}^d$ of class $C^{0,1}$. The problem consists in finding $\lambda \in L^\infty(\tilde{\Gamma})$ such that

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = g_0 & \text{on } \Gamma, \\ \partial_\nu u = g_1 & \text{on } \Gamma, \\ \partial_\nu u + \lambda u = 0 & \text{on } \tilde{\Gamma}, \end{cases} \tag{1.11}$$

where Γ is a non-empty open subset of $\partial\Omega$, $\tilde{\Gamma} = \partial\Omega \setminus \overline{\Gamma}$, ν is the outward unit normal of Ω and (g_0, g_1) in $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$.
Such inverse problem corresponds to the identification of the impedance λ on an inaccessible part $\tilde{\Gamma}$ of the boundary from the measurements (g_0, g_1) on the accessible part Γ of the boundary. A simple application in the field of electrostatic non destructive testing can be found: this problem consists in finding some corrosion on the inaccessible boundary by measuring both the potential and the current on the accessible boundary.

To simplify the problem, we assume that $g_1 := g \neq 0$ is a fixed parameter in $L^2(\Gamma)$, and we define the mapping Ψ as follows. For $\lambda \in L^\infty_+(\tilde{\Gamma})$ with

$$L^\infty_+(\tilde{\Gamma}) := \{\lambda \in L^\infty(\tilde{\Gamma}),\ \exists m > 0,\ \lambda(x) \geq m \ a.e.\},$$

$\Psi(\lambda) = u|_\Gamma \in L^2(\Gamma)$, where $u \in H^1(\Omega)$ is the solution of the well-posed problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \partial_\nu u = g & \text{on } \Gamma, \\ \partial_\nu u + \lambda u = 0 & \text{on } \tilde{\Gamma}. \end{cases} \tag{1.12}$$

Problem (1.12) is clearly equivalent to the following weak formulation : find $u \in H^1(\Omega)$ such that for all $v \in H^1(\Omega)$,

$$\int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\tilde{\Gamma}} \lambda uv \, ds = \int_\Gamma gv \, ds, \tag{1.13}$$

and well-posedness of problem (1.13) follows from Poincaré-Friedrichs inequality, which implies the equivalence between the standard norm of $H^1(\Omega)$ and the norm $\|\cdot\|$ defined by

$$\|u\|^2 = \int_\Omega |\nabla u|^2 \, dx + \int_{\tilde{\Gamma}} u^2 \, ds.$$

The mapping $\Psi : L^\infty_+(\tilde{\Gamma}) \to L^2(\Gamma)$ is clearly non linear. Moreover, the problem of solving $\Psi(\lambda) = f$ in $L^\infty_+(\tilde{\Gamma})$ for $f \in L^2(\Gamma)$ is ill-posed since formally,

$$\lambda = -\frac{\partial_\nu u|_{\tilde{\Gamma}}}{u|_{\tilde{\Gamma}}},$$

and we known from the analysis of the Cauchy problem for the Laplace equation that the identification of u in Ω from the Cauchy data (g_0, g_1) on Γ is ill-posed. In the sequel, we will study uniqueness for this inverse Robin problem and how to solve it in some sense.

1.7 The inverse obstacle problem

We complete this short review of elementary inverse problems by a geometric inverse problem, namely the inverse obstacle problem. We again consider a bounded, connected and open domain $D \in \mathbb{R}^d$ of class $C^{0,1}$. The problem consists in finding an open domain O of class $C^{0,1}$ such that $O \Subset D$ and such that there exists $u \in H^1(\Omega)$, with $\Omega = D \setminus \overline{O}$, such that

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = g_0 & \text{on } \Gamma, \\ \partial_\nu u = g_1 & \text{on } \Gamma, \\ u = 0 & \text{on } \partial O, \end{cases} \tag{1.14}$$

where Γ is a non-empty open subset of $\partial\Omega$, ν is the outward unit normal of Ω and (g_0, g_1) in $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$. The obstacle O is then characterized by a Dirichlet boundary condition. This problem can be seen as a simple model of non destructive testing in thermography for the steady case. It consists in identifying an insulating cavity inside a a body from measurements of both the temperature and the flux on a subpart of the boundary.

The inverse obstacle problem is obviously non linear, even more non linear than the inverse Robin problem, in the sense that the parameter to retrieve, that is the open domain O, does not lie in a vector space. However, it lies in a metric space, namely the set of all open domains contained in D, which can be for example equipped with the Hausdorff distance for open domains. In the sequel, we will study uniqueness for such inverse problem and the way we can solve it in some sense.

Chapter 2
The Tikhonov regularization and the Morozov principle

2.1 Introduction

We consider a continuous operator $\Psi : \mathcal{Z} \to \mathcal{Y}$, where $\mathcal{Z}$ and $\mathcal{Y}$ are Hilbert spaces. In the following we will denote $\mathcal{Z}^*$ and $\mathcal{Y}^*$ the dual spaces of $\mathcal{Z}$ and $\mathcal{Y}$, and $\Psi^* : \mathcal{Y}^* \to \mathcal{Z}^*$ the adjoint operator associates with Ψ. We also identify $\mathcal{Y}^*$ to itself. We assume that Ψ is injective and has a dense range, which is an usual situation in many ill-posed problems. However, we assume that the range of Ψ is not closed, so that the problem: find $u \in \mathcal{Z}$ such that

$$\Psi z = y \tag{2.1}$$

with $y \in \mathcal{Y}$ may have no solution for some y and is therefore a ill-posed problem.
In general, for ill-posed problems, the data y comes from measurements and is then contaminated by some noise of amplitude δ. We hence assume that we know some noisy data $y^\delta \in \mathcal{Y}$ such that $\|y^\delta - y\|_{\mathcal{Y}} \le \delta$. In general is it reasonable to assume that $\|y^\delta\|_{\mathcal{Y}} > \delta$, which means that the data is bigger than the amplitude of noise.
The inverse problem is as follows : *we assume that for exact and unknown data f, problem (2.1) has a (unique) unknown solution $z \in \mathcal{Z}$. The objective is, given some known noisy data $y^\delta \in \mathcal{Y}$, to find some approximate solution $\tilde{z} \in \mathcal{Z}$ of the exact solution z.*

2.2 The Tikhonov regularization

A classical tool for that is the Tikhonov regularization. It consists in solving the following problem for some $\epsilon > 0$: find $z_\epsilon^\delta \in \mathcal{Z}$ such that

$$\Psi^*\Psi z_\epsilon^\delta + \epsilon\, z_\epsilon^\delta = \Psi^* y^\delta. \tag{2.2}$$

Two other charactizations of z_ϵ^δ may be easily obtained. First, z_ϵ^δ is the solution of the variational formulation: find $z_\epsilon^\delta \in \mathcal{Z}$ such that

$$(\Psi z_\epsilon^\delta, \Psi v)_\mathcal{Y} + \epsilon(z_\epsilon^\delta, v)_\mathcal{Z} = (y^\delta, Av) \quad \forall v \in \mathcal{Z}. \tag{2.3}$$

Secondly z_ϵ^δ is the unique minimizor of the functional

$$\mathscr{J}_\epsilon(z) = \frac{1}{2}\,\|\Psi z - y^\delta\|_\mathcal{Y}^2 + \frac{\epsilon}{2}\,\|z\|_\mathcal{Z}^2. \tag{2.4}$$

The fact that the problems (2.2), (2.3) and (2.4) are equivalent is obvious and well-posedness of (2.3) is an immediate consequence of Lax-Milgram's theorem.

Remark 2.1. Note that the well-posedness of (2.3) is also obtained when Ψ is neither surjective nor injective. Therefore, the Tikhonov is also adapted to circumvent a potential lack of injectivity in the inverse problem of interest.

In the case of exact data y, the Tikhonov solution z_ϵ converges to the exact solution z.

Theorem 2.1. *We have $z_\epsilon \to z$ in $\mathcal{Z}$ when $\epsilon \to 0$.*

Proof (First approach). By using both (2.1) and (2.3) we obtain

$$(\Psi(z_\epsilon - z), \Psi v)_\mathcal{Y} + \epsilon(z_\epsilon, z)_\mathcal{Z} = 0, \quad \forall v \in \mathcal{Z}. \tag{2.5}$$

By choosing $v = z_\epsilon - z$, we obtain that

$$(z_\epsilon, z_\epsilon - z)_\mathcal{Z} \leq 0$$

and that

$$\|z_\epsilon - z\|_\mathcal{Z}^2 = (z_\epsilon, z_\epsilon - z)_\mathcal{Z} - (u, z_\epsilon - z)_\mathcal{Z} \leq -(u, z_\epsilon - z)_\mathcal{Z}. \tag{2.6}$$

By using Cauchy-Schwarz's inequality, it follows that

$$\|z_\epsilon\|_\mathcal{Z} \leq \|u\|_\mathcal{Z}, \quad \|z_\epsilon - z\|_\mathcal{Z} \leq \|u\|_\mathcal{Z}. \tag{2.7}$$

From z_ϵ we can hence extract a "subsequence", still denoted z_ϵ, that weakly converges to some $w \in \mathcal{Z}$ when ϵ tends to 0. From (2.5) and (**??**) we also obtain that

$$\|\Psi(z_\epsilon - z)\|_\mathcal{Y}^2 = -\epsilon(z_\epsilon, z_\epsilon - z)_\mathcal{Z} \leq \epsilon\|z\|_\mathcal{Z}^2,$$

that is

$$\|\Psi(z_\epsilon - z)\|_\mathcal{Y} \leq \sqrt{\epsilon}\,\|z\|_\mathcal{Z},$$

and then Ψz_ϵ tends to Ψz. Since Ψz_ϵ weakly converges to Ψw in $\mathcal{Y}$ we have $\Psi z = \Psi w$ and $z = w$ from injectivity of Ψ. It remains to remark from (2.6) that the weak convergence of the subsequence in $\mathcal{Z}$ implies its strong convergence in $\mathcal{Z}$. The strong convergence of all the sequence (not only the subsequence) follows from a classical contradiction argument. □

We now present an alternative proof of theorem 2.1, which is based on the spectral theorem for self-adjoint operators.

Proof (Second approach). We note that $\Lambda = \Psi^*\Psi : \mathcal{Z} \to \mathcal{Z}^*$, if we identify $\mathcal{Z}^*$ with itself, is a self-adjoint continuous operator on $\mathcal{Z}$. We can then apply the spectral theorem to T. If we consider the corresponding spectral family $E(\lambda)$ we can formally write

$$\Lambda = \int_0^{+\infty} \lambda \, dE(\lambda).$$

Here we have used the fact that Λ is positive so that we have restricted the real line to its positive values. From (2.1) and (2.2), we obtain

$$\Psi^*\Psi z_\epsilon + \epsilon \, z_\epsilon = \Psi^*\Psi z,$$

that is

$$z_\epsilon - z = \left((\Lambda + \epsilon \mathbb{1}_{\mathcal{Z}})^{-1}\Lambda - \mathbb{1}_{\mathcal{Z}}\right) u.$$

Since

$$(T + \epsilon I_{\mathcal{Z}})^{-1}T - I_{\mathcal{Z}} = \int_0^{+\infty} \left(\frac{\lambda}{\lambda+\epsilon} - 1\right) dE(\lambda) = -\int_0^{+\infty} \frac{\epsilon}{\lambda+\epsilon}\, dE(\lambda),$$

it follows that

$$\|z_\epsilon - z\|_{\mathcal{Z}}^2 = \int_0^{+\infty} \frac{\epsilon^2}{(\lambda+\epsilon)^2}\, (dE(\lambda)u, u), \tag{2.8}$$

where $(dE(\lambda)z, z)$ defines a positive measure on $\mathbb{R}^+$. It remains to apply the Lebesgue theorem. In this view we remark that unless $\lambda = 0$, $\epsilon^2/(\lambda+\epsilon)^2$ tends to 0 when ϵ tends to 0 and $\{\lambda = 0\}$ is a set the measure of which is 0 since 0 is not an eigenvalue of Λ (remember that Ψ is injective). We also remark that $\epsilon^2/(\lambda+\epsilon)^2$ is bounded by 1, and the result follows from the Lebesgue theorem. □

Remark 2.2. It should be noticed from (2.8) that $\|z_\epsilon - z\|_{\mathcal{Z}}$ is an increasing fonction of $\epsilon > 0$.

Now let us have a look at the realistic situation when the (noisy) data is y^δ in the Tikhonov regularization. Remember we have denoted z_ϵ^δ the Tikhonov solution associated with data y^δ. The difference $z_\epsilon^\delta - z_\epsilon$ is then the solution associated with data $y^\delta - y$ and, if we choose $v = z_\epsilon^\delta - z_\epsilon$ in (2.3) with such data, we obtain ($\|y^\delta - y\|_{\mathcal{Y}} \le \delta$)

$$\|z_\epsilon^\delta - z_\epsilon\|_{\mathcal{Z}} \leq \frac{\delta}{\sqrt{\epsilon}}.$$

Thus the difference between the Tikhonov solution with noisy data and the exact solution is

$$\|z_\epsilon^\delta - z\|_{\mathcal{Z}} \leq \|z_\epsilon - z\|_{\mathcal{Z}} + \frac{\delta}{\sqrt{\epsilon}}.$$

As a conclusion, for $\delta \neq 0$, there is no evidence that $z_\epsilon^\delta \to z$ when $\epsilon \to 0$. In fact, the situation is even more critical since we have the following theorem.

Theorem 2.2. *If data y^δ is not in the range of operator Ψ, then $\|z_\epsilon^\delta\|_{\mathcal{Z}} \to +\infty$ when $\epsilon \to 0$.*

Proof. By contradiction, if we assume that the assertion $\|z_\epsilon^\delta\|_{\mathcal{Z}} \to +\infty$ when $\epsilon \to 0$ is not true, this means that we can can find a subsequence of z_ϵ^δ which is bounded in $\mathcal{Z}$, from which we can extract a subsequence that weakly converges to $w \in \mathcal{Z}$. We have from (2.3)

$$(\Psi z_\epsilon^\delta, \Psi v)_{\mathcal{Y}} + \epsilon (z_\epsilon^\delta, v)_{\mathcal{Z}} = (y^\delta, \Psi v), \quad \forall v \in \mathcal{Z}$$

Passing to the limit when $\epsilon \to 0$, we obtain

$$(\Psi w, \Psi v)_{\mathcal{Y}} = (y^\delta, \Psi v), \quad \forall v \in \mathcal{Z}$$

and since Ψ has dense range $\Psi w = y^\delta$, which contradicts the fact that y^δ is not in the range of Ψ. □

2.3 The Morozov's principle

From Theorem 2.1 and Theorem 2.2, we conclude that the choice of ϵ in the presence of noisy data is intricate. The idea of Morozov's principle consists in choosing ϵ such a way that the error $\|\Psi z_\epsilon^\delta - y^\delta\|_{\mathcal{Y}}$ we make by using the Tikhonov regularization is of the same order as the error on the data $\|y^\delta - y\|_{\mathcal{Y}}$, that is δ.

We have the following theorem.

Theorem 2.3. *If the noisy data y^δ satisfies*

$$\|y^\delta - y\|_{\mathcal{Y}} \leq \delta < \|y^\delta\|_{\mathcal{Y}},$$

then there exists a unique choice $\epsilon > 0$ such that the solution of (2.3) associated with data y^δ satisfies

$$\|\Psi z_\epsilon^\delta - y^\delta\|_{\mathcal{Y}} = \delta.$$

Proof (First proof). Let us denote $g^\delta(\epsilon) = \|\Psi z_\epsilon^\delta - y^\delta\|_{\mathcal{Y}}^2$. The function g^δ is differentiable with respect to $\epsilon > 0$ and

$$\frac{\mathrm{d}g^\delta}{\mathrm{d}\epsilon}(\epsilon) = 2(\Psi z_\epsilon^\delta - y^\delta, \Psi v_\epsilon^\delta)_{\mathcal{Y}},$$

where $v_\epsilon^\delta \in \mathcal{Z}$ is uniquely defined by

$$(\Psi v_\epsilon^\delta, \Psi v)_{\mathcal{Y}} + \epsilon(v_\epsilon^\delta, v)_{\mathcal{Z}} = -(z_\epsilon^\delta, v)_{\mathcal{Z}}, \quad \forall v \in \mathcal{Z}. \tag{2.9}$$

On the other hand, by definition of z_ϵ^δ, we have

$$(\Psi z_\epsilon^\delta, \Psi v)_{\mathcal{Y}} + \epsilon(z_\epsilon^\delta, v)_{\mathcal{Z}} = (y^\delta, \Psi v)_{\mathcal{Z}}, \quad \forall v \in \mathcal{Z}. \tag{2.10}$$

By choosing $v = v_\epsilon^\delta$ in (2.10) we obtain

$$(\Psi z_\epsilon^\delta - y^\delta, \Psi v_\epsilon^\delta)_{\mathcal{Y}} = -\epsilon(z_\epsilon^\delta, v_\epsilon^\delta)_{\mathcal{Z}},$$

and by choosing $v = v_\epsilon^\delta$ in (2.9), we obtain $(z_\epsilon^\delta, v_\epsilon^\delta)_{\mathcal{Z}} \leq 0$. More precisely, we have in fact $(z_\epsilon^\delta, v_\epsilon^\delta)_{\mathcal{Z}} < 0$. Indeed, let us assume that $(z_\epsilon^\delta, v_\epsilon^\delta)_{\mathcal{Z}} = 0$. From (2.9) we obtain that $v_\epsilon^\delta = 0$, and then $z_\epsilon^\delta = 0$, which is not possible. We hence conclude that $\frac{\mathrm{d}g^\delta}{\mathrm{d}\epsilon} > 0$ for all $\epsilon > 0$. Then g^δ is a strictly increasing function. Now let us prove that

$$\lim_{\epsilon \to 0} g^\delta(\epsilon) = 0, \qquad \lim_{\epsilon \to +\infty} g^\delta(\epsilon) = \|y^\delta\|_{\mathcal{Y}}^2.$$

The second statement is obtained by taking $v = z_\epsilon^\delta$ in (2.10), which implies that

$$\|z_\epsilon^\delta\|_{\mathcal{Z}} \leq \frac{\|y^\delta\|_{\mathcal{Y}}}{\sqrt{\epsilon}},$$

and then $\Psi z_\epsilon^\delta \to 0$ in $\mathcal{Y}$ when $\epsilon \to +\infty$. Now let us prove the first statement. We also have $\|\Psi z_\epsilon^\delta\|_{\mathcal{Y}} \leq \|y^\delta\|_{\mathcal{Y}}$, hence we can extract from (z_ϵ^δ) a subsequence that we still denote (z_ϵ^δ) such that Ψz_ϵ^δ weakly converges to some $h \in \mathcal{Y}$. We have

$$(\Psi z_\epsilon^\delta - y^\delta, \Psi v)_{\mathcal{Y}} = -\epsilon(z_\epsilon^\delta, v)_{\mathcal{Z}} \leq \sqrt{\epsilon}\|y^\delta\|_{\mathcal{Y}}\|v\|_{\mathcal{Z}} \to 0$$

when $\epsilon \to 0$. We also have

$$(\Psi z_\epsilon^\delta - y^\delta, \Psi v)_{\mathcal{Y}} \to (h - y^\delta, \Psi v)_{\mathcal{Y}} = 0, \quad \forall v \in \mathcal{Z}.$$

Since Ψ has a dense range, $h = y^\delta$.
Lastly, we write

$$\begin{aligned} g^\delta(\epsilon) = \|\Psi z_\epsilon^\delta - y^\delta\|_{\mathcal{Y}}^2 &= (\Psi z_\epsilon^\delta - y^\delta, \Psi z_\epsilon^\delta)_{\mathcal{Y}} - (\Psi z_\epsilon^\delta - y^\delta, y^\delta)_{\mathcal{Y}} \\ &= -\epsilon\|z_\epsilon^\delta\|_{\mathcal{Z}}^2 - (\Psi z_\epsilon^\delta - y^\delta, y^\delta)_{\mathcal{Y}} \leq -(\Psi z_\epsilon^\delta - y^\delta, y^\delta)_{\mathcal{Y}} \to 0 \end{aligned}$$

when $\epsilon \to 0$, which completes the proof of the first statement.
We complete the proof by using the fact that g^δ is a continuous function

which increases from 0 to $\|y^\delta\|^2 > \delta^2$, then there exists a unique $\epsilon > 0$ such that $g^\delta(\epsilon) = \delta^2$. □

Here again a second is possible based on the spectral theorem.

Proof (Second proof). We again apply the spectrum theorem to the self-adjoint operator $\Lambda = \Psi\Psi^* : \mathcal{Y} \to \mathcal{Y}$. We consider the corresponding spectral family $E(\lambda)$ and write

$$\Lambda = \int_0^{+\infty} \lambda \, dE(\lambda).$$

The solution z_ϵ^δ solves

$$(\Psi\Psi^*)\Psi z_\epsilon^\delta + \epsilon \Psi z_\epsilon^\delta = (\Psi\Psi^*) y^\delta,$$

that is

$$\Psi z_\epsilon^\delta - y^\delta = \left((\Lambda + \epsilon \mathbb{1}_{\mathcal{Y}})^{-1} Q - \mathbb{1}_{\mathcal{Y}}\right) y^\delta.$$

Since

$$(\Lambda + \epsilon \mathbb{1}_{\mathcal{Y}})^{-1}\Lambda - \mathbb{1}_{\mathcal{Y}} = -\int_0^{+\infty} \frac{\epsilon}{\lambda + \epsilon} \, dE(\lambda),$$

we have that

$$\|\Psi z_\epsilon^\delta - y^\delta\|_{\mathcal{Y}}^2 = \int_0^{+\infty} \frac{\epsilon^2}{(\lambda+\epsilon)^2} \, (dE(\lambda) y^\delta, y^\delta) := y^\delta(\epsilon)$$

Clearly, y^δ is a stricly increasing function, and by using the Lebesgue theorem, y^δ is a continuous function, $y^\delta(\epsilon) \to 0$ when $\epsilon \to 0$ (here we have used the fact that 0 is not an eigenvalue of Λ, which is due to the fact that Ψ has dense range), and $y^\delta(\epsilon) \to \|y^\delta\|_{\mathcal{Y}}^2$ when $\epsilon \to +\infty$. The assumption $\|y^\delta\|_{\mathcal{Y}} > \delta$ guarantees that $y^\delta(\epsilon) = \delta^2$ for a unique value of $\epsilon > 0$. □

A natural question is to wonder if the choice $\epsilon(\delta)$ given by Morozov's principle is actually the best possible choice among all values of ϵ, for a given amplitude of noise δ. There is no positive answer to that question. However, in some very special cases, with the so-called source conditions, we have estimates for $\|z_\epsilon^\delta - z\|_{\mathcal{Z}}$ when $\epsilon(\delta)$ is the Morozov's choice. The following theorem is borrowed from Kirsch (1996).

Theorem 2.4. *In the case when z is in the range of operator Ψ^*, that is there exists a (unique) $p \in \mathcal{Y}$ such that $z = \Psi^* p$, then if $\epsilon = \epsilon(\delta)$ is the Morozov's choice associated with data y^δ and z_ϵ^δ the corresponding solution of* (2.3)*, then*

$$\|z_\epsilon^\delta - z\|_{\mathcal{Z}} \le 2\sqrt{\|p\|_{\mathcal{Y}}}\sqrt{\delta}.$$

Proof. If z_ϵ^δ is the Tikhonov solution for the Morozov's choice $\epsilon(\delta)$, in view of the minimization problem (2.4), we have

$$\|\Psi z_\epsilon^\delta - y^\delta\|_{\mathcal{Y}}^2 + \epsilon(\delta) \, \|z_\epsilon^\delta\|_{\mathcal{Z}}^2 \le \|\Psi z - y^\delta\|_{\mathcal{Y}}^2 + \epsilon(\delta) \, \|z\|_{\mathcal{Z}}^2.$$

Since $\|\Psi z_\epsilon^\delta - y^\delta\|_\mathcal{Y} = \delta$, $\Psi z = y$ and $\|y^\delta - y\|_\mathcal{Y} \le \delta$, we obtain

$$\delta^2 + \epsilon(\delta)\,\|z_\epsilon^\delta\|_\mathcal{Z}^2 \le \delta^2 + \epsilon(\delta)\,\|z\|_\mathcal{Z}^2,$$

which implies

$$\|z_\epsilon^\delta\|_\mathcal{Z} \le \|z\|_\mathcal{Z}. \tag{2.11}$$

Now, since

$$\|z_\epsilon^\delta - z\|_\mathcal{Z}^2 = \|z_\epsilon^\delta\|_\mathcal{Z}^2 - 2(z_\epsilon^\delta, u)_\mathcal{Z} + \|u\|_\mathcal{Z}^2,$$

it follows from (2.11) that

$$\|z_\epsilon^\delta - z\|_\mathcal{Z}^2 \le 2(u - z_\epsilon^\delta, u)_\mathcal{Z}.$$

On the other hand, we have

$$(z - z_\epsilon^\delta, z)_\mathcal{Z} = (z - z_\epsilon^\delta, \Psi^* p)_\mathcal{Z} = (\Psi(z - z_\epsilon^\delta), p)_\mathcal{Y} = (y - \Psi z_\epsilon^\delta, p)_\mathcal{Y}.$$

Hence

$$\begin{aligned}(z - z_\epsilon^\delta, u)_\mathcal{Z} &= (y - y^\delta, p)_\mathcal{Y} + (y^\delta - \Psi z_\epsilon^\delta, p)_\mathcal{Y} \\ &\le \left(\|y - y^\delta\|_\mathcal{Y} + \|y^\delta - \Psi z_\epsilon^\delta\|_\mathcal{Y}\right)\|p\|_\mathcal{Y} \le 2\delta\,\|p\|_\mathcal{Y}.\end{aligned}$$

We finally have

$$\|z_\epsilon^\delta - z\|_\mathcal{Z}^2 \le 4\delta\,\|p\|_\mathcal{Y},$$

which completes the proof. □

Remark 2.3. Note that the source condition $z \in R(\Psi^*)$ cannot be verified in practice since the exact solution z is unknown. On the other hand, even if the Morozov's principle is not justified from a theoretical point of view, it gives very good results in practice.

2.4 Interpretation with duality in optimization

We complete this chapter by an interpretation of the Morozov's principle with the help of duality in optimization. It will also enable us to obtain a practical method to compute the Morozov's value of $\epsilon(\delta)$ and the corresponding Tikhonov solution z_ϵ^δ.

In this view we consider the optimization problem (compare with problem (2.4)) :

$$\inf_{v \in K} L(v) \tag{P}$$

with

$$L(v) = \frac{1}{2}\|v\|_\mathcal{Z}^2, \quad K = \{v \in \mathcal{Z}\,;\, \|Av - y^\delta\|_\mathcal{Y} \le \delta\}.$$

We have the following theorem.

Theorem 2.5. *For $y^\delta \in \mathcal{Y}$, Problem (P) has a unique solution $z^\delta \in \mathcal{Z}$.*

Proof. We first use the classical existence theorem for optimization problems. The set K is a convex, closed and not empty subset of $\mathcal{Z}$. This last result follows from the fact that Ψ has dense range. The function L is convex and continuous. Since L is coercive, that is $L(v) \to +\infty$ when $\|v\|_{\mathcal{Z}} \to +\infty$, Problem (P) has at least one solution. Such solution is unique since L is strictly convex. $\square$

Remark 2.4. We note that if $\|y^\delta\|_{\mathcal{Y}} \leq \delta$, we have $z^\delta = 0$, while if $\|y^\delta\|_{\mathcal{Y}} > \delta$, $z^\delta \neq 0$.

Now we recall some notions of duality in optimization, following Ekeland & Temam (1974). We consider an optimization problem (Q) denoted the primal problem :

$$\inf_{v \in \mathcal{Z}} F(v), \tag{Q}$$

where $\mathcal{Z}$ is a Hilbert space, $F : \mathcal{Z} \to \overline{\mathbb{R}}$ a function $\neq +\infty$. Here we have denoted $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. We recall what is the conjugate function.

Definition 2.1. The conjugate function $F^* : \mathcal{Z}^* \to \overline{\mathbb{R}}$ of F is defined, for $u^* \in \mathcal{Z}^*$, by

$$F^*(u^*) = \sup_{u \in \mathcal{Z}} (\langle u, u^* \rangle_{\mathcal{Z},\mathcal{Z}^*} - F(u)).$$

Then we introduce the notion of perturbed problem. We consider a function $\Phi : \mathcal{Z} \times \mathcal{Y} \to \overline{\mathbb{R}}$, where $\mathcal{Y}$ is another Hilbert space, and Φ satisfies

$$\Phi(v, 0) = F(v).$$

For all $q \in \mathcal{Y}$, we consider the perturbed problem (Q_q):

$$\inf_{v \in \mathcal{Z}} \Phi(v, q). \tag{Q_q}$$

Then we define the dual problem of Problem (Q) with respect to the perturbation Φ. Let $\Phi^* : \mathcal{Z}^* \times \mathcal{Y}^* \to \overline{\mathbb{R}}$ be the conjugate function of Φ. The dual problem, denoted (Q^*), is the following optimization problem:

$$\sup_{q^* \in \mathcal{Y}^*} -\Phi^*(0, q^*). \tag{Q^*}$$

We have the following proposition.

Proposition 2.1.

$$(-\infty \leq) \quad \sup(Q^*) \leq \inf(Q) \quad (\leq +\infty).$$

Proof. For $q^* \in \mathcal{Y}^*$, we have

$$\Phi^*(0, q^*) = \sup_{u \in \mathcal{Z},\, q \in \mathcal{Y}} (\langle u, 0\rangle_{\mathcal{Z},\mathcal{Z}^*} + \langle q, q^*\rangle_{\mathcal{Y},\mathcal{Y}^*} - \Phi(u, q)),$$

so that, for all $u \in \mathcal{Z}$,

$$\Phi^*(0, q^*) \geq \langle 0, q^*\rangle_{\mathcal{Y},\mathcal{Y}^*} - \Phi(u, 0) = -\Phi(u, 0).$$

We hence have $\forall u \in \mathcal{Z}$, $\forall q^* \in \mathcal{Y}^*$, $-\Phi^*(0, q^*) \leq \Phi(u, 0)$, and finally $\sup(P^*) \leq \inf(P)$. $\square$

Remark 2.5. Equality is not satisfied in general: when $\sup(Q^*) \neq \inf(Q)$, we say that there is a duality gap.

We admit the following theorem, which is proved in Ekeland & Temam (1974), and which guarantees equality.

Theorem 2.6. *We assume that Φ is convex and that $\inf(P) < +\infty$. If there exists $u_0 \in \mathcal{Z}$ such that $q \to \Phi(u_0, q)$ is finite and continuous at point 0, then $\inf(Q) = \sup(Q^*) < +\infty$ and Problem (Q^*) has solutions. We say that Problem (Q) is stable.*

Now let us apply the above theorem to our Problem (P). With previous notations, we define

$$F(z) = L(z) + \chi_{B_\delta}(\Psi z),$$

and

$$\Phi(z, q) = L(z) + \chi_{B_\delta}(\Psi z - q), \tag{2.12}$$

where $B_\delta \subset \mathcal{Y}$ is the closed ball of center y^δ and radius δ, χ_{B_δ} is the indicator function defined by

$$\begin{cases} \chi_{B_\delta}(q) = 0, & \text{if } q \in B_\delta, \\ \chi_{B_\delta}(q) = +\infty, & \text{if } q \notin B_\delta. \end{cases}$$

The function F is defined on $\mathcal{Z}$ while Φ is defined on $\mathcal{Z} \times \mathcal{Y}$. Let us form the dual problem (P^*) which, after some simple computations, is defined by

$$\sup_{q* \in \mathcal{Y}} -\Phi^*(0, q^*) = \sup_{q* \in \mathcal{Y}} \Big(- L^*(\Psi^* q^*) - \chi^*_{B_\delta}(-q^*)\Big). \tag{P^*}$$

It remains to compute L^* and $\chi^*_{B_\delta}$.

It is easy to see that

$$L^*(v^*) = \frac{1}{2}\|v^*\|^2_{\mathcal{Z}^*}.$$

For $q^* \in \mathcal{Y}$, we have

$$\chi^*_{B_\delta}(q^*) = \sup_{q \in \mathcal{Y}}((q, q^*)_{\mathcal{Y}} - \chi_{B_\delta}(q)),$$

that is

$$\chi^*_{B_\delta}(q^*) = \sup_{\substack{q\in\mathcal{Y} \\ \|q-y^\delta\|_\mathcal{Y}\le\delta}} (q,q^*)_\mathcal{Y} = (y^\delta,q^*)_\mathcal{Y} + \delta \sup_{\substack{q\in\mathcal{Y}, \\ \|q\|_\mathcal{Y}\le 1}} (q,q^*)_\mathcal{Y}$$
$$= (y^\delta,q^*)_\mathcal{Y} + \delta\,\|q^*\|_\mathcal{Y}.$$

Therefore, Problem (P^*) corresponds to

$$\sup_{q^*\in\mathcal{Y}} -\Phi^*(0,q^*) = \sup_{q^*\in\mathcal{Y}} \Big(-\frac{1}{2}\|\Psi^* q^*\|^2_{\mathcal{Z}^*} - \delta\,\|q^*\|_\mathcal{Y} + (y^\delta,q^*)_\mathcal{Y}\Big)$$

which finally rewrites into

$$\inf_{q^*\in\mathcal{Y}} G(q^*) = \inf_{q^*\in\mathcal{Y}} \Big(\frac{1}{2}\|\Psi^* q^*\|^2_{\mathcal{Z}^*} + \delta\,\|q^*\|_\mathcal{Y} - (y^\delta,q^*)_\mathcal{Y}\Big). \qquad (\tilde{P}^*)$$

Remark 2.6. Is is remarkable that while the primal problem (P) is a constrained minimization problem, the dual problem (P^*) is an unconstrained minimization problem. Therefore, (P^*) – *i.e.* $(\tilde{P}^*)$ – is easier to solve than (P).

Now let us check that we satisfy the assumptions of Theorem 2.6. Clearly Φ given by (2.12) is a convex function of (u,q) and $\inf(P) < +\infty$ since Problem P has a (unique) solution. Let is choose $z_0 \in \mathcal{Z}$ such that $\|\Psi z_0 - y^\delta\|_\mathcal{Y} \le \delta/2$. For any $q\in\mathcal{Y}$ such that $\|q\|_\mathcal{Y}\le\delta/2$, we have $\|\Psi z_0 - q - y^\delta\|_\mathcal{Y}\le\delta$, that is $q \to \Phi(z_0,q) = L(z_0) < +\infty$ is constant in a neighborhood of point 0. We can then apply Theorem 2.6. In particular, it implies that (P^*) has solutions and that

$$\inf(P) = \sup(P^*) < +\infty.$$

Let p^* be such a solution of (P^*), the above relationship implies that

$$\frac{1}{2}\|z^\delta\|^2_\mathcal{Z} = -\frac{1}{2}\|\Psi^* p^*\|^2_{\mathcal{Z}^*} - \delta\,\|p^*\|_\mathcal{Y} + (y^\delta,p^*)_\mathcal{Y}. \qquad (2.13)$$

We now use the fact that G is Fréchet-differentiable at any point $q\neq 0$ (because of the term $\delta\,\|q\|_\mathcal{Y}$) and express the optimality at point p^*. In this view we have to verify that $p^*\neq 0$. Actually, let us take $y_0^\delta = y^\delta/\|y^\delta\|_\mathcal{Y}$ and $q^* = \epsilon y_0^\delta \in \mathcal{Y}$. For $\epsilon>0$, we have

$$G(\epsilon y_0^\delta) = \frac{\epsilon^2}{2}\|\Psi^* y_0^\delta\|^2_{\mathcal{Z}^*} + \epsilon(\delta - \|y^\delta\|_\mathcal{Y}).$$

For small ϵ, $G(\epsilon y_0^\delta)$ has the sign of $\delta - \|y^\delta\|_\mathcal{Y} < 0$, hence there exists $q^*\in\mathcal{Y}$ such that $G(q^*) < 0 = G(0)$, and the solutions p^* do not vanish. Optimality writes $G'(p^*) = 0$ with

$$G'(p^*) = \Psi(\Psi^* p^*) + \delta \frac{p^*}{\|p^*\|_{\mathcal{Y}}} - y^\delta,$$

We have in particular

$$\|\Psi(\Psi^* p^*) - y^\delta\|_{\mathcal{Y}} = \delta.$$

We conclude from (2.13) that

$$\frac{1}{2}\|z^\delta\|_{\mathcal{Z}}^2 = \frac{1}{2}\|\Psi^* p^*\|_{\mathcal{Z}^*}^2,$$

that is the Riesz representant in $\mathcal{Z}$ of $\Psi^* p^* \in \mathcal{Z}^*$ solves the primal problem (P), which has the unique solution z^δ. We conclude that $z^\delta = \Psi^* p^*$ – provided we identify $\Psi^* p^* \in \mathcal{Z}^*$ with its representant in $\mathcal{Z}$ – which gives uniqueness of the solution p^* to Problem (P^*) since Ψ has dense range. Such solution is denoted p^δ. Incidentally, we note that

$$\|\Psi z^\delta - y^\delta\|_{\mathcal{Y}} = \delta.$$

Lastly, we come back to the Tikhonov solution $z_\epsilon^\delta \in \mathcal{Z}$ of problem (2.2) with data y^δ and ϵ chosen with Morozov's principle. The solution $z_\epsilon^\delta \in \mathcal{Z}$ solves

$$\Psi^*(\Psi z_\epsilon^\delta) + \epsilon z_\epsilon^\delta - \Psi^* y^\delta = 0.$$

On the other hand, the solution z^δ of problem (P) satisfies

$$\Psi^*(\Psi z^\delta) + \frac{\delta}{\|p^\delta\|_{\mathcal{Y}}} z^\delta - \Psi^* y^\delta = 0.$$

We hence conclude that z^δ is in fact the Tikhonov solution z_ϵ^δ with $\epsilon = \epsilon(\delta)$ given by

$$\epsilon(\delta) = \frac{\delta}{\|p^\delta\|_{\mathcal{Y}}},$$

which provides an explicit expression of the value $\epsilon(\delta)$ given by the Morozov's principle. All these results are summarized in the following theorem.

Theorem 2.7. *If the noisy data y^δ satisfies*

$$\|y^\delta - y\|_{\mathcal{Y}} \le \delta < \|y^\delta\|_{\mathcal{Y}},$$

then Problem (P) and Problem (P^*) have a unique solution z^δ and p^δ, respectively, such that $z^\delta = \Psi^* p^\delta$. Furthermore, z^δ coincides with the Tikhonov solution z_ϵ^δ of problem (2.2) with $\epsilon(\delta)$ chosen following Morozov's principle. Lastly, $p^\delta \neq 0$ and

$$\epsilon(\delta) = \frac{\delta}{\|p^\delta\|_{\mathcal{Y}}}.$$

The above theorem hence provides a strategy to find the Tikhonov-Morozov solution associated with noisy data y^δ. It consists in solving first ($\tilde{P}^*$). Our solution is then obtained by applying Ψ^* to the solution of ($\tilde{P}^*$).

Remark 2.7. It can be proved directly – namely without using Theorem 2.6 – that the dual problem ($\tilde{P}^*$) has a unique solution $p^\delta \in \mathcal{Y}$.

2.5 Time-dependent problems

We will now apply the Tikhonov regularization method to estimate quantities associated with evolution equation, for instance a heat equation or a wave equation observed on a subdomain. As presented in the introduction, this problems can be rewritten in an abstract form using the semigroup theory – see Appendix A – namely we consider a dynamical system

$$\begin{cases} \dot{z}(t) = Az(t), \quad t \in \mathbb{R}^+ \\ z(0) = \zeta \end{cases} \tag{2.14}$$

where A is an unbounded operator of domain $D(A)$, generator of a continuous semigroup of contraction $(\Phi(t))_{t\geq 0}$. We additionally consider to have at our disposal some measurements $\{y(t), t \geq 0\}$ associated with a given trajectory of (2.14). We thus denote by C a bounded observation operator such that

$$C : \left| \begin{array}{l} \mathcal{Z} \to \mathcal{Y} \\ z \to y \end{array} \right. \tag{2.15}$$

From this collection of measurements obtained through the combination of the trajectory – *i.e.* the dynamics – and the observation operator, we want to recover the initial condition of (2.14). Therefore, we define an operator Ψ to be inverted as the operator such that for noiseless data we have

$$\mathcal{Y} \ni y(t) = \Psi(t)\zeta, \quad t \in \mathbb{R}^+.$$

and we directly infer that

$$\Psi(t) = Ce^{tA}. \tag{2.16}$$

For all $T \geq 0$, we now define by Ψ_T the operator

$$\Psi_T : \left| \begin{array}{l} \mathcal{Z} \to \mathcal{Y}_T \\ \zeta \to (y(t))_{t\in[0,T]} \end{array} \right. \tag{2.17}$$

where $\mathcal{Y}_T = L^2((0,T);\mathcal{Y})$. The question of proving the injectivity of (2.17) will be discussed in the next chapters. Here we assume, typically that Ψ_T is injective but not surjective. However, as mentioned in Remark 2.1, the lack of injectivity will also be handled by the Tikhonov regularization procedure which consists in minimizing (2.4), namely here

$$\begin{aligned}
\mathscr{J}_T(\zeta) &= \frac{1}{2}\,\|\Psi_T\zeta - y^\delta\|^2_{\mathcal{Y}_T} + \frac{\epsilon}{2}\,\|\lambda\|^2_{\mathcal{Z}}, && (2.18a)\\
&= \int_0^T \frac{1}{2}\|\Psi\zeta - y^\delta(t)\|^2_{\mathcal{Y}}\,\mathrm{d}t + \frac{\epsilon}{2}\,\|\lambda\|^2_{\mathcal{Z}} && (2.18b)\\
&= \int_0^T \frac{1}{2}\|Cz_\zeta(t) - y^\delta(t)\|^2_{\mathcal{Y}}\,\mathrm{d}t + \frac{\epsilon}{2}\,\|\lambda\|^2_{\mathcal{Z}} && (2.18c)
\end{aligned}$$

where in the last identity $z_\zeta(t)$ should be understood as a solution of (2.14) associated with ζ, namely the trajectory "knowing" ζ. Note that the minimization $\min_{\zeta\in\mathcal{Z}} \mathscr{J}_T$ is associated with its corresponding normal equation (2.2). Namely here, $\bar{\zeta}^\delta = \mathrm{argmin}_{\zeta\in\mathcal{Z}}\, \mathscr{J}_T$ is given by

$$\int_0^T e^{tA^*}C^*Ce^{tA}\bar{\zeta}^\delta\,\mathrm{d}t + \epsilon\,\lambda^\delta_\epsilon = \int_0^T e^{tA^*}C^*y^\delta(t)\,\mathrm{d}t. \tag{2.19}$$

One difficulty associated with the use of (2.19) is the computing cost as we need to compute operators defined over the whole time window $[0,T]$. In fact, it is possible to simplify drastically the expression (2.19) by relying on the so-called adjoint equation, namely the solution of

$$\begin{cases}\dot{q}(t) + A^*q(t) = -C^*(y(t) - Cz(t)), & t\in(0,T)\\ q(T) = 0\end{cases} \tag{2.20}$$

defined for any given $\{y(t), t\in[0,T]\}$. Note here that q is defined only on $[0,T]$ and in reverse-time. The dynamics (2.20) is well-posed because $-A^*$ is the generator of a semigroup in reverse-time as soon as A^* is the generator of a semigroup in forward-time which is the case as soon as A is the generator of a semigroup in forward-time. From Duhamel formula – see Appendix A – we have for any given $\{y(t), t\in[0,T]\}$,

$$q(t) = \int_t^T e^{(s-t)A^*}C^*(y(s) - Cz(s))\,\mathrm{d}s, \quad t\in[0,T],$$

Hence, recalling that $z(s) = e^{tA}\zeta$ we find that (2.19) is equivalent to

$$\bar{\zeta}^\delta = \frac{1}{\epsilon}\bar{q}^\delta(0). \tag{2.21}$$

where $\bar{q}^\delta$ is associated with the measurements y^δ and the optimal trajectory $\bar{z}^\delta = e^{tA}\bar{\zeta}^\delta$. Therefore, the identity (2.19) is very seducing but hides a self-linked identity. Indeed, we may want to solve an adjoint equation to compute $\bar{\zeta}^\delta$. However, the adjoint equation is defined backward in time with the use of the forward solution z^δ defined from $\bar{\zeta}^\delta$ itself. In other words, the optimality system, associated with $\bar{\zeta}^\delta$ is a two-end problem given by

$$\begin{cases} \dot{\bar{z}}^\delta = A\bar{z}^\delta, & \text{in } (0,T) \\ \dot{\bar{q}}_\epsilon^\delta + A^*\bar{q}_\epsilon^\delta = -C^*(y^\delta - C\bar{z}^\delta), & \text{in } (0,T) \\ \bar{z}^\delta(0) = \dfrac{1}{\epsilon}\bar{q}^\delta(0) \\ \bar{q}^\delta(T) = 0 \end{cases} \tag{2.22}$$

To circumvent this difficulty and propose a practical approach for computing $\bar{\zeta}^\delta$, we will now propose a gradient descent algorithm. From its definition (2.18a), $\mathscr{J}_T$ is a quadratic functional, hence we have for all $(\zeta,\xi) \in \mathcal{Z}^2$

$$\mathscr{J}_T(\zeta+\xi) - \mathscr{J}_T(\zeta) = \ell(\xi) + b(\xi,\xi),$$

where – considering that C is a bounded operator – we have

$$b(\xi,\xi) = \int_0^T (Ce^{tA}\xi, Ce^{tA}\xi)_{\mathcal{Y}}\,\mathrm{d}t + \epsilon\,(\xi,\xi)_{\mathcal{Z}} = O(\|\xi\|^2).$$

Therefore, $\mathscr{J}_T$ admits a Fréchet derivative given by

$$\begin{aligned} \mathrm{d}\mathscr{J}_T(\zeta)(\xi) &= \ell(\xi) \\ &= \int_0^T (Ce^{tA}\xi, Ce^{tA}\zeta - y^\delta)_{\mathcal{Y}}\,\mathrm{d}t + \epsilon\,(\xi,\zeta)_{\mathcal{Z}} \\ &= \int_0^T (\xi, e^{tA^*}C^*(Cz - y^\delta))_{\mathcal{Y}}\,\mathrm{d}t + \epsilon\,(\xi,\zeta)_{\mathcal{Z}} \\ &= -(q(0),\xi)_{\mathcal{Y}} + \epsilon(\zeta,\xi). \end{aligned} \tag{2.23}$$

Hence, the gradient of $\mathscr{J}_T$ is given by

$$\nabla\mathscr{J}_T(\zeta) = -q_\zeta^\delta(0) + \epsilon\zeta. \tag{2.24}$$

where q_ζ^δ is associated with the trajectory z_ζ and the measurements y^δ through the dynamics (2.20). Therefore, a descent gradient algorithm consists in solving, from $\zeta^0 = 0$ and until convergence, the recursive relation

$$\zeta^{k+1} = \zeta^k - \rho_k\nabla\mathscr{J}_T(\zeta^k), \quad k \geq 0 \tag{2.25}$$

for an adequately chosen relaxation sequence $(\rho_k)_{k\geq 0}$ as specified in the next theorem. More precisely, (2.25) consists in recursively solving – from $z^0 \equiv 0$ and for $k \geq 0$ – the back and forth dynamics

$$\begin{cases} \dot{z}_{k+1} = Az_{k+1}, & \text{in } (0,T) \\ z_{k+1}(0) = (\mathbb{1} - \rho_k \epsilon) z_k(0) + \rho_k \epsilon q_k(0) \end{cases} \tag{2.26a}$$

and

$$\begin{cases} \dot{q}_{k+1} + A^* q_{k+1} = -C^*(y^\delta - Cz_{k+1}), & \text{in } (0,T) \\ q_{k+1}(T) = 0 \end{cases} \tag{2.26b}$$

where typically we would expect to be able to choose $(\rho_k)_{k\geq 0}$ close to $1/\epsilon$. However, we are limited by the following result.

Theorem 2.8. *Let consider a Hilbert space $\mathcal{Z}$ and a functional $\mathscr{J} : \mathcal{Z} \to \mathbb{R}$ with a Frechet derivative in $\mathcal{Z}$. We assume that there exist $\alpha, m > 0$ such that*

$$\forall (z_2, z_1) \in \mathcal{Z}^2, \quad (\nabla \mathscr{J}(z_2) - \nabla \mathscr{J}(z_1), z_2 - z_1) \geq \alpha \|z_2 - z_1\|^2,$$

and

$$\forall (z_2, z_1) \in \mathcal{Z}^2, \quad \|\nabla \mathscr{J}(z_2) - \nabla \mathscr{J}(z_1)\| \leq m \|z_2 - z_1\|.$$

Then, if there exists $a, b > 0$ such that, for all $k > 0$, $a \leq \rho_k \leq b < \frac{2\alpha}{m^2}$, then the gradient descent method defined from any z_0 by

$$z_{k+1} = z_k - \rho_k \nabla \mathscr{J}(z_k), \quad k \geq 0$$

converges geometrically to $\bar{z} = \operatorname{argmin} \mathscr{J}$. Namely, there exists $\beta(\alpha, m, a, b)$ such that

$$\beta < 1, \text{ and } \|z_k - \bar{z}\| \leq \beta^k \|z_0 - \bar{z}\|.$$

Proof. Using the optimality condition $\nabla \mathscr{J}(\bar{z}) = 0$, we have

$$z_{k+1} - \bar{z} = (z_k - \bar{z}) - \rho_k \big(\nabla \mathscr{J}(z_k) - \nabla \mathscr{J}(\bar{z})\big)$$

We then obtain with $\rho_k > 0$,

$$\begin{aligned} \|z_{k+1} - \bar{z}\|^2 = \|z_k - \bar{z}\|^2 &- 2\rho_k \big(\nabla \mathscr{J}(z_k) - \nabla \mathscr{J}(\bar{z}), z_k - \bar{z}\big) \\ &+ \rho_k^2 \|\nabla \mathscr{J}(z_k) - \nabla \mathscr{J}(\bar{z})\|^2 \\ \leq (1 - 2\alpha\rho_k + m^2\rho_k^2) &\|z_k - \bar{z}\|^2. \end{aligned}$$

From the study of the polynomial $\pi : \rho \mapsto 1 - 2\alpha\rho + m^2\rho^2$, (see Figure 2.1), we find that with $\beta = \max(\pi(a), \pi(b))$,

$$a \leq \rho_k \leq b < \frac{2\alpha}{m^2} \Rightarrow \sqrt{1 - 2\alpha\rho_k + m^2\rho_k^2} \leq \beta < 1,$$

which concludes the proof $\square$.

Remark 2.8. The presented strategies generalizes easily to more general formulations, typically a dynamics of the form

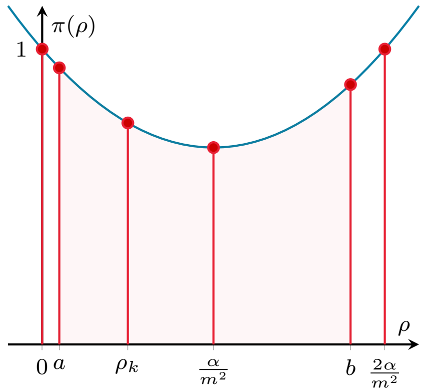

Fig. 2.1 relazation parameter choice from polynomial $\pi : \rho \mapsto 1-2\alpha\rho+m^2\rho^2$ study

$$\begin{cases} \dot{z} = Az + \beta \\ z(0) = z_\diamond + \zeta \end{cases}$$

Chapter 3
Unique continuation results

3.1 Introduction

Let us consider a connected open domain $\Omega \subset \mathbb{R}^2$ and $\omega \subset \Omega$, with $\omega \neq \emptyset$. Assume that some harmonic distribution $u \in \mathcal{D}'(\Omega)$, that is $\Delta u = 0$ in Ω, satisfies $u = 0$ in ω. Since u is infinitely smooth in Ω, we may consider the complex function of complex variable

$$f(x+iy) = \tilde{f}(x,y) = \frac{\partial u}{\partial x}(x,y) - i\frac{\partial u}{\partial y}(x,y).$$

It is clear that $\tilde{f}$ is infinitely smooth with respect to (x,y) and that the Cauchy-Riemann relationships are satisfied, namely

$$\frac{\partial \tilde{f}}{\partial x} + i\frac{\partial \tilde{f}}{\partial y} = 0.$$

Hence, the function f is holomorphic (or, equivalently, analytic) in Ω. Since f vanishes in the open set ω, it vanishes in the connected set Ω. Then $\partial_x u$ and $\partial_y u$ vanish in Ω, which means that u is a constant. Such constant is 0 because u vanishes in ω. The conclusion is that any harmonic distribution which vanishes in ω vanishes in Ω.

The above argument is based on complex analysis and therefore restricted to the case of the Laplace operator in dimension 2. For higher dimension and for more general operators (though with constant coefficients), we can extend the previous unique continuation result with the help of the Holmgren's theorem.

3.2 The Holmgren's theorem

Let us begin with the following Holmgren's theorem (see Hormander (1976) for a proof). For we need to introduce the notion of characteristic plane with respect to an operator.

Definition 3.1. Let $P(D)$ be a differential operator with constant coefficients, $D = (\partial_{x_1}, \partial_{x_2}, \cdots, \partial_{x_d})$, and let us consider $P_0(D)$ its principal part. A plane of normal vector $N \in \mathbb{R}^d \backslash \{0\}$ is said to be characteristic with respect to P if $P_0(N) = 0$.

Let us see a few fundamental examples.

1. Case of the Laplacian in $\mathbb{R}^d$: for $x = (x_1, x_2, \cdots, x_d)$, we have $P(D) = \Delta = \sum_{i=1}^d \partial_{x_i}^2 = P_0(D)$, so that
$$P_0(N) = \sum_{i=1}^d N_i^2 = |N|^2.$$
Since $P_0(N) \neq 0$ for any $N \neq 0$, there are no characteristic plane for the Laplacian operator.
2. Case of the heat operator in $\mathbb{R}^{d+1}$: for $(x,t) = (x_1, x_2, \cdots, x_d, t)$ we have $P(D) = \partial_t - \Delta = \partial_t - \sum_{i=1}^d \partial_{x_i}^2$, so that $P_0(D) = -\sum_{i=1}^d \partial_{x_i}^2$. For $N = (N_1, N_2, \cdots, N_d, N_{d+1})$, we have
$$P_0(N) = -\sum_{i=1}^d N_i^2.$$
Hence $P_0(N) = 0$ if and only of $N_i = 0$, $i = 1, \cdots, d$. The characteristic planes for the heat equation are the planes of equation $t = c$, where c is a constant.
3. Case of the wave operator in $\mathbb{R}^{d+1}$: $P(D) = \partial_t^2 - \Delta = \partial_t^2 - \sum_{i=1}^d \partial_{x_i}^2 = P_0(D)$. We have
$$P_0(N) = N_{d+1}^2 - \sum_{i=1}^d N_i^2,$$
so that the characteristic planes for the wave equation are determined by $N_{d+1}^2 = \sum_{i=1}^d N_i^2$.

The Holmgren's theorem is the following statement.

Theorem 3.1. *Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be open convex domains in $\mathbb{R}^d$ such that $\mathcal{O}_1 \subset \mathcal{O}_2$, and let $P(D)$ be a differential operator with constant coefficients such that every plane which is characteristic with respect to P and intersects $\mathcal{O}_2$ also meets $\mathcal{O}_1$. Every $u \in \mathcal{D}'(\mathcal{O}_2)$ satisfying the equation $P(D)u = 0$ and vanishing in $\mathcal{O}_1$ then vanishes in $\mathcal{O}_2$.*

3.3 Unique continuation

3.3.1 Propagation of uniqueness

In order to in use Holmgren's theorem for non convex sets $\mathcal{O}_1$ and $\mathcal{O}_2$, we need to introduce the notion of geodesic distance in a domain Ω of $\mathbb{R}^d$.

Definition 3.2. If $\Omega \subset \mathbb{R}^d$ is a connected open domain, for $x, y \in \Omega$, the geodesic distance between x and y is defined by

$$d_\Omega(x,y) = \inf\{\ell(g),\ g : [0,1] \to \Omega,\ g(0) = x,\ g(1) = y\},$$

where g is a continuous path in Ω of length $\ell(g)$. Here, the length of g is defined as

$$\ell(g) = \sup \left\{ \sum_{j=0}^{n-1} |g(t_j) - g(t_{j+1})|,\ n \in \mathbb{N},\ 0 = t_0 \le t_1 \le \cdots \le t_n = 1 \right\},$$

where the sup is taken over all decompositions of $[0,1]$ into an arbitrary (finite) number of intervals.

We also need to introduce the notion of sequence of balls as in Robbiano (1991).

Lemma 3.1. *Consider two points x_0 and x in the open and connected domain Ω. For all $\epsilon, \delta_0 > 0$, there exists some $\delta \in (0, \delta_0)$ and a δ-sequence of balls $B(q_n, \delta)$ for $n = 0, \cdots, N$ that links x_0 to x, that is*

$$\begin{cases} q_0 = x_0, \\ B(q_{n+1}, \delta) \subset B(q_n, 2\delta), \quad n = 0, \cdots, N-1, \\ B(q_n, 3\delta) \subset \Omega, \quad n = 0, \cdots, N, \\ q_N = x, \end{cases}$$

with $|q_n - q_{n+1}| \le \delta$ for $n = 0, \cdots, N-1$ and such that $N\delta \le d_\Omega(x_0, x) + \epsilon$.

Proof. We denote by g a continuous path joining x_0 to x and included in Ω such that $\ell(g) \le d_\Omega(x_0, x) + \epsilon$. We hence have $g\ :\ [0,1] \to \Omega$ with $g(0) = x_0$ et $g(1) = x$. Let us define

$$\mu = \inf_{t \in [0,1]} d(g(t), \Omega^c) > 0.$$

We choose $\delta < \mu/3$, and we construct a δ-sequence of balls as follows. We define the sequence (α_j) by induction : $\alpha_0 = 0$ and while $\alpha_j < 1$,

$$\alpha_{j+1} = \sup\{\alpha \in [0,1], \quad |g(\alpha) - g(\alpha_j)| < \delta\}.$$

The sequence (α_j) is increasing. Let us prove that for sufficiently large j, $\alpha_j = 1$. We remark that if $\alpha_{j+1} < 1$, then $|g(\alpha_{j+1}) - g(\alpha_j)| = \delta$. Indeed, by definition of α_{j+1}, $|g(\alpha_{j+1}) - g(\alpha_j)| \leq \delta$. On the other hand, for all $\alpha > \alpha_{j+1}$, $|g(\alpha) - g(\alpha_j)| \geq \delta$, and g being continuous, $|g(\alpha_{j+1}) - g(\alpha_j)| \geq \delta$.
Assume that $\alpha_j < 1$ for all j. Then the sequence (α_j) converges and $|g(\alpha_{j+1}) - g(\alpha_j)| \to 0$ when $j \to +\infty$, which contradicts the above remark. We hence have $\alpha_j = 1$ for $j \geq N$, and we have

$$\alpha_0 = 0 < \alpha_1 < \cdots < \alpha_{N-1} < \alpha_N = 1.$$

It remains to prove that if $q_j = g(\alpha_j)$ pour $j = 0, \cdots, N$, the $B(q_j, \delta)$ form a δ-sequence of balls joining x_0 to x.
Obviously $q_0 = g(\alpha_0) = x_0$ and $q_N = g(\alpha_N) = x$. Besides, if $y \in B(g(\alpha_{j+1}), \delta)$ for $j = 0, \cdots, N-1$, we have

$$|y - g(\alpha_j)| \leq |y - g(\alpha_{j+1})| + |g(\alpha_{j+1}) - g(\alpha_j)| < \delta + \delta = 2\delta,$$

hence $B(q_{j+1}, \delta) \subset B(q_j, 2\delta)$.
Lastly we have $B(q_j, 3\delta) \subset \Omega$ for $j = 0, \cdots, N$, since $d(g(\alpha_j), \Omega^c) \geq \mu > 3\delta$.
Now we have to prove that $\delta N \leq d_\Omega(x_0, x) + \epsilon$. We note that for $j = 0, \cdots, N-2$ the segment $[g(\alpha_j), g(\alpha_{j+1})]$ is included in Ω. We hence have

$$\sum_{j=0}^{N-2} |g(\alpha_{j+1}) - g(\alpha_j)| \ \leq \ell(g) \leq d_\Omega(x_0, x) + \epsilon,$$

that is, since $|g(\alpha_{j+1}) - g(\alpha_j)| = \delta$ for $j = 0, \cdots, N-2$, $\delta(N-1) \leq D + \epsilon$.
□

Let us now derive from Holmgren's theorem some unique continuation results of practical use for the Laplace, heat and wave equations.

3.3.2 Case of the Laplace and heat equations

Theorem 3.2. *Let ω and Ω be open domains in $\mathbb{R}^d$ such that $\omega \subset \Omega$. We assume that $\omega \neq \emptyset$ and that Ω is connected. Every $u \in \mathcal{D}'(\Omega)$ satisfying the equation $\Delta u = 0$ and vanishing in ω then vanishes in Ω.*

Proof. Let us consider some $x_0 \in \omega$ and $\delta_0 > 0$ such that $B(x_0, \delta_0) \subset \omega$. We consider now any $x \in \Omega$. From Lemma 3.1, there exists some $\delta \in (0, \delta_0)$ and a δ-sequence of balls joining x_0 to x. We apply N times Holmgren's theorem for operator $P = \Delta$ with $\mathcal{O}_1 = B(q_n, \delta)$ and $\mathcal{O}_2 = B(q_n, 2\delta)$, for $n = 1, \cdots, N-1$. Since u vanishes in $B(q_0, \delta)$, it vanishes in $B(q_0, 2\delta)$, and since $B(q_1, \delta) \subset B(q_0, 2\delta)$, it vanishes in $B(q_1, \delta)$. We conclude that u vanishes in $B(q_N, \delta)$, that is in $B(x, \delta)$. □

Remark 3.1. It is clear that Theorem 3.2 still holds if the Laplacian is perturbed by a term of lower order, that is for operator $P = \Delta + A \cdot \nabla + b$, where $A \in \mathbb{R}^d$ and $b \in \mathbb{R}$ are constants.

Theorem 3.3. *Let ω and Ω be open domains in $\mathbb{R}^d$ such that $\omega \subset \Omega$. We assume that $\omega \neq \emptyset$ and that Ω is connected. Let us consider some $T > 0$. Every $u \in \mathcal{D}'(\Omega \times (0,T))$ satisfying the equation $\partial_t u - \Delta u = 0$ and vanishing in $\omega \times (0,T)$ then vanishes in $\Omega \times (0,T)$.*

Proof. The proof begins like the one of Theorem 3.2. We introduce some $x_0 \in \omega$ and $\delta_0 > 0$ such that $B(x_0, \delta_0) \subset \omega$, some $x \in \Omega$ as well as a δ-sequence of balls joining x_0 to x. We apply N times Holmgren's theorem for operator $P = \partial_t - \Delta$ with $\mathcal{O}_1 = B(q_n, \delta) \times (0,T)$ and $\mathcal{O}_2 = B(q_n, 2\delta) \times (0,T)$, for $n = 1, \cdots, N-1$. It is clear that any plane in $\mathbb{R}^{d+1}$ of equation $t = c$ (c is a constant) which intersects $B(q_n, 2\delta) \times (0,T)$ also intersects $B(q_n, \delta) \times (0,T)$. Since u vanishes in $B(q_0, \delta) \times (0,T)$, it vanishes in $B(q_0, 2\delta) \times (0,T)$, and since $B(q_1, \delta) \subset B(q_0, 2\delta)$, it vanishes in $B(q_1, \delta) \times (0,T)$. We conclude that u vanishes in $B(q_N, \delta) \times (0,T)$, that is in $B(x, \delta) \times (0,T)$. $\square$

3.3.3 Case of the wave equation

Considering the wave equation, we first give the following local result Lattès & Lions (1967).

Lemma 3.2. *Let us consider $\delta, \tau > 0$ and $x_1, x_2 \in \mathbb{R}^d$ such that $\tau > |x_1 - x_2|$. Let us define the open and convex domains*

$$\mathcal{O}_1 = B(x_1, \delta) \times (-\tau, \tau)$$

and

$$\mathcal{O}_2 = \bigcup_{\lambda \in [0,1]} B((1-\lambda)x_1 + \lambda x_2, \delta) \times (-\tau + \lambda |x_1 - x_2|, \tau - \lambda |x_1 - x_2|).$$

If $u \in \mathcal{D}'(\mathcal{O}_2)$ satisfies $\partial_t^2 u - \Delta u = 0$ in $\mathcal{O}_2$ and $u = 0$ in $\mathcal{O}_1$, then $u = 0$ in $\mathcal{O}_2$.

Proof. The proof relies on the Homlgren's theorem for the operator $P = \partial_t^2 - \Delta$ with domains $\mathcal{O}_1 \subset \mathcal{O}_2$. Let us consider some plane π of normal $N \neq 0$ which is characteristic with respect to P, that is $N_{d+1} = \pm \sqrt{N_1^2 + \cdots + N_d^2}$. We observe that if π intersects $\mathcal{O}_2$, then it intersects $\mathcal{O}_1$. $\square$

Proposition 3.1. *Let Ω be an open and connected domain of $\mathbb{R}^d$. Let $u \in \mathcal{D}'(\Omega \times (-T,T))$ satisfy $\partial_t^2 u - \Delta u = 0$. Let us consider $x_0, x \in \Omega$ such that $T > d_\Omega(x_0, x)$. Let $\delta_0 > 0$ such that $u = 0$ in $B(x_0, \delta_0) \times (-T,T)$. For all $\epsilon \in (-T + d_\Omega(x_0,x), T - d_\Omega(x_0,x))$ there exists some $\delta \in (0, \delta_0)$ such that $u = 0$ in $B(x,\delta) \times (-T + d_\Omega(x_0,x) + \epsilon, T - d_\Omega(x_0,x) - \epsilon)$.*

Proof. Let us take δ and a sequence of balls depending on ϵ and δ_0 as in the lemma 3.1. Now we apply N times Lemma 3.2 with

$$\mathcal{O}_1 = B(q_n, \delta) \times (-T + S_n, T - S_n)$$

and

$$\mathcal{O}_2 = \bigcup_{\lambda \in [0,1]} B((1-\lambda)q_n + \lambda q_{n+1}, \delta) \times (-T + S_n + \lambda |q_n - q_{n+1}|, T - S_n - \lambda |q_n - q_{n+1}|).$$

Here we have denoted

$$S_n = \sum_{j=0}^{n-1} |q_j - q_{j+1}|.$$

It is straightforward to check that $\mathcal{O}_2 \subset B(q_n, 2\delta) \times (-T, T) \subset \Omega \times (-T, T)$. Since $u = 0$ in $B(q_0, \delta) \times (-T, T)$, then $u = 0$ in $B(q_1, \delta) \times (-T + S_1, T - S_1), \cdots$, then $u = 0$ in $B(q_N, \delta) \times (-T + S_N, T - S_N)$. We note that

$$T - S_N = T - \sum_{j=0}^{N-1} |q_j - q_{j+1}| \geq T - N\delta \geq T - d_\Omega(x_0, x) - \epsilon,$$

Hence $u = 0$ in $B(x, \delta) \times (-T + d_\Omega(x_0, x) + \epsilon, T - d_\Omega(x_0, x) - \epsilon)$, which completes the proof. □

We now state our main continuation uniqueness result.

Theorem 3.4. *Let ω and Ω be open domains in $\mathbb{R}^d$ such that $\omega \subset \Omega$. We assume that $\omega \neq \emptyset$ and that Ω is connected. Let us consider some $T > 0$. Every $u \in \mathcal{D}'(\Omega \times (-T, T))$ satisfying the equation $\partial_t^2 u - \Delta u = 0$ and vanishing in $\omega \times (-T, T)$ then vanishes in*

$$Q_0 = \{(x, t) \in \Omega \times (-T, T),\ d_\Omega(x, \omega) < T - |t|\}.$$

Proof. Let us pick some $x \in \Omega$. We recall that $d_\Omega(x, \omega) = \inf_{x_0 \in \omega} d_\Omega(x, x_0)$. Let us take any $\epsilon \in (0, \epsilon_0(x))$ with $\epsilon_0(x) = (T - d_\Omega(x, \omega))/2$. One may find $x_0 \in \omega$ such that $d_\Omega(x, \omega) \geq d_\Omega(x, x_0) - \epsilon$. There exists some δ_0 such that $u = 0$ in $B(x_0, \delta_0) \times (-T, T)$. By using Proposition 3.1, we obtain that there exists some $\delta < \delta_0$ such that $u = 0$ in $B(x, \delta) \times (-T + d_\Omega(x, x_0) + \epsilon, T - d_\Omega(x, x_0) - \epsilon)$. As a conclusion, for any small ϵ (depending on x), there exists δ such that $u = 0$ in $B(x, \delta) \times (-T + d_\Omega(x, \omega) + 2\epsilon, T - d_\Omega(x, \omega) - 2\epsilon)$. Since ϵ is arbitrarily small, the conclusion follows. □

For some applications, we will need uniqueness in a subdomain the spatial section of which is the whole Ω. For a connected open domain Ω and some $x_0 \in \Omega$, we define

$$D(\Omega, x_0) = \sup_{x \in \Omega} d_\Omega(x, x_0).$$

For $d \geq 2$ we may have $D(\Omega, x_0) = +\infty$, even if Ω is a bounded domain, as can be seen in the following example in $2D$.

Remark 3.2. For $n \in \mathbb{N}^*$, let us define the segments S_n and S'_n by

$$S_n = \left\{ (x, y) \in \mathbb{R}^2, \quad x = \frac{1}{n}, 0 \leq y \leq \frac{3}{2} \right\}$$

and

$$S'_n = \left\{ (x, y) \in \mathbb{R}^2, \quad x = \frac{1}{2}\left(\frac{1}{n} + \frac{1}{n+1}\right), \frac{1}{2} \leq y \leq 2 \right\}.$$

It is readily seen that the open domain

$$\Omega = (0,2) \times (0,2) \setminus \left(\bigcup_{n=1}^{+\infty} S_n \cup \bigcup_{n=1}^{+\infty} S'_n \right)$$

is connected, bounded, and satisfies $D(\Omega, x_0) = +\infty$ for any $x_0 \in \Omega$.

However we have $D(\Omega, x_0) < +\infty$ as soon as Ω is a bounded Lipschitz domain, which is a consequence of the following theorem Oudot et al. (2010), since $\overline{\Omega}$ is a compact set.

Theorem 3.5. *If $\Omega \subset \mathbb{R}^d$ is a connected, bounded, open domain of class $C^{0,1}$, then for all $x_0 \in \Omega$, the function $x \in \Omega \to d_\Omega(x_0, x)$ can be extended to a continuous function in $\overline{\Omega}$ (for the euclidian norm).*

Proof. First, let us prove that $x \in \Omega \to d_\Omega(x_0, x)$ is continuous. We hence consider $\epsilon > 0$ and $x \in \Omega$. There exists $\delta < \epsilon$ such that $B(x, \delta) \subset \Omega$, where $B(x, \delta)$ is associated with the euclidian norm. For all $\tilde{x} \in B(x, \delta)$, $d_\Omega(x, \tilde{x}) = |x - \tilde{x}| < \delta$ and by the triangle inequality $|d_\Omega(x_0, \tilde{x}) - d_\Omega(x_0, x)| \leq d_\Omega(x, \tilde{x}) < \delta < \epsilon$, which proves the continuity at point $x \in \Omega$.
Now we consider $x \in \partial\Omega$. From the definition **??** of a domain of class $C^{0,1}$, there exists a neighborhood V of x in $\mathbb{R}^d$ and a Lipschitz homeomorphism $f : \mathbb{R}^d \to \mathbb{R}^d$ such that $f^{-1}(0) = x$, $f^{-1}(\mathbb{R}^{d-1} \times \{0\}) \cap V = \partial\Omega \cap V$ and $f^{-1}(\mathbb{R}^{d-1} \times \{x_d > 0\}) \cap V = \Omega \cap V$. It suffices to set, for $x' \in \mathbb{R}^{d-1}$ and $x_d \in \mathbb{R}$, $f(x', x_d) = (x', x_d - \varphi(x'))$. Let us denote by L the Lipschitz constant of φ, which is also the Lipschitz constant of f and of f^{-1}. Without loss of generality, V may be chosen such that $f(\Omega \cap V)$ is the intersection of $\mathbb{R}^d_+$ and of the open ball $B(0, \epsilon/L)$. Then for all $\tilde{x} \in \overline{\Omega} \cap V$, we consider the path $g : s \in [0,1] \to f^{-1}(s\tilde{y})$ with $\tilde{y} = f(\tilde{x})$. Since $f(\overline{\Omega} \cap V)$ is convex, $g([0,1])$ is included in $\overline{\Omega} \cap V$, that is in $\overline{\Omega}$, and joins x to $\tilde{x}$. This enables us to define $d_\Omega(x_0, x)$ for $x \in \partial\Omega$. It remains to prove continuity of $d_\Omega(x_0, x)$ at x. Since f^{-1} is $L-$Lipschitz and $|\tilde{y}| < \epsilon/L$,

$$d_\Omega(x, \tilde{x}) \leq |g| = \int_0^1 |g'(s)| \, ds = \int_0^1 |\tilde{y} \cdot \nabla f^{-1}(s\tilde{y})| \, ds \leq L|\tilde{y}| < \epsilon.$$

Hence $|d_\Omega(x_0, \tilde{x}) - d_\Omega(x_0, x)| \leq d_\Omega(x, \tilde{x}) < \epsilon$, which completes the proof. $\square$

We also define, for $\omega \subset \Omega$, with $\omega \neq \emptyset$,

$$D(\Omega, \omega) = \sup_{x \in \Omega} \inf_{x_0 \in \omega} d_\Omega(x, x_0) = \sup_{x \in \Omega} d_\Omega(x, \omega).$$

It is clear that if Ω is a bounded Lipschitz domain, this quantity if finite. Indeed,

$$D(\Omega, \omega) \leq \inf_{x_0 \in \omega} \sup_{x \in \Omega} d_\Omega(x, x_0) = \inf_{x_0 \in \omega} D(\Omega, x_0) < +\infty.$$

From Theorem 3.4, we obtain the following straightforward consequence.

Corollary 3.1. *Let ω and Ω be open domains in $\mathbb{R}^d$ such that $\omega \subset \Omega$. We assume that $\omega \neq \emptyset$ and that Ω is connected and Lipschitz. Let us consider some T such that $T > T_0 \coloneqq D(\Omega, \omega)$. Every $u \in \mathcal{D}'(\Omega \times (-T, T))$ satisfying the equation $\partial_t^2 u - \Delta u = 0$ and vanishing in $\omega \times (-T, T)$ then vanishes in $\Omega \times (-T + T_0, T - T_0)$.*

3.3.4 Case of boundary data

Now, let us consider some similar results when the function is not supposed to vanish in a subset $\omega \subset \Omega$ but the Cauchy data $u|_\Gamma$ and $\partial_\nu u|_\Gamma$ are supposed to vanish on a subset Γ of the boundary $\partial\Omega$. Let us begin with the Laplace operator.

Theorem 3.6. *Assume that $\Omega \subset \mathbb{R}^d$ is a bounded open and connected domain of class $C^{0,1}$, and Γ a non-empty open subpart of $\partial\Omega$. If $u \in H^1(\Omega)$ satisfies*

$$\begin{cases} \Delta u = 0, & \text{in } \Omega \\ u = 0, & \text{in } \Gamma \\ \partial_\nu u = 0, & \text{in } \Gamma \end{cases} \tag{3.1}$$

then $u = 0$ in Ω.

Proof. For some $x_0 \in \Gamma$ and ϵ with $B(x_0, \epsilon) \cap \partial\Omega \subset \Gamma$ (such open ball exists since Γ is open and non-empty), let us consider the open domain $\tilde{\Omega} = \Omega \cup B(x_0, \epsilon)$. Let us define $\tilde{u}$ as the extension of u by 0 in $\tilde{\Omega}$. Since $u|_\Gamma = 0$, $\tilde{u} \in H^1(\tilde{\Omega})$. Now let us prove that $\Delta\tilde{u} = 0$ in $\tilde{\Omega}$ in the sense of distributions. For some test function $\varphi \in C_0^\infty(\tilde{\Omega})$, we have

$$\langle \Delta\tilde{u}, \varphi \rangle = \langle \tilde{u}, \Delta\varphi \rangle = \int_{\tilde{\Omega}} \tilde{u} \Delta\varphi \, dx = \int_\Omega u \Delta\varphi \, dx.$$

By using integration by parts in Ω, we obtain

$$\langle \Delta\tilde{u}, \varphi \rangle = \int_\Omega \Delta u \varphi \, dx + \left\langle u, \frac{\partial\varphi}{\partial\nu} \right\rangle_{\partial\Omega} - \left\langle \frac{\partial u}{\partial\nu}, \varphi \right\rangle_{\partial\Omega},$$

where the brackets stand for duality pairing between $H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$. In the right-hand side, with use the fact that $\Delta u = 0$ in Ω, $u|_\Gamma = 0$ and $\partial_\nu u|_\Gamma = 0$, and φ vanishes in a volume vicinity of $\partial\Omega \setminus \Gamma$. We hence have $\Delta\tilde{u} = 0$ in $\tilde{\Omega}$.
Considering some non-empty open domain $\omega \Subset \tilde{\Omega} \setminus \overline{\Omega}$, we have that $\tilde{u} = 0$ in ω and applying theorem 3.2 to $\tilde{u}$ in domain $\tilde{\Omega}$ implies $\tilde{u} = 0$ in $\tilde{\Omega}$, that is $u = 0$ in Ω. $\square$

In the same vein, we could easily prove the following results for the heat and the wave equations.

Theorem 3.7. *Assume that $\Omega \subset \mathbb{R}^d$ is a bounded open connected domain of class $C^{0,1}$, and Γ a non-empty open subpart of $\partial\Omega$. If $u \in L^2(0,T;H^1(\Omega))$ satisfies*

$$\begin{cases} \partial_t u - \Delta u = 0, & \text{in } \Omega \times (0,T) \\ u = 0, & \text{on } \Gamma \times (0,T) \\ \partial_\nu u = 0, & \text{on } \Gamma \times (0,T) \end{cases} \tag{3.2}$$

then $u = 0$ in $\Omega \times (0,T)$.

Theorem 3.8. *Assume that $\Omega \subset \mathbb{R}^d$ is a bounded open connected domain of class $C^{0,1}$, and Γ a non-empty open subpart of $\partial\Omega$. If $u \in L^2(-T,T;H^1(\Omega)) \cap H^1(-T,T;L^2(\Omega))$ satisfies*

$$\begin{cases} \partial_t^2 u - \Delta u = 0, & \text{in } \Omega \times (-T,T) \\ u = 0, & \text{on } \Gamma \times (-T,T) \\ \partial_\nu u = 0, & \text{on } \Gamma \times (-T,T) \end{cases} \tag{3.3}$$

then $u = 0$ in

$$Q_0 = \{(x,t) \in \Omega \times (-T,T),\ d_\Omega(x,\Gamma) < T - |t|\}.$$

In particular, if $T > T_0 := D(\Omega,\Gamma)$, then $u = 0$ in $\Omega \times (-T + T_0, T - T_0)$.

Chapter 4
Data completion problems – The Laplace equation case

4.1 Introduction

In this chapter we apply the Tikhonov regularization to data completion problems, which are viewed as linear ill-posed problems governed by partial differential equations. As we will see, different ways of choosing the non invertible operator A in chapter 2 will lead to different methods. In order to illustrate those different methods, we will apply them on the same example presented in chapter **??**, which is the Cauchy problem for the Laplace equation (1.3).

4.2 First method with regularity assumptions

Let Ω be a bounded, connected open domain of class $C^{1,1}$ in $\mathbb{R}^d$ (see definition **??**), $d > 1$, and Γ a non-empty open subpart of $\partial\Omega$. We consider the following problem: for a pair of data $(g_0, g_1) \in H^{\frac{3}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)$, find $u \in H^2(\Omega)$ such that

$$\begin{cases} \Delta u = 0, & \text{in } \Omega \\ u = g_0, & \text{on } \Gamma \\ \partial_\nu u = g_1, & \text{on } \Gamma, \end{cases} \tag{4.1}$$

where ν is the outward unit normal $\partial\Omega$. Such problem coincides with Problem (1.3), the only difference is that the regularity order of the domain and the function is one level up.

Thanks to this higher regularity, let us remark that up to a slight modification, Problem (1.3) is a particular instance of the abstract ill-posed problem described in Chapter 2. Indeed, if U is a function in $H^2(\Omega)$ such that $U|_\Gamma = g_0$ and $\partial_\nu U|_\Gamma = g_1$ – note that such function exits following Lions & Magenes (1968), and is of course not unique – and if we define $y = -\Delta U \in L^2(\Omega)$, the change of variable $z = u - U \in H^2(\Omega)$ implies that solving Problem 4.1 is equivalent to solve

$$\begin{cases} \Delta z = y, & \text{in } \Omega \\ z = 0, & \text{on } \Gamma \\ \partial_\nu z = 0, & \text{on } \Gamma \end{cases} \tag{4.2}$$

And Problem (4.2) amounts, for $y \in \mathcal{Y}$, to find $z \in \mathcal{Z}$ such that $\Psi z = y$, with

$$\left| \begin{array}{l} \mathcal{Z} = \{v \in H^2(\Omega),\ v|_\Gamma = 0,\ \partial_\nu v|_\Gamma = 0\} \\ \mathcal{Y} = L^2(\Omega) \\ \Psi\ :\ z \mapsto \Delta z. \end{array} \right. \tag{4.3}$$

First of all, we note that Spaces $\mathcal{Z}$ and $\mathcal{Y}$ are Hilbert spaces, while Ψ is a continuous operator. Moreover, we already know that Ψ is injective – see Theorem 3.6 – and is not onto since the Cauchy problem for the Laplace equation is ill-posed. Let us prove that Ψ has a dense range. In this respect, we need the following uniqueness result for $L^2(\Omega)$ functions.

Theorem 4.1. *Assume that $\Omega \subset \mathbb{R}^d$ is a bounded open connected domain of class $C^{1,1}$, and Γ a non-empty open subpart of $\partial\Omega$. If $u \in L^2(\Omega)$ satisfies problem (4.1) with $(g_0, g_1) = (0, 0)$, then $u = 0$ in Ω.*

Before giving the proof of the two above uniqueness result, let us recall the following integration by parts formula Lions & Magenes (1968):
For a $C^{1,1}$ domain Ω, for any $u \in H^0(\Omega, \Delta) \coloneqq \{u \in L^2(\Omega),\ \Delta u \in L^2(\Omega)\}$ and $v \in H^2(\Omega)$, we have

$$(\Delta u, v)_{L^2(\Omega)} - (u, \Delta v)_{L^2(\Omega)} = \left\langle \frac{\partial u}{\partial \nu}, v \right\rangle_{\partial\Omega} - \left\langle u, \frac{\partial v}{\partial \nu} \right\rangle_{\partial\Omega}, \tag{4.4}$$

where the first bracket has the meaning of duality between $H^{-\frac{3}{2}}(\partial\Omega)$ and $H^{\frac{3}{2}}(\partial\Omega)$ while the second one has the meaning of duality between $H^{-\frac{1}{2}}(\partial\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$.

Proof (Proof of theorem 4.1 :). The proof is based on theorem 3.6 and on a regularity argument. We consider a fonction $\alpha \in C^\infty(\overline{\Omega})$ that vanishes in a (volume) vicinity of $\partial\Omega \setminus \Gamma$. Let us prove that $\alpha u \in H^2(\Omega)$. We will use the following statement that for all $s \in \mathbb{R}$, if $u \in H^s(\mathbb{R}^d)$ and $\Delta u \in H^s(\mathbb{R}^d)$, then $u \in H^{s+2}(\mathbb{R}^d)$, which immediately follows from Fourier analysis. This result will be denoted (R_s) in the following proof.
Again let us denote by $\tilde{u}$ the trivial extension of u in $\mathbb{R}^d$ and $\tilde{\alpha}$ an extension in $C_0^\infty(\mathbb{R}^d)$ of α. We have in $\mathbb{R}^d$

$$\Delta(\tilde{\alpha}\tilde{u}) = \tilde{u}\,\Delta\tilde{\alpha} + 2\sum_{i=1}^{d} \frac{\partial\tilde{\alpha}}{\partial x_i}\frac{\partial\tilde{u}}{\partial x_i} + \tilde{\alpha}\,\Delta\tilde{u}. \tag{4.5}$$

It is clear that $\tilde{u}\,\Delta\tilde{\alpha} \in L^2(\mathbb{R}^d)$. If $\partial\tilde{\alpha}/\partial x_i$ is denoted by $\tilde{\beta}_i$, we have

$$\tilde{\beta}_i \frac{\partial \tilde{u}}{\partial x_i} = \frac{\partial(\tilde{\beta}_i \tilde{u})}{\partial x_i} - \frac{\partial \tilde{\beta}_i}{\partial x_i}\tilde{u}, \tag{4.6}$$

which implies that $\tilde{\beta}_i \partial\tilde{u}/\partial x_i \in H^{-1}(\mathbb{R}^d)$. Now, let us prove that $\tilde{\alpha}\,\Delta\tilde{u} = 0$ in $\mathbb{R}^d$. For any test function $\varphi \in C_0^\infty(\mathbb{R}^d)$, we have, in the sense of distributions in $\mathbb{R}^d$,

$$\langle \tilde{\alpha}\Delta\tilde{u}, \varphi\rangle = \langle \tilde{u}, \Delta(\tilde{\alpha}\varphi)\rangle = \int_\Omega u\Delta(\alpha\varphi)\,dx.$$

Since $u \in H^0(\Omega,\Delta)$ and $\alpha\varphi$ is a smooth function, we can perform the above integration by parts (4.4).

$$\int_\Omega u\Delta(\alpha\varphi)\,dx = \Big\langle u, \frac{\partial(\alpha\varphi)}{\partial\nu}\Big\rangle_{\partial\Omega} - \Big\langle \frac{\partial u}{\partial\nu}, \alpha\varphi\Big\rangle_{\partial\Omega} + \int_\Omega \Delta u(\alpha\varphi)\,dx.$$

Since $u|_\Gamma = 0$, $\partial_\nu u|_\Gamma = 0$, $(\alpha\varphi)$ vanishes near $\partial\Omega\setminus\Gamma$ and $\Delta u = 0$ in Ω, we have

$$\langle \tilde{\alpha}\Delta\tilde{u}, \varphi\rangle = 0.$$

It follows that $\tilde{\alpha}\Delta\tilde{u} = 0$ in $\mathbb{R}^d$ and from (4.5) that $\Delta(\tilde{\alpha}\tilde{u}) \in H^{-1}(\mathbb{R}^d)$. From the statement (R_s) for $s = -1$, it follows that $\tilde{\alpha}\tilde{u} \in H^1(\mathbb{R}^d)$, for any $\tilde{\alpha} \in C_0^\infty(\mathbb{R}^d)$ that vanishes near $\partial\Omega\setminus\Gamma$ in Ω. This result implies that we have now $\tilde{\beta}_i\partial\tilde{u}/\partial x_i \in L^2(\mathbb{R}^d)$ from (4.6), and hence that $\Delta(\tilde{\alpha}\tilde{u}) \in L^2(\mathbb{R}^d)$ from (4.5). By using once again (R_s) for $s = 0$, it follows that $\tilde{\alpha}\tilde{u} \in H^2(\mathbb{R}^d)$, and hence $\alpha u \in H^2(\Omega)$.
It remains to apply Theorem 3.6 and we obtain that u vanishes any compact $K \subset \Omega$. Hence $u = 0$ in Ω. □

Theorem 4.2. *The operator A defined by (4.3) has a dense range.*

Proof. Assume that for some $f \in L^2(\Omega)$,

$$(\Delta u, f)_{L^2(\Omega)} = 0, \quad \forall u \in V = \{v \in H^2(\Omega),\ v|_\Gamma = 0,\ \partial_\nu v|_\Gamma = 0\}.$$

By choosing $u = \varphi \in C_0^\infty(\Omega)$, we obtain that $\Delta f = 0$, in particular $f \in H^0(\Omega,\Delta)$. For all $u \in V$, we can hence use the integration by parts formula (4.4)

$$(\Delta u, f)_{L^2(\Omega)} - (u, \Delta f)_{L^2(\Omega)} = \Big\langle \frac{\partial u}{\partial\nu}, f\Big\rangle_{\partial\Omega} - \Big\langle u, \frac{\partial f}{\partial\nu}\Big\rangle_{\partial\Omega},$$

This is for all $u \in V$

$$\Big\langle \frac{\partial u}{\partial\nu}, f\Big\rangle_{\partial\Omega} - \Big\langle u, \frac{\partial f}{\partial\nu}\Big\rangle_{\partial\Omega} = 0,$$

that is, since $u|_\Gamma = 0$ and $\partial_\nu u|_\Gamma = 0$,

$$\Big\langle \frac{\partial u}{\partial\nu}, f\Big\rangle_{\tilde{\Gamma}} - \Big\langle u, \frac{\partial f}{\partial\nu}\Big\rangle_{\tilde{\Gamma}} = 0,$$

where $\tilde{\Gamma} = \partial\Omega \setminus \overline{\Gamma}$, the first bracket has the meaning of duality pairing between $\tilde{H}^{\frac{1}{2}}(\tilde{\Gamma})$ and $H^{-\frac{1}{2}}(\tilde{\Gamma})$ while the second bracket has the meaning of duality pairing between $\tilde{H}^{\frac{3}{2}}(\tilde{\Gamma})$ and $H^{-\frac{3}{2}}(\tilde{\Gamma})$. Since the operator $V \to \tilde{H}^{\frac{3}{2}}(\tilde{\Gamma}) \times \tilde{H}^{\frac{1}{2}}(\tilde{\Gamma})$ such that $u \in V \mapsto (u|_{\tilde{\Gamma}}, \partial_\nu u|_{\tilde{\Gamma}})$ is onto, we obtain that $f|_{\tilde{\Gamma}} = 0$ and $\partial_\nu f|_{\tilde{\Gamma}} = 0$. We conclude that function $f \in L^2(\Omega)$ solves the problem (3.1) with $\tilde{\Gamma}$ playing the role of Γ and we conclude from theorem 4.1 that $f = 0$, wich proves that A has dense range. □

We hence conclude that we are exactly in the situation of Chapter 2, which means that we can apply the Tikhonov regularization to operator A as well as the Morozov principle. In view of (**??**) the Tikhonov regularization amounts, for $\epsilon > 0$, to find $v_\epsilon \in V = \{v \in H^2(\Omega),\ v|_\Gamma = 0,\ \partial_\nu v|_\Gamma = 0\}$ such that

$$(\Delta v_\epsilon, \Delta v)_{L^2(\Omega)} + \epsilon(v_\epsilon^\delta, v)_{H^2(\Omega)} = (f, \Delta v)_{L^2(\Omega)}, \quad \forall v \in V. \tag{4.7}$$

Such formulation is a particular form of the method of quasi-reversibility introduced in Lattès & Lions (1967). It consists in transforming the initial second-order ill-posed problem into a family, depending on the small parameter $\epsilon > 0$, of fourth-order well-posed problems. We remark that (4.7) is a weak formulation in the terminology of elliptic partial differential equations, and can therefore be directly discretized with the help of a finite element method. We have proved in Theorem 2.1 that in the case of exact data ($\delta = 0$) ,

$$\lim_{\epsilon \to 0} \|(v_\epsilon + U) - u\|_{H^2(\Omega)} = 0$$

and in the presence of noisy data, for some constant $c > 0$, we have

$$\|(v_\epsilon^\delta + U^\delta) - u)\|_{H^2(\Omega)} \le \|(v_\epsilon + U) - u\|_{H^2(\Omega)} + c\,\frac{\delta}{\sqrt{\epsilon}}.$$

Of course since we are exactly in the framework of chapter 2, we can choose $\epsilon > 0$ as a function of δ following the Morozov's principle.

Note however that the practical computation of f^δ from noisy Cauchy data (g_0^δ, g_1^δ) which is required in the homogeneous formulation (4.7) may be not easy and an alternative formulation of quasi-reversibility may be the non-homogeneous one : for some $\epsilon > 0$, find $u_\epsilon \in V_g$, with

$$V_g = \{v \in H^2(\Omega),\ v|_\Gamma = g_0,\ \partial_\nu v|_\Gamma = g_1\},$$

such that

$$(\Delta u_\epsilon, \Delta v)_{L^2(\Omega)} + \epsilon(u_\epsilon, v)_{H^2(\Omega)} = 0, \quad \forall v \in V. \tag{4.8}$$

The fact that Problem (4.8) is well-posed is a simple consequence of the Lax-Milgram's theorem and the use of extension U. Following the same lines as in the proof of Theorem 2.1, we would easily prove that in the case of exact data ($\delta = 0$)

$$\lim_{\epsilon \to 0} \|u_\epsilon - u\|_{H^2(\Omega)} = 0,$$

and in the presence of noisy data, for some constant $c > 0$ we have

$$\|u_\epsilon^\delta - u\|_{H^2(\Omega)} \leq \|u_\epsilon - u\|_{H^2(\Omega)} + c\,\frac{\delta}{\sqrt{\epsilon}}.$$

Remark 4.1. It may be easily verified that the formulations (4.7) and (4.8) are not equivalent, in the sense that $v_\epsilon^\delta + U^\delta$ does not coincide with u_ϵ^δ.

The formulations (4.7) and (4.8) have two main drawbacks.

1. First, those formulations require the domain and the exact solution u to be rather smooth. A more usual framework concerning regularity is a domain of class $C^{0,1}$ instead of $C^{1,1}$ and a solution in $H^1(\Omega)$ instead of $H^2(\Omega)$.
2. Secondly those formulations correspond to fourth-order problems. From the finite element point of view, we hence have to use some complicated finite elements (Hermite finite elements) instead of simple ones (Lagrange finite elements).

This is why we now present a mixed version of the Tikhonov regularization which avoids these two drawbacks.

4.3 A mixed version of Tikhonov regularization

4.3.1 A general variational setting

Let us consider three Hilbert spaces $\mathcal{V}$, $\mathcal{M}$ and $\mathcal{H}$, as well as a continuous onto operator $C : \mathcal{V} \to \mathcal{H}$ and the corresponding affine space $V_g = \{u \in \mathcal{V},\ Cu = g\}$ for $g \in \mathcal{H}$. For a continuous bilinear form b on $\mathcal{V} \times \mathcal{M}$ and a continuous linear form ℓ on M, let us consider the weak formulation: find $u \in \mathcal{V}_g$ such that for all $\mu \in \mathcal{M}$,

$$b(u, \mu) = \ell(\mu). \tag{4.9}$$

The bilinear form b is said to satisfy the $\inf - \sup$ property on $\mathcal{V}_0 \times \mathcal{M}$ if

Assumption 1 *There exists $\alpha > 0$ such that*

$$\inf_{\substack{u \in \mathcal{V}_0 \\ u \neq 0}} \sup_{\substack{\mu \in \mathcal{M} \\ \mu \neq 0}} \frac{b(u, \mu)}{||u||_{\mathcal{V}} ||\mu||_{\mathcal{M}}} \geq \alpha.$$

The bilinear form b is said to satisfy the solvability property on $\mathcal{V}_0 \times \mathcal{M}$ if

Assumption 2 *For all $\mu \in \mathcal{M}$,*

$$\forall u \in \mathcal{V}_0, \quad b(u,\mu) = 0 \Longrightarrow \mu = 0.$$

Lastly, b is said to satisfy the uniqueness property on $\mathcal{V}_0 \times \mathcal{M}$ if

Assumption 3 *For all $u \in \mathcal{V}_0$,*

$$\forall \mu \in \mathcal{M}, \quad b(u,\mu) = 0 \Longrightarrow u = 0.$$

From the Brezzi-Nečas-Babuška theorem (see for example Ern & Guermond (2004)), we know that problem (4.9) is well-posed if and only if both conditions 1 and 2 are satisfied. Clearly, assumption 1 implies assumption 3, but the converse implication is false. In what follows, and to retrieve the framework of chapter 2, it is assumed that the bilinear form b does not satisfy the $\inf - \sup$ condition 1, which from the Brezzi-Nečas-Babuška theorem implies that the problem (4.9) for a given ℓ is in general ill-posed. We assume however that the weaker property (3) is satisfied, which means that problem (4.9) has at most one solution.

4.3.2 The mixed-type Tikhonov regularization

A regularized formulation of ill-posed problem (4.9) is the following: for $\epsilon > 0$, find $(u_\epsilon, \lambda_\epsilon) \in \mathcal{V}_g \times \mathcal{M}$ such that for all $(v,\mu) \in \mathcal{V}_0 \times \mathcal{M}$,

$$\begin{cases} \epsilon(u_\epsilon, v)_V + b(v,\lambda_\epsilon) = 0 \\ b(u_\epsilon,\mu) - (\lambda_\epsilon,\mu)_M = \ell(\mu). \end{cases} \tag{4.10}$$

We have the following theorem.

Theorem 4.3. *For any $f \in \mathcal{H}$ and $\ell \in \mathcal{M}'$, the problem (4.10) has a unique solution. For some $g \in H$ and $\ell \in \mathcal{M}'$ such that (4.9) has a (unique) solution u, then the solution $(u_\epsilon, \lambda_\epsilon) \in \mathcal{V}_g \times \mathcal{M}$ satisfies $(u_\epsilon, \lambda_\epsilon) \to (u, 0)$ in $\mathcal{V} \times \mathcal{M}$ when $\epsilon \to 0$.*

Proof. Let us introduce some $U \in \mathcal{V}$ such that $CU = g$, which exists since C is onto, and let us set $\hat{u}_\epsilon = u_\epsilon - U$, so that problem (4.10) is equivalent to: find $(\hat{u}_\epsilon, \lambda_\epsilon) \in \mathcal{V}_0 \times \mathcal{M}$ such that for all $(v,\mu) \in \mathcal{V}_0 \times \mathcal{M}$,

$$\begin{cases} \epsilon(\hat{u}_\epsilon, v)_\mathcal{V} + b(v,\lambda_\epsilon) = -\epsilon(U,v)_\mathcal{V} \\ b(\hat{u}_\epsilon,\mu) - (\lambda_\epsilon,\mu)_\mathcal{M} = \ell(\mu) - b(U,\mu), \end{cases} \tag{4.11}$$

which is itself equivalent to: find $(\hat{u}_\epsilon, \lambda_\epsilon) \in \mathcal{V}_0 \times \mathcal{M}$ such that for all $(v,\mu) \in \mathcal{V}_0 \times \mathcal{M}$,

$$B_\epsilon((\hat{u}_\epsilon, \lambda_\epsilon); (v,\mu)) = L_\epsilon((v,\mu)),$$

where the bilinear form B_ϵ and the linear form L_ϵ are given on $\mathcal{V}_0 \times \mathcal{M}$ by

$$B_\epsilon((u,\lambda);(v,\mu)) = \epsilon(u,v)_{\mathcal{V}} + b(v,\lambda) - b(u,\mu) + (\lambda,\mu)_{\mathcal{M}}$$

and

$$L_\epsilon((v,\mu)) = -\epsilon(U,v)_{\mathcal{V}} - \ell(\mu) + b(U,\mu).$$

Since for $(u,\lambda) \in \mathcal{V}_0 \times \mathcal{M}$,

$$B_\epsilon((u,\lambda);(u,\lambda)) \geq \epsilon||u||^2_{\mathcal{V}} + ||\lambda||^2_{\mathcal{M}},$$

B_ϵ is coercive on $\mathcal{V}_0 \times \mathcal{M}$, which implies from the Lax-Milgram lemma that the problem (4.10) is well-posed for all $\epsilon > 0$.

Now let us assume that $g \in \mathcal{H}$ and $\ell \in \mathcal{M}'$ are such that (4.9) has a (unique) solution u. By subtracting (4.9) to the second equation of (4.10), we obtain that for all $(v,\mu) \in \mathcal{V}_0 \times \mathcal{M}$,

$$\begin{cases} \epsilon(u_\epsilon,v)_{\mathcal{V}} + b(v,\lambda_\epsilon) = 0 \\ b(u_\epsilon - u,\mu) - (\lambda_\epsilon,\mu)_{\mathcal{M}} = 0. \end{cases}$$

Choosing $v = u_\epsilon - u \in \mathcal{V}_0$, $\mu = \lambda_\epsilon \in \mathcal{M}$ and subtracting the two obtained equations we end up with

$$\epsilon(u_\epsilon, u_\epsilon - u)_{\mathcal{V}} + ||\lambda_\epsilon||^2_{\mathcal{M}} = 0.$$

This in particular implies that

$$||u_\epsilon||_{\mathcal{V}} \leq ||u||_{\mathcal{V}}, \ ||\lambda_\epsilon||_{\mathcal{M}} \leq \sqrt{\epsilon}||u||_{\mathcal{V}}.$$

The second inequality directly implies that $\lambda_\epsilon \to 0$ in $\mathcal{M}$ when $\epsilon \to 0$. From the first inequality, there exists some subsequence of u_ϵ, still denoted u_ϵ, such that $u_\epsilon \rightharpoonup w$ in $\mathcal{V}$ for some $w \in \mathcal{V}$. Since the affine set $\mathcal{V}_g$ is convex and closed, it is weakly closed, that is $w \in \mathcal{V}_g$. Moreover, by passing to the limit in the second equation of (4.10), we obtain that for all $\mu \in \mathcal{M}$, $b(w,\mu) = \ell(\mu)$. Since problem (4.9) has a unique solution, we conclude that $w = u$. We lastly remark that

$$||u_\epsilon - u||^2_{\mathcal{V}} \leq -(u, u_\epsilon - u)_{\mathcal{V}},$$

so that weak convergence in $\mathcal{V}$ implies strong convergence in $\mathcal{V}$. By a standard contradiction argument, all the sequence u_ϵ (not only a subsequence), converges to u in $\mathcal{V}$. □

Now let us show the link between our regularized formulation (4.10) and the standard Tikhonov regularization. More precisely, we can interpret (4.10) as a mixed formulation, in the sense of Brezzi-Fortin **?** for instance, of the Tikhonov regularization. Indeed, by the Riesz theorem, there exists a unique continuous operator $B : \mathcal{V} \longrightarrow \mathcal{M}$ and a unique $L \in \mathcal{M}$ such that for all

$u \in \mathcal{V}$ and all $\mu \in \mathcal{M}$,

$$(Bu, \mu)_{\mathcal{M}} = b(u, \mu), \tag{4.12}$$

and

$$(L, \mu)_{\mathcal{M}} = \ell(\mu). \tag{4.13}$$

Hence problem (4.9) is equivalent to find $u \in \mathcal{V}_g$ such that $Bu = L$. The Tikhonov regularization of such ill-posed problem consists in solving, for $\epsilon > 0$, the well-posed minimization problem

$$\inf_{v \in \mathcal{V}_g} \left(||Bv - L||^2_{\mathcal{M}} + \epsilon ||v||^2_{\mathcal{V}} \right). \tag{4.14}$$

The following proposition specifies the relationship between problems (4.10) and (4.14):

Proposition 4.1. *Let us denote by v_ϵ the unique solution to problem (4.14) and set $\mu_\epsilon = Bv_\epsilon - L$. Then $(v_\epsilon, \mu_\epsilon)$ coincides with the unique solution $(u_\epsilon, \lambda_\epsilon)$ to problem (4.10).*

Proof. Let us denote v_ϵ the solution to problem (4.14). Such solution is characterized by $v_\epsilon \in \mathcal{V}_g$ and

$$(Bv_\epsilon - L, Bv)_{\mathcal{M}} + \epsilon (v_\epsilon, v)_{\mathcal{V}} = 0, \ \forall v \in \mathcal{V}_0,$$

that is by setting $\mu_\epsilon = Bv_\epsilon - L \in \mathcal{M}$, for all $(v, \mu) \in \mathcal{V}_0 \times \mathcal{M}$,

$$\begin{cases} \epsilon(v_\epsilon, v)_{\mathcal{V}} + (Bv, \mu_\epsilon)_{\mathcal{M}} = 0 \\ (Bv_\epsilon, \mu)_{\mathcal{M}} - (\mu_\epsilon, \mu)_{\mathcal{M}} = (L, \mu)_{\mathcal{M}}, \end{cases}$$

that is $(v_\epsilon, \mu_\epsilon) \in \mathcal{V}_g \times \mathcal{M}$ solves problem (4.10) by using the definitions of B and L given by (4.12) and (4.13). We conclude that $(v_\epsilon, \mu_\epsilon) = (u_\epsilon, \lambda_\epsilon)$, which completes the proof. □

4.3.3 Application to the Cauchy problem for the Laplace equation

Let us consider a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d > 1$, the boundary $\partial\Omega$ of which is partitioned into two sets Γ and $\tilde{\Gamma}$. We come back to the Cauchy problem for the Laplace equation (1.3), that is: for some data $(g_0, g_1) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$, find $u \in H^1(\Omega)$ such that

$$\begin{cases} \Delta u = 0 \ \text{ in } \ \Omega \\ u = g_0 \ \text{ on } \ \Gamma \\ \partial_\nu u = g_1 \ \text{ on } \ \Gamma, \end{cases} \tag{4.15}$$

where ν is the outward unit normal to Ω. The problem (4.15) is equivalent to a weak formulation of type (4.9).

Lemma 4.1. *For $(g_0, g_1) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$, the function $u \in H^1(\Omega)$ is a solution to problem (4.15) if and only if $u|_\Gamma = g_0$ and for all $\mu \in H^1(\Omega)$ with $\mu|_{\tilde{\Gamma}} = 0$,*

$$\int_\Omega \nabla u \cdot \nabla \mu \ dx = \langle g_1, \mu|_\Gamma \rangle_{H^{-1/2}(\Gamma), \tilde{H}^{1/2}(\Gamma)}, \tag{4.16}$$

where the brackets stand for duality pairing between $H^{-1/2}(\Gamma)$ and $\tilde{H}^{1/2}(\Gamma)$.

Proof. First, let us assume that $u \in H^1(\Omega)$ and satisfies the weak formulation (4.16). We have $u = g_0$ on Γ and by first choosing $\mu = \varphi \in C_0^\infty(\Omega)$, we obtain $\Delta u = 0$ in Ω in the distributional sense. By using the classical Green formula, we have for all $\mu \in H^1(\Omega)$,

$$\int_\Omega \nabla u \cdot \nabla \mu \, dx = -\int_\Omega \Delta u \, \mu \, dx + \langle \partial_\nu u, \mu \rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)}.$$

If in addition $\mu|_{\tilde{\Gamma}} = 0$ and using the fact that $\Delta u = 0$ in Ω, we obtain that for all $\mu \in H^1(\Omega)$ with $\mu|_{\tilde{\Gamma}} = 0$,

$$\int_\Omega \nabla u \cdot \nabla \mu \ dx = \langle \partial_\nu u, \mu \rangle_{H^{-1/2}(\Gamma), \tilde{H}^{1/2}(\Gamma)},$$

and by comparison with (4.16) we obtain that for all $\mu \in H^1(\Omega)$ with $\mu|_{\tilde{\Gamma}} = 0$,

$$\langle \partial_\nu u, \mu \rangle_{H^{-1/2}(\Gamma), \tilde{H}^{1/2}(\Gamma)} = \langle g_1, \mu \rangle_{H^{-1/2}(\Gamma), \tilde{H}^{1/2}(\Gamma)},$$

which implies that $\partial_\nu u = g_1$ in $H^{-1/2}(\Gamma)$. We conclude that u satisfies (4.15). Conversely, we would prove the same way that if $u \in H^1(\Omega)$ satisfies (4.15), then it satisfies (4.16). □

The weak formulation (4.16) is hence a particular instance of abstract problem (4.9) with $\mathcal{V} = H^1(\Omega)$, $\mathcal{H} = H^{1/2}(\Gamma)$, $\mathcal{M} = \{\mu \in H^1(\Omega),\ \mu|_{\tilde{\Gamma}} = 0\}$, $C : H^1(\Omega) \to H^{1/2}(\Gamma)$ is the trace operator on Γ (which is onto), $\mathcal{V}_{g_0} = \{u \in H^1(\Omega),\ u|_\Gamma = g_0\}$ while for $(u, \mu) \in \mathcal{V} \times \mathcal{M}$,

$$b(u, \mu) = \int_\Omega \nabla u \cdot \nabla \mu \, dx, \quad \ell(\mu) = \langle g_1, \mu|_\Gamma \rangle_{H^{-1/2}(\Gamma), \tilde{H}^{1/2}(\Gamma)}. \tag{4.17}$$

For this particular bilinear form b, only the two last conditions 2 and 3 are satisfied.

Proposition 4.2. *For the bilinear form b given by (4.17), the conditions 2 and 3 are satisfied while the condition 1 is not.*

Proof. We start by condition 2. For $\mu \in \mathcal{M} = \{\mu \in H^1(\Omega),\ \mu|_{\tilde{\Gamma}} = 0\}$, let us assume that for all $u \in \mathcal{V}_0 = \{u \in H^1(\Omega),\ u|_\Gamma = 0\}$,

$$\int_\Omega \nabla\mu\cdot\nabla u\,dx = 0.$$

Choosing $u=\varphi\in C_0^\infty(\Omega)$, we obtain that $\Delta\mu=0$ in the distributional sense in Ω. The Green formula then gives that for all $u\in\mathcal{V}_0$,

$$\langle\partial_\nu\mu,u\rangle_{H^{-1/2}(\partial\Omega),H^{1/2}(\partial\Omega)} = \langle\partial_\nu\mu,u\rangle_{H^{-1/2}(\tilde\Gamma),\tilde H^{1/2}(\tilde\Gamma)} = 0.$$

We conclude that $\mu\in H^1(\Omega)$ satisfies the homogeneous Cauchy problem

$$\begin{cases}\Delta\mu &= 0 \ \text{ in } \ \Omega\\ \mu &= 0 \ \text{ on } \ \tilde\Gamma\\ \partial_\nu\mu &= 0 \ \text{ on } \ \tilde\Gamma,\end{cases}$$

so that $\mu=0$ by Theorem 3.6. Similarly, condition 3 amounts to prove that if u solves the Cauchy problem (4.15) with $(g_0,g_1)=(0,0)$, then $u=0$. Besides, we know that the problem (4.15) is ill-posed, which by contradiction proves from the Brezzi-Nečas-Babuška theorem that the $\inf-\sup$ condition 1 is not satisfied. $\square$

The mixed formulation (4.10) of the Tikhonov regularization can be applied, that is: for $\epsilon>0$, find $(u_\epsilon,\lambda_\epsilon)\in\mathcal{V}_{g_0}\times\mathcal{M}$ such that for all $(v,\mu)\in\mathcal{V}_0\times\mathcal{M}$,

$$\begin{cases}\epsilon\displaystyle\int_\Omega\nabla u_\epsilon\cdot\nabla v\,dx+\int_\Omega\nabla v\cdot\nabla\lambda_\epsilon\,dx=0\\ \displaystyle\int_\Omega\nabla u_\epsilon\cdot\nabla\mu\,dx-\int_\Omega\nabla\lambda_\epsilon\cdot\nabla\mu\,dx=\langle g_1,\mu|_\Gamma\rangle_{H^{-1/2}(\Gamma),\tilde H^{1/2}(\Gamma)}.\end{cases} \tag{4.18}$$

Compared to the abstract formulation (4.10), in formulation (4.18) we have used the scalar product associated with the semi-norm in $H^1(\Omega)$ instead of the full norm in $H^1(\Omega)$, which is possible thanks to Poincaré's inequality. From Theorem 4.3 and since condition 3 is satisfied, the problem (4.18) is well-posed for any Cauchy data $(g_0,g_1)\in H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)$ and for (g_0,g_1) such that problem (4.15) has a (unique) solution u, we have $(u_\epsilon,\lambda_\epsilon)\to(u,0)$ in $H^1(\Omega)\times H^1(\Omega)$.

4.4 Data completion method

Let us come back to the notations of section **??** and assume that $\overline{\Gamma}\cap\overline{\tilde\Gamma}=\emptyset$. From Proposition 1.1 and Theorem 1.2, solving problem (4.15) is equivalent to solve the following variational problem: find $\lambda\in H^{1/2}(\tilde\Gamma)$ such

$$s(\lambda,\mu)=\ell(\mu),\quad\forall\mu\in H^{\frac12}(\tilde\Gamma),$$

where the bilinear form s and the linear form ℓ are given by (**??**) and (**??**), respectively. In particular, s is symmetric. Let us prove some other properties of s.

Proposition 4.3. *We have that* $s(\mu,\mu) \geq 0$ *for all* $\mu \in H^{\frac{1}{2}}(\tilde{\Gamma})$ *and* $s(\mu,\mu) = 0$ *implies that* $\mu = 0$.

Proof. For $\mu \in H^{\frac{1}{2}}(\tilde{\Gamma})$, the function $u_N(\mu)$ is the unique minimizer in the set $\{v \in H^1(\Omega),\ v|_{\tilde{\Gamma}} = \mu\}$ of the cost function $\int_\Omega |\nabla v|^2\,dx$. We hence obtain, since $u_D(\mu) \in H^1(\Omega)$ and $v|_{\tilde{\Gamma}} = \mu$, that

$$\int_\Omega |\nabla u_D(\mu)|^2\,dx \geq \int_\Omega |\nabla u_N(\mu)|^2\,dx,$$

which implies the first part of the statement.
Let us assume now that $s(\mu,\mu) = 0$. Because the above minimization problem has a unique solution, we have $u_D(\mu) = u_N(\mu) = u$, which means that u solves the Cauchy problem which trivial data (3.1), then $u = 0$ in Ω and $\mu = 0$, which completes the proof. $\square$

Solving the Cauchy problem (1.3) is equivalent to solve problem (1.8). Since S is compact, injective and has dense range (because S is self-adjoint), we are exactly in the framework of chapter 2 for ill-posed problems. We are hence able, for any data $(g_0, g_1) \in H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$, to apply the Tikhonov regularization and the Morozov's discrepancy principle to the operator S.
In our particular case where the operator S is positive, an alternative method to the Tikhonov regularization is the Lavrentiev regularization associated with operator S, that is for $\epsilon > 0$: find $\lambda \in H^{\frac{1}{2}}(\tilde{\Gamma})$ such that

$$s(\lambda,\mu) + \epsilon(\lambda,\mu)_{H^{\frac{1}{2}}(\tilde{\Gamma})} = \ell(\mu), \quad \forall \mu \in H^{\frac{1}{2}}(\tilde{\Gamma}). \tag{4.19}$$

The weak formulation (4.19) amounts to minimize in $H^{\frac{1}{2}}(\tilde{\Gamma})$ the functional

$$J(\mu) + \frac{\epsilon}{2}\|\mu\|^2_{H^{\frac{1}{2}}(\tilde{\Gamma})}, \quad J(\mu) := \frac{1}{2}s(\mu,\mu) - \ell(\mu).$$

We complete this section with a comparison between the functional J and the so-called Kohn-Vogelius functional KV, which is defined by

$$KV(\mu) = \frac{1}{2}\int_\Omega |\nabla u_D(\mu, g_0) - \nabla u_N(\mu, g_1)|^2\,dx.$$

Precisely, we have the following result.

Proposition 4.4. *For all* $\mu \in H^{\frac{1}{2}}(\tilde{\Gamma})$,

$$KV(\mu) = J(\mu) + C, \quad C := \frac{1}{2}\int_\Omega |\nabla \breve{u}_D(g_0) - \nabla \breve{u}_N(g_1)|^2\,dx,$$

that is functionals KV and J coincide up to a constant which depends only on the data.

Proof. we first prove that the gradient of KV and of J are equal. The gradient of J at point λ is given by

$$\langle \nabla J(\lambda), \mu \rangle_{\tilde{\Gamma}} = \int_{\Omega} \nabla u_D(\lambda, g_0) \cdot \nabla u_D(\mu)\, dx - \int_{\Omega} \nabla u_N(\lambda, g_1) \cdot \nabla u_N(\mu)\, dx + \langle g_1, u_N(\mu) \rangle_{\Gamma},$$

while the gradient of KV is given by

$$\langle \nabla KV(\lambda), \mu \rangle_{\tilde{\Gamma}} = \int_{\Omega} \nabla (u_D(\lambda, g_0) - u_N(\lambda, g_1)) \cdot \nabla (u_D(\mu) - u_N(\mu))\, dx.$$

We then obtain that

$$\begin{aligned}\langle \nabla J(\lambda), \mu \rangle_{\tilde{\Gamma}} &- \langle \nabla KV(\lambda), \mu \rangle_{\tilde{\Gamma}} \\ &= \int_{\Omega} \nabla u_D(\lambda, g_0) \cdot \nabla u_N(\mu)\, dx - \int_{\Omega} \nabla u_N(\lambda, g_1) \cdot \nabla u_N(\mu)\, dx \\ &\quad + \int_{\Omega} \nabla u_N(\lambda, g_1) \cdot \nabla (u_D(\mu) - u_N(\mu))\, dx + \langle g_1, u_N(\mu) \rangle_{\Gamma}.\end{aligned}$$

It is easy to see that the two terms in the right-hand side vanish separately, which proves that the gradient of KV and of J are equal. The value of the constant C follows from the definition of $\breve{u}_D(g_0)$ and $\breve{u}_N(g_1)$. □

The weak formulation (4.19) is directly in a suitable form for some discretization with the help of a finite element method. It is equivalent to minimize the Kohn-Vogelius functional (with regularization) in a discretized space.

Chapter 5

Data completion problems – The wave equation case

5.1 Introduction

5.1.1 Problem setting

Let Ω be a bounded, connected open domain of class C^2 in $\mathbb{R}^d$, $d > 1$. We consider the following wave equation

$$\begin{cases} \partial_t^2 u(x,t) - \Delta u(x,t) = 0, & (x,t) \in \Omega \times (0,\infty) \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x), \quad \dot{u}(x,0) = v_0(x), & x \in \Omega \end{cases} \tag{5.1}$$

and we recall that if $(u_0, v_0) \in H_0^1(\Omega) \times L^2(\Omega)$ then $(u, \dot{u}) \in C((0,T); H_0^1(\Omega)) \times C((0,T); L^2(\Omega))$ for any $T > 0$ (see below). Denoting $z(t) = \begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix}$ belonging to $\mathcal{Z} =$ equipped with the norm, we can rewrite (5.1) as the first-order system

$$\begin{cases} \dot{z} = Az \\ z(0) = \zeta \end{cases} \tag{5.2}$$

where $\zeta = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$ and $A : \mathcal{D}(A) \to \mathcal{Z}$ with

$$A = \begin{pmatrix} 0 & \mathbb{1} \\ \Delta & 0 \end{pmatrix}, \quad \mathcal{D}(A) = \mathcal{D}(-\Delta) \times H_0^1(\Omega), \quad \mathcal{Z} = H_0^1(\Omega) \times L^2(\Omega).$$

The operator A is skew-symmetric hence it generates a semigroup and for all $\zeta \in \mathcal{Z}$ then $(u, \dot{u}) \in C((0,T); \mathcal{Z})$ for any $T > 0$, see Section A.6. Moreover, we equipped $\mathcal{Z}$ with the norm

$$\forall z = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{Z}, \quad \|z\|_{\mathcal{Z}}^2 = \|\nabla u\|_{L^2(\Omega)}^2 + \|v\|_{L^2(\Omega)}^2,$$

and we have the energy conservation property in $C^1([0,T];\mathcal{D}(A))$

$$\forall t \geq 0, \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|z(t)\right\|_{\mathcal{Z}}^2 = (z, Az)_{\mathcal{Z}} = 0 \Rightarrow \mathcal{E}(t) := \frac{1}{2}\left\|z(t)\right\|_{\mathcal{Z}}^2 = \mathcal{E}(0).$$

of conclusion still valid in $C^0([0,T];\mathcal{Z})$ by density.

For this system, we consider to have at our disposal, at every time t, some measurements in $\omega \subset \Omega$ an open and non-empty subset of Ω. We denote

$$\Gamma = \overline{\partial\omega} \cap \overline{\partial\Omega},$$

which can be an empty set if ω is strictly included in Ω. Moreover we will assume that $\Omega \setminus \omega$ has a boundary $(\partial\Omega \cup \partial\omega) \setminus \Gamma$ which is Lipchitz. The measurements can be either of the form

$$\forall t \geq 0, \quad \left| \begin{array}{l} H_0^1(\Omega) \rightarrow H^1(\omega) \\ u(\cdot,t) \mapsto z(t) = u(\cdot,t)|_\omega \end{array} \right. \tag{5.3}$$

or alternatively

$$\forall t \geq 0, \quad \left| \begin{array}{l} L^2(\Omega) \rightarrow L^2(\omega) \\ \dot{u}(\cdot,t) \mapsto z(t) = \partial_t u(\cdot,t)|_\omega \end{array} \right. \tag{5.4}$$

In each case, accordingly introducing the observation space $\mathcal{Y} = H^1(\omega)$ or $\mathcal{Y} = L^2(\omega)$, we can define an observation operator $C \in \mathcal{L}(\mathcal{Z};\mathcal{Y})$ by

$$C = \begin{pmatrix} \gamma_\omega^0 & 0 \end{pmatrix}, \text{ or } C = \begin{pmatrix} 0 & \gamma_\omega^0 \end{pmatrix}, \tag{5.5}$$

respectively, with γ_ω^0 the restriction operator on ω.

The inverse problem of interest is here the following. Defining the linear operator

$$\Psi_T \, : \left| \begin{array}{rl} \mathcal{Z} & \rightarrow \mathcal{Y}_T \\ \zeta = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} & \mapsto y \end{array} \right. \tag{5.6}$$

our objective is to inverse Ψ_T, namely from $y \in \mathcal{Y}_T$ to recover the initial condition $\zeta \in \mathcal{Z}$. This again will depend of our choice of space $\mathcal{Y}_T$.

5.1.2 First characterization of the Ψ_T

In this section, we enumerate some fundamental first properties of Ψ_T.

Adjoint – In oder to define Ψ_T^*, we need first to define the adjoint of the observation operators defined in (5.5). The case where we measure $\partial_t u(\cdot,t)|_\omega$

does not present any difficulty. We define

$$C = \begin{pmatrix} 0 & \gamma^0_\omega \end{pmatrix} : \left| \begin{array}{l} H^1_0(\Omega) \times L^2(\Omega) \to L^2(\omega) \\ \qquad z = \begin{pmatrix} u \\ v \end{pmatrix} \mapsto y = v_{|\omega} \end{array} \right. \tag{5.7}$$

such that we directly infer that

$$C^* = \begin{pmatrix} 0 \\ \mathbb{1}_\omega \end{pmatrix} : \left| \begin{array}{l} L^2(\omega) \to H^1_0(\Omega) \times L^2(\Omega) \\ \qquad y \mapsto \begin{pmatrix} 0 \\ \mathbb{1}_\omega y \end{pmatrix} \end{array} \right. \tag{5.8}$$

where $\mathbb{1}_\omega$ naturally defines the extension by zero on Ω of a function defined on ω, namely

$$\mathbb{1}_\omega y : \Omega \ni x \mapsto \begin{cases} y(x), & x \in \omega, \\ 0, & x \in \Omega \setminus \omega. \end{cases}$$

The second case is a little more intricate with

$$C = \begin{pmatrix} \gamma_\omega & 0 \end{pmatrix} : \left| \begin{array}{l} H^1(\Omega) \times L^2(\Omega) \to H^1_{\Gamma,0}(\omega) \\ \qquad z = \begin{pmatrix} u \\ v \end{pmatrix} \mapsto y = u_{|\omega} \end{array} \right. \tag{5.9}$$

First, we equipped $H^1_{\Gamma,0}(\omega)$ with a useful norm. To that purpose, we define the operator

$$L_\omega : H^1_{\Gamma,0}(\omega) \to H^1_0(\Omega), \qquad L_\omega \varphi = \psi, \tag{5.10}$$

where ψ is the solution of the following elliptic equation

$$\left| \begin{array}{ll} \Delta\psi = 0, & \text{in } \Omega \setminus \omega \\ \psi = 0, & \text{on } \partial\Omega \\ \psi = \varphi, & \text{in } \overline{\omega} \end{array} \right. \tag{5.11}$$

namely, L_ω is a harmonic lifting operator. Solving (5.10) for $\Omega \setminus \omega$ of Lipchitz boundary consists in solving

$$\left| \begin{array}{ll} \Delta\psi = 0, & \text{in } \Omega \setminus \omega \\ \psi = 0, & \text{on } \partial\Omega \setminus \partial\omega \\ \psi = \varphi, & \text{in } \partial\omega \setminus \partial\Omega \end{array} \right. \tag{5.12}$$

This problem is well posed using Lax-Milgram theorem and we easily verify that L_ω is bounded from $H^1(\omega)$ to $H^1_0(\Omega)$ using

$$\|L_\omega \varphi\|^2_{H^1_0(\Omega)} \leq \|\nabla\varphi\|^2_{L^2(\omega)} + C^{\mathrm{st}}_1 \, \|\varphi_{|\partial\omega}\|^2_{H^{\frac{1}{2}}(\partial\omega)} \leq C^{\mathrm{st}}_2 \, \|\varphi\|^2_{H^1(\omega)} \, . \tag{5.13}$$

Then, we consider the space H^1_ω defined by $H^1_{\Gamma,0}(\omega)$ equipped with the norm

$$\|\varphi\|_{H^1_\omega} = \|\nabla(L_\omega\varphi)\|_{L^2(\Omega)}.$$

Indeed, using first (5.13) and secondly

$$\begin{aligned}\|\varphi\|^2_{H^1(\omega)} &= \|\nabla\varphi\|^2_{L^2(\omega)} + \|\varphi\|^2_{L^2(\omega)}\\ &\leq \|\nabla(L_\omega\varphi)\|^2_{L^2(\Omega)} + \|L_\omega\varphi\|^2_{L^2(\Omega)} \leq (1 + C^{\mathrm{st}})\,\|\nabla(L_\omega\varphi)\|^2_{L^2(\Omega)}\,,\end{aligned}$$

with C^{st} given by the Poincaré inequality, it is straightforward to see that $\|\cdot\|_{H^1_\omega}$ is a norm equivalent to the usual norm of $H^1_{\Gamma,0}(\omega)$

Considering now L_ω as a bounded operator from H^1_ω to $H^1_0(\Omega)$, let us identify L^*_ω. We directly infer from the first equation of (5.12) an orthogonality property of the form:

$$\forall\varphi \in H^1_\omega, \quad \forall\psi \in H^1_0(\Omega) \text{ s.t. } \psi_{|\omega} = 0, \quad (\nabla(L_\omega\varphi),\, \nabla\psi)_{L^2(\Omega)} = 0.$$

Therefore, we have that for all $\psi \in H^1_0(\Omega)$ and $\varphi \in H^1_\omega$,

$$(\nabla(L_\omega\varphi),\, \nabla\psi)_{L^2(\Omega)} = (\nabla(L_\omega\varphi),\, \nabla(L_\omega\psi_{|\omega}))_{L^2(\Omega)} = (\varphi, \psi_{|\omega})_{H^1_\omega},$$

showing that – via Riesz representation – the adjoint is then

$$L^*_\omega : \left|\begin{aligned} H^1_0(\Omega) &\to H^1_\omega\\ \psi &\mapsto \psi|_\omega\end{aligned}\right. \tag{5.14}$$

Eventually, we have proven that the operator C introduced in (5.9) has an adjoint given by

$$C^* = \begin{pmatrix} L_\omega \\ 0\end{pmatrix} : \left|\begin{aligned} H^1_\omega &\to H^1(\Omega) \times L^2(\Omega)\\ y &\mapsto \begin{pmatrix} L_\omega y\\ 0\end{pmatrix}\end{aligned}\right. \tag{5.15}$$

From the definition of the adjoint of the observation operator, we finally find

$$(\Psi_T(t))^* = C^* e^{tA^*},$$

and

$$\Psi^*_T = \int_0^t e^{tA^*} C^*\,\mathrm{d}t,$$

since, indeed, we have forall $y \in \mathcal{Y}_T = L^2((0,T);\mathcal{Y})$ and $\zeta \in \mathcal{Z}$,

$$\begin{aligned}(\Psi^*_T y, \zeta)_{\mathcal{Z}} &= (y, \Psi_T\zeta)_{\mathcal{Y}_T}\\ &= \int_0^T (y(t), \Psi_T(t)\zeta)\,\mathrm{d}t = \int_0^T (y(t), Ce^{tA}\zeta)_{\mathcal{Y}}\,\mathrm{d}t.\end{aligned}$$

Continuity – For all $t \geq 0$, the operator $\Psi_T(t) \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$, since C is bounded. Then we have for all $z_0 \in \mathcal{Z}$

$$\begin{aligned}\|\Psi_T\zeta\|^2_{\mathcal{Y}_T} &= (\zeta, \Psi_T^*\Psi_T\zeta)_{\mathcal{Z}} = \int_0^T \left\|Ce^{tA}\zeta\right\|^2_{\mathcal{Y}} \,\mathrm{d}t \\ &\leq \|C\|^2_{\mathcal{L}(\mathcal{Z},\mathcal{Y})} \int_0^T \left\|e^{tA}\zeta\right\|^2_{\mathcal{Z}} \,\mathrm{d}t = \|C\|^2_{\mathcal{L}(\mathcal{Z},\mathcal{Y})} \int_0^T \left\|z_{|\zeta}(t)\right\|^2_{\mathcal{Z}} \,\mathrm{d}t.\end{aligned}$$

From the energy-conservation of the wave equation – see also following Lemma 5.1 – we have that for all $t \geq 0$, $\left\|z_{|\zeta}(t)\right\|_{\mathcal{Z}} = \|\zeta\|_{\mathcal{Z}}$. Therefore, we find that

$$\|\Psi_T\zeta\|_{\mathcal{Y}_T} = \Big(\int_0^T \left\|Ce^{tA}\zeta\right\|^2_{\mathcal{Y}} \,\mathrm{d}t\Big)^{\frac{1}{2}} \leq \sqrt{T}\,\|C\|_{\mathcal{L}(\mathcal{Z},\mathcal{Y})}\,\|\zeta\|_{\mathcal{Z}}\,,$$

which proves that Ψ_T is bounded.

Injectivity – Let us finish by an injectivity result using one of the Holmgrem's theorems presented in Chapter 3.

Theorem 5.1. *There exists T_0 such that for all $T > T_0$, the operator Ψ_T is injective from $\mathcal{Z} = H^1_0(\Omega) \times L^2(\Omega)$ to $\mathrm{Ran}\,\Psi_T$.*

Proof. Let us first consider the case of measurements of u in $L^2(\omega)$ or $H^1(\omega)$. We thus have $u = 0$ in $\omega \times (0, T)$. From Corrolary 3.1, with applied on the interval $(0, T)$ with $T_0 \coloneqq 2D(\Omega, \omega)$, we have that $u = 0$ in $\Omega \times (\frac{T_0}{2}, T - \frac{T_0}{2})$. Therefore, $2\mathcal{E}(t) = \|z_\zeta(t)\|^2_{\mathcal{Z}} = 0$ in $(\frac{T_0}{2}, T - \frac{T_0}{2})$. By energy conservation for the wave equation we deduce that $2\mathcal{E}(0) = \|\zeta\|^2_{\mathcal{Z}} = 0$, hence $\zeta = 0$.

The case of velocity measurements $\partial_t u$ in $L^2(\omega)$ deserves a specific treatment. Indeed, we need to apply the previous reasoning to $\partial_t u$ which is also a solution of the wave equation in $\mathcal{D}'(\Omega \times (0, T))$. We deduce that $\partial_t u = 0$ in $\Omega \times (\frac{T_0}{2}, T - \frac{T_0}{2})$. Which means that $\partial_t^2 u = 0$ in $\Omega \times (\frac{T_0}{2}, T - \frac{T_0}{2})$, hence for almost all $t \in (T_0, T - T_0)$

$$\left|\begin{aligned} -\Delta u &= 0, \quad && \text{in } \Omega \\ u &= 0, && \text{on } \partial\Omega \end{aligned}\right.$$

which means $u = 0$ in $\Omega \times (T_0, T - T_0)$. Again by energy conservation, we end up with $\zeta = 0$.

5.2 Observability conditions associated with Ψ_T

We now study the observability condition associated with wave equation namely we want to establish if there exists a time T_0 and a constant C^{st} such that for all $T \geq T_0$, and for all weak solution of the wave equation from an initial condition $z(0)$, we have

$$\int_0^T \|y\|_{\mathcal{Y}}^2 \, \mathrm{d}t = \|\Psi_T \zeta\|_{\mathcal{Y}_T}^2 = \int_0^T \left\|C e^{tA}\zeta\right\|_{\mathcal{Y}}^2 \, \mathrm{d}t \geq C^{\mathrm{st}} \|\zeta\|_{\mathcal{Z}}^2 . \tag{5.16}$$

Namely observing $y \in \mathcal{Y}$ on $(0,T)$ brings as information as knowing $z(0) \in \mathcal{Z}$. This is exactly equivalent to

$$(\zeta, \Psi_T^* \Psi_T \zeta)_{\mathcal{Z}} = \int_0^T \left\|C e^{tA}\zeta\right\|_{\mathcal{Y}}^2 \, \mathrm{d}t \geq C^{\mathrm{st}} \|\zeta\|_{\mathcal{Z}}^2 . \tag{5.17}$$

Therefore the observability condition is a coercivity condition of $\Psi_T^*\Psi_T \in \mathcal{L}(\mathcal{Z})$. To establish such condition, several approaches are possible. We propose to follow Lions (1988) and based our estimations on the multiplier method. Then we will discuss briefly more general approaches based on microlocal analysis in order to derive the most general condition of observability called the Geometric Control Condition (GCC). On this difficult topic, we refer the interest reader to Burq & Gérard (2002) which further complete our presentation.

5.2.1 The multiplier method

The multiplier method is based on energy repartition results between the whole domain, its boundary and eventually subdomains. It will allows to establish observability condition for observation domain surrounding a large enough part of the boundary $\partial\Omega$. We will in the see in Section **??** that the fact that the observation domain touches the domain boundaries is necessary even if more general domain ω can be envisioned for the observability condition.

Following (Lions, 1988, Chapter 7), we first introduce a part of the boundary on which our subdomain will lean on. We define

$$\forall x_0 \in \mathbb{R}^d, \quad \Gamma(x_0) \coloneqq \big\{x \in \partial\Omega \,\big|\, (x - x_0)\cdot\nu(x) > 0\big\},$$

and consider a subdomain, illustrated in Figure **??**, and defined by

$$\omega_\rho(x_0) = \mathcal{O}_\rho \cap \Omega \text{ where } \mathcal{O}_\rho = \bigcup_{x\in\Gamma(x_0)} B(x,\rho).$$

Our objective in this section is to prove an observability results of the form: *There exists, T_0 and C^{st} such that for all $T \geq T_0$, such that for all weak solution of* (5.1) *with initial condition* $(u_0, v_0) \in H_0^1(\Omega)\times L^2(\Omega)$

$$\int_0^T \int_{\omega_\rho(x_0)} \big[|\partial_t u|^2 + |u|^2\big] \,\mathrm{d}x\,\mathrm{d}t \geq C^{\mathrm{st}} \big[\|u_0\|_{H_0^1(\Omega)}^2 + \|v_0\|_{L^2(\Omega)}^2\big].$$

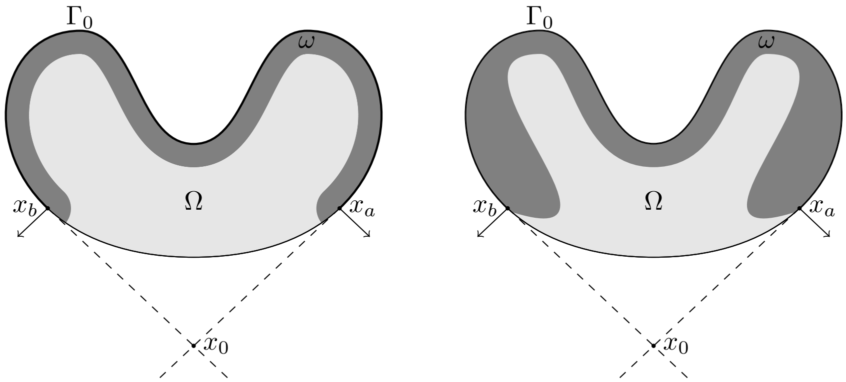

Fig. 5.1: (Left)An example of domain $\omega_\rho(x_0)$. (Right) An example of domain ω where there exists ρ s.t. $\omega_\rho(x_0) \subset \omega$

or

$$\int_0^T \int_{\omega_\rho(x_0)} \left[|\nabla u|^2 + |u|^2\right] \mathrm{d}x \, \mathrm{d}t \geq C^{\mathrm{st}} \left[\|u_0\|^2_{H^1_0(\Omega)} + \|v_0\|^2_{L^2(\Omega)}\right].$$

5.2.1.1 Energy repartition identities

Before proving the observability condition, we first need to establish some useful identities. We first recall that

Lemma 5.1. *For all weak solution of* (5.1) *with initial condition* $(u_0, v_0) \in H^1(\Omega) \times L^2(\Omega)$*, we have for all* $t \geq 0$

$$\begin{aligned} \mathcal{E}(0) &:= \frac{1}{2} \|u_0\|^2_{H^1_0(\Omega)} + \frac{1}{2} \|u_0\|^2_{L^2(\Omega)} \\ &= \frac{1}{2} \|u(t)\|^2_{H^1_0(\Omega)} + \frac{1}{2} \|\dot{u}(t)\|^2_{L^2(\Omega)} =: \mathcal{E}(t). \end{aligned} \tag{5.18}$$

Proof. We consider regular initial conditions $(u_0, v_0) \in H^2(\Omega) \times H^1(\Omega)$, hence $u \in C^0((0,T); H^2(\Omega)) \cap C^1((0,T); H^1(\Omega))$. We multiply the strong form (5.1) by $\partial_t u$ and integrate in space and time to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\|\partial_t u\|^2_{L^2(\Omega)} + \|u\|^2_{H^1(\Omega)}\right] = 0.$$

We conclude the proof by density. □

Lemma 5.2. *For all weak solution of* (5.1) *with initial condition* $(u_0, v_0) \in H^1(\Omega) \times L^2(\Omega)$*, we have for all* $T \geq 0$

$$\int_0^T \|\dot u(t)\|^2_{L^2(\Omega)}\ \mathrm{d}t - \int_0^T \|u(t)\|^2_{H^1_0(\Omega)}\ \mathrm{d}t = \big(\dot u(t), u(t)\big)_{L^2(\Omega)}\Big|_0^T. \tag{5.19}$$

Proof. Again, we consider regular initial conditions $(u_0, v_0) \in H^2(\Omega) \times H^1(\Omega)$. We multiply the strong form (5.1) by u and integrate in space and time to get

$$\begin{aligned}0 &= \int_0^T \int_\Omega (\partial_t^2 u - \Delta u) u \,\mathrm{d}x\,\mathrm{d}t \\ &= -\int_0^T \int_\Omega |\partial_t u|^2 \,\mathrm{d}x\,\mathrm{d}t + \big(\dot u(t), u(t)\big)_{L^2(\Omega)}\Big|_0^T + \int_0^T \int_\Omega |\nabla u|^2 \mathrm{d}x\ \mathrm{d}t.\end{aligned}$$

We conclude the proof by density. □

Lemma 5.3. *Let consider a function $m \in C^1(\bar\Omega \times [0,T])^d$. For all weak solution of* (5.1) *with initial condition $(u_0, v_0) \in H^1(\Omega) \times L^2(\Omega)$, we have for all $T \geq 0$*

$$\begin{aligned}\frac{1}{2}\int_0^T \int_{\partial\Omega} m\cdot\nu |\partial_\nu u|^2 \,\mathrm{d}x\,\mathrm{d}t &= \big(\dot u(t), m\cdot\nabla u(t)\big)\Big|_0^T \\ &\quad + \frac{1}{2}\int_0^T \int_\Omega \nabla\cdot m\big[|\partial_t u|^2 - |\nabla u|^2\big]\,\mathrm{d}x\,\mathrm{d}t \\ &\quad - \int_0^T \int_\Omega \partial_t u\, \partial_t m\cdot\nabla u \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega \nabla u \cdot \boldsymbol{\nabla} m \cdot \nabla u \,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{5.20}$$

Proof. We consider regular initial conditions $(u_0, v_0) \in H^2(\Omega) \times H^1(\Omega)$, multiply the strong form (5.1) by $m\cdot\nabla u$ and integrate in space and time to get

$$\begin{aligned}0 &= \int_0^T \int_\Omega (\partial_t^2 u - \Delta u) m\cdot\nabla u \,\mathrm{d}x\,\mathrm{d}t \\ &= \big(\dot u(t), m\cdot\nabla u(t)\big)_{L^2(\Omega)}\Big|_0^T - \int_0^T \int_\Omega \partial_t u\, \partial_t (m\cdot\nabla u)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_0^T \int_\Omega \nabla u\cdot\nabla(m\cdot\nabla u)\mathrm{d}x\ \mathrm{d}t + \int_0^T \int_{\partial\Omega} (m\cdot\nabla u)(\nabla u\cdot\nu)\mathrm{d}\sigma\mathrm{d}t.\end{aligned} \tag{5.21}$$

Then, we remark that

$$\begin{aligned}\int_\Omega \partial_t u \,\big(m\cdot\nabla(\partial_t u)\big)\,\mathrm{d}x &= \frac{1}{2}\int_\Omega \big(m\cdot\nabla(\partial_t u)^2\big)\,\mathrm{d}x \\ &= -\frac{1}{2}\int_\Omega (\nabla\cdot m)(\partial_t u)^2\,\mathrm{d}x,\end{aligned} \tag{5.22}$$

since $\partial_t u$ is null on the boundary. Likewise, we show that

$$\begin{aligned}\int_\Omega m\cdot\boldsymbol{\nabla}^2 u\cdot\nabla u\,\mathrm{d}x &= \frac{1}{2}\int_\Omega m\cdot\nabla|\nabla u|^2\,\mathrm{d}x\\ &= -\frac{1}{2}\int_\Omega(\nabla\cdot m)|\nabla u|^2\,\mathrm{d}x+\frac{1}{2}\int_{\partial\Omega} m\cdot\nu|\nabla u|^2\,\mathrm{d}\sigma\end{aligned}\tag{5.23}$$

Finally, we remind that u is identically null on the boundary, hence $\nabla u\big|_{\partial\Omega}= \partial_\nu u\big|_{\partial\Omega}$, and combine (5.21) with (5.22) and (5.23) to obtain (5.20). □

5.2.1.2 A boundary observability condition

In fact from Lemma 5.3, we can derive a fundamental boundary observability condition.

Theorem 5.2. *Let Ω a bounded domain of $\mathbb{R}^d$ of boundary $\partial\Omega$ of class C^2. For all $T > T(x_0) = 2r(x_0)$ with*

$$r(x_0) \coloneqq \max_{x\in\bar\Omega}|x-x_0|$$

and for all weak solution of (5.1) *with initial condition* $(u_0, v_0) \in H^1(\Omega)\times L^2(\Omega)$

$$(T-T(x_0))\mathcal{E}(0) \le \frac{r(x_0)}{2}\int_0^T\int_{\Gamma(x_0)}|\partial_\nu u|^2\,\mathrm{d}\sigma\,\mathrm{d}t. \tag{5.24}$$

Proof. We consider regular initial conditions $(u_0, v_0) \in H^2(\Omega)\times H^1(\Omega)$, and start by applying the identity (5.20) with $m = (x - x_0)$ and get

$$\begin{aligned}\mathcal{T}_1+\frac{d}{2}\int_0^T\int_\Omega\left[|\partial_t u|^2-|\nabla u|^2\right]\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_\Omega|\nabla u|^2\,\mathrm{d}x\,\mathrm{d}t&\\ =\frac{1}{2}\int_0^T\int_{\partial\Omega}(x-x_0)\cdot\nu|\partial_\nu u|^2\,\mathrm{d}\sigma\,\mathrm{d}t&,\end{aligned}$$

with

$$\mathcal{T}_1=\left(\dot u, q\cdot\nabla u(t)\right)_{L^2(\Omega)}\Big|_0^T.$$

As $(x-x_0)\cdot\nu \le 0$ on $\partial\Omega\setminus\Gamma(x_0)$ and $|(x-x_0)\cdot\nu| \le r(x_0)$ on $\Gamma(x_0)$, we have

$$\int_0^T\int_{\partial\Omega}(x-x_0)\cdot\nu|\partial_\nu u|^2\,\mathrm{d}\sigma\,\mathrm{d}t\le r(x_0)\int_0^T\int_{\Gamma(x_0)}|\partial_\nu u|^2\,\mathrm{d}\sigma\,\mathrm{d}t.$$

whence give

$$\mathcal{T}_1+\frac{d-1}{2}\mathcal{T}_2+\frac{1}{2}\int_0^T\int_\Omega\left[|\partial_t u|^2+|\nabla u|^2\right]\mathrm{d}x\,\mathrm{d}t\le\frac{r(x_0)}{2}\int_0^T\int_{\Gamma(x_0)}|\partial_\nu u|^2\,\mathrm{d}\sigma\,\mathrm{d}t,$$

where from (5.19)

$$\mathcal{T}_2 = \int_0^T \int_\Omega \left[|\partial_t u|^2 - |\nabla u|^2\right] \mathrm{d}x \, \mathrm{d}t = \big(\dot{u}(t), u(t)\big)_{L^2(\Omega)}\Big|_0^T.$$

From the energy conservation – see Lemma 5.1 – we have the estimation

$$\int_0^T \int_\Omega \left[|\partial_t u|^2 + |\nabla u|^2\right] \mathrm{d}x \, \mathrm{d}t = T\mathcal{E}(0),$$

which gives

$$\mathcal{T}_1 + \frac{d-1}{2}\mathcal{T}_2 + T\mathcal{E}(0) \leq \frac{r(x_0)}{2} \int_0^T \int_{\Gamma(x_0)} |\partial_\nu u|^2 \, \mathrm{d}\sigma \, \mathrm{d}t. \tag{5.25}$$

Let us finally give an estimation of the term

$$\left|\mathcal{T}_1 + \frac{d-1}{2}\mathcal{T}_2\right| = \left(\dot{u}(t), m \cdot \nabla u(t) + \frac{d-1}{2} u(t)\right)_{L^2(\Omega)}\Big|_0^T.$$

First, we have

$$\left|\mathcal{T}_1 + \frac{d-1}{2}\mathcal{T}_2\right| \leq 2 \left\| \left(\dot{u}(t), m \cdot \nabla u(t) + \frac{d-1}{2} u(t)\right) \right\|_{L^\infty(0,T)}.$$

Then, using Young's inequality,

$$\begin{aligned}\big(\dot{u}(t), m \cdot \nabla u(t) + \frac{d-1}{2} u(t)\big) \leq \frac{r(x_0)}{2} \, &\|\dot{u}(t)\|^2_{L^2(\Omega)} \\ &+ \frac{1}{2r(x_0)} \left\| m \cdot \nabla u(t) + \frac{d-1}{2} u(t) \right\|^2_{L^2(\Omega)}\end{aligned}$$

Considering now

$$\begin{aligned}\left\| m \cdot \nabla u(t) + \frac{d-1}{2} u(t) \right\|^2_{L^2(\Omega)} = \|m \cdot \nabla u(t)\|^2_{L^2(\Omega)} &+ \frac{(d-1)^2}{4} \|u(t)\|^2_{L^2(\Omega)} \\ &+ (d-1)\Big(m \cdot \nabla u(t), u(t)\Big)_{L^2(\Omega)},\end{aligned}$$

with

$$\begin{aligned}\Big(m \cdot \nabla u(t), u(t)\Big)_{L^2(\Omega)} &= \int_\Omega u(x - x_0) \cdot \nabla u \, \mathrm{d}x \\ &= \frac{1}{2} \int_\Omega (x - x_0) \cdot \nabla |u|^2 \, \mathrm{d}x = -\frac{d}{2} \|u\|^2_{L^2(\Omega)},\end{aligned}$$

we obtain

$$\left\|m \cdot \nabla u(t) + \frac{d-1}{2} u(t)\right\|^2_{L^2(\Omega)} \leq r(x_0)^2 \left\|\nabla u(t)\right\|^2_{L^2(\Omega)} + \Big(\frac{(d-1)^2}{4} - \frac{d(d-1)}{2}\Big) \left\|u(t)\right\|^2_{L^2(\Omega)}.$$

The function $\mathbb{N}^* \ni d \mapsto \frac{(d-1)^2}{4} - \frac{d(d-1)}{2}$ is negative, hence, we finally get

$$\left|\mathcal{T}_1 + \frac{d-1}{2}\mathcal{T}_2\right| \leq \left\|r(x_0)\mathcal{E}(t)\right\|_{L^\infty(0,T)} \leq T(x_0)\mathcal{E}(0). \tag{5.26}$$

We now combine (5.25) with (5.26) to get (5.24). We finally conclude by density of the trajectory obtained from regular initial conditions. □

Remark 5.1 (Mixed Dirichlet and Neumann conditions). Theorem 5.2 is a fundamental observability condition from boundary information. If $T > T(x_0)$, it gives that disposing of the Dirichlet boundary condition – here $u = 0$ on $\Gamma(x_0)$ – and the Neumann boundary condition on $\partial_\nu u$ on $\Gamma(x_0)$, gives uniquely the initial condition on the whole domain Ω. In other words, we could have chosen the observation problem defined from the following observation operator

$$C = \Big(0 \ \gamma^1_{\Gamma(x_0)}\Big) : \left| \begin{array}{l} H^1_0(\Omega) \times L^2(\Omega) \to L^2(\omega) \\ \qquad z = \begin{pmatrix} u \\ v \end{pmatrix} \mapsto y = \partial_\nu u \end{array}\right.$$

and numerous results of this chapter extend to this configuration. However, this boundary case is a little bit more general as C is, in this case, not bounded but defined on $\mathcal{D}(A)$. The extension to this case then necessitate to extend the results to so-called admissible operators Tucsnak & Weiss (2009). In fact, C could then be proved to be admissible from the following proposition.

Proposition 5.1. *Let Ω a bounded domain of $\mathbb{R}^d$ of boundary $\partial\Omega$ of class C^2. For all weak solution of (5.1) with initial condition $(u_0, v_0) \in H^1(\Omega) \times L^2(\Omega)$, we have for all $T \geq 0$,*

$$\int_0^T \int_{\partial\Omega} |\partial_\nu u|^2 \, \mathrm{d}\sigma \, \mathrm{d}t \leq C^{st}(T+1)\mathcal{E}(0).$$

Proof. We again use (5.20) with $m \in C^1(\Omega)^d$ such that $m \cdot \nu = 1$ on $\partial\Omega$ – see (Lions, 1988, Lemma 3.1) for such a construction. Then, there exists C^{st} such that

$$\int_0^T \int_{\partial\Omega} m \cdot \nu |\partial_\nu u|^2 \, \mathrm{d}t \leq C^{\mathrm{st}}(T+1)\Big[\left\|\partial_t u\right\|^2_{L^\infty(0,T;L^2(\Omega))} + \left\|\nabla u\right\|^2_{L^\infty(0,T;L^2(\Omega))}\Big],$$

which by energy conservation gives the expected results. □

5.2.1.3 Observability condition for subdomain measurements

The main idea in order to obtain the desired observability conditions on a subdomain $\omega_\rho(x_0)$ is to control $\partial_\nu u$ on $\Gamma(x_0)$ by the interior quantities $\partial_t u$ on $\omega_\rho(x_0)$ and/or ∇u on $\omega_\rho(x_0)$.

First we start with a useful lemma

Lemma 5.4. *Let Ω a bounded domain of $\mathbb{R}^d$ of boundary $\partial\Omega$ of class C^2. There exists $m \in C^1(\overline{\Omega})^d$ such that*

$$\tilde{m}\cdot\nu = 1 \text{ on } \Gamma(x_0), \quad \tilde{m}\cdot\nu \geq 0 \text{ on } \partial\Omega, \quad \sup \tilde{m} \subset \omega_\rho(x_0).$$

Proof. One possible choice can be found in (Lions, 1988, Remark 3.2) . We could consider a domain $\omega_m \subset \omega_\rho(x_0)$ of C^2 boundary, such that $\Gamma(x_0) \subset \partial\omega_m$. Then let g a $C^\infty(\partial\omega_m)$ such that $g = 1$ on $\Gamma(x_0)$. We solve

$$\left|\begin{array}{ll} -\boldsymbol{\Delta} m = 0, & \text{in } \omega_m \\ m\cdot\nu = g, & \text{on } \partial\omega_m \\ m\cdot\tau = 0, & \text{on } \partial\omega_m, \tau\cdot\nu = 0 \end{array}\right.$$

□

Using such a multiplier will help us to prove the following proposition.

Proposition 5.2. *Let Ω a bounded domain of $\mathbb{R}^d$ of boundary $\partial\Omega$ of class C^2. There exists C^{st}, such that for all weak solution of (5.1) with initial condition $(u_0, v_0) \in H^1(\Omega)\times L^2(\Omega)$, we have for all $T > 0$ and $0 < eps < T$,*

$$\int_\epsilon^{T-\epsilon}\int_{\Gamma(x_0)} |\partial_\nu u|^2 \,\mathrm{d}\sigma\,\mathrm{d}t \leq C^{st}\int_0^T\int_{\omega_\rho(x_0)} \left[|\partial_t u|^2 + |\nabla u|^2\right]\mathrm{d}x\,\mathrm{d}t. \quad (5.27)$$

Proof. Let us choose $m : \Omega\times(0,T) \ni (x,t) \mapsto t(T-t)\tilde{m}(x)$ where $\tilde{m}$ is built from Lemma 5.4. We first have that there exists $C^{\mathrm{st}}(\epsilon)$ such that

$$\int_\epsilon^{T-\epsilon}\int_{\Gamma(x_0)} |\partial_\nu u|^2 \,\mathrm{d}\sigma\,\mathrm{d}t \leq C^{\mathrm{st}}(\epsilon)\int_0^T\int_{\Gamma(x_0)} m\cdot\nu|\partial_\nu u|^2\,\mathrm{d}\sigma\,\mathrm{d}t. \quad (5.28)$$

Then, from (5.20), we have

$$\begin{aligned}\int_0^T\int_{\Gamma(x_0)} m\cdot\nu|\partial_\nu u|^2\,\mathrm{d}\sigma\,\mathrm{d}t \leq \frac{1}{2}\int_0^T\int_{\omega_\rho(x_0)} \nabla\cdot m\left[|\partial_t u|^2 - |\nabla u|^2\right]\mathrm{d}x\,\mathrm{d}t \\ + \int_0^T\int_{\omega_\rho(x_0)} \partial_t u\,\partial_t m\cdot\nabla u\,\mathrm{d}x\mathrm{d}t \\ + \int_0^T\int_{\omega_\rho(x_0)} \nabla u\cdot\boldsymbol{\nabla} m\cdot\nabla u\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

As $m, \partial_t m, \nabla m \in L^\infty((0,T); L^\infty(\omega_\rho(x_0)))$, using $2ab \leq a^2 + b^2$, we get that there exists C^{st} depending only on $\omega_\rho(x_0)$ through the $\|\cdot\|_{L^\infty(\omega)}$ of $\partial_t m$ and ∇m such that

$$\int_0^T \int_{\partial\Omega} m|\partial_\nu u|^2 \,\mathrm{d}t \leq C^{\mathrm{st}} \int_0^T \int_{\omega_\rho(x_0)} \big[|\partial_t u|^2 + |\nabla u|^2\big] \,\mathrm{d}x \,\mathrm{d}t. \qquad (5.29)$$

We finally get (5.27) from (5.28) combined with (5.29) $\square$

We can now prove the observability condition result.

Theorem 5.3. *Let Ω a bounded domain of $\mathbb{R}^d$ of boundary $\partial\Omega$ of class C^2. There exists C^{st} such that for all $T > T(x_0)$ given by Theorem 5.2 and for all weak solution of (5.1) with initial condition $(u_0, v_0) \in H^1(\Omega) \times L^2(\Omega)$, we have*

$$\int_0^T \int_{\omega_\rho(x_0)} \big[|\partial_t u|^2 + |\nabla u|^2\big] \,\mathrm{d}x \,\mathrm{d}t \geq C^{st}\big[\|u_0\|^2_{H^1_0(\Omega)} + \|v_0\|^2_{L^2(\Omega)}\big]. \qquad (5.30)$$

Proof. Let ϵ such that $T - 2\epsilon > T(x_0)$. From (5.24) obtained in Theorem 5.24, we get

$$(T - 2\epsilon - T(x_0))\mathcal{E}(0) \leq (T - 2\epsilon - T(x_0))\mathcal{E}(\epsilon) \leq \frac{r(x_0)}{2} \int_\epsilon^{T-\epsilon} |\partial_\nu u|^2 \,\mathrm{d}\sigma \mathrm{d}t.$$

Therefore there exists $C^{\mathrm{st}} > 0$ such that

$$\mathcal{E}(0) \leq C^{\mathrm{st}} \int_\epsilon^{T-\epsilon} |\partial_\nu u|^2 \,\mathrm{d}\sigma \mathrm{d}t.$$

Then from (5.27), obtained in Proposition 5.2, we directly infer (5.30). $\square$

We can now proceed to the statement of main theorem of this section.

Theorem 5.4. *Let $\rho > 0$. There exists, T_0 and C^{st} such that for all $T > T_0$,*

$$\int_0^T \int_{\omega_\rho(x_0)} \big[|\partial_t u|^2 + |u|^2\big] \,\mathrm{d}x \,\mathrm{d}t \geq C^{st}\big[\|u_0\|^2_{H^1_0(\Omega)} + \|v_0\|^2_{L^2(\Omega)}\big]. \qquad (5.31)$$

or

$$\int_0^T \int_{\omega_\rho(x_0)} \big[|\nabla u|^2 + |u|^2\big] \,\mathrm{d}x \,\mathrm{d}t \geq C^{st}\big[\|u_0\|^2_{H^1_0(\Omega)} + \|v_0\|^2_{L^2(\Omega)}\big]. \qquad (5.32)$$

for all weak solution of (5.1) with initial condition (u_0, v_0).

Proof. First, let us prove (5.31) for a given $\omega_\rho(x_0)$. We define $\breve{\rho} < \rho$, and $\psi \in C_c^\infty(\overline{\Omega})$ be a cutoff function satisfying

$$\psi(x) = \begin{cases} 0, & \text{if } x \in \Omega \backslash \omega_\rho \\ 1, & \text{if } x \in \omega_{\check{\rho}}(x_0) \end{cases}$$

and $0 \leq \psi(x) \leq 1$ for every $x \in \overline{\Omega}$, and $\Delta\psi \in \mathrm{L}^\infty(\Omega)$. Denote also $\varphi(t) = t^2(T-t)^2$. Then, by successive integrations by parts, we obtain

$$\begin{aligned} 0 &= \int_0^T \int_{\omega_\rho(x_0)} \varphi\psi(\partial_t^2 u - \Delta u) u \,\mathrm{d}x \,\mathrm{d}t \\ &= \int_0^T \int_{\omega_\rho(x_0)} \ddot{\varphi}\psi \frac{|u|^2}{2} \,\mathrm{d}x \,\mathrm{d}t - \int_0^T \int_{\omega_\rho(x_0)} \varphi\psi |\partial_t u|^2 \,\mathrm{d}x \,\mathrm{d}t \\ &\quad + \int_0^T \int_{\partial\omega} \varphi \frac{\partial\psi}{\partial n} \frac{|u|^2}{2} \,\mathrm{d}\sigma \,\mathrm{d}t - \int_0^T \int_{\omega_\rho(x_0)} \varphi \Delta\psi \frac{|u|^2}{2} \,\mathrm{d}x \,\mathrm{d}t \\ &\quad + \int_0^T \int_{\omega_\rho(x_0)} \varphi\psi |\nabla u|^2 \,\mathrm{d}x \,\mathrm{d}t. \end{aligned}$$

The definition of ψ entails $(\partial\psi/\partial n)|_{\partial\omega} = 0$, hence we obtain a kind of local equirepartition of the energy of the form

$$\begin{aligned} \int_0^T \int_{\omega_\rho(x_0)} \varphi\psi |\partial_t u|^2 \,\mathrm{d}x \,\mathrm{d}t + \int_0^T \int_{\omega_\rho(x_0)} \varphi\Delta\psi \frac{|u|^2}{2} \,\mathrm{d}x \,\mathrm{d}t \\ = \int_0^T \int_{\omega_\rho(x_0)} \varphi\psi |\nabla u|^2 \,\mathrm{d}x \,\mathrm{d}t + \int_0^T \int_{\omega_\rho(x_0)} \ddot{\varphi}\psi \frac{|u|^2}{2} \,\mathrm{d}x \,\mathrm{d}t \end{aligned} \tag{5.33}$$

From (5.33), we can now obtain either (5.31) or (5.32). For the first inequality, we use (5.30) for a time $T > T_0$ and ϵ such that $T - 2\epsilon > T_0$. Hence, we find

$$\begin{aligned} \mathcal{E}(0) = \mathcal{E}(\epsilon) &\leq C_1^{\text{st}} \int_\epsilon^{T-\epsilon} \int_{\omega_{\check{\rho}}(x_0)} \left[|\partial_t u|^2 + |\nabla u|^2 \right] \mathrm{d}x \,\mathrm{d}t \\ &\leq C_2^{\text{st}} \int_\epsilon^{T-\epsilon} \int_{\omega_{\check{\rho}}(x_0)} \left[|\partial_t u|^2 + \varphi\psi \, |\nabla u|^2 \right] \mathrm{d}x \,\mathrm{d}t \\ &\leq C_3^{\text{st}} \int_0^T \int_{\omega_\rho(x_0)} \left[|\partial_t u|^2 + |u|^2 \right] \mathrm{d}x \,\mathrm{d}t, \end{aligned}$$

which justify (5.31). Identically, we find

$$\begin{aligned}\mathcal{E}(0) = \mathcal{E}(\epsilon) &\leq C_1^{\mathrm{st}} \int_{\epsilon}^{T-\epsilon} \int_{\omega_{\breve{\rho}}(x_0)} \left[|\partial_t u|^2 + |\nabla u|^2\right] \mathrm{d}x \, \mathrm{d}t \\ &\leq C_2^{\mathrm{st}} \int_{\epsilon}^{T-\epsilon} \int_{\omega_{\breve{\rho}}(x_0)} \left[\varphi\psi \, |\partial_t u|^2 + |\nabla u|^2\right] \mathrm{d}x \, \mathrm{d}t \\ &\leq C_3^{\mathrm{st}} \int_0^T \int_{\omega_\rho(x_0)} \left[|\partial_t u|^2 + |u|^2\right] \mathrm{d}x \, \mathrm{d}t,\end{aligned}$$

hence justifying (5.32).

Theorem 5.4 allows to conclude about a coercivity property of the form (5.17). Indeed, we directly infer that for $T > T_0$, Ψ_T defined by (5.6) as a linear operator from $\mathcal{Z} = H_0^1(\Omega) \times L^2(\Omega)$ into $\mathcal{Y}_T = L^2((0,T); H^1(\omega))$ or into $\mathcal{Y}_T = H^1((0,T); L^2(\omega))$ is such that $\Psi_T^* \Psi_T \in \mathcal{L}(\mathcal{Z})$ is invertible of bounded inverse.

5.2.2 Weaker norms

From the previous section results, we can derive weaker observability conditions, allowing to rely on less informative data.

Theorem 5.5. *Considering a subdomain ω such that the following observability condition holds: There exist T_0, and C^{st} such that for all T greater than $T_0 \geq D(\Omega, \omega)$ and for all weak solution of* (5.1) *with initial condition $(u_0, v_0) \in H_0^1(\Omega) \times L^2(\Omega)$,*

$$\int_0^T \int_\omega \left[|\partial_t u|^2 + |u|^2\right] \mathrm{d}x \, \mathrm{d}t \geq C^{st} \left[\|u_0\|^2_{H_0^1(\Omega)} + \|v_0\|^2_{L^2(\Omega)}\right], \tag{5.34}$$

then, we also have

$$\int_0^T \int_\omega |\partial_t u|^2 \, \mathrm{d}x \, \mathrm{d}t \geq C^{st} \left[\|u_0\|^2_{H_0^1(\Omega)} + \|v_0\|^2_{L^2(\Omega)}\right]. \tag{5.35}$$

Proof. We proceed by contradiction. Assuming that (5.35) is not satisfied, we consider a sequence of initial condition $\{z_0^m = (u_0^m, v_0^m)\}_{m \geq 0}$ such that

$$\|u_0^m\|^2_{H_0^1(\Omega)} + \|v_0^m\|^2_{L^2(\Omega)} = 1. \tag{5.36}$$

and such that the corresponding weak solution of (5.1) with initial condition (u_0^m, v_0^m) denoted by $\{z^m = (u^m, \partial_t u^m)\}_{m \geq 0}$ satisfies

$$\int_0^T \int_\omega |\partial_t u^m|^2 \, \mathrm{d}x \, \mathrm{d}t \xrightarrow{m \to +\infty} 0.$$

As (5.36) is satisfied, we can assume (up to a subextraction) that

$$z_0^m = (u_0^m, v_0^m) \xrightharpoonup{m\to+\infty} z_0 = (u_0, v_0), \qquad \text{in } \mathcal{Z}.$$

Then, we have

$$\int_0^T \int_\omega |\partial_t u|^2 \,\mathrm{d}x\,\mathrm{d}t = \lim_{m\to+\infty} \int_0^T (\partial_t u, \partial_t u^m)_{L^2(\omega)} = 0,$$

hence, $\partial_t u \equiv 0$ on ω. From Theorem 5.1 (*i.e.* from the injectivity of Ψ_T in the case of velocity measurements). We remark that

$$\forall z \in \mathcal{Z}, \quad (e^{tA} z_0^m, z) = (z_0^m, e^{tA^*} z) \xrightarrow{m\to+\infty} 0$$

which implies that $e^{tA} z_0^m \rightharpoonup 0$ when m tends to infinity, namely that

$$\partial_t u^m \xrightharpoonup{m\to+\infty} 0, \quad \nabla u^m \xrightharpoonup{m\to+\infty} 0, \quad u^m \xrightharpoonup{m\to+\infty} 0 \quad \text{in } L^2(\Omega\times(0,T))$$

By the compact embedding theorem the convergence is strong in $L^2(\Omega \times (0,T))$ and we directly infer that the limit is 0, namely

$$u^m \xrightarrow{m\to+\infty} 0, \qquad \text{in } L^2(\Omega\times(0,T)).$$

Therefore, there exists C^{st} such that

$$\|u_0^m\|^2_{H_0^1(\Omega)} + \|v_0^m\|^2_{L^2(\Omega)} \le C^{\mathrm{st}} \int_0^T \int_\omega \big[|\partial_t u^m|^2 + |u^m|^2\big] \,\mathrm{d}x\,\mathrm{d}t \xrightarrow{m\to+\infty} 0$$

which contradicts (5.36). $\square$

A second important result allows to weaken the norm on step further to the price of weakening the norm on $\mathcal{Z}$, hence to accept to reconstruct solution on less regular spaces.

Theorem 5.6. *Considering a subdomain ω such that the observability condition (5.35) holds. Then, we also have*

$$\int_0^T \int_\omega |u|^2 \,\mathrm{d}x\,\mathrm{d}t \ge C^{st}\big[\|u_0\|^2_{L^2(\Omega)} + \|v_0\|^2_{H^{-1}(\Omega)}\big]. \tag{5.37}$$

Proof. We consider a solution defined by

$$w(t) = \int_0^t u\mathrm{d}s + \chi,$$

such that χ is solution of

$$\left| \begin{aligned} -\Delta\chi &= v_0, \quad \text{in } \Omega \\ \chi &= 0, \quad \text{on } \partial\Omega \end{aligned} \right.$$

We have

$$\begin{aligned} \partial_t^2 w - \Delta w &= \partial_t u - \int_0^t \Delta u \, \mathrm{d}s - \Delta\chi \\ &= \partial_t u - \int_0^t \Delta u \, \mathrm{d}s - v_0 = 0, \end{aligned}$$

and also $w = 0$ on $\partial\Omega$, $w(0) = \chi$, $\partial_t w(0) = u_0$. The operator $-\Delta_0$ – *i.e, the Laplace operator with Dirichlet boundary conditions* – defines an isomorphism from H_1^0 to H^{-1} – see Theorem A.6 in Appendix A – and we infer directly that (5.35) applies to w implies (5.37) and conversely. $\square$

5.2.3 The Geometric Control Condition (GCC)

The multiplier method allows to characterize subdomains where the observability condition is satisfied. However, this characterization is too restrictive, namely there exist other subdomain ω such that the observability condition (5.31) or (5.32) hence (5.35) or (5.37) are satisfied. The most general condition is called the *Geometric Control Condition* and can be summarized into the following rule: *a subdomain ω is compatible with the observability condition for a given T_0 if any optic ray "following the Descartes rules" enters the subdomain before T_0.*

Proving such results goes beyond the present chapter and we refer to Bardos et al. in (Lions, 1988, Appendix 2) for a brief presentation of all the necessary concepts behind the GCC, to Burq & Gérard (2002) for a very complete and pedagogical presentation for subdomain touching the boundary with regular stabilization coefficient, to Bardos et al. (1988) for the original reference about GCC conditions for a control subdomain, and finally to Burq & Gérard (2017) for a complete statement of the strong geometric control condition (SGCC) allowing to consider L^∞ damping coefficients typically introduced when considering all the observations in ω with an extension $\mathbb{1}_\omega$. In this section, we limit ourselves to formulate (1) a complete statement of the SGCC condition and associated observability results and (2) gives a intuition on why the $\omega(x_0)$ obtained by the multiplier methods are compatible with the GCC albeit the GCC defines more general subdomains.

5.2.3.1 Statement

For a domain Ω of analytic boundary $\partial\Omega$ we define a ray are defined as segments $\{x_\tau(s), s \in [0,T]\}$ of direction denoted by τ in the interior of Ω with three different behavior when encountering the boundary $\partial\Omega$:

hyperbolic points: the ray meets the boundary transversely – namely $\tau \cdot \nu \neq 0$ – and reflects following the Descartes rules, see the blue line in Figure 5.2;

diffractive points: if the ray kisses with $\tau \cdot \nu \neq 0$ and its extension reenters the domain, then the ray follows the its direction τ, see the blue line in Figure 5.2;

glancing points: if the ray kisses with $\tau \cdot \nu \neq 0$ and its extension escape the domain, then the ray follows the boundary until it reenter the domain Ω by following a segment see the red line in Figure 5.2.

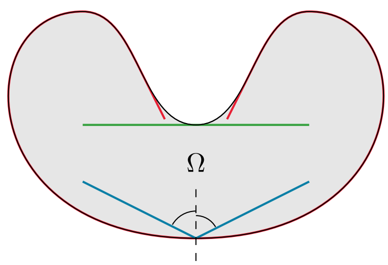

Fig. 5.2 The 3 types of rays: hyperbolic (blue line), diffractive (green line) and glancing (red line)

We then state the following result which gives necessary and sufficient conditions for observability conditions to hold.

Claim (Strong Geometric Control Condition). Let Ω be a domain of analytic boundary $\partial\Omega$ and $\omega \subset \Omega$. We assume that there exists T_0

$$\text{(SGCC)}: \quad \forall \xi \in \Omega, \exists s \in (0,T), \exists \delta > 0, B(x_\xi(s), \delta) \subset \omega_{\text{gcc}}. \tag{5.38}$$

Then we have (5.35) or equivalently (5.37). Conversely if for T_0, there exists one ray that does not encounter ω, then (5.35) and (5.37) are not satisfied.

5.2.3.2 From GCC to the multiplier method

One essential question is to understand the link between the GCC condition and the multiplier method. As Theorem 5.2.3.1 gives necessary and sufficient conditions for defining ω, the question is to understand how a subdomain

$\omega_\rho(x_0)$ defined from the multiplier method follows the GCC condition. Again, we will remain descriptive in this paragraph to only point out important bibliography ressources for the interested reader.

First, we recall that $\omega_\rho(x_0)$ contains $\Gamma(x_0)$. Moreover, the GCC condition on ω_ρ extends to the case of boundary observations on a subset $\Gamma \subset \partial\Omega$, see for example the result in Lions (1988) and how this condition can be understood as an asymptotic limit of interior measurements (Joly, 2006), at least in the case of velocity measurements. Then, in Miller & 2002 (2002), it has been proved that any acceptable domain $\Gamma(x_0)$ satisfies the a GCC condition, namely there exists a time T_0 such that all the rays encounter $\Gamma(x_0)$. Therefore $\omega_\rho(x_0)$ also satisfies the SGCC condition.

Conversely, there exists domains compatible with the SGCC that are not of the form $\omega_\rho(x_0)$ or does not contain a subdomain of the form $\omega_\rho(x_0)$. Again the idea is to find boundary domain Γ compatible with the GCC but not of the form $\Gamma(x_0)$. A simple but very intuitive result has been given by Gagnon (2017) on an ellipsoid domain. Indeed in this case, any connected boundary domain Γ of length $|\Gamma| \geq \frac{1}{2}|\partial\Omega|$ is of the form $\Gamma(x_0)$ and conversely any domain $\Gamma(x_0)$ has a length $|\Gamma| \geq \frac{1}{2}|\partial\Omega|$. However, there exist boundary

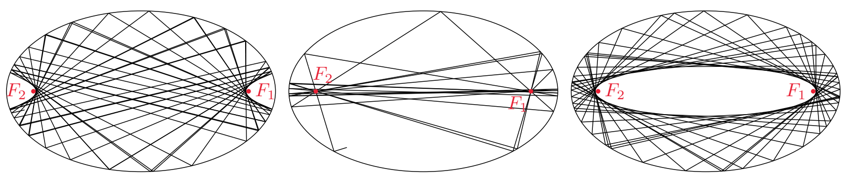

Fig. 5.3: Types of rays in elliptic domain: hyperbolic (left), going through one foci (center), elliptic (right)

domains Γ of length $|\Gamma| < \frac{1}{2}|\partial\Omega|$ which satisfies a GCC. To define such an example, we should first determine how optic rays travel in an ellipse domain. There are 3 types of rays for this specific elliptic domain, which are depicted in Figure 5.3. The first type corresponds to hyperbolic caustics. All the rays are contained between two hyperbolas of maximal semi-major axis given by the linear eccentricity of the ellipse domain. The second type corresponds to rays which go through one foci. This rays can be proved to converge to the major axis. The third type corresponds to elliptic caustics. All the rays are here contained between the ellipse boundary and an ellipse of minimal semi-major axis given by the linear eccentricity of the ellipse domain. The gliding rays on the ellipse boundary can be seen to pertaining to this third class as it corresponds to the limit case of an elliptic caustic confined to the boundary. Finally, the major and minor axis are rays. Considering these types of rays, let us now introduce a point x_1 on $\partial\Omega$, without loss of generality with negative coordinates. Then, we introduce the point x_2 also on $\partial\Omega$, such that (x_1, x_2) pass through the focus intersecting F_1. The domain Γ_{gcc} is then defined as the portion of $\partial\Omega$ x_2 from x_1 in the trigonometric order. This boundary domain is not of the form $\Gamma(x_0)$ as its length $|\Gamma|$ can be less than $\frac{1}{2}|\partial\Omega|$. However, all the rays encounter this boundary. From one such $\Gamma_{\rm gcc}$, we easily define

$$\omega_{\rm gcc} = \mathcal{O}_\rho \cap \Omega \text{ where } \mathcal{O}_\rho = \bigcup_{x\in\Gamma_{\rm gcc}} B(x,\rho),$$

with ρ small enough such that there is no $\omega(x_0)$ satisfying $\omega(x_0) \subset \omega_{\rm gcc}$.

From the previous paragraph we understand that the domain $\omega(x_0)$ are very particular examples. In fact, they correspond to a rather general class of acceptable domain. In particular, the condition that $\omega_\rho(x_0)$ is defined from a boundary domain $\Gamma(x_0)$ is not as restrictive as we could imagine, as every subdomain following the GCC should at least touches the boundary. This avoid indeed the existence of a sort of *whispering gallery* effect[1]. Namely, rays that are almost tangent to the boundary avoiding to enter the observation domain as illustrated in Figure 5.4.

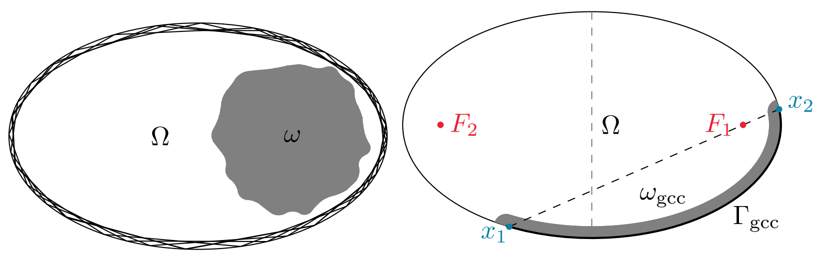

Fig. 5.4: On an elliptic domain Ω, an interior subdomain of observation ω which does not satisfy the GCC condition (left) or which does (right)

5.3 Reconstruction strategy based on asymptotic observer

In this section, we propose an alternative strategy of initial reconstruction strategy to the least square minimization seen in Section 2.5. The approach presented below was first introduced and analyzed in Ramdani et al. (2010). We also refer to Imperiale et al. (2018) for complementary developments and a generalization in the case of wave equations in unbounded domain Ω.

[1] In a whispering gallery, a faint sound could be heard round the gallery entire circumference, but not from the center

5.3.1 Asymptotic observer design

We consider again the target wave model, here of the form

$$\begin{cases} \partial_t^2 u(x,t) - \Delta u(x,t) = 0, & (x,t) \in \Omega \times (0,\infty) \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x), \quad \dot{u}(x,0) = v_0(x), & x \in \Omega \end{cases} \tag{5.39}$$

rewritten in its first order form

$$\begin{cases} \dot{z} = Az, \text{ in } (0,\infty) \\ z(0) = \zeta \end{cases} \tag{5.40}$$

We assume that on this class of model (5.40), we observ a target solution $\check{z}$ of unknown initial condition. The ideal observation would have been $\check{y}$ such that $\check{y} = C\check{z}$ where C is one of the previously defined observation operator. In fact, $\check{y}$ are not available instead we have at our disposal $y^\delta \in \mathcal{Y}_T = L^2((0,T);\mathcal{Y})$ such that for instance

$$\forall T > 0, \quad \left\| \check{y} - y^\delta \right\|_{\mathcal{Y}_T} \leq \delta T$$

Benefiting from the measurement $y^\delta \in \mathcal{Y}_T$ and the associated observation operator C, we can design a modified system that use the measurements in order to converge asymptotically in time to the target trajectory even when initialized from inaccurate initial conditions. We call such system an asymptotic observer (Krener, 1998).

Definition 5.1. An asymptotic observer of (5.40) is a causal function $\mathbb{R}^+ \ni t \mapsto \hat{z}(t; \{y^\delta(s), s \leq t\})$

(i) For all $0 \leq s \leq t$, and for all $\epsilon > 0$, $\exists \alpha(\epsilon), \delta(\epsilon)$ such that

$$\left.\begin{aligned} \|\hat{z}(s) - \check{z}(s)\|_{\mathcal{Z}} \leq \alpha(\epsilon), \\ \left\|\check{y}(t) - y^\delta\right\|_{L^2((s,t);\mathcal{Y})} \leq \delta(\epsilon) \end{aligned}\right\} \Rightarrow \|\check{z}(t) - \hat{z}(t)\|_{\mathcal{Z}} \leq \epsilon.$$

(ii) For all $0 \leq t \leq T$, and for all $\epsilon, \alpha, \delta > 0$, $\exists T_0 > 0$

$$\left.\begin{aligned} \|\hat{z}(s) - \check{z}(s)\|_{\mathcal{Z}} \leq \alpha, \\ \left\|\check{y}(t) - y^\delta\right\|_{L^2((s,t);\mathcal{Y})} \leq \delta \\ \operatorname{supp}(\check{y}(t) - y^\delta) \subset (s,t) \end{aligned}\right\} \Rightarrow \|\check{z}(T) - \hat{z}(T)\|_{\mathcal{Z}} \leq \epsilon, \quad \forall T \geq t + T_0$$

Remark 5.2. Note that an asymptotic observer satisfies for all $s \geq 0$

$$\hat{z}(s) = \check{z}(s), \quad \big(y^\delta(t) = \check{y}(t),\, t \geq s\big) \Rightarrow \big(\hat{z}(t) = \check{z}(t),\ t \geq s\big), \tag{5.41}$$

Having in mind the Definition 5.1, we thus propose the tentative observer

$$\begin{cases} \dot{\hat{z}} = A\hat{z} + \gamma C^*(y^\delta - C\hat{z}), \text{ in } (0,T) \\ \hat{z}(0) = \hat{\zeta} \end{cases} \tag{5.42}$$

for a given $\hat{\zeta}$ (typically $\hat{\zeta} = 0$).

Proposition 5.3. *Assuming that $\hat{\zeta} \in \mathcal{Z}$, $y^\delta \in L^2((0,T);\mathcal{Y})$, the system defined by (5.42) is well-posed in the sense that there exists a mild solution $\hat{z} \in C^0([0,T];\mathcal{Z})$.*

Proof. Let us show that $A^\gamma = A - \gamma C^* C$ of domain $\mathcal{D}(A)$ is maximal dissipative. First, we have the dissipativeness since

$$\forall z \in \mathcal{Z}, \quad (z, A^\gamma z)_{\mathcal{Z}} = (z, Az)_{\mathcal{Z}} - \gamma(Cz, Cz)_{\mathcal{Z}} \leq 0.$$

Then for $\lambda > 0$ and $\beta \in \mathcal{Z}$, we seek $z \in \mathcal{D}(A)$ such that

$$(1 - \lambda A^\gamma)z = \beta \Rightarrow (1 - \lambda A)z = (\beta - + \lambda\gamma C^* C z).$$

The application $F : \mathcal{D}(A) \ni z = (1 - \lambda A)^{-1}(\beta - + \lambda\gamma C^* C z) \in z$ satisfies

$$\|F(z_1) - F(z_2)\|_z \leq \lambda\gamma \|C\|^2 \|z_1 - z_2\|.$$

Therefore, there exists $\lambda < (\gamma \|C\|^2)^{-1}$ such that F is a contraction. By the contraction mapping principle, $(1 - \lambda A^\gamma)$ is surjective from $\mathcal{Z}$ to $\mathcal{D}(A)$.

Finally $\gamma C^* y^\delta \in L^2((0,T);\mathcal{Y})$, hence by Theorem A.5, there exists a unique mild solution $\hat{z} \in C^0([0,T];\mathcal{Z})$ of (5.42). This mild solution is also a weak solution. □

By uniqueness of the solution of (5.42) and (5.40) we have that

$$\dot{\check{z}} = A\check{z} + \gamma C^*(\check{y} - C\check{z}),$$

whence justifies that $\hat{z}$ and $\check{z}$ coincide for $t \geq s$ as soon as $\hat{z}(s) = \check{z}(s)$ and $\hat{z}$ is fed with the perfect data $\check{y}$.

Let us now understand how the error between the two systems can be controlled and even decreases with perfect measurements. To this end, we introduce the error $\tilde{z} = \check{z} - \hat{z}$, we have

$$\begin{aligned} \dot{\tilde{z}} &= \dot{\check{z}} - \dot{\hat{z}} \\ &= A\check{z} - A\hat{z} - \gamma C^*(C\check{z} + \chi - C\hat{z}) \\ &= A\tilde{z} - \gamma C^* C\tilde{z} - \gamma\chi. \end{aligned}$$

Therefore, the error system is solution of

$$\begin{cases} \dot{\tilde{z}} = (A - \gamma C^* C)\tilde{z} - \gamma\chi \\ \tilde{z}(0) = \tilde{\zeta} = \check{z}(0) - \hat{z}(0) \end{cases} \tag{5.43}$$

which is well-defined in $C^0([0,T];\mathcal{Z})$, as proved in Proposition 5.3. We then demonstrate that $\hat{z}$ is an asymptotic observer.

Theorem 5.7. *Assuming that $y^\delta \in L^2((0,T);\mathcal{Y})$, the system defined by (5.42) is an asymptotic observer if $A-\gamma C^* C$ is an exponentially stable semigroup.*

Proof. We assume that there exist C^{st} and $\lambda > 0$ such that the semigroup $\tilde{\Phi}$ associated with $A - \gamma C^* C$ satisfies for all $t \geq 0$,

$$\|\tilde{\Phi}(t)\| \leq C^{\text{st}} e^{-\lambda t}.$$

Applying the Duhamel formula, we get

$$\tilde{z}(t) = \tilde{\Phi}(t-s)\tilde{z}(t) + \int_s^t \tilde{\Phi}(t-\tau)\chi(\tau)\,\mathrm{d}\tau,$$

hence we have

$$\begin{aligned} \|\tilde{z}(t)\|_{\mathcal{Z}} &\leq C_1^{\text{st}} e^{-\lambda(t-s)} \|\tilde{z}(s)\|_{\mathcal{Z}} + C_2^{\text{st}} \int_s^t e^{-\lambda(t-\tau)} \|\chi(\tau)\|_{\mathcal{Y}} \,\mathrm{d}\tau \\ &\leq C_1^{\text{st}} e^{-\lambda(t-s)} \|\tilde{z}(s)\|_{\mathcal{Z}} + C_2^{\text{st}} \Big(\int_s^t e^{-2\lambda(t-\tau)}\mathrm{d}\tau\Big)^{\frac{1}{2}} \|\chi\|_{L^2((s,t);\mathcal{Y})} \\ &\leq C_1^{\text{st}} e^{-\lambda(t-s)} \|\check{z}(s) - \hat{z}(s)\|_{\mathcal{Z}} + C_2^{\text{st}} \left(\frac{(1-e^{2\lambda(t-s)})}{2\lambda}\right)^{\frac{1}{2}} \|\chi\|_{L^2((s,t);\mathcal{Y})}. \end{aligned}$$

This justifies Definition 5.1-(i). Moreover, we then compute for $T \geq t$.

$$\|\tilde{z}(T)\|_{\mathcal{Z}} \leq C_1^{\text{st}} e^{-\lambda(T-t)} \|\check{z}(t) - \hat{z}(t)\|_{\mathcal{Z}},$$

whence justifies Definition 5.1-(ii). □

Remark 5.3. The robustness to noise can be refined by considering other norms on the measurement noise. Typically, if we assume $\chi \in L^1((s,t),\mathcal{Z})$, we have this time

$$\|\tilde{z}(t)\|_{\mathcal{Z}} \leq C_1^{\text{st}} e^{-\lambda(t-s)} \|\check{z}(s) - \hat{z}(s)\|_{\mathcal{Z}} + C_2^{\text{st}} \|\chi\|_{L^1((s,t);\mathcal{Y})}.$$

or with $\|\chi(t)\|_{L^\infty((0,T);\mathcal{Y}} \leq \delta$,

$$\begin{aligned} \|\tilde{z}(t)\|_{\mathcal{Z}} &\leq C_1^{\text{st}} e^{-\lambda t} \|\tilde{z}(s)\|_{\mathcal{Z}} + C^{\text{st}} \Big(\int_s^t e^{-\lambda(t-\tau)}\delta\mathrm{d}\tau\Big) \\ &\leq C_1^{\text{st}} e^{-\lambda t} \|\check{z}(s) - \hat{z}(s)\| + C^{\text{st}}(1-e^{\lambda T})\frac{\delta}{\lambda}. \end{aligned}$$

Therefore, the noise impact is alleviated with a factor λ. The more stabilization we have, the more noise control we get.

Now we can demonstrate that the semigroup is exponentially stable under the observability condition.

Theorem 5.8. *We assume that there exists $T_0 > 0$ and C^{st} such that*

$$\forall T \geq T_0, \quad \forall z_0 \in \mathcal{Z}, \quad \int_0^T \left\| C e^{tA} z_0 \right\|_{\mathcal{Y}}^2 \, \mathrm{d}t \geq \|z_0\|_{\mathcal{Z}}^2 . \tag{5.44}$$

Then, the semigroup $\tilde{\Phi}$ associated with $A - \gamma C^* C$ is exponentially stable.

Proof. i) First let us prove that there exists C^{st} such that

$$\forall T \geq T_0, \quad \forall z_0 \in \mathcal{Z}, \quad \int_0^T \|C\tilde{z}(t)\|_{\mathcal{Y}}^2 \, \mathrm{d}t \geq C^{\mathrm{st}} \|z_0\|_{\mathcal{Z}}^2 ,$$

where $\tilde{z} = e^{t(A - C^* C)z} z_0$. In this respect, we decompose $\tilde{z}$ solution of

$$\begin{cases} \dot{\tilde{z}} = A\tilde{z} \\ \tilde{z}(0) = z_0 \end{cases}$$

into $\tilde{z} = \psi + \eta$ where

$$\begin{cases} \dot{\psi} = A\psi \\ \psi(0) = z_0 \end{cases} \quad \text{and} \quad \begin{cases} \dot{\eta} = A\eta - \gamma C^* C\tilde{z} \\ \eta(0) = 0 \end{cases}$$

The observability condition (5.44) applied to ψ gives

$$C^{\mathrm{st}} \|z_0\|_{\mathcal{Z}}^2 \leq \int_0^T \|C\psi\|_{\mathcal{Y}}^2 \, \mathrm{d}t \leq 2 \int_0^T \|C\tilde{z}\|_{\mathcal{Y}}^2 + 2 \int_0^T \|C\eta\|_{\mathcal{Y}}^2 .$$

Then, we have for η

$$\eta(t) = -\gamma \int_0^t e^{(t-s)A} C^* C \tilde{z} \, \mathrm{d}s,$$

leading to the estimation

$$\begin{aligned} \|\eta(t)\| &\leq \gamma \int_0^t e^{(t-s)\lambda} \|C\| \, \|C\tilde{z}(s)\| \, \mathrm{d}s \\ &\leq C^{\mathrm{st}} \|C\|^2 \left(\frac{1 - e^{-\lambda T}}{2\lambda} \right)^{\frac{1}{2}} \|C\tilde{z}\|_{L^2((0,T);\mathcal{Z})} \end{aligned}$$

Finally, we get the expected result, namely there exist $\kappa > 0$ such that

$$\|z_0\|_{\mathcal{Z}}^2 \leq \kappa \int_0^T \|C\tilde{z}(t)\|_{\mathcal{Z}}^2 \,\mathrm{d}t, \tag{5.45}$$

ii) Now we have for solution in $\mathcal{D}(A)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\tilde{z}(t)\|_{\mathcal{Z}}^2 = (A^\gamma \tilde{z}(t), \tilde{z}(t)) = -\|C\tilde{z}(t)\|_{\mathcal{Z}}^2$$

which gives for all mild solution , for all $T \geq T_0$ fixed, $N \leq 0$ and $NT \leq t \leq (N+1)T$,

$$\frac{1}{2}\|\tilde{z}(t)\|_{\mathcal{Z}}^2 - \frac{1}{2}\|\tilde{z}(t-T)\|_{\mathcal{Z}}^2 = -\int_{t-T}^t \|C\tilde{z}(t)\|_{\mathcal{Z}}^2 \,\mathrm{d}t$$

Combined with (5.45), we find

$$\frac{\alpha}{2}\|\tilde{z}(t-T)\|_{\mathcal{Z}}^2 \leq \int_{t-T}^t \|C\tilde{z}(t)\|_{\mathcal{Z}}^2 \,\mathrm{d}t \leq \frac{1}{2}\|\tilde{z}(t-T)\|_{\mathcal{Z}}^2,$$

with $\alpha \leq 1$ by construction. Therefore we have

$$\|\tilde{z}(t)\|_{\mathcal{Z}}^2 \leq (1-\rho)\|\tilde{z}(t-T)\|_{\mathcal{Z}}^2 \leq (1-\rho)^N \|\tilde{z}(t-NT)\|_{\mathcal{Z}}^2$$

Introducing $\lambda = -\ln(1-\rho)/T \geq 0$, we finally find that

$$\|\tilde{z}(t)\|_{\mathcal{Z}}^2 e^{\lambda t} \leq e^{\lambda t - N\lambda T} \leq e^{\lambda T},$$

which ensures that there exit $C^{\mathrm{st}}, \lambda > 0$ such that for all t, $\|\tilde{\Phi}(t)\| \leq C^{\mathrm{st}} e^{-\lambda t}$. □

5.3.2 Back and forth observer reconstruction strategy

We now introduce the so-called backward observer defined by

$$\begin{cases} \dot{\hat{z}}_b = A\hat{z}_b - \gamma C^*(y^\delta - C\hat{z}_b), \text{ in } (0,T) \\ \hat{z}_b(T) = \hat{z}(T) \end{cases} \tag{5.46}$$

The dynamics is well posed as soon as $-A$ is the generator of semigroup and C is bounded, which is the case here as $A^* = -A$, namely A is skew-adjoint.

The backaward dynamics allows to define the following reconstruction algorithm

$$
\text{(B\&F)}:\begin{cases}
\dot{\hat{z}}^{k+1} = A\hat{z}^{k+1} + \gamma C^*\left(y^\delta - C\hat{z}^{k+1}\right), & \text{in } [0,T],\\
\hat{z}^{k+1}(0) = \hat{z}_b^k(0),\\
\dot{\hat{z}}_b^{k+1} = A\hat{z}_b^{k+1} - \gamma C^*\left(y^\delta - C\hat{z}_b^{k+1}\right), & \text{in } [0,T],\\
\hat{z}_b^{k+1}(T) = \hat{z}^{k+1}(T),
\end{cases} \tag{5.47}
$$

that will proved to be an original reconstruction strategy of the target initial condition $\breve{z}(0)$ of an observed corresponding target trajectory $\breve{z} : t \mapsto e^{tA}\breve{z}_0$.

5.3.2.1 A free-noise result

Let us start by considering a free-noise situation, $\breve{y}$ replacing y^δ in (5.47). Let us introduce the forward error state $\tilde{z} = \breve{z} - \hat{z}$ solution to, in the absence of noise,

$$
\begin{cases}
\dot{\tilde{z}}^{k+1} = (A - \gamma C^*C)\tilde{z}^{k+1}, & \text{in } [0,T],\\
\tilde{z}^{k+1}(0) = \tilde{z}_b^k(0),
\end{cases} \tag{5.48}
$$

and the backward error state $\tilde{z}_b = \breve{z} - \hat{z}_b$ solution to, in the absence of noise,

$$
\begin{cases}
\dot{\tilde{z}}_b^{k+1} = (A + \gamma C^*C)\tilde{z}_b^{k+1}, & \text{in } [0,T],\\
\tilde{z}_b^{k+1}(T) = \tilde{z}^{k+1}(T).
\end{cases} \tag{5.49}
$$

We denote by $A^\gamma = A - \gamma C^*C$ the generator of the contraction semigroup driving the forward error. We have

$$
\tilde{z}^{k+1}(t) = e^{tA^\gamma}\tilde{z}^{k+1}(0). \tag{5.50}
$$

Similarly, we get for the backward error state that $\tilde{z}_b$ is given by

$$
\tilde{z}_b^{k+1}(t) = e^{(T-t)A^{\gamma *}}\tilde{z}_b^{k+1}(T). \tag{5.51}
$$

Let us now define Λ_γ, the linear reconstruction-error operator, that maps the initial state error $\tilde{z}_0$ to $\tilde{z}_b(0)$. From (5.50) and (5.51), we have the explicit characterization

$$
\Lambda_\gamma : \mathcal{Z} \ni z \mapsto \Lambda_\gamma z = e^{TA^{\gamma *}}e^{tA^\gamma} z \ \in \mathcal{Z}. \tag{5.52}
$$

We then have for all $n \in \mathbb{N}^*$ the following reconstruction formula

$$
\forall n \geq 1, \quad \tilde{z}^{k+1}(0) = \Lambda_\gamma \tilde{z}^k(0). \tag{5.53}
$$

Therefore, the convergence of (5.47), is related to a contracting property of the operator Λ_γ. Let us then better specify Λ_γ.

Proposition 5.4. *The operator Λ_γ is a bounded self-adjoint positive operator from $\mathcal{Z}$ to $\mathcal{Z}$ defined by*

$$\Lambda_\gamma = \mathbb{1} - 2\gamma \int_0^T e^{sA^{\gamma *}} C^* C e^{sA^\gamma} \, \mathrm{d}s, \tag{5.54}$$

where $\mathbb{1}$ denotes the identity operator. Moreover, $\|\Lambda_\gamma\|_{\mathcal{L}(\mathcal{Z},\mathcal{Z})} \leq 1$.

Proof. Using (5.52), we easily deduce that the operator Λ_γ is symmetric and positive since, from (5.52), we have

$$\forall \zeta \in \mathcal{Z}, \quad (\Lambda_\gamma \zeta, \zeta)_{\mathcal{Z}} = \|e^{tA^\gamma}\zeta\|^2_{\mathcal{Z}} \geq 0.$$

To show (5.54), let us first assume that ζ is sufficiently smooth, namely $\zeta_0 \in D(A)$, so that $\tilde{z} : t \mapsto e^{tA^\gamma}\zeta \in C^0([0,T], \mathcal{D}(A)) \cap C^0([0,T], \mathcal{Z})$. Then, using (5.52) and Stokes' formula, we get that

$$\begin{aligned}
\Lambda_\gamma \zeta &= \zeta + \int_0^T \partial_s \left(e^{A^{\gamma *}s} e^{A^\gamma s} \right) \zeta \, \mathrm{d}s \\
&= \zeta + \int_0^T e^{sA^{\gamma *}} \left(A^{\gamma *} + A^\gamma \right) e^{sA^\gamma} \zeta \, \mathrm{d}s \\
&= \zeta - 2\gamma \int_0^T e^{sA^{\gamma *}} C^* C e^{sA^\gamma} \zeta \, \mathrm{d}s,
\end{aligned}$$

recalling that $A^\gamma = A - \gamma C^* C$ with A skew-adjoint. Finally, to prove that the norm of Λ_γ is less or equal to one, we proceed in two steps. First, we note that

$$0 \leq (\Lambda_\gamma \zeta, \zeta)_{\mathcal{Z}} = \|\zeta\|^2_{\mathcal{Z}} - 2 \underbrace{\gamma \int_0^T \|C e^{A^\gamma s} \zeta\|^2_{\mathcal{Y}} \, \mathrm{d}s}_{\geq 0} \leq \|\zeta\|^2_{\mathcal{Z}}. \tag{5.55}$$

Second, we introduce the (self-adjoint) square root $\Lambda_\gamma^{\frac{1}{2}}$ of the positive self-adjoint operator Λ_γ. We get

$$\|\Lambda_\gamma \zeta\|^2_{\mathcal{Z}} = (\Lambda_\gamma \zeta, \Lambda_\gamma \zeta)_{\mathcal{Z}} = \left(\Lambda_\gamma \Lambda_\gamma^{\frac{1}{2}} \zeta, \Lambda_\gamma^{\frac{1}{2}} \zeta \right)_{\mathcal{Z}}.$$

Using (5.55), it comes

$$\|\Lambda_\gamma \zeta\|^2_{\mathcal{Z}} \ \leq \ \|\Lambda_\gamma^{\frac{1}{2}} \zeta\|^2_{\mathcal{Z}} \ = \ (\Lambda_\gamma \zeta, \zeta)_{\mathcal{Z}} \ \leq \ \|\zeta\|^2_{\mathcal{Z}}, \tag{5.56}$$

which proves the result. We extend the result for all $\zeta \in \mathcal{Z}$ by density of $D(A)$ into $\mathcal{Z}$.

We deduce from Proposition 5.4 and (5.53) that the error in (5.47) cannot grow as it is controlled by the initial error. In fact, we go further and prove in the next proposition that the error strictly decreases.

Proposition 5.5. *If the observability inequality* (5.44) *of Theorem 5.8 is satisfied, then* $\|\Lambda_\gamma\|_{\mathcal{L}(\mathcal{Z},\mathcal{Z})} < 1$.

Proof. From the proof of Theorem 5.8, we know that there exists a constant $\tilde{\kappa} > 0$ such that

$$\forall z \in \mathcal{Z}, \quad \int_0^T \| Ce^{A^\gamma s} z \|_{\mathcal{Y}}^2 \, \mathrm{d}s \geq \tilde{\kappa} \| z \|_{\mathcal{Z}}^2, \tag{5.57}$$

Therefore, we have

$$\begin{aligned} 0 \leq (\Lambda_\gamma z, z)_{\mathcal{Z}} &\leq (1 - 2\gamma\tilde{\kappa})\|z\|_{\mathcal{Z}}^2 \\ &\leq \underbrace{(1 - 2\gamma\tilde{\kappa})}_{<1}\|z\|_{\mathcal{Z}}^2. \end{aligned} \tag{5.58}$$

We would then conclude as in (5.56) by introducing the square root $(\Lambda_\gamma)^{\frac{1}{2}}$ and using (5.58) so that we have

$$\begin{aligned} \|\Lambda_\gamma \zeta\|_{\mathcal{Z}}^2 &= \left(\Lambda_\gamma \Lambda_\gamma^{\frac{1}{2}} z, \Lambda_\gamma^{\frac{1}{2}} z\right)_{\mathcal{Z}} \\ &\leq (1 - 2\gamma\tilde{\kappa})\|\Lambda_\gamma^{\frac{1}{2}} z\|_{\mathcal{Z}}^2 \\ &\leq (1 - 2\gamma\tilde{\kappa})\,(\Lambda_\gamma z, z)_{\mathcal{Z}} \\ &\leq (1 - 2\gamma\tilde{\kappa})^2\|z\|_{\mathcal{Z}}^2. \end{aligned}$$

Proposition 5.5 shows the convergence of the sequence defined by (5.53)

$$\begin{aligned} \|\tilde{z}^{k+1}(0)\|_{\mathcal{Z}} &= \|\Lambda_\gamma \tilde{z}^k(0)\|_{\mathcal{Z}} \\ &\leq \alpha\|\tilde{z}^k(0)\|_{\mathcal{Z}} \leq \alpha^n\|\tilde{z}^0(0)\|. \end{aligned}$$

where $\alpha = \|\Lambda_\gamma\|_{\mathcal{L}(\mathcal{Z},\mathcal{Z})} < 1$. Therefore, in the absence of noise, the strategy is then converging exponentially fast.

5.3.2.2 Noise impact on the reconstruction

We now discuss the robustness of the algorithm (5.47) with respect to noisy data. We again denote by $\chi(t) \in \mathcal{Y}$ the difference between the measurements at hand and the measurements that would have been produced by the target trajectory $\check{z}t$

$$\forall t \in [0,T], \quad y(t) = C\breve{z}(t) + \chi(t), \tag{5.59}$$

and we assume that $\chi \in \mathrm{L}^1(0,T;\mathcal{Y})$. We want to assess if our reconstruction algorithm (5.47) still converges, with a controlled error with respect to some norm on χ.

Considering the observation given by (5.59), the error $\tilde{z} = \breve{z} - \hat{z}$ now satisfies

$$\begin{cases} \dot{\tilde{z}} = A\tilde{z} - \gamma C^* C \tilde{z} - \gamma C^* \chi, \quad \text{in} \quad [0,T], \\ \tilde{z}(0) = \tilde{z}_0. \end{cases} \tag{5.60}$$

The term $\gamma C^* \chi$ acts as a source term, and using Duhamel's formula we have

$$\tilde{z}(t) = e^{tA^\gamma}\tilde{z}_0 - \gamma \int_0^t e^{(T-s)A^\gamma} C^* \chi(s)\,\mathrm{d}s. \tag{5.61}$$

Similarly, the error $\tilde{z}_b$ satisfies

$$\begin{cases} \dot{\tilde{z}}_b = A\tilde{z}_b + \gamma C^* C \tilde{z}_b + \gamma C^* \chi \quad \text{in} \quad [0,T], \\ \tilde{z}_b(T) = \tilde{z}(T) \end{cases} \tag{5.62}$$

and we get, thanks to Duhamel's formula (reverse in time):

$$\tilde{z}_b(t) = e^{A^{\gamma *}(T-t)}\tilde{z}_b(T) - \gamma \int_t^T e^{A^{\gamma *}(s-t)} C^* \chi(s)\,\mathrm{d}s. \tag{5.63}$$

Then combining (5.61) and (5.63), we get

$$\tilde{z}_b(0) = \Lambda_\gamma \tilde{z}_0 + \gamma e^{TA^{\gamma *}} \int_0^T e^{(T-s)A^\gamma} C^* \chi(s)\,\mathrm{d}s - \gamma \int_0^T e^{A^{\gamma *} s} C^* \chi(s)\,\mathrm{d}s. \tag{5.64}$$

Recalling that $\Lambda_\gamma = e^{TA^{\gamma *}} e^{tA^\gamma}$ (see (5.52)), we then deduce that the error in Algorithm (5.47) is given by

$$\begin{aligned} \tilde{z}^{k+1}(0) &= \Lambda_\gamma \tilde{z}^k(0) \\ &\quad + \gamma \left(\int_0^T e^{A^{\gamma *} s} C^* \chi(s)\,\mathrm{d}s - e^{TA^{\gamma *}} \int_0^T e^{(T-s)A^\gamma} C^* \chi(s)\,\mathrm{d}s \right). \end{aligned} \tag{5.65}$$

Then a triangular inequality gives

$$\begin{aligned} \left\| \tilde{z}^{k+1}(0) \right\|_{\mathcal{Z}} &\leq \left\| \Lambda_\gamma \, \tilde{z}^k(0) \right\|_{\mathcal{Z}} \\ &\quad + \gamma \left(\int_0^T \left\| e^{A^{\gamma *} s} C^* \chi(s) \right\|_{\mathcal{Z}} \,\mathrm{d}s + \left\| e^{TA^{\gamma *}} \int_0^T e^{A^{\gamma *}(T-s)} C^* \chi(s)\,\mathrm{d}s \right\|_{\mathcal{Z}} \right). \end{aligned}$$

The operators A^γ and $A^{\gamma *}$ generate semigroups of contraction, hence we have

$$\left\| e^{A^{\gamma *} s} C^* \chi(s) \right\|_{\mathcal{Z}} \leq \| C^* \chi(s) \|_{\mathcal{Z}} = \|C\| \, \|\chi(s)\|_{\mathcal{Y}} ,$$

and

$$\left\| e^{TA^{\gamma *}} \int_0^T e^{(T-s)A^\gamma} C^* \chi(s) \, \mathrm{d}s \right\|_{\mathcal{Z}} \leq \|C\| \int_0^T \|\chi(s)\|_{\mathcal{Y}} \; \mathrm{d}s.$$

We know that if the observability inequality is satisfied, Proposition 5.5 gives that $\|\Lambda_\gamma\| = \alpha < 1$. Thus, we deduce that

$$\left\| \tilde{z}^{k+1}(0) \right\|_{\mathcal{Z}} \leq \alpha \parallel \tilde{z}^k(0) \parallel_{\mathcal{Z}} + 2\gamma \|C\| \int_0^T \|\chi(s)\|_{\mathcal{Y}} \; \mathrm{d}s,$$

from which we get by induction

$$\left\| \tilde{z}^{k+1}(0) \right\|_{\mathcal{Z}} \leq \alpha^n \parallel \tilde{z}^0(0) \|_{\mathcal{Z}} + 2\gamma \|C\| \left(\sum_{j=0}^n \alpha^j \right) \int_0^T \|\chi(s)\|_{\mathcal{Y}} \; \mathrm{d}s.$$

Finally, noticing that

$$\sum_{j=0}^n \alpha^j = \frac{1-\alpha^{n+1}}{1-\alpha} \; \leq \; \frac{1}{1-\alpha},$$

we get

$$\left\| \tilde{z}^{k+1}(0) \right\|_{\mathcal{Z}} \leq \alpha^n \left\| \tilde{z}^0(0) \right\|_{\mathcal{Z}} + C^{\mathrm{st}} \frac{\gamma}{1-\alpha} \int_0^T \|\chi(s)\|_{\mathcal{Y}} \; \mathrm{d}s. \tag{5.66}$$

The above inequality proves that the algorithm (5.47) converges to a ball of center the target initial condition and of radius controlled by the noise norm in $L^1(0,T;\mathcal{Y})$. It can be refined if $\|\chi\|_{L^\infty((0,T);\mathcal{Z})} \leq \delta$. In this case we get from (5.65), we get

$$\begin{aligned} \left\| \tilde{z}^{(n+1)}(0) \right\|_{\mathcal{Z}} \leq \alpha \parallel \tilde{z}^{k+1}(0) \parallel_{\mathcal{Z}} + \gamma\delta \|C\| \int_0^T e^{-\lambda s} \, \mathrm{d}s \\ + \gamma\delta \|C\| \, e^{-\lambda T} \int_0^T e^{-\lambda(T-s)} \, \mathrm{d}s, \end{aligned}$$

whence finally yields

$$\left\| \tilde{z}^{k+1}(0) \right\|_{\mathcal{Z}} \leq \alpha^n \left\| \tilde{z}^0(0) \right\|_{\mathcal{Z}} + C^{\mathrm{st}} \frac{\gamma\delta}{\lambda(1-\alpha)}. \tag{5.67}$$

Chapter 6
Data completion problems – The heat equation case

6.1 Introduction

6.1.1 Problem setting

We consider a regular bounded domain $\Omega \subset \mathbb{R}^d$ where we will define solutions of a heat equation with an unknown source term error. We consider a known function $f \in L^2(\Omega)$ but a potentially unknown time dependent $\mu \in L^2(0,T)$

$$\begin{cases} \partial_t u(x,t) - \Delta u(x,t) = f(x)\mu(t), & (x,t) \in \Omega \times (0,T) \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) & x \in \Omega \end{cases} \tag{6.1}$$

The model defined by the dynamics (6.1) enters the framework of (A.1) in Appendix A with $\mathcal{Z} = L^2(\Omega)$,

$$A = \Delta_0, \quad D(A) = H^2(\Omega) \cap H_0^1(\Omega).$$

and the associated semigroup Φ is a contraction. In the following, we will denote $\mathcal{V} = H_0^1(\Omega)$. Moreover, we introduce the model error operator

$$B\,:\,\mathbb{R} \ni \mu \mapsto f(x)\mu \in L^2(\Omega),$$

and its adjoint operator is given by

$$B^*\,:\,L^2(\Omega) \ni f(x) \mapsto \int_\Omega f(x)\psi(x) \in \mathbb{R}.$$

For this system, we consider to have at our disposal, at every time, some measurements y and want to reconstruct, for a target trajectory $\breve{z}$, the initial condition $\breve{\mu}_0 \in \mathcal{Z}$ but also potentially $\breve{\nu} \in L^2(0,T)$. The available noisy measurement are y^δ and they are a perturbation of the unavailable perfect measurement $\breve{y} = C\breve{z}$ such that, for all t,

$$\chi(t) = y^\delta(t) - \breve{y}(t) \Rightarrow \|\chi(t)\|_{\mathcal{Y}} = O(\delta).$$

For the sake of simplicity, we will restrict the presentation to bounded observation operator as in Chapter 5, but more general unbounded operators could also be considered (Bensoussan et al., 2007; Tucsnak & Weiss, 2009). Considering our heat equation example, we typically consider to observe the system over a subdomain ω. Therefore, we have

$$C = \gamma_\omega^0 \,:\, L^2(\Omega) \ni z \mapsto z_{|\omega} \in L^2(\omega).$$

and for the adjoint

$$C^* \,:\, L^2(\omega) \ni \varphi \mapsto \mathbb{1}_\omega(x)\varphi(x) \in L^2(\Omega).$$

6.2 A least square estimation strategy

We want to follow a similar strategy to what we studied in Chapter 2, namely define a least-square criterion to be minimized. We first recall that Δ_0 can also be considered as an isomorphism from $\mathcal{V} = H_0^1(\Omega)$ into $\mathcal{V}' = H^{-1}(\Omega)$, see Section A.5. Then, considering the type of uncertainties, we introduce three parameters $\alpha, \kappa, \gamma > 0$ and propose the following criteria

$$\min_{\substack{\zeta\in\mathcal{V}\\ \mu\in L^2(0,T)}} \Big\{ \mathscr{J}_T(\zeta,\mu) = -\frac{\alpha}{2}\langle \Delta_0\zeta,\zeta\rangle_{\mathcal{V}',\mathcal{V}} + \int_0^T \Big[\frac{\kappa}{2}|\mu(t)|^2 + \frac{\gamma}{2}\left\|y^\delta(t) - Cz_{\zeta,\mu}(t)\right\|_{\mathcal{Y}}^2\Big]\,\mathrm{d}t, \Big\}, \quad (6.2)$$

where we denote by $z_{\zeta,\mu}(s)$, a trajectory of (A.1) for a corresponding initial condition $z(0) = \zeta$. Note again – as in Section 2.5 – that minimizing $\mathscr{J}_T$ must be understood as a minimization under the constraint that $z_{\zeta,\mu}$ follows the dynamics (A.1). We also point out that we slightly depart from Chapter 2, as we here consider $\langle -\Delta_0\zeta,\zeta\rangle_{\mathcal{V}',\mathcal{V}}$ instead of the more classical Tikhonov regularization therm $\|\zeta\|_{\mathcal{Z}}^2$. We here refer to the generalized Tikhonov regularization principle, see for instance , which in our case enforces the initial condition to be in $\mathcal{V} = H_0^1(\Omega)$ for a reconstruction problem which is known to be severely ill-posed.

Theorem 6.1. *The criterion $\mathscr{J}_T$ admits one and only one minimizer*

$$(\bar{\zeta}_T, \bar{\mu}_T) = \operatorname*{argmin}_{\mathcal{V}\times L^2(0,T)} \mathscr{J}_T(\zeta,\mu).$$

given by

$$\bar{\zeta} = P_0 \bar{q}_T(0), \quad \bar{\mu} = \kappa^{-1} B^* \bar{q}_T \tag{6.3}$$

where $P_0 = (-\alpha \Delta_0)^{-1}$ *and* $\bar{q}_T$ *coupled to* $\bar{z}_T = z_{|\bar{\zeta}_T, \bar{\mu}_T}$ *is the mild solution of*

$$\begin{cases} \dot{\bar{z}}_T = A\bar{z}_T + \kappa B B^* \bar{q}_T, & \text{in } (0,T) \\ \dot{\bar{q}}_T + A^* \bar{q}_T = -\gamma C^*(y^\delta - C\bar{z}_T(t)), & \text{in } (0,T) \\ \bar{z}_T(0) = z_0 + \epsilon \bar{q}_T(0) \\ \bar{q}_T(T) = 0 \end{cases} \tag{6.4}$$

Proof. First of all, $z_{\zeta,\mu}$, as a mild solution of (6.1) and from Duhamel formula, is a linear function of (ζ, μ). Therefore, (6.2) defines a quadratic functional in the Hilbert space $\mathcal{Z} \times L^2((0,T);\mathcal{Q})$. Moreover , there exists a constant m and M such that

$$\forall (\zeta, \mu) \in \mathcal{V} \times L^2((0,T);\mathcal{Q}), \quad \mathscr{J}_T(\zeta,\mu) \leq M\big(\|\zeta\|^2_{\mathcal{V}} + \|\mu\|^2_{L^2(0,T)}\big)$$

since $\|\Phi\| \leq 1$, C and B are bounded, and

$$\forall (\zeta, \mu) \in \mathcal{V} \times L^2((0,T);\mathcal{Q}), \quad \mathscr{J}_T(\zeta,\mu) \geq m\big(\|\zeta\|^2_{\mathcal{V}} + \|\mu\|^2_{L^2(0,T)}\big),$$

whence ensures that there exists one and only one minimizer.

The criterion $\mathscr{J}_T$ is differentiable again because Φ is a semigroup C and B are bounded. To compute its derivative, we introduce – as in Section 2.5 – for any mild solution $z_{\zeta,\mu}$ of (6.1), the mild solution of the adjoint dynamics

$$\begin{cases} \dot{q} + A^* q = -\gamma C^*(y^\delta - C z_{\zeta,\mu}) \\ q(T) = 0 \end{cases}$$

and we recall that

$$q(t) = \gamma \int_t^T e^{(s-t)A^*} C^*(y^\delta(s) - Cz(s))\, \mathrm{d}s, \quad t \in [0,T],$$

We then compute

$$\langle \mathrm{D}_\zeta \mathscr{J}_T(\zeta,\mu), (\xi,\eta)\rangle = -\alpha \langle \Delta_0 \zeta, \xi\rangle_{\mathcal{V}',\mathcal{V}} - \int_0^T \gamma (Ce^{tA}\xi, y^\delta(t) - Cz_{\zeta,\mu}(t))_{\mathcal{Y}}\, \mathrm{d}t.$$

We thus find for all $(\xi,\eta) \in \mathcal{V} \times L^2(0,T)$

$$\langle \mathrm{D}_\zeta \mathscr{J}_T(\zeta,\mu), (\zeta,\eta)\rangle = -\alpha \langle \Delta_0 \zeta, \xi\rangle_{\mathcal{V}',\mathcal{V}} - (q(0),\xi)_{\mathcal{Z}} \tag{6.5}$$

Identically, we compute

$$\langle \mathrm{D}_\mu \mathscr{J}_T(\zeta,\mu),(\xi,\eta)\rangle = \int_0^T \kappa\mu(t)\eta(t)\,\mathrm{d}t \\ - \int_0^T \gamma \int_0^t (Ce^{(t-s)A}B\eta(s), y^\delta(t) - Cz_{\zeta,\mu}(s))_{\mathcal{Y}}\,\mathrm{d}s\,\mathrm{d}t.$$

which gives by Fubini

$$\begin{aligned}\langle \mathrm{D}_\mu \mathscr{J}_T(\zeta,\mu),(\xi,\eta)\rangle &= \int_0^T \kappa\mu(t)\eta(t)\,\mathrm{d}t - \int_0^T (B\eta(t), q(t))_{\mathcal{Z}}\,\mathrm{d}t \\ &= \int_0^T (\kappa\mu(t) - B^*q(t))\eta(t)\,\mathrm{d}t.\end{aligned} \tag{6.6}$$

The optimality condition gives (6.3), and the gradient descent strategies

$$\begin{cases}\zeta^{k+1} = \zeta^k - \rho_k P_0 \nabla_\zeta \mathscr{J}_T(\zeta^k,\mu^k), & k \geq 0\\ \mu^{k+1} = \mu^k - \rho_k \nabla_\mu \mathscr{J}_T(\zeta^k,\mu^{k+1}), & k \geq 0\end{cases}$$

from $(\zeta_0,\mu_0) = (0,0)$ can be proved to be convergent for an adequate relaxation parameter chosen as in Theorem 2.8. The gradient descent consists in solving, from $(z_0,q_0) = (0,0)$ and for $k \geq 0$, the weakly coupled system

$$\begin{cases}\dot z_{k+1} = Az_{k+1} + \kappa^{-1}BB^*q_k, & \text{in } (0,T)\\ z_{k+1}(0) = (\mathbb{1} - \rho_k)z_k(0) + \rho_k P_0 q_k(0)\end{cases} \tag{6.7a}$$

and

$$\begin{cases}\dot q_{k+1} + A^*q_{k+1} = -\gamma C^*(y - Cz_{k+1}), & \text{in } (0,T)\\ q_{k+1}(T) = 0\end{cases} \tag{6.7b}$$

and (6.7a), (6.7b) admit mild solutions. Therefore, the two-ends limiting system (6.4) also admits a mild solution, which is also a weak solution. □

6.3 Riccati equation and solution in Hilbert spaces

In the next section, we are going to present a strategy avoiding to solve the two-ends problem (6.4), based on the solution of the operator evolution equation

$$\begin{cases}\dot P = AP + PA^* - \gamma PC^*CP + \kappa^{-1}BB^*, & t > 0\\ P(0) = P_0\end{cases} \tag{6.8}$$

called Riccati equation. In this section, we start by briefly presenting existence results for such non-linear operator evolution equation as presented in (Bensoussan et al., 2007, Part II, Chap 1.).

6.3.1 Mild and weak solutions

We define

$$\mathcal{S}(\mathcal{Z}) = \left\{ L \in \mathcal{L}(\mathcal{Z}) \,\middle|\, Q^* = Q \right\},$$

and

$$\mathcal{S}^+(\mathcal{Z}) = \left\{ L \in \mathcal{S}(\mathcal{Z}) \,\middle|\, (z, Qz)_{\mathcal{Z}} \leq 0, \forall z \in \mathcal{Z} \right\}.$$

where a relation order can be defined by

$$P \leq Q \quad \text{iif} \quad \forall z \in \mathcal{Z},\ (z, Pz)_{\mathcal{Z}} \leq (z, Qz)_{\mathcal{Z}}.$$

First, we are going to introduce the notion of solution to (6.8) in $\mathcal{S}^+(\mathcal{Z})$. As in the semigroup theory seen in Appendix A – there here exist strict solutions – namely C^1 solutions – mild solutions and weak solutions. And as in the semigroup theory, the question is wether P_0, as an operator, belongs to a certain domain. In this respect, let us consider – for any $Q \in \mathcal{S}(\mathcal{Z})$ – the bilinear form

$$b_Q : \left| \begin{aligned} &\mathcal{D}(A^*) \times \mathcal{D}(A^*) \to \mathbb{R} \\ &\qquad (z, v) \mapsto (Qz, A^*v)_{\mathcal{Z}} + (Qv, A^*z)_{\mathcal{Z}} \end{aligned} \right.$$

We then introduce

$$\Upsilon : \left| \begin{aligned} &\mathcal{D}(\Upsilon) \to \mathcal{S}(\mathcal{Z}) \\ &\quad Q \mapsto AQ + QA^*. \end{aligned} \right.$$

with

$$\mathcal{D}(\Upsilon) = \left\{ Q \in \mathcal{S}(\mathcal{Z}) \,\middle|\, b_Q \in \mathcal{L}(\mathcal{Z} \times \mathcal{Z}; \mathbb{R}) \right\}. \tag{6.9}$$

Indeed if $Q \in \mathcal{D}(\Upsilon)$, b_Q has a unique extension in $\mathcal{Z} \times \mathcal{Z}$ and there exists a unique operator $\Upsilon(Q) \in \mathcal{L}(\mathcal{Z})$ such that

$$(\Upsilon(Q)z, v)_{\mathcal{Z}} = b_Q(z, v), \quad (z, v) \in \mathcal{D}(A^*) \times \mathcal{D}(A^*).$$

We can now introduce the various solution definitions.

Definition 6.1 (Riccati solutions).

(i) A *strict solution* of 6.8 in $[0, T]$ is a function $t \mapsto P(t) \in C^1([0, T]; \mathcal{S}(\mathcal{Z})) \cap C^0([0, T]; \mathcal{D}(\Upsilon))$
(ii) A *mild solution* of 6.8 in $[0, T]$ is a function $t \mapsto P(t) \in C^0([0, T]; \mathcal{S}(\mathcal{Z}))$ that verifies

$$P(t)z = e^{tA}P_0e^{tA^*}z + \kappa^{-1}\int_0^t e^{sA}BB^*e^{sA^*}z\,\mathrm{d}s \\ - \gamma\int_0^t e^{(t-s)A}P(s)C^*CP(s)e^{(t-s)A^*}z\,\mathrm{d}s. \tag{6.10}$$

(iii) A *weak solution* of 6.8 in $[0,T]$ is a function $t \mapsto P(t) \in C^0([0,T];\mathcal{S}(\mathcal{Z}))$ such that $P(0) = P_0$ and for any z and v in $\mathcal{D}(A^*)$, $(P(t)z, v)$ is differentiable in $[0,T]$ and verifies

$$\frac{\mathrm{d}}{\mathrm{d}t}(P(t)z,v) = (P(t)z, A^*v)_{\mathcal{Z}} + (P(t)A^*z,v)_{\mathcal{Z}} \\ + \kappa^{-1}(B^*z)(B^*v) - \gamma(CP(t)z, CP(t)v)_{\mathcal{Y}}. \tag{6.11}$$

Proposition 6.1. *Let $P \in C^0([0,T];\mathcal{S}(\mathcal{Z}))$. Then, P is a mild solution of* (6.8) *if and only if P is also a weak solution of* (6.8).

Proof. P is a mild solution of (6.8), hence for all $z, v \in \mathcal{D}(A^*)$,

$$(P(t)z,v) = (P_0e^{tA^*}z, e^{tA^*}v) + \\ \int_0^t \big([\kappa^{-1}BB^* - \gamma P(s)CC^*P(s)]e^{(t-s)A^*}z, e^{(t-s)A^*}v\big)\,\mathrm{d}s, \tag{6.12}$$

therefore $t \mapsto (P(t)z, v)$ is differentiable with respect to t and (6.13) holds.
Conversely, if P is a weak solution of (6.8), then

$$\frac{\mathrm{d}}{\mathrm{d}t}(P(t)e^{(t-s)A^*}z, e^{(t-s)A^*}v) = \kappa^{-1}(B^*e^{(t-s)A^*}z)(B^*e^{(t-s)A^*}v) \\ - \gamma(CP(t)e^{(t-s)A^*}z, CP(t)e^{(t-s)A^*}v)_{\mathcal{Y}}, \tag{6.13}$$

which integrating from 0 to t gives (6.10). We conclude by density of $\mathcal{D}(A^*)$ ◻

Now, if P is a strict solution, we directly infer that P is a weak solution. From Proposition 6.1, we therefore directly prove the following proposition.

Proposition 6.2. *Let $P \in C^1((0,T);\mathcal{S}(\mathcal{Z}))$. If P is a strict solution of* (6.8) *then it is a mild and weak solution of* (6.8).

6.3.2 Local and global existence

We consider the following assumption, satisfied by our formulation (6.1)

Assumption 4 *The operator A is the generator of strongly continuous e^{tA} on $\mathcal{Z}$, $C \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$, $B \in \mathcal{L}(\mathcal{Z}, \mathbb{R})$ and $P_0 \in \mathcal{L}(\mathcal{Z})$.*

We recall that there exists M, $\lambda \in \mathbb{R}$ such that

$$\left\|e^{tA}\right\| \leq Me^{vt},$$

and we denote by A_λ the *Yosida approximation* of A. In the proof of Theorem A.3 (Lummer-Philips) or in Lemma A.1, we have

$$\forall z \in \mathcal{Z}, e^{tA_\lambda}z \xrightarrow{\lambda \to 0} e^{tA}z, \tag{6.14}$$

and we can introduce

$$M_T = \sup_{\substack{t\in[0,T]\\ \lambda\in[0,1]}} \left\|e^{tA_\lambda}\right\|. \tag{6.15}$$

As in Lummer-Philips' theorem, a way to find solution for (6.8), is to first seek solution of

$$\begin{cases} \dot{P}_\lambda = A_\lambda P_\lambda + P_\lambda A_\lambda^* - \gamma P_\lambda C^* C P_\lambda + \kappa^{-1} BB^*, \quad t > 0 \\ P_\lambda(0) = \epsilon \mathbb{1} = P_0 \end{cases} \tag{6.16}$$

where here all the operators are bounded.

Theorem 6.2 (Local existence). *Under Assumption 4, let $T > 0$, $r = 2M_T^2 \|P_0\| = 2M_T^2 \epsilon$ and $\tau \in (0, T)$ such that*

$$\tau(\kappa \|B\| + r^2\gamma^2 \|C\|) \leq \|P_0\|, \text{ and } 2rM_T^2\tau \leq \frac{1}{2}.$$

Then (6.16) and (6.8) have a unique mild solution P and P_λ in the ball

$$\mathcal{B}_{r,\tau} = \left\{P \in C([0,\tau]; \mathcal{S}(\mathcal{Z})), \|P\|_{L^\infty([0,T];\mathcal{S}(\mathcal{Z}))} \leq 1\right\}.$$

Proof. We introduce the mapping F defined by

$$F(P)(t)z = e^{tA}P_0e^{tA^*} + \kappa^{-1}\int_0^t e^{sA}BB^*e^{sA^*}z \,\mathrm{d}s \\ - \gamma\int_0^t e^{(t-s)A}P(s)C^*CP(s)e^{(t-s)A^*}z \,\mathrm{d}s.$$

First we bound

$$\|F(P)(t)z\|_{\mathcal{Z}} \leq M_T^2\left[\|P_0\| + \tau(\|B\| + r^2\|C\|)\right]\|z\| \leq 2M_T^2\|P_0\|\|z\|.$$

It follows that

$$\forall t \in [0, T], \forall P \in \mathcal{B}_{r,\tau} \quad \|F(P)(t)\|_{\mathcal{Z}} \leq r,$$

namely F maps $\mathcal{B}_{r,\tau}$ into $\mathcal{B}_{r,\tau}$.
Then for $P, Q \in \mathcal{B}_{r,\tau}$, we have

$$\begin{aligned} &F(P)(t)z - F(Q)(t)z \\ &\qquad = \gamma \int_0^t e^{(t-s)A} \big[PC^*C(Q-P) + (Q-P)C^*CQ\big] e^{(t-s)A^*} z \, \mathrm{d}s, \end{aligned}$$

which gives

$$\|F(P)(t) z F(Q)(t)\| \leq 2r M_T^2 \tau \|C\|^2 \|P - Q\| \leq \frac{1}{2} \|P - Q\| .$$

Therefore by the contraction mapping theorem, there is a solution P such that $P = F(P)$, ensuring the local existence of mild solution (6.12) in the sense of (6.8). Obviously, the same reasoning ensures the local existence of P_λ a mild solution in the sense of (6.16).

Proposition 6.3. *Under the conditions of application of Theorem 6.2, we have*

$$\forall z \in \mathcal{Z}, \quad \|P_\lambda z - Pz\|_{L^\infty([0,T];\mathcal{Z})} \xrightarrow{\lambda \to 0} 0.$$

Proof. We consider $P^n = F^n(P_0)$, and $P^n_\lambda = F^n_\lambda(P_0)$ with

$$\|P^n - P\|_{L^\infty([0,T];\mathcal{S}(\mathcal{Z}))} \xrightarrow{n \to 0} 0 \text{ and } \|P^n_\lambda - P_\lambda\|_{L^\infty([0,T];\mathcal{S}(\mathcal{Z}))} \xrightarrow{n \to 0} 0$$

Moreover we have with $\rho = \frac{1}{2}$

$$\|P^n - P\|_{L^\infty([0,T];\mathcal{S}(\mathcal{Z}))} = \Big\| \sum_{k \geq n} P^{k+1} - P^k \Big\| \leq \sum_{k \geq n} \rho^k \|F(P_0)\| = \frac{\rho^n}{1-\rho} \|F(P_0)\| ,$$

and identically

$$\|P^n_\lambda - P_\lambda\|_{L^\infty([0,T];\mathcal{S}(\mathcal{Z}))} \leq \frac{\rho^n}{1-\rho} \|F(P_0)\| .$$

Then, we fix $z \in \mathcal{Z}$ and $\varepsilon > 0$. There exists N such that for all $n \geq N$,

$$\|P_\lambda z - P^n_\lambda z\|_{\mathcal{Z}} \leq \|P^n_\lambda - P_\lambda\|_{L^\infty([0,T];\mathcal{S}(\mathcal{Z}))} \|z\|_{\mathcal{Z}} \leq \frac{\varepsilon}{3},$$

and

$$\|Pz - P^n z\|_{\mathcal{Z}} \leq \frac{\varepsilon}{3}.$$

We finally compute

$$\begin{aligned}\|P_\lambda z - Pz\|_{\mathcal{Z}} &\le \|P_\lambda z - P_\lambda^n z\|_{\mathcal{Z}} + \|P_\lambda^n z - P^n z\|_{\mathcal{Z}} + \|P^n z - Pz\|_{\mathcal{Z}} \\ &\le \frac{2\varepsilon}{3} + \|F_\lambda^n(P_0)z - F^n(P_0)z\|_{\mathcal{Z}}\end{aligned}$$

Finally from (6.14), we directly infer that

$$F_\lambda^n(P_0)z \xrightarrow{\lambda\to 0} F^n(P_0)z$$

which concludes the proof. □

One strong interest of relying on P_λ approximation of P is that some properties are easier to obtain on P_λ.

Proposition 6.4. *Under Assumption 4. Let $\lambda \to 0$, we have for all $\lambda \in (0,1)$*

$$0 \le P_\lambda \le M_T^2(\|P_0\| + T\|B\|^2)\mathbb{1}.$$

Proof. Let us introduce $A_\lambda^\gamma = A_\lambda - \frac{\gamma}{2}P_\lambda(t)C^*C$. The operator P_λ is in fact solution of

$$\begin{cases}\dot{P}_\lambda = A_\lambda^\gamma P_\lambda + P_\lambda A_\lambda^{\gamma *} + \kappa^{-1}BB^*, & t > 0 \\ P(0) = P_0\end{cases}$$

The operator A_λ^γ is bounded, hence by Cauchy-Lipchitz for Banach spaces Brezis (1983) (see also Theorem 6.5 in the following), there exists an evolution operator[1] $\phi_\lambda^\gamma(t,s)$ associated with the solutions of

$$\dot{z}(\tau) = A_\lambda^\gamma(\tau)z(\tau), \quad \tau \in [s,t] \tag{6.17}$$

We easily verify that P_λ is in fact solution of

$$P_\lambda = \phi_\lambda^\gamma(t,s)P_0\phi_\lambda^{\gamma *}(t,s) + \kappa^{-1}\int_0^t \phi_\lambda^\gamma(t,s)BB^*\phi_\lambda^{\gamma *}(t,s)\,\mathrm{d}s,$$

which ensures that $P_\lambda \ge 0$. Then, using the mild form (6.12) applied to P_λ with A_λ, we get

$$(z, P_\lambda z) = (P_0 e^{tA_\lambda^*}z, e^{tA_\lambda^*}z) + \int_0^t (B^* e^{sA_\lambda^*}z)^2\,\mathrm{d}s - \int_0^t \left\|CP_\lambda e^{(t-s)A_\lambda^*}z\right\|^2\mathrm{d}s.$$

Finally, we can establish a global existence and uniqueness result.

Theorem 6.3. *Assume that Assumption 4 is satisfied. Then, Problem 6.8 (Problem 6.16 resp.) has a unique mild solution in $P \in C^0([0,T];\mathcal{S}^+(\mathcal{Z}))$ ($P_\lambda \in C^0([0,T];\mathcal{S}^+(\mathcal{Z}))$ resp.). Moreover we have the convergence*

$$\forall z \in \mathcal{Z}, \quad \|P_\lambda z - Pz\|_{L^\infty([0,T];\mathcal{Z})} \xrightarrow{\lambda\to 0} 0.$$

[1] Note that A_λ^γ is a time-dependent operator, hence (6.17) is not autonomous

Proof. Let fix $T > 0$ and $m = M_T^2(\|P_0\| + T\|B\|^2)$. Now, we choose $r > 0$ and $\tau \in (0, T)$ such that

$$r = 2mM_T^2, \quad \tau(\kappa\|B\| + r^2\gamma^2\|C\|) \leq m, \text{ and } 2rM_T^2\tau \leq \frac{1}{2}$$

We have the local existence of P, $P_\lambda \in C([0,\tau]; \mathcal{S}^+(\mathcal{Z}))$ from Theorem 6.2. Moreover from Proposition 6.4, we have

$$0 \leq P_\lambda \leq m\mathbb{1},$$

and the "pointwise" convergence of P_λ to P obtained in Theorem 6.2 gives also

$$0 \leq P \leq m\mathbb{1}.$$

Therefore, we can repeat the local existence and uniqueness argument from $P_n(\tau)$ and $P(\tau)$ to the time 2τ. This concludes the proof. □

To finish this section, we now move to the condition of existence of a strict solution.

Theorem 6.4. *Assume the Assumption 4 is satisfied and that* $P_0 \in \mathcal{D}(\Upsilon)$*, then there exist a unique strict solution of the Riccati dynamics* (6.8)*.*

Proof. The uniqueness has already been proved in Theorem **??**, so we focus on the existence. Let P and P_λ the mild-solution of (6.8) and (6.16), and let us note

$$\Upsilon_\lambda : \mathcal{S}(\mathcal{Z}) \ni Q \mapsto A_\lambda Q + QA^*.$$

We have P_λ is a C^1 solution from Cauchy-Lipchitz

$$\begin{cases} \dot{P}_\lambda = \Upsilon_\lambda(P_\lambda) - \gamma P_\lambda C^*CP_\lambda + \kappa^{-1}BB^* \\ P_n(0) = P_0 \end{cases}$$

and we can further derivate this dynamics to get $Q_\lambda = \dot{P}_n$ a C^1 solution of

$$\begin{cases} \dot{Q}_\lambda = \Upsilon_\lambda(Q_\lambda) - \gamma P_\lambda C^*CQ_\lambda - \gamma Q_\lambda C^*CP_\lambda \\ P_\lambda(0) = \Upsilon_\lambda(P_0) - \gamma P_0C^*CP_0 + \kappa^{-1}BB^* \end{cases}$$

The initial condition $P_\lambda(0) \in \mathcal{S}(\mathcal{Z})$, hence from Theorem 6.3 applied to the dynamics of Q_λ, we have

$$Q_\lambda z \xrightarrow{\lambda\to 0} Qz = e^{tA}Q(0)e^{tA^*} \\ - \gamma\int_0^t e^{(t-s)A}\left[P(s)C^*CQ(s) + Q(s)C^*CP(s)\right]e^{(t-s)A^*} ds.$$

As $Q_\lambda \to Q$ and $P_\lambda \to P$ in $C^0([0,T]; \mathcal{S}(\mathcal{Z}))$, then $P \in C^1([0,T]; \mathcal{S}(\mathcal{Z}))$ and $\dot{P} = Q$. Finally for $t > 0$, $z, v \in \mathcal{D}(A^*)$, we have

$$\begin{aligned} b_{P(t)}(z,v) &= (P(t)z, A^*v)_{\mathcal{Z}} + (A^*z, P(t)v)_{\mathcal{Z}} \\ &= \lim_{\lambda\to 0}(P_\lambda(t)z, A^*v)_{\mathcal{Z}} + (A^*z, P_\lambda(t)v)_{\mathcal{Z}} \\ &= \lim_{\lambda\to 0}(\Upsilon_\lambda z, v)_{\mathcal{Z}} \\ &= \lim_{\lambda\to 0}(Q_\lambda z, v)_{\mathcal{Z}} + \gamma(P(t)C^*CPz, v)_{\mathcal{Z}} - (BB^*z, v)_{\mathcal{Z}} \\ &= (Qz, v)_{\mathcal{Z}} + \gamma(P(t)C^*CPz, v)_{\mathcal{Z}} - \kappa^{-1}(BB^*z, v)_{\mathcal{Z}} \end{aligned}$$

Therefore $P(t) \in \mathcal{D}(\Upsilon)$ and

$$\Upsilon(P(t)) = \dot{P}(t) + \gamma P(t)C^*CP - \kappa^{-1}BB^*.$$

Finally, we conclude by a useful property of strict solution.

Proposition 6.5. *Let $Q \in \mathcal{D}(\Upsilon)$, Then for any $z \in \mathcal{D}(A^*)$, we have $Qz \in \mathcal{D}(A)$ and*

$$\Upsilon z = AQz + QA^*z.$$

Proof. For any $z, v \in \mathcal{D}(A^*)$, we have

$$(Qz, A^*v)_{\mathcal{Z}} = b_Q(z,v) - (A^*z, Qv)_{\mathcal{Z}}.$$

Fixing $z \in \mathcal{D}(A^*)$, the linear application

$$\ell : \left| \begin{array}{r} \mathcal{D}(\mathcal{A}^*) \to \mathbb{R} \\ v \mapsto (Qz, A^*v) \end{array} \right.$$

can be extended in $\mathcal{L}(\mathcal{Z})$. Therefore $Qz \in \mathcal{D}(A)$ (see Lemma A.2), and for all $v \in \mathcal{Z}$

$$(\Upsilon(Q)z, v)_{\mathcal{Z}} = b_Q(z,v) = (AQz, v)_{\mathcal{Z}} + (QA^*z, v)_{\mathcal{Z}}.$$

□

6.4 An equivalent Kalman-based reconstruction strategy

Our objective is here to propose a way to compute $\bar{z}$, trajectory associated with the minimization of the functional 6.2 without solving the two-ends problem (6.4). In a sense, we propose to decouple the system (6.4) using a Riccati dynamics of the form (6.8). This will lead us to the definition of a famous Kalman-Bucy filter, introduced for finite dimensional systems in Kalman & Bucy (1961). Our presentation for partial differential equations echos Bensoussan (1971).

6.4.1 The Kalman estimator

We start by a definition of the Kalman estimator.

Definition 6.2. The Kalman estimator is defined by

$$\forall t \geq 0, \quad \hat{z}(t) = \bar{z}_t(t), \tag{6.18}$$

where $\bar{z}_t$ is the solution of (6.4) for the minimizer of (6.2)

Theorem 6.5. *Let A the generator of semigroup Φ satisfying the estimation $\|\Phi(t)\| \leq Me^{\omega t}$ for two given constants $M, \upsilon \in \mathbb{R}^+$. Let C a bounded operator and P a mild solution of the* (6.8)*. There exists one and only one* mild *solution in $C^0([0,T];\mathcal{Z})$ of the dynamics*

$$\begin{cases} \dot{\hat{z}} = A\hat{z} + \gamma PC^*(y^\delta - C\hat{z}), & \textit{in } (0,T) \\ \hat{z}(0) = 0 \end{cases} \tag{6.19}$$

and this solution belongs to $C^0([0,T];\mathcal{Z})$. Moreover, this solution is the unique weak *solution in $L^2([0,T];\mathcal{Z})$ of the form defined for all $v \in \mathcal{D}(A^*)$:*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}(\hat{z}, v) = (\hat{z}, A^* v) + \gamma(y^\delta - C\hat{z}, CP(t)v), & \textit{in } (0,T) \\ (\hat{z}(0), v) = 0, \quad v \end{cases} \tag{6.20}$$

Proof. Let us denote $\beta : t \mapsto \gamma P(t)C^* y^\delta(t) \in L^2((0,T),\mathcal{Z})$, and $G : t \mapsto \gamma P(t)C^*C \in C^0([0,T];\mathcal{L}(\mathcal{Z}))$. The dynamics (6.19) is a specific case of

$$\begin{cases} \dot{\hat{z}} = A\hat{z} + G(t)\hat{z}(t) + \beta(t), & t \geq 0 \\ \hat{z}(0) = \hat{z}_0 \end{cases} \tag{6.21}$$

Therefore, we extend the notion of a mild solution from Appendix A in an obvious way by here seeking a solution of the form

$$\hat{z}(t) = e^{tA}\hat{z}_0 + \int_0^t e^{(t-s)A}\big[G(s)\hat{z}(s) + \beta(s)\big]\,\mathrm{d}s. \tag{6.22}$$

We then introduce the non-linear mapping

$$F : \left| \begin{array}{l} C^0([0,T];\mathcal{Z}) \to C^0([0,T];\mathcal{Z}) \\ \Big[t \mapsto z(t)\Big] \mapsto \Big[t \mapsto \displaystyle\int_0^t \Phi(t-s)\big[G(s)z(s) + \beta(s)\big]\,\mathrm{d}s\Big] \end{array} \right.$$

We recursively define a sequence $(z^k)_{k\geq 0}$ in $C^0([0,T];\mathcal{Z})$, by $z^{k+1} = F(z^k)$ from $z^0 = 0$.

From Proposition 6.4, there exists K such that $\|P\|_{C^0((0,T);\mathcal{Z})} \leq K$. Therefore for all $z_1, z_2 \in \mathcal{Z}$,

$$\begin{aligned}
&\left\|F^k(z_1) - F^k(z_2)\right\|_{C^0([0,T];\mathcal{Z})} \\
&= \Bigg\| \int_0^{t_k=t} \Phi(t_k - t_{k-1}) G(t_{k-1}) \Bigg[\int_0^{t_{k-1}} \Phi(t_k - t_{k-2}) G(t_{k-1}) \\
&\qquad \cdots \Bigg[\int_0^{t_1} \Phi(t_1 - t_0) G(t_0) \|z_1 - z_2\| \, \mathrm{d}t_0 \Bigg] \cdots \mathrm{d}t_{k-2} \Bigg] \mathrm{d}t_{k-1} \Bigg\|_{C^0([0,T];\mathcal{Z})} \\
&\leq M^k K^k e^{k\upsilon T} \|z_1 - z_2\|_{C^0([0,T];\mathcal{Z})} \int_0^{t_k} \int_0^{t_{k-1}} \cdots \int_0^{t_1} \mathrm{d}t_0 \cdots \mathrm{d}t_{k-2} \, \mathrm{d}t_{k-1} \\
&\leq \frac{(MKTe^{\upsilon T})^k}{k!} \|z_1 - z_2\|_{C^0([0,T];\mathcal{Z})} \xrightarrow{k \to +\infty} 0.
\end{aligned}$$

Therefore, there exist k large enough and $\rho < 1$ such that

$$\left\|F^k(z_1) - F^k(z_2)\right\|_{C^0([0,T];\mathcal{Z})} \leq \rho \|z_1 - z_2\|_{C^0([0,T];\mathcal{Z})},$$

which ensure that $(z^k)_{k \geq 0}$ from the Contraction Mapping Principle. The existence of the weak solution is straightforward from the mild solution following the same arguments as in Theorem A.5. For the uniqueness, it is sufficient to consider $\beta = 0$ and $\hat{z}(0) = 0$, Considering $v \in \mathcal{D}(A^*)$ and $q_T(t) = \Phi^*(T-t)v$ which also belongs to $\mathcal{D}(A^*)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(q_T(t), \hat{z}(t)\big)_{\mathcal{Z}} = \big(q_T(t), G(t)\hat{z}(t)\big)_{\mathcal{Z}}$$

which gives

$$\big(q_T(T), \hat{z}(T)\big)_{\mathcal{Z}} = \big(q_T(0), u(0)\big)_{\mathcal{Z}} + \int_0^T \big(q_T(t), G(t)\hat{z}(t)\big)_{\mathcal{Z}} \, \mathrm{d}t.$$

So for all $v \in \mathcal{D}(A^*)$,

$$\big(v, \hat{z}(T)\big)_{\mathcal{Z}} = \int_0^T \big(v, \Phi(T-t)G(t)\hat{z}(t)\big)_{\mathcal{Z}} \, \mathrm{d}t \Rightarrow \hat{z}(T) = \int_0^T \Phi(T-t)G(t)\hat{z}(t) \, \mathrm{d}t.$$

We conclude by uniqueness of the mild solution. $\square$

Theorem 6.6. *The Kalman observer is the unique solution of* (6.19). *Moreover, we have the fundamental identity*

$$\forall t \in [0,T], \quad \bar{z}_T(t) = \hat{z}(t) + P(t)\bar{q}_T(t). \tag{6.23}$$

Proof. From Theorem 6.1, we have the existence of a weak solution of the two-ends problem (6.4). From Theorem 6.4, we have the existence of a strict solution of the covariance operator $P \in C^1([0,\infty[;\mathcal{S}^+(\mathcal{Z}))$.

Finally from Theorem 6.5, the Kalman estimator exists.

Now let introduce $\eta = \hat{z} - \bar{z}_T + P\bar{q}_T$ and $v \in D(A)$, and compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\eta(t), v)_{\mathcal{Z}} = \big(\hat{z}(t), A^* v\big)_{\mathcal{Z}} + \gamma\big(y^\delta(t) - C\hat{z}(t), CP(t)v\big)_{\mathcal{Y}} \\ - \big(\bar{z}_T(t), A^* v\big)_{\mathcal{Z}} - \kappa^{-1}\big(B^*\bar{q}_T(t)\big)\big(B^* v\big) \\ + \frac{\mathrm{d}}{\mathrm{d}t}\big(P(t)\bar{q}_T(s), v\big)\Big|_{s=t} + \frac{\mathrm{d}}{\mathrm{d}t}\big(\bar{q}_T(t), P(s)v\big)\Big|_{s=t}.\end{aligned} \tag{6.24}$$

Moreover, as P is a weak solution of (6.8), we have from (6.13)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(P(t)\bar{q}_T(s), v)\Big|_{s=t} = (P(t)\bar{q}_T(t), A^* v)_{\mathcal{Z}} + (\bar{q}_T(t), A^* P(t)v)_{\mathcal{Z}} \\ + \kappa^{-1}(B^*\bar{q}_T(t))(B^* v) - \gamma(CP(t)\bar{q}_T(t), CP(t)v)_{\mathcal{Y}},\end{aligned} \tag{6.25}$$

and, as $\bar{q}$ is a weak solution of (6.4),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\bar{q}_T(t), P(s)v)\Big|_{s=t} = -(\bar{q}_T(t), A^* P(t)v)_{\mathcal{Z}} \\ - \gamma\big(y^\delta(t) - C\bar{z}_T(t), CP(t)v\big)_{\mathcal{Y}}.\end{aligned} \tag{6.26}$$

Gathering (6.24), (6.25) and (6.26), we finally obtain

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}t}(\eta(t), v)_{\mathcal{Z}} = (\eta, A^* - \gamma C^* CP(t)v)_{\mathcal{Z}}, \quad t \in [0, T] \\ \eta(0) = 0\end{cases}$$

whence, by Theorem 6.5, $\eta = 0$ in $[0, T]$.

Therefore we have 6.23 whence gives in particular

$$\forall T > 0, \quad \bar{z}_T(T) = \hat{z}(T),$$

which is exactly the definition of the Kalman estimator (6.18). □

Remark 6.1 (The Kalman estimator is an observer). Under additional conditions – called observability and controllability conditions – it is possible to prove that the Kalman estimator converges asymptotically to the target solution in the absence of noise with additional robustness in the presence of noise. In this case, the Kalman estimator is in fact an observer in the sense of Definition 5.1.

6.4.2 Reconstruction of the initial condition

Our initial objective is to estimate the initial condition, hence to compute $\bar{\zeta}_T$ which is not computed by the Kalman estimator $\hat{z}$. However, injecting (6.23) in the adjoint dynamics of (6.4), we get

$$\begin{cases} \dot{\bar{q}}_T + \big(A^* - \gamma C^* C P(t)\big)\bar{q}_T = -\gamma C^* \big(y^\delta - C\hat{z}\big), \quad t \in (0,T) \\ \bar{q}_T(T) = 0 \end{cases} \tag{6.27}$$

with still $\bar{\zeta}_T = P_0 \bar{q}_T(0)$, namely in our case

$$\left| \begin{aligned} -\alpha \Delta \bar{\zeta}_T &= \bar{q}_T(0), \qquad && \text{in } \Omega \\ \bar{\zeta}_T &= 0, && \text{on } \partial\Omega \end{aligned} \right.$$

In other word, we found a backward dynamics which reconstructs the initial condition in only one iteration. Unfortunately, the dynamics (6.27) is not of practical use as it should be computed backward but the Riccati solution P is computed forward in time.

We first introduce the operator $t \mapsto Q(t) \in \mathcal{L}(\mathcal{Z})$ solution of

$$\begin{cases} \dot{Q} = AQ - \gamma P C^* C Q \quad \text{in } (0,T) \\ Q(0) = P_0 \end{cases} \tag{6.28}$$

For all $z \in \mathcal{Z}$, we remark that $r(t) = Q(t)z$ is solution of (6.21) with $r(0) \in \mathcal{D}(A)$. Hence $r \in C^0([0,T];\mathcal{Z}) \cap C^1([0,T];\mathcal{D}(A))$. Therefore $Q \in C^0([0,T];\mathcal{L}(\mathcal{Z}))$ such that

$$\forall z \in \mathcal{Z}, \quad Q(t)z = e^{tA} P_0 z - \gamma \int_0^t e^{(t-s)A} P(s) C^* C Q(s) z \, \mathrm{d}s.$$

This defines a *mild* solution of (6.28). Moreover Q is also a strict solution in the senses that for all $z \in \mathcal{D}(A)$, $Qz \in C^1([0,T];\mathcal{D}(A))$.

Then, we introduce the estimator

$$\begin{cases} \dot{\hat{\zeta}} = \gamma Q C^* (y^\delta - C\hat{z}), \quad \text{in } (0,T) \\ \hat{\zeta}(0) = 0 \end{cases} \tag{6.29}$$

which admits a weak solution in $L^2((0,T);\mathcal{Z})$ as $t \mapsto Q(t)C^*(y^\delta(t) - C\hat{z}(t)) \in L^2((0,T);\mathcal{Z})$. This new operator and dynamics allow to reconstruct the initial condition as stated in the next proposition. In other words, by solving one additional forward Riccati equation and one additional forward dynamics, we will be able to directly estimate $\bar{\zeta}_T$ without any iteration.

Proposition 6.6. *For all $T \geq 0$, we have the following identity*

$$\forall t \in [0,T], \quad \bar{\zeta}_T = \hat{\zeta}(t) + Q^*(t)\bar{q}_T(t). \tag{6.30}$$

Hence $\hat{\zeta}$ is a Kalman estimator of the initial condition in the following sense:

$$\forall t \geq 0, \quad \hat{\zeta}(t) = \bar{\zeta}_t. \tag{6.31}$$

Proof. We denote $\eta : t \mapsto \hat{\zeta}(t) + Q^*(t)\bar{q}_T(t)$, and consider $v \in \mathcal{D}(A)$ giving

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\eta, v)_{\mathcal{Z}} &= \frac{\mathrm{d}}{\mathrm{d}t}\big(\hat{\zeta}(t), v\big)_{\mathcal{Z}} + \frac{\mathrm{d}}{\mathrm{d}t}\big(\bar{q}_T(s), Q(t)v\big)_{\mathcal{Z}}\Big|_{s=t} + \frac{\mathrm{d}}{\mathrm{d}t}\big(\bar{q}_T(t), Q(s)v\big)_{\mathcal{Z}}\Big|_{s=t} \\
&= \big(\gamma(y^\delta - C\hat{z}), CQ(t)v\big)_{\mathcal{Z}} - \big(\bar{q}_T(t), (A - PC^*C)Q(t)v\big)_{\mathcal{Z}} \\
&\qquad - \big(\gamma(y^\delta - C\hat{z}), CQ(t)v\big)_{\mathcal{Z}} + \big(\bar{q}_T(t), (A - PC^*C)Q(t)v\big)_{\mathcal{Z}} \\
&= 0.
\end{aligned}$$

Therefore, $\eta \equiv \eta(0) = P_0\bar{q}_T(t) = \bar{\zeta}_T$, whence justifies (6.30). Finally $q_T(T) = 0$ in (6.30) yields $\hat{\zeta}(T) = \bar{\zeta}_T$. $\square$

Appendix A
Linear evolution equation in Hilbert spaces

In this appendix, we consider a real value Hilbert space $\mathcal{Z}$ and seek solution of linear evolution equations of the form

$$\begin{cases} \dot{z}(t) = Az(t) + \beta(t), & t > 0 \\ z(0) = z_0 \end{cases} \tag{A.1}$$

where A a linear operator on $\mathcal{Z}$, z_0 an initial condition in $\mathcal{Z}$ and β a function from $\mathbb{R}^+$ to $\mathcal{Z}$. To this end, we propose to rely on semigroup theory and briefly present the fundamental results useful in this manuscript. The interested reader could find similar presentations in Brezis (1983), (Joly, 2003, Appendix A), Burq & Gérard (2002) or Raymond (2018) and a more exhaustive presentation about semigroup theory can be found in Pazy (1983).

A.1 Two first examples

A.1.1 The bounded operator case

We denote $\mathcal{L}(\mathcal{Z})$ the space of bounded operators equipped with the norm

$$\forall A \in \mathcal{L}(\mathcal{Z}), \quad \|A\| = \|A\|_{\mathcal{L}(\mathcal{Z})} = \sup_{z\in\mathcal{Z}, z\neq 0} \frac{\|Az\|}{\|z\|}.$$

For a given $A \in \mathcal{L}(\mathcal{Z})$, there exists C^{st} such that

$$\forall z \in \mathcal{Z}, \quad \|Az\| \leq C^{\mathrm{st}} \|z\|.$$

As recalled in (Brezis, 1983, Theorem 7.3), the Cauchy-Lipchitz theorem extends to infinite dimensional spaces, and gives one and only one solution to (A.1). We recall that the proof is based on the Contraction Mapping

Principle applied to

$$F : z \mapsto z_0 + \int_0^\tau Az(t)dt$$

for τ small enough.

Then, we can define the exponential operator $e^{tA} \in \mathcal{L}(\mathcal{Z})$ from the series

$$e^{tA} := \sum_0^{+\infty} \frac{t^n}{n!} A^n,$$

which converges absolutely in $\mathcal{L}(\mathcal{Z})$ since

$$\left\| e^{tA} \right\| \leq e^{t\|A\|}.$$

and satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{tA} = Ae^{tA}.$$

Having defined the exponential, we easily find that the unique solution of (A.1) is given by the *Duhamel* formula

$$z(t) = e^{tA} z_0 + \int_0^T e^{(t-s)A} \beta(s) \, \mathrm{d}s.$$

Moreover, the time-regularity of z is conditioned by the time-regularity of β, with

$$\beta \in \mathcal{C}^k(\mathbb{R}^+, \mathcal{Z}) \Rightarrow z \in \mathcal{C}^{k+1}(\mathbb{R}^+, \mathcal{Z}).$$

A.1.2 A diagonalisable case

Let $-A$ be a self-adjoint positive operator on $\mathcal{Z}$. We further assume that A is diagonalizable with countable positive eigenvalues $\{\mu_n\}_{n \geq 0}$. Let $\{\varphi_n\}_{n \geq 0}$ be the complete orthonormal system of associated eigenfunctions such that

$$-A\varphi_n = \mu_n \varphi_n, \quad \mu_n \geq 0, \quad (\varphi_n, \varphi_m) = \delta_{n,m} = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases}$$

Any $z \in \mathcal{Z}$ is developed in the basis $\{\varphi_n\}_{n \geq 0}$ using a sequence of coefficient $(\alpha_n)_{n \geq 0}$ as

$$z = \sum_{n \geq 0} \alpha_n \varphi_n, \quad \sum_{n \geq 0} |\alpha_n|^2 \leq +\infty.$$

We note that A is not necessarily bounded since, in order to define Az, we need that $z \in D(A)$, the domain of A, defined by

$$\mathcal{D}(A) := \Big\{ z = \sum_{n\geq 0} \alpha_n \varphi_n, \; \Big| \; \sum_{n\geq 0} |\mu_n|^2 |\alpha_n|^2 < +\infty \Big\}.$$

However, even if A is not bounded, we can still define e^{tA} by

$$\forall v = \sum_{n\geq 0} v_n \varphi_n \in \mathcal{Z}, \quad e^{tA} v := \sum_{n\geq 0} v_n e^{-\mu_n t} \varphi_n,$$

which defines en element of $\mathcal{Z}$ as we have indeed

$$\left\| e^{tA} v \right\|^2 = \sum_{n\geq 0} |e^{-2\mu_n t}| |v_n|^2 \leq \sum_{n\geq 0} |v_n|^2 = \|v\|^2 .$$

And again, a solution of (A.1) is given by the *Duhamel* formula with a time-regularity conditioned by the time regularity of β.

A.1.3 Some common properties

Both examples define an operator $\Phi(t) = e^{tA} \in \mathcal{L}(\mathcal{Z})$ which satisfies the so-called semigroup property

$$\forall (t,s) \in \mathbb{R}^+, \quad \Phi(t+s) = \Phi(t)\Phi(s).$$

Moreover for all $t \in \mathbb{R}^+$, $\Phi(t)$ and A commute, and the unique solution of (A.1) is given by the Duhamel formula

$$z(t) = \Phi(t) z_0 + \int_0^T \Phi(t-s)\beta(s)\,\mathrm{d}s.$$

In the sequel, the semigroup theory will allow us to generalize these properties to a large class of unbounded operator A.

A.2 Definition of a semigroup

Definition A.1 (semigroup). The set of operator $(\Phi(t))_{t\geq 0}$ is a C^0-semigroup – also called strongly continuous (one-parameter) semigroup – if

(i) For al $t \in \mathbb{R}^+, \Phi(t) \in \mathcal{L}(\mathcal{Z})$
(ii) $\Phi(0) = \mathbb{1}$
(iii) For all $(t,s) \in \mathbb{R}^+, \quad \Phi(t+s) = \Phi(t)\Phi(s)$
(iv) For all $z \in \mathcal{Z}, \quad \lim_{t\to 0^+} \Phi(t) z = z$

Note that in Definition A.3, the condition (iv) is equivalent to: for all $z \in \mathcal{Z}$, the mapping

$$t \mapsto \Phi(t)z \in C^0(\mathbb{R}^+, \mathcal{Z}).$$

and the continuity has to be understood for z fixed.

Proposition A.1 (A priori estimate). *Let $(\Phi(t))_{t\geq 0}$ be a C^0-semigroup, then there exists $M > 0$ and $\upsilon \geq 0$ such that*

$$\forall t > 0, \quad \|\Phi(t)\| \leq M e^{\upsilon t}. \tag{A.2}$$

Proof. From the continuity property of the semigroup, namely $\Phi(t)z \in C^0(\mathbb{R}^+, \mathcal{Z})$, we have that

$$\forall z \in \mathcal{Z}, \quad \sup_{t\in[0,1]} \|\Phi(t)z\| < +\infty.$$

From the uniform boundedness principle[1] (Brezis, 1983, Theorem 2.2), there exists M such that

$$\sup_{t\in[0,1]} \|\Phi(t)\| \leq M.$$

Then, we have

$$\|\Phi(t)\| = \|\Phi(t - \lfloor t \rfloor)\Phi(\lfloor t \rfloor)\| \leq M M^{\lfloor t \rfloor} \leq M e^{\mu t}$$

with $\mu = \ln M \geq 0$ since $M \geq \|\Phi(0)\| = 1$.

This *a priori* estimate can be further specified in certain cases.

Definition A.2. A semigroup is said to be a contraction if

$$\forall t > 0, \quad \|\Phi(t)\| \leq 1.$$

A semigroup is said to be an exponentially stable if there exist C^{st} and $\lambda > 0$ such that

$$\forall z \in \mathcal{Z}, \quad \|\Phi(t)z\| \leq C^{\mathrm{st}} e^{-\lambda t} \|z\|_{\mathcal{Z}}.$$

Now, we are going to see that a semigroup is necessarily the resolvent of a Cauchy problem of the form $\dot{z}(t) = Az(t)$. We thus need to link a semigroup Φ to a model dynamics operator A.

Definition A.3 (Semigroup generator). Let Φ be a semigroup. The generator of Φ is the operator $A : \mathcal{D}(A) \to \mathcal{Z}$ defined by

$$Az = \lim_{t\to 0^+} \frac{\Phi(t)z - z}{t},$$

with

$$\mathcal{D}(A) = \Big\{ z \in \mathcal{Z} \;\Big|\; \lim_{t\to 0^+} \frac{\Phi(t)z - z}{t} \text{ exists} \Big\}.$$

[1] also called the Banach-Steinhauss theorem

We have the following proposition

Proposition A.2. *Let A be the generator of a semigroup on $\mathcal{Z}$. Then for all $t \geq 0$, $\mathcal{D}(A)$ is stable by $\Phi(t)$ and for all $z \in \mathcal{D}(A)$,*

$$A\Phi(t)z = \Phi(t)Az.$$

Moreover $\mathcal{D}(A)$ is dense in $\mathcal{Z}$.

Proof. For all $z \in \mathcal{D}(A)$ and $t > 0$,

$$\Big[\frac{\Phi(\tau) - \mathbb{1}}{\tau}\Big]\Phi(t)z = \Phi(t)\Big[\frac{\Phi(\tau) - \mathbb{1}}{\tau}\Big]z,$$

and taking the limit $\tau \to 0$ gives $\Phi(t)z \in \mathcal{D}(A)$ together with $A\Phi(t)z = \Phi(t)Az$.

Then, let us define

$$\Upsilon_\epsilon z = \frac{1}{\epsilon}\int_0^\epsilon \Phi(t)z\,\mathrm{d}t = \int_0^1 \Phi(\epsilon s)z\,\mathrm{d}s,$$

whence – from Lebesgue's Dominated Convergence Theorem – converges to z as ϵ tends to 0. Moreover, we easily compute

$$\begin{aligned}\Big[\frac{\Phi(\tau) - \mathbb{1}}{\tau}\Big]\Upsilon_\epsilon z &= \frac{1}{\tau\epsilon}\int_0^\epsilon \Phi(t+\tau) - \Phi(t)z\,\mathrm{d}t \\ &= \frac{1}{\tau\epsilon}\Big[\int_0^{\epsilon+\tau} \Phi((\epsilon+\tau)t)z\,\mathrm{d}t - \int_0^\epsilon \Phi(t)z\,\mathrm{d}t - \int_0^\tau \Phi(t)z\,\mathrm{d}t\Big].\end{aligned}$$

Therefore, we have

$$\Big[\frac{\Phi(\tau) - \mathbb{1}}{\tau}\Big]\Upsilon_\epsilon z = \Upsilon_\tau\Big[\frac{\Phi(\epsilon) - \mathbb{1}}{\epsilon}\Big]z,$$

which gives when taking the limit $\tau \to 0$,

$$A\Upsilon_\epsilon z = \Big[\frac{\Phi(\epsilon) - \mathbb{1}}{\epsilon}\Big]z \Rightarrow \mathcal{D}(A) \ni \Upsilon_\epsilon z \xrightarrow{\epsilon\to 0} z,$$

and ends the proof. □

We then have the following theorem.

Theorem A.1. *Let $A : \mathcal{D}(A) \subset \mathcal{Z} \longrightarrow A$ an unbounded operator on $\mathcal{Z}$ generator of a semigroup Φ. For all initial condition $z_0 \in \mathcal{D}(A)$, there exist a unique solution $t \mapsto z(t) \in \mathcal{C}^1(\mathbb{R}_+, \mathcal{Z})$ of*

$$\begin{cases}\dot{z} = Az, \ \textit{in } \mathbb{R}^+ \\ z(0) = z_0\end{cases}$$

Proof. Let $z(t) = \Phi(t)z_0$. We have for all $h > 0$

$$\frac{z(t+\tau) - z(t)}{\tau} = \frac{\Phi(t+\tau) - \Phi(t)}{\tau} z_0 = \frac{\Phi(\tau) - \mathbb{1}}{\tau} z(t) \xrightarrow{\tau \to 0^+} Az(t).$$

Likewise, we find

$$\frac{z(t-\tau) - z(t)}{-\tau} = \frac{\Phi(t) - \Phi(t-\tau)}{\tau} z_0 = \Phi(t-\tau)\frac{\Phi(\tau) - \mathbb{1}}{\tau} z_0 = \Phi(t-\tau)Az_0 + r_\tau,$$

with

$$\|r_\tau\| = \left\| \Phi(t-\tau)\Big[\frac{\Phi(\tau) - \mathbb{1}}{\tau} - Az_0\Big] \right\| \xrightarrow{\tau \to 0} 0.$$

Therefore, we also have

$$\frac{z(t-\tau) - z(t)}{-\tau} \xrightarrow{\tau \to 0^+} \Phi(t)Az_0 = Az(t).$$

We thus have that $z \in C^1(\mathbb{R}^+, \mathcal{Z})$ and $\dot{z} = Az$.

For the uniqueness, let $z \in C^1(\mathbb{R}^+, \mathcal{Z})$ such that $\dot{z} = Az$. We are going to show that for all $v \in C^1(\mathbb{R}^+, \mathcal{Z})$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t)v(t)\Big|_{t=t_0} = \Phi(t_0)\big[\dot{v}(t_0) + Av(t_0)\big].$$

Indeed, we easily verify for all $\tau \in \mathbb{R}$

$$\begin{aligned}\frac{\Phi(t_0+\tau)v(t_0+\tau) - \Phi(t_0)v(t_0)}{\tau} = \Phi(t_0+\tau)&\frac{v(t_0+\tau) - v(t_0)}{\tau} \\ &+ \frac{\Phi(t_0+\tau) - \Phi(t_0)}{\tau} v(t_0),\end{aligned}$$

hence for $t_1 \geq 0$ and $t \in [0, t_1]$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t)z(t_1 - t) = \big[-\dot{z}(t_1 - t) + Az(t_1 - t)\big],$$

which implies that $z(t_1) = \Phi(t_1)z(0)$. □

We thus have proved that $z(t) = \Phi(t)z_0$ is the expected solution of $\dot{z} = Az$ starting from $z_0 \in \mathcal{D}(A)$. We therefore denote $\Phi(t) = e^{tA}$. More generally, the expression $z(t) = \Phi(t)z_0$ can extend the notion of solution for $z_0 \in \mathcal{Z}$ with the following result.

Theorem A.2. *For all $z_0 \in \mathcal{Z}$, the function $t \mapsto z(t) = \Phi(t)z_0$ satisfies for all $\psi \in C_c^1(\mathbb{R}^+, \mathbb{R})$,*

$$\int_0^\infty \psi z \,\mathrm{d}t \in \mathcal{D}(A) \text{ and } A\Big(\int_0^\infty \psi z \,\mathrm{d}t\Big) + \int_0^\infty \dot{\psi} z \,\mathrm{d}t = 0. \tag{A.3}$$

Conversely if z satisfies (A.3)*, then* $z(t) = \Phi(t)z(0)$.

Proof. Let us assume $z(t) = \Phi(t)z_0$ and consider for all $z \in \mathcal{Z}$,

$$\Upsilon_\epsilon z \coloneqq \frac{1}{\epsilon}\int_0^\epsilon \Phi(t)z\,\mathrm{d}t,$$

which satisfies (see the proof of Proposition A.2)

$$A\Upsilon_\epsilon z = \Big[\frac{\Phi(\epsilon) - \mathbb{1}}{\epsilon}\Big]z,$$

which implies that for all $z \in \mathcal{Z}$, $\Upsilon_\epsilon z \in D(\mathcal{A})$. Let us now define the function $\mathcal{D}(A) \ni z_\epsilon(t) \coloneqq \Upsilon_\epsilon z(t) = \Phi(t)\Upsilon_\epsilon z_0$. We have $\dot{z}_\epsilon = Az_\epsilon$. For all $\psi \in C_c^1(\mathbb{R}^+, \mathbb{R})$, we thus get

$$\int_0^\infty \psi\dot{z}_\epsilon\,\mathrm{d}t = \int_0^\infty Az_\epsilon\psi\,\mathrm{d}t = \int_0^\infty \Big[\frac{\Phi(\epsilon) - \mathbb{1}}{\epsilon}\Big]z\psi\,\mathrm{d}t = \Big[\frac{\Phi(\epsilon) - \mathbb{1}}{\epsilon}\Big]\Big(\int_0^\infty z\psi\,\mathrm{d}t\Big).$$

Taking the limit $\epsilon \to 0$, we finally obtain $\int_0^\infty z\psi\,\mathrm{d}t \in \mathcal{D}(A)$, and

$$A\Big(\int_0^\infty \psi z\,\mathrm{d}t\Big) = -\int_0^\infty \dot{\psi}z\,\mathrm{d}t.$$

Conversely, let $z(t)$ satisfying (A.3) and $z_\epsilon : t \mapsto \Upsilon_\epsilon z(t)$. We have

$$\begin{aligned}\int_0^\infty \psi Az_\epsilon\,\mathrm{d}t &= A\int_0^\infty \psi z_\epsilon\,\mathrm{d}t = A\Upsilon_\epsilon\int_0^\infty \psi z\,\mathrm{d}t\\ &= \Upsilon_\epsilon A\int_0^\infty \psi z\,\mathrm{d}t = -\int_0^\infty \dot{\psi}z_\epsilon\,\mathrm{d}t.\end{aligned}$$

Therefore $w_\epsilon : t \mapsto z_\epsilon(t) - \int_0^t Az\,\mathrm{d}s$ satisfies after integration by part

$$\forall\psi \in C_c^1((0,T);\mathbb{R}), \quad \int_0^\infty \psi w_\epsilon\,\mathrm{d}t = 0. \tag{A.4}$$

Applying (A.4) to

$$\psi = \frac{1}{\delta}\chi\Big(\frac{\tau - t_0}{\delta}\Big) + \frac{1}{\delta}\chi\Big(\frac{\tau - t_1}{\delta}\Big),$$

for $(t_0, t_1) \in (0,T)^2$ and $\chi \in C_c^1(\mathbb{R})$ such that $\int_0^\infty \chi(s)\,\mathrm{d}s = 1$, we get, when taking the limit $\delta \to 0$, $w_\epsilon(t_0) = w_\epsilon(t_1)$. Therefore, w_ϵ is constant in $[0,T]$, meaning that

$$z_\epsilon(t) = z(0) + \int_0^t Az_\epsilon\,\mathrm{d}s.$$

By consequence, $z_\epsilon \in C^1([0,T];\mathcal{Z})$ and $\dot{z}_\epsilon = Az_\epsilon$. From Theorem A.1, we have for all $t \in [0,T]$ that $z_\epsilon(t) = \Phi(t)z_\epsilon(0)$, which gives $z(t) = \Phi(t)z(0)$ when $\epsilon \to 0$. □

A.3 Maximally dissipative operators

We now need to give simple conditions such that $(A, \mathcal{D}(A))$ is the generator of semigroup of contraction. To this end, we introduce the notion of dissipative operator.

Definition A.4. An operator $(A, \mathcal{D}(A))$ is dissipative if

$$\forall z \in \mathcal{D}(A), \quad \forall \lambda > 0,\ \|\lambda z - Az\| \geq \lambda \|z\|. \tag{A.5}$$

In other words, if $(A, \mathcal{D}(A))$ is dissipative, then $\forall \lambda > 0$, $(\lambda \mathbb{1} - A)$ is injective. We have also the following characterization.

Proposition A.3. *The operator $(A, \mathcal{D}(A))$ in the Hilbert space $\mathcal{Z}$ is dissipative if and only if*

$$\forall z \in \mathcal{D}(A), \quad (z, Az) \leq 0. \tag{A.6}$$

Proof. Let assume that (A.6) is satisfied then for all $z \in \mathcal{D}(A)$,

$$\big((\lambda \mathbb{1} - A)z, (\lambda \mathbb{1} - A)z\big) = \lambda^2 \|z\|^2 - 2\lambda(z, Az) + \|Az\|^2 \geq \lambda^2 \|z\|^2 .$$

Conversely if (A.5) is satisfied, then with $\lambda > 0$

$$0 \leq \big((\lambda \mathbb{1} - A)z, (\lambda \mathbb{1} - A)z\big) - \lambda^2 \|z\|^2 = -2\lambda(z, Az) + \|Az\|^2 .$$

giving

$$(z, Az) \leq \frac{\|Az\|^2}{2\lambda} \xrightarrow{\lambda \to +\infty} 0.$$

□

Now, we can define maximal dissipative operators.

Definition A.5. A dissipative operator $(A, \mathcal{D}(A))$ is maximal-dissipative (or m-dissipative) if for all $\lambda > 0$, $(\lambda \mathbb{1} - A)$ is surjective.

Proposition A.4. *If $(A, \mathcal{D}(A))$ is maximal-dissipative in the Hilbert space $\mathcal{Z}$, then A is closed and $\mathcal{D}(A)$ is dense.*

Proof. Let first prove that $\mathcal{D}(A)$ is dense. Let $v \in \overline{\mathcal{D}(A)}^{\perp}$. There exists $z \in \mathcal{D}(A)$ such that $v = z - Az$ because $(\mathbb{1} - A)$ is bijective. Then, we have

$$0 = (v, z) = (Az, z) + \|z\|^2 \geq \|z\|^2 ,$$

giving that $z = 0$, hence $v = 0$.

Let now prove that A is closed. We consider $u_n \to u$ and $Au_n \to v$. As $(\mathbb{1} - A)$ is bijective, there exists $z \in \mathcal{D}(A)$ such that $u - v = z - Az$. We then compute

$$\begin{aligned}\|u_n - z\|^2 &\le \|u_n - z\|^2 - \big(A(u_n - z), u_n - z\big)\\ &= (u_n - Au_n - (u - v), u_n - z)\\ &\le \|u_n - z\| \left(\|Au_n - v\| + \|u_n - u\|\right),\end{aligned}$$

which justifies that $u_n \to z$, hence $u = z$ and $v = Az$. We obtain $Au_n \to Az$. □

From a m-dissipative operator we can introduce two important approximation operators.

Definition A.6. Let $(A, \mathcal{D}(A))$ a m-dissipative operator, the *resolvent* operator from $\mathcal{Z}$ to $D(A)$ is defined by

$$R_\lambda(A) = (\lambda\mathbb{1} - A)^{-1}, \tag{A.7}$$

and the *Yosida approximation of* A is the linear operator defined from $\mathcal{Z}$ to $\mathcal{Z}$ by

$$A_\lambda(A) = \frac{1}{\lambda} A R_{\frac{1}{\lambda}}(A) = A J_\lambda(A), \tag{A.8}$$

with

$$J_\lambda(A) = \frac{1}{\lambda} R_{\frac{1}{\lambda}}(A) = (\mathbb{1} - \lambda A)^{-1}. \tag{A.9}$$

We will prove that J_λ is a *regularizing approximation* of the identity. In fact, we summarize properties of the resolvent and of the Yosida approximation in the next proposition.

Proposition A.5. *We have the following properties:*

(i) $\forall \lambda > 0$, $J_\lambda, R_\lambda \in \mathcal{L}(\mathcal{Z})$ *and* $\|J_\lambda\| \le 1$, $\|R_\lambda\| \le \frac{1}{\lambda}$,
(ii) $\forall \lambda > 0$, $A_\lambda \in \mathcal{L}(\mathcal{Z})$ *and* $\|A_\lambda\| \le \frac{1}{\lambda}$
(iii) $\forall z \in \mathcal{D}(A)$, $A_\lambda z = J_\lambda A z$,
(iv) $\forall z \in \mathcal{D}(A)$, $\|J_\lambda z - z\| \le \lambda \|Az\|$,
(v) $\forall z \in \mathcal{Z}$, $\lim_{\lambda\to 0} J_\lambda z = z$,
(vi) $\forall z \in \mathcal{D}(A)$, $\lim_{\lambda\to 0} A_\lambda z = Az$,
(vii) $\forall z \in \mathcal{Z}$, $(-A_\lambda z, z) \ge 0$.

Proof. (i) We have

$$z \in \mathcal{Z}, \quad J_\lambda z - \lambda A J_\lambda z = z, \tag{A.10}$$

which gives

$$\|J_\lambda z\|^2 \le (z, J_\lambda z) \le \|z\| \, \|J_\lambda z\| \,.$$

which gives directly $\|J_\lambda\| \le 1$, hence $\|R_\lambda\| \le \frac{1}{\lambda}$.

(ii) From (A.10) we also get

$$-\left(AJ_\lambda z, J_\lambda z\right) + \lambda \left\|AJ_\lambda z\right\|^2 = -(AJ_\lambda z, z) \leq \|AJ_\lambda z\| \, \|z\| \tag{A.11}$$

leading to $\|AJ_\lambda\| \leq \frac{1}{\lambda}$.

(iii) We have for all $z \in \mathcal{D}(A)$, the identity

$$(\mathbb{1} - \lambda A)J_\lambda z = J_\lambda(\mathbb{1} - \lambda A)z = z \tag{A.12}$$

which gives $J_\lambda A z = AJ_\lambda z$.

(iv) For all $z \in \mathcal{D}(A)$, we obtain from (A.12)

$$\|J_\lambda z - z\| \leq \lambda \|J_\lambda\| \, \|Az\| \leq \lambda \|Az\| \, .$$

(v) For all $z \in \mathcal{Z}$, let $\mathcal{D}(A) \ni z_n \xrightarrow{n \to +\infty} z$. We have

$$J_\lambda z - z = J_\lambda z - J_\lambda z_n + J_\lambda z_n - z_n + z_n - z.$$

Then for all ϵ, we choose n such that $\|z_n - z\| \leq \epsilon$ and $\lambda \leq \epsilon \|Az_n\|^{-1}$, we obtain

$$\|J_\lambda z - z\| \leq 2 \|z_n - z\| + \lambda \|Az_n\| \leq \epsilon.$$

(vi) We then infer

$$A_\lambda z - Az = J_\lambda A z - Az \xrightarrow{\lambda \to 0} 0.$$

(vii) From (A.11), we get

$$(-A_\lambda z, z) = (-AJ_\lambda z, J_\lambda z) + \lambda \|AJ_\lambda z\|^2 \geq 0.$$

□

To verify that an operator is m-dissipative, it is convenient to seek the surjectivity of $(\mathbb{1} - \lambda_0 A)$ for a particular λ_0 as stated in the next proposition.

Proposition A.6. *The operator $(\mathbb{1} - \lambda A)$ is surjective for all $\lambda > 0$, if and only if there exists $\lambda_0 > 0$ such that $(\mathbb{1} - \lambda_0 A)$ is surjective.*

Proof. Let $\lambda_0 > 0$ such that $(\mathbb{1} - \lambda_0 A)$ is surjective. For $\lambda > 0$ and $f \in \mathcal{Z}$, we want to find z such that

$$z - \lambda A z = f.$$

This is equivalent to solve

$$\frac{\lambda}{\lambda_0}(z - \lambda_0 A z) - \frac{\lambda}{\lambda_0} z + z = f,$$

which rewrites into

$$z = (\mathbb{1} - \lambda_0 A)^{-1}\Big[\frac{\lambda_0}{\lambda} f + \Big(1 - \frac{\lambda_0}{\lambda}\Big) z\Big]. \tag{A.13}$$

The application

$$F : \mathcal{Z} \ni z \mapsto (\mathbb{1} - \lambda_0 A)^{-1}\Big[\frac{\lambda_0}{\lambda} f + \Big(\mathbb{1} - \frac{\lambda_0}{\lambda}\Big) z\Big]$$

is such that for all z_1, z_2

$$\|F(z_1) - F(z_2)\| = \Big|1 - \frac{\lambda_0}{\lambda}\Big| \, \|J_{\lambda_0}(z_1 - z_2)\| \leq \Big|1 - \frac{\lambda_0}{\lambda}\Big|.$$

Hence, F is contracting if $\left|1 - \frac{\lambda_0}{\lambda}\right| < 1$, namely if $\lambda > \frac{\lambda_0}{2}$. In this case, the contraction mapping theorem ensures that there exists a unique solution to (A.13). Therefore $(\mathbb{1} - \lambda A)$ is surjective for all $\lambda > \frac{\lambda_0}{2}$, and recursively for all n, for all $\lambda > \frac{\lambda_0^n}{2^n} \xrightarrow{n \to \infty} 0$. □

Very useful characterizations of m-dissipative operators are given by the following theorem.

Proposition A.7. *In the Hilbert space $\mathcal{Z}$, we have*

- *if $(A, \mathcal{D}(A))$ is dissipative, self-adjoint, with a dense domain*[2] *then it is m-dissipative,*
- *if $(A, \mathcal{D}(A))$ is skew-adjoint, with a dense domain then it is m-dissipative,*

Proof. Note first that when A is skew-adjoint we have

$$(z, Az) = -(z, Az) = 0,$$

hence A and $A^* = -A$ are dissipative.

Moreover in both cases, we have

$$\overline{\operatorname{Ran}(\mathbb{1} - A)}^{\perp} = \operatorname{Ker}(\mathbb{1} - A^*) = \{0\},$$

since in both cases A^* is also dissipative. Finally, a self-adjoint or skew-adjoint operator is closed, therefore $\operatorname{Ran}(\mathbb{1} - A) = \overline{\operatorname{Ran}(\mathbb{1} - A)} = \mathcal{Z}$. □

A fundamental result finally links m-dissipative operators to semigroup generators. The Lummer-Philips' theorem is in fact a variant of a more famous results known as the Hille Yosida theorem and we refer to Pazy (1983) for a more complete presentation.

Theorem A.3 (Lummer-Philips). *Let $A : \mathcal{D}(A) \subset \mathcal{Z} \longrightarrow \mathcal{Z}$ an unbounded operator defined on the Hilbert space $\mathcal{Z}$. We have the following equivalence:*

[2] the domain density is required to define the adjoint. We also recall that a self-adjoint operator is a symmetric operator with the same domain Brezis (1983)

- $(A, \mathcal{D}(A))$ *m-dissipative,*
- $(A, \mathcal{D}(A))$ *is the generator of semigroup of contraction.*

Proof. Let $(A, \mathcal{D}(A))$ be the generator of semigroup of contraction Φ. We have for all $z \in \mathcal{Z}$

$$(\Phi(t)z, z) \leq \|\Phi(t)\| \, \|z\|^2 \leq \|z\|^2 \,,$$

hence

$$(Az, z) = \lim_{t \to 0^+} \frac{(\Phi(t)z, z) - \|z\|^2}{t} \leq 0,$$

which ensures that A is dissipative.

Now let introduce for all $\lambda > 0$, the Laplace transform of $\Phi(t)z$,

$$L_\lambda \, : \, \mathcal{Z} \ni z \mapsto \int_0^\infty e^{-\lambda t} \Phi(t) z \, \mathrm{d}t \in \mathcal{Z},$$

which is well-defined since $\left\| e^{-\lambda t} \Phi(t) z \right\| \leq e^{-\lambda t} \|z\|$ is integrable. Then, we compute for all $z \in \mathcal{Z}$

$$\Big(\frac{\Phi(\epsilon) - \mathbb{1}}{\epsilon} \Big) L_\lambda z = \frac{e^{\lambda \epsilon} - 1}{\epsilon} \int_\epsilon^\infty e^{-\lambda t} \Phi(t) z \, \mathrm{d}t - \frac{1}{\epsilon} \int_0^\epsilon e^{-\lambda t} \Phi(t) z \, \mathrm{d}t.$$

Then by taking the limit $\epsilon \to 0$, we obtain that $L_\lambda z \in \mathcal{D}(A)$ and

$$A L_\lambda z = \lambda L_\lambda z - z.$$

We thus prove that $(\lambda \mathbb{1} - A) L_\lambda z = z$. Identically we find for all $z \in \mathcal{D}(A)$, $L_\lambda (\lambda \mathbb{1} - A) z = z$ ensuring that $(\lambda \mathbb{1} - A)$ is bijective and $R_\lambda(A) = L_\lambda(A)$.

Conversely, let assume that $(A, \mathcal{D}(A))$ is m-dissipative. The Yosida approximation $A_\lambda \in \mathcal{L}(\mathcal{Z})$, hence we can defined its exponential $\Phi_\lambda(t) = e^{t A_\lambda}$. Moreover, A_λ is dissipative, hence $\|\Phi_\lambda(t)\| \leq 1$. Then, we have

$$\begin{aligned}
\Phi_\lambda(t) - \Phi_\mu(t) &= e^{t A_\mu} \left[e^{t(A_\lambda - A_\mu)} - \mathbb{1} \right] \\
&= e^{t A_\mu} \int_0^t e^{s(A_\lambda - A_\mu)} (A_\lambda - A_\mu) \mathrm{d}s \\
&= \int_0^t \Phi_\mu(t-s) \Phi_\lambda(s) (A_\lambda - A_\mu) \mathrm{d}s,
\end{aligned}$$

which gives the estimation

$$\forall z \in \mathcal{Z}, \quad \sup_{t \in [0,T]} \|\Phi_\lambda(t) z - \Phi_\mu(t) z\| \leq T \, \|A_\lambda z - A_\mu z\| \,.$$

The sequence $(A_\lambda z)_{\lambda \geq 0}$ is convergent when $\lambda \to 0$, hence it is Cauchy sequence in $\mathcal{Z}$. Therefore, $(t \to \Phi_\lambda(t) z)_{\lambda \geq 0}$ is a Cauchy sequence in the complete metric space $C^0([0,T]; \mathcal{Z})$, thus a convergent sequence to a limit $t \mapsto \Phi(t) z$.

Then, it is straightforward to prove that this limit has the properties of a semigroup of contraction. Finally, by computing for all $zin\mathcal{D}(A)$,

$$\Big[\frac{\Phi_\lambda(\epsilon)-\mathbb{1}}{\epsilon}\Big]z = \frac{1}{\epsilon}\int_0^\epsilon \Phi_\lambda(t)A_\lambda z\,\mathrm{d}t.$$

and taking the limit $\lambda \to 0$ followed by the limit $\epsilon \to 0$, we justify that A is the generator of Φ. □

Remark A.1. In the proof of Lummer-Philips' theorem, we have seen that the resolvent of a generator A is the Laplace transform of the associated semigroup Φ, namely

$$R_\lambda(A)z = (\lambda\mathbb{1} - A)^{-1} = \int_0^\infty e^{-\lambda t}\Phi(t)z\,\mathrm{d}t. \tag{A.14}$$

Now, it is remarkable that the Yosida approximation allows also to define an approximation of the semigroup, which generalize a classical approximation of the exponential.

Lemma A.1. *Let $(A, \mathcal{D}(A))$ a m-dissipative operator in the Hilbert space $\mathcal{Z}$, and $(\Phi(t))_{t\geq 0}$ the associated semigroup. Then, for all $z \in \mathcal{Z}$,*

$$\Phi(t)z = \lim_{n\to+\infty}\Big(\mathbb{1} - \frac{t}{n}A\Big)^{-n}z = \lim_{n\to+\infty}\big(J_{\frac{t}{n}}(A)\big)^n z.$$

Proof. As proved in the Lummer-Philips theorem, we have

$$\forall z \in \mathcal{Z}, \quad \Phi(t)z = \lim_{n\to+\infty} e^{tA_{\frac{t}{n}}}z,$$

where, from (A.10), $A_{\frac{t}{n}} = \frac{n}{t}(J_{\frac{t}{n}}(A) - \mathbb{1})$ and $\left\|J_{\frac{t}{n}}\right\| \leq 1$. Denoting $L = J_{\frac{t}{n}}(A)$, we compute for all $z \in \mathcal{Z}$

$$\begin{aligned}
\left\|e^{tA_{\frac{t}{n}}}z - \big(J_{\frac{t}{n}}(A)\big)^n z\right\| &= \left\|e^{n(L-\mathbb{1})}z - L^n z\right\| \\
&\leq e^{-n}\sum_{k\geq 0}\frac{n^k}{k!}\left\|L^k z - L^n z\right\| \\
&\leq e^{-n}\sum_{k\geq 0}\frac{n^k}{k!}\sum_{k\leq j\leq n}\left\|L^{j+1}z - L^j z\right\| \\
&\leq e^{-n}\Big(\sum_{k\geq 0}\frac{n^k}{k!}\,|k-n|\Big)\,\|z - Lz\| \\
&\leq e^{-\frac{n}{2}}\Big(\sum_{k\geq 0}\frac{n^k}{k!}\,|k-n|^2\Big)^{\frac{1}{2}}\,\|z - Lz\|,
\end{aligned}$$

where the last inequality is obtained from Cauchy-Schwarz inequality. Then, we proceed to the straightforward manipulation

$$\begin{aligned}\sum_{k\geq 0}\frac{n^k}{k!}\left|k-n\right|^2 &= n^2\sum_{k\geq 0}\frac{n^k}{k!}+\sum_{k\geq 0}\frac{n^k}{k!}k^2-2n\sum_{k\geq 0}\frac{n^k}{k!}k\\ &= n^2\sum_{k\geq 0}\frac{n^k}{k!}+n\sum_{k\geq 1}\frac{n^{k-1}}{(k-1)!}(k-1+1)-2n^2\sum_{k\geq 1}\frac{(n-1)^k}{(k-1)!}\\ &= (n^2+n^2+n-2n^2))e^n = ne^n.\end{aligned}$$

Therefore, we obtain for all $z \in mathcal D(A)$,

$$\left\|e^{tA\frac{n}{t}}z-\left(J_{\frac{t}{n}}(A)\right)^n z\right\| \leq \sqrt{n}\left\|z-J_{\frac{t}{n}}(A)z\right\| \leq \frac{t}{\sqrt{n}}\left\|Az\right\| \xrightarrow{n\to+\infty} 0.$$

Finally, we conclude by density. □

It is worth noticing that in the previous lemma, $J_{\frac{t}{n}}(A)$ corresponds to a discretization of the dynamics using an implicit time scheme. Therefore, we see that this time-scheme is unconditionally stable and very useful for approximating the semigroup as, for instance, in the proof of the next theorem.

Theorem A.4. *Let $(A,\mathcal{D}(A))$ a m-dissipative operator in the Hilbert space $\mathcal{Z}$, generator of the semigroup $(\Phi(t))_{t\geq 0}$. Then, $(A^*,\mathcal{D}(A^*))$ is a m-dissipative operator generator of $(\Phi(t)^*)_{t\geq 0}$.*

Proof. First, if $(A,\mathcal{D}(A))$ is dissipative, then

$$\forall z\in\mathcal{D}(A^*)\cap\mathcal{D}(A),\quad (A^*z,z)=(z,Az)\leq 0,$$

which is still valid for all $z\in\mathcal{D}(A^*)$ by density of $\mathcal{D}(A)$ in $\mathcal{Z}$.

Second, for all $z\in\mathcal{Z}$ and for all $v\in\mathcal{D}(A)$

$$\begin{aligned}(z,v) &= (z,(\lambda\mathbb{1}-A)^{-1}(\lambda\mathbb{1}-A)v)\\ &= (((\lambda\mathbb{1}-A)^{-1})^*z,(\lambda\mathbb{1}-A)v)\\ &= ((\lambda\mathbb{1}-A^*)((\lambda\mathbb{1}-A)^{-1})^*z,v),\end{aligned}$$

which by by density of $\mathcal{D}(A)$ in $\mathcal{Z}$ gives

$$(\lambda\mathbb{1}-A^*)\left((\lambda\mathbb{1}-A)^{-1}\right)^*z=z.$$

In other words, $\lambda\mathbb{1}-A^*$ is bijective and

$$(\lambda\mathbb{1}-A^*)^{-1}=\left((\lambda\mathbb{1}-A)^{-1}\right)^*.$$

Finally, we denote $\bar{\Phi}$ the semigroup associated with $(A^*,\mathcal{D}(A^*))$. We have from Lemma A.1,

$$\bar{\Phi}(t)z = \lim_{n\to+\infty}\Big(\mathbb{1} - \frac{t}{n}A^*\Big)^{-n} z = \lim_{n\to+\infty}\bigg(\Big(\mathbb{1} - \frac{t}{n}A\Big)^{-n} z\bigg)^* = \Phi(t)^* z,$$

which concludes the proof. □

A.4 Existence of solutions for the non-homogeneous case

We can now consider the solutions of (A.1), but before that we only need a technical lemma.

Lemma A.2. *Let $(A, \mathcal{D}(A))$ a closed operator with dense domain in the Hilbert space $\mathcal{Z}$. Suppose that $z \in \mathcal{Z}$ and $w \in \mathcal{Z}$ satisfy*

$$\forall v \in \mathcal{D}(A^*), \quad (A^* v, z) = (v, w),$$

then $z \in \mathcal{D}(A)$ and $w = Az$[3].

Proof. Let consider $z_n \in \mathcal{D}(A)$ a sequence converging to z. We have

$$(v, w) = \lim_{n\to\infty}(A^* v, z_n) = \lim_{n\to\infty}(v, Az_n) = (v, Az),$$

as A is closed.

Theorem A.5. *Let A be the generator of a semigroup in the Hilbert space $\mathcal{Z}$. We have*

(i) For all $z_0 \in \mathcal{D}(A)$ and $\beta \in C^1([0,T];\mathcal{Z})$, there exist a unique strong *solution $z \in C^1((0,T),\mathcal{Z})$ such that $z(0) = z_0$ and*

$$\dot{z} = Az + \beta;$$

(ii) For all $z_0 \in \mathcal{Z}$, and for all $\beta \in L^p((0,T);\mathcal{Z})$, there exists a unique mild-*solution $z \in C^0((0,T);\mathcal{Z})$ such that $z(0) = z_0$ and $\forall \psi \in C^1_c((0,T);\mathbb{R})$, $\int_0^T \psi(t) z(t)\,\mathrm{d}t \in \mathcal{D}(A)$ and*

$$\int_0^T \dot{\psi}(t) z(t)\,\mathrm{d}t + A\Big(\int_0^T \psi(t) z(t)\,\mathrm{d}t\Big) + \int_0^T \beta(t)\psi(t)\,\mathrm{d}t = 0. \tag{A.15}$$

Moreover, z is given by the Duhamel formula

$$z(t) = e^{tA} z_0 + \int_0^t e^{(t-s)A}\beta(s)\,\mathrm{d}s, \tag{A.16}$$

[3] In other words $(A^*)^* = A$.

which hence prove that z is absolutely continuous[4].

(iii) For all $z_0 \in \mathcal{Z}$, and for all $\beta \in L^p((0,T),\mathcal{Z}), p \geq 1$, the mild solution is also a weak *solution, namely $z \in L^p((0,T),\mathcal{Z})$, for all $v \in D(A^*)$, $(v, z(\cdot)) \in W^{1,p}(0,T)$ and*

$$\begin{cases} \dfrac{d}{dt}(v, z(t)) = (A^* v, z(t)) + (\beta, v), \quad t \in (0,T) \ a.e. \\ z(0) = z_0 \end{cases} \tag{A.17}$$

Proof (of (i) and (iii)). (i) By linearity of the equation, we only need to prove that the solution of

$$\begin{cases} \dot{z} = Az + \beta \\ z(0) = 0 \end{cases} \tag{A.18}$$

with $\beta \in C^1((0,T);\mathcal{Z})$ is given by $z(t) = \int_0^t e^{(t-s)A}\beta(s)\,\mathrm{d}s$.

Let $z \in C^1((0,T),\mathcal{Z})$ solution of (A.18), and $w(t) = \Phi(t_1 - t)z(t)$ for $t \in [0, t_1]$.

$$\frac{\mathrm{d}}{\mathrm{d}t} w(t) = \Phi(t_1 - t)\dot{z}(t) - \Phi(t_1 - t)Az(t) = \Phi(t_1 - t)\beta(t),$$

hence

$$z(t_1) = w(t_1) = w(t_1) - w(0) = \int_0^{t_1} \Phi(t_1 - t)\beta(t)\mathrm{d}t.$$

Conversely, let consider

$$z(t) = \int_0^t \Phi(t-s)\beta(s)\,\mathrm{d}s = \int_0^t \Phi(s)\beta(t-s)\,\mathrm{d}s.$$

First, we show that $z \in C^1(\mathbb{R}^+;\mathcal{Z})$. Indeed, when computing

$$\begin{aligned} \frac{z(t+\tau) - z(t)}{\tau} &= \frac{1}{\tau}\int_t^{t+\tau} \Phi(s)\beta(t+\tau-s)\,\mathrm{d}s + \int_0^t \Phi(s)\frac{\beta(t+\tau-s) - \beta(t-s)}{\tau}, \\ &\xrightarrow{\tau\to 0} \Phi(t)\beta(0) + \int_0^t \Phi(s)\dot{\beta}(t-s)\,\mathrm{d}s, \end{aligned}$$

we find $z \in C^1(\mathbb{R}^+;\mathcal{Z})$ and $\dot{z}(t) = \Phi(t)\beta(0) + \int_0^t \Phi(s)\dot{\beta}(t-s)\,\mathrm{d}s$. Second, we show that $z \in C^0((0,T);\mathcal{D}(A))$ and $\dot{z} = Az + \beta$. This time, we write

$$\begin{aligned} \frac{z(t+\tau) - z(t)}{\tau} &= \int_0^t \Big[\frac{\Phi(t+\tau-s) - \Phi(t-s)}{\tau}\Big]\beta(s)\,\mathrm{d}s + \int_t^{t+\tau} \Phi(t+\tau-s)\beta(s)\,\mathrm{d}s. \\ &= \Big[\frac{\Phi(\tau) - \mathbb{1}}{\tau}\Big]z(t) + \int_t^{t+\tau} \Phi(t+\tau-s)\beta(s)\,\mathrm{d}s. \end{aligned}$$

[4] there exists a Lebesgue integral function f such that $z(t) = z(0) + \int_0^t f(s)\,\mathrm{d}s$

and by taking the limit $\tau \to 0$, we get $z \in \mathcal{D}(A)$ and $\dot{z} = Az + \beta$. Using the fact that $z \in C^1(\mathbb{R}^+; \mathcal{Z})$, we therefore infer that $z \in C^0((0,T); \mathcal{D}(A))$.

(iii) Now, let $z \in C^0((0,T); \mathcal{Z})$ solution of (A.16). By injection of L^p into L^∞, $z \in L^p((0,T); \mathcal{Z})$. We compute for all $\psi \in C_c^\infty((0,T))$ and for all $v \in D(A^*)$,

$$\begin{aligned}\Big\langle \frac{\mathrm{d}}{\mathrm{d}t}(z(\cdot), v), \psi \Big\rangle &= -\int_0^T (z(t), v)\dot{\psi}(t)\,\mathrm{d}t \\ &= -\int_0^T (\Phi(t)z_0, v)\dot{\psi}(t)\,\mathrm{d}t - \int_0^T \int_s^T (\Phi(t-s)\beta(s), v)\dot{\psi}(t)\,\mathrm{d}t\mathrm{d}s.\end{aligned}$$

Then, we remark that

$$\forall z_0 \in \mathcal{D}(A), \quad \frac{\mathrm{d}}{\mathrm{d}t}(\Phi(t)z_0, v) = (A\Phi(t)z_0, v) = (\Phi(t)z_0, A^*v),$$

which yields to

$$-\int_0^T (\Phi(t)z_0, v)\dot{\psi}\,\mathrm{d}t = \int_0^T (\Phi(t)z_0, A^*v)\psi\,\mathrm{d}t,$$

and is still valid by density for all $z_0 \in \mathcal{Z}$. Identically, we find

$$-\int_s^T (\Phi(t-s)\beta(s), v)\dot{\psi}\,\mathrm{d}t = (\beta(s), v)\psi(s) + \int_s^T (\Phi(t-s)\beta(s), A^*v)\psi\,\mathrm{d}t.$$

Gathering the last two identities, we finally get

$$\Big\langle \frac{\mathrm{d}}{\mathrm{d}t}(z(\cdot), v), \psi \Big\rangle = \int_0^T \Big[(z(t), A^*v) + (\beta(t), v)\Big]\psi(t)\,\mathrm{d}t,$$

which gives $(z(\cdot), v) \in W^{1,p}(0,T)$ and (A.17).

To complete the proof of (iii), we only need to assert that the solution of (A.17) is unique. In this respect, by linearity we only need to prove that 0 is the unique solution of

$$\begin{cases} \forall v \in D(A^*), & \dfrac{\mathrm{d}}{\mathrm{d}t}(\tilde{z}(t), v) = (\tilde{z}(t), A^*v), \quad \text{in } (0,T) \\ \forall v \in D(A^*), & (\tilde{z}(0), v) = 0 \end{cases}$$

Indeed in that case, for all $v \in D(A^*)$, by integration

$$(\tilde{z}(t), v) = \Big(A^*v, \int_0^t z(s)\,\mathrm{d}s\Big) = (A^*v, \lambda(s)),$$

with $\lambda(s) = \int_0^t z(s)\,\mathrm{d}s$. From Lemma A.2, for all $s \in (0,T)$, we have $\lambda(s) \in \mathcal{D}(A)$ and $z = A\lambda(s)$. Therefore, from Theorem A.1, $\lambda \in C^1((0,T), \mathcal{D}(A))$ is solution of

$$\begin{cases} \dot{\lambda} = A\lambda, & \text{in } (0,T) \\ \lambda(0) = 0 \end{cases}$$

hence $\lambda = 0$ which concludes the proof □

.

Remark A.2. Theorem A.5-(i) – namely the existence of strict solution – can be extended to the case where $\beta \in W^{1,1}_{\mathrm{loc}}([0,T];\mathcal{Z})$ or $\beta \in L^1_{\mathrm{loc}}([0,T];\mathcal{D}(A)) \cap C^0([0,T];\mathcal{Z})$.

A.5 The heat equation case

Theorem A.6. *For all $f \in H^{-1}(\Omega)$*[5]*, and for all $\lambda \geq 0$, there exists a unique $u \in H^1_0(\Omega)$ solution of*

$$-\Delta u + \lambda u = f, \qquad \text{in } \mathcal{D}'(\Omega) \tag{A.19}$$

Proof. The problem (A.19) is equivalent to the variational formulation

$$\forall v \in H^1_0(\Omega), \quad (\nabla u, \nabla v)_{L^2(\Omega)} + \lambda (u,v)_{L^2(\Omega)} = \langle f, v\rangle_{H^{-1}(\Omega),H^1_0(\Omega)},$$

which admits a unique solution from Lax-Milgram theorem. □

Theorem A.7. *Let $\mathcal{Z}$ a Hilbert space and T a self-adjoint compact operator. Their exists a unique real sequence $(\lambda_i)_{i\in I}$ of eigenvalues of T with I at most countable, such that, if I is infinite, $(\lambda_i)_{i\in I}$ tends to 0. Moreover, the associated eigenvectors $(e_i)_{i\in I}$ form an orthogonal sequence of $\mathcal{Z}$.*

We now consider the operator $T : L^2(\Omega) \to L^2(\Omega)$ such that $Tu \in H^1_0(\Omega)$ and

$$\forall v \in H^1_0(\Omega), \quad (\nabla Tu, \nabla v)_{L^2(\Omega)} = (u,v)_{L^2(\Omega)}.$$

In other words, T is the inverse of the Laplace operator with Dirichlet boundary conditions $-\Delta_0$. We have

$$\|\nabla Tu\|^2_{L^2(\Omega)} = (\nabla Tu, \nabla Tu)_{L^2(\Omega)} = (u, Tu)_{L^2(\Omega)} \leq \|u\|_{L^2(\Omega)} \|Tu\|_{L^2(\Omega)},$$

hence by Poincaré inequality

$$\|\nabla Tu\|_{L^2(\Omega)} \leq C^{\mathrm{st}} \|u\|_{L^2(\Omega)}.$$

Therefore, T is bounded from $L^2(\Omega)$ to $H^1_0(\Omega)$. It is then bounded compact from $L^2(\Omega)$ to $L^2(\Omega)$. Moreover T is clearly injective, and self-adjoint since

[5] equipped with the norm $\|f\|_{H^{-1}(\Omega)} = \sup_{v\in H^1_0(\Omega)} \left|\langle f, v\rangle_{H^{-1}(\Omega),H^1_0(\Omega)}\right|$.

$$(Tu, v)_{L^2(\Omega)} = (\nabla Tu, \nabla Tv)_{L^2(\Omega)} = (u, \nabla Tv)_{L^2(\Omega)}.$$

There exists $(e_n)_{n\geq 0}$ an orthonormal basis of $L^2(\Omega)$ associated with eigenvalues λ_n of T, strictly positive and converging to 0. Then $\mu_n = \frac{1}{\lambda_n}$ are the eigenvalues of $-\Delta_0$.

We conclude that as $-\Delta_0$ is skew adjoint, it is the generator of a semigroup. By the way, we are also in the diagonalizable case.

A.6 The wave equation case

Considering the wave equation, The semigroup theory furnish the following existence result.

Theorem A.8. *Let Ω be a bounded domain, $\gamma \in L^\infty(\Omega)$, and $T > 0$. For all $(u_0, v_0) \in H_0^1(\Omega) \times L^2(\Omega)$, there exist a unique function*

$$u \in C^1((0,T); L^2(\Omega)) \times C^0((0,T); H_0^1(\Omega))$$

solution of the dynamics

$$\begin{cases} \partial_t^2 u(x,t) + \gamma(x)u(x,t) - \Delta u(x,t) = 0, & (x,t) \in \Omega \times (0,\infty) \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x), \quad \dot{u}(x,0) = v_0(x), & x \in \Omega \end{cases} \tag{A.20}$$

moreover the energy defined by

$$\mathcal{E}(t) = \frac{1}{2}\int_\Omega \left[|\partial_t u|^2 + |\nabla u|^2\right] \mathrm{d}x$$

satisfies

$$\forall t \geq 0, \quad \mathcal{E}(t) \leq \mathcal{E}(0).$$

Proof. We define $\mathcal{Z} = H_0^1(\Omega) \times L^2(\Omega)$, equipped with the norm

$$\forall z = (u,v) \in \mathcal{Z}, \quad \|z\|_{\mathcal{Z}}^2 = \|\nabla u\|_{L^2(\Omega)}^2 + \|v\|_{L^2(\Omega)}^2$$

We introduce the operator $A_0 = -\Delta_0$ as the Laplace operator with Dirichlet conditions. Namely, we introduce on $H_0^1(\Omega) \times H_0^1(\Omega)$, the bilinear form

$$a(u_1, u_2) = \int_\Omega \nabla v_1 \cdot \nabla v_2 \,\mathrm{d}x$$

and define

$$\begin{cases} D(A_0) = \left\{u \in H_0^1(\Omega) \,\middle|\, \exists f \in L^2(\Omega) \,:\, a(u,v) = (f,v)_{L^2(\Omega)}, \forall v \in H_0^1(\Omega)\right\} \\ A_0 u = f, \quad \forall u \in D(A_0) \end{cases}$$

Again, we have $D(A_0) = H^1(\Omega, \Delta)$ which corresponds to $H^2(\Omega)$ for C^2 domain. We then introduce

$$A = \begin{pmatrix} 0 & \mathbb{1} \\ -A_0 & -\gamma \mathbb{1} \end{pmatrix} \text{ with } D(A) = D(A_0) \times H_0^1(\Omega).$$

We have

$$\begin{aligned} \left(A \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right)_{\mathcal{Z}} &= (\nabla v, \nabla u)_{L^2(\Omega)} - (A_0 u + \gamma v, v)_{L^2(\Omega)} \\ &= (\nabla v, \nabla u)_{L^2(\Omega)} - (\nabla u, \nabla v)_{L^2(\Omega)} - (\gamma v, v)_{L^2(\Omega)} \end{aligned}$$

Therefore A is dissipative. Moreover for $(f, g) \in \mathcal{Z}$ let us seek $(u, v) \in \mathcal{D}(A)$ such that

$$\lambda \begin{pmatrix} u \\ v \end{pmatrix} - A \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \Leftrightarrow \begin{cases} \lambda u - v = f \\ (\gamma + \lambda) v + A_0 u = g \end{cases}$$

By combining the two equations we have to find $u \in H_0^1(\Omega)$ such that

$$\begin{aligned} \forall w \in H_0^1(\Omega), \quad & a(u, w) + (\lambda \gamma u, w)_{L^2(\Omega)} \\ & \qquad + \lambda^2 (u, w)_{L^2(\Omega)} = (g + (\lambda + \gamma) f, w)_{L^2(\Omega)} \end{aligned}$$

Therefore, by Lax-Milgram theorem there exists a unique solution of this problem in $H_0^1(\Omega)$ and $u \in D(A_0)$ as $g + (\lambda + \gamma) f \in L^2(\Omega)$. Therefore $\lambda \mathbb{1} - A$ is surjective, hence A defines a semigroup. Finally, we have the estimation

$$\forall t > 0, \quad \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{2} \|z\|_{\mathcal{Z}}^2 \right) = (z, Az) = -\gamma(v, v),$$

which gives $\mathcal{E}(t) \leq \mathcal{E}(0)$ for all $t > 0$. $\square$

References

Andrieux, S., Baranger, T. N., & Ben Abda, A. (2006). Solving Cauchy problems by minimizing an energy-like functional. *Inverse Problems*, *22*(1), 115–133.
URL http://dx.doi.org/10.1088/0266-5611/22/1/007

Azaïez, Mejdi, Ben Belgacem, Faker, & El Fekih, Henda (2006). On Cauchy's problem. II. Completion, regularization and approximation. *Inverse Problems*, *22*(4), 1307–1336.
URL http://dx.doi.org/10.1088/0266-5611/22/4/012

Bardos, Claude, Lebeau, Gilles, & Rauch, J (1988). Un exemple d'utilisation des notions de propagation pour le contrôle et la stabilisation des problèmes hyperboliques. *Rendiconti del Seminario Matematico del Universita Politecnico Torino*, *Fascicolo speciale*(Hyperbolic Equations (1987)), 12–31.

Ben Belgacem, F., & El Fekih, H. (2005). On cauchy's problem: I. a variational steklov-poincaré theory. *Inverse problems*, *21*, 1915–1936.

Bensoussan, A. (1971). *Filtrage optimal des systèmes linéaires*. Dunod.

Bensoussan, A., Delfour, M. C., Da Prato, G., & Mitter, S. K. (2007). *Representation and Control of Infinite Dimensional Systems*. Birkhauser Verlag, second edition ed.

Brezis, H. (1983). *Analyse fonctionnelle, Théorie et applications*. Masson.

Burq, N., & Gérard, P. (2002). Contrôle optimal des equations aux derivees partielles. Cours de l'Ecole Polytechnique.
URL https://www.math.u-psud.fr/~nb/articles/coursX.pdf

Burq, Nicolas, & Gérard, Patrick (2017). Stabilisation of wave equations on the torus with rough dampings. Working paper or preprint.
URL https://hal.archives-ouvertes.fr/hal-01672371

Ekeland, Ivar, & Temam, Roger (1974). *Analyse convexe et problèmes variationnels*. Dunod. Collection Études Mathématiques.

Ern, Alexandre, & Guermond, Jean-Luc (2004). *Theory and practice of finite elements*, vol. 159 of *Applied Mathematical Sciences*. New York: Springer-Verlag.

Gagnon, Ludovick (2017). Sufficient Conditions for the Controllability of Wave Equations with a Transmission Condition at the Interface. *arXiv.org*.

Grisvard, P. (1985). *Elliptic problems in nonsmooth domains*, vol. 24 of *Monographs and Studies in Mathematics*. Boston, MA: Pitman (Advanced Publishing Program).

Hormander, L. (1976). *Linear Partial Differential Operators, Fourth Printing*. Springer-Verlag.

Imperiale, Sebastien, Moireau, Philippe, & Tonnoir, A (2018). Analysis of an observer strategy for initial state reconstruction of wave-like systems in unbounded domains. Working paper or preprint.
URL https://hal.inria.fr/hal-01898440

Jerison, David S., & Kenig, Carlos E. (1981). The Neumann problem on Lipschitz domains. *Bull. Amer. Math. Soc. (N.S.)*, *4*(2), 203–207.

Joly, P. (2003). Analyse et approximation de modèles de propagation d'ondes. Cours de l'Ecole Polytechnique.
URL http://perso.ensta-paristech.fr/~ciarlet/C7-1/Ondes-Joly01.pdf

Joly, R (2006). Convergence of the wave equation damped on the interior to the one damped on the boundary. *J. Differential Equations*, *229*(2), 588–653.

Kalman, Rudolph Emil, & Bucy, R S (1961). New Results in Linear Filtering and Prediction Theory. *J. Basic Eng*, *83*(1), 95–108.

Kirsch, Andreas (1996). *An introduction to the mathematical theory of inverse problems*, vol. 120 of *Applied Mathematical Sciences*. New York: Springer-Verlag.

Krener, Arthur J (1998). A Lyapunov theory of nonlinear observers. In G. G. Yin, & Q. Zhang (Eds.) *Stochastic analysis, control, optimization and applications*, (pp. 409–420). Springer.

Lattès, R., & Lions, J.-L. (1967). *Méthode de quasi-réversibilité et applications*. Travaux et Recherches Mathématiques, No. 15. Paris: Dunod.

Lions, J.-L. (1988). *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome 1*, vol. 8 of *Recherches en Mathématiques Appliquées [Research in Applied Mathematics]*. Masson, Paris. Contrôlabilité exacte. [Exact controllability], With appendices by E. Zuazua, C. Bardos, G. Lebeau and J. Rauch.

Lions, J.-L., & Magenes, E. (1968). *Problèmes aux limites non homogènes et applications. Vol. 1*. Travaux et Recherches Mathématiques, No. 17. Paris: Dunod.

Miller, L, & 2002 (2002). Escape function conditions for the observation, control, and stabilization of the wave equation. *SIAM J of Control Optim*, *41*(5), 1154–1566.

Oudot, Steve Y., Guibas, Leonidas J., Gao, Jie, & Wang, Yue (2010). Geodesic Delaunay triangulations in bounded planar domains. *ACM Trans. Algorithms*, *6*(4), Art. 67, 47.
URL http://dx.doi.org/10.1145/1824777.1824787

Pazy, A. (1983). *Semigroups of Linear Operators and Applications to Partial Differential Equations*, vol. 44 of *Applied Mathematical Sciences*. New York: Springer-Verlag.

Ramdani, Karim, Tucsnak, M, & Weiss, G. (2010). Recovering the initial state of an infinite-dimensional system using observers. *Automatica*, *46*(10), 1616–1625.

Raymond, J. P. (2018). équations d'évolution. Master Lecture.
URL https://www.math.univ-toulouse.fr/~raymond/book-evolution.pdf

Robbiano, L. (1991). Théorème d'unicité adapté au contrôle des solutions des problèmes hyperboliques. *Communication in Partial Differential Equations*, *16*, 789–800.

Tucsnak, M., & Weiss, G. (2009). *Observation and control for operator semigroups*. Birkhäuser Advanced Texts: Basler Lehrbücher. Basel: Birkhäuser Verlag.
URL http://dx.doi.org/10.1007/978-3-7643-8994-9