Abel inversion using fast Fourier transforms

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A fast Fourier transform based Abel inversion technique is proposed. The method is faster than previously used techniques, potentially very accurate (even for a relatively small number of points), and capable of handling large data sets. The technique is discussed in the context of its use with 2-D digital interferogram analysis algorithms. Several examples are given.

I. Introduction

In many areas of physics and engineering it is necessary to determine the radial distribution of some spherically or cylindrically symmetric quantity. Typically, one is forced to deduce this distribution from measurements of the projection of these quantities onto a plane, in which case the radial distribution is linked to the values of these projections via the Abel transform. Thus, there have been a plethora of techniques developed to perform the inverse Abel transform.[1–10]

Recently, in our laboratory, we have been studying the properties of self-generated magnetic fields in laser-induced plasmas and to aid in this study we have developed a Fourier transform based interferogram analysis technique which is able to simultaneously provide information about both the phase and amplitude of the fringes and thereby give accurate measurements of the Faraday rotation and phase shift of a probe beam.[11] This technique provides large quantities of information which need to be analyzed using an Abel inversion algorithm. Thus, we were led to seek an Abel inversion technique that is fast, so as to be able to handle the large quantities of data that are available, and preferably based on Fourier transforms so as to be easily combined with the interferogram analysis algorithm. The considerable recent interest in the development of Fourier transform based interferogram analysis techniques[11–17] suggests that the development of such an Abel inversion technique will be of current importance in a number of other studies.

In this paper, then, we describe a Fourier transform based Abel inversion technique which, although related to a technique recently proposed by Tatekura,[10] offers substantial advantages in speed over previously proposed techniques. The relatively small number of computations involved also provides for greater numerical precision and the algorithm works particularly well in the case of the approximately Gaussian profiles encountered in laser-produced plasmas.

II. Fourier Expansion Method

The Abel transform may be written as

$$S(y) = 2\int_y^R f(r)(r^2 - y^2)^{-1/2} r\,dr, \tag{1}$$

where $S(y)$ represents the observed data and $f(r)$ is the function to be determined. The inversion formula is well known and is given by

$$f(r) = -\frac{1}{\pi}\int_r^R \frac{dS(y)}{dy}(y^2 - r^2)^{-1/2} dy. \tag{2}$$

Inversion techniques may be based on either Eq. (1) or Eq. (2) or by expanding either $S(y)$ or $f(r)$ into a sum of orthogonal basis functions. The program of the work described in this paper is to perform a Fourier decomposition of the data $S(y)$ and calculate the Abel inversion of each spatial frequency component. In this way we obtain a direct connection between $S(y)$ and its Abel inversion via the Fourier coefficients.

To analyze this approach, assume $S(y)$ is continuous and symmetric around $y = 0$ and has zero values at the boundaries and outside the interval $\langle -R,R\rangle$. Under these assumptions, $S(y)$ may be written as a cosine expansion:

$$S(y) = a_0 + \sum_{k=1}^{+\infty} a_k \cos\frac{k\pi y}{R}, \tag{3}$$

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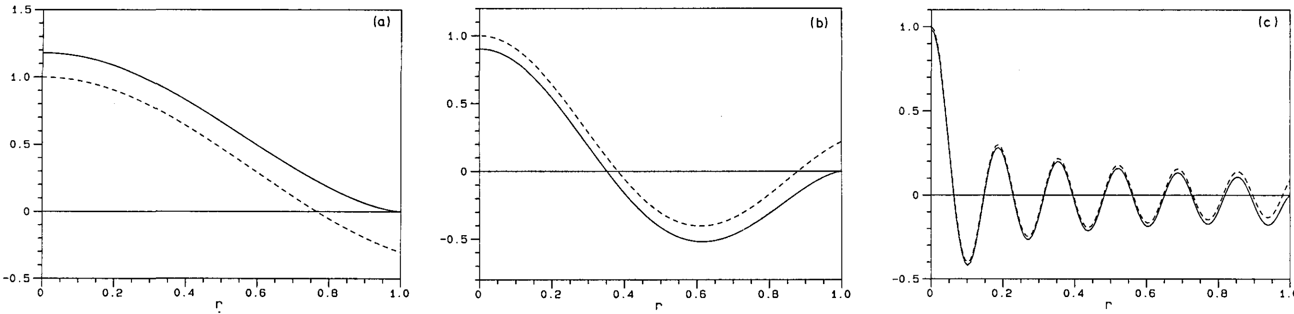

Fig. 1. Comparison of graphs of functions $g_k(r)$ (solid lines) and $J_0(k\pi r)$ (dashed lines): (a) $k = 1$; (b) $k = 2$; (c) $k = 12$.

where a_k are, obviously, the appropriate Fourier coefficients. On substituting Eq. (3) into the Abel inversion formula, Eq. (2), we obtain

$$f(r) = \sum_{k=1}^{+\infty} \frac{ka_k}{R} \int_r^R (y^2 - r^2)^{-1/2} \sin \frac{k\pi y}{R} dy. \tag{4}$$

If we make the transformation of variables

$$t = \frac{(y^2 - r^2)^{1/2}}{R}, \tag{5}$$

Eq. (4) may be written as

$$f(r) = \frac{\pi}{2R} \sum_{k=1}^{+\infty} ka_k g_k \left(\frac{r}{R}\right), \tag{6}$$

where

$$g_k(\rho) = \frac{2}{\pi} \int_0^{(1-\rho^2)^{1/2}} (t^2 + \rho^2)^{-1/2} \sin k\pi (t^2 + \rho^2)^{1/2} dt. \tag{7}$$

which is defined in the interval $\langle -1,1 \rangle$.

Thus, it is clear that the Abel inversion of $S(y)$ may be written directly as an expansion of the functions $g_k(\rho)$ with expansion coefficients which are simply proportional to the Fourier coefficients of $S(y)$. This contrasts with Ref. 10 where each Fourier coefficient of $f(r)$ is a summation over the Fourier coefficients of $S(y)$; thus a double summation is required to calculate each value of $f(r)$. In our case, one of these summations is effectively incorporated into the function $g_k(\rho)$ which is defined so as to be valid for all data.

In the limit of data with infinite support, the data must be expressed as a Fourier integral and the corresponding inversion is given by

$$f(r) = \frac{1}{2} \int_0^\infty a(\omega) J_0(\omega r) \omega d\omega, \tag{8}$$

where

$$a(\omega) = \frac{2}{\pi} \int_0^\infty \cos(\omega y) S(y) dy \tag{9}$$

is the Fourier transform of the data $S(y)$ and $J_0(\omega r)$ is a zero-order Bessel function of the first kind. In this sense, then, the $g_k(\rho)$ tend to the functions $J_0(k\pi\rho)$ when the data become very large. Both the function $g_k(\rho)$ and $J_0(k\pi\rho)$ have been plotted in Fig. 1 for $k = 1$, 2, and 12 and their similarity is evident, particularly for the large k.

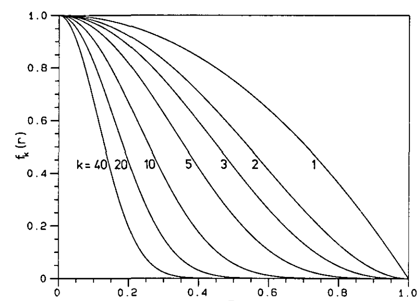

Fig. 2. Functions $f_k(r)$ for a range of k. As k becomes larger these functions closely approximate a Gaussian.

The following properties may be readily confirmed for $g_k(\rho)$:

$$g_k(\pm 1) = 0, \tag{10}$$

$$\left.\frac{dg_k}{d\rho}\right|_{\rho=0} = 0, \tag{11}$$

$$\left.\frac{dg_k}{d\rho}\right|_{\rho=\pm 1} = 0, \tag{12}$$

$$\lim_{k\to+\infty} g_k(0) = 0. \tag{13}$$

Note also that Eq. (6) implies that the condition

$$\lim_{k\to+\infty} ka_k = 0 \tag{14}$$

must obtain for the inversion technique to be useful.

III. Implementation and Tests

The advantage of the approach outlined here is that the Abel inversion may be calculated directly from the Fourier coefficients using the basis functions $g_k(\rho)$. It is important, therefore, to be able to precalculate these basis functions. This is indeed possible since their values are only required at discrete intervals between zero and one so a given matrix of values will serve for any array of data of a given size.

The algorithm was tested using the functions

$$f_k(r) = (1 - r^2)^k, \quad |r| \leq 1, \tag{15}$$

for k ranging from 1 to 40. Some of these functions are drawn in Fig. 2. The Abel transform of these functions was calculated for forty points in the interval

Table I. Example of Standard Deviations (SD) for the Reconstruction of Functions $f_k(r) = (1 - r^2)^k$ for Different k Using the Abel Inversion Technique Described in This Paper

k	SD	k	SD
1	0.315E-02	11	0.305E-08
2	0.158E-03	12	0.302E-08
3	0.169E-04	13	0.298E-08
4	0.147E-05	14	0.293E-08
5	0.358E-06	15	0.289E-08
6	0.295E-07	20	0.272E-08
7	0.157E-07	25	0.260E-08
8	0.351E-08	30	0.250E-08
9	0.327E-08	35	0.243E-08
10	0.317E-08	40	0.236E-08

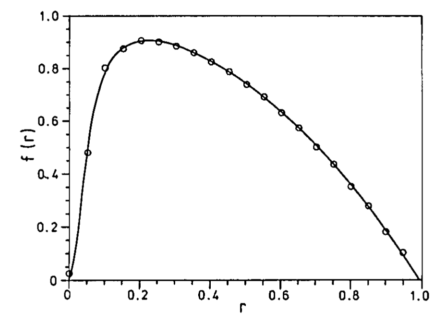

Fig. 3. Test function with a deep central dip (solid line) and the inversion of the Abel transform of this curve (open circles).

$\langle -1,1 \rangle$ and the inversion technique was then applied and the standard deviation of the inversion from the original function was calculated. The results are summarized in Table I. Note that the standard deviation is substantially smaller for large k, corresponding to profiles similar to the Gaussian profiles typical of laser-produced plasmas, but even for $k = 2$ the standard deviation is comparable with that obtained using other techniques.[6,7] Note, however, that Eq. (12) indicates that the algorithm described here will not be able to handle data accurately unless the derivative vanishes at the boundary of the data. This would normally be the case, to a very good approximation, for experimental data. It is also true for all $f_k(r)$, $k > 1$. For $k = 1$, however, the inversion is rather poor since the function $f_1(r)$ has a large derivative at the boundary.

The technique was also tested on a function with a large central dip.[10] The function and the reconstructed points are shown in Fig. 3. The agreement can be seen to be excellent with only minor deviations on the central two or three points.

As a demonstration of the technique with experimental data, the algorithm was used to analyze the interferometer data shown in Fig. 4(a). These data were obtained using a digital analysis code, reported elsewhere.[15] The center of symmetry y_c must be estimated automatically for each transverse slice through the data if the technique is to be applied with such a digital interferogram analysis algorithm. This was done using

$$y_c = \frac{\int_{-R}^{R} yS(y)dy}{\int_{-R}^{R} S(y)dy} \tag{16}$$

as an estimator, where the data $S(y)$ on a given slice is over the interval $\langle -R,R \rangle$. The Abel inversion around this center was calculated and results are shown in Fig. 4(b). As can be seen, very clean results were obtained over the 2-D set of data. It is worth noting that the direct use of Fourier coefficients in the inversion allows linear smoothing techniques to be directly applied during reconstruction. This has additional advantages in the speed of data processing possible.

Finally, it should be noted that the proposed technique may also be easily applied to functions of the form $S(y) = T(y)/y$, where $T(y)$ is the measured antisymmetric function and $S(y)$ is to be Abel inverted. This functional form is encountered in the study of magnetic fields in plasmas using Faraday rotation.[11] In particular, it is easy to show that

$$S(0) = \frac{\pi}{R} \sum_{k=1}^{+\infty} kb_k, \tag{17}$$

where b_k are the Fourier coefficients of the measured function $T(y)$. Thus, the apparent singularity at $y = 0$ may be easily treated with this Abel inversion approach and the data may be satisfactorily analyzed.

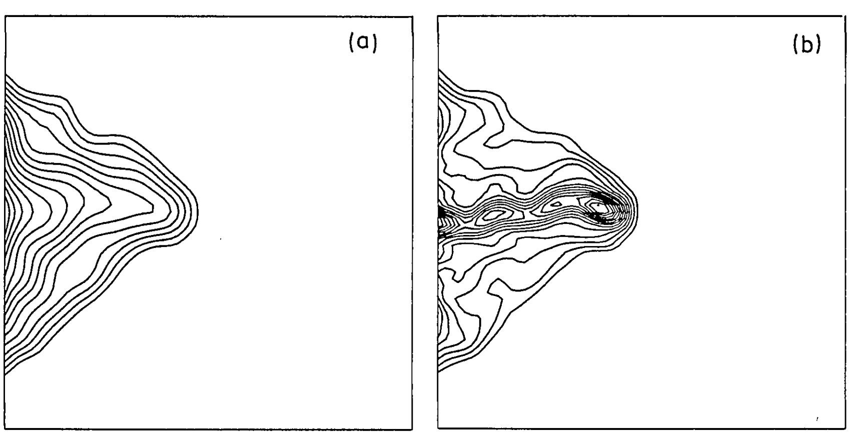

Fig. 4. (a) Phase shift recovered from an experimental interferogram obtained during a study of the density structure in a laser-produced plasma. (b) Plasma density profile obtained with the Abel inversion technique.

IV. Discussion

In this paper we have described a fast Abel transform algorithm based directly on the Fourier coefficients obtained from a fast Fourier transform program. Such an approach is appropriate due to the considerable current interest in developing Fourier transform based programs for automatic analysis of interferograms[11–17] which will yield large 2-D sets of data rapidly and accurately. An Abel inversion algorithm such as that presented here is ideally suited for use with such programs.

The technique is somewhat related to the work of Tatekura.[10] However, our formulation requires only a single summation for each data point whereas that of Tatekura requires a double summation. Thus our more direct approach possesses a substantial advantage in speed, which is usually an important factor in on-line experiments.

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Joseph L. Horner (Rome Air Development Center)—left, Brahm Javidi (Michigan State University), and Francis T. S. Yu (Penn State University) at the 1987 OSA Annual Meeting. *Photo:* F. S. Harris, Jr.