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NUMERICAL SOLUTION OF THE ABEL INTEGRAL EQUATION

R. PIESSENS and P. VERBAETEN

Abstract.

A numerical method for the solution of the Abel integral equation is presented. The known function is approximated by a sum of Chebyshev polynomials. The solution can then be expressed as a sum of generalized hypergeometric functions, which can easily be evaluated, using a simple recurrence relation.

Key words and phrases: Abel integral equation, Chebyshev polynomial, recurrence relation.

1. Introduction.

The Abel integral equation

$$\int_0^x \varphi(y)(x-y)^{-\alpha}dy = f(x) \quad (0 < \alpha < 1) \tag{1}$$

occurs in a number of engineering problems. If it is assumed that $f(x)$ is differentiable, the solution of (1) is explicitly given by [1]

$$\varphi(x) = \frac{\sin(\alpha\pi)}{\pi}\left[\frac{f(0)}{x^{1-\alpha}} + \int_0^x \frac{f'(y)}{(x-y)^{1-\alpha}}dy\right]. \tag{2}$$

However, this formula is not of practical value in problems where no explicit mathematical expression for $f(x)$ is known or where the derivative of $f(x)$ is very difficult to calculate. This is the reason why other methods are presented in the literature [2]-[6].

The purpose of this paper is to describe a new and simple method for solving (1) in the case that $f(x)$ can be approximated accurately by an expression of the form

$$f(x) \cong x^\beta \sum_{k=0}^{N}{}' c_k T_k(1-2x) \tag{3}$$

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where the prime denotes that the first term is taken with the factor $\frac{1}{2}$. $T_k(x)$ is the Chebyshev polynomial of the first kind and degree k and $\beta > -\alpha$ is a free parameter.

Here it is assumed that $0 \leqq x \leqq 1$, but this restriction is not essential.

2. Method of solution.

The solution of

$$\int_0^x g_n(y)(x-y)^{-\alpha}dy = x^\beta T_n(1-2x)$$

can easily be derived using Laplace transform techniques. Indeed [7]

$$\mathscr{L}\{x^\beta T_n(1-2x)\} = \frac{\Gamma(\beta+1)}{p^{\beta+1}}\, {}_3F_1\left(\begin{matrix}-n,n,\beta+1\\ \frac{1}{2}\end{matrix};\frac{1}{p}\right)$$

where $\mathscr{L}$ denotes the Laplace transform. Here

$${}_pF_q\left[\begin{matrix}\alpha_1,\alpha_2,\ldots\alpha_p;\\ \beta_1,\beta_2,\ldots\beta_q;\end{matrix}Z\right] = 1+\sum_{n=1}^{\infty}\left(\prod_{i=1}^{p}(\alpha_i)_n\right)\Big/\left(\prod_{i=1}^{q}(\beta_i)_n\right)\cdot Z^n/n!$$

where as usual $(\alpha)_n = \alpha(\alpha+1)\ldots(\alpha+n-1) = \Gamma(\alpha+n)/\Gamma(\alpha)$.

Thus

$$\mathscr{L}\{g_n(x)\} = \frac{\Gamma(1+\beta)}{\Gamma(1-\alpha)}\frac{1}{p^{\alpha+\beta}}\, {}_3F_1\left(\begin{matrix}-n,n,\beta+1\\ \frac{1}{2}\end{matrix};\frac{1}{p}\right)$$

and consequently

$$g_n(x) = \frac{x^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\frac{\Gamma(1+\beta)}{\Gamma(1-\alpha)}\, {}_3F_2\left(\begin{matrix}-n,n,\beta+1\\ \frac{1}{2},\ \alpha+\beta\end{matrix};x\right). \tag{4}$$

Thus, if $f(x)$ is approximated by (3), the solution of (1) is approximately

$$\varphi(x) \cong \frac{x^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\frac{\Gamma(1+\beta)}{\Gamma(1-\alpha)}\sum_{k=0}^{\infty}{}' c_k f_k(x) \tag{5}$$

where

$$f_n(x) = {}_3F_2\left(\begin{matrix}-n,n,\beta+1\\ \frac{1}{2},\ \alpha+\beta\end{matrix};x\right). \tag{6}$$

Using Fasenmyer's technique [8], a recurrence formula for the computation of $f_n(x)$ can be derived, namely

$$f_n(x)+(A_n+B_nx)f_{n-1}(x)+(C_n+D_nx)f_{n-2}(x)+E_nf_{n-3}(x) = 0 \tag{7}$$

where

$$A_n = -\frac{1}{n-2}\left[n-3+\frac{(n-1)(2n-3)}{n+\alpha+\beta-1}\right]$$

$$B_n = 4\,\frac{n+\beta}{n+\alpha+\beta-1}$$

$$C_n = \frac{1}{n-2}\left[-1+\frac{n-1}{n+\alpha+\beta-1}(3n-\alpha-\beta-5)\right]$$

$$D_n = -4\frac{(n-\beta-3)(n-1)}{(n+\alpha+\beta-1)(n-2)}$$

$$E_n = -\frac{(n-\alpha-\beta-2)(n-1)}{(n+\alpha+\beta-1)(n-2)}$$

Starting-values for (7) are

$$f_0(x) = 1$$

$$f_1(x) = 1-\frac{2(\beta+1)}{\alpha+\beta}x$$

$$f_2(x) = 1-\frac{8(\beta+1)}{\alpha+\beta}x+\frac{8(\beta+1)(\beta+2)}{(\alpha+\beta)(\alpha+\beta+1)}x^2 .$$

The recurrence formula (7) is a difference equation of Poincaré's type [9]. When n is large, it approximates to the form

$$f_n+(4x-3)f_{n-1}-(4x-3)f_{n-2}-f_{n-3} = 0 .$$

The characteristic function of this associated difference equation has three zeros with modulus 1. Using a method described in [10] we can find the asymptotic behaviour of three linearly independent solutions $y^{(1)}$, $y^{(2)}$, and $y^{(3)}$ of (7). For $x=0$, we have

$$y_n^{(1)} = 1, \quad y_n^{(2)} = n, \quad y_n^{(3)} \sim n^{2(1-\alpha-\beta)}, \quad n\to\infty .$$

For $0<x<1$, we have

$$y_n^{(1)} \sim n^{1-\alpha}\cos(n\theta), \quad y_n^{(2)} \sim n^{1-\alpha}\sin(n\theta), \quad y_n^{(3)} \sim n^{-2(\beta+1)}$$

for $n\to\infty$, where $\theta = bg\cos(1-2x)$.

For $x=1$, we have

$$y_n^{(1)} = (-1)^n n, \quad y_n^{(2)} \sim (-1)^n n^{\frac{3}{2}(\alpha-1)}, \quad y_n^{(3)} \sim n^{-2(\beta+1)} \quad \text{for } n\to\infty .$$

This means that (7) cannot have any solution which increases or decreases strongly, if β is not too large.

Thus f_n must be computed using the forward recursion technique. We have ascertained experimentally the numerical stability of this technique.

If $f(x)$ is known as an explicit mathematical expression, the coefficients c_k in the right hand member of (3) can be calculated using a method for the numerical computation of coefficients of the Chebyshev series expansion of $x^{-\beta} f(x)$ over the interval $[0,1]$. Such methods, based on the discrete orthogonality property of the Chebyshev polynomials, are well known and excellently described in [11].

In the case that $f(x)$ is obtainable only from measured data, Clenshaw's curve fitting method [12] can be used to construct an approximation of $f(x)$ in the form (3).

In both cases the value of β must be chosen so that $x^{-\beta} f(x)$ is as smooth as possible on $[0,1]$. Formula (3) then gives a good approximation of $f(x)$, even with a relatively small value of N.

Since in most cases the sequence $\{|c_k|,\ k=0,1,2,\ldots\}$ is strongly decreasing to zero and since we know that the sequence $\{|f_k(x)|,\ k=0,1,2,\ldots\}$ cannot increase strongly, the error in the approximation of (5) is generally less than the last term in the right member of (5).

3. Numerical examples.

Example 1. We consider the equation

$$\int_0^x \varphi(y)(x-y)^{-\frac{1}{2}}dy = e^x - 1 .$$

The exact solution is

$$\varphi(x) = \frac{e^x}{\sqrt{\pi}} Erf(\sqrt{x}) .$$

Using $\beta=1$ and $N=12$, formula (5) gives results with an absolute error less than 10^{-14} for all $x \in [0,1]$.

Example 2. We consider an application which is of great interest in astrophysics and plasmaphysics, namely the calculation of the radial distribution of the emission coefficient of an axially symmetric radiation source [3]-[6]. Let $E(r)$ be the emission coefficient at the distance r from the origin of the source. If the assumption is made that the self-absorption of the medium is negligible and that the radius beyond which $E(r)$ is negligible, is equal to 1, the radiance emitted by the source along the edge of a circular slab in a cross section taken perpendicular to the axis of symmetry, is given by

$$\gamma(y) = 2\int_y^1 \frac{r\,E(r)}{\sqrt{r^2-y^2}}\,dr, \quad y \geqq 0 . \tag{8}$$

This equation can be transformed into Abel's integral equation

$$f(x) = \int_0^x \frac{\varphi(u)}{\sqrt{x-u}}\,du \tag{9}$$

where

$$\begin{aligned} 1-y^2 &= x \\ 1-r^2 &= u \\ \varphi(u) &= E(\sqrt{1-u}) \\ f(x) &= \gamma(\sqrt{1-x}) . \end{aligned}$$

As an example of our method, we shall solve (8) with

$$\gamma(y) = \frac{\sqrt{\pi}}{1.1}\,(1-y^2)^{-\frac{1}{2}} \exp\left[1.21\left(1-\frac{1}{1-y^2}\right)\right] . \tag{10}$$

The exact solution is

$$E(r) = (1-r^2)^{-3/2} \exp\left[1.21\left(1-\frac{1}{1-r^2}\right)\right] . \tag{11}$$

This example is also considered by Minerbo and Levy [3] and used to compare other solution techniques described by several authors [4]-[6].

In this case there is no optimal value of β and we have chosen arbitrarily $\beta=1$. In table 1, we give three series of results I, II and III. For the first series of results the coefficients c_k are calculated using the formulae

$$c_k \cong \frac{2}{N}\sum_{r=0}^{N}{}'' \gamma\left(\sqrt{\frac{1+x_r}{2}}\right)\left(\frac{1-x_r}{2}\right)^{-\beta} T_k(x_r), \quad x_r = \cos\frac{\pi r}{N}$$

for $k=0,1,2,\ldots,N-1$ and

$$c_N \cong \frac{1}{N}\sum_{r=0}^{N}{}'' \gamma\left(\sqrt{\frac{1+x_r}{2}}\right)\left(\frac{1-x_r}{2}\right)^{-\beta} T_N(x_r), \quad x_r = \cos\frac{\pi r}{N}$$

where the double prime indicates that both the first and last terms of the sum are taken with the factor $\frac{1}{2}$.

For the second series of results, the coefficients c_k are calculated by Clenshaw's curve fitting technique, using 21 values of $\gamma(y)$ in the points $y_k = 0.05k$, $k = 0, 1, 2, \ldots, 20$. These results must be compared to the accuracy test of [3], and demonstrate the superiority of our method.

The third series of results is calculated in the same way as the second series but, in order to study the amplification of random noise on the data, a normally distributed random error of 1% (relative error) is superimposed on the values $\gamma(y_k)$. The calculations were carried out on the IBM 370/155 computer of the Computing Centre of the University of Leuven, using double precision arithmetic.

Table 1. *Numerical results of example* 2.

r	$E(r)$ Exact	Errors of numerical results I ($N=30$)	II ($N=14$)	III ($N=9$)
0.0	1.00000000	0.12×10^{-4}	-0.36×10^{-4}	0.14×10^{-1}
0.1	1.002785736	0.16×10^{-5}	0.56×10^{-5}	0.65×10^{-2}
0.2	1.01087503	0.28×10^{-6}	-0.18×10^{-4}	-0.82×10^{-2}
0.3	1.02203495	-0.18×10^{-6}	0.74×10^{-6}	-0.12×10^{-1}
0.4	1.03154109	-0.11×10^{-7}	-0.17×10^{-4}	-0.81×10^{-3}
0.5	1.02859087	-0.87×10^{-6}	-0.84×10^{-5}	0.52×10^{-2}
0.6	0.98886808	0.20×10^{-5}	-0.91×10^{-5}	0.35×10^{-2}
0.7	0.85853126	0.33×10^{-6}	-0.28×10^{-4}	0.45×10^{-2}
0.8	0.53867964	-0.66×10^{-6}	-0.73×10^{-4}	0.15×10^{-2}
0.9	0.06943778	-0.15×10^{-5}	-0.50×10^{-3}	-0.32×10^{-2}
1.0	0.0	-0.17×10^{-13}	0.54×10^{-18}	-0.19×10^{-12}

4. Conclusion.

We have presented a new inversion method for the Abel integral equation, which has several practical advantages:

(i) it is a simple but very accurate method.

(ii) unlike other methods [3]-[6], this method does not impose any restriction on the value of α.

(iii) it can be used as a discrete method and the data points do not have to be equally spaced. However, if a mathematical expression for $f(x)$ is available, still higher accuracy can be achieved. In this case, as distinct from the method given by Fettis [2], the number of evaluations of $f(x)$ is independent of the number of points in which the solution must be calculated.

(iv) random noise on the data is only slightly amplified.

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