Laurent Bourgeois, Philippe Moireau

Data completion and identification in problems governed by PDEs

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Contents

Acronyms

Here is a list of symbols used in this book:

Ψ	Operator to be inversed
A	Semi-group generator
C	Observation operator

Chapter 1
Inverse and ill-posed problems

1.1 Introduction

Solving an inverse problems generally consists in identifying some hidden parameters or defects in a system with the help of accessible data. More precisely, we impose some solicitations to the system and measure the corresponding responses in order to recover those parameters or defects. The inverse problems may be linear or not, in the sense that the mapping from the parameter/defect to the data may be linear or not. Most often, those inverse problems are ill-posed in the sense that a small perturbation of the data implies a strong error on the parameter/defect to identify. More precisely, the French Mathematician Jacques Hadamard gave the following definition of a well-posed problem in mathematical physics:

- existence of a solution
- uniqueness of the solution
- continuous dependance of this solution on the data.

A problem is ill-posed if one of this properties at least is not satisfied. Let $\Psi : \mathcal{Z} \to \mathcal{Y}$ be a mapping from a normed space $\mathcal{Z}$ to a normed space $\mathcal{Y}$. The equation $\Psi(z) = y$ is called well-posed if Ψ is bijective and $\Psi^{-1} : \mathcal{Y} \to \mathcal{Z}$ is continuous. The equation is ill-posed otherwise. A particular and frequent case is when $\mathcal{Z}$ and $\mathcal{Y}$ are both Banach spaces and Ψ is linear and continuous. Then if Ψ is bijective, its inverse Ψ^{-1} is continuous, as a consequence of the open mapping theorem. In other words, the first and second properties of a well-posed problem automatically implies the third one. An important case is when Ψ is a compact operator: then the equation $\Psi z = y$ is always ill-posed unless $\mathcal{Z}$ is finite dimensional. Indeed, if we assume that Ψ^{-1} exists and is continuous, then $\Psi^{-1}\Psi = I : \mathcal{Z} \to \mathcal{Z}$ is a compact operator by composition of a continuous and a compact operator. But the identity operator is a compact operator on $\mathcal{Z}$ if and only if $\mathcal{Z}$ is finite dimensional. Let us now present several elementary examples of inverse and ill-posed problems.

1.2 A toy example: differentiation operator

We consider the operator $\Psi : L^2(0,1) \to L^2(0,1)$ defined by

$$(\Psi z)(t) = \int_0^t z(s)\, ds.$$

Clearly, the inverse of Ψ is differentiation. Let us show that the equation $\Psi z = y$ is ill-posed. For it suffices to prove that Ψ is a compact operator. If we set $y = \Psi z$, we have $y' = z \in L^2(0,1)$, so that Ψ is a continuous operator from $L^2(0,1)$ to $H^1(0,1)$. Since the embedding $H^1(0,1) \to L^2(0,1)$ is compact, then $\Psi : L^2(0,1) \to L^2(0,1)$ is compact. Note, however, that Ψ is injective. We also observe that another expression of Ax is

$$(\Psi z)(t) = \int_0^1 K(t,s) z(s)\, ds,$$

where K is defined by $K(t,s) = 1$ of $s \le t$ and 0 otherwise. We readily see that $K \in L^2((0,1)\times(0,1))$ and it is a general result that an integral operator on $L^2(a,b)$, the kernel of which is in $L^2((a,b)\times(a,b))$, is compact.

Theorem 1.1. *Let us consider an operator $\Psi : L^2(a,b) \mapsto L^2(a,b)$ defined by*

$$(\Psi u)(t) = \int_a^b K(s,t) u(s)\, dt, \quad t \in (a,b)$$

for $K \in L^2((a,b)\times(a,b))$ with $a < b$. Then Ψ is compact.

Proof. Proving that Ψ is compact amounts to prove that if a sequence (z_n) weakly converges to z in $L^2(a,b)$ then (Ψz_n) strongly converges to Ψz in $L^2(a,b)$. We have

$$(\Psi z_n)(t) = \int_0^1 K(t,s) z_n(s)\, ds.$$

For almost all $t \in (a,b)$, the function $s \mapsto K(t,s)$ belongs to $L^2(a,b)$. This is a consequence of Fubini's theorem. Hence for almost $t \in (a,b)$, we have $(Ax_n)(t) \to (Ax)(t)$. In addition, by the Cauchy-Schwarz inequality, for almost all $t \in (a,b)$,

$$\begin{aligned} |\Psi z_n(t) - \Psi z(t)|^2 &\le \int_0^1 |K(t,s)|^2\, ds \int_0^1 |z_n(s) - z(s)|^2\, ds \\ &\le C \int_0^1 |K(t,s)|^2\, ds, \end{aligned}$$

since the weakly convergent sequence (z_n) is bounded. We observe that the right-hand side function of t is integrable in (a,b) and independent of n. From the Lebesgue's theorem, we conclude that (Ψz_n) converges to Ψz in $L^2(a,b)$. □

Since the operator Ψ is compact, the problem of finding z from y such that $\Psi z = y$ is ill-posed. This has a practical consequence on the instability of numerical differentiation. If one wants to differentiate a noisy function which is given at discrete values one has to proceed carefully, that is by introducing a regularization, because such procedure is strongly unstable.

1.3 The backward heat equation

Let us first consider the heat equation for $(x,t) \in (0,\pi) \times (0,T)$ with initial condition

$$\begin{cases} \partial_t u - \partial_x^2 u = 0 & \text{in } (0,\pi) \times (0,T), \\ u(0,t) = u(\pi,t) = 0 & \text{on } (0,T), \\ u(x,0) = \phi(x) & \text{on } (0,\pi). \end{cases} \tag{1.1}$$

As soon as $\phi \in L^2(0,\pi)$, such problem is well-posed in $C^0(0,T;L^2(0,\pi)) \cap L^2(0,T;H^1(0,\pi))$. To show that, it is natural to decompose the solution with respect to x in the complete basis $\sin(nx)$ of $L^2(0,\pi)$. Doing so, we obtain that

$$u(x,t) = \sum_{n=1}^{+\infty} \phi_n e^{-n^2 t} \sin(nx), \phi_n := \frac{2}{\pi} \int_0^\pi \phi(y) \sin(ny)\, dy.$$

Now, let us consider the same problem with a final condition, which is the so-called backward heat equation

$$\begin{cases} \partial_t u - \partial_x^2 u = 0 & \text{in } (0,\pi) \times (0,T), \\ u(0,t) = u(\pi,t) = 0 & \text{on } (0,T), \\ u(x,T) = f(x) & \text{on } (0,\pi). \end{cases} \tag{1.2}$$

By using the previous decomposition, we obtain

$$u(x,t) = \sum_{n=1}^{+\infty} f_n e^{n^2(T-t)} \sin(nx), \quad f_n := \frac{2}{\pi} \int_0^\pi f(y) \sin(ny)\, dy.$$

In particular,

$$\|u(x,0)\|^2_{L^2(0,\pi)} = \frac{\pi}{2} \sum_{n=1}^{+\infty} |f_n|^2 e^{2n^2 T},$$

which means that $\phi(x) = u(x,0)$ is in general undefined in $L^2(0,\pi)$ for some $f \in L^2(0,\pi)$, unless the sequence of (f_n) decreases extremely fast. To be more specific, we observe that finding ϕ from f consists in solving $\Psi \phi = f$ with $\Psi : L^2(0,\pi) \to L^2(0,\pi)$ and

$$(\Psi\phi)(x) = \int_0^\pi K(x,y)\phi(y)\,dy, \quad K(x,y) := \frac{2}{\pi}\sum_{n=1}^{+\infty} e^{-n^2 T}\sin(nx)\sin(ny).$$

Obviously, the function K belongs to $L^2((0,\pi)\times(0,\pi))$, which implies from Theorem 1.1 that the operator Ψ is compact, and hence the problem of finding the initial condition ϕ from the final one f by solving (1.2) is ill-posed. However, it can be proved that such problem has at most one solution.

1.4 The Cauchy problem for the Laplace equation

Assume that $\Omega \subset \mathbb{R}^d$ is a bounded open and connected domain of class $C^{0,1}$, and Γ a non-empty open subpart of $\partial\Omega$. We consider the following problem: for a pair of data $(g_0, g_1) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$, find $u \in H^1(\Omega)$ such that

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = g_0 & \text{on } \Gamma, \\ \partial_\nu u = g_1 \text{on } \Gamma, \end{cases} \tag{1.3}$$

where ν is the outward unit normal $\partial\Omega$.
Such problem is the Cauchy problem for the Laplace's equation, which arises in many practical situations : the (g_0, g_1) are measurements on the accessible part Γ on the domain and are redundant, while no data is available on the complementary part $\tilde{\Gamma}$ of $\partial\Omega$, which is not accessible. Note that, from Theorem 3.6, problem (1.3) has at most one solution.

A practical method to solve the Cauchy problem (1.3) is to view it as a data completion problem, following the ideas given in Andrieux et al. (2006); Azaïez et al. (2006); Ben Belgacem & El Fekih (2005). In particular, the proofs that follow are borrowed from Azaïez et al. (2006); Ben Belgacem & El Fekih (2005). In order to simplify the presentation, we assume that $\overline{\Gamma} \cap \overline{\tilde{\Gamma}} = \emptyset$, which implies in particular that space $H^{1/2}(\Gamma)$ coincides with space $\tilde{H}^{1/2}(\Gamma)$, which is defined as the dual space of $H^{-1/2}(\Gamma)$. The idea is to complete the boundary data on the inaccessible part of the boundary $\tilde{\Gamma}$ so that the solutions to the Dirichlet problem and the Neumann problem formed in Ω with data g_0 and g_1 on Γ coincide with each other. More precisely, for $\mu \in H^{1/2}(\tilde{\Gamma})$ we consider $u_D(\mu, g_0)$ and $u_N(\mu, g_1)$ the solutions in $H^1(\Omega)$ of well-posed problems

$$\begin{cases} \Delta u_D = 0 & \text{in } \Omega, \\ u_D = g_0 & \text{on } \Gamma, \\ u_D = \mu & \text{on } \tilde{\Gamma}, \end{cases} \tag{1.4}$$

and

$$\begin{cases} \Delta u_N = 0 & \text{in } \Omega, \\ \partial_\nu u_N = g_1 & \text{on } \Gamma, \\ u_N = \mu \text{on } \tilde{\Gamma}. \end{cases} \tag{1.5}$$

Remark 1.1. Let us remark that if we do not assume that $\overline{\Gamma} \cap \overline{\tilde{\Gamma}} = \emptyset$, problem (1.4) is undefined in general for $g_0 \in H^{1/2}(\Gamma)$ and $\mu \in H^{1/2}(\tilde{\Gamma})$.

We have the following proposition.

Proposition 1.1. *For Cauchy data $(g_0, g_1) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$, the Cauchy problem (1.3) has a solution in $H^1(\Omega)$ if and only if there exists $\mu \in H^{1/2}(\tilde{\Gamma})$ such that*

$$\partial_\nu u_D(\mu, g_0)|_{\tilde{\Gamma}} = \partial_\nu u_N(\mu, g_1)|_{\tilde{\Gamma}}. \tag{1.6}$$

Proof. First we assume that equation (1.6) is satisfied. Then the function $u_D - u_N \in H^1(\Omega)$ solves the homogeneous Cauchy problem (3.1) with $\tilde{\Gamma}$ playing the role of Γ and then $u_D = u_N$ in Ω in view of Theorem 3.6. The solution $u \coloneqq u_D = u_N$ is a solution of the Cauchy problem (1.3).
Conversely, if $u \in H^1(\Omega)$ is the solution to the Cauchy problem, we just have to set $\mu \coloneqq u|_{\Gamma} \in H^{1/2}(\tilde{\Gamma})$. □

Next, we establish an equivalence between equation (1.6) and a Steklov-Poincaré problem. In this view we define $u_D(\mu) \coloneqq u_D(\mu, 0)$ and $\breve{u}_D(g_0) = u_D(0, g_0)$ and we adopt similar notations for u_N. We define the bilinear form on $H^{1/2}(\tilde{\Gamma}) \times H^{1/2}(\tilde{\Gamma})$

$$s(\lambda, \mu) = \int_\Omega \nabla u_D(\lambda) \cdot \nabla u_D(\mu)\, dx - \int_\Omega \nabla u_N(\lambda) \cdot \nabla u_N(\mu)\, dx$$

and the linear form on $H^{1/2}(\tilde{\Gamma})$

$$\ell(\mu) = -\int_\Omega \nabla \breve{u}_D(g_0) \cdot \nabla u_D(\mu)\, dx + \langle g_1, u_N(\mu) \rangle_\Gamma,$$

where the bracket denotes duality pairing between $H^{-1/2}(\Gamma)$ and $\tilde{H}^{1/2}(\Gamma)$.
We have the following theorem.

Theorem 1.2. *The function $\lambda \in H^{1/2}(\tilde{\Gamma})$ solves equation (1.6) if and only if it solves the weak formulation*

$$s(\lambda, \mu) = \ell(\mu), \quad \forall \mu \in H^{\frac{1}{2}}(\tilde{\Gamma}).$$

Proof. let us assume that λ in $H^{1/2}(\tilde{\Gamma})$ satisfies (1.6). Then for all μ in $H^{1/2}(\Gamma)$,

$$\int_{\tilde{\Gamma}} \partial_\nu u_D(\lambda, g_0) \mu\, ds = \int_{\tilde{\Gamma}} \partial_\nu u_N(\lambda, g_1) \mu\, ds.$$

Hence

$$\int_{\partial\Omega} \partial_\nu u_D(\lambda, g_0) u_D(\mu)\, ds = \int_{\partial\Omega} \partial_\nu u_N(\lambda, g_1) u_N(\mu)\, ds - \int_\Gamma g_1 u_N(\mu)\, ds.$$

Integration by parts implies

$$\int_\Omega \nabla u_D(\lambda, g_0) \cdot \nabla u_D(\mu)\, dx = \int_\Omega \nabla u_N(\lambda, g_1) \cdot \nabla u_N(\mu)\, dx + \int_\Gamma g_1 u_N(\mu)\, ds.$$

Remarking that

$$u_D(\lambda, g_0) = u_D(\lambda) + \breve{u}_D(g_0), \quad u_N(\lambda, g_1) = u_N(\lambda) + \breve{u}_N(g_1),$$

we obtain

$$\begin{aligned} \int_\Omega \nabla u_D(\lambda) \cdot \nabla u_D(\mu)\, dx + \int_\Omega \nabla \breve{u}_D(g_0) \cdot \nabla u_D(\mu)\, dx \\ = \int_\Omega \nabla u_N(\lambda) \cdot \nabla u_N(\mu)\, dx + \int_\Omega \nabla \breve{u}_N(g_1) \cdot \nabla u_N(\mu)\, dx \\ + \int_\Gamma g_1 u_N(\mu)\, ds. \end{aligned} \tag{1.7}$$

On the other hand,

$$\int_\Omega \nabla \breve{u}_N(g_1) \cdot \nabla u_N(\mu)\, dx = \int_\Gamma \breve{u}_N(g_1) \partial_\nu u_N(\mu)\, ds + \int_{\tilde\Gamma} \breve{u}_N(g_1) \partial u_N(\mu)\, ds = 0.$$

We hence obtain

$$s(\lambda, \mu) = \ell(\mu),$$

which completes the proof. The converse assertion follows the same lines. □

The weak formulation of Theorem 1.2 is equivalent to

$$S\lambda = L, \tag{1.8}$$

where $S : H^{1/2}(\tilde\Gamma) \to H^{-1/2}(\tilde\Gamma)$ and $L \in H^{-1/2}(\tilde\Gamma)$ are defined with the help of the bilinear form s and the linear form ℓ.

Let us now show that the operator S is compact. In order to highlight this fact, we notice that

$$s(\lambda, \mu) = \langle \partial_\nu (u_D - u_N)(\lambda), \mu \rangle_{\tilde\Gamma},$$

That is

$$S\lambda = \partial_\nu (u_D - u_N)(\lambda)|_{\tilde\Gamma}\,, \tag{1.9}$$

which justifies the fact that we call S the Steklov-Poincaré operator.

Proposition 1.2. *The operator $S : H^{1/2}(\tilde\Gamma) \to H^{-1/2}(\tilde\Gamma)$ is compact.*

Proof. For the sake of simplicity, we assume that Ω is a $C^{1,1}$ domain in order to use some simple regularity results for the Laplace equation. The case when Ω is a polygonal domain ($d=2$) or polyhedral ($d=3$) would be treated with the help of Grisvard (1985) and the general case of Lipschitz domains for $d=2,3$ would be treated with the help of Jerison & Kenig (1981).
In view of the expression (1.9) we focus on the regularity of solutions u_D and u_N of problems (1.4) and (1.5). We have supposed that $\overline{\Gamma}\cap\overline{\tilde{\Gamma}}=\emptyset$. We hence may find a smooth cut-off function ϕ such that $\phi=1$ in a neighborhood of Γ and $\phi=0$ in a neighborhood of $\tilde{\Gamma}$. By denoting $v=\phi\, u_N(\lambda)$, the function v solves the problem

$$\begin{cases} \Delta v = f & \text{in } \Omega, \\ \partial_\nu v = 0 & \text{on } \Gamma, \\ v = 0 & \text{on } \tilde{\Gamma}, \end{cases}$$

with $f=(u_N(\lambda)\Delta\phi+2\nabla u_N(\lambda)\cdot\nabla\phi)\in L^2(\Omega)$. From standard regularity results for the Laplace equation, $v\in H^2(\Omega)$ and

$$\|v\|_{H^2(\Omega)} \leq C\,\|f\|_{L^2(\Omega)} \leq C\,\|u_N(\lambda)\|_{H^1(\Omega)} \leq C\,\|\lambda\|_{H^{\frac{1}{2}}(\tilde{\Gamma})},$$

and lastly

$$\|u_N(\lambda)\|_{H^{\frac{3}{2}}(\Gamma)} \leq C\,\|\lambda\|_{H^{\frac{1}{2}}(\tilde{\Gamma})}.$$

In view of (1.9) we now consider $w=u_D(\lambda)-u_N(\lambda)$, which solves the problem

$$\begin{cases} \Delta w = 0 & \text{in } \Omega, \\ w = -u_N(\lambda) & \text{on } \Gamma, \\ w = 0 & \text{on } \tilde{\Gamma}. \end{cases}$$

We conclude that

$$\|w\|_{H^2(\Omega)} \leq C\,\|u_N(\lambda)\|_{H^{\frac{3}{2}}(\Gamma)} \leq C\,\|\lambda\|_{H^{\frac{1}{2}}(\tilde{\Gamma})}.$$

Lastly

$$\|S\lambda\|_{H^{\frac{1}{2}}(\tilde{\Gamma})} \leq C\,\|\lambda\|_{H^{\frac{1}{2}}(\tilde{\Gamma})}.$$

We complete the proof by recalling that the embedding $H^{1/2}(\tilde{\Gamma})\to H^{-1/2}(\tilde{\Gamma})$ is compact. □

From Proposition 1.1 and Theorem 1.2, we see that solving the Cauchy problem for the Laplace equation (1.3) is equivalent to invert the operator S, which from Proposition 1.2 happens to be compact. This means that problem (1.3) is ill-posed. We will see in the sequel how such problem can be regularized in some sense.

1.5 The wave equation with interior measurements

Let us now consider the wave equation for $(x,t) \in \Omega \times (0,T)$ where Ω is a regular bounded domain of boundary $\Gamma = \partial\Omega$. We denote (u_0, v_0) the initial condition and consider no external solicitation so that the wave system reads

$$\begin{cases} \partial_t^2 u - \Delta u = 0 & \text{in } \Omega \times (0,T) \\ u = 0 & \text{on } \Gamma \times (0,T) \\ u = u_0 & \text{in } \Omega \\ v = v_0 & \text{in } \Omega \end{cases} \tag{1.10}$$

We denote by

$$\mathcal{D}_0(\Delta, \Omega) = \Big\{ u \in H_0^1(\Omega) \,|\, \Delta u \in L^2(\Omega) \},$$

the space allowing to consider strong solutions of (5.1). Indeed from Appendix **??**, two types of solutions can be expected. If

$$(u_0, v_0) \in \mathcal{D}_0(\Delta, \Omega) \times H_0^1(\Omega),$$

then the problem (5.1) admits a strong solution

$$u \in C^2([0,T]; L^2(\Omega)) \cap C^1([0,T]; H_0^1(\Omega)) \cap C^0([0,T]; \mathcal{D}_0(\Delta, \Omega)).$$

However, if we consider an initial condition in the energy space

$$(u_0, v_0) \in H_0^1(\Omega) \times L^2(\Omega),$$

then the problem (5.1) admits a weak solution

$$u \in C^1([0,T]; L^2(\Omega)) \cap C^0([0,T]; H_0^1(\Omega)).$$

Let us now consider that we have at our disposal some measurements of a target solution of (5.1) initialized from an unknown initial condition (u_0, v_0). Typically, we assume to measure $u \in \omega \times (0,T)$ where ω is a subdomain of Ω. Our inverse problem is then to assess if we can reconstruct the unknown (u_0, v_0) from the measurement y of u denoted in $\omega \times (0,T)$. Keeping in mind Hadamard's definition, two questions arise: (1) do we have enough information in y to reconstruct (u_0, v_0) and (2) is this reconstruction stable with respect to the data at hand? The first question has been studied by numerous authors since the pioneer work of Lions (1988). In essence, there is enough information in the observation if T is large enough so that the all the information contained in the initial solution propagates up to the subdomain ω. The most precise result was given by Bardos et al. (1988) clarifying what we have meant by "information".

Definition 1.1. (GCC) Let $\omega \subset \Omega$ and $T_0 > 0$. We say that the couple (ω, T_0) satisfies GCC if every generalized geodesic (i.e. ray of geometric optics) traveling at speed one in Ω meets ω in a time $t < T_0$. We say that ω satisfies the GCC if there exists $T_0 > 0$ such that satisfies the GCC.

Let us then introduce the linear operator

$$\Psi_T : \left| \begin{array}{ll} \mathcal{Z} & \to \mathcal{Y}_T \\ z_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} & \mapsto y = u_\omega \end{array} \right.$$

and $\mathcal{Y}_T = H^1([0,T], L^2(\omega))$ or $\mathcal{Y}_T = L^2([0,T], H^1(\omega))$ – namely $\mathcal{Y}_T = H^k([0,T], H^{1-k}(\omega))$ for $k = 0, 1$. Note that assuming $\mathcal{Y}_T = H^1([0,T], L^2(\omega))$ leads to equivalently consider the derivative in the sense of distribution $\Psi_T z_0 = \partial_t u|_\omega \in L^2([0,T], L^2(\omega))$. Then, we have the following coercivity result.

Claim. Let $k = 0, 1$ and assuming that ω satisfies the GCC, then Ψ_T is coercive from $H^1_0(\Omega) \times L^2(\Omega)$ to $H^k([0,T], H^{1-k}(\omega))$ for large time, namely there exists (T, α) such that for every $(u_0, v_0) \in H^1_0(\Omega) \times L^2(\Omega)$

$$\int_0^T \int_\omega |\partial_t u|^2 \geq \alpha \Big(\|u_0\|^2_{H^1_0(\Omega)} + \|v_0\|^2_{L^2(\Omega)} \Big),$$

or

$$\int_0^T \int_\omega |\nabla u|^2 \geq \alpha \Big(\|u_0\|^2_{H^1_0(\Omega)} + \|v_0\|^2_{L^2(\Omega)} \Big).$$

Therefore, establishing the injectivity and/or surjectivity of Ψ_T depends of our choices of spaces $\mathcal{Z}$ and $\mathcal{Y}_T$. Namely assuming $\mathcal{Z} = H^1_0(\Omega) \times L^2(\Omega)$, if $\mathcal{Y}_T = H^k([0,T], H^{1-k}(\omega))$, the problem is injective as soon as (ω, T_0) satisfies the GCC and $T > T_0$. However, Ψ is not injective for $T < T_0$. However the question of its surjectivity is not obvious and it is certainly not satisfied when considering the – rather natural– space $\mathcal{Y}_T = L^2([0,T], L^2(\omega))$.

Remark 1.2. Note finally that other kinds of measurements can be envisioned, in particular boundary measurements. Considering the model (5.1), we could consider boundary forces measurements

$$y = \partial_n u, \quad \text{in } \Gamma \times (0,T).$$

Another possibility could be to replace partially the Dirichlet boundary conditions in (5.1) by Neumann boundary conditions

$$\begin{cases} \partial_t^2 u - \Delta u = 0 & \text{in } \Omega \times (0,T) \\ \partial_n u = 0 & \text{on } \Gamma_N \times (0,T) \\ u = 0 & \text{on } \Gamma_D \times (0,T) \\ v = v_0 & \text{in } \Omega \end{cases}$$

Then the measurements could be of the Dirichlet type

$$y = u, \quad \text{in } \Gamma_N \times (0, T).$$

1.6 The inverse Robin problem

Up to now we have stuck to linear inverse problems. Let us now address a non linear one, that is the inverse Robin problem. We hence consider a bounded, connected and open domain $\Omega \in \mathbb{R}^d$ of class $C^{0,1}$. The problem consists in finding $\lambda \in L^\infty(\tilde{\Gamma})$ such that

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = g_0 & \text{on } \Gamma, \\ \partial_\nu u = g_1 & \text{on } \Gamma, \\ \partial_\nu u + \lambda u = 0 & \text{on } \tilde{\Gamma}, \end{cases} \tag{1.11}$$

where Γ is a non-empty open subset of $\partial\Omega$, $\tilde{\Gamma} = \partial\Omega \setminus \overline{\Gamma}$, ν is the outward unit normal of Ω and (g_0, g_1) in $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$.
Such inverse problem corresponds to the identification of the impedance λ on an inaccessible part $\tilde{\Gamma}$ of the boundary from the measurements (g_0, g_1) on the accessible part Γ of the boundary. A simple application in the field of electrostatic non destructive testing can be found: this problem consists in finding some corrosion on the inaccessible boundary by measuring both the potential and the current on the accessible boundary.

To simplify the problem, we assume that $g_1 := g \neq 0$ is a fixed parameter in $L^2(\Gamma)$, and we define the mapping Ψ as follows. For $\lambda \in L^\infty_+(\tilde{\Gamma})$ with

$$L^\infty_+(\tilde{\Gamma}) := \{\lambda \in L^\infty(\tilde{\Gamma}),\ \exists m > 0,\ \lambda(x) \geq m \ a.e.\},$$

$\Psi(\lambda) = u|_\Gamma \in L^2(\Gamma)$, where $u \in H^1(\Omega)$ is the solution of the well-posed problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \partial_\nu u = g & \text{on } \Gamma, \\ \partial_\nu u + \lambda u = 0 & \text{on } \tilde{\Gamma}. \end{cases} \tag{1.12}$$

Problem (1.12) is clearly equivalent to the following weak formulation : find $u \in H^1(\Omega)$ such that for all $v \in H^1(\Omega)$,

$$\int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\tilde{\Gamma}} \lambda u v \, ds = \int_\Gamma g v \, ds, \tag{1.13}$$

and well-posedness of problem (1.13) follows from Poincaré-Friedrichs inequality, which implies the equivalence between the standard norm of $H^1(\Omega)$

and the norm $\|\cdot\|$ defined by

$$\|u\|^2 = \int_{\Omega} |\nabla u|^2 \, dx + \int_{\tilde{\Gamma}} u^2 \, ds.$$

The mapping $\Psi : L^\infty_+(\tilde{\Gamma}) \to L^2(\Gamma)$ is clearly non linear. Moreover, the problem of solving $\Psi(\lambda) = f$ in $L^\infty_+(\tilde{\Gamma})$ for $f \in L^2(\Gamma)$ is ill-posed since formally,

$$\lambda = -\frac{\partial_\nu u|_{\tilde{\Gamma}}}{u|_{\tilde{\Gamma}}},$$

and we known from the analysis of the Cauchy problem for the Laplace equation that the identification of u in Ω from the Cauchy data (g_0, g_1) on Γ is ill-posed. In the sequel, we will study uniqueness for this inverse Robin problem and how to solve it in some sense.

1.7 The inverse obstacle problem

We complete this short review of elementary inverse problems by a geometric inverse problem, namely the inverse obstacle problem. We again consider a bounded, connected and open domain $D \in \mathbb{R}^d$ of class $C^{0,1}$. The problem consists in finding an open domain O of class $C^{0,1}$ such that $O \Subset D$ and such that there exists $u \in H^1(\Omega)$, with $\Omega = D \setminus \overline{O}$, such that

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = g_0 & \text{on } \Gamma, \\ \partial_\nu u = g_1 & \text{on } \Gamma, \\ u = 0 & \text{on } \partial O, \end{cases} \tag{1.14}$$

where Γ is a non-empty open subset of $\partial\Omega$, ν is the outward unit normal of Ω and (g_0, g_1) in $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$. The obstacle O is then characterized by a Dirichlet boundary condition. This problem can be seen as a simple model of non destructive testing in thermography for the steady case. It consists in identifying an insulating cavity inside a a body from measurements of both the temperature and the flux on a subpart of the boundary.

The inverse obstacle problem is obviously non linear, even more non linear than the inverse Robin problem, in the sense that the parameter to retrieve, that is the open domain O, does not lie in a vector space. However, it lies in a metric space, namely the set of all open domains contained in D, which can be for example equipped with the Hausdorff distance for open domains. In the sequel, we will study uniqueness for such inverse problem and the way we can solve it in some sense.

Chapter 2
The Tikhonov regularization and the Morozov principle

2.1 Introduction

We consider a continuous operator $\Psi : \mathcal{Z} \to \mathcal{Y}$, where $\mathcal{Z}$ and $\mathcal{Y}$ are Hilbert spaces. In the following we will denote $\mathcal{Z}^*$ and $\mathcal{Y}^*$ the dual spaces of $\mathcal{Z}$ and $\mathcal{Y}$, and $\Psi^* : \mathcal{Y}^* \to \mathcal{Z}^*$ the adjoint operator associates with Ψ. We also identify $\mathcal{Y}^*$ to itself. We assume that Ψ is injective and has a dense range, which is an usual situation in many ill-posed problems. However, we assume that the range of Ψ is not closed, so that the problem: find $u \in \mathcal{Z}$ such that

$$\Psi z = y \tag{2.1}$$

with $y \in \mathcal{Y}$ may have no solution for some y and is therefore a ill-posed problem.
In general, for ill-posed problems, the data y comes from measurements and is then contaminated by some noise of amplitude δ. We hence assume that we know some noisy data $y^\delta \in \mathcal{Y}$ such that $\|y^\delta - y\|_{\mathcal{Y}} \leq \delta$. In general is it reasonable to assume that $\|y^\delta\|_{\mathcal{Y}} > \delta$, which means that the data is bigger than the amplitude of noise.
The inverse problem is as follows : *we assume that for exact and unknown data f, problem (2.1) has a (unique) unknown solution $z \in \mathcal{Z}$. The objective is, given some known noisy data $y^\delta \in \mathcal{Y}$, to find some approximate solution $\tilde{z} \in \mathcal{Z}$ of the exact solution z.*

2.2 The Tikhonov regularization

A classical tool for that is the Tikhonov regularization. It consists in solving the following problem for some $\epsilon > 0$: find $z_\epsilon^\delta \in \mathcal{Z}$ such that

$$\Psi^*\Psi z_\epsilon^\delta + \epsilon\, z_\epsilon^\delta = \Psi^* y^\delta. \tag{2.2}$$

Two other charactizations of z_ϵ^δ may be easily obtained. First, z_ϵ^δ is the solution of the variational formulation: find $z_\epsilon^\delta \in \mathcal{Z}$ such that

$$(\Psi z_\epsilon^\delta, \Psi v)_{\mathcal{Y}} + \epsilon (z_\epsilon^\delta, v)_{\mathcal{Z}} = (y^\delta, Av) \quad \forall v \in \mathcal{Z}. \tag{2.3}$$

Secondly z_ϵ^δ is the unique minimizor of the functional

$$\mathscr{J}_\epsilon(z) = \frac{1}{2} \|\Psi z - y^\delta\|_{\mathcal{Y}}^2 + \frac{\epsilon}{2} \|z\|_{\mathcal{Z}}^2. \tag{2.4}$$

The fact that the problems (2.2), (2.3) and (2.4) are equivalent is obvious and well-posedness of (2.3) is an immediate consequence of Lax-Milgram's theorem.

Remark 2.1. Note that the well-posedness of (2.3) is also obtained when Ψ is neither surjective nor injective. Therefore, the Tikhonov is also adapted to circumvent a potential lack of injectivity in the inverse problem of interest.

In the case of exact data y, the Tikhonov solution z_ϵ converges to the exact solution z.

Theorem 2.1. *We have $z_\epsilon \to z$ in $\mathcal{Z}$ when $\epsilon \to 0$.*

Proof (First approach). By using both (2.1) and (2.3) we obtain

$$(\Psi(z_\epsilon - z), \Psi v)_{\mathcal{Y}} + \epsilon (z_\epsilon, z)_{\mathcal{Z}} = 0, \quad \forall v \in \mathcal{Z}. \tag{2.5}$$

By choosing $v = z_\epsilon - z$, we obtain that

$$(z_\epsilon, z_\epsilon - z)_{\mathcal{Z}} \leq 0$$

and that

$$\|z_\epsilon - z\|_{\mathcal{Z}}^2 = (z_\epsilon, z_\epsilon - z)_{\mathcal{Z}} - (u, z_\epsilon - z)_{\mathcal{Z}} \leq -(u, z_\epsilon - z)_{\mathcal{Z}}. \tag{2.6}$$

By using Cauchy-Schwarz's inequality, it follows that

$$\|z_\epsilon\|_{\mathcal{Z}} \leq \|u\|_{\mathcal{Z}}, \quad \|z_\epsilon - z\|_{\mathcal{Z}} \leq \|u\|_{\mathcal{Z}}. \tag{2.7}$$

From z_ϵ we can hence extract a "subsequence", still denoted z_ϵ, that weakly converges to some $w \in \mathcal{Z}$ when ϵ tends to 0. From (2.5) and (**??**) we also obtain that

$$\|\Psi(z_\epsilon - z)\|_{\mathcal{Y}}^2 = -\epsilon (z_\epsilon, z_\epsilon - z)_{\mathcal{Z}} \leq \epsilon \|z\|_{\mathcal{Z}}^2,$$

that is

$$\|\Psi(z_\epsilon - z)\|_{\mathcal{Y}} \leq \sqrt{\epsilon}\, \|z\|_{\mathcal{Z}},$$

and then Ψz_ϵ tends to Ψz. Since Ψz_ϵ weakly converges to Ψw in $\mathcal{Y}$ we have $\Psi z = \Psi w$ and $z = w$ from injectivity of Ψ. It remains to remark from (2.6) that the weak convergence of the subsequence in $\mathcal{Z}$ implies its strong convergence in $\mathcal{Z}$. The strong convergence of all the sequence (not only the subsequence) follows from a classical contradiction argument. □

We now present an alternative proof of theorem 2.1, which is based on the spectral theorem for self-adjoint operators.

Proof (Second approach). We note that $\Lambda = \Psi^*\Psi : \mathcal{Z} \to \mathcal{Z}^*$, if we identify $\mathcal{Z}^*$ with itself, is a self-adjoint continuous operator on $\mathcal{Z}$. We can then apply the spectral theorem to T. If we consider the corresponding spectral family $E(\lambda)$ we can formally write

$$\Lambda = \int_0^{+\infty} \lambda \, dE(\lambda).$$

Here we have used the fact that Λ is positive so that we have restricted the real line to its positive values. From (2.1) and (2.2), we obtain

$$\Psi^*\Psi z_\epsilon + \epsilon \, z_\epsilon = \Psi^*\Psi z,$$

that is

$$z_\epsilon - z = \big((\Lambda + \epsilon \mathbb{1}_{\mathcal{Z}})^{-1}\Lambda - \mathbb{1}_{\mathcal{Z}})\big) u.$$

Since

$$(T + \epsilon I_{\mathcal{Z}})^{-1} T - I_{\mathcal{Z}} = \int_0^{+\infty} \left(\frac{\lambda}{\lambda + \epsilon} - 1 \right) dE(\lambda) = - \int_0^{+\infty} \frac{\epsilon}{\lambda + \epsilon} \, dE(\lambda),$$

it follows that

$$\|z_\epsilon - z\|_{\mathcal{Z}}^2 = \int_0^{+\infty} \frac{\epsilon^2}{(\lambda + \epsilon)^2} \, (dE(\lambda)u, u), \tag{2.8}$$

where $(dE(\lambda)z, z)$ defines a positive measure on $\mathbb{R}^+$. It remains to apply the Lebesgue theorem. In this view we remark that unless $\lambda = 0$, $\epsilon^2/(\lambda+\epsilon)^2$ tends to 0 when ϵ tends to 0 and $\{\lambda = 0\}$ is a set the measure of which is 0 since 0 is not an eigenvalue of Λ (remember that Ψ is injective). We also remark that $\epsilon^2/(\lambda + \epsilon)^2$ is bounded by 1, and the result follows from the Lebesgue theorem. □

Remark 2.2. It should be noticed from (2.8) that $\|z_\epsilon - z\|_{\mathcal{Z}}$ is an increasing fonction of $\epsilon > 0$.

Now let us have a look at the realistic situation when the (noisy) data is y^δ in the Tikhonov regularization. Remember we have denoted z_ϵ^δ the Tikhonov solution associated with data y^δ. The difference $z_\epsilon^\delta - z_\epsilon$ is then the solution associated with data $y^\delta - y$ and, if we choose $v = z_\epsilon^\delta - z_\epsilon$ in (2.3) with such data, we obtain ($\|y^\delta - y\|_{\mathcal{Y}} \leq \delta$)

$$\|z_\epsilon^\delta - z_\epsilon\|_\mathcal{Z} \le \frac{\delta}{\sqrt{\epsilon}}.$$

Thus the difference between the Tikhonov solution with noisy data and the exact solution is

$$\|z_\epsilon^\delta - z\|_\mathcal{Z} \le \|z_\epsilon - z\|_\mathcal{Z} + \frac{\delta}{\sqrt{\epsilon}}.$$

As a conclusion, for $\delta \neq 0$, there is no evidence that $z_\epsilon^\delta \to z$ when $\epsilon \to 0$. In fact, the situation is even more critical since we have the following theorem.

Theorem 2.2. *If data y^δ is not in the range of operator Ψ, then $\|z_\epsilon^\delta\|_\mathcal{Z} \to +\infty$ when $\epsilon \to 0$.*

Proof. By contradiction, if we assume that the assertion $\|z_\epsilon^\delta\|_\mathcal{Z} \to +\infty$ when $\epsilon \to 0$ is not true, this means that we can can find a subsequence of z_ϵ^δ which is bounded in $\mathcal{Z}$, from which we can extract a subsequence that weakly converges to $w \in \mathcal{Z}$. We have from (2.3)

$$(\Psi z_\epsilon^\delta, \Psi v)_\mathcal{Y} + \epsilon (z_\epsilon^\delta, v)_\mathcal{Z} = (y^\delta, \Psi v), \quad \forall v \in \mathcal{Z}$$

Passing to the limit when $\epsilon \to 0$, we obtain

$$(\Psi w, \Psi v)_\mathcal{Y} = (y^\delta, \Psi v), \quad \forall v \in \mathcal{Z}$$

and since Ψ has dense range $\Psi w = y^\delta$, which contradicts the fact that y^δ is not in the range of Ψ. $\square$

2.3 The Morozov's principle

From Theorem 2.1 and Theorem 2.2, we conclude that the choice of ϵ in the presence of noisy data is intricate. The idea of Morozov's principle consists in choosing ϵ such a way that the error $\|\Psi z_\epsilon^\delta - y^\delta\|_\mathcal{Y}$ we make by using the Tikhonov regularization is of the same order as the error on the data $\|y^\delta - y\|_\mathcal{Y}$, that is δ.

We have the following theorem.

Theorem 2.3. *If the noisy data y^δ satisfies*

$$\|y^\delta - y\|_\mathcal{Y} \le \delta < \|y^\delta\|_\mathcal{Y},$$

then there exists a unique choice $\epsilon > 0$ such that the solution of (2.3) associated with data y^δ satisfies

$$\|\Psi z_\epsilon^\delta - y^\delta\|_\mathcal{Y} = \delta.$$

Proof (First proof). Let us denote $g^\delta(\epsilon) = \|\Psi z_\epsilon^\delta - y^\delta\|_\mathcal{Y}^2$. The function g^δ is differentiable with respect to $\epsilon > 0$ and

$$\frac{\mathrm{d}g^\delta}{\mathrm{d}\epsilon}(\epsilon) = 2(\Psi z_\epsilon^\delta - y^\delta, \Psi v_\epsilon^\delta)_\mathcal{Y},$$

where $v_\epsilon^\delta \in \mathcal{Z}$ is uniquely defined by

$$(\Psi v_\epsilon^\delta, \Psi v)_\mathcal{Y} + \epsilon(v_\epsilon^\delta, v)_\mathcal{Z} = -(z_\epsilon^\delta, v)_\mathcal{Z}, \quad \forall v \in \mathcal{Z}. \tag{2.9}$$

On the other hand, by definition of z_ϵ^δ, we have

$$(\Psi z_\epsilon^\delta, \Psi v)_\mathcal{Y} + \epsilon(z_\epsilon^\delta, v)_\mathcal{Z} = (y^\delta, \Psi v)_\mathcal{Z}, \quad \forall v \in \mathcal{Z}. \tag{2.10}$$

By choosing $v = v_\epsilon^\delta$ in (2.10) we obtain

$$(\Psi z_\epsilon^\delta - y^\delta, \Psi v_\epsilon^\delta)_\mathcal{Y} = -\epsilon(z_\epsilon^\delta, v_\epsilon^\delta)_\mathcal{Z},$$

and by choosing $v = v_\epsilon^\delta$ in (2.9), we obtain $(z_\epsilon^\delta, v_\epsilon^\delta)_\mathcal{Z} \leq 0$. More precisely, we have in fact $(z_\epsilon^\delta, v_\epsilon^\delta)_\mathcal{Z} < 0$. Indeed, let us assume that $(z_\epsilon^\delta, v_\epsilon^\delta)_\mathcal{Z} = 0$. From (2.9) we obtain that $v_\epsilon^\delta = 0$, and then $z_\epsilon^\delta = 0$, which is not possible. We hence conclude that $\frac{\mathrm{d}g^\delta}{\mathrm{d}\epsilon} > 0$ for all $\epsilon > 0$. Then g^δ is a strictly increasing function. Now let us prove that

$$\lim_{\epsilon \to 0} g^\delta(\epsilon) = 0, \qquad \lim_{\epsilon \to +\infty} g^\delta(\epsilon) = \|y^\delta\|_\mathcal{Y}^2.$$

The second statement is obtained by taking $v = z_\epsilon^\delta$ in (2.10), which implies that

$$\|z_\epsilon^\delta\|_\mathcal{Z} \leq \frac{\|y^\delta\|_\mathcal{Y}}{\sqrt{\epsilon}},$$

and then $\Psi z_\epsilon^\delta \to 0$ in $\mathcal{Y}$ when $\epsilon \to +\infty$. Now let us prove the first statement. We also have $\|\Psi z_\epsilon^\delta\|_\mathcal{Y} \leq \|y^\delta\|_\mathcal{Y}$, hence we can extract from (z_ϵ^δ) a subsequence that we still denote (z_ϵ^δ) such that Ψz_ϵ^δ weakly converges to some $h \in \mathcal{Y}$. We have

$$(\Psi z_\epsilon^\delta - y^\delta, \Psi v)_\mathcal{Y} = -\epsilon(z_\epsilon^\delta, v)_\mathcal{Z} \leq \sqrt{\epsilon}\|y^\delta\|_\mathcal{Y}\|v\|_\mathcal{Z} \to 0$$

when $\epsilon \to 0$. We also have

$$(\Psi z_\epsilon^\delta - y^\delta, \Psi v)_\mathcal{Y} \to (h - y^\delta, \Psi v)_\mathcal{Y} = 0, \quad \forall v \in \mathcal{Z}.$$

Since Ψ has a dense range, $h = y^\delta$.
Lastly, we write

$$\begin{aligned} g^\delta(\epsilon) &= \|\Psi z_\epsilon^\delta - y^\delta\|_\mathcal{Y}^2 = (\Psi z_\epsilon^\delta - y^\delta, \Psi z_\epsilon^\delta)_\mathcal{Y} - (\Psi z_\epsilon^\delta - y^\delta, y^\delta)_\mathcal{Y} \\ &= -\epsilon\|z_\epsilon^\delta\|_\mathcal{Z}^2 - (\Psi z_\epsilon^\delta - y^\delta, y^\delta)_\mathcal{Y} \leq -(\Psi z_\epsilon^\delta - y^\delta, y^\delta)_\mathcal{Y} \to 0 \end{aligned}$$

when $\epsilon \to 0$, which completes the proof of the first statement.
We complete the proof by using the fact that g^δ is a continuous function

which increases from 0 to $\|y^\delta\|^2 > \delta^2$, then there exists a unique $\epsilon > 0$ such that $g^\delta(\epsilon) = \delta^2$. □

Here again a second is possible based on the spectral theorem.

Proof (Second proof). We again apply the spectrum theorem to the self-adjoint operator $\Lambda = \Psi\Psi^* : \mathcal{Y} \to \mathcal{Y}$. We consider the corresponding spectral family $E(\lambda)$ and write

$$\Lambda = \int_0^{+\infty} \lambda \, dE(\lambda).$$

The solution z_ϵ^δ solves

$$(\Psi\Psi^*)\Psi z_\epsilon^\delta + \epsilon \Psi z_\epsilon^\delta = (\Psi\Psi^*) y^\delta,$$

that is

$$\Psi z_\epsilon^\delta - y^\delta = \left((\Lambda + \epsilon \mathbb{1}_\mathcal{Y})^{-1} Q - \mathbb{1}_\mathcal{Y}\right) y^\delta.$$

Since

$$(\Lambda + \epsilon \mathbb{1}_\mathcal{Y})^{-1}\Lambda - \mathbb{1}_\mathcal{Y} = -\int_0^{+\infty} \frac{\epsilon}{\lambda + \epsilon} \, dE(\lambda),$$

we have that

$$\|\Psi z_\epsilon^\delta - y^\delta\|_\mathcal{Y}^2 = \int_0^{+\infty} \frac{\epsilon^2}{(\lambda+\epsilon)^2} \, (dE(\lambda) y^\delta, y^\delta) := y^\delta(\epsilon)$$

Clearly, y^δ is a stricly increasing function, and by using the Lebesgue theorem, y^δ is a continuous function, $y^\delta(\epsilon) \to 0$ when $\epsilon \to 0$ (here we have used the fact that 0 is not an eigenvalue of Λ, which is due to the fact that Ψ has dense range), and $y^\delta(\epsilon) \to \|y^\delta\|_\mathcal{Y}^2$ when $\epsilon \to +\infty$. The assumption $\|y^\delta\|_\mathcal{Y} > \delta$ guarantees that $y^\delta(\epsilon) = \delta^2$ for a unique value of $\epsilon > 0$. □

A natural question is to wonder if the choice $\epsilon(\delta)$ given by Morozov's principle is actually the best possible choice among all values of ϵ, for a given amplitude of noise δ. There is no positive answer to that question. However, in some very special cases, with the so-called source conditions, we have estimates for $\|z_\epsilon^\delta - z\|_\mathcal{Z}$ when $\epsilon(\delta)$ is the Morozov's choice. The following theorem is borrowed from Kirsch (1996).

Theorem 2.4. *In the case when z is in the range of operator Ψ^*, that is there exists a (unique) $p \in \mathcal{Y}$ such that $z = \Psi^* p$, then if $\epsilon = \epsilon(\delta)$ is the Morozov's choice associated with data y^δ and z_ϵ^δ the corresponding solution of* (2.3)*, then*

$$\|z_\epsilon^\delta - z\|_\mathcal{Z} \le 2\sqrt{\|p\|_\mathcal{Y}}\sqrt{\delta}.$$

Proof. If z_ϵ^δ is the Tikhonov solution for the Morozov's choice $\epsilon(\delta)$, in view of the minimization problem (2.4), we have

$$\|\Psi z_\epsilon^\delta - y^\delta\|_\mathcal{Y}^2 + \epsilon(\delta)\,\|z_\epsilon^\delta\|_\mathcal{Z}^2 \le \|\Psi z - y^\delta\|_\mathcal{Y}^2 + \epsilon(\delta)\,\|z\|_\mathcal{Z}^2.$$

Since $\|\Psi z_\epsilon^\delta - y^\delta\|_{\mathcal{Y}} = \delta$, $\Psi z = y$ and $\|y^\delta - y\|_{\mathcal{Y}} \leq \delta$, we obtain

$$\delta^2 + \epsilon(\delta)\, \|z_\epsilon^\delta\|_{\mathcal{Z}}^2 \leq \delta^2 + \epsilon(\delta)\, \|z\|_{\mathcal{Z}}^2,$$

which implies

$$\|z_\epsilon^\delta\|_{\mathcal{Z}} \leq \|z\|_{\mathcal{Z}}. \tag{2.11}$$

Now, since

$$\|z_\epsilon^\delta - z\|_{\mathcal{Z}}^2 = \|z_\epsilon^\delta\|_{\mathcal{Z}}^2 - 2(z_\epsilon^\delta, u)_{\mathcal{Z}} + \|u\|_{\mathcal{Z}}^2,$$

it follows from (2.11) that

$$\|z_\epsilon^\delta - z\|_{\mathcal{Z}}^2 \leq 2(u - z_\epsilon^\delta, u)_{\mathcal{Z}}.$$

On the other hand, we have

$$(z - z_\epsilon^\delta, z)_{\mathcal{Z}} = (z - z_\epsilon^\delta, \Psi^* p)_{\mathcal{Z}} = (\Psi(z - z_\epsilon^\delta), p)_{\mathcal{Y}} = (y - \Psi z_\epsilon^\delta, p)_{\mathcal{Y}}.$$

Hence

$$\begin{aligned}(z - z_\epsilon^\delta, u)_{\mathcal{Z}} &= (y - y^\delta, p)_{\mathcal{Y}} + (y^\delta - \Psi z_\epsilon^\delta, p)_{\mathcal{Y}} \\ &\leq \left(\|y - y^\delta\|_{\mathcal{Y}} + \|y^\delta - \Psi z_\epsilon^\delta\|_{\mathcal{Y}}\right) \|p\|_{\mathcal{Y}} \leq 2\delta\, \|p\|_{\mathcal{Y}}.\end{aligned}$$

We finally have

$$\|z_\epsilon^\delta - z\|_{\mathcal{Z}}^2 \leq 4\delta\, \|p\|_{\mathcal{Y}},$$

which completes the proof. $\square$

Remark 2.3. Note that the source condition $z \in R(\Psi^*)$ cannot be verified in practice since the exact solution z is unknown. On the other hand, even if the Morozov's principle is not justified from a theoretical point of view, it gives very good results in practice.

2.4 Interpretation with duality in optimization

We complete this chapter by an interpretation of the Morozov's principle with the help of duality in optimization. It will also enable us to obtain a practical method to compute the Morozov's value of $\epsilon(\delta)$ and the corresponding Tikhonov solution z_ϵ^δ.

In this view we consider the optimization problem (compare with problem (2.4)) :

$$\inf_{v \in K} L(v) \tag{P}$$

with

$$L(v) = \frac{1}{2}\|v\|_{\mathcal{Z}}^2, \quad K = \{v \in \mathcal{Z}\,;\, \|Av - y^\delta\|_{\mathcal{Y}} \leq \delta\}.$$

We have the following theorem.

Theorem 2.5. *For $y^\delta \in \mathcal{Y}$, Problem (P) has a unique solution $z^\delta \in \mathcal{Z}$.*

Proof. We first use the classical existence theorem for optimization problems. The set K is a convex, closed and not empty subset of $\mathcal{Z}$. This last result follows from the fact that Ψ has dense range. The function L is convex and continuous. Since L is coercive, that is $L(v) \to +\infty$ when $\|v\|_{\mathcal{Z}} \to +\infty$, Problem (P) has at least one solution. Such solution is unique since L is strictly convex. $\square$

Remark 2.4. We note that if $\|y^\delta\|_{\mathcal{Y}} \leq \delta$, we have $z^\delta = 0$, while if $\|y^\delta\|_{\mathcal{Y}} > \delta$, $z^\delta \neq 0$.

Now we recall some notions of duality in optimization, following Ekeland & Temam (1974). We consider an optimization problem (Q) denoted the primal problem :

$$\inf_{v \in \mathcal{Z}} F(v), \tag{Q}$$

where $\mathcal{Z}$ is a Hilbert space, $F : \mathcal{Z} \to \overline{\mathbb{R}}$ a function $\neq +\infty$. Here we have denoted $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. We recall what is the conjugate function.

Definition 2.1. The conjugate function $F^* : \mathcal{Z}^* \to \overline{\mathbb{R}}$ of F is defined, for $u^* \in \mathcal{Z}^*$, by

$$F^*(u^*) = \sup_{u \in \mathcal{Z}} (\langle u, u^* \rangle_{\mathcal{Z},\mathcal{Z}^*} - F(u)).$$

Then we introduce the notion of perturbed problem. We consider a function $\Phi : \mathcal{Z} \times \mathcal{Y} \to \overline{\mathbb{R}}$, where $\mathcal{Y}$ is another Hilbert space, and Φ satisfies

$$\Phi(v, 0) = F(v).$$

For all $q \in \mathcal{Y}$, we consider the perturbed problem (Q_q):

$$\inf_{v \in \mathcal{Z}} \Phi(v, q). \tag{Q_q}$$

Then we define the dual problem of Problem (Q) with respect to the perturbation Φ. Let $\Phi^* : \mathcal{Z}^* \times \mathcal{Y}^* \to \overline{\mathbb{R}}$ be the conjugate function of Φ. The dual problem, denoted (Q^*), is the following optimization problem:

$$\sup_{q^* \in \mathcal{Y}^*} -\Phi^*(0, q^*). \tag{Q^*}$$

We have the following proposition.

Proposition 2.1.

$$(-\infty \leq) \quad \sup(Q^*) \leq \inf(Q) \quad (\leq +\infty).$$

Proof. For $q^* \in \mathcal{Y}^*$, we have

$$\Phi^*(0,q^*) = \sup_{u\in\mathcal{Z},\, q\in\mathcal{Y}} (\langle u,0\rangle_{\mathcal{Z},\mathcal{Z}^*} + \langle q,q^*\rangle_{\mathcal{Y},\mathcal{Y}^*} - \Phi(u,q)),$$

so that, for all $u \in \mathcal{Z}$,

$$\Phi^*(0,q^*) \geq \langle 0,q^*\rangle_{\mathcal{Y},\mathcal{Y}^*} - \Phi(u,0) = -\Phi(u,0).$$

We hence have $\forall u \in \mathcal{Z}$, $\forall q^* \in \mathcal{Y}^*$, $-\Phi^*(0,q^*) \leq \Phi(u,0)$, and finally $\sup(P^*) \leq \inf(P)$. $\square$

Remark 2.5. Equality is not satisfied in general: when $\sup(Q^*) \neq \inf(Q)$, we say that there is a duality gap.

We admit the following theorem, which is proved in Ekeland & Temam (1974), and which guarantees equality.

Theorem 2.6. *We assume that Φ is convex and that $\inf(P) < +\infty$. If there exists $u_0 \in \mathcal{Z}$ such that $q \to \Phi(u_0,q)$ is finite and continuous at point 0, then $\inf(Q) = \sup(Q^*) < +\infty$ and Problem (Q^*) has solutions. We say that Problem (Q) is stable.*

Now let us apply the above theorem to our Problem (P). With previous notations, we define

$$F(z) = L(z) + \chi_{B_\delta}(\Psi z),$$

and

$$\Phi(z,q) = L(z) + \chi_{B_\delta}(\Psi z - q), \tag{2.12}$$

where $B_\delta \subset \mathcal{Y}$ is the closed ball of center y^δ and radius δ, χ_{B_δ} is the indicator function defined by

$$\begin{cases} \chi_{B_\delta}(q) = 0, & \text{if } q \in B_\delta, \\ \chi_{B_\delta}(q) = +\infty, & \text{if } q \notin B_\delta. \end{cases}$$

The function F is defined on $\mathcal{Z}$ while Φ is defined on $\mathcal{Z}\times\mathcal{Y}$. Let us form the dual problem (P^*) which, after some simple computations, is defined by

$$\sup_{q*\in\mathcal{Y}} -\Phi^*(0,q^*) = \sup_{q*\in\mathcal{Y}} \Big(- L^*(\Psi^* q^*) - \chi^*_{B_\delta}(-q^*)\Big). \tag{P^*}$$

It remains to compute L^* and $\chi^*_{B_\delta}$.

It is easy to see that

$$L^*(v^*) = \frac{1}{2}\|v^*\|^2_{\mathcal{Z}^*}.$$

For $q^* \in \mathcal{Y}$, we have

$$\chi^*_{B_\delta}(q^*) = \sup_{q\in\mathcal{Y}}((q,q^*)_{\mathcal{Y}} - \chi_{B_\delta}(q)),$$

that is

$$\chi^*_{B_\delta}(q^*) = \sup_{\substack{q\in\mathcal{Y}\\ \|q-y^\delta\|_{\mathcal{Y}}\le\delta}} (q,q^*)_{\mathcal{Y}} = (y^\delta,q^*)_{\mathcal{Y}} + \delta \sup_{\substack{q\in\mathcal{Y},\\ \|q\|_{\mathcal{Y}}\le 1}} (q,q^*)_{\mathcal{Y}}$$
$$= (y^\delta,q^*)_{\mathcal{Y}} + \delta\,\|q^*\|_{\mathcal{Y}}.$$

Therefore, Problem (P^*) corresponds to

$$\sup_{q^*\in\mathcal{Y}} -\Phi^*(0,q^*) = \sup_{q^*\in\mathcal{Y}} \Big(-\frac{1}{2}\|\Psi^* q^*\|^2_{\mathcal{Z}^*} - \delta\,\|q^*\|_{\mathcal{Y}} + (y^\delta,q^*)_{\mathcal{Y}}\Big)$$

which finally rewrites into

$$\inf_{q^*\in\mathcal{Y}} G(q^*) = \inf_{q^*\in\mathcal{Y}} \Big(\frac{1}{2}\|\Psi^* q^*\|^2_{\mathcal{Z}^*} + \delta\,\|q^*\|_{\mathcal{Y}} - (y^\delta,q^*)_{\mathcal{Y}}\Big). \qquad (\tilde{P}^*)$$

Remark 2.6. Is is remarkable that while the primal problem (P) is a constrained minimization problem, the dual problem (P^*) is an unconstrained minimization problem. Therefore, (P^*) – *i.e.* ($\tilde{P}^*$) – is easier to solve than (P).

Now let us check that we satisfy the assumptions of Theorem 2.6. Clearly Φ given by (2.12) is a convex function of (u,q) and $\inf(P) < +\infty$ since Problem P has a (unique) solution. Let is choose $z_0 \in \mathcal{Z}$ such that $\|\Psi z_0 - y^\delta\|_{\mathcal{Y}} \le \delta/2$. For any $q \in \mathcal{Y}$ such that $\|q\|_{\mathcal{Y}} \le \delta/2$, we have $\|\Psi z_0 - q - y^\delta\|_{\mathcal{Y}} \le \delta$, that is $q \to \Phi(z_0,q) = L(z_0) < +\infty$ is constant in a neighborhood of point 0. We can then apply Theorem 2.6. In particular, it implies that (P^*) has solutions and that

$$\inf(P) = \sup(P^*) < +\infty.$$

Let p^* be such a solution of (P^*), the above relationship implies that

$$\frac{1}{2}\|z^\delta\|^2_{\mathcal{Z}} = -\frac{1}{2}\|\Psi^* p^*\|^2_{\mathcal{Z}^*} - \delta\,\|p^*\|_{\mathcal{Y}} + (y^\delta,p^*)_{\mathcal{Y}}. \qquad (2.13)$$

We now use the fact that G is Fréchet-differentiable at any point $q \neq 0$ (because of the term $\delta\,\|q\|_{\mathcal{Y}}$) and express the optimality at point p^*. In this view we have to verify that $p^* \neq 0$. Actually, let us take $y^\delta_0 = y^\delta/\|y^\delta\|_{\mathcal{Y}}$ and $q^* = \epsilon y^\delta_0 \in \mathcal{Y}$. For $\epsilon > 0$, we have

$$G(\epsilon y^\delta_0) = \frac{\epsilon^2}{2}\|\Psi^* y^\delta_0\|^2_{\mathcal{Z}^*} + \epsilon(\delta - \|y^\delta\|_{\mathcal{Y}}).$$

For small ϵ, $G(\epsilon y^\delta_0)$ has the sign of $\delta - \|y^\delta\|_{\mathcal{Y}} < 0$, hence there exists $q^* \in \mathcal{Y}$ such that $G(q^*) < 0 = G(0)$, and the solutions p^* do not vanish. Optimality writes $G'(p^*) = 0$ with

$$G'(p^*) = \Psi(\Psi^* p^*) + \delta \frac{p^*}{\|p^*\|_{\mathcal{Y}}} - y^\delta,$$

We have in particular

$$\|\Psi(\Psi^* p^*) - y^\delta\|_{\mathcal{Y}} = \delta.$$

We conclude from (2.13) that

$$\frac{1}{2}\|z^\delta\|_{\mathcal{Z}}^2 = \frac{1}{2}\|\Psi^* p^*\|_{\mathcal{Z}^*}^2,$$

that is the Riesz representant in $\mathcal{Z}$ of $\Psi^* p^* \in \mathcal{Z}^*$ solves the primal problem (P), which has the unique solution z^δ. We conclude that $z^\delta = \Psi^* p^*$ – provided we identify $\Psi^* p^* \in \mathcal{Z}^*$ with its representant in $\mathcal{Z}$ – which gives uniqueness of the solution p^* to Problem (P^*) since Ψ has dense range. Such solution is denoted p^δ. Incidentally, we note that

$$\|\Psi z^\delta - y^\delta\|_{\mathcal{Y}} = \delta.$$

Lastly, we come back to the Tikhonov solution $z_\epsilon^\delta \in \mathcal{Z}$ of problem (2.2) with data y^δ and ϵ chosen with Morozov's principle. The solution $z_\epsilon^\delta \in \mathcal{Z}$ solves

$$\Psi^*(\Psi z_\epsilon^\delta) + \epsilon z_\epsilon^\delta - \Psi^* y^\delta = 0.$$

On the other hand, the solution z^δ of problem (P) satisfies

$$\Psi^*(\Psi z^\delta) + \frac{\delta}{\|p^\delta\|_{\mathcal{Y}}} z^\delta - \Psi^* y^\delta = 0.$$

We hence conclude that z^δ is in fact the Tikhonov solution z_ϵ^δ with $\epsilon = \epsilon(\delta)$ given by

$$\epsilon(\delta) = \frac{\delta}{\|p^\delta\|_{\mathcal{Y}}},$$

which provides an explicit expression of the value $\epsilon(\delta)$ given by the Morozov's principle. All these results are summarized in the following theorem.

Theorem 2.7. *If the noisy data y^δ satisfies*

$$\|y^\delta - y\|_{\mathcal{Y}} \leq \delta < \|y^\delta\|_{\mathcal{Y}},$$

then Problem (P) and Problem (P^*) have a unique solution z^δ and p^δ, respectively, such that $z^\delta = \Psi^* p^\delta$. Furthermore, z^δ coincides with the Tikhonov solution z_ϵ^δ of problem (2.2) with $\epsilon(\delta)$ chosen following Morozov's principle. Lastly, $p^\delta \neq 0$ and

$$\epsilon(\delta) = \frac{\delta}{\|p^\delta\|_{\mathcal{Y}}}.$$

The above theorem hence provides a strategy to find the Tikhonov-Morozov solution associated with noisy data y^δ. It consists in solving first $(\tilde{P}^*)$. Our solution is then obtained by applying Ψ^* to the solution of $(\tilde{P}^*)$.

Remark 2.7. It can be proved directly – namely without using Theorem 2.6 – that the dual problem $(\tilde{P}^*)$ has a unique solution $p^\delta \in \mathcal{Y}$.

2.5 Time-dependent problems

We will now apply the Tikhonov regularization method to estimate quantities associated with evolution equation, for instance a heat equation or a wave equation observed on a subdomain. As presented in the introduction, this problems can be rewritten in an abstract form using the semi-group theory, namely we consider a dynamical system

la theorie des semi-groupes n'est pas évoquée avant dans ce poly. est-ce que les élèves sont déjà familiers ?

$$\begin{cases} \dot{z} = Az, & \text{in}\mathbb{R}^+ \\ z(0) = \lambda \end{cases} \tag{2.14}$$

where A is an unbounded operator of domain $D(A)$, generator of a continuous semi-group of contraction $(\Phi(t))_{t\geq 0}$. We additionally consider to have at our disposal some measurements $\{y(t), t \geq 0\}$ associated to a given trajectory of (2.14). We thus denote by C a bounded observation operator such that

$$C : \left| \begin{array}{l} \mathcal{Z} \to \mathcal{Y} \\ z \to y \end{array} \right. \tag{2.15}$$

From this collection of measurements obtained through the combination of the trajectory – *i.e.* the dynamics – and the observation operator, we want to recover the initial condition of (2.14). Therefore, we define an operator Ψ to be inverted as the operator such that for noiseless data we have

$$\mathcal{Y} \ni y(t) = \Psi(t)\lambda, \quad \text{in}\mathbb{R}^+.$$

and we directly infer that

$$\Psi(t) = Ce^{At}. \tag{2.16}$$

je conseille dans ce cas de préciser pourquoi c'est abusif, ou de ne rien mettre si c'est trop un détail

With a slight abuse of notation, for all $T \geq 0$, we define also by Ψ_T the operator

$$\Psi_T : \left| \begin{array}{l} \mathcal{Z} \to \mathcal{Y}_T \\ \lambda \to (y(t))_{t\in[0,T]} \end{array} \right. \tag{2.17}$$

where $\mathcal{Y}_T = L^2((0,T);\mathcal{Y})$. The question of proving the injectivity of (2.17) will be discussed in the next chapters. Here we assume, typically that Ψ_T is injective but not surjective. However, as mentioned in Remark 2.1, the lack of injectivity will also be handled by the Tikhonov regularization procedure which consists in minimizing (2.4), namely here

$$\mathscr{J}_{\epsilon,T}(z) = \frac{1}{2} \|\Psi_T \lambda - y^\delta\|^2_{\mathcal{Y}_T} + \frac{\epsilon}{2} \|z\|^2_{\mathcal{Z}}, \tag{2.18a}$$

$$= \int_0^T \frac{1}{2} \|\Psi \lambda - y^\delta(t)\|^2_{\mathcal{Y}} \,\mathrm{d}t + \frac{\epsilon}{2} \|z\|^2_{\mathcal{Z}} \tag{2.18b}$$

$$= \int_0^T \frac{1}{2} \|Cz(t) - y^\delta(t)\|^2_{\mathcal{Y}} \,\mathrm{d}t + \frac{\epsilon}{2} \|z\|^2_{\mathcal{Z}} \tag{2.18c}$$

where in the last identity $z_{|\lambda}(t)$ should be understood as a solution of (2.14) associated to λ, namely the trajectory "knowing" λ. Note that the minimization $\min_{\lambda \in \mathcal{Z}} \mathscr{J}_{\epsilon,T}$ is associated to its corresponding normal equation (2.2) which here reads

il peut être lecteur-friendly de redéfinir ce que l'on entend par cette notation

On pourrait rappeler les conditions pour pouvoir écrire cette équation. L2 ? C1 ? Linéarité ?

$$\int_0^T e^{A^* t} C^* C e^{At} \lambda^\delta_\epsilon \,\mathrm{d}t + \epsilon\, \lambda^\delta_\epsilon = \int_0^T e^{A^* t} C^* y^\delta(t) \,\mathrm{d}t. \tag{2.19}$$

One difficulty associated to the use of (2.19) is the computing cost as we need to compute operators defined over the whole time window $[0,T]$. In fact, it is possible to simplify drastically the expression (2.19) by relying on the so-called adjoint equation, namely the solution of

$$\begin{cases} \dot{q}(t) + A^* q(t) = -C^*(y(t) - Cz(t)), & t \in (0,T) \\ q(T) = 0 \end{cases} \tag{2.20}$$

adjoint q ?

defined for any given $\{y(t), t \in [0,T]\}$. Note here that p is defined only on $[0,T]$ and in reverse-time. The dynamics (2.22) is well-posed because $-A^*$ is the generator of a semi-group in reverse-time as soon as A^* is the generator of a semi-group in forward-time which is the case as soon as A is the generator of a semi-group in forward-time. From Duhamel formulae – see Appendix **??** – we have for any given $\{y(t), t \in [0,T]\}$,

ref 2.20 ?

$$q(t) = \int_t^T e^{A^* t} C^*(y(s) - Cz(s)) \,\mathrm{d}s, \quad t \in [0,T]$$

Hence, recalling that $z(s) = e^{At}\lambda$ we find that (2.19) is equivalent to

$$\lambda^\delta_\epsilon = \frac{1}{\epsilon} q^\delta_\epsilon(0). \tag{2.21}$$

qui minimise la fonctionnelle J ?

where q^δ_ϵ is associated to the measurements y^δ and the solution z^δ_ϵ. The formulae (2.19) is very seducing but hides an intricate relation. We could envision to solve only the adjoint equation to compute λ^δ_ϵ. However, the adjoint equation is defined backward in time with the use of the forward solution z^δ_ϵ defined from λ^δ_ϵ itself. In other words, the optimality system,

je préciserais pourquoi : est-ce que c'est seulement numérique ?

associated to λ_ϵ^δ is a two-end problem given by

avec epsilon et lambda ?

$$\begin{cases} \dot{z}_\epsilon^\delta = A z_\epsilon^\delta, & \text{in } (0,T) \\ \dot{q}_\epsilon^\delta + A^* q_\epsilon^\delta = -C^*(y^\delta - C z_\epsilon^\delta), & \text{in } (0,T) \\ z(0) = \dfrac{1}{\epsilon} q_\epsilon^\delta(0) \\ q(T) = 0 \end{cases} \tag{2.22}$$

To circumvent this difficulty and propose a practical approach, we will now propose a gradient descent approach based. From its definition (2.18a), $\mathscr{J}_{\epsilon,T}$ is a quadratique functional, hence we have

$$\mathscr{J}_{\epsilon,T}(\lambda+\xi) - \mathscr{J}_{\epsilon,T}(\lambda+\xi) \underset{=0}{=} \ell(\xi) + b(\xi,\xi)$$

where – considering that C is a bounded operator – we have

$$b(\xi,\xi) = \int_0^T (Ce^{At}\xi, Ce^{At}\xi)_{\mathcal{Y}}\,\mathrm{d}t + \epsilon\,(\xi,\xi)_{\mathcal{Z}} + O(\|\xi\|^2)$$

Therefore, $\mathscr{J}_{\epsilon,T}$ admits a Fréchet derivative given by

$$\begin{aligned} \mathrm{d}\mathscr{J}_{\epsilon,T}(\lambda)(\xi) &= \ell(\xi) \\ &= \int_0^T (Ce^{At}\xi, Ce^{At}\lambda - y^\delta)_{\mathcal{Y}}\,\mathrm{d}t + \epsilon\,(\xi,\lambda)_{\mathcal{Z}} \\ &= \int_0^T (\xi, e^{A^* t}C^*(Cz - y^\delta))_{\mathcal{Y}}\,\mathrm{d}t + \epsilon\,(\xi,\lambda)_{\mathcal{Z}} \\ &= -(q(0),\xi)_{\mathcal{Y}} + \epsilon(\lambda,\xi) \end{aligned} \tag{2.23}$$

Hence the gradient of $\mathscr{J}_{\epsilon,T}$ is given by

$$\nabla\mathscr{J}_{\epsilon,T}(\lambda) = -q_{|\lambda}^\delta(0) + \epsilon\lambda. \tag{2.24}$$

where $q_{|\lambda}^\delta$ is associated to the trajectory $z_{|\lambda}$ and the measurements y^δ through the dynamics (2.22). The (2.24), is very useful to set up a descent gradient algorithm which consists in solving from $\lambda^0 = 0$ the sequence

k est l'incrément du gradient
the numerical indicator ?

$$\lambda^{k+1} = \lambda^k - \rho_k \nabla\mathscr{J}_{\epsilon,T}(\lambda^k), \tag{2.25}$$

for an adequately chosen relaxation sequence $(\rho_k)_{k\geq 0}$ as specified in the next theorem. More precisely, (2.25) consists in solving – from $z^0 \equiv 0$ and for $k \geq 0$ – the back and forth dynamics

$$
\begin{cases}
\dot{z}^\delta_{k+1} = Az^\delta_{k+1}, & \text{in } (0,T) \\
\dot{q}^\delta_{k+1} + A^* q^\delta_{k+1} = -C^*(y^\delta - Cz^\delta_{k+1}), & \text{in } (0,T) \\
z^{k+1}(0) = (1-\rho_k \epsilon) z^k(0) + \rho_k q^\delta_{k+1}(0) & \\
q^{k+1}(T) = 0 &
\end{cases}
\tag{2.26}
$$

where typically we would expect to be able to choose $(\rho_k)_{k \geq 0}$ close to $1/\epsilon$. However, we are limited by the following result.

Theorem 2.8. *Let consider a function ...*

In the strategy we didn't had any information about the lambda. But the reasoning is compatible with…

Remark 2.8. The presented strategies is compatible with more general formulations, typically model of the form je propose de donner un argument

$$
\begin{cases}
\dot{z} = Az \\
z(0) = z_\diamond + \lambda
\end{cases}
$$

Chapter 3
Unique continuation results

3.1 Introduction

Let us consider a connected open domain $\Omega \subset \mathbb{R}^2$ and $\omega \subset \Omega$, with $\omega \neq \emptyset$. Assume that some harmonic distribution $u \in \mathcal{D}'(\Omega)$, that is $\Delta u = 0$ in Ω, satisfies $u = 0$ in ω. Since u is infinitely smooth in Ω, we may consider the complex function of complex variable

$$f(x+iy) = \tilde{f}(x,y) = \frac{\partial u}{\partial x}(x,y) - i\frac{\partial u}{\partial y}(x,y).$$

It is clear that $\tilde{f}$ is infinitely smooth with respect to (x,y) and that the Cauchy-Riemann relationships are satisfied, namely

$$\frac{\partial \tilde{f}}{\partial x} + i\frac{\partial \tilde{f}}{\partial y} = 0.$$

Hence, the function f is holomorphic (or, equivalently, analytic) in Ω. Since f vanishes in the open set ω, it vanishes in the connected set Ω. Then $\partial_x u$ and $\partial_y u$ vanish in Ω, which means that u is a constant. Such constant is 0 because u vanishes in ω. The conclusion is that any harmonic distribution which vanishes in ω vanishes in Ω.
The above argument is based on complex analysis and therefore restricted to the case of the Laplace operator in dimension 2. For higher dimension and for more general operators (though with constant coefficients), we can extend the previous unique continuation result with the help of the Holmgren's theorem.

3.2 The Holmgren's theorem

Let us begin with the following Holmgren's theorem (see Hormander (1976) for a proof). For we need to introduce the notion of characteristic plane with respect to an operator.

Definition 3.1. Let $P(D)$ be a differential operator with constant coefficients, $D = (\partial_{x_1}, \partial_{x_2}, \cdots, \partial_{x_d})$, and let us consider $P_0(D)$ its principal part. A plane of normal vector $N \in \mathbb{R}^d \setminus \{0\}$ is said to be characteristic with respect to P if $P_0(N) = 0$.

Let us see a few fundamental examples.

1. Case of the Laplacian in $\mathbb{R}^d$: for $x = (x_1, x_2, \cdots, x_d)$, we have $P(D) = \Delta = \sum_{i=1}^d \partial^2_{x_i} = P_0(D)$, so that
$$P_0(N) = \sum_{i=1}^d N_i^2 = |N|^2.$$
Since $P_0(N) \neq 0$ for any $N \neq 0$, there are no characteristic plane for the Laplacian operator.
2. Case of the heat operator in $\mathbb{R}^{d+1}$: for $(x,t) = (x_1, x_2, \cdots, x_d, t)$ we have $P(D) = \partial_t - \Delta = \partial_t - \sum_{i=1}^d \partial^2_{x_i}$, so that $P_0(D) = -\sum_{i=1}^d \partial^2_{x_i}$. For $N = (N_1, N_2, \cdots, N_d, N_{d+1})$, we have
$$P_0(N) = -\sum_{i=1}^d N_i^2.$$
Hence $P_0(N) = 0$ if and only of $N_i = 0$, $i = 1, \cdots, d$. The characteristic planes for the heat equation are the planes of equation $t = c$, where c is a constant.
3. Case of the wave operator in $\mathbb{R}^{d+1}$: $P(D) = \partial_t^2 - \Delta = \partial_t^2 - \sum_{i=1}^d \partial^2_{x_i} = P_0(D)$. We have
$$P_0(N) = N_{d+1}^2 - \sum_{i=1}^d N_i^2,$$
so that the characteristic planes for the wave equation are determined by $N_{d+1}^2 = \sum_{i=1}^d N_i^2$.

The Holmgren's theorem is the following statement.

Theorem 3.1. *Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be open convex domains in $\mathbb{R}^d$ such that $\mathcal{O}_1 \subset \mathcal{O}_2$, and let $P(D)$ be a differential operator with constant coefficients such that every plane which is characteristic with respect to P and intersects $\mathcal{O}_2$ also meets $\mathcal{O}_1$. Every $u \in \mathcal{D}'(\mathcal{O}_2)$ satisfying the equation $P(D)u = 0$ and vanishing in $\mathcal{O}_1$ then vanishes in $\mathcal{O}_2$.*

3.3 Unique continuation

3.3.1 Propagation of uniqueness

In order to in use Holmgren's theorem for non convex sets $\mathcal{O}_1$ and $\mathcal{O}_2$, we need to introduce the notion of geodesic distance in a domain Ω of $\mathbb{R}^d$.

Definition 3.2. If $\Omega \subset \mathbb{R}^d$ is a connected open domain, for $x, y \in \Omega$, the geodesic distance between x and y is defined by

$$d_\Omega(x,y) = \inf\{\ell(g),\ g : [0,1] \to \Omega,\ g(0) = x,\ g(1) = y\},$$

where g is a continuous path in Ω of length $\ell(g)$. Here, the length of g is defined as

$$\ell(g) = \sup \left\{ \sum_{j=0}^{n-1} |g(t_j) - g(t_{j+1})|,\ n \in \mathbb{N},\ 0 = t_0 \leq t_1 \leq \cdots \leq t_n = 1 \right\},$$

where the sup is taken over all decompositions of $[0,1]$ into an arbitrary (finite) number of intervals.

We also need to introduce the notion of sequence of balls as in Robbiano (1991).

Lemma 3.1. *Consider two points x_0 and x in the open and connected domain Ω. For all $\epsilon, \delta_0 > 0$, there exists some $\delta \in (0, \delta_0)$ and a δ-sequence of balls $B(q_n, \delta)$ for $n = 0, \cdots, N$ that links x_0 to x, that is*

$$\begin{cases} q_0 = x_0, \\ B(q_{n+1}, \delta) \subset B(q_n, 2\delta), \quad n = 0, \cdots, N-1, \\ B(q_n, 3\delta) \subset \Omega, \quad n = 0, \cdots, N, \\ q_N = x, \end{cases}$$

with $|q_n - q_{n+1}| \leq \delta$ for $n = 0, \cdots, N-1$ and such that $N\delta \leq d_\Omega(x_0, x) + \epsilon$.

Proof. We denote by g a continuous path joining x_0 to x and included in Ω such that $\ell(g) \leq d_\Omega(x_0, x) + \epsilon$. We hence have $g\ :\ [0,1] \to \Omega$ with $g(0) = x_0$ et $g(1) = x$. Let us define

$$\mu = \inf_{t \in [0,1]} d(g(t), \Omega^c) > 0.$$

We choose $\delta < \mu/3$, and we construct a δ-sequence of balls as follows. We define the sequence (α_j) by induction : $\alpha_0 = 0$ and while $\alpha_j < 1$,

$$\alpha_{j+1} = \sup\{\alpha \in [0,1], \quad |g(\alpha) - g(\alpha_j)| < \delta\}.$$

The sequence (α_j) is increasing. Let us prove that for sufficiently large j, $\alpha_j = 1$. We remark that if $\alpha_{j+1} < 1$, then $|g(\alpha_{j+1}) - g(\alpha_j)| = \delta$. Indeed, by definition of α_{j+1}, $|g(\alpha_{j+1}) - g(\alpha_j)| \leq \delta$. On the other hand, for all $\alpha > \alpha_{j+1}$, $|g(\alpha) - g(\alpha_j)| \geq \delta$, and g being continuous, $|g(\alpha_{j+1}) - g(\alpha_j)| \geq \delta$.
Assume that $\alpha_j < 1$ for all j. Then the sequence (α_j) converges and $|g(\alpha_{j+1}) - g(\alpha_j)| \to 0$ when $j \to +\infty$, which contradicts the above remark. We hence have $\alpha_j = 1$ for $j \geq N$, and we have

$$\alpha_0 = 0 < \alpha_1 < \cdots < \alpha_{N-1} < \alpha_N = 1.$$

It remains to prove that if $q_j = g(\alpha_j)$ pour $j = 0, \cdots, N$, the $B(q_j, \delta)$ form a δ-sequence of balls joining x_0 to x.
Obviously $q_0 = g(\alpha_0) = x_0$ and $q_N = g(\alpha_N) = x$. Besides, if $y \in B(g(\alpha_{j+1}), \delta)$ for $j = 0, \cdots, N-1$, we have

$$|y - g(\alpha_j)| \leq |y - g(\alpha_{j+1})| + |g(\alpha_{j+1}) - g(\alpha_j)| < \delta + \delta = 2\delta,$$

hence $B(q_{j+1}, \delta) \subset B(q_j, 2\delta)$.
Lastly we have $B(q_j, 3\delta) \subset \Omega$ for $j = 0, \cdots, N$, since $d(g(\alpha_j), \Omega^c) \geq \mu > 3\delta$.
Now we have to prove that $\delta N \leq d_\Omega(x_0, x) + \epsilon$. We note that for $j = 0, \cdots, N-2$ the segment $[g(\alpha_j), g(\alpha_{j+1})]$ is included in Ω. We hence have

$$\sum_{j=0}^{N-2} |g(\alpha_{j+1}) - g(\alpha_j)| \ \leq \ell(g) \leq d_\Omega(x_0, x) + \epsilon,$$

that is, since $|g(\alpha_{j+1}) - g(\alpha_j)| = \delta$ for $j = 0, \cdots, N-2$, $\delta(N-1) \leq D + \epsilon$.
□

Let us now derive from Holmgren's theorem some unique continuation results of practical use for the Laplace, heat and wave equations.

3.3.2 Case of the Laplace and heat equations

Theorem 3.2. *Let ω and Ω be open domains in $\mathbb{R}^d$ such that $\omega \subset \Omega$. We assume that $\omega \neq \emptyset$ and that Ω is connected. Every $u \in \mathcal{D}'(\Omega)$ satisfying the equation $\Delta u = 0$ and vanishing in ω then vanishes in Ω.*

Proof. Let us consider some $x_0 \in \omega$ and $\delta_0 > 0$ such that $B(x_0, \delta_0) \subset \omega$. We consider now any $x \in \Omega$. From Lemma 3.1, there exists some $\delta \in (0, \delta_0)$ and a δ-sequence of balls joining x_0 to x. We apply N times Holmgren's theorem for operator $P = \Delta$ with $\mathcal{O}_1 = B(q_n, \delta)$ and $\mathcal{O}_2 = B(q_n, 2\delta)$, for $n = 1, \cdots, N-1$. Since u vanishes in $B(q_0, \delta)$, it vanishes in $B(q_0, 2\delta)$, and since $B(q_1, \delta) \subset B(q_0, 2\delta)$, it vanishes in $B(q_1, \delta)$. We conclude that u vanishes in $B(q_N, \delta)$, that is in $B(x, \delta)$. □

Remark 3.1. It is clear that Theorem 3.2 still holds if the Laplacian is perturbed by a term of lower order, that is for operator $P = \Delta + A \cdot \nabla + b$, where $A \in \mathbb{R}^d$ and $b \in \mathbb{R}$ are constants.

Theorem 3.3. *Let ω and Ω be open domains in $\mathbb{R}^d$ such that $\omega \subset \Omega$. We assume that $\omega \neq \emptyset$ and that Ω is connected. Let us consider some $T > 0$. Every $u \in \mathcal{D}'(\Omega \times (0,T))$ satisfying the equation $\partial_t u - \Delta u = 0$ and vanishing in $\omega \times (0,T)$ then vanishes in $\Omega \times (0,T)$.*

Proof. The proof begins like the one of Theorem 3.2. We introduce some $x_0 \in \omega$ and $\delta_0 > 0$ such that $B(x_0, \delta_0) \subset \omega$, some $x \in \Omega$ as well as a δ-sequence of balls joining x_0 to x. We apply N times Holmgren's theorem for operator $P = \partial_t - \Delta$ with $\mathcal{O}_1 = B(q_n, \delta) \times (0,T)$ and $\mathcal{O}_2 = B(q_n, 2\delta) \times (0,T)$, for $n = 1, \cdots, N-1$. It is clear that any plane in $\mathbb{R}^{d+1}$ of equation $t = c$ (c is a constant) which intersects $B(q_n, 2\delta) \times (0,T)$ also intersects $B(q_n, \delta) \times (0,T)$. Since u vanishes in $B(q_0, \delta) \times (0,T)$, it vanishes in $B(q_0, 2\delta) \times (0,T)$, and since $B(q_1, \delta) \subset B(q_0, 2\delta)$, it vanishes in $B(q_1, \delta) \times (0,T)$. We conclude that u vanishes in $B(q_N, \delta) \times (0,T)$, that is in $B(x, \delta) \times (0,T)$. $\square$

3.3.3 Case of the wave equation

Considering the wave equation, we first give the following local result Lattès & Lions (1967).

Lemma 3.2. *Let us consider $\delta, \tau > 0$ and $x_1, x_2 \in \mathbb{R}^d$ such that $\tau > |x_1 - x_2|$. Let us define the open and convex domains*

$$\mathcal{O}_1 = B(x_1, \delta) \times (-\tau, \tau)$$

and

$$\mathcal{O}_2 = \bigcup_{\lambda \in [0,1]} B((1-\lambda)x_1 + \lambda x_2, \delta) \times (-\tau + \lambda|x_1 - x_2|, \tau - \lambda|x_1 - x_2|).$$

If $u \in \mathcal{D}'(\mathcal{O}_2)$ satisfies $\partial_t^2 u - \Delta u = 0$ in $\mathcal{O}_2$ and $u = 0$ in $\mathcal{O}_1$, then $u = 0$ in $\mathcal{O}_2$.

Proof. The proof relies on the Homlgren's theorem for the operator $P = \partial_t^2 - \Delta$ with domains $\mathcal{O}_1 \subset \mathcal{O}_2$. Let us consider some plane π of normal $N \neq 0$ which is characteristic with respect to P, that is $N_{d+1} = \pm\sqrt{N_1^2 + \cdots + N_d^2}$. We observe that if π intersects $\mathcal{O}_2$, then it intersects $\mathcal{O}_1$. $\square$

Proposition 3.1. *Let Ω be an open and connected domain of $\mathbb{R}^d$. Let $u \in \mathcal{D}'(\Omega \times (-T,T))$ satisfy $\partial_t^2 u - \Delta u = 0$. Let us consider $x_0, x \in \Omega$ such that $T > d_\Omega(x_0, x)$. Let $\delta_0 > 0$ such that $u = 0$ in $B(x_0, \delta_0) \times (-T,T)$. For all $\epsilon \in (-T + d_\Omega(x_0,x), T - d_\Omega(x_0,x))$ there exists some $\delta \in (0, \delta_0)$ such that $u = 0$ in $B(x,\delta) \times (-T + d_\Omega(x_0,x) + \epsilon, T - d_\Omega(x_0,x) - \epsilon)$.*

Proof. Let us take δ and a sequence of balls depending on ϵ and δ_0 as in the lemma 3.1. Now we apply N times Lemma 3.2 with

$$\mathcal{O}_1 = B(q_n, \delta) \times (-T + S_n, T - S_n)$$

and

$$\mathcal{O}_2 = \bigcup_{\lambda \in [0,1]} B((1-\lambda)q_n + \lambda q_{n+1}, \delta) \times (-T + S_n + \lambda |q_n - q_{n+1}|, T - S_n - \lambda |q_n - q_{n+1}|).$$

Here we have denoted

$$S_n = \sum_{j=0}^{n-1} |q_j - q_{j+1}|.$$

It is straightforward to check that $\mathcal{O}_2 \subset B(q_n, 2\delta) \times (-T, T) \subset \Omega \times (-T, T)$. Since $u = 0$ in $B(q_0, \delta) \times (-T, T)$, then $u = 0$ in $B(q_1, \delta) \times (-T + S_1, T - S_1)$, $\cdots$, then $u = 0$ in $B(q_N, \delta) \times (-T + S_N, T - S_N)$. We note that

$$T - S_N = T - \sum_{j=0}^{N-1} |q_j - q_{j+1}| \geq T - N\delta \geq T - d_\Omega(x_0, x) - \epsilon,$$

Hence $u = 0$ in $B(x, \delta) \times (-T + d_\Omega(x_0, x) + \epsilon, T - d_\Omega(x_0, x) - \epsilon)$, which completes the proof. $\square$

We now state our main continuation uniqueness result.

Theorem 3.4. *Let ω and Ω be open domains in $\mathbb{R}^d$ such that $\omega \subset \Omega$. We assume that $\omega \neq \emptyset$ and that Ω is connected. Let us consider some $T > 0$. Every $u \in \mathcal{D}'(\Omega \times (-T, T))$ satisfying the equation $\partial_t^2 u - \Delta u = 0$ and vanishing in $\omega \times (-T, T)$ then vanishes in*

$$Q_0 = \{(x, t) \in \Omega \times (-T, T),\ d_\Omega(x, \omega) < T - |t|\}.$$

Proof. Let us pick some $x \in \Omega$. We recall that $d_\Omega(x, \omega) = \inf_{x_0 \in \omega} d_\Omega(x, x_0)$. Let us take any $\epsilon \in (0, \epsilon_0(x))$ with $\epsilon_0(x) = (T - d_\Omega(x, \omega))/2$. One may find $x_0 \in \omega$ such that $d_\Omega(x, \omega) \geq d_\Omega(x, x_0) - \epsilon$. There exists some δ_0 such that $u = 0$ in $B(x_0, \delta_0) \times (-T, T)$. By using Proposition 3.1, we obtain that there exists some $\delta < \delta_0$ such that $u = 0$ in $B(x, \delta) \times (-T + d_\Omega(x, x_0) + \epsilon, T - d_\Omega(x, x_0) - \epsilon)$. As a conclusion, for any small ϵ (depending on x), there exists δ such that $u = 0$ in $B(x, \delta) \times (-T + d_\Omega(x, \omega) + 2\epsilon, T - d_\Omega(x, \omega) - 2\epsilon)$. Since ϵ is arbitrarily small, the conclusion follows. $\square$

For some applications, we will need uniqueness in a subdomain the spatial section of which is the whole Ω. For a connected open domain Ω and some $x_0 \in \Omega$, we define

$$D(\Omega, x_0) = \sup_{x \in \Omega} d_\Omega(x, x_0).$$

For $d \geq 2$ we may have $D(\Omega, x_0) = +\infty$, even if Ω is a bounded domain, as can be seen in the following example in $2D$.

Remark 3.2. For $n \in \mathbb{N}^*$, let us define the segments S_n and S'_n by

$$S_n = \left\{ (x,y) \in \mathbb{R}^2, \quad x = \frac{1}{n}, 0 \leq y \leq \frac{3}{2} \right\}$$

and

$$S'_n = \left\{ (x,y) \in \mathbb{R}^2, \quad x = \frac{1}{2}\left(\frac{1}{n} + \frac{1}{n+1}\right), \frac{1}{2} \leq y \leq 2 \right\}.$$

It is readily seen that the open domain

$$\Omega = (0,2) \times (0,2) \setminus \left(\bigcup_{n=1}^{+\infty} S_n \cup \bigcup_{n=1}^{+\infty} S'_n \right)$$

is connected, bounded, and satisfies $D(\Omega, x_0) = +\infty$ for any $x_0 \in \Omega$.

However we have $D(\Omega, x_0) < +\infty$ as soon as Ω is a bounded Lipschitz domain, which is a consequence of the following theorem Oudot et al. (2010), since $\overline{\Omega}$ is a compact set.

Theorem 3.5. *If $\Omega \subset \mathbb{R}^d$ is a connected, bounded, open domain of class $C^{0,1}$, then for all $x_0 \in \Omega$, the function $x \in \Omega \to d_\Omega(x_0, x)$ can be extended to a continuous function in $\overline{\Omega}$ (for the euclidian norm).*

Proof. First, let us prove that $x \in \Omega \to d_\Omega(x_0, x)$ is continuous. We hence consider $\epsilon > 0$ and $x \in \Omega$. There exists $\delta < \epsilon$ such that $B(x,\delta) \subset \Omega$, where $B(x,\delta)$ is associated to the euclidian norm. For all $\tilde{x} \in B(x,\delta)$, $d_\Omega(x,\tilde{x}) = |x - \tilde{x}| < \delta$ and by the triangle inequality $|d_\Omega(x_0,\tilde{x}) - d_\Omega(x_0,x)| \leq d_\Omega(x,\tilde{x}) < \delta < \epsilon$, which proves the continuity at point $x \in \Omega$.
Now we consider $x \in \partial\Omega$. From the definition **??** of a domain of class $C^{0,1}$, there exists a neighborhood V of x in $\mathbb{R}^d$ and a Lipschitz homeomorphism $f : \mathbb{R}^d \to \mathbb{R}^d$ such that $f^{-1}(0) = x$, $f^{-1}(\mathbb{R}^{d-1} \times \{0\}) \cap V = \partial\Omega \cap V$ and $f^{-1}(\mathbb{R}^{d-1} \times \{x_d > 0\}) \cap V = \Omega \cap V$. It suffices to set, for $x' \in \mathbb{R}^{d-1}$ and $x_d \in \mathbb{R}$, $f(x', x_d) = (x', x_d - \varphi(x'))$. Let us denote by L the Lipschitz constant of φ, which is also the Lipschitz constant of f and of f^{-1}. Without loss of generality, V may be chosen such that $f(\Omega \cap V)$ is the intersection of $\mathbb{R}^d_+$ and of the open ball $B(0, \epsilon/L)$. Then for all $\tilde{x} \in \overline{\Omega} \cap V$, we consider the path $g : s \in [0,1] \to f^{-1}(s\tilde{y})$ with $\tilde{y} = f(\tilde{x})$. Since $f(\overline{\Omega} \cap V)$ is convex, $g([0,1])$ is included in $\overline{\Omega} \cap V$, that is in $\overline{\Omega}$, and joins x to $\tilde{x}$. This enables us to define $d_\Omega(x_0, x)$ for $x \in \partial\Omega$. It remains to prove continuity of $d_\Omega(x_0, x)$ at x. Since f^{-1} is $L-$Lipschitz and $|\tilde{y}| < \epsilon/L$,

$$d_\Omega(x,\tilde{x}) \leq |g| = \int_0^1 |g'(s)|\,ds = \int_0^1 |\tilde{y} \cdot \nabla f^{-1}(s\tilde{y})|\,ds \leq L|\tilde{y}| < \epsilon.$$

Hence $|d_\Omega(x_0,\tilde{x}) - d_\Omega(x_0,x)| \leq d_\Omega(x,\tilde{x}) < \epsilon$, which completes the proof. $\square$

We also define, for $\omega \subset \Omega$, with $\omega \neq \emptyset$,

$$D(\Omega, \omega) = \sup_{x \in \Omega} \inf_{x_0 \in \omega} d_\Omega(x, x_0) = \sup_{x \in \Omega} d_\Omega(x, \omega).$$

It is clear that if Ω is a bounded Lipschitz domain, this quantity if finite. Indeed,

$$D(\Omega, \omega) \leq \inf_{x_0 \in \omega} \sup_{x \in \Omega} d_\Omega(x, x_0) = \inf_{x_0 \in \omega} D(\Omega, x_0) < +\infty.$$

From Theorem 3.4, we obtain the following straightforward consequence.

Corollary 3.1. *Let ω and Ω be open domains in $\mathbb{R}^d$ such that $\omega \subset \Omega$. We assume that $\omega \neq \emptyset$ and that Ω is connected and Lipschitz. Let us consider some T such that $T > T_0 := D(\Omega, \omega)$. Every $u \in \mathcal{D}'(\Omega \times (-T, T))$ satisfying the equation $\partial_t^2 u - \Delta u = 0$ and vanishing in $\omega \times (-T, T)$ then vanishes in $\Omega \times (-T + T_0, T - T_0)$.*

3.3.4 Case of boundary data

Now, let us consider some similar results when the function is not supposed to vanish in a subset $\omega \subset \Omega$ but the Cauchy data $u|_\Gamma$ and $\partial_\nu u|_\Gamma$ are supposed to vanish on a subset Γ of the boundary $\partial\Omega$. Let us begin with the Laplace operator.

Theorem 3.6. *Assume that $\Omega \subset \mathbb{R}^d$ is a bounded open and connected domain of class $C^{0,1}$, and Γ a non-empty open subpart of $\partial\Omega$. If $u \in H^1(\Omega)$ satisfies*

$$\begin{cases} \Delta u = 0, & \text{in } \Omega \\ u = 0, & \text{in } \Gamma \\ \partial_\nu u = 0, & \text{in } \Gamma \end{cases} \tag{3.1}$$

then $u = 0$ in Ω.

Proof. For some $x_0 \in \Gamma$ and ϵ with $B(x_0, \epsilon) \cap \partial\Omega \subset \Gamma$ (such open ball exists since Γ is open and non-empty), let us consider the open domain $\tilde{\Omega} = \Omega \cup B(x_0, \epsilon)$. Let us define $\tilde{u}$ as the extension of u by 0 in $\tilde{\Omega}$. Since $u|_\Gamma = 0$, $\tilde{u} \in H^1(\tilde{\Omega})$. Now let us prove that $\Delta\tilde{u} = 0$ in $\tilde{\Omega}$ in the sense of distributions. For some test function $\varphi \in C_0^\infty(\tilde{\Omega})$, we have

$$\langle \Delta\tilde{u}, \varphi \rangle = \langle \tilde{u}, \Delta\varphi \rangle = \int_{\tilde{\Omega}} \tilde{u} \Delta\varphi \, dx = \int_\Omega u \Delta\varphi \, dx.$$

By using integration by parts in Ω, we obtain

$$\langle \Delta\tilde{u}, \varphi \rangle = \int_\Omega \Delta u \varphi \, dx + \left\langle u, \frac{\partial\varphi}{\partial\nu} \right\rangle_{\partial\Omega} - \left\langle \frac{\partial u}{\partial\nu}, \varphi \right\rangle_{\partial\Omega},$$

where the brackets stand for duality pairing between $H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$. In the right-hand side, with use the fact that $\Delta u = 0$ in Ω, $u|_\Gamma = 0$ and $\partial_\nu u|_\Gamma = 0$, and φ vanishes in a volume vicinity of $\partial\Omega \setminus \Gamma$. We hence have $\Delta\tilde{u} = 0$ in $\tilde{\Omega}$.
Considering some non-empty open domain $\omega \Subset \tilde{\Omega} \setminus \overline{\Omega}$, we have that $\tilde{u} = 0$ in ω and applying theorem 3.2 to $\tilde{u}$ in domain $\tilde{\Omega}$ implies $\tilde{u} = 0$ in $\tilde{\Omega}$, that is $u = 0$ in Ω. $\square$

In the same vein, we could easily prove the following results for the heat and the wave equations.

Theorem 3.7. *Assume that $\Omega \subset \mathbb{R}^d$ is a bounded open connected domain of class $C^{0,1}$, and Γ a non-empty open subpart of $\partial\Omega$. If $u \in L^2(0,T;H^1(\Omega))$ satisfies*

$$\begin{cases} \partial_t u - \Delta u = 0, & \text{in } \Omega \times (0,T) \\ u = 0, & \text{on } \Gamma \times (0,T) \\ \partial_\nu u = 0, & \text{on } \Gamma \times (0,T) \end{cases} \tag{3.2}$$

then $u = 0$ in $\Omega \times (0,T)$.

Theorem 3.8. *Assume that $\Omega \subset \mathbb{R}^d$ is a bounded open connected domain of class $C^{0,1}$, and Γ a non-empty open subpart of $\partial\Omega$. If $u \in L^2(-T,T;H^1(\Omega)) \cap H^1(-T,T;L^2(\Omega))$ satisfies*

$$\begin{cases} \partial_t^2 u - \Delta u = 0, & \text{in } \Omega \times (-T,T) \\ u = 0, & \text{on } \Gamma \times (-T,T) \\ \partial_\nu u = 0, & \text{on } \Gamma \times (-T,T) \end{cases} \tag{3.3}$$

then $u = 0$ in

$$Q_0 = \{(x,t) \in \Omega \times (-T,T),\ d_\Omega(x,\Gamma) < T - |t|\}.$$

In particular, if $T > T_0 := D(\Omega,\Gamma)$, then $u = 0$ in $\Omega \times (-T+T_0, T-T_0)$.

Chapter 4

Data completion problems – The Laplace equation case

4.1 Introduction

In this chapter we apply the Tikhonov regularization to data completion problems, which are viewed as linear ill-posed problems governed by partial differential equations. As we will see, different ways of choosing the non invertible operator A in chapter 2 will lead to different methods. In order to illustrate those different methods, we will apply them on the same example presented in chapter **??**, which is the Cauchy problem for the Laplace equation (1.3).

4.2 First method with regularity assumptions

Let Ω be a bounded, connected open domain of class $C^{1,1}$ in $\mathbb{R}^d$ (see definition **??**), $d > 1$, and Γ a non-empty open subpart of $\partial\Omega$. We consider the following problem: for a pair of data $(g_0, g_1) \in H^{\frac{3}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)$, find $u \in H^2(\Omega)$ such that

$$\begin{cases} \Delta u = 0, & \text{in } \Omega \\ u = g_0, & \text{on } \Gamma \\ \partial_\nu u = g_1, & \text{on } \Gamma, \end{cases} \tag{4.1}$$

where ν is the outward unit normal $\partial\Omega$. Such problem coincides with Problem (1.3), the only difference is that the regularity order of the domain and the function is one level up.

Thanks to this higher regularity, let us remark that up to a slight modification, Problem (1.3) is a particular instance of the abstract ill-posed problem described in Chapter 2. Indeed, if U is a function in $H^2(\Omega)$ such that $U|_\Gamma = g_0$ and $\partial_\nu U|_\Gamma = g_1$ – note that such function exits following Lions & Magenes (1968), and is of course not unique – and if we define $y = -\Delta U \in L^2(\Omega)$, the change of variable $z = u - U \in H^2(\Omega)$ implies that solving Problem 4.1 is equivalent to solve

$$\begin{cases} \Delta z = y, & \text{in } \Omega \\ z = 0, & \text{on } \Gamma \\ \partial_\nu z = 0, & \text{on } \Gamma \end{cases} \tag{4.2}$$

And Problem (4.2) amounts, for $y \in \mathcal{Y}$, to find $z \in \mathcal{Z}$ such that $\Psi z = y$, with

$$\left| \begin{array}{l} \mathcal{Z} = \{v \in H^2(\Omega),\ v|_\Gamma = 0,\ \partial_\nu v|_\Gamma = 0\} \\ \mathcal{Y} = L^2(\Omega) \\ \Psi\ :\ z \mapsto \Delta z. \end{array} \right. \tag{4.3}$$

First of all, we note that Spaces $\mathcal{Z}$ and $\mathcal{Y}$ are Hilbert spaces, while Ψ is a continuous operator. Moreover, we already know that Ψ is injective – see Theorem 3.6 – and is not onto since the Cauchy problem for the Laplace equation is ill-posed. Let us prove that Ψ has a dense range. In this respect, we need the following uniqueness result for $L^2(\Omega)$ functions.

Theorem 4.1. *Assume that $\Omega \subset \mathbb{R}^d$ is a bounded open connected domain of class $C^{1,1}$, and Γ a non-empty open subpart of $\partial\Omega$. If $u \in L^2(\Omega)$ satisfies problem (4.1) with $(g_0, g_1) = (0, 0)$, then $u = 0$ in Ω.*

Before giving the proof of the two above uniqueness result, let us recall the following integration by parts formula Lions & Magenes (1968):
For a $C^{1,1}$ domain Ω, for any $u \in H^0(\Omega, \Delta) \coloneqq \{u \in L^2(\Omega),\ \Delta u \in L^2(\Omega)\}$ and $v \in H^2(\Omega)$, we have

$$(\Delta u, v)_{L^2(\Omega)} - (u, \Delta v)_{L^2(\Omega)} = \langle \frac{\partial u}{\partial \nu}, v \rangle_{\partial\Omega} - \langle u, \frac{\partial v}{\partial \nu} \rangle_{\partial\Omega}, \tag{4.4}$$

where the first bracket has the meaning of duality between $H^{-\frac{3}{2}}(\partial\Omega)$ and $H^{\frac{3}{2}}(\partial\Omega)$ while the second one has the meaning of duality between $H^{-\frac{1}{2}}(\partial\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$.

Proof (Proof of theorem 4.1 :). The proof is based on theorem 3.6 and on a regularity argument. We consider a fonction $\alpha \in C^\infty(\overline{\Omega})$ that vanishes in a (volume) vicinity of $\partial\Omega \setminus \Gamma$. Let us prove that $\alpha u \in H^2(\Omega)$. We will use the following statement that for all $s \in \mathbb{R}$, if $u \in H^s(\mathbb{R}^d)$ and $\Delta u \in H^s(\mathbb{R}^d)$, then $u \in H^{s+2}(\mathbb{R}^d)$, which immediately follows from Fourier analysis. This result will be denoted (R_s) in the following proof.
Again let us denote by $\tilde{u}$ the trivial extension of u in $\mathbb{R}^d$ and $\tilde{\alpha}$ an extension in $C_0^\infty(\mathbb{R}^d)$ of α. We have in $\mathbb{R}^d$

$$\Delta(\tilde{\alpha}\tilde{u}) = \tilde{u}\,\Delta\tilde{\alpha} + 2\sum_{i=1}^{d} \frac{\partial\tilde{\alpha}}{\partial x_i}\frac{\partial\tilde{u}}{\partial x_i} + \tilde{\alpha}\,\Delta\tilde{u}. \tag{4.5}$$

It is clear that $\tilde{u}\,\Delta\tilde{\alpha} \in L^2(\mathbb{R}^d)$. If $\partial\tilde{\alpha}/\partial x_i$ is denoted by $\tilde{\beta}_i$, we have

$$\tilde{\beta}_i \frac{\partial \tilde{u}}{\partial x_i} = \frac{\partial(\tilde{\beta}_i \tilde{u})}{\partial x_i} - \frac{\partial \tilde{\beta}_i}{\partial x_i} \tilde{u}, \tag{4.6}$$

which implies that $\tilde{\beta}_i \partial\tilde{u}/\partial x_i \in H^{-1}(\mathbb{R}^d)$. Now, let us prove that $\tilde{\alpha}\,\Delta\tilde{u} = 0$ in $\mathbb{R}^d$. For any test function $\varphi \in C_0^\infty(\mathbb{R}^d)$, we have, in the sense of distributions in $\mathbb{R}^d$,

$$\langle \tilde{\alpha}\Delta\tilde{u}, \varphi\rangle = \langle \tilde{u}, \Delta(\tilde{\alpha}\varphi)\rangle = \int_\Omega u\Delta(\alpha\varphi)\,dx.$$

Since $u \in H^0(\Omega, \Delta)$ and $\alpha\varphi$ is a smooth function, we can perform the above integration by parts (4.4).

$$\int_\Omega u\Delta(\alpha\varphi)\,dx = \langle u, \frac{\partial(\alpha\varphi)}{\partial\nu}\rangle_{\partial\Omega} - \langle \frac{\partial u}{\partial \nu}, \alpha\varphi\rangle_{\partial\Omega} + \int_\Omega \Delta u(\alpha\varphi)\,dx.$$

Since $u|_\Gamma = 0$, $\partial_\nu u|_\Gamma = 0$, $(\alpha\varphi)$ vanishes near $\partial\Omega \setminus \Gamma$ and $\Delta u = 0$ in Ω, we have

$$\langle \tilde{\alpha}\Delta\tilde{u}, \varphi\rangle = 0.$$

It follows that $\tilde{\alpha}\Delta\tilde{u} = 0$ in $\mathbb{R}^d$ and from (4.5) that $\Delta(\tilde{\alpha}\tilde{u}) \in H^{-1}(\mathbb{R}^d)$. From the statement (R_s) for $s = -1$, it follows that $\tilde{\alpha}\tilde{u} \in H^1(\mathbb{R}^d)$, for any $\tilde{\alpha} \in C_0^\infty(\mathbb{R}^d)$ that vanishes near $\partial\Omega \setminus \Gamma$ in Ω. This result implies that we have now $\tilde{\beta}_i \partial\tilde{u}/\partial x_i \in L^2(\mathbb{R}^d)$ from (4.6), and hence that $\Delta(\tilde{\alpha}\tilde{u}) \in L^2(\mathbb{R}^d)$ from (4.5). By using once again (R_s) for $s = 0$, it follows that $\tilde{\alpha}\tilde{u} \in H^2(\mathbb{R}^d)$, and hence $\alpha u \in H^2(\Omega)$.
It remains to apply Theorem 3.6 and we obtain that u vanishes any compact $K \subset \Omega$. Hence $u = 0$ in Ω. □

Theorem 4.2. *The operator A defined by (4.3) has a dense range.*

Proof. Assume that for some $f \in L^2(\Omega)$,

$$(\Delta u, f)_{L^2(\Omega)} = 0, \quad \forall u \in V = \{v \in H^2(\Omega),\ v|_\Gamma = 0,\ \partial_\nu v|_\Gamma = 0\}.$$

By choosing $u = \varphi \in C_0^\infty(\Omega)$, we obtain that $\Delta f = 0$, in particular $f \in H^0(\Omega, \Delta)$. For all $u \in V$, we can hence use the integration by parts formula (4.4)

$$(\Delta u, f)_{L^2(\Omega)} - (u, \Delta f)_{L^2(\Omega)} = \langle \frac{\partial u}{\partial\nu}, f\rangle_{\partial\Omega} - \langle u, \frac{\partial f}{\partial\nu}\rangle_{\partial\Omega},$$

This is for all $u \in V$

$$\langle \frac{\partial u}{\partial\nu}, f\rangle_{\partial\Omega} - \langle u, \frac{\partial f}{\partial\nu}\rangle_{\partial\Omega} = 0,$$

that is, since $u|_\Gamma = 0$ and $\partial_\nu u|_\Gamma = 0$,

$$\langle \frac{\partial u}{\partial\nu}, f\rangle_{\tilde{\Gamma}} - \langle u, \frac{\partial f}{\partial\nu}\rangle_{\tilde{\Gamma}} = 0,$$

where $\tilde{\Gamma} = \partial\Omega \setminus \overline{\Gamma}$, the first bracket has the meaning of duality pairing between $\tilde{H}^{\frac{1}{2}}(\tilde{\Gamma})$ and $H^{-\frac{1}{2}}(\tilde{\Gamma})$ while the second bracket has the meaning of duality pairing between $\tilde{H}^{\frac{3}{2}}(\tilde{\Gamma})$ and $H^{-\frac{3}{2}}(\tilde{\Gamma})$. Since the operator $V \to \tilde{H}^{\frac{3}{2}}(\tilde{\Gamma}) \times \tilde{H}^{\frac{1}{2}}(\tilde{\Gamma})$ such that $u \in V \mapsto (u|_{\tilde{\Gamma}}, \partial_\nu u|_{\tilde{\Gamma}})$ is onto, we obtain that $f|_{\tilde{\Gamma}} = 0$ and $\partial_\nu f|_{\tilde{\Gamma}} = 0$. We conclude that function $f \in L^2(\Omega)$ solves the problem (3.1) with $\tilde{\Gamma}$ playing the role of Γ and we conclude from theorem 4.1 that $f = 0$, wich proves that A has dense range. □

We hence conclude that we are exactly in the situation of chapter 2, which means that we can apply the Tikhonov regularization to operator A as well as the Morozov principle. In view of (**??**) the Tikhonov regularization amounts, for $\epsilon > 0$, to find $v_\epsilon \in V = \{v \in H^2(\Omega),\ v|_\Gamma = 0,\ \partial_\nu v|_\Gamma = 0\}$ such that

$$(\Delta v_\epsilon, \Delta v)_{L^2(\Omega)} + \epsilon(v_\epsilon^\delta, v)_{H^2(\Omega)} = (f, \Delta v)_{L^2(\Omega)}, \quad \forall v \in V. \tag{4.7}$$

Such formulation is a particular form of the method of quasi-reversibility introduced in Lattès & Lions (1967). It consists in transforming the initial second-order ill-posed problem into a family, depending on the small parameter $\epsilon > 0$, of fourth-order well-posed problems. We remark that (4.7) is a weak formulation in the terminology of elliptic partial differential equations, and can therefore be directly discretized with the help of a finite element method. We have proved in Theorem 2.1 that in the case of exact data ($\delta = 0$) ,

$$\lim_{\epsilon \to 0} \|(v_\epsilon + U) - u\|_{H^2(\Omega)} = 0$$

and in the presence of noisy data, for some constant $c > 0$, we have

$$\|(v_\epsilon^\delta + U^\delta) - u)\|_{H^2(\Omega)} \leq \|(v_\epsilon + U) - u\|_{H^2(\Omega)} + c\,\frac{\delta}{\sqrt{\epsilon}}.$$

Of course since we are exactly in the framework of chapter 2, we can choose $\epsilon > 0$ as a function of δ following the Morozov's principle.

Note however that the practical computation of f^δ from noisy Cauchy data (g_0^δ, g_1^δ) which is required in the homogeneous formulation (4.7) may be not easy and an alternative formulation of quasi-reversibility may be the non-homogeneous one : for some $\epsilon > 0$, find $u_\epsilon \in V_g$, with

$$V_g = \{v \in H^2(\Omega),\ v|_\Gamma = g_0,\ \partial_\nu v|_\Gamma = g_1\},$$

such that

$$(\Delta u_\epsilon, \Delta v)_{L^2(\Omega)} + \epsilon(u_\epsilon, v)_{H^2(\Omega)} = 0, \quad \forall v \in V. \tag{4.8}$$

That problem (4.8) is well-posed is a simple consequence of the Lax-Milgram's theorem and the use of extension U. Following the same lines as in the proof of theorem 2.1, we would easily prove that in the case of exact data ($\delta = 0$)

$$\lim_{\epsilon \to 0} \|u_\epsilon - u\|_{H^2(\Omega)} = 0,$$

and in the presence of noisy data, for some constant $c > 0$ we have

$$\|u_\epsilon^\delta - u\|_{H^2(\Omega)} \leq \|u_\epsilon - u\|_{H^2(\Omega)} + c\,\frac{\delta}{\sqrt{\epsilon}}.$$

Remark 4.1. It may be easily verified that the formulations (4.7) and (4.8) are not equivalent, in the sense that $v_\epsilon^\delta + U^\delta$ does not coincide with u_ϵ^δ.

The formulations (4.7) and (4.8) have two main drawbacks.

1. First, those formulations require the domain and the exact solution u to be rather smooth. A more usual framework concerning regularity is a domain of class $C^{0,1}$ instead of $C^{1,1}$ and a solution in $H^1(\Omega)$ instead of $H^2(\Omega)$.
2. Secondly those formulations correspond to fourth-order problems. From the finite element point of view, we hence have to use some complicated finite elements (Hermite finite elements) instead of simple ones (Lagrange finite elements).

This is why we now present a mixed version of the Tikhonov regularization which avoids these two drawbacks.

4.3 A mixed version of Tikhonov regularization

4.3.1 A general variational setting

Let us consider three Hilbert spaces V, M and H, as well as a continuous onto operator $C : V \to H$ and the corresponding affine space $V_g = \{u \in V,\ Cu = g\}$ for $g \in H$. For a continuous bilinear form b on $V \times M$ and a continuous linear form ℓ on M, let us consider the weak formulation: find $u \in V_g$ such that for all $\mu \in M$,

$$b(u, \mu) = \ell(\mu). \tag{4.9}$$

The bilinear form b is said to satisfy the $\inf - \sup$ property on $V_0 \times M$ if

Assumption 1 *There exists $\alpha > 0$ such that*

$$\inf_{\substack{u\in V_0\\ u\neq 0}} \sup_{\substack{\mu\in M\\ \mu\neq 0}} \frac{b(u,\mu)}{||u||_V||\mu||_M} \geq \alpha.$$

The bilinear form b is said to satisfy the solvability property on $V_0 \times M$ if

Assumption 2 *For all $\mu \in M$,*

$$\forall u \in V_0, \quad b(u, \mu) = 0 \Longrightarrow \mu = 0.$$

Lastly, b is said to satisfy the uniqueness property on $V_0 \times M$ if

Assumption 3 *For all $u \in V_0$,*

$$\forall \mu \in M, \quad b(u,\mu) = 0 \Longrightarrow u = 0.$$

From the Brezzi-Nečas-Babuška theorem (see for example Ern & Guermond (2004)), we know that problem (4.9) is well-posed if and only if both conditions 1 and 2 are satisfied. Clearly, assumption 1 implies assumption 3, but the converse implication is false. In what follows, and to retrieve the framework of chapter 2, it is assumed that the bilinear form b does not satisfy the $\inf-\sup$ condition 1, which from the Brezzi-Nečas-Babuška theorem implies that the problem (4.9) for a given ℓ is in general ill-posed. We assume however that the weaker property (3) is satisfied, which means that problem (4.9) has at most one solution.

4.3.2 The mixed-type Tikhonov regularization

A regularized formulation of ill-posed problem (4.9) is the following: for $\epsilon > 0$, find $(u_\epsilon, \lambda_\epsilon) \in V_g \times M$ such that for all $(v,\mu) \in V_0 \times M$,

$$\begin{cases} \epsilon(u_\epsilon, v)_V + b(v,\lambda_\epsilon) = 0 \\ b(u_\epsilon,\mu) - (\lambda_\epsilon,\mu)_M = \ell(\mu). \end{cases} \tag{4.10}$$

We have the following theorem.

Theorem 4.3. *For any $f \in H$ and $\ell \in M'$, the problem (4.10) has a unique solution. For some $g \in H$ and $\ell \in M'$ such that (4.9) has a (unique) solution u, then the solution $(u_\epsilon, \lambda_\epsilon) \in V_g \times M$ satisfies $(u_\epsilon, \lambda_\epsilon) \to (u, 0)$ in $V \times M$ when $\epsilon \to 0$.*

Proof. Let us introduce some $U \in V$ such that $CU = g$, which exists since C is onto, and let us set $\hat{u}_\epsilon = u_\epsilon - U$, so that problem (4.10) is equivalent to: find $(\hat{u}_\epsilon, \lambda_\epsilon) \in V_0 \times M$ such that for all $(v,\mu) \in V_0 \times M$,

$$\begin{cases} \epsilon(\hat{u}_\epsilon, v)_V + b(v,\lambda_\epsilon) = -\epsilon(U,v)_V \\ b(\hat{u}_\epsilon,\mu) - (\lambda_\epsilon,\mu)_M = \ell(\mu) - b(U,\mu), \end{cases} \tag{4.11}$$

which is itself equivalent to: find $(\hat{u}_\epsilon, \lambda_\epsilon) \in V_0 \times M$ such that for all $(v,\mu) \in V_0 \times M$,

$$B_\epsilon((\hat{u}_\epsilon, \lambda_\epsilon);(v,\mu)) = L_\epsilon((v,\mu)),$$

where the bilinear form B_ϵ and the linear form L_ϵ are given on $V_0 \times M$ by

$$B_\epsilon((u,\lambda);(v,\mu)) = \epsilon(u,v)_V + b(v,\lambda) - b(u,\mu) + (\lambda,\mu)_M$$

and

$$L_\epsilon((v,\mu)) = -\epsilon(U,v)_V - \ell(\mu) + b(U,\mu).$$

Since for $(u,\lambda) \in V_0 \times M$,

$$B_\epsilon((u,\lambda);(u,\lambda)) \geq \epsilon||u||_V^2 + ||\lambda||_M^2,$$

B_ϵ is coercive on $V_0 \times M$, which implies from the Lax-Milgram lemma that the problem (4.10) is well-posed for all $\epsilon > 0$.

Now let us assume that $g \in H$ and $\ell \in M'$ are such that (4.9) has a (unique) solution u. By subtracting (4.9) to the second equation of (4.10), we obtain that for all $(v,\mu) \in V_0 \times M$,

$$\begin{cases} \epsilon(u_\epsilon,v)_V + b(v,\lambda_\epsilon) = 0 \\ b(u_\epsilon - u,\mu) - (\lambda_\epsilon,\mu)_M = 0. \end{cases}$$

Choosing $v = u_\epsilon - u \in V_0$, $\mu = \lambda_\epsilon \in M$ and subtracting the two obtained equations we end up with

$$\epsilon(u_\epsilon, u_\epsilon - u)_V + ||\lambda_\epsilon||_M^2 = 0.$$

This in particular implies that

$$||u_\epsilon||_V \leq ||u||_V,\ ||\lambda_\epsilon||_M \leq \sqrt{\epsilon}||u||_V.$$

The second inequality directly implies that $\lambda_\epsilon \to 0$ in M when $\epsilon \to 0$. From the first inequality, there exists some subsequence of u_ϵ, still denoted u_ϵ, such that $u_\epsilon \rightharpoonup w$ in V for some $w \in V$. Since the affine set V_g is convex and closed, it is weakly closed, that is $w \in V_g$. Moreover, by passing to the limit in the second equation of (4.10), we obtain that for all $\mu \in M$, $b(w,\mu) = \ell(\mu)$. Since problem (4.9) has a unique solution, we conclude that $w = u$. We lastly remark that

$$||u_\epsilon - u||_V^2 \leq -(u, u_\epsilon - u)_V,$$

so that weak convergence in V implies strong convergence in V. By a standard contradiction argument, all the sequence u_ϵ (not only a subsequence), converges to u in V. $\square$

Now let us show the link between our regularized formulation (4.10) and the standard Tikhonov regularization. More precisely, we can interpret (4.10) as a mixed formulation, in the sense of Brezzi-Fortin **?** for instance, of the Tikhonov regularization. Indeed, by the Riesz theorem, there exists a unique continuous operator $B : V \longrightarrow M$ and a unique $L \in M$ such that for all $u \in V$ and all $\mu \in M$,

$$(Bu,\mu)_M = b(u,\mu), \tag{4.12}$$

and

$$(L,\mu)_M = \ell(\mu). \tag{4.13}$$

Hence problem (4.9) is equivalent to find $u \in V_g$ such that $Bu = L$. The Tikhonov regularization of such ill-posed problem consists in solving, for $\epsilon > 0$, the well-posed minimization problem

$$\inf_{v\in V_g}\left(||Bv-L||_M^2 + \epsilon||v||_V^2\right). \tag{4.14}$$

The following proposition specifies the relationship between problems (4.10) and (4.14):

Proposition 4.1. *Let us denote by v_ϵ the unique solution to problem (4.14) and set $\mu_\epsilon = Bv_\epsilon - L$. Then $(v_\epsilon, \mu_\epsilon)$ coincides with the unique solution $(u_\epsilon, \lambda_\epsilon)$ to problem (4.10).*

Proof. Let us denote v_ϵ the solution to problem (4.14). Such solution is characterized by $v_\epsilon \in V_g$ and

$$(Bv_\epsilon - L, Bv)_M + \epsilon(v_\epsilon, v)_V = 0,\ \forall v \in V_0,$$

that is by setting $\mu_\epsilon = Bv_\epsilon - L \in M$, for all $(v,\mu) \in V_0 \times M$,

$$\begin{cases} \epsilon(v_\epsilon, v)_V + (Bv, \mu_\epsilon)_M = 0 \\ (Bv_\epsilon, \mu)_M - (\mu_\epsilon, \mu)_M = (L,\mu)_M, \end{cases}$$

that is $(v_\epsilon, \mu_\epsilon) \in V_g \times M$ solves problem (4.10) by using the definitions of B and L given by (4.12) and (4.13). We conclude that $(v_\epsilon, \mu_\epsilon) = (u_\epsilon, \lambda_\epsilon)$, which completes the proof. □

4.3.3 Application to the Cauchy problem for the Laplace equation

Let us consider a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d > 1$, the boundary $\partial\Omega$ of which is partitioned into two sets Γ and $\tilde{\Gamma}$. We come back to the Cauchy problem for the Laplace equation (1.3), that is: for some data $(g_0, g_1) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$, find $u \in H^1(\Omega)$ such that

$$\begin{cases} \Delta u = 0 \text{ in } \Omega \\ u = g_0 \text{ on } \Gamma \\ \partial_\nu u = g_1 \text{ on } \Gamma, \end{cases} \tag{4.15}$$

where ν is the outward unit normal to Ω. The problem (4.15) is equivalent to a weak formulation of type (4.9).

Lemma 4.1. *For* $(g_0, g_1) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$*, the function* $u \in H^1(\Omega)$ *is a solution to problem (4.15) if and only if* $u|_\Gamma = g_0$ *and for all* $\mu \in H^1(\Omega)$ *with* $\mu|_{\tilde{\Gamma}} = 0$*,*

$$\int_\Omega \nabla u \cdot \nabla \mu \;\, dx = \langle g_1, \mu|_\Gamma \rangle_{H^{-1/2}(\Gamma), \tilde{H}^{1/2}(\Gamma)}, \tag{4.16}$$

where the brackets stand for duality pairing between $H^{-1/2}(\Gamma)$ *and* $\tilde{H}^{1/2}(\Gamma)$*.*

Proof. First, let us assume that $u \in H^1(\Omega)$ and satisfies the weak formulation (4.16). We have $u = g_0$ on Γ and by first choosing $\mu = \varphi \in C_0^\infty(\Omega)$, we obtain $\Delta u = 0$ in Ω in the distributional sense. By using the classical Green formula, we have for all $\mu \in H^1(\Omega)$,

$$\int_\Omega \nabla u \cdot \nabla \mu \, dx = -\int_\Omega \Delta u \, \mu \, dx + \langle \partial_\nu u, \mu \rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)}.$$

If in addition $\mu|_{\tilde{\Gamma}} = 0$ and using the fact that $\Delta u = 0$ in Ω, we obtain that for all $\mu \in H^1(\Omega)$ with $\mu|_{\tilde{\Gamma}} = 0$,

$$\int_\Omega \nabla u \cdot \nabla \mu \;\, dx = \langle \partial_\nu u, \mu \rangle_{H^{-1/2}(\Gamma), \tilde{H}^{1/2}(\Gamma)},$$

and by comparison with (4.16) we obtain that for all $\mu \in H^1(\Omega)$ with $\mu|_{\tilde{\Gamma}} = 0$,

$$\langle \partial_\nu u, \mu \rangle_{H^{-1/2}(\Gamma), \tilde{H}^{1/2}(\Gamma)} = \langle g_1, \mu \rangle_{H^{-1/2}(\Gamma), \tilde{H}^{1/2}(\Gamma)},$$

which implies that $\partial_\nu u = g_1$ in $H^{-1/2}(\Gamma)$. We conclude that u satisfies (4.15). Conversely, we would prove the same way that if $u \in H^1(\Omega)$ satisfies (4.15), then it satisfies (4.16). $\square$

The weak formulation (4.16) is hence a particular instance of abstract problem (4.9) with $V = H^1(\Omega)$, $H = H^{1/2}(\Gamma)$, $M = \{\mu \in H^1(\Omega),\ \mu|_{\tilde{\Gamma}} = 0\}$, $C : H^1(\Omega) \to H^{1/2}(\Gamma)$ is the trace operator on Γ (which is onto), $V_{g_0} = \{u \in H^1(\Omega),\ u|_\Gamma = g_0\}$ while for $(u, \mu) \in V \times M$,

$$b(u, \mu) = \int_\Omega \nabla u \cdot \nabla \mu \, dx, \quad \ell(\mu) = \langle g_1, \mu|_\Gamma \rangle_{H^{-1/2}(\Gamma), \tilde{H}^{1/2}(\Gamma)}. \tag{4.17}$$

For this particular bilinear form b, only the two last conditions 2 and 3 are satisfied.

Proposition 4.2. *For the bilinear form* b *given by (4.17), the conditions 2 and 3 are satisfied while the condition 1 is not.*

Proof. We start by condition 2. For $\mu \in M = \{\mu \in H^1(\Omega),\ \mu|_{\tilde{\Gamma}} = 0\}$, let us assume that for all $u \in V_0 = \{u \in H^1(\Omega),\ u|_\Gamma = 0\}$,

$$\int_\Omega \nabla\mu \cdot \nabla u \, dx = 0.$$

Choosing $u = \varphi \in C_0^\infty(\Omega)$, we obtain that $\Delta\mu = 0$ in the distributional sense in Ω. The Green formula then gives that for all $u \in V_0$,

$$\langle \partial_\nu \mu, u \rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)} = \langle \partial_\nu \mu, u \rangle_{H^{-1/2}(\tilde{\Gamma}), \tilde{H}^{1/2}(\tilde{\Gamma})} = 0.$$

We conclude that $\mu \in H^1(\Omega)$ satisfies the homogeneous Cauchy problem

$$\begin{cases} \Delta\mu &= 0 \text{ in } \Omega \\ \mu &= 0 \text{ on } \tilde{\Gamma} \\ \partial_\nu \mu &= 0 \text{ on } \tilde{\Gamma}, \end{cases}$$

so that $\mu = 0$ by Theorem 3.6. Similarly, condition 3 amounts to prove that if u solves the Cauchy problem (4.15) with $(g_0, g_1) = (0, 0)$, then $u = 0$. Besides, we know that the problem (4.15) is ill-posed, which by contradiction proves from the Brezzi-Nečas-Babuška theorem that the $\inf - \sup$ condition 1 is not satisfied. □

The mixed formulation (4.10) of the Tikhonov regularization can be applied, that is: for $\epsilon > 0$, find $(u_\epsilon, \lambda_\epsilon) \in V_{g_0} \times M$ such that for all $(v, \mu) \in V_0 \times M$,

$$\begin{cases} \epsilon \int_\Omega \nabla u_\epsilon \cdot \nabla v \, dx + \int_\Omega \nabla v \cdot \nabla \lambda_\epsilon \, dx = 0 \\ \int_\Omega \nabla u_\epsilon \cdot \nabla \mu \, dx - \int_\Omega \nabla \lambda_\epsilon \cdot \nabla \mu \, dx = \langle g_1, \mu|_\Gamma \rangle_{H^{-1/2}(\Gamma), \tilde{H}^{1/2}(\Gamma)}. \end{cases} \tag{4.18}$$

Compared to the abstract formulation (4.10), in formulation (4.18) we have used the scalar product associated with the semi-norm in $H^1(\Omega)$ instead of the full norm in $H^1(\Omega)$, which is possible thanks to Poincaré's inequality. From Theorem 4.3 and since condition 3 is satisfied, the problem (4.18) is well-posed for any Cauchy data $(g_0, g_1) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ and for (g_0, g_1) such that problem (4.15) has a (unique) solution u, we have $(u_\epsilon, \lambda_\epsilon) \to (u, 0)$ in $H^1(\Omega) \times H^1(\Omega)$.

4.4 Data completion method

Let us come back to the notations of section **??** and assume that $\overline{\Gamma} \cap \overline{\tilde{\Gamma}} = \emptyset$. From Proposition 1.1 and Theorem 1.2, solving problem (4.15) is equivalent to solve the following variational problem: find $\lambda \in H^{1/2}(\tilde{\Gamma})$ such

$$s(\lambda, \mu) = \ell(\mu), \quad \forall \mu \in H^{\frac{1}{2}}(\tilde{\Gamma}),$$

where the bilinear form s and the linear form ℓ are given by (**??**) and (**??**), respectively. In particular, s is symmetric. Let us prove some other properties of s.

Proposition 4.3. *We have that $s(\mu,\mu) \geq 0$ for all $\mu \in H^{\frac{1}{2}}(\tilde{\Gamma})$ and $s(\mu,\mu) = 0$ implies that $\mu = 0$.*

Proof. For $\mu \in H^{\frac{1}{2}}(\tilde{\Gamma})$, the function $u_N(\mu)$ is the unique minimizer in the set $\{v \in H^1(\Omega),\ v|_{\tilde{\Gamma}} = \mu\}$ of the cost function $\int_\Omega |\nabla v|^2\, dx$. We hence obtain, since $u_D(\mu) \in H^1(\Omega)$ and $v|_{\tilde{\Gamma}} = \mu$, that

$$\int_\Omega |\nabla u_D(\mu)|^2\, dx \geq \int_\Omega |\nabla u_N(\mu)|^2\, dx,$$

which implies the first part of the statement.
Let us assume now that $s(\mu,\mu) = 0$. Because the above minimization problem has a unique solution, we have $u_D(\mu) = u_N(\mu) = u$, which means that u solves the Cauchy problem which trivial data (3.1), then $u = 0$ in Ω and $\mu = 0$, which completes the proof. $\square$

Solving the Cauchy problem (1.3) is equivalent to solve problem (1.8). Since S is compact, injective and has dense range (because S is self-adjoint), we are exactly in the framework of chapter 2 for ill-posed problems. We are hence able, for any data $(g_0, g_1) \in H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$, to apply the Tikhonov regularization and the Morozov's discrepancy principle to the operator S.
In our particular case where the operator S is positive, an alternative method to the Tikhonov regularization is the Lavrentiev regularization associated with operator S, that is for $\epsilon > 0$: find $\lambda \in H^{\frac{1}{2}}(\tilde{\Gamma})$ such that

$$s(\lambda,\mu) + \epsilon(\lambda,\mu)_{H^{\frac{1}{2}}(\tilde{\Gamma})} = \ell(\mu), \quad \forall \mu \in H^{\frac{1}{2}}(\tilde{\Gamma}). \tag{4.19}$$

The weak formulation (4.19) amounts to minimize in $H^{\frac{1}{2}}(\tilde{\Gamma})$ the functional

$$J(\mu) + \frac{\epsilon}{2}\|\mu\|^2_{H^{\frac{1}{2}}(\tilde{\Gamma})}, \quad J(\mu) := \frac{1}{2}s(\mu,\mu) - \ell(\mu).$$

We complete this section with a comparison between the functional J and the so-called Kohn-Vogelius functional KV, which is defined by

$$KV(\mu) = \frac{1}{2}\int_\Omega |\nabla u_D(\mu, g_0) - \nabla u_N(\mu, g_1)|^2\, dx.$$

Precisely, we have the following result.

Proposition 4.4. *For all $\mu \in H^{\frac{1}{2}}(\tilde{\Gamma})$,*

$$KV(\mu) = J(\mu) + C, \quad C := \frac{1}{2}\int_\Omega |\nabla \breve{u}_D(g_0) - \nabla \breve{u}_N(g_1)|^2\, dx,$$

that is functionals KV and J coincide up to a constant which depends only on the data.

Proof. we first prove that the gradient of KV and of J are equal. The gradient of J at point λ is given by

$$\langle \nabla J(\lambda), \mu \rangle_{\tilde{\Gamma}} = \int_{\Omega} \nabla u_D(\lambda, g_0) \cdot \nabla u_D(\mu)\, dx \\ - \int_{\Omega} \nabla u_N(\lambda, g_1) \cdot \nabla u_N(\mu)\, dx + \langle g_1, u_N(\mu) \rangle_{\Gamma},$$

while the gradient of KV is given by

$$\langle \nabla KV(\lambda), \mu \rangle_{\tilde{\Gamma}} = \int_{\Omega} \nabla(u_D(\lambda, g_0) - u_N(\lambda, g_1)) \cdot \nabla(u_D(\mu) - u_N(\mu))\, dx.$$

We then obtain that

$$\langle \nabla J(\lambda), \mu \rangle_{\tilde{\Gamma}} - \langle \nabla KV(\lambda), \mu \rangle_{\tilde{\Gamma}} \\ = \int_{\Omega} \nabla u_D(\lambda, g_0) \cdot \nabla u_N(\mu)\, dx - \int_{\Omega} \nabla u_N(\lambda, g_1) \cdot \nabla u_N(\mu)\, dx \\ + \int_{\Omega} \nabla u_N(\lambda, g_1) \cdot \nabla(u_D(\mu) - u_N(\mu))\, dx + \langle g_1, u_N(\mu) \rangle_{\Gamma}.$$

It is easy to see that the two terms in the right-hand side vanish separately, which proves that the gradient of KV and of J are equal. The value of the constant C follows from the definition of $\breve{u}_D(g_0)$ and $\breve{u}_N(g_1)$. $\square$

The weak formulation (4.19) is directly in a suitable form for some discretization with the help of a finite element method. It is equivalent to minimize the Kohn-Vogelius functional (with regularization) in a discretized space.

Chapter 5

Data completion problems – The wave equation case

5.1 Introduction

5.1.1 Problem setting

Let Ω be a bounded, connected open domain of class $C^{1,1}$ in $\mathbb{R}^d$ (see definition **??**), $d > 1$. We consider the following wave equation

$$\begin{cases} \partial_t^2 u(x,t) - \Delta u(x,t) = 0, & (x,t) \in \Omega \times (0,\infty) \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x), \quad \dot{u}(x,0) = v_0(x), & x \in \Omega \end{cases} \tag{5.1}$$

and we recall that if $(u_0, v_0) \in H_0^1(\Omega) \times L^2(\Omega)$ then $(u, \dot{u}) \in C((0,T); H_0^1(\Omega)) \times C((0,T); L^2(\Omega))$ for any $T > 0$, see e.g. **?**. Denoting $z(t) = \begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix}$, we can rewrite (5.1) as the first-order system

$$\begin{cases} \dot{z} = Az \\ z(0) = \lambda \end{cases} \tag{5.2}$$

where $\lambda = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$ and $A : \mathcal{D}(A) \to \mathcal{Z}$ with

$$A = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}, \quad \mathcal{D}(A) = \mathcal{D}(-\Delta) \times H_0^1(\Omega), \quad \mathcal{Z} = H_0^1(\Omega) \times L^2(\Omega).$$

The operator A generates a group and for all $\lambda \in \mathcal{Z}$ then $(u, \dot{u}) \in C((0,T); \mathcal{Z})$ for any $T > 0$, see e.g. **?** and Appendix **??**.

For this system, we consider to have at our disposal some measurements in $\omega \subset \Omega$ an open and non-empty subset of Ω and at every time t. These measurements can be either of the form

$$\forall t \geq 0, \quad \left| \begin{array}{l} H_0^1(\Omega) \to H^1(\omega) \\ u(\cdot,t) \mapsto z(t) = u(\cdot,t)|_\omega \end{array} \right. \tag{5.3}$$

or alternatively

$$\forall t \geq 0, \quad \left| \begin{array}{l} L^2(\Omega) \to L^2(\omega) \\ \dot{u}(\cdot,t) \mapsto z(t) = \dot{u}(\cdot,t)|_\omega \end{array} \right. \tag{5.4}$$

In each case, accordingly introducing the observation space $\mathcal{Y} = H^1(\omega)$ or $\mathcal{Y} = L^2(\omega)$, we can define an observation operator $C \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$ by

$$C = \begin{pmatrix} \gamma_\omega & 0 \end{pmatrix}, \text{ or } C = \begin{pmatrix} 0 & \gamma_\omega \end{pmatrix}, \tag{5.5}$$

respectively, with γ_ω the restriction operator on ω.

The inverse problem of interest is here the following. Defining the linear operator

$$\Psi_T : \left| \begin{array}{rl} \mathcal{Z} & \to \mathcal{Y}_T \\ \lambda = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} & \mapsto y = u_\omega \end{array} \right.$$

our objective is to inverse Ψ_T, namely from $y \in \mathcal{Y}_T$ to recover the initial condition $\lambda \in \mathcal{Z}$. This again will depend of our choice of space $\mathcal{Y}_T$. However, we can first establish an injectivity result using one the Holmgrem theorems presented in Chapter 3.

Theorem 5.1. *There exists T_0 such that for all $T > T_0$, the operator Ψ_T is injective from $\mathcal{Z} = H_0^1(\Omega) \times L^2(\Omega)$ to its image.*

Proof. Let us consider u a solution of (5.1). We then define

$$\tilde{u}$$

We then apply Corrolary 3.1 to get that $\tilde{u}$ is null on

5.1.2 Injectivity of Ψ_T

5.1.3 Observation operator adjoint

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