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Global existence for fully parabolic boundary value problems

Adrian CONSTANTIN
School of Mathematics
Trinity College
Dunlin 2, Ireland
e-mail: adrian@maths.tcd.ie
and
Department of Mathematics
Lund University
PO Box 118
22100 Lund, Sweden
e-mail: adrian.constantin@math.lu.se

Joachim ESCHER
Institute of Applied Mathematics
University of Hannover
Welfengarten 1
30167 Hannover, Germany
e-mail: escher@ifam.uni-hannover.de

Abstract. We present some results on the global existence of classical solutions for quasilinear parabolic equations with nonlinear dynamic boundary conditions in bounded domains with a smooth boundary.

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1 Introduction

In this paper we consider quasilinear initial boundary value problems of the type

$$\left.\begin{aligned} u_t - \operatorname{div}(a(t,x,u)\nabla u) &= f(t,x,u,\nabla u), && t>0,\ x\in\Omega,\\ u(t,x) &= 0, && t\geq 0,\ x\in\Gamma_0\subset\partial\Omega,\\ u_t + \langle a(t,x,u)\nabla u, \nu(x)\rangle &= g(t,x,u), && t>0,\ x\in\Gamma_1\subset\partial\Omega,\\ u(0,x) &= u_0(x), && x\in\overline{\Omega}, \end{aligned}\right\} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ ($n \geq 1$) with a smooth boundary $\partial\Omega$ consisting of two disjoint and open (with respect to $\partial\Omega$) components Γ_0 and Γ_1 with $\Gamma_1 \neq \emptyset$ (we do not exclude the possibility that $\Gamma_0 = \emptyset$), and where ν is the outer unit normal on $\partial\Omega$ and $\langle\cdot,\cdot\rangle$ stays for the inner product in $\mathbb{R}^n$. We assume that $a \in C^2(\mathbb{R}_+ \times \overline{\Omega} \times \mathbb{R}, \mathbb{R}^{n\times n})$ is such that

$$\langle a(t,x,r)\eta, \eta\rangle \geq c|\eta|^2, \qquad (t,x,r) \in \mathbb{R}_+ \times \overline{\Omega} \times \mathbb{R},\ \eta \in \mathbb{R}^n,$$

while

$$f \in C^1(\mathbb{R}_+ \times \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n, \mathbb{R}), \qquad g \in C^2(\mathbb{R}_+ \times \Gamma_1 \times \mathbb{R}, \mathbb{R}).$$

Under these assumptions we prove in Section 2 that, given any initial datum $u_0 \in C^2(\overline{\Omega})$ with $u_0|\Gamma_0 = 0$, there exists a unique classical solution u to (1.1) with

$$u \in C^1((0,t^+), C(\overline{\Omega})) \cap C((0,t^+), C^2(\overline{\Omega})),$$

where $t^+ = t^+(u_0) > 0$ stands for the maximal interval of existence of u (see Theorem 2.5). Furthermore, if f satisfies the following growth condition

$$\left|\frac{\partial}{\partial\eta} f(t,x,r,\eta)\right| \leq h(t,r)(1+|\eta|^\alpha), \qquad (t,x,r,\eta) \in \mathbb{R}_+ \times \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \tag{1.2}$$

for some $h \in C(\mathbb{R}^2, \mathbb{R}_+)$ and some $\alpha \in [0,1)$, we show that an $L_\infty(\Omega)$-a priori bound for u guarantees the global existence of classical solutions to (1.1) in the semilinear case, i.e., when the coefficient a in (1.1) is independent of u. In the general situation, we have to assume that $\alpha = 1$, see Theorem 2.6 and Theorem 2.11 below. Observe that (1.2) implies that there exists $h_0 \in C(\mathbb{R}^2, \mathbb{R}_+)$ such that

$$|f(t,x,r,\eta)| \leq h_0(t,r)(1+|\eta|^{1+\alpha}), \qquad (t,x,r,\eta) \in \mathbb{R}_+ \times \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n. \tag{1.3}$$

We show at the end of Section 2 that (1.3) is close to optimal. Indeed, if $n = 1$, $a \equiv 1$, and $f(t,x,u,\eta) = 1+|\eta|^{2+\delta}$ for some $\delta > 0$, it follows from Proposition 2.14 below that there are classical solutions to (1.1) which blow-up in finite time, although their $L_\infty(\Omega)$-norm stays bounded. In Section 3 we specify conditions which guarantee, under the growth condition (1.2) with an appropriate function h, the global existence of solutions to (1.1). We also show that these conditions are quite sharp by means of a blow-up result. Finally, the Appendix contains the derivation of a quasilinear physical model of type (1.1). For other quasilinear problems of type (1.1) arising as models in chemical kinetics we refer to [26].

Problems of type (1.1) have already been considered by different authors. In [9] the classical solvability for linear problems of type (1.1) is established, while the papers [8], [16],[17], and [19] are devoted to various aspects of semilinear problems. Quasilinear problems with a right-hand side f which does not depend upon the gradient have been considered in [15] and [21]. In this paper we consider problems of type (1.1) in a more general quasilinear situation than these previous investigations. We prove the local existence and uniqueness of classical solutions while the references given above are only concerned with weak or mild solutions (in the nonlinear case). We regard the results on the blow-up scenario as being some of the most important contributions to be brought forward by this paper.

2 Local existence and blow-up scenario

In this section we present results on the local existence and uniqueness of classical solutions to problems of type (1.1). For technical reasons it is convenient to treat separately the quasilinear and semilinear cases. We also give sufficient conditions to ensure that blow-up can occur only if the $L_\infty(\Omega)$-norm of the solution becomes unbounded in finite time. A simple class of equations of type (1.1), having classical solutions for which the spatial derivative blows-up while the solution itself stays bounded, shows that our conditions for the L_∞ blow-up scenario are quite sharp.

2.1 Quasilinear equations

Let Ω be a domain in $\mathbb{R}^n$ and assume that its boundary $\partial\Omega$ is decomposed as described in Section 1. Throughout this paper we fix $p > 2(n+1)$.

Given two Banach spaces E and F, we write $\mathcal{L}(E,F)$ for the Banach space of all linear and bounded operators from E into F, endowed with the usual operator norm. The subset of all isomorphisms in $\mathcal{L}(E,F)$ is denoted by $\mathrm{Isom}(E,F)$.

Furthermore, given $\tau \geq 0$, we write $W_p^\tau(\Omega)$ and $B_{pp}^\tau(\Gamma_i)$ for the Sobolev-Slobodeckii spaces over Ω and the Besov spaces over Γ_i, $i = 0, 1$, respectively, cf. [1, 27, 28]. We mention that in the case that τ is an integer the spaces $W_p^\tau(\Omega)$ coincide with the usual Sobolev spaces, cf. formula 3.4.2(1) in [28]. In the case $\tau \in (0,1)$, a norm for which $W_p^\tau(\Omega)$ is complete is given by

$$\left(\int_\Omega |v(x)|^p dx\right)^{1/p} + \left(\int_{\Omega\times\Omega} \frac{|v(x)-v(y)|^p}{|x-y|^{n+\tau p}}\, dx\, dy\right)^{1/p}, \quad v \in W_p^\tau(\Omega), \tag{2.1}$$

cf. Proposition 3.4.2 in [28]. Given $s > 1/p$, let

$$\gamma_{\Gamma_i,s} \in \mathcal{L}(W_p^s(\Omega), B_{pp}^{s-1/p}(\Gamma_i)), \quad i = 0,\, 1, \tag{2.2}$$

denote the trace operator on $W_p^s(\Omega)$. It follows from Theorem 3.3.3 in [28] that there exists a coretraction $\gamma^c_{\Gamma_1,s}$ to $\gamma_{\Gamma_1,s}$, i.e., there is

$$\gamma^c_{\Gamma_1,s} \in \mathcal{L}(B^{s-1/p}_{pp}(\Gamma_1), W^s_p(\Omega)) \quad \text{with} \quad \gamma_{\Gamma_1,s}\gamma^c_{\Gamma_1,s} = id_{B^{s-1/p}_{pp}(\Gamma_1)} \tag{2.3}$$

Multiplying with a test function that is identically one in a neighbourhood of Γ_1 and identically zero in a neighbourhood of Γ_0, we may modify $\gamma^c_{\Gamma_1,s}$ to accomodate the relation $\gamma_{\Gamma_0,s}\gamma^c_{\Gamma_1,s} = 0$. If no ambiguity seems likely, we write γ_{Γ_1} for $\gamma_{\Gamma_1,s}$. Let us now introduce the spaces

$$\begin{aligned}\mathbb{E}_0 &:= L_p(\Omega) \times B^{1-1/p}_{pp}(\Gamma_1),\\ \mathbb{E}_1 &:= \{(v, \gamma_{\Gamma_1} v) \,:\, v \in W^2_p(\Omega),\ \gamma_{\Gamma_0} v = 0\}.\end{aligned}$$

By (2.2), $\mathbb{E}_1$ is a closed subspace of $W^2_p(\Omega) \times B^{2-1/p}_{pp}(\Gamma_1)$, and thus a well-defined Banach space. In order to prove various embedding and interpolation results for the pair $(\mathbb{E}_0, \mathbb{E}_1)$, we need some preparation. Let

$$K^s_p(\Omega) := \ker(\gamma_{\Gamma_0,s}) \cap \ker(\gamma_{\Gamma_1,s}), \quad s > 1/p,$$

and let $[\cdot,\cdot]_\theta$ denote the usual complex interpolation function. Then we have

$$[L_p(\Omega), K^s_p(\Omega)]_\theta = \begin{cases} W^{s\theta}_p(\Omega), & 0 < s\theta < 1/p,\\ K^{s\theta}_p(\Omega) = W^{s\theta}_{p,0}(\Omega), & 1/p < s\theta < 1 + 1/p,\\ K^{s\theta}_p(\Omega) = W^{s\theta}_p(\Omega) \cap W^1_{p,0}(\Omega), & 1 + 1/p < s\theta \le 2,\end{cases} \tag{2.4}$$

where $W^\tau_{p,0}(\Omega)$ stands for the closure of the test functions $\mathcal{D}(\Omega)$ in $W^\tau_p(\Omega)$, $\tau \ge 0$. This follows from Theorem 4.1 in [25] and the reiteration theorem for complex interpolation, cf. e.g. (I.2.8.4) in [7].

Proposition 2.1 *Let*

$$J \,:\, \mathbb{E}_0 \to \mathbb{E}_0, \quad (v, w) \mapsto (v - \gamma^c_{\Gamma_1} w, w). \tag{2.5}$$

Then we have that $J \in \mathrm{Isom}(\mathbb{E}_0, \mathbb{E}_0) \cap \mathrm{Isom}(\mathbb{E}_1, K^2_p(\Omega) \times B^{2-1/p}_{pp}(\Gamma_1))$.

Proof. It follows from the first part of (2.3) that

$$J \in \mathcal{L}(\mathbb{E}_0, \mathbb{E}_0) \cap \mathcal{L}(\mathbb{E}_1, W^2_p(\Omega) \times B^{2-1/p}_{pp}(\Gamma_1)),$$

while the second part of (2.3) yields

$$J(\mathbb{E}_1) \subset K^2_p(\Omega) \times B^{2-1/p}_{pp}(\Gamma_1).$$

Since the inverse of J is obviously given by $J^{-1}(f, g) = (f + \gamma^c_{\Gamma_1} g, g)$, the assertion follows from the open mapping theorem. $\square$

In order to economize our notation we set

$$\mathbb{E}_\theta := [\mathbb{E}_0, \mathbb{E}_1]_\theta \quad \text{for} \quad \theta \in (0,1).$$

Our next result describes the interpolation spaces $\mathbb{E}_\theta$, provided $\theta \in [0,1]$ is sufficiently small. For large values of θ an explicit characterization of $\mathbb{E}_\theta$ seems not to be known. However, we prove appropriate embeddings for theses spaces. These turn out to be sufficient for our purposes.

Corollary 2.2 (i) $\mathbb{E}_1$ *is dense in* $\mathbb{E}_0$.

(ii) *If* $0 < s_0 < 1/p$ *then*

$$W_p^{s_0}(\Omega) \times B_{pp}^{1+s_0/2-1/p}(\Gamma_1) = \mathbb{E}_{s_0/2}. \tag{2.6}$$

(iii) *Let* $\mathbb{V}_\delta := \{(v, \gamma_{\Gamma_1} v)\,;\, v \in W_p^{2\delta}(\Omega), \gamma_{\Gamma_0} v = 0\}$ *for* $\delta \in (1/2p, 1]$. Then

$$\mathbb{E}_{1-1/2p} \hookrightarrow \mathbb{V}_{1-1/2p} \hookrightarrow \mathbb{E}_{1-1/p} \hookrightarrow \mathbb{V}_{1-1/p}. \tag{2.7}$$

Proof. (i) Let $\mathcal{D}(\Omega)$ denote the set of all test functions on Ω. Note that $\mathcal{D}(\Omega) \times C^\infty(\Gamma_1) \subset K_p^2(\Omega) \times B_{pp}^{2-1/p}(\Gamma_1)$ is dense in $\mathbb{E}_0$. Applying J^{-1} to the previous inclusion, we see that Proposition 2.1 implies the assertion.

(ii) Let $s_0 \in (0, 1/p)$. Then we conclude from (2.4) and the fact that Besov spaces are stable under complex interpolation, cf. Theorem 3.3.6 in [28], that

$$[\mathbb{E}_0, K_p^2(\Omega) \times B_{pp}^{2-1/p}(\Gamma_1)]_{s_0/2} = W_p^{s_0}(\Omega) \times B_{pp}^{1+s_0/2-1/p}(\Gamma_1).$$

But, by its explicit formula, J is obviously a continuous automorphism on $W_p^{s_0}(\Omega) \times B_{pp}^{1+s_0/2-1/p}(\Gamma_1)$. Applying J to the above identity, the assertion follows from Proposition 2.1 since interpolation carries over to the images under J.

(iii) Observe that $1 - 1/2p > 1/2p$. Thus (2.4) implies that

$$[L_p(\Omega) \times B_{pp}^{1-1/p}(\Gamma_1), K_p^2(\Omega) \times B_{pp}^{2-1/p}(\Gamma_1)]_{1-1/2p} = K_p^{2-1/p}(\Omega) \times B_{pp}^{2-3/2p}(\Gamma_1).$$

Invoking Proposition 2.1, we obtain by interpolation that

$$\begin{aligned}[\mathbb{E}_0, \mathbb{E}_1]_{1-1/2p} = \{(v,w) \in W_p^{2-1/p}(\Omega) \\ \times B_{pp}^{2-3/2p}(\Gamma_1)\,;\, \exists v_0 \in K_p^{2-1/p}(\Omega) : v = v_0 + \gamma_{\Gamma_1}^c w\}.\end{aligned}$$

Since $\gamma_{\Gamma_1}\gamma_{\Gamma_1}^c = id_{B_{pp}^{2-3/2p}(\Gamma_1)}$ and $B_{pp}^{2-3/2p}(\Gamma_1) \hookrightarrow B_{pp}^{2-2/p}(\Gamma_1)$, the first embedding of (2.7) is established. Similarly, we have

$$\begin{aligned}[\mathbb{E}_0, \mathbb{E}_1]_{1-1/p} = \{(v,w) \in W_p^{2-2/p}(\Omega) \\ \times B_{pp}^{2-2/p}(\Gamma_1)\,;\, \exists v_0 \in K_p^{2-2/p}(\Omega) : v = v_0 + \gamma_{\Gamma_1}^c w\}.\end{aligned}$$

Let now $v \in W_p^{2-1/p}(\Omega)$ with $\gamma_{\Gamma_0} v = 0$ be given. Set $w := \gamma_{\Gamma_1} v \in B_{pp}^{2-2/p}(\Gamma_1)$ and $v_0 := v - \gamma_{\Gamma_1}^c w \in K_p^{2-1/p}(\Omega) \subset K_p^{2-2/p}(\Omega)$. Then we have $(v,w) \in [\mathbb{E}_0, \mathbb{E}_1]_{1-1/p}$, which gives the second embedding in (2.7). The last assertion is obtained analogously. ☐

Let us introduce now operators which allow us to re-express the original problem (1.1) as an abstract quasilinear evolution equation in $\mathbb{E}_0$. Given $t \in \mathbb{R}_+$ and $V = (v, \gamma_{\Gamma_1} v) \in \mathbb{V}_{1-1/p}$, define a linear operator $\mathbb{A}(t, V)$ from $\mathbb{E}_1$ to $\mathbb{E}_0$ by setting

$$\mathbb{A}(t,V)W := (-\mathrm{div}(a(t,\cdot,v)\nabla w),\ \langle a(t,\cdot,\gamma_{\Gamma_1} v)\nabla w, \nu(\cdot)\rangle),$$

where $W = (w, \gamma_{\Gamma_1} w) \in \mathbb{E}_1$. Our next result shows that, given any $(t, V) \in \mathbb{R}_+ \times \mathbb{V}_{1-1/p}$, the operator $-\mathbb{A}(t, V)$ with domain $\mathbb{E}_1$ generates a strongly continuous analytic semigroup on $\mathbb{E}_0$. The set of all such operators is denoted by $\mathcal{H}(\mathbb{E}_1, \mathbb{E}_0)$. It is well-known that $\mathcal{H}(\mathbb{E}_1, \mathbb{E}_0)$ is an open subset of $\mathcal{L}(\mathbb{E}_1, \mathbb{E}_0)$, cf. Proposition I.1.3.1 in [7]. Writing C^{1-} for locally Lipschitz mappings, we have

Lemma 2.3 $\mathbb{A} \in C^{1-}(\mathbb{R}_+ \times \mathbb{V}_{1-1/p}, \mathcal{H}(\mathbb{E}_1, \mathbb{E}_0))$.

Proof. Given $(t, V) \in \mathbb{R}_+ \times \mathbb{V}_{1-1/p}$ it follows from Theorem 2.3 in [15] that $\mathbb{A}(t, V)$ belongs to $\mathcal{H}(\mathbb{E}_1,\, \mathbb{E}_0)$. Since $a \in C^2(\mathbb{R}_+ \times \overline{\Omega} \times \mathbb{R})$, we conclude that

$$[(t,v) \mapsto a(t,\cdot,v)] \in C^{1-}(\mathbb{R} \times W_p^{2-2/p}(\Omega), [C^1(\overline{\Omega})]^{n\times n}).$$

Moreover, we know from Theorem 2.8.2 in [28] that pointwise multiplication maps $C^1(\overline{\Omega}) \times B_{pp}^{1-1/p}(\Gamma_1)$ continuously into $B_{pp}^{1-1/p}(\Gamma_1)$. Examining the difference $A(t, V_1) - A(t, V_2)$ and using the above locally Lipschitz property, we can now easily get the assertion. □

We introduce the operator

$$F : \mathbb{R}_+ \times \mathbb{V}_{1-1/p} \to \mathbb{E}_0, \quad F(t,V) := (f(t,\cdot,v,\nabla v), g(t,\cdot,\gamma_{\Gamma_1} v)), \tag{2.8}$$

where $V = (v, \gamma_{\Gamma_1} v) \in \mathbb{V}_{1-1/p}$. We remark that F is designed to take care of the nonlinear right-hand sides f and g of (1.1). Observe furthermore that $\mathbb{V}_{1-1/p} \hookrightarrow C(\overline{\Omega}) \times C(\Gamma_1)$, by Sobolev's embedding theorem. In the following we write

$$\|V\|_\infty = \max_{x\in\overline{\Omega}} |v(x)| \quad \text{for} \quad V = (v, \gamma_{\Gamma_1} v) \in \mathbb{V}_{1-1/p}.$$

Moreover, we fix $s_0 \in (0, 1/p)$. Recalling that $p > 2(n + 1)$, we conclude that

$$0 < s_0 < \frac{1}{p} < 1 - \frac{n+2}{p}. \tag{2.9}$$

The following mapping properties of F will be useful in our approach.

Lemma 2.4 (i) $F \in C^{1-}(\mathbb{R}_+ \times \mathbb{V}_{1-1/p}, \mathbb{E}_{s_0/2})$. *Moreover, F is bounded on bounded subsets of $\mathbb{R}_+ \times \mathbb{V}_{1-1/p}$.*

(ii) *Assume that f satisfies* (1.2) *with $\alpha = 0$. Then, given $T > 0$ and $M > 0$, there is a constant $C = C(T, M) > 0$, such that*

$$\|F(t,V)\|_{\mathbb{E}_{s_0/2}} \le C(1 + \|V\|_{\mathbb{E}_{1-1/2p}}), \tag{2.10}$$

for all $(t, V) \in [0, T] \times \mathbb{V}_{1-1/2p}$ such that $\|V\|_\infty \le M$.

Proof. (i) Recall that $s_0 < 1 - (n+2)/p$ in view of (2.9). Hence we may choose τ such that $s_0 < \tau < 1 - (n+2)/p$. The Sobolev embedding theorem and Theorem 3.3.1 in [28] ensure that

$$W_p^{1-2/p}(\Omega) \hookrightarrow C^\tau(\overline{\Omega}) \hookrightarrow W_p^{s_0}(\Omega). \tag{2.11}$$

The first embedding of (2.11) implies that the mapping

$$\mathbb{V}_{1-1/p} \to C^1(\overline{\Omega}) \times \left[C^\tau(\overline{\Omega})\right]^n, \quad V = (v, \gamma_{\Gamma_1} v) \mapsto (v, \nabla v),$$

is well-defined and linear. Since f is of class C^1, we conclude that the mapping that

$$\mathbb{R}_+ \times C^1(\overline{\Omega}) \times \left[C^\tau(\overline{\Omega})\right]^n \to C^\tau(\overline{\Omega}), \quad (t, v, \nabla v) \mapsto f(t, \cdot, v, \nabla v),$$

is locally Lipschitz continuous and bounded on bounded sets. Combining the previous two assertions, we infer that $f \in C^{1-}(\mathbb{R}_+ \times \mathbb{V}_{1-1/p}, C^\tau(\overline{\Omega}))$.

In order to treat the second component g of F we proceed similarly. By our choice of s_0 we have

$$1 + \frac{s_0}{2} - \frac{1}{p} < 1 < 2 - \frac{2}{p} - \frac{n-1}{p},$$

and therefore

$$B_{pp}^{2-2/p}(\Gamma_1) \hookrightarrow C^1(\Gamma_1) \hookrightarrow B_{pp}^{1+s_0/2-1/p}(\Gamma_1), \tag{2.12}$$

cf. Theorem 3.3.1 in [28]. By assumption, we know that g is of class C^2. Hence the mean value theorem implies that the mapping

$$\mathbb{R}_+ \times \mathbb{V}_{1-1/p} \to C^1(\Gamma_1), \quad (t, (v, \gamma_{\Gamma_1} v)) \mapsto g(t, \cdot, \gamma_{\Gamma_1} v),$$

is locally Lipschitz continuous and bounded on bounded sets.

Finally, using the second embeddings in (2.11) and (2.12), we get the statement by observing Corollary 2.2(ii) and the previously established properties of each component of $F = (f, g)$.

(ii) Let M be given and assume that $V = (v, \gamma_{\Gamma_1} v) \in \mathbb{V}_{1-1/2p}$ satisfies $\|V\|_\infty \leq M$. Then (1.2) with $\alpha = 0$, (2.1), and the mean value theorem imply that there is a $C = C(M, T)$ such that

$$\|f(t, \cdot, v, \nabla v)\|_{W_p^{s_0}(\Omega)} \leq C(1 + \|\nabla v\|_{W_p^{s_0}(\Omega)}) \leq C(1 + \|v\|_{W_p^{1+s_0}(\Omega)}). \tag{2.13}$$

Increasing C if necessary, we conclude by the mean value theorem that

$$\|g(t, \cdot, \gamma_{\Gamma_1} v)\|_{C^1(\Gamma_1)} \leq C(1 + \|\gamma_{\Gamma_1} v\|_{C^1(\Gamma_1)}). \tag{2.14}$$

Observe that $s_0 < 1 - 1/p$. Hence we have $W_p^{2-1/p}(\Omega) \hookrightarrow W_p^{1+s_0}(\Omega)$. Combing this with (2.13) and (2.14) with (2.12), we get the assertion from Corollary 2.2(ii). □

We are now ready to prove the following local existence and uniqueness result. Writing C^ε for locally Hölder continuous functions, we have

Theorem 2.5 (i) *Given $U_0 \in \mathbb{E}_{1-1/2p}$, there exists a $t^+ > 0$ and a unique solution U in*

$$C([0,t^+),\mathbb{E}_{1-1/2p}) \cap C^1((0,t^+),\mathbb{E}_0) \cap C^{1/2p}([0,t^+),\mathbb{V}_{1-1/p}) \tag{2.15}$$

of the abstract quasilinear evolution equation

$$V' + \mathbb{A}(t,V)V = F(t,V), \quad t>0, \quad V(0) = U_0. \tag{2.16}$$

If the maximal interval of existence is bounded then $\limsup_{t\to t^+} \|U(t)\|_{\mathbb{E}_{1-1/2p}} = \infty$.

(ii) *Let u denote the first component of U. Then*

$$u \in C^{1/2p}([0,t^+),W_p^{2-2/p}(\Omega)) \cap C^1((0,t^+),C(\overline{\Omega})) \cap C((0,t^+),C^2(\overline{\Omega})) \tag{2.17}$$

and u is a classical solution to (1.1).

(iii) *Let $u_0 \in W_p^{2-1/p}(\Omega)$ with $\gamma_{\Gamma_0} u_0 = 0$ be given and assume that*

$$u \in C([0,t^+),W_p^{2-1/p}(\Omega)) \cap C^1((0,t^+),C(\overline{\Omega})) \cap C((0,t^+),C^2(\overline{\Omega}))$$

is a classical solution to (1.1). *Then $U := (u,\gamma_{\Gamma_1}u)$ solves* (2.16) *with initial datum $U_0 := (u_0,\gamma_{\Gamma_1}u_0)$.*

Proof. (i) Let $\alpha := 1-1/2p$, $\beta := 1-1/p$, $\delta := s_0/2$. Then Corollary 2.2(iii), Lemma 2.3, and Lemma 2.4(i) imply that

$$(\mathbb{A},F) \in C^{1-}(\mathbb{R}_+ \times \mathbb{E}_\beta, \mathcal{H}(\mathbb{E}_1,\mathbb{E}_0) \times \mathbb{E}_\delta).$$

Hence, given $U_0 \in \mathbb{E}_\alpha \hookrightarrow \mathbb{E}_\beta$, it follows from Theorem 12.1 and Remark 12.2(a) in [6] and from Theorem 7.1 in [3] that (2.16) possesses a unique solution

$$U \in C([0,t^+),\mathbb{E}_\alpha) \cap C^{\alpha-\beta}([0,t^+),\mathbb{E}_\beta) \cap C^1((0,t^+),\mathbb{E}_0).$$

Recalling that $\mathbb{E}_\beta = \mathbb{E}_{1-1/p} \hookrightarrow \mathbb{V}_{1-1/p}$ in view of (2.7), we get (2.15).

(ii) Let u denote the first component of the solution U of (2.16) constructed in (i). Fix t_0, $T \in (0,t^+)$ with $t_0 < T$ and let $I := [t_0,T]$. By the construction of $\mathbb{A}$ and F it suffices to show that

$$u \in C(I,C^2(\overline{\Omega})) \cap C^1(I,C(\overline{\Omega})). \tag{2.18}$$

For this fix $\tau_0 \in (0,t_0)$, $T' \in (T,t^+)$, set $J := [\tau_0,T']$, and choose $\eta \in \mathcal{D}((\tau_0,T'),\mathbb{R})$ such that $\eta|I = 1$. Moreover, given $v \in C^2(\overline{\Omega})$, define

$$\mathcal{A}(t,\cdot)v := -\mathrm{div}(a(t,\cdot,u(t,\cdot))\nabla v), \quad \mathcal{B}(t,\cdot)v := \langle a(t,\cdot,\gamma_{\Gamma_1}v)\nabla v,\nu(\cdot)\rangle, \quad t \in J.$$

Further, for $t \in J$, we set

$$\begin{aligned} f_0(t,\cdot) &:= \eta(t) f(t,\cdot,u(t,\cdot),\nabla u(t,\cdot)) + \eta'(t)u(t,\cdot),\\ g_0(t,\cdot) &:= \eta(t) g(t,\cdot,\gamma_{\Gamma_1}u(t,\cdot)) + \eta'(t)\gamma_{\Gamma_1}u(t,\cdot), \end{aligned}$$

and we consider the following linear parabolic initial boundary value problem with dynamic boundary condition

$$\left.\begin{aligned} v_t + \mathcal{A}(t,\cdot)v &= f_0(t,x), && t\in(\tau_0,T'),\ x\in\Omega,\\ v(t,x) &= 0, && t\in(\tau_0,T'),\ x\in\Gamma_0\subset\partial\Omega,\\ v_t + \mathcal{B}(t,\cdot)v &= g_0(t,x), && t\in(\tau_0,T'),\ x\in\Gamma_1\subset\partial\Omega,\\ v(\tau_0,x) &= 0, && x\in\overline{\Omega}. \end{aligned}\right\} \tag{2.19}$$

We shall see that the linear problem (2.19) is solvable in a classical sense. For this we have to guarantee that the coefficients of $(\mathcal{A}, \mathcal{B})$ and the right-hand side (f_0, g_0) are sufficiently regular. First, it follows from (2.15) that

$$u \in C^{1/2p}([0,t^+), W_p^{2-2/p}(\Omega)), \quad \gamma_{\Gamma_1}u \in C^1((0,t^+), B_{pp}^{1-1/p}(\Gamma_1)). \tag{2.20}$$

Using the trace theorem we infer that

$$\gamma_{\Gamma_1}u \in C^{1/2p}([0,t^+), B_{pp}^{2-3/p}(\Gamma_1)) \cap C^1((0,t^+), B_{pp}^{1-1/p}(\Gamma_1)).$$

The interpolation inequality

$$\|w\|_{B_{pp}^{3/2-2/p}(\Gamma_1)} \le c\,\|w\|^{1/2}_{B_{pp}^{1-1/p}(\Gamma_1)} \|w\|^{1/2}_{B_{pp}^{2-3/p}(\Gamma_1)}, \ \ w \in B_{pp}^{2-3/p}(\Gamma_1),$$

yields now

$$\gamma_{\Gamma_1}u \in C^{1/2+1/4p}([0,t^+), B_{pp}^{3/2-2/p}(\Gamma_1)). \tag{2.21}$$

Consider now the Banach spaces

$$\begin{aligned} C^{\alpha,0}(J\times\Gamma_1) = \{&h\in C(J\times\Gamma_1)\\ &: \ h(\cdot,x)\in C^\alpha(J) \text{ for every } x\in\Gamma_1 \text{ and } \|h\|_{\alpha,0}<\infty\},\\ C^{0,\beta}(J\times\Gamma_1) = \{&h\in C(J\times\Gamma_1)\\ &: \ h(t,\cdot)\in C^\beta(\Gamma_1) \text{ for every } t\in J \text{ and } \|h\|_{0,\beta}<\infty\}, \end{aligned}$$

with norms

$$\|h\|_{\alpha,0} := \sup_{x\in\Gamma_1} \|h(\cdot,x)\|_{C^\alpha(J)},$$

respectively $\|h\|_{0,\beta} := \sup\limits_{t\in J} \|h(t,\cdot)\|_{C^\beta(\Gamma_1)}$. We define

$$C^{\alpha,\beta}(J\times\Gamma_1) := C^{\alpha,0}(J\times\Gamma_1) \cap C^{0,\beta}(J\times\Gamma_1).$$

Note that for every $\sigma \in (0,1)$ we have

$$C^{(1+\sigma)/2}(J, C^{1+\sigma}(\Gamma_1)) \hookrightarrow C^{(1+\sigma)/2, 1+\sigma}(J \times \Gamma_1).$$

Since $p > 2(n+1)$ we can find some $\sigma \in (0, 1/2p)$ such that $3/2 - 2/p > 1 + \sigma + (n-1)/p$. Then, by Sobolev's embedding theorem, we have $B_{pp}^{3/2-2/p}(\Gamma_1) \hookrightarrow C^{1+\sigma}(\Gamma_1)$. On the other hand, since $\sigma < 1/2p$, we have that $1/2 + 1/4p > (1+\sigma)/2$. Hence

$$C^{1/2+1/4p}(J, B_{pp}^{3/2-2/p}(\Gamma_1)) \hookrightarrow C^{(1+\sigma)/2}(J, C^{1+\sigma}(\Gamma_1)).$$

The previous relation in combination with (2.21) yields

$$\gamma_{\Gamma_1} u \in C^{(1+\sigma)/2, 1+\sigma}(J \times \Gamma_1).$$

Since g is of class C^2, we deduce that

$$g_0 \in C^{(1+\sigma)/2, 1+\sigma}(J \times \Gamma_1).$$

Similarly, we get

$$f_0 \in C^{\sigma/2, \sigma}(J \times \overline{\Omega})$$

while for the coefficients of $(\mathcal{A}, \mathcal{B})$ we obtain

$$a(t, \cdot, u(t, \cdot)) \in C^{\sigma/2, 1+\sigma}(J \times \overline{\Omega}), \quad a(t, \cdot, \gamma_1 u(t, \cdot)) \in C^{(1+\sigma)/2, 1+\sigma}(J \times \Gamma_1).$$

Hence Theorem 1.1 in [9] implies that (2.19) possesses a unique classical solution

$$v \in C^{1+\sigma/2}(J, C(\overline{\Omega})) \cap C^{\sigma/2}(J, C^2(\overline{\Omega})).$$

Clearly, $(v, \gamma_1 v)$ is the unique solution to

$$V' + \mathbb{A}(t, U(t))V = (f_0, g_0), \quad t \in (\tau_0, T'), \quad V(\tau_0) = (0, 0). \tag{2.22}$$

But ηU is also a solution of (2.22). Hence we conclude that $v = \eta u$, which gives (2.18).

(iii) This is clear from the construction of $\mathbb{A}$ and F. $\square$

The next result provides a criterion for the global existence of a solution to (1.1).

Theorem 2.6 *Let $u_0 \in W_p^2(\Omega)$ with $\gamma_{\Gamma_0} u_0 = 0$ be given and let u denote the solution to* (1.1). *Moreover, assume that* (1.2) *holds true with $\alpha = 0$. Then $t^+ < \infty$ implies that $\limsup_{t \to t^+} \|u(t, \cdot)\|_{L_\infty(\Omega)} = \infty$.*

Proof. Let $U_0 := (u_0, \gamma_{\Gamma_1} u_0) \in \mathbb{E}_1 \hookrightarrow \mathbb{E}_{1-1/2p}$ and let U be the solution to (2.16). Then we have that $U = (u, \gamma_{\Gamma_1} u)$. Assume that $\limsup_{t \to t^+} \|u(t, \cdot)\|_{L_\infty(\Omega)} < \infty$.

In the following we show that U is a global solution. First we note that from (2.15) and Lemma 2.3 we get

$$[t \mapsto \mathbb{A}(t, U(t))] \in C^{1/2p}([0, t^+), \mathcal{H}(\mathbb{E}_1, \mathbb{E}_0)).$$

Hence it follows from Theorem 5.1 in [3] that there exists a unique parabolic fundamental solution

$$\mathbb{U} : \{(t,s) \in [0, t^+) \times [0, t^+);\ s \leq t\} \to \mathcal{L}(\mathbb{E}_1, \mathbb{E}_0)$$

for the family $\{\mathbb{A}(t, U(t))\,;\ t \in [0, t^+)\}$ and that the solution U to (2.16) is represented by the following integral equation:

$$U(t) = \mathbb{U}(t,0)U_0 + \int_0^t \mathbb{U}(t,s)F(s, U(s))\, ds \quad \text{in} \quad \mathbb{E}_{1-1/2p}.$$

Furthermore we infer from Theorem A.1 in [4] that there are positive constants K and ω such that

$$\|\mathbb{U}(t,s)\|_{\mathcal{L}(\mathbb{E}_{1-1/2p})} + (t-s)^{1-s_0/2}\|\mathbb{U}(t,s)\|_{\mathcal{L}(\mathbb{E}_{s_0/2}, \mathbb{E}_{1-1/2p})} \leq Ke^{\omega(t-s)}$$

for $(t,s) \in [0, t^+) \times [0, t^+)$ with $s < t$. Let now $T \in (0, t^+)$ be given. Then

$$\|U(t)\|_{\mathbb{E}_{1-1/2p}} \leq Ke^{\omega t}\|U_0\|_{\mathbb{E}_{1-1/2p}} + K\int_0^t (t-s)^{s_0/2-1}e^{\omega(t-s)}\|F(s, U(s))\|_{\mathbb{E}_{s_0/2}}.$$

Using Lemma 2.4(ii), we therefore get

$$\|U(t)\|_{\mathbb{E}_{1-1/2p}} \leq \alpha(t) + CK\int_0^t (t-s)^{s_0/2-1}e^{\omega(t-s)}\|U(s)\|_{\mathbb{E}_{1-1/2p}},$$

where $\alpha(t) := Ke^{\omega t}\|U_0\|_{\mathbb{E}_{1-1/2p}} + CK\int_0^t (t-s)^{s_0/2-1}e^{\omega(t-s)}ds$. Hence a generalized Gronwall inequality (see [14]) ensures the existence of a $\beta(T) > 0$ such that $\|U(t)\|_{\mathbb{E}_{1-1/2p}} \leq \beta(T)$ for $t \in [0,T]$. A standard continuation argument based on Theorem 2.5(i) now implies that $t^+ = \infty$ (see also Theorem 12.3 in [6]). □

2.2 Semilinear equations

In this section we consider the particular situation when the coefficient a does neither depend on the solution u nor on the time variable t. This means that we consider the following semilinear problem

$$\left.\begin{aligned} u_t - \operatorname{div}(a(x)\nabla u) &= f(t,x,u,\nabla u), && t>0,\ x \in \Omega,\\ u(t,x) &= 0, && t>0,\ x \in \Gamma_0 \subset \partial\Omega,\\ u_t + \langle a(x)\nabla u, \nu(x)\rangle &= g(t,x,u), && t>0,\ x \in \Gamma_1 \subset \partial\Omega,\\ u(0,x) &= u_0(x), && x \in \overline{\Omega}, \end{aligned}\right\} \tag{2.23}$$

Our goal is to prove a result for (2.23) which is analogous to Theorem 2.6 but which allows a stronger growth rate of the nonlinearity f with respect to ∇u. To achieve this we need an appropriate extension of the operator $\mathbb{A} \in \mathcal{H}(\mathbb{E}_1, \mathbb{E}_0)$, given by

$$\mathbb{A}U = (-\mathrm{div}(a(x)\nabla u), \langle a(x)\nabla u, \nu(x)\rangle), \quad U = (u, \gamma_{\Gamma_1} u) \in \mathbb{E}_1.$$

For this we write p' for the dual exponent to p, i.e., $1/p + 1/p' = 1$, and we introduce the following spaces

$$\begin{aligned}\mathbb{F}_1 &:= \{(v, \gamma_{\Gamma_1} v);\ v \in W_p^1(\Omega),\ \gamma_{\Gamma_0} v = 0\},\\ \mathbb{F}_0^\sharp &:= \{(\varphi, \gamma_{\Gamma_1}\varphi);\ \varphi \in W_{p'}^1(\Omega),\ \gamma_{\Gamma_0}\varphi = 0\}.\end{aligned}$$

Let $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ denote the duality pairing between $L_{p'}(\Omega) \times L_{p'}(\Gamma_1)$ and $L_p(\Omega) \times L_p(\Gamma_1)$, i.e.,

$$\langle\!\langle \Phi, V \rangle\!\rangle := \int_\Omega \varphi v \, dx + \int_{\Gamma_1} \psi w \, d\sigma$$

for $\Phi = (\varphi, \psi) \in L_{p'}(\Omega) \times L_{p'}(\Gamma_1)$ and $V = (v, w) \in L_p(\Omega) \times L_p(\Gamma_1)$. Further, $d\sigma$ stands for the surface volume element on Γ_1. Let now $\mathbb{F}_0 := [\mathbb{F}_0^\sharp]'$ be the dual space of $\mathbb{F}_0^\sharp$ with respect to $\langle\!\langle \cdot, \cdot \rangle\!\rangle$. Then it follows from Corollary 3.4 and Corollary 4.2 in [15] that there exists an extension $\overline{\mathbb{A}}$ of $\mathbb{A}$ such that $\overline{\mathbb{A}} \in \mathcal{H}(\mathbb{F}_1, \mathbb{F}_0)$. However, it turns out that this extension is too large for our purposes here. For this reason we introduce the interpolation spaces

$$\mathbb{X}_1 := [\mathbb{F}_1, \mathbb{E}_1]_{1-1/p}, \quad \mathbb{X}_0 := [\mathbb{F}_0, \mathbb{E}_0]_{1-1/p}.$$

Moreover, we set $\mathbb{X}_\theta := [\mathbb{X}_0, \mathbb{X}_1]_\theta$ for $\theta \in (0,1)$ and we write $\mathbb{A}_0$ for the $\mathbb{X}_0$-realization of $\overline{\mathbb{A}}$. Then we have to following result.

Proposition 2.7 (i) $\mathbb{A}_0 \in \mathcal{H}(\mathbb{X}_1, \mathbb{X}_0)$.

(ii) $\mathbb{X}_0 = W_p^{-1/p}(\Omega) \times B_{pp}^{1-2/p}(\Gamma_1)$, $\mathbb{X}_1 = \{(v, \gamma_{\Gamma_1}); v \in W_p^{2-1/p}(\Omega),\ \gamma_{\Gamma_0} v = 0\}$.

(iii) $L_p(\Omega) \times B_{pp}^{1-3/2p}(\Gamma_1) \hookrightarrow \mathbb{X}_{1/2p}$.

Proof. (i) Recall that $\mathbb{A} \in \mathcal{H}(\mathbb{E}_1, \mathbb{E}_0)$ and that $\overline{\mathbb{A}} \in \mathcal{H}(\mathbb{F}_1, \mathbb{F}_0)$. Hence, using a well-know characterization of the class $\mathcal{H}$, see Theorem I.1.2.2 in [7], the assertion follows by interpolation.

(ii) Observe that $0 < 1/p < 1/p' = 1 - 1/p$. Hence we conclude from (2.4), with p replaced by p', that

$$\begin{aligned}&[L_{p'}(\Omega) \times B_{p'p'}^{-1/p'}(\Gamma_1), K_{p'}^1(\Omega) \times B_{p'p'}^{1-1/p'}(\Gamma_1)]_{1/p}\\ &\quad = W_{p'}^{1/p}(\Omega) \times B_{p'p'}^{1/p-1/p'}(\Gamma_1) \qquad (2.24)\end{aligned}$$

Moreover it follows from (1.4) in [15] that there is an extension of γ_{Γ_1}, again denoted by γ_{Γ_1}, and an extension of $\gamma^c_{\Gamma_1}$, also denoted by $\gamma^c_{\Gamma_1}$, such that

$$\gamma^c_{\Gamma_1} \in \mathcal{L}(B^{-1/p'}_{p'p'}(\Gamma_1), L_{p'}(\Omega)) \cap \mathcal{L}(B^{1/p-1/p'}_{p'p'}(\Gamma_1), W^{1/p}_{p'}(\Omega)),$$

and $\gamma^c_{\Gamma_1}$ is a right-inverse of γ_{Γ_1}. From this we infer that there is an extension of the operator J introduced in Proposition 2.1 (to see this, repeat the arguments from its proof) such that J is a continuous automorphism on each of the spaces $L_{p'}(\Omega) \times B^{-1/p'}_{p'p'}(\Gamma_1)$ and $W^{1/p}_{p'}(\Omega) \times B^{1/p-1/p'}_{p'p'}(\Gamma_1)$, and

$$J \in \mathrm{Isom}(\mathbb{F}^\sharp_0, K^1_{p'}(\Omega) \times B^{1-1/p'}_{p'p'}(\Gamma_1)). \tag{2.25}$$

Now we conclude from (2.24) and (2.25) that

$$[L_{p'}(\Omega) \times B^{-1/p'}_{p'p'}(\Gamma_1), \mathbb{F}^\sharp_0]_{1/p} = W^{1/p}_{p'}(\Omega) \times B^{1/p-1/p'}_{p'p'}(\Gamma_1),$$

by the same argument used in the proof of Corollary 2.2 (ii). But Theorem 4.8.2 in [27] and Theorem 2.11.2 in [28] imply that

$$[W^{1/p}_{p'}(\Omega)]' = W^{-1/p}_{p}(\Omega) \quad \text{and} \quad [B^{1/p-1/p'}_{p'p'}(\Gamma_1)]' = B^{1-2/p}_{pp}(\Gamma_1).$$

Invoking finally the duality theorem for the complex interpolation function, we get the first assertion of (ii). The second assertion is shown as in Corollary 2.2(iii).

(iii) It follows from Theorem 3.3.6(i) in [28] that

$$[B^{1-2/p}_{pp}(\Gamma_1), B^{2-2/p}_{pp}(\Gamma_1)]_{1/2p} = B^{1-3/2p}_{pp}(\Gamma_1),\ [W^{-1/p}_{p}(\Omega), K^{2-1/p}_{p}]_{1/2p} = W^0_p(\Omega).$$

But Theorem 7.63 and Remark 7.66 in [1] ensure that $L_p(\Omega) \hookrightarrow W^0_p(\Omega)$. Hence, using the isomorphism J, the assertion follows as in the proof of Corollary 2.2(ii). □

Remark 2.8 (i) Observe that $\mathbb{X}_1 = \mathbb{V}_{1-1/2p}$ and that $\mathbb{E}_{s_0/2} \hookrightarrow \mathbb{X}_{1/2p}$. Hence we conclude from Lemma 2.4(i) that $F \in C^{1-}(\mathbb{R}_+ \times \mathbb{X}_1, \mathbb{X}_{1/2p})$ and F is bounded on bounded sets.

(ii) It follows from Proposition 2.7(i) and from (i) that, given $U_0 \in \mathbb{X}_1$, the semilinear evolution equation

$$V' + \mathbb{A}_0 V = F(t, V),\ t > 0,\ V(0) = U_0, \tag{2.26}$$

possesses a unique solution

$$U \in C([0, t^+), \mathbb{X}_1) \cap C^1([0, t^+), \mathbb{X}_0),$$

which is represented by the following variation of constant formula:

$$U(t) = e^{-t\mathbb{A}_0} U_0 + \int_0^t e^{-(t-s)\mathbb{A}_0} F(s, U(s))\, ds, \quad t \in [0, t^+).$$

This implies that, given $U_0 \in \mathbb{E}_1$, the solution to (2.16) constructed in Theorem 2.5(i) coincides in the semilinear case with the corresponding solution to (2.26).

(iii) It is known that $W_p^0(\Omega) = L_p(\Omega)$ if and only if $p = 2$. Hence the embedding $L_p(\Omega) \hookrightarrow W_p^0(\Omega)$ is proper if $p > 2$. For $p \in (1,2)$ one has $W_p^0(\Omega) \hookrightarrow L_p(\Omega)$, see Remark 7.69 in [1].

Proposition 2.9 *Assume that f satisfies (1.2). Then, given $T > 0$ and $M > 0$, there exist numbers $C(T,M) > 0$, $p > 2(n+1)$, and $\beta \in (0,1]$ such that*

$$\|f(t,v,\nabla v)\|_{L_p(\Omega)} \leq C(T,M)(1 + \|v\|^{\beta}_{W_p^{2-1/p}(\Omega)})$$

for all $(t,v) \in [0,T] \times W_p^{2-1/p}(\Omega)$ such that $\|v\|_{L_\infty(\Omega)} \leq M$.

Proof. Let $t \in [0,T]$ and assume that $v \in W_p^{2-1/p}(\Omega)$ satisfies $\|v\|_{L_\infty(\Omega)} \leq M$. It follows from (1.2) that there is a positive constant $C = C(T,M)$ such that

$$|f(t,x,v(x),\nabla v(x)| \leq C(1 + |\nabla v(x)|^{1+\alpha}), \quad x \in \overline{\Omega}.$$

Integrating over Ω and increasing C if necessary, we get

$$\|f(t,\cdot,v,\nabla v)\|_{L_p(\Omega)} \leq C(1 + \| \, |\nabla v| \, \|^{1+\alpha}_{L_{p(1+\alpha)}(\Omega)}) \leq C(1 + \|v\|^{1+\alpha}_{W^1_{p(1+\alpha)}(\Omega)}). \quad (2.27)$$

Assume that $\theta \in (0,1)$ is such that

$$1 - \frac{n}{p(1+\alpha)} < (1-\theta)\left(-\frac{n}{p}\right) + \theta\left(2 - \frac{1}{p} - \frac{n}{p}\right) = -\frac{n}{p} + \theta\left(2 - \frac{1}{p}\right). \quad (2.28)$$

Then we have the following interpolation inequality (see [2])

$$\|v\|_{W^1_{p(1+\alpha)}(\Omega)} \leq C\|v\|^{1-\theta}_{L_p(\Omega)}\|v\|^{\theta}_{W_p^{2-1/p}(\Omega)}, \qquad v \in W_p^{2-1/p}(\Omega). \quad (2.29)$$

To choose suitable $\theta \in (0,1)$ and $p > 2(n+1)$ satisfying (2.28), observe that (2.28) is equivalent to

$$\left[1 + \frac{n}{p} - \frac{n}{p(1+\alpha)}\right] \cdot \frac{p}{2p-1} < \theta. \quad (2.30)$$

Since $1 + \alpha < 2$, we have

$$\left[1 + \frac{n}{p} - \frac{n}{p(1+\alpha)}\right] \cdot \frac{p}{2p-1} < \frac{1}{2} \cdot \frac{2p+n}{2p-1}.$$

It follows also from $1 + \alpha < 2$ that we can find $p > 2(n+1)$ such that

$$\frac{1}{2} \cdot \frac{2p+n}{2p-1} \leq \frac{1}{1+\alpha}.$$

We now choose θ such that

$$\Big[1+\frac{n}{p}-\frac{n}{p(1+\alpha)}\Big]\cdot\frac{p}{2p-1}<\theta\le\frac{1}{2}\cdot\frac{2p+n}{2p-1}\le\frac{1}{1+\alpha}.$$

and set $\beta := \theta(1+\alpha) \in (0,1]$. It then remains to combine (2.27) and (2.29) to complete the proof. □

As in Subsection 2.1, we set

$$\|V\|_\infty := \max_{x\in\overline{\Omega}} |v(x)| \quad \text{for} \quad V=(v,\gamma_{\Gamma_1}v)\in\mathbb{X}_1.$$

This is meaningful, since $\mathbb{X}_1 \hookrightarrow C(\overline{\Omega})\times C(\Gamma_1)$.

Corollary 2.10 *Assume that f satisfies* (1.2). *Then, given $T>0$ and $M>0$, there exist numbers $C(T,M)>0$ and $p>2(n+1)$ such that*

$$\|F(t,V)\|_{\mathbb{X}_{1/2p}} \le C(T,M)(1+\|V\|_{\mathbb{X}_1})$$

for all $t\in[0,T]$ and all $V\in\mathbb{X}_1$ such that $\|V\|_\infty \le M$.

Proof. From Proposition 2.7(iii) we know that

$$L_p(\Omega)\times B^{1-3/2p}_{pp}(\Gamma_1)\hookrightarrow \mathbb{X}_{1/2p}.$$

Further, as in the proof of Lemma 2.4(i), there is a positive constant C such that

$$\|g(t,\cdot,\gamma_{\Gamma_1}v)\|_{B^{1-3/2p}_{pp}(\Gamma_1)} \le C(1+\|V\|_{\mathbb{X}_1}).$$

Now the assertion follows from Proposition 2.9. □

Theorem 2.11 *Let $u_0\in W^2_p(\Omega)$ with $\gamma_{\Gamma_1}u_0=0$ be given and let u denote the classical solution to* (2.23). *Moreover, assume that* (1.2) *holds true for some $\alpha\in[0,1)$. Then $t^+<\infty$ implies that $\limsup_{t\to t^+}\|u(t,\cdot)\|_{L_\infty(\Omega)}=\infty$.*

Proof. Let $U_0 := (u_0,\gamma_{\Gamma_1}u_0)\in\mathbb{E}_1\subset\mathbb{X}_1$. Then $U:=(u,\gamma_{\Gamma_1}u)$ is the solution of

$$U(t)=e^{-t\mathbb{A}_0}U_0+\int_0^t e^{-(t-s)\mathbb{A}_0}F(s,U(s))\,ds$$

in $C([0,t^+),\mathbb{X}_1)$, see Remark 2.8(ii). Furthermore, by interpolation, it follows from Proposition 2.7(i) that there are positive constants K and ω such that

$$\|e^{-t\mathbb{A}_0}\|_{\mathcal{L}(\mathbb{X}_1)}+t^{1-(1/2p)}\|e^{-t\mathbb{A}_0}\|_{\mathcal{L}(\mathbb{X}_{1/2p},\mathbb{X}_1)}\le Ke^{t\omega},\quad t>0.$$

Using this estimate together with Corollary 2.10, the assertion follows as in the proof of Theorem 2.6. □

2.3 Critical blow-up exponent

Consider the problem

$$\left.\begin{aligned} u_t - u_{xx} &= f(u_x), && t>0,\ 0<x<L,\\ u(t,0) &= 0, && t \geq 0,\\ u_t(t,L)+u_x(t,L) &= 0, && t>0,\\ u(0,x) &= u_0(x), && 0 \leq x \leq L, \end{aligned}\right\} \tag{2.31}$$

where $f \in C^2(\mathbb{R},(0,\infty))$ and $L>0$ are given. Given $u_0 \in C^2[0,L]$ with $u_0(0)=0$, we know by Theorem 2.5 that there is some $t^+>0$ such that (2.31) has a unique solution

$$u \in C^1((0,t^+),C[0,L]) \ \cap\ C((0,t^+),C^2[0,L]) \ \cap\ C([0,t^+),C^1[0,L]),$$

since $p>2$ ensures that $W_p^{2-1/p}(0,L) \subset C^1[0,L]$. Moreover, if $t^+<\infty$, then $\limsup_{t\uparrow t^+} \|u(t,\cdot)\|_{W_p^{2-1/p}(0,L)} = \infty$.

The following result will be useful in our investigations.

Lemma 2.12 [11] *Let $T>0$ and $u \in W^{1,1}((0,T),C(\overline{\Omega}))$. Then for every $t \in (0,T)$ there exists at least one pair of points $\xi(t),\ \zeta(t) \in \overline{\Omega}$ with*

$$m(t) := \min_{x\in\overline{\Omega}}[u(t,x)] = u(t,\xi(t)), \quad M(t) := \max_{x\in\overline{\Omega}}[u(t,x)] = u(t,\zeta(t)),$$

and the functions $m(t),\ M(t)$ are absolutely continuous on $(0,T)$ with

$$\frac{dm}{dt}(t) = u_t(t,\xi(t)) \quad \text{and} \quad \frac{dM}{dt}(t) = u_t(t,\zeta(t)) \quad \text{a. e. on } (0,T).$$

The previous result allows us to investigate the time evolution of the maxima and minima of classical solutions for nonlinear partial differential equations. This leads in some cases to criteria for global existence of solutions [12] or to blow-up results [11].

Proposition 2.13 *For every $u_0 \in C^2[0,L]$ with $u_0(0)=0$ the corresponding classical solution of (2.31) stays uniformly bounded in finite time.*

Proof. Let $t^+>0$ be the maximal existence time of the unique solution to (2.30) with initial data u_0. Lemma 2.12 guarantees that the functions

$$m(t) = \min_{x\in[0,L]}\{u(t,x)\} \leq 0, \ \ M(t) = \max_{x\in[0,L]}\{u(t,x)\} \geq 0,$$

are absolutely continuous on $(0,t^+)$ and satisfy

$$\frac{dm}{dt}(t) = u_t(t,\xi_1(t)) \quad \text{and} \quad \frac{dM}{dt}(t) = u_t(t,\xi_2(t)) \quad \text{a. e. on} \quad (0,t^+),$$

where $\xi_1(t)$, $\xi_2(t) \in [0, L]$ are such that $m(t) = u(t, \xi_1(t))$ and $M(t) = u(t, \xi_2(t))$ for $t \in (0, t^+)$. Using the partial differential equation in (2.31), we deduce that if m is differentiable at some $t \in (0, t^+)$ for which $\xi_1(t) \in (0, L)$, then $m_t \geq f(0)$ since then $u_{xx}(t, \xi_1(t)) \geq 0$ while $u_x(t, \xi_1(t)) = 0$. If m is differentiable at some $t \in (0, t^+)$ and $m(t) = u(t, 1)$, from (2.31) we deduce that $m_t \geq 0$ since $u_x(t, 1) \leq 0$ in this case. Finally, if $m(t) = u(t, 0)$, then $m(t) = 0$ by (2.31). We conclude that at all points $t \in (0, t^+)$ where m is differentiable we either have $m_t \geq 0$ or $m = 0$. A similar argument shows that at all points $t \in (0, t^+)$ where M is differentiable we either have $M_t \leq 0$ or $M = 0$. From here we infer the statement. □

Proposition 2.14 *Assume that*

$$\liminf_{s \to \infty} f(s) > 0, \tag{2.32}$$

and

$$K_0 = \int_0^\infty \frac{s\,ds}{f(s)} < \infty. \tag{2.33}$$

If $L > 0$ *is such that*

$$\int_{-K_0}^\infty \frac{ds}{f(s)} < L < \int_{\mathbb{R}} \frac{ds}{f(s)} = L_0, \tag{2.34}$$

then for every nonnegative initial data $u_0 \in C^2[0, L]$ *satisfying* $u_0(0) = 0$*, the corresponding classical solution of* (2.31) *ceases to exist after finite time despite the fact that its supremum norm does not blow-up.*

Proof. In our approach to prove blow-up we use some ideas devised in [18] for the case of homogeneous Dirichlet boundary conditions. Let us introduce the strictly decreasing function $G : \mathbb{R} \to (0, L_0)$ defined by

$$G(s) = \int_s^\infty \frac{dr}{f(r)}, \qquad s \in \mathbb{R}.$$

Further, let $\varphi : (0, L_0) \to (-\infty, K_0)$ be the C^2-function given by

$$\varphi(s) = \int_{G^{-1}(s)}^\infty \frac{r\,dr}{f(r)}, \quad s \in (0, L_0).$$

Note that φ has a continuous extension to $[0, L_0)$ and

$$\varphi''(s) + f(\varphi'(s)) = 0, \qquad 0 < s < L_0, \tag{2.35}$$

with

$$\varphi(0) = 0 \quad \text{and} \quad \varphi'(0) = \lim_{\varepsilon \downarrow 0} \varphi'(\varepsilon) = \infty. \tag{2.36}$$

Since $L < L_0$ in view of (2.34), the last relation in (2.36) ensures the existence of some constant $A > 0$ with

$$\varphi'(x) \geq -A, \qquad x \in [0, L]. \tag{2.37}$$

According to (2.32) there is some $\eta > 0$ with

$$f(s) \geq \eta, \qquad s \geq -A. \tag{2.38}$$

On the other hand, in view of (2.34), we may choose σ, $\delta \in (0, 1)$ such that for $K = K_0/\sigma$ we have

$$1 > \sigma > \frac{1}{L} \int_{-K}^{\infty} \frac{ds}{f(s)}, \tag{2.39}$$

and

$$0 < \delta \leq \min\{1,\, K,\, (1-\sigma)\eta\}. \tag{2.40}$$

For a given $u_0 \in C^1[0, L]$ satisfying $u_0(0) = 0$ we claim that the maximal existence time t^+ of the solution satisfies $t^+ \leq \frac{K}{\delta}$. To prove this we show that the assumption $t^+ > \frac{K}{\delta}$ leads to a contradiction.

Since $\varphi(s) \leq K_0$ for all $s \in (0, L_0)$, we deduce that

$$\frac{1}{\sigma}\, \varphi(\sigma x) \leq K, \;\; x \in [0, L].$$

Define

$$\phi(t, x) = \frac{1}{\sigma}\, \varphi(\sigma x) - K + \delta t, \;\; t > 0, \;\; x \in [0, L]. \tag{2.41}$$

Then

$$\phi(0, x) \leq u_0(x), \;\; x \in [0, L], \tag{2.42}$$

while (2.36) ensures that

$$\phi(t, 0) = \delta t - K \leq 0, \;\; 0 < t \leq \frac{K}{\delta}. \tag{2.43}$$

On the other hand, from (2.41) we infer

$$\phi_t(t, L) + \phi_x(t, L) = \delta + \varphi'(\sigma L) \leq 0, \;\; 0 < t \leq \frac{K}{\delta}, \tag{2.44}$$

since by (2.39) we have $\sigma L > G(-K)$ and this forces $\varphi'(\sigma L) = G^{-1}(\sigma L) < -K$ in view of the fact that $G : \mathbb{R} \to (0, L_0)$ is nondecreasing. Therefore (2.44) follows from the choice of δ satisfying (2.40). Finally,

$$\phi_t - \phi_{xx} - f(\phi_x) \leq 0, \;\; 0 < t < \frac{K}{\delta}, \;\; 0 < x < L. \tag{2.45}$$

Indeed, we have that

$$\begin{aligned}\phi_t - \phi_{xx} - f(\phi_x) &= \delta - \sigma\varphi''(\sigma x) - f(\varphi'(\sigma x)) \\ &= \delta - \sigma(\varphi''(\sigma x) + f(\varphi'(\sigma x))) - (1-\sigma) f(\varphi'(\sigma x)) \\ &= \delta - (1-\sigma) f(\varphi'(\sigma x)) \le \delta - (1-\sigma)\eta \le 0,\end{aligned}$$

if we take into account (2.35), (2.37), (2.38), and (2.40).

From (2.31) and (2.42)–(2.45) we deduce that the function $v \in W_1^1((0, \frac{K}{\delta}), C[0,L])$ defined by $v(t,x) = u(t,x) - \phi(t,x)$ satisfies

$$\left.\begin{array}{rcll} v_t - v_{xx} - f(u_x) + f(\phi_x) & \ge & 0, & 0 < x < L,\ 0 < t \le \frac{K}{\delta}, \\ v(t,0) & \ge & 0, & 0 < t \le \frac{K}{\delta}, \\ v_t(t,L) + v_x(t,L) & \ge & 0, & 0 < t \le \frac{K}{\delta}, \\ v(0,x) & \ge & 0, & 0 \le x \le L. \end{array}\right\} \tag{2.46}$$

An argumentation similar to the one used in the proof of Proposition 2.13, and consisting in tracking the time evolution of the minimum of $v(t,\cdot)$, shows that

$$\phi(t,x) \le u(t,x), \ \ 0 \le t \le \frac{K}{\delta}, \ \ 0 \le x \le L. \tag{2.47}$$

But then, since $\phi(\frac{K}{\delta}, 0) = 0$ in view of (2.36) and (2.41), we would have that

$$\begin{aligned} u_x\left(\frac{K}{\delta}, 0\right) &= \lim_{\varepsilon\downarrow 0} \frac{u(\frac{K}{\delta},\varepsilon) - u(\frac{K}{\delta},0)}{\varepsilon} = \lim_{\varepsilon\downarrow 0} \frac{u(\frac{K}{\delta},\varepsilon)}{\varepsilon} \\ &\ge \liminf_{\varepsilon\downarrow 0} \frac{\phi(\frac{K}{\delta},\varepsilon)}{\varepsilon} = \liminf_{\varepsilon\downarrow 0} \frac{\phi(\frac{K}{\delta},\varepsilon) - \phi(\frac{K}{\delta},0)}{\varepsilon} \\ &= \liminf_{\varepsilon\downarrow 0} \frac{\varphi(\sigma\varepsilon) - \varphi(0)}{\sigma\varepsilon} = \varphi'(0) = \infty. \end{aligned}$$

The obtained contradiction shows that the solution blows-up in finite time $t^+ < \frac{K}{\delta}$. The proof is complete. □

Remark 2.15 As pointed out in the Introduction, the growth condition (1.3) with $\alpha \in [0,1)$ is close to optimal for the occurence of blow-up in the L_∞-norm cf. Theorem 2.11. Indeed, if $f(\eta) = 1 + |\eta|^{2+\delta}$ for some $\delta > 0$, it follows from Proposition 2.14 that there are classical solutions to (2.31) which blow up in finite time while staying uniformly bounded. □

3 Global existence

In this section we give a criterion for the global existence of all classical solutions to certain problems of type (1.1). To show that our global existence result is quite sharp, we also address the blow-up issue.

Throughout this section we consider the problem (1.1) in the setting presented in the Introduction. In the quasilinear case we assume that (1.2) holds with $\alpha = 0$, while for semilinear problems of type (2.23) we assume (1.2) with some $\alpha \in [0,1)$. Therefore, in view of Theorem 2.6, respectively Theorem 2.11, blow-up can occur only if the classical solution becomes unbounded in finite time.

The following result on differential inequalities of Carathéodory type will be useful.

Lemma 3.1 [13] *For $w(t,z) \in C(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$, let $r(t)$ be the maximal solution and let $\rho(t)$ be the minimal solution of the ordinary differential equation*

$$z'(t) = w(t,z), \quad z(0) = z_0, \tag{3.1}$$

defined on some interval $[0,T]$ with $T > 0$.

(i) *If $q(t) : [0,T] \to \mathbb{R}$ is absolutely continuous and satisfies*

$$q'(t) \le w(t,q(t)) \quad a.e. \ \ on \ \ [0,T],$$

with $q(0) \le z_0 = r(0)$, then $q(t) \le r(t)$, $t \in [0,T]$.

(ii) *If $q(t) : [0,T] \to \mathbb{R}$ is absolutely continuous and satisfies*

$$q'(t) \ge w(t,q(t)) \quad a.e. \ \ on \ \ [0,T],$$

with $q(0) \ge z_0 = \rho(0)$, then $q(t) \ge \rho(t)$, $t \in [0,T]$.

Recall [24] that if $w \in C(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$ is given, a solution $r \in C^1([0,t_0), \mathbb{R})$ with $t_0 > 0$, of the scalar differential equation $z' = w(t,z(t))$ is said to be a maximal solution if for every solution z of the equation on $[0,t_0)$ with $z(0) = r(0)$, the inequality

$$z(t) \le r(t), \quad t \in [0,t_0)$$

holds. A minimal solution is defined similarily by reversing the above inequality. It is known [24] that for every $w \in C(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$ and every $z_0 \in \mathbb{R}$, there is some $t_0 = t_0(z_0) > 0$ such that there exists unique maximal and minimal solutions of the Cauchy problem

$$z' = w(t,z(t)), \quad z(0) = z_0,$$

defined on $[0,t_0)$.

Remark 3.2 In order to apply Lemma 3.1 to prevent blow-up in finite time, it is useful to provide families of functions $w \in C(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$ for which the maximal and minimal solutions to (3.1) are global.

(i) Let $w(t,r) = \beta(t)\,\omega(r)$ with $\beta \in C(\mathbb{R}_+, (0,\infty))$ and $\omega \in C(\mathbb{R}, (0,\infty))$. Then all solutions of (3.1) are global if and only if $\int_0^\infty \frac{ds}{\omega(s)} = \infty$, cf. [10]. On the other hand, if $\int_0^\infty \frac{ds}{\omega(s)} < \infty$, then all solutions of (3.1) blow up in finite time.

(ii) If $w(t,r) = -\beta(t)\,\omega(r)$ with $\beta \in C(\mathbb{R}_+,(0,\infty))$ and $\omega \in C(\mathbb{R},(0,\infty))$, then all solutions of (3.1) are global if and only if $\int_{-\infty}^{0} \frac{ds}{\omega(s)} = \infty$.

(iii) If $w \in C(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$ satisfies

$$z\,w(t,z) \le \beta(t)\,\omega(z^2), \qquad (t,z) \in \mathbb{R}_+ \times \mathbb{R}, \tag{3.2}$$

where $\beta \in C(\mathbb{R}_+,\mathbb{R}_+)$, and $\omega \in C(\mathbb{R}_+,\mathbb{R}_+)$ is nondecreasing with $\int_0^{\infty} \frac{ds}{1+\omega(s)} = \infty$, then all solutions to (3.1) are global. This follows by using (3.2) to derive a differential inequality for $z^2(t)$, where $z(t)$ is a solution, and concluding by (i). □

We proceed now with a global existence criterion for problems of type (1.1).

Theorem 3.2 *Assume that for all $(t,x,u) \in \mathbb{R}_+ \times \overline{\Omega} \times \mathbb{R}$ we have*

$$w_1(t,u) \le f(t,x,u,0) \le w_2(t,u), \tag{3.3}$$

and

$$w_1(t,u) \le g(t,x,u) \le w_2(t,u), \tag{3.4}$$

for some w_1, $w_2 \in C(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$ satisfying

$$w_1(t,0) \le 0 \le w_2(t,0), \qquad t \in \mathbb{R}_+. \tag{3.5}$$

Then for any initial data $u_0 \in W_p^2(\Omega)$, $p > 2(n+1)$, with $u_0(x) = 0$ for all $x \in \Gamma_0$, and such that the minimal solution $\rho(t)$ of the differential equation

$$z_t = w_1(t,z), \quad z(0) = \min_{x\in\overline{\Omega}}\{u_0(x)\} \le 0,$$

exists globally in time, and the maximal solution $r(t)$ of the differential equation

$$z_t = w_2(t,z), \quad z(0) = \max_{x\in\overline{\Omega}}\{u_0(x)\} \ge 0,$$

exists globally in time, the corresponding unique classical solution to (1.1) *is defined globally in time.*

Proof. Let u be the classical solution of (1.1) with the initial data u_0 and let $t^+ > 0$ be its maximal existence time. Set

$$m(t) := \min_{x\in\overline{\Omega}}[u(t,x)], \quad M(t) := \max_{x\in\overline{\Omega}}[u(t,x)], \qquad t \in [0,t^+).$$

Define the open set $I = \{t \in (0,t^+) : M(t) > 0\}$. If $I \ne \emptyset$, then for $t \in I$, $M(t)$ is attained at some point $\zeta(t) \in \Omega \cup \Gamma_1$. If $\zeta(t) \in \Omega$, then $\nabla u(t,\zeta(t)) = 0$ and

$$\sum_{i,j=1}^{n} a_{ij}(t,\zeta(t),u(t,\zeta(t)))\,\frac{\partial^2 u(t,\zeta(t))}{\partial x_i \partial x_j} \le 0$$

cf. [20]. Applying Lemma 2.12 and using (3.3), we get

$$\begin{aligned}\frac{dM(t)}{dt} = u_t(t,\zeta(t)) &= \sum_{i,j=1}^{n} a_{ij}(t,\zeta(t),u(t,\zeta(t)))\frac{\partial^2 u(t,\zeta(t))}{\partial x_i \partial x_j} \\ &+f(t,\zeta(t),u(t,\zeta(t)),0) \\ &\leq f(t,\zeta(t),u(t,\zeta(t)),0) \leq w_2(t,M(t)) \quad \text{a.e. } \{t\in I:\ \zeta(t)\in\Omega\}.\end{aligned}$$

If $t \in I$ but $\zeta(t) \in \Gamma_1$, then $u_\nu(t,\zeta(t)) \geq 0$. Therefore, the boundary condition on Γ_1 and Lemma 2.12 show that $M_t \leq g(t,\zeta(t),M(t))$ if M is differentiable at t. Using (3.4), we deduce that

$$\frac{dM(t)}{dt} \leq w_2(t,M(t)) \quad \text{a.e. } \{t\in I:\zeta(t)\in\Gamma_1\}.$$

Finally, if $t \notin I$, then it follows that $t \in J = (0,t^+)\setminus I = \{t\in(0,t^+): M(t)=0\}$. As in the proof of Theorem 3.1 in [12], we see that $M'(t) = 0$ a.e. on J so that (3.5) forces $M'(t) \leq \omega_2(t,M(t))$ a.e. on J. Therefore

$$M'(t) \leq w_2(t,M(t)) \quad \text{a.e. } t\in(0,t^+). \tag{3.6}$$

Note that if $I=\emptyset$, then $M(t)\equiv 0$, and (3.5) yields again the above inequality.

Similarly, we can prove that

$$m'(t) \geq w_1(t,m(t)) \quad \text{a.e. } t\in(0,t^+). \tag{3.7}$$

Assume that $t^+ < \infty$. Then it follows that $\limsup_{t\to t^+}\|u(t,\cdot)\|_{C(\overline{\Omega})} = \infty$. Thus, we obtain a sequence $\{t_k\} \subset (0,t^+)$ converging to t^+ such that

$$\lim_{t_k\to t^+} M(t_k) = \infty \quad \text{or} \quad \lim_{t_k\to t^+} m(t_k) = -\infty.$$

Assume first that $\lim_{t_k\to t^+} M(t_k) = \infty$.

By hypothesis, the maximal solution $r(t)$ of the ordinary differential equation

$$z_t = w_2(t,z), \quad z(0) = M(0) \geq 0,$$

exists globally in time. Applying Lemma 3.1, in view of (3.6), we obtain

$$M(t) \leq r(t), \quad t\in[0,t^+).$$

This contradicts $\lim_{t_k\to t^+} M(t_k) = \infty$.

Similarly, if $\lim_{t_k\to t^+} m(t_k) = -\infty$, using Lemma 3.1 and (3.7), we also obtain a contradiction. Therefore, $t^+ = \infty$ and the solution u exists globally in time. This completes the proof. □

Example 3.4 Let $\alpha \in [0,1)$ and $\varepsilon_0 \geq 0$ be given. Then all classical solutions of the boundary value problem

$$\begin{aligned} u_t - \Delta u &= (u - \varepsilon_0 u^3)\,(1 + |\nabla u|^{1+\alpha})\,\ln(1+u^2), && t > 0,\ 1 < |x| < 2, \\ u(t,x) &= 0, && t \geq 0,\ |x| = 1, \\ u_t + \tfrac{1}{2}\,\nabla u \cdot x &= (u - \varepsilon_0 u^3)\,\ln(1+u^2), && t > 0,\ |x| = 2, \\ u(0,x) &= u_0(x), && 1 \leq |x| \leq 2, \end{aligned}$$

are global. To see this, it suffices to apply Theorem 3.3 with

$$w_1(t,u) = w_2(t,u) = (u - \varepsilon_0 u^3)\,\ln(1+u^2), \qquad (t,u) \in \mathbb{R}_+ \times \mathbb{R},$$

in view of Remark 3.2 (iii). □

To illustrate the importance of the conditions (3.3)–(3.4) with suitable functions $w_1, w_2 \in C(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$, we consider the problem

$$\left.\begin{aligned} u_t - \Delta u &= \omega(u) + b(t,x)\,|\nabla u|^{3/2}, && t > 0,\ x \in \Omega, \\ u(t,x) &= 0, && t \geq 0,\ x \in \Gamma_0 \subset \partial\Omega, \\ u_t + \partial_\nu u &= \omega(u), && t > 0,\ x \in \Gamma_1 \subset \partial\Omega, \\ u(0,x) &= u_0(x), && x \in \overline{\Omega}, \end{aligned}\right\} \tag{3.8}$$

of type (1.1), where $\omega \in C^2(\mathbb{R}, \mathbb{R})$ and $b \in C(\mathbb{R}_+ \times \overline{\Omega}, \mathbb{R}_+)$.

Proposition 3.5 *If the restriction of $\omega \in C^2(\mathbb{R}, \mathbb{R})$ to $\mathbb{R}_+$ is positive, nondecreasing, and convex, then the condition $\int_0^\infty \frac{ds}{\omega(s)} < \infty$ ensures that there are classical solutions to* (3.8) *which blow-up in finite time.*

Proof. Let $\lambda \geq 0$ and ϕ denote the first eigenvalue and a corresponding eigenfunction of the following eigenvalue problem

$$\left.\begin{aligned} \Delta\phi + \lambda\phi &= 0 \quad \text{in} \quad \Omega, \\ \phi &= 0 \quad \text{on} \quad \Gamma_0 \subset \partial\Omega, \\ \phi_\nu &= 0 \quad \text{on} \quad \Gamma_1 \subset \partial\Omega. \end{aligned}\right\}$$

We choose ϕ such that $\phi(x) > 0$ in Ω and $\int_\Omega \phi(x)\,dx = 1$. Let $\beta = \int_{\Gamma_1} \phi(x)\,d\sigma$. It follows from the strong maximum principle that $\beta > 0$.

Since $\omega : \mathbb{R}_+ \to (0,\infty)$ is convex and nondecreasing, we deduce that $\omega' : \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing. The condition $\int_0^\infty \frac{ds}{\omega(s)} < \infty$ shows that $\lim_{r\to\infty} \omega'(r) = \infty$, as otherwise the restriction of ω to $\mathbb{R}_+$ would be sublinear. Choose $K > 0$ such that

$$\omega(r) \geq \frac{2\lambda}{\beta + 1}\,r, \quad r \geq K. \tag{3.9}$$

If $u_0 \in C^2(\overline{\Omega})$ is a nonnegative function such that $\gamma_{\Gamma_0} u_0 = 0$ and $\int_\Omega u_0(x)\phi(x)\,dx > K$, let us prove that the maximal existence time $t^+ > 0$ of the corresponding classical solution u to (3.8) is finite.

Looking at the time evolution of the minimum of $u(t,\cdot)$, an argument similar to the one pursued in the proof of Theorem 3.3 will yield that $u(t,x) \geq 0$ for all $(t,x) \in [0,t^+) \times \overline{\Omega}$. Our aim is to show that $t^+ < \infty$. To this end, multiplying the differential equation in (3.8) by $\phi(x)$ and using Green's second identity [20], we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega u\,\phi\,dx &= \int_\Omega u_t\,\phi\,dx = \int_\Omega \phi\,\Delta u\,dx + \int_\Omega \phi\,\omega(u)\,dx + \int_\Omega \phi\,b(t,x)\,|\nabla u|^{3/2}\,dx \\ &\geq \int_\Omega u\,\Delta\phi\,dx + \int_{\Gamma_1} \phi\,u_\nu d\sigma + \int_\Omega \phi\,\omega(u)\,dx \\ &= -\lambda\int_\Omega u\,\phi\,dx + \int_{\Gamma_1}\phi\,u_\nu d\sigma + \int_\Omega \phi\,\omega(u)\,dx, \qquad 0<t<t^+.\end{aligned}$$

On the other hand,

$$\frac{d}{dt}\int_{\Gamma_1} u\,\phi\,d\sigma = \int_{\Gamma_1} u_t\phi\,d\sigma, \qquad 0<t<t^+,$$

so that

$$\begin{aligned}\frac{d}{dt}\left(\int_\Omega u\,\phi\,dx + \int_{\Gamma_1} u\,\phi\,d\sigma\right) &\geq -\lambda\int_\Omega u\,\phi\,dx + \int_{\Gamma_1}\phi\,(u_t+u_\nu)d\sigma + \int_\Omega\phi\,\omega(u)\,dx \\ = -\lambda\int_\Omega u\,\phi\,dx + \int_{\Gamma_1}\phi\,\omega(u)d\sigma + \int_\Omega \phi\,\omega(u)\,dx, &\qquad 0<t<t^+,\end{aligned}$$

if we take into account (3.8). Using Jensen's inequality, we infer that for all $t \in (0,t^+)$ we have

$$\begin{aligned}\frac{d}{dt}\left(\int_\Omega u\,\phi\,dx + \int_{\Gamma_1} u\,\phi\,d\sigma\right) \geq -\lambda\int_\Omega u\,\phi\,dx + \omega\left(\int_\Omega u\,\phi\,dx\right) \\ + \beta\omega\left(\frac{1}{\beta}\int_{\Gamma_1} u\,\phi\,d\sigma\right).\end{aligned} \tag{3.10}$$

The convexity of ω on $\mathbb{R}_+$ yields

$$\begin{aligned}&\omega\left(\int_\Omega u\,\phi\,dx\right) + \beta\omega\left(\frac{1}{\beta}\int_{\Gamma_1} u\,\phi\,d\sigma\right) \\ &= (\beta+1)\left(\frac{1}{\beta+1}\,\omega\left(\int_\Omega u\,\phi\,dx\right) + \frac{\beta}{\beta+1}\,\omega\left(\frac{1}{\beta}\int_{\Gamma_1} u\,\phi\,d\sigma\right)\right) \\ &\geq (\beta+1)\,\omega\left(\frac{1}{\beta+1}\int_\Omega u\,\phi\,dx + \frac{1}{\beta+1}\int_{\Gamma_1} u\,\phi\,d\sigma\right).\end{aligned} \tag{3.11}$$

Let us denote

$$v(t) := \frac{1}{\beta+1}\left(\int_\Omega u\,\phi\,dx + \int_{\Gamma_1} u\,\phi\,d\sigma\right), \qquad t \in (0,t^+).$$

By (3.10)–(3.11) we get

$$\frac{d}{dt}\,v(t) \geq -\,\frac{\lambda}{\beta+1}\,v(t) + \omega(v(t)), \qquad t \in (0,t^+). \tag{3.12}$$

Since $v(0) > K$ by assumption, we claim that $v(t) > K$ for all $t \in [0,t^+)$. Indeed, if this does not hold true, then there is some $t_0 \in (0,t^+)$ such that $v(t_0) = K$ and $v(t) > K$ for all $t \in [0,t_0)$. But then (3.12) together with (3.9) show that $v(t)$ is increasing on $(0,t_0)$. Hence $v(t_0) > K$. The obtained contradiction proves our claim. Knowing this, in view of (3.9), we obtain from (3.12) that

$$\frac{d}{dt}\,v(t) \geq \frac{1}{2}\,\omega(v(t)), \qquad t \in (0,t^+).$$

Taking into account Remark 3.2 (i) we deduce from the above and Lemma 3.1 that $v(t)$ blows-up in finite time. This shows that $t^+ < \infty$. □

Remark 3.6 Observe that in accordance to Remark 3.2 (i), all solutions of the differential equation $r' = \omega(r)$ with initial data $r(0) \geq 0$ will blow-up in finite time. Note the contrast to the hypotheses of Theorem 3.3.

Example 3.7 As a simple consequence of Proposition 3.5, note that, given $b \in C(\mathbb{R}_+,\mathbb{R}_+)$, for every $q > 1$ there are classical solutions of the problem

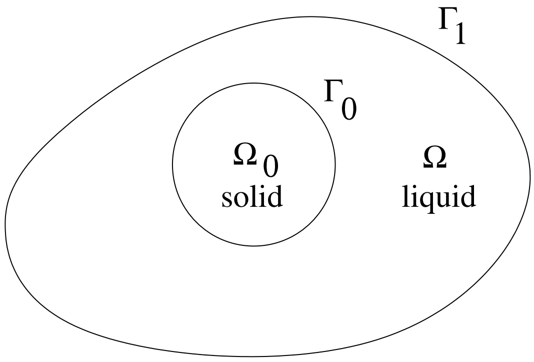
$$\begin{aligned} u_t - \Delta u &= (1+u^2)^{q/2} + b(t)\,|\nabla u|^{3/2}, && t>0,\ x\in\Omega,\\ u(t,x) &= 0, && t\geq 0,\ x\in\Gamma_0\subset\partial\Omega,\\ u_t + \partial_\nu u &= (1+u^2)^{q/2}, && t>0,\ x\in\Gamma_1\subset\partial\Omega,\\ u(0,x) &= u_0(x), && x\in\overline{\Omega}, \end{aligned}$$

which blow-up in finite time.

4 Appendix: A physical model

We present a phenomenological deduction of a problem of type (1.1). For further discussions of models of type (1.1) we refer to [17], [22], and [26].

Consider a medium occupying a bounded domain $\Omega_1 \subset \mathbb{R}^n$ ($n = 2, 3$) with smooth boundary $\Gamma_1 = \partial\Omega_1$, and in which a solid and a liquid coexist. We assume that the solid is immersed in the liquid. That is, if Ω_0 is the solid, the smooth hypersurface $\Gamma_0 = \partial\Omega_0$ is disjoint from Γ_1. Let the medium be surrounded by a sufficiently thin material having large thermal conductivity. We neglect the



thickness of the boundary material. For instance we can consider an iron tube filled with water and containing a cylindrical rod. We consider the problem of finding the interior temperature distribution $u = u(t,x)$ in the liquid $\Omega = \Omega_1 - \overline{\Omega_0}$ if the solid Ω_0 is kept at zero temperature.

In the liquid the heat equation is satisfied,

$$u_t - \operatorname{div}(c(t,x,u)\,\nabla u) = f(t,x,u) \quad \text{in} \quad \Omega, \tag{4.1}$$

where c is the diffusion coefficient (or thermal conductivity) and f is an internal heat source. The vector $c\,\nabla u$ is called the heat flux. By the thermal contact of the liquid with the solid we have the boundary condition

$$u(t,x) = 0 \quad \text{on} \quad \Gamma_0. \tag{4.2}$$

On the other hand, the heat flow from inside Ω to the boundary is $-\,cu_\nu$, where ν is the outward unit normal to Ω on Γ_1. The accumulation rate of heat on the boundary is u_t so that, if $h(t,x,u)$ is a source term on Γ_1, we must have that

$$u_t + cu_\nu = h(t,x,u) \quad \text{on} \quad \Gamma_1. \tag{4.3}$$

Prescribing the initial temperature $u(0,x) = u_0(x)$ on $\overline{\Omega} = \Omega \cup \Gamma_0 \cup \Gamma_1$, we see that (4.1)–(4.3) amounts to a problem of type (1.1).

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References

[1] R.A. ADAMS, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] H. AMANN, Global existence for semilinear parabolic systems, *J. Reine Angew. Math.* **360** (1985), 57–83.

[3] H. AMANN, Dynamic theory of quasilinear parabolic equations - I. Abstract evolution equations, *Nonlinear Analysis* **12** (1988), 895–919.

[4] H. AMANN, Dynamic theory of quasilinear parabolic equations - II Reaction-Diffusion Systems, *Differential Integral Equations* **3** (1990), 13–75.

[5] H. AMANN, Dynamic theory of quasilinear parabolic systems - III Global existence, *Math. Z.* **202** (1989), 219–250.

[6] H. AMANN, Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems, in H. J. Schmeisser, H. Triebel, editors, *Function Spaces, Differential Operators and Nonlinear Analysis*, Teubner, Stuttgart, Leipzig, 1993, 9–126.

[7] H. AMANN, *Linear and Quasilinear Parabolic Problems, Volume I: Abstract Linear Theory*, Birkhäuser, Basel, 1995.

[8] J. ARRIETA, P. QUITTNER, and A. RODRIGUEZ-BERNAL, Parabolic problems with nonlinear dynamical boundary conditions and singular initial data, *Differential Integral Equations* **14** (2001), 1487–1510.

[9] G.I. BIZHANOVA and V.A. SOLONNIKOV, On the solvability of the initial boundary value problem with time derivative in the boundary condition for a second order parabolic equation in a weighted function space (in Russian), *Algebra Analiz* **5** (1993), 109–142.

[10] A. CONSTANTIN, Global existence of solutions for perturbed differential equations, *Annali Mat. Pura Appl.* **168** (1995), 237–299.

[11] A. CONSTANTIN and J. ESCHER, Wave breaking for nonlinear nonlocal shallow water equations, *Acta Mathematica* **181** (1998), 229–243.

[12] A. CONSTANTIN and J. ESCHER, Global solutions for quasilinear parabolic problems, *J. Evolution Equations* **2** (2002), 97–111.

[13] A. CONSTANTIN, J. ESCHER and Z. YIN, Global solutions for quasilinear parabolic systems, *J. Differential Equations* **197** (2004), 73–84.

[14] A. CONSTANTIN and S. PESZAT, Global existence of solutions of semilinear parabolic evolution equations, *Differential Integral Equations* **13** (2000), 99–114.

[15] J. ESCHER, Quasilinear parabolic systems with dynamical boundary conditions, *Comm. P.D.E.* **17** (1993), 1309–1365.

[16] J. ESCHER, On quasilinear fully parabolic boundary value problems, *Differential Integral Equations* **7** (1994), 1325–1343.

[17] J. ESCHER, On the qualitative behaviour of some semilinear parabolic problems, *Differential Integral Equations* **8** (1995), 247–267.

[18] M. FILA and G. LIEBERMAN, Derivative blow-up and beyond for quasilinear parabolic equations, *Differential Integral Equations* **7** (1994), 811–821.

[19] M. FILA and P. QUITTNER, Large time behavior of solutions of a semilinear parabolic equation with a nonlinear dynamical boundary condition, in *Topics in Nonlinear Analysis* PNLDE 35, Birkhäuser Verlag, Basel 1999, 251–272.

[20] D. GILBARG and N.S. TRUDINGER, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, Heidelberg, New York, 1997.

[21] T. HINTERMANN, Evolution equations with dynamic boundary conditions, *Proc. Roy. Soc. Edinburgh* **113** (1989), 43–60.

[22] R. LANGER, A problem in diffusion or in the flow of heat for a solid in contact with a fluid, *Tohoku Math. J.* **35** (1932), 260–275.

[23] A. LUNARDI, *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, Birkhäuser, Basel, 1995.

[24] B.G. PACHPATTE, *Inequalities for Differential and Integral Equations*, Academic Press, New York, 1998.

[25] R. SEELEY, Interpolation in L^p with boundary conditions, *Studia Math.* **64** (1972), 47–60.

[26] M. SLINKO and K. HARTMANN, *Methoden und Programme zur Berechnung Chemischer Reaktoren*, Akademie-Verlag, Berlin, 1972.

[27] H. TRIEBEL, *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland, Amsterdam, 1978.

[28] H. TRIEBEL, *Theory of Function Spaces*, Birkhäuser, Basel, 1983.

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