Systems & Control: Foundations & Applications

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Representation and Control of Infinite Dimensional Systems

Second Edition

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In memory of Andrée

Preface to the Second Edition

A new edition in a single volume

Over the past decade, more and more sophisticaced mathematical tools and approaches have been incorporated in the field of *Control of infinite dimensional systems*. This was motivated by a whole range of challenging applications arising from new phenomenological studies, technological developments, and more stringent design requirements. At the same time, researchers and advanced engineers have been steadily using an impressive amount of very sophisticated mathematics in their analysis, synthesis, and design of systems. What was regarded as too abstract, specialized, or theoretical in 1990 has now become a standard part of the toolkit.

The decision to produce a second edition of the original 1992–1993 two-volume edition is further motivated by several other factors. Over the years the book has been recognized as a key reference in the field, and a revised and corrected edition was desirable. Even if some good books on the control of infinite dimensional linear systems have appeared since then, we felt that the original material has not aged too much and that the breadth of its presentation is still attractive and very competitive. The result is a completely revised and corrected second edition in a single convenient volume with integrated bibliography and index.

The book has been restructured into five parts and each part into several chapters. The most significant changes occurred in Part I, which now provides a very broad account of *finite dimensional linear systems*. It serves as a background and a motivation for the other parts. The scope of this part has been expanded by adding topics such as *dissipative systems* in Chapter 1 and introducing a new Chapter 2 on *linear quadratic two-person zero-sum differential games* that provides an example of a solution to the matrix Riccati differential equation that is not necessarily positive semi-definite and some connections with the H^∞-theory and dissipative systems of Chapter 1.

Description of the five parts

Part I on the *control of finite dimensional linear dynamical systems* and *linear quadratic two-person zero-sum differential games* presents a broad review of the field. It is an introduction and a motivation for the book, so that the reader understands the background from which the infinite dimensional theory is developed and also obtains an appreciation of the substantial conceptual and technical difficulties that had to be overcome to give a satisfactory treatment of the subject in the infinite dimensional context. Chapter 1 is concerned with the theory of controllability and observability of linear systems, and the role these concepts play in the study of the quadratic cost problem over an infinite time horizon. It has been expanded to include, in addition to the updated section on the H^∞-theory, a new section on the theory of *dissipative systems* of J. C. Willems [2].

A new Chapter 2 on *linear quadratic two-person zero-sum differential games* has been included to broaden the perspective. The pioneering work in that area has been done in the paper of P. Bernhard [2] in 1979 and the seminal book of T. Başar and P. Bernhard [1] in 1991 and 1995. This chapter is self-contained with complete mathematical proofs. It focuses on the open loop case. New results using the *invariant embedding approach* of R. Bellman in the style of J. L. Lions [3] are included to provide further insight into the associated *inf sup* (open loop upper value) and *sup inf* (open loop lower value) problems. This is combined with developments by P. Zhang [1] in 2005 who established that in the linear quadratic case the duality gap is either zero or infinite. This means that only three cases can occur: (a) *sup inf* $= -\infty$ and *inf sup* finite; (b) *sup inf* finite and *inf sup* $= +\infty$; and (c) *sup inf* and *inf sup* both finite; in which case, there is equality and the existence of a saddle point. In particular it illustrates the occurrence of symmetrical solutions to the matrix Riccati differential equation that are not necessarily positive semidefinite. It also connects with the glimpse of H^∞-theory and the new section on the theory of *dissipative systems* at the end of Chapter 1.

Part II deals with the *representation of infinite dimensional systems.* It develops *semigroup theory and variational methods* for the representation of infinite dimensional systems such as dynamical partial differential equations and delay differential systems. Chapter 1 gives a unique presentation of the *theory of semigroups of linear operators* integrated with *interpolation theory.* It brings together advanced concepts and techniques that are usually treated independently. Chapter 2 provides a basic introduction to the *variational theory of parabolic systems.* It nicely describes the explicit and illuminating connection between the early work of T. Kato and J.-L. Lions and the later results of P. Auscher, S. Hofmann, J. L. Lewis, and Ph. Tchamitchian [1] in 2001 on the Kato's conjecture on the domain of the square root operators. Chapter 3 contains the basic constructions to effectively deal with unbounded control and observation operators via *semigroup methods.*

Chapter 4 on the modeling of differential systems with delays in the state, control, and observation is self-contained. To our knowledge, it is possibly the only book where the state space theory is completely developed using the fundamental *structural operators* introduced by Delfour and Manitius in 1976–1977 and further extended to the neutral case by M. C. Delfour and J. Karrakchou [1,2] in 1987.

Part III is devoted to the generic qualitative properties of controlled systems. It studies the *controllability for an infinite dimensional abstract linear dynamical system*, which can be specialized to obtain results for controllability of parabolic and hyperbolic partial differential equations both when control is exercised in the interior of the domain and when control is exercised through the boundary. The important problem of exact controllability of hyperbolic equations in appropriate spaces, which leads to stabilizability properties for these systems (and hence verification of the *finite cost* condition for infinite time problems) is discussed in detail in this chapter. The systematic use of eigenvalues and eigenfunctions of appropriate differential operators to obtain the results is a somewhat novel aspect of this chapter.

Parts IV and V present a dynamical programming approach to the *optimal linear quadratic control problem* over a finite and an infinite time horizon of certain classes of infinite dimensional systems. *Dynamic programming* is a deep conceptual idea. Its applications are manifold. For the philosophically minded reader, we have provided a quotation from Soren Kierkegaard on page xvi of the *Preface to Volume II of the First Edition* as an indication that philosophy might often anticipate developments in science.

Part IV is devoted to the *quadratic cost optimal control problem* over a *finite time horizon.* We develop the theory for an abstract dynamical model satisfying certain semigroup (group) assumptions, and then we treat concrete situations by verifying these assumptions using differential equations (partial, functional-differential) methods. This chapter presents a reasonably complete treatment of the subject, which includes both boundary control and boundary observation (not necessarily simultaneously) for parabolic and hyperbolic systems. Technically, these are the more difficult parts of the book because they involve unbounded control and observation operators. Many of the results presented here appeared in book form for the first time in 1993. The approach we adopt is *dynamic programming*, which leads to a synthesis of the optimal control in feedback form via a study of an operator Riccati differential equation. The systematic use of *dynamic programming* gives a unified view of this topic.

Part V, the final part of the book, is concerned with the *quadratic cost optimal control problem* over an *infinite time horizon.* Here the concepts of stabilizability and detectability play an essential role. The properties of stabilizability and detectability have to be verified, in some sense directly, for the parabolic case (which has a finite dimensional unstable part) and via the theory of exact controllability (for example) in the hyperbolic case. Thus, there is a close relation between Part V and Parts I and III of the book. The approach here is again via *dynamic programming* and a study of an algebraic matrix Riccati equation. We also prove the stability of the closed loop system. In many (but not all) situations, the theory of the infinite time quadratic cost problem in the infinite dimensional case is as complete as for the finite dimensional situation. Again, many results here appeared for the first time in book form in 1993.

Acknowledgments

It is a pleasant task to acknowledge the support of the Centre de recherches mathématiques of the Université de Montréal and the Natural Sciences and Engineering Council through Discovery Grant–8730 in the "re-engineering" of the first edition of the book into this new second edition.

We are very grateful to Thomas Grasso of Birkhäuser Boston Inc. for his cooperation and efforts since the 2003 IEEE CDC Meeting to promote the book and for his kind invitation to consider a second edition. It is also a pleasure to thank Tamer Başar, Editor-in-Chief, for accepting it in the book series *Systems & Control: Foundations & Applications.*

Most sincere thanks to Louise Letendre of the Centre de Recherches Mathématiques for her colossal, patient, and truly professional work of completely reorganizing and upgrading the old LaTeX files of the original two-volume book to the latest AMS LaTex standards in a single volume. Very special thanks also go to André Montpetit of the Centre de Recherches Mathématiques who provided his technical support, experience, and talent to adapt, upgrade, and ajust the TeX files and the special format of the bibliography to the latest Birkhäuser/Springer macros.

Additional earlier acknowledgments can be found in the Prefaces to Volume I (pages xii to xiii) and II (pages xvi to xvii) of the first edition.

October, 2006

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Preface to Volume I of the First Edition

At the end of the 1960s, the state space theory of linear systems both for time-invariant and time-varying systems had essentially been worked out. The basic concepts of controllability, observability, and their relationship to the theory of minimal realizations received almost a complete treatment in the work of Kalman and others. The feedback solution of the quadratic cost optimal control problem, both in a finite and an infinite time interval were well understood. The concepts of controllability and observability and the weaker notions of stabilizability and detectability play an essential role through stability ideas in the solution of the infinite time quadratic cost optimal control problem. The ideas of optimal control from the Hamilton–Jacobi–Caratheodory, *dynamic programming, calculus of variations* (and in its modern form the *McShane–Pontryagin maximum principle*) points of view had received definitive treatments. As far as deterministic control problems for finite dimensional linear systems are concerned, attention shifted away from optimality to synthesis problems, such as decoupling, regulator, and tracking with internal stability, and to a unification of the time and frequency domain points of view. A qualitative theory of nonlinear control of finite dimensional dynamical systems began its development at the same time.

It is a fact that most lumped parameter systems are approximations of distributed parameter systems, and hence, the study of infinite dimensional systems such as control of systems governed by partial differential equations and functional differential equations are both of intrinsic interest and potentially important for application areas such as chemical process control, control of elastic structures, and even for challenging issues such as the stabilization of plasma instabilities. It was, in some sense, natural to concentrate on optimal control problems for infinite dimensional systems so that the well-developed theory of *calculus of variations and Hamilton–Jacobi* theory could be suitably generalized and used. Clearly the first step in such a development is a reasonably complete understanding of the *linear quadratic* problem both over a finite and an infinite time interval. This book attempts to do this with a reasonable degree of completeness.

This book has been many years in the making, and the original technical reports date back to 1975. It must also be one of the most cited books with the appended words “to appear.” The reasons for this delay in publication are complex and could be the subject of psycho-analytical research or a study into the origins of human frailties. It started out as a collaborative effort of Alain Bensoussan, Michel C. Delfour, and Sanjoy Mitter. From the outset it was our intention to write a book that would go beyond what was then available in the literature of optimal control of infinite dimensional systems and cover such topics as the feedback boundary control of hyperbolic systems when control is exercised through Dirichlet boundary conditions, feedback control of hyperbolic systems over an infinite time interval, and the control of neutral functional differential equations. In retrospect, we seriously underestimated the difficulty of carrying out such a program. Indeed the solution of many of these control problems required results that were then not available in the partial differential equations literature. For example, the solution of certain problems of optimal control of hyperbolic equations with a quadratic cost but over an infinite time interval required a result that would enable one to conclude that the set of admissible controls is nonempty. Such a result is a byproduct of the theory of exact controllability of hyperbolic systems, a theory that was only developed in the 1980s. Our research interests had also changed in the mid-1970s, and although we continued to think about the subject matter of this book, we did not return to it seriously until the mid-1980s or so. Even then the book would never have been completed but for the efforts of Giuseppe Da Prato who joined the team approximately two years ago (one of the coauthors takes full credit for having had the brilliant idea of enlisting Da Prato in the project). The result is a book by four authors, from four different countries, who met in Pisa, Paris, Montréal, and Cambridge, Mass. to bring this long endeavor to completion—surely a model of international cooperation. Readers of the book will have to judge whether this effort has been successful.

Earlier versions of some chapters of this book have been used for graduate courses at the Scuola Normale Superiore (Chapter 1) and at the Université de Montréal (Chapters 1 and 4). Other parts have been used for graduate seminars.

It is a pleasant task to thank the many institutions that have made this work possible—Université de Paris–Dauphine and INRIA, France; Scuola Normale Superiore, Pisa, Italy; Centre de recherches mathématiques and Département de mathématiques et de statistique, Université de Montréal, Canada; Department of Electrical Engineering and Computer Science and the Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, Cambridge, MA, USA. The research of Giuseppe Da Prato has been partially supported by the Italian National Project MURST “Equazioni di Evoluzione e Applicazioni Fisico-Matematiche.” The research of Michel C. Delfour has been supported by the Natural Sciences and Engineering Council operating grant OGP–8730 and infrastructure grant INF–7939, the “Ministère de l'Éducation du Québec" through a FCAR Team Grant, the France–Québec exchange program, and a Killam Fellowship from Canada Council. The research of Sanjoy Mitter has been supported by the U.S. National Science Foundation in the 1970s, the U.S. Air Force Office of Scientific Research over a long period, and the U.S. Army Research Office through the Center for Intelligent Control Systems for the last six years. This support is gratefully acknowledged.

It is a pleasure to thank Edwin Beschler of Birkhäuser Boston Inc. for his cooperation, Ann Kostant also of Birkhäuser for expert help in word processing, and Margaret Flaherty of the Laboratory for Information and Decision (M.I.T.) and Diane de Filippis of the Centre de recherches mathématiques (U. de M.) for typing large parts of the manuscript with precision and care.

The Earth,
April 15, 1992

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Giuseppe Da Prato, Pisa, Italia
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Preface to Volume II of the First Edition

Volume I of this two-volume book has dealt with the question of modeling and representation of infinite dimensional systems. Volume II is concerned with the optimal control of certain classes of infinite dimensional systems with a quadratic cost criterion, both over a finite and an infinite time horizon. The knowledge of Volume I is a prerequisite for reading Volume II.

Volume II consists of three parts. Chapter 1 of Part I is concerned with the theory of controllability and observability of finite dimensional linear systems, and the role these concepts play in the study of the quadratic cost problem over an infinite time horizon. This chapter is included so that the reader understands the background from which the infinite dimensional theory developed and obtains an appreciation of the substantial conceptual and technical difficulties that had to be overcome to give a satisfactory treatment of the subject in the infinite dimensional context. Chapter 2 is devoted to the study of controllability for an infinite dimensional abstract linear dynamical system that can be specialized to obtain results for controllability of parabolic and hyperbolic partial differential equations both when control is exercised in the interior of the domain and when control is exercised through the boundary. The important problem of exact controllability of hyperbolic equations in appropriate spaces, which leads to stabilizability properties for these systems (and hence verification of the "finite cost" condition for infinite time problems), is discussed in detail in this chapter. The systematic use of eigenvalues and eigenfunctions of appropriate differential operators to obtain the results is a somewhat novel aspect of this chapter.

Part II of the book is devoted to the *quadratic cost optimal control problem* over a finite time horizon. We develop the theory for an abstract dynamical model satisfying certain semigroup (group) assumptions, and then we treat concrete situations by verifying these assumptions using differential equations (partial, functional-differential) methods. This chapter presents a reasonably complete treatment of the subject, which includes both boundary control and boundary observation (not necessarily simultaneously) for parabolic and hyperbolic systems. Technically, these are the more difficult parts of the book because they involve unbounded control and observation operators. Many of the results presented here appear in book form for the first time.

The approach we adopt is *dynamic programming*, which leads to a synthesis of the optimal control in feedback form via a study of an operator Riccati differential equation. The systematic use of *dynamic programming* gives a unified view of this topic.

Part III, the final part of the book, is concerned with the *quadratic cost optimal control problem* over a infinite time horizon. Here the concepts of stabilizability and detectability play an essential role. The properties of stabilizability and detectability have to be verified, in some sense directly, for the parabolic case (which has a finite dimensional unstable part) and via the theory of exact controllability (for example) in the hyperbolic case. Thus, there is a close relation between Parts I and III of the book. The approach here is again via *dynamic programming* and a study of an algebraic Riccati equation. We also prove the stability of the closed loop system. In many (but not all) situations the theory of the infinite time quadratic cost problem in the infinite dimensional case is as complete as for the finite dimensional situation. Again, many results here appear for the first time in book form.

Dynamic programming is a deep conceptual idea. Its applications are manifold. For the philosophically minded reader, we have provided the following quotation from Soren Kierkegaard as an indication that philosophy might often anticipate developments in science:

> *It is perfectly true, as philosophers say, that life must be understood backwards. But they forget the other proposition, that it must be lived forwards ... And if one thinks over the proposition it becomes more and more evident that life can never really be understood in time simply because at no particular moment can I find the necessary resting-place from which to understand it–backwards.*

From entry in Soren Kierkegaard's Journal for the Year 1843.
Quoted in Richard Wollheim, *Thread of life.*
Harvard University Press, Cambridge, MA, 1984.

Earlier versions of some chapters of this book have been used for graduate courses at the Scuola Normale Superiore and at the Université de Montréal. Other parts have been used for graduate seminars.

It is a pleasant task to thank the many institutions that have made this work possible—Université de Paris–Dauphine and INRIA, France; Scuola Normale Superiore, Pisa, Italy; Centre de recherches mathématiques and Département de mathématiques et de statistique, Université de Montréal, Canada; Department of Electrical Engineering and Computer Science and the Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, Cambridge, MA, USA. The research of Giuseppe Da Prato has been partially supported by the Italian National Project MURST "Equazioni di Evoluzione e Applicazioni Fisico-Matematiche." The research of Michel

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It is a pleasure to thank Georg Schmidt for a critical reading of Part I, Chapter 2 of the book. His comments have led to many improvements in this chapter. We thank Edwin Beschler of Birkhäuser Boston Inc. for his cooperation, Ann Kostant also of Birkhäuser for expert help in word processing, and Margaret Flaherty of the Laboratory for Information and Decision Systems (M.I.T.) for typing large parts of the manuscript with precision and care.

The Earth,
March 26, 1993

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Introduction

1 Scope of the book

The primary concern of this book[1] is the control of linear infinite dimensional systems, that is, systems whose state space is infinite dimensional and its evolution is typically described by a linear partial differential equation, linear functional differential equation or linear integral equation.

We focus on two aspects of the control problem:

(i) qualitative properties such as stability, controllability, and observability;
(ii) optimal feedback control of such systems when the performance is measured by a quadratic cost criterion, both over a finite and an infinite time interval.

However, before such a study can be carried out, a detailed investigation of the problem of representation of infinite dimensional systems is needed. This is done in Part II of the book, and Parts III to V are devoted to a study of the control problem. This endeavor is initiated by a broad review of finite dimensional systems in Part I so that the reader understands the background from which the infinite dimensional theory is developed and also obtains an

[1] The *numbering* of equations, theorems, propositions, lemmas, corollaries, definitions, examples, notations, and remarks is by chapter. When a reference to another chapter within a part is necessary, it is always followed by the words *in Chapter* and the *number of the chapter*: for instance "in equation (2.1) of Chapter 3". When a reference to another chapter in a different part is necessary, it is always followed by the words *of Chapter*, the *number of the chapter*, *of Part*, and the *number of the part*: for instance "equation (2.1) in Chapter 1 of Part IV". The text of theorems, propositions, lemmas, and corollaries is slanted; the text of definitions, examples, notations, and remarks is normal shape ended by a square □. The bibliography is by author in alphabetical order. For each author or group of coauthors, there is a numbering in square brackets starting with one. A reference to an item by a single author is of the form J. L. Lions [4], and a reference to an item with several coauthors R. Bellman and K. L. Cooke [1].

appreciation of the substantial conceptual and technical difficulties that had to be overcome to give a satisfactory treatment of the subject in the infinite dimensional context.

2 From finite to infinite dimensional sytems

It is the purpose of this introductory chapter to give an idea of the problematic and explain the reason for the necessary mathematical development in functional analysis and differential equations that is carried out in this book. In the process we provide both a qualitative summary of the differences between finite dimensional systems and infinite dimensional systems as well as provide a detailed outline of the book. It should be mentioned that technological problems of importance such as the control of large-space structures, analysis and control of plasma fusion, and chemical process control where significant time delays are present need an understanding of the theory presented in this book for their rigorous analysis. We do not present the frequency domain point of view in this book. If finite dimensional theory is any guide, then the engineering solution of technological problems will undoubtedly require a synthesis of the time and frequency domain viewpoints.

To get an appreciation of the technical problems associated with the control of infinite dimensional systems, consider the distributed control of wave motion. Let $y(x,t)$ denote the transverse displacement at time $t \geq 0$ of a vibrating medium in an n-dimensional bounded open region Ω with smooth boundary $\partial\Omega \subset \mathbb{R}^n$. Let us assume that

$$y(x,t) = 0 \quad \text{for } x \in \partial\Omega \quad \text{and} \quad t > 0$$

and let the initial data at time $t = 0$ be

$$y(x,0) = y_0 \quad \text{and} \quad \frac{\partial}{\partial t} y(x,0) = y_1,$$

for some sufficiently smooth functions y_0 and y_1 defined on Ω. Consider the partial differential equation

$$\frac{\partial^2 y}{\partial t^2}(t) + Ay(t) = Bu(t) \tag{2.1}$$

or the equivalent first-order system

$$\begin{cases} \dfrac{\partial y}{\partial t}(t) - z(t) = 0, \\ \dfrac{\partial z}{\partial t}(t) + Ay(t) = Bu(t), \end{cases} \tag{2.2}$$

where the operator

$$A \stackrel{\text{def}}{=} -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right) + a_0(x), \quad a_0(x) \geq \alpha > 0$$

is a uniformly elliptic operator with C^∞-coefficients in $\overline{\Omega}$, that is, $a_{ij} \in C^\infty(\overline{\Omega})$.[2] Let the controllers $u(\cdot) \in L^2(0,\infty;\mathbb{R}^m)$ and let

$$B(x) \stackrel{\text{def}}{=} [B_1(x) \cdots B_m(x)]$$

be an $n \times m$ matrix with each column in $C^\infty(\overline{\Omega})$. We consider the state space

[2] Let Ω be an open subset of $\mathbb{R}^n$. Denote by $C(\Omega)$ or $C^0(\Omega)$ the space of continuous functions from Ω to $\mathbb{R}$, and for an integer $k \geq 1$,

$$C^k(\Omega) \stackrel{\text{def}}{=} \left\{ f \in C^{k-1}(\Omega) \,:\, \partial^\alpha f \in C(\Omega),\, \forall \alpha,\, |\alpha| = k \right\},$$

where $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}^N$ is a multi-index, $|\alpha| = \alpha_1 + \cdots + \alpha_N$ is the order of the derivative, and

$$\partial^\alpha f \stackrel{\text{def}}{=} \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}}. \tag{2.3}$$

By convention $\partial^0 f$ will be the function f in order to make sense of the case $\alpha = 0$. When $|\alpha| = 1$, we also use the standard notation $\partial_i f$ or $\partial f/\partial x_i$. $\mathcal{D}^k(\Omega)$ or $C^k_c(\Omega)$ (resp., $\mathcal{D}(\Omega)$ or $C^\infty_c(\Omega)$) will denote the space of all k-times (resp., infinitely) continuously differentiable functions with compact support contained in the open set Ω.

A function $f : \Omega \to \mathbb{R}$ is *uniformly continuous* if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that for all x and y in Ω such that $|x-y| < \delta$ we have $|f(x)-f(y)| < \varepsilon$.

If a function f is bounded and uniformly continuous on Ω, it possesses a unique, continuous extension to the closure $\overline{\Omega}$ of Ω. Denote by $C^k(\overline{\Omega})$ the space of functions f in $C^k(\Omega)$ for which $\partial^\alpha f$ is bounded and uniformly continuous on Ω for all α, $0 \leq |\alpha| \leq k$. A function f in $C^k(\Omega)$ is said to *vanish at the boundary* of Ω if for every α, $0 \leq |\alpha| \leq k$, and $\varepsilon > 0$, there exists a compact subset K of Ω such that, for all $x \in \Omega \cap \complement K$, $|\partial^\alpha f(x)| \leq \varepsilon$. Denote by $C^k_0(\Omega)$ the space of all such functions. Clearly $C^k_0(\Omega) \subset C^k(\overline{\Omega}) \subset C^k(\Omega)$. Endowed with the norm

$$\|f\|_{C^k(\Omega)} \stackrel{\text{def}}{=} \max_{0 \leq |\alpha| \leq k} \sup_{x \in \Omega} |\partial^\alpha f(x)|, \tag{2.4}$$

$C^k_0(\Omega)$ and $C^k(\overline{\Omega})$ are Banach spaces. Finally

$$C^\infty(\Omega) \stackrel{\text{def}}{=} \bigcap_{k \geq 0} C^k(\Omega),\ C^\infty(\overline{\Omega}) \stackrel{\text{def}}{=} \bigcap_{k \geq 0} C^k(\overline{\Omega}),\ \text{and } C^\infty_0(\Omega) \stackrel{\text{def}}{=} \bigcap_{k \geq 0} C^k_0(\Omega).$$

When f is a vector function from Ω to $\mathbb{R}^m$, the corresponding spaces will be denoted $C^k_0(\Omega)^m$ or $C^k_0(\Omega,\mathbb{R}^m)$, $C^k(\overline{\Omega})^m$ or $C^k(\overline{\Omega},\mathbb{R}^m)$, $C^k(\Omega)^m$ or $C^k(\Omega,\mathbb{R}^m)$, etc.

In dimension one, $n = 1$, and for a bounded open interval $]a,b[$, the notation and the definition of $C(\overline{]a,b[})$ coincide with the ones of $C([a,b])$. For simplicity we shall also often write $C(a,b)$ with the implicit convention that $C(a,b)$ stands for $C([a,b])$.

$$w \stackrel{\text{def}}{=} \begin{bmatrix} y \\ z \end{bmatrix} \in \mathcal{W} \stackrel{\text{def}}{=} H_0^1(\Omega) \times L^2(\Omega),$$

where $H_0^1(\Omega)$ denotes the usual Sobolev space of L^2-functions, with derivatives, in the distribution sense, belonging to $L^2(\Omega)$ and vanishing at the boundary. Then (2.2) can be formally written as

$$\frac{dw}{dt}(t) + \tilde{A}w(t) = \tilde{B}u(t), \tag{2.5}$$

where

$$\tilde{A} \stackrel{\text{def}}{=} \begin{bmatrix} 0 & -I \\ A & 0 \end{bmatrix} \quad \text{and} \quad \tilde{B} \stackrel{\text{def}}{=} \begin{bmatrix} 0 \\ B \end{bmatrix}.$$

Here $\tilde{B} \in \mathcal{L}(\mathbb{R}^m; \mathcal{W})$ and the operator $\tilde{A}$ is an *unbounded* operator with a dense domain

$$\mathcal{D}(\tilde{A}) = \left[H_0^1(\Omega) \cap H^2(\Omega)\right] \times H_0^1(\Omega) \subset \mathcal{W}.$$

Because of the presence of the unbounded operator $\tilde{A}$, it is clear that the concept of a solution for (2.5) is not immediate. Intuitively, if we want (2.5) to have a classical solution, then we would need

$$w_0 = \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} \in \mathcal{D}(\tilde{A})$$

and $u(\cdot) \in C^1(\Omega)$. On the other hand, $-\tilde{A}$ generates a strongly continuous semigroup $S(t)$ on $\mathcal{W}$, i.e., $S(t)$ is a bounded linear operator on $\mathcal{W}$ satisfying

(i) $S(0) = I$.
(ii) $S(t_1 + t_2) = S(t_1)S(t_2), \quad 0 < t_1, t_2 < \infty$.
(iii) $t \mapsto S(t)w\colon [0, \infty) \to \mathcal{W}$ is continuous for each $w \in \mathcal{W}$.

Then we may say that $w(t)$ is a solution of (2.5) if it satisfies

$$w(t) = S(t)w_0 + \int_0^t S(t-s)\tilde{B}u(s)\, ds, \tag{2.6}$$

where the integral is interpreted in the Bochner sense.

Now, given a strongly continuous semigroup $S(t)$ on X (say a Banach space), it has a unique infinitesimal generator A, a closed unbounded operator that is defined on some dense domain $\mathcal{D}(A)$. The converse question, namely, when does A generate a strongly continuous semigroup, is a far more difficult issue and is the content of the Hille–Yosida theorem. Furthermore, there are different kinds of semigroups, and each plays an important role in the study of partial differential and functional differential equations. We give an informal listing of some of these below.

(i) *Differentiable*: $S(t)X \subset \mathcal{D}(A)$ for $t > 0$. Then $S(t)x \in C^\infty(X)$ for all $t > 0$ and each fixed $x \in X$. Partial differential equations of diffusion type on a bounded domain with homogeneous Dirichlet boundary conditions give rise to such semigroups. (§2.8, Chapter 1 of Part II).
(ii) *Analytic*: $S(t)$ is differentiable and admits an analytic extension into the complex t-plane satisfying certain bounds. Abstract parabolic equations in variational form give rise to such semigroups. (§2.7, Chapter 1 of Part II).
(iii) $S(t)$ is a *group* (orthogonal or unitary). The operator $-\tilde{A}$ considered previously generates a group. This is the content of Stone's theorem. (Theorem 2.9, Chapter 1 of Part II).
(iv) $S(t)$ is a *compact* operator for t large. Hereditary differential systems give rise to such semigroups. An extensive study of hereditary differential systems is carried out in Chapter 4 of Part II.
(v) *Contraction* $\|S(t)\| \le 1$ on $0 \le t < \infty$. Dissipative operators give rise to such semigroups. This is the content of the Lumer–Phillips theorem (Theorem 2.6, Chapter 1 of Part II).

An important problem in the control of linear systems is the study of its stability. Consider the stability problem for the linear system

$$\frac{dy}{dt}(t) = Ay(t), \quad A\colon \mathbb{R}^n \to \mathbb{R}^n. \tag{2.7}$$

It is well known that the following are equivalent:

(a) (2.7) is asymptotically stable.
(b) (2.7) is exponentially stable.
(c) $\forall y_0 \in \mathbb{R}^n$, $\int_0^\infty |e^{At}y_0|^2_{\mathbb{R}^n}\, dt < \infty$.

The stability of (2.7) can be tested by looking for a quadratic Lyapunov function:

$$V(x) = (x, Px)_{\mathbb{R}^n}, \quad P = P^* > 0.$$

This leads to a study of the Lyapunov equation

$$A^*P + PA = -Q, \quad Q > 0, \tag{2.8}$$

and if A is a stability matrix $\operatorname{Re}\lambda < 0$, then the solution of (2.8) exists and can be written as

$$P = \int_0^\infty e^{A^*t} Q e^{At}\, dt, \tag{2.9}$$

and conversely if (2.8) has a positive definite solution, then A is a stability matrix.

The corresponding question of infinite dimensional systems

$$\frac{dy}{dt}(t) = Ay(t), \quad y(t) \in X \text{ a Banach space} \tag{2.10}$$

and A the infinitesimal generator of a strongly continuous semigroup $S(t)$ leads to a study of the asymptotic behavior of $S(t)$ as t goes to infinity. The theorems characterizing the equivalence of conditions (a) and (c) above and related issues are discussed in §2.2 of Chapter 1 of Part II, where Theorem 2.2 is perhaps the most important. The Lyapunov method leading to the study of the Lyapunov equation is the content of Theorem 2.4. The relation between the spectrum of A and the stability of the semigroup $S(t)$ is subtle (see §2.9, Chapter 1 of Part II).

In general, the questions related to the generation of semigroups receive a thorough description in §2, Chapter 1 of Part II.

As we have just discussed, not only do we have to deal with homogeneous equations of the form (2.10), where A is the infinitesimal generator of a semigroup, but we also have to deal with nonhomogeneous equations

$$\frac{dy}{dt}(t) = Ay(t) + f(t), \quad \text{where } f \in L^p(0, T; X). \tag{2.11}$$

When the nonhomogeneous equation arises from a feedback control problem, we have to consider equations of the form

$$\begin{cases} \dfrac{dy}{dt}(t) = Ay(t) + F(t)y(t) + f(t) \quad \text{on } [0, T], \\ y(0) = y, \end{cases} \tag{2.12}$$

where A is the infinitesimal generator of a strongly continuous semigroup and $F\colon [0, T] \to \mathcal{L}(X)$ is strongly continuous ($\mathcal{L}(X)$ is the space of continuous linear maps from X to X). This equation leads to the consideration of a perturbation of the infinitesimal generator A.

There are now many concepts of a solution (see Definition 3.1, Chapter 1 of Part II). Existence and uniqueness of solutions as well as equivalence of solutions is discussed in §3.1 to §3.5 of Chapter 1 of Part II. The key ingredient in proving the various existence and uniqueness results is the Yosida approximation.

The question of regularity of solutions is discussed in §3.6 and §3.7. In §3.6, the assumption is made that X is a Hilbert space and A is the generator of an analytic semigroup and we concern ourselves with maximal regularity, namely, that

$$\frac{dy}{dt} \quad \text{and} \quad Ay$$

have the same regularity as f. Consideration of the nonzero initial condition case leads to the study of interpolation spaces (studied in detail in §4, Chapter 1 of Part II) and the main Isomorphism Theorem (Theorem 3.1, Chapter 1 of Part II). Regularity can also be studied in $C([0, T]; X)$[3] with X a Banach space, and this is carried out in §3.7.

We often have to study nonhomogeneous equations

[3] $C([0, T]; X)$ is the linear space of continuous functions $f : [0, T] \to X$ endowed with the sup norm. It will also be denoted as $C(0, T; X)$.

$$\frac{dy}{dt}(t) = Ay(t) + f(t) \tag{2.13}$$

on the infinite time interval $[0, \infty[$. These situations arise from consideration of feedback control problems and related stability issues. When A has a pure point spectrum and the infinite part of the spectrum is stable and the unstable part is finite dimensional, these equations can be effectively studied in $[0, \infty[$ (§3.9, Chapter 1 of Part II).

In Parts IV and V of the book where we study optimal control problems with a quadratic cost function where control is exercised through the boundary (or pointwise), we shall need the concept of fractional powers of closed operators and the semigroups they might generate. This is the subject of §5 and §6 of Chapter 1 of Part II.

Control of partial differential equations where control is exercised through the boundary leads to considerable technical difficulties. To obtain an understanding of these difficulties, consider again a vibrating flexible string

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} \tag{2.14}$$

for the transversal displacement

$$y(t, x) \quad \text{on } 0 \leq x \leq 1, \ t \geq 0.$$

Assume that

$$y(0, t) = 0 \quad \text{on } t \geq 0$$

and control is exercised through the end point $x = 1$

$$y(1, t) = u(t), \quad t \geq 0. \tag{2.15}$$

We would like to control the pair

$$\left(y(\cdot, t), \frac{dy}{dt}(\cdot, t) \right)$$

so that the initial state

$$y(x, 0) = y_0(x), \quad \frac{dy}{dt}(x, 0) = v_0(x)$$

on $0 \leq x \leq 1$ is brought to

$$y(x, T) = 0, \quad \frac{dy}{dt}(x, T) = 0 \qquad \text{on } 0 \leq x \leq 1 \tag{2.16}$$

in some finite time $T > 0$.

We try to transform the system to a distributed control framework

$$\frac{dw}{dt}(t) = \tilde{A}w(t) + \tilde{B}u(t) \tag{2.17}$$

by defining the state $w(\cdot)$ and the operators $\tilde{A}$ and $\tilde{B}$ appropriately. For this purpose, define

$$Y(x,t) = y(x,t) - \int_{1-}^{1} \delta_1(x)u(t)(x)\,dx \tag{2.18}$$

where the Dirac measure δ_1 has a unit weight at $x = 0$. Then formal calculations will show that

$$y(1,t) - y(1_-,t) = \delta_1 u(t),$$

or equivalently,

$$y(1,t) = u(t) \quad \text{with } y(1_-,t) = 0.$$

Hence we obtain the distributed control system

$$\frac{\partial^2 Y}{\partial t^2} = \frac{\partial^2 Y}{\partial x^2} - \delta_1' u(t) \tag{2.19}$$

(δ_1' is the distributional derivative of δ_1) with the boundary conditions

$$Y(0,t) = 0 \quad \text{and} \quad Y(1_-,t) = 0 \quad \text{on } t \geq 0,$$

and this can be formally put in the form (2.17).

In general, boundary control problems for partial differential equations lead to boundary operators B that are unbounded and may involve distributions as in this example. The dual of boundary control problems are boundary observation problems, and in the most general situation, both control and observation are exercised through the boundary leading to control and observation operators that are both unbounded.

In some sense, when control of partial differential equations is exercised through the boundary, the parabolic case (the corresponding semigroup is analytic) and the hyperbolic case (the corresponding semigroup is a group) have to be treated using different methods. Chapters 2 and 3 deal with the control of parabolic systems when control and observation is exercised through the boundary, leading to control and observation operators that are unbounded.

Chapter 2 of Part II is a summary exposition of variational theory of parabolic systems. This theory can also be applied to certain wave equations where damping is present. A systematic exposition of this method can be found in J. L. Lions and E. Magenes [1], and this theory was extensively used by Lions in his book *Control of Partial Differential Equations* (cf. J. L. Lions [3]). The main reason for including this chapter in the book is to explain the *Method of Transposition*, which is later used in various parts of the book (for example, in the study of delay systems in Chapter 4 of Part II and in the study of controllability in Chapter 1 of Part III). The idea of the Method of Transposition is to define an integral or weak version of the original equation involving the adjoint of the operator A, using Green's formula and integration by parts (§2.3 and in particular (2.5) of this chapter) and then to obtain a smooth adjoint isomorphism of the form

$$y \mapsto \left(A^*y - \frac{dy}{dt}, y(T)\right)$$

between suitable spaces. It should be noted that the Method of Transposition when combined with the change of variable idea of the previous chapter allows one to treat parabolic systems with control exercised via Dirichlet boundary conditions. As the boundary control operator becomes "rougher" (e.g., Dirichlet case), the solution also becomes "rougher" and it becomes more difficult to perform a "smooth" observation. The balance between the relative "roughness" or "unboundedness" of the control and observation operators is a key technical issue in the solution of the quadratic control problem as we shall see in Parts IV and V.

Chapter 3 of Part II is concerned with semigroup methods, and one of the main theorems in this chapter is an Isomorphism Theorem (Theorem 2.3), which uses regularity results for analytic semigroups and interpolation spaces (§4 of Chapter 1). The other key idea used in this chapter is the use of a change of variable to define a new state and make sense of the state space system as an input–output map (see Theorem 2.3 and §3.4, Chapter 3).

The final chapter of this part of the book is concerned with the representation problem for linear functional differential equations with general control and observation mechanisms. A fairly complete modern treatment of this subject that covers both retarded and neutral Functional Differential Equations (FDE), difference equations, integro-differential equations, and other equations with a hereditary structure is now available (cf. M. C. Delfour and J. Karrakchou [1, 2]) . They can now all be treated in the same framework. However, we have chosen to concentrate on FDEs of the retarded type to provide a better understanding of the basic ideas and constructions, but everything is readily extendable to more general hereditary structures. The prototype problem here can be written as

$$\begin{cases} \dfrac{dx}{dt}(t) = \displaystyle\sum_{i=0}^{N} \left[x(t-i) + u(t-i)\right], \\ y(t) = a_0 x(t) + a_1 x(t-N) + b_0 u(t) + b_1 u(t-N), \end{cases} \tag{2.20}$$

where $u(\cdot)$ denotes the control variable and $y(\cdot)$ denotes the observation. Two questions need to be resolved to obtain a "current" theory of existence, uniqueness, and representation of solutions to such systems. The first is the choice of an appropriate function space for initial conditions, and the second is the choice of a minimal state space for these systems. The choice of initial conditions, namely space of continuous functions $C(-h, 0; \mathbb{R}^n)$ and the product space $M^p(-h, 0; \mathbb{R}^n) = \mathbb{R}^n \times L^p(-h, 0; \mathbb{R}^n)$, and the corresponding questions of existence and uniqueness of solutions in appropriate function spaces are discussed in §3 of Chapter 4 of Part II. §4 is concerned with the state space representation of FDEs. The key concept of structural operators (§4.4), which intuitively describes the way the systems combine and transform initial conditions over the initial interval is introduced here. A complete modern treatment of the various adjoint systems and semigroups arising out of the original system is described in this section. Finally, §5 and §6 give a state space treatment of linear FDEs of the retarded type when control and observation in various forms are present. All this readily extends to other types of systems with a delay structure. It should be mentioned that the state space point of view is essential for the study of optimal control problems considered in Parts III to V.

Notes: some related books on the control of linear systems that have appeared since 1992

Several books related to this one have appeared in the ensuing period between the publication of the first edition of the book and its current revision.

In optimal control, the two-volume book of I. Lasiecka and R. Triggiani [15,16] that appeared in 2000 covers in greater detail and provides new results on optimal control of parabolic and hyperbolic systems with quadratic cost functions. The book of X. Li and J. Yong [1] published in 1995 is concerned with optimal control of partial differential equations, and in the last chapter, the author discusses linear quadratic optimal control problems, both over a finite and an infinite time horizon.

The books of R. F. Curtain and H. Zwart [1] published in 1995 and O. J. Staffans [2] in 2005 are concerned with state space realization theory of linear input–output infinite dimensional systems as well as qualitative properties such as stability, controllability, and observability. The book by Staffans has considerable overlap with our treatment of semigroup theory as well as representations of partial differential equations in terms of semigroups. Staffans' work also presents a detailed treatment of the theory of B. Sz.-Nagy and C. Foias [7,8] on *contractive semigroups* and their *unitary dilations* from 1965 to 1967. These ideas are intimately connected with *scattering theory* as developed by P. Lax and R. S. Phillips [1]. Finally, the *systems* point of view as exemplified in the Russian work of V. M. Adamjan, D. Z. Arov, and M. G. Kreĭn [1 to 5] receives a detailed treatment here.

The book of H. O. Fattorini [5] published in 1999 also deals with optimal control of nonlinear partial differential equations via the Maximum Principle. Finally, the reader is referred to the book of A. V. Fursikov and O. Yu. Imanuvilov [1] in 1996 that studies null controllability via Carleman estimates.

We have not cited other references because we confined ourselves to linear control problems.

Part I

Finite Dimensional Linear Control Dynamical Systems

1

Control of Linear Differential Systems

1 Introduction

This Part I serves the purpose of an introduction to Parts III to V of the book, which are mainly concerned with the quadratic cost optimal control problem for distributed parameter systems and systems with time delay, both over a finite and an infinite time interval. For problems over a finite time interval, the main tool used is *Dynamic Programming*, which leads to a Hamilton–Jacobi equation for the value function. For the class of control problems considered, the Hamilton–Jacobi equation can be explicitly solved via the study of an operator Riccati equation. The study of the operator Riccati equation when control is exercised through the boundary in the case of distributed parameter systems or when delays are present in the control in the case of systems with time delay poses additional technical difficulties. The results of Part II are needed to overcome these difficulties. For problems over an infinite time interval, the concepts of controllability and observability (and the weaker concepts of stabilizability and detectability) play an essential role in the development of the theory.

Problems of optimal control with a quadratic cost function, not necessarily definite, are of interest in H^∞-theory and differential games. These problems are related to the theory of dissipative systems. The last two sections of this chapter present an introduction to these problems.

2 Controllability, observability, stabilizability, and detectability

In this section we present the theory of controllability and observability for finite dimensional linear systems in a manner that has implications in the study of controllability and observability for infinite dimensional systems. In particular, we show that if the system is controllable, then the transfer from the zero-state to any other state can be carried out by means of minimum energy controls. It is the same idea, which is later used in the theory of exact controllability of hyperbolic (second order) systems. The mathematical techniques needed to accomplish it, however, are far more difficult. Throughout this book $\mathbb{N}$ denotes the set of integers and $\mathbb{R}$ and $\mathbb{C}$ the respective fields of real and complex numbers. In this chapter the norm in $\mathbb{R}^d$ will be denoted $|\cdot|$ irrespective of the dimension d of the space. We shall use $\|\cdot\|$ to denote the norm in various Hilbert spaces, $(\cdot,\cdot)$ to denote scalar product and * denotes the adjoint of an operator (and also the dual space).

Consider the linear finite dimensional control system in the interval $[0,T]$:

$$\begin{cases} \dfrac{dx}{dt}(t) = Ax(t) + Bu(t) \text{ in } [0,T], \\ y(t) = Cx(t) \end{cases} \tag{2.1}$$

where the *state* $x(t) \in \mathbb{R}^n$, the *control* functions $\big(t \mapsto u(t)\big) \in L^2(0,T;\mathbb{R}^m)$, the *output* $y(t) \in \mathbb{R}^p$, and $A \in \mathcal{L}(\mathbb{R}^n;\ \mathbb{R}^n)$, $B \in \mathcal{L}(\mathbb{R}^m;\mathbb{R}^n)$, and $C \in \mathcal{L}(\mathbb{R}^n;\mathbb{R}^p)$.

We assume that the initial state $x(0) = 0$. The solution of the linear differential equation (2.1) can be written as

$$L_t u \stackrel{\text{def}}{=} x(t;u) = \int_0^t e^{A(t-s)} Bu(s)\, ds, \tag{2.2}$$

where L_t is a linear bounded transformation from $L^2(0,t;\mathbb{R}^m)$ into $\mathbb{R}^n$.

2.1 Controllability

Definition 2.1. We say that the system is *controllable* (from $0 \in \mathbb{R}^n$) in $[0,T]$ if given any $\bar{x} \in \mathbb{R}^n$, there exists a control function $u(\cdot) \in L^2(0,T;\mathbb{R}^m)$ such that

$$x(T;u) = \bar{x}. \qquad \square$$

Controllability in $[0,T]$ is therefore equivalent to the surjectivity of the map $u \mapsto L_T u$. We now give necessary and sufficient conditions for controllability using elementary facts from the theory of linear transformations on Hilbert spaces.

Let H and K be Hilbert spaces, and let $A \in \mathcal{L}(H;K)$. Let $A^* \in \mathcal{L}(K^*;H^*)$ be the adjoint operator. In the sequel we identify H and H^* and K and K^*. Let $\mathcal{R}(A)$ denote the range of the operator A and $\mathcal{N}(A)$ denote the null space of A. $\perp$ denotes the orthogonal complement of a closed subspace.

Proposition 2.1. *The following relations are true*:

(i) $\mathcal{R}(A) \subseteq \mathcal{N}(A^*)^\perp$ *and* $\mathcal{R}(A) = \mathcal{N}(A^*)^\perp \iff \mathcal{R}(A)$ *is closed.*
(ii) $\mathcal{R}(A^*) \subseteq \mathcal{N}(A)^\perp$ *and* $\mathcal{R}(A^*) = \mathcal{N}(A)^\perp \iff \mathcal{R}(A)$ *is closed.*
($\mathcal{R}(A)$ *closed* $\iff \mathcal{R}(A^*)$ *is closed.*)

(iii) $\mathcal{N}(A) = \mathcal{R}(A^*)^\perp$.
(iv) $\mathcal{N}(A^*) = \mathcal{R}(A)^\perp$.
(v) *Furthermore*

$$\mathcal{R}(A) \textit{ is closed } \iff \exists c > 0 \textit{ such that } \|h\| \leq c\|Ah\|, \quad \forall h \in H.$$

Using the above, we can prove the following proposition.

Proposition 2.2. *The following are true*:

(i) $\mathcal{R}(A)$ *is dense in* $K \iff \mathcal{N}(A^*) = \{0\}$.
(ii) $\mathcal{N}(A) = \{0\} \iff \mathcal{R}(A^*)$ *is dense in* H.
(iii) $\mathcal{R}(A)$ *is dense in* $K \iff AA^* \colon K \to K$ *satisfies* $AA^* > 0$.
(iv) $\mathcal{N}(A) = \{0\} \iff A^*A \colon H \to H$ *satisfies* $A^*A > 0$.
(v) $A \in \mathcal{L}(H;K)$ *is invertible* $\iff \mathcal{R}(A) = K$, $\mathcal{N}(A) = \{0\}$

$$\begin{aligned} &\iff \exists c > 0, \textit{ such that } \|h\| \leq c\|Ah\|, \qquad \forall h \in H, \\ &\iff \exists c > 0, \textit{ such that } \|k\| \leq c\|AA^*k\|, \quad \forall k \in K. \end{aligned}$$

Remark 2.1. Much of the above extends to operators A that are densely defined and closed to spaces H and K, which are Banach spaces. For proofs of these facts, see, for example, M. Schechter [1]. For a use of these ideas in a systems context, see M. C. Delfour and S. K. Mitter [7]. □

We can immediately use the above to get a necessary and sufficient condition for controllability. Using (2.2) and noting that

$$\begin{aligned} L_T^* &\colon \mathbb{R}^n \to L^2(0,T;\mathbb{R}^m) \\ &\colon y \mapsto B^* e^{(T-\cdot)} y, \end{aligned}$$

we get

$$W(0,T) \stackrel{\text{def}}{=} L_T L_T^* = \int_0^T e^{A(T-s)} B B^* e^{A^*(T-s)}\, ds. \tag{2.3}$$

Hence we have proved.

Theorem 2.1. *The system* (2.1) *is controllable in* $[0,T]$ *if and only if* $W(0,T) > 0$ (*positive definite*).

We can transform this criteria of controllability to an algebraic criterion by noting that

$$\mathcal{N}\big(W(0,T)\big) = \mathcal{N}(\mathcal{C}), \tag{2.4}$$

where

$$\mathcal{C} = \Big[B \vdots AB \vdots \cdots \vdots A^{n-1}B\Big] \Big[B \vdots AB \vdots \cdots \vdots A^{n-1}B\Big]^*. \tag{2.5}$$

This can be proved as follows.

Let $x \in \mathcal{N}\big(W(0,T)\big)$. Then

$$0 = \big(x, W(0,T)x\big) = \int_0^T |B^* e^{A^*(T-t)}x|^2 \, dt,$$

and hence $B^* e^{A^*(T-t)}x = 0$, $\forall t \in [0,T]$.

Differentiating $(n-1)$-times with respect to t and setting $t = 0$, we get

$$B^* x = 0, \quad B^* A^* x = 0, \ldots, B^* A^{*n-1} x = 0,$$

and hence

$$x \in \mathcal{N}(\mathcal{C}).$$

Conversely, suppose $x \in \mathcal{N}(\mathcal{C})$. By the Cayley–Hamilton theorem,

$$e^{A(T-t)} = \sum_{i=0}^{n-1} \alpha_i (T-t) A^i.$$

Hence

$$x^* W(0,T) = \int_0^T \left[\sum_{i=0}^{n-1} \alpha_i(T-t) x^* A^i B \right] B^* e^{A^*(T-t)} \, dt.$$

Now $x^* A^i B = 0$, $i = 0, \ldots, n-1$, because $x \in \mathcal{N}(\mathcal{C})$. Therefore, $x^* W(0,T) = 0$ and because $W(0,T)$ is symmetric,

$$x \in \mathcal{N}\big(W(0,T)\big).$$

Hence using the fact that $W(0,T)$ is a symmetric matrix

$$\mathcal{R}\big(W(0,T)\big) = \mathcal{R}(\mathcal{C}). \tag{2.6}$$

But

$$\mathcal{R}(\mathcal{C}) = \mathcal{R}([B \vdots AB \vdots \cdots \vdots A^{n-1}B]), \tag{2.7}$$

and hence we have proved the following theorem.

Theorem 2.2. *The following conditions are equivalent*:

(i) *System* (2.1) *is controllable in* $[0,T]$.
(ii) $W(0,T) > 0$ (*positive-definite*).
(iii) rank $\left[B \vdots AB \vdots \cdots \vdots A^{n-1}B \right] = n$.

When the system is controllable, we shall refer to (A,B) as a *controllable pair.*

It is of some interest to establish that if the system is controllable, then we can transfer the state x_0 at time t_0 to the state x_1 at time t_1 by using a control u^*, which has the least energy

$$\int_{t_0}^{t_1} |u^*(s)|^2 \, ds$$

among all controls that transfer the phase (x_0, t_0) to the phase (x_1, t_1).

To establish this, let us use the notation

$$\Phi(t, s) = e^{A(t-s)}.$$

Introduce the adjoint system

$$\begin{cases} \dfrac{dp}{dt}(t) = -A^* p(t) \text{ in } [t_0, t_1[, \\ p(t_1) = \eta, \end{cases} \tag{2.8}$$

where η will be specified later. The solution of (2.8) is given by

$$p(t) = \Phi^*(t_1, t)\eta. \tag{2.9}$$

Now choose the control

$$u^*(t) = -B^* p(t) = -B^* \Phi^*(t_1, t)\eta. \tag{2.10}$$

If the control is to transfer (x_0, t_0) to (x_1, t_1), we must have

$$x_1 = x(t_1) = \Phi(t_1, t_0)x_0 - \int_{t_0}^{t_1} \Phi(t_1, t)BB^*\Phi^*(t_1, t)\, dt\, \eta,$$

and hence if the system is controllable and we choose

$$\eta = W^{-1}(t_0, t_1)[\Phi(t_1, t_0)x_0 - x_1],$$

the desired transfer will be effected. Let us now prove that $u^*(t)$ is the minimum energy control.

Let $\bar{u}(t)$ be some other control that effects the desired transfer. Then

$$\begin{aligned} x_1 &= \Phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1, t)B\bar{u}(t)\, dt \\ &= \Phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1, t)Bu^*(t)\, dt. \end{aligned}$$

Therefore

$$\int_{t_0}^{t_1} \Phi(t_1, t)B[\bar{u}(t) - u^*(t)]\, dt = 0.$$

Hence

$$\int_{t_0}^{t_1} ([\bar{u}(t) - u^*(t)], -B^*\Phi^*(t_1, t)\eta)\, dt = 0$$

and

$$\int_{t_0}^{t_1} \big(\bar{u}(t) - u^*(t), u^*(t)\big)\, dt = 0.$$

Therefore

$$\begin{aligned}\int_{t_0}^{t_1} |\bar{u}(t)|^2\,dt &= \int_{t_0}^{t_1} |\bar{u}(t) - u^*(t) + u^*(t)|^2\,dt \\ &= \int_{t_0}^{t_1} |u^*(t)|^2\,dt + \int_{t_0}^{t_1} |\bar{u}(t) - u^*(t)|^2\,dt \\ &> \int_{t_0}^{t_1} |u^*(t)|^2\,dt \quad \text{if} \quad \bar{u} \neq u^*.\end{aligned}$$

Suppose now that U and X are infinite dimensional Hilbert spaces, $A\colon X \to X$ is an unbounded linear operator with domain $D(A)$, and $B\colon U \to X$ is a bounded linear operator. Then there is a difficulty with carrying through the arguments given earlier (for example, $\mathcal{R}(L_T)$ need not be a closed subspace of X). Nevertheless, $\mathcal{R}(L_T)$ can be given the structure of a Hilbert space with continuous injection in X (Proposition 2.1 in Chapter 1 of Part II). Furthermore, there is a need to make a distinction between approximate controllability ($\mathcal{R}(L_T)$ is dense in X) and exact controllability (whose definition is somewhat subtle, see Definition 2.2 in Chapter 1 of Part III); indeed $\mathcal{R}(L_T) = X$ will usually not hold. However, an abstract criterion for exact controllability is related to an estimate on $L_T L_T^*$ (see Proposition 2.2; this chapter). It is also of interest to note that the minimum energy control viewpoint of verifying exact controllability as outlined in this section extends to infinite dimensional situations, but of course at the cost of using much more elaborate technical machinery. Chapter 1 of Part III gives a treatment of exact controllability for parabolic and hyperbolic partial differential equations when control is exercised in a distributed manner and when control is exercised through the boundary.

2.2 Observability

We now discuss the dual concept of observability. For this purpose we may assume that the input is known, and because we are dealing with linear time-invariant finite dimensional systems, we may take $u(t) \equiv 0$, $t \in [t_0, \infty[$. Consider therefore the system

$$\begin{cases} \dfrac{dx}{dt}(t) = Ax(t), \\ y(t) = Cx(t), \end{cases} \tag{2.11}$$

and let the initial state $x(t_0) = x_0$, for some arbitrary x_0.

The solution of (2.11) can be written as

$$y(t) = Ce^{A(t-t_0)}x_0. \tag{2.12}$$

Definition 2.2. The system (2.1) is said to be *observable* in $[t_0, t_1]$ if the map

$$\mathcal{O}\colon \mathbb{R}^n \to L^2(t_0, t_1; \mathbb{R}^p)\colon x_0 \to y(\cdot) = Ce^{A(\cdot - t_0)}x_0$$

is injective. □

The definition expresses the fact that we can recover uniquely the initial state from a knowledge of the output $y(\cdot)$ in the time interval $[t_0, t_1]$. The interval $[t_0, t_1]$ can in fact be taken to be arbitrarily small. Now,

$$\mathcal{O} \text{ is injective } \Longleftrightarrow \mathcal{N}(\mathcal{O}) = \{0\} \Longleftrightarrow \mathcal{N}(\mathcal{O}^*\mathcal{O}) = \{0\}.$$

An easy computation shows that

$$\Sigma(t_0, t_1) \stackrel{\text{def}}{=} \mathcal{O}^*\mathcal{O} = \int_{t_0}^{t_1} e^{(t_1 - t)A^*} C^* C e^{(t_1 - t)A}\, dt,$$

and clearly

$$\mathcal{N}(\mathcal{O}^*\mathcal{O}) = \{0\} \Longleftrightarrow \Sigma(t_0, t_1) > 0 \text{ (positive-definite)}.$$

It is also easy to show that

$$\begin{aligned}\mathcal{N}(\mathcal{O}^*\mathcal{O}) &= \mathcal{N}([C^* \vdots A^*C^* \vdots \cdots \vdots A^{*n-1}C^*]^*[C^* \vdots A^*C^* \vdots \cdots \vdots A^{*n-1}C^*]),\\ &= \mathcal{N}([C^* \vdots A^*C^* \cdots A^{*n-1}C^*])\end{aligned}$$

and hence, we have proved the following theorem.

Theorem 2.3. *The system* (2.1) *is observable if and only if*

$$\operatorname{rank}[C^* \vdots A^*C^* \vdots \cdots \vdots A^{*n-1}C^*] = n.$$

When the system is observable, we shall refer to (A, C) as an observable pair.

2.3 Duality

There is a formal duality between the concepts of controllability and observability. For this purpose, we introduce the dual system

$$\begin{cases} \dfrac{d\xi}{dt}(t) = -A^*\xi(t) - C^*\gamma(t), \\ \eta(t) = -B^*\xi(t), \end{cases} \tag{2.13}$$

which evolves backward in time.

Then mathematically,

- The system (2.1) is observable $\Longleftrightarrow$ The system (2.13) is controllable.
- The system (2.13) is observable $\Longleftrightarrow$ The system (2.1) is controllable.

In this book we do not study observability of partial differential equations but exploiting the above duality ideas, much of the theory of controllability developed in Chapter 1 of Part III can be used to develop a theory of observability for partial differential equations (even with unbounded observation operators).

Conceptually, controllability of a system permits the choice of feedback controls resulting in certain desirable (e.g., stability) properties of closed loop systems, whereas observability of a system permits the design of state estimators with desirable properties. We shall discuss this in a later section.

2.4 Canonical structure for linear systems

In this section we wish to obtain a decomposition (structure) theorem for the system (2.1) that is not necessarily controllable and observable. Let us denote by

$$V := \mathbb{R}(\mathcal{C}) = \mathcal{R}([B \vdots AB \vdots \cdots \vdots A^{n-1}B]).$$

V is the smallest A-invariant subspace containing the image of B. Let us also denote by

$$W := \mathcal{N}(\mathcal{O}) = \mathcal{N}([C^* \vdots A^*C^* \cdots \vdots A^{*n-1}C^*]),$$

and clearly W is the smallest A^*-invariant subspace containing the image of C^*.

Now we may write the state space $X := \mathbb{R}^n$ as

$$X = (V \cap W^\perp) \oplus (V^\perp \cap W^\perp) \oplus (V \cap W) \oplus (V^\perp \cap W).$$

Then there exists a basis in which we may write the system as

$$\frac{d}{dt}\begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \\ \bar{x}_3(t) \\ \bar{x}_4(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & 0 \\ 0 & A_{22} & A_{23} & A_{24} \\ 0 & 0 & A_{33} & 0 \\ 0 & 0 & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \\ \bar{x}_3(t) \\ \bar{x}_4(t) \end{bmatrix} + \begin{bmatrix} B_1 u_1(t) \\ 0 \\ B_3 u_3(t) \\ 0 \end{bmatrix}, \tag{2.14}$$

$$y(t) = [C_1 \quad C_2 \quad 0 \quad 0] \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \\ \bar{x}_3(t) \\ \bar{x}_4(t) \end{bmatrix}. \tag{2.15}$$

In the above, (A_{11},B_{11}), (A_{33},B_3) are controllable pairs and (A_{11},C_1), (A_{22},C_2) are observable pairs.

2.5 The pole-assignment theorem

An important consequence of a linear system possessing the controllability property is that by means of linear state feedback control, the poles of the corresponding closed loop system can be placed in arbitrary locations.

By state feedback control, we mean that the control law is of the form

$$u(t) = Kx(t) + v(t),$$

where $K\colon \mathbb{R}^n \to \mathbb{R}^m$ and $v(\cdot)$ is a reference input.

Theorem 2.4. *Let $\Sigma = \{\lambda_i \in \mathbb{C}, i = 1, 2, \ldots, n\}$ with the proviso that if λ_i is complex, then its conjugate $\bar{\lambda}_i \in \Sigma$. Then the system is controllable if and only if there exists a feedback control $u(t) = Kx(t)$ for some feedback matrix K such that the spectrum of $A + BK$ is equal to Σ.*

Proof. We start by noting two facts:

(i) Controllability is invariant under non-singular linear transformations of state space. In case $m = 1$, that is, we are dealing with the system:

$$\frac{dx}{dt}(t) = Ax(t) + bu(t), \qquad u(t) \in \mathbb{R} \quad \text{and} \quad b \in \mathbb{R}^n,$$

then assuming (A, b) is a controllable pair, there exists a non-singular linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$ such that the above system transforms to

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \ldots & 0 & 1 \\ \alpha_1 & \alpha_2 & \ldots & \alpha_{n-1} & \alpha_n \end{bmatrix}, \quad \tilde{b} = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ 1 \end{bmatrix}.$$

It is then easy to see that if the characteristic polynomial corresponding to the desired spectrum Σ is

$$\chi = s^n - \beta_n s^{n-1} \cdots - \beta_2 s - \beta_1,$$

then the feedback law

$$\tilde{u}(t) = K\tilde{x}(t) + v(t), \quad K = \operatorname{diag}(\beta_1 - \alpha_1, \ldots, \beta_n - \alpha_n)$$

transforms

$$\frac{d\tilde{x}}{dt} = \tilde{A}\tilde{x}(t) + \tilde{b}\tilde{u}(t)$$

to

$$\frac{d\tilde{x}}{dt} = (\tilde{A} + \tilde{b}k)\tilde{x}(t) + \tilde{b}v(t)$$

such that the spectrum of $(\tilde{A} + \tilde{b}K)$ in Σ.

(ii) Controllability is not destroyed by state feedback. That is

$$\mathcal{R}\left[B \vdots (A + BK)B \vdots \cdots \vdots (A + BK)^{n-1}B\right] = \mathcal{R}\left[B \vdots AB \vdots \cdots \vdots A^{n-n}B\right].$$

Now the general case is reduced to the case $m = 1$ by virtue of the following lemma.

Lemma 2.1. *Let $0 \neq b \in \mathcal{R}(B)$. If (A, B) is a controllable pair, there exists a matrix $K \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n)$ such that $(A + BK, b)$ is a controllable pair.*

Let $b_1 = b$ and let $n_1 =$ dimension of the cyclic subspace $\mathcal{X}_1$ generated by (A, b_1). Let $x_1 = b_1$ and define $x_j = Ax_{j-1} + b_1 (j = 2, \dots, n_1)$. Then the x_j, $j = 1, \dots, n_1$, forms a basis for the cyclic subspace generated by (A, b_1). If $n_1 < n$, choose $b_2 \in \mathcal{R}(B)$ such that $b_2 \notin \mathcal{X}_1$. As (A, B) is controllable, such a b_2 exists. Let n_2 be the largest integer such that

$$x_1, x_2, \dots, x_{n_1}, \quad b_2, Ab_2, \dots, A^{n_2-1}b_2$$

are independent.

Define

$$x_{n_1+i} = Ax_{n_1+i-1} + b_2, \quad i = 1, 2, \dots, n_2.$$

Then $x_1, x_2, \dots, x_{n_1+n_2}$ is a basis for the cyclic subspace generated by $(A, b_1 + b_2)$. Continuing in this way, we eventually get an independent set of vectors $x_1, \dots, x_n$, and

$$x_{i+1} = Ax_i + \tilde{b}_i, \quad i = 1, \dots, n-1,$$

and $\tilde{b}_i \in \mathcal{R}(B)$.

As $\tilde{b}_i = Bu_i$ for some $u_i \in \mathbb{R}^m$ (the control space) and the $x_1, \dots, x_n$ are independent, there exists a K_1 such that

$$BK_1 x_i = \tilde{b}_i, \quad i = 1, \dots, n,$$

where $\tilde{b}_i \in \mathcal{R}(B)$ is arbitrary. Therefore

$$(A + BK_1)x_i = x_{i+1}, \quad i = 1, 2, \dots, n-1,$$

and hence

$$x_i = (A + BK_1)^{i-1}b, \quad i = 1, \dots, n.$$

Therefore $(A + BK_1, b)$ is a controllable pair.

Hence the problem has been reduced to the case of $n = 1$ and the control law $K = K_1 + bK^*$ achieves the desired property.

To prove the theorem in the other direction, let λ_i, $i = 1, \dots, n$, be real and distinct with $\lambda_i \notin$ spectrum of A. Choose $K \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n)$ such that spectrum $(A + BK) = \{\lambda_1, \dots, \lambda_n\}$. Let x_i, $i = 1, \dots, n$, be the corresponding eigenvectors. Hence

$$x_i = (\lambda_i I - A)^{-1} BKx_i, \quad i = 1, 2, \dots, n.$$

Now

$$(\lambda I - A)^{-1} = \sum_{j=1}^{n} \rho_j(\lambda) A^{j-1},$$

where $\rho_j(\lambda)$ are rational functions defined on the complement $\mathbb{C} \backslash \sigma(A)$ of the spectrum $\sigma(A)$ of A. Hence

$$x_i = \sum_{j=1}^{n} \rho_j(\lambda_i) A^{j-1} B F x_i \in \mathcal{R}([B \vdots AB \vdots \cdots \vdots A^{n-1}B]).$$

Now the x_i's span $\mathbb{R}^n$ and hence $\mathcal{R}([B \vdots AB \vdots \cdots \vdots A^{n-1}B]) = n$. $\square$

No satisfactory result similar to the *pole-assignment theorem* is known in infinite dimensions. It would be interesting to prove a result of this kind for delay systems

$$\begin{cases} \dfrac{dx}{dt}(t) = A_1 x(t) + A_2 x(t-h) + Bu(t), \\ x(0) = \phi_0 \in \mathbb{R}^n, \\ x(0) = \phi_1(\theta) \text{ a.e. } \theta \in [-h, 0), \quad \phi_1 \in L^2(-h, 0; \mathbb{R}^n). \end{cases}$$

Corollary 2.1. *If the system is controllable, then there exists a feedback control $u(t) = Kx(t)$ such that the closed loop system*

$$\frac{dx}{dt}(t) = (A + BK)x(t)$$

is asymptotically stable.

2.6 Stabilizability and detectability

The structure theorem of linear systems and the pole-assignment theorem motivate the introduction of the concepts of stabilizability and detectability. Consider the input–state part of the control system (2.1):

$$\frac{dx}{dt}(t)(t) = Ax(t) + Bu(t).$$

Then from the structure theorem, we know that there is a coordinate system in which the above system has a representation

$$\frac{d}{dt}\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}(t) = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix}\begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix} + \begin{bmatrix} \bar{B}_{11} \\ 0 \end{bmatrix} u(t) \tag{2.16}$$

and $(\bar{A}_{11}, \bar{B}_{11})$ is a controllable pair. Hence, from the corollary of the pole-assignment system, we can stabilize the system by means of the control $u(t) = K\bar{x}_1(t)$, provided

$$\frac{d\bar{x}_2}{dt}(t) = \bar{A}_{22}\bar{x}_2(t)$$

is asymptotically stable.

Definition 2.3. (i) We say that the system (2.1) is *open loop stabilizable* if

$$\forall x_0 \in \mathbb{R}^n, \quad \exists u \in L^2(0, \infty; \mathbb{R}^m)$$

such that the solution $x(t; x_0, u)$ of (2.1) satisfies

$$\int_0^\infty |x(t; x_0, u)|^2 \, dt < \infty. \tag{2.17}$$

(ii) We say that the system (2.1) is *stabilizable by feedback* if there exists a feedback matrix K such that

$$u(t) = Kx(t)$$

and

$$\frac{dx}{dt} = (A + BK)x(t)$$

is asymptotically stable. □

Remark 2.2. For finite dimensional linear systems, if the system is open loop stabilizable, then it is stabilizable by feedback and conversely. Note that by the previous discussion, stabilizability is a weaker concept than controllability, since stabilizability only requires that the uncontrollable part of the system be asymptotically stable. □

The dual of the concept of stabilizability is that of detectability. This is best discussed by considering the dual system

$$\frac{d\xi}{dt}(t) = -A^*\xi(t) - C^*y(t), \tag{2.18}$$

which involves only the state–output part of the system.

Definition 2.4. The system (2.1) is said to be *detectable* if the dual system (2.18) is stabilizable by feedback. □

Remark 2.3. Detectability requires that the unobservable part of the system be asymptotically stable and hence is a weaker concept than observability. □

A matrix test for stabilizability and detectability can be given.

Theorem 2.5. *The following properties are equivalent*:

(i) (A, B) *is a controllable pair.*

(ii) $\mathrm{rank}[\lambda I - A \vdots B] = n$, $\forall \lambda \in \mathbb{C}$.

(iii) $\mathrm{rank}[\lambda I - A \vdots B] = n$, *for each eigenvalue* λ *of* A.

Proof. Note that (ii) and (iii) are equivalent because the first $n \times n$ block of the $n \times (n+m)$ matrix $[\lambda I - A \vdots B]$ has full rank whenever λ is not an eigenvalue of A.

We now prove that (i) $\Longrightarrow$ (ii). For this purpose, assume that $\mathrm{rank}[\lambda I - A \vdots B] < n$ for some $\lambda \in \mathbb{C}$. Hence the space spanned by the rows of $[\lambda I - A \vdots B]$ has a dimension less than n and therefore there exists some vector $z^* \in \mathbb{R}^n$ and some λ such that $z^*[\lambda I - A \vdots B] = 0$. Hence

$$z^*A = \lambda z^* \quad \text{and} \quad z^*B = 0,$$

and therefore $z^* A^k B = \lambda^k z^* B = 0$, $\forall K$, and hence

$$z^*[B \vdots AB \vdots \cdots \vdots A^{n-1}B] = 0,$$

which contradicts controllability.

To prove (i) $\Longrightarrow$ (ii) assume (A, B) is not a controllable pair. Then from the *structure theorem*, there exists a non singular linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$ such that

$$(A, B) \mapsto \left(\bar{A} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \bar{B} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \right),$$

where A_{11}, B_1 is a controllable pair and A_{11} is $r \times r$ with $r < n$. Let λ be an eigenvalue of A_{22}^* with the corresponding eigenvector v so that

$$v^*(\lambda I - A_{22}) = 0. \text{ Then the } n\text{-vector } (\neq 0),$$

$$w = \begin{bmatrix} 0 \\ v \end{bmatrix} \text{ is such that}$$

$w^* \bar{A} = \lambda w^*$ and $w^* \bar{B} = 0$. Hence $z = (T^*)^{-1} w \neq 0$ satisfies $z^*(\lambda I - A)T^{-1} \vdots B = 0$, and therefore, $z^*(\lambda I - A \vdots B) = 0$ contradicting (ii). □

Corollary 2.2. (i) (A, B) *is* stabilizable *if and only if*

$$\operatorname{rank}[\lambda I - A \vdots B] = n, \quad \forall \lambda \in \mathbb{C}, \ \operatorname{Re} \lambda \geq 0.$$

(ii) *Dually* (A, C) *is detectable if and only if*

$$\operatorname{rank} \begin{bmatrix} \lambda I - A^* \\ C^* \end{bmatrix} = n, \quad \forall \lambda \in \mathbb{C} \textit{ such that } \operatorname{Re} \lambda \geq 0.$$

Corollary 2.2 is generalized to an infinite dimensional setting when A is the generator of a *strongly continuous semigroup*[1] on a Hilbert space H with e^{tA} compact for $t > 0$ (and indeed for more general situations) and $B \in \mathcal{L}(U, H)$ and $C \in \mathcal{L}(H; Y)$ and U and Y are Hilbert spaces (Proposition 3.3 of Chapter 1 in Part V).

2.7 Applications of controllability and observability

The concepts of controllability, observability, and the slightly weaker notions of stabilizability and detectability have important applications in diverse areas of systems theory and optimal control. In this section we present two such applications. The first application is for the theory of stability and tests for stability via the Lyapunov equations. Our second application is for regulator theory, where we show that if a system is controllable and observable, then we can build a compensator to regulate the system in a satisfactory way.

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[1] cf. Definition 2.1 on page 89.

2.7.1 Stability

Our first application is to the study of stability of linear differential equations via the Lyapunov method.

Consider the stability problem for the equation

$$\frac{dx}{dt}(t) = Ax(t), \quad A\colon \mathbb{R}^n \to \mathbb{R}^n. \tag{2.19}$$

It is well known (cf. Chapter 1 in Part II) that the following are equivalent:

(i) (2.19) is asymptotically stable.
(ii) (2.19) is exponentially stable.
(iii) $\forall x_0 \in \mathbb{R}^n$, $\int_0^\infty |e^{At}x_0|^2\,dt < \infty$.

To test for stability of (2.19), we look for a Lyapunov function of the form

$$V(x) = (x, Px), \quad P^* = P > 0 \text{ positive definite.}$$

Computing the derivative of V along trajectories of (2.19), we get

$$\frac{dV}{dt} = (x, A^*P + PAx),$$

and if it is to be negative, we require that

$$A^*P + PA = -Q, \quad Q > 0 \text{ positive definite.} \tag{2.20}$$

It is well known that if A is a stability matrix (all its eigenvalues have a strictly negative real part), then the solution of (2.20) can be written as

$$P = \int_0^\infty e^{A^*t}Qe^{At}\,dt, \tag{2.21}$$

and conversely if (2.20) has a positive definite solution, then A is a stability matrix. A generalization of these results in infinite dimensional spaces has been extensively discussed in Chapter 1 of Part II.

Now suppose that $Q \geq 0$ and factorize Q as $Q = C^*C$, $C \in \mathcal{L}(\mathbb{R}^p; \mathbb{R}^n)$. Introduce the state–output system:

$$\begin{cases} \dfrac{dx}{dt}(t) = Ax(t), \\ y(t) = Cx(t). \end{cases} \tag{2.22}$$

Definition 2.5. The state–output system is said to be *asymptotically output stable* if

$$\forall x \in \mathbb{R}^n, \quad \int_0^\infty |Ce^{At}x|^2\,dt < \infty. \qquad \square$$

Now it is easy to see that if (A, C) is an observable pair and A is asymptotically stable, then the equation

$$A^*P + PA = -C^*C$$

has a solution and

$$P = \int_0^\infty e^{A^*t} C^* C e^{At}\, dt > 0 \text{ (positive definite)},$$

the positive-definiteness being a consequence of observability. Conversely if the state–output system is asymptotically output stable and if (A, C) is an observable pair, then A is asymptotically stable. We can now weaken this further.

Theorem 2.6. *If the state–output system is asymptotically output stable (by some feedback matrix K) and if (A, C) is a detectable pair, then the system*

$$\frac{dx}{dt}(t) = (A + BK)x(t)$$

is asymptotically stable.

Proof. By hypothesis there exists a $K^* \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^p)$ such that $(A^* + C^*K^*)$ is a stability matrix and hence $(A + KC)$ is a stability matrix. Now write

$$\frac{dx}{dt}(t) = Ax(t)$$

as

$$\frac{dx}{dt}(t) = (A + KC)x(t) - KCx(t),$$

and we can write its solution as

$$x(t) = e^{(A+KC)t}x + \int_0^t e^{(A+KC)(t-s)} KCx(s)\, ds. \tag{2.23}$$

Hence

$$\left[\int_0^\infty |x(t)|^2\, dt\right]^{1/2} \le \left[\int_0^\infty |e^{(A+KC)t}x|^2\, dt\right]^{1/2} + \left[\int_0^\infty \left|\int_0^t e^{(A+KC)(t-s)} KCx(s)\, ds\right|^2 dt\right]^{1/2}. \tag{2.24}$$

Furthermore, as $(A+KC)$ is a stability matrix, there exists $\alpha > 0$ and $M \ge 1$ such that

$$\forall x \in \mathbb{R}^n, \quad |e^{(A+KC)t}x| \le Me^{-\alpha t}|x|,$$

and the second term of the right-hand side of inequality (2.24) can be majorized by

$$\int_0^\infty \left[\int_0^t Me^{-\alpha(t-s)}\|K\| \, |Cx(s)| \, ds\right]^2 dt.$$

Now introduce the function

$$f(s) = \begin{cases} M\|K\|e^{-\alpha s}, & s \geq 0, \\ 0, & \text{otherwise} \end{cases}$$

and

$$g(s) = \begin{cases} |Cx(s)|, & s \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Now $f \in L^1(-\infty, \infty; \mathbb{R})$ and $g \in L^2(-\infty, \infty; \mathbb{R})$. Hence by Young's inequality

$$\|f * g\|_{L^2} \leq \|f\|_{L^1} . \|g\|_{L^2},$$

where

$$(f * g)(t) = \int_{-\infty}^\infty f(t-s)g(s) \, ds = \int_0^t M\|K\|e^{-\alpha(t-s)}|Hx(s)| \, ds.$$

This proves the theorem. □

Remark 2.4. This theorem and its proof generalize to certain infinite dimensional Hilbert space situations and have implications in the study of the algebraic Riccati equation. See §3.4, Chapter 1 of Part V. □

2.7.2 Compensators for linear systems

Consider the linear control system

$$\begin{cases} \dfrac{dx}{dt}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t). \end{cases} \tag{2.25}$$

We have seen that if (A, B) is a controllable pair, then the spectrum of the closed loop system can be made arbitrary by state feedback. We now wish to discuss the situation where full-state feedback is not available and the control has to be of the form

$$u(t) = \int_0^t \Lambda(t-s)y(s) \, ds + v(t), \tag{2.26}$$

where the kernel Λ corresponds to some finite dimensional system and $v(t)$ is a reference input. We would like to investigate the properties of the corresponding closed loop system. We shall construct such a control by

(i) constructing a state estimator $\hat{x}(\cdot)$ from the observations $y(\cdot)$,

(ii) and by constructing an appropriate control as a linear function of the estimate $\hat{x}(t)$.

We first construct such an estimator in the form

$$\begin{cases} \dfrac{d\hat{x}}{dt}(t) = A\hat{x}(t) + M\nu(t) + Bu(t), \\ \nu(t) = y(t) - C\hat{x}(t), \\ \text{and } M \text{ is to be chosen later.} \end{cases} \tag{2.27}$$

The vector $\nu(\cdot)$ has the interpretation as an "innovation" function; that is, $\nu(t)$ is the new information in the output $y(t)$ at time t not contained in $C\hat{x}(t)$. Let $e(t) = x(t) - \hat{x}(t)$ be the error between the state and its estimate. We see that

$$\frac{de}{dt}(t) = (A - MC)e(t). \tag{2.28}$$

Now if we assume that (A, C) is an observable pair, then the spectrum of $(A - MC)$ can be made arbitrary by a suitable choice of M. In particular, by a suitable choice of M, $(A - MC)$ can be made a stability matrix. Note that for the requirement of stability of the $(A - MC)$, it is enough to assume that (A, C) is a detectable pair.

Now consider a feedback control of the form

$$u(t) = -K\hat{x}(t), \tag{2.29}$$

giving us a closed loop system,

$$\frac{dx}{dt}(t) = Ax(t) - BK\hat{x}(t) = (A - BK)x(t) + BKe(t). \tag{2.30}$$

By the pole-assignment theorem, the spectrum of $(A - BK)$ can be made arbitrary by a suitable choice of K, provided (A, B) is a controllable pair (stabilizability is enough).

Consider the pair of differential equations:

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - MC \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}. \tag{2.31}$$

It is clear that the spectrum of the above block matrix is determined by that of $A - BK$ and $A - MC$ and hence can be made arbitrary by proper choices of K and M and in particular can be made a stability matrix. This is one of the fundamental results of linear feedback control.

We now turn to a discussion of quadratic-cost optimal control and show how this theory gives rise to optimal feedback controllers that render the closed loop system asymptotically stable. It turns out that the theory of controllability and observability plays an important role in the study of optimal control over an infinite time horizon.

3 Optimal control

Consider the *linear control system*

$$\begin{cases} \dfrac{dx}{dt}(t) = Ax(t) + Bu(t), \quad x(s) = x, \\ y(t) = Cx(t) \end{cases} \tag{3.1}$$

on the time interval $[s, T]$, $0 \le s < T$, where x is an arbitrary initial state in $\mathbb{R}^n$. The notation and terminology are the same as the ones defined in the section on controllability and observability.

We shall be primarily concerned with the following *optimal control problem*: to choose a control $\hat{u}(\cdot) \in L^2(s, T; \mathbb{R}^m)$ that minimizes the cost functional

$$J(u; s, x) = \frac{1}{2} \int_s^T \left[\big(u(t), Ru(t)\big) + \big(x(t), Qx(t)\big) \right] dt + \frac{1}{2} \big(x(T), Sx(T)\big), \tag{3.2}$$

where $R = R^* > 0$, $Q = Q^* = C^*C \ge 0$, and $S = S^* \ge 0$. A control $\hat{u}(\cdot)$ minimizing $J(u; x)$ will be called an *optimal control.* We shall not be concerned with the question of existence and uniqueness of solutions but with the characterization of the optimal control $\hat{u}(\cdot)$ and the corresponding optimal trajectory $\hat{x}(\cdot)$.

Even though we are interested in solving the optimal control problem over a fixed interval $[0, T]$ and for the fixed initial condition x_0, it will turn out to be conceptually important to solve the problem for all initial points (s, x), $0 \le s < T$.

We shall also be interested in the infinite time problem: Find a control $\hat{u}(\cdot) \in L^2(s, \infty; \mathbb{R}^m)$ that minimizes:

$$J(u; s, x) = \frac{1}{2} \int_s^\infty \left[\big(u(t), Ru(t)\big) + \big(x(t), Qx(t)\big) \right] dt. \tag{3.3}$$

3.1 Finite time horizon

We first discuss the finite time situation. We are interested in obtaining the optimal control $\hat{u}(\cdot)$ in feedback form $\hat{u}(t) = K(t)x(t)$, where $K(t) \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^m)$. There are three basic approaches to the solution of this problem. The first is via the *Calculus of Variations* or equivalently the *Maximum Principle* and makes use of the adjoint equation. It can be shown that the optimal control is characterized by the solution of the following system:

$$\begin{cases} \dfrac{d\hat{x}}{dt}(t) = A\hat{x}(t) + B\hat{u}(t), \\ \hat{x}(s) = x, \end{cases} \tag{3.4}$$

$$\begin{cases} \dfrac{dp}{dt}(t) = -A^*p(t) - Q\hat{x}(t), \\ p(T) = S\hat{x}(T), \end{cases} \tag{3.5}$$

$$R\hat{u}(t) + B^* p(t) = 0 \quad s \le t \le T. \tag{3.6}$$

We recognize this as a *two-point boundary value problem.* The system (3.4)–(3.6) can be decoupled as follows. It is here the idea of invariant embedding, namely, the idea of considering the optimal control problem for all initial (s, x) where $0 < s < T$ and $x \in \mathbb{R}^n$ arbitrary becomes important. One first shows that the system of equations

$$\begin{cases} \dfrac{d\xi}{dt}(t) = A\xi(t) - BR^{-1}B^*\eta(t), \\ \xi(s) = x, \end{cases} \tag{3.7}$$

$$\begin{cases} \dfrac{d\eta}{dt}(t) = -A^*\eta(t) - Q\xi(t), \\ \eta(T) = S\xi(T) \end{cases} \tag{3.8}$$

admits a unique solution. We can then prove that the mapping

$$x \mapsto \big(\xi(\cdot), \eta(\cdot)\big) \colon \mathbb{R}^n \to W(s, T; \mathbb{R}^n) \times W(s, T; \mathbb{R}^n), \tag{3.9}$$

where

$$W(s, T; \mathbb{R}^n) = \left\{ z \colon z \in L^2(s, T; \mathbb{R}^n), \frac{dz}{dt} \in L^2(s, T; \mathbb{R}^n) \right\}$$

is an affine continuous map and finally that the mapping

$$x \mapsto \eta(s) \colon \mathbb{R}^n \to \mathbb{R}^n \text{ is an affine mapping.} \tag{3.10}$$

From this it follows that we can write

$$\eta(s) = P(s)x + r(s), \tag{3.11}$$

where

$$P(s) \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n) \quad \text{and} \quad r(s) \in \mathbb{R}^n. \tag{3.12}$$

Let $\big(x(\cdot), p(\cdot)\big)$ be a solution of (3.4)–(3.6) in the interval $[0, T]$. Then

$$p(t) = P(t)x(t) + r(t), \quad \forall t \in [0, T],$$

and a calculation shows that $P(t)$ and $r(t)$ satisfy:

$$\begin{cases} \dfrac{dP}{dt}(t) + A^*P(t) + P(t)A - P(t)BR^{-1}B^*P(t) + Q = 0, \\ P(T) = S, \end{cases} \tag{3.13}$$

$$\begin{cases} \dfrac{dr}{dt}(t) + A^*r(t) - P(t)BR^{-1}B^*r(t) = 0, \\ r(T) = 0. \end{cases} \tag{3.14}$$

Clearly $r(t)$ is identically zero and standard results of ordinary differential equations prove that (3.12) has a unique global solution.

Therefore we have obtained the optimal control in feedback from:

$$\hat{u}(t) = -R^{-1}B^*P(t)\hat{x}(t).$$

The same result can be obtained using the method of Dynamic Programming. Conceptually, the Calculus of Variations gives necessary conditions of optimality, whereas the method of Dynamic Programming gives sufficient conditions of optimality. But for the strictly convex quadratic cost problem, both methods give necessary and sufficient conditions of optimality.

Let

$$V(s,x) = \inf_u J(u; s, x).$$

Then an application of the Principle of Optimality leads to the following partial differential equation for V:

$$\begin{cases} \dfrac{\partial V}{\partial s}(s,x) + \min\limits_u \Big[\dfrac{1}{2}\big(u(s), Ru(s)\big) + \dfrac{1}{2}\big(x(s), Qx(s)\big) \\ \qquad\qquad\qquad + \big(\nabla_x V(x,s), Ax(s) + Bu(s)\big) \Big] = 0, \\ V(T,x) = (Sx, x), \end{cases} \tag{3.15}$$

where $\nabla_x V(x,s)$ is the gradient of $V(x,s)$ with respect to the vector x. Carrying out the minimization, we obtain

$$u(s) = -R^{-1}B^*\nabla_x V(x,s), \quad \forall s \in [0,T]. \tag{3.16}$$

Hence we obtain the so-called *Bellman–Hamilton–Jacobi equation*

$$\frac{\partial V}{\partial s}(s,x) + \frac{1}{2}\big(\nabla_x V(x,s), BR^{-1}B^*\nabla_x V(x,s)\big) \\ + \frac{1}{2}(x, Qx) + (A^*\nabla_x V(s,x), x) = 0. \tag{3.17}$$

Now, (3.17) can be solved by looking for a solution of the form:

$$V(s,x) = \tfrac{1}{2}\big(x, P(s)x\big) + \big(x, r(s)\big). \tag{3.18}$$

We can check that $P(s)$ and $r(s)$ satisfies (3.12) and (3.14) previously obtained, and we recover the same results as obtained by Calculus of Variation arguments.

There is a third approach, in some sense related to the *Dynamic Programming approach*, to the solution of quadratic cost optimal control problems. This is the *method of completing the squares*. The main idea here is to observe that if

$$\frac{dx}{dt} = Ax(t) + Bu(t),$$

and if $P(t) = P^*(t)$ be such that dP/dt exists in the interval $[s,T]$, then defining

$$z(t) = \begin{bmatrix} u(t) \\ x(t) \end{bmatrix} \quad \text{and} \quad L(t) = \begin{bmatrix} 0 & R^{-1}B^*P(t) \\ P(t)B & \dfrac{dP}{dt} + A^*P(t) + P(t)A \end{bmatrix},$$

we get

$$\int_s^T \big(z(t), L(t)z(t)\big)\,dt - \big(x(T), P(T)x(T)\big) + \big(x(s), P(s)x(s)\big) = 0. \tag{3.19}$$

Now let $P(t)$ be the global solution on the interval $[s,T]$ of the equation

$$\begin{cases} \dfrac{dP}{dt}(t) + A^*P(t) + P(t)A - P(t)BR^{-1}B^*P(t), +Q = 0, \\ P(T) = S. \end{cases} \tag{3.20}$$

Then adding the identity (3.20) to $J(u;s,x)$ and doing some algebraic manipulations, we obtain

$$J(u;s,x) = \int_s^T |u(t) + R^{-1}B^*P(t)x(t)|^2\,dt + \big(x(s), P(s)x(s)\big). \tag{3.21}$$

Hence the optimal control is given by

$$\hat{u}(t) = -R^{-1}B^*P(t)x(t), \tag{3.22}$$

and the optimal cost $V(s,x)$ by

$$V(s,x) = \big(x, P(s)x\big). \tag{3.23}$$

In this book, we use the latter two methods in infinite dimensional situations to obtain results on optimal control for the quadratic cost problem in a finite time interval. Apparently, one obtains more general results using this methodology.

In Part IV, Chapters 1 to 3 develop the optimal control of parabolic, hyperbolic, and delay equations over a finite time interval. The observation operator is usually considered to be also bounded, but the case of unbounded observation operators is also treated. Chapter 2 deals with parabolic equations with unbounded control operators, whereas Chapter 3 deals with hyperbolic equations with unbounded control operators. The latter two chapters are the most technical, and the full power of the theory developed in Part II needs to be used.

3.2 Infinite time horizon

The study of the quadratic cost problem over a finite and an infinite time interval is essentially a study of the Riccati differential equation over the finite time interval $[s,T]$

$$\frac{dP}{dt}(t) + A^*P(t) + P(t)A - P(t)BR^{-1}B^*P(t) + C^*C = 0 \tag{3.24}$$

subject to the terminal condition

$$P(T) = P_T \geq 0, \tag{3.25}$$

its asymptotic behavior and the corresponding matrix quadratic equation

$$A^*P + PA - PBR^{-1}B^*P + C^*C = 0. \tag{3.26}$$

The following theorem is proved (in certain infinite dimensional situations) in §2 of Chapter 1 in Part IV, and in §2 to §4 of Chapter 1 in Part V.

Theorem 3.1. *There exists a matrix $P(t)$ satisfying the following*:

(i) $P(\cdot)$ *is defined and belongs to* $C^1\big([s,T];\mathcal{L}(\mathbb{R}^n)\big)$ *and satisfies* (3.24) *and* (3.25).
(ii) $P(t) \geq 0$, $s \leq t \leq T$ *and is the unique solution of* (3.24)–(3.25).
(iii) *Let $\tilde{K}(t)$ be a continuous function on $[s,T]$, and let $\tilde{P}(t)$ be the solution of the linear differential equation*

$$\begin{aligned} \frac{d\tilde{P}}{dt}(t) + \big(A - B\tilde{K}(t)\big)^*\tilde{P}(t) + \tilde{P}(t)\big(A - B\tilde{K}(t)\big) \\ + C^*C + \tilde{K}(t)^*R\tilde{K}(t) = 0, \end{aligned} \tag{3.27}$$

$$\tilde{P}(T) = P_T \geq 0. \tag{3.28}$$

If $P(t)$ is a solution to (3.24) *and* (3.25), *then*

$$P(t) \leq \tilde{P}(t), \quad s \leq t \leq T.$$

(iv) *Consider* (3.24) *with $s \to -\infty$ and $T = 0$. If (A,B) is stabilizable, then $P(t)$ is bounded on $]-\infty,0]$. If (A,C) is detectable, then*

$$P_\infty = \lim_{t\to-\infty} P(t)$$

exists and is positive semi-definite.
In that case, P_∞ is the unique positive semi-definite solution of (3.26) *and*

$$A - BR^{-1}B^*P_\infty$$

is a stability matrix.

The optimal control over the infinite time interval is then given by

$$\hat{u}(t) = -R^{-1}B^*P_\infty\hat{x}(t). \tag{3.29}$$

The study of these control problems when control is exercised through the boundary is most complicated for hyperbolic equations and is taken up in Chapter 3 of Part V. The study of exact controllability comes into the picture here to ensure that the space of admissible controls is nonempty.

4 A glimpse into H^∞-theory: state feedback case

4.1 Introduction

In many control problems the quadratic criterion is not the most appropriate. We consider a *disturbance attenuation problem* when the full state vector can be measured and introduce the so-called H^∞-optimal control problem. There is now a vast literature on this topic. For two textbook presentations, the reader is referred to T. Başar and P. Bernhard [1] and B. A. Francis [1].

Consider the finite dimensional linear time-invariant system

$$\begin{cases} \dfrac{dx}{dt}(t) = Ax(t) + Lw(t) + Bu(t), \\ \quad z(t) = Cx(t) + Du(t), \end{cases} \tag{4.1}$$

where u is interpreted as the controls and z as controlled outputs that are to be made small in the presence of exogenous disturbances w. The matrices and vectors have appropriate dimensions.

Let $S(A, B)$ denote the set of all constant gain stabilizing state feedback matrices. That is,

$$S(A, B) = \{K\colon \operatorname{Re}[\lambda_i(A + BK)] < 0\}.$$

Then for $u \in S(A, B)$, the closed loop dynamics take the form

$$\begin{cases} \dfrac{dx}{dt}(t) = (A + BK)x(t) + Lw(t), \\ \quad z(t) = (C + DK)x(t), \end{cases} \tag{4.2}$$

or in transfer function form

$$z(s) = T_K(s)w(s),$$

where

$$T_K(s) = (C + DK)\big(sI - (A + BK)\big)^{-1}L.$$

The problem of H^∞-optimal state feedback is now stated as

$$\inf_{K \in S(A,B)} \|T_K\|_{H^\infty},$$

where

$$\|T_K\|_{H_\infty} = \sup_{\omega} \sigma_{\max}\|T(j\omega)\|$$

and $\sigma_{\max}$ denotes the maximum singular value of the matrix $T(j\omega)$.

This H^∞-optimal state feedback problem represents a special case of the more general H^∞- optimal output feedback disturbance rejection problem.

4.2 Main results

The following assumptions are made on (4.1):

(A1) The pair (A, B) is stabilizable.
(A2) The pair (A, C) is observable.
(A3) $D^*[C\ D] = [0\ I]$.

(A1) is necessary for the stabilization of (4.1). Assumption (A2) is a technical assumption that guarantees the invertibility of certain algebraic Riccati equation solutions. The orthogonality assumption (A3) is analogous to no cross weighting between the state and control in the standard LQ problem. In case (A3) is violated, the change of control variables from u to v given by

$$u(t) = (D^*D)^{-1/2}v(t) - (D^*D)^{-1}Cx(t)$$

yields a new system that satisfies (A3). Furthermore, it can be shown that a controlled output of the form

$$z(t) = Cx(t) + Du(t) + Eu(t)$$

may be transformed to the form of (4.1). Thus, (A3) is made with minimal loss of generality.

The main result is now stated.

Theorem 4.1. *Consider the linear system* (4.1) *under assumptions* (A1) *to* (A3). *Under these conditions, there exists a* $K \in S(A, B)$ *such that* $\|T_K\|_{H^\infty} < \gamma$ *if and only if there exists an* $X = X^* > 0$ *that satisfies*

$$XA + A^*X + C^*C + X\left(\frac{1}{\gamma^2}LL^* - BB^*\right)X = 0 \tag{4.3}$$

with $A + (1/\gamma^2 LL^* - BB^*)X$ *stable* (*i.e., all eigenvalues in the open left-half complex plane*).

The remainder of this section is devoted to the proof of Theorem 4.1 for $\gamma = 1$. By linearity, this simplification is made without loss of generality.

First, some preliminary results from linear quadratic optimization theory are stated for the linear system

$$\begin{cases} \dfrac{dx}{dt}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \\ y(t) = Cx(t). \end{cases}$$

Lemma 4.1. *Let* (A, B) *be stabilizable and* (A, C) *detectable* (*resp. observable*). *Then there exists a unique* $P = P^* \geq 0$ (*resp.* > 0) *that satisfies*

$$PA + A^*P + C^*C - PBB^*P = 0,$$

with $A - BB^*P$ stable. Furthermore, the state feedback control law $u(t) = -B^*Px(t)$ is a minimizing solution for the problem

$$\inf_u \frac{1}{2}\int_0^\infty \left(|y(t)|^2 + |u(t)|^2\right) dt.$$

This is Theorem 3.1 restated.

Lemma 4.2. *Let $G(s) = C(sI - A)^{-1}B$ with A stable. Then $\|G\|_\infty < 1$ if and only if there exists an $X = X^* \geq 0$ that satisfies*

$$XA + A^*X + C^*C + XBB^*X = 0$$

with $A + BB^*X$ stable. Furthermore, the state feedback $u(t) = BB^*Xx(t)$ solves

$$\sup_u \frac{1}{2}\int_0^\infty (|y(t)|^2 - |u(t)|^2)\, dt.$$

For a proof of the above lemma see J. C. Willems [1].

The proof of Theorem 4.1 is now presented. To prove the "if" direction, assume that $X = X^* > 0$ satisfies (4.3) with $A + (LL^* - BB^*)X$ stable. A manipulation of (4.3) yields

$$\begin{aligned} X(A - BB^*X) &+ (A - BB^*X)^*X \\ &+ (C - DB^*X)^*(C - DB^*X) + XLL^*X = 0. \end{aligned} \tag{4.4}$$

Now assumptions (A2) to (A3) imply that the pair $(A - BB^*X, C - DB^*X)$ is observable. This observability, (4.4), and $X > 0$ together imply that the matrix $A - BB^*X$ is stable via standard Lyapunov stability theory. Thus, $K = -B^*X \in S(A, B)$. Finally, using (4.4) and the stability of $A + (LL^* - BB^*)X$ in Lemma 4.2 implies that $\|T_K\|_{H^\infty} < 1$, which is the desired result.

To prove the "only if" direction requires the following preliminary result:

Lemma 4.3. *There exists an $X = X^* > 0$ that satisfies*

$$XA + A^*X + C^*C + X(LL^* - BB^*)X = 0 \tag{4.5}$$

with $A + (LL^* - BB^*)X$ stable if and only if there exists a $P = P^* > 0$ that satisfies

$$AP + PA^* + PC^*CP + LL^* - BB^* = 0 \tag{4.6}$$

with $-A^* - C^*CP$ stable.

Proof. Let $P = X^{-1}$. This establishes an equivalence of the existence of positive definite solutions to either (4.5) or (4.6). To show equivalence of the stability conditions, a manipulation of either (4.5) or (4.6) yields the similarity condition

$$A + (LL^* - BB^*)X = P(-A^* - C^*CP)P^{-1}. \qquad \square$$

Thus to prove the "only if" portion of Theorem 4.1, it suffices to prove the equivalent condition associated with (4.6).

Toward this end, let $K \in S(A,B)$ be such that $\|T_K\|_{H^\infty} < 1$. Then from Lemma 4.2, there exists an $\tilde{X} = \tilde{X}^* \geq 0$ that satisfies

$$\tilde{X}(A+BK) + (A+BK)^*\tilde{X} + (C+DK)^*(C+DK) + \tilde{X}LL^*\tilde{X} = 0, \quad (4.7)$$

with $A + BK + LL^*\tilde{X}$ stable. A straightforward manipulation of (4.7) yields

$$\tilde{X}A + A^*\tilde{X} + C^*C + \tilde{X}(LL^* - BB^*)\tilde{X} = -(K + B^*\tilde{X})^*(K + B^*\tilde{X}). \quad (4.8)$$

Let $\tilde{P} = \tilde{X}^{-1}$. Note that assumptions (A2)–(A3) imply observability of the pair $(A + BK, C + DK)$, which in turn implies $\tilde{X} > 0$, thus allowing the inversion of $\tilde{X}$. In terms of $\tilde{P}$, (4.8) now takes the form

$$A\tilde{P} + \tilde{P}A^* + \tilde{P}C^*C\tilde{P} + LL^* - BB^* = -\tilde{V}^*\tilde{V}, \quad (4.9)$$

where

$$\tilde{V} = K\tilde{P} + B^*.$$

Now (4.9) is not quite in the desired form of (4.6). Thus, as in J. C. Willems [1], let $\Delta P = P - \tilde{P}$. Then subtracting (4.9) from (4.6) yields (after some straightforward manipulations)

$$(-\tilde{A})^*\Delta P + \Delta P(-\tilde{A}) + \tilde{V}^*\tilde{V} - \Delta P C^*C\Delta P = 0, \quad (4.10)$$

where $\tilde{A} = A^* + C^*C\tilde{P}$. Clearly, if one can find a symmetric positive semi-definite solution, ΔP, to (4.10), then $P = \tilde{P} + \Delta P$ satisfies the desired (4.6). However, (4.6) is of the form of a standard LQ Riccati equation as in Lemma 4.1. The requisite stabilizability and detectability in Lemma 4.1 is now shown as follows:

1. The matrix pair $(-\tilde{A}, C^*)$ is controllable. This follows directly from assumption (A2) and that $-\tilde{A} = -A^* - C^*C\tilde{P}$.
2. The matrix pair $(-\tilde{A}, \tilde{V})$ is detectable. To see this, a straightforward manipulation of (4.9) yields the similarity condition

$$-\tilde{A} - K^*\tilde{V} = \tilde{P}(A + BK + LL^*\tilde{X})\tilde{P}^{-1}.$$

However, recall that $\tilde{X}$ is such that $A+BK+LL^*\tilde{X}$ is stable, thus proving the desired detectability.

These observations along with Lemma 4.1 guarantee the existence and uniqueness of a positive definite ΔP, which satisfies (4.10). Thus, $P = \tilde{P} + \Delta P > 0$ and satisfies (4.6).

It then remains to be shown that $-A^* - C^*CP$ is stable. However, from Lemma 4.1, one has that $-\tilde{A} - C^*C\Delta P$ is stable. But

$$-\tilde{A} - C^*C\Delta P = -A^* - C^*CP,$$

which, via Lemma 4.3, completes the proof.

It is worthwhile remarking that the current proof exploits the fact that the condition $\|T_K\| \leq 1$ is a characterization of dissipativity in terms of scattering variables. It is well known that the dissipative property in turn can be characterized in terms of a quadratic Lyapunov function obtained from an indefinite quadratic cost problem (cf. Lemma 4.2). The exposition here essentially exploits these ideas to obtain the desired result. We have included a section on dissipative systems for this reason.

It should be remarked that the state space theory of H^∞ should only be considered as a computational device and not as a substitute for the operator theory solution of H^∞-problems. The important questions of approximation and robustness should be treated from the input–output viewpoint, and it is unclear whether it is meaningful in the state space description.

The general case, when only measurement feedback is available, was treated in the important paper of J. C. Doyle, K. Glover, P. P. Khargonekar, and B. A. Francis [1] in 1989. These results were generalized to certain Infinite Dimensional Systems (the so-called Pritchard–Salamon systems) by B. van Keulen [1] in 1993. The new element here is that the feedback solution is determined in terms of two coupled Ricatti equations.

5 Dissipative systems

5.1 Definitions and preliminary results

Consider the linear, stationary dynamical system

$$\begin{cases} \dfrac{dx}{dt} = Ax + Bu, \quad x(0) = x_0, \\ y = Cx + Du, \end{cases} \tag{5.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, and A, B, C, D are constant matrices of dimensions $(n \times n)$, $(n \times m)$, $(p \times n)$, and $(p \times m)$, respectively. We shall assume that the input functions $t \mapsto u(t) : \mathbb{R} \to \mathbb{R}^m$ are locally square integrable.

Given $x(t_0) = x_0 \in \mathbb{R}^n$, there exists a unique absolutely continuous function $t \mapsto x(t) : [t_0, \infty) \to \mathbb{R}^n$ that satisfies (5.1) almost everywhere. Correspondingly, the output function $t \mapsto y(t) : [t_0, \infty) \to \mathbb{R}^n$ is locally square integrable and the state trajectory $t \mapsto x(t) : [t_0, \infty) \to \mathbb{R}^n$ is also locally square integrable. In terms of the above, we denote the input space by $\mathcal{U}$ and the output space by $\mathcal{Y}$.

We make the assumption that (5.1) is reachable (and controllable) and observable. Hence, the realization of the input–output map

$$y(t) = Ce^{A(t-t_0)}x_0 + \int_{t_0}^{t} Ce^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t) \tag{5.2}$$

given by (5.1) is minimal.

In the sequel, we wish to examine the dissipativeness of system (5.1) with respect to the supply rate $w = (u, y) = u^* y$, where we are assuming that $m = p$ and $(\cdot, \cdot)$ represents the scalar product on $\mathbb{R}^n$.

Definition 5.1. A function $S : \mathbb{R}^n \to \mathbb{R}^+$ is called a *storage* function. The system (5.1) is said to be *dissipative* with respect to the *supply rate* $w(t) = (u(t), y(t))$ if there exists a storage function $S : \mathbb{R}^n \to \mathbb{R}^+$ such that for all $x_0 \in \mathbb{R}^n$ and $u \in \mathcal{U}$, the *dissipation inequality*

$$S(x_0) + \int_{t_0}^{t_1} (u(t), y(t))\, dt \geq S(x(t_1)) \tag{5.3}$$

holds for all $t_1 \geq t_0$ in $\mathbb{R}$, where

$$x(t_1) = e^{A(t_1 - t_0)} x_0 + \int_{t_0}^{t_1} e^{A(t_1 - \tau)} Bu(\tau)\, d\tau \tag{5.4}$$

and

$$y(t) = Ce^{A(t-t_0)} x_0 + \int_{t_0}^{t_1} e^{A(t-\tau)} Bu(\tau)\, d\tau + Du(t). \tag{5.5}$$

Note that if S is smooth, then the dissipation inequality may be written as

$$\begin{gathered} S(x) \geq 0, \\ (\nabla S(x), Ax + Bu) \leq (u, Cx) + (u, Du)\ , \end{gathered} \tag{5.6}$$

$\forall\ x \in \mathbb{R}^n$, and $u \in \mathbb{R}^m$. □

The form of (5.6) (note the similarity to the Hamilton–Jacobi equation) suggests that the identification of storage functions arises by defining appropriate variational problems associated with (5.1). We start with the following lemma (J. C. Willems [1, Lemma 1]).

Lemma 5.1. *Let the dynamical system defined by* (5.1) *be minimal and dissipative with respect to the supply rate* $w(t) = (u(t), y(t))$*. Let* S *be a storage function. Then there exists an* $\varepsilon > 0$ *such that*

$$S(x) > S(0) + \varepsilon \|x\|^2\ ,\ \forall\, x \in \mathbb{R}^n\, .$$

5.2 Associated variational problems

Now, define the following variational problems:

$$S_a(x_0) = - \lim_{t_1 \to \infty} \inf_{u \in \mathcal{U}} \int_0^{t_1} (u(t), y(t))\, dt, \tag{5.7}$$

$$S_r(x_0) = - \lim_{t_{-1} \to -\infty} \inf_{u \in \mathcal{U}} \int_{t_{-1}}^{0} (u(t), y(t))\, dt, \tag{5.8}$$

subject to the constraint (5.1) and the boundary conditions $x(t_{-1}) = 0, x(0) = x_0$. We adopt the normalization $\min_{x\in\mathbb{R}^n} S(x) = S(0) = 0$. S_a is referred to as the *available storage* and S_r the *required supply.* Note that (5.7) and (5.8) are quadratic variational problems, but they may be singular when $D + D^*$, which is nonnegative from dissipativity, is singular, and an optimal control may not exist even though the desired infimum exists.

The variational problems characterizing the available storage and required supply may then be solved by considering appropriate solutions of the algebraic Riccati equation:

$$KA + A^*K + (KB - C^*)(D + D^*)^{-1}(B^*K - C) = 0. \tag{5.9}$$

Theorem 5.1 (J. C. Willems [2]). *The algebraic Riccati equation* (5.9) *has a negative definite solution if and only if* (5.1) *is dissipative with respect to the supply rate $w = (u, y)$. In that case, there exists only one real symmetric solution K^- with the property* $\operatorname{Re}\lambda(A^-) \le 0$, *where $A^- = A+B(D+D^*)^{-1}(B^*K^-C)$, and only one real symmetric solution K^+ with the property $\lambda(A^+) \ge 0$, where $A^+ = A^- + (KB - C^*)(D + D^*)^{-1}B^*$. Moreover, $0 < K^- \le K^+$ and every real symmetric solution satisfies $K^{-1} \le K \le K^+$. Hence, all real symmetric solution are positive definite, $\lambda(A^-) < 0$, and $\lambda(A^+) > 0$.*

Theorem 5.2. *Under the assumptions of Theorem* 5.1,

$$S_a(x) = \frac{1}{2}(x, K^-x) \text{ and } S_r(x) = \frac{1}{2}(x, K^+x).$$

Proof. Let us first consider the case when $D + D^*$ is nonsingular. Let K be a real symmetric solution of the algebraic Riccati equation (5.9). Then

$$\frac{d}{dt}(x, Kx)$$

along the trajectories of $\dot{x} = Ax + Bu$ is given by

$$\begin{aligned}&(u, Cx + Du)\\&-\frac{1}{2}(u - (D + D^*)^{-1}(B^*K - C)x, (D + D^*)(u - (D + D^*)^*(B^*K - C)x).\end{aligned}$$

Now assume that $u \in \mathcal{U}$, transfers x_0 at t_0 to x_1 at t_1. Then

$$\begin{aligned}&(x_1, Kx_1) - (x_0, Kx_0)\\&= 2\int_{t_0}^{t_1} (u(t), y(t))\,dt\\&\quad - \int_{t_0}^{t_1} \big(u - (D + D^*)^{-1}(B^*K - C)x,\\&\qquad\qquad (D + D^*)(u - (D + D^*)^{-1}(B^*K - C)x\big)\,dt.\end{aligned}$$

Let us put $t_0 = 0$ and $K = K^-$ in the above, and noting that $K^- \geq 0$, we obtain

$$-\int_0^{t_1} (u(t), y(t))\, dt \leq \frac{1}{2}(x_0, K^- x_0), \quad \forall\, u \in \mathcal{U}.$$

Hence, $S_a(x_0) \leq \frac{1}{2}(x_0, K^- x_0)$. Now consider the control

$$u = (D + D^*)^{-1}(B^* K^- - C)x.$$

The closed loop system is then $\dot{x} = A^- x$, and because $\operatorname{Re}\lambda(A^-) < 0$,

$$\lim_{t\to\infty} x(t) = 0.$$

Hence, along this trajectory,

$$\frac{1}{2}(x_0, K^- x_0) + \int_0^\infty (u(t), y(t))\, dt = 0,$$

and hence, $S_a(x_0) = \frac{1}{2}(x_0, K^- x_0)$. A similar argument shows that

$$S_r(x_0) = \frac{1}{2}(x_0, K^+ x_0).$$

We now look at the case when $D + D^*$ is singular. The idea is to replace D by $(D + \frac{\varepsilon}{2} I)$, where I is the identity matrix. It is clear that the new dynamical system with D replaced by $(D + \frac{\varepsilon}{2} I)$ is also dissipative with respect to $w = (u, y)$. We now proceed to prove the result through a limiting argument by considering

$$\lim_{\varepsilon\downarrow 0} \Big(D + \frac{\varepsilon}{2} I\Big).$$

Therefore, consider the algebraic Riccati equation (5.9) with $D + \varepsilon/2\, I$ in place of D

$$K_\varepsilon A + A^* K_\varepsilon + (K_\varepsilon B - C^*)(D + D^* + \varepsilon I)^{-1}(B^* K_\varepsilon - C^*) = 0$$

and consider the real symmetric solutions K_ε^- and K_ε^+ with the properties $\operatorname{Re}\lambda(A_\varepsilon^-) \leq 0$ and $\operatorname{Re}\lambda(A_\varepsilon^+) \geq 0$, with A_ε^- and A_ε^+ defined analogously as in Theorem 5.2. We then have $0 < K_\varepsilon^- \leq K_\varepsilon^+$ and all real symmetric solutions K_ε satisfy $K_\varepsilon^- \leq K_\varepsilon \leq K_\varepsilon^+$. Indeed, we have $K_\varepsilon^- < K_\varepsilon^+$ and $\operatorname{Re}\lambda(A_\varepsilon^-) < 0$ and $\operatorname{Re}\lambda(A_\varepsilon^+) > 0$. Now define

$$K^- = \lim_{\varepsilon\downarrow 0} K_\varepsilon^- \quad \text{and} \quad K^+ = \lim_{\varepsilon\downarrow 0} K_\varepsilon^+.$$

Then, Theorem 5.2 holds for K^- and K^+ so defined. To see this, note that K_ε^- and K_ε^+ are monotone nondecreasing, and hence, the limit $K^- = \lim_{\varepsilon\downarrow 0} K_\varepsilon^-$ and $K^+ = \lim_{\varepsilon\downarrow 0} K_\varepsilon^+$ exist.

Now we have

$$S_a^\varepsilon(x) = (x, K_\varepsilon^- x) \quad \text{and} \quad S_r^\varepsilon(x) = (x, K_\varepsilon^+ x).$$

We need to prove that S_a^ε and S_r^ε are continuous functions of ε, $\varepsilon \geq 0$.

Consider $S_a^\varepsilon(x_0)$ and assume that

$$\lim_{\varepsilon \downarrow \varepsilon_0} S_a^\varepsilon(x_0) \neq S_a^{\varepsilon_0}(x_0).$$

Now $S_a^\varepsilon(x_0)$ is a monotone nonincreasing function of ε. Hence,

$$\lim_{\varepsilon \downarrow \varepsilon_0} S_a^\varepsilon(x_0) < S_a^{\varepsilon_0}(x_0).$$

By the definition of $S_a^{\varepsilon_0}(x_0)$, it follows that there exists a $u \in \mathcal{U}$ and $t_1 \geq 0$, such that

$$\lim_{\varepsilon \downarrow \varepsilon_0} S_a^\varepsilon(x_0) < -\int_0^{t_1} (u(t), y(t))\, dt,$$

with $\dot{x} = Ax + Bu$, $y = Cx + (D + \varepsilon_0/2\, I)u$ and $x(0) = x_0$. Now, for fixed u and t_1 and for all ε, $\varepsilon_0 \leq \varepsilon \leq \varepsilon + \delta$,

$$-\int_0^{t_1} (u(t), y(t)) dt,$$

with constraints $\dot{x} = Ax + Bu$, $y = Cx + (D + \varepsilon_0/2\, I)u$ and $x(0) = x_0$, is a continuous function of ε, which contradicts the inequality

$$\lim_{\varepsilon \downarrow \varepsilon_0} S_a^\varepsilon(x_0) < S_a^{\varepsilon_0}(x_0).$$

Hence, $S_a^\varepsilon(x_0)$ is a right continuous function of ε, and hence, $\frac{1}{2}(x, K^- x) = S_a(x)$. In a similar manner, we can prove $S_r(x) = \frac{1}{2}(x, K^+ x)$. Hence, we have proved that Theorem 5.1 remains true when $D + D^*$ is singular. $\square$

Remark 5.1. Note that when $D + D^*$ is singular, the optimal control may not exist. $\square$

5.3 Quadratic storage functions

Given a linear, stationary setting, it is natural to consider storage functions that are quadratic, $\frac{1}{2}(x, Qx)$, $Q = Q^*$. The following theorem follows easily from Theorem 5.2.

Theorem 5.3. *The stationary minimal linear dynamical system given by* (5.1) *is dissipative with respect to the supply rate* $w = (u, y)$ *if and only if the matrix inequalities*

$$\begin{bmatrix} A^*Q + QA & QB - C^* \\ B^*Q - C & -D - D^* \end{bmatrix} \leq 0, \tag{5.10}$$

$$Q = Q^* \geq 0, \tag{5.11}$$

have a solution. The function $\frac{1}{2}(x, Qx)$ *is a storage function if and only if* Q *satisfies* (5.10) *and* (5.11). *Consequently* K^- *and* K^+*, as defined earlier, satisfy these inequalities and* $0 < K^- \leq Q \leq K^+$.

Recent research on the synthesis of controllers using linear matrix inequalities and semi-definite programming (see, for example, the recent book of S. Boyd and L. Vandenberghe [1]) is related to the ideas presented here.

6 Final remarks

If one were to write another volume it should be concerned with a generalization of the following finite dimensional problem to infinite dimensions:

$$\inf_u \int_s^T w\big(x(t), u(t)\big)\, dt, \tag{6.1}$$

where $w(x,u) = \frac{1}{2}[(x,Qx) + (u,Ru) + 2(u,Cx)]$, $R = R^* \geq 0$, and $Q = Q^*$ but without any definiteness condition on Q, subject to the dynamical constraint

$$\begin{cases} \dfrac{dx}{dt} = Ax(t) + Bu(t), \\ x(s) = x \end{cases} \tag{6.2}$$

and the related infinite time problem

$$\inf_u \int_s^\infty w\big(x(t), u(t)\big)\, dt \tag{6.3}$$

with the dynamical constraint (6.2) and final value conditions $x(\infty) = 0$ or $x(\infty) = \text{free}$.

A variety of problems such as linear quadratic differential games, the Kalman–Yacubovich–Popov lemma of stability theory, and H^∞-theory (in state space form) is related to this topic.

Notes

The results of this chapter are for the most part classical. The concepts of controllability and observability were introduced by R. E. Kalman [1, 3], where the criteria of controllability and observability and the structure theorem were developed. In this connection, see also the early paper of E. Gilbert [1]. The minimum energy control for transfer from the origin to a final state was first discussed in R. E. Kalman, Y. C. Ho, and K. S. Narendra [1]. For a textbook study of these questions as well as a treatment of stability theory, see R. W. Brockett [1]. The concepts of stabilizability and detectability were introduced by Wonham in his study of the algebraic Riccati equation (cf. W. M. Wonham [2]). The criteria for stabilizability and detectability are due to M. L. J. Hautus [1]. The pole-assignment theorem in the real case was proved by W. M. Wonham [1] and independently by J. D. Simon and S. K. Mitter [1]. The proof presented here follows W. M. Wonham [3]. Theorem 2.6 is due to J. Zabczyk [3]. The idea of a state estimator was first introduced by D. G. Luenberger [1]. For the treatment of compensators for linear systems as estimator-controller and the use of the pole-assignment theorem in this context, see J. D. Simon and S. K. Mitter [1]. For a discussion of the quadratic cost optimal control problem, see the book by R. W. Brockett [1] where the completing the square argument is developed. The idea of *invariant embedding* used in §3.1 is due to R. Bellman [1]. Its use in decoupling the Hamiltonian equations in the form described here is due to J. L. Lions [3]. It was formally used in the same way by S. K. Mitter [1]. Theorem 3.1 is due to W. M. Wonham [1] (under the slightly stronger hypothesis of observability). The results of §4 follow an unpublished manuscript of S. K. Mitter and J. S. Shamma [1] but are essentially an application of the results given in the paper of J. C. Willems [1]. There is now a vast literature on the so-called H^∞-approach to the control of linear systems. For textbook presentations, see T. Başar and P. Bernhard [1], B. A. Francis [1], and B. van Keulen [1].

2

Linear Quadratic Two-Person Zero-Sum Differential Games

1 Introduction

In this chapter we broaden the general perspective of the book and consider *two-player zero-sum games* with *linear dynamics* and a *quadratic utility function* over a *finite time horizon*. They can be viewed as a natural extension of the single player *linear quadratic optimal control problem*. In particular they illustrate the occurrence of symmetrical solutions to the matrix Riccati differential equation that are not necessarily positive semi-definite. It also connects with the glimpse of H^∞-theory given in the previous chapter.

Since Game Theory arises from a different context than the one in Control Theory, we quickly review some background and terminology. For instance, the notion of *utility function* has a long history. GABRIEL CRAMER [1] (1728) and then DANIEL BERNOUILLI [2] (1738) first proposed as a decision criterion the maximization of expected *utility* rather than of expected wealth. Daniel Bernouilli introduced as *utility function*, the log, to resolve the St. Petersberg Paradox (Risk Aversion) in 1760. Much later, in an appendix to their classic work, J. VON NEUMANN and O. MORGENSTERN [1] (1943) set out for the first time an axiomatic justification for this criterion.

We consider two players (most definitions will carry over to N players). In a *static game*, there is only one shot; that is, the game is played only once. Player 1 decides a value v_1, which can be a vector, a real number, or a discrete variable. Similarly, Player 2 decides v_2. Each of them has a utility function

$$C_i(v_1, v_2), \quad i = 1, 2,$$

that he/she wants to minimize. A *Cournot–Nash equilibrium* (cf. A. COURNOT [1] in 1838 and J. NASH [1] in 1950[1]) is a pair $(\hat{v}_1, \hat{v}_2)$ such that

[1] The concept of the Nash equilibrium is not exactly original to John F. Nash. Antoine Augustin Cournot showed how to find what we now call the Nash equilibrium of the Cournot duopoly game. Consequently, some authors refer to it as

$$C_1(\hat{v}_1, \hat{v}_2) \leq C_1(v_1, \hat{v}_2), \quad \forall v_1 \in U_{ad}^1,$$
$$C_2(\hat{v}_1, \hat{v}_2) \leq C_2(\hat{v}_1, v_2), \quad \forall v_2 \in U_{ad}^2,$$

where U_{ad}^1 and U_{ad}^2 represent the sets of constraints for both players.

In a two-person zero-sum game, one has

$$C_1(v_1, v_2) = C(v_1, v_2),$$
$$C_2(v_1, v_2) = -C(v_1, v_2).$$

So there is in fact only one functional, $C(v_1, v_2)$. Player 1 wants to minimize it, and player 2 wants to maximize it. It is a noncooperative game. A *Nash equilibrium* satisfies the condition

$$C(\hat{v}_1, v_2) \leq C(\hat{v}_1, \hat{v}_2) \leq C(v_1, \hat{v}_2), \quad \forall v_1 \in U_{ad}^1 \text{ and } \forall v_2 \in U_{ad}^2.$$

In this context the pair $(\hat{v}_1, \hat{v}_2)$ is called a *saddle point* and $C(\hat{v}_1, \hat{v}_2)$ the *value of the game*. The existence of a saddle point is not so common for games. In general, the so-called *lower* and *upper values* are different

$$\sup_{v_2} \inf_{v_1} C(v_1, v_2) < \inf_{v_1} \sup_{v_2} C(v_1, v_2),$$

and they can respectively be $-\infty$ and $+\infty$.

In this chapter we study the *dynamical* version on a finite time horizon of the static zero-sum two-player game by introducing a time-dependent *state vector* whose dynamics are governed by a system of linear differential equations. At that level there are *open loop* and *closed loop strategies* for the players and sometimes an imperfect knowledge of the state. We shall restrict ourselves to the open loop case and a perfect knowledge of the state. In that context, the Min Sup problem was studied in 1969 by M. C. Delfour and S. K. Mitter [1]. The fundamental theory of closed loop two-player zero-sum LQ games was given in 1979 by P. Bernhard [2] followed by the seminal book of T. Başar and P. Bernhard [1] in 1991 and 1995 that covered the H^∞-theory. The very nice work by P. Zhang [1] in 2005 established the equivalence between the finiteness of the open loop value of a two-player zero-sum LQ game and the finiteness of its open loop lower and upper values. It means that the *duality gap*, that is the difference between the upper and the lower values of the game, is either 0 or $+\infty$. The reader is referred to the above references for a detailed bibliography of the vast and rich literature on dynamical games.

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a Nash–Cournot equilibrium. However, Nash showed for the first time in his dissertation, *Non-cooperative games* (1950), that Nash equilibria must exist for all finite games with any number of players. Until Nash, this had only been proved for two-player zero-sum games by John von Neumann and Oskar Morgenstern [1] (2nd edition, 1947).

This new chapter is self-contained with mathematical proofs. Section 2 gives the definitions, notation, properties, semi-derivatives, and convexity/concavity characterizations of the utility function.

Section 3 gives necessary and sufficient conditions for the existence of a saddle point of the game and introduces the *coupled system* that will also arise in the characterization of the open loop lower and upper values of the game. This will later be completed in Theorem 5.1 in §5 with the equivalent condition that the value $v(x_0)$ of the game is finite.

Section 4 is devoted to the case where the open loop lower value is finite. By duality this also covers the case where the open loop upper value is finite. It emphasizes the role of the *feasibility condition.*

Section 5 combines the results of §4 to characterize the finiteness of the open loop value of the game, and Theorem 5.1 shows its equivalence with the existence of an open loop saddle point, completing the results of Theorem 3.1 in §3. It also includes the equivalence between the finiteness of the upper and lower values and the finiteness of the value of P. Zhang [1]. Therefore, in the linear-quadratic case, the *duality gap* is always zero or infinite.

Section 6 constructs the associated Riccati differential equation in the open loop saddle point case via invariant embedding with respect to the initial time. It is also shown that under the assumption of an open loop saddle point in the time horizon $[0, T]$ for all initial states, there is an open loop saddle point in the time horizon $[s, T]$ for all s, $0 \leq s < T$, and all initial states (Theorem 6.3 (iii)). From this we get an *optimality principle* and adapt the *invariant embedding approach* of R. Bellman in the style of J. L. Lions [3] to construct the decoupling symmetrical matrix function $P(s)$ and show that it is an $H^1(0, T)$ solution of the matrix Riccati differential equation. Thence an open loop saddle point in $[0, T]$ yields closed loop optimal strategies for both players that achieves a closed loop-closed loop saddle point in the sense of P. Bernhard [2]. Furthermore, a necessary and sufficient set of conditions for the existence of an open loop saddle point in $[0, T]$ is the convexity–concavity of the utility function and the existence of an $H^1(0, T)$ solution to the matrix Riccati differential equation. This leaves the cases where either the open loop lower value is $-\infty$ or the open loop upper value is $= +\infty$. Two informative examples of solutions to the matrix Riccati differential equation where the open loop saddle point does not exist are worked out, even when the solution of the Riccati differential equation is strictly positive and infinitely differentiable. A third example is given where the solution of the Riccati differential equation is infinitely differentiable and strictly positive and the open loop lower and upper values are $+\infty$. The chapter also briefly discusses the open/close loop upper/lower value of the game and generalized solutions of the Riccati differential equation.

This chapter is an expanded version of M. C. Delfour [16].

2 Definitions, notation, and preliminary results

2.1 System, utility function, and values of the game

Given a finite dimensional Euclidean space $\mathbb{R}^d$ of dimension $d \geq 1$, the *norm* and *inner product* will be denoted by $|x|$ and $x \cdot y$, respectively and irrespective of the dimension d of the space. Given $T > 0$, the norm and inner product in $L^2(0,T;\mathbb{R}^n)$ will be denoted $\|f\|$ and (f,g). The norm in the Sobolev space $H^1(0,T;\mathbb{R}^n)$ will be written $\|f\|_{H^1}$.

Consider the following two-player zero-sum game over the finite time interval $[0,T]$ characterized by the quadratic *utility function*

$$C_{x_0}(u,v) \stackrel{\text{def}}{=} Fx(T) \cdot x(T) + \int_0^T Q(t)x(t) \cdot x(t) + |u(t)|^2 - |v(t)|^2 \, dt, \quad (2.1)$$

where x is the solution of the linear differential system, the so-called *state equation*

$$\frac{dx}{dt}(t) = A(t)x(t) + B_1(t)u(t) + B_2(t)v(t) \quad \text{a.e. in } [0,T], \qquad x(0) = x_0, \quad (2.2)$$

x_0 is the *initial state* at time $t = 0$, $u \in L^2(0,T;\mathbb{R}^m)$, $m \geq 1$, is the strategy of the first player, and $v \in L^2(0,T;\mathbb{R}^k)$, $k \geq 1$, is the strategy of the second player. We assume that F is an $n \times n$-matrix and that A, B_1, B_2, and Q are matrix-functions of appropriate order that are measurable and bounded almost everywhere in $[0,T]$. Moreover $Q(t)$ and F are symmetrical (but not necessarily positive semi-definite). The above assumptions on F, A, B_1, B_2, and Q will be used throughout this chapter.

It will be convenient to use the following compact notation and drop the a.e. in $[0,T]$ wherever no confusion arises

$$C_{x_0}(u,v) \stackrel{\text{def}}{=} Fx(T) \cdot x(T) + \int_0^T Qx \cdot x + |u|^2 - |v|^2 \, dt$$

and

$$x' = Ax + B_1u + B_2v \quad \text{a.e. in } [0,T], \qquad x(0) = x_0. \quad (2.3)$$

Remark 2.1. The following more general quadratic structure involving cross terms:

$$C_{x_0}(u,v) \stackrel{\text{def}}{=} Fx(T) \cdot x(T) + \int_0^T (x,u,v) \cdot \begin{bmatrix} Q & S & T \\ S^* & N_1 & 0 \\ T^* & 0 & -N_2 \end{bmatrix} \begin{bmatrix} x \\ u \\ v \end{bmatrix} dt, \quad (2.4)$$

can be brought back to the above form by the folllowing change of variables

$$\begin{bmatrix} x \\ u \\ v \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ -N_1^{-1}S^* & N_1^{-1/2} & 0 \\ N_2^{-1}T^* & 0 & N_2^{-1/2} \end{bmatrix} \begin{bmatrix} x \\ \bar{u} \\ \bar{v} \end{bmatrix}, \quad (2.5)$$

where the new matrix functions N_1, N_2, S, and T are all assumed to be measurable and bounded and $N_1(t)$ and $N_2(t)$ are symmetrical positive definite matrices such that

$$\begin{aligned}
&\exists \nu_1 > 0 \text{ such that } \forall u \in \mathbb{R}^m \text{ and almost all } t, \quad N_1(t)u \cdot u \geq \nu_1 \, |u|^2, \\
&\exists \nu_2 > 0 \text{ such that } \forall v \in \mathbb{R}^k \text{ and almost all } t, \quad N_2(t)v \cdot v \geq \nu_2 \, |v|^2.
\end{aligned} \tag{2.6}$$

This yields

$$Fx(T) \cdot x(T) + \int_0^T \left[Q - SN_1^{-1}S^* + TN_2^{-1}T^*\right] x \cdot x + |\bar{u}|^2 - |\bar{v}|^2 \, dt$$

and the system

$$\begin{aligned}
x' &= Ax + B_1(-N_1^{-1}S^*x + N_1^{-1/2}\bar{u}) + B_2(N_2^{-1}T^*x + N_2^{-1/2}\bar{v}) \\
&= (A - B_1N_1^{-1}S^* + B_2N_2^{-1}T^*)x + B_1N_1^{-1/2}\bar{u} + B_2N_2^{-1/2}\bar{v}.
\end{aligned}$$

By introducing the new matrices

$$\begin{aligned}
&\overline{Q} = Q - SN_1^{-1}S^* + TN_2^{-1}T^*, \\
&\bar{A} = A - B_1N_1^{-1}S^* + B_2N_2^{-1}T^*, \\
&\overline{B}_1 = B_1N_1^{-1/2}, \quad \overline{B}_2 = B_2N_2^{-1/2},
\end{aligned}$$

we are back to the simpler initial structure

$$\overline{C}_{x_0}(\bar{u}, \bar{v}) \stackrel{\text{def}}{=} Fx(T) \cdot x(T) + \int_0^T \overline{Q}x \cdot x + |\bar{u}|^2 - |\bar{v}|^2 \, dt,$$

$$\begin{cases} x' = \bar{A}x + \overline{B}_1\bar{u} + \overline{B}_2\bar{v}, \\ x(0) = x_0. \end{cases}$$

It is readily seen that

$$\begin{aligned}
\inf_{u \in L^2(0,T;\mathbb{R}^m)} \sup_{v \in L^2(0,T;\mathbb{R}^k)} C_{x_0}(u, v) &= \inf_{\bar{u} \in L^2(0,T;\mathbb{R}^m)} \sup_{\bar{v} \in L^2(0,T;\mathbb{R}^k)} \overline{C}_{x_0}(\bar{u}, \bar{v}), \\
\sup_{v \in L^2(0,T;\mathbb{R}^k)} \inf_{u \in L^2(0,T;\mathbb{R}^m)} C_{x_0}(u, v) &= \sup_{\bar{v} \in L^2(0,T;\mathbb{R}^k)} \inf_{\bar{u} \in L^2(0,T;\mathbb{R}^m)} \overline{C}_{x_0}(\bar{u}, \bar{v}).
\end{aligned}$$

For the H^∞-theory, $N_1(t) = I$ and $N_2(t) = \gamma^2 I$, where $\gamma \neq 0$ and I is the identity matrix. As for the classical optimal minimizing control problem, it can be recovred by choosing $N_2(t) = I$ and $B_2(t) = 0$. □

We shall also study the existence of a symmetrical solution to the associated *matrix Riccati differential equation*

$$P' + PA + A^*P - PRP + Q = 0 \quad \text{a.e. in } [0, T], \qquad P(T) = F, \tag{2.7}$$

where $R = B_1B_1^* - B_2B_2^*$, in relation with the following objectives of the game.

Definition 2.1. Let x_0 be an initial state in $\mathbb{R}^n$ at time $t = 0$.

(i) The game is said to achieve its *open loop lower value* (resp. *upper value*) if

$$v^-(x_0) \overset{\text{def}}{=} \sup_{v \in L^2(0.T;\mathbb{R}^k)} \inf_{u \in L^2(0.T;\mathbb{R}^m)} C_{x_0}(u,v) \tag{2.8}$$

$$(\text{resp. } v^+(x_0) \overset{\text{def}}{=} \inf_{u \in L^2(0.T;\mathbb{R}^m)} \sup_{v \in L^2(0.T;\mathbb{R}^k)} C_{x_0}(u,v)) \tag{2.9}$$

is finite. By definition $v^-(x_0) \le v^+(x_0)$.

(ii) The game is said to achieve its *open loop value* if its open loop lower value $v^-(x_0)$ and upper value $v^+(x_0)$ are finite and

$$v^-(x_0) = v^+(x_0). \tag{2.10}$$

The *open loop value* of the game will be denoted by $v(x_0)$.

(iii) A pair $(\bar{u}, \bar{v})$ in $L^2(0,T;\mathbb{R}^m) \times L^2(0,T;\mathbb{R}^k)$ is an *open loop saddle point* of $C_{x_0}(u,v)$ in $L^2(0,T;\mathbb{R}^m) \times L^2(0,T;\mathbb{R}^k)$ if for all u in $L^2(0,T;\mathbb{R}^m)$ and all v in $L^2(0,T;\mathbb{R}^k)$

$$C_{x_0}(\bar{u}, v) \le C_{x_0}(\bar{u}, \bar{v}) \le C_{x_0}(u, \bar{v}). \tag{2.11}$$

□

In general, (ii) does not necessarily imply (iii), but we shall see that it does for linear-quadratic games.

Definition 2.2. Associate with $x_0 \in \mathbb{R}^n$ the sets and the functions

$$V(x_0) \overset{\text{def}}{=} \left\{ v \in L^2(0,T;\mathbb{R}^k) : \inf_{u \in L^2(0,T;\mathbb{R}^m)} C_{x_0}(u,v) > -\infty \right\}, \tag{2.12}$$

$$U(x_0) \overset{\text{def}}{=} \left\{ u \in L^2(0,T;\mathbb{R}^m) : \sup_{v \in L^2(0,T;\mathbb{R}^k)} C_{x_0}(u,v) < +\infty \right\}, \tag{2.13}$$

$$J^-_{x_0}(v) \overset{\text{def}}{=} \inf_{u \in L^2(0,T;\mathbb{R}^m)} C_{x_0}(u,v), \quad J^+_{x_0}(u) \overset{\text{def}}{=} \sup_{v \in L^2(0,T;\mathbb{R}^k)} C_{x_0}(u,v). \tag{2.14}$$

□

By definition, $V(x_0) \neq \varnothing$ if and only if $v^-(x_0) > -\infty$ and $U(x_0) \neq \varnothing$ if and only if $v^+(x_0) < +\infty$. In this chapter we only consider the open loop case. Corresponding definitions can be given in the *closed loop* case, and the reader is referred to P. Bernhard [2] and T. Başar and P. Bernhard [1].

2.2 Properties, semi-derivatives, and convexity/concavity of $C_{x_0}(u,v)$

The functional $C_{x_0}(u,v)$ is infinitely differentiable, and since it is quadratic, its Hessian of second order derivatives is independent of the point (u,v). More precisely[2]

$$\tfrac{1}{2}dC_{x_0}(u,v;\bar{u},\bar{v}) = Fx(T)\cdot\bar{y}(T) + (Qx,\bar{y}) + (u,\bar{u}) - (v,\bar{v}), \tag{2.15}$$

where x is the solution of (2.3) and $\bar{y}$ is the solution of

$$\bar{y}' = A\bar{y} + B_1\bar{u} + B_2\bar{v}, \quad \bar{y}(0) = 0. \tag{2.16}$$

It is customary to introduce the *adjoint state equation*

$$p' + A^*p + Qx = 0, \quad p(T) = Fx(T) \tag{2.17}$$

and rewrite (2.15) for the gradient in the following form:

$$\tfrac{1}{2}dC_{x_0}(u,v;\bar{u},\bar{v}) = (B_1^*p + u,\bar{u}) + (B_2^*p - v,\bar{v}). \tag{2.18}$$

As predicted, the Hessian is independent of (u,v):

$$\tfrac{1}{2}d^2C_{x_0}(u,v;\bar{u},\bar{v};\tilde{u},\tilde{v}) = F\tilde{y}(T)\cdot\bar{y}(T) + (Q\tilde{y},\bar{y}) + (\tilde{u},\bar{u}) - (\tilde{v},\bar{v}), \tag{2.19}$$

where $\bar{y}$ is the solution of (2.16) and $\tilde{y}$ is the solution of

$$\tilde{y}' = A\tilde{y} + B_1\tilde{u} + B_2\tilde{v}, \quad \tilde{y}(0) = 0. \tag{2.20}$$

In particular, for all x_0, u, v, $\bar{u}$, and $\bar{v}$

$$d^2C_{x_0}(u,v;\bar{u},\bar{v};\bar{u},\bar{v}) = 2\,C_0(\bar{u},\bar{v}), \tag{2.21}$$

and this yields the following characterizations of the u-convexity, v-concavity, and (u,v)-convexity–concavity.

Lemma 2.1. *The following statements are equivalent:*

(i) *the map* $u \mapsto C_0(u,0) : L^2(0,T;\mathbb{R}^m) \to \mathbb{R}$ *is convex;*
(ii) *for all* $u \in L^2(0,T;\mathbb{R}^m)$, $C_0(u,0) \geq 0$*;*

[2] Given a real function f defined on a Banach space B, the *first directional semi-derivative* at x in the direction v (when it exists) is defined as

$$df(x;v) \overset{\text{def}}{=} \lim_{t\searrow 0} \frac{f(x+tv)-f(x)}{t}.$$

When the map $v \mapsto df(x;v) : B \to \mathbb{R}$ is linear and continuous, it defines the *gradient* $\nabla f(x)$ as an element of the dual B^* of B. The *second order bidirectional derivative* at x in the directions (v,w) (when it exists) is defined as

$$d^2f(x;v,w) \overset{\text{def}}{=} \lim_{t\searrow 0} \frac{df(x+tw;v)-df(x;v)}{t}.$$

When the map $(v,w) \mapsto d^2f(x;v,w) : B\times B \to \mathbb{R}$ is bilinear and continuous, it defines the *Hessian operator* $Hf(x)$ as a continuous linear operator from B to B^*.

(iii) $\inf_{u\in L^2(0,T;\mathbb{R}^m)} C_0(u,0) = C_0(0,0)$;
(iv) *for all* v *and* x_0 *the map* $u \mapsto C_{x_0}(u,v) : L^2(0,T;\mathbb{R}^m) \to \mathbb{R}$ *is convex.*

Proof. (i) $\Rightarrow$ (ii) If $u \mapsto C_0(u,0) : L^2(0,T;\mathbb{R}^m) \to \mathbb{R}$ is convex, then its Hessian is positive or zero and the result follows from identity (2.21). (ii) $\Rightarrow$ (iii) From (ii) and the fact that $C_0(0,0) = 0$. (iii) $\Rightarrow$ (iv) Since $C_{x_0}(u,v)$ is quadratic in u, for all u and u'

$$\begin{aligned}&C_{x_0}(u',v) - C_{x_0}(u,v) - dC_{x_0}(u,v;u'-u,0)\\&= \frac{1}{2}d^2C_{x_0}(u,v;u'-u,0;u'-u,0) = C_0(u'-u,0) \geq C_0(0,0) = 0\end{aligned}$$

from identity (2.21). (iv) $\Rightarrow$ (i) Pick $x_0 = 0$ and $v = 0$. □

Corollary 2.1. *The following statements are equivalent:*

(i) *the map* $v \mapsto C_0(0,v) : L^2(0,T;\mathbb{R}^k) \to \mathbb{R}$ *is concave;*
(ii) *for all* $v \in L^2(0,T;\mathbb{R}^k)$, $C_0(0,v) \leq 0$;
(iii) $\sup_{v\in L^2(0,T;\mathbb{R}^k)} C_0(0,v) = C_0(0,0)$;
(iv) *for all* u *and* x_0, *the map* $v \mapsto C_{x_0}(u,v) : L^2(0,T;\mathbb{R}^k) \to \mathbb{R}$ *is concave.*

Corollary 2.2. *The following statements are equivalent:*

(i) *the map* $(u,v) \mapsto C_0(u,v) : L^2(0,T;\mathbb{R}^m) \times L^2(0,T;\mathbb{R}^k) \to \mathbb{R}$ *is* (u,v)*-convex–concave; that is,*

$$\begin{aligned}&\forall v \in L^2(0,T;\mathbb{R}^k), \quad u \mapsto C_0(u,v) \text{ is convex and}\\&\forall u \in L^2(0,T;\mathbb{R}^m), \quad v \mapsto C_0(u,v) \text{ is concave;}\end{aligned} \tag{2.22}$$

(ii) *the pair* $(0,0)$ *is a saddle point of* $C_0(u,v)$:

$$\sup_{v\in L^2(0,T;\mathbb{R}^k)} C_0(0,v) = C_0(0,0) = \inf_{u\in L^2(0,T;\mathbb{R}^m)} C_0(u,0); \tag{2.23}$$

(iii) $\sup_{v\in L^2(0,T;\mathbb{R}^k)} C_0(0,v) = C_0(0,0) = \inf_{u\in L^2(0,T;\mathbb{R}^m)} C_0(u,0)$;
(iv) *for all* x_0 *the map* $(u,v) \mapsto C_{x_0}(u,v) : L^2(0,T;\mathbb{R}^m) \times L^2(0,T;\mathbb{R}^k) \to \mathbb{R}$ *is* (u,v)*-convex–concave; that is,*

$$\begin{aligned}&\forall v \in L^2(0,T;\mathbb{R}^k), \quad u \mapsto C_{x_0}(u,v) \text{ is convex and}\\&\forall u \in L^2(0,T;\mathbb{R}^m), \quad v \mapsto C_{x_0}(u,v) \text{ is concave.}\end{aligned} \tag{2.24}$$

3 Saddle point and coupled state–adjoint state system

We first obtain necessary and sufficient conditions for the existence of a saddle point of the game and introduces the *coupled (state–adjoint state) system* (cf. Notation 3.1 on page 56) that will also arise in the characterization of the open loop lower and upper values of the game in §4. Theorem 5.1 in §5 will later complete this theorem with the equivalent condition that the value $v(x_0)$ of the game is finite.

Theorem 3.1. *The following conditions are equivalent*:

(i) *There exists an open loop* saddle point *of* $C_{x_0}(u,v)$.
(ii) *There exists a solution* $(\hat{u},\hat{v})$ *in* $L^2(0,T;\mathbb{R}^m)\times L^2(0,T;\mathbb{R}^k)$ *of the system*

$$\forall u \in L^2(0,T;\mathbb{R}^m),\ \forall v \in L^2(0,T;\mathbb{R}^k), \quad dC_{x_0}(\hat{u},\hat{v};u,v)=0, \tag{3.1}$$

and C_{x_0} *is convex–concave in the sense of* (2.24).
(iii) *There exists a solution* $(x,p)\in H^1(0,T;\mathbb{R}^n)^2$ *of the coupled system*

$$\begin{cases} x' = Ax - B_1B_1^*p + B_2B_2^*p, & x(0)=x_0 \\ p' + A^*p + Qx = 0, & p(T) = Fx(T), \end{cases} \tag{3.2}$$

$$\hat{u} = -B_1^*p, \quad \hat{v} = B_2^*p, \tag{3.3}$$

and

$$\sup_{v\in L^2(0,T;\mathbb{R}^k)} C_0(0,v) = C_0(0,0) = \inf_{u\in L^2(0,T;\mathbb{R}^m)} C_0(u,0). \tag{3.4}$$

Under any one of the above conditions, the value of the game is given by

$$v(x_0) = C_{x_0}(\hat{u},\hat{v}) = p(0)\cdot x_0. \tag{3.5}$$

Proof. (i) ⇒ (ii). Let $(\bar{u},\bar{v})$ in $L^2(0,T;\mathbb{R}^m)\times L^2(0,T;\mathbb{R}^k)$ be an open loop saddle point of $C_{x_0}(u,v)$ in $L^2(0,T;\mathbb{R}^m)\times L^2(0,T;\mathbb{R}^k)$. Then by Definition 2.1

$$\sup_{L^2(0,T;\mathbb{R}^k)} C_{x_0}(\bar{u},v) = C_{x_0}(\bar{u},\bar{v}) = \inf_{L^2(0,T;\mathbb{R}^m)} C_{x_0}(u,\bar{v}). \tag{3.6}$$

Since $C_{x_0}(u,v)$ is infinitely differentiable, the minimizing point $\bar{u}$ of $C_{x_0}(u,\bar{v})$ with respect to u is characterized by the first order condition $dC_{x_0}(\bar{u},\bar{v};u,0)=0$ for all u and the second order condition $d^2C_{x_0}(\bar{u},\bar{v};u,0;u,0)\geq 0$ for all u. Since $d^2C_{x_0}(\bar{u},\bar{v};u,0;u,0)$ is independent of $(\bar{u},\bar{v})$ and x_0, $C_{x_0}(u,v)$ is convex in u for all x_0 and all v. A similar argument for the maximum yields $dC_{x_0}(\bar{u},\bar{v};,w)=0$ and $d^2C_{x_0}(\bar{u},\bar{v};0,w;0,w)\leq 0$ for all w and the concavity of $C_{x_0}(u,v)$ in v.

(ii) ⇒ (i). By assumption $C_{x_0}(\hat{u},\hat{v})$ is convex–concave and infinitely differentiable and there is a solution to the two first order conditions. By I. Ekeland and R. Temam [1], there exists a saddle point.

(ii) ⇔ (iii) From the previous computations of the gradient and Corollary 2.2.

Finally, for the computation of the value

$$\begin{aligned} C_{x_0}(\hat{u},\hat{v}) &= Fx(T)\cdot x(T) + (Qx,x) + \|B_1^*p\|^2 - \|B_2^*p\|^2 \\ &= p(T)\cdot x(T) - (p'+A^*p,x) + ([B_1B_1^* - B_2B_2^*]p,p) \\ &= p(0)\cdot x(0) + (p, x' - Ax + ([B_1B_1^* - B_2B_2^*]p) = p(0)\cdot x_0. \end{aligned}$$

□

Notation 3.1. System (3.2) will be referred to as the *coupled state–adjoint state system* or simply the *coupled system*. It will also be useful to denote by $\mathcal{N}_{x,p}$ the set of all solutions (y, q) of the associated homogeneous coupled system

$$\begin{cases} y' = Ay - B_1 B_1^* q + B_2 B_2^* q, & y(0) = 0, \\ q' + A^* q + Qy = 0, & q(T) = Fy(T). \end{cases} \tag{3.7}$$

Thus system (3.2) has a solution up to an additive pair of $\mathcal{N}_{x,p}$. □

4 Finite open loop lower value

In this section we give a set of necessary and sufficient conditions for the finiteness of the open loop lower value of the game. By dual assumptions, we also get a set of necessary and sufficient conditions for the finiteness of the open loop upper value of the game. We first state the main theorems. Then, we proceed in three steps: existence and characterization of a minimizer for each $v \in V(x_0)$, formulation of the resulting maximization problem with respect to $v \in V(x_0)$, and existence and characterization of the pair that achieves the finite open loop lower value of the game. Finally, we prove the uniqueness of the solution of the coupled system under the assumption that the open loop lower value is finite for all initial states in §4.6.

4.1 Main theorems

The quadratic character of the problem yields surprising equivalences that reduce the complexity of its solution. We start with the open loop lower value of the game.

Theorem 4.1. *The following conditions are equivalent.*

(i) *There exist $\hat{u}$ in $L^2(0,T;\mathbb{R}^m)$ and $\hat{v}$ in $L^2(0,T;\mathbb{R}^k)$ such that*

$$C_{x_0}(\hat{u}, \hat{v}) = \inf_{u \in L^2(0,T;\mathbb{R}^m)} C_{x_0}(u, \hat{v}) = \sup_{v \in L^2(0,T;\mathbb{R}^k)} \inf_{u \in L^2(0,T;\mathbb{R}^m)} C_{x_0}(u, v). \tag{4.1}$$

(ii) *The open loop lower value $v^-(x_0)$ of the game is finite.*

(iii) *There exists a solution $(x, p) \in H^1(0,T;\mathbb{R}^n) \times H^1(0,T;\mathbb{R}^n)$ of the* coupled system (3.2) *such that $B_2^* p \in V(x_0)$, the solution pairs $(\hat{u}, \hat{v})$ is given by the expressions*

$$\hat{u} = -B_1^* p, \quad \hat{v} = B_2^* p, \tag{4.2}$$

and

$$\sup_{v \in V(0)} \inf_{u \in L^2(0,T;\mathbb{R}^m)} C_0(u, v) = \inf_{u \in L^2(0,T;\mathbb{R}^m)} C_0(u, 0) = C_0(0, 0). \tag{4.3}$$

Under any one of the above conditions, the open loop lower value is given by

$$v^-(x_0) = C_{x_0}(\hat{u}, \hat{v}) = p(0) \cdot x_0. \tag{4.4}$$

The proof of this main theorem will be given in §4 and §4.5.

Remark 4.1. The above necessary and sufficient conditions for the finiteness of the open loop value of the game complete the results of P. ZHANG [1] by introducing the *feasibility condition* (4.3) that is equivalent to saying that the open loop lower value of the game is zero and that $(0,0)$ is a solution for the zero initial state. It also recasts the results in the more intuitive state/adjoint state framework. Condition (4.3) is equivalent to the convexity of $C_{x_0}(u,v)$ with respect to u and the concavity of $J^-_{x_0}(v) = \inf_{u\in L^2(0,T;\mathbb{R}^m)} C_{x_0}(u,v)$ with respect to $v \in V(x_0)$. □

This theorem has a counterpart for the upper value $v^+(x_0)$ of the game.

Theorem 4.2. *The following conditions are equivalent*:

(i) *There exist* $\hat{u}$ *in* $L^2(0,T;\mathbb{R}^m)$ *and* $\hat{v}$ *in* $L^2(0,T;\mathbb{R}^k)$ *such that*

$$C_{x_0}(\hat{u}, \hat{v}) = \sup_{v\in L^2(0,T;\mathbb{R}^k)} C_{x_0}(\hat{u}, v) = \inf_{u\in L^2(0,T;\mathbb{R}^m)} \sup_{v\in L^2(0,T;\mathbb{R}^k)} C_{x_0}(u,v). \tag{4.5}$$

(ii) *The open loop upper value* $v^+(x_0)$ *of the game is finite.*

(iii) *There exists a solution* $(x,p) \in H^1(0,T;\mathbb{R}^n)\times H^1(0,T;\mathbb{R}^n)$ *of the* coupled system (3.2) *such that* $-B_1^*p \in U(x_0)$, *the solution pairs* $(\hat{u},\hat{v})$ *is given by the expressions*

$$\hat{u} = -B_1^* p, \quad \hat{v} = B_2^* p \tag{4.6}$$

and

$$\inf_{u\in U(0)} \sup_{v\in L^2(0,T;\mathbb{R}^k)} C_0(u,v) = \sup_{v\in L^2(0,T;\mathbb{R}^k)} C_0(0,v) = C_0(0,0). \tag{4.7}$$

Under any one of the above conditions, the open loop upper value is given by

$$v^+(x_0) = C_{x_0}(\hat{u}, \hat{v}) = p(0) \cdot x_0. \tag{4.8}$$

Condition (4.7) says that $C_{x_0}(u,v)$ is concave with respect to v and that $J^+_{x_0}(u) = \sup_{v\in L^2(0,T;\mathbb{R}^k)} C_{x_0}(u,v)$ is convex with respect to $u \in U(x_0)$.

4.2 Abstract operators and a preliminary lemma

The differential equation (2.2) has a unique solution $x = x(\cdot; x_0, u, v)$ in $H^1(0,T;\mathbb{R}^n)$ and the solution map

$$x_0, u, v \mapsto x(\cdot; x_0, u, v) \colon \mathbb{R}^n \times L^2(0,T;\mathbb{R}^m) \times L^2(0,T;\mathbb{R}^k) \to H^1(0,T;\mathbb{R}^n)$$

is linear and continuous. Thus we can write

$$x(\cdot; x_0, u, v) = K_0 x^0 + K_1 u + K_2 v \tag{4.9}$$

by introducing the following continuous linear operators:

$$\begin{aligned} &K_0\colon \mathbb{R}^n \to H^1(0,T;\mathbb{R}^n),\\ &K_1\colon L^2(0,T;\mathbb{R}^m) \to H^1(0,T;\mathbb{R}^n),\\ &K_2\colon L^2(0,T;\mathbb{R}^k) \to H^1(0,T;\mathbb{R}^n). \end{aligned} \tag{4.10}$$

Finally introduce the continuous self-adjoint linear map

$$\Lambda(x, x_T) \overset{\text{def}}{=} (Qx, F x_T)\colon L^2(0,T;\mathbb{R}^n) \times \mathbb{R}^n \to L^2(0,T;\mathbb{R}^n) \times \mathbb{R}^n, \tag{4.11}$$

the continuous linear compact injection

$$i(x) \overset{\text{def}}{=} \big(x, x(T)\big)\colon H^1(0,T;\mathbb{R}^n) \to L^2(0,T;\mathbb{R}^n) \times \mathbb{R}^n, \tag{4.12}$$

and the continuous bilinear form

$$\begin{aligned} q_{x_0}\big((u,v),(\bar u,\bar v)\big) \overset{\text{def}}{=} \big(\Lambda i(K_0 x^0 + K_1 u + K_2 v), i(K_0 x^0 + K_1 \bar u + K_2 \bar v)\big)_{L^2\times\mathbb{R}^n}&\\ + (u,\bar u)_{L^2} - (v,\bar v)_{L^2}.& \end{aligned} \tag{4.13}$$

Then

$$C_{x_0}(u,v) = q_{x_0}\big((u,v),(u,v)\big).$$

Lemma 4.1. *Let U be a Hilbert space, $M : U \to U$ a continuous linear self-adjoint compact operator, $f \in U$, c a constant, and $j(u) = c + 2(f,u) + ([I + M]u, u)$.*

(i) *Then the following conditions are equivalent:*

a) $$\exists \hat u \in U, \quad j(\hat u) = \inf_{u\in U} j(u), \tag{4.14}$$

b) $$\inf_{u\in U} j(u) > -\infty, \tag{4.15}$$

c) $$\exists \hat u \in U \text{ such that } [I + M]\hat u + f = 0 \tag{4.16}$$

$$\text{and } \forall u \in U, \quad ([I + M]u, u) \ge 0. \tag{4.17}$$

(ii) *Condition* (4.16) *is equivalent to*

$$\forall w \in \ker[I + M], \quad (f, w) = 0. \tag{4.18}$$

(iii) *Condition* (4.17) *is equivalent to the convexity of j.*

Proof. (i) a) $\Rightarrow$ b). Obvious. b) $\Rightarrow$ c). We proceed by contradiction. Since M is compact, $I + M$ has closed range $\mathrm{Im}[I + M]$.

If (4.16) is not true, then $f \notin \mathrm{Im}[I+M] = (\ker[I+M])^\perp$ since $\mathrm{Im}[I+M]$ is closed. Let P be the orthogonal projection onto $\mathrm{Im}[I + M]$. For $\lambda > 0$, let $u_\lambda = -\lambda(f - Pf)$. Then

$$j(u_\lambda) = c + 2(f, u_\lambda) + ([I + M]u_\lambda, u_\lambda) = -\lambda\|f - Pf\|^2.$$

Since $f \notin \mathrm{Im}[I + M]$, then $f - Pf \neq 0$, and as λ goes to infinity, $j(u_\lambda)$ goes to $-\infty$ that contradicts condition (4.15).

If condition (4.17) is not verified, then there exists $w \neq 0$ such that

$$([I + M]w, w) < 0.$$

For $\lambda > 0$ let $u_\lambda = \hat{u} + \lambda w$. Then

$$\begin{aligned} j(u_\lambda) &= c + 2(f, u_\lambda) + ([I + M]u_\lambda, u_\lambda) \\ &= j(\hat{u}) + 2\lambda([I + M]\hat{u} + f, w) + \lambda^2([I + M]w, w) \\ &= j(\hat{u}) + \lambda^2([I + M]w, w), \end{aligned}$$

and $j(u_\lambda)$ goes to $-\infty$ as λ goes to infinity since the coefficient of λ^2 is strictly negative. This contradiction yields (4.15).

c) $\Rightarrow$ a). From conditions (4.16) and (4.17), there exists a $\hat{u} \in U$ such that

$$j(\hat{u}) = \inf_{u\in H} j(u).$$

(ii) Since $I + M$ has a closed range, condition (4.16) is equivalent to

$$f \in \mathrm{Im}[I + M] = (\ker[I + M])^\perp \Longrightarrow \forall w \in \ker[I + M], \quad (f, w) = 0.$$

(iii) is obvious. □

4.3 Existence and characterization of the minimizers

Theorem 4.3. *Given $x_0 \in \mathbb{R}^n$ and $v \in L^2(0, T; \mathbb{R}^k)$, the following statements are equivalent:*

(i) *There exists $\hat{u} \in L^2(0, T; \mathbb{R}^m)$ such that*

$$C_{x_0}(\hat{u}, v) = J^-_{x_0}(v) = \inf_{u\in L^2(0,T;\mathbb{R}^m)} C_{x_0}(u, v). \tag{4.19}$$

(ii) $\inf_{u\in L^2(0,T;\mathbb{R}^m)} C_{x_0}(u, v) > -\infty$ *(that is, $v \in V(x_0)$).*

(iii) *There exists a pair $(x, p) \in H^1(0, T; \mathbb{R}^n) \times H^1(0, T; \mathbb{R}^n)$ solution of the system*

$$\begin{cases} x' = Ax - B_1B_1^*p + B_2v, & x(0) = x_0, \\ p' + A^*p + Qx = 0, & p(T) = Fx(T), \end{cases} \tag{4.20}$$

$$\hat{u}(t) = -B_1^*(t)p(t), \tag{4.21}$$

and

$$\inf_{u\in L^2(0,T;\mathbb{R}^m)} C_0(u,0)\ge 0. \tag{4.22}$$

(iv) *The convexity condition* (4.22) *is verified and*

$$\forall q\in N_p, \quad x_0\cdot q(0)+\int_0^T v\cdot B_2^* q\,dt=0, \tag{4.23}$$

where

$$N_p \overset{\text{def}}{=} \left\{q\in H^1(0,T;\mathbb{R}^n) : \forall (y,q)\in N_{x,p}\right\} \tag{4.24}$$

and $N_{x,p}$ *denotes the set of all solutions* (y,q) *of the homogeneous system*

$$\begin{cases} y'=Ay-B_1\,B_1^* q, & y(0)=0,\\ q'+A^*q+Qy=0, & q(T)=Fy(T). \end{cases} \tag{4.25}$$

Under any one of the avove conditions, for all (x,p) *solution of* (4.20)

$$J_{x_0}^-(v)=p(0)\cdot x_0+\int_0^T B_2^*p\cdot v-|v|^2\,dt. \tag{4.26}$$

Proof. The proof of the theorem will require Lemma 4.1. By definition

$$\begin{aligned} q_{x_0}\big((u,v),(\bar u,\bar v)\big) = {} & \big(\Lambda i(K_0x^0+K_2v), i(K_0x^0+K_2\bar v)\big)_{L^2\times\mathbb{R}^n}-(v,\bar v)_{L^2}\\ & +\big(\Lambda i(K_1u), i(K_0x^0+K_2\bar v)\big)_{L^2\times\mathbb{R}^n}\\ & +\big(\Lambda i(K_0x^0+K_2v), i(K_1\bar u)\big)_{L^2\times\mathbb{R}^n}\\ & +\big(\Lambda i(K_1u), i(K_1\bar u)\big)_{L^2\times\mathbb{R}^n}+(u,\bar u)_{L^2}. \end{aligned} \tag{4.27}$$

Let $M=(iK_1)^*\Lambda(iK_1)$ and $f=(iK_1)^*\Lambda i(K_0x^0+K_2v)$ and $c=\big(\Lambda i(K_0x^0+K_2v), i(K_0x^0+K_2v)\big)_{L^2\times\mathbb{R}^n}-(v,v)_{L^2}$. Then

$$C_{x_0}(u,v)=j(u)\overset{\text{def}}{=} c+2(f,u)+([I+M]u,u),$$

where M is linear, continuous, and compact. So we can use Lemma 4.1.

(i) $\Leftrightarrow$ (ii) From the equivalence of a) and b) in Lemma 4.1.

(ii) $\Leftrightarrow$ (iii) From the equivalence of b) and c) in Lemma 4.1 using the computation of first and second order derivatives (2.18) and (2.21) in §2.2.

Condition (4.16) of the set of necessary and sufficient conditions of Lemma 4.1 becomes

$$\begin{aligned} &\exists\hat u,\ \forall u\in L^2(0,T;\mathbb{R}^m),\ ([I+(iK_1)^*\Lambda(iK_1)]\hat u+(iK_1)^*\Lambda i(K_0x^0+K_2v),u)=0,\\ &\exists\hat u,\ \forall u\in L^2(0,T;\mathbb{R}^m),\ (\hat u,u)+(\Lambda i(K_0x^0+K_1\hat u+K_2v),iK_1u)=0,\\ &\exists\hat u,\ \forall u\in L^2(0,T;\mathbb{R}^m),\ (\hat u,u)+(\Lambda ix,iy_u)=0, \end{aligned} \tag{4.28}$$

where

$$x' = Ax + B_1\hat{u} + B_2 v, \quad x(0) = x_0. \tag{4.29}$$

By introducing the adjoint system

$$p' + A^* p + Qx = 0, \quad p(T) = Fx(T), \tag{4.30}$$

condition (4.28) becomes

$$\begin{aligned}
&\exists \hat{u},\, \forall u \in L^2(0,T;\mathbb{R}^m), \quad Fx(T)\cdot y_u(T) + \int_0^T Qx\cdot y_u + \hat{u}\cdot u\, dt = 0 \\
&\qquad \iff \exists \hat{u},\, \forall u \in L^2(0,T;\mathbb{R}^m), \quad \int_0^T [B_1^* p + \hat{u}]\cdot u\, dt = 0 \iff \hat{u} = -B_1^* p.
\end{aligned}$$

This means that system (4.20) has at least one solution (x,p).

Condition (4.22) follows from condition (4.17) and identity (2.21)

$$\forall \bar{u}, \quad 2\, C_0(\bar{u},0) = d^2 C_{x_0}(u,v;\bar{u},0;\bar{u},0) = 2([I+M]\bar{u},\bar{u}) \geq 0. \tag{4.31}$$

(iii) $\Leftrightarrow$ (iv) We use the equivalence between conditions (4.18) and (4.16) in Lemma 4.1 (ii). So it is sufficient to explicit condition (4.18). Note that the spaces $N_{x,p}$ and N_p are both linear.

Computation of $N_u = \ker[I + (iK_1)^* \Lambda (iK_1)]$. Any $w \in N_u$ is solution of

$$\begin{aligned}
&\forall u, \quad \big((iK_1)^* \Lambda (iK_1) w, u\big) + (w,u) = 0, \\
&\forall u, \quad (\Lambda(iK_1 w), iK_1 u) + (w,u) = 0, \\
&\forall u, \quad Fy_w(T)\cdot y_u(T) + \int_0^T Qy_w\cdot y_u + w\cdot u\, dt = 0,
\end{aligned}$$

where y_w is the solution of the equation

$$y_w' = Ay_w + B_1 w = 0, \quad y_w(0) = 0. \tag{4.32}$$

Introduce the solution q_w of the adjoint system

$$q_w' + A^* q_w + Qy_w = 0, \quad q_w(T) = Fy_w(T).$$

By substitution of Qy_w and integration by parts

$$\begin{aligned}
&\forall u, \quad Fy_w(T)\cdot y_u(T) + \int_0^T Qy_w\cdot y_u + w\cdot u\, dt = 0 \\
&\qquad \iff \forall u, \quad \int_0^T [B_1^* q_w + w]\cdot u\, dt = 0 \iff w = -B_1^* q_w.
\end{aligned}$$

By substitution of w in (4.32), the pair (y_w, q_w) is a solution of system (4.25), $q_w \in N_p$, and $N_u \subset -B_1^* N_p$, where the linear subspace N_p is given by (4.24).

Conversely, if $w \in B_1^* N_p$, there exists a solution (y, q) of system (4.25) such that $w = -B_2^* q$ and hence $y_w = y$. So by reverting the above sequence of equivalences, we conclude that $w \in N_u$, and since N_p is a linear subspace, $N_u = B_2^* N_p$.

Condition (4.18) now becomes

$$\begin{aligned}
&\forall w \in N_u, \quad \big((iK_1)^* \Lambda i(K_0 x^0 + K_2 v), w\big) = 0,\\
&\forall w \in N_u, \quad (\Lambda i(K_0 x^0 + K_2 v), iK_1 w) = 0,\\
&\forall w \in N_u, \quad (\Lambda ix, iy_w) = 0,
\end{aligned}$$

where (x, p) is the solution of the system

$$\begin{cases} x' = Ax + B_2 v = 0, & x(0) = x_0,\\ p' + A^* p + Qx = 0, & p(T) = Fx(T). \end{cases} \tag{4.33}$$

This yields

$$\forall w \in N_u, \quad Fx(T) \cdot y_w(T) + \int_0^T Qx \cdot y_w \, dt = 0,$$

$$\forall w \in N_u, \quad x_0 \cdot q_w(0) + (v, B_2^* q_w) = 0 \quad \Rightarrow \forall q \in N_p, \quad x_0 \cdot q(0) + (v, B_2^* q) = 0,$$

and hence condition (4.23). □

Notation 4.1. Given $x_0 \in \mathbb{R}^n$ such that $V(x_0) \neq \varnothing$ and $v \in V(x_0)$, denote by $\mathcal{P}(v, x_0)$ the set of all solutions (x, p) of system (4.20). It is readily checked that for all $p \in \mathcal{P}(v, x_0)$, $\mathcal{P}(v, x_0) = p + N_p$. □

4.4 Intermediary results

Theorem 4.4.

(i) *The sets $N_{x,p}$, N_p, and $B_2^* N_p$ are finite dimensional linear subspaces of $H^1(0, T; \mathbb{R}^n)^2$, $H^1(0, T; \mathbb{R}^n)$, and $L^2(0, T; \mathbb{R}^k)$, respectively. $\mathcal{P}(v, x_0)$ is a finite dimensional affine subspace of $H^1(0, T; \mathbb{R}^n)$.*

(ii) *If $V(x_0) \neq \varnothing$ for some $x_0 \in \mathbb{R}^n$, $V(x_0)$ is a closed affine subspace of $L^2(0, T; \mathbb{R}^k)$, $V(0)$ is a non-empty closed linear subspace of $L^2(0, T; \mathbb{R}^k)$,*

$$V(0) = (B_2^* N_p)^\perp, \tag{4.34}$$

$$\forall v \in V(x_0), \quad V(x_0) = v + V(0). \tag{4.35}$$

(iii) *Given $v \in V(x_0)$ and $p \in \mathcal{P}(v, x_0)$, define*

$$v^* \overset{\text{def}}{=} v + P_{V(0)}(B_2^* p - v), \tag{4.36}$$

where $P_{V(0)}$ is the orthogonal projection onto $V(0)$ in $L^2(0, T; \mathbb{R}^k)$. Then v^* is independent of the choice of p, v^* is unique in $V(x_0) \cap B_2^* \mathcal{P}(v, x_0)$, and there exists $p^* \in \mathcal{P}(v, x_0)$ such that $v^* = B_2^* p^*$. If, in addition, $B_2^* p - v \in V(0)^\perp$, then $v = v^* = B_2^* p^*$.

Analogs of Theorems 4.3 and 4.4 hold for the open loop upper value.

Remark 4.2. This theorem is a key element in the proof of the existence of a maximizer of the inf problem. The basic ideas and the arguments are due to P. ZHANG [1]. For completeness, we have added part (i) to show that the subspace $B_2^* N_p$ is finite dimensional and hence closed. This is critical in the proof of part (ii). □

Proof of Theorem 4.4. (i) From system (4.25), $N_{x,p}$ is a closed linear subspace of $H^1(0,T;\mathbb{R}^n)^2$ as the kernel of the continuous linear map

$$\begin{aligned}(x,p) &\mapsto \mathcal{A}(x,p)\\ &\overset{\text{def}}{=} (-x' + Ax - B_1\, B_1^* p, -x(0), p' + A^* p + Qx, Fx(T) - p(T))\\ &: H^1(0,T;\mathbb{R}^n)^2 \to \left(L^2(0,T;\mathbb{R}^n) \times \mathbb{R}^n\right)^2.\end{aligned}$$

We now use the fact that a topological vector space is finite dimensional if and only if every closed bounded set is compact. Indeed let K be a closed bounded subset of points (y,q) in $N_{x,p}$ for the $L^2(0,T;\mathbb{R}^n)^2$-topology. Since all the matrices in system (4.25) are bounded, the right-hand sides are bounded and the derivatives (y',q') are also bounded in $L^2(0,T;\mathbb{R}^n)^2$ and, a fortiori, in $H^1(0,T;\mathbb{R}^n)^2$. Since the injection of $H^1(0,T;\mathbb{R}^n)^2$ into $L^2(0,T;\mathbb{R}^n)^2$ is compact, then the closure of K in $L^2(0,T;\mathbb{R}^n)^2$ is compact. But, by assumption, we already know that K is closed. Thence K is compact in $L^2(0,T;\mathbb{R}^n)^2$ and $N_{x,p}$ is finite dimensional.

(ii) Since $V(x_0) \neq \varnothing$, then, by definition, for all v_1, v_2 in $V(x_0)$, condition (ii) of Theorem 4.3 is verified and condition (iii) is also verified for some pairs (x_1,p_1) and (x_2,p_2) verifying system (4.20). Therefore, for any $\alpha \in \mathbb{R}$, the pair $(x_\alpha,p_\alpha) = (\alpha x_1 + (1-\alpha)x_2, \alpha p_1 + (1-\alpha)p_2)$ is also a solution of system (4.20) for x_0 and $v_\alpha = \alpha v_1 + (1-\alpha)v_2 \in V(x_0)$. Identity (4.35) follows from the fact that $V(x_0)$ is an affine subspace. Moreover, from (4.35), $V(x_0) \neq \varnothing$ necessarily implies that $V(0) \neq \varnothing$. Finally, from condition (4.23) with $x_0 = 0$

$$v \in V(0) \quad \Leftrightarrow \quad \forall q \in N_p, \quad \int_0^T v \cdot B_2^* q \, dt = 0 \quad \Leftrightarrow v \in (B_2^* N_p)^\perp$$

and $V(0) = (B_2^* N_p)^\perp$, a non-empty closed linear subspace.

(iii) Given p_1, p_2 in $\mathcal{P}(v,x_0)$, $p_2 - p_1 \in N_p$ and

$$v + P_{V(0)}(B_2^* p_2 - v) - (v + P_{V(0)}(B_2^* p_1 - v)) = P_{V(0)}(B_2^*(p_2 - p_1)) = 0,$$

since $B_2^* N_p = V(0)^\perp$. So v^* is independent of the choice of $p \in \mathcal{P}(v,x_0)$. Since $V(x_0)$ is affine, then for all $v \in V(x_0)$,

$$\begin{gathered}v^* = v + P_{V(0)}(B_2^* p - v) \in v + V(0) = V(x_0), \qquad (4.37)\\ v^* - B_2^* p = v - B_2^* p - P_{V(0)}(v - B_2^* p) \in V(0)^\perp = B_2^* N_p\\ \Rightarrow \exists q \in N_p \text{ such that } v^* - B_2^* p = B_2^* q \quad \Rightarrow v^* = B_2^*(p+q) \in B_2^* \mathcal{P}(v,x_0),\end{gathered}$$

and $v^* \in V(x_0) \cap B_2^* \mathcal{P}(v, x_0)$. This element is unique since for v_1^* and v_2^* in $V(x_0) \cap B_2^* \mathcal{P}(v, x_0)$, $v_2^* - v_1^* \in V(0) \cap B_2^* N_p = V(0) \cap V(0)^\perp = \{0\}$. Finally, if $B_2^* p - v \in V(0)^\perp$, then from (4.37) we get $v = v^*$. □

4.5 Existence and characterization of maximizers of the minimum

Assume that $v^-(x_0)$ is finite. By definition of $V(x_0)$

$$v^-(x_0) = \sup_{v \in L^2(0,T;\mathbb{R}^k)} \inf_{u \in L^2(0,T;\mathbb{R}^m)} C_{x_0}(u, v) = \sup_{v \in V(x_0)} J_{x_0}^-(v), \tag{4.38}$$

where $V(x_0)$ is a closed affine subspace of $L^2(0, T; \mathbb{R}^k)$ and by (4.26) and condition (4.23)

$$J_{x_0}^-(v) = \inf_{u \in L^2(0,T;\mathbb{R}^m)} C_{x_0}(u, v) = p(0) \cdot x_0 + \int_0^T B_2^* p \cdot v - |v|^2 \, dt, \tag{4.39}$$

or, equivalently,

$$J_{x_0}^-(v) = Fx(T) \cdot x(T) + \int_0^T Q(t)x(t) \cdot x(t) + |B_1^*(t)p(t)|^2 - |v(t)|^2 \, dt, \tag{4.40}$$

for all solutions (x, p) of system (4.20). Define the equivalence class $[(x, p)] = (x, p) + N_{x,p}$. Then for each pair $v \in V(x_0)$, $[(x, p)]$ is the unique solution in $H^1(0, T; \mathbb{R}^n) \times H^1(0, T; \mathbb{R}^n)/N_{x,p}$ of system (4.20). So the map

$$v \mapsto [(x, p)] : V(x_0) \to \frac{H^1(0, T; \mathbb{R}^n) \times H^1(0, T; \mathbb{R}^n)}{N_{x,p}} \tag{4.41}$$

is affine and continuous, and the map

$$\begin{aligned} &(x, p) \mapsto (x(T), x, p) \\ &: H^1(0, T; \mathbb{R}^n) \times H^1(0, T; \mathbb{R}^n) \to \mathbb{R}^n \times L^2(0, T; \mathbb{R}^n) \times L^2(0, T; \mathbb{R}^n) \end{aligned} \tag{4.42}$$

is continuous and compact.

So we are back to a continuous linear quadratic function $J_{x_0}^-(v)$ that is to be maximized over the closed affine subspace $V(x_0)$. The state is now the pair (x, p) solution of (4.20), but the structure is the same. Lemma 4.1 readily extends to the case of a sup over a closed affine subspace and the following conditions are equivalent:

$$a) \qquad \exists \hat{v} \in V(x_0), \quad J_{x_0}^-(\hat{v}) = \sup_{v \in V(x_0)} J_{x_0}^-(v), \tag{4.43}$$

$$b) \qquad \sup_{v \in V(x_0)} J_{x_0}^-(v) < +\infty, \tag{4.44}$$

$$c) \qquad \exists \hat{v} \in V(x_0) \text{ such that } [I + M]\hat{v} + f \in V(0)^\perp \tag{4.45}$$

$$\text{and } \forall w \in V(0), \quad ([I + M]w, w) \leq 0, \tag{4.46}$$

for the new compact operator M corresponding to the new state (x,p).

It remains to compute the directional derivative of $J^-_{x_0}(v)$ at $v \in V(x_0)$ in the direction $w \in V(0)$. By direct computation from formula (2.18)

$$\frac{1}{2} dC_{x_0}(-B_1^* p, v; 0, w) = \int_0^T (B_2^* p - v) \cdot w \, dt \tag{4.47}$$

that is independent of $p \in \mathcal{P}(v, x_0)$ for all $w \in V(0)$ from formula (4.23) of Theorem 4.3 (iv). Hence

$$\begin{aligned} dJ^-_{x_0}(v; w) &= dC_{x_0}(-B_1^* p, v; 0, w) \\ &= 2 \int_0^T (B_2^* p - v) \cdot w \, dt, \quad \forall p \in \mathcal{P}(v, x_0). \end{aligned} \tag{4.48}$$

As for the second order derivative:

$$\begin{aligned} &\frac{1}{2} d^2 C_{x_0}(-B_1^* p, v; 0, w; 0, w') \\ &= F y_{w'}(T) \cdot y_w(T) + \int_0^T Q y_{w'} \cdot y_w + B_1^* q_{w'} \cdot B_1^* q_w - w' \cdot w \, dt \end{aligned} \tag{4.49}$$

$$\begin{aligned} \Rightarrow \ \frac{1}{2} d^2 J^-_{x_0}(v; w; w) &= \frac{1}{2} d^2 C_{x_0}(-B_1^* p, v; 0, w; 0, w) \\ &= \frac{1}{2} J_0^-(w) = \frac{1}{2} \inf_{u \in L^2(0.T;\mathbb{R}^m)} C_0(u, w), \end{aligned} \tag{4.50}$$

where the last term must be negative or zero for all $w \in V(0)$. But, from Theorem 4.3 (iii), $C_0(u, 0)$ is convex in u. By using the equivalent condition of Lemma 2.1 (ii) for the u-convexity of $C_0(u, 0)$, we finally get the following two-part condition:

$$\sup_{v \in V(0)} \inf_{u \in L^2(0,T;\mathbb{R}^m)} C_0(u, v) \leq 0 \leq \inf_{u \in L^2(0,T;\mathbb{R}^m)} C_0(u, 0). \tag{4.51}$$

This condition is equivalent to condition (4.3) since $C_0(0,0) = 0$. Indeed condition (4.3) implies condition (4.51). Conversely from condition (4.51)

$$C_0(0,0) = 0 \leq \inf_{u \in L^2(0,T;\mathbb{R}^m)} C_0(u, 0) \leq \sup_{v \in V(0)} \inf_{u \in L^2(0,T;\mathbb{R}^m)} C_0(u, v)$$

$$\sup_{v \in V(0)} \inf_{u \in L^2(0,T;\mathbb{R}^m)} C_0(u, v) \leq \inf_{u \in L^2(0,T;\mathbb{R}^m)} C_0(u, 0) \leq C_0(0,0) = 0,$$

and this yields condition (4.3).

Proof of Theorem 4.1. (i) $\Rightarrow$ (ii) is obvious. (ii) $\Rightarrow$ (iii) From the previous discussion, the finiteness of $v^-(x_0)$ is equivalent to

$$\exists \hat{v} \in V(x_0) \text{ such that } dJ^-_{x_0}(\hat{v}; w) = 2 \int_0^T (B_2^* \hat{p} - \hat{v}) \cdot w \, dt = 0, \quad \forall w \in V(0),$$

$$d^2 J^-_{x_0}(\hat{v}; w; w) = 2 \inf_{u \in L^2(0.T;\mathbb{R}^m)} C_0(u, w) \leq 0, \quad \forall w \in V(0).$$

The second order condition combined with the fact that $V(x_0) \neq \varnothing$ (the equivalence of part (ii) and (iii) in Theorem 4.3 and hence the convexity inequality (4.22)) yields condition (4.3). The first order condition says that $B_2^*\hat{p} - \hat{v} \in V(0)^\perp$. By Theorem 4.4 (iii) there exists $\hat{p}^* \in \mathcal{P}(\hat{v}, x_0)$ such that $\hat{v} = B_2^*\,\hat{p}^*$, where $(\hat{x}^*, \hat{p}^*)$ is a solution of (4.20). Since $\hat{v} = B_2^*\hat{p}^*$, the coupled system (3.2) has a solution unique up to an element of $\mathcal{N}_{x,p}$. After substitution of $\hat{v} = B_2^*\hat{p}^*$ in (4.20), $(\hat{x}^*, \hat{p}^*)$ becomes a solution of the coupled system (3.2). This also yields the identities (4.2). (iii) $\Rightarrow$ (i) By assumption $\hat{v} = B_2^* p \in V(x_0)$. The existence of a solution (x, p) to system (3.2) yields the existence of a solution to system (4.20) of Theorem 4.3 (iii) with $\hat{u} = -B_1^* p$ as a minimizer. For all $v \in V(x_0)$

$$J_{x_0}^-(v) = J_{x_0}^-(B_2^*p) + dJ_{x_0}^-(B_2^*p; v - B_2^*p) + \frac{1}{2}d^2J_{x_0}^-(B_2^*p; v - B_2^*p; v - B_2^*p).$$

The second order term is negative by condition (4.3) since, by assumption, $B_2^*p \in V(x_0)$ and hence $v - B_2^*p \in V(0)$ for all $v \in V(x_0)$. As for the first order term, recall that, in view of (4.2), for all $w \in V(0)$

$$dJ_{x_0}^-(B_2^*p; w) = \int_0^T (B_2^*p - v) \cdot w \, dt = 0.$$

Thus $dJ_{x_0}^-(B_2^*p; v - B_2^*p) = 0$ since $v - B_2^*p \in V(0)$. Hence B_2^*p is a maximizer of $J_{x_0}^-$. □

4.6 Finite open loop lower value for all initial states and uniqueness of solution of the coupled system

In this section we sharpen the results of the previous section when the open loop lower value, value, or upper value of the game is finite for *all initial state* $x_0 \in \mathbb{R}^n$. In each case this global assumption yields the *uniqueness of solution.*

Theorem 4.5. *The following conditions are equivalent:*

(i) *For each* $x_0 \in \mathbb{R}^n$*, there exist* $\hat{u}$ *in* $L^2(0,T;\mathbb{R}^m)$ *and* $\hat{v}$ *in* $L^2(0,T;\mathbb{R}^k)$ *such that*

$$C_{x_0}(\hat{u}, \hat{v}) = \inf_{u \in L^2(0,T;\mathbb{R}^m)} C_{x_0}(u, \hat{v}) = \sup_{v \in L^2(0,T;\mathbb{R}^k)} \inf_{u \in L^2(0,T;\mathbb{R}^m)} C_{x_0}(u, v). \quad (4.52)$$

(ii) *For each* $x_0 \in \mathbb{R}^n$*, the open loop lower value* $v^-(x_0)$ *of the game is finite.*

(iii) *For each* $x_0 \in \mathbb{R}^n$*, there exists a unique pair* $(x, p) \in H^1(0,T;\mathbb{R}^n)^2$ *solution of the* coupled system (3.2) *such that* $B_2^*p \in V(x_0)$*, there exists a unique pair* $(\hat{u}, \hat{v})$ *that verifies* (3.3)*, and*

$$\sup_{v \in V(0)} \inf_{u \in L^2(0,T;\mathbb{R}^m)} C_0(u, 0) = \inf_{u \in L^2(0,T;\mathbb{R}^m)} C_0(u, 0) = C_0(0, 0). \quad (4.53)$$

Note that under any one of the above conditions, $(0,0)$ is the unique solution of (4.53).

Remark 4.3. The uniqueness under condition (i) was originally given by P. Zhang, G. Zheng, Y. Xu, and J. Xi [1] by a different argument. Our short and transparent proof seems to be new. The same proof can readily be used in the context of Optimal Control (cf. J. L. Lions [3]) . □

Proof. (i) ⇒ (ii) is obvious. (ii) ⇒ (iii). From Theorem 4.1 where condition (4.53) is condition (4.3). We only need to show the uniqueness of solution to the coupled system (3.2). By linearity, this amounts to prove that the solution (y,q) of the homogeneous system (3.7) such that $B_2^* q \in V(0)$ is $(0,0)$. Given an arbitrary x_0 consider the expression

$$\begin{aligned}
q(0)\cdot x_0 &= q(T)\cdot x(T) - \int_0^T q'\cdot x + q\cdot x'\,dt \\
&= Fx(T)\cdot y(T) + \int_0^T Qx\cdot y + B_1^* p\cdot B_1^* q - B_2^* p\cdot B_2^* q\,dt \\
&= \frac{1}{2} dC_{x_0}(\hat{u},\hat{v}; -B_1^* q, B_2^* q) = 0
\end{aligned}$$

from (2.15), (3.2), (4.2), and the fact that $B_2^* q \in V(0)$. Since this identity is true for all $x_0 \in \mathbb{R}^n$, then $q(0)=0$. But now we can look at the coupled system (3.7) as a linear differential system of $2n$ equations in (x,p) with zero initial condition $(y(0),q(0)) = (0,0)$ whose unique solution is $(y,q)=(0,0)$. This proves the uniqueness. (iii) ⇒ (i). Again from Theorem 4.1 since the conditions are verified for each $x_0 \in \mathbb{R}^n$. □

We readily have the dual result.

Theorem 4.6. *The following conditions are equivalent:*

(i) *For each $x_0 \in \mathbb{R}^n$, there exist $\hat{u}$ in $L^2(0,T;\mathbb{R}^m)$ and $\hat{v}$ in $L^2(0,T;\mathbb{R}^k)$ such that*

$$C_{x_0}(\hat{u},\hat{v}) = \sup_{v\in L^2(0,T;\mathbb{R}^k)} C_{x_0}(\hat{u},v) = \inf_{u\in L^2(0,T;\mathbb{R}^m)} \sup_{v\in L^2(0,T;\mathbb{R}^k)} C_{x_0}(u,v). \quad (4.54)$$

(ii) *For each $x_0 \in \mathbb{R}^n$, the open loop upper value $v^+(x_0)$ of the game is finite.*

(iii) *For each $x_0 \in \mathbb{R}^n$, there exists a unique pair $(x,p) \in H^1(0,T;\mathbb{R}^n)^2$ solution of the* coupled system (3.2) *such that $-B_1^* p \in U(x_0)$, there exists a unique pair $(\hat{u},\hat{v})$ that verifies* (3.3), *and*

$$\inf_{u\in U(0)} \sup_{v\in L^2(0,T;\mathbb{R}^k)} C_0(u,v) = \sup_{v\in L^2(0,T;\mathbb{R}^k)} C_0(0,v) = C_0(0,0). \quad (4.55)$$

Again, under any one of the above conditions, $(0,0)$ is the unique solution of (4.55).

5 Finite open loop value and open loop saddle point

In this section we show that the necessary and sufficient condition for the finiteness of the value $v(x_0)$ of the game is the existence of a saddle point of the utility function or a Nash equilibrium (cf. footnote 1 on page 48). In the process, we complete the characterization of an open loop saddle point in Theorem 3.1 by showing that the, a priori, convexity–concavity of the utility function is in fact necessary.

Theorem 5.1. *The following conditions are equivalent:*

(i) *There exists an open loop* saddle point *of* $C_{x_0}(u, v)$.
(ii) *The open loop value* $v(x_0)$ *of the game is finite.*
(iii) *There exists a solution* $(x, p) \in H^1(0, T; \mathbb{R}^n) \times H^1(0, T; \mathbb{R}^n)$ *of the* coupled system (3.2), *the solution pair* $(\hat{u}, \hat{v})$ *is given by the expressions* (3.3), *and the convexity–concavity* (3.4) *is verified.*

Under any one of the above conditions, the open loop value is given by (3.5).

Proof. (i) $\Rightarrow$ (ii). Since the utility function has a saddle point, the value of the game is finite. (ii) $\Rightarrow$ (iii). From Theorems 4.1 and 4.2, there exists a solution to the coupled system (3.2) and the convexity–concavity condition (3.4) readily follows from (4.7) and (4.3). (iii) $\Rightarrow$ (i). From condition (3.4) $C_{x_0}(u, v)$ is convex–concave. Moreover the coupled system (3.2) has a solution. Therefore by Theorem 3.1 (iii) $C_{x_0}(u, v)$ has an open loop saddle point. □

Remark 5.1. The common part of the necessary condition for the finiteness of the lower value $v^-(x_0)$, value $v(x_0)$, and upper value $v^+(x_0)$ of the game is the existence of a solution of the coupled system (3.2). The difference is in the respective *feasibility conditions* (4.3), (3.4), and (4.7): $v^-(0) = 0$, $v(0) = 0$, and $v^+(0) = 0$. □

We conclude with the enlightening result proved by P. Zhang [1] (Theorem 4.1) that has shed new light on the characterization of a game with finite value. One of the consequences is that only three cases can occur: (i) $v^+(x_0)$ finite and $v^-(x_0) = -\infty$, (ii) $v^+(x_0) = +\infty$ and $v^-(x_0)$ finite, and (iii) $v(x_0)$ finite. So the duality gap is either $+\infty$ or 0.

Theorem 5.2. *Given* $x_0 \in \mathbb{R}^n$, *the following statements are equivalent:*

(i) *There exists an open loop saddle point of* $C_{x_0}(u, v)$.
(ii) *The open loop value of the game of* $C_{x_0}(u, v)$ *is finite.*
(iii) *Both the open loop lower and upper values of* $C_{x_0}(u, v)$ *are finite.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are obvious. It remains to prove that (iii) $\Rightarrow$ (i). From condition (4.3) of Theorem 4.1 and condition (4.7) of Theorem 4.2, we get condition (3.4) of Theorem 5.1. Finally, Theorems 4.1 and 4.2 together give the existence of a pair $(x, p) \in H^1(0, T; \mathbb{R}^n) \times H^1(0, T; \mathbb{R}^n)$ solution of the coupled system (3.2). Therefore by Theorem 5.1 the utility function has a saddle point. □

We now get uniqueness of solution to the coupled system when the value of the game is finite for *all initial state* $x_0 \in \mathbb{R}^n$ by combining Theorems 4.6 and 4.5 of §4.6.

Theorem 5.3. *The following conditions are equivalent:*

(i) *For each* $x_0 \in \mathbb{R}^n$*, there exists an open loop* saddle point *of* $C_{x_0}(u,v)$*.*
(ii) *For each* $x_0 \in \mathbb{R}^n$*, the open loop value* $v(x_0)$ *of the game is finite.*
(iii) *For each* $x_0 \in \mathbb{R}^n$*, there exists a unique pair* $(x,p) \in H^1(0,T;\mathbb{R}^n)^2$ *solution of the* coupled system (3.2)*, there exists a unique pair* $(\hat{u},\hat{v})$ *that verifies* (3.3)*, and the convexity–concavity condition* (3.4) *is verified.*

Under any one of the above conditions, $(0,0)$ is the unique solution of (3.4).

6 Riccati differential equation in the open loop saddle point case

6.1 Invariant embedding with respect to the initial time

Consider the linear quadratic game on the time interval $[s,T]$, $0 \le s < T$, with initial state $h \in \mathbb{R}^n$ at time s

$$C_h^s(u,v) \stackrel{\text{def}}{=} Fx(T)\cdot x(T) + \int_s^T Qx\cdot x + |u|^2 - |v|^2\,dt, \tag{6.1}$$

$$x' = Ax + B_1u + B_2v \quad \text{a.e. in } [s,T], \quad x(s) = h. \tag{6.2}$$

Definition 6.1. Let $h \in \mathbb{R}^n$ be an initial state at time s, $0 \le s < T$.

(i) The game is said to achieve its *open loop lower value* (resp. *upper value*) if

$$v_s^-(h) \stackrel{\text{def}}{=} \sup_{v\in L^2(s.T;\mathbb{R}^k)} \inf_{u\in L^2(s.T;\mathbb{R}^m)} C_h^s(u,v), \tag{6.3}$$

$$(\text{resp. } v_s^+(h) \stackrel{\text{def}}{=} \inf_{u\in L^2(s.T;\mathbb{R}^m)} \sup_{v\in L^2(s,T;\mathbb{R}^k)} C_h^s(u,v)) \tag{6.4}$$

is finite.
(ii) The game is said to achieve its *open loop value* if its open loop lower value $v_s^-(h)$ and upper value $v_s^+(h)$ are achieved and $v_s^-(h) = v_s^+(h)$. The open loop value of the game will be denoted by $v_s(h)$.
(iii) A pair $(\bar{u},\bar{v})$ in $L^2(s,T;\mathbb{R}^m) \times L^2(s,T;\mathbb{R}^k)$ is a *open loop saddle point* of $C_h^s(u,v)$ if for all u in $L^2(s,T;\mathbb{R}^m)$ and all v in $L^2(s,T;\mathbb{R}^k)$

$$C_h^s(\bar{u},v) \le C_h^s(\bar{u},\bar{v}) \le C_h^s(u,\bar{v}). \tag{6.5}$$

□

6.2 From convexity/concavity in $[0, T]$ to $[s, T]$

The first result is that, if the utility function $C_{x_0}(u, v)$ is convex, concave, or convex–concave for some x_0 on $[0, T]$, so is $C_h^s(u, v)$ for all $h \in \mathbb{R}^n$ on $[s, T]$ and all s, $0 \le s < T$.

Theorem 6.1. (i) *If, for all $(x_0, v) \in \mathbb{R}^n \times L^2(0, T; \mathbb{R}^k)$, the map $u \mapsto C_{x_0}(u, v)$ is convex, then for all s, $0 \le s < T$, and all $(h, v) \in \mathbb{R}^n \times L^2(s, T; \mathbb{R}^k)$ the map $u \mapsto C_h^s(u, v)$ is convex.*

(ii) *If, for all $(x_0, u) \in \mathbb{R}^n \times L^2(0, T; \mathbb{R}^m)$, the map $v \mapsto C_{x_0}(u, v)$ is concave, then for all s, $0 \le s < T$, and all $(h, u) \in \mathbb{R}^n \times L^2(s, T; \mathbb{R}^m)$ the map $v \mapsto C_h^s(u, v)$ is concave.*

Proof. We only prove (i). From identities (2.19) and (2.21) for all $(u, v) \in L^2(0, T; \mathbb{R}^m) \times L^2(0, T; \mathbb{R}^k)$

$$\begin{aligned} \forall \bar{u} \in L^2(0, T; \mathbb{R}^m), \quad & d^2 C_{x_0}(u, v; \bar{u}, 0; \bar{u}, 0) \\ & = F\bar{y}(T) \cdot \bar{y}(T) + (Q\bar{y}, \bar{y}) + (\bar{u}, \bar{u}) \ge 0, \end{aligned} \tag{6.6}$$

where $\bar{y}$ is the solution of

$$\bar{y}' = A\bar{y} + B_1\bar{u}, \quad \bar{y}(0) = 0. \tag{6.7}$$

To prove the same result on $[s, T]$, associate with each $\bar{u} \in L^2(s, T; \mathbb{R}^m)$ its extension by zero $\tilde{\bar{u}}$ from $[s, T]$ to $[0, T]$. Therefore

$$\forall \bar{u} \in L^2(s, T; \mathbb{R}^m), \quad F\bar{y}(T) \cdot \bar{y}(T) + \int_0^T Q\bar{y} \cdot \bar{y}) + \tilde{\bar{u}} \cdot \tilde{\bar{u}}\, dt \ge 0, \tag{6.8}$$

where $\bar{y}$ is the solution of

$$\bar{y}' = A\bar{y} + B_1\tilde{\bar{u}}, \quad \bar{y}(0) = 0. \tag{6.9}$$

Notice that, since $\tilde{\bar{u}}$ is zero in $[0, s]$, $\bar{y} = 0$ in $[0, s]$ and $\bar{y}$ is also solution of

$$\bar{y}' = A\bar{y} + B_1\bar{u}, \quad \bar{y}(s) = 0, \tag{6.10}$$

$$\Rightarrow \forall \bar{u} \in L^2(s, T; \mathbb{R}^m), \quad F\bar{y}(T) \cdot \bar{y}(T) + \int_s^T Q\bar{y} \cdot \bar{y}) + \bar{u} \cdot \bar{u}\, dt \ge 0. \tag{6.11}$$

Hence for all $h \in \mathbb{R}^n$, all $(u, v) \in L^2(s, T; \mathbb{R}^m) \times L^2(s, T; \mathbb{R}^k)$, and all $\bar{u} \in L^2(s, T; \mathbb{R}^m)$

$$\begin{aligned} d^2 C_h^s(u, v; \bar{u}, 0; \bar{u}, 0) &= F\bar{y}(T) \cdot \bar{y}(T) + \int_s^T Q\bar{y} \cdot \bar{y}) + \bar{u} \cdot \bar{u}\, dt \\ &= d^2 C_{x_0}(0, 0; \tilde{\bar{u}}, 0; \tilde{\bar{u}}, 0) \ge 0. \end{aligned}$$

Thus for all s and all (h, v), the map $u \mapsto C_h^s(u, v)$ is convex. □

6.3 Open loop saddle point optimality principle

At this juncture, it is important to notice that the necessary conditions (4.3) and (4.7) associated with the respective finiteness of the lower and upper values of the game on $[0,T]$ do not generally survive on $[s,T]$. However the convexity–concavity condition (3.4) does.

Theorem 6.2. *Assume that $v(x_0)$ is finite for some $x_0 \in \mathbb{R}^n$, denote by $(x(\cdot;x_0), p(\cdot;x_0))$ a solution of the coupled system* (3.2) *in $[0,T]$, and let s, $0 \le s < T$.*

(i) *The value $v_s(x(s;x_0))$ of the game is finite.*
(ii) *The restriction of (x,p) to $[s,T]$ is a solution of the coupled system*

$$\begin{cases} x_s' = Ax_s - B_1B_1^*p_s + B_2B_2^*p_s \text{ a.e. in } [s,T], & x_s(s) = x(s;x_0), \\ p_s' + A^*p_s + Qx_s = 0, & p_s(T) = Fx_s(T), \end{cases} \tag{6.12}$$

the restrictions $(u_s, v_s) = (u|_{[s,T]}, v_{[s,T]})$ of the controls (u,v) on $[0,T]$ to $[s,T]$ verify

$$u_s = -B_1^*p_s \text{ and } v_s = B_2^*p_s, \quad v_s(x(s;x_0)) = p_s(s)\cdot x(s;x_0), \tag{6.13}$$

$$v(x_0) = v_s(x(s;x_0)) + \int_0^s Qx\cdot x + |u|^2 - |v|^2\, dt, \tag{6.14}$$

and

$$\sup_{v\in L^2(s,T;\mathbb{R}^k)} C_0^s(0,v) = C_0^s(0,0) = \inf_{u\in L^2(s,T;\mathbb{R}^m)} C_0^s(u,0). \tag{6.15}$$

Proof. From Theorem 5.1 on $[s,T]$, part (i) is equivalent to part (ii) and it is sufficient to prove part (ii). From Theorem 6.1, the convexity–concavity conditions on $[0,T]$ survive on $[s,T]$ and we get (6.15). Moreover if $(x(\cdot;x_0), p(\cdot;x_0))$ is a solution of the coupled system (3.2) in $[0,T]$ with initial state x_0 at time 0 and the controls (u,v) verify identities (3.3), then the restrictions $(x_s,p_s) = (x|_{[s,T]}, p|_{[s,T]})$ is a solution to the coupled system (6.12) and the restrictions $(u_s,v_s) = (u|_{[s,T]}, v_{[s,T]})$ of the the controls on $[0,T]$ verify (6.13). So, by the analog of Theorems 5.1, we get the finiteness of the value of the game on $[s,T]$. □

Theorem 6.3. *Assume that $v(x_0)$ is finite for all $x_0 \in \mathbb{R}^n$.*

(i) *The solution (x_s,p_s) of the coupled system* (6.12) *and the controls (u_s,v_s) on $[s,T]$ in* (6.13) *are unique.*
(ii) *The map*

$$x_0 \mapsto X(s)x_0 \stackrel{\text{def}}{=} x(s;x_0) : \mathbb{R}^n \to \mathbb{R}^n \tag{6.16}$$

is a linear bijection, where $(x(\cdot;x_0), p(\cdot;x_0))$ is the unique solution of the coupled system (3.2) *in $[0,T]$.*

(iii) *For all $h \in \mathbb{R}^n$, the utility function $C_h^s(u,v)$ has a unique open loop saddle point $(\hat{u}_s, \hat{v}_s) \in L^2(s,T;\mathbb{R}^m) \times L^2(s,T;\mathbb{R}^k)$ and there exists a unique solution $(\hat{x}_s, \hat{p}_s)$ of the coupled system*

$$\begin{cases} \hat{x}_s' = A\hat{x}_s - B_1B_1^*\hat{p}_s + B_2B_2^*\hat{p}_s \ \textit{a.e. in } [s,T], & \hat{x}_s(s) = h, \\ \hat{p}_s' + A^*\hat{p}_s + Q\hat{x}_s = 0 \ \textit{a.e. in } [s,T], & \hat{p}_s(T) = F\hat{x}_s(T), \end{cases} \tag{6.17}$$

$$\textit{such that } \hat{u}_s = -B_1^*\hat{p}_s \textit{ and } \hat{v}_s = B_2^*\hat{p}_s. \tag{6.18}$$

Proof. Recall that under the assumption that $v(x_0)$ is finite, $C_h^s(u,v)$ is convex–concave for all s. This means that a saddle point of C_h^s is completely characterized by the existence of a solution of the coupled system on $[s,T]$.

(i) Assume that the pair $(\hat{u}_s, \hat{v}_s)$ is a saddle point of $C_{\hat{x}(s)}^s$ on the time interval $[s,T]$. Denote by $(\hat{x}_s, \hat{p}_s)$ the corresponding solution to the coupled system (6.12). Consider the following new pair on the interval $[0,T]$:

$$\tilde{u} \stackrel{\text{def}}{=} \begin{cases} \hat{u}, & \text{in } [0,s] \\ \hat{u}_s, & \text{in } [s,T] \end{cases} \quad \tilde{v} \stackrel{\text{def}}{=} \begin{cases} \hat{v}, & \text{in } [0,s], \\ \hat{v}_s, & \text{in } [s,T], \end{cases} \tag{6.19}$$

and the corresponding solution $(\tilde{x}, \tilde{p})$ to the state–adjoint state system (2.3)-(2.17).

If it can be shown that the pair $(\tilde{u}, \tilde{v})$ is a saddle point of $C_{x_0}(u,v)$ on $[0,T]$, then by uniqueness of the saddle point on $[0,T]$, we can conclude that $(\tilde{u}, \tilde{v}) = (\hat{u}, \hat{v})$ and hence $(\hat{u}_s, \hat{v}_s) = (\hat{u}|_{[s,T]}, \hat{v}|_{[s,T]})$. From this we get the uniqueness of the saddle point of $C_{\hat{x}(s)}^s$ on $[s,T]$ and the uniqueness of solution to the coupled system (6.12). The first remark is that $\tilde{x}(s) = \hat{x}(s)$ and from (6.14)

$$\begin{aligned} C_{x_0}(\hat{u}, \hat{v}) &= v(x_0) = \int_0^s Q\hat{x} \cdot \hat{x} + |\hat{u}|^2 - |\hat{v}|^2 \, dt + v_s(\hat{x}(s)) \\ &= \int_0^s Q\hat{x} \cdot \hat{x} + |\hat{u}|^2 - |\hat{v}|^2 \, dt + F\hat{x}_s(T) \cdot \hat{x}_s(T) + \int_s^T Q\hat{x}_s \cdot \hat{x}_s + |\hat{u}_s|^2 - |\hat{v}_s|^2 \, dt \\ &\Rightarrow C_{x_0}(\hat{u}, \hat{v}) = C_{x_0}(\tilde{u}, \tilde{v}). \end{aligned}$$

Yet, this is not sufficient to conclude that $(\tilde{u}, \tilde{v})$ is a saddle point of $C_{x_0}(u,v)$. We must show that

$$\sup_{v \in L^2(0,T;\mathbb{R}^k)} C_{x_0}(\tilde{u}, v) = C_{x_0}(\tilde{u}, \tilde{v}) = \inf_{u \in L^2(0,T;\mathbb{R}^m)} C_{x_0}(u, \tilde{v}). \tag{6.20}$$

The second remark is that, since $(\tilde{u} - \hat{u}, \tilde{v} - \hat{v})$ is equal to $(0,0)$ on $[0,s]$, $(\hat{u}_s - \hat{u}, \hat{v}_s - \hat{v})$ is a saddle point of $C_0^s(u_s, v_s)$. Combining this with the fact that, by (6.15), $(0,0)$ is also a saddle point of $C_0^s(u_s, v_s)$, the pairs $(\hat{u}_s - \hat{u}, 0)$ and $(0, \hat{v}_s - \hat{v})$ are also saddle points of $C_0^s(u_s, v_s)$ and $C_0^s(\hat{u}_s - \hat{u}, 0) = C_0^s(0, \hat{v}_s - \hat{v}) = 0$. The third remark is that

$$\begin{aligned} C_{x_0}(\hat{u}, \tilde{v}) &= C_{x_0}(\hat{u}, \hat{v}) + dC_{x_0}(\hat{u}, \hat{v}; 0, \tilde{v} - \hat{v}) + C_0(0, \tilde{v} - \hat{v}) \\ &= C_{x_0}(\hat{u}, \hat{v}) + C_0(0, \tilde{v} - \hat{v}). \end{aligned}$$

But, since $\tilde{v}-\hat{v}$ is equal to 0 on $[0,s]$,

$$C_0(0,\tilde{v}-\hat{v}) = C_0^s(0,\hat{v}_s-\hat{v}) = 0 \quad \Rightarrow C_{x_0}(\hat{u},\tilde{v}) = C_{x_0}(\hat{u},\hat{v}) = C_{x_0}(\tilde{u},\tilde{v}).$$

We now prove the second part of identity (6.20):

$$C_{x_0}(u,\tilde{v}) = C_{x_0}(\hat{u},\tilde{v}) + dC_{x_0}(\hat{u},\tilde{v};u-\hat{u},0) + C_0(u-\hat{u},0). \tag{6.21}$$

Since $(0,0)$ is a saddle point of $C_0(u,v)$

$$\inf_{u\in L^2(0,T;\mathbb{R}^m)} C_0(u-\hat{u},0) = \inf_{u\in L^2(0,T;\mathbb{R}^m)} C_0(u,0) = 0.$$

It remains to prove that for all $u \in L^2(0,T;\mathbb{R}^m)$, $dC_{x_0}(\hat{u},\tilde{v};u-\hat{u},0) = 0$. First observe that

$$\begin{aligned} dC_{x_0}(\hat{u},\tilde{v};u-\hat{u},0) &= dC_{x_0}(\hat{u},\hat{v};u-\hat{u},0) + dC_0(0,\tilde{v}-\hat{v};u-\hat{u},0) \\ &= dC_0(0,\tilde{v}-\hat{v};u-\hat{u},0). \end{aligned}$$

Since $(0,\hat{v}_s-\hat{v})$ is a saddle point of C_0^s on $[s,T]$, there exists a pair (ξ,π) solution of the coupled system

$$\begin{cases} \xi' = A\xi - B_1B_1^*\pi + B_2B_2^*\pi \text{ a.e. in } [s,T], & \xi(s)=0, \\ \pi' + A^*\pi + Q\xi = 0, & \pi(T) = F\xi(T), \end{cases} \tag{6.22}$$

$$0 = -B_1^*\pi, \quad \hat{v}_s - \hat{v} = B_2^*\pi. \tag{6.23}$$

The first equation can also be written as

$$\xi' = A\xi + B_2(\hat{v}_s-\hat{v}) \text{ a.e. in } [s,T], \quad \xi(s)=0.$$

Denote by $\tilde{\xi}$ the solution of the state equation (2.3) on $[0,T]$ corresponding to the initial state 0 and the control pair $(0,\tilde{v}-\hat{v})$

$$\tilde{\xi}' = A\tilde{\xi} + B_2(\tilde{v}-\hat{v}) \text{ a.e. in } [0,T], \quad \tilde{\xi}(0)=0,$$

and observe that, since the restriction of $\tilde{v}-\hat{v}$ to $[0,s]$ is 0, $\tilde{\xi}=0$ on $[0,s]$ and $\tilde{\xi}=\xi$ on $[s,T]$. Denoting by y the solution of

$$y' = Ay + B_1(u-\hat{u}) \text{ a.e. in } [0,T], \quad y(0)=0,$$

we get the following expression (cf. (2.15) and (2.18) for the directional derivative):

$$\begin{aligned} & dC_0(0,\tilde{v}-\hat{v};u-\hat{u},0) \\ &= F\tilde{\xi}(T)\cdot y(T) + \int_0^T Q\tilde{\xi}\cdot y + 0\cdot(u-\hat{u}) + (\tilde{v}-\hat{v})\cdot 0\,dt \\ &= F\tilde{\xi}(T)\cdot y(T) + \int_0^T Q\tilde{\xi}\cdot y\,dt = F\tilde{\xi}(T)\cdot y(T) + \int_s^T Q\tilde{\xi}\cdot y\,dt \\ &= F\xi(T)\cdot y(T) + \int_s^T Q\xi\cdot y\,dt = \int_s^T B_1^*\pi\cdot(u-\hat{u})\,dt = 0, \end{aligned}$$

since $B_1^*\pi = 0$ on $[s,T]$ from (6.23). This establishes the second part of expression (6.20).

The proof of the first part is dual to the proof of the second part. This yields the uniqueness and completes the proof of part (i).

(ii) The map (6.16) is clearly linear (and continuous). Assume that it is not bijective, then there exists some $x_0 \in \mathbb{R}^n$, $x_0 \neq 0$, such that $\hat{x}(s) = 0$. The restriction of $(\hat{x}, \hat{p})$ to the interval $[s,T]$ is a solution of the system

$$\begin{cases} \xi' = A\xi - B_1B_1^*\pi + B_2B_2^*\pi \text{ a.e. in } [s,T], & \xi(s) = 0 = \hat{x}(s), \\ \pi' + A^*\pi + Q\xi = 0 \text{ a.e. in } [s,T], & \pi(T) = F\xi(T). \end{cases} \tag{6.24}$$

But from part (i) the unique solution of system (6.24) is $(0,0)$. Hence

$$\begin{aligned} &(\hat{x}, \hat{p}) = (0,0) \text{ in } [s,T] \quad \Rightarrow (\hat{x}(s), \hat{p}(s)) = (0,0), \\ \Rightarrow &\begin{cases} \hat{x}' = A\hat{x} - B_1B_1^*\hat{p} + B_2B_2^*\hat{p} \text{ a.e. in } [0,s], & \hat{x}(s) = 0, \\ \hat{p}' + A^*\hat{p} + Q\hat{x} = 0 \text{ a.e. in } [0,s], & \hat{p}(s) = 0, \end{cases} \\ &\Rightarrow (\hat{x}, \hat{p}) = (0,0) \text{ in } [0,s] \quad \Rightarrow x_0 = \hat{x}(0) = 0. \end{aligned}$$

This contradicts our initial conjecture that $x_0 \neq 0$, and we conclude that the linear map (6.16) is injective and, a fortiori, bijective.
(iii) From part (i) for each $h \in \mathbb{R}^n$ and each s, $0 \leq s < T$, there exists a unique $h_0 \in \mathbb{R}^n$ such that $h = X(s)h_0$. But $C_{h_0}(u,v)$ has a unique open loop saddle point in $[0,T]$. From part (i), $C^s_{X(s)h_0}(u,v)$ has a unique open loop saddle point in $[s,T]$. The result now follows from the fact that $h = X(s)h_0$. The equations and the identities follow from theorems from Theorem 6.2 (ii). □

Remark 6.1. The proof of part (i) is not trivial. It is one of the key elements to get the result of part (iii) that says that $C^s_h(u,v)$ has a saddle point for all initial state h and all initial times s. □

The next example illustrates that even when the coupled system has a unique solution in the time interval $[0,T]$, the uniqueness is not necessarily preserved on a smaller interval $[s,T]$.

Example 6.1. Consider the one-dimensional example of P. Bernhard [2, Example 5.1, page 67] in $[0,2]$ with $x_0 \in \mathbb{R}$

$$\begin{cases} x' = (2-t)u + tv \text{ in } [0,2], \\ x(0) = x_0, \end{cases} \tag{6.25}$$

$$C_{x_0}(u,v) = \frac{1}{2}|x(2)|^2 + \int_0^2 |u|^2 - |v|^2\, dt. \tag{6.26}$$

The solution of the coupled loop system

$$\begin{cases} \hat{x}' = \left[-(2-t)^2 + t^2\right] p_1 \text{ in } [0,2], \\ \hat{x}(0) = x_0, \end{cases} \quad \begin{cases} \hat{p}' = 0 \text{ in } [0,2], \\ \hat{p}(2) = \dfrac{1}{2}\hat{x}(2), \end{cases} \tag{6.27}$$

is given by the expressions

$$\hat{x}(t) = (t-1)^2 x_0, \quad \hat{p}(t) = \frac{1}{2} x_0. \tag{6.28}$$

So it exists and is unique for all $x_0 \in \mathbb{R}$. At time $t = 1$, $x(1) = 0$ for all $x_0 \in \mathbb{R}$.

Now consider the closed loop system in the time interval $[1, 2]$ for the initial condition 0 at time $t = 1$

$$\begin{cases} \hat{x}_1' = \left[-(2-t)^2 + t^2\right] p_1 \text{ in } [1,2], \\ \hat{x}_1(1) = 0, \end{cases} \quad \begin{cases} \hat{p}_1' = 0 \text{ in } [1,2], \\ \hat{p}_1(2) = \dfrac{1}{2}\hat{x}_1(2). \end{cases} \tag{6.29}$$

Its solution is given by the expression

$$\hat{x}_1(t) = 2(t-1)^2 c, \quad \hat{p}_1(t) = c, \tag{6.30}$$

up to an arbitrary constant $c \in \mathbb{R}$. So the solution of this homogeneous system is not unique. The concatenation $\tilde{u}$ of $\hat{u}$ on $[0, 1]$ and $\hat{u}_1$ on $[1, 2]$ associated with $c \neq x_0/2$ as defined in (6.19) yields

$$\begin{cases} \tilde{x}' = (2-t)\,\tilde{u} + t\,\tilde{v} \text{ in } [0,2], \\ \tilde{x}(0) = x_0, \end{cases} \quad \begin{cases} \tilde{p}' = 0 \text{ in } [0,2], \\ \tilde{p}(2) = \dfrac{1}{2}\tilde{x}(2). \end{cases} \tag{6.31}$$

It is readily seen that $\tilde{x}(1) = \hat{x}(1)$ at time $t = 1$, but that the adjoint systems do not coincide: $\hat{p}(1) = x_0/2 \neq c = \tilde{p}(1)$ (and in fact for all $t \in [0, 2]$). □

6.4 Decoupling of the coupled system

We need the following lemma.

Lemma 6.1. *Assume that the open loop saddle point value $v(x_0)$ is finite for all $x_0 \in \mathbb{R}^n$. Let s, $0 \le s < T$, and $(\hat{x}_s, \hat{p}_s)$ be the unique solution of the coupled system* (6.17) *with initial state h at time s. Then the map $P(s)$*

$$h \mapsto P(s)h \stackrel{\text{def}}{=} \hat{p}_s(s) : \mathbb{R}^n \to \mathbb{R}^n \tag{6.32}$$

is linear, continuous, and symmetrical.

Proof. By definition, $P(s)$ is linear and continuous. For the symmetry, let (x, p) and $(\bar{x}, \bar{p})$ be the solutions of the coupled system (6.17) for the respective initial states h and $\bar{h}$ at time s. By symmetry of F, $Q(t)$ and $B_1(t)B_1^*(t) - B_2(t)B_2^*(t)$

$$
\begin{aligned}
P(s)h \cdot \bar{h} &= p(s) \cdot \bar{x}(s) = p(T) \cdot \bar{x}(T) - \int_s^T p' \cdot \bar{x} + p \cdot \bar{x}' \, dt \\
&= Fx(T) \cdot \bar{x}(T) \\
&\quad - \int_s^T -(A^* p + Qx) \cdot \bar{x} + p \cdot (A\bar{x} - B_1 B_1^* \bar{p} + B_2 B_2^* \bar{p}) \, dt \\
&= Fx(T) \cdot \bar{x}(T) + \int_s^T Qx \cdot \bar{x} + p \cdot (B_1 B_1^* - B_2 B_2^*) \bar{p} \, dt = P(s)\bar{h} \cdot h
\end{aligned}
$$

and $P(s)^* = P(s)$. □

In view of Lemma 6.1, we use invariant embedding to get more a priori information on the decoupling matrix function $P(s)$.

Theorem 6.4. *Assume that $v(x_0)$ is finite for all $x_0 \in \mathbb{R}^n$.*

(i) *Given the solution of the coupled system* (3.2) *in* $[0, T]$ *for* $x_0 \in \mathbb{R}^n$,

$$\hat{p}(s) = P(s)\hat{x}(s), \quad 0 \le s \le T. \tag{6.33}$$

(ii) *The elements of the matrix function P are $H^1(0,T)$-functions, the elements of the matrix functions*

$$A_P \stackrel{\text{def}}{=} A - RP, \quad R \stackrel{\text{def}}{=} B_1 B_1^* - B_2 B_2^*, \tag{6.34}$$

belong to $L^\infty(0,T)$, and the closed loop system

$$\hat{x}' = [A - (B_1 B_1^* - B_2 B_2^*)\, P]\, \hat{x} \text{ a.e. in } [0,T], \quad \hat{x}(0) = x_0, \tag{6.35}$$

has a unique solution in $H^1(0,T;\mathbb{R}^n)$. For all (t,s), $0 \le s \le t \le T$, the fundamental matrix solution $\Phi_P(t,s)$ associated with the closed loop system (6.35) *and its inverse $\Phi_P(t,s)^{-1}$ are continuous in $\{(t,s) : 0 \le s \le t \le T\}$. For all pairs $0 \le s \le t \le T$*

$$\frac{\partial}{\partial s}\Phi_P(t,s) + \Phi_P(t,s)A_P(s) = 0 \text{ a.e. in } [0,t], \quad \Phi_P(t,t) = I. \tag{6.36}$$

Proof. (i) From Theorem 6.2 and Theorem 6.3 (i)

$$\hat{x}_s = \hat{x}|_{[s,T]}, \quad \hat{p}_s = \hat{p}|_{[s,T]} \quad \Rightarrow \hat{p}(s) = \hat{p}_s(s) = P(s)\hat{x}_s(s) = P(s)\hat{x}(s),$$

and we get (6.33). The closed loop system is obtained by direct substitution of the identity (6.33) for $\hat{p}$ into the first equation of the coupled system (3.2) in $[0, T]$.

(ii) Associate with the solution of the coupled system (3.2) in $[0, T]$, the matrix function

$$\Lambda(s)x_0 \stackrel{\text{def}}{=} \hat{p}(s; x_0), \quad \forall x_0 \in \mathbb{R}^n,\, 0 \le s \le T. \tag{6.37}$$

From (6.33) in part (i) and the invertibility of $X(s)$

$$\Lambda(s)x_0 \stackrel{\text{def}}{=} P(s)X(s)x_0, \quad \forall x_0 \in \mathbb{R}^n,\, 0 \le s \le T$$
$$\Rightarrow \Lambda(s) = P(s)X(s), \Rightarrow P(s) = \Lambda(s)X(s)^{-1}, \quad 0 \le s \le T.$$

Since $X(s)$ is invertible and the elements of the matrices X and Λ are $H^1(0,T)$-functions

$$P'(s) = \Lambda(s)'X(s)^{-1} - \Lambda(s)X(s)^{-1}X(s)'X(s)^{-1}. \tag{6.38}$$

In particular the elements of the matrix function P are $H^1(0,T)$-functions. Then the matrix function $A_P(t)$ in (6.34) belongs to $L^\infty(0,T)$ and the closed loop system (6.35) has a unique solution in $H^1(0,T;\mathbb{R}^n)$. From this Φ_P has the usual properties of a fundamental matrix solution Φ_P in $\{(t,s) : 0 \le s \le t \le T\}$, $\Phi_P(t,0) = \Phi_P(t,s)\Phi_P(s,0)$, and

$$\frac{\partial \Phi_P}{\partial s}(t,s) + \Phi_P(t,s)A_P(s) \text{ a.e. in } [0,T], \quad \Phi_P(t,t) = I. \tag{6.39}$$

□

6.5 Riccati differential equation

Under the assumption of the finiteness of the open loop value of the game in $[0,T]$ for each initial state, we have used *invariant embedding* to introduce the *decoupling symmetrical matrix* function $P(s)$. To show that it is a solution of the *matrix Riccati differential equation* (2.7), we follow the technique of P. Bernhard [2, Lemma 3.1] since, from Theorem 6.3 (ii), the matrix function $X(s)$ is invertible for all s, $P(s) = \Lambda(s)X(s)^{-1}$, and $P'(s)$ is given by identity (6.38).

Theorem 6.5. *Assume that the open loop value $v(x_0)$ is finite for all $x_0 \in \mathbb{R}^n$.*

(i) *There exists a unique symmetrical solution with elements in $H^1(0,T)$ of the matrix Riccati differential equation*

$$P' + PA + A^*P - PRP + Q = 0, \quad P(T) = F, \tag{6.40}$$

where $R = B_1B_1^* - B_2B_2^*$. Moreover,

$$\hat{p}(t) = P(t)\,\hat{x}(t),\ 0 \le t \le T,\ \text{and } C_{x_0}(\hat{u},\hat{v}) = P(0)x_0 \cdot x_0, \tag{6.41}$$

where $(\hat{x},\hat{p}) \in H^1(0,T;\mathbb{R}^n)^2$ is the unique solution of the coupled system (3.2).

(ii) *The optimal strategies of the two players are closed loop*

$$\hat{u} = -B_1^*P\hat{x} \text{ and } \hat{v} = B_2^*P\hat{x}, \tag{6.42}$$

and they achieve a closed loop–closed loop saddle point in the sense of P. Bernhard [2].

(iii) *For all* $x_0 \in \mathbb{R}^n$ *the function* $C_{x_0}(u, v)$ *is convex–concave.*

Proof. (i) From identity (6.38) in the proof of part (ii) of Theorem 6.4 a straightforward computation yields that the matrix function P is a solution of the matrix Riccati differential equation (6.40). This solution is unique. Indeed if $\bar{P}$ is another solution of the Riccati equation, the closed loop system with $\bar{P}$ has a unique solution $\bar{x}$ and it is easy to check that $\bar{p} = \bar{P}\bar{x}$ is a solution of the associated adjoint equation. But there is a unique solution to the coupled system. By definition of P via invariant embedding we get that $\bar{P} = P$. (ii) and (iii) From identities (4.2) and (4.3) in Theorem 4.1. □

Remark 6.2. We shall see in Example 7.1 that the fact that the elements of P are $H^1(0, T)$ is necessary but not sufficient to get an open loop saddle point. □

6.6 Open loop saddle point and Riccati differential equation

The existence of a symmetrical solution in $H^1(0, T)$ to the matrix Riccati differential equation (6.40) implies that, for all $x_0 \in \mathbb{R}^n$, there exists a solution $(\hat{x}, \hat{p}) \in H^1(0, T; \mathbb{R}^n) \times H^1(0, T; \mathbb{R}^n)$ of the coupled system (3.2). However, as we shall see in Example 6.2, this is not sufficient to get an open loop saddle point of the utility function $C_{x_0}(u, v)$.

Theorem 6.6. *A set of necessary and sufficient conditions for the existence of an open loop saddle point of the utility function* $C_{x_0}(u, v)$ *for all* $x_0 \in \mathbb{R}^n$ *is*

a) *the utility function* $C_{x_0}(u, v)$ *is convex in* u *and concave in* v *for some* x_0,
b) *and there exists a (unique) symmetrical solution in* $H^1(0, T)$ *to the matrix Riccati differential equation* (6.40).

Proof. From Theorem 6.5 we get a) and b). Conversely, from a) if P is a solution of the Riccati differential equation, the closed loop system has a unique solution x_P and $p_P = Px_P$ is the solution of the adjoint system. It is then easy to check that the pair (x_P, p_P) is indeed a solution of the coupled system (3.2) in $[0, T]$. Finally from the convexity–concavity property b) we get the existence of the open loop saddle point. □

Remark 6.3. The method of *completion of the squares* (cf. for instance T. Başar and P. Bernhard [1, Chapter 9, Theorem 9.4] can also be used here to obtain

$$
\begin{aligned}
\sup_{v \in L^2(0,T;\mathbb{R}^k)} \inf_{u \in L^2(0,T;\mathbb{R}^m)} C_{x_0}(u, v) &\leq P(0)x_0 \cdot x_0 \\
&\leq \inf_{u \in L^2(0,T;\mathbb{R}^m)} \sup_{v \in L^2(0,T;\mathbb{R}^k)} C_{x_0}(u, v).
\end{aligned}
$$

□

So it would be tempting to conclude that there is a saddle point *without* condition a). But this is not true as will be illustrated in Example 6.2 below. It is a game without open loop saddle point, where the solution of the Riccati differential equation (6.40) is unique strictly positive and *infinitely differentiable*. In that example it is shown that $U(x_0) = \varnothing$ for all x_0. In order to get a saddle point, $v^-(x_0)$ and $v^+(x_0)$ must both be finite. Therefore the open loop lower value of the game will be finite if b) is verified and $V(x_0) \neq \varnothing$; the open loop upper value of the game will be finite if b) is verified and $U(x_0) \neq \varnothing$.

Example 6.2. Consider the utility function and linear dynamics

$$C_{x_0}(u,v) = \int_0^1 2x^2 + u^2 - v^2\, dt, \quad x' = x + u + v, \quad x(0) = x_0 \tag{6.43}$$

given by P. Zhang [1]. Here $A = B_1 = B_2 = 1$, $F = 0$, and $Q = 2$. Now $R = B_1 B_1^* - B_2 B_2^* = 0$ and the associated Riccati differential equation (6.40) reduces to

$$P' + 2P + 2 = 0 \text{ in } [0,1], \quad P(1) = 0.$$

It has a unique infinitely differentiable solution $P(t) = e^{2(1-t)} - 1$ that is strictly positive in $[0,1)$.

We now extend the result of P. Zhang [1] on the nonexistence of an open loop saddle point from the initial state $x_0 = 0$ to any initial state. For all $x_0 \in \mathbb{R}$ the open loop lower value $v^-(x_0)$ of the game is finite, but the open loop upper value $v^+(x_0)$ is $+\infty$. Indeed for each $v \in L^2(0,T;\mathbb{R})$

$$\begin{aligned} &\inf_{u \in L^2(0,T;\mathbb{R})} C_{x_0}(u,v) \le C_{x_0}(-v,v) = \int_0^1 2\left(x_0\, e^t\right)^2\, dt = \left(e^2-1\right)(x_0)^2 \\ &\quad \Rightarrow v^-(x_0) = \sup_{v \in L^2(0,T;\mathbb{R})} \inf_{u \in L^2(0,T;\mathbb{R})} C_{x_0}(u,v) \le \left(e^2-1\right)(x_0)^2 . \end{aligned}$$

By definition of the sup,

$$\begin{aligned} v^-(x_0) &= \sup_{v \in L^2(0,T;\mathbb{R})} \inf_{u \in L^2(0,T;\mathbb{R})} C_{x_0}(u,v) \\ &\ge \inf_{u \in L^2(0,T;\mathbb{R})} C_{x_0}(u,0) = \inf_{u \in L^2(0,T;\mathbb{R})} \int_0^1 2x^2 + u^2\, dt \ge 0 \\ &\quad \Rightarrow \forall x_0 \in \mathbb{R}, \quad 0 \le v^-(x_0) \le \left(e^2-1\right)(x_0)^2 . \end{aligned}$$

For the open loop upper value, associate with each $u \in L^2(0,T;\mathbb{R})$ the sequence of functions $v_n(t) = -u(t) + n$, $n \ge 1$. The corresponding sequence of states is

$$x_n(t) = e^t x_0 + n \int_0^t e^{t-s}\, ds = e^t x_0 + n(e^t - 1)$$

and utility functions

$$\begin{aligned}
C_{x_0}(u, v_n) &= n^2 \int_0^1 2(e^t - 1)^2 - 1\, dt + 2n \int_0^1 u(t)\, dt \\
&\quad + \int_0^1 (e^t x_0)^2\, dt + 2n\, x_0 \int_0^1 e^t\, (e^t - 1)\, dt \\
&= n^2 \int_0^1 1 + 2e^{2t} - 4e^t\, dt + 2n \int_0^1 u(t)\, dt \\
&\quad + \int_0^1 (e^t x_0)^2\, dt + 2n\, x_0\, (e - 1)^2 \\
&\geq (e-2)^2 n^2 + 2n \left[x_0\, (e-1)^2 + \int_0^1 u(t)\, dt \right] + \int_0^1 (e^t x_0)^2\, dt \\
&\Rightarrow \sup_{v \in L^2(0,T;\mathbb{R})} C_{x_0}(u, v) \geq C_{x_0}(u, v_n) \to +\infty
\end{aligned}$$

as n goes to infinity. Therefore for all $x_0 \in \mathbb{R}^n$ and all $u \in L^2(0,T;\mathbb{R})$,

$$\sup_{v \in L^2(0,T;\mathbb{R})} C_{x_0}(u, v) = +\infty \quad \Rightarrow v^+(x_0) = +\infty \quad \text{and} \quad U(x_0) = \varnothing,$$

and there is no open loop saddle point. □

6.7 The general case of Remark 2.1

It is interesting to look at the feedback strategies in the case of the general utility function with mixed terms (2.4) of Remark 2.1. With the change of variable (2.5)

$$\begin{bmatrix} x \\ u \\ v \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ -N_1^{-1} S^* & N_1^{-1/2} & 0 \\ N_2^{-1} T^* & 0 & N_2^{-1/2} \end{bmatrix} \begin{bmatrix} x \\ \bar{u} \\ \bar{v} \end{bmatrix}$$

and the matrices $\bar{A}$, $\overline{B}_1$, $\overline{B}_2$, and $\overline{Q}$, the matrix function $\overline{P}$ is the unique solution of the Riccati differential equation

$$\overline{P}' + \overline{P}\bar{A} + \bar{A}^* \overline{P} - \overline{P}\,\overline{R}\,\overline{P} + \overline{Q} = 0 \text{ a.e. in } [0,T], \quad \overline{P}(T) = F, \tag{6.44}$$

where $\overline{R} = \overline{B}_1 \overline{B}_1^* - \overline{B}_2 \overline{B}_2^*$ and the feedback strategies are given by

$$\bar{u} = -\overline{B}_1^* \overline{P} x \text{ and } \bar{v} = \overline{B}_2^* \overline{P} x \tag{6.45}$$

that yields in terms of the original variables

$$u = -N_1^{-1} \left[S^* + B_1^* \overline{P} \right] x \text{ and } v = N_2^{-1} \left[T^* + B_2^* \overline{P} \right] x \tag{6.46}$$

and the closed loop system

$$\begin{cases} x' = \left[A - B_1 N_1^{-1} \left(S^* + B_1^* \overline{P} \right) + B_2 N_2^{-1} \left(T^* + B_2^* \overline{P} \right) \right] x \\ \quad = \left[\bar{A} - B_1 N_1^{-1} B_1^* \overline{P} + B_2 N_2^{-1} B_2^* \overline{P} \right] x, \\ x(0) = x_0. \end{cases} \tag{6.47}$$

7 Riccati differential equation and open/closed loop upper/lower value of the game

In the literature, an important issue is the connection between the existence of a symmetrical solution to the *matrix Riccati differential equation*

$$P' + PA + A^*P - PRP + Q = 0 \text{ a.e. in } [0,T], \quad P(T) = F, \tag{7.1}$$

where $R = B_1B_1^* - B_2B_2^*$, in relation with the existence of either an open or a closed loop lower value, upper value, or saddle point of the game. For instance, in the closed loop case, quoting P. Bernhard [2] in his introduction

> "It has long been known that, for the two-person, zero-sum differential game with linear dynamics, quadratic payoff, fixed end-time, and free end-state (*standard LQ game*), the existence of a solution to a Riccati equation is a sufficient condition for the existence of a saddle point within the class of instantaneous state feedback strategies (Refs. 1-2), and therefore within any wider class (Ref. 3)."[3]

In the open loop case, the above statements are incomplete (cf. Example 6.2), even under the assumptions

$$F \geq 0 \text{ and } Q(t) \geq 0 \text{ a.e. in } [0,T]$$

used in P. Bernhard [2] that necessarily imply the convexity and the coercivity of $C_{x_0}(u,v)$ with respect to u and $V(x_0) = L^2(0,T;\mathbb{R}^k)$ for all $x_0 \in \mathbb{R}^n$. Even when the solution of the Riccati differential equation (7.1) is $H^1(0,T)$ or bounded (Remark 6.3 and Example 6.2), it is also *necessary* that the utility function be convex in u and concave in v (Theorem 6.6). In fact, the existence of a strictly positive and infinitely differentiable solution to the Riccati differential equation (7.1) does not even imply that either the open loop lower value or the open loop upper value of the game be finite as shown in Example 7.2 below.

This leaves the cases where either the open loop lower or the upper value of the game explodes. In such cases, the solution of the Riccati differential equation might have a blow-up time as illustrated in Example 7.1 below (cf. P. Bernhard [2], Example 5.1, page 67)

> "The following game has a saddle point that *survives* a conjugate point."

where he means a closed loop-closed loop saddle point. The conjugate point corresponds to a *blow-up time* of the solution of the Riccati equation (7.1) where the solution is not of the $H^1(0,T)$-type. Finally an open loop saddle point yields closed loop optimal strategies that achieve a closed loop-closed loop saddle point in the sense of P. Bernhard [2] (Theorem 6.5), but the *converse is not necessarily true*. It is informative to detail the example of P. Bernhard.

[3] Ref. 1 is Y. Ho, A.E. Bryson, and S. Baron [1], Ref. 2 is P. Faurre [1], and Ref. 3 is P. Bernhard [1].

Example 7.1. Consider the dynamics and utility function in the time interval $[0,2]$

$$x'(t) = (2-t)\,u(t) + t\,v(t), \text{ a.e. in } [0,2], \quad x(0) = x_0, \tag{7.2}$$

$$C_{x_0}(u,v) = \frac{1}{2}|x(2)|^2 + \int_0^2 |u(t)|^2 - |v(t)|^2\,dt. \tag{7.3}$$

Here $A = 0$, $B_1(t) = 2-t$, $B_2(t) = t$, $F = 1/2$, $Q = 0$, and $R = B_1B_1^* - B_2B_2^* = 4(1-t)$. It is shown in P. Bernhard [2] that the Riccati equation reduces to

$$P' - 4(1-t)P^2 = 0, \quad P(2) = 1/2 \quad \Rightarrow P(t) = \frac{1}{2(t-1)^2}.$$

Its solution is positive and blows up at $t = 1$. It is not an element of $H^1(0,2)$. We now show that there is no open loop saddle point in the time interval $[0,2]$. For the open loop lower value of the game, the minimization with respect to u has a unique solution for all (x_0, v) since the utility function $u \mapsto C_{x_0}(u,v)$ is convex and bounded below by $-\|v\|^2_{L^2}$. The minimizer is completely characterized by the coupled system

$$\begin{cases} x'(t) = (2-t)\,\hat{u}(t) + t\,v(t) \text{ a.e. in } [0,2], & x(0) = x_0, \\ p'(t) = 0 \text{ a.e. in } [0,2], & p(2) = \dfrac{1}{2}x(2), \end{cases}$$
$$\hat{u}(t) = -(2-t)\,p(t).$$

From this

$$x(2) = \frac{3}{7}\left[x_0 + \int_0^2 s\,v(s)\,ds\right] \text{ and } p(t) = \frac{1}{2}x(2)$$

and

$$\begin{aligned} J^-_{x_0}(v) &\overset{\text{def}}{=} \inf_{u \in L^2(0,2;\mathbb{R})} C_{x_0}(u,v) \\ &= C_{x_0}(\hat{u},v) = \frac{1}{2}x(2)^2 + \frac{1}{4}\,x(2)^2 \int_0^2 (2-t)^2\,dt - \int_0^2 |v(t)|^2\,dt \\ &= \frac{7}{6}x(2)^2 - \int_0^2 |v(t)|^2\,dt = \frac{3}{14}\left[x_0 + \int_0^2 s\,v(s)\,ds\right]^2 - \int_0^2 |v(t)|^2\,dt. \end{aligned}$$

It is readily seen that $J^-_{x_0}$ is concave in v and that the supremum with respect to v of $J^-_{x_0}(v)$ exists. Indeed, from the first order condition

$$\forall v,\ \frac{1}{2}dJ^-_{x_0}(\hat{v};v) = \frac{3}{14}\left[x_0 + \int_0^2 s\,\hat{v}(s)\,ds\right]\int_0^2 s\,v(s)\,ds - \int_0^2 \hat{v}(t)\,v(t)\,dt = 0,$$

there is a unique stationary point $\hat{v}(t) = t\,x_0/2$, the Hessian is negative

$$\begin{aligned}
\frac{1}{2}d^2 J^-_{x_0}(\hat{v};v;v) &= \frac{3}{14}\left[\int_0^2 s\,v(s)\,ds\right]^2 - \int_0^2 |v(t)|^2\,dt \\
&\le \frac{3}{14}\left[\int_0^2 s^2\,ds\right]\left[\int_0^2 |v(s)|^2\,ds\right] - \int_0^2 |v(t)|^2\,dt \\
&\le \left[\frac{3}{14}\,\frac{2^3}{3} - 1\right]\int_0^2 |v(t)|^2\,dt = -\frac{3}{7}\int_0^2 |v(t)|^2\,dt \le 0,
\end{aligned}$$

and the open loop lower value of the game is $v^-(x_0) = J^-_{x_0}(\hat{v}) = (x_0)^2/2$.

However the open loop upper value of the game is $v^+(x_0) = +\infty$ for all $x_0 \in \mathbb{R}$. Indeed pick the sequence of controls $\{v_n\}$, $n \ge 1$, $v_n(t) = 0$ in $[0,1]$ and $v_n(t) = n$ in $[1,2]$. The corresponding sequence of states at time $t = 2$ is

$$x_n(2) = x_0 + \int_0^2 (2-t)\,u(t)\,dt + n\int_1^2 t\,dt = \left[x_0 + \int_0^2 (2-t)\,u(t)\,dt\right] + \frac{3}{2}n.$$

Denote by X the square bracket that does not depend on n. Then

$$\begin{aligned}
C_{x_0}(u,v_n) &= \frac{1}{2}\left|X + \frac{3}{2}n\right|^2 + \int_0^2 |u(t)|^2\,dt - \int_1^2 n^2\,dt \\
&= \frac{1}{8}n^2 + \frac{3}{2}n\,X + \frac{X^2}{2} + \int_0^2 |u(t)|^2\,dt \to +\infty \text{ as } n \to +\infty.
\end{aligned}$$

Thus for all $x_0 \in \mathbb{R}$ and $u \in L^2(0,T;\mathbb{R})$

$$\sup_{v \in L^2(0,T;\mathbb{R})} C_{x_0}(u,v) = +\infty \quad \Rightarrow v^+(x_0) = +\infty \quad \text{and} \quad U(x_0) = \varnothing.$$

Therefore, whatever is the initial state x_0, $C_{x_0}(u,v)$ has no open loop saddle point. □

By changing the weight $F = 1/2$ to $1/3$ in the final term $Fx(2)^2$ in the utility function of the above Example 7.1, we get an example of a convex coercive utility function with respect to u for which there is an infinitely differentiable positive solution to the Riccati differential equation (7.1), but the open loop upper and lower values are both equal to $+\infty$.

Example 7.2. Consider the state equation (7.2) with the new utility function

$$C_{x_0}(u,v) = \frac{1}{3}|x(2)|^2 + \int_0^2 |u(t)|^2 - |v(t)|^2\,dt. \tag{7.4}$$

The Riccati differential equation (7.1) reduces to

$$P' - 4(1-t)P^2 = 0 \text{ in } [0,2], \quad P(2) = 1/3 \quad \Rightarrow P(t) = \frac{1}{2(t-1)^2+1}$$

and its solution is unique, strictly positive, and infinitely differentiable. Feedbacks of the form $\hat{u} = -B_1^* P x$ and $\hat{v} = B_2^* P x$ would yield $C_{x_0}(\hat{u},\hat{v}) = P(0)x_0^2 = x_0^2/3$.

As in Example 7.1, the utility function (7.4) is both convex and coercive in the first variable u. Therefore, given v, there exists a unique solution $\hat{u}$ to the minimization of $C_{x_0}(u,v)$ with respect to u and

$$\begin{aligned} J^-_{x_0}(v) &\stackrel{\text{def}}{=} \inf_{u \in L^2(0,2;\mathbb{R})} C_{x_0}(u,v) \\ &= C_{x_0}(\hat{u},v) = 3\left[x_0 + \int_0^2 s\,v(s)\,ds\right]^2 - \int_0^2 |v(t)|^2\,dt. \end{aligned}$$

It is readily seen that $J^-_{x_0}$ is not concave in v and that the supremum with respect to v of $J^-_{x_0}(v)$ is $+\infty$. Indeed, pick the sequence $v_n(t) = n$ and let n go to $+\infty$

$$J_{x_0}(v_n) = 10\,n^2 + 12\,n\,x_0 + 3x_0^2 \to +\infty.$$

Therefore, the open loop lower value $v^-(x_0)$ and, a fortiori, the open loop upper value $v^+(x_0)$ are both equal to $+\infty$.

Finally, it is interesting to note that the corresponding coupled system on $[0,2]$

$$\begin{cases} \hat{x}' = \left[-(2-t)^2 + t^2\right]\hat{p} \text{ in } [0,2], \\ \hat{x}(0) = x_0, \end{cases} \qquad \begin{cases} \hat{p}' = 0 \text{ in } [0,2], \\ \hat{p}(2) = \dfrac{1}{3}\hat{x}(2), \end{cases}$$

has a unique solution given by

$$\hat{x}(t) = \left[2(t-1)^2 + 1\right]\frac{x_0}{3}, \quad \hat{p}(t) = \frac{1}{3}x_0.$$

So what is missing to get a finite open loop lower value is the concavity of the function $v \mapsto J^-_{x_0}(v)$. $\square$

Part II

Representation of Infinite Dimensional Linear Control Dynamical Systems

1

Semigroups of Operators and Interpolation

1 Notation

We shall denote by X a complex Banach space of norm $|\cdot|$, and by $\mathcal{L}(X)$ the Banach algebra of all linear continuous mappings $T\colon X \to X$. The linear space $\mathcal{L}(X)$ is endowed with the usual norm: For any $T \in \mathcal{L}(X)$

$$\|T\| = \sup\{|Tx|\colon x \in X, |x| \le 1\}. \tag{1.1}$$

Given a set $S \subset \mathbb{R}$, $C(S;X)$ will denote the set of all continuous mappings from S into X. For a closed bounded interval $S = [a,b]$, the space $C([a,b];X)$ endowed with the norm

$$\|f\|_{C([a,b];X)} = \sup\{|f(t)|\colon t \in [a,b]\} \tag{1.2}$$

is a Banach space. The spaces of k-times continuously differentiable mappings will be denoted by $C^k(S;X)$ and $C^k([a,b];X)$, $k \in \mathbb{N}$.

$L^p(a,b;X)$ will be the Banach space of equivalent classes of strongly measurable (in the Böchner sense) mappings $[a,b] \to X$ that are p-integrable, $1 \le p < \infty$, (resp. essentially bounded, $p = \infty$), with norm

$$\begin{gathered}\|f\|_{L^p(a,b;X)} = \left\{\int_a^b |f(s)|^p\, ds\right\}^{1/p}, \\ (\text{resp. } \|f\|_{L^\infty(a,b;X)} = \text{ess.sup}\{|f(t)|, t \in [a,b]\}).\end{gathered} \tag{1.3}$$

By $W^{1,p}(a,b;X)$ we shall denote the set of all mappings f in $L^p(a,b;X)$ with a vector distributional derivative in $L^p(a,b;X)$. This derivative will be denoted df/dt or f'. For each $f \in W^{1,p}(a,b;X)$, there exists a unique absolutely continuous function $\tilde{f}\colon [a,b] \to X$ such that $\tilde{f} = f$ and $\tilde{f}' = f'$ a.e. in $[a,b]$. Therefore we shall always identify f and $\tilde{f}$, which is the integral of its derivative.

A *linear operator* is a linear map $A\colon D(A) \subset X \to X$ defined on a *domain* $D(A)$ that is assumed to be a linear subspace of X. The *image* or *range* of A is denoted $R(A)$. A linear operator is said to be *closed* if its *graph*

$$G(A) = \{(x, Ax)\colon x \in D(A)\}$$

is closed in the product space $X \times X$. For a closed linear operator A, the graph norm operator topology of $D(A)$ is defined by the norm

$$|x|_{D(A)} = |x| + |Ax|, \quad x \in D(A). \tag{1.4}$$

When $D(A)$ is endowed with the graph norm topology, it is a Banach space and A becomes a continuous linear operator from $D(A)$ to X.

Associate with the closed linear operator A the operators

$$A_\lambda = \lambda I - A\colon D(A) \to X \tag{1.5}$$

for arbitrary complex numbers λ. The *resolvent set* $\rho(A)$ is the set of all complex numbers λ such that the operator A_λ has a bounded inverse

$$R(\lambda, A) \stackrel{\text{def}}{=} [\lambda I - A]^{-1} \tag{1.6}$$

in X. It is known that for closed linear operators A, $\lambda \in \rho(A)$ if and only if $R(\lambda, A)$ exists and $\mathrm{Im}(\lambda I - A) = X$ (cf. E. Hille and R. S. Phillips [1, Theorem 2.16.3, p. 55]). The complement of $\rho(A)$ is called the *spectrum* of the operator A and is denoted by $\sigma(A)$. The spectrum of A is made up of three disjoint parts: the *continuous spectrum* $\sigma_C(A)$, which consists of all values of λ such that A_λ has an unbounded inverse whose domain is dense in X; the *residual spectrum* $\sigma_R(A)$, which consists of all values of λ such that A_λ has an inverse whose domain is not dense in X; and the *point spectrum* $\sigma_P(A)$, which consists of all values of λ for which no inverse exists. In other words:

(i) $\lambda \in \sigma_C(A)$ if A_λ is one-to-one with a dense image in X not equal to X,
(ii) $\lambda \in \sigma_R(A)$ if A_λ is one-to-one, but its image is not dense in X,
(iii) $\lambda \in \sigma_P(A)$ if A_λ is not one-to-one.

2 Linear evolution equations and strongly continuous semigroups

2.1 Definitions and preliminary results

Let $A\colon D(A) \subset X \to X$ be a closed linear operator. Consider the initial value problem

$$\begin{cases} y'(t) = Ay(t), & t \geq 0, \\ y(0) = x, & x \in X. \end{cases} \tag{2.1}$$

If A is bounded (that is $A \in \mathcal{L}(X)$) the solution of problem (2.1) is given by the expression

$$y(t) = e^{tA}x \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \frac{t^k A^k x}{k!}; \tag{2.2}$$

moreover, setting $S_A(t) = e^{tA}$, the following properties hold:

$$\forall t, s \in \mathbb{R}, \quad S_A(t+s) = S_A(t)S_A(s), \quad \text{and} \quad S_A(0) = I. \tag{2.3}$$

There is a one-to-one correspondence between the abelian group $(\mathbb{R}, +)$ and the subset of transformations $S_A = \{S_A(t) \colon t \in \mathbb{R}\}$ in $\mathcal{L}(X)$ under composition "$\circ$". So we shall say that $(S_A, \circ)$ is a *group* and denote it $\{S_A(t)\}$ or simply S_A. Moreover the mapping $t \mapsto S_A(t) \colon \mathbb{R} \to \mathcal{L}(X)$ is continuous. We say that S_A is *uniformly continuous*.

When A is unbounded it is still possible to construct a solution of (2.1) of the form $y(t) = S_A(t)x$ in specific applications. However, in general, $S_A(t)$ is only defined for $t \geq 0$ and (2.3) is only verified for $t \geq 0$ and $s \geq 0$ (as for parabolic problems). Thus we say that S_A is a *semigroup*. In addition, S_A is in general not uniformly continuous, but for each $x \in X$, the function $t \mapsto S_A(t)x$ is continuous. We say that S_A is *strongly continuous*. This naturally leads to the following definition, which will be followed by a characterization of the properties of the operator A, which generates the family of operators S_A.

Definition 2.1. A mapping $S \colon [0, +\infty[\to \mathcal{L}(X)$ (resp. $\mathbb{R} \to \mathcal{L}(X)$) is said to be a *strongly continuous semigroup* (resp. *group*) on X if the following properties hold:

(i) $S(0) = I$, $S(t+s) = S(t)S(s)$, $\forall t, s \geq 0$ (resp. $\forall t$, $s \in \mathbb{R}$),
(ii) for all $x \in X$, $S(\cdot)x$ is continuous on $[0, \infty[$ (resp. $\mathbb{R}$). □

Remark 2.1. A strongly continuous semigroup (resp. group) on X is also referred to as a *semigroup* (resp. *group*) *of class* C_0 on X in the terminology of E. Hille and R. S. Phillips [1, Chapter X, §10.6, p. 321]. □

Remark 2.2. Assume that S is a strongly continuous semigroup on X and that for all $t > 0$, $S(t)^{-1}$ exists in $\mathcal{L}(X)$. If $\forall x \in X$, $S(\cdot)^{-1}x$ is continuous on $[0, +\infty[$, then setting

$$\bar{S}(t) = \begin{cases} S(t)^{-1} & \text{if } t \leq 0, \\ S(t) & \text{if } t \geq 0, \end{cases}$$

$\bar{S}$ is a strongly continuous group. □

The *infinitesimal generator* A of S is the linear operator in X defined by

$$\begin{aligned} D(A) &= \left\{ x \in X \colon \text{ such that the } \lim_{h \searrow 0^+} \frac{1}{h}[S(h)x - x] \text{ exists} \right\}, \\ Ax &= \lim_{h \searrow 0^+} \frac{1}{h}[S(h)x - x], \quad \forall x \in D(A). \end{aligned} \tag{2.4}$$

Proposition 2.1. *Let S be a strongly continuous semigroup on X with infinitesimal generator A. Then:*

(i) $D(A)$ *is dense in* X,
(ii) $\forall x \in D(A)$, $S(\cdot)x \in C^1([0,\infty[;X) \cap C\big([0,\infty[;D(A)\big)$ *and*

$$\frac{d}{dt}S(t)x = AS(t)x = S(t)Ax, \quad t \geq 0. \tag{2.5}$$

(iii) A *is closed.*

Proof. (i) Set

$$Q_h = \frac{1}{h}(S(h) - I) \quad \text{and} \quad M_{a,h}x = \frac{1}{h}\int_a^{a+h} S(s)x\,ds, \quad a \geq 0,\ h > 0.$$

Then

$$\lim_{h \to 0^+} M_{a,h}x = S(a)x \quad \text{and} \quad Q_h M_{0,t} = \frac{1}{t}(M_{t,h} - M_{0,h}).$$

It follows that, for any $x \in X$, we have $M_{0,t}x \in D(A)$ and that

$$AM_{0,t}x = \frac{1}{t}(S(t) - I)x, \quad t > 0.$$

Since $\lim_{t \to 0^+} M_{0,t}x = x$ we see that $D(A)$ is dense in X so that (i) is proved.

(ii) If $x \in D(A)$ we have $Q_h S(t)x = S(t)Q_h x \to S(t)Ax$ as $h \to 0$. Thus $S(t)x \in D(A)$ and $AS(t)x = S(t)Ax$. Let us now prove that $S(\cdot)x$ is right differentiable for $x \in D(A)$. If $t_0 \geq 0$, and $h > 0$, we have

$$\frac{1}{h}[S(t_0 + h) - S(t_0)x] = Q_h S(t_0)x \to AS(t_0)x \quad \text{as } h \to 0.$$

To prove the left differentiability, fix $t_0 > 0$ and let $h \in\,]0, t_0[$. Then we have

$$\frac{S(t_0 - h)x - S(t_0)x}{(t_0 - h) - t_0} = S(t_0 - h)Q_h x \to S(t_0)Ax = AS(t_0)x \quad \text{as } h \to 0,$$

where we have used the result that $S(\cdot)x$ is bounded on bounded subsets of $[0,\infty[$ by the *uniform boundedness theorem*. The C^1-continuity with values in X and the continuity with values in $D(A)$ follow from (2.5). Thus (ii) is proved.

(iii) Consider a sequence $\{x_n\} \subset D(A)$ such that $x_n \to x \in X$ and $Ax_n \to y \in X$. By (2.5) it follows that $S(t)x_n - x_n = \int_0^t S(s)Ax_n\,ds$, so that, as n goes to infinity, $S(t)x - x = \int_0^t S(s)y\,ds$. Thus by definition of A,

$$Q_t x = \frac{1}{t}\int_0^t S(s)y\,ds \to y \quad \text{as } t \to 0 \quad \text{and} \quad Ax = y. \qquad \square$$

Theorem 2.1. *Let S be a strongly continuous semigroup on X with infinitesimal generator A. Then for each x in $D(A)$, the system*

$$\frac{dx}{dt}(t) = Ax(t),\ t \geq 0, \quad x(0) = x \in D(A) \tag{2.6}$$

has a unique solution x in $C^1([0,\infty[;X) \cap C\big([0,\infty[;D(A)\big)$ and

$$x(t) = S(t)x, \quad \forall t \geq 0. \tag{2.7}$$

Proof. The existence follows from Proposition 2.1. To prove uniqueness let v be another solution in $C^1([0,\infty[;X) \cap C\big([0,\infty[;D(A)\big)$. Fix $t > 0$ and set

$$z(s) = S(t-s)v(s), \quad s \in [0,t].$$

From Proposition 2.1 and the properties of

$$v \quad \text{and} \quad z \in C^1([0,\infty[;X) \cap C\big([0,\infty[;D(A)\big),$$

we have for all s in $[0,t]$

$$\frac{dz}{ds}(s) = -AS(t-s)v(s) + S(t-s)\frac{dv}{ds}(s) = S(t-s)\left[\frac{dv}{ds}(s) - Av(s)\right] = 0.$$

As a result $z \in C^1([0,t];X)$ and $z(s) = z(0)$ for all $s \in [0,t]$. This implies $v(t) = z(t) = S(t)v(0) = S(t)x$ for all $t \geq 0$. □

2.2 Asymptotic behavior of $\mathbf{S(t)}$

Let S be a strongly continuous semigroup on X. Define

$$\omega_0(S) = \inf_{t>0} \frac{1}{t} \log \|S(t)\|. \tag{2.8}$$

$\omega_0(S)$ is said to be the *type* of S.

Proposition 2.2. *The type $\omega_0 = \omega_0(S)$ of the semigroup S is finite or equal to $-\infty$, and moreover,*

$$\omega_0 = \lim_{t\to\infty} \frac{1}{t} \log \|S(t)\|. \tag{2.9}$$

Proof. By definition ω_0 is finite or $-\infty$ and

$$\omega_0 \leq \liminf_{t\to\infty} \frac{1}{t} \log \|S(t)\|.$$

So it is sufficient to show that

$$\limsup_{t\to\infty} \frac{1}{t} \log \|S(t)\| \leq \omega_0.$$

For each $\omega > \omega_0$, there exists $t_\omega > 0$ such that

$$\frac{1}{t_\omega} \log \|S(t_\omega)\| < \omega.$$

Any $t \geq 0$ can be written in the form

$$t = n(t)t_\omega + r(t), \quad n(t) \in \mathbb{N},\ 0 \leq r(t) < t_\omega.$$

Then

$$S(t) = S\big(n(t)t_\omega + r(t)\big) = S(t_\omega)^{n(t)} S\big(r(t)\big)$$

and

$$\log \|S(t)\| \leq n(t) \log \|S(t_\omega)\| + \log \big\|S\big(r(t)\big)\big\|.$$

But $\|S(t)\|$ is bounded on the compact interval $[0, t_\omega]$ by some constant $M > 0$ and

$$\begin{aligned} \frac{\log \|S(t)\|}{t} &\leq \frac{n(t) \log \|S(t_\omega)\| + \log M}{n(t)t_\omega + r(t)} \leq \frac{\log \|S(t_\omega)\|}{t_\omega + r(t)/n(t)} + \frac{\log M}{t} \\ &\leq \frac{\log \|S(t_\omega)\|}{t_\omega} + \frac{\log M}{t} \leq \omega + \frac{\log M}{t}. \end{aligned}$$

As t goes to ∞,

$$\limsup_{t \to \infty} \frac{1}{t} \log \|S(t)\| \leq \omega.$$

Since $\omega > \omega_0$ is arbitrary and finite, the above inequality holds with $\omega = \omega_0$. □

Corollary 2.1. *If S is a strongly continuous semigroup of type ω_0, then, for any $\omega > \omega_0$, there exists $M_\omega \geq 1$ such that*

$$\|S(t)\| \leq M_\omega e^{\omega t}, \quad t \geq 0. \tag{2.10}$$

Proof. Choose t_ω as in the proof of Proposition 2.2 and set $t = n(t)t_\omega + r(t)$. If $\omega_0 \geq 0$, then $\omega > \omega_0 \geq 0$ and we have

$$\|S(t)\| \leq \|S(t_\omega)\|^{n(t)} \big\|S\big(r(t)\big)\big\| \leq M_\omega \exp(t_\omega n(t)\omega) \leq M_\omega e^{\omega t},$$

where $M_\omega = \sup\{\|S(s)\| \colon s \in [0, t_\omega]\}$. If $\omega_0 < 0$, then we consider the semigroup $S_0(t) = e^{-\omega_0 t} S(t)$ for which the type is 0. So for each $\omega > \omega_0$, $\omega - \omega_0 > 0$ and there exists $M_\omega > 1$ such that

$$e^{-\omega_0 t} \|S(t)\| = \|e^{-\omega_0 t} S(t)\| \leq M_\omega e^{(\omega - \omega_0)t}, \quad t \geq 0.$$

This completes the proof. □

Semigroups S of negative type can be completely characterized through the asymptotic behavior of their trajectories. This characterization was first introduced by R. Datko [2] in 1970 who extended the Lyapunov's theorem from finite dimensional spaces to Hilbert spaces H and showed that exponential decay is equivalent to the fact that all trajectories belong to $L^2(0,\infty;H)$. This last result was generalized by A. Pazy [1] in 1972 from Hilbert spaces H to Banach spaces X and from L^2-spaces to L^p-spaces, $1 \le p < \infty$. The simplified proof of (ii) $\Longrightarrow$ (iii) below is due to A. J. Pritchard and J. Zabczyk [1] in 1981. To our knowledge condition (v) was first given by A. Bensoussan, M. C. Delfour, and S. K. Mitter [1] in 1976 (see also M. C. Delfour [7] in 1978).

In the following we first give the general results in L^p for an arbitrary Banach space X. Then we specialize to Hilbert spaces and introduce the Lyapunov's operator equation.

Theorem 2.2. *Let S be a strongly continuous semigroup in a Banach space X and $p \in [1,\infty[$, a real number. The following properties are equivalent:*

(i) *the property*

$$\forall x \in X, \quad \int_0^\infty |S(t)x|^p \, dt < \infty; \tag{2.11}$$

(ii) *there exists a constant $c > 0$ such that*

$$\forall x \in X, \quad \int_0^\infty |S(t)x|^p \, dt \le c^p |x|^p; \tag{2.12}$$

(iii) *the type $\omega_0(S)$ of S verifies the condition*

$$\omega_0(S) < 0; \tag{2.13}$$

(iv) *there exists $\alpha > 0$ and $M \ge 1$ such that*

$$\forall x \in X, \quad \forall t \ge 0, \quad |S(t)x| \le M e^{-\alpha t} |x|; \tag{2.14}$$

(v) *S is asymptotically stable in $\mathcal{L}(X)$, that is*

$$\lim_{t \to \infty} \|S(t)\| = 0. \tag{2.15}$$

Corollary 2.2. *Let $\{S(t)\}$ be a strongly continuous semigroup of type $\omega_0(S)$:*

(i) *For all $\omega > \omega_0(S)$, $A - \omega I$ is the infinitesimal generator of the exponentially stable semigroup $\{S_\omega(t)\}$,*

$$S_\omega(t) = e^{-\omega t} S(t), \quad t \ge 0.$$

(ii) *For all x in X, there exists a unique element*

$$y = -\int_0^\infty e^{-\omega s} S(s) x \, ds \in D(A)$$

such that

$$[A - \omega I] y = x.$$

In other words, for all $\omega > \omega_0(S)$, the operator $A - \omega I \colon D(A) \to X$ has a bounded inverse and for all x in X

$$\int_0^\infty e^{-\omega s} S(s)x\, ds = -[A - \omega I]^{-1} x = R(\omega, A)x$$

and $R(\omega, A)x$ is the Laplace transform of $S(\cdot)x$.

Remark 2.3. Statements (iii), (iv), and (v) are independent of p, $1 \le p < \infty$. As a result (i) and (ii) are true for all p, $1 \le p < \infty$, and conversely it is sufficient to establish (i) or (ii) for some p, $1 \le p < \infty$, to obtain (iii), (iv), and (v). □

Remark 2.4. In infinite dimension asymptotic stability,

$$\forall x \in X, \quad S(t)x \to 0 \quad \text{as } t \to \infty$$

does not imply exponential stability. R. DATKO [2] has given the following simple example for a group $\{S(t)\}$ in the Hilbert space ℓ^2 with a bounded infinitesimal generator A. Let $H = \ell^2$,

$$\ell^2 = \left\{ x = (x_1, \dots, x_n, \dots) \colon \sum_{i=1}^\infty x_i^2 < \infty \right\}.$$

The group is

$$(S(t)x)_n = e^{-t/n} x_n, \quad n = 1, 2, \dots.$$

It can be shown that

$$\begin{gathered}
\forall x \in \ell^2, \quad S(t)x \to 0 \quad \text{as } t \to \infty, \\
\forall t, \quad \|S(t)x\| = 1 \quad \text{and} \quad \omega_0(S) = 0, \\
(Ax)_n = -\frac{x_n}{n}, \quad n = 1, 2, \dots, D(A) = \ell^2, \\
\sigma(A) = \left\{ -\frac{1}{n} \colon n = 1, 2, \dots \right\} \cup \{0\}.
\end{gathered}$$

□

Remark 2.5. Relaxations of conditions (2.11) from the p-th power of the norm in X to other functions can be found in several places in the literature. For instance J. ZABCZYK [3, §5, Theorem 5.1 and Remark 5.2] used a continuous strictly increasing convex function $N \colon [0, \infty[\, \to [0, \infty[$ such that $N(0) = 0$, and the condition

$$\forall x \in X, \quad \exists \alpha > 0, \quad \int_0^\infty N(\alpha(|S(t)x|)\, dt < \infty.$$

This condition implies condition (iv) in Theorem 2.2. In 1986 S. ROLEWICZ [1] generalized the above result to evolution operators and showed that the convexity assumption can be dropped. It seems that this result was independently re-discovered in 1987 in L. MARKUS [1] and W. LITTMAN [1]. □

Remark 2.6. For p, $1 \le p < \infty$, and a strongly continuous semigroup S of negative type

$$\forall x \in D(A), \quad S(\cdot)x \in W^{1,p}(0,\infty;X) \cap L^p\big(0,\infty;D(A)\big). \tag{2.16}$$

$\square$

Proof of Theorem 2.2. (i) $\Longrightarrow$ (ii). Define for each integer $k \ge 1$, the set

$$U_k = \left\{ x \in X\colon \int_0^\infty |S(t)x|^p \, dt < k \right\}.$$

By hypothesis $X = \bigcup_{k=1}^\infty U_k$ and because X is a complete metric space

$$\exists k_0 > 0, \quad \exists x_0 \in U_{k_0}, \quad \exists r_0 > 0 \text{ such that } B(x_0, r_0) \subset U_{k_0},$$

where $B(x_0, r_0)$ is the ball of center x_0 and radius r_0 (cf., for instance, J. Horváth [1, Corollary to Baire's theorem, p. 62]. Hence

$$\int_0^\infty |S(t)(x_0 + y)|^p \, dt \le k_0^p, \quad \forall y \in B(0, r_0),$$

and for all y in $B(0, r_0)$

$$\|S(\cdot)y\|_{L^p(0,\infty;X)} \le \|S(\cdot)(x_0 + y)\|_{L^p(0,\infty;X)} + \|S(\cdot)(x_0)\|_{L^p(0,\infty;X)} \le 2k_0.$$

So for all x in X, $x \ne 0$, we can apply the above inequality to $y = r_0 x/|x|$ and

$$\|S(\cdot)x\|_{L^p(0,\infty;X)} \le \frac{2k_0}{r_0}|x|.$$

(ii) $\Longrightarrow$ (iii). We know that for all $\omega > \omega_0$ there exists $M \ge 1$ such that

$$\forall t \ge 0, \quad \forall x \in X, \quad |S(t)x| \le M e^{\omega t}|x|.$$

Choose $\omega > \max\{1, \omega_0\}$

$$\begin{aligned}\frac{1}{p\omega}(1 - e^{-p\omega t})|S(t)x|^p &= \int_0^t e^{-p\omega r}|S(t)x|^p \, dr \\ &= \int_0^t e^{-p\omega r}\|S(r)\|^p |S(t-r)x|^p \, dr \le M^p \int_0^\infty |S(t)x|^p \, dt \le M^p c^p |x|^p.\end{aligned}$$

As a result

$$\forall t > 0, \quad |S(t)x|^p \le \frac{p\omega}{1 - e^{-p\omega t}} M^p c^p |x|^p.$$

Fix $T > 0$. As $\|S(t)\|$ is bounded on any compact interval $[0, T]$, $T > 0$, and $\omega > 0$, then

$$\exists K > 0 \text{ such that } \forall t \ge 0, \quad |S(t)x| \le K|x|,$$

where

$$K = \max\left\{\max\{\|S(t)\| \colon 0 \le t \le T\}, Mc\left[p\omega/(1-e^{-p\omega T})\right]^{1/p}\right\}.$$

As K is independent of x, we have $\|S(t)\| \le K$ for all $t \ge 0$.

We now use a second estimate for $t > 0$

$$t|S(t)x|^p = \int_0^t |S(t)x|^p\,dr \le \int_0^t \|S(r)\|^p |S(t-r)x|^p\,dr \le K^p c^p |x|^p,$$

which yields for all $t \ge 0$

$$\|S(t)\| \le Kct^{-1/p}.$$

But by definition of ω_0 as an infimum over $t > 0$

$$\forall t > 0, \quad \frac{1}{t}\log\|S(t)\| \ge \omega_0$$

and necessarily

$$\forall t > 0, \quad e^{\omega_0 t} \le \|S(t)\| \le Kct^{-1/p}.$$

But this only holds for $\omega_0 < 0$.

(iii) $\Longrightarrow$ (iv). It is sufficient to choose $\alpha = 1$ when $\omega_0 = -\infty$ and $\alpha = -\frac{1}{2}\omega_0$ when ω_0 is finite.

(iv) $\Longrightarrow$ (i) and (iv) $\Longrightarrow$ (v) are obvious. To complete the proof we prove that (v) $\Longrightarrow$ (iv). By definition of ω_0

$$\forall t \ge 0, \quad \|S(t)\| \ge e^{\omega_0 t}.$$

But by hypothesis $S(t) \to 0$ as t goes to ∞ and necessarily

$$0 = \lim_{t\to\infty} \|S(t)\| \ge \lim_{t\to\infty} e^{\omega_0 t},$$

which in turn implies that $\omega_0 < 0$. □

We now prove the corollary.

Proof of Corollary 2.2. Notice that the type $\omega_0(S_\omega) = \omega_0 - \omega < 0$. So from the equivalence between (i) and (iii) in Theorem 2.2

$$\forall x \in X, \quad S_\omega(\cdot)x \in L^1(0,\infty;X)$$

and

$$y(t) = -\int_0^t S_\omega(s)x\,ds \to y = -\int_0^\infty S_\omega(s)x\,ds \quad \text{in } X$$

as t goes to $+\infty$. Consider the expression

$$\begin{aligned}
S_\omega(t)y - y &= -S_\omega(t)\int_0^\infty S_\omega(s)x\,ds + \int_0^\infty S_\omega(s)x\,ds \\
&= -\int_0^\infty S_\omega(t+s)x\,ds + \int_0^\infty S_\omega(s)x\,ds \\
&= -\int_t^\infty S_\omega(s)x\,ds + \int_0^\infty S_\omega(s)x\,ds = \int_0^t S_\omega(s)x\,ds
\end{aligned}$$

and
$$\lim_{t\searrow 0}\frac{S_\omega(t)y-y}{t}=x.$$
Therefore
$$y\in D(A)\quad\text{and}\quad [A-\omega I]y=x.$$
So $A-\omega I\colon D(A)\to X$ is onto. To show that it is one-to-one, consider $y\in D(A)$ such that $[A-\omega I]y=0$. Then
$$\frac{d}{dt}S_\omega(t)y=S_\omega(t)[A-\omega I]y=0,\quad \forall t\geq 0$$
and this means that there exists a constant $c\in X$ such that
$$\forall t\geq 0,\quad S_\omega(t)y=c.$$
But $\{S_\omega(t)\}$ is exponentially stable and
$$c=y=S_\omega(0)y=\lim_{t\to\infty}S_\omega(t)y=0\Longrightarrow y=0.$$
This completes the proof. □

A. Pazy [1] pointed out that for semigroups of negative type the quantity
$$|x|_p=\|S(\cdot)x\|_{L^p(0,\infty;X)}\tag{2.17}$$
defines a norm on X and that the property
$$\exists c>0,\quad \forall x\in X,\qquad |x|_p\leq c|x|\tag{2.18}$$
defines a continuous embedding of X into $L^p(0,\infty;X)$. So it is natural to ask when are the two norms equivalent.

Theorem 2.3 (A. Pazy [1])**.** *Let X be a Banach space, p, $1\leq p<\infty$, a real number and $S(\cdot)$ a semigroup of negative type. Then the following conditions are equivalent*:

(i) *the norms $|\cdot|$ and $|\cdot|_p$ are equivalent on X, that is,*
$$\exists m>0 \text{ such that } |x|_p\geq m|x|;$$
(ii) *the condition*
$$\exists t_0>0,\quad \exists \bar{c}>0 \text{ such that } \forall x\in X,\quad |S(t_0)x|\geq \bar{c}|x|.$$

Proof. (i) $\Longrightarrow$ (ii). For all x in X and $t>0$
$$\begin{aligned}m^p|x|^p\leq\int_0^\infty |S(r)x|^p\,dr &\leq \int_0^t |S(r)x|^p\,dr+\int_0^\infty |S(t+r)x|^p\,dr\\ &\leq \int_0^t \|S(r)\|^p|x|^p\,dr+c^p|S(t)x|^p\\ &\leq tM^p|x|^p+c^p|S(t)x|^p.\end{aligned}$$

Choose $t_0 = m^p/2M^p$ and $\bar{c} = m/2c$.

(ii) $\Longrightarrow$ (i). Use the identity

$$t_0 \, |S(t_0)x|^p = \int_0^{t_0} |S(t_0 - t)S(t)x|^p \, dt$$
$$\leq \max_{[0,t_0]} \|S(t)\|^p \int_0^{\infty} |S(t)x|^p \, dt \leq M^p c^p |x|_p^p$$

and

$$\bar{c}|x| \leq |S(t_0)x| \leq \frac{Mc}{t_0^{1/p}}|x|. \qquad \square$$

Remark 2.7. When condition (ii) is verified for some $t_0 > 0$, it is also verified for all t in $[0, t_0]$. Indeed

$$c|x| \leq |S(t_0)x| = |S(t_0 - t)S(t)x| \leq M|S(t)x|.$$

Any $t \geq 0$ can be decomposed as

$$t = nt_0 + \tau, \quad n \text{ an integer and } 0 \leq \tau < t_0.$$

Then

$$|S(t)x| = |S(t_0)^n S(\tau)x| \geq c^n |S(\tau)x| \geq c^{n+1}|x|. \qquad \square$$

Corollary 2.3. *Assume that the hypotheses of the previous theorem are verified, that S is of negative type and that for all $t \geq 0$ the image* $\operatorname{Im} S(t)$ *of* $S(t)$ *is dense in X. Then $|\cdot|_p$ defines a norm equivalent to the norm $|\cdot|$ on X if and only if the semigroup S on X has an extension to a group of bounded operators on X.*

Theorem 2.4. *Let $X = H$ be a Hilbert space with inner product $(\cdot, \cdot)$, and let S be a strongly continuous semigroup on H with infinitesimal generator A. Then each statement in the Theorem* 2.2 *is equivalent to*:

(vi) *there exists a positive symmetric operator $P \in \mathcal{L}(H)$ such that*

$$\forall x, y \in D(A), \quad (PAx, y) + (Px, Ay) + (x, y) = 0. \tag{2.19}$$

Proof. It is sufficient to show that (i) with $p = 2$ implies (vi) and that (vi) implies (ii) with $p = 2$.

(i) $\Longrightarrow$ (vi). For all $t \geq 0$ and x and y in H define

$$(P(t)x, y) = \int_0^t (S(r)x, S(r)y) \, dr, \quad (Px, y) = \int_0^{\infty} (S(r)x, S(r)y) \, dr.$$

By hypothesis P and $P(t)$ are well-defined elements of $\mathcal{L}(H)$. In addition they are symmetric and positive. Moreover since (i) $\Longrightarrow$ (ii)

$$\begin{aligned}\big|\big(\big(P-P(t)\big)x,y\big)\big| &= \left|\int_t^\infty (S(r)x,S(r)y)\,dr\right| \\ &\le \|S(\cdot)x\|_{L^2(t,\infty;H)}\|S(\cdot)y\|_{L^2(t,\infty;H)} \le c|y|\,\|S(\cdot)x\|L^2(t,\infty;H)\end{aligned}$$

and, as t goes to ∞, $P(t)x \to Px$ in H for each x in H.

For all x and y in $D(A)$

$$\begin{aligned}(P(t)Ax,y)+(P(t)x,Ay) &= \int_0^t \{(S(r)Ax,S(r)y)+(S(r)x,S(r)Ay)\}\,dr \\ &= \int_0^t \frac{d}{dr}(S(r)x,S(r)y)\,dr = (S(t)x,S(t)y)-(x,y).\end{aligned}$$

Therefore for $y=x$

$$\lim_{t\to\infty} |S(t)x|^2 = |x|^2 + \langle PAx,x\rangle + \langle Px,Ax\rangle.$$

But

$$\int_0^\infty |S(t)x|^2\,dt < \infty \Longrightarrow \liminf_{t\to\infty} |S(t)x|^2 = 0$$

and because the limit exists it coincides with its lim inf and

$$\forall x\in D(A), \quad \lim_{t\to\infty} |S(t)x| = 0.$$

We now go back to our previous computation for x and y

$$(P(t)Ax,y)+(P(t)x,Ay) = (S(t)x,S(t)y)+(x,y)$$

and let t go to $+\infty$ to obtain (vi).

(vi) $\Longrightarrow$ (ii). For each $x\in D(A)$, $S(t)x\in D(A)$, $\forall t\ge 0$, and

$$(PAS(t)x,S(t)x)+(PS(t)x,AS(t)x)+(S(t)x,S(t)x)=0$$

or equivalently

$$\frac{d}{dt}(PS(t)x,S(t)x)+|S(t)x|^2=0.$$

By integrating from 0 to T

$$(PS(T)x,S(T)x)-(Px,x) = -\int_0^T |S(t)x|^2\,dt.$$

Since P is positive for all $x\in D(A)$

$$\int_0^T |S(t)x|^2\,dt \le (Px,x) \Longrightarrow \int_0^\infty |S(t)x|^2\,dt \le (Px,x)$$

and we obtain (ii) for $p=2$ with $c=\|P\|$. By density of $D(A)$ in H, the result also holds for all x in H. $\square$

Remark 2.8. Notice that in the Hilbertian case, it is not necessary to use Baire's theorem to show (i) $\Longrightarrow$ (ii). Instead we show that (i) $\Longrightarrow$ (vi) $\Longrightarrow$ (ii). □

Corollary 2.4. *If $\bar{P}$ is a positive and symmetric solution of the Lyapunov equation in $\mathcal{L}(H)$, then*

$$\forall x \in H, \quad (\bar{P}x, x) \geq (Px, x), \tag{2.20}$$

where P in $\mathcal{L}(H)$ is defined as

$$\forall x, y \in H, \quad (Px, y) = \int_0^\infty (S(t)x, S(t)y)\, dt. \tag{2.21}$$

Proof. For all x in $D(A)$ we proceed as in the proof of the previous theorem and

$$\frac{d}{dt}(\bar{P}S(t)x, S(t)x) + |S(t)x|^2 = 0.$$

Hence

$$\int_0^T |S(t)x|^2\, dt \leq (\bar{P}x, x) \Longrightarrow (Px, x) = \int_0^\infty |S(t)x|^2\, dt \leq \|\bar{P}\|\, |x|^2$$

where the transformation P, as defined in the proof of Theorem 2.4, is well defined and for all x in $D(A)$, $(Px, x) \leq (\bar{P}x, x)$. So by density of $D(A)$ in H the corollary holds for all x in H. □

2.3 Spectral properties of the infinitesimal generator

Proposition 2.3. *Let S be a strongly continuous semigroup in X. Assume that there exist constants $M > 0$ and $\omega \in \mathbb{R}$ be such that*

$$\|S(t)\| \leq Me^{\omega t}, \quad \forall t \geq 0. \tag{2.22}$$

Then the infinitesimal generator A of S has the following properties:

(i) $C_\omega = \{\lambda \in \mathbb{C} \colon \operatorname{Re}\lambda > \omega\} \subset \rho(A)$,
(ii) *for any $\lambda \in C_\omega$ the resolvent of A is given by*

$$R(\lambda, A)y = \int_0^\infty e^{-\lambda t} S(t)y\, dt, \quad y \in X. \tag{2.23}$$

Proof. We have to show that, for any $\lambda \in C_\omega$ and any $y \in X$, the equation

$$\lambda x - Ax = y \tag{2.24}$$

has a unique solution given by x equal to the right-hand-side of (2.23).

Existence.

For $\lambda \in C_\omega$ and $y \in X$, x is well-defined as an elements of X by the right-hand side of (2.23). Then we have

$$Q_h x = \frac{1}{h}(S(h)x - x) = \frac{1}{h}(e^{\lambda h} - 1)x - \frac{1}{h}e^{\lambda h}\int_0^h e^{-\lambda t}S(t)y\,dt.$$

As $h \to 0$, we find that $Q_h x \to Ax$ and that the right-hand side goes to $\lambda x - y$. Therefore $\lambda x - y = Ax$ so that $x \in D(A)$ and x is a solution of (2.24).

Uniqueness.

Let $\bar{x} \in D(A)$ be another solution of (2.24). Then we have

$$x = \int_0^\infty e^{-\lambda t}S(t)(\lambda\bar{x} - A\bar{x})\,dt = -\int_0^\infty \frac{d}{dt}[e^{-\lambda t}S(t)\bar{x}]\,dt = \bar{x}.$$

Thus $x = \bar{x}$. □

2.4 Hille–Yosida–Miyadera–Feller–Phillips theorem

We now give necessary and sufficient conditions on a linear operator A to be the infinitesimal generator of a strongly continuous semigroup. For early versions of this theorem, the reader is referred to E. Hille [2, p. 238] and K. Yosida [1] and for modern versions to W. Feller [1], I. Miyadera [1], and R. S. Phillips [1].

Theorem 2.5. *Let $A\colon D(A) \subset X \to X$ be a linear operator. Then the following statements are equivalent:*

(i) *$D(A)$ is dense in X, there exist real numbers $M > 0$ and $\omega \in \mathbb{R}$ such that $\rho(A) \supset \{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda > \omega\}$ and the following inequalities hold:*

$$\|R^k(\lambda, A)\| \le M(\operatorname{Re}\lambda - \omega)^{-k}, \quad \forall k \in \mathbb{N}, \quad \forall \lambda, \operatorname{Re}\lambda > \omega. \tag{2.25}$$

(ii) *A is the infinitesimal generator of a strongly continuous semigroup S and there exist real numbers $\omega \in \mathbb{R}$ and $M > 0$ such that*

$$\|S(t)\| \le Me^{\omega t}, \quad \forall t \ge 0. \tag{2.26}$$

Proof. (ii) $\Longrightarrow$ (i). From Proposition 2.3 $\{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda > \omega\} \subset \rho(A)$ and from formula (2.23) and inequality (2.26), we get condition (2.25) for $k = 1$. From Proposition 2.1, $D(A)$ is dense in X and A is closed. By differentiating (2.23) k times with respect to λ, we get

$$R^{(k)}(\lambda, A)y = (-1)^k \int_0^\infty t^k e^{-\lambda t}S(t)y\,dt, \quad y \in X.$$

It follows that

$$\|R^{(k)}(\lambda, A)\| \le Mk!(\operatorname{Re}\lambda - \omega)^{-k-1}, \quad k \in \mathbb{N}.$$

But from Proposition 2.3(ii) for all $y \in X$

$$R^k(\lambda, A)y = \int_0^\infty \frac{t^k}{k!} e^{-\lambda t} S(k)y\,dt = \frac{(-1)^k}{k!} R^{(k)}(\lambda, A)y$$

and the conclusion follows.

(i) $\Longrightarrow$ (ii). We proceed in four steps.

Step 1. Construction of an approximate semigroup S_n.

For any integer $n \in \mathbb{N}$ such that $n > \omega$, we set

$$J_n = nR(n, A), \quad A_n = AJ_n \tag{2.27}$$

and recall that

$$A_n = n^2 R(n, A) - nI.$$

The bounded operators A_n are called the *Yosida approximations* of A. We claim that the following identities hold:

$$\lim_{n\to\infty} J_n x = x, \qquad x \in X, \tag{2.28}$$

$$\lim_{n\to\infty} A_n x = Ax, \quad x \in D(A). \tag{2.29}$$

We first prove (2.28). For $x \in D(A)$, $J_n x - x = R(n, A)Ax \to 0$ as $n \to \infty$ from (2.25). But $\|J_n\| \le Mn(n-\omega)^{-1}$ is less than a constant and by density (2.28) is true for all x in X. Now (2.29) is an immediate consequence of (2.28) since $A_n x = J_n Ax$ for any $x \in D(A)$. We now set

$$S_n(t) = \exp(tA_n) = e^{-nt} \sum_{k=0}^\infty \frac{n^{2k} t^k}{k!} R^k(n, A). \tag{2.30}$$

By (2.25) it follows that

$$\|S_n(t)\| \le M \exp\big(nt\omega/(n-\omega)\big) \le Me^{2\omega t}, \quad \text{if } n > 2\omega,\ t > 0. \tag{2.31}$$

Step 2. Uniform convergence of $S_n(t)x$ to $S(t)x$ for all x in X on compact intervals in $[0, \infty[$.

Fix $x \in X$ and set $u_n(t) = S_n(t)x$. Then, if $n, m > \omega$, we have

$$\begin{cases} \dfrac{d}{dt}(u_n - u_m) = A_n(u_n - u_m) + (A_n - A_m)u_m, \\ (u_m - u_m)(0) = 0, \end{cases}$$

$$u_n(t) - u_m(t) = \int_0^t \exp\big((t-s)A_n\big) \exp(sA_m)(A_n - A_m)x\,ds,$$

and, using (2.31), the estimate

$$|u_n(t) - u_m(t)| \le M^2 e^{2t\omega} t |A_n x - A_m x|, \quad m, n > 2\omega \tag{2.32}$$

follows. In view of (2.29), for all $x \in D(A)$ $\{u_n(t)\}$ is a Cauchy sequence. But the same is true for any $x \in X$ by virtue of (2.31). Thus the following limit exists:

$$\lim_{n\to\infty} u_n(t) = \lim_{n\to\infty} S_n(t)x = u(t), \quad x \in X, \tag{2.33}$$

uniformly in t on the bounded intervals of $[0, \infty[$. Set now $S(t)x = u(t)$. It is easy to check that S is a strongly continuous semigroup in X and that (2.26) holds.

Step 3. If $x \in D(A)$, then $S(\cdot)x$ is differentiable and

$$\frac{d}{dt} S(t)x = S(t)Ax = AS(t)x. \tag{2.34}$$

In fact by (2.29) and (2.33) it follows that

$$\frac{d}{dt} u_n(t) = \exp(tA_n)A_n x \to S(t)Ax \quad \text{as } n \to \infty.$$

Step 4. A is the infinitesimal generator of S.

Let B be the infinitesimal generator of S. By (2.34) it follows that B is an extension of A. Then it is sufficient to prove that if $x \in D(B)$ then $x \in D(A)$. Let, in fact, $x \in D(B)$, $\operatorname{Re}\lambda > \omega$, and $z = \lambda x - Bx$. Then $R(\lambda, A)z \in D(A)$ and

$$(\lambda - B)R(\lambda, A)z = (\lambda - A)R(\lambda, A)z = z$$

so that

$$x = R(\lambda, B)z = R(\lambda, A)z \in D(A). \qquad \square$$

2.5 Adjoint semigroups and their generators

Given a semigroup of continuous linear transformations $S(t)$, $t \ge 0$, on X, it is always possible to define the adjoint transformations $S^*(t)$, $t \ge 0$, on X' and it is easy to check that

$$\forall t \ge 0, \quad \forall s \ge 0, \quad S^*(t+s) = S^*(t)S^*(s), \quad S^*(0) = I,$$
$$\forall x^* \in X', \quad t \to S^*(t)x^* : [0, \infty[\to X' \text{ (weak) is continuous .}$$

We shall say that S^* is the *adjoint semigroup* associated with S. However S^* is not necessarily a strongly continuous semigroup on X. Fortunately we have the following result in reflexive Banach spaces.

Proposition 2.4. *If S is a strongly continuous semigroup on a reflexive Banach space X, the adjoint semigroup S^* is also a strongly continuous semigroup on X.*

Proof. Let A be the infinitesimal generator of S and let A^* be the adjoint operator of A. As $D(A)$ is dense in X, $R(\lambda, A)^* = R(\bar{\lambda}, A^*)$ and $\rho(A) = \rho(A^*)$ (cf. E. HILLE and R. S. PHILLIPS [1, Theorem 2.6.5, p. 56]). For all λ in $\rho(A)$, $R(\lambda, A)$, and $R(\bar{\lambda}, A^*)$ are bounded and

$$\|R(\bar{\lambda}, A^*)\|_{\mathcal{L}(X')} = \|R(\lambda, A^*\|_{\mathcal{L}(X')} = \|R(\lambda, A)\|_{\mathcal{L}(X)}.$$

So if we can show that $D(A^*)$ is dense in X', all the estimates for A in Theorem 2.5 will be true for A^* and we shall conclude that A^* is the infinitesimal generator of a strongly continuous semigroup on X'. For x^* in X' we construct the approximations $x_n^* = nR(n, A^*)x^*$, $n \geq 1$. For all x in X

$$\lim_{n\to\infty} \langle x_n^*, x\rangle = \lim_{n\to\infty} \langle x^*, nR(n, A)x\rangle = \langle x^*, x\rangle.$$

As X is reflexive we conclude that $x_n^* \to x^*$ in X' (weak). But in a reflexive Banach space the strong closure $\overline{D(A^*)}$ of $D(A^*)$ is weakly closed because $\overline{D(A^*)}$ is a closed linear subspace of X'. So we have proved that $D(A^*)$ is (strongly) dense in X' because the strong and weak closures of $D(A^*)$ coincide (cf. K. YOSIDA [2, Theorem 11, p. 125]). □

2.6 Semigroups of contractions and dissipative operators

A *semigroup of contractions* is a strongly continuous semigroup S such that

$$\|S(t)\| \leq 1, \quad \forall t \geq 0. \tag{2.35}$$

If S is a semigroup of contractions, then condition (2.25) reduces to

$$\|R(\lambda, A)\| \leq \frac{1}{\operatorname{Re}\lambda}, \quad \forall\lambda,\ \operatorname{Re}\lambda > 0, \tag{2.36}$$

because we can choose $M = 1$ and $\omega = 0$.

In this section we shall give another characterization of linear operators that generate semigroups of contractions. For this we need the definition of *dissipative operators.*

We first recall that for any $x \in X$ the *sub-differential* $\partial|x|$ is defined by

$$\partial|x| = \{x' \in X' \colon |x'| = 1, \langle x, x'\rangle = |x|\}, \tag{2.37}$$

where X' is the topological dual of X. By the Hahn–Banach theorem, it follows that $\partial|x|$ is never empty.

Definition 2.2. (i) We say that the linear operator $A\colon D(A) \subset X \to X$ *is dissipative* if

$$\forall x \in D(A), \quad \exists x' \in \partial|x| \text{ such that } \operatorname{Re}\langle Ax, x'\rangle \leq 0.$$

(ii) The linear operator $A\colon D(A) \subset X \to X$ is said to be *maximal dissipative* if it is dissipative and has no proper dissipative extension. □

Proposition 2.5. *Let A be the infinitesimal generator of a strongly continuous semigroup of contractions. Then A and its adjoint operator A^* are maximal dissipative. Moreover*

$$\forall \lambda > 0, \quad R(\lambda I - A) = X \quad \textit{and} \quad R(\lambda I - A^*) = X'.$$

Proof. Let $x \in D(A)$, $x' \in \partial|x|$, and $h > 0$. Then in view of (2.37) and (2.35)

$$\mathrm{Re}\langle S(h)x - x, x' \rangle = \mathrm{Re}\langle S(h)x, x' \rangle - |x| \leq 0$$

since $\|S(h)\| \leq 1$. It follows that

$$\mathrm{Re}\langle Ax, x' \rangle = \mathrm{Re} \lim_{h \to 0} \frac{1}{h}(\langle S(h)x - x, x' \rangle) \leq 0.$$

By hypothesis, A is a closed densely defined linear operator on X. Hence its adjoint A^* is a well-defined closed linear operator on X' with domain $D(A^*)$ (cf. E. Hille and R. S. Phillips [1, Theorem 2.11.8, p. 43]). But from (2.36) for all $\lambda > 0$, $\lambda \in \rho(A)$, $\lambda I - A$ has a bounded inverse $R(\lambda, A)$, $R(\lambda I - A) = X$, and

$$\|R(\lambda, A)\| \leq \frac{1}{\lambda}, \quad \forall \lambda > 0.$$

As $D(\bar{A}) = X$ and $\lambda I - A$ has a linear bounded inverse, then

$$R(\lambda I - A^*) = X'$$

(cf. E. Hille and R. S. Phillips [1, Theorem 2.11.15, p. 45]) and

$$R(\lambda, A^*) = (\lambda I - A^*)^{-1} = (\lambda I - A)^{-1*} = R(\lambda, A)^*$$

(cf. E. Hille and R. S. Phillips [1, Theorem 2.11.14, p. 44]). So for $\lambda > 0$, $\lambda I - A^*$ has a bounded inverse and

$$\|R(\lambda, A^*)\| = \|R(\lambda, A)^*\| = \|R(\lambda, A)\| \leq \frac{1}{\lambda}, \quad \forall \lambda > 0.$$

Hence for all $y^* \in X'$ and $\lambda > 0$

$$|R(\lambda, A^*)y^*|_{X'} \leq \frac{1}{\lambda}|y^*|_{X'}.$$

For all $x^* \in D(A^*)$

$$R(\lambda, A^*)(\lambda I - A^*)x^* = x^*$$

and for all $x^* \in D(A^*)$, $\lambda > 0$

$$|x^*|_{X'} \leq \frac{1}{\lambda}|(\lambda I - A^*)x^*|_{X'}.$$

The dissipativity of A^* now follows from the following lemma.

Lemma 2.1 (A. Pazy [1, Theorem 4.2, p. 14]). *A linear operator T is dissipative if and only if*

$$|(\lambda I - T)x| \geq \lambda |x|, \quad \forall x \in D(T), \quad \forall \lambda > 0.$$

Finally both A and A^* are maximal dissipative because in both cases

$$\forall \lambda > 0, \quad \lambda I - A\colon D(A) \to X \quad \text{and} \quad \lambda I - A^*\colon D(A^*) \to X'$$

are bijective maps (one-to-one and onto). □

If A is maximal dissipative, it is generally not the generator of a strongly continuous semigroup since its domain is not necessarily dense (cf. A. Pazy [2, Example 4.7, p. 16–17]). Also if A is a dissipative closed densely defined operator that is not maximal, then it will not necessarily be a generator. A necessary and sufficient condition is given by the following theorem.

Theorem 2.6 (G. Lumer and R. S. Phillips [1]). *Let $A\colon D(A) \subset X \to X$ be a linear operator defined on a Banach space X. A is the infinitesimal generator of a semigroup of contractions on X if and only if*:

(i) *A is a closed linear operator with dense domain in X and*
(ii) *A and its adjoint operator A^* are dissipative.*

Proof. ($\Longleftarrow$) By Theorem 2.5 it is sufficient to prove that for any $y \in X$ and for any λ with $\operatorname{Re}\lambda > 0$ the equation

$$\lambda x - Ax = y \tag{2.38}$$

has a unique solution and that the following estimate holds:

$$|x| \leq \frac{1}{\operatorname{Re}\lambda}|y|. \tag{2.39}$$

Now, since A is dissipative we easily obtain the following a priori estimate: there exists $x' \in \partial|x|$ such that

$$\begin{aligned}\operatorname{Re}\lambda|x| = \operatorname{Re}\langle\lambda x, x'\rangle &= \operatorname{Re}\langle Ax, x'\rangle + \operatorname{Re}\langle y, x'\rangle \\ &\leq \operatorname{Re}\langle y, x'\rangle \leq |y|.\end{aligned}$$

This proves the uniqueness of the solution of (2.38) as well as estimate (2.39). It remains to prove that $(\lambda - A)\big(D(A)\big) = X$ if $\operatorname{Re}\lambda > 0$.

Step 1. $(\lambda - A)\big(D(A)\big)$ is dense in X if $\operatorname{Re}\lambda > 0$.

Let $x' \in X'$ and $\operatorname{Re}\lambda > 0$ such that $\langle \lambda x - Ax, x'\rangle = 0$ for any $x \in D(A)$. We have $\langle Ax, x'\rangle = \lambda\langle x, x'\rangle$, so that $x' \in D(A^*)$ and $\langle x, \bar{\lambda}x' - A^*x'\rangle$ for any $x \in D(A)$. Since $D(A)$ is dense in X we have $\bar{\lambda}x' - A^*x' = 0$, which implies (by (2.39) with A replaced by A^*) that $x' = 0$ so that $(\lambda - A)\big(D(A)\big)$ is dense in X.

Step 2. $(\lambda - A)\big(D(A)\big) = X$ if $\operatorname{Re}\lambda > 0$.

Let $\operatorname{Re}\lambda > 0$ and $y \in X$. Since $(\lambda - A)\big(D(A)\big)$ is dense in X there exists a sequence $\{x_n\} \subset D(A)$ such that $y_n = \lambda x_n - Ax_n \to y$ as $n \to \infty$. By (2.39) it follows that

$$|x_n - x_m| \leq \frac{1}{\operatorname{Re}\lambda}|y_n - y_m|$$

so that $\{x_n\}$ is a Cauchy sequence. Hence there exists $x \in X$ such that $x_n \to x$ in X, $Ax_n = \lambda x_n - y_n \to \lambda x - y$. Since A is closed we have $x \in D(A)$ and $\lambda x - Ax = y$.

($\Longrightarrow$). The converse is true by Proposition 2.5. This completes the proof. □

In the last theorem the dissipativity of both A and A^* are necessary to obtain a semigroup of contractions. This arises from the fact that in a Banach space a closed linear maximal dissipative operator A is not necessarily densely defined even when $R(\lambda I - A) = X$ for all $\lambda > 0$. This difficulty disappears when X is a reflexive Banach space (cf. A. Pazy [2, Theorem 4.6, p. 16]) and we can restate Theorem 2.6 as follows.

Theorem 2.7. *Let $A\colon D(A) \subset X \to X$ be a linear operator defined on a reflexive Banach space X. A is the infinitesimal generator of a semigroup of contractions on X if and only if*:

(i) *A is dissipative, and*
(ii) *$\exists \lambda_0 > 0$ such that $R(\lambda_0 I - A) = X$.*

Proof. ($\Longrightarrow$). From Proposition 2.5

($\Longleftarrow$) By invoking Lemma 2.1, it is easy to show that for a linear dissipative operator, A is closed if and only if its range, $R(\lambda_0 I - A)$, is closed for some $\lambda_0 > 0$. Then from (ii) and (iii) by A. Pazy [2, Theorem 4.6, p. 16], $\overline{D(A)} = X$. Finally the result follows from A. Pazy [2, Theorem 4.3a]. □

To complete the picture we also quote the Hilbert space version of the two previous theorems.

Theorem 2.8 (R. S. Phillips [2, Theorem 1.1.3, p. 203]**).** *Let $A\colon D(A) \subset H \to H$ be a linear operator defined on a Hilbert space H. Then the following conditions are equivalent*:

(i) *A is the infinitesimal generator of a semigroup of contractions on H*;
(ii) *A is maximal dissipative*;
(iii) *A^* is maximal dissipative.*

In the above theorem we have made use of the fact that in a Hilbert space a maximal dissipative operator is closed (cf. H. Tanabe [1, Theorem 2.1.1, p. 20]). The original theorem proved by R. S. Phillips [2, Theorem 1.1.3, p. 203] used the conditions

(ii) A is maximal dissipative and densely defined,

(iii) A^* is maximal dissipative and densely defined,

which are equivalent to

(ii) A is closed and maximal dissipative,
(iii) A^* is closed and maximal dissipative.

We complete this section with the important subclass of semigroups of contractions that preserve the norm

$$\forall x, \quad \forall t \geq 0, \qquad |S(t)x| = |x|.$$

Theorem 2.9 (M. H. Stone [1]**).** *Assume that $X = H$ where H is a Hilbert space:*

(i) *If A is a self-adjoint operator on H, then $B = iA$ is the infinitesimal generator of a strongly continuous group of unitary transformations.*
(ii) *Conversely if $S(t)$ is a strongly continuous group of unitary transformations with infinitesimal generator B, then iB is self-adjoint.*

Proof. (i) By definition of B

$$\operatorname{Re}\langle Bx, x\rangle = \operatorname{Re}\langle iAx, x\rangle = 0, \quad \forall x \in D(A).$$

It follows that the linear operators B and $B^* = -B$ are dissipative. The conclusion follows from Theorem 2.6.

(ii) Since $S(t)$ is a unitary transformation

$$|S(t)x|^2 = |x|^2, \tag{2.40}$$

and if $x \in D(B)$, B the infinitesimal generator of S, then by differentiating (2.40) at $t = 0$ we have $\operatorname{Re}\langle Bx, x\rangle = 0$ so that

$$\operatorname{Im}\langle iBx, x\rangle = \operatorname{Re}\langle Bx, x\rangle = 0$$

and iB is self-adjoint. □

2.7 Analytic semigroups

Assumption A *Let A be a closed operator with dense domain $D(A)$ in X. Assume that there exist $\omega \in \mathbb{R}$ and θ_0, $\pi/2 < \theta_0 < \pi$, such that*

A-1.

$\rho(A)$ contains a sector S_{ω,θ_0},

$$S_{\omega,\theta_0} = \{\lambda \in \mathbb{C} \colon \lambda \neq \omega \text{ and } |\arg(\lambda - \omega)| < \theta_0\},$$

A-2.

There exists $M > 0$ such that

$$\|R(\lambda, A)\| \leq \frac{M}{|\lambda - \omega|}, \quad \lambda \in S_{\omega,\theta_0}. \tag{2.41}$$

Now define the operator

$$S(t) = \frac{1}{2\pi i} \int_{\gamma_{\varepsilon,\theta}} e^{\lambda t} R(\lambda, A)\, d\lambda, \quad t > 0,\ S(0) = I, \tag{2.42}$$

where $\gamma_{\varepsilon,\theta}$ is the path (oriented in the increasing direction of $\operatorname{Im}\lambda$) defined in the following way (see Figure 1.1):

$$\begin{aligned}
\gamma_{\varepsilon,\theta} &= \gamma^+_{\varepsilon,\theta} \cup \gamma^-_{\varepsilon,\theta} \cup \gamma^0_{\varepsilon,\theta}, \quad \theta \in]\tfrac{\pi}{2}, \theta_0], \\
\gamma^\pm_{\varepsilon,\theta} &= \{z \in \mathbb{C} \colon z = \omega + \mathrm{r}e^{\pm i\theta}, r \geq \varepsilon\}, \\
\gamma^0_{\varepsilon,\theta} &= \{z \in \mathbb{C} \colon z = \omega + \varepsilon e^{\pm i\eta}, |\eta| \leq \theta\}.
\end{aligned}$$

We notice that the integral in (2.42) is convergent because $\theta > \pi/2$; moreover it does not depend on the choice of ε and θ as can easily be checked by using the Cauchy theorem for holomorphic functions.

Theorem 2.10 (E. Hille [1]**).** *Let A be a closed linear operator in X with dense domain $D(A)$ in X such that assumptions* A1–A2 *are verified and let S be defined by* (2.42). *Then the following statements hold*:

(i) *S is a strongly continuous semigroup with infinitesimal generator A.*

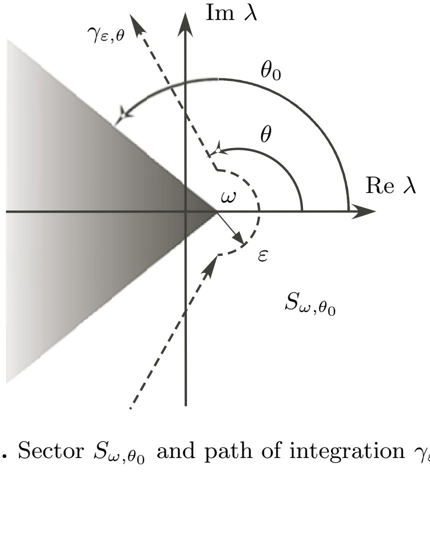

Fig. 1.1. Sector S_{ω,θ_0} and path of integration $\gamma_{\varepsilon,\theta}$.

(ii) *There exist* $M > 0$, $N > 0$ *such that*

$$\|S(t)\| \le Me^{\omega t}, \quad \forall t > 0, \tag{2.43}$$

$$\|(A - \omega I)S(t)\| \le \frac{N}{t} e^{\omega t}, \quad \forall t > 0, \tag{2.44}$$

$$S \in C^1\big(]0, \infty[; \mathcal{L}(X)\big) \quad \textit{and} \quad S'(t) = AS(t), \quad \forall t > 0. \tag{2.45}$$

(iii) S *has an analytic extension in a sector* $S_{0,\theta_0 - \pi/2}$ *and*

$$\begin{aligned} S(z) &= \frac{1}{2\pi i} \int_{\gamma_{\varepsilon,\theta+\pi/2}} e^{\lambda z} R(\lambda, A)\, d\lambda, \\ z &= \rho e^{i\theta}, \quad \theta \in [0, \theta_0 - \frac{\pi}{2}[\ \textit{ and } \rho \ge 0. \end{aligned} \tag{2.46}$$

Proof. We can assume that $\omega = 0$ (otherwise we change A to $A - \omega I$ and $S(t)$ to $e^{-\omega t}S(t)$). The proof is divided into five steps.

Step 1. $S \in C^1\big(]0, \infty[; \mathcal{L}(X)\big)$ and $S'(t) = AS(t)$, $t > 0$.

It is clear that, from (2.42), S is of class C^∞ in S_{ω,θ_0} and that

$$\begin{aligned} S'(t) &= \frac{1}{2\pi i} \int_{\gamma_{\varepsilon,\theta}} \lambda e^{\lambda t} R(\lambda, A)\, d\lambda \\ &= \frac{1}{2\pi i} \int_{\gamma_{\varepsilon,\theta}} e^{\lambda t} d\lambda + \frac{1}{2\pi i} \int_{\gamma_{\varepsilon,\theta}} A e^{\lambda t} R(\lambda, A)\, d\lambda = AS(t) \end{aligned}$$

because $\int_{\gamma_{\varepsilon,\theta}} e^{\lambda t} d\lambda = 0$.

Step 2. There exist $M > 0$ and $N > 0$ such that $\|S(t)\| \le M$, $\|AS(t)\| \le N/t$.

Setting $\lambda t = \xi$, (2.42) becomes

$$\begin{aligned} S(t) &= \frac{1}{2\pi i} \int_{t\gamma_{\varepsilon,\theta}} e^{\xi} R(\xi/t, A)\, d\xi/t = \frac{1}{2\pi i} \int_{\gamma_{\varepsilon,\theta}} e^{\xi} R(\xi/t, A)\, d\xi/t \\ &= \frac{1}{2\pi i} \bigg\{ \int_{\varepsilon}^{+\infty} \exp(\mathrm{r}e^{i\theta}) R(\mathrm{r}e^{i\theta}/t, A) e^{i\theta}\, dr/t \\ &\qquad - \int_{\varepsilon}^{+\infty} \exp(\mathrm{r}e^{-i\theta}) R(\mathrm{r}e^{-i\theta}/t, A) e^{-i\theta}\, dr/t \\ &\qquad\qquad + \int_{-\theta}^{\theta} \exp(\varepsilon e^{i\eta}) R(\varepsilon e^{i\eta}/t, A) i\varepsilon e^{i\eta}\, d\eta/t \bigg\} \end{aligned}$$

from which

$$\begin{aligned} \|S(t)\| &\le \frac{1}{2\pi} \left\{ 2 \int_{\varepsilon}^{\infty} M e^{r\cos\theta}\, dr/r + \int_{-\theta}^{\theta} M e^{\varepsilon \cos\eta}\, d\eta \right\}, \\ \|S'(t)\| &\le \frac{1}{2\pi t} \left\{ 2 \int_{\varepsilon}^{\infty} M e^{r\cos\theta}\, dr + \varepsilon \int_{-\theta}^{\theta} M e^{\varepsilon\cos\eta}\, d\eta \right\} \end{aligned}$$

as $\varepsilon \to 0$, we obtain

$$\|S'(t)\| \le \frac{M}{\pi t|\cos\theta|}.$$

Step 3. S is strongly continuous.

As $S(t)$ is bounded (by the second step), it is sufficient to prove that $\lim_{t\to 0} S(t)x = x$ for any $x \in D(A)$. Let $x \in D(A)$ and set $y = x - Ax$. Then $x = R(1,A)y$ and we have

$$\begin{aligned}
S(t)x = S(t)R(1,A)y &= \frac{1}{2\pi i}\int_{\gamma_{\varepsilon,\theta}} e^{\lambda t}R(\lambda,A)R(1,A)y\,d\lambda \\
&= \frac{1}{2\pi i}\int_{\gamma_{\varepsilon,\theta}} e^{\lambda t}R(\lambda,A)y\frac{d\lambda}{1-\lambda} - \frac{1}{2\pi i}\int_{\gamma_{\varepsilon,\theta}} e^{\lambda t}R(1,A)y\frac{d\lambda}{1-\lambda} \\
&= \frac{1}{2\pi i}\int_{\gamma_{\varepsilon,\theta}} e^{\lambda t}R(\lambda,A)y\frac{d\lambda}{1-\lambda}.
\end{aligned}$$

As $t \to 0$ we have

$$\lim_{t\searrow 0} S(t)x = \frac{1}{2\pi i}\int_{\gamma_{\varepsilon,\theta}} R(\lambda,A)y\frac{d\lambda}{1-\lambda} = R(1,A)y = x.$$

Step 4. $S(t+s) = S(t)S(s), t, s, > 0$.

We have

$$S(t)S(s) = -\frac{1}{4\pi^2}\int_{\gamma_{\varepsilon,\theta}} e^{\lambda t}R(\lambda,A)\,d\lambda \int_{\gamma_{2\varepsilon,\theta'}} e^{\mu s}R(\mu,A)\,d\mu,$$

where $\theta' \in\,]\pi/2, \theta[$. It follows, using the resolvent identity, that

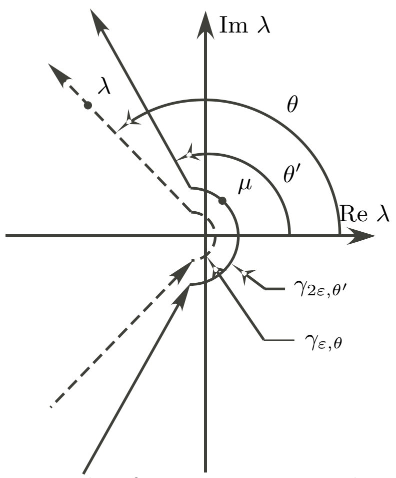

Fig. 1.2. Paths of integration $\gamma_{\varepsilon,\theta}$ and $\gamma_{2\varepsilon,\theta'}$.

$$
\begin{aligned}
S(t)S(s) &= -\frac{1}{4\pi^2}\int_{\gamma_{\varepsilon,\theta}\times\gamma_{2\varepsilon,\theta'}} e^{\lambda t+\mu s}[R(\lambda,A)-R(\mu,A)]\frac{d\lambda\, d\mu}{\mu-\lambda} \\
&= -\frac{1}{4\pi^2}\int_{\gamma_{\varepsilon,\theta}} e^{\lambda t}R(\lambda,A)\,d\lambda \int_{\gamma_{2\varepsilon,\theta'}} e^{\mu s}\frac{d\mu}{\mu-\lambda} \\
&\qquad\qquad + \frac{1}{4\pi^2}\int_{\gamma_{2\varepsilon,\theta'}} e^{\mu s}R(\mu,A)\,d\mu \int_{\gamma_{\varepsilon,\theta}} e^{\lambda t}\frac{d\lambda}{\mu-\lambda} = S(t+s)
\end{aligned}
$$

because

$$
\frac{1}{2\pi i}\int_{\gamma_{2\varepsilon,\theta'}} e^{\mu s}\frac{d\mu}{\mu-\lambda} = e^{\lambda t} \quad \text{and} \quad \frac{1}{2\pi i}\int_{\gamma_{\varepsilon,\theta}} e^{\lambda t}\frac{d\lambda}{\mu-\lambda} = 0.
$$

Step 5. A is the infinitesimal generator of S.

Let B be the infinitesimal generator of S. By the first step it follows that B is an extension of A. To prove that $B = A$ we follow the same argument as for the fourth step of the proof of Theorem 2.5.

Finally the proof of statement (iii) of the theorem is straightforward and it is left to the reader.

□

Theorem 2.11. *Let $\{S(t)\colon t \geq 0\}$ be a strongly continuous semigroup on X, A be its infinitesimal generator, and $\omega \in \mathbb{R}$ and $M > 0$ be such that*

$$
\|S(t)\| \leq Me^{\omega t}, \quad \forall t \geq 0. \tag{2.47}
$$

Then the following statements are equivalents:

(i) *A verifies the following conditions:*

$$
\exists\theta_0 > \frac{\pi}{2}, \quad \rho(A) \supset S_{\omega,\theta_0} = \{\lambda \in \mathbb{C}\colon |\arg(\lambda-\omega)| < \theta_0\} \tag{2.48}
$$

and

$$
\exists M > 0,\ \forall\theta \in\,]0,\theta_0[\,, \quad \|R(\lambda,A)\| \leq \frac{M}{|\lambda-\omega|},\ \forall\lambda \in S_{\omega,\theta_0}. \tag{2.49}
$$

(ii) *The map*

$$
t \mapsto S(t)\colon [0,\infty[\, \to \mathcal{L}(X) \tag{2.50}
$$

belongs to $C^1\big(]0,\infty[\,;\mathcal{L}(X)\big)$ and

$$
\exists N > 0, \quad \forall t > 0, \qquad \|(A-\omega I)S(t)e^{-\omega t}\| \leq \frac{N}{t}. \tag{2.51}
$$

(iii) *S has an analytic extension in a sector $S_{0,\theta'}$, $0 < \theta' < \pi/2$ and $e^{-\omega t}S(t)$ is bounded in every closed subsector of $S_{0,\theta'}$.*

Proof. We have already shown in Theorem 2.8 that (i) $\Longrightarrow$ (ii).

(ii) $\Longrightarrow$ (iii). Define $G(t) = e^{-\omega t} S(t)$ and $B = A - \omega I$. By hypothesis G is differentiable and for all $n \geq 1$,

$$G^{(n)}(t) = B^n G(t) = [BG(t/n)]^n = [G'(t/n)]^n,$$

and G necessarily belongs to $C^\infty\big(]0, \infty[\,; \mathcal{L}(X)\big)$. Now from (2.51)

$$\|G^{(n)}(t)\| \leq n^n N^n t^{-n}$$

and the series

$$F(z) = \sum_{n=0}^{\infty} \frac{(z-t)^n}{n!} G^{(n)}(t)$$

converges in the disk

$$C_t = \left\{ z \in \mathbb{C}\colon |z - t| < \frac{t}{Ne} \right\} \tag{2.52}$$

because the coefficients $a_n = (nN/t)^n/n!$ are such that

$$\frac{a_{n+1}}{a_n} = \frac{N}{t} \left[\frac{(n+1)}{n} \right]^n \to \frac{Ne}{t}.$$

By the Principle of Identity for analytic functions, one sees that F is analytic in the sector $S_{0,\theta} (\sin\theta = 1/eN)$, which is the envelope of the disks C_t, $t > 0$. As S is analytic in $S_{0,\theta}$ so is $S(t) = e^{\omega t} G(t)$. For any $\varepsilon, 0 < \varepsilon < \theta$, it is now easy to show that $F(z)$ is bounded in the closure of $S_{0,\theta-\varepsilon}$ and hence on any closed subsector of $S_{0,\theta}$.

(iii) $\Longrightarrow$ (i). Again it is sufficient to prove that for $\omega = -\varepsilon, \varepsilon > 0$, there exists $\omega \in \mathbb{R}$ and θ_0, $\pi/2 < \theta_0 < \pi$, for which assumptions A-1 and A-2 are verified in the sector S_{0,θ_0}. The resolvent operator is given by the identity

$$R(\lambda, A) = \int_0^\infty e^{-\lambda t} G(t)\, dt, \quad \forall \lambda,\ \operatorname{Re}\lambda > 0.$$

By hypothesis $S(z)$ is analytic in a sector $S_{0,\theta}$ for some $\theta, 0 < \theta < \pi/2$. Consider the sector

$$S_{0,\pi+\theta/2} = \left\{ \lambda \in \mathbb{C}\colon \lambda \neq 0, 0 < \arg\lambda < \frac{\theta + \pi}{2} \right\} \tag{2.53}$$

and observe that for $\lambda = |\lambda| e^{i\alpha}$ in $S_{0,(\pi+\theta)/2}$ ($\lambda \neq 0$ and $0 < \alpha < \theta + \pi/2$) one can get $R(\lambda, A)$ by integrating along the line $\{te^{-3i\theta/4}\colon t \geq 0\}$

$$R(\lambda, A) = \int_0^\infty e^{-\lambda t e^{-3i\theta/4}} S(te^{-3i\theta/4}) e^{-3i\theta/4}\, dt \tag{2.54}$$

because that line is contained in the sector $S_{0,\theta}$. We know that $S(z)$ is bounded in any subsector. Hence there exists $M > 0$ such that

$$\|R(\lambda, A)\| \leq \int_0^\infty e^{-t\,\mathrm{Re}[\lambda e^{-3i\theta/4}]} M\,dt. \tag{2.55}$$

But

$$\beta = \mathrm{Re}[\lambda e^{-3i\theta/4}] = |\lambda| \cos\left[\alpha - 3\frac{\theta}{4}\right]$$

and

$$-3\frac{\theta}{4} < \alpha - 3\frac{\theta}{4} < \frac{\pi}{2} - \frac{\theta}{4} \Longrightarrow \cos\left[\alpha - 3\frac{\theta}{4}\right] \geq \min\left\{\cos\left(3\frac{\theta}{4}\right), \sin\left(\frac{\theta}{4}\right)\right\} = \beta_0 > 0.$$

Therefore

$$\beta \geq |\lambda|\beta_0 > 0$$

and

$$\|R(\lambda, A)\| \leq \frac{M}{\beta_0|\lambda|}, \quad \forall\lambda \in S_{0,\pi+\theta/2}, \quad \mathrm{Im}\,\lambda \geq 0.$$

By repeating the same construction for $\mathrm{Im}\,\lambda \leq 0$ with the line $\{te^{-3i\theta/4} \colon t \geq 0\}$, we obtain the same inequality. Assumptions A-1 and A-2 are then verified. □

Definition 2.3. A strongly continuous semigroup $\{S(t) \colon t \geq 0\}$ on X is said to be analytic if it verifies any one of the conditions of Theorem 2.9. □

The variational case (which will be studied in detail in Chapter 2) is the following situation. There exists a Hilbert space V such that

$$V \subset H \text{ algebrically and topologically, and } V \text{ is dense in } H. \tag{2.56}$$

The space H is identified with its dual. So we cannot identify V', the dual of V, to V. On the other hand we have the sequence of embeddings

$$V \subset H \equiv H' \subset V', \tag{2.57}$$

each space being dense in the next one, with continuous injection. We denote by $\langle \cdot, \cdot \rangle$ the duality between V and V', and by $\|\cdot\|$ the norm in V (recall that $|\cdot|$ represents the norm and $(\cdot, \cdot)$ the scalar product in H).

Let a be a continuous bilinear form on V, which is V–H coercive

$$\exists\alpha > 0, \quad \exists\beta \in \mathbb{R}, \quad \forall v \in V, \qquad a(v, v) + \beta|v|^2 \geq \alpha\|v\|^2. \tag{2.58}$$

Associate with a the operator A

$$\begin{aligned} &D(A) = \{v \in V \colon w \mapsto a(v, w) \text{ is } H\text{-continuous}\}, \\ &(Av, w) = -a(v, w), \quad \forall v \in D(A), \quad \forall w \in V. \end{aligned} \tag{2.59}$$

In the variational literature it is customary to use $-A$ instead of A. The choice that we have made in (2.59) is more in line with *semigroup theory*.

We shall check that A generates an analytic semigroup and, thus, that there exists a unique classical solution of (2.1). Notice that the change of variable

$$y_\beta(t) = e^{-\beta t} y(t)$$

in the equation

$$\frac{dy}{dt} = Ay, \quad y(0) = y_0,$$

yields

$$\frac{dy_\beta}{dt} = (A - \beta I)y_\beta, \quad y_\beta(0) = y_0$$

and that

$$\langle(-A + \beta I)v, v\rangle_V = a(v, v) + \beta|v|^2 \geq \alpha\|v\|^2.$$

So it is sufficient to prove the result for $\beta = 0$.

Theorem 2.12. *Assume* (2.58) *holds; then A generates an analytic semigroup $\{S(t)\}$ in H such that*

$$\|S(t)\| \leq e^{\beta t}. \tag{2.60}$$

Proof. Cf. H. Tanabe [1, §5.4 and §3.6, Theorem 3.6.1]. □

2.8 Differentiable semigroups

Definition 2.4. Let S be a strongly continuous semigroup in X. We say that S is differentiable at $t_0 > 0$ if the limit

$$S'(t_0) = \lim_{h \searrow 0} \frac{1}{h}\big(S(t_0 + h) - S(t_0)\big) \tag{2.61}$$

exists in $\mathcal{L}(X)$. □

Proposition 2.6. *Let S be a strongly continuous semigroup with infinitesimal generator A. Assume that S is differentiable at t_0. Then the following statements hold*:

(i) *$S(t_0)x \in D(A)$ for any $x \in X$ and $S'(t_0) = AS(t_0)$,*
(ii) *S is differentiable at any $t \geq t_0$,*
(iii) *for any $n \in \mathbb{N}, S$ is n times differentiable at $t \geq nt_0$ and*

$$S^{(n)}(t) = A^n S(t), \quad t \geq nt_0.$$

Proof. (i) Due to Proposition 2.1 we have

$$S'(t_0)x = AS(t_0)x \quad \text{for all } x \in D(A).$$

Let now $x \in X$ and $\{x_n\} \subset D(A)$ be such that $x_n \to x$ in X. As $n \to \infty$ we have

$$AS(t_0)x_n = S'(t_0)x_n \to S'(t_0)x \quad \text{and} \quad S(t_0)x_n \to S(t_0)x.$$

As A is closed $S(t_0)x \in D(A)$ and $AS(t_0)x = S'(t_0)x$.
(ii) For any $t > t_0$ we have

$$\lim_{h \searrow 0} \frac{1}{h}\big(S(t+h) - S(t)\big) = \lim_{h \searrow 0} \frac{1}{h}[S(t_0+h) - S(t_0)]S(t-t_0) = S'(t_0)S(t-t_0).$$

(iii) We only consider the case $n = 2$; the case $n > 2$ is treated analogously by recurrence. We have

$$\begin{aligned} \lim_{h \searrow 0} \frac{1}{h}\big(S'(2t_0 + h) - S'(2t_0)\big) &= \lim_{h \searrow 0} AS(t_0)\frac{1}{h}\big(S(t_0 + h) - S(2t_0)\big) \\ &= AS(t_0)S'(t_0) = A^2 S(2t_0) \end{aligned}$$

and $S^{(2)}(2t_0) = A^2 S(2t_0)$. For $t > 2t_0$ we proceed as in part (ii). □

We shall now study some spectral properties of differentiable semigroups.

Proposition 2.7 (A. Pazy [1]). *Let S be a strongly continuous semigroup and assume that there exists a constant $M > 0$ such that for all $t \geq 0$, $\|S(t)\| \leq M$. If S is differentiable at $t_0 > 0$, then the resolvent set $\rho(A)$ of its infinitesimal generator A contains the set*

$$\Sigma = \{\lambda \in \mathbb{C} \colon \operatorname{Re}\lambda \leq 0, |\operatorname{Im}\lambda| e^{t_0 \operatorname{Re}\lambda} \geq 2\|S'(t_0)\|\}. \tag{2.62}$$

Moreover there exists $N > 0$ such that

$$\|R(\lambda, A)\| \leq N(1 + |\lambda|), \quad \lambda \in \Sigma. \tag{2.63}$$

Proof. For any $t \geq t_0$ we set

$$B_\lambda(t)x = \int_0^t e^{\lambda(t-s)} S(s)x\, ds, \quad \lambda \in \mathbb{C},\ x \in X. \tag{2.64}$$

We have

$$B'_\lambda(t)x = S(t)x + \lambda \int_0^t e^{\lambda(t-s)} S(s)x\, ds, \tag{2.65}$$

$$B_\lambda(t)(\lambda - A)x = e^{\lambda t}x - S(t)x. \tag{2.66}$$

By differentiating (2.66) with respect to t, we obtain

$$B'_\lambda(t)(\lambda - A)x = \lambda e^{\lambda t}x - S'(t)x = \lambda e^{\lambda t}\left[I - \frac{1}{\lambda}e^{-\lambda t}S'(t)\right]x. \tag{2.67}$$

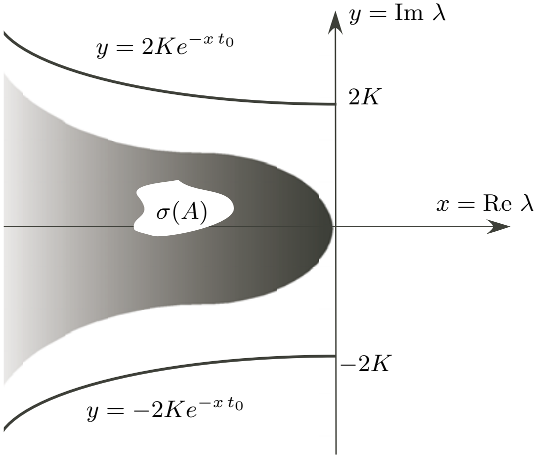

Fig. 1.3. Set Σ contained in $\rho(A)$ with $K = \|S'(t_0)\|$.

But if we choose t and λ such that

$$\left\| \frac{1}{\lambda} e^{-\lambda t} S'(t) \right\| \leq \frac{1}{2}, \tag{2.68}$$

then $[I - 1/\lambda e^{-\lambda t} S'(t)]^{-1}$ exists and is bounded. From (2.67) and (2.65)

$$\begin{aligned} R(\lambda, A)x &= \frac{1}{\lambda} e^{-\lambda t} \left[I - \frac{1}{\lambda} e^{-\lambda t} S'(t) \right]^{-1} B'_\lambda(t)x \\ &= \left[I - \frac{1}{\lambda} e^{-\lambda t} S'(t) \right]^{-1} \left\{ \frac{1}{\lambda} e^{-\lambda t} S(t)x + \int_0^t e^{-\lambda s} S(s)x \, ds \right\}. \end{aligned} \tag{2.69}$$

The right-hand side of the above identity is bounded and we conclude that for t and λ verifying (2.68) $\lambda \in \rho(A)$. But for $t = t_0$ and $\lambda \in \Sigma$, (2.68) is verified. Hence $\Sigma \subset \rho(A)$.

It remains to verify inequality (2.63). Set $K = \|S'(t_0)\|$. For $\lambda \in \Sigma$

$$\frac{1}{|\lambda|} \exp(-t_0 \operatorname{Re} \lambda) \leq \frac{K}{2}. \tag{2.70}$$

From (2.69) with $t = t_0$

$$\|R(\lambda, A)\| \leq \frac{2M}{|\lambda|} \exp(-t_0 \operatorname{Re} \lambda) + 2M \int_0^{t_0} \exp(-\operatorname{Re} \lambda s) \, ds.$$

But for all $\lambda \in \Sigma$, $\operatorname{Re} \lambda \leq 0$ and

$$\|R(\lambda, A)\| \leq 2M \exp(-t_0 \operatorname{Re} \lambda) \left[\frac{1}{|\lambda|} + t_0 \right].$$

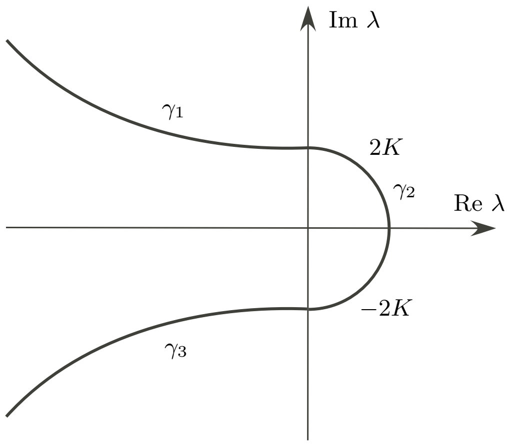

Fig. 1.4. Path of integration $\gamma_1 \cup \gamma_2 \cup \gamma_3$.

Therefore from (2.70) we finally find

$$\|R(\lambda, A)\| \leq MK[1 + t_0|\lambda|]. \qquad \square$$

We now prove the converse.

Theorem 2.13 (A. Pazy [1]). *Let A be the infinitesimal generator of a strongly continuous semigroup S for which there exists a constant $M > 0$ such that, for all $t > 0$, $\|S(t)\| \leq M$. Moreover assume that there exist three positive constants a, K, N, such that*

$$\rho(A) \supset \Sigma = \{\lambda \in \mathbb{C} \colon |\operatorname{Im}\lambda| e^{a \operatorname{Re}\lambda} \geq 2K\}, \tag{2.71}$$

$$\|R(\lambda, A)\| \leq N(1 + |\lambda|), \quad \lambda \in \Sigma. \tag{2.72}$$

Then S is differentiable for $t > 3a$ and we have for $t > 2a$

$$S(t) = \sum_{i=1}^{3} G_i(t) = \sum_{i=1}^{3} \frac{1}{2\pi i} \int_{\gamma_i} e^{\lambda t} R(\lambda, A)\, d\lambda, \tag{2.73}$$

where γ_i is defined by

$$\begin{aligned} \gamma_1 &= \{\lambda \in \mathbb{C} \colon \operatorname{Im}\lambda = 2Ke^{-a \operatorname{Re}\lambda}\}, \\ \gamma_2 &= \left\{\lambda \in \mathbb{C} \colon \lambda = 2Ke^{-i\theta}, \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right\}, \\ \gamma_3 &= \{\lambda \in \mathbb{C} \colon \operatorname{Im}\lambda = -2Ke^{-a \operatorname{Re}\lambda}\}. \end{aligned} \tag{2.74}$$

Proof. Set

$$\tilde{S}(t) = \sum_{i=1}^{3} \tilde{S}_i(t), \quad \tilde{S}_i(t) = \frac{1}{2\pi_i} \int_{\gamma_i} e^{\lambda t} R(\lambda, A)\, d\lambda, \quad i = 1, 2, 3. \tag{2.75}$$

The integrals in (2.75) are meaningful for $t > 2a$. To check this for the first one, set $\lambda = x + iy$. We have

$$\tilde{S}_1(t) = \frac{1}{2\pi i} \int_0^{-\infty} e^{tx+ity} R(x+iy, A)(1 - 2iKae^{-ax})\, dx, \tag{2.76}$$

and for $y = 2Ke^{-ax}$, we obtain

$$\|\tilde{S}_1(t)\| = \frac{N}{2\pi} \int_0^{-\infty} e^{tx}(1 + |x| + 2Kae^{-ax})\, dx, \tag{2.77}$$

which is meaningful if $t > 2a$.

Let us now show that $\tilde{S}_1(t)$ is differentiable for $t > 3a$. In fact we have

$$\frac{d}{dt}\tilde{S}_1(t) = \frac{1}{2\pi i} \int_0^{-\infty} (x + 2Kie^{-ax}) e^{tx+ity} R(x + 2Kie^{-ax}, A)(1 - 2iKae^{-ax})\, dx \tag{2.78}$$

from which

$$\left\| \frac{d}{dt}\tilde{S}_1(t) \right\| = \frac{N}{2\pi} \int_0^{-\infty} e^{tx}[1 + |x| + 2Ke^{-ax}] [|x| + 2Ke^{-ax}][1 + 2Kae^{-ax}]\, dx, \tag{2.79}$$

which is meaningful for $t > 3a$.

It remains to show that $S = \tilde{S}$. By a direct computation (using several times the argument employed to prove (2.62)) it is not difficult to check that

$$\tilde{S}'(t)x = A\tilde{S}(t)x, \text{ for any } x \in D(A^3).$$

As $D(A^3)$ is dense in X, by the uniqueness of the solution of the Cauchy problem (cf. Proposition 2.6), it follows that $S = \tilde{S}$. □

2.9 Spectral determining growth condition

Let A be the infinitesimal generator of a strongly continuous semigroup S of type $\omega_0(S)$. We set

$$s(A) = \begin{cases} \sup\{\operatorname{Re}\lambda\colon \lambda \in \sigma(A)\}, & \text{if } \sigma(A) \neq \emptyset, \\ -\infty, & \text{if } \sigma(A) = \emptyset. \end{cases} \tag{2.80}$$

If $s(A) = \omega_0(S)$ we say that A verifies the *spectral determining growth condition* or *spectrum determined growth assumption*. In this case, by the Corollary to Proposition 2.2, we can determine the asymptotic behavior of $\|S(t)\|$ by the knowledge of the spectrum $\sigma(A)$ of A.

We start with a general result.

Proposition 2.8. *Let A be the infinitesimal generator of a strongly continuous semigroup $\{S(t)\}$ on a Banach space X and let $\omega_0(S)$ denote the type of $\{S(t)\}$. Then*

$$\forall t > 0, \quad \{e^{\lambda t} \colon \lambda \in \sigma(A)\} \subset \sigma\big(S(t)\big), \tag{2.81}$$

and

$$s(A) \leq \omega_0(S). \tag{2.82}$$

Proof. Given an arbitrary λ, consider the new semigroup

$$S_\lambda(t)x = e^{-\lambda t} S(t)$$

with infinitesimal generator $A - \lambda I$. For all $x \in D(A)$,

$$[S(t) - e^{\lambda t} I]x = e^{\lambda t}[S_\lambda(t) - I]x,$$

$$[S(t) - e^{\lambda t} I]x = e^{\lambda t} \int_0^t S(r) e^{-\lambda r}[A - \lambda I]x\, dr, \tag{2.83}$$

$$[S(t) - e^{\lambda t} I]x = e^{\lambda t}[A - \lambda I] \int_0^t S(r) e^{-\lambda r} x\, dr. \tag{2.84}$$

If $\lambda \in \sigma_P(A)$, then $[A - \lambda I]$ is not one-to-one and necessarily, from (2.83), $[S(t) - e^{\lambda t} I]$ is not one-to-one. Hence $e^{\lambda t} \in \sigma_P\big(S(t)\big)$, $\forall t \geq 0$. If $\lambda \in \sigma_R(A) \cup \sigma_C(A)$, then $[A - \lambda I]$ is one-to-one but $R(A - \lambda I) \neq X$ and necessarily, from (2.84), $R([S(t) - e^{\lambda t} I]) \neq X$. So $e^{\lambda t} \in \sigma_P\big(S(t)\big) \cup \sigma_R\big(S(t)\big) \cup \sigma_C\big(S(t)\big)$, $\forall t \geq 0$. This proves (2.81).

To verify (2.82), we use the *spectral radius theorem*

$$\lim_{n \to \infty} \|S(1)^n\|^{1/n} = \sup\big\{|\lambda| \colon \lambda \in \sigma\big(S(1)\big)\big\}.$$

In particular from (2.81)

$$\sup\{|e^\lambda| \colon \lambda \in \sigma(A)\} \leq \sup\{|\lambda| \colon \lambda \in S(1)\} = \lim_{n \to \infty} \|S(n)\|^{1/n}.$$

Note that $|e^\lambda| = e^{\operatorname{Re} \lambda}$ and that the direction of the above inequality is not changed by taking the ln of both sides:

$$s(A) = \sup_{\lambda \in \sigma(A)} \operatorname{Re} \lambda \leq \lim_{n \to \infty} \frac{\ln \|S(n)\|}{n} = \omega_0(S).$$

This proves (2.82) and concludes the proof of the proposition. □

In general inequality (2.82) is not true in the other direction except for special classes of semigroups.

Proposition 2.9 (R. Triggiani [2]). *Let S be an analytic semigroup. Then $\omega_0(S) = s(A)$.*

Proof. Assume that (2.41) holds and that $\omega_0(S) = 0$. This is not a restriction because if $\omega_0(S) \neq 0$ we can change $S(t)$ to $e^{-\omega_0(S)t}S(t)$. Suppose now, by contradiction, that there exists $\varepsilon > 0$ such that $s(A) = -2\varepsilon$. By assumptions A-1 and A-2 in Assumption A there exists $\theta \in]\pi/2, \pi[$ such that $\rho(A)$ contains the sector $S_{\varepsilon,\pi/2}$ and by Cauchy's theorem

$$S(t) = \frac{1}{2\pi i}\int_{\gamma_+} e^{\lambda t}R(\lambda, A)\,d\lambda + \frac{1}{2\pi i}\int_{\gamma_-} e^{\lambda t}R(\lambda, A)\,d\lambda,$$

where

$$\gamma_\pm = \{\lambda \in C\colon \lambda = -\varepsilon + \rho e^{\pm i\eta}, \rho \geq 0\}$$

and η is chosen such that $\pi/2 < \eta < \theta$ and $\rho(A) \supset S_{-\varepsilon,\eta}$. Clearly $\|R(\lambda, A)\|$ is bounded in the sector $S_{-\varepsilon,\eta}$, and moreover, its behavior at infinity is as $1/|\lambda|$. Now it is easy to check that $\|S(t)\| \leq$ constant $e^{-\varepsilon t}$, which is a contradiction since $\omega_0(S) = 0$. □

The most general available results at the moment seem to be the following ones.

Proposition 2.10. (i) *If X is a Banach space and S is a strongly continuous semigroup that is eventually uniformly continuous, that is*

$$\exists t_0 > 0, \quad t \mapsto S(t)\colon [t_0, \infty[\to \mathcal{L}(X) \text{ is continuous},$$

then $s(A) = \omega_0(S)$.

(ii) *If X is a Banach lattice and $S(t_0)$ is compact for some $t_0 > 0$, then $s(A) = \omega_0(S)$.*

The proof of (i) can be found in G. Greiner and R. Nagel [1, p. 87] and the proof of (ii) in F. Neurander [1, p. 205].

Corollary 2.5. *If X is a Banach space, the condition $s(A) = \omega_0(S)$ is verified in any of the following situations*:

(i) *A is bounded,*
(ii) *$\exists t_0 > 0$, $S(t_0)$ is compact,*
(iii) *S is a differentiable semigroup,*
(iv) *S is nilpotent,*
(v) *S is analytic.*

Remark 2.9. The property $s(A) = \omega_0(S)$ was known for X a Hilbert space and A bounded (see I. Daleckii and M. Krein [1]). It was proved in 1975 for analytic semigroups by R. Triggiani [2, comments following Proposition 2.2] and for *eventually compact* semigroups (that is, there exists some $t_0 > 0$ for which $S(t)$ is compact for all $t \geq t_0$) by J. Zabczyk [2]. This last result was a generalization of what was already known for delay systems (cf. J. K. Hale [2]). The generalization to semigroups that are eventually uniformly continuous seems to be due to R. Nagel [1] and L. A. Monauni [1] in 1980–1981. For other conditions using the spectral mapping theorem see E. HILLE and R. S. PHILLIPS [1, Section 16.7]. Several counterexamples are available (for instance E. HILLE and R. S. PHILLIPS [1, Section 23.16] where $s(A) = -\infty$ and $\omega_0(S)$ is an arbitrary real number). This example is constructed in an indirect way using fractional integration. Another example has been given by J. ZABCZYK [2] where he shows that for any two real numbers $a < b$, one can construct a complex group $S(t)$ with generator A such that

$$s(A) = a < b = \omega_0(S).$$

One question that remains open is to determine whether a similar example can be constructed for a real group or semigroup. □

2.10 Examples of semigroups

2.10.1 Parabolic equations

Let H be a Hilbert space and A a self-adjoint closed operator in H with dense domain $D(A)$ in H. We assume that there exists $\omega \in \mathbb{R}$ such that

$$\langle Ax, x\rangle \le \omega |x|^2, \quad \forall x \in D(A). \tag{2.85}$$

By Lumer–Phillips' theorem, A is the infinitesimal generator of a strongly continuous semigroup S in H such that

$$\|S(t)\| \le e^{\omega t}, \quad t \ge 0. \tag{2.86}$$

Proposition 2.11. *S is an analytic semigroup.*

Proof. First notice that, as $A - \omega I$ is self-adjoint negative, then $\sigma(A) \subset]-\infty, \omega]$. Then for any $\lambda \in \mathbb{C}\backslash]-\infty,\omega]$ and any $y \in H$ there exists $x \in D(A)$ such that

$$\lambda x - Ax = y. \tag{2.87}$$

Set $\lambda = \omega + \rho e^{i\theta}$. Then multiply the identity $\rho e^{i\theta} x - (A-\omega)x = y$ by $e^{-i\theta/2}x$ and take the real part. We obtain

$$\cos\frac{\theta}{2}|x|^2 - \cos\frac{\theta}{2}\langle (A-\omega I)x, x\rangle = \mathrm{Re}[e^{-i\theta/2}\langle x, y\rangle].$$

From (2.85) it follows that $|x| \le (\rho\cos\theta/2)^{-1}|y|$ so that

$$\|R(\lambda, A)\| \le \frac{1}{|\lambda - \omega|\cos\theta/2}.$$

So pick a sector S_{ω,θ_0}, for some θ_0, $\pi/2 < \theta_0 < \pi$, and $\theta, 0 \le \theta < \theta_0$. Then $\cos(\theta/2) > \cos(\theta_0/2)$ and

$$\|R(\lambda, A)\| \le \frac{1}{|\lambda - \omega|\cos(\theta_0/2)} \quad \text{for all } \lambda \quad \text{in } S_{\omega,\theta_0}.$$

Thus S is an analytic semigroup. □

Consider now an example.

Let Ω be an open set of $\mathbb{R}^n$ with regular boundary $\partial\Omega$. Consider the initial value problem for $t \geq 0$

$$\begin{aligned} \frac{\partial}{\partial t}u(t,x) &= \sum_{j,k=1}^{n} \frac{\partial}{\partial x_k}\left\{a_{jk}(x)\frac{\partial}{\partial x_j}u(t,x)\right\}, \quad x \in \Omega, \\ u(t,x) &= 0, \quad x \in \partial\Omega, \\ u(0,x) &= u_0(x), \quad x \in \Omega, \end{aligned} \tag{2.88}$$

where a_{jk} are real continuous functions in $\bar{\Omega}$. Assume that there exists $\nu > 0$, such that (strong ellipticity)

$$\sum_{j,k=1}^{n} a_{jk}\xi_j\xi_k \geq \nu|\xi|^2, \quad \xi \in \mathbb{R}^n. \tag{2.89}$$

Set $H = L^2(\Omega)$ and let A be the linear operator in H defined as

$$\begin{aligned} D(A) &= H^2(\Omega) \cap H_0^1(\Omega), \\ (Au)(x) &= \sum_{j,k=1}^{n} \frac{\partial}{\partial x_k}\left\{a_{jk}(x)\frac{\partial}{\partial x_j}u(t,x)\right\}. \end{aligned} \tag{2.90}$$

Then A is self-adjoint and hypothesis (2.85) is fulfilled with $\omega = \lambda_0$, where λ_0 is the first eigenvalue of A (see for instance S. Agmon [2]).

By setting $u(t) = u(t,\cdot)$, problem (2.88) can be written as

$$u' = Au, \quad u(0) = u_0$$

and we can solve it by Proposition 2.1. We shall give details in §3 for a more general situation.

Remark 2.10. If A is an elliptic operator of order $2m$, with general boundary conditions, then, under suitable hypotheses, A generates an analytic semigroup (see S. Agmon [1]). □

2.10.2 Schrödinger equation

Let Ω be as in the previous examples. Consider for $t \geq 0$ the problem

$$\begin{aligned} \frac{\partial}{\partial t}u(t,x) &= i\sum_{j,k=1}^{n} \frac{\partial}{\partial x_k}\left\{a_{jk}(x)\frac{\partial}{\partial x_j}u(t,x)\right\}, \quad x \in \Omega, \\ u(t,x) &= 0, \quad x \in \partial\Omega, \\ u(0,x) &= u_0(x), \quad x \in \Omega, \end{aligned} \tag{2.91}$$

then, setting $B = iA$ (where A is defined by (2.90)), we can write problem (2.91) in the form

$$u' = Bu, \quad u(0) = u_0. \tag{2.92}$$

By Stone's Theorem 2.9, B generates a contraction semigroup and problem (2.91) has a solution.

2.10.3 Wave equation

Let Λ be a strictly positive self-adjoint operator in the Hilbert space K. Consider the Hilbert space $H = D(\sqrt{\Lambda}) \oplus K$ and denote by

$$Y = \begin{bmatrix} y \\ y_1 \end{bmatrix}$$

the generic element of H. The inner product in H is defined by

$$\langle Y, Z \rangle_H = \langle \sqrt{\Lambda} y, \sqrt{\Lambda} x \rangle_K + \langle y_1, z_1 \rangle_K. \tag{2.93}$$

Let A be the linear operator in H defined as follows

$$\begin{aligned} &D(A) = D(\Lambda) \oplus D(\sqrt{\Lambda}), \\ &AY = \begin{bmatrix} y_1 \\ -\Lambda y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\Lambda & 0 \end{bmatrix} \begin{bmatrix} y \\ y_1 \end{bmatrix}. \end{aligned} \tag{2.94}$$

Proposition 2.12. *A is the infinitesimal generator of a strongly continuous semigroup of contractions S on H, and S is given by*

$$S(t) = \begin{bmatrix} \cos(\sqrt{\Lambda} t\ ) & \Lambda^{1/2} \sin(\sqrt{\Lambda} t) \\ -\Lambda^{1/2} \sin(\sqrt{\Lambda} t) & \cos(\sqrt{\Lambda} t) \end{bmatrix}. \tag{2.95}$$

Moreover $A^* = -A$.

Finally assume that $\sigma(\Lambda)$ consists of a sequence $\{\lambda_k\}$ of eigenvalues and that $\{e_k\}$ is a corresponding complete orthonormal set of eigenvectors. Then we have

$$\sigma(A) = \{\pm i \sqrt{\lambda_k}\}, \quad AE_k^{\pm} = \pm i \sqrt{\lambda_k} E_k^{\pm}, \quad k \in N, \tag{2.96}$$

where $E_k^{\pm} = (e_k, \pm\sqrt{\lambda_k}\, e_k)$.

Proof. Let $\operatorname{Re} \lambda > 0$, $Y \in H$, then the equation $\lambda X - AX = Y$ is equivalent to the system

$$\begin{cases} \lambda x - x_1 = y, \\ \lambda x_1 + \Lambda x = y_1, \end{cases}$$

whose solution is given by

$$\begin{cases} x = \lambda(\lambda^2 + \Lambda)^{-1} y + (\lambda^2 + \Lambda)^{-1} y_1, \\ x_1 = -\Lambda(\lambda^2 + \Lambda)^{-1} y + \lambda(\lambda^2 + \Lambda)^{-1} y_1. \end{cases}$$

Thus $\lambda \in \rho(A)$ and

$$R(\lambda, A) = \begin{bmatrix} \lambda(\lambda^2 + \Lambda)^{-1} & (\lambda^2 + \Lambda)^{-1} \\ -\Lambda(\lambda^2 + \Lambda)^{-1} & \lambda(\lambda^2 + \Lambda)^{-1} \end{bmatrix}. \tag{2.97}$$

Formula (2.95) and the identity $-A = A^*$ are easily verified as well as the last statement. □

We now give an example. Let Ω be a bounded set of $\mathbb{R}^n$ with regular boundary $\partial\Omega$, and let $\{a_{jk}\}$ be real continuously differentiable functions in $\bar{\Omega}$ such that (2.89) holds. Consider the following problem for $t \geq 0$:

$$\begin{aligned} \frac{\partial^2}{\partial t^2} u(t,x) &= \sum_{j,k=1}^{n} \frac{\partial}{\partial x_k} \left\{ a_{jk}(x) \frac{\partial}{\partial x_j} u(t,x) \right\}, \quad x \in \Omega, \\ u(t,x) &= 0, \quad x \in \partial\Omega, \\ u(0,x) = u_0(x), &\quad \frac{\partial}{\partial t} u(0,x) = u_1(x), \quad x \in \Omega. \end{aligned} \tag{2.98}$$

Let Λ be the linear operator in $K = L^2(\Omega)$ defined by

$$\begin{aligned} D(\Lambda) &= H^2(\Omega) \cap H_0^1(\Omega), \\ (\Lambda u)(x) &= -\sum_{j,k=1}^{n} \frac{\partial}{\partial x_k} \left[a_{jk}(x) \frac{\partial}{\partial x_j} u(t,x) \right]. \end{aligned} \tag{2.99}$$

Then Λ is self-adjoint and strictly positive. Setting

$$U(t) = \begin{bmatrix} u(t,\cdot) \\ \dfrac{\partial}{\partial t} u(t,\cdot) \end{bmatrix},$$

problem (2.98) is equivalent to

$$U' = AU, \quad U(0) = U_0 = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \tag{2.100}$$

and can be solved by Proposition 2.12.

2.10.4 Delay equations

Let C and D be given in $\mathcal{L}(\mathbb{C}^n)$ and consider the problem

$$\begin{aligned} z'(t) &= Cz(t) + Dz(t-r), \quad t \geq 0, \\ z(0) &= h_0 \in \mathbf{C}^n, \\ z(\theta) &= h_1(\theta), \quad \text{a.e. } \theta \in [-r, 0], \end{aligned} \tag{2.101}$$

where $h_1 \in L^2(-r, 0; \mathbb{C}^n)$ and $r > 0$ is the *delay*.

Problem (2.101) can be easily solved in successive steps on each time interval of length r. We have

$$z(t) = \begin{cases} h_1(t) & \text{if } t \in [-r, 0[, \\ e^{tC}h_0 + \int_0^t e^{(t-s)C} Dh_1(s-r)\,ds & \text{if } t \in [0, r], \\ e^{(t-r)C}z(r) + \displaystyle\int_r^t e^{(t-s)C} Dz(s-r)\,ds & \text{if } t \in [r, 2r], \end{cases} \tag{2.102}$$

and so on.

We consider the Hilbert space $H = \mathbb{C}^n \times L^2(-r, 0; \mathbb{C}^n)$ and denote by $h = (h_0, h_1)$ its generic element. We define a semigroup S on H by setting

$$S(t)h = (z(t), z_t), \quad h = (h_0, h_1) \in H, \tag{2.103}$$

where z is the solution of (2.101) and $z_t(\theta) = z(t+\theta)$ for $\theta \in [-r, 0]$. The semigroup properties $S(t+s) = S(t)S(s)$, $S(0) = I$ are easily checked.

Proposition 2.13. *S is a strongly continuous semigroup in H that is differentiable for any $t \geq r$. The infinitesimal generator A of S is given by*

$$\begin{gathered} D(A) = \{h = (h_0, h_1) \in H : h_1 \in W^{1,2}(-r, 0; \mathbb{C}^n),\ h_0 = h_1(0)\} \\ Ah = (Ch_1(0) + Dh_1(-r), h_1') \end{gathered} \tag{2.104}$$

and we have

$$\sigma(A) = \{\lambda \in C\colon\ \det(\lambda - C - e^{-\lambda r}D) = 0\}. \tag{2.105}$$

Proof. In several steps.

Step 1. S is strongly continuous.

For any $h = (h_0, h_1) \in H$, we have $S(t)h - h = (z(t) - h_0, z_t - h_1)$. But if $-r < \theta < -t$

$$(z_t - h_1)(\theta) = h_1(t+\theta) - h_1(\theta),$$

and if $\theta \geq -t$

$$(z_t - h_1)(\theta) = e^{(t+\theta)C}h_0 + \int_0^{t+\theta} e^{(t+\theta-s)C} Dh_1(s-r)ds - h_1(\theta).$$

Thus we can easily check that $S(t)h \to h$ as $t \geq 0$.

Step 2. $S(t)$ is differentiable for $t \geq r$.

Let $t \in [r, 2r]$, $h = (h_0, h_1)$; then $S(t)h = (z(t), z_t)$ where

$$z(t) = e^{tC}h_0 + \int_0^t e^{(t-s)C} Dz(s-r)\,ds,$$

$$z_t(\theta) = e^{(t+\theta)C}h_0 + \int_0^{t+\theta} e^{(t+\theta-s)C} Dz(s-r)\,ds.$$

It follows that $S'(t)h = (z'(t), (z_t)')$, where

$$z'(t) = Cz(t) + Dz(t-r), \quad (z_t)'(\theta) = Cz_t(\theta) + Dz(t+\theta-r).$$

Thus S is differentiable for $t \geq r$.

Step 3. Let B be the infinitesimal generator of $S(t)$. We first show that $A \subset B$. Indeed A is given by formula (2.104).

Let $h = (h_0, h_1)$ belong to the right-hand side of (2.104). Since $h_1 \in W^{1,2}(-r, 0; \mathbb{C}^n)$ and $h_1(0) = h_0 = z(0)$, the new function

$$w(t) = \begin{cases} z(t), & t \in [0, T], \\ h_1(t), & t \in [-r, 0] \end{cases}$$

belongs to $W^{1,2}(-r, T; \mathbb{C}^n)$. For each $t \in [0, T]$, let

$$w_t(\theta) = w(t+\theta), \quad \theta \in [-r, 0].$$

Then

$$\begin{aligned} \frac{S(t)h - h}{t} &= \frac{(w(t), w_t) - (h_0, h_1)}{t} \\ &= \left(\frac{w(t) - w(0)}{t}, \frac{w_t - w_0}{t} \right) \\ &\qquad \to (w'(0), w_0') = (Ch_1(0) + Dh_1(-r), h_1') \end{aligned}$$

in $\mathbb{C}^n \times L^2(-r, 0; \mathbb{C}^n)$. Hence we have proved that if $h \in D(A)$ then $(S(t)h - h)/t$ goes to Ah as t goes to 0, Therefore $B \supset A$. Finally we can check that for λ sufficiently large, we have $(\lambda - A)\big(D(A)\big) = H$ so that $A = B$.

Step 4. Equation (2.105) holds.

It is sufficient to prove that $\lambda \in \rho(A)$ if and only if the matrix

$$\Delta(\lambda) = \lambda I - C - e^{\lambda r} D$$

is invertible. The complex number λ belongs to $\rho(A)$ if and only if

$$\forall k = (k_0, k_1) \in H, \quad \exists h \in D(A), \text{ such that } \lambda h - Ah = k. \tag{2.106}$$

Given $k = (k_0, k_1) \in H$, the equation $\lambda h - Ah = k$ is equivalent to the system

$$\lambda h_0 - Ch_0 - Dh_1(-r) = k_0, \tag{2.107}$$

$$\lambda h_1 - h_1' = k_1, \quad h_1(0) = h_0. \tag{2.108}$$

This is also equivalent to

$$h_1(\theta) = e^{\lambda\theta} h_0 + \int_\theta^0 e^{\lambda(\theta - s)} k_1(s)\, ds, \quad \theta \in [-r, 0], \tag{2.109}$$

$$\lambda h_0 - Ch_0 - Dh_1(-r) = k_0.$$

By substituting the value of $h_1(-r)$ given by (2.109) in the last equation, we obtain

$$(\lambda - C - e^{-\lambda r} D) h_0 = k_0 + D \int_{-r}^{0} e^{-\lambda(r+s)} k_1(s)\, ds. \tag{2.110}$$

Thus, if (2.106) is true, then for each $k = (k_0, k_1)$ in H, (2.110) has a unique solution h_0. Therefore $\Delta(\lambda)$ is invertible. Conversely if $\Delta(\lambda)$ is invertible, then for each k there exists a unique h_0 given by (2.109) and a unique h_1 given by (2.109) such that (2.106) be verified. □

The following corollary is an immediate consequence of Proposition 2.9.

Corollary 2.6. *The type of S is given by*

$$\omega_0(S) = s(A) = \sup\{\operatorname{Re}\lambda\colon\ \det(\lambda - C - e^{-\lambda r} D) = 0\}. \tag{2.111}$$

Remark 2.11. For more general examples and results for delay systems, see Chapter 4. □

3 Nonhomogeneous linear evolution equations

3.1 Setting of the problem and definitions

In this section A represents the infinitesimal generator of a strongly continuous semigroup S in X. We shall consider the following initial value problem:

$$\begin{cases} u'(t) = Au(t) + f(t), \quad t \in [0, T], \\ u(0) = x, \end{cases} \tag{3.1}$$

for $x \in X$ and $f \in L^1(0, T; X)$ and approximations u_n to its solution u given by the following approximating initial value problem:

$$\begin{cases} u_n'(t) = A_n u_n(t) + f(t), \\ u_n(0) = x, \end{cases} \tag{3.2}$$

where $A_n = n^2 R(n, A) - nI$ is the *Yosida approximation* of A. As $A_n \in \mathcal{L}(X)$, problem (3.2) has a unique solution given by

$$u_n(t) = e^{tA_n} x + \int_0^t e^{(t-s)A_n} f(s)\, ds. \tag{3.3}$$

We now give several definitions of a solution of problem (3.1) for $x \in X$ and $f \in L^1(0, T; X)$. The concept of solution for rougher data (x, f) will be discussed in Chapters 2 and 3.

Definition 3.1. (i) u is a *strict solution* of problem (3.1) in $L^p(0,T;X)$ if u belongs to $W^{1,p}(0,T;X) \cap L^p\big(0,T;D(A)\big)$ and

$$\begin{cases} u'(t) = Au(t) + f(t) & \text{a.e. in } [0,T], \\ u(0) = x. \end{cases}$$

(ii) u is a *strong solution* of problem (3.1) in $L^p(0,T;X)$ if there exists a sequence $\{u_k\}$ in $W^{1,p}(0,T;X) \cap L^p\big(0,T;D(A)\big)$ such that

$$\begin{gathered} u_k \to u, \quad \text{and} \quad u_k' - Au_k \to f \quad \text{in } L^p(0,T;X), \\ u_k(0) \to x \quad \text{in } X, \text{ as } k \to \infty. \end{gathered} \tag{3.4}$$

(iii) u is a *classical solution* of problem (3.1) in $L^p(0,T;X)$ if for all $\varepsilon > 0$ $u \in W^{1,p}(\varepsilon,T;X) \cap L^p\big(\varepsilon,T;D(A)\big) \cap C([0,T];X)$ and

$$\begin{cases} u'(t) = Au(t) + f(t) & \text{a.e. in } [0,T], \\ u(0) = x. \end{cases}$$

(iv) The function

$$u(t) = S(t)x + \int_0^t S(t-s)f(s)\,ds$$

is called the *mild solution* of problem (3.1) if u belongs to $C([0,T];X)$.

(v) u is a *weak solution* of problem (3.1) if u belongs to $L^p(0,T;X)$, for all k in $D(A^*)$, $\langle k, u(\cdot)\rangle_X$ belongs to $W^{1,p}(0,T)$, and for almost all t in $[0,T]$ and k in $D(A^*)$

$$\begin{cases} \dfrac{d}{dt}\langle k, u(t)\rangle = \langle A^*k, u(t)\rangle + \langle k, f(t)\rangle, \\ u(0) = x, \end{cases}$$

where $A^* \colon D(A^*) \to X'$ is the dual operator of A. □

Remark 3.1. The notion of weak solution for arbitrary Banach spaces was introduced by J. Ball [1] with the difference that he requires that u belongs to $C([0,T];X)$ rather than $L^p(0,T;X)$. □

When X is a reflexive Banach space, strong solutions can be characterized in a slightly different way. Recall that for a reflexive Banach space, the family of adjoint transformations $\{S^*(t) \in \mathcal{L}(X') \colon t \geq 0\}$ is a strongly continuous semigroup on X' and that the domain $D(A^*)$ of its infinitesimal generator A^* is dense in X'. The injection $D(A^*) \subset X'$ is continuous and dense when $D(A^*)$ is endowed with the graph norm topology defined by the norm

$$|x|_{D(A^*)} = |x|_{X^*} + |A^*x|_{X'}.$$

As a result the injection $X \equiv X'' \subset D(A^*)'$ is also continuous and dense since the elements of the bidual X'' of X can be identified with those of X.

Remark 3.2. When X is a reflexive Banach space A^* is the generator of the adjoint semigroup $S^*(t)$ and the notion of weak solution is equivalent to the existence of a function u in $W^{1,p}(0,T;D(A^*)') \cap L^p(0,T;X)$ such that

$$\begin{cases} u' = (A^*)^* u + f \quad \text{in } L^p(0,T;D(A^*)'), \\ u(0) = x \quad \text{in } X. \end{cases}$$

This is the usual setup for parabolic equations in the variational framework, where the operator $-A$ arises from a variational problem. □

3.2 Existence and uniqueness of a strong solution

Proposition 3.1. *Assume that $x \in X$ and $f \in L^p(0,T;X)$. Then problem* (3.1) *has a unique strong solution u in $L^p(0,T;X)$. Moreover u belongs to $C([0,T];X)$ and is given by the formula*

$$u(t) = S(t)x + \int_0^t S(t-s)f(s)\,ds. \tag{3.5}$$

Finally if u_n is the solution of problem (3.2)*, then*

$$u_n \to u \text{ in } L^p(0,T;X) \text{ as } n \to \infty.$$

Proof. Let u be the function defined by (3.5). Clearly u is continuous. To check that u is a strong solution of (3.1), set

$$u_k(t) = kR(k,A)u(t), \quad f_k(t) = kR(k,A)f(t), \quad x_k = kR(k,A)x.$$

By (3.5) it follows that

$$u_k(t) = S(t)x_k + \int_0^t S(t-s)f_k(s)\,ds;$$

by differentiating with respect to t we see that u_k is a strict solution of the problem

$$\begin{cases} u_k'(t) = Au_k(t) + f_k(t), \\ u_k(0) = x_k. \end{cases}$$

By (2.28) in Theorem 2.5 it follows that $u_k \to u$, $f_k \to f$ in $L^p(0,T;X)$ and $x_k \to x$ in X. Thus u is a strong solution.

We now prove uniqueness. Let u be a strong solution of (3.1) and let $\{u_k\}$ be a sequence such that (3.4) holds. We set $f_k = u_k' - Au_k$ and $x_k = u_k(0)$. By integrating the equation

$$\frac{d}{ds}\big(S(t-s)u_k(s)\big) = S(t-s)f_k(s), \quad s \in [0,t]$$

between 0 and t and letting k go to infinity, we find that u is given by (3.5).

As for the last statements we have

$$|u(t) - u_n(t)| \le |S(t)x - e^{tA_n}x| + \int_0^t |S(t-s)f(s) - e^{(t-s)A_n}f(s)|\,ds$$

and the conclusion follows from Theorem 2.5. □

Remark 3.3. In the literature a strong solution is generally called a mild solution. We shall also use this terminology in Parts IV and V. □

Proposition 3.2. *Given $x \in X$ and $f \in L^p(0,T;X)$, $1 \le p < \infty$, problem* (3.1) *has a unique weak solution u that coincides with the strong solution given by* (3.5).

Existence.

Let u be given by (3.5). By definition $u \in C([0,T];X) \subset L^p(0,T;X)$. Given k in $D(A^*)$ and ϕ in $\mathcal{D}(]0,T[)$, the vectorial distributional derivative of u is given by

$$\begin{aligned}
-\int_0^T \langle k, u(t)\rangle_{D(A^*)}\phi'(t)\,dt &= -\int_0^T \left[\left\langle k, S(t)k + \int_0^t S(t-s)f(s)\,ds\right\rangle\right]\phi'(t)\,dt \\
&= -\int_0^T \langle k, S(t)k\rangle\phi'(t)\,dt \\
&\qquad -\int_0^T ds\int_s^T dt\langle k, S(t-s)f(s)\rangle\phi'(t).
\end{aligned}$$

Now for all x in $D(A)$ and k in $D(A^*)$

$$\frac{d}{dt}\langle k, S(t)x\rangle = \langle k, AS(t)x\rangle_X = \langle A^*k, S(t)x\rangle_X$$

and by density of $D(A)$ in X, this extends to all x in X. In view of this technical result, we can now integrate by parts in the previous equations

$$-\int_0^T \langle k, S(t)k\rangle\phi'(t)\,dt = -\int_0^T \langle A^*k, S(t)x\rangle\phi(t)\,dt$$

and

$$-\int_s^T \langle k, S(t-s)f(s)\rangle\phi'(t)\,dt = \int_s^T \langle A^*k, S(t-s)f(s)\rangle\phi(t)\,dt + \langle k, f(s)\rangle\phi(s).$$

So finally

$$-\int_0^T \langle k, u(t)\rangle_{D(A^*)}\phi'(t)\,dt = \int_0^T \langle A^*k, u(t)\rangle + \langle k, f(t)\rangle]\phi(t)\,dt.$$

As $u \in C([0,T];X)$ and $f \in L^p(0,T;X)$, $\langle k, u(\cdot)\rangle$ belongs to $W^{1,p}(0,T)$ and for each t in $[0,T]$

$$\frac{d}{dt}\langle k, u(t)\rangle = \langle A^*k, u(t)\rangle + \langle k, f(t)\rangle.$$

So u is a weak solution of problem (3.1).

Uniqueness.

It is sufficient to prove that $u = 0$ when $x = 0$ and $f = 0$. By definition of a weak solution for all k in $D(A^*)$ and t in $[0, T]$

$$\frac{d}{dt}\langle k, u(t)\rangle = \langle A^* k, u(t)\rangle$$

and

$$\lim_{t \searrow 0}\langle k, u(t)\rangle = 0.$$

Therefore

$$\langle k, u(t)\rangle = \left\langle A^* k, \int_0^t u(s)\, ds \right\rangle = \langle A^* k, z(t)\rangle,$$

where

$$z(t) = \int_0^t u(s)\, ds.$$

We need the following lemma.

Lemma 3.1 (Cf. S. Goldberg [1, p. 127]**).** *Assume that A is a closed linear densely defined operator on X. Let u and z in X satisfy*

$$\langle k, u\rangle = \langle A^* k, z\rangle, \quad \forall k \in D(A^*).$$

Then $z \in D(A)$ and $u = Az$.

In view of the lemma for each t, $z(t) \in D(A)$ and $u(t) = Az(t)$. Moreover since $u \in L^p(0, T; X)$, $z \in L^p\big(0, T; D(A)\big)$ and

$$\frac{dz}{dt}(t) = u(t) = Az(t) \in L^p(0, T; X).$$

Therefore $z \in L^p\big(0, T; D(A)\big) \cap W^{1,p}(0, T; X)$ and

$$\begin{cases} \dfrac{dz}{dt}(t) = Az(t), & t \in [0, T], \\ z(0) = 0. \end{cases}$$

So z is a strict and hence a strong solution. By Proposition 3.1 this solution is unique and equal to zero. The uniqueness now follows from the fact that $u(t) = Az(t) = 0$ for all t in $[0, T]$.

Remark 3.4. The key points of the above proof have been borrowed from J. Ball [1]. We see that for initial conditions x in X and right-hand-sides f in $L^p(0, T; X)$ the two definitions of weak solutions coincide. In fact J. Ball [1] proved the following theorem for a closed linear densely defined operator A on X: For each $x \in X$, there exists a unique weak solution $u(t)$ of (3.1) $\big(u \in C(0, T]; X)\big)$ satisfying $u(0) = x$ if and only if A is the generator of a strongly continuous semigroup $\{S(t)\}$ on X. □

Remark 3.5. The last two propositions show the equivalence of a solution in $L^p(0,T;X)$ and a strong solution that is the limit in $C([0,T];X)$ of strict solutions. For reflexive Banach spaces the application

$$u \to \big(u' - (A^*)^* u, u(0)\big)$$
$$: W^{1,p}\big(0,T;D(A^*)'\big) \cap C([0,T];X) \to L^p(0,T;D(A^*)') \times X$$

is injective and invertible on the subspace

$$L^p(0,T;X) \times X \quad \text{of } L^p(0,T;D(A^*)') \times X.$$

So it is not an isomorphism, but its image is at least dense. □

3.3 Existence of a strict solution

In order to obtain a strict solution of problem (3.1), we need more regularity in x and f.

Proposition 3.3. (i) *If $x \in D(A)$ and $f \in L^p\big(0,T;D(A)\big)$ then problem* (3.1) *has a unique strict solution that belongs to $W^{1,p}(0,T;X) \cap C\big([0,T];D(A)\big)$.*
(ii) *If $x \in D(A)$ and $f \in W^{1,p}(0,T;X)$, then problem* (3.1) *has a unique strict solution that belongs to $C^1([0,T];X) \cap C\big([0,T];D(A)\big)$.*

Proof. (i) Let u be the strong solution of problem (3.1) and let the solution u_n of (3.2) be its associated approximating sequence. By hypotheses the problem

$$\begin{cases} v'(t) = Av(t) + Af(t), \\ v(0) = Ax \end{cases} \tag{3.6}$$

has a strong solution $v \in C([0,T];X)$. Let v_n be the solution of the problem

$$\begin{cases} v_n'(t) = A_n v_n(t) + A_n f(t), \\ v_n(0) = A_n x. \end{cases} \tag{3.7}$$

Clearly, $v_n = A_n u_n$. By Proposition 3.1, we have

$$u_n \to u, \quad A_n u_n = v_n \to v \quad \text{in } C([0,T];X),$$

that is,

$$nR(n,A)u_n \to u \quad \text{and} \quad A_n u_n = AnR(n,A)u_n \to v \quad \text{in } C([0,T];X).$$

Since A is closed this implies that $v = Au$. Then $u \in C\big([0,T];D(A)\big)$. Moreover

$$u_n \in W^{1,p}(0,T;X) \quad \text{and} \quad u_n' = A_n u_n + f \to v \quad \text{in } C([0,T];X).$$

So $u_n \to u$ in $W^{l,p}(0,T;X)$ and $u' = Au + f$.
(ii) Due to the hypotheses the problem

$$\begin{cases} z'(t) = Az(t) + f'(t), \\ z(0) = Ax + f(0) \end{cases}$$

has a strong solution $z \in C([0,T];X)$. Moreover let z_n be the solution of the problem

$$\begin{cases} z_n'(t) = A_n z_n(t) + f'(t), \\ z_n(0) = A_n x + f(0). \end{cases}$$

Clearly $z_n = u_n'$ so that, by Proposition 3.1,

$$u_n \to u, \quad u_n' = z_n \to z \quad \text{in } C([0,T];X).$$

It follows that $u \in C^1([0,T];X)$ and $u' = z$. Moreover $Au = u' - f$ belongs to $C([0,T];X)$ so that $u \in C\big([0,T];D(A)\big)$. □

3.4 Perturbations of infinitesimal generators

Consider the problem

$$\begin{cases} u'(t) = Au(t) + F(t)u(t) + f(t), \quad t \in [0,T], \\ u(0) = x \end{cases} \tag{3.8}$$

under the following hypotheses:

$$\begin{array}{ll} \text{(i)} & A \text{ is the infinitesimal generator of a strongly continuous} \\ & \text{semigroup } S, \text{ and} \\ \text{(ii)} & F \text{ is a mapping from } [0,T] \text{ into } \mathcal{L}(X) \\ & \text{which is strongly continuous, that is } F(\cdot)x \text{ belongs to} \\ & C([0,T];X) \text{ for any } x \in X. \end{array} \tag{3.9}$$

Set $S_n(t) = \exp(tA_n)$. By Theorem 2.5 there exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$\|S(t)\| \leq Me^{\omega t}, \quad \|S_n(t)\| \leq Me^{\omega t}, \qquad t \geq 0. \tag{3.10}$$

Moreover, by the *uniform boundedness theorem*, there exists $K > 0$ such that

$$\|F(t)\| \leq K, \quad t \in [0,T]. \tag{3.11}$$

If f belongs to $L^p(0,T;X)$, $p \geq 1$, then the function Ff, $(Ff)(t) = F(t)f(t)$, is Böchner measurable, as easily checked, and belongs to $L^p(0,T;X)$, whereas if f is continuous then Ff is continuous.

The definitions of strict, strong, classical, and weak solutions in Definition 3.1 can be extended in an obvious way to problem (3.8). We say, in addition, that u is a *mild solution* of problem (3.8) if u is continuous in $[0,T]$ and verifies the integral equation

$$u(t) = S(t)x + \int_0^t S(t-s)\{F(s)u(s) + f(s)\}\,ds. \tag{3.12}$$

Lemma 3.2. *The following statements are equivalent*:

(i) *u is a strong solution of problem* (3.8) *in* $L^p(0,T;X)$.
(ii) *u is a mild solution of problem* (3.8).

Proof. (i) $\Longrightarrow$ (ii). Let u be a strong solution, then there exists a sequence $\{u_k\} \subset W^{1,p}(0,T;X) \cap L^p\big(0,T;D(A)\big)$ such that

$$\begin{gathered} u_k \to u, \quad u_k' - Au_k - Fu_k = f_k \to f \quad \text{in } L^p(0,T;X), \\ u_k(0) \to x \quad \text{in } X. \end{gathered}$$

Then u_k is a strict solution of problem (3.1) with $x = x_k$ and $f = f_k + Fu_k$; by Proposition 3.1 it follows that

$$u_k(t) = S(t)x_k + \int_0^t S(t-s)\{F(s)u_k(s) + f_k(s)\}\,ds$$

and, as $k \to \infty$, we find that u verifies (3.12).

(ii) $\Longrightarrow$ (i). Let $u \in C([0,T];X)$ be a mild solution. Then, again by Proposition 3.1, u is the strong solution of problem (3.1) with f replaced by $f + Fu$. Thus, there exists a sequence

$$\{u_k\} \subset W^{1,p}(0,T;X) \cap L^p\big(0,T;D(A)\big)$$

such that

$$\begin{gathered} u_k \to u, \quad u_k' - Au_k - Fu_k \to f + Fu, \quad \text{in } L^p(0,T;X), \\ u_k(0) \to x \quad \text{in } X. \end{gathered}$$

It follows that

$$u_k' - Au_k - Fu_k \to f$$

so that u is a strong solution of (3.8). □

We now solve problem (3.12). For this it is useful to introduce the approximate equation

$$u_n(t) = S_n(t)x + \int_0^t S_n(t-s)\{F(s)u_n(s) + f(s)\}\,ds, \tag{3.13}$$

which is clearly equivalent to the system

$$\begin{cases} u_n'(t) = A_n u_n(t) + F(t)u_n(t) + f(t), & t \in [0,T], \\ u_n(0) = x. \end{cases} \tag{3.14}$$

Proposition 3.4. *Assume that F verifies* (3.9), *$x \in X$ and $f \in L^p(0,T;X)$. Then the following results are true*:

(i) *Problem* (3.8) *has a unique solution u in $L^p(0,T;X)$ that belongs to $C([0,T];X)$. Moreover for each n, problem* (3.14) *has a unique solution $u_n \in C([0,T];X)$ and $u_n \to u$ in $C([0,T];X)$.*

(ii) *If X is a reflexive Banach space, problem* (3.8) *has a unique weak solution in $W^{1,p}(0,T;D(A^*)') \cap C([0,T];X)$ which coincides with the strong solution in $L^p(0,T;X)$.*

Proof. (i) Problems (3.8) and (3.14) are respectively equivalent to the equations

$$u = w(u), \quad u_n = w_n(u_n), \tag{3.15}$$

where the mappings w and w_n are defined as

$$w(u)(t) = S(t)x + \int_0^t S(t-s)\{F(s)u(s) + f(s)\}\, ds,$$
$$w_n(u)(t) = S_n(t)x + \int_0^t S_n(t-s)\{F(s)u(s) + f(s)\}\, ds.$$

Now w and w_n map $C[0,T];X)$ into itself and, if $k \in N$,

$$|w^k(u) - w^k(\bar{u})|_{C([0,T];X)} \le \frac{1}{n!}(MKe^{\omega^+ T})^n |u - \bar{u}|_{C([0,T];X)},$$
$$|w_n^k(u) - w_n^k(\bar{u})|_{C([0,T];X)} \le \frac{1}{n!}(MKe^{\omega^+ T})^n |u - \bar{u}|_{C([0,T];X)},$$

where $\omega^+ = \sup\{\omega, 0\}$. Now the conclusion follows by the Contraction Mapping Principle.

(ii) *Existence.* We know from part (i) that (3.12) has a unique solution u in $C([0,T];X)$. We show that u belongs to $W^{1,p}(0,T;D(A^*)')$ and that

$$u' = [(A^*)^* + F(t)]u + f \quad \text{in } L^p(0,T;D(A^*)').$$

We compute the vectorial distributional derivative of u: for k in $D(A^*)$ and $\phi \in \mathcal{D}(]0,T[)$

$$\begin{aligned}
-\int_0^T \langle k, u(t)\rangle \phi'(t)\, dt &= -\int_0^T \left\langle k, S(t)x + \int_0^t S(t-s)[F(s)u(s) + f(s)]\, ds \right\rangle \phi'(t)\, dt \\
&= -\int_0^T \langle S^*(t)k, x\rangle \phi'(t)\, dt \\
&\quad - \int_0^T dt \int_0^t ds \langle S^*(t-s)k, F(s)u(s) + f(s)\rangle \phi'(t) \\
&= \int_0^T \left\langle \frac{d}{dt} S^*(t)k, x \right\rangle \phi(t)\, dt
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^T dt \int_s^T ds\left\langle \frac{d}{dt}S^*(t-s)k, F(s)u(s)+f(s)\right\rangle\phi(t)\\
&=\int_0^T \langle S^*(t)A^*k, x\rangle\phi(t)\,dt\\
&+\int_0^T dt\int_s^T ds\langle S^*(t-s)A^*k,F(s)u(s)+f(s)\rangle\phi(t)\\
&+\int_0^T \langle k, F(s)u(s)+f(s)\rangle\phi(s)\,ds\\
&=\int_0^T \left\langle k, (A^*)^*\left[S(t)x+\int_0^t S(t-s)[F(s)u(s)+f(s)]\,ds\right.\right.\\
&\qquad\qquad\left.\left. + F(t)u(t)+f(t)\right\rangle\phi(t)\,dt\right].
\end{aligned}
$$

So finally

$$
u' = (A^*)^*u(t) + F(t)u(t) + f(t) \in L^p(0,T;D(A^*)') \tag{3.16}
$$

and $u \in W^{1,p}([0,T];D(A^*)') \cap C([0,T];X)$.

Uniqueness.

The proof is similar to the one of Proposition 3.2. It is sufficient to prove that for $x = 0$ and $f = 0$, (3.16) has only the solution $u = 0$. So first note that $u = 0$ is a solution of (*) with $u(0) = 0$ in $W^{1,p}(0,T;D(A^*)') \cap C([0,T];X)$. We fix $T > 0$ and k in $D(A^*)$ and introduce the function $v(T,t) = S^*(T-t)k$. Then

$$
\frac{d}{dt}\langle v(T,t), u(t)\rangle_X = \langle v(T,t), F(t)u(t)\rangle_X
$$

and

$$
\langle v(T,T), u(T)\rangle_X = \langle v(T,0), u(0)\rangle + \int_0^T \langle v(T,t), F(t)u(t)\rangle\,dt.
$$

So for all k in $D(A^*)$

$$
\langle k, u(T)\rangle_X = \int_0^T \langle k, S(T-t)F(t)u(t)\rangle\,dt
$$

and necessarily

$$
u(T) = \int_0^T S(T-t)F(t)u(t)\,dt.
$$

But this means that u is a mild solution for $x = 0$ and $f = 0$. So u and 0 are mild solutions. However since the mild solution is unique $u(T) = 0$, $\forall T \geq 0$. This completes the proof. □

We now study strict solutions. The proof of the following proposition is the same as the one of Proposition 3.3 and will be left to the reader.

Proposition 3.5. *The following statements hold*:

(i) *Assume that* F *verifies* (3.9) *and that* $F(\cdot)x \in C\big([0,T];D(A)\big)$ *for any* $x \in D(A)$. *If, in addition,* $x \in D(A)$ *and* $f \in L^p\big(0,T;D(A)\big)$, *then problem* (3.8) *has a unique strict solution*

$$u \in W^{1,p}(0,T;X) \cap C\big([0,T];D(A)\big).$$

(ii) *Assume that* F *verifies* (3.9) *and that* $F(\cdot)x \in C^1([0,T];X)$ *for any* $x \in X$. *If, in addition,* $x \in D(A)$ *and* $f \in W^{1,p}(0,T;X)$, *then problem* (3.8) *has a unique strict solution*

$$u \in C^1([0,T];X) \cap C\big([0,T];D(A)\big).$$

3.5 Evolution operators

Consider the systems

$$\begin{cases} u'(t) = Au(t) + F(t)u(t) + f(t), \quad t \in [s,T], \\ u(s) = x, \quad s \in [0,T], \end{cases} \tag{3.17}$$

$$\begin{cases} u_n'(t) = A_n u_n(t) + F(t)u_n(t) + f(t), \quad t \in [s,T], \\ u_n(s) = x, \quad s \in [0,T]. \end{cases} \tag{3.18}$$

The generalization of Propositions 3.4 and 3.5 to these systems is straightforward. Then, by Proposition 3.4 we know that system (3.17) (resp. (3.18)) has a unique strong solution. We define the transformations U and U_n of X as follows:

$$U(t,s)x = u(t), \quad U_n(t,s)x = u_n(t), \qquad t \ge 0, \ x \in X.$$

As easily seen $U(t,s)$ and $U_n(t,s)$ are linear bounded operators in X. The mapping U

$$(t,s) \mapsto U(t,s) \colon \Delta_T \to \mathcal{L}(X),$$

where

$$\Delta_T = \{(t,s) \colon s \in [0,T], t \in [s,T]\},$$

is called the *evolution operator* associated with $A + F$.

Proposition 3.6. *Assume that* A *is the infinitesimal generator of a strongly continuous semigroup on* X *and that* F *verifies* (3.9). *Let* U *be the evolution operator relative to* $A + F$ *and* U_n *the evolution operator relative to* $A_n + F$. *Then the following statements are true*:

(i) $U(\cdot)x$ *is continuous in* Δ_T *for any* $x \in X$.
(ii) *For any* $x \in X$, $\lim_{n\to\infty} U_n(t,s)x = U(t,s)x$ *uniformly in* $(t,s) \in \Delta_T$.
(iii) *If* $T \ge t \ge s \ge r \ge 0$, *then* $U(t,s)U(s,r) = U(t,r)$, $U(s,s) = I$.

(iv) *For any* $(t,x) \in \Delta_T$, *we have*

$$\|U(t,s)| \leq Me^{(\omega+K)(t-s)}, \quad \|U_n(t,s)\| \leq Me^{(\omega+K)(t-s)}, \tag{3.19}$$

where $K = \sup\{\|F(t)\|\colon 0 \leq t \leq T\}$.

(v) *If* $f \in L^p(0,T;X)$ *the strong solution of* (3.8) *is given by*

$$u(t) = U(t,0)x + \int_0^t U(t,s)f(s)\,ds. \tag{3.20}$$

Proof. Let w_s and w_{sn} be the mappings in $C([s,T];X)$ defined by

$$w_s(u)(t) = S(t-s)x + \int_s^t S(t-r)F(r)u(r)\,dr,$$

$$w_{sn}(u)(t) = S_n(t-s)x + \int_s^t S_n(t-r)F(r)u(r)\,dr.$$

If $k \in N$ we have

$$|w_s^k(u) - w_s^k(\bar{u})|_{C([s,T];X)} \leq \frac{1}{n!}(MKe^{\omega^+T})^n|u-\bar{u}|_{C([s,T];X)},$$

$$|w_{sn}^k(u) - w_{sn}^k(\bar{u})|_{C([s,T];X)} \leq \frac{1}{n!}(MKe^{\omega^+T})^n|u-\bar{u}|_{C([s,T];X)},$$

where $\omega^+ = \sup\{\omega, 0\}$. Thus the statements (i) and (ii) follow by the Contraction Mapping Principle. Moreover (iii) and (v) follow from (ii) (since the analogous statements for U_n are well known). Finally (iv) is a consequence of the Gronwall lemma. □

3.6 Maximal regularity results in Hilbert spaces and main isomorphism

We go back to the problem

$$\begin{cases} u'(t) = Au(t) + f(t), & t \in [0,T], \\ u(0) = x. \end{cases} \tag{3.21}$$

By *Maximal Regularity* we mean that u' and Au have the same regularity as f. We assume that

(i) $H = X$ is a Hilbert space and that
(ii) A is the infinitesimal generator of an analytic semigroup of negative type. (3.22)

The hypothesis that the type of S be negative is not restrictive since by the transformation $v = e^{\theta t}u$ we can change A to $A - \theta I$ (cf. Theorem 2.11). Therefore since S is an analytic semigroup of negative type, there exists $M > 0$ such that

$$\|R(\lambda, A)\| \leq \frac{M}{|\lambda|}, \quad \text{if } \operatorname{Re}\lambda > 0. \tag{3.23}$$

Remark 3.6. We shall see in §3.7 that maximal regularity can also be studied in a general Banach space. □

In the first part of this section we study problem (3.21) with $u(0) = 0$, that is

$$\begin{cases} u'(t) = Au(t) + f(t), & t \in [0, T], \\ u(0) = 0. \end{cases} \tag{3.24}$$

In the second part we will consider the problem (3.21) with $u(0) = x$ and we shall give conditions on x and f such that maximal regularity holds. This will naturally lead us to *interpolation spaces*, which will be studied in detail in §4 and the construction of the *main isomorphism.*

Assume that $f \in L^2(0, T; H)$, then by Proposition 3.1, problem (3.24) has a unique strong solution given by

$$u(t) = \int_0^t S(t-s)f(s)\,ds. \tag{3.25}$$

In order to study the regularity of u it is convenient to use the Fourier transform in H.

For any $u \in \mathcal{D}(\mathbb{R}, H)$ we set

$$\hat{u}(k) = \int_{-\infty}^{+\infty} e^{-ikt} u(t)\,dt. \tag{3.26}$$

Let $\{e_\alpha\}_{\alpha\in\Gamma}$ be a complete orthonormal system in H. By Parseval's equality applied to the scalar function $\langle u(t), e_\alpha\rangle$, we have

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} |\langle \hat{u}(k), e_\alpha\rangle|^2\,dk = \int_{-\infty}^{+\infty} |\langle u(t), e_\alpha\rangle|^2\,dt, \tag{3.27}$$

which implies that

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} |\hat{u}(k)|^2\,dk = \int_{-\infty}^{+\infty} |u(t)|^2\,dt. \tag{3.28}$$

Then the mapping

$$u \mapsto \gamma(u) = \hat{u} \colon \mathcal{D}(\mathbb{R}, H) \to L^2(\mathbb{R}; H)$$

is continuous with respect to the $L^2(\mathbb{R}, H)$–topology and can be uniquely extended to all $L^2(\mathbb{R}, H)$. We shall denote by $\gamma(u)$ or $\hat{u}$ such an extension.

3.6.1 A priori estimates for Au

We first assume that $f \in \mathcal{D}(]0, \infty[\,; H)$ and consider the problem

$$\begin{cases} \bar{u}'(t) = A\bar{u}(t) + \bar{f}(t), & t \in \mathbb{R}, \\ \bar{u}(0) = 0, \end{cases} \tag{3.29}$$

where

$$\bar{f}(t) = \begin{cases} f(t) & \text{if } t \in]0,T[, \\ 0 & \text{if } t \notin]0,T[. \end{cases}$$

Due to Proposition 3.1, problem (3.29) has a unique strong solution given by

$$\bar{u}(t) = \begin{cases} u(t), & \text{if } t \in [0,T], \\ 0, & \text{if } t \leq 0, \\ e^{(T-t)A}u(T), & \text{if } t \geq T, \end{cases} \tag{3.30}$$

where u is the strong solution of problem (3.24). Since S is of negative type, we have

$$\bar{u}, \quad A\bar{u}, \quad \bar{u}' \in L^2(\mathbb{R}; H). \tag{3.31}$$

Lemma 3.3. *We have for all f in $W^{1,2}(0,\infty;H)$ and $T > 0$*

$$\int_0^T |Au(t)|^2\,dt \leq (M+1)^2 \int_0^T |f(t)|^2\,dt. \tag{3.32}$$

Proof. By (3.29) it follows that

$$\int_0^{+\infty} e^{-ikt}\bar{u}'\,dt = \int_0^{+\infty} e^{-ikt}A\bar{u}(t)\,dt + \int_0^{+\infty} e^{ikt}\bar{f}\,dt.$$

Denote by $\hat{u}$ and $\hat{f}$ the Fourier transforms of u and f, respectively. Then

$$ik\hat{u}(k) = A\hat{u}(k) + \hat{f}(k),$$

so that

$$\hat{u}(k) = R(ik, A)\hat{f}(k),$$

which implies that

$$|A\hat{u}(k)| \leq (M+1)|\hat{f}(k)|,$$

and by Parseval's inequality, we obtain (3.32). □

3.6.2 Main result for the case $u(0) = 0$

As S is a semigroup of negative type, $|Ax|$ is a norm on $D(A)$ equivalent to the graph norm and A is an isomorphism for $D(A)$ onto H. As a result

$$\|u\|^2_{L^2(0,\infty;H)} = \|Au\|^2_{L^2(0,\infty;H)} + \|u'\|^2_{L^2(0,\infty;H)}$$

is a norm on the space

$$W(2,0;D(A),H) = W^{1,2}(0,\infty;H) \cap L^2\big(0,\infty;D(A)\big). \tag{3.33}$$

Such spaces will be studied in more detail in §4.

In view of the previous a priori estimates, we have the following result.

Proposition 3.7. *Assume that* (3.22) *holds and that* f *belongs to* $L^2(0,\infty;H)$:

(i) *Problem* (3.24) *has a unique strict solution* u *in* $W(2,0;D(A),H)$ *and the following inequality is verified:*

$$\|u\|_W \le \sqrt{2}(M+2)\|f\|_{L^2(0,\infty;H)}. \tag{3.34}$$

(ii) *The linear map*

$$u \mapsto u' - Au$$
$$: W_0 = \{v \in W\big(2,0;D(A),H\big) \colon v(0)=0\} \to L^2(0,\infty;H) \tag{3.35}$$

is an isomorphism.

Proof. (i) Given f in $L^2(0,\infty;H)$, we know that (3.24) has a unique strong solution in all intervals $[0,T]$, $T>0$, for which inequality (3.32) holds. In particular we can let T go to ∞ and

$$\|Au\|^2_{L^2(0,\infty;H)} \le (M+1)\|f\|^2_{L^2(0,\infty;H)}.$$

Moreover because $u' = Au + f$

$$\|u'\|^2_{L^2(0,\infty;H)} \le (M+2)\|f\|^2_{L^2(0,\infty;H)}$$

and we get inequality (3.34). So u is a strict solution because

$$u \in W^{1,2}(0,\infty;H) \cap L^2\big(0,\infty;D(A)\big).$$

(ii) As the map $u \mapsto u(0)\colon W^{1,2}(0,\infty;H) \to H$ is linear and continuous, W_0 is a closed subspace of W. The map (3.35) is clearly linear and continuous. It is bijective from part (i) because for all f in $L^2(0,\infty;H)$, (3.24) has a unique solution in W_0. Hence (3.35) is an isomorphism. □

3.6.3 The case u(0) = x and the main isomorphism

We now consider the problem

$$\begin{cases} u'(t) = Au(t) + f(t), & t \in [0,T], \\ u(0) = x. \end{cases} \tag{3.36}$$

In order to complete the isomorphism (3.35) and stay in the space of solutions $W(2,0;D(A),H)$, it is necessary to introduce appropriate interpolation spaces, which will be studied in details in §4.

Consider the following subspace:

$$D_A\left(\tfrac{1}{2},2\right) \overset{\text{def}}{=} \{u(0)\colon u \in W(2,0;D(A),H)\} \tag{3.37}$$

of H endowed with the quotient norm

$$\|x\|_D = \inf\{\|u\|_W : u \in W(2,0;D(A),H) \text{ and } u(0)=x\}. \tag{3.38}$$

$D_A(1/2,2)$ is the space of traces $T(2,0;D(A),H)$, that will be studied in §4.2. It will be shown in Proposition 4.3 that this space of traces is isomorphic to the space of averages $(D(A),H\ )_{1/2,2}$ as defined in (4.14). In the variational literature this space is also written $[D(A),H]_{1/2}$. It is readily seen that the map

$$u \mapsto u(0) : W(2,0;D(A),H) \to D_A\left(\tfrac{1}{2},2\right)$$

is linear continuous and onto.

Theorem 3.1. *The following statements hold*:

(i) *Assume that* (3.22) *holds; then the map*

$$u \mapsto \big(u' - Au, u(0)\big) : W(2,0;D(A),H) \to L^2(0,\infty;H) \times D_A\left(\tfrac{1}{2},2\right) \tag{3.39}$$

is an isomorphism. In particular for all f in $L^2(0,\infty;H)$ and x in $D_A(1/2,2)$, problem (3.36) *has a unique strict solution in $L^2(0,\infty;H)$.*

(ii) *Assume that S is an analytic semigroup in the sense of Definition* 2.3 *in* §2.7*. Then for each $T>0$, the map*

$$\begin{aligned} u \mapsto {} & \big(u' - Au, u(0)\big) \\ & : W^{1,2}(0,T;H) \cap L^2\big(0,T;D(A)\big) \to L^2(0,T;H) \times D_A\left(\tfrac{1}{2},2\right) \end{aligned} \tag{3.40}$$

is an isomorphism. In particular for all f in $L^2(0,T;H)$ and x in $D_A(1/2,2)$, problem (3.36) *has a unique strict solution in $L^2(0,T;H)$.*

Proof. (i) The map (3.39) is clearly linear and continuous by construction. We show that it is surjective and hence an isomorphism by the *Open Mapping Theorem* (cf. K. Yosida [2, §5, pp.77]). This is equivalent to prove that for each (f,x) in $L^2(0,\infty;H) \times D_A(1/2,2)$, there exists a unique solution in $W(2,0;D(A),H)$ to problem (3.36). So for each x in $D_A(1/2,2)$, there exists u_1 in $W(2,0;D(A),H)$ such that $u_l(0) = x$ and $u_1' - Au_1 \in L^2(0,\infty;H)$. For each f in $L^2(0,\infty;H)$, there exists a unique strict solution u_2 in $W(2,0;D(A),H)$ to

$$u_2' - Au_2 = f - (u_1' - Au_1), \quad u_2(0) = 0$$

by Proposition 3.7. The sum $u = u_1 + u_2$ belongs to $W(2,0;D(A),H)$, $u(0)=x$ and

$$u' = u_1' + u_2' = u_1' + Au_2 + f - (u_1' - Au_1) = A(u_1+u_2) + f = Au + f.$$

We have constructed a solution u in $W(2,0;D(A),H)$ to problem (3.36). Therefore the map (3.39) is surjective. It is injective if for $f=0$ and $x=0$ the only solution u in $W(2,0;D(A),H)$ is $u=0$. But this is true by Proposition 3.7 (ii).

(ii) The term $u \mapsto u' - Au$ is well-defined linear and continuous. We only have to check that for u in $W^{1,2}(0,\infty;H) \cap L^2\big(0,\infty;D(A)\big)$, $u(0) \in D_A(1/2,2)$ or equivalently that there exists $\tilde{u}$ in $W^{1,2}(0,\infty;H) \cap L^2\big(0,\infty;D(A)\big)$, such that $u(0) = \tilde{u}$. It turns out that

$$\tilde{u}(t) = \frac{\sqrt{t}}{1+t} u\left(\frac{tT}{1+t}\right)$$

is such a function since

$$\|A\tilde{u}\|^2_{L^2(0,\infty;H)} = \|Au\|^2_{L^2(0,T;H)}$$

and moreover

$$\begin{aligned}
\tilde{u}'(t) &= \frac{T\sqrt{T}}{(1+t)^3} u'\left(\frac{tT}{1+t}\right) - \frac{\sqrt{T}}{(1+t)^2} u\left(\frac{tT}{1+t}\right), \\
|\tilde{u}'(t)| &\leq T\left|\frac{\sqrt{T}}{1+t} u'\left(\frac{tT}{1+t}\right)\right| + \left|\frac{\sqrt{T}}{1+1} u\left(\frac{tT}{1+t}\right)\right|.
\end{aligned}$$

So $\tilde{u}' \in L^2(0,\infty;H)$ since each of two terms on the right-hand side does.

Now first assume that S is a semigroup of negative type. The map (3.40) is bijective since for all $x \in D_A(1/2,2)$ and $f \in L^2(0,\infty;H)$ there exists a unique solution u_T in $W^{1,2}(0,\infty;H) \cap L^2\big(0,\infty;D(A)\big)$ to the equation

$$u_T' - Au_T = f_T, \quad u_T(0) = x,$$

where f_T in $L^2(0,\infty;H)$ is given by

$$f_T(t) = \begin{cases} f(t) & \text{if } 0 \leq t \leq T, \\ 0 & \text{if } t > T. \end{cases}$$

Its restriction u to $[0,T]$ belongs to $W^{1,2}(0,T;H) \cap L^2\big(0,T;D(A)\big)$ and

$$u' = Au + f, \quad u(0) = x.$$

The existence implies the surjectivity and the uniqueness the injectivity.

When $S(t)$ is not of negative type, $\omega_0 = \omega_0(S) \geq 0$, fix $\omega > \omega_0$ and define $S_\omega(t) = S(t)e^{-(\omega+1)t}$. Then $\omega_0(S_\omega) < -1$ and $A_\omega = A - (\omega+1)I$. Consider for f in $L^2(0,T;H)$ and x in $D_A(1/2,2)$ the problem

$$u_\omega' - Au_\omega = f_\omega, \quad u_\omega(0) = x,$$

where

$$f_\omega(t) = e^{-(\omega+1)t} f(t), \quad 0 \leq t \leq T.$$

Since S_ω is of negative type we are back to the previous situation and we know that there exists a unique solution u_ω in $W^{1,2}(0,T;H) \cap L^2\big(0,T;D(A)\big)$ since $D(A_\omega) = D(A)$. Define

$$u(t) = e^{(\omega+1)t} u_\omega(t), \quad 0 \le t \le T.$$

Then u belongs to $W^{1,2}(0,T;H) \cap L^2\big(0,T;D(A)\big)$, and it is readily seen that

$$u' = Au + f, \quad u(0) = x.$$

So again we have a bijection and an isomorphism. □

Corollary 3.1. *When S is an analytic semigroup in the sense of Definition 2.3 in §2.7, then for all $T > 0$*

$$\big\{u(0) \colon u \in W^{1,2}(0,T;H) \cap L^2\big(0,T;D(A)\big)\big\} = D_A\left(\tfrac{1}{2}, 2\right).$$

Concerning classical solutions, we have the following results.

Proposition 3.8. *Assume that $f \in L^2(0,T;H)$ and $x \in H$; then problem (3.36) has a unique classical solution in $L^2(0,T;H)$.*

Proof. Let u be the strong solution of (3.36). Set

$$u(t) = S(t)x + v(t).$$

Then, due to Theorem 3.1, v is the strict solution of the problem

$$\begin{cases} v'(t) = Av(t) + f(t), \quad t \ge 0, \\ v(0) = 0. \end{cases}$$

As $S(t)x$ is analytic for $t > 0$, it follows that u is a classical solution of problem (3.36).

To show uniqueness, it is sufficient to show that 0 is the unique classical solution of the problem

$$\begin{cases} z'(t) = Az(t), \\ z(0) = 0, \quad t \ge 0. \end{cases} \tag{3.41}$$

Let z be a classical solution of (3.41); then z verifies (3.41) for any $t > 0$. We have

$$\frac{d}{ds}\big(S(t-s)z(s)\big) = 0 \quad \text{for all } 0 < s < t.$$

By integrating this between ε and $t - \varepsilon$, we obtain

$$S(\varepsilon)z(t-s) = S(t-\varepsilon)z(\varepsilon).$$

As z is continuous, letting ε tend to 0 we find $z = 0$. □

More information on problem (3.36) will be given in §4.

3.7 Regularity results in $C([0,T];X)$

Let X be a Banach space and let $A\colon D(A) \subset X \to X$ be the generator of an analytic semigroup S. Let $M > 0$, $N > 0$, and $\omega \in \mathbb{R}$ be such that

$$\begin{cases} \|S(t)\| \le Me^{\omega t}, & \|(A-\omega I)S(t)\| \le \dfrac{N}{t}e^{\omega t}, \quad t>0, \\ \|S_n(t)\| \le Me^{\omega t}, & \|(A_n-\omega I)S_n(t)\| \le \dfrac{N}{t}e^{\omega t}, \quad t>0. \end{cases} \tag{3.42}$$

We consider here the problem

$$\begin{cases} u'(t) = Au(t) + f(t), \\ u(0) = x \in X, \end{cases} \tag{3.43}$$

where f belongs to $C([0,T];X)$. We know that the mild solution of (3.43) is given by

$$u(t) = S(t)x + \int_0^t S(t-s)f(s)\,ds. \tag{3.44}$$

The following results are well known (see, for instance, A. Pazy [1]).

Proposition 3.9. *Assume that $f \in C^\alpha([0,T];X)$, $\alpha \in \,]0,1[$, and $x \in D(A)$. Then problem* (3.43) *has a unique strict solution*

$$u \in C^1([0,T];X) \cap C\big([0,T];D(A)\big).$$

Moreover

$$u'(t) = S(t)\big(Ax + f(t)\big) + \int_0^t AS(t-s)\big(f(s)-f(t)\big)\,ds. \tag{3.45}$$

Proof. By making the change of variable $v(t) = u(t)e^{-\omega t}$, $t \ge 0$, it is sufficient to prove the theorem for a semigroup $S(t)$ such that $\omega = 0$. We first note that formula (3.44) is meaningful because, by (3.42), we have

$$\big|AS(t-s)\big(f(s)-f(t)\big)\big| \le N\|f\|_{C^\alpha([0,T];X)}|t-s|^{\alpha-1}.$$

Now set

$$u_n(t) = S_n(t)x + \int_0^t S_n(t-s)\,ds.$$

Then it is easy to check that

$$u_n'(t) = S_n(t)\big(A_nx + f(t)\big) + \int_0^t A_nS_n(t-s)\big(f(s)-f(t)\big)\,ds.$$

As $x \in D(A)$ and $\big|A_nS_n(t-s)\big(f(s)-f(t)\big)\big| \le N\|f\|_{C^\alpha([0,T];X)}|t-s|^{\alpha-1}$, we have, by the Dominated Convergence theorem,

$$\lim_{n\to\infty} u_n'(t) = S(t)\big(Ax + f(t)\big) + \int_0^t AS(t-s)(f(s) - f(t)r)\,ds$$

uniformly in $t \in [0,T]$. It follows that $u \in C^1([0,T];X)$ and (3.45) holds. It remains to prove that $u \in C\big([0,T];D(A)\big)$. We have in fact $A_n u_n(t) = u_n'(t) - f(t)$ so that $A\big(nR(n,A)u_n(t)\big) \to u'(t) - f(t)$ as $n \to \infty$ uniformly in $t \in [0,T]$. As A is closed and $nR(n,A)u_n(t) \to u$ in $C([0,T];X)$, it follows that $u \in C\big([0,T];D(A)\big)$ and $u' - Au = f$. □

Proposition 3.10. *If $f \in C([0,T];X)$ and $x = 0$, we have $u \in C^\alpha([0,T];X)$ for any $\alpha \in]0,1[$.*

Proof. Let $t > s > 0$; then by (3.44), we have

$$u(t) - u(s) = \int_s^t S(t-r)f(r)\,dr + \int_0^s dr \int_{s-r}^{t-r} S'(\sigma)f(r)\,d\sigma.$$

By (3.42) it follows that

$$|u(t) - u(s)| \le Me^{|\omega|T}\|f\|_{C([0,T];X)}|t-s| + NLe^{|\omega|T}\|f\|_{C([0,T];X)},$$

where

$$\begin{aligned} L = \int_0^s dr \int_{s-r}^{t-r} \frac{d\sigma}{\sigma} &\le \int_0^s (s-r)^{-\alpha} \int_{s-r}^{t-r} \sigma^{1-\alpha}\,d\sigma \\ &\le \frac{1}{\alpha}\int_0^s (s-r)^{-\alpha}\,dr|t-s|^\alpha \le \frac{T^{1-\alpha}}{1-\alpha}\frac{(t-s)^\alpha}{\alpha}. \end{aligned}$$

Thus, the conclusion follows. □

Consider now the special case $f = 0$. We have the following results.

Proposition 3.11. *Set $u(t) = S(t)x$, and let $\alpha \in]0,1[$,$T > 0$. Then the following statements are equivalent*:

(i) $u \in C^\alpha([0,T];X)$,
(ii) $\sup_{t\in]0,T]} t^{-\alpha}|S(t)x - x| < \infty$.

Proof. (i) $\Longrightarrow$ (ii) follows from

$$t^{-\alpha}|S(t)x - x| \le \|u\|_{C^\alpha([0,T];X)}.$$

(ii) $\Longrightarrow$ (i). If $t > s > 0$, we have

$$S(t)x - S(s)x = S(s)\big(S(t-s)x - x\big)$$

so that there exists a constant $C > 0$ such that

$$|S(t)x - S(s)x| \le C|t-s|^\alpha.$$ □

We set

$$|x|_\alpha = \sup_{t\in]0,T]} \{t^{-\alpha}|S(t)x - x| \colon \alpha \in]0,1[\}, \tag{3.46}$$

$$D_A(\alpha,\infty) = \{x \in X \colon |x|_\alpha < \infty\}. \tag{3.47}$$

$D_A(\alpha,\infty)$ is an interpolation space (see §4).

The following maximal regularity result was proved by E. Sinestrari [1] (see also G. Da Prato and P. Grisvard [1]).

Theorem 3.2. *Assume that $f \in C^\alpha([0,T];X)$, $\alpha \in]0,1[$, $x \in D(A)$. Let u be the solution of* (3.43). *Then the following statements are equivalent*:

(i) $Ax + f(0) \in D_A(\alpha,\infty)$,

(ii) u', Au *belongs to* $C^\alpha([0,T];X)$. *Moreover, if* (i) *or* (ii) *holds, we have* $u'(t) \in D_A(\alpha,\infty)$ *and*

$$\sup_{t\in]0,T]} |u'(t)|_\alpha < \infty.$$

We complete this section with a simple analyticity result.

Proposition 3.12. *Let $f \in C([0,\infty[;X)$ and $x \in X$. Assume that f has an analytic extension in a sector $S_\theta = \{\lambda \in \mathbb{C} \colon |\arg\lambda| \le \theta\}$ for some $\theta > 0$. Let $\theta_0 \in]0,\pi/2]$ such that the semigroup S is analytic in S_{θ_0}. Then if u is the solution of* (3.43), *u has an analytic extension in $S_{\theta\wedge\theta_0}$.*

Proof. Follows immediately from (3.44). □

3.8 Examples of nonhomogeneous problems

3.8.1 Parabolic equations

Let Ω be an open set of $\mathbb{R}$ with smooth boundary $\partial\Omega$. Consider the following initial value problem with Dirichlet boundary conditions: For all $t \ge 0$

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= \sum_{j,k=l}^{n} \frac{\partial u}{\partial x_k}\left(a_{jk}(x)\frac{\partial}{\partial x_j}(t,x)\right) + f(t,x), \quad x \in \Omega, \\ u(t,x) &= 0, \quad x \in \partial\Omega, \\ u(0,x) &= u_0(x), \quad x \in \Omega, \end{aligned} \tag{3.48}$$

where a_{jk} are real continuously differentiable functions in $\bar{\Omega}$ and verify the ellipticity condition (2.89).

Let $H = L^2(\Omega)$ and let A be the linear operator defined by (2.94). As proved in §2.9, A is the infinitesimal generator of an analytic semigroup in $H = L^2(\Omega)$. The interpolation space $D_A(1/2,2)$ coincides with $H_0^1(\Omega)$ (cf. J. L. Lions and E. Magenes [1]).

By using Propositions 3.7 and 3.8, we find the following result.

Proposition 3.13. *The following statements hold*:

(i) *If* $f \in L^2([0,T] \times \Omega)$ *and* $u_0 \in L^2(\Omega)$*; then problem* (3.48) *has a unique classical solution* u *that belongs to*

$$C\big([0,T]; L^2(\Omega)\big) \cap W^{1,2}\big(\varepsilon, T; L^2(\Omega)\big) \cap L^2\big(\varepsilon, T; H^2(\Omega) \cap H_0^1(\Omega)\big),$$

for any $\varepsilon \in \,]0,T[$.

(ii) *If* $f \in L^2([0,T] \times \Omega)$ *and* $u_0 \in H_0^1(\Omega)$*, then the solution* u *is strict and*

$$u \in W^{1,2}\big(0,T; L^2(\Omega)\big) \cap L^2\big(0,T; H^2(\Omega) \cap H_0^1(\Omega)\big).$$

For characterization of several interpolation spaces between the domain of an elliptic operator and $L^p(\Omega)$ or $C^\alpha(\Omega)$, see respectively P. Grisvard [2] and P. Acquistapace and B. Terreni [2]. Similar results can be obtained for Neumann or more general boundary conditions.

3.8.2 Schrödinger equation

We shall use here the notation of §2.10. Consider the problem: For all $t \geq 0$

$$\begin{gathered}
\frac{\partial}{\partial t} u(t,x) = i \sum_{j,k=l}^{n} \frac{\partial}{\partial x_k}\left(a_{jk}(x) \frac{\partial}{\partial x_j} u(t,x)\right) + f(t,x), \quad x \in \Omega, \\
u(t,x) = 0, \quad x \in \partial\Omega, \\
u(0,x) = u_0(x), \quad x \in \Omega.
\end{gathered} \tag{3.49}$$

By using Propositions 3.1 and 3.3, we find the following results.

Proposition 3.14. *The following statements hold*:

(i) *If* $f \in L^2([0,T] \times \Omega)$ *and* $u_0 \in L^2(\Omega)$*, the problem* (3.49) *has a unique* strong *solution*

$$u \in C\big([0,T]; L^2(\Omega)\big).$$

(ii) *If* $f \in L^2\big(0,T; H^2(\Omega) \cap H_0^1(\Omega)\big)$ *and* $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$*, then the solution* u *is* strict *and*

$$u \in W^{1,2}\big(0,T; L^2(\Omega)\big) \cap C\big([0,T]; H^2(\Omega) \cap H_0^1(\Omega)\big).$$

(iii) *If* $f \in W^{1,2}\big(0,T; L^2(\Omega)\big)$ *and* $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$*, then the solution* u *is* strict *and*

$$u \in C^1\big([0,T]; L^2(\Omega)\big) \cap C\big([0,T]; H^2(\Omega) \cap H_0^1(\Omega)\big).$$

3.8.3 Wave equation

We use here the notation of §2.10 and consider the problem: For all $t \geq 0$

$$\begin{aligned} \frac{\partial^2}{\partial t^2} u(t,x) = \sum_{j,k=l}^{n} \frac{\partial}{\partial x_k}\left(a_{jk}(x)\frac{\partial}{\partial x_j}u(t,x)\right), \quad x \in \Omega, \\ u(t,x) = 0, x \in \partial\Omega, \\ u(0,x) = u_0(x), \\ \frac{\partial}{\partial t}u(0,x) = u_1(x), \quad x \in \Omega. \end{aligned} \tag{3.50}$$

We now prove the following result.

Proposition 3.15. *The following statements hold:*

(i) *Assume that* $f \in L^2([0,T] \times \Omega)$, $u_0 \in H_0^1(\Omega)$, *and* $u_1 \in L^2(\Omega)$. *Then problem* (3.50) *has a unique strong solution* u *that belongs to*

$$C\big([0,T]; H_0^1(\Omega)\big) \cap C^1\big([0,T]; L^2(\Omega)\big).$$

(ii) *Assume that* $f \in L^2\big(0,T; H^2(\Omega) \cap H_0^1(\Omega)\big)$, $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, *and* $u_1 \in H_0^1(\Omega)$. *Then* u *is a strict solution that belongs to*

$$C\big([0,T]; H^2(\Omega) \cap H_0^1(\Omega)\big) \cap C^1\big([0,T]; H_0^1(\Omega)\big) \cap W^{2,2}\big(0,T; L^2(\Omega)\big).$$

(iii) *Assume that* $f \in W^{1,2}\big(0,T; L^2(\Omega)\big)$, $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, *and* $u_1 \in H_0^1(\Omega)$. *Then* u *is a strict solution that belongs to*

$$C\big([0,T]; H^2(\Omega) \cap H_0^1(\Omega)\big) \cap C^1\big([0,T]; H_0^1(\Omega)\big) \cap C^2\big([0,T]; L^2(\Omega)\big).$$

Proof. Setting

$$U(t) = \begin{bmatrix} 0 \\ f(t,\cdot) \end{bmatrix}, \quad F(t) = \begin{bmatrix} u(t,\cdot) \\ \frac{\partial}{\partial t}u(t,\cdot) \end{bmatrix}, \quad U_0 = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix},$$

problem (3.50) is equivalent to

$$U' = AU + F(t), \quad U(0) = U_0,$$

where the operator A is defined by (2.94). Now all conclusions follow from Propositions 3.1 and 3.3. □

3.9 Point spectrum operators

If $L : D(L) \subset X \to X$ is a linear operator, we set

$$\sigma^-(L) = \{\lambda \in \sigma(L) : \operatorname{Re}\lambda < 0\}, \tag{3.51}$$

$$\sigma^+(L) = \{\lambda \in \sigma(L) \colon \operatorname{Re}\lambda > 0\}, \tag{3.52}$$

$$\sigma^0(L) = \{\lambda \in \sigma(L) \colon \operatorname{Re}\lambda = 0\}. \tag{3.53}$$

The elements of $\sigma^-(L)$ are called the *stable points* and those of $\sigma^+(L)$ the *unstable points* of the spectrum of L.

In this section we are interested in solutions of the equation

$$u'(t) = Au(t) + f(t) \tag{3.54}$$

either on all the real line or in $[0, +\infty[$. Given

$$f \in L^2(\mathbb{R}; X) \quad \big(\text{resp. } L^2([0, +\infty[\,; X)\big),$$

we say that $u \in L^2(\mathbb{R}; X)$ $\big(\text{resp. } L^2(0, +\infty; X)\big)$ is a *mild* solution of (3.54) if

$$u(t) = e^{(t-a)A}u(a) + \int_a^t e^{(t-s)A} f(s)\, ds \tag{3.55}$$

for all a and all t in $\mathbb{R}$ (resp. in $[0, +\infty[$) such that $a \le t$. We will study this problem under hypothesis $(\mathcal{P})$ below.

We say that a linear operator $L\colon D(L) \subset X \to X$ in X verifies *Assumption* $(\mathcal{P})$ if the following conditions are verified:

Assumption $(\boldsymbol{\mathcal{P}})$.

(i) L is the infinitesimal generator of a strongly continuous semigroup e^{tL} on X,
(ii) $\sigma^0(L)$ is empty and $\sigma^+(L)$ consists of a finite set of eigenvalues of finite algebraic multiplicity,
(iii) there exists $\varepsilon > 0$, $N_L > 0$ such that

$$\sup\{\operatorname{Re}\lambda \colon \lambda \in \sigma^-(L)\} < -\varepsilon$$

and

$$\|e^{tL}(I - \Pi_L)\| \le N_L e^{-\varepsilon t} \quad \forall t \ge 0, \tag{3.56}$$

where Π_L is the projector on the subspace of all eigenvectors corresponding to the eigenvalues in $\sigma^+(L)$ defined by

$$\Pi_L = \frac{1}{2\pi i} \int_\gamma R(\lambda, L)\, d\lambda \tag{3.57}$$

and γ is a simple Jordan curve around $\sigma^+(L)$ in the open half plane of all λ such that $\operatorname{Re}\lambda > 0$.

We set

$$X_L^+ = \Pi_L X, \quad X_L^- = (I - \Pi_L)X. \tag{3.58}$$

Remark 3.7. Assume that L fulfills $(\mathcal{P})$. Then by hypothesis (ii), X_L^+ is finite dimensional and there exists $\eta > 0$ such that

$$\operatorname{Re}\lambda > \eta, \quad \text{for all } \lambda \in \sigma^+(L). \tag{3.59}$$

Moreover the subspace X_L^+ is stable for the semigroup e^{tL}, which can be extended for $t < 0$ by the formula

$$e^{tL}\Pi_L = \frac{1}{2\pi i}\int_\gamma e^{t\lambda}R(\lambda, L)\,d\lambda, \quad t \in \mathbb{R}. \tag{3.60}$$

Finally, by (3.59)–(3.60), it follows that there exists $N_L' > 0$ such that

$$\|e^{tL}\Pi_L\| \le N_L' e^{\eta t} \quad \forall t \le 0. \tag{3.61}$$

□

Remark 3.8. Assumption $(\mathcal{P})$ holds in the following cases:

(a) H has a finite dimension and $\sigma(A) \cap i\mathbb{R} = \emptyset$.
(b) e^{tL} is compact for any $t > 0$ and $\sigma(A) \cap i\mathbb{R} = \emptyset$,

where (a) is a special case of (b). □

We first study solutions of (3.54) on all the real line.

Proposition 3.16. *Assume that A verifies hypothesis $(\mathcal{P})$. Then if $f \in L^2(\mathbb{R}, X)$, equation* (3.54) *has a unique mild solution $u \in L^2(\mathbb{R}; X)$ that is given by the formula*

$$u(t) = \int_{-\infty}^t e^{(t-s)A}(I - \Pi_A)f(s)\,ds - \int_t^{+\infty} e^{(t-s)A}\Pi_A f(s)\,ds. \tag{3.62}$$

Proof. First of all, we notice that formula (3.62) is meaningful due to (3.56) and (3.61).

Step 1 *(Existence).* Assume first that $f \in L^2\big(\mathbb{R}; D(A)\big)$ and let u be the function defined by (3.62). Then we have

$$\begin{aligned} u'(t) = (I - \Pi_A)f(t) + A\int_{-\infty}^t e^{(t-s)A}(I - \Pi_A)f(s)\,ds \\ + \Pi_A f(t) - A\int_t^{-\infty} e^{(t-s)A}\Pi_A f(s)\,ds = Au(t) + f(t). \end{aligned}$$

Thus we have proved that if $f \in L^2\big(\mathbb{R}; D(A)\big)$, then u is a solution of (3.54). To prove the existence in general, it is sufficient to approximate $f(t)$ by $kR(k, A)f(t)$ and let k tend to infinity.

Step 2 (Uniqueness). It is sufficient to prove that if

$$u \in W^{1,2}(\mathbb{R};X) \cap L^2\big(\mathbb{R};D(A)\big)$$

is such that $u'(t) = Au(t)$ for all $t \in \mathbb{R}$, then $u(t) = 0$ for all t. In fact for such a u, we have $u(t) = e^{tA}u(0)$, so we must have $\varPi_A u(0) = 0$ and then $\varPi_A u(t) = 0$ for all t. It follows that for all $t > s$

$$u(t) = e^{(t-s)A}(I - \varPi_A)u(s),$$

and, as $t \to +\infty$, we find $u(t) = 0$ by (3.56).

□

Now we consider solutions of (3.54) in $[0, +\infty[$.

Proposition 3.17. *Assume that A verifies hypothesis $(\mathcal{P})$. Then if $f \in L^2(0,+\infty;X)$, the mild solutions of (3.54) in $L^2(0,+\infty;X)$ are given by the formula*

$$u(t) = e^{tA}(I - \varPi_A)x + \int_0^t e^{(t-s)A}(I - \varPi_A)f(s)\,ds - \int_t^{+\infty} e^{(t-s)A}\varPi_A f(s)\,ds, \quad x \in X. \tag{3.63}$$

Proof. The fact the function in (3.63) defines a solution of (3.53) in $L^2(0,+\infty;X)$ can be easily checked. Conversely, let $u \in L^2(0,+\infty;X)$ be a mild solution of (3.54) in $L^2(0,+\infty;X)$. Then we have

$$\begin{aligned} u(t) &= e^{tA}u(0) + \int_0^t e^{(t-s)A}f(s)ds \\ &= e^{tA}(I - \varPi_A)u(0) \\ &\quad + \int_0^t e^{(t-s)A}(I - \varPi_A)f(s)\,ds - \int_t^{+\infty} e^{(t-s)A}\varPi_A f(s)\,ds + v(t), \end{aligned} \tag{3.64}$$

where

$$v(t) = e^{tA}\left(\varPi_A u(0) + \int_0^{+\infty} e^{-sA}\varPi_A f(s)\,ds\right). \tag{3.65}$$

Now, v is bounded in $L^2(-\infty,0;X)$ and by (3.64), v in $L^2(0,+\infty;X)$. Thus v is a bounded solution of (3.54) with $f(t) = 0$, and, by the uniqueness proved in Proposition 3.16, it follows that $v(t) = 0$ for all t. In conclusion u is given by (3.63) and the proof is complete. □

Corollary 3.2. *Assume that A verifies Assumption $(\mathcal{P})$. Let $x \in X$, $f \in L^2(0,+\infty;X)$ and let u be the mild solution of problem* (3.1) *in* $L^2(0,+\infty;X)$. *Then the following statements are equivalent*:

(i) *$f \in L^2(0,+\infty;X)$;*

(ii) *the following compatibility condition holds*

$$\Pi_A x + \int_0^{+\infty} e^{-sA} \Pi_A f(s)\, ds = 0. \tag{3.66}$$

Proof. The conclusion follows from (3.64)–(3.65) and Proposition 3.17. □

4 Interpolation spaces

4.1 Notation

We shall consider two Banach spaces X_0 and X_1 with X_0 continuously and densely embedded in X_1. Denote by $|\cdot|_0$ and $|\cdot|_1$ the respective norms of X_0 and X_1. Let $k > 0$ be a constant such that

$$|x|_1 \leq k|x|_0. \tag{4.1}$$

For any $p \geq 1$ and $i = 0$ or 1 we shall denote by $L^p_*(X_i)$ the Banach space of all Bochner measurable mappings $u\colon [0, +\infty[\to X_1$ such that

$$\int_0^\infty |u(t)|^p_{X_i} \frac{dt}{t} < +\infty. \tag{4.2}$$

We shall use Hardy's inequality, which we recall below.

Lemma 4.1. *Let* $u\colon [0, +\infty[\to \mathbb{R}$ *be measurable and such that* $t^\alpha u \in L^p(0, \infty)$ *for some* $\alpha \in]0, 1[$. *Setting*

$$M(u)(t) = \frac{1}{t} \int_0^t u(\sigma)\, d\sigma, \tag{4.3}$$

we have $t^\alpha M(u) \in L^p(0, \infty)$ *and there exists a constant* $c(\alpha, p) > 0$ *such that*

$$\|t^\alpha M(u)\|_{L^p(0,\infty)} \leq c(\alpha, p)\|t^\alpha u\|_{L^p(0,\infty)}. \tag{4.4}$$

If Z and W are two Banach spaces, we write $Z \cong W$ to indicate that they are isomorphic and $Z \subset W$ if Z is continuously embedded in W.

We shall now give two equivalent definitions of interpolation spaces.

4.2 Spaces of traces $T(p, \alpha, X_0, X_1)$

For any $\alpha \in \mathbb{R}$ and $p \in [1, \infty[$ we shall denote by $W(p, \alpha, X_0, X_1)$ the set of all mappings $u\colon [0, \infty[\to X_1$ such that

$$t^\alpha u \in L^p(0, \infty; X_0), \quad t^\alpha u' \in L^p(0, \infty; X_1), \tag{4.5}$$

where

$$u' = \frac{du}{dt}.$$

When $W(p, \alpha, X_0, X_1)$ is endowed with the norm

$$\|u\|^p_{W(p,\alpha,X_0,X_1)} = \|t^\alpha u\|^p_{L^p(0,\infty,X_0)} + \|t^\alpha u'\|^p_{L^p(0,\infty,X_1)}, \tag{4.6}$$

it is a Banach space.

Clearly if $u \in W(p, \alpha, X_0, X_1)$, then u is a.e. equal to a continuous function u in $]0, \infty[$ with values in X_1. We prove now the stronger result.

Proposition 4.1. *Let $p \geq 1$, $0 < \alpha + 1/p < 1$, $u \in W(p, \alpha, X_0, X_1)$. Then the limit of $u(t)$ exists, as $t \to 0$, in X_1. Moreover, for any $T > 0$, there exists a constant $C_T > 0$ such that for all t, $0 \leq t \leq T$,*

$$|u(t)|_{X_1} \leq C_T \|u\|_{W(p,\alpha,X_0,X_1)}. \tag{4.7}$$

Proof. We first prove (4.7). Let $0 < \alpha + p^{-1} < 1$, $q^{-1} = 1 - p^{-1}$, $T > 0$ fixed, s, $t \in [0, T]$. By integrating with respect to s, between 0 and T the identity

$$s^\alpha u(t) = s^\alpha u(s) + s^\alpha \int_s^t u'(\sigma)\, d\sigma,$$

we obtain

$$u(t) = T^{-\alpha-1} \left\{ (\alpha+1) \int_0^T s^\alpha u(s)\, ds + \int_0^T \sigma^{\alpha+1} u'(\sigma)\, d\sigma \right.$$
$$\left. - \int_t^T [T^{\alpha+1} - \sigma^{\alpha+1}] u'(\sigma)\, d\sigma \right\}$$

from which

$$|u(t)|_1 \leq T^{-\alpha-1}(\alpha+1) \int_0^T |s^\alpha u(s)|\, ds + 2 \int_0^T |u'(\sigma)|\, d\sigma.$$

As $s^{-\alpha} \in L^q(0, T)$, by using the Hölder inequality, we find

$$|u(t)|_1 \leq kT^{-\alpha-1/p}(\alpha+1)\|s^\alpha u\|_{L^p(0,\infty;X_0)} + \|s^{-\alpha}\|_{L^q(0,T)} \|s^\alpha u'\|_{L^p(0,\infty;X_1)}$$

and (4.7) is proved. Set now $u_n(t) = u(t + 1/n)$; then, as easily checked, u_n belongs to $W(p, \alpha, X_0, X_1)$ and $u_n \to u$ in $W(p, \alpha, X_0, X_1)$. Now by (4.7) it follows that $\{u_n\}$ is a Cauchy sequence in $C([0, T]; X_1)$ and the limit of $u(t)$, as $t \to 0$, does exist. $\square$

We now assume that $0 < \alpha + p^{-1} < 1$ and consider the mapping

$$u \mapsto \gamma u = u(0) \colon W(p, \alpha, X_0, X_1) \to X_1, \tag{4.8}$$

which is well-defined by Proposition 4.1. We define the *spaces of traces* $T(p, \alpha, X_0, X_1)$ by setting

$$T(p,\alpha,X_0,X_1) = \{x \in X_1 \colon \exists u \in W(p,\alpha,X_0,X_1), u(0) = x\}.$$

$T(p,\alpha,X_0,X_1)$, endowed with the norm

$$|x|_{T(p,\alpha,X_0,X_1)} = \inf\{\|u\|_{W(p,\alpha,X_0,X_1)} \colon u(0) = x\}, \tag{4.9}$$

is a Banach space. We clearly have

$$X_0 \subset T(p,\alpha,X_0,X_1) \subset X_1. \tag{4.10}$$

Remark 4.1. If $u \in W(p,\alpha,X_0,X_1)$, then clearly, for any $t \geq 0$, $u(t) \in T(p,\alpha,X_0,X_1)$ and

$$|u(t)|_{T(p,\alpha,X_0,X_1)} \leq \|u\|_{W(p,\alpha,X_0,X_1)}. \tag{4.11}$$

It follows by a density argument that

$$u \in C\big([0,\infty[;T(p,\alpha,X_0,X_1)\big) \quad \text{for any } u \in W(p,\alpha,X_0,X_1). \tag{4.12}$$

□

Proposition 4.2. *Let* $p \geq 1$, $0 < \alpha + 1/p < 1$. *The following statements hold*:

(i) *The embedding of* $T(p,\alpha,X_0,X_1)$ *in* X_1 *is continuous.*
(ii) *The embedding of* X_0 *in* $T(p,\alpha,X_0,X_1)$ *is continuous and dense.*

Proof. (i) Due to the Closed Graph theorem, it is sufficient to prove that the embedding

$$x \mapsto x \colon T(p,\alpha,X_0,X_1) \to X_1$$

is closable. Let $x_n \to 0$ in $T(p,\alpha,X_0,X_1)$ and $x_n \to y$ in X_1. We have to show that $y = 0$. Let in fact $u_n \in W(p,\alpha,X_0,X_1)$ be such that

$$u_n(0) = x_n, \quad \|u_n\|_{W(p,\alpha,X_0,X_1)} \leq \left(1 + \frac{1}{n}\right)|x_n|_{T(p,\alpha,X_0,X_1)}.$$

This can always be done by definition of the norm in $T(p,\alpha,X_0,X_1)$ as an inf (cf. (4.9)). Then $u_n \to 0$ in $W(p,\alpha,X_0,X_1)$, so that by (4.7) $x_n = u_n(0) \to 0 = y$ in X_1.

(ii) Let $x \in X_0$, $\phi \in C^\infty([0,\infty[\colon \mathbb{R})$ such that $\phi(t) = 1$ if $t \in [0,\frac{1}{2}]$ and $\phi(t) = 0$ if $t \geq 1$. Setting $u = \phi x$ we have

$$\|u\|^p_{W^p(p,\alpha,X_0,X_1)} = \int_0^1 |t^\alpha \phi(t) x|_0^p \, dt + \int_0^1 |t^\alpha \phi'(t) x|_1^p \, dt \leq \text{ const. } |x|^p_{X_0}.$$

It follows that $|x|_{T(p,\alpha,X_0,X)} \leq$ const.$|x|_0$. This proves the continuity of the embedding $X_0 \to T(p,\alpha,X_0,X_1)$.

We now prove the density. Let $x \in T(p,\alpha,X_0,X_1)$. By definition there exists $u \in W(p,\alpha,X_0,X_1)$ such that $u(0) = x$ and we can construct a sequence $\varepsilon_n > 0$, $\varepsilon_n \to 0$, such that $u(\varepsilon_n) \in X_0$. For any such $\varepsilon > 0$ define the function

$$z_\varepsilon(t) = \begin{cases} u_\varepsilon(t) - u(t), & 0 \le t \le \varepsilon, \\ 0, & t > 0. \end{cases}$$

By construction

$$z'_\varepsilon(t) = -u'(t)\chi_{[0,\varepsilon]}(t),$$

where $\chi_{[0,\varepsilon]}$ is the characteristic function of the interval $[0,\varepsilon]$. Moreover

$$\|t^\alpha z'_\varepsilon\|_{L^p} = \|t^\alpha u'\|_{L^p(0,\varepsilon)}$$

and $t^\alpha z'_\varepsilon \to 0$ in $L^p(0,\infty;X_1)$ as $\varepsilon \to 0$. Similarly z_ε is a measurable function from $[0,\infty[$ to X_0 and we wish to show that the L^p-norm of $t^\alpha z_\varepsilon$ goes to zero as $\varepsilon \to 0$. By definition

$$\|t^\alpha z_\varepsilon\|_{L^p}^p = \int_0^\infty \left| t^\alpha \int_t^\varepsilon u'(s)\,ds\,\chi_{[0,\varepsilon]}(t)\right|^p dt = \int_0^\varepsilon t^{\alpha p} \left| \int_t^\varepsilon u'(s)\,ds \right|^p dt$$

and for $0 < t \le \varepsilon$, $1 < p < \infty$, and $p^{-1} + q^{-1} = 1$

$$\left| \int_t^\varepsilon u'(s)\,ds \right| \le \left\{ \int_t^\varepsilon s^{-\alpha q}\,ds \right\}^{1/q} \|s^a u'\|_{L^p(t,\varepsilon)}.$$

But $1 - \alpha q > 0$ and

$$\int_t^\varepsilon s^{-\alpha q}\,ds \le \frac{\varepsilon^{1-\alpha q}}{1-\alpha q}$$

because by hypothesis

$$1 - \alpha q = \frac{p}{p-1}[1 - p^{-1} - \alpha] > 0.$$

Therefore

$$\|t^\alpha z_\varepsilon\|_{L^p}^p \le \left[\frac{\varepsilon^{1-\alpha q}}{1-\alpha q} \right]^{p/q} \int_0^\varepsilon t^{\alpha p}\,dt\,\|s^\alpha u'\|_{L^p}^p.$$

Again by hypothesis $\alpha p + 1 > 0$ and

$$\int_0^\varepsilon t^{\alpha p}\,dt = \frac{\varepsilon^{\alpha p+1}}{\alpha p + 1}$$

because

$$\alpha p + 1 = p(\alpha + p^{-1}) > 0.$$

Finally

$$\|t^\alpha z_\varepsilon\|_{L^p} \le \frac{\varepsilon^{(1-\alpha q)1/q}}{(1-\alpha q)^{1/q}} \frac{\varepsilon^{(\alpha p+1)1/q}}{(\alpha p+1)^{1/q}} \|s^\alpha u'\|_{L^p}$$

and

$$\frac{1-\alpha q}{q} + \frac{\alpha p + 1}{p} = p^{-1} + q^{-1} = 1.$$

This shows that $t^\alpha z_\varepsilon \in L^p(0,\infty;X_0,X_1)$ and

$$\|t^\alpha z_\varepsilon\|_{L^p} \leq \varepsilon(1-\alpha q)^{-1/q}(\alpha p+1)^{-1/q}\|s^\alpha u'\|_p.$$

Therefore $z_\varepsilon \in W(p,\alpha;X_0,X_1)$ and $z_\varepsilon \to 0$. Hence by Remark 4.1

$$x_n - x = u(\varepsilon_n) - u(0) = z_{\varepsilon_n}(0) \to 0$$

in $T(p,\alpha;X_0,X_1)$ as $\varepsilon_n \to 0$.

When $p=1$, we have by hypothesis $-1<\alpha<0$

$$\begin{aligned}\|t^\alpha z_\varepsilon\|_{L^1} &= \int_0^\varepsilon t^\alpha \left|\int_t^\varepsilon u'(s)\,ds\right| dt \\ &\leq \int_0^\varepsilon t^\alpha \max_{[t,\varepsilon]} s^{-\alpha}\,dt\,\|s^\alpha u'\|_{L^1} \\ &\leq \int_0^\varepsilon t^\alpha\,\varepsilon^{-\alpha}\,dt\,\|s^\alpha u'\|_{L^1}\end{aligned}$$

because

$$t \leq s \leq \varepsilon \Longrightarrow t^{-\alpha} \leq s^{-\alpha} \leq \varepsilon^{-\alpha}.$$

Therefore because $\alpha+1>0$

$$\|t^\alpha z_\varepsilon\|_{L^1} \leq \frac{\varepsilon^{\alpha+1}}{\alpha+1}\varepsilon^{-\alpha}\|s^\alpha u'\|_{L^1} = \frac{\varepsilon}{\alpha+1}\|s^\alpha u'\|_{L^1},$$

and we reach the same conclusion as for $p>1$. This proves the density of X_0 in $T(p,\alpha;X_0,X_1)$ □

Example 4.1. Let $X_0 = H^1(\mathbb{R})$, $X_1 = L^2(\mathbb{R})$, $p=2$, $\alpha=0$. Then

$$W(2,0,H^1(\mathbb{R}),L^2(\mathbb{R})) = \{u \in L^2\big(0,\infty;H^1(\mathbb{R})\colon u' \in L^2(0,\infty;L^2(\mathbb{R})\big)\}.$$

For any $u \in W\big(2,0,H^1(\mathbb{R}),L^2(\mathbb{R})\big)$ we set $\tilde{u}(t,x) = u(t)(x)$ and $\mathbb{R}^2_+ = \{(t,x)\in\mathbb{R}^2\colon t\geq 0\}$. Then, by Fubini's theorem, we have

$$\tilde{u}, \quad \frac{\partial\tilde{u}}{\partial t}, \frac{\partial\tilde{u}}{\partial x} \in L^2(\mathbb{R}^2_+)$$

so that $\tilde{u} \in H^1(\mathbb{R}^2_+)$ and $W\big(2,0,H^1(\mathbb{R}),L^2(\mathbb{R})\big) \cong H^1(\mathbb{R}^2_+)$, whereas

$$T\big(2,0,H^1(\mathbb{R}),L^2(\mathbb{R})\big) \cong H^{1/2}(\mathbb{R}^2_+).$$ □

We now give the Interpolation Theorem.

Theorem 4.1. *Let $X_0 \to X_1$, $Y_0 \to Y_1$ be Banach spaces. Let*

$$\pi \in \mathcal{L}(X_0,Y_0)\cap\mathcal{L}(X_1,Y_1).$$

Then if $p\geq 1$ and $0<\alpha+p^{-1}<1$, we have

$$\pi \in \mathcal{L}\big(T(p,\alpha,X_0,X_1),T(p,\alpha,Y_0,Y_1)\big)$$

and

$$\|\pi\|_{\mathcal{L}\left(T(p,\alpha,X_0,X_1),T(p,\alpha,Y_0,Y_1)\right)} \leq \max\{\|\pi\|_{\mathcal{L}(X_0,Y_0)},\|\pi\|_{\mathcal{L}(X_1,Y_1)}\}. \tag{4.13}$$

4.3 Spaces of averages $(X_0, X_1)_{\theta,p}$

For any $\theta \in]0,1[$ and any $p \in [1,\infty[$ set

$$(X_0, X_1)_{\theta,p} = \left\{ x \in X_1 : \begin{array}{l} \exists u_i \colon [0,\infty[\to X_i, \quad i = 0,1, \\ t^{-\theta} u_0 \in L^p_*(X_0), \\ t^{1-\theta} u_1 \in L^p_*(X_1), \\ x = u_0(t) = u_1(t) \quad \text{a.e.} \end{array} \right\} \tag{4.14}$$

and moreover

$$\begin{aligned} |x|_{(X_0,X_1)_{\theta,p}} = \inf\{ &\|t^{-\theta} u_0\|_{L^p_*(X_0)} + \|t^{1-\theta} u_1\|_{L^p_*(X_1)} \\ &: t^{-\theta} u_0 \in L^p_*(X_0), t^{1-\theta} u_1 \in L^p_*(X_1)\}. \end{aligned} \tag{4.15}$$

In the variational literature, it is customary to denote by $[X_0, X_1]_\theta$ the space $(X_0, X_1)_{\theta,2}$. This notation will be used in Chapter 2 and occasionally in the other parts of the book.

Proposition 4.3. *Let $p \geq 1$, $0 < \alpha + p^{-1} < 1, \theta = \alpha + p^{-1}$. Then we have*

$$(X_0, X_1)_{\theta,p} \cong T(p, \alpha, X_0, X_1). \tag{4.16}$$

Proof. We proceed in two steps.

Step 1. We have

$$T(p, \alpha, X_0, X_1) \subset (X_0, X_1)_{\theta,p}. \tag{4.17}$$

Let $x \in T(p, \alpha, X_0, X_1)$ and $u \in W(p, a, X_0, X_1)$ such that $x = u(0)$. Set

$$= u_0(t) = u\left(\frac{1}{t}\right), \quad u_1(t) = -\int_0^{1/t} u'(s)\, ds = x - u\left(\frac{1}{t}\right). \tag{4.18}$$

We first check that

$$t^{-\theta} u_0 \in L^p_*(X_0). \tag{4.19}$$

In fact we have

$$\begin{aligned} \int_0^\infty |t^{-\theta} u_0(t)|_0^p \frac{dt}{t} &= \int_0^\infty \left| t^{-\theta} u\left(\frac{1}{t}\right)\right|_0^p \frac{dt}{t} \\ &= \int_0^\infty |s^\theta u(s)|_0^p \frac{ds}{s} \\ &= \int_0^\infty |s^\alpha u(s)|_0^p \, ds \leq \|u\|^p_{W(p,\alpha,X_0,X_1)} \end{aligned}$$

and (4.19) is proved. Then we check that

$$t^{1-\theta} u_1 \in L^p_*(X_1). \tag{4.20}$$

We have

$$\begin{aligned}\int_0^\infty |t^{1-\theta}u_1(t)|_1^p \frac{dt}{t} &= \int_0^\infty \left| t^{1-\theta} \int_0^{1/t} u'(\sigma) \right|_1^p \frac{dt}{t} \\ &= \int_0^\infty \left| s^{\theta-1} \int_0^s u'(\sigma)\, d\sigma \right|_1^p \frac{ds}{s} \\ &= \int_0^\infty \left| s^\alpha \frac{1}{s} \int_0^s u'(\sigma)\, \sigma \right|_1^p ds \\ &\leq \text{ const. } \int_0^\infty |s^\alpha u'(s)|_1^p\, ds \\ &\leq \text{ const. } \|u\|^p_{W(p,\alpha,X_0,X_1)}\end{aligned}$$

due to Hardy's inequality. Thus the inclusion (4.17) is proved.

Step 2. We have

$$(X_0, X_1)_{\theta,p} \subset T(p, \alpha, X_0, X_1). \tag{4.21}$$

Assume that $x = u_0(t) + u_1(t)$ with $t^{-\theta}u_0 \in L^p_*(X_0)$, $t^{1-\theta}u_1 \in L^p_*(X_1)$. We first regularize u_0 and u_1 by setting

$$\tilde{u}_0(t) = \frac{1}{t}\int_0^t u_0(s)\, ds, \quad \tilde{u}_1(t) = \frac{1}{t}\int_0^t u_1(s)\, ds. \tag{4.22}$$

We still have $x = \tilde{u}_0(t) + \tilde{u}_1(t)$. We now set

$$u(t) = \tilde{u}_0\left(\frac{1}{t}\right) = x - \tilde{u}_1\left(\frac{1}{t}\right) = t\int_0^{1/t} u_0(s)\, ds \tag{4.23}$$

and prove successively that for $\theta = \alpha + p^{-1}$

$$t^\alpha u \in L^p(0,\infty; X_0), \quad t^\alpha u' \in L^p(0,\infty; X_1), \quad u(0) = x.$$

This will prove inclusion (4.21).

We show (4.24). We have

$$\begin{aligned}\int_0^\infty |t^\alpha u(t)|_0^p\, dt &= \int_0^\infty |t^{\theta-1/p}u(t)|_0^p\, dt \\ &= \int_0^\infty |t^{\theta-1/p+1}\int_0^{1/t} u_0(\sigma)\, d\sigma|_0^p\, dt \\ &= \int_0^\infty \left|\tau^{1/p-\theta-1}\int_0^\tau u_0(\sigma)\, d\sigma\right|_0^p \tau^{-2}\, d\tau \\ &= \int_0^\infty \left|\tau^{-\theta}\int_0^\tau u_0(\sigma)\, d\sigma\right|_0^p \tau^{-1}\, d\tau \leq \text{ const. } \int_0^\infty |t^{-\theta}u_0(t)|_0^p \frac{dt}{t}\end{aligned}$$

and (4.24) is proved.

To show (4.24) consider the following identity:

$$u'(t) = -\int_0^{1/t} u_1(s)\, ds + \frac{1}{t} u_1\left(\frac{1}{t}\right) \tag{4.24}$$

and set

$$J = \int_0^\infty |t^\alpha u'(t)|_1^p \, dt.$$

We have $J \le A^{1/p} + B^{1/p}$, where

$$A = \int_0^\infty \left| t^\alpha \int_0^{1/t} u_1(\sigma)\, d\sigma \right|_1^p dt, \quad B = \int_0^\infty \left| t^{\alpha-1} u_1\left(\frac{1}{t}\right)\right|_1^p dt.$$

Now, setting $\tau = t^{-1}$ and using once again Hardy's inequality, we have

$$\begin{aligned} A &= \int_0^\infty \left|\tau^{1-\theta}\frac{1}{\tau}\int_0^\tau u_1(\sigma)\, d\sigma\right|_1^p \tau^{-1}\, d\tau \le \text{ const. } \int_0^\infty |\tau^{1-\theta}\tilde{u}_1(\tau)|_1^p \tau^{-1}\, d\tau \\ &= \text{ const. } \|t^{1-\theta}\tilde{u}_1\|_{L^p_*(X_1)}. \\ B &= \int_0^\infty |\tau^{1-\theta} u_1(\tau)|_1^p \tau^{-1}\, d\tau = \|t^{1-\theta} u_1\|_{L^p_*(X_1)}. \end{aligned}$$

Thus (4.24) follows.

It remains to show (4.24). By Proposition 4.1 the limit $\alpha = \lim_{t\to 0} u(t)$ exists in X_1. By (4.23) it follows that $\lim_{t\to 0} u_1(t) = x - \alpha = \beta$. So we have to prove that $\beta = 0$. By Hardy's inequality we have $t^{1-\theta}\tilde{u}_1 \in L^p_*(X_1)$ so that $\int_0^\infty t^{p(1-\theta)-1}|\tilde{u}_1(t)|_1^p\, dt < \infty$, which implies $\beta = 0$. □

We now prove some inclusions.

Proposition 4.4. *Let* $p, q \in [1,\infty[\,, \theta, \omega \in]0,1[$*; then*:

(i) *if* $p < q$, *we have* $(X_0, X_1)_{\theta,p} \subset (X_0, X_1)_{\theta,q'}$,
(ii) *if* $\theta < \omega$, *we have* $(X_0, X_1)_{\theta,p} \subset (X_0, X_1)_{\omega,q'}$.

Proof. Let $x \in (X_0, X_1)_{\theta,p}$, $x = u_0(t) + u_1(t)$ with $t^{-\theta}u_0 \in L^p_*(X_0)$, $t^{1-\theta}u_1 \in L^p_*(X_1)$. Choose $\phi \in \mathcal{D}(]0,\infty[)$ such that $\phi(t) \ge 0$, $\int_0^\infty \phi(s)s^{-1}\, ds = 1$ and set

$$v_j(t) = \int_0^\infty \phi(s) u_j\left(\frac{t}{s}\right)\frac{ds}{s}, \quad j = 0, 1.$$

We have $x = v_0(t) + v_1(t)$. Moreover, due to Young's inequality,

$$\begin{aligned} t^{-\theta}v_0 &\in L^p_*(X_0) \cap L^\infty(X_0) \subset L^q_*(X_0), \\ t^{1-\theta}v_1 &\in L^p_*(X_1) \cap L^\infty(X_1) \subset L^q_*(X_1), \end{aligned}$$

and (i) is proved. Let us now prove (ii). Due to (i) it is sufficient to prove that

$$(X_0, X_1)_{\theta,p} \subset (X_0, X_1)_{\omega,1}. \tag{4.25}$$

Let $x \in (X_0, X_1)_{\theta,p}$, $x - u(0), t^\alpha u \in L^p(0,\infty; X_0)$, $t^\alpha u' \in L^p(0,\infty; X_1)$, for $\alpha + p^{-1} = \theta$. We can assume $u(t) = 0$ for any $t \geq 1$. Using Hölder's inequality we have

$$t^{\omega-1} u \in L^1(0,\infty; X_0), \quad t^{\omega-1} u' \in L^1(0,\infty; X_1). \tag{4.26}$$

In fact we have, for instance

$$\int_0^1 |t^{\omega-1} u(t)|_0 \, dt = \int_0^1 t^{\omega-1-\alpha} |t^\alpha u(t)|_0 \, dt$$

$$\int_0^1 |t^\alpha u(t)|_0^p \, dt \left[\int_0^1 \Big| \, s^{\omega-1-\alpha)q} \, ds \right]^{1/q} = \left(\frac{p(\omega - \theta)}{p-1} \right)^{1/q} \int_0^1 |t^\alpha u(t)|_0^p \, dt < \infty.$$

Now (4.26) implies that $x \in T(p, \omega - 1, X_0, X_1) \cong (X_0, X_1)_{\omega,1}$. □

4.4 Interpolation spaces between the domain of a linear operator *A* and the space *X*

Let X be a Banach space and A be a closed linear operator in X with dense domain $D(A)$ in X. We assume that $\rho(A) \supset \,]0, +\infty[$ and that there exists a constant $C > 0$ such that

$$\|R(\lambda, A)\| \leq \frac{C}{\lambda}, \quad \lambda > 0. \tag{4.27}$$

Proposition 4.5. *Let $p \geq 1$, $\theta \in \,]0,1[$; then we have*

$$(D(A), X)_{\theta,p} = \{x \in X \colon \lambda^{1-\theta} A R(\lambda, A) x \in L^p_*(X)\}. \tag{4.28}$$

Moreover the norm $|x|_{(D(A),X)_{\theta,p}}$ is equivalent to

$$|x|_X + \|\lambda^{1-\theta} A R(\lambda, A) x\|_{L^p(X)}. \tag{4.29}$$

Proof. The proof proceeds in several steps.

Step 1.

We have the following inclusion:

$$(D(A), X)_{\theta,p} \subset \{x \in X \colon \lambda^{1-\theta} A R(\lambda, A) x \in L^p_*(X)\}. \tag{4.30}$$

Let $x = u_0(t) + u_1(t)$ with $t^{-\theta} u_0 \in L^p_*\big(D(A)\big)$, $t^{1-\theta} u_1 \in L^p_*(X)$. We have

$$t^{1-\theta} A R(t, A) x = t^{-\theta} \big(t R(t, A)\big) A u_0(t) + t^{1-\theta} \big(A R(t, A)\big) u_1(t).$$

It follows that

$$\|t^{-\theta} A R(t, A) x\|^p_{L^p_*(X)} \leq C^p \|t^{-\theta} u_0\|^p_{L^p_*\left(D(A)\right)} + (1 + C) \|t^{1-\theta} u_1\|^p_{L^p_*(X)}$$

and (4.29) is proved.

Step 2.

We have

$$\{x \in X \colon \lambda^{1-\theta} AR(\lambda, A)x \in L^p_*(X)\} \subset (D(A), X)_{\theta,p}. \tag{4.31}$$

Assume that $t^{1-\theta} AR(t, A)x \in L^p_*(X)$ and set

$$u_0(t) = \begin{cases} 0 & \text{if } t < 1, \\ tR(t, A)x & \text{if } t \geq 1, \end{cases} \quad u_1(t) = \begin{cases} x & \text{if } t < 1, \\ -AR(t, A)x & \text{if } t \geq 1, \end{cases}$$

so that $u_0(t) + u_1(t) = x$. We have $t^{-\theta} u_0 \in L^p_*\big(D(A)\big)$. In fact

$$\begin{aligned} \int_0^\infty |t^{-\theta} u_0(t)|^p_{D(A)} \frac{dt}{t} &= \int_1^\infty |t^{1-\theta} AR(t, A)x|^p \frac{dt}{t} \\ &= \|t^{1-\theta} AR(t, A)x\|^p_{L^p_*(X)}. \end{aligned}$$

We finally prove that $t^{1-\theta} u_1 \in L^p_*(X)$. We have in fact

$$\begin{aligned} \int_0^\infty |t^{1-\theta} u_1(t)|^p_X \frac{dt}{t} &= \int_0^1 t^{p(1-\theta)-1} |x|_X \, dt + \int_1^\infty |t^{1-\theta} AR(t, A)x|^p \frac{dt}{t} \\ &\leq \|t^{1-\theta} AR(t, A)x\|^p_{L^p_*(X)} + \frac{1}{p(1-\theta)} |x|_X. \end{aligned}$$

□

4.5 The case of a strongly continuous semigroup

Proposition 4.6. *Assume that A is the infinitesimal generator of a strongly continuous semigroup S in X of negative type. Then, if $p \geq 1$ and $\theta \in]0, 1[$, we have*

$$(D(A), X)_{\theta,p} = \{x \in X \colon t^{\theta-1}[S(t)x - x] \in L^p_*(X)\}. \tag{4.32}$$

Moreover the norm $|x|_{(D(A),X)_{\theta,p}}$ is equivalent to

$$|x|_X + \|t^{\theta-1}(S(t)x - x)\|_{L^p_*(X)}. \tag{4.33}$$

Proof. Again the proof proceeds in several steps.

Step 1.

We have

$$x \in (D(A), X)_{\theta,p} \Longrightarrow t^{\theta-1}(S(t)x - x) \in L^p_*(X). \tag{4.34}$$

We set

$$\frac{1}{t}(S(t)x - x) = \frac{1}{t} \int_0^t g(s) \, ds, \tag{4.35}$$

where $g(s) = \big(I - S(t-s)\big)u'(s) + S(t-s)Au(s)$ and u is a function in $W(\alpha, p, D(A), X)$, $\alpha + p^{-1} = \theta$, such that $u(0) = x$. We have

$$t^\alpha u \in L^p\big(0, \infty; D(A)\big),$$

which is equivalent to $t^\theta u \in L^p_*\big(D(A)\big)$. Therefore $t^\theta g \in L^p_*(X)$ so that $t^{\theta-1}(S(t)x - x) \in L^p_*(X)$ by virtue of Hardy's inequality and (4.34) is proved.

Step 2.

We have

$$t^{\theta-1}(S(t)x - x) \in L^p_*(X) \Longrightarrow x \in (D(A), X)_{\theta,p}. \tag{4.36}$$

Let $v(t) = t^{\theta-1}(S(t)x - x) \in L^p_*(X)$; we have to prove (Proposition 4.5 that $\lambda^{1-\theta}AR(\lambda, A) \in L^p_*(X)$, which is equivalent to show that

$$AR(\lambda, A)x = \lambda \int_0^\infty e^{-\lambda s}(S(s)x - x)\, ds.$$

We have

$$t^{\theta-1}AR\left(\frac{1}{t}, A\right)x = t^{\theta-2}\int_0^\infty e^{-s/t}s^{1-\theta}v(s)\, ds = \int_0^\infty v(s)\psi\left(\frac{t}{s}\right)\frac{ds}{s},$$

where $\psi(\xi) = \xi^{\theta-2}e^{-\xi} \in L^1_*(X)$. By Young's inequality it follows that $t^{\theta-1}AR(1/t, A)x \in L^p_*(X)$ and the proof is complete. $\square$

4.6 The case of an analytic semigroup

Proposition 4.7. *Assume that A is the infinitesimal generator of an analytic semigroup S of negative type. Then if $p \geq 1$ and $\theta \in\,]0, 1[$, we have*

$$(D(A), X)_{\theta,p} = \{x \in X : t^\theta AS(t)x \in L^p_*(X)\}. \tag{4.37}$$

Moreover the norm $|x|_{(D(A),X)_{\theta,p}}$ is equivalent to

$$|x|_X + \|t^\theta AS(t)x\|_{L^p_*(X)}. \tag{4.38}$$

Proof. Again we proceed in several steps.

Step 1.

We have

$$t^\theta AS(t)x \in L^p_*(X) \Longrightarrow t^{-1}(S(t)x - x) \in L^p_*(X). \tag{4.39}$$

As

$$t^{-1}(S(t)x - x) = \frac{1}{t}\int_0^t AS(s)x\, ds,$$

if $t^\theta AS(t)x \in L^p_*(X)$, we have, by Hardy's inequality, identity (4.37).

Step 2.

If $v(\lambda) = \lambda^{1-\theta} AR(\lambda, A) \in L^p_*(X)$, we have $t^{-\theta} AS(1/t)x \in L^p_*(X)$ and then $t^\theta AS(t)x \in L^p_*(X)$. The conclusion follows from Young's inequality and the following identity:

$$t^{-\theta} AS\left(\frac{1}{t}\right)x = \frac{1}{2\pi i} \int_\gamma e^{-\lambda/t} (\lambda/t)^\theta v(\lambda) \frac{d\lambda}{\lambda},$$

where the path γ is chosen as in §2.7 (Theorem 2.10). □

Remark 4.2. By (4.12) in §4.1, if H is a Hilbert space and A is the infinitesimal generator of an analytic semigroup, then

$$W^{1,2}(0, T; H) \cap L^2\big(0, T; D(A)\big) \subset C\big([0, T]; D_A(\tfrac{1}{2}, 2)\big),$$

where $D_A(\frac{1}{2}, 2) = T(2, 0; D(A), H) \cong (D(A), H)_{1/2,2}$. □

Example 4.2. Let X be a Banach space and set $E = L^p(\mathbb{R}; X)$, $p \geq 1$. Then the linear operator on E defined as

$$Au = u', \quad D(A) = W^{1,p}(\mathbb{R}; X) \tag{4.40}$$

is the infinitesimal generator of the semigroup

$$(S(t)u)(s) = u(t + s). \tag{4.41}$$

By Proposition 4.6, it follows that $u \in \big(D(A), X\big)_{\theta,p}$ if and only if

$$\int_0^\infty \int_{-\infty}^{+\infty} t^{\theta p - p - 1} |u(t + s) - u(s)|^p \, dt \, ds < +\infty. \tag{4.42}$$

We set

$$\big(W^{1,p}(\mathbb{R}; X), L^p(\mathbb{R}; X)\big)_{\theta,p} = W^{1-\theta,p}(\mathbb{R}; X), \quad p \geq 1, \ \theta \in]0, 1[. \tag{4.43}$$

□

Example 4.3. Let H be a Hilbert space and A be a self-adjoint negative operator on H. Then A is the infinitesimal generator of an analytic semigroup S (see §2.2.1). Given the spectral family E_λ associated with A, we have

$$AS(t)x = \int_{-\infty}^0 \lambda e^{\lambda t} dE_\lambda(x), \quad t > 0, \ x \in H \tag{4.44}$$

$$|AS(t)x|^2 = \int_{-\infty}^0 \lambda^2 e^{2\lambda t} d|E_\lambda(x)|^2, \quad t > 0, \ x \in H. \tag{4.45}$$

Let $x \in (D(A), H)_{\theta,2}$. Due to Proposition 4.7, this is equivalent to

$$\int_0^{+\infty} t^{2\theta-1}|AS(t)x|^2\,dt < \infty. \tag{4.46}$$

On the other hand, we have

$$\begin{aligned}\int_0^{+\infty} t^{2\theta-1}|AS(t)x|^2\,dt &= \int_{-\infty}^0 d|E_\lambda(x)|^2 \int_0^{+\infty} \lambda^2 e^{2\lambda t} t^{2\theta-1}\,dt \\ &= \Gamma(2\theta)2^{-2\theta}\int_{-\infty}^0 (-\lambda)^{2-2\theta} d|E_\lambda(x)|^2 \\ &= \Gamma(2\theta)2^{-2\theta}|(-A)^{1-\theta}x|^2, \end{aligned} \tag{4.47}$$

that is

$$(D(A), H)_{\theta,2} \cong D\big((-A)^{1-\theta}\big), \tag{4.48}$$

where $(-A)^{1-\theta}$ is the $(1-\theta)$-th fractional power of the positive self-adjoint operator $(-A)$ (cf. F. Riesz and B. Sz.-Nagy [1]). □

4.7 The interpolation space $[X, Y]_\theta$

In §4 we have described the so-called "real interpolation methods" to construct the interpolation space $(X, Y)_{\theta,p}$. "Complex interpolation methods" can also be used to obtain interpolation spaces that are denoted by $[X, Y]_\theta$. The nice feature of $[X, Y]_\theta$ is that it only depends on one parameter. In general real and complex methods yield different interpolation spaces (cf. H. Triebel [1, p. 15 and pp. 55–59]). However, for $p = 2$ and two Hilbert spaces X and Y such that $X \subset Y$, the spaces coincide

$$[X, Y]_\theta = (X, Y)_{\theta,2}$$

(cf. H. Triebel [1, Remarks 3 and 4, p. 143]).

This discussion is important because in the variational literature, the notation $[X, Y]_\theta$ is widely used. For instance, in J. L. Lions and E. Magenes [1, Volume I, Chapter 1, §15, p. 108 and comments in §17, pp. 113–114], the authors use as a definition of $[X, Y]_\theta$ for two Banach spaces X and Y (which are contained in a locally convex topological vector space Φ, $X \subset \Phi$, $Y \subset \Phi$, with continuous injection of X into Y)

$$\{a\colon a \in X + Y, t^{-(\theta+1/2)}K(t, a; X, Y) \in L^2(0, \infty)\},$$

where

$$K(t, a; X, Y) = \inf_{a_0+a_1=a} [\|a_0\|_X^2 + t^2\|a_1\|_Y^2]^{1/2}$$

for $a_0 \in X$ and $a_1 \in Y$. But this coincides with the definition of $(X, Y)_{\theta,2}$ given by H. Triebel [1, §1.4.2, p. 29].

As a result in Remark 4.2

$$D_A(\tfrac{1}{2}, 2) \cong (D(A), H)_{1/2,2} = [D(A), H]_{1/2}$$

and in Example 4.3 (cf. Equation (4.48))

$$[D(A), H]_\theta = (D(A), H)_{\theta,2} \cong D\big((-A)^{1-\theta}\big).$$

5 Fractional powers of dissipative operators

We shall not attempt to duplicate the general theory, which is available elsewhere. We simply quote a certain number of definitions and results from H. Tanabe [1, Chapter 2, §2.3] and A. Pazy [2, §2.6] for operators defined on a Hilbert space H. Our objective is to use this material in §6 to establish the connection between the interpolation space $[D(A), H]_\theta = (D(A), H)_{\theta,2}$ and the domain $D\big((-A)^{1-\theta}\big)$ of the fractional power of $(-A)$ for the infinitesimal generator A of a strongly continuous semigroup. It turns out that when A is the infinitesimal generator of a strongly continuous semigroup, we shall always speak of the fractional powers of the operator $-A$. To avoid carrying a minus sign everywhere, it is customary to present the theory for a general operator A instead of $-A$. With this warning we proceed in the traditional manner.

Definition 5.1 (H. Tanabe [1, Definition 2.3.1, p. 32]**).** Let A be a closed linear operator densely defined in a Hilbert space H. The operator A is said to be of *type* (ω, M) if:

(i) $\exists\omega$, $0 < \omega < \pi$, $\exists M \geq 1$, such that

$$\rho(A) \supset \{\lambda\colon |\arg\lambda| > \omega\}, \quad \forall\lambda < 0, \quad \|\lambda(A-\lambda)^{-1}\| \leq M,$$

(ii) and $\forall\varepsilon > 0$, $\exists M_\varepsilon > 0$ such that

$$\|\lambda(A-\lambda)^{-1}\| \leq M_\varepsilon, \quad \forall\lambda,\ |\arg\lambda| > \omega + \varepsilon. \qquad \square$$

Remark 5.1. The type (ω, M) of an operator A is not to be confused with the type $\omega_0(S)$ of a strongly continuous semigroup $\{S(t)\}$ as defined in (2.8) of §2.2. $\square$

For analytic semigroups $\{S(t)\}$ such that

$$\exists c > 0, \quad \exists\alpha > 0, \quad |S(t)x| \leq ce^{-\alpha t}|x|, \quad \forall x \in H, \tag{5.1}$$

with infinitesimal generator A, $(-A)$ is of type (ω, M) for some $\omega < \pi/2$ and $M > 0$ (cf. Assumption A in §2.7). It is also known that an operator A is maximal dissipative if and only if $(-A)$ is of type $(\pi/2, 1)$. Thus the class of operators of type (ω, M) is certainly not empty.

Assume that A is a linear operator of type (ω, M) on H and that A^{-1} is bounded. In this case, there exists a neighbourhood U of 0 such that

$$\rho(A) \supset S \stackrel{\text{def}}{=} \{\lambda\colon |\arg\lambda| > \omega\} \cup U. \tag{5.2}$$

With this assumption we can construct a contour Γ similar to the one in §2.7 and define for all $\alpha > 0$ the bounded linear operator

$$A^{-\alpha} = \frac{1}{2\pi i}\int_\Gamma \lambda^{-\alpha}(A-\lambda)^{-1}\,d\lambda. \tag{5.3}$$

It is easy to see that $\mathrm{Ker}[A^{-\alpha}] = \{0\}$ and that the following definition is meaningful for $\alpha > 0$:

$$D(A^\alpha) = R(A^{-\alpha}), \quad A^\alpha = (A^{-\alpha})^{-1}.$$

For $\alpha = 0$, we set $A^0 = I$.

Proposition 5.1 (H. Tanabe [1, pp. 35–39]). *Given a linear operator A of type (ω, M) such that A^{-1} is bounded, then*:

(i) *A^α is a closed operator with a dense domain,*
(ii) *if $0 < \alpha < \beta$, then $D(A^\alpha) \supset D(A^\beta)$,*
(iii) *for all $\alpha > 0$, $\beta > 0$*

$$A^{\alpha+\beta} = A^\alpha A^\beta = A^\beta A^\alpha, \tag{5.4}$$

(iv) *if $0 < \alpha < 1$, then A^α is of type $(\alpha\omega, M)$,*
(v) *for $0 \leq \alpha < \beta \leq 1$, there exists $c_{\alpha,\beta} > 0$ (dependent only on M, α, β) such that*

$$\forall u \in D(A^\beta), \quad \|A^\alpha u\| \leq c_{\alpha,\beta} \|A^\beta u\|^{\alpha/\beta} \|u\|^{1-\alpha/\beta}. \tag{5.5}$$

We now specialize the above results.

Definition 5.2. The operator A is said to be *accretive* (resp. *maximal accretive*) if -A is dissipative (resp. maximal dissipative). □

The reader is referred to §2.6 for a study of the properties of dissipative operators.

Lemma 5.1 (T. Kato [3, Lemma A.6, p. 272]). *If A is a closed and maximal accretive operator in the Hilbert space H and*

$$\exists \delta > 0 \text{ such that } \forall u \in D(A), \quad \mathrm{Re}(Au, u) \geq \delta(u, u), \tag{5.6}$$

then for α, $0 < \alpha < 1$,

$$\forall u \in D(A^\alpha), \quad \mathrm{Re}(A^\alpha u, u) \geq \delta^\alpha (u, u). \tag{5.7}$$

Theorem 5.1 (T. Kato [3, Theorem 3.1, p. 258]). *If A is a closed maximal accretive operator in the Hilbert space H, then for α, $0 \leq \alpha < \frac{1}{2}$, A^α is a closed maximal accretive operator and*

$$D(A^\alpha) = D(A^{*\alpha}) \tag{5.8}$$

and there exist constants $m_1 > 0$ and $m_2 > 0$ such that

$$\forall u \in D(A), \quad m_1 \|A^\alpha u\| \leq \|A^{*\alpha} u\| \leq m_2 \|A^\alpha u\|. \tag{5.9}$$

For analytic semigroup such that (5.1) is verified, we have the following results.

Theorem 5.2 (A. Pazy [2, Theorem 6.13, p. 74])**.** *Let $-A$ be the infinitesimal generator of an analytic semigroup $T(t)$. If $0 \in \rho(A)$; then*:

(i) $T(t)\colon X \to D(A^\alpha)$, $\forall t > 0$, $\forall \alpha \geq 0$,
(ii) $\forall x \in D(A^\alpha)$, $T(t)A^\alpha x = A^\alpha T(t)x$,
(iii) $\forall t > 0$, $A^\alpha T(t)$ *is bounded and there exist* $M_\alpha > 0$ *such that*

$$\|A^\alpha T(t)\| \leq M_\alpha t^{-\alpha} e^{-\delta t}, \tag{5.10}$$

(iv) *for all* α, $0 < \alpha \leq 1$,

$$\exists c_\alpha > 0, \quad \forall x \in D(A^\alpha), \quad \|T(t)x - x\| \leq c_\alpha t^\alpha \|A^\alpha x\|. \tag{5.11}$$

6 Interpolation spaces and domains of fractional powers of an operator

In this section we use the notation $[X, Y]_\theta$ as introduced in §4.7 with X and Y two Hilbert spaces with continuous injection of X into Y.

We have seen that for a negative self-adjoint operator on a Hilbert space, it is possible to define the α-th fractional power $(-A)^\alpha$, $0 < \alpha < 1$, of $-A$. Example 4.3 in §4.6 shows that the domain $D((-A)^\alpha)$ of the operator $(-A)^\alpha$ is isomorphic to the interpolation space

$$[D(A), H]_{1-\alpha} = (D(A), H)_{1-\alpha,2}. \tag{6.1}$$

The same type of construction can be repeated for maximal dissipative operators, and J. L. Lions [2] showed that

$$D\big((-A)^\alpha\big) \cong [D(A), H]_{1-\alpha}, \quad 0 < \alpha < 1. \tag{6.2}$$

Moreover it was also known (cf. T. Kato [2, Theorem 3.1, p. 258]) that for that class of operators

$$D\big((-A)^\alpha\big) = D\big((-A^*)^\alpha\big), \quad 0 < \alpha < \tfrac{1}{2}. \tag{6.3}$$

These results were extended by A. Yagi [1] to include differential operators that are not generally maximal dissipative.

In this section we essentially give an English translation of A. Yagi [1]'s note to the *Comptes Rendus*. It is useful to recall the following definitions, notation, and relations, which have been used in §3 through §5 (cf. (3.37)–(3.38), (4.14)–(4.15)), for a Hilbert space H

$$\begin{aligned} D_A(\theta, 2) &\overset{\text{def}}{=} T(2, \theta; D(A), H), \quad 0 \leq \theta \leq 1, \\ [D(A), H]_\theta &= (D(A), H)_{\theta.2}, \quad 0 \leq \theta \leq 1. \end{aligned}$$

Moreover by Proposition 4.3 we know that for all θ, $0 \leq \theta <\leq 1$

$$D_A(\theta, 2) = T(2, \theta; D(A), H) \cong (D(A), H)_{\theta.2} = [D(A), H]_\theta. \tag{6.4}$$

Theorem 6.1 (A. Yagi [1]). *Let A be a linear operator of type (ω, M) on a Hilbert space H such that A^{-1} is bounded in H. Denote by A^* the adjoint of A. Then the following conditions are equivalent*:

(i) *for all θ, $0 \leq \theta \leq 1$,*

$$D(A^\theta) = [D(A), H]_{1-\theta} \quad \textit{and} \quad D(A^{*\theta}) = [D(A^*), H]_{1-\theta}; \tag{6.5}$$

(ii) *there exists α, β, $0 < \alpha, \beta < 1$, such that*

$$D(A^\alpha) \subset [D(A), H]_{1-\alpha} \quad \textit{and} \quad D(A^{*\beta}) \subset [D(A^*), H]_{1-\beta}; \tag{6.6}$$

(iii) *there exists α, β, $0 < \alpha, \beta < 1$, and there exists constants $M_\alpha \geq 0$ and $M^*_\beta \geq 0$ such that*

$$\left\{\int_0^\infty \lambda^{2\alpha-1} \|A^{1-\alpha}(\lambda + A)^{-1} f\|^2_H \, ds\right\}^{1/2} \leq M_\alpha \|f\|_H, \quad \forall f \in H; \tag{6.7}$$

$$\left\{\int_0^\infty \lambda^{2\beta-1} \|A^{*(1-\beta)}(\lambda + A)^{-1} g\|^2_H \, dx\right\}^{1/2} \leq M^*_\beta \|g\|_H, \quad \forall g \in H. \tag{6.8}$$

(iv) *the holomorphic function $A^{-z}\colon \{z : \operatorname{Re} z > 0\} \to \mathcal{L}(H, H)$ (and hence the holomorphic function $A^{*(-z)}$) can be extended to a strongly continuous function from $\{z\colon \operatorname{Re} z \geq 0\}$ to $\mathcal{L}(H, H)$.*

Theorem 6.2 (A. Yagi [1]). *Assume that A is a linear operator that verifies the conditions of Theorem* 6.1:

(i) *If there exists γ, $0 < \gamma < 1$, such that*

$$\forall \theta, \quad 0 < \theta < \gamma, \quad [D(A), H]_{1-\theta} = [D(A^*), H]_{1-\theta}, \tag{6.9}$$

then condition (ii) of Theorem 6.1 *is verified.*

(ii) *If there exists γ, $0 < \gamma < 1$, such that*

$$\forall \theta, \quad 0 < \theta < \gamma, \quad D(A^\theta) = D(A^{*\theta}), \tag{6.10}$$

then condition (iii) of Theorem 6.1 *is verified.*

Corollary 6.1. *Under the hypotheses of Theorem* 6.1 *and for some γ, $0 < \gamma < 1$, the following statements are equivalent*:

(i) $\forall \theta$, $0 < \theta < \gamma$, $[D(A), H]_{1-\theta} = [D(A^*), H]_{1-\theta}$,
(ii) $\forall \theta$, $0 < \theta < \gamma$, $D(A^\theta) = D(A^{*\theta})$.

With the above results and Theorem 5.1 in §5 we can now complete the picture for closed maximal accretive operators.

Proposition 6.1. *Assume that A is a closed maximal accretive operator in the Hilbert space H for which A^{-1} is bounded in H. Then*

$$D(A^\theta) = [D(A), H]_{1-\theta}, D(A^{*\theta}) = [D(A^*), H]_{1-\theta}, \quad 0 \le \theta \le 1, \tag{6.11}$$

and

$$[D(A), H]_{1-\theta} = D(A^\theta) = D(A^{*\theta}) = [D(A^*), H]_{1-\theta}, \quad \forall\theta,\ 0 \le \theta < \tfrac{1}{2}. \tag{6.12}$$

Proof. From Theorem 5.1 $D(A^\theta) = D(A^{*\theta})$, $0 \le \theta < 1/2$. Thus from Theorem 6.2(ii) and the equivalence of (i) and (iii) in Theorem 6.1 we obtain (6.11) and a portion of (6.12). □

Therefore Proposition 6.1 generalizes the result of Example 4.3 in §4 for negative self-adjoint operators that generate an analytic semigroup on a Hilbert space.

Using the results of P. Grisvard [1], we can now complete the theory on elliptic operators corresponding to regular boundary problems. We quote from A. Yagi [1]. Let Ω be a bounded open domain in $\mathbb{R}^n$ with an infinitely differentiable boundary Γ. Let $A(x; D)$ be a differentiable operator of order $2m$ with infinitely differentiable coefficients in Ω, and let $\{B_j(x; D)\}_{1\le j\le m}$ be m boundary differential operators of order $m_j \le 2m - 1$ with infinitely differentiable coefficients on Γ. Assume that

(a) $A(x; D)$ is properly elliptic in Ω (cf. J. L. Lions and E. Magenes [1, Volume 1, Chapter 2, §1.2, Definition 1.2]),
(b) $\{B_j(x; D)\}_{1\le j\le m}$ is a normal system on Γ (cf. J. L. Lions and E. Magenes [1, Volume 1, Chapter 2, §1.4, Definition 1.4]),
(c) $A(x; D)$ and $\{\beta_j(x; D)\}_{1\le j\le m}$ verify the conditions in H. Tanabe [1, Chapter 3, Theorem 3.8.1] for $\theta = \pi$.

Under the above assumptions define an operator A in $L^2(\Omega)$ as follows:

$$\begin{aligned} &D(A) = \{u \in H^{2m}(\Omega) \colon B_j(x; D)u = 0 \quad \text{on } \Gamma, 1 \le j \le m\}, \\ &Au = A(x; D)u. \end{aligned} \tag{6.13}$$

A is called a *realization* of $A(x; D)$ in $L^2(\Omega)$ under the boundary conditions $\{B_j(x; D)\}_{1\le j\le m}$. Its adjoint is given by

$$\begin{aligned} &D(A^*) = \{v \in H^{2m}(\Omega) \colon C_j(x; D)v = 0 \quad \text{on } \Gamma, 1 \le j \le m\}, \\ &A^*v = A(x; D)^*v, \end{aligned} \tag{6.14}$$

where $\{C_j(x; D)\}_{1\le j\le m}$ is the adjoint system of $\{B_j(x; D)\}_{1\le j\le m}$. The space $D(A)$ (resp. $D(A^*)$) will also be denoted $H^{2m}_B(\Omega)$ (resp. $H^{2m}_C(\Omega)$) when they are identified with a subspace of $H^{2m}(\Omega)$. In fact from the a priori estimates for elliptic operators the two spaces coincide because the graph norm of $D(A)$ (resp. $D(A^*)$) is equivalent to the one of $H^{2m}(\Omega)$.

To check condition (i) in Theorem 6.2, it is sufficient to find a constant γ, $0 < \gamma < 1$, such that

$$\left[H_B^{2m}(\Omega), H^0(\Omega)\right]_{1-\theta} = \left[H_C^{2m}(\Omega), H^0(\Omega)\right]_{1-\theta}, \quad 0 < \theta < \gamma. \tag{6.15}$$

But according to P. Grisvard [1] or J. L. Lions [3, Volume 2, Chapter 4,Theorem 14.4], the interpolation spaces are specifically characterized by

$$\begin{aligned} &\left[H_B^{2m}(\Omega), H^0(\Omega)\right]_{1-\theta} \\ &= \left\{ u \in H^{2m\theta}(\Omega) : \begin{array}{ll} B_j(x;D)u = 0 \quad \text{on } \Gamma, & m_j < 2m\theta - \frac{1}{2} \\ B_j(x;D)u \in L^2_{-1/2}(\Omega), & m_j = 2m\theta - \frac{1}{2} \end{array} \right\}, \end{aligned}$$

where $L^2_{-1/2}(\Omega)$ denotes the space of functions φ such that $d(x,\Gamma)^{-1/2}\varphi \in L^2(\Omega)$. A similar characterization holds for $[H_C^{2m}(\Omega), H^0(\Omega)]_{1-\theta}$. So in particular for all $\theta, 0 < \theta < 1/4m$,

$$\left[H_B^{2m}(\Omega), H^0(\Omega)\right]_{1-\theta} = H^{2m\theta}(\Omega) = \left[H_C^{2m}(\Omega), H^0(\Omega)\right]_{1-\theta}. \tag{6.16}$$

Hence from Theorem 6.2 (i)

$$\forall \theta, \quad 0 < \theta < \frac{1}{4m}, \quad D(A^\theta) = D(A^{*\theta}) \tag{6.17}$$

and from the equivalence of (i) and (ii) in Theorem 6.1 for all θ, $0 \le \theta \le 1$,

$$D(A^\theta) = \left[H_B^{2m}(\Omega), H^0(\Omega)\right]_{1-\theta}, \quad D(A^{*\theta}) = \left[H_C^{2m}(\Omega), H^0(\Omega)\right]_{1-\theta}.$$

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2

Variational Theory of Parabolic Systems

1 Variational differential equations

A complete treatment of variational differential equations is beyond the scope of this book. We shall mainly quote some results from the books of J. L. Lions and E. Magenes [1]. In order to motivate the chosen constructions and models, we give a series of classical examples. We assume that the reader is familiar with Sobolev spaces and their properties.

In this chapter we use a notation that is slightly different from Chapter 1. This notation is standard in books using the "variational theory."

Notation 1.1. Let X and Y be two Hilbert spaces such that $X \subset Y$. In that case the interpolation spaces $(X,Y)_{\theta,p}$ with $p=2$ and $[X,Y]_\theta$ coincide for $0<\theta<1$ (cf. Chapter 1 §4.7). We shall use the notation $[X,Y]_\theta$, $0<\theta<1$. □

In addition recall that the operator A that will be used throughout this chapter will correspond to $-A$ in Chapter 1 whenever it describes the same system (cf. Chapter 1, §5 and §6).

1.1 Distributed control

Let $T>0$ be a real number and let Ω be a bounded open subset of $\mathbb{R}^n$ ($n \geq 1$, an integer) with "sufficiently smooth" boundary $\varGamma$. We use $\varGamma$ rather than $\partial\Omega$ as was done in Chapter 1 because the notation is more in line with the literature on variational equations. Following J. L. Lions and E. Magenes [1], introduce the cylinder

$$Q = \Omega \times]0,T[, \quad \varSigma = \varGamma \times]0,T[, \tag{1.1}$$

the coefficients

$$a_{ij} \in L^\infty(Q), \quad 1 \leq i,\ j \leq n, \qquad a_0 \in L^\infty(Q) \tag{1.2}$$

and the bilinear form

$$a(t;\varphi,\psi) = \sum_{i,j=1}^{n} \int_\Omega a_{ij}(x,t)\frac{\partial \varphi}{\partial x_j}\frac{\partial \psi}{\partial x_i}\,dx + \int_\Omega a_0(x,t)\varphi\psi\,dx. \tag{1.3}$$

1.1.1 Homogeneous Dirichlet boundary conditions

Choose $L^2(Q)$ or equivalently $L^2\big(0,T;L^2(\Omega)\big)$ as space of control functions u. Consider the boundary value problem

$$\begin{cases} A(t)y + \dfrac{\partial y}{\partial t} = u \quad \text{in } Q, \\ y = 0 \quad \text{on } \Sigma, \\ y(x,0) = y^0(x) \quad \text{in } \Omega, \end{cases} \tag{1.4}$$

where

$$A(t)\phi = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial \phi}{\partial x_j}\right) + a_0(x,t)\phi. \tag{1.5}$$

Introduce the spaces

$$V = H_0^1(\Omega), \quad H = L^2(\Omega) \tag{1.6}$$

and the continuous dense injection

$$i\colon V \hookrightarrow H. \tag{1.7}$$

Identify the elements of the dual H' of H with those of H. This results in the injections

$$V \overset{i}{\hookrightarrow} H \equiv H' \overset{i^*}{\hookrightarrow} V', \tag{1.8}$$

where i^* denotes the (topological) transposed of the continuous linear injection i.

The system of equations (1.4) can now be written as a "variational differential equation" in V', the topological dual of V. A more precise definition of this terminology will be given in §2. To see this multiply both sides of the first equation (1.4) by an arbitrary element v in V and use Green's formula for the term $A(t)y$

$$\int_\Omega A(t)y(t)v\,dx = a(t;y(t),v). \tag{1.9}$$

For all v in V

$$\begin{aligned} a(t;y(t),v) + \int_\Omega \frac{\partial y}{\partial t}v\,dx &= (u(t),iv), \\ y(0) &= y^0, \end{aligned} \tag{1.10}$$

where $(\cdot,\cdot)$ denotes the inner product in H. This readily suggests to interpret the time derivative of y as an element dy/dt of V' because the linear functional

$$v \mapsto \int_\Omega \frac{\partial y}{\partial t} v\, dx = \langle i^* u(t), v\rangle_V - a(t; y(t), v) \colon V \to \mathbb{R} \tag{1.11}$$

is continuous ($\langle \cdot, \cdot \rangle_V$ denotes the duality pairing between V' and V). Thus (1.10) is equivalent to the following equation in V':

$$A(t)y + \frac{dy}{dt} = i^* u, \quad y(0) = y^0, \tag{1.12}$$

where $A(t)$ is now interpreted as the continuous linear map from V to V' defined by

$$\langle A(t)\phi, \psi \rangle_V = a(t, \phi, \psi). \tag{1.13}$$

The idea is now to look for a solution y of (1.12) in the space

$$W(0,T) = \left\{ \phi \in L^2(0,T;V) \colon \frac{d\phi}{dt} \in L^2(0,T;V') \right\}, \tag{1.14}$$

where the time derivative must be carefully interpreted as a distributional derivative with values in V'. It can be shown that system (1.12) has a unique solution y in $W(0,T)$ for every y^0 in H under the following "coercivity" hypothesis:

$$\begin{gathered} \exists \alpha > 0 \quad \text{such that } \forall \xi \in \mathbb{R}^n, \\ \sum_{i,j=1}^n a_{ij}(x,t)\xi_i \xi_j \geq \alpha \sum_{i=1}^n \xi_i^2, \quad \text{a.e. in } Q. \end{gathered} \tag{1.15}$$

From the control point of view, the space of controls is $U = L^2(\Omega) = H$, the space of control functions is $\mathcal{U} = L^2(0,T;U)$, and the control operator

$$B \colon U \to H \tag{1.16}$$

is the identity operator. With this notation system (1.12) can be rewritten as

$$A(t)y + \frac{dy}{dt} = i^* B u, \quad y(0) = y^0. \tag{1.17}$$

In general when the control operator is continuous on $H = L^2(\Omega)$ we say that it is a "distributed control operator."

1.1.2 Homogeneous Neumann boundary condition

As in the first example we choose control functions u in $L^2\big(0,T;L^2(\Omega)\big)$ and consider the following boundary value problem:

$$\begin{cases} A(t)y + \dfrac{\partial y}{\partial t} = u \quad \text{in } Q, \\ \dfrac{\partial y}{\partial \nu_A} = 0 \quad \text{on } \Sigma, \\ y(x,0) = y^0(x) \quad \text{in } \Omega, \end{cases} \tag{1.18}$$

where $A(t)$ is given by (1.5),

$$\nu = (\nu_1, \dots, \nu_n)$$

is the unit external normal to $\varGamma$, and

$$\frac{\partial y}{\partial \nu_A} = \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial y}{\partial x_j} \nu_j \tag{1.19}$$

is the conormal derivative of y with respect to the operator A. Here we choose

$$V = H^1(\Omega), \quad H = L^2(\Omega) \tag{1.20}$$

and still denote by i the continuous injection of V into H. We proceed as in the first example and obtain the variational differential equation (1.17). To obtain the existence of a solution here, we need to add to hypothesis (1.15) that

$$a_0(x,t) \geq \alpha, \quad \text{a.e. in } Q. \tag{1.21}$$

1.2 Boundary control condition

1.2.1 Control through a Neumann condition

In this example the space of control functions u is $L^2(\varSigma)$ or equivalently $L^2(0,T;U)$ with $U = L^2(\varGamma)$. The associated boundary value problem is

$$\begin{cases} A(t)y + \dfrac{\partial y}{\partial t} = 0 & \text{in } Q, \\ \dfrac{\partial y}{\partial \nu_A} = u & \text{on } \varSigma, \\ y(x,0) = y^0(x) & \text{in } \Omega, \end{cases} \tag{1.22}$$

where $A(t)$ is given by (1.5) and $\partial y / \partial \nu_A$ is the conormal derivative (1.19) with respect to the operator A. We choose the same space V and H as in the example of §1.1.2.

To transform system (1.22) into a variational differential equation in V' we again use Green's formula: For all v in $V = H^1(\Omega)$

$$\begin{aligned} \int_\Omega A(t)y(t)v\,dx &= a(t; y(t), v) - \int_\varGamma \frac{\partial y}{\partial \nu_A} v\,d\varGamma \\ &= a(t; y(t), v) - \int_\varGamma u(t)v\,d\varGamma. \end{aligned}$$

So for all v in V

$$a(t; y(t), v) + \int_\Omega \frac{\partial y}{\partial t} v\,dx = \int_\varGamma u(t)v\,d\varGamma, \quad y(0) = y^0. \tag{1.23}$$

Here we define the control operator $B\colon U \to V'$ as follows:

$$\langle Bu, \phi \rangle_V = \int_\Gamma u\phi_{|\Gamma}\, d\Gamma, \quad u \in L^2(\Gamma),\ \phi \in V. \tag{1.24}$$

It is linear and continuous because the trace operator

$$\phi \to \phi|_\Gamma \colon V = H^1(\Omega) \to H^{1/2}(\Gamma) \tag{1.25}$$

and the injection of $H^{1/2}(\Gamma)$ into $L^2(\Gamma)$ are continuous. The final result is the variational differential equation

$$A(t)y + \frac{dy}{dt} = Bu, \quad y(0) = y^0. \tag{1.26}$$

By its very nature, B is a "boundary control operator." It is never a continuous linear map on $L^2(\Omega)$. System (1.26) has a unique solution in $W(0,T)$ under hypotheses (1.15) and (1.21).

1.2.2 Control through a Dirichlet condition

Consider the boundary value problem

$$\begin{cases} A(t)y + \dfrac{\partial y}{\partial t} = 0 \quad \text{in } Q, \\ y = u \quad \text{on } \Sigma, \\ y(x,0) = y^0(x) \quad \text{in } \Omega, \end{cases}$$

where $A(t)$ is given by (1.5). Here the control enters as a Dirichlet condition on Γ. This problem cannot be transformed into a variational problem by integration by parts. We shall see in §2.1 and §2.4 that this more delicate problem can be handled by the Method of Transposition. However, after a change of variable, it can also be put in variational form.

1.2.3 Point controls

Another boundary value problem that is not amenable to a variational equation by integration by parts is when the control is achieved through a finite number of points (or locations) in the domain Ω. We shall see in §2.2 and §2.4 that this problem can be handled by the Method of Transposition for domains in $\mathbb{R}^n$, $n \le 3$.

1.3 Main theorem

The main theorem that we shall quote is one of several isomorphism theorems, which can be found in J. L. Lions and E. Magenes [1], Volume 1, Chapter 3, Theorem 1.1 and Examples in §4.7].

Let V and H be two Hilbert spaces with the following notation:

$|\cdot|$ (resp. $\|\cdot\|$) is the norm in H (resp. V),
H' (resp. V') is the topological dual of H (resp. V),
$(\cdot,\cdot)$ is the inner product in H,
$\langle\cdot,\cdot\rangle_V$ is the duality pairing on $V' \times V$.

We assume that there is a continuous injection $i\colon V \to H$ and we identify the elements of H' with those of H:

$$V \overset{i}{\hookrightarrow} H \equiv H' \hookrightarrow V'. \tag{1.27}$$

Given a fixed real number $T > 0$ and a family of continuous linear operators

$$A(t) \in \mathcal{L}(V, V'), \quad 0 \le t \le T, \tag{1.28}$$

we make the following assumptions:

$$\forall v, w \in V, \quad t \mapsto \langle A(t)v, w\rangle_V \text{ is measurable on }]0,T[\tag{1.29}$$

and for all $t \in [0,T]$, $\exists c > 0$, such that

$$|\langle A(t)v, w\rangle_V| \le c\|v\| \,:\, \|w\| \tag{1.30}$$

and $\exists \alpha > 0$ and $\exists \lambda \in \mathbb{R}$ such that

$$\forall v \in V, \quad \langle A(t)v, v\rangle_V + \lambda|v|^2 \ge \alpha\|v\|^2. \tag{1.31}$$

Assumption (1.31) is known as the V–H *coercivity* of A. Define the space

$$W(0,T) = \left\{ y \in L^2(0,T;V)\colon \frac{dy}{dt} \in L^2(0,T;V') \right\}, \tag{1.32}$$

where the derivative is to be interpreted as a vectorial distributional derivative with values in V'.

Given

$$f \in L^2(0,T;V'), \quad y^0 \in H, \tag{1.33}$$

we consider the *variational differential equation*

$$A(t)y + \frac{dy}{dt} = f, \quad y(0) = y^0. \tag{1.34}$$

Notation 1.2. In this chapter we use the notation $A(t)$ of the variational literature. When $A(t) = A$ (independent of t), the infinitesimal generator of the associated semigroup will be $-A$ (cf. Chapter 1, §2.7, Thm 2.10). □

Theorem 1.1. *Under assumptions* (1.27) *through* (1.33)*, equation* (1.34) *has a unique solution y in $W(0,T)$ and the map*

$$y \mapsto \left(A(t)y + \frac{dy}{dt}, y(0) \right) : W(0,T) \to L^2(0,T;V') \times H \tag{1.35}$$

is an (*algebraic and topological isomorphism*)*.*

Proof. See J. L. Lions [4, p. 116, Theorem 1.2] for the case where V is separable. For a more general and modern proof, the reader is referred to J. L. Lions and E. Magenes [1, Volume 1, Chapter 3, §4, Theorem 4.1]. □

Remark 1.1. It is important to notice that the operators $A(t)$ are neither assumed to be self-adjoint nor invertible. In particular when $H = \mathbb{R}^n$ ($n \geq 1$, an integer) and the elements of the matrix A are L^∞-functions, the V–H coercivity assumption (1.31) is verified along with all other hypotheses. □

Each element of $W(0,T)$ can be identified with a unique function in $C([0,T];H)$ and the injection

$$W(0,T) \hookrightarrow C([0,T];H) \tag{1.36}$$

is continuous (cf. J. L. Lions and E. Magenes [1, Volume 1, Proposition 2.1, p. 18 and Theorem 3.1, p. 19]). This gives a pointwise meaning to $y(t)$, $0 \leq t \leq T$, as an element of the space H so that $y(0)$ in (1.34) is well defined.

If we define the adjoint operator $A^*(t)$

$$\langle A^*(t)v, w\rangle_V = \langle A(t)w, v\rangle_V, \tag{1.37}$$

then the map

$$v \mapsto \left(A^*(t)v - \frac{dv}{dt}, v(T)\right): W(0,T) \to L^2(0,T;V') \times H \tag{1.38}$$

is also an isomorphism. In §2, we shall see how a smooth version of the "adjoint isomorphism'" can be "transposed" to deal with examples that cannot be directly modeled by the variational differential equation (1.34).

1.4 A perturbation theorem

Theorem 1.2. *Assume that the hypotheses of Theorem* 1.1 *are verified. Let* $\theta \in [0,1]$ *and*

$$K(t)\colon [V,H]_\theta \to [V,H]'_{1-\theta}, \quad 0 \leq t \leq T, \tag{1.39}$$

be a family of continuous linear operators such that

$$\forall v \in [V,H]_\theta, \quad t \mapsto K(t)v \text{ belongs to } L^\infty(0,T;[V,H]'_{1-\theta}). \tag{1.40}$$

Then the operator $A + K$ *is* V–H *coercive and the variational differential equation*

$$[A(t) + K(t)]y + \frac{dy}{dt} = f \in L^2(0,T;V'), \quad y(0) = y^0 \in H \tag{1.41}$$

has a unique solution y *in* $W(0,T)$.

Proof. It is sufficient to show that $A+K$ is V–H coercive: there exist $\beta > 0$ and $\mu \in \mathbb{R}$ such that

$$\forall v \in V, \quad \langle [A(t)+K(t)]v, v\rangle_V + \mu|v|^2 \geq \beta\|v\|^2.$$

By hypothesis on A

$$\exists \alpha > 0, \quad \exists \lambda \in \mathbb{R} \quad \text{such that } \; \forall v \in V, \qquad \langle A(t)v, v\rangle_V + \lambda|v|^2 \geq \alpha\|v\|^2.$$

So for any $\mu \in \mathbb{R}$

$$\begin{aligned} a_\mu &= \langle [A(t)+K(t)]v, v\rangle_V + \mu|v|^2 \\ &\qquad\qquad \geq \alpha\|v\|^2 + (\mu-\lambda)|v|^2 - \|K(t)\|_{\mathcal{L}([V,H]_\theta, [V,H]'_{1-\theta})}|v|_\theta|v|_{1-\theta}, \end{aligned}$$

where $|v|_\theta$ is the norm in $[V,H]_\theta$. By hypothesis (1.40), there exists a constant $k > 0$ such that

$$\operatorname*{ess\,sup}_{[0,T]} |K(t)|_{\mathcal{L}([V,H]_\theta, [V,H]'_{1-\theta})} \leq k.$$

Moreover from J. L. Lions and E. Magenes [1, Volume 1, Proposition 2.3] or Proposition 5.1(v) in Chapter 1 (cf. also §4.3)

$$\forall v \in V, \quad |v|_\theta = |v|_{[V,H]_\theta} \leq c\|v\|^\theta |v|^{1-\theta}.$$

Therefore there exists $c > 0$ such that

$$k|v|_\theta|v|_{1-\theta} \leq c\|v\||v|$$

and

$$c\|v\||v| = 2c\frac{1}{2}\sqrt{\frac{\alpha}{c}}\|v\|\sqrt{\frac{c}{\alpha}}|v| \leq c\left[\frac{\alpha}{4}\|v\|^2 + \frac{c}{\alpha}|v|^2\right] = \frac{\alpha}{4}\|v\|^2 + \frac{c^2}{\alpha}|v|^2.$$

Finally

$$\begin{aligned} a_\mu \geq \alpha\|v\|^2 + (\mu-\lambda)|v|^2 - \frac{\alpha}{4}\|v\|^2 - \frac{c^2}{\alpha}|v|^2 &= \frac{3\alpha}{4}\|v\|^2 + \left(\mu - \lambda - \frac{c^2}{\alpha}\right)|v|^2 \\ &\geq \frac{3}{4}\alpha\|v\|^2 \end{aligned}$$

for $\mu \geq \lambda + c^2/\alpha$. So we can choose $\beta = 3\alpha/4$, and this completes the proof of the V–H coercivity of $A+K$. □

1.5 A regularity theorem

We now present a regularity result for variational differential equations due to C. Bardos [1]. This type of result is useful at various stages in the analysis of optimal control problems. It can also be used in the construction of "smooth adjoint isomorphisms" that are the primary ingredients in the "Method of Transposition."

We go back to the definitions, notations, and hypotheses (1.27) to (1.32) of §1.3. We denote by

$$A_\lambda(t) = A(t) + \lambda I \tag{1.42}$$

the operator associated with the λ for which the V–H coercivity assumption is verified. So for each t

$$A_\lambda(t) \in \mathcal{L}(V, V') \tag{1.43}$$

is an isomorphism that also induces an isomorphism

$$A_\lambda \in \mathcal{L}\big(L^2(0,T;V), L^2(0,T;V')\big), \quad (A_\lambda v)(t) = A_\lambda(t)v(t). \tag{1.44}$$

For each $t \in [0,T]$, we define the domains of $A(t)$, A, $A_\lambda(t)$, and A_λ

$$\begin{aligned} D\big(A(t)\big) &= \{v \in V \colon A(t)v \in H\},\\ D(A) &= \{w \in L^2(0,T;V) \colon Aw \in L^2(0,T;H)\},\\ D\big(A_\lambda(t)\big) &= \{v \in V \colon A_\lambda(t)v \in H\},\\ D(A_\lambda) &= \{w \in L^2(0,T;V) \colon A_\lambda w \in L^2(0,T;H)\}. \end{aligned}$$

It is readily seen that

$$D\big(A_\lambda(t)\big) = D\big(A(t)\big), \quad D(A_\lambda) = D(A). \tag{1.45}$$

The unbounded operator

$$A_\lambda(t) \colon D\big(A(t)\big) \to H\big(\text{resp. } A_\lambda \colon D(A) \to L^2(0,T;H)\big) \tag{1.46}$$

is an isomorphism when $D\big(A(t)\big)$ (resp. $D(A)$) is embedded with the graph norm topology of $A(t)$ (resp. A) (cf. T. Kato [2], T. Kato [3], J. L. Lions [2]).

Define the adjoints $A^*(t)$ and A^* of $A(t)$ and A

$$\begin{aligned} \langle A^*(t)v, w\rangle_V &= \langle A(t)w, v\rangle_V, \quad \forall v, w \in V,\\ \langle A^* f, g\rangle_{L^2(0,T;V)} &= \langle Ag, f\rangle_{L^2(0,T;V)}, \quad \forall f,\ g \in L^2(0,T;V). \end{aligned} \tag{1.47}$$

Similar definitions for $A_\lambda(t)$ and A_λ yield

$$A^*_\lambda(t) = A^*(t) - \lambda I, \quad A^*_\lambda = A^*_\lambda - \lambda I. \tag{1.48}$$

Moreover, for $t \in [0,T]$, $A_\lambda(t)$ and A_λ are isomorphisms

$$A^*_\lambda(t) \in \mathcal{L}(V, V'), \quad A^*_\lambda \in \mathcal{L}\big(L^2(0,T;V), L^2(0,T;V')\big). \tag{1.49}$$

As we did for A define

$$\begin{aligned} D\big(A^*(t)\big) &= \{v \in V \colon A^*(t)v \in H\},\\ D(A^*) &= \{f \in L^2(0,T;V) \colon A^* f \in L^2(0,T;H)\}. \end{aligned} \tag{1.50}$$

Analogous definitions for $A_\lambda(t)$ and A_λ yield

$$D\big(A_\lambda^*(t)\big) = D\big(A^*(t)\big), \quad D(A_\lambda^*) = D(A^*), \tag{1.51}$$

and the unbounded linear operator

$$A_\lambda^*(t) \colon D\big(A^*(t)\big) \to H, \quad A_\lambda^* \colon D(A^*) \to L^2(0,T;H) \tag{1.52}$$

are isomorphisms when $D\big(A^*(t)\big)$ and $D(A^*)$ are endowed with their respective graphs norm topology.

The adjoint of $A_\lambda(t)^*$ (resp. A_λ^*) can be considered as an extension of $A_\lambda(t)$ (resp. A_λ) because it coincides with it on $D\big(A_\lambda(t)\big)$ (resp. $D(A_\lambda)$). With that convention the linear maps

$$\begin{aligned} A_\lambda(t) &\in \mathcal{L}\big(D\big(A(t)\big), H\big) \cap \mathcal{L}(V, V') \cap \mathcal{L}\big(H, D\big(A^*(t)\big)'\big), \\ A_\lambda &\in \mathcal{L}\big(D(A), L^2(0,T;H)\big) \cap \mathcal{L}\big(L^2(0,T;V), L^2(0,T;V')\big) \\ &\qquad \cap \mathcal{L}(L^2(0,T;H), D(A^*)') \end{aligned} \tag{1.53}$$

are isomorphisms. Similarly,

$$\begin{aligned} A_\lambda^*(t) &\in \mathcal{L}(D(A^*(t), H) \cap \mathcal{L}(V, V') \cap \mathcal{L}\big(H, D\big(A(t)\big)'\big), \\ A_\lambda^* &\in \mathcal{L}\big(D(A^*), L^2(0,T;H)\big) \cap \mathcal{L}\big(L^2(0,T;V), L^2(0,T;V')\big) \\ &\qquad \cap \mathcal{L}(L^2(0,T;H), D(A)') \end{aligned} \tag{1.54}$$

are isomorphisms.

Remark 1.2. It is possible to interpolate between the different isomorphisms (1.53). So for each $\alpha \in [0,1]$ the maps

$$\begin{aligned} A_\lambda(t) &\in \mathcal{L}\big(\big[D\big(A(t)\big), H\big]_\alpha, \big[D\big(A^*(t)\big), H\big]'_{1-\alpha}\big), \\ A_\lambda &\in \mathcal{L}([D(A), L^2(0,T;H)]_\alpha, [D(A^*), L^2(0,T;H)]'_{1-\alpha}) \end{aligned} \tag{1.55}$$

are isomorphisms. However it is generally not true that

$$\big[D\big(A_\lambda(t)\big), H\big]_{1/2} = \big[D\big(A_\lambda^*(t)\big), H\big]_{1/2} = V$$

or

$$[D(A_\lambda), L^2(0,T;H)]_{1/2} = [D(A_\lambda^*), L^2(0,T;H)]_{1/2} = L^2(0,T;V). \qquad \square$$

As $A_\lambda(t)$ is maximal positive (that is, $-A_\lambda(t)$ is the generator of a strongly continuous contraction semigroup), the fractional powers $A_\lambda^\alpha(t)$ and $A_\lambda^{*\alpha}(t)$, $0 < \alpha < 1$, are well defined (cf. Part II, Chapter 1, §5 and T. Kato [1, T. Kato [2]). They are also isomorphisms

$$\begin{aligned} &A_\lambda^\alpha(t) \colon D\big(A_\lambda^\alpha(t)\big) \to H, && A_\lambda^{*\alpha}(t) \colon D\big(A_\lambda^{*\alpha}(t)\big) \to H, \\ &A_\lambda^\alpha \colon D(A_\lambda^\alpha) \to L^2(0,T;H), && A_\lambda^{*\alpha} \colon D(A_\lambda^{*\alpha}) \to L^2(0,T;H) \end{aligned} \tag{1.56}$$

whose domains coincide with the appropriate interpolation spaces (cf. Part II, Chapter 1, §6, Proposition 6.1)

$$\begin{aligned} D\big(A_\lambda^\alpha(t)\big) &= \big[D\big(A(t)\big), H\big]_{1-\alpha}, \quad D(A_\lambda^\alpha) = [D(A), L^2(0,T;H)]_{1-\alpha}, \\ D\big(A_\lambda^{*\alpha}(t)\big) &= \big[D\big(A^*(t)\big), H\big]_{1-\alpha}, \quad D(A_\lambda^{*\alpha}) = [D(A^*), L^2(0,T;H)]_{1-\alpha}. \end{aligned} \tag{1.57}$$

Remark 1.3. In view of Remark 1.3 and (1.56)–(1.57), for all $\alpha \in [0,1]$

$$A_\lambda(t) \in \mathcal{L}\big(D\big(A_\lambda^{1-\alpha}(t)\big), D\big(A_\lambda^{*\alpha}(t)\big)'\big), \quad A_\lambda \in \mathcal{L}\big(D(A_\lambda^{1-\alpha}), D(A_\lambda^{*\alpha})'\big). \tag{1.58}$$

□

J. L. Lions [2] has shown that

$$D\big(A_\lambda^\alpha(t)\big) = D\big(A_\lambda^{*\alpha}(t)\big), \quad 0 \le \alpha < \tfrac{1}{2} \tag{1.59}$$

and given sufficient conditions under which

$$D\big(A_\lambda^{1/2}(t)\big) = D\big(A_\lambda^{*1/2}(t)\big) = V. \tag{1.60}$$

One of them is

$$D\big(A_\lambda^*(t)\big) = D\big(A_\lambda(t)\big); \tag{1.61}$$

another one is more technical but covers the so-called *regular* elliptic boundary value problems.

Theorem 1.3. *Assume that there exists a Hilbert space X, $X \subset H$, such that*

$$V \textit{ is a closed subspace of } [X,H]_{1/2}, \ D\big(A_\lambda(t)\big) \subset X, \ D\big(A_\lambda^*(t)\big) \subset X. \tag{1.62}$$

Then

$$\begin{aligned} \big[D\big(A_\lambda(t)\big), H\big]_{1/2} &= D\big(A_\lambda^{1/2}(t)\big) = V, \\ \big[D\big(A_\lambda^*(t)\big), H\big]_{1/2} &= D\big(A_\lambda^{*1/2}(t)\big) = V. \end{aligned} \tag{1.63}$$

Proof. Cf. J. L. Lions [2, Theorem 6.1, p. 238] and Proposition 6.1 in Part II, Chapter 1, §6. □

Example 1.1. Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary Γ. Let V be a closed subspace of $H^m(\Omega)$ such that

$$H_0^m(\Omega) \subset V \subset H^m(\Omega). \tag{1.64}$$

Assume that $A(t) = A$ is generated by the bilinear form

$$a(u,v) = \sum_{|p|,|q| \le m} \int_\Omega a_{pq}(x) D^q u D^p v \, dx,$$
$$\langle Au, v\rangle_V = a(u,v),$$
$$\exists \alpha > 0, \quad \forall v \in V, \ a(v,v) \ge \alpha \|v\|^2$$

for sufficiently smooth functions a_{pq} on Ω and an appropriate choice of V (cf. J. L. LIONS [2, Section 6.2, p. 239] for more details).

The triplet $\{V, H, a(u,v)\}$ is said to be "regular" if

$$D(A) \subset H^{2m}(\Omega), \quad D(A^*) \subset H^{2m}(\Omega). \tag{1.65}$$

□

Remark 1.4. (1962, J. L. LIONS [2, Remark 6.1, p. 240]) It is important to recall that for mixed boundary value problems the relationship (1.63) does not hold. For instance it is not known whether

$$D(A^{1/2}) = D(A^{*1/2})$$

is true for a second-order elliptic operator that is not self-adjoint with a Dirichlet condition on part of the boundary and a Neumann condition on the other part. This also applies to the Dirichlet boundary value problem with an irregular boundary. □

Remark 1.5. In 2002, P. AUSCHER, S. HOFMANN, J. L. LEWIS, and PH. TCHAMITCHIAN [1] proved Kato's conjecture for elliptic operators on $\mathbb{R}^n$. More precisely, let $M = M(x)$ be an $n \times n$ matrix of complex, L^∞ coefficients, defined on $\mathbb{R}^n$, and satisfying the ellipticity (or "accretivity") condition

$$\lambda\, |\xi|^2 \le \operatorname{Re} M\xi \cdot \xi^* \text{ and } |M\xi \cdot \zeta^*| \le \Lambda\, |\xi|\, |\zeta|, \tag{1.66}$$

for ξ, ζ in $\mathbb{C}^n$ and for some λ, Λ such that $0 < \lambda \le \Lambda < \infty$; that is, $M\xi \cdot \zeta^* = \sum_{j,k} a_{j,k}(x)\xi_k \overline{\zeta}_j$. Define

$$Af \overset{\text{def}}{=} -\operatorname{div}(M\nabla f), \tag{1.67}$$

interpreted in the usual weak sense via a sesquilinear form. Under the accretivity condition (1.66), one can define a square root $A^{1/2} = \sqrt{A}$. They establish that the domain of the square root of A is the Sobolev space $H^1(\mathbb{R}^n)$ in any dimension with the estimate $\|\sqrt{A}f\|_2 \sim \|\nabla f\|_2$. A version of that result also holds for an operator A with lower order terms (Theorem 6.1). □

In view of identities (1.59), we can rewrite the isomorphisms (1.58) as follows: For $0 \le \alpha < \frac{1}{2}$

$$A_\lambda(t) \in \mathcal{L}\big(D\big(A_\lambda^{1-\alpha}(t)\big), D\big(A_\lambda^{\alpha}(t)\big)'\big), \quad A_\lambda \in \mathcal{L}(D(A_\lambda^{1-\alpha}), D(A_\lambda^{\alpha})'). \tag{1.68}$$

$$A_\lambda^*(t) \in \mathcal{L}\big(D(A_\lambda^{*(1-\alpha)}(t), D\big(A_\lambda^{*\alpha}(t)\big)'\big), A_\lambda^* \in \mathcal{L}\big(D(A_\lambda^{*(1-\alpha)}), D(A_\lambda^{*\alpha})'\big). \tag{1.69}$$

The next proposition completes the picture.

Proposition 1.1. *Assume that*

$$D\big(A_\lambda^{1/2}(t)\big) = D\big(A_\lambda^{*1/2}(t)\big). \tag{1.70}$$

Then

$$\begin{aligned} \big[D\big(A(t)\big), H\big]_{1/2} &= D\big(A_\lambda^{1/2}(t)\big) = V, \\ \big[D\big(A^*(t)\big), H\big]_{1/2} &= D\big(A_\lambda^{*1/2}(t)\big) = V, \end{aligned} \tag{1.71}$$

and

$$A_\lambda^{1/2}(t) \in \mathcal{L}(V, H), \quad A_\lambda^{*1/2}(t) \in \mathcal{L}(V, H) \tag{1.72}$$

are isomorphisms.

Proof. Cf. J. L. Lions [2, Corollary 5.2]. □

Corollary 1.1. *The following maps are isomorphisms*:

$$\begin{aligned} A_\lambda^{1/2}(t) &\in \mathcal{L}\big(D\big(A(t)\big), V\big) \cap \mathcal{L}(V, H) \cap \mathcal{L}(H, V') \cap \mathcal{L}\big(V', D\big(A^*(t)\big)'\big), \\ A_\lambda^{*1/2}(t) &\in \mathcal{L}\big(D\big(A^*(t)\big), V\big) \cap \mathcal{L}(V, H) \cap \mathcal{L}(H, V') \cap \mathcal{L}\big(V', D\big(A(t)\big)'\big), \end{aligned} \tag{1.73}$$

$$\begin{aligned} A_\lambda^{1/2} &\in \mathcal{L}(D(A), \mathcal{V}) \cap \mathcal{L}(\mathcal{V}, \mathcal{H}) \cap \mathcal{L}(\mathcal{H}, \mathcal{V}') \cap \mathcal{L}(\mathcal{V}', D(A^*)'), \\ A_\lambda^{*1/2} &\in \mathcal{L}(D(A^*), \mathcal{V}) \cap \mathcal{L}(\mathcal{V}, \mathcal{H}) \cap \mathcal{L}(\mathcal{H}, \mathcal{V}') \cap \mathcal{L}(\mathcal{V}', D(A)'), \end{aligned} \tag{1.74}$$

where

$$\mathcal{V} = L^2(0, T; V), \quad \mathcal{H} = L^2(0, T; H) \equiv \mathcal{H}', \quad \mathcal{V}' = L^2(0, T; V'). \tag{1.75}$$

Given two Banach spaces X and Y and a continuous dense injection i from X to Y, define

$$W(0, T; X, Y) = \left\{ y \in L^2(0, T; X) \colon \frac{dy}{dt} \in L^2(0, T; Y) \right\}, \tag{1.76}$$

where dy/dt is to be interpreted as the vectorial distributional derivative of the function iy, $(iy)(t) = iy(t)$. Similarly let

$$W(0, T; D(A), H) = \left\{ y \in D(A) \colon \frac{dy}{dt} \in L^2(0, T; H) \right\}. \tag{1.77}$$

Our next objective is to show that under appropriate conditions on $A(t)$, the linear operator

$$A_\lambda^{1/2} \colon W(0, T; D(A), H) \to W(0, T; V, V') \tag{1.78}$$

is an isomorphism. This will first necessitate giving a meaning to the time derivative $A_\lambda(t)'$ of the operator valued function $A_\lambda(t)$.

Proposition 1.2. *Assume that hypotheses* (1.27) *to* (1.31) *and hypothesis* (1.71) *are verified. Assume that the operator-valued function*

$$A \colon [0, T] \to \mathcal{L}(V, V') \tag{1.79}$$

has a derivative $A(t)'$ in $\mathcal{L}(V, V')$ almost everywhere in $[0, T]$ such that

$$\exists c > 0, \quad \|A(t)'\|_{\mathcal{L}(V,V')} \leq c, \quad a.e. \text{ in } [0,T] \tag{1.80}$$

and for all v and w in V

$$t \mapsto \langle A(t)v, w\rangle_V \text{ belongs to } W^{1,\infty}(0,T;\mathbb{R}), \tag{1.81}$$

$$\frac{d}{dt}\langle A(t)v, w\rangle_V = \langle A(t)'v, w\rangle_V. \tag{1.82}$$

Then $A_\lambda^{1/2} \colon [0,T] \to \mathcal{L}(V,V')$ has a derivative $A_\lambda^{1/2}(t)' \in \mathcal{L}(V,V')$ almost everywhere in $[0,T]$ such that

$$\exists c > 0, \quad \|A_\lambda^{1/2}(t)'\|_{\mathcal{L}(V,V')} \leq c \tag{1.83}$$

and for all v and w in V

$$t \mapsto \langle A_\lambda^{1/2}(t)v, w\rangle_V \text{ belongs to } W^{1,\infty}(0,T;\mathbb{R}), \tag{1.84}$$

$$\frac{d}{dt}\langle A_\lambda^{1/2}(t)v, w\rangle_V = \langle A_\lambda^{1/2}(t)'v, w\rangle_V. \tag{1.85}$$

Proof. Minor adaptation of C. Bardos [1, Proposition 1.1]. □

This proposition gives a meaning to the derivative $A_\lambda^{1/2}(t)'$ of the operator valued function $A_\lambda^{1/2}(t)$.

Example 1.2. When $A(t)$ is given by a bilinear form

$$a(t;u,v) = \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i} + a_0(x,t)uv,$$

where

$$a_0, \frac{\partial a_0}{\partial t} \in L^\infty(Q), \quad a_{ij} \in W^{1,\infty}(Q),$$

then

$$\langle A'(t)u, v\rangle_V = \sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial t}(x,t)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i} + \frac{\partial a_0}{\partial t}(x,t)uv$$

and conditions (1.73) and (1.74) are verified. □

Let y belong to $W(0,T;D(A_\lambda),H)$. Then from Proposition 1.1, $A_\lambda^{1/2}y \in L^2(0,T;V)$. In view of Propositions 1.2 and 1.1

$$\frac{d}{dt}A_\lambda^{1/2}y = A_\lambda^{1/2}\frac{dy}{dt} + (A_\lambda^{1/2})'y \in L^2(0,T;V') \tag{1.86}$$

because

$$\frac{dy}{dt} \in L^2(0,T;H) \implies A_\lambda^{1/2}\frac{dy}{dt} \in L^2(0,T;V'),$$

$$y \in L^2\big(0,T;D(A_\lambda)\big) \subset L^2(0,T;V) \implies (A_\lambda^{1/2})'y \in L^2(0,T;H) \subset L^2(0,T;V').$$

By construction, the mapping is continuous. The surjectivity is obvious.

Proposition 1.3. *Under hypotheses* (1.27) *to* (1.31) *and* (1.71) *on A and hypotheses* (1.80) *to* (1.82) *on A', the linear map*

$$A_\lambda^{1/2} \colon W(0,T;D(A_\lambda),H) \to W(0,T;V,V') \tag{1.87}$$

is an isomorphism.

Corollary 1.2. *Under the hypotheses of Proposition* 1.3*, $A_\lambda^{1/2}$ is also an isomorphism from $W(0,T;V,V')$ to $W(0,T;H,D(A_\lambda^*)')$, where*

$$W(0,T;H,D(A_\lambda^*)') = \left\{ y \in L^2(0,T;H) \colon \frac{dy}{dt} \in D(A_\lambda^*)' \right\}. \tag{1.88}$$

We finally quote the regularity result from C. Bardos [1].

Theorem 1.4. *Assume that hypotheses* (1.27) *to* (1.31) *and* (1.71) *on A and hypotheses* (1.80) *to* (1.82) *on A' are verified. Consider the variational differential equation*

$$A(t)y + \frac{dy}{dt} = f, \quad y(0) = y^0 \tag{1.89}$$

for f in $L^2(0,T;H)$ and y_0 in V:

(i) *Then* (1.89) *has a unique solution y in $W(0,T;D(A_\lambda),H)$ and the mapping*

$$y \mapsto \left(A(t)y + \frac{dy}{dt}, y(0) \right) \colon W(0,T;D(A_\lambda),H) \to L^2(0,T;H) \times V \tag{1.90}$$

is an isomorphism.

(ii) *Moreover if $D\big(A_\lambda(t)\big)$ is independent of t, the solution y of* (1.89) *belongs to $C([0,T];V)$.*

Proof. Slight adaptation of the proof of C. Bardos [1] or directly by using the isomorphism A_λ of Proposition 1.3, Theorem 1.2 and Theorem 1.1. □

Corollary 1.3. *Assume that the hypothesis of Theorem* 1.3 *are verified. Then the mapping*

$$v \mapsto \left(A^*(t)v - \frac{dv}{dt}, v(T) \right) \colon W(0,T;D(A_\lambda^*),H) \to L^2(0,T;H) \times V \tag{1.91}$$

is an isomorphism.

Remark 1.6. The isomorphism in part (i) for time-varying systems is to be compared with the ones in Theorem 3.1 in Chapter 1 and Theorem 1.1 in this chapter for time-invariant analytic semigroups in a Hilbert space and the so-called maximal regularity results. □

Example 1.3. Let Ω be a bounded open subset of $\mathbb{R}^n$ with a C^∞ boundary. Let $a(t;u,v)$ be the bilinear form (1.3),

$$V = H_0(\Omega), \quad \langle A(t)u, v\rangle_V = a(t;u,v).$$

Assume that

$$a_0, \quad \frac{\partial a_0}{\partial t} \in L^\infty(Q), \quad a_{ij} \in W^{1,\infty}(Q), \qquad 1 \le i,\ j \le n,$$

and

$$\exists \alpha > 0, \quad \forall \xi \in \mathbb{R}^n, \quad \sum_{i,j=1}^n a_{ij}(x,t)\xi_j\xi_i \ge \alpha \sum_{\imath=1}^n \xi_i^2.$$

The operator A is V–H coercive. Moreover for all u, v in V

$$\big\langle \big(A(t)\big)' u, v\big\rangle_V = \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial t}(x,t)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i} + \frac{\partial a_0}{\partial t}(x,t)uv$$

belongs to $L^\infty(0,T;\mathbb{R})$ and $\big(A(t)\big)'$ verifies (1.79) and (1.80). Moreover

$$H_0^1(\Omega) = V \subset H^1(\Omega)$$

and

$$D\big(A_\lambda^*(t)\big) = D\big(A_\lambda(t)\big) = H^2(\Omega) \cap H_0^1(\Omega) \subset H^2(\Omega).$$

So the last corollary applies to this problem. Here

$$D(A_\lambda^*) = L^2(0,T;D), \quad D = H^2(\Omega) \cap H_0^1(\Omega)$$

and the isomorphism (1.90) goes between the spaces

$$W(0,T;D,H) \to L^2(0,T;H) \times V. \qquad \square$$

The reader interested in additional regularity theorems is referred to Theorems 1.2 and 1.3 in C. Bardos [1], which essentially say that under appropriate hypotheses on $A(t)'$, additional regularity in the space or time variables yields additional regularity in the solution. When $A(t) = A$ is time-invariant, we recover the usual results.

2 Method of Transposition

2.1 Control through a Dirichlet boundary condition

As in the example of §1.2.2, the space of control functions u is $L^2(\Sigma)$ or equivalently $L^2(0,T;U)$ with $U = L^2(\Gamma)$. The boundary value problem is

$$\begin{cases} A(t)y + \dfrac{\partial y}{\partial t} = 0 & \text{in } Q, \\ y = u & \text{on } \Sigma, \\ y(x,0) = y^0(x) & \text{in } \Omega, \end{cases} \tag{2.1}$$

where $A(t)$ is given by (1.5) with the following hypotheses on the coefficients:

$$a_0 \in L^\infty(\Omega), \quad a_{ij} \in W^{1,\infty}(Q), \qquad 1 \le i,\ j \le n, \tag{2.2}$$

and the coercivity (1.15). We also set $V = H_0(\Omega)$, $H = L^2(\Omega)$, and define the injection i as in (1.7) and (1.8).

In this example it is not possible to directly obtain a variational differential equation in V'. The method that naturally arises in this context is the "Method of Transposition." To see this it is useful to perform the following formal computation. Pick an arbitrarily sufficiently smooth function $v(x,t)$ defined on Q such that $v(x,t) = 0$ on Σ. Multiply the first equation (2.1) by v and integrate over Q. Use Green's formula twice on the term $A(t)y$ and integration by parts on the term $\partial y/\partial t$. The result is as follows for all appropriate functions v:

$$\begin{aligned} \int_Q y\left[A^*(t)v - \frac{\partial v}{\partial t}\right] dx\,dt + \int_\Omega y(T)v(T)\,dx \\ = -\int_\Sigma u \frac{\partial v}{\partial \nu_{A^*}}\,d\Sigma + \int_\Omega y^0 v(0)\,dx, \end{aligned} \tag{2.3}$$

where

$$A^*(t)\psi = -\sum_{i,j=1}^n \frac{\partial}{\partial x_j}\left(a_{ij}(x,t)\frac{\partial \psi}{\partial x_i}\right) + a_0(x,t)\psi \quad \text{in } \Omega \tag{2.4}$$

and the co-normal derivative with respect to the operator A^* is given by

$$\frac{\partial v}{\partial \nu_{A^*}}(x,t) = \sum_{i,j=1}^n a_{ij}(x,t)\frac{\partial v}{\partial x_i}(x,t)\nu_i(x) \quad \text{on } \Sigma, \tag{2.5}$$

with $\nu = (\nu_1, \ldots, \nu_n)$ the unit external normal to Γ. In the form (2.3), the data y^0 and u explicitly appear in the equation and the unknowns are

$$y, \quad y(T). \tag{2.6}$$

In the process we have exhibited the two terms that usually appear in the adjoint system

$$A^*(t)v - \frac{\partial v}{\partial t}, \quad v(T). \tag{2.7}$$

If we introduce the space

$$H^{2,1}(Q) = L^2\big(0,T;H^2(\Omega)\big) \cap H^1\big(0,T;L^2(\Omega)\big) \tag{2.8}$$

and its subspace

$$\Phi = \{v \in H^{2,1}(Q) \colon v|_{\Sigma} = 0\}, \tag{2.9}$$

then under appropriate assumptions we shall see that the map

$$v \mapsto \left(A^*(t)v - \frac{\partial v}{\partial t}, v(T)\right) \colon \Phi \to L^2(Q) \times H_0^1(\Omega) \tag{2.10}$$

is an isomorphism. Moreover for

$$u \in L^2(\Sigma) \quad \text{and} \quad y^0 \in \big(H_0^1(\Omega)\big)' = H^{-1}(\Omega), \tag{2.11}$$

the linear map

$$v \mapsto \int_{\Sigma} u \frac{\partial v}{\partial \nu_{A^*}} \, d\Sigma + \int_{\Omega} y^0 v(0) \, d\Omega \colon \Phi \to \mathbb{R} \tag{2.12}$$

is continuous.

If we denote by $\mathcal{D}_a$ the isomorphism (2.10) and by ℓ the continuous linear functional (2.12) on Φ, equation (2.3) is now equivalent to find

$$\tilde{y} = (y, y_T) \in L^2(Q) \times H^{-1}(\Omega) \tag{2.13}$$

such that

$$\langle \tilde{y}, \mathcal{D}_a v \rangle_{L^2(Q) \times H_0^1(\Omega)} = \ell(v), \quad \forall v \in \Phi. \tag{2.14}$$

But this is equivalent to find $\tilde{y}$ in $\big(L^2(Q) \times H_0(\Omega)\big)'$ such that

$$(\mathcal{D}_a)^* \tilde{y} = \ell \quad \text{in } \Phi'. \tag{2.15}$$

This problem has a unique solution because the transposed of the isomorphism $\mathcal{D}_a$ is also an isomorphism.

So the idea behind the Method of Transposition is to construct a "smooth" adjoint isomorphism and transpose it in order to make sense of the original boundary value problem for "rougher" data. When data are smoother we naturally recover the usual results.

The other ingredient associated with the Method of Transposition is the theory of interpolation, which enables us to "interpolate" between a "smooth" and a "rough" version of the same isomorphism.

2.2 Point control

Let $\{x_1, x_2, \ldots, x_N\}$ be a set of points in the domain Ω. Consider the boundary value problem

$$\begin{cases} A(t)y + \dfrac{\partial y}{\partial t} = \displaystyle\sum_{i=1}^{N} \delta(x_i) u_i(t), \\ y|_{\Sigma} = 0, \\ y(x,0) = y^0(x), \end{cases} \tag{2.16}$$

where $A(t)$ is given by (1.15) with hypotheses (2.2) and (1.15), $\delta(x_i)$ is the Dirac delta function at x_i, $1 \le i \le N$, and $u_i\colon [0,T] \to \mathbb{R}$, $1 \le i \le N$, are the components of the control function $u\colon [0,T] \to U$ with $U = \mathbb{R}^N$.

By repeating the construction in the previous section, it is easy to show that (2.16) is equivalent to

$$\int_Q y\left[A^*(t)v - \frac{\partial v}{\partial t}\right] dx\,dt + \int_\Omega y(T)v(T)\,dx \\ = \int_0^T \sum_{i=1}^N u_i(t)v(x_i,t)\,dt \; + \int_\Omega y^0 v(0)\,dx \quad (2.17)$$

for all v in Φ. The right-hand-side of (2.17) is a continuous linear functional on Φ when the map

$$v \mapsto \int_0^T \sum_{i=1}^N u_i(t)v(x_i,t)\,dt + \int_\Omega y^0 v(0)\,d\Omega\colon \Phi \to \mathbb{R} \quad (2.18)$$

is continuous. This happens when y^0 belongs to $H^{-1}(\Omega)$ and when the evaluation map

$$w \mapsto w(x_i)\colon\, H^2(\Omega) \to \mathbb{R} \quad (2.19)$$

is continuous, that is when the dimension n of the space $\mathbb{R}^n$ that contains Ω is less or equal to 3 (cf. J. L. Lions and E. Magenes [1, Volume 1, Theorem 9.8, p. 45] . As a result for dimensions $n = 1, 2$, and 3, we can use the Method of Transposition.

2.3 Main result

In its crudest form the technique used in the previous examples rests on the following classical lemma.

Lemma 2.1. *Given an (algebraic and topological) isomorphism $\mathcal{L}\colon X \to Y$ between two real Banach spaces X and Y, its transpose $\mathcal{L}^*: Y' \to X'$ is also an isomorphism between the topological dual spaces Y' and X' for Y and X endowed with their respective norm topologies. Hence for each x' in X', the* variational equation

$$\langle \mathcal{L}^* y', x\rangle_X = \langle x', x\rangle_X \quad \forall x \in x \quad (2.20)$$

has a unique solution y' in Y' that is equal to $(\mathcal{L}^*)^{-1}x'$. Moreover the solution is continuous with respect to the datum x'.

Proof. Cf. N. Dunford and R. S. Schwartz [2, Vol. 1, p. 479, Lem. 7]. □

There are many ways to exploit the transposition of a smooth adjoint isomorphism. For instance, the reader is referred to J. L. Lions and E. Magenes [1, Volume 1] for applications to elliptic (Chapter 2, §6, p. 166) and parabolic problems (Chapter 3, p. 225). Recently this technique has also been used by M. C. Delfour [1] for systems with delays in state and control variables.

2.4 Application of transposition to the examples of §2.1 and §2.2

We now make precise the constructions of §2.1 and §2.2. Define

$$\begin{gathered} H^{2,1}(Q) = L^2\big(0,T;H^2(\Omega)\big) \cap H^1\big(0,T;L^2(\Omega)\big), \\ \Phi = \{v \in H^{2,1}(Q)\colon v|_\Sigma = 0\} \end{gathered} \tag{2.21}$$

and consider the continuous linear map

$$v \mapsto \mathcal{L}_a v = \left(A^*(t)v - \frac{\partial v}{\partial t}, v(T)\right) : \Phi \to L^2(Q) \times H_0^1(\Omega), \tag{2.22}$$

where for all ψ in $H^2(\Omega)$

$$A^*(t)\psi = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(a_{ij}(x,t)\frac{\partial \psi}{\partial x_i}\right) + a_0(x,t)\psi. \tag{2.23}$$

Assume that

$$a_0, \frac{\partial a_0}{\partial t} \in L^\infty(Q), \quad a_{ij} \in W^{1,\infty}(Q), \qquad 1 \le i,j \le n, \tag{2.24}$$

and

$$\exists \alpha > 0, \quad \forall \xi \in \mathbb{R}^n, \quad \sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \ge \alpha \sum_{i=1}^{n} \xi_i^2 \quad \text{a.e. in } Q. \tag{2.25}$$

Then A^* is V–H coercive.

Lemma 2.2. *Assume that Ω is bounded C^∞ domain and that the a's verify properties* (2.24) *to* (2.25). *Then the map $\mathcal{L}_a$ is an isomorphism.*

Proof. Cf. C. Bardos [1], L. Tartar [1], and A. Bensoussan and J. L. Lions [1]. □

The map $\mathcal{L}_a$ is the isomorphism we shall transpose. Introduce the additional notation

$$V = H_0^1(\Omega), \quad D = H^2(\Omega) \cap H_0^1(\Omega), \quad H = L^2(\Omega). \tag{2.26}$$

Given any two Hilbert spaces X and Y with a continuous linear injection from X into Y, we define the space

$$W(0,T;X,Y) = \left\{v \in L^2(0,T;X)\colon \frac{dv}{dt} \in L^2(0,T;Y)\right\}. \tag{2.27}$$

Notice that Φ coincides with the space $W(0,T;D,H)$.

Remark 2.1. In fact Lemma 2.2 can be obtained directly from the Corollary to Theorem 1.4 and, in addition, because

$$D\big(A_\lambda(t)\big) = D \text{ (independent of } t)$$

the solution of

$$A^*(t)v - \frac{\partial v}{\partial t} = f \in L^2(0,T;H), \quad v(T) = v^T \in V$$

belongs to $C(0,T;V)$. □

We also know from interpolation theory (cf. J. L. Lions [2]) that

$$[D,H]_{1/2} = [H^2(\Omega) \cap H_0^1(\Omega), L^2(\Omega)]_{1/2} = H_0^1(\Omega) = V. \tag{2.28}$$

Consider the following functional ω on Φ:

$$\omega(v) = \int_0^T \langle f(t), v(t)\rangle_D \, dt + \langle y^0, v(0)\rangle_V \tag{2.29}$$

for f in $L^2(0,T;D')$ and y^0 in V', where $\langle\cdot,\cdot\rangle_X$ denotes the duality pairing on $X' \times X$. The functional ω is linear and continuous. So we conclude from Lemma 2.1 that

$$\exists \text{ a unique } (y, y_T) \in \big(L^2(Q) \times H_0^1(\Omega)\big)' \equiv L^2(Q) \times V' \tag{2.30}$$

such that for all v in Φ

$$\int_0^T \left(y, A^*(t)v - \frac{\partial v}{\partial t}\right) dt + \langle y_T, v(T)\rangle_V = \int_0^T \langle f, v\rangle_D \, dt + \langle y^0, v(0)\rangle_V. \tag{2.31}$$

So we are now ready to refine this first result.

Proposition 2.1. *Under the hypotheses of Lemma 2.2, the map*

$$y \mapsto \mathcal{D}y = \left(\big(A^*(t)\big)^* y + \frac{dy}{dt}, y(0)\right) : W(0,T;H,D') \\ \to L^2(0,T;D') \times V' \tag{2.32}$$

is an isomorphism. Moreover, we have the following identity:

$$\textit{for all } y \in W(0,T;H,D') \quad \textit{and} \quad v \in W(0,T;D,H)$$

$$\int_0^T \left(y, A^*(t)v - \frac{\partial v}{\partial t}\right) dt + \langle y(T), v(T)\rangle_V \\ = \int_0^T \left\langle \big(A^*(t)\big)^* y + \frac{dy}{dt}, v \right\rangle_D dt + \langle y(0), v(0)\rangle_V, \tag{2.33}$$

where $y(0)$ and $y(T)$ are to be interpreted as values of the function y in $C(0,T;V')$[1], which is almost everywhere equal to y in $W(0,T;H,D')$ (here $\big(A^*(t)\big)^*$ denotes the topological transposed of the linear map $A^*(t)\colon D \to H$).

[1] Recall the notation $C(0,T;V')$ for $C([0,T];V')$.

Proof. We have seen that the pair (y, y_T) is the unique solution of the equation

$$\int_0^T \left(y, A^*(t)v - \frac{\partial v}{\partial t}\right)_H dt + \langle y_T, v(T)\rangle_V = \int_0^T \langle f, v\rangle_D \, dt + \langle y^0, v(0)\rangle_V, \quad \forall v \in \Phi. \quad (2.34)$$

Let $v(t) = \varphi(t)w$ for some w in D and φ in $\mathcal{D}(]0, T[)$. The function v belongs to Φ and its substitution in (2.34) yields

$$-\int_0^T (y(t), w)_H \frac{d\varphi}{dt}(t) \, dt = \int_0^T \langle f(t) - \big(A^*(t)\big)^* y, w\rangle \varphi(t) \, dt.$$

But the right-hand-side is precisely the definition of the distributional derivative $D_t y$ for the vector distribution y:

$$\langle y(\varphi), w\rangle_D = \int_0^T (y(t), w)_H \varphi(t) \, dt, \;\; \langle D_t y(\varphi), w\rangle_D = -\int_0^T (y(t), w)_H \frac{d\varphi}{dt}(t) \, dt.$$

From (2.34) we conclude that

$$\frac{d}{dt}(y(t), w)_H = \big\langle f(t) - \big(A^*(t)\big)^* y, w\big\rangle_D \implies D_t y = f(t) - \big(A^*(t)\big)^* y \in L^2(0, T; D'). \quad (2.35)$$

Therefore y belongs to $W(0, T; H, D')$ as predicted. Moreover the injection

$$W(0, T; H, D') \to C(0, T; V') \quad (2.36)$$

is continuous (cf. J. L. LIONS and E. MAGENES [1 , Volume 2, p. 34, Theorem 6.2]) because

$$[H, D']_{1/2} = [D, H]'_{1,2} = [H^2(\Omega) \cap H_0^1(\Omega), L^2(\Omega)]_{1/2} = H_0^1(\Omega)'$$

(cf. J. L. LIONS and E. MAGENES [1, Volume 1, p. 204 and p. 178]). As a result y in $W(0, T; H, D')$ is almost everywhere equal to a continuous function $\underline{y}$ in $C(0, T; V')$ and $y(0)$ and $y(T)$ make sense as elements of V'.

Going back to (2.34) and substituting (2.35) into it we obtain

$$\int_0^T \left\{ \left(y, \frac{\partial v}{\partial t}\right)_H + \langle D_t y, v\rangle_D \right\} dt = \langle y_T, v(T)\rangle_V - \langle y^0, v(T)\rangle_V.$$

Choose $v(t) = w\psi(t)$ for some w in D and ψ in $H^1(0, T; \mathbb{R})$ and substitute it into the previous equation:

$$\int_0^T \left\{ (y(t), w)_H \frac{d\psi}{dt}(t) + \langle D_t y, w\rangle_D \psi(t) \right\} dt = \langle y_T, w\rangle_V \psi(T) - \langle y^0, v(0)\rangle_V \psi(0). \quad (2.37)$$

Recall from (2.35) that

$$\langle D_t y, w\rangle_D = \frac{d}{dt}(y(t), w)_H \in L^2(0,T;\mathbb{R})$$

so that (2.37) reduces to

$$\int_0^T \frac{d}{dt}\{(y(t), w)_H \psi(t)\}\, dt = \langle y_T, w\rangle_V \psi(T) - \langle y^0, v(0)\rangle_V \psi(0).$$

As for each w the map

$$t \mapsto (y(t), w)_H$$

belongs to $H^1(0,T;\mathbb{R})$, we conclude that

$$\langle y_T, w\rangle_V = (y(t), w)_{H|t=T}, \quad \langle y_0, w\rangle_V = (y(t), w)_{H|t=0}. \tag{2.38}$$

It remains to interpret the right-hand sides in (2.38) correctly because in principle $y \in L^2(0,T;H)$ has no pointwise character. However we know that the injection (2.36) is continuous and that the function

$$t \mapsto \underline{y}(t) = i_V^* y(t)$$

is continuous on $[0,T]$ with values in V'. So (2.38) yields

$$y_T = \underline{y}(T) \in V' \text{ and } \underline{y}(0) = y^0 \in V'. \tag{2.39}$$

This shows that $\mathcal{D}$ as defined in (2.32) is indeed an isomorphism. This completes the proof of the Proposition. □

If we go back to the Dirichlet boundary control to §2.1 and to (2.3), we have to show that the linear map

$$v \mapsto -\int_\Sigma u \frac{\partial v}{\partial \nu_{A^*}}\, d\Sigma \colon L^2(0,T;D) \to \mathbb{R}$$

is continuous. Indeed for almost all t, the map

$$w \mapsto \frac{\partial w}{\partial \nu_{A^*(t)}} \colon D = H^2(\Omega) \cap H^1(\Omega) \to H^{1/2}(\Gamma)$$

and the injection of $H^{1/2}(\Gamma)$ into $L^2(\Gamma)$ are continuous. This defines the control operator $B(t) \colon U = L^2(\Gamma) \to D'$ as follows:

$$\langle B(t)u, w\rangle_D = \left(u, \frac{\partial w}{\partial \nu_{A^*(t)}}\right)_{L^2(\Gamma)} \tag{2.40}$$

and the adjoint operator $B^*(t) \colon D \to L^2(\Gamma)$

$$B^*(t)w = \frac{\partial w}{\partial \nu_{A^*(t)}}. \tag{2.41}$$

They are both linear continuous and strongly measurable and bounded as a function of t because the a_{ij}'s belong to $W^{1,\infty}(Q)$ and Ω is bounded and C^∞.

With the above definitions we see that the solution y in $W(0,T;H,D')$ of the operational differential equation

$$\begin{aligned} \big(A^*(t)\big)^* y + \frac{dy}{dt} = B(t)u, \quad u \in L^2(0,T;U), U = L^2(\Gamma), \\ y(0) = y^0 \in V' \end{aligned} \tag{2.42}$$

coincides with the solution of problem (2.1).

Similarly for the example of §2.2, the control functions are

$$\underline{u} = (u_1, \dots, u_N) \in L^2(0,T;U), \quad u = \mathbb{R}^N$$

and the control operator is

$$\langle B\underline{u}, w\rangle_D = \sum_{i=1}^{N} u_i w(x_i), \quad w \in D, \quad \underline{u} \in \mathbb{R}^N. \tag{2.43}$$

In other words

$$B\underline{u}(t) = \sum_{i=1}^{N} u_i(t)\delta_{x_i},$$

where δ_{x_i} is the Dirac delta function in x_i. For domains Ω in $\mathbb{R}^n$, $1 \le n \le 3$, the operator $B\colon U \to D'$ is continuous and the operational differential equation (2.42) is a good model for point control.

Notation 2.1. The operator $\big(A^*(t)\big)^* \in \mathcal{L}(H, D')$ is an extension of the operator $A(t) \in \mathcal{L}(D, H)$ as defined in (1.5). In the sequel we shall write $A(t)$ instead of $\big(A^*(t)\big)^*$ and keep in mind that $A(t) \in \mathcal{L}(D,H) \cap \mathcal{L}(V,V') \cap \mathcal{L}(H,D')$. □

2.5 A change of variable

It is a natural question to ask whether the operational differential equation

$$\begin{aligned} A(t)y + \frac{dy}{dt} = f \in L^2(0,T;D'), \\ y(0) = y^0 \in V', \quad y \in W(0,T;H,D') \end{aligned} \tag{2.44}$$

is fundamentally different from a variational differential equation of the form

$$\begin{aligned} \tilde{A}(t)\tilde{y} + \frac{d\tilde{y}}{dt} = \tilde{f} \in L^2(0,T;V'), \\ \tilde{y}(0) = \tilde{y}^0 \in H, \quad \tilde{y} \in W(0,T;V,V') \end{aligned} \tag{2.45}$$

of the type we studied in §1.3.

It turns out that we can go from (2.44) to (2.45) by making the following change of variable:

$$\tilde{y}(t) = [A_\lambda^{1/2}(t)]^{-1} y(t) = A_\lambda^{-1/2}(t) y(t), \tag{2.46}$$

where

$$A_\lambda(t) = A(t) + \lambda I$$

and $\lambda \in \mathbb{R}$ is the number for which the V–H coercivity is verified. By Proposition 1.3,

$$\tilde{y} \in W(0, T; V, V').$$

Moreover

$$\frac{d\tilde{y}}{dt} = A_\lambda^{-1/2}(t) \frac{dy}{dt} + \big(A_\lambda^{-1/2}(t)\big)' y.$$

But

$$\big(A_\lambda^{-1/2}(t)\big)' = -A_\lambda^{-1/2}(t) \big(A_\lambda^{1/2}(t)\big)' A_\lambda^{1/2}(t)$$

and the derivative of $A_\lambda^{1/2}(t)$ makes sense as an element of $\mathcal{L}(H, V')$ and

$$y \in L^2(0, T; H) \implies \big(A_\lambda^{-1/2}(t)\big)' y \in L^2(0, T; V').$$

As for the first term

$$A_\lambda^{-1/2}(t) \in \mathcal{L}(D', V'),$$
$$\frac{dy}{dt} \in L^2(0, T; D') \implies A_\lambda^{-1/2}(t) \frac{dy}{dt} \in L^2(0, T; V').$$

Multiply the first equation (2.44) by $A_\lambda^{-1/2}(t)$

$$A_\lambda^{-1/2}(t) A(t) y + A_\lambda^{-1/2}(t) \frac{dy}{dt} = A_\lambda^{-1/2}(t) f \in L^2(0, T; V'),$$
$$A_\lambda^{-1/2}(0) y(0) = A_\lambda^{-1/2}(0) y^0 \in H.$$

Then

$$A_\lambda^{-1/2}(t) A(t) = A(t) A_\lambda^{-1/2}(t)$$

and

$$A(t)\tilde{y}(t) + \frac{d\tilde{y}}{dt} - \big(A_\lambda^{-1/2}(t)\big)' A_\lambda^{1/2}(t) \tilde{y} = \tilde{f},$$
$$\tilde{y}(0) = \tilde{y}^0,$$

where

$$\tilde{f} = A_\lambda^{-1/2}(t) f \in L^2(0, T; V), \quad \tilde{y}^0 = A_\lambda^{-1/2}(0) y^0 \in H. \tag{2.47}$$

The operator

$$K(t) = -\big(A_\lambda^{-1/2}(t)\big)' A_\lambda^{1/2}(t) = A_\lambda^{-1/2}(t) \big(A_\lambda^{1/2}(t)\big)' \in \mathcal{L}(V, H)$$

is a perturbation of the operator $A(t)$, and we know from Theorem 1.2 that for such a perturbation, $A + K$ is V–H coercive.

From this quick analysis we conclude that $\tilde{y}$ given by (2.46) is the unique solution in $W(0,T;V,V')$ of (2.44) with

$$\tilde{A}(t) = A(t) + A_\lambda^{-1/2}(t)\big(A_\lambda^{1/2}(t)\big)'. \tag{2.48}$$

To complete the picture, the control operators in the two examples of §2.1 and §2.2

$$B(t) \in \mathcal{L}(U, D') \tag{2.49}$$

are transformed into

$$\tilde{B}(t) = A_\lambda^{-1/2}(t)B(t) \in \mathcal{L}(U, V') \tag{2.50}$$

and we are back to variational differential equation (1.26).

Thus the examples of §2, although quite different from those of §1, do not require a fundamentally new theory and appropriate changes in the state variable bring a significant simplification in the general analysis.

2.6 Other isomorphisms

Other isomorphisms can be used as starting points under the hypothesis

$$a_0, \quad \frac{\partial a_0}{\partial t} \in L^\infty(Q), \quad a_{ij} \in W^{1,\infty}(Q), \qquad 1 \le i,\ j \le n,$$

and the coercivity hypothesis

$$\exists \alpha > 0, \quad \sum_{i,j=1}^{n} a_{ij}(x,t)\xi_j\xi_i \ge \alpha \sum_{i=1}^{n} \xi_i^2, \qquad \forall \xi \in \mathbb{R}^n.$$

For instance the isomorphism

$$v \mapsto \left(A^*(t)v - \frac{\partial v}{\partial t}, v|_\Sigma, v(T)\right) : H^{2,1}(Q) \to L^2(Q) \times H^{3/2,3/4}(\Sigma) \times H^1(\Omega), \tag{C.R.}$$

where C.R. stands for the “compatibility relations” among the three spaces, or

$$v \mapsto \left(A^*(t)v - \frac{\partial v}{\partial t}, \frac{\partial v}{\partial \nu_{A^*}}, v(T)\right) : H^{2,1}(Q) \to L^2(Q) \times H^{1/2,1/4}(\Sigma) \times H^1(\Omega)$$

(cf. A. Bensoussan and J. L. Lions [1, Chapter 2, §6]).

3 Second order problems

Second order problems are usually not variational and do not fall within the framework of this chapter. For example the wave equation

$$\begin{cases} y'' + A(t)y = f, \\ y(0) = y_0, \quad y'(0) = y_1 \end{cases} \tag{3.1}$$

or Schrödinger's equation are not variational. However in some cases when damping is added, some second order problems become variational.

The simplest example

$$\begin{cases} y'' + 2\gamma Ay' + Ay = f, \\ y(0) = y_0, \quad y'(0) = y_1 \end{cases} \tag{3.2}$$

for $\gamma > 0$ falls into this category. In fact (3.2) is known as the parabolic regularization of (3.1). J. L. Lions [4, Chapter 5, §1, p. 380] shows that as γ goes to zero the solutions in (3.2) converge to the solutions of (3.1). In *structure theory*, (3.2) is often referred to as a structure with *viscous damping*. To see that (3.2) is variational we first rewrite (3.2) as a first order equation by introducing the new variables

$$y^0 = y, \quad y^1 = y' \tag{3.3}$$

and

$$\frac{d}{dt}\begin{bmatrix} y^0 \\ y^1 \end{bmatrix} + \begin{bmatrix} 0 & -I \\ A & 2\gamma A \end{bmatrix}\begin{bmatrix} y^0 \\ y^1 \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}. \tag{3.4}$$

Let V and H be two Hilbert spaces with continuous injection of V into H. So without loss of generality assume that

$$|v| \le \|v\|, \quad \forall v \in V.$$

This defines a new operator

$$\mathcal{A} = \begin{bmatrix} 0 & -I \\ A & 2\gamma A \end{bmatrix}, \tag{3.5}$$

where $A \in \mathcal{L}(V, V')$ is generated by the bilinear continuous form on V

$$\langle Av, w \rangle = a(v, w), \tag{3.6}$$

which is symmetrical, positive, and V–H coercive

$$a(v, w) = a(w, v), \quad a(v, v) \ge 0, \tag{3.7}$$

$$\exists \alpha > 0, \ \exists \lambda \in \mathbb{R}, \quad a(v, v) + \lambda |v|_H^2 \ge \alpha \|v\|_V^2. \tag{3.8}$$

As we have two components we need two new spaces $\mathcal{V}$ and $\mathcal{H}$ that will play the role of V and H. The space $\mathcal{H}$ will be the product space $V \times H$ endowed with the inner product: For all $\vec{v} = (v^0, v^1)$ and $\vec{w} = (w^0, w^1)$

$$[\vec{v}, \vec{w}] \stackrel{\text{def}}{=} a(v^0, w^0) + \lambda (v^0, w^0)_H + (v^1, w^1)_H. \tag{3.9}$$

The space $\mathcal{H}$ will be our new pivot space, and it remains to specify the space $\mathcal{V}$. To do that we look at the continuity of the bilinear form

$$\vec{a}(\vec{v},\vec{w}) = [\mathcal{A}\vec{v},\vec{w}]. \tag{3.10}$$

By construction

$$\begin{aligned}\vec{a}(\vec{v},\vec{w}) &= a(-v^1,w^0) + \lambda(-v^1,w^0) + (Av^0 + 2\gamma Av^1, w^1) && (3.11)\\ &= -a(v^1,w^0) - \lambda(v^1,w^0) + a(v^0,w^1) + 2\gamma a(v^1,w^1). && (3.12)\end{aligned}$$

It is clearly continuous on $\mathcal{V}\times\mathcal{V}$ for $\mathcal{V} = V\times V$. Moreover,

$$\vec{a}(\vec{v},\vec{v}) = -\lambda(v^1,v^0) + 2\gamma a(v^1,v^1) \tag{3.13}$$

and

$$\begin{aligned}\vec{a}(\vec{v},\vec{v}) + \mu[\vec{v},\vec{v}] &= -\lambda(v^1,v^0) + 2\gamma a(v^1,v^1) + \mu[a(v^0,v^0)\\ &\qquad + \lambda|v^0|^2 + |v^1|^2]\\ &= 2\gamma[a(v^1,v^1) + \lambda|v^1|^2] + \mu[a(v^0,v^0) + \lambda|v^0|^2]\\ &\qquad + [\mu - 2\gamma\lambda]|v^1|^2 - \lambda(v^1,v^0)\\ &\geq \alpha[2\gamma\|v^1\|_V^2 + \mu\|v^0\|_V^2] + (\mu - 2\gamma\lambda)|v^1|^2 - \lambda(v^1,v^0)\\ &\geq \gamma\alpha[\|v^1\|_V^2 + \|v^0\|_V^2]\end{aligned} \tag{3.14}$$

for $\mu \geq \max\{2\gamma\lambda, \gamma + \lambda^2\ (4\gamma\alpha^2)\}$. Thus $\vec{a}$ is $\mathcal{V}-\mathcal{H}$ coercive and all the results of the previous sections apply.

Another example that is of special interest is the so-called "structural damping," which is of the form

$$\begin{cases} y'' + 2\gamma A^{1/2}y' + Ay = f,\\ y(0) = y_0, \quad y'(0) = y_1, \end{cases} \tag{3.15}$$

and more generally for $0 \leq \alpha \leq 1$

$$\begin{cases} y'' + 2\gamma A^{\alpha}y' + Ay = f,\\ y(0) = y_0, \quad y'(0) = y_1. \end{cases} \tag{3.16}$$

It is known (cf. S. Chen and R. Triggiani [1, 2, 3]), that for $1/2 \leq \alpha \leq 1$, the corresponding semigroup is analytic, but it is not yet known whether the problem is variational for $1/2 \leq \alpha < 1$.

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3

Semigroup Methods for Systems With Unbounded Control and Observation Operators

1 Complements on semigroups

Let $\{S(t)\}$ be a strongly continuous semigroup on the Hilbert space H. Let $|\cdot|$ and $(\cdot,\cdot)$ be the norm and inner product in H. Denote by A the infinitesimal generator of $\{S(t)\}$ and by $D(A)$ its domain. When $D(A)$ is endowed with the graph norm of A

$$\|h\|_A^2 = |h|^2 + |Ah|^2, \quad h \in D(A), \tag{1.1}$$

it becomes a Hilbert space and

$$A\colon D(A \to H \tag{1.2}$$

is a continuous linear operator.

1.1 Notation

Let X and Y be two Hilbert spaces such that $X \subset Y$. In that case the interpolation spaces $(X,Y)_{\theta,p}$ with $p=2$ and $[X,Y]_\theta$ coincide for $0<\theta<1$ (cf. Chapter 1, §4.7). We shall use the notation $[X,Y]_\theta$, θ, $0<\theta<1$.

For all data of the form

$$y^0 \in D(A), \quad f \in L^2\big(0,T;D(A)\big), \tag{1.3}$$

the function

$$y(t) = S(t)y^0 + \int_0^t S(t-s)f(s)\,ds \tag{1.4}$$

is the unique solution in the space

$$\mathcal{V}(0,T;D(A),H) = \left\{ y \in C\big([0,T];D(A)\big) \colon \frac{dy}{dt} \in L^2(0,T;H) \right\} \tag{1.5}$$

of the differential equation

$$\begin{cases} \dfrac{dy}{dt} = Ay + f, \\ y(0) = y^0. \end{cases} \tag{1.6}$$

The notation in (1.5) will be used on several occasions. Given two Banach spaces with a continuous linear injection of X into Y,

$$\mathcal{V}(0,T;X,Y) = \left\{ y \in C([0,T];X) \colon \frac{dy}{dt} \in L^2(0,T;Y) \right\}, \tag{1.7}$$

where dy/dt is to be interpreted as the vectorial distributional derivative of y in $C([0,T];X)$ considered as an element of $L^2(0,T;Y)$.

It is well known that, by density of $D(A)$ in H, the variation of constants formula (1.4) makes sense for data

$$y^0 \in H, \quad f \in L^2(0,T;H), \tag{1.8}$$

and such a solution is usually referred to as a "mild solution" of system (1.6) (cf. Chapter 1, Definition 3.1 (iv)). It is useful to be more explicit and precise about this. We first identify the elements of the dual H' of H with those of H,

$$H \equiv H'.$$

Let $\{S^*(t)\}$ be the adjoint semigroup on $H' = H$ and let A^* be its infinitesimal generator with domain $D(A^*)$. When $D(A^*)$ is endowed with the graph norm of A^*, the restriction of $\{S^*(t)\}$ to $D(A^*)$ is also a strongly continuous semigroup on $D(A^*)$

$$S^*(t) \in \mathcal{L}\big(D(A^*), D(A^*)\big). \tag{1.9}$$

For simplicity we still denote by $\{S^*(t)\}$ the semigroup on $D(A^*)$. Its topological transposed

$$S^*(t)^* \in \mathcal{L}(D(A^*)', D(A^*)') \tag{1.10}$$

is also a strongly continuous semigroup on $D(A^*)'$. It can be viewed as an extension to $D(A^*)'$ of the semigroup $\{S(t)\}$ on H. For this reason we shall use the same notation $\{S(t)\}$ and keep in mind that it can be viewed as a strongly continuous semigroup on either one of the three spaces

$$D(A) \to H \equiv H' \to D(A^*)'. \tag{1.11}$$

For data as specified in (1.8), let y in $C([0,T];H)$ be the function given by the variation of constants formula (1.4). For an arbitrary k in $D(A^*)$

$$\big(k, y(t)\big) = (S^*(t)k, y^0) + \int_0^t \big(S^*(t-s)k, f(s)\big)\, ds.$$

This function is differentiable because for all k in $D(A^*)$ and $s \in [0,T[$

$$t \mapsto \frac{\partial S^*}{\partial t}(t-s)k = A^* S^*(t-s)k = S^*(t-s)A^* k$$

belongs to $C([s,T];H)$. To see that we compute the distributional derivative: For all $\varphi \in \mathcal{D}(]0,T[)$ consider the expression

$$\tau = \int_0^T \big(k, y(t)\big) \frac{d\varphi}{dt}(t)\, dt.$$

We can assume that $y^0 = 0$. For the second term

$$\begin{aligned}
\tau &= -\int_0^T dt \int_0^t ds \big(S^*(t-s)k, f(s)\big) \frac{d\varphi}{dt}(t) \\
&= -\int_0^T ds \int_s^T dt\, \big(S^*(t-s)k, f(s)\big) \frac{d\varphi}{dt}(t) \\
&= \int_0^T ds \int_s^T dt \left(\frac{d}{dt} S^*(t-s)k, f(s)\right) \varphi(t) + \int_0^T \big(k, f(t)\big)\varphi(t)\, dt \\
&= \int_0^T dt \left[\left(A^* k, \int_0^t S(t-s) f(s)\, ds\right) + \big(k, f(t)\big)\right] \varphi(t).
\end{aligned}$$

As a result

$$\begin{aligned}
\frac{d}{dt}\big(k, y(t)\big) = (S^*(t)A^*k, y^0) + \int_0^t \big(S^*(t-s)A^*k, f(s)\big)\, ds + \big(k, f(t)\big) \\
= \big(A^*k, y(t)\big) + \big(k, f(t)\big)
\end{aligned}$$

and

$$\begin{cases} \dfrac{dy}{dt} = (A^*)^* y + f, \\ y(0) = y_0, \end{cases} \tag{1.12}$$

where

$$(A^*)^* \colon H \to D(A^*)' \tag{1.13}$$

is the transposed of the continuous linear operator

$$A^* \colon D(A^*) \to H, \tag{1.14}$$

when $D(A^*)$ is endowed with the graph norm of A^*. As for the semigroup $\{S(t)\}$, $(A^*)^*$ is to be viewed as an extension of the operator A because they coincide on $D(A)$. So we adopt the notation

$$A \in \mathcal{L}(D(A), H) \cap \mathcal{L}(H, D(A^*)') \tag{1.15}$$

in both cases and conclude that for data given by (1.8) the function y given by (1.4) is the unique solution in $\mathcal{V}(0,T;H,D(A^*)')$ to the differential equation (1.6) (cf. Chapter 1, Remarks 3.1 and 3.2).

It is always possible to interpolate pairwise between the various spaces for A and $S(t)$

$$A \in \mathcal{L}([D(A), H]_\alpha, [D(A^*), H]'_{1-\alpha}), \quad 0 < \alpha < 1, \tag{1.16}$$

$$S(t) \in \mathcal{L}([D(A), D(A^*)']_\beta, [D(A), D(A^*)']_\beta), \quad 0 < \beta < 1. \tag{1.17}$$

For some types of boundary control problems, it is necessary to also make sense of the solution of the differential equation (1.6) for data

$$y^0 \in D(A^*)', \quad f \in L^2(0, T; D(A^*)'). \tag{1.18}$$

The starting point is again the variation of constants formula

$$y(t) = S(t)y^0 + \int_0^t S(t-s)f(s)\,ds, \tag{1.19}$$

where $\{S(t)\}$ is now considered as a semigroup on $D(A^*)'$ rather than H.

By definition y belongs to $C([0, T]; D(A^*)')$. We can now repeat the previous construction with k in $D(A^{*2})$, the domain of A^{*2}, which coincides with the domain of the infinitesimal generator A^* of $\{S^*(t)\}$ considered as a semigroup on $D(A^*)$

$$D\big(A^*; D(A^*)\big) = D(A^{*2}; H) = D(A^{*2}), \tag{1.20}$$

where $D(L; X)$ denotes the domain of the unbounded linear operator L considered as an operator on the space X.

The end result is that for data of the form (1.18), the function y given by expression (1.19) belongs to $\mathcal{V}(0, T; D(A^*)', D(A^{*2})')$ and is the unique solution of the differential equation (1.6). The following theorem is the analog of Proposition 3.3 in Chapter 1.

Theorem 1.1. *Let $\{S(t)\}$ be a strongly continuous semigroup and let $\lambda \in \mathbb{R}$ be such that*

$$A_\lambda = A - \lambda I \in \mathcal{L}(D(A), H) \tag{1.21}$$

be an isomorphism. The solution y of (1.6) for $y^0 \in H$ and $f \in H^1(0, T; D(A^*)')$ belongs to $C^1([0, T]; D(A^*)') \cap C([0, T]; H)$ and for all t, $0 \le t \le T$,

$$y(t) = -A_\lambda^{-1} f(t) + S(t)[y^0 + A_\lambda^{-1} f(0)] + \int_0^t S(t-s)A_\lambda^{-1}\left[\frac{df}{ds}(s) - \lambda f(s)\right] ds. \tag{1.22}$$

Proof. The solution y of (1.6) belongs to $C([0, T]; D(A^*)')$ and is given by the usual formula

$$y(t) = S(t)y^0 + \int_0^t S(t-s)f(s)\,ds.$$

We already know that for $y^0 \in H$, the first term belongs to $C([0,T];H)$ $\cap C^1([0,T];D(A^*)')$. So we only need to prove the theorem for $y^0 = 0$. Recall that $A_\lambda \in \mathcal{L}(D(A),H) \cap \mathcal{L}(H,D(A^*)')$. Hence

$$y(t) = \int_0^t A_\lambda S(t-s) A_\lambda^{-1} f(s)\, ds. \tag{1.23}$$

But

$$-\frac{d}{ds} S(t-s) = AS(t-s)$$

and

$$y(t) = -\int_0^t S(t-s) A_\lambda^{-1} \lambda f(s)\, ds - \int_0^t \frac{d}{ds}[S(t-s)] A_\lambda^{-1} f(s)\, ds,$$
$$y(t) = -A_\lambda^{-1} f(t) + S(t) A_\lambda^{-1} f(0) + \int_0^t S(t-s) A_\lambda^{-1} [f'(s) - \lambda f(s)]\, ds.$$

By assumption $f \in H^1(0,T;D(A^*)')$ implies that

$$A_\lambda^{-1} f \in H^1(0,T;H) \subset C([0,T];H)$$

and necessarily each term and, a fortiori y, belongs to $C([0,T];H)$. Moreover

$$\frac{dy}{dt} = Ay + f \in C([0,T];D(A^*)')$$

and

$$y \in C([0,T];H) \cap C^1([0,T];D(A^*)').$$

This completes the proof. □

Remark 1.1. Notice that we also recover from identity (1.22) the usual results

$$y^0 \in D(A), \quad f \in H^1(0,T;H) \implies y \in \mathcal{V}(0,T;D(A),H).$$

□

Remark 1.2. For rough data identity (1.23) will often be used with the A_λ outside the integral

$$y(t) = S(t)y^0 + A_\lambda \int_0^t S(t-s) A_l^{-1} f(s)\, ds \tag{1.24}$$

in order to take full advantage of the eventual smoothing properties of the integral. □

This technique readily extends to higher derivatives of the function f.

Corollary 1.1. *Assume that, in addition to the assumptions of Theorem* 1.1, $y^0 \in D(A)$ *and* $f \in C([0,T];H) \cap H^2(0,T;D(A^*)')$. *The solution* y *of* (1.6) *belongs to* $C^1([0,T];H) \cap C\big([0,T];D(A)\big)$ *and for* t, $0 \le t \le T$,

$$y(t) = -A_\lambda^{-1}\big[f(t) + A_\lambda^{-1}\big(f'(t) - \lambda f(t)\big)\big] + S(t)\big[y^0 + A_\lambda^{-1} f(0) + A_\lambda^{-2}\big(f'(0) - \lambda f(0)\big)\big] + \int_0^t S(t-s) A_\lambda^{-2} [f''(s) - 2\lambda f'(s) + \lambda^2 f(s)]\, ds. \quad (1.25)$$

Proof. It is convenient to introduce the following change of variable:

$$y_\lambda(t) = e^{-\lambda t} y(t), \quad f_\lambda(t) = e^{-\lambda t} f(t), \quad S_\lambda(t) = e^{-\lambda t} S(t).$$

Then

$$y_\lambda(t) = S_\lambda(t) y^0 + \int_0^t S_\lambda(t-s) f_\lambda(s)\, ds$$

and as in the proof of the theorem

$$\begin{aligned} y_\lambda(t) &= S_\lambda(t) y^0 + \int_0^t A_\lambda S_\lambda(t-s) A_\lambda^{-1} f_\lambda(s)\, ds \\ &= S_\lambda(t)[y^0 + A_\lambda^{-1} f_\lambda(0)] - A_\lambda^{-1} f_\lambda(t) + \int_0^t S_\lambda(t-s) A_\lambda^{-1} f'_\lambda(s)\, ds. \end{aligned}$$

Now the same construction is used for the integral term

$$\int_0^t A_\lambda S_\lambda(t-s) A_\lambda^{-2} f'_\lambda(s)\, ds = \int_0^t S_\lambda(t-s) A_\lambda^{-2} f''_\lambda(s)\, ds + S_\lambda(t) A_\lambda^{-2} f'_\lambda(0) - A_\lambda^{-2} f'_\lambda(t).$$

Finally

$$y_\lambda(t) = -A_\lambda^{-1}[f(t) + A_\lambda^{-1} f'_\lambda(t)] + S_\lambda(t)[y^0 + A_\lambda^{-1} f_\lambda(0) + A_\lambda^{-2} f'_\lambda(0)] + \int_0^t S_\lambda(t-s) A_\lambda^{-2} f''_\lambda(s)\, ds.$$

From the last identity it is easy to conclude that y has the expected regularity and that expression (1.25) can be easily recovered by a change of variable. □

The above statements are quite general. Sharper results are available for specific classes of systems such as analytic semigroups.

2 Complements on analytic semigroups

In this section we recall the main regularity result for analytic semigroups. With the help of interpolation theory we also give intermediary regularity results. Finally we show how the basic regularity result for analytic semigroups is at the origin of various methods of change of variable.

2.1 Regularity results

Given a semigroup $\{S(t)\}$ of type $\omega_0(S)$, for each $\omega > \omega_0(S)$, there exists $M \geq 1$ such that

$$|S(t)h| \leq Me^{\omega t}|h|, \quad \forall h \in H,\ \forall t \geq 0. \tag{2.1}$$

The semigroup

$$S_\omega(t) = e^{-\omega t}S(t) \tag{2.2}$$

is a stable semigroup with infinitesimal generator

$$A_\omega = A - \omega I\colon D(A) \to H \tag{2.3}$$

(cf. Chapter 1, Corollary to Theorem 2.2). Similarly for the adjoint semigroup $\{S_\omega^*(t)\}$, the associated infinitesimal generator

$$A_\omega^* = A^* - \omega I\colon D(A^*) \to H \tag{2.4}$$

is also an isomorphism. Therefore the transposed of A_ω^*

$$(A_\omega^*)^*\colon H \to D(A^*)' \tag{2.5}$$

is again an isomorphism that can be viewed as an extension of A_ω from $D(A)$ to H because they coincide on $D(A)$ and $D(A)$ is dense in H. Hence

$$A_\omega \in \mathcal{L}(D(A), H) \cup \mathcal{L}(H, D(A^*)') \tag{2.6}$$

and from Interpolation Theory for all α, $0 \leq \alpha \leq 1$, the map

$$A_\omega\colon [D(A), H]_\alpha \to [D(A^*), H]'_{1-\alpha} \tag{2.7}$$

is also an isomorphism.

Recall the following regularity theorem (cf. Chapter 1, §3, Theorem 3.1).

Theorem 2.1. *Let $\{S(t)\}$ be an analytic strongly continuous semigroup. Then for all $T > 0$ the linear map*

$$y \mapsto \mathcal{D}y = \left(\frac{dy}{dt} - Ay, y(0)\right)\colon W(0,T; D(A), H) \to L^2(0,T;H) \times [D(A), H]_{1/2} \tag{2.8}$$

is an isomorphism.

Corollary 2.1. *Let $\{S(t)\}$ be an analytic strongly continuous semigroup. Then for all $T > 0$ the linear map*

$$y \mapsto \mathcal{D}y = \left(\frac{dy}{dt} - Ay, y(0)\right) \colon W(0,T; H, D(A^*)') \to L^2(0,T; D(A^*)') \times [D(A^*), H]'_{1/2} \tag{2.9}$$

is an isomorphism.

Proof. The map (2.9) is clearly linear and continuous. To show that it is bijective, it is sufficient to show that for all

$$f \in L^2(0,T;D(A^*)') \quad \text{and} \quad y^0 \in [D(A^*),H]'_{1/2},$$

there exists a unique solution $y \in W(0,T;H,D(A^*)')$ such that

$$y' - Ay = f, \quad y(0) = y^0. \tag{2.10}$$

Define for ω verifying (2.1)

$$F = A_\omega^{-1} f \in L^2(0,T;H) \quad \text{and} \quad Y^0 = A_\omega^{-1} y^0 \in [D(A),H]_{1/2}.$$

By the previous theorem, the equation

$$Y' - AY = F, \quad Y(0) = Y^0 \tag{2.11}$$

has a unique solution $Y \in W(0,T;D(A),H)$. Now define

$$y(t) = A_\omega Y(t), \quad t \in [0,T].$$

It is easy to check that

$$\begin{gathered} y = A_\omega Y \in L^2(0,T;H), \quad y' = A_\omega Y' \in L^2(0,T;D(A^*)'), \\ y(0) = A_\omega Y(0) \in [D(A^*),H]'_{1/2}. \end{gathered} \tag{2.12}$$

Therefore $y \in W(0,T;H,D(A^*)')$ and

$$y' - Ay = A_\omega[Y' - AY] = A_\omega F = f, \quad y(0) = A_\omega Y(0) = A_\omega Y^0 = y^0. \tag{2.13}$$

This completes the proof. □

We can now interpolate between isomorphisms (2.8) and (2.9).

Theorem 2.2. *Let $\{S(t)\}$ be an analytic strongly continuous semigroup. Then for all $T > 0$, all α, $0 \le \alpha \le 1$, the linear map*

$$\begin{cases} y \mapsto \mathcal{D}y = \left(\dfrac{dy}{dt} - Ay, y(0)\right) \\ W(0,T;[D(A),H]_\alpha,[D(A^*),H]'_{1-\alpha}) \\ \to L^2(0,T;[D(A^*),H]'_{1-\alpha}) \times [[D(A),H]_\alpha,[D(A^*),H]'_{1-\alpha}]_{1/2} \end{cases} \tag{2.14}$$

is an isomorphism.

If we make the additional hypothesis

$$[D(A),H]_{1/2} = [D(A^*),H]_{1/2} \tag{2.15}$$

and denote this space by V, then, for $\alpha = 1/2$, $\mathcal{D}$ is an isomorphism

$$W(0,T;V,V') \to L^2(0,T;V') \times H \tag{2.16}$$

because $[V,V']_{1/2} = H$. This is to be compared with Theorem 1.1 in Chapter 2.

There is virtually no limit to the up and down interpolation process because for any integer $n \geq 1$

$$A_\omega^n : D(A^n) \to D(A^{n-1}), \quad A_\omega^{*n} : D(A^{*n}) \to D(A^{*n-1}) \tag{2.17}$$

are also isomorphisms. So the map

$$\begin{aligned} \mathcal{D} \colon & W(0,T; D(A^{*n})', D(A^{*(n+1)})') \\ & \to L^2(0,T; D(A^{*(n+1)})') \times [D(A^{*n})', D(A^{*(n+1)})']_{1/2} \end{aligned}$$

is also an isomorphism and this can be combined with the results of Theorem 2.2 to obtain a continuum of isomorphisms. Define for $r = n + \alpha$, $n \geq 0$ an integer and α, $0 \leq \alpha \leq 1$ a real number,

$$H^r = [D(A^{n+1}), D(A^n)]_{1-\alpha}, \quad H^{-r} = [D(A^{*(n+1)}), D(A^{*n})]'_{1-\alpha}. \tag{2.18}$$

Then for all r in $\mathbb{R}$, the map

$$\mathcal{D} \colon W(0,T; H^r, H^{r-1}) \to L^2(0,T; H^{r-1}) \times [H^r, H^{r-1}]_{1/2} \tag{2.19}$$

is an isomorphism.

2.2 Other representations and the method of change of variable

It is well known that some non-homogeneous boundary value problems can be transformed into simpler homogeneous boundary value problems by an appropriate change of the unknown or state variable y (cf. F. Treves [1, pp. 426–428]). This technique also finds its analog in Control Theory in the semigroup setting (cf. H. O. Fattorini [3], A. V. Balakrishnan [3], D. C. Washburn [1], J. Zabczyk [4], and A. Chojnowska-Michalik [1]). Specific examples will be given in appropriate chapters.

For analytic semigroups, the regularity property of the map

$$\mathcal{S} \colon L^2(0,T; D(A^*)') \to W(0,T; H, D(A^*)'), \quad (\mathcal{S}f)(t) = \int_0^t S(t-s) f(s)\, ds$$

can be used to exhibit the underlying mechanism behind these methods.

Theorem 2.3. *Assume that $\{S(t)\}$ is an analytic strongly continuous semigroup and that $\lambda \in \mathbb{R}$ is such that*

$$A_\lambda = A - \lambda I \in \mathcal{L}(D(A), H) \tag{2.20}$$

is an isomorphism. For $T > 0$, f in $L^2(0,T; D(A^*)')$ and h in $[D(A^*), H]'_{1/2}$, let

$$y(t) = S(t)h + (\mathcal{S}f)(t) \tag{2.21}$$

be the solution of (2.10) *in $W(0,T; H, D(A^*)')$*

(i) *The solution y is equivalently given by the expression*

$$y(t) = S(t)h + A\int_0^t S(t-s)f_\lambda(s)\,ds - \lambda\int_0^t S(t-s)f_\lambda(s)\,ds, \quad (2.22)$$

where

$$f_\lambda(t) = A_\lambda^{-1}f(t), \quad 0 \le t \le T. \quad (2.23)$$

(ii) *If, in addition, $f \in W(0,T;H,D(A^*)')$, then*

$$y(t) = Y(t) - f_\lambda(t), \quad (2.24)$$

where Y in $W(0,T;H,D(A^*)')$ is given by the expression

$$\begin{cases} Y(t) = S(t)Y(0) + \displaystyle\int_0^t S(t-s)\frac{df_\lambda}{ds}(s)\,ds, \\ Y(0) = h + f_\lambda(0). \end{cases} \quad (2.25)$$

The proof is obvious and only uses integration by parts and the previous theorems.

This theory directly applies to parabolic systems with control through a boundary condition (cf. §4).

3 Unbounded control and observation operators

The space of *controls* will be a real Hilbert space U and the space of *control functions* will be $L^2(0,T;U)$ for some $T > 0$. The *control operator* is an element

$$B \in \mathcal{L}(U, D(A^*)'), \quad (3.1)$$

where $D(A^*)$ is endowed with the graph norm of A^*. We say that B is *bounded* if

$$B \in \mathcal{L}(U, H). \quad (3.2)$$

This terminology can be justified as follows. Consider the operator

$$B^* \in \mathcal{L}(D(A^*), U'),\ \forall u \in U,\ \forall v \in D(A^*),\ \langle B^*v, u\rangle_U = \langle Bu, v\rangle_{D(A^*)}. \quad (3.3)$$

It is readily seen that B is "bounded" in the sense of our definition if B^* is bounded as an operator from H' into U'

$$B^* \in \mathcal{L}(H', U'). \quad (3.4)$$

We have seen in the chapters on parabolic systems and delay systems in Part I how this control structure naturally arises.

The space of *observations* will be a real Hilbert space Z, and the space of *observation functions* will be $L^2(0,T;Z)$ for some $T > 0$. The *observation operator* is an element

$$C \in \mathcal{L}(D(A), Z), \tag{3.5}$$

where $D(A)$ is endowed with the graph norm of A. We say that C is *bounded* if

$$C \in \mathcal{L}(H, Z). \tag{3.6}$$

The operator C^* is the transposed of C

$$C^* \in \mathcal{L}(Z', D(A)'). \tag{3.7}$$

3.1 Analytic systems

The system

$$\begin{cases} \dfrac{dy}{dt} = Ay + f, \quad t > 0, \\ y(0) = y^0 \end{cases} \tag{3.8}$$

will be referred to as an *analytic system* if the operator A is the infinitesimal generator of an analytic strongly continuous semigroup $\{S(t)\}$. We have seen in §2 that we have a strong regularity theorem for this class of systems (cf. Theorem 2.2 and the discussion thereafter). For all α, $0 \le \alpha \le 1$, the linear map

$$\mathcal{S}\colon L^2(0,T;[D(A^*),H]'_{1-\alpha}) \to W(0,T;[D(A),H]_\alpha,[D(A^*),H]'_{1-\alpha}) \tag{3.9}$$

is continuous. If we choose for some α, $0 \le \alpha \le 1$,

$$B^* \in \mathcal{L}([D(A^*),H]_{1-\alpha}, U'), \quad C \in \mathcal{L}([D(A),H]_\alpha, Z), \tag{3.10}$$

then the linear map

$$C\mathcal{S}B\colon L^2(0,T;U) \to L^2(0,T;Z) \tag{3.11}$$

is continuous. As a result the system makes sense as an *input–output* system.

To get a complete picture we can add the initial condition or state at time 0

$$\begin{aligned} h, u \mapsto CS(\cdot)h + C\mathcal{S}Bu\colon \big[[D(A),H]_\alpha,[D(A^*),H]'_{1-\alpha}\big]_{1/2} \\ \times L^2(0,T;U) \to L^2(0,T;Z). \end{aligned} \tag{3.12}$$

This model deals with an observation in the $L^2(0,T;Z)$ sense. Other types of observations can be contemplated, but the philosophy is always the same: to make sense of the input–output map in appropriate function spaces.

Notice that when identity (2.15) in §2 is verified the above discussion amounts to choose

$$B \in \mathcal{L}(U, V'), \quad C \in \mathcal{L}(V, Z) \tag{3.13}$$

and the continuous linear map

$$\begin{cases} h, u \mapsto CS(\cdot)h + C\mathcal{S}Bu \\ \colon H \times L^2(0,T;U) \to L^2(0,T;Z). \end{cases} \tag{3.14}$$

3.2 Unbounded control operators

In §1 we made sense of the system

$$\begin{cases} \dfrac{dy}{dt} = Ay + Bu, \\ y(0) = y^0 \end{cases} \tag{3.15}$$

for the control and control functions

$$B \in \mathcal{L}(U, D(A^*)'), \quad u \in L^2(0,T;U),\ y^0 \in D(A^*)'. \tag{3.16}$$

From that analysis the linear map

$$\mathcal{S}B\colon L^2(0,T;U) \to \mathcal{V}(0,T;D(A^*)',D(A^{*2})') \subset C([0,T];D(A^*)') \tag{3.17}$$

is continuous. We have seen in §3.1 that for analytic systems, the above map is continuous with values in $W(0,T;H,D(A^*)')$ and

$$\mathcal{S}B\colon L^2(0,T;U) \to L^2(0,T;H). \tag{3.18}$$

For other families of semigroups such as *systems with delays*, we typically have

$$B^*\colon [D(A^*),H]_{1/2-\varepsilon} \to U' \text{ continuous } \forall \varepsilon > 0 \tag{3.19}$$

and the continuity of the linear map

$$\mathcal{S}B\colon L^2(0,T;U) \to C([0,T];H). \tag{3.20}$$

There are different ways to formulate hypothesis (3.20). We choose to formulate it in a "dual way" and then show it is equivalent to (3.20).

Hypothesis 3.1. Given $T > 0$, the linear map

$$h \mapsto B^*S^*(\cdot)h\colon D(A^*) \to C([0,T];U') \tag{3.21}$$

can be extended to a continuous linear map

$$h \mapsto B^*S^*(\cdot)\colon H \to L^2(0,T;U'), \tag{3.22}$$

where B^* is given by (3.3). □

Proposition 3.1. *The following statements are equivalent*:

(i) *Hypothesis* (3.1).
(ii) *The linear map*

$$u \mapsto (\mathcal{S}Bu)(T)\colon L^2(0,T;U) \to H \tag{3.23}$$

is continuous.

(iii) *The linear map*

$$u \mapsto (\mathcal{S}Bu)\colon L^2(0,T;U) \to C([0,T];H) \tag{3.24}$$

is continuous.

Proof. (iii) $\Longrightarrow$ (i) is trivial.

(i) $\Longrightarrow$ (ii). For each u in $L^2(0,T;U)$, denote by u^T the function

$$u^T(t) = u(T-t), \quad 0 \le t \le T.$$

For all h in $D(A^*)$

$$\begin{aligned}\langle B^* S^*(\cdot)h, u^T\rangle_{L^2(0,T;U)} &= \left\langle \int_0^T S(T-s)Bu(s)\,ds, h\right\rangle_{D(A^*)} \\ &= \langle (\mathcal{S}Bu)(T), h\rangle_{D(A^*)}.\end{aligned}$$

By Hypothesis 3.1

$$\begin{aligned}|\langle (\mathcal{S}Bu)(T), h\rangle| &\le \|B^* S^*(\cdot)h\|_{L^2(0,T;U')} \|u^T\|_{L^2(0,T;U)} \\ &\le c|h|\, \|u\|_{L^2(0,T;U)}\end{aligned}$$

and necessarily

$$(\mathcal{S}Bu)(T) \in H, \quad |(\mathcal{S}Bu)(T)| \le c\|u\|_{L^2(0,T;U)}.$$

Therefore the map (3.23) is well defined and continuous.

(ii) $\Longrightarrow$ (iii) Note that Hypothesis (3.1) for T implies that the same hypothesis is true for all t, $0 < t \le T$. As a result we can repeat the previous argument and obtain

$$\begin{aligned}|\langle (\mathcal{S}Bu)(t), h\rangle| &\le \|B^* S^*(\cdot)h\|_{L^2(0,t;U')} \|u\|_{L^2(0,t;U)} \\ &\le \|B^* S^*(\cdot)h\|_{L^2(0,T;U')} \|u\|_{L^2(0,T;U)} \\ &\le c|h|\, \|u\|_{L^2(0,T;U)}.\end{aligned}$$

Hence $\mathcal{S}Bu$ is a well-defined function from $[0,T]$ into H and

$$\sup_{[0,T[} |(\mathcal{S}Bu)(t)| \le c\|u\|_{L^2(0,T;U)}.$$

To prove the continuity of $\mathcal{S}Bu$ for each u, define $\underline{u} \in L^2(0,\infty;U)$

$$\underline{u}(r) = \begin{cases} u(r), & r \in [0,T], \\ 0, & r > T.\end{cases}$$

Then for h in $D(A^*)$

$$\langle (\mathcal{S}Bu)(t), h\rangle_{D(A^*)} = \int_0^T \langle B^* S^*(T-s)h, [\tau(T-t)\underline{u}](s)\rangle\, ds,$$

where $\tau(t)$ is the translation semigroup on $L^2(0,\infty;U)$

$$[\tau(t)\underline{u}](s) = \underline{u}(t+s), \quad t \geq 0,\ s \geq 0.$$

So for any t' and t in $[0,T]$

$$\begin{aligned}
&|(\mathcal{S}Bu)(t') - (\mathcal{S}Bu)(t), h >_{D(A^*)}| \\
&\qquad \leq \|B^* S^*(T-\cdot)h\|_{L^2(0,T;U)} \|\tau(T-t')\underline{u} - \tau(T-t)\underline{u}\|_{L^2(0,T;U)} \\
&\qquad \leq \|B^* S^*(\cdot)h\|_{L^2(0,T;U)} \|\tau(T-t')\underline{u} - \tau(T-t)\underline{u}\|_{L^2(0,T;U)} \\
&\qquad \leq c|h|\, \|\tau(T-t')\underline{u} - \tau(T-t)\underline{u}\|_{L^2(0,\infty;U)} \\
&\qquad\qquad\qquad\qquad \Longrightarrow\ |(\mathcal{S}Bu)(t') - (\mathcal{S}Bu)(t)| \\
&\qquad \leq c\|\tau(T-t')\underline{u} - \tau(T-t)\underline{u}\|_{L^2(0,\infty;U)}.
\end{aligned}$$

By strong continuity of the translation semigroup, we conclude that $\mathcal{S}Bu$ is continuous and that (iii) is true. This completes the proof of Proposition 3.1. □

Theorem 3.1. *Fix $T \geq 0$ and assume that Hypothesis* (3.1) *is verified. Then for data*

$$y^0 \in H, \quad u \in L^2(0,T;U), \tag{3.25}$$

the function

$$y(t) = S(t)y^0 + (\mathcal{S}Bu)(t) \tag{3.26}$$

is the unique solution in $\mathcal{V}(0,T;H,D(A^*)')$ to the differential equation

$$\begin{cases} \dfrac{dy}{dt} = Ay + Bu, \\ y(0) = y^0, \end{cases} \tag{3.27}$$

and there exists a constant $c > 0$ such that

$$\|y\|_{C([0,T];H)} + \left\|\frac{dy}{dt}\right\|_{L^2(0,T;D(A^*)')} \leq c[|y^0|_H + \|u\|_{L^2(0,T;U)}]. \tag{3.28}$$

The above results are general and essentially require an hypothesis on the free adjoint system

$$\begin{cases} -\dfrac{dv}{dt} = A^* v, \\ v(T) = h \in H, \end{cases} \tag{3.29}$$

with observation operator B^*

$$z(t) = B^* v(t) \tag{3.30}$$

because

$$z(t) = B^* S^*(T-t)h. \tag{3.31}$$

When system (3.15) is time varying (that is, with $A^*(t)$ instead of A^*), Hypothesis 3.1 would require that the linear map

$$h \mapsto z \colon H \to L^2(0,T;U') \tag{3.32}$$

be continuous.

We can also use the method of change of variable in Theorem 2.3 to transform expression (3.26) into

$$y(t) = S(t)h + A_\lambda(\mathcal{S}A_\lambda^{-1}Bu)(t). \tag{3.33}$$

Define

$$B_\lambda = A_\lambda^{-1}B \in \mathcal{L}(U,H) \tag{3.34}$$

and introduce the change of variable

$$Y(t) = y(t) + B_\lambda u(t), \quad 0 \le t \le T. \tag{3.35}$$

Then we have the following equivalent representations:

$$y(t) = S(t)h + A\int_0^t S(t-s)B_\lambda u(s)\,ds + \int_0^t S(t-s)[-\lambda B_\lambda]u(s)\,ds, \tag{3.36}$$

$$\begin{cases} y(t) = Y(t) - B_\lambda u(t), \\ Y(t) = S(t)Y(0) + \displaystyle\int_0^t S(t-s)B_\lambda \frac{dy}{ds}(s)\,ds \\ \qquad\qquad\qquad\qquad + \displaystyle\int_0^t S(t-s)[-\lambda B_\lambda]u(s)\,ds, \\ Y(0) = h + B_\lambda u(0), \quad u \in H^1(0,T;U). \end{cases} \tag{3.37}$$

This is to be compared with Theorem 2.3.

The next theorem relates expressions (3.26) and (3.36).

Theorem 3.2. *Assume that λ is such that A_λ is an isomorphism.*

(i) *The map*

$$u \mapsto \mathcal{S}Bu \colon L^2(0,T;U) \to C([0,T];H) \tag{3.38}$$

is continuous if and only if the map

$$u \mapsto \mathcal{S}B_\lambda u \colon L^2(0,T;U) \to C\big([0,T];D(A)\big) \tag{3.39}$$

is continuous.

(ii) *Hypothesis* 3.1 *is equivalent to the existence of the continuous linear extension*

$$h \mapsto B_\lambda^* S^*(\cdot)h \colon D(A)' \to L^2(0,T;U') \tag{3.40}$$

to the map

$$h \mapsto B_\lambda^* S^*(\cdot)h \colon H \to C([0,T];U'). \tag{3.41}$$

Proof. We use the fact that $\mathcal{S}Bu = A_\lambda(\mathcal{S}B_\lambda u)$. □

It is always possible to make a change of variable to make sense of a control operator $B \in \mathcal{L}(U, D(A^*)')$. For instance, by using the resolvent

$$R(\lambda, A) = [\lambda I - A]^{-1},$$

we have

$$R(\lambda, A)\mathcal{S}B = \mathcal{S}R(\lambda, A)B.$$

Define the new control operator

$$\tilde{B} = R(\lambda, A)B \in \mathcal{L}(U, H),$$
$$\tilde{y}(t) = R(\lambda, A)y(t) \implies \tilde{y}(t) = (\mathcal{S}\tilde{B}u)(t).$$

It is easy to check that Hypothesis 3.1

$$h \mapsto \tilde{B}^* S^*(\cdot)h \colon H \to L^2(0, T; U')$$

is equivalent to

$$h \mapsto B^* S^*(\cdot)h \colon D(A^*) \to L^2(0, T; U'),$$

which is always verified! The price to pay for that change of variable is that the new state variable belongs to $C([0, T]; D(A^*)')$ rather than $C([0, T]; H)$.

3.3 Unbounded observation operators

This section is "dual" to §3.2. We start with the *free system*

$$y(t) = S(t)h, \quad h \in H, \ t \geq 0, \tag{3.42}$$

$$\begin{cases} \dfrac{dy}{dt} = Ay, \\ y(0) = h. \end{cases} \tag{3.43}$$

The observation equation is

$$z(t) = Cy(t), \quad 0 \leq t \leq T. \tag{3.44}$$

From now on, everything we said for B^* and $\{S^*(t)\}$ in §3.2 is true for C and $\{S(t)\}$.

Hypothesis 3.2. Given $T > 0$, the linear map

$$h \mapsto CS(\cdot)h \colon D(A) \to C([0, T]; Z) \tag{3.45}$$

can be extended to a continuous linear map

$$h \mapsto CS(\cdot)h \colon H \to L^2(0, T; Z). \tag{3.46}$$

□

Define for S^* the analog of $\mathcal{S}$ in (3.9) for S:

$$\mathcal{S}^*\colon L^2(0,T;H) \to L^2(0,T;H), \tag{3.47}$$

$$(\mathcal{S}^*g)(t) = \int_0^t S^*(t-s)g(s)\,ds. \tag{3.48}$$

This operator is not to be confused with the topological dual $(\mathcal{S})^*$ of

$$\mathcal{S}\colon L^2(0,T;H) \to L^2(0,T;H). \tag{3.49}$$

They are however related in the following way:

$$[(\mathcal{S})^* g^T](t) = (\mathcal{S}^*g)(T-t), \quad 0 \le t \le T, \tag{3.50}$$

where

$$g^T(t) = g(T-t), \quad 0 \le t \le T. \tag{3.51}$$

Proposition 3.2. *The following statements are equivalent*:

(i) *Hypothesis* 3.2.

(ii) *The linear map*

$$z \mapsto (\mathcal{S}^*C^*z)(T)\colon L^2(0,T;Z') \to H \tag{3.52}$$

is continuous.

(iii) *The linear map*

$$z \mapsto \mathcal{S}^*C^*z\colon L^2(0,T;Z') \to C([0,T];H) \tag{3.53}$$

is continuous.

Theorem 3.3. *Fix $T > 0$ and assume that Hypothesis 3.2 is verified. Then for data*

$$w^0 \in H, \quad z \in L^2(0,T;Z'), \tag{3.54}$$

the function

$$w(t) = S^*(t)w^0 + (\mathcal{S}^*C^*z)(t) \tag{3.55}$$

is the unique solution in $\mathcal{V}(0,T;H,D(A)')$ to the differential equation

$$\begin{cases} \dfrac{dw}{dt} = A^*w + C^*z, \\ w(t) = w^0, \end{cases} \tag{3.56}$$

and there exists a constant $c > 0$ such that

$$\|w\|_{C([0,T];H)} + \left\|\frac{dw}{dt}\right\|_{L^2(0,T;D(A)')} \le c[|w^0|_H + \|z\|_{L^2(0,T;Z')}]. \tag{3.57}$$

Theorem 3.4. *Assume that $\lambda \in \mathbb{R}$ is such that A_λ is an isomorphism.*

(i) *The map*

$$z \mapsto \mathcal{S}^* C^* z \colon L^2(0,T;Z') \to C([0,T];H) \tag{3.58}$$

is continuous if and only if the map

$$z \mapsto \mathcal{S}^* C_\lambda^* z \colon L^2(0,T;Z') \to C\big([0,T];D(A^*)\big) \tag{3.59}$$

is continuous, where

$$C_\lambda = CA_\lambda^{-1} \in \mathcal{L}(H,Z). \tag{3.60}$$

(ii) *Hypothesis* 3.2 *is equivalent to the existence of the continuous linear extension*

$$h \mapsto C_\lambda S(\cdot)h \colon D(A^*)' \to L^2(0,T;Z) \tag{3.61}$$

to the map

$$h \mapsto C_\lambda S(\cdot)h \colon H \to C([0,T];Z). \tag{3.62}$$

3.4 Unbounded control and observation operators

When we consider the controlled system

$$\begin{cases} \dfrac{dy}{dt} = Ay + Bu, \\ y(0) = y^0, \end{cases} \tag{3.63}$$

with the observation

$$z(t) = Cy(t), \tag{3.64}$$

the natural hypothesis that comes is the following one.

Hypothesis 3.3. The linear map

$$C\mathcal{S}B \colon L^2(0,T;U) \to L^2(0,T;Z) \tag{3.65}$$

is continuous. □

By duality, this is equivalent to the continuity of the map

$$\begin{cases} B^*\mathcal{S}^*C^* \colon L^2(0,T;Z') \to L^2(0,T;U'), \\ (\mathcal{S}^* g)(t) = \displaystyle\int_t^T S(s-t)^* g(s)\, ds. \end{cases} \tag{3.66}$$

Unfortunately this last statement does not really provide additional information on the specific connections among C, A, and B to produce the map (3.65).

In order to gain some insight into this problem we turn to the method of change of variable introduced in §2.3 (Theorem 2.3) and use it twice to simultaneously handle the unboundedness of B and C (cf. also Theorem 1.1 and its corollary).

We proceed formally. The state y is given by the formula

$$y(t) = S(t)h + \int_0^t S(t-s)Bu(s)\,ds. \tag{3.67}$$

Let $\lambda \in \mathbb{R}$ be the number for which

$$A_\lambda = -[\lambda I - A] \in \mathcal{L}(D(A), H) \cap \mathcal{L}(H, D(A^*)')$$

is an isomorphism. Define for all $t \geq 0$

$$\begin{cases} y_\lambda(t) = e^{-\lambda t} y(t), \\ u_\lambda(t) = e^{-\lambda t} u(t), \\ S_\lambda(t) = S(t) e^{-\lambda t}. \end{cases} \tag{3.68}$$

It is easy to check that $\{S_\lambda(t)\}$ is also a semigroup and that

$$\frac{d}{dt} S_\lambda(t)h = A_\lambda S_\lambda(t)h, \quad \forall h \in D(A). \tag{3.69}$$

Equation (3.67) is equivalent to

$$y_\lambda(t) = S_\lambda(t)h + \int_0^t S_\lambda(t-s)Bu_\lambda(s)\,ds \tag{3.70}$$

and the observation

$$z_\lambda(t) = e^{-\lambda t} z(t) = C y_\lambda(t). \tag{3.71}$$

In order to remove the unboundedness of $B \in \mathcal{L}(U, D(A^*)')$ we proceed as in Theorem 1.1. Assume that

$$u_\lambda \in H^1(0,T;U) \tag{3.72}$$

and introduce the new operator B_λ and the new variable $Y_1(t)$

$$\begin{cases} B_\lambda = A_\lambda^{-1} B \in \mathcal{L}(U, H), \\ Y_1(t) = y_\lambda(t) + B_\lambda u_\lambda(t) \in L^2(0,T;H). \end{cases} \tag{3.73}$$

By performing integration by parts we get

$$\begin{cases} Y_1(t) = S_\lambda(t) Y_1(0) + \displaystyle\int_0^t S_\lambda(t-s) B_\lambda \frac{du_\lambda}{ds}(s)\,ds, \\ Y_1(0) = h + B_\lambda u_\lambda(0) \in H, \end{cases} \tag{3.74}$$

and we conclude that (cf. Theorem 1.1)

$$Y_1 \in C([0,T];H) \cap C^1([0,T];D(A^*)'). \tag{3.75}$$

To remove the unboundedness of $C \in \mathcal{L}(D(A),Z)$, we assume that

$$u_\lambda \in H^2(0,T;U), \quad h \in D(A) \quad \text{and} \quad CA_\lambda^{-1}B \in \mathcal{L}(U,Z) \tag{3.76}$$

and introduce the new operator C_λ and the new variable $Y_2(t)$

$$\begin{cases} C_\lambda = CA_\lambda^{-1} \in \mathcal{L}(H,Z), \\ Y_2(t) = A_\lambda Y_1(t) + B_\lambda \dfrac{du_\lambda}{dt}(t). \end{cases} \tag{3.77}$$

This situation is slightly different from the corollary to Theorem 1.1 because it cannot be a priori assumed that $Bu(\cdot) \in C([0,T];H)$. So we assume that $u_\lambda(0) = 0$ in (3.74). By integration by parts we get

$$\begin{cases} Y_2(t) = S_\lambda(t)Y_2(0) + \displaystyle\int_0^t S_\lambda(t-s)B_\lambda \frac{d^2u_\lambda}{ds^2}(s)\,ds, \\ Y_2(0) = A_\lambda h + B_\lambda \dfrac{du_\lambda}{dt}(0). \end{cases} \tag{3.78}$$

The observation equation is

$$z_\lambda(t) = Cy_\lambda(t) = C[Y_1(t) - B_\lambda u_\lambda(t)] = CA_\lambda^{-1}[A_\lambda Y_1(t) - A_\lambda B_\lambda u_\lambda(t)]$$

and

$$z_\lambda(t) = C_\lambda \left[Y_2(t) - B_\lambda \frac{du_\lambda}{dt}(t) \right] - C_\lambda A_\lambda B_\lambda u_\lambda(t).$$

Finally

$$z_\lambda(t) = C_\lambda Y_2(t) - C_\lambda B_\lambda u_\lambda'(t) - C_\lambda A_\lambda B_\lambda u_\lambda(t). \tag{3.79}$$

The second term belongs to $H^1(0,T;Z)$ and the last term to $H^2(0,T;Z)$ because we have assumed that

$$C_\lambda A_\lambda B_\lambda = CA_\lambda^{-1}B \in \mathcal{L}(U,Z). \tag{3.80}$$

As for the first term

$$C_\lambda Y_2(t) = CS_\lambda(t)h + C_\lambda S_\lambda(t)Bu_\lambda'(0) + \int_0^t C_\lambda S_\lambda(t-s)B_\lambda u_\lambda''(s)\,ds,$$

it is well defined for $h \in D(A)$ and all $u_\lambda \in H^2(0,T;U)$ such that $u_\lambda(0) = 0$. Finally $z_\lambda \in C([0,T];Z)$ for all $h \in D(A)$ and $u \in H^2(0,T;U)$ such that $u(0) = 0$. We summarize our conclusions in the next theorem.

Theorem 3.5. *Assume that $\lambda \in \mathbb{R}$ is such that A_λ be an isomorphism. If the* compatibility relation

$$CA_\lambda^{-1}B \in \mathcal{L}(U,Z) \tag{3.81}$$

is verified, then the linear map

$$u \mapsto z\colon \{u \in H^2(0,T;U)\colon u(0)=0\} \to L^2(0,T;Z) \tag{3.82}$$

is continuous, where

$$z(t) = e^{\lambda t} z_\lambda(t), \quad u_\lambda(t) = e^{-\lambda t} u(t), \tag{3.83}$$

$$\begin{aligned} z_\lambda(t) = -C_\lambda A_\lambda B_\lambda u_\lambda(t) + C_\lambda \int_0^t S_\lambda(t-s) B_\lambda \frac{d^2 u_\lambda}{ds^2}(s)\,ds \\ - C_\lambda B_\lambda u'_\lambda(t) + C_\lambda S_\lambda(t) B_\lambda u'_\lambda(0). \end{aligned} \tag{3.84}$$

Hypothesis 3.3 *is verified if and only if the map* (3.82) *can be continuously extended to all* $L^2(0,T;U)$*.*

Remark 3.1. Hypothesis (3.82)–(3.84) is verified under the compatibility relation (3.81). However, it is not clear whether Hypothesis (3.82)–(3.84) implies (3.81). □

When the control and observation operators are "more unbounded"

$$B \in \mathcal{L}(U, D(A^{*n})')C \in \mathcal{L}(D(A^m), Z), \quad CA_\lambda^{-\ell} B \in \mathcal{L}(U,Z), \qquad 1 \le \ell \le m+n, \tag{3.85}$$

the previous constructions can be repeated. For the control operator we introduce the new variable

$$Y_n(t) = y_\lambda(t) + \sum_{i=0}^{n+1} A_\lambda^{n-(i+1)} B_\lambda \frac{d^i u_\lambda}{dt^i}(t) \tag{3.86}$$

is continuous and the operator

$$B_\lambda = A_\lambda^{-n} B \in \mathcal{L}(U,H). \tag{3.87}$$

The new variable is given by the expression

$$Y_n(t) = S_\lambda(t) Y_n(0) + \int_0^t S_\lambda(t-s) B_\lambda \frac{d^n u_\lambda}{ds^n}(s)\,ds. \tag{3.88}$$

Similarly on the observation side we define the new state variable

$$Y_{n+m}(t) = A_\lambda^m Y_n(t) + \sum_{\ell=0}^{m-1} A_\lambda^{m-\ell-1} B_\lambda \frac{d^{n+\ell} u_\lambda}{dt^{n+\ell}}(t) \tag{3.89}$$

and the operator

$$C_\lambda = CA_\lambda^{-m} \in \mathcal{L}(H,Z). \tag{3.90}$$

The new state is given by

$$Y_{n+m}(t) = S_\lambda(t) Y_{n+m}(0) + \int_0^t S_\lambda(t-s) \frac{d^{n+m} u_\lambda}{ds^{n+m}}(s)\,ds \tag{3.91}$$

and the observation by

$$z_\lambda(t) = Cy_\lambda(t) = CA_\lambda^{-m} A_\lambda^m y_\lambda(t) = C_\lambda A_\lambda^m y_\lambda(t) \tag{3.92}$$

or

$$z_\lambda(t) = C_\lambda Y_{n+m}(t) - \sum_{\ell=0}^{m+n-1} C_\lambda A_\lambda^{m+n-1-\ell} B_\lambda \frac{d^\ell u_\lambda}{dt^\ell}(t). \tag{3.93}$$

Notice that in view of hypothesis (3.85) the terms in u_λ all make sense for $h \in D(A^m)$, $u \in H^{m+n}(0,T;U)$, and $u^{(\ell)}(0) = 0$, $0 \le \ell \le m+n-1$, and that a statement similar to the one of Theorem 3.5 can be made.

4 Time-invariant variational parabolic systems

In this section we specialize the results of §1 and §2 in Chapter 2 of Part II to the *time-invariant* case, compare them with the ones that can be obtained by semigroup methods, and relate them to the semigroup model for control of §3. The starting points are Theorem 1.1 in §1.3 and Theorem 1.4 in §1.5 (Chapter 2 of Part II). More precisely we start with a continuous linear operator

$$A \in \mathcal{L}(V, V') \tag{4.1}$$

for which the V–H coerciveness hypothesis

$$\exists \alpha > 0,\ \exists \lambda \in \mathbb{R}, \qquad \forall v \in V,\ \langle Av, v\rangle_V + \lambda |v|^2 \ge \alpha \|v\|^2 \tag{4.2}$$

is verified. Theorem 1.1 says that, for all $T > 0$, the variational differential equation

$$Ay + \frac{dy}{dt} = f \in L^2(0,T;V'), \quad y(0) = y^0 \in H \tag{4.3}$$

has a unique solution y in $W(0,T;V,V') \subset C([0,T];H)$. For $f = 0$ the solutions of the above equation generate a strongly continuous semigroup $\{S(t)\}$ on the Hilbert space H

$$S(t)y^0 = y(t), \quad t \ge 0, \qquad \forall y^0 \in H. \tag{4.4}$$

Its infinitesimal generator coincides with $-A$, the domain of which will be written $D(A)$. *We emphasize the fact that in other chapters on semigroup the infinitesimal generator is usually written A. The notation $-A$ is exceptionally used to be consistent throughout this section.*

In §1 we have seen that the function

$$y(t) = S(t)y^0 + \int_0^t S(t-s)f(s)\,ds \quad \text{in } D(-A^*)' \tag{4.5}$$

is also the solution of system (4.3) in $\mathcal{V}(0,T;D(-A^*)', D(-A^{*2})')$, which is a larger space. Theorem 1.1 contains the following regularity property: For all $T > 0$, the linear map

$$\mathcal{S}\colon L^2(0,T;V') \to W(0,T;V,V'), \quad (\mathcal{S}f)(t) = \int_0^t S(t-s)f(s)\,ds \tag{4.6}$$

is continuous.

From semigroup theory in Chapter 1 of Part II (cf. §2, Theorem 2.12), the semigroup $\{S(t)\}$ generated by $-A$ on the Hilbert space H is *analytic.* Moreover for the λ of the coercivity assumption (4.2) the operator

$$-A_\lambda = -A - \lambda I \tag{4.7}$$

is the infinitesimal generator of the strongly continuous analytic contraction semigroup

$$S_\lambda(t) = e^{-\lambda t}S(t), \quad t \geq 0 \tag{4.8}$$

on H. As a consequence the linear map

$$\begin{gathered}\mathcal{S}_\lambda\colon L^2(0,T;H) \to W(0,T;D(-A),H) \subset C([0,T];[D(-A),H]_{1/2}),\\ (\mathcal{S}_\lambda f)(t) = \int_0^t S_\lambda(t-s)f(s)\,ds\end{gathered} \tag{4.9}$$

is continuous and there exists a constant $M > 0$ independent of $T > 0$ such that

$$\|-A_\lambda \mathcal{S}_\lambda f\|_{L^2(0,T;H)} \leq M\|f\|_{L^2(0,T;H)}. \tag{4.10}$$

But under assumption (4.2), $-A_\lambda$ is invertible and

$$\|\mathcal{S}_\lambda f\|_{W(0,T;D(-A),H)} \leq M'\|f\|_{L^2(0,T;H)} \tag{4.11}$$

for some constant $M' > 0$ independent of T and f. From the above property and the identity

$$(\mathcal{S}f)(t) = e^{\lambda t}[\mathcal{S}_\lambda(e^{-\lambda\bullet}f)](t), \tag{4.12}$$

we can obtain for $\mathcal{S}f$ an inequality similar to (4.10) but with a constant $M(T,\lambda) > 0$, that now depends on T and λ (monotonically increasing with $\lambda \geq 0$):

$$\|\mathcal{S}f\|_{W(0,T;D(-A),H)} \leq M(T,\lambda)\|f\|_{L^2(0,T;H)}. \tag{4.13}$$

This last identity shows that we have the regularity result

$$\mathcal{S} : L^2(0,T;H) \to W(0,T;D(-A),H) \tag{4.14}$$

for the map (4.6). Now $-A_\lambda = -A - \lambda I$ is an isomorphism from $D(-A)$ onto H and from H onto $D(-A^*)'$. It follows from the identity

$$A_\lambda^{-1}\mathcal{S}f = \mathcal{S}A_\lambda^{-1}f \tag{4.15}$$

that

$$\mathcal{S}\colon L^2(0,T;D(-A^*)') \to W(0,T;H,D(-A^*)') \tag{4.16}$$

is also an isomorphism which can be regarded as an extension of (4.14). In particular for α, $0 \le \alpha \le 1$,

$$\begin{aligned}\mathcal{S}\colon L^2(0,T;[H,D(-A^*)']_\alpha)& \\ &\to W(0,T;[D(-A),H]_\alpha,[H,D(-A^*)']_\alpha)\end{aligned} \tag{4.17}$$

is an isomorphism that can also be expressed in terms of the domains of the fractional powers A_λ^α of A_λ (cf. Chapter 1, §6, Proposition 6.1)

$$D(A_\lambda^\alpha) = [D(-A),H]_{1-\alpha} = [D(A),H]_{1-\alpha}. \tag{4.18}$$

Finally

$$\mathcal{S} : L^2(0,T;D(A_\lambda^{*\alpha})') \to W(0,T;D(A_\lambda^{1-\alpha}),D(A_\lambda^{*\alpha})') \tag{4.19}$$

and in particular for $\alpha = \frac{1}{2}$

$$\mathcal{S}\colon L^2(0,T;D(A_\lambda^{*1/2})') \to W(0,T;D(A_\lambda^{1/2}),D(A_\lambda^{*1/2})'). \tag{4.20}$$

Comparing the above map to (4.6), we cannot conclude that they coincide. However if we assume that

$$D(A_\lambda^{1/2}) = [D(A),H]_{1/2} = [D(A^*),H]_{1/2} = D(A_\lambda^{*1/2}), \tag{4.21}$$

then

$$[D(A),H]_{1/2} = V = [D(A^*),H]_{1/2} \tag{4.22}$$

and we recover (4.6) (cf. Chapter 2, Remark 1.2 and Theorem 1.3). It says that in the variational case (4.6) is generally true without assumption (4.22).

For the control through a Neumann boundary condition, we have seen in §1.2 (Chapter 2, Part II) that the control operator B is of the form

$$\langle Bu,v\rangle_V = \int_\Gamma uv\big|_\Gamma \, d\Gamma, \quad u \in L^2(\Gamma),\ v \in V, \tag{4.23}$$

where

$$\begin{aligned} H = L^2(\Omega), \quad V = H^1(\Omega), \quad D(A^*) &= \left\{ v \in H^2(\Omega)\colon \frac{\partial v}{\partial \nu_A}\bigg|_\Gamma = 0 \right\}, \\ D(A) &= \left\{ v \in H^2(\Omega)\colon \frac{\partial v}{\partial \nu_A^*}\bigg|_\Gamma = 0 \right\}. \end{aligned} \tag{4.24}$$

Here the hypotheses of Theorem 1.3 (Chapter 2 in Part II) are verified with $X = H^2(\Omega)$ and

$$D(A^{1/2}) = D(A^{*1/2}) = V. \tag{4.25}$$

Notice also that not only $B \in \mathcal{L}(U,V')$ but also

$$B \in \mathcal{L}(U,H^{1/2+2\varepsilon}(\Omega)'), \quad \varepsilon > 0 \tag{4.26}$$

because the trace operator

$$v \mapsto v|_{\Gamma} \colon H^{1/2+2\varepsilon}(\Omega) \to U = L^2(\Gamma) \tag{4.27}$$

is continuous. Moreover

$$H^{1/2+2\varepsilon}(\Omega) = [D(A^*), H]_{3/4-\varepsilon} = D(A^{*(1/4+\varepsilon)}) \tag{4.28}$$

and

$$B \in \mathcal{L}(U, D(A^{*(1/4+\varepsilon)})'). \tag{4.29}$$

For the control through a Dirichlet boundary condition, we have shown in §2.1 (Chapter 2 in Part II) that the control operator B is

$$\langle Bu, v\rangle_{D(A^*)} = \int_{\Gamma} u \frac{\partial v}{\partial \nu_{A^*}}\bigg|_{\Gamma} d\Gamma, \quad u \in L^2(\Gamma),\ v \in D(A^*), \tag{4.30}$$

where

$$H = L^2(\Omega), \quad V = H_0^1(\Omega), \quad D(-A) = D(-A^*) = H^2(\Omega) \cap H_0^1(\Omega), \tag{4.31}$$

which necessarily implies (4.23). Here the trace operator

$$v \mapsto \frac{\partial v}{\partial v_{A^*}}\bigg|_{\Gamma} \colon H^{3/2+2\varepsilon}(\Omega) \cap H_0^1(\Omega) \to L^2(\Gamma), \quad \varepsilon > 0 \tag{4.32}$$

is continuous,

$$H^{3/2+2\varepsilon}(\Omega) \cap H_0^1(\Omega) = [D(A^*), H]_{1/4-\varepsilon} = D(A^{*(3/4+\varepsilon)}), \tag{4.33}$$

and

$$B \in \mathcal{L}(U, D(A^{*(3/4+\varepsilon)})'), \quad U = L^2(\Gamma). \tag{4.34}$$

In both cases hypothesis (4.22) is verified and

$$\mathcal{S}B \colon L^2(0,T;U) \to W(0,T;H,D(A^*)') \subset L^2(0,T;H). \tag{4.35}$$

Sharper results can also be obtained in view of (4.27) and (4.32): For Neumann

$$\mathcal{S}B \colon L^2(0,T;U) \to W(0,T;D(A^{3/4-\varepsilon}),D(A^{*(1/4+\varepsilon)})'), \tag{4.36}$$

and for Dirichlet

$$\mathcal{S}B \colon L^2(0,T;U) \to W(0,T;D(A^{1/4-\varepsilon}),D(A^{*(3/4+\varepsilon)})'). \tag{4.37}$$

In §2 we have shown that under Hypothesis 3.1 the map

$$\mathcal{S}B \colon L^2(0,T;U) \to C([0,T];H) \tag{4.38}$$

is continuous. Hypothesis 3.1 amounts to prove that the map

$$h \mapsto B^* S^*(\cdot)h\colon H \to L^2(0,T;U') \tag{4.39}$$

or, equivalently, that the map

$$h \mapsto B^* v\colon H \to L^2\big(0,T;L^2(\Gamma)\big) \equiv L^2(\Sigma) \tag{4.40}$$

is continuous for v the solution of the adjoint system

$$A^* v + \frac{\partial v}{\partial t} = 0 \quad \text{on }]0,T[, \quad v(T) = h. \tag{4.41}$$

For the Neumann case

$$\begin{aligned} A^* v + \frac{\partial v}{\partial t} &= 0 \quad \text{in } Q, \\ \frac{\partial v}{\partial \nu_{A^*}} &= 0 \quad \text{on } \Sigma, \\ v(T) &= h \quad \text{in } \Omega, \end{aligned} \tag{4.42}$$

and

$$B^* v = v_{|\Sigma}. \tag{4.43}$$

Here for h in H, $v \in W(0,T;V,V') \subset L^2(0,T;V)$ and

$$v \mapsto v|_{\Sigma}\colon L^2\big(0,T;H^1(\Omega)\big) \to L^2\big(0,T;L^2(\Gamma)\big) = L^2(\Sigma) \tag{4.44}$$

is linear and continuous. Therefore, Hypothesis 3.1 is verified.

For the Dirichlet case

$$\begin{aligned} A^* v + \frac{\partial v}{\partial t} &= 0 \quad \text{in } Q, \\ v &= 0 \quad \text{on} \quad \Sigma, \\ v(T) &= h \quad \text{in} \quad \Omega, \end{aligned} \tag{4.45}$$

and

$$B^* v = \frac{\partial v}{\partial \nu_{A^*}}\bigg|_{\Sigma}. \tag{4.46}$$

For h in V, the map

$$v \mapsto \frac{\partial v}{\partial \nu_{A^*}}\bigg|_{\Sigma} : W(0,T;D(A^*),H) \to H^{1/2,1/4}(\Sigma) \tag{4.47}$$

is continuous, but for all h in H,

$$v \in H^{1/2,1/4}(\Sigma) \quad \text{and} \quad \frac{\partial v}{\partial \nu_{A^*}}\bigg|_{\Sigma} \notin L^2(\Sigma). \tag{4.48}$$

From this Hypothesis 3.1 fails and we would conclude that the *elegant framework of this chapter is not adequate for parabolic systems with control* *through a Dirichlet boundary condition.* However if we first perform the change of variable of §2.5 (Chapter 2 in Part II)

$$\tilde{y}(t) = A^{-1/2} y(t), \quad \tilde{v}(t) = A^{1/2} v(t), \tag{4.49}$$

then

$$\begin{gathered} \frac{d\tilde{y}}{dt} = -A\tilde{y} + \tilde{B}u, \quad \tilde{B} = A^{-1/2} B \in \mathcal{L}(U, V'), \\ \tilde{y}(0) = A^{-1/2} y(0) \in H, \end{gathered} \tag{4.50}$$

and

$$\begin{gathered} \frac{d\tilde{v}}{dt} = -A\tilde{v}, \\ \tilde{v}(T) = A^{1/2} v(T) \in H \implies v(T) = h \in V. \end{gathered} \tag{4.51}$$

Thus in view of (4.45), *Hypothesis* (3.1) *is verified and the framework of this chapter is now appropriate for a parabolic system with control through a Dirichlet boundary condition.* In addition,

$$\tilde{y} \in L^2(0, T; V) \implies A^{-1/2} y \in L^2(0, T; V) \implies y \in L^2(0, T; H), \tag{4.52}$$

but we only have

$$y \in C([0, T]; V'). \tag{4.53}$$

In fact we do not need the full force of $A^{-1/2}$ and $A^{-1/4+\varepsilon}$ is sufficient because $B \in \mathcal{L}(U, D(A^{*(1/4+\varepsilon)})')$ and

$$\tilde{B} = A^{-1/4+\varepsilon} B \in \mathcal{L}(U, D(A^{*1/2})') = \mathcal{L}(U, V')$$

because

$$A^{*(1/4-\varepsilon)} \in \mathcal{L}\big(D(A^{*1/2}), D(A^{*(1/4+\varepsilon)})\big).$$

4

State Space Theory of Differential Systems With Delays

1 Introduction

In this chapter our objective is to present a modern approach that provides a unifying framework for a large family of differential and integro-differential systems with delays. This point of view is of paramount importance for the Control Theory, Filtering Theory, and Realization Theory of such systems. The material presented in this chapter is an outgrowth of the lecture notes in French presented at the INRIA School on Representation and Control of Delay Systems in June 1984 by M. C. Delfour [14]. The original material has been restructured, the proofs have been streamlined, and new results have been introduced in §6.

According to A.D. Myshkis [1], the origin of differential equations with delays or "differential equations with a retarded argument" goes back to Condorcet in a "Mémoire de l'Académie des Sciences" dated 1771 about a problem studied by Euler in 1740. So it is a topic with a relatively long history.

To appreciate the specificity of a delay differential equation, recall that ordinary differential equations are essentially local (in time) relations of the form

$$F\big(t, x(t), x^{(1)}(t), x^{(2)}(t), \dots, x^{(n)}(t)\big) = 0 \tag{1.1}$$

between the function x and its derivatives $x^{(i)}$, $1 \leq i \leq n$. Equation (1.1) relates x and its derivatives at time t. However other types of relations F can be constructed that relate x and its derivatives at different times. For instance

$$x^{(1)}(t) - x(t-1) = 0 \quad \text{or } x^{(1)}(t) - x(\tfrac{t}{2}) = 0. \tag{1.2}$$

Such relations belong to the general class of "Functional Differential Equations or Differential Equations with a Deviating Argument (in papers from U.R.S.S.)." They essentially define a *functional relation* G between the function x and its derivative,

$$G(x, x^{(1)}, \dots, x^{(n)}) = 0, \tag{1.3}$$

as opposed to the local relation F in (1.1) at time t. At that level of generality everything is possible and the variable x need not be interpreted as a function of time. However it is customary to call the variable t the *time* and we shall follow this convention.

Functional Differential Equations (FDEs) have often been classified into three categories: *retarded type* as in (1.2) *advanced type*

$$x^{(1)}(t) = x(t+2) \tag{1.4}$$

or *neutral type*

$$x^{(1)}(t) = x^{(1)}(t-1) + x(t-1) + x(t). \tag{1.5}$$

In this chapter we shall take the down-to-earth point of view that a real system is nonanticipatory and limits our analysis to systems where the solution $x(t')$, $t' > t$, only depends on the past values $x(s)$, $s \leq t$, of the variable x. This definition does not only cover differential equations with delays but also important classes of integral or difference equations as we shall see in §1.2. This broad family of systems will be referred to as *Delay Systems* or sometimes as *Hereditary Systems*.

Such systems enter into the modelization of many problems: technological systems (e.g., chemical processes, rolling mills), the classical two-body problem in electrodynamics, the dynamics of nuclear reactors, the transmission line model, population models, biomedical systems (e.g., control of the human respiratory system), the propagation of diseases in a population, the control of epidemics, economic systems, and even the famous macro-economic models "à la " Forrester" (cf. M. C. Delfour and A. Manitius [1] for details and specific references).

Detailed bibliographies on FDEs have been initiated by A.D. Myshkis [2] in 1949 and periodically updated by N. H. Choksy [1]. The first books written in English were probably those of E. Pinney [1] in 1958, R. Bellman and K. L. Cooke [2] in 1959 and 1963, J. A. Nohel [1] in 1964, A. Halanay [1] in 1966, and the translations in 1966 of the 1964 book of Elsgolc and the one of A.D. Myshkis [2] in 1951. This topic has been widely studied by numerous mathematicians and engineers everywhere in the world. We recommend the book of J. K. Hale [3] and its bibliography for a detailed treatment of delay systems up to 1977. Several references to the engineering literature have been included in our bibliography because problems in Control Theory have often been the motivation behind the various state space formulations we shall encounter in this chapter.

This chapter will emphasize the *product space approach*, which seems to have been independently introduced in the late sixties–early seventies by M. Artola [1, 2, 3] for parabolic partial differential equations with delays, M. C. Delfour and S. K. Mitter [2, 3, 4, 5, 9] for nonlinear time-varying delay differential equations, and by Ju. G. Borisovic and A. S. Turbabin [1] for nonhomogeneous linear time-invariant delay differential equations. At that time our objective was to bring the theory of systems with delays more in line with the modern theory of partial differential equations in Hilbert spaces and build up a unified Control Theory of Infinite Dimensional Systems. In that regard delays systems were very appealing because they were halfway between ordinary differential equations and partial differential equations.

2 Examples and orientation

2.1 Examples

In this section we provide a series of simple examples to introduce and motivate the model and the constructions, which will be used in this chapter.

Example 2.1. Delay differential equations.

$$\dot{y}(t) = \sum_{i=0}^{N} y(t-i), \quad t > 0,\ y(\theta) = \phi(\theta), \qquad -N \leq \theta \leq 0, \tag{2.1}$$

where $N \geq 0$ is an integer. □

Example 2.2. Volterra integro-differential equations.

$$\dot{y}(t) = \int_0^t A(r-t)y(r)\,dr, \quad t > 0,\ y(0) = \phi^0, \tag{2.2}$$

where A is an L^1-function. □

Example 2.3. Integro-differential equations.

$$\begin{cases} \dot{y}(t) &= \displaystyle\int_{-h}^{0} A_1(\theta)y(t+\theta)\,d\theta + \int_{-h}^{0} A_2(\theta)\dot{y}(t+\theta)\,d\theta, \quad t > 0, \\ y(\theta) &= \phi(\theta), \quad -h \leq \theta \leq 0, \end{cases} \tag{2.3}$$

where A_1 and A_2 are square integrable functions. □

Example 2.4. Functional differential equations of neutral type.

$$\dot{x}(t) = \dot{x}(t-1), \quad t > 0,\ x(\theta) = \phi(\theta), \qquad -1 \leq \theta \leq 0. \tag{2.4}$$

□

Example 2.5. Volterra integral equations.

$$x(t) = \int_0^t A(s-t)x(s)\,ds + f(t), \quad t \geq 0, \tag{2.5}$$

where A and f are locally integrable. □

Example 2.6. Difference equations.

$$\begin{cases} x(t) = \sum_{i=1}^{N} A_i x(t-i), & t > 0, \\ x(\theta) = \phi(\theta), & -N \le \theta \le 0, \end{cases} \tag{2.6}$$

where $N \ge 1$ is an integer. □

Example 2.7. Delay-differential equations with delays in the trajectory, control and observation variables.

$$\begin{cases} \dot{y}(t) = \sum_{i=0}^{N} [y(t-i) + u(t-i)], \\ x(t) = C_0 y(t) + C_1 y(t-N) + K_0 u(t) + K_1 u(t-N), \end{cases} \tag{2.7}$$

where $N \ge 0$ is an integer. □

The first observation is that in Examples 2.1, 2.9, and 2.10 (resp. 2.11, 2.12 and 2.13), $y(t)$ (resp. $x(t)$) is not a good candidate for the state at time $t \ge 0$. The intuitively natural one is

$$\begin{gathered} y_t \quad (\text{resp. } x_t)\colon I(-h,0) \to \mathbb{R}, \\ y_t(\theta) = y(t+\theta) \quad \left(\text{resp. } x_t\big(\theta\big) = x(t+\theta)\right), \end{gathered} \tag{2.8}$$

where $h, 0 < h \le +\infty$ is the *length of the memory* of the system and

$$I(-h,0) = \mathbb{R} \cap [-h,0]. \tag{2.9}$$

So in Examples 2.8 and 2.9, the differential equation is of the form

$$\dot{y}(t) = H y_t, \quad t \ge 0, \tag{2.10}$$

for an appropriate real-valued map H defined on the space of real-valued continuous functions on $I(-h,0)$ or, as we shall see later, on some subspace in the case of infinite memory ($h = +\infty$).

Example 2.8.

$$h = N, \quad H\psi = \sum_{i=1}^{N} \psi(-i), \quad \begin{cases} \dot{y}(t) = H y_t, & t > 0, \\ y_0 = \phi. \end{cases} \tag{2.11}$$

□

Example 2.9.

$$h = +\infty, \quad H\psi = \int_{-\infty}^{0} A(\theta)\psi(\theta)\, d\theta, \quad \begin{cases} \dot{y}(t) = H y_t, & t > 0, \\ y(0) = \phi^0, & y_0 = 0. \end{cases}$$ □

Proceeding in this way with Example 2.10, it is readily seen that the map H must be defined on the Sobolev space $W^{1,p}(-h,0;\mathbb{R})$.

Example 2.10.

$$0 < h < +\infty, \quad H\psi = \int_{-h}^{0} [A_1(\theta)\psi(\theta) + A_2(\theta)\dot{\psi}(\theta)]\, d\theta. \qquad \square$$

A complete theory is available in the product space

$$M^p = \mathbb{R} \times L^p(-h,0;\mathbb{R}) \tag{2.12}$$

for systems of the form

$$\dot{y}(t) = Hy_t + f(t), \quad t > 0, \qquad y(0) = \phi^0, \; y_0 = \phi^1, \tag{2.13}$$

where $f \in L^p_{\mathrm{loc}}(0,\infty;\mathbb{R})$, $(\phi^0,\phi^1) \in M^p$, and H is a continuous linear map

$$H\colon W^{1,p}(-h,0;\mathbb{R}) \to \mathbb{R}, \quad 1 \le p < \infty, \tag{2.14}$$

for finite or infinite memory, $0 < h \le +\infty$ (cf. M. C. Delfour [8]). Moreover this class of maps cannot be enlarged in this framework.

The construction used in Examples 2.8 to 2.10 does not apply to Example 2.11 because the map

$$H\psi = \dot{\psi}(-1)$$

would have to be defined on the smaller space $C^1(-1,0;\mathbb{R})$[1] of continuously differentiable functions. To get around this difficulty, we group terms involving a derivative of x:

$$\frac{d}{dt}[x(t) - x(t-1)] = 0$$

and note that the above expression is of the form

$$\frac{d}{dt} Mx_t = 0, \quad M\psi = \psi(0) - \psi(-1).$$

This last construction suggests the more general model

$$\frac{d}{dt}[Mx_t] = Lx_t + f(t), \quad t > 0, \; x_0 = \phi. \tag{2.15}$$

Equivalently by introducing a new variable $y(t)$

$$\begin{cases} Mx_t - y(t) = g(t), \quad t > 0, \\ x_0 = \phi^2, \\ \dot{y}(t) - Hy_t - Lx_t = f(t), \quad t > 0, \\ y(0) = \phi^0, \quad y_0 = \phi^1, \end{cases} \tag{2.16}$$

we obtain a model that can handle Examples 2.11 to 2.13.

[1] Recall the notation $C(-1,0;\mathbb{R})$ for $C([-1,0];\mathbb{R})$.

Example 2.11. $h = 1$, $M\psi = \psi(0) - \psi(-1)$, $L = 0$, and $H = 0$,

$$\begin{cases} Mx_t - y(t) = 0, \quad t > 0, \\ x_0 = \phi, \\ \dot{y}(t) = 0, \quad t > 0, \\ y(0) = M\phi. \end{cases} \tag{2.17}$$

□

Example 2.12. $h = +\infty$, $L = 0$, and $H = 0$,

$$M\psi = \psi(0) - \int_{-\infty}^{0} A(\theta)\psi(\theta)\, d\theta, \tag{2.18}$$

$$\begin{cases} Mx_t - y(t) = f(t), \quad t > 0, \\ x_0 = 0, \\ \dot{y}(t) = 0, \quad t > 0, \\ y(0) = 0. \end{cases} \tag{2.19}$$

□

Example 2.13. $h = N$, $L = 0$, and $H = 0$,

$$M\psi = \psi(0) - \sum_{i=1}^{N} A_i \psi(-i),$$

$$\begin{cases} Mx_t - y(t) = 0, \quad t > 0, \\ x_0 = \phi, \\ \dot{y}(t) = 0, \quad t > 0, \\ y(0) = 0. \end{cases} \tag{2.20}$$

□

The last example requires an additional operator N in model (2.16):

$$\begin{cases} Mx_t - Ny_t = B_1 u_t + g(t), \quad t > 0, \\ \dot{y}(t) - Hy_t - Lx_t = B_0 u_t + f(t), \quad t > 0, \\ x_0 = \phi^2, \quad y_0 = \phi^1, \quad u_0 = w. \end{cases} \tag{2.21}$$

Example 2.14. 2.7 $h = N$, $L = 0$,

$$\begin{cases} M\psi = \psi(0), \quad N\psi = C_0\psi(0) + C_1\psi(-N), \\ H\psi = \displaystyle\sum_{i=0}^{N} \psi(-i), \quad B_0\psi = \sum_{i=0}^{N} \psi(-i), \\ B_1\psi = K_0\psi(0) + K_1\psi(-N). \end{cases} \tag{2.22}$$

The interesting feature of the application of model (2.21) to Example 2.7 is that the observation x becomes one of the unknown variables similar to y:

$$\begin{cases} \dot{y}(t) - Hy_t = B_0 u_t, \quad t > 0, \\ y(0) = \phi^0, \quad y_0 = \phi^1, \quad u_0 = w, \\ x(t) - Ny_t = B_1 u_t, \quad t > 0. \end{cases} \tag{2.23}$$

□

So the final mathematical model we shall adopt for our analysis will be

$$\begin{cases} Mx_t - Ny_t = B_1 u_t + f(t), \quad t > 0, \\ \dot{y}(t) - Hy_t - Lx_t = B_0 u_t + g(t), \quad t > 0, \\ (y(0), y_0, x_0) = (\phi^0, \phi^1, \phi^2), \quad u_0 = w. \end{cases} \tag{2.24}$$

This model has been studied in detail by M. C. Delfour and J. Karrakchou [1, 2], with appropriate assumptions on the various maps.

2.2 Orientation

Model (2.24) is generic of a very large class of delay systems with finite or infinite memories. Our objective is to discuss the choice of state and state space for linear control systems with observation. We first consider Lipschitzian delay-differential equations of the form

$$\begin{cases} \dot{x}(t) = f(t, x_t), \quad t > 0, \\ x_0 = \phi \end{cases} \tag{2.25}$$

and then specialize to time-invariant linear control systems of the form

$$\begin{cases} \dot{x}(t) = Lx_t + B_0 u_t + f(t), \quad t > 0, \\ x_0 = \phi \end{cases} \tag{2.26}$$

with observation equation

$$y(t) = Cx_t + B_0 u_t. \tag{2.27}$$

The complete theory for the general model (2.24) can be found in M. C. Delfour and J. Karrakchou [1, 2]. It was felt that the specialization to system (2.26)–(2.27) was sufficient as a first introduction and will better illustrate the ideas and constructions behind our approach using product spaces and structural operators.

In §3 we discuss the existence and uniqueness of the solution for Lipschitzian systems of the form (2.25). We compare the classical assumptions for initial conditions in the space of continuous functions with those required for initial conditions, which are only L^p-functions. In this last case we also need an initial point to define the starting point of the trajectory. This is the so-called *product space* approach because the initial datum is a pair $(\phi^0, \phi^1) \in \mathbb{R}^n \times L^p(-h, 0; \mathbb{R}^n)$. A key result is that all time-invariant linear systems can be studied in the product space framework under the same assumptions as in the continuous functions framework. This last result is of paramount importance for control problems because the state space can be chosen as a Hilbert or a reflexive Banach space bringing delay systems in line with partial differential equations in Hilbertian or reflexive Sobolev spaces.

In §4 we construct a first state and a state equation, study transposed and adjoint systems, introduce structural operators, and intertwining operators, and characterize adjoint semigroups and their infinitesimal generators. This is the extension of the traditional state in the continuous function framework. In particular the reader will find the *implicit characterization* of the adjoint of the infinitesimal generator associated with the state for systems with both finite or infinite memory. This question was initially raised by R. B. VINTER [2], but its importance has not always been fully appreciated. We shall see how fundamental it is in the following sections. This section also contains many other key technical results, which will be extensively used in subsequent sections

Section 5 introduces the control variable and the two fundamental choices of definition of the state. Section 6 incorporates a delayed observation equation and extends the two definitions of state to finally obtain an evolution equation and an observation equation in a product space $\mathbb{R}^n \times L^p \times L^p$ without delays but with unbounded control and observation operators.

This chapter will emphasize the importance of the construction of a *state*, which is fundamental in Control Theory. For delay systems this key development seems to have come from N. N. KRASOVSKII [1, 2] in 1959. Yet the use of the nonreflexive Banach space of continuous functions at the level of the classical theory of semigroups was not completely satisfactory for a purely technical reason. Around 1969 it was then understood that for a very large class of delay systems the state space could be enlarged from continuous functions to an initial point and an initial L^p-function. This led to the *product space approach*, which was independently introduced in the late sixties-early seventies by M. ARTOLA [1, 2, 3] for parabolic partial differential equations with delays, M. C. DELFOUR [1, 3, 5, 8] and M. C. DELFOUR and S. K. MITTER [2, 3, 4, 5, 9] for nonlinear time-varying delay differential equations, and JU. G. BORISOVIC and A. S. TURBABIN [1] for nonhomogeneous linear time-invariant delay differential equations.

At that time our objective was to bring the theory of systems with delays in line with the modern theory of partial differential equations in Hilbert spaces and build up a unified Control Theory of Infinite Dimensional Systems. In that regard delays systems were very appealing as they were halfway between ordinary differential equations and partial differential equations. This technical contribution made it possible to give a complete mathematical solution to the linear quadratic optimal control problem over finite and infinite time horizons (cf. M. C. Delfour and S. K. Mitter [3, 6, 8]; M. C. Delfour, C. McCalla, and S. K. Mitter [1]; M. C. Delfour [3, 6, 9, 10, 12, 13, 15]; M. C. Delfour, E. B. Lee, and A. Manitius [1]). In the context of numerical solutions the product space also turned out to be the appropriate framework because it was disconnecting the initial point and the initial function (cf. M. C. Delfour [6]). For subsequent work by many other authors (Banks, Burns, Gibson, Herdman, Ito, Kappel, Kunisch, Rosen, Schappacher, Tran, and several others), the reader is referred for instance to the papers of D. Salamon 4], and I. Lasiecka and A. Manitius [1] and their bibliographies.

Another important contribution was the construction of the *structural operators*, which first clarified the relationship between the *true adjoint* and the *transposed system* and provided the natural concepts of observability and controllability. They were developed in Montréal in the 1975–78 period and first announced in December 1976 at the CDC by A. Manitius [5] and at the INRIA by M. C. Delfour and A. Manitius [1]. For control systems without delays in the control variable, the transformation of the state by the structural operator was introducing a new state, the *structural state*, which was more natural and better adjusted to the specific delay structure of the system. A complete treatment of structural operators along with their implications in spectral theory was given by M. C. Delfour and A. Manitius [2, 3] for arbitrary delay functionals L on the space of continuous functions (see also C. Bernier and A. Manitius [1], and A. Manitius [5, 6, 7, 8]) immediately saw and exploited their potential in the study of the notions controllability and observability. An extention of structural operators to time-varying systems was later done by F. Colonius, A. Manitius, and D. Salamon [1].

But it turned out that structural operators and the associated construction of the structural state also play a fundamental role in the transformation of systems with delays in the control variable into an evolution equation without delays. The appropriate extention of the definition of *structural state*[2] in $\mathbb{R}^n \times L^p$ was introduced by R. B. Vinter and R. H. Kwong [1] and generalized by M. C. Delfour [15] to general delayed control operators. We still call it structural state because it roughly corresponds to the transformation of the usual state and the segment of the control function by the structural operators associated with the delay structure of the system and the control. Another state in $\mathbb{R}^n \times L^p \times L^p$, the *extended state*, was introduced by A. Ichikawa [1, 2, 3]. This state follows the evolution of the pieces of trajectory x_t and control u_t. He added u_t to the original choice $(x(t), x_t)$ for the state in the product

[2] In the recent literature the terminology "forcing function state" has been used (cf. for instance the book by G. Grippenberg, S. O. Londen, and O. Staffans [1]). This emphasizes one of the many properties of the underlying semigroup. It is obviously a matter of taste. For instance we shall see in the proof of Theorem 5.1 that this state can also be seen as the resulting product of transposition techniques. In fact the underlying semigroup is nothing but the adjoint of the semigroup associated with the state of the transposed system.

space. This theory will be completed and extended in §5.2. To our knowledge those two definitions of the state seem to be the only interesting and natural ones for linear control systems with delays. In retrospective—and it is now more striking because we have a unified framework for the time-invariant system (see §2.1)—it is extremely interesting to see that these two notions of state had been discovered by D. H. MILLER [1] around 1974 in the context of Volterra equations.

All this was fine but not quite complete. From the Control Theory point of view it was necessary to be able to deal with a delayed observation equation. This is relatively simple to do with the extended state because we have direct access to x_t and u_t. However this was far from being obvious with the structural state. The appropriate construction came from D. SALAMON [2] who added a third component to the structural state. This *extended structural state* does not require a full knowledge of x_t and u_t but only of some linear combinations through appropriate structural operators associated with the control and observation delay structure. This construction was extended and generalized to system (2.24) by J. KARRAKCHOU [1] and M. C. DELFOUR and J. KARRAKCHOU [1, 2]. Similar developments using a slightly different approach were also done by D. SALAMON [1, 3] for neutral systems.

We would like to emphasize that our objective in this chapter is to give a systematic introductory but sufficiently complete treatment of the above-mentioned material, which is not available in standard textbooks. We feel that it is now a fundamental part of the theory of delay systems, which brings them in line with the general theory of control systems in infinite dimension.

The optimal control problem will be covered in the second part of this book. However it will appear as a special case of the general theory. Nevertheless many references have been included at the end of this chapter. For early papers using various forms of the Maximum Principle, Dynamic Programming, and abstract variational theories, the reader is referred to the first books of G. L. KHARATISHVILI [1] and M. N. OGUZTÖRELI [1] in 1966 and the papers by H. T. BANKS and A. MANITIUS [1] in 1974, F. COLONIUS [1] in 1982, and the lecture notes of A. MANITIUS [4] in 1976.

Many papers deal with the theory of partial differential equations with delays. It was unfortunately not realistic to include this material here. The reader is referred to M. ARTOLA [1, 2, 3, 4], and especially the bibliography in M. ARTOLA [4] on the work of the Italian School; A. ARDITO and P. RICCIARDI [1]; G. DI BLASIO, K. KUNISCH, and E. SINESTRARI [1]; J. DYSON and R. VILLELLA-BRESSAN [2]; K. KUNISCH and W. SCHAPPACHER [1]; and C. C. TRAVIS and G. F. WEBB [1, 2].

2.3 Notation

$\mathbb{R}$ denotes the field of real numbers, and for an arbitrary integer $n \geq 1$, $\mathbb{R}^n$ will be the n- dimensional Euclidean space. The norm of x in $\mathbb{R}^n$ and the inner product of x and y will be written $|x|$ and $x \cdot y$, respectively.

Given $-\infty \le a \le b \le +\infty$,

$$I(a,b) = \mathbb{R} \cap [a,b].$$

For a real Banach space X, $L^p(a,b;X)$ will denote the space of all equivalence classes of p-integrable (resp. essentially bounded) Lebesgue measurable functions on $I(a,b)$ into X for $1 \le p < \infty$ (resp. $p = \infty$). The derivative of a function x on $I(a,b)$ into X will be denoted $\dot{x}$, dx/dt, Dx, or $D_t x$ (in the distributional sense). The Sobolev space of all y in $L^p(a,b;X)$ with distributional derivatives $D_t^j y, j = 1, \dots, m$, in $L^p(a,b;X)$ will be written $W^{m,p}(a,b;X)$.

$C(a,b;X)$ will be the Banach space of all bounded continuous functions x from $I(a,b)$ into X. For $m \ge 1$, $C^m(a,b;X)$ will be the space of all m-times bounded continuous differentiable functions on $I(a,b)$. $C_0(a,b;X) = \{x \in C(a,b;X) \colon \forall \varepsilon > 0, \exists$ a compact subset K of $I(a,b)$ such that $|x(t)| < \varepsilon$, $\forall t \in K^c\}$, where K^c is the complement of K with respect to $I(a,b)$,

$$K^c = \{t \in I(a,b) \colon t \notin K\};$$

when a and b are finite $I(a,b) = [a,b]$ and C_0 and C coincide. $C_c(a,b;X)$ will be the subspace of functions of $C(a,b;X)$ with compact support in $]a,b[$. It is not to be confused with the space of bounded continuous functions with support in $I(a,b)$. In general the two spaces do not coincide except on $I(-\infty,\infty)$. In addition to the above function spaces, we shall also use the notation.

$$\mathcal{F}_{\mathrm{loc}}(a,\infty,X) = \{y : I(a,\infty) \to X \colon \forall T > a, y|_{I(a,T)} \in \mathcal{F}(a,T;X)\}$$

for any function space $\mathcal{F}$ (for instance $\mathcal{F}$ can be $C, L^2, W^{1,p}, \text{etc.} \dots$). $\mathcal{D}(]a,b[;\mathbb{R}^n)$ will denote the vector space of all infinitely continuously differentiable functions from $]a,b[$ into $\mathbb{R}^n$. $W_0^{m,p}(a,b;\mathbb{R}^n)$ will be the closure of $\mathcal{D}(]a,b[;\mathbb{R}^n)$ in $W^{m,p}(a,b;\mathbb{R}^n)$.

Given the integers $n \ge 1$ and $k \ge 1$, and real numbers $p, 1 \le p < \infty$, and h, $0 < h \le +\infty$ (possibly $h = +\infty$), we shall use the following notation for the two product spaces, which will frequently occur in this chapter:

$$M^p = \mathbb{R}^n \times L^p(-h,0;\mathbb{R}^n),$$
$$Z^p = \mathbb{R}^n \times L^p(-h,0;\mathbb{R}^n) \times L^p(-h,0;\mathbb{R}^k).$$

Whenever confusion is possible, subscripts n and/or k will be added.

Given a real number $p, 1 < p < \infty$, and an integer $\ell \ge 1$, the elements of the topological dual $L^p(a,b;\mathbb{R}^\ell)'$ of $L^p(a,b;\mathbb{R}^\ell)$ will be identified with those of $L^q(a,b;\mathbb{R}^\ell)$, where q is the conjugate of p, $p^{-1} + q^{-1} = 1$. Similarly $(M^p)'$ and $(Z^p)'$ will be identified with M^q and $(Z^q)'$, respectively.

Given a real measure μ on a σ-algebra of subsets of a set S, $|\mu|$ will denote the total variation of μ (cf. W. Rudin [1, pp. 117–118]. The total variation of an $n \times m$ matrix β of real measures $\{\beta_{ij} \colon 1 \le i \le n, 1 \le j \le m\}$ is defined as

$$|\beta| = \left\{ \sum_{i=1}^{n} \sum_{j=1}^{m} |\beta_{ij}|^2 \right\}^{1/2},$$

where $|\beta_{ij}|$ is the total variation of β_{ij}.

3 Existence theorems for Lipschitzian systems

All examples in §2 implicitly suggest the introduction of a product space as the space of initial conditions. In this chapter we present some general existence theorems for Lipschitzian systems in both the continuous function and the product space framework. In this way the reader will be in a better position to appreciate the relative advantages and limitations and, more importantly, the complementarity of the two approaches.

3.1 Continuous functions framework

We have seen that a delay system is characterized by the length of its *memory* $h, 0 < h \leq +\infty$. When $h < +\infty$, we say that the system has a *finite memory* and when $h = +\infty$ an *infinite memory.* The notation

$$I(h,0) = [-h,0] \cap \mathbb{R} \tag{3.1}$$

will be very convenient to simultaneously deal with $[-h,0]$ when h is finite and $]-\infty,0]$ when h is infinite. Denote by $C(-h,0;\mathbb{R}^n)$ the space of bounded continuous functions from $I(-h,0) \to \mathbb{R}^N$. When $h = +\infty$ it is not a Banach space and its elements are not necessarily uniformly continuous. So we could use one of the subspaces

$$\begin{aligned} C_\ell(-\infty,0;\mathbb{R}^n) &= \left\{ \phi \in C(-\infty,0;\mathbb{R}^n) \colon \lim_{\theta \to -\infty} \phi(\theta) \text{ exists} \right\}, \\ C_0(-\infty,0;\mathbb{R}^n) &= \left\{ \phi \in C(-\infty,0;\mathbb{R}^n) \colon \lim_{\theta \to -\infty} \phi(\theta) = 0 \right\}. \end{aligned} \tag{3.2}$$

In this chapter we choose the second subspace and introduce the following uniform notation for the space of initial conditions:

$$K(-h,0;\mathbb{R}^n) = \begin{cases} C(-h,0;\mathbb{R}^n), & \text{if } h < \infty, \\ C_0(-\infty,0;\mathbb{R}^n), & \text{if } h = \infty. \end{cases} \tag{3.3}$$

We shall see in Lemma 3.5 that this is a natural choice because for p, $1 \leq p < \infty$, $W^{1,p}(-\infty,0;\mathbb{R}^n) \subset C_0(-\infty,0;\mathbb{R}^n)$. Given a function

$$f \colon [0,\infty[\times K(-h,0;\mathbb{R}^n) \to \mathbb{R}^n, \tag{3.4}$$

we consider the differential equation

$$\begin{cases} \dfrac{dx}{dt}(t) = f(t, x_t), \quad t \geq 0, \\ x_0 = \phi \in K(-h, 0; \mathbb{R}^n), \end{cases} \tag{3.5}$$

where $x_t \colon I(-h, 0) \to \mathbb{R}^n$ is defined from the solution $x \colon [0, \infty[\to \mathbb{R}^n$ and the initial condition ϕ as follows:

$$x_t(\theta) = \begin{cases} x(t+\theta), & -t \leq \theta \leq 0, \\ \phi(t+\theta), & \theta < -t \end{cases} \tag{3.6}$$

for θ in $I(-h, 0)$.

Theorem 3.1. *Assume that the function f in* (3.4) *verifies the following assumptions*:

(H1) *for each ϕ in $K(-h, 0; \mathbb{R}^n)$, the function*

$$t \mapsto f(t, \phi) \colon [0, \infty[\to \mathbb{R}^n \tag{3.7}$$

is Lebesque measurable,

(H2) *there exists a non-negative locally integrable real function n such that for all ϕ_1 and ϕ_2 in $K(-h, 0; \mathbb{R}^n)$*

$$|f(t, \phi_2) - f(t, \phi_1)| \leq n(t) \|\phi_2 - \phi_1\|_C, \tag{3.8}$$

(H3) *and the function*

$$t \mapsto f(t, 0) \colon [0, \infty[\to \mathbb{R}^n \tag{3.9}$$

is locally integrable.

Then, given any initial condition ϕ in $K(-h, 0; \mathbb{R}^n)$, there exists a unique absolutely continuous solution $x = x(\cdot; \phi)$ to system (3.5) *on* $[0, \infty[$. *Moreover for each $T > 0$, there exists a constant $c(T) > 0$ such that for all ϕ_1 and ϕ_2 in $K(-h, 0; \mathbb{R}^n)$*

$$\|x(\cdot; \phi_2) - x(\cdot; \phi_1)\|_{W^{1,1}(0,T;\mathbb{R}^n)} \leq c(T) \|\phi_2 - \phi_1\|_C, \tag{3.10}$$

where $W^{1,1}(0, T; \mathbb{R}^n)$ denotes the space of absolutely continuous functions from $[0, T]$ *to $\mathbb{R}^n$ with a derivative in $L^1(0, T; \mathbb{R}^n)$.*

This theorem is not the only existence theorem that is available. It has been presented with a set of assumptions that contains as a special case *linear systems*; that is, when the map $f(t, \phi)$ is affine in ϕ

$$f(t, \phi) = L(t)\phi + f(t). \tag{3.11}$$

However it is easy to modify the set of assumptions (H1) to (H3) to obtain local versions. Similarly for non-Lipschitzian functions f, it is possible to obtain the analog of the classical Carathéodory conditions and local existence theorems (cf. J. K. Hale [1, 3]).

The proof of Theorem 3.1 necessitates the following two classical lemmas.

Lemma 3.1. *Let $T > 0$ be fixed and assume that assumptions* (H1) *to* (H3) *are verified*:

(i) *For each z in $K(-h,T;\mathbb{R}^n)$ the function*

$$t \mapsto f(t, z_t)\colon [0,T] \to \mathbb{R}^n \tag{3.12}$$

belongs to $L^1(0,T;\mathbb{R}^n)$, where

$$z_t(\theta) = z(t+\theta), \quad \theta \in I(-h,0), \qquad t \geq 0. \tag{3.13}$$

(ii) *For all pairs*

$$(\phi^1, x^1), \quad (\phi^2, x^2) \in K(-h,0;\mathbb{R}^n) \times C(0,T;\mathbb{R}^n) \tag{3.14}$$

such that $\phi^1(0) = x^1(0)$ and $\phi^2(0) = x^2(0)$ and all t in $[0,T]$

$$\begin{cases} \displaystyle\int_0^t |f(s,(x^2)_s) - f(s,(x^1)_s)|\, ds, \\ \displaystyle\leq \int_0^t n(s) \max\{\|\phi^2 - \phi^1\|_C + \|x^2 - x^1\|_{C(0,s)}\}\, ds. \end{cases} \tag{3.15}$$

Proof. (i) By definition of the space $K(-h,0;\mathbb{R}^n)$ each of its element is a uniformly continuous function and the function $t \mapsto \bar{z}(t) = z_t$ belongs to $C\big(0,T;K(-h,0;\mathbb{R}^n)\big)$. In particular

$$\bar{z} \in L^1\big(0,T;K(-h,0;\mathbb{R}^n)\big)$$

and there exists a sequence of step functions $s_n\colon [0,T] \to K(-h,0;\mathbb{R}^n)$ that converges pointwise to $\bar{z}$ for almost all t in $[0,T]$ and converges globally in the L^1 norm. To show that the function (3.12) is measurable, it is sufficient to establish the following two properties:

1. $f_n(t) = f\big(t, s_n(t)\big) \to f(t) = f\big(t, \bar{z}(t)\big)$, a.e. in $[0,T]$,
2. for all n, f_n is Lebesque measurable on $[0,T]$.

Then the function f will be Lebesque measurable as a pointwise limit of a sequence of Lebesque measurable functions.

By assumptions (H2) for each t the map $\phi \mapsto f(t,\phi)$ is continuous and necessarily

$$f_n(t) = f\big(t, s_n(t)\big) \to f(t) = f\big(t, \bar{z}(t)\big) \tag{3.16}$$

as n goes to infinity. This proves property 1). For property 2) it is sufficient to establish that for each step function s, the function $f_s(t) = f\big(t, s(t)\big)$ is measurable. By definition a step function is of the form

$$s(t) = \sum_{i=1}^k a_i \chi_{A_i}(t), \quad a_i \in K(-h,0;\mathbb{R}^n), \tag{3.17}$$

where $r \geq 1$ is a positive integer and the A_i's are disjoint measurable subsets of $[0,T]$ such that the measure of their union

$$A = \bigcup_{i=1}^{r} A_i$$

is finite (χ_{A_i} is the characteristic function of A_i). So f_s can be rewritten as

$$f\big(t, s(t)\big) = f(t,0)[1 - \chi_A(t)] + \sum_{i=1}^{r} f(t,a_i)\chi_{A_i}(t),$$

which is the sum of $r+1$ measurable functions by assumptions (H1).

To show that $f(t)$ is integrable, we evaluate its L^1- norm

$$\int_0^T |f(t,z_t)|\, dt \leq \int_0^T |f(t,0)|\, dt + \int_0^t |f(t,z_t) - f(t,0)|\, dt$$

and use assumption (H3) for the first term and assumption (H2) for the second term, which is bounded by

$$\int_0^T n(t)\|z_t - 0\|_C\, dt \leq \|n\|_{L^1(0,T;\mathbb{R})}\|z\|_{C(-h,T;\mathbb{R}^n)}.$$

(ii) By choice of ϕ^i and x^i, $s \mapsto (x^i)_s$ belongs to $C\big(0,T;K(-h,0;\mathbb{R}^n)\big)$ and the conclusions follow from part (i) and the Lipschitzian assumption (H2). $\square$

Lemma 3.2. *Let $\alpha, 0 < \alpha < 1$, be a real number and n be a non-negative function in $L^1_{\text{loc}}(0,\infty;\mathbb{R})$:*

(i) *The function*

$$g_\alpha(t) = \exp\left\{\frac{1}{\alpha}\int_0^t n(s)\, ds\right\}, \quad t \geq 0, \tag{3.18}$$

is monotonically increasing and greater than or equal to 1.

(ii) *For all $t \geq 0$*

$$\int_0^t n(s) g_\alpha(s)\, ds \leq \alpha g_\alpha(t). \tag{3.19}$$

(iii) *For all $T > 0$ and x in $C(0,T;\mathbb{R}^n)$ the quantity*

$$\|x\|_{C_\alpha(0,T;\mathbb{R}^n)} = \sup_{t\in[0,T]} \{\|x(t)/g_\alpha(t)\}\tag{3.20}$$

is an equivalent norm on the Banach space $C(0,T;\mathbb{R}^n)$ and

$$\frac{1}{g_\alpha(T)}\|x\|_C \leq \|x\|_{C_\alpha} \leq \|x\|_C. \tag{3.21}$$

Proof. (i) is obvious, (ii) is obtained by differentiating $g_\alpha(t)$, and the equivalence of the norms in (3.21) follows by definition of the norm (3.20). □

Remark 3.1. The introduction of the function g_α for global existence problems is due to A. BIELECKI [1]. It plays the same role as Gronwall's inequality. □

Proof of Theorem 3.1. The initial condition ϕ is fixed. For each x in $C(0,T;\mathbb{R}^N)$ such that $x = \phi(0)$, define the function Fx

$$(Fx)(t) = \phi(0) + \int_0^t f(s, x_s)\, ds, \quad 0 \le t \le T.$$

In view of Lemma 3.1, F maps the closed subset

$$S = \{x \in C(0,T;\mathbb{R}^n) \colon x(0) = \phi(0)\}$$

of $C(0,T;\mathbb{R}^n)$ onto itself. We now prove that for all $\alpha, 0 < \alpha < 1, F$ is a contracting map on S. So by the Banach fixed point theorem we get existence and uniqueness of the solution to $Fx = x$ or

$$x(t) = \phi(0) + \int_0^t f(s, x_s)\, ds, \quad 0 \le t \le T. \tag{3.22}$$

Given x and y in S

$$|(Fy)(t) - (Fx)(t)| \le \int_0^t |f(s, y_s) - f(s, x_s)|\, ds.$$

By assumption (H2) and inequality (3.15) in Lemma 3.1, the right-hand side is bounded by

$$\max_{r \in [0,t]} \frac{|y(r) - x(r)|}{g_\alpha(r)} \int_0^t n(s) g_\alpha(s)\, ds$$

and in view of (3.19) in Lemma 3.2,

$$|(Fy)(t) - (Fx)(t)| \le \alpha g_\alpha(t) \|y - x\|_{C_\alpha(0,t;\mathbb{R}^n)}.$$

Finally

$$\|Fy - Fx\|_{C_\alpha(0,T;\mathbb{R}^n)} \le \alpha \|y - x\|_{C_\alpha(0,T;\mathbb{R}^n)}$$

and (3.22) has a unique solution in S. Moreover $s \mapsto f(s, x_s)$ belongs to $L^1(0,T;\mathbb{R}^n)$ and necessarily x belongs to $W^{1,1}(0,T;\mathbb{R}^n)$.

To obtain the estimate (3.10) let x^1 and x^2 be the respective solutions of (3.22) for ϕ^1 and ϕ^2. Then

$$|x^2(t) - x^1(t)| \le |\phi^2(0) - \phi^1(0)| + \int_0^t |f(s, x_s^2) - f(s, x_s^1)|\, ds,$$

and by using inequality (3.15) the integral term is bounded by

$$\|n\|_{L^1(0,T;\mathbb{R})}\|\phi^2-\phi^1\|_C+\int_0^t n(s)\max_{0\le r\le s}\{\|x^2(r)-x^1(r)\|\}\,ds.$$

Now use (3.21) on $[0,s]$ for the last integral, which is bounded by

$$\int_0^t n(s)g_\alpha(s)\|x^2-x^1\|_{C_\alpha(0,s;\mathbb{R}^n)}\,ds\le \alpha g_\alpha(t)\|x^2-x^1\|_{C_\alpha(0,t;\mathbb{R}^n)}.$$

Finally

$$|x^2(t)-x^1(t)|\le[1+\|n\|_{L^1(0,T)}]\|\phi^2-\phi^1\|_C+\alpha g_\alpha(t)\|x^2-x^1\|_{C_\alpha(0,T;\mathbb{R}^n)},$$
$$(1-\alpha)\|x^2-x^1\|_{C_\alpha(0,T;\mathbb{R}^n)}\ \le(1+\|n\|_{L^1(0,T)})\|\phi^2-\phi^1\|_C,$$

and in view of (3.21)

$$\|x^2-x^1\|_{C(0,T;\mathbb{R}^n)}\le\frac{g_\alpha(t)}{1-\alpha}[1+\|n\|_{L^1(0,T)}]\|\phi^2-\phi^1\|_C. \tag{3.23}$$

For the derivative we again use (3.15)

$$\begin{aligned}\|\dot x^2-\dot x^1\|_{L^1}&\le\int_0^T\ |f(t,x_t^2)-f(t,x_t^1)|\,dt\\&\le\|n\|_{L^1}\max\{\|x^2-x^1\|_C,\|\phi^2-\phi^1\|_C\},\end{aligned} \tag{3.24}$$

and by combining (3.23) and (3.24), we obtain (3.10). □

Remark 3.2. The proof of Lemma 3.1 is essentially the original proof given by C. Carathéodory [1] in 1927 generalized to Banach spaces. □

3.2 L^p or product space framework

We shall see now that in many cases, it is possible to separate the initial point $x(0)$ of the solution x to the differential equation (3.5) from the piece of function x_0 on $I(-h,0)$, which is necessary to make sense of the right-hand side of (3.5). This will lead us to consider an initial condition as a point and a function

$$\phi=(\phi^0,\phi^1)\in M^p\ =\mathbb{R}^n\times L^p(-h,0;\mathbb{R}^n) \tag{3.25}$$

in the product space that will be denoted M^p, where $p, 1\le p<\infty$, is a fixed real number. For the reader interested in the origin of the letter M, it was just the letter next to L.

As in §3.1 we start with a function $f(t,\phi)$ that is defined for ϕ in $K(-h,0;\mathbb{R}^n)$. When ϕ^1 is an L^p function the segment

$$x_t(\theta)=\begin{cases}x(t+\theta), & -t\le\theta\le 0,\\ \phi^1(t+\theta), & \theta<-t,\end{cases} \tag{3.26}$$

$\theta\in I(-h,0)$, is generally not a continuous function in $K(-h,0;\mathbb{R}^n)$, but only an $L^p(-h,0;\mathbb{R}^n)$ function. So we cannot expect to give a pointwise meaning to the function $t\mapsto f(t,x_t)\colon[0,\infty[\,\to\mathbb{R}^n$.

Example 3.1. Consider the function

$$f(t,\phi) = a(t)\phi(-2), \quad \phi \in C(-2,0;\mathbb{R}), \tag{3.27}$$

where a is measurable and bounded on $[0,\infty[$. Then the function

$$t \mapsto f(t,x_t) = a(t)x(t-2) \colon [0,2] \to \mathbb{R} \tag{3.28}$$

belongs to $L^p(0,2;\mathbb{R})$ for all x in $L^p(-2,2;\mathbb{R})$. □

Unfortunately not all functions $f(t,\phi)$ are extendable as can be seen from the next example.

Example 3.2. Consider for $\phi \in C(-2,0;\mathbb{R})$ the function

$$f(t,\phi) = \begin{cases} \phi(t-1), & 0 \le t \le 1, \\ 0, & t > 1. \end{cases} \tag{3.29}$$

For all continuous functions x in $C(-2,2;\mathbb{R})$, the function

$$t \mapsto f(t,x_t) = \begin{cases} x(-1), & 0 \le t \le 1, \\ 0, & 1 < t \le 2 \end{cases} : [0,2] \to \mathbb{R} \tag{3.30}$$

is well defined and discontinuous at $t = 1$. However it has no extension to functions x in $L^p(-2,2;\mathbb{R})$. □

Nevertheless we shall see later that the family of functions f that can be extended to L^p initial conditions includes all linear time-invariant systems that are of the form

$$f(t,\phi) = L\phi + f(t), \tag{3.31}$$

where L is linear and continuous from $K(-h,0;\mathbb{R}^n)$ into $\mathbb{R}^n$ and f belongs to $L^1_{\mathrm{loc}}(0,\infty;\mathbb{R}^n)$.

The following theorem is the counterpart of Theorem 3.1.

Theorem 3.2. *Fix the real number* $p, 1 \le p < \infty$, *and* $h, 0 < h \le +\infty$. *Let the map* f *given in* (1.4) *verify assumptions* (H1) *to* (H3) *in Theorem* 3.1 *and the following additional assumption*:

(H4) *there exists a real non-negative monotonically increasing function such that for all* $t \ge 0$ *and all* z^1 *and* z^2 *in* $C_c(-h,t;\mathbb{R}^n)$

$$\int_0^t |f(s,(z^2)_s) - f(s,(z^1)_s)|\, ds \le m(t)\left[\int_{-h}^t |z^2(s) - z^1(s)|^p \, ds\right]^{1/p}. \tag{3.32}$$

Then, given any $\phi = (\phi^0, \phi^1)$ *in* $M^p = \mathbb{R}^n \times L^p(-h, 0; \mathbb{R}^n)$, *there exists a unique absolutely continuous solution* $x = x(\cdot; \phi)$ *to the system*

$$\begin{cases} \dfrac{dx}{dt}(t) = f(t, x_t), \quad t > 0, \\ (x(0), x_0) = (\phi^0, \phi^1) = \phi \in M^p. \end{cases} \tag{3.33}$$

Moreover for each $T > 0$, *there exists a constant* $c(T) > 0$ *such that for all* ϕ_1 *and* ϕ_2 *in* M^p

$$\|x(\cdot; \phi_2) - x(\cdot; \phi_1)\|_{W^{1,1}(0,T;\mathbb{R}^n)} \leq c(T)\|\phi_2 - \phi_1\|_{M^p}, \tag{3.34}$$

where

$$\|\phi\|_{M^p} = \|(\phi^0, \phi^1)\|_{M^p} = [|\phi^0|^p + \|\phi^1\|_{L^p}^p]^{1/p}. \tag{3.35}$$

To complete this section we now give the proof of the last two theorems. Theorem 3.2 necessitates two lemmas similar to the ones used in the proof of Theorem 3.1.

Lemma 3.3. *Fix the real numbers* p, $1 \leq p < +\infty$, $T > 0$ *and* h, $0 < h \leq \infty$. *Assume that assumptions* (H1) *to* (H4) *are verified. Associate with each* z *in* $C_c(-h, T; \mathbb{R}^n)$ *the function* f_z *in* $L^1(0, T; \mathbb{R}^n)$ *defined as*

$$f_z(t) = f(t, z_t), \quad 0 \leq t \leq T. \tag{3.36}$$

Then the map

$$z \mapsto f_z \colon C_c(-h, T; \mathbb{R}^n) \to L^1(0, T; \mathbb{R}^n) \tag{3.37}$$

extends to a unique uniformly continuous map from $L^p(-h, T; \mathbb{R}^n)$ *into* $L^1(0, T; \mathbb{R}^n)$. *Moreover for all* t *in* $[0, T]$ *and* z^1 *and* z^2 *in* $L^p(-h, T; \mathbb{R}^n)$,

$$\int_0^t |f(s, (z^2)_s) - f(s, (z^1)_s)|\, ds \leq \ m(t)\|z^2 - z^1\|_{L^p(-h,t;\mathbb{R}^n)}. \tag{3.38}$$

Proof. By Lemma 3.1 to each z in $C_c(-h, T; \mathbb{R}^n)$, we can associate a function f_z in $L^1(0, T; \mathbb{R}^n)$ defined by (3.36). By assumption (H4) for all z^1 and z^2 in $C_c(-h, T; \mathbb{R}^n)$

$$\|f_{z^2} - f_{z^1}\|_{L^1(0,T;\mathbb{R}^n)} \leq m(T)\|z^2 - z^1\|_{L^p(-h,T;\mathbb{R}^n)}.$$

Thus the map (3.37) is uniformly continuous for the $L^p(-h, T; \mathbb{R}^n)$ topology. By density of $C_c(-h, T; \mathbb{R}^n)$ in $L^p(-h, T; \mathbb{R}^n)$, it has a unique uniformly continuous extension to all $L^p(-h, T; \mathbb{R}^n)$. As for inequality (3.38) we use assumption (H4) on $[0, t]$ and bound $m(t)$ by $m(T)$ for $0 \leq t \leq T$. □

Lemma 3.4. *Let* α, $0 < \alpha < 1$, $p, 1 \leq p < \infty$, *and* $c > 0$ *be fixed real numbers*:

(i) *The function*

$$g_\alpha(t) = \exp\left[\left(\frac{c}{\alpha}\right)^p \frac{t}{p}\right], \quad t \geq 0, \tag{3.39}$$

is monotonically increasing and greater than or equal to 1.

(ii) *For all* $t \geq 0$

$$c\|g_\alpha\|_{L^p(0,t;\mathbb{R})} \leq \alpha g_\alpha(t). \tag{3.40}$$

By introducing the norm (3.20) *with* g_α *given by* (3.39)*, we also have the equivalence of the norms on* $C(0,T;\mathbb{R}^n)$ *and inequalities* (3.21)*.*

Proof. Differentiate $g_\alpha(s)^p$ with respect to s and integrate with respect to s from 0 to t

$$\frac{d}{ds} g_\alpha(s)^p = \left(\frac{c}{\alpha}\right)^p g_\alpha(s)^p,$$

$$\int_0^t c^p g_\alpha(s)^p \, ds = \alpha^p[g_\alpha(t)^p - 1] \leq \alpha^p g_\alpha(t)^p.$$ □

Proof of Theorem 3.2. The proof follows the same pattern as the one of Theorem 3.1. The initial condition $\phi = (\phi^0, \phi^1)$ is fixed. For each x in the closed subset

$$S = \{x \in C(0,T;\mathbb{R}^n) \colon x(0) = \phi^0\}$$

of $C(0,T;\mathbb{R}^n)$, define the function

$$(Fx)(t) = \phi^0 + \int_0^t f(s, x_s) \, ds, \quad 0 \leq t \leq T.$$

The map $x \mapsto Fx$ is well defined from S to S. We prove that for any $\alpha, 0 < \alpha < 1$, F is contracting on S and apply Banach fixed point theorem to obtain the existence of a unique x in S such that $Fx = x$ or equivalently

$$x(t) = \phi^0 + \int_0^t f(s, x_s) \, ds, \quad 0 \leq t \leq T.$$

Given x and y in S

$$|(Fy)(t) - (Fx)(t)| \leq \int_0^t |f(s, y_s) - f(s, x_s)| \, ds.$$

By (3.38) in Lemma 3.3, the right-hand side is bounded by

$$m(t)\|y - x\|_{L^p(0,t;\mathbb{R}^n)}.$$

Construct the function $g_\alpha(t)$ in (3.39) with $c = m(T)$. Then

$$\begin{aligned} |(Fy)(t) - (Fx)(t)| &\leq m(T)\|y - x\|_{C_\alpha(0,t;\mathbb{R}^n)}\|g_\alpha\|_{L^p(0,t;\mathbb{R})} \\ &\leq \alpha g_\alpha(t)\|y - x\|_{C_\alpha(0,t;\mathbb{R}^n)} \end{aligned}$$

and F is contracting for the norm C_α. From this point on the proof is essentially the same as the one of Theorem 3.1. □

3.3 Linear time-invariant systems

The price to pay for initial conditions in L^p is the extra assumption (H4). However (H4) is always verified for linear time-invariant systems of the form (3.31).

Theorem 3.3. *Assume that the map $f(t,\phi)$ is of the form* (3.31) *for f in $L^1_{\mathrm{loc}}(0,\infty;\mathbb{R}^n)$ and a continuous linear map*

$$L\colon K(-h,0;\mathbb{R}^n) \to \mathbb{R}^n : \tag{3.41}$$

(i) *There exists a $n \times n$ matrix of regular Borel measures such that*

$$L\phi = \int_{-h}^{0} d_\theta\, \eta\phi(\theta), \quad \forall \phi \in K(-h,0;\mathbb{R}^n). \tag{3.42}$$

(ii) *The four assumptions* (H1) *to* (H4) *are verified for all real numbers p, $1 \le p < \infty$. Moreover there exists a constant $c > 0$ such that for all (t,s), $0 \le s \le t$, and all z in $C_c(-h,t;\mathbb{R}^n)$*

$$\left[\int_s^t |Lz_r|^p\, dr\right]^{1/p} \le c\|\eta\| \sup_{\theta \in I(-h,0)} \left[\int_{s+\theta}^{t+\theta} |z(r)|^p\, dr\right]^{1/p}, \tag{3.43}$$

where $\|\eta\|$ is the total variation of the matrix η.

(iii) *Given $T > 0$, introduce the continuous function $\mathcal{L}z$,*

$$(\mathcal{L}z)(r) = Lz_r, \quad r \ge 0, \tag{3.44}$$

for each z in $C_c(-h,T;\mathbb{R}^n)$ and the map

$$z \mapsto \mathcal{L}z\colon C_c(-h,T;\mathbb{R}^n) \to L^p(0,T;\mathbb{R}^n). \tag{3.45}$$

Then for any $T > 0$ the map $\mathcal{L}$ has a continuous linear extension to $L^p(-h,T;\mathbb{R}^n)$

$$\mathcal{L}\colon L^p(-h,T;\mathbb{R}^n) \to L^p(0,T;\mathbb{R}^n) \tag{3.46}$$

and for all pairs (s,T), $0 \le s \le T$,

$$\|\mathcal{L}z\|_{L^p(s,T)} \le \|\eta\| \sup_{\theta \in I(-h,0)} \{\|x\|_{L^p(s+\theta,T+\theta)}. \tag{3.47}$$

(iv) *For all x in $W^{1,p}(-h,T;\mathbb{R}^n)$ and $t \ge 0$*

$$\int_0^t (\mathcal{L}\dot{x})(s)\, ds = Lx_t - Lx_0, \tag{3.48}$$

where $\dot{x}$ is the first derivative of x.

Proof. (i) By the Riesz's representation theorem (cf. W. RUDIN [1, p. 131, Theorem 6.19]).

(ii) It is easy to check that assumptions (H1) to (H3) are verified. As for assumption (H4) we use Fubini's theorem applied to complex measures (cf. W. RUDIN [1, Chapter 6 and p. 140, Theorem 7.8]). For all z^1 and z^2 in $C_c(-h,t;\mathbb{R}^n)$

$$\begin{aligned}\int_0^t |f(s,z_s^2)-f(s,z_s^1)|\,ds &= \int_0^t |L(z_s^2-z_s^1)|\,ds\\ &\le \int_0^t ds \int_{-h}^0 d_\theta\,|\eta|\;|z^2(s+\theta)-z^1(s+\theta)|.\end{aligned}$$

As the integrand is continuous with compact support in $I(h,0)\times[0,t]$, it is $m\times\eta$ measurable (m, the Lebesgue measure) and after changing the order of integration

$$\int_{-h}^0 d_\theta\,|\eta| \int_0^t ds|z^2(s+\theta)-z^1(s+\theta)| \le m(t)\|z^2-z^1\|_{L^p(-h,t;\mathbb{R}^n)},$$

where for $t\ge 0$

$$m(t)=\|\eta\|\begin{cases} t^{1/q}, & 1<p<\infty,\\ 1, & p=1,\end{cases}$$

and $\|\eta\|$ is the total variation of η on $I(-h,0)$.

We have established (3.43) for $p=1$ because it is a special case of (H4). For $1<p<\infty$ consider for any f in $C(0,t;\mathbb{R}^n)$ the following expression for $0\le s\le t$:

$$\left|\int_s^t f(r)\cdot Lz_r\,dr\right| \le \int_s^t dr \int_{-h}^0 d_\theta\,|\eta|\;|f(r)|\;|z(r+\theta)|.$$

Again the integrand is a continuous function on $I(-h,0)\times[0,t]$ with compact support. Hence it is measurable and Fubini's theorem can again be used to change the order of integration in the last integral

$$\int_{-h}^0 d_\theta\,|\eta| \int_s^t dr|f(r)|\;|z(r+\theta)| \le \int_{-h}^0 d_\theta\,|\eta|\;\|f\|_{L^q(s,t)}\|z\|_{L^p(s+\theta,t+\theta)}$$

and

$$\left|\int_s^t f(r)\cdot Lz_r\,dr\right| \le \|f\|_{L^q(s,t)} \int_{-h}^0 d_\theta\,|\eta|\;\|z\|_{L^p(s+\theta,t+\theta)}.$$

By density of $C(0,t;\mathbb{R}^n)$ in $L^q(0,t;\mathbb{R}^n)$, we obtain (3.43) for all z in $C_c(-h,t;\mathbb{R}^n)$.

(iii) The continuous linear extension of (3.45) again follows by density of $C_c(-h,t;\mathbb{R}^n)$ in $L^p(-h,t;\mathbb{R}^n)$ and inequality (3.43).

(iv) The proof of this part requires the following lemma.

Lemma 3.5. *Let $\ell \geq 1$ be an integer and p, $1 \leq p < \infty$, $T > 0$ and h, $0 < h \leq +\infty$ (possibly $+\infty$), be real numbers. Then*

$$W^{1,p}(-h,T;\mathbb{R}^n) \subset K(-h,T;\mathbb{R}^n) \tag{3.49}$$

with continuous injection.

For x in $W^{1,p}(-h,T;\mathbb{R}^n)$, the difference

$$Lx_t - Lx_0 = L(x_t - x_0)$$

is well defined and continuous. For x in $W^{2,p}(-h,T;\mathbb{R}^n)$,

$$(x_t - x_0)(\theta) = x(t+\theta) - x(\theta) = \int_\theta^{t+\theta} \dot{x}(s)\, ds$$

and

$$\int_{-h}^0 d_\theta\, \eta \int_\theta^{t+\theta} \dot{x}(s)\, ds = \int_{-h}^0 d_\theta\, \eta \int_0^t \dot{x}(s+\theta)\, ds = \int_0^t ds \int_{-h}^0 d_\theta\, \eta \dot{x}_s(\theta).$$

Hence for all t in $[0,T]$

$$Lx_t - Lx_0 = \int_0^t (\mathcal{L}\dot{x})(s)\, ds, \quad \forall x \in W^{2,p}(-h,T;\mathbb{R}^n).$$

Now by density and continuity this last result is true for all x in $W^{1,p}(-h,T;\mathbb{R}^n)$. □

Proof of Lemma 3.5. Any element x of $W^{1,p}(-h,T;\mathbb{R}^n)$ is almost everywhere equal to a bounded continuous function $\bar{x}$ on $I(-h,T)$ (cf. R. A. Adams [1, p. 97, Theorem 5.4]) and the injection in the space of bounded continuous functions is continuous. This proves the lemma for h finite, and we only have to deal with the case $h = +\infty$.

To complete the proof it is now sufficient to show that $\bar{x}(t)$ goes to zero as t goes to $-\infty$. For all pairs (s,s'), $-\infty < s \leq s' \leq t$

$$\bar{x}(t) = \bar{x}(s') + \int_{s'}^t \dot{x}(r)\, dr.$$

Take the L^p norm with respect to s' from s to t

$$(t-s)^{1/p}|\bar{x}(t)| \leq \|\bar{x}\|_{L^p(s,t;\mathbb{R}^\ell)} + (t-s)\|\dot{x}\|_{L^p(s,t;\mathbb{R}^\ell)}.$$

But $\bar{x} = x$ in $L^p(s,t;\mathbb{R}^\ell)$ and for all $t \leq T$ and $s = t-1$

$$|\bar{x}(t)| \leq 2^{1-1/p}\|x\|_{W^{1,p}(-\infty,t;\mathbb{R}^\ell)}.$$

For each $\varepsilon > 0$, there exists $T' < T$ such that

$$\forall\ t \leq T', \quad \|x\|_{W^{1,p}(-\infty,t;\mathbb{R}^\ell)} \leq \varepsilon$$

and by combining the last two inequalities

$$\forall \varepsilon > 0, \quad \exists T' \leq T \quad \text{such that } \forall t \leq T,\ |\bar{x}(t)| \leq \varepsilon.$$

Therefore $\bar{x}$ belongs to $C_0(-\infty,T;\mathbb{R}^\ell)$. □

4 State space theory of linear time-invariant systems

In this section we fix p, $1 \le p < \infty$, and specialize to linear systems

$$\begin{cases} \dot{x}(t) = Lx_t + f(t), \quad t \ge 0, \\ (x(0), x_0) = (\phi^0, \phi^1) \in M^p \equiv \mathbb{R}^n \times L^p(-h, 0; \mathbb{R}^n), \end{cases} \tag{4.1}$$

where f belongs to $L^p_{\text{loc}}(0, \infty; \mathbb{R}^n)$ and L is a continuous map

$$L\colon K(-h, 0; \mathbb{R}^n) \to \mathbb{R}^n. \tag{4.2}$$

4.1 Preliminary results and smoothness of the solution

We have seen in §3.2 that the right-hand side of the first equation (4.1) makes sense for x in $L^p_{\text{loc}}(-h, \infty; \mathbb{R}^n)$ even if L is only defined on the space of continuous functions. This situation is not really surprising because we are dealing with initial conditions in an infinite dimensional space and the real state of the system at time t is an appropriately defined piece of function on the time interval $I(t-h, t)$. In fact two states are considered that are fundamental in the theory of delay systems. A traditional difficulty in the various theorems and proofs we shall encounter is the notation. A good notation will simplify the computations and make all the arguments precise in both the finite and the infinite memory case. For instance we have introduced the notation $\mathcal{L}x$ in Theorem 3.3, which is more precise and compact than the traditional notation Lx_t. In addition we shall need the following notation to restrict or extend functions by 0.

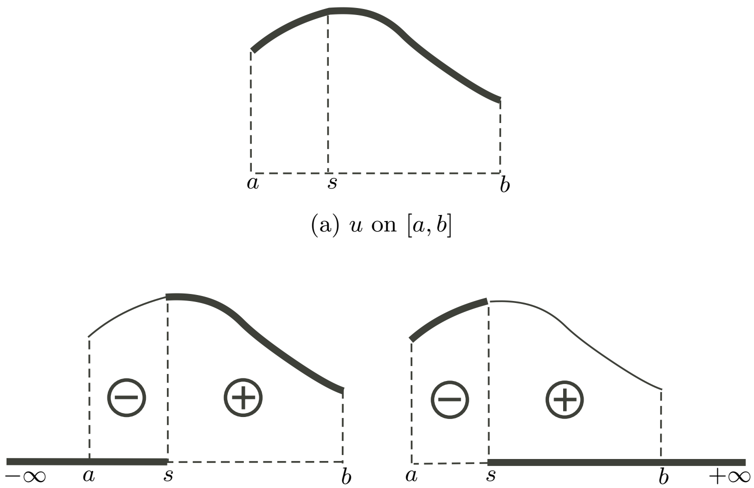
Definition 4.1. Let $\ell \ge 1$ be an integer and a and b, $a < b$, be two real numbers. Let $\mathcal{F}(a, b; \mathbb{R}^\ell)$ be a set of functions from $[a, b]$ to $\mathbb{R}^\ell$. For each u in $\mathcal{F}(a, b; \mathbb{R}^\ell)$ and all s, $a \le s \le b$, define the functions $e^s_- u$ and $e^s_+ u$ as follows:

$$e^s_-\colon I(a, \infty) \to \mathbb{R}^\ell, \ (e^s_- u)(t) = \begin{cases} u(t), & a \le t \le s, \\ 0, & s < t < \infty, \end{cases} \tag{4.3}$$

$$e^s_+\colon I(-\infty, b) \to \mathbb{R}^\ell, \ (e^s_+ u)(t) = \begin{cases} 0, & -\infty < t < s, \\ u(t), & s \le t \le b. \end{cases} \tag{4.4}$$

□

In other words the subscript "+" indicates that we keep the function u on the right (positive direction) of s up to b and set it equal to zero on the left of s down to $-\infty$. Similarly the subscript " −" indicates that we keep the function u on the left (negative direction) of s from a and set it equal to zero on the right of s up to $+\infty$. The construction is illustrated in Figure 4.1.



(a) u on $[a, b]$

(b) $e_+^s u$ on $I(-\infty, b)$

(c) $e_-^s u$ on $I(a, \infty)$

Fig. 4.1. The functions u, $e_+^s u$, and $e_-^s u$.

Remark 4.1. In the sequel we shall often make use of the composition $\mathcal{L}e_+^0$ of the maps

$$e_+^0 : L^p(0, T; \mathbb{R}^n) \to L^p(-\infty, T; \mathbb{R}^n)$$

and

$$\mathcal{L}\colon L^p(-\infty, T; \mathbb{R}^n) \to L^p(0, T; \mathbb{R}^n).$$

□

Theorem 4.1. *Let p, $1 \le p < \infty$, and h, $0 < h \le +\infty$, be real numbers and*

$$f \in L^p_{\text{loc}}(0, \infty; \mathbb{R}^n) \quad \text{and} \quad \phi = (\phi^0, \phi^1) \in M^p \tag{4.5}$$

be given:

(i) *The system*

$$\dot{x} = \mathcal{L}x + f, \quad (x(0), x_0) = \phi \tag{4.6}$$

has a unique solution x in $W^{1,p}_{\text{loc}}(0, \infty; \mathbb{R}^n)$. Moreover, for all $T > 0$, there exists a constant $c(T) > 0$ such that

$$\|x\|_{W^{1,p}(0,T;\mathbb{R}^n)} \le c(T)[\|\phi\|_{M^p} + \|f\|_{L^p(0,T;\mathbb{R}^n)}]. \tag{4.7}$$

(ii) *When*

$$\phi^1 \in W^{1,p}(-h, 0; \mathbb{R}^n) \quad \text{and} \quad \phi^0 = \phi^1(0), \tag{4.8}$$

the solution x of (4.6) belongs to $W^{1,p}(-h, T; \mathbb{R}^n)$ for all $T > 0$ and there exists $c(T) > 0$ such that

$$\|x\|_{W^{1,p}(-h,T;\mathbb{R}^n)} \le c(T)[\|\phi^1\|_{W^{1,p}} + \|f\|_{L^p(0,T;\mathbb{R}^n)}]. \tag{4.9}$$

Remark 4.2. In fact part (i) can be slightly improved. It is possible to show that for any continuous linear map

$$L\colon W^{1,p}(-h,0;\mathbb{R}^n) \to \mathbb{R}^n, \tag{4.10}$$

system (4.6) has a unique solution x in $C_{\text{loc}}(0,\infty;\mathbb{R}^n)$ when (4.6) is interpreted in an appropriate weak sense (cf. M. C. DELFOUR [8]). □

4.2 First state equation

Starting with the homogeneous system

$$\dot{x} = \mathcal{L}x, \quad (x(0), x_0) = \phi \in M^p, \tag{4.11}$$

we can construct the following semigroup of continuous linear transformations on

$$\phi \mapsto S(t)\phi = (x(t), x_t)\colon M^p \to M^p, \quad t \geq 0. \tag{4.12}$$

Theorem 4.2. *Let p, $1 \leq p < \infty$, and h, $0 < h \leq +\infty$, be real numbers and $L\colon K(-h,0;\mathbb{R}^n) \to \mathbb{R}^n$ be linear and continuous*:

(i) *The family $\{S(t)\colon t \geq 0\}$ of transformations of M^p defined by* (4.12) *forms a strongly continuous semigroup on M^p.*
(ii) *Its infinitesimal generator is characterized by*

$$A(\phi^0, \phi^1) = (L\phi^1, D\phi^1) \tag{4.13}$$

for all $\phi = (\phi^0, \phi^1)$ in the domain $D(A)$ of A

$$D(A) = \left\{ (\phi^0, \phi^1) \in M^p : \begin{array}{l} \phi^1 \in W^{1,p}(-h,0;\mathbb{R}^n) \\ \textit{and} \quad \phi^0 = \phi^1(0) \end{array} \right\}, \tag{4.14}$$

where $D\phi^1$ denotes the first derivative of ϕ^1.

Notation 4.1. In view of the structure of the elements of $D(A)$ we shall use the notation ψ for both the function ψ in $W^{1,p}(-h,0;\mathbb{R}^n)$ and the element $(\psi(0),\psi)$ of $D(A)$. □

Notice that from the definition of D(A) and A, the largest family of maps L that generates a strongly continuous semigroup $\{S(t)\}$ on M^p is precisely the one of Remark 4.2. This result clearly indicated that the state space theory of other types of delay systems such as the neutral type could not be obtained from systems of the form (4.11) (cf. M. C. DELFOUR [8] for more details). To complete the picture we immediately give the second theorem for the nonhomogeneous case, which will require an important technical lemma. All proofs will be given at the end of the section.

Theorem 4.3. *Assume that the assumptions of Theorem* 4.2 *are verified and that x is the solution in $W^{1,p}_{\text{loc}}(0,\infty;\mathbb{R}^n)$ to* (4.6) *for $f \in L^p_{\text{loc}}(0,\infty;\mathbb{R}^n)$*:

(i) *For all* $t \geq 0$ *the state*

$$\tilde{x}(t) \stackrel{\text{def}}{=} (x(t), x_t)$$

is well defined and

$$\tilde{x}(t) = S(t)\phi + \int_0^t S(t-s)(f(s), 0)\, ds. \tag{4.15}$$

If $\phi \in D(A)$, *then for all* $t \geq 0, \tilde{x}(t) \in D(A)$.

(ii) *For* $1 < p < \infty$ *and* q *its conjugate,* $q^{-1} + p^{-1} = 1, \tilde{x}$ *is a solution of the system*

$$\begin{cases} \dfrac{d}{dt} < \psi, \tilde{x}(t) > + \langle A^*\psi, \tilde{x}(t)\rangle + \psi^0 \cdot f(t), \quad t > 0, \\ \tilde{x}(0) = \phi, \quad \forall \psi \in D(A^*), \end{cases} \tag{4.16}$$

where A^* *is the infinitesimal generator of the semigroup* $\{S^*(t)\}$ *on* M^q.

(iii) *Given* p *and* q *as in part* (ii) *for all* $T > 0$, $\tilde{x}$ *is the unique solution in*

$$\mathcal{V}(0, T; M^p, D(A^*)') = \left\{ z \in C(0, T; M^p) \colon \frac{d}{dt} i^* z \in L^p(0, T; D(A^*)') \right\} \tag{4.17}$$

to the following equation in $D(A^*)'$:

$$\begin{cases} \dfrac{d}{dt} i^*\tilde{x}(t) = (A^*)^*\tilde{x}(t) + i^*(f(t), 0), \quad t > 0, \\ \tilde{x}(0) = \phi, \end{cases} \tag{4.18}$$

where i^* *and* $(A^*)^*$ *are the topological dual maps of the continuous linear maps*

$$i \colon D(A^*) \to M^q, \quad A^* \colon D(A^*) \to M^q,$$

where i *is the canonical dense injection of* $D(A^*)$ *into* M^q *and* $D(A^*)$ *is endowed with the graph norm topology defined by*

$$\|\psi\|_{D(A^*)} = [\|\psi\|^q_{M^q} + \|A^*\psi\|^q_{M^q}]^{1/q}. \tag{4.19}$$

Remark 4.3. This theorem will be generalized in §5.2 to systems with delays in the control variable. □

Equation (4.17) is the first example of a state equation for delay systems. It is to be interpreted in the weak sense (4.16). This result is an abstract one because we have not yet characterized $D(A^*)$, which is quite different from $D(A)$. In most situations (4.15), (4.16), or (4.17) will be sufficient. However we shall give other definitions of states and state equations that are different from $\tilde{x}(t)$ in (4.15).

Terminology 1. The pair $(x(t), x_t)$ associated with the solution x of system (4.6) will be called the *state* of system (4.6) and denoted $\tilde{x}(t)$. The corresponding semigroup $\{S(t)\}$ defined by (4.12) will be referred to as the *semigroup* associated with the state.

This is the natural extension of the traditional terminology for the state x_t in the space of continuous functions. Fundamentally $\{S(t)\}$ is a translation semigroup acting on the concatenation of the initial function and the solution x of system (4.6). The terminology *initial function state* and *initial function semigroup* is also used in the recent literature as in the book of G. GRIPPENBERG, S. O. LONDEN, and O. STAFFANS [1].

We now proceed to the proof of the two theorems, which will necessitate the following important technical lemma.

Lemma 4.1. *Let p, $1 \le p < \infty$, q, $q^{-1} + p^{-1} = 1$, and $T > 0$, be real and $\ell \ge 1$ be an integer*:

(i) *The function u belongs to $L^p(-h, T; \mathbb{R}^\ell)$ if and only if the function*

$$t \mapsto (u_\bullet)(t) = u_t \colon [0, T] \to L^p(-h, 0; \mathbb{R}^\ell) \tag{4.20}$$

is continuous. Moreover for all u in $L^p(-h, T; \mathbb{R}^\ell)$, all t, $0 \le t \le T$, and all ψ in $W_0^{1,q}(-h, 0; \mathbb{R}^\ell)$,

$$D_t \langle \psi, u_t \rangle_{L^q \times L^p} = -\langle D_\theta\, \psi, u_t \rangle_{L^q \times L^p}. \tag{4.21}$$

(ii) *The following three properties are equivalent*:

$$u \in W^{1,p}(-h, T; \mathbb{R}^\ell), \tag{4.22}$$

$$u_\bullet \in C^1\big(0, T; L^p(-h, 0; \mathbb{R}^\ell)\big), \tag{4.23}$$

$$u_\bullet \in C\big(0, T; W^{1,p}(-h, 0; \mathbb{R}^\ell)\big). \tag{4.24}$$

Moreover in each case

$$D_t u_t = D_\theta\, u_t, \quad 0 \le t \le T, \tag{4.25}$$

and the equality holds in $C(0, T; L^p(-h, 0); \mathbb{R}^\ell)$.

Proof. (i) For any u in $C_c(-h, T; \mathbb{R}^\ell)$ the function (4.20) is well defined and continuous and the map

$$u \mapsto u_\bullet \colon C_c(-h, T; \mathbb{R}^\ell) \to C\big(0, T; L^p(-h, 0; \mathbb{R}^\ell)\big) \tag{4.26}$$

is linear and continuous for the L^p-topology:

$$\int_{-h}^0 |u_t(\theta)|^p \, d\theta = \int_{-h}^0 |u(t+\theta)|^p \, d\theta \le \|u\|^p_{L^p(-h,T;\mathbb{R}^\ell)}$$
$$\implies \|u_\bullet\|_{C\left(0,T;L^p(-h,0;\mathbb{R}^\ell)\right)} \le \|u\|_{L^p(-h,T;\mathbb{R}^\ell)}.$$

As a result the map (4.26) has a unique continuous linear extension to all $L^p(-h,T;\mathbb{R}^\ell)$. Conversely if $u_\bullet \in C\big(0,T;L^p(-h,0;\mathbb{R}^\ell)\big)$,

$$\forall t \in [0,T], \quad u_t \in L^p(-h,0;\mathbb{R}^\ell) \Longrightarrow u \in L^p(-h,T;\mathbb{R}^\ell).$$

To compute the vectorial distribution derivative of $u_\bullet$ consider the following expression for φ in $\mathcal{D}(]0,T[)$ and ψ in $W_0^{1,q}(-h,0;\mathbb{R}^\ell)$:

$$\begin{aligned} E &= -\int_0^T \langle \psi, u_t\rangle_{L^q\times L^p} \frac{d\varphi}{dt}(t)\,dt \\ &= -\int_0^T dt \int_{-h}^0 d\theta \psi(\theta)\cdot u(t+\theta)\frac{d\varphi}{dt}(t). \end{aligned}$$

Notice that $e_+^{-h}\psi$ is the extension by 0 of the function ψ on $I(-h,0)$ to $]-\infty,0]$. As $\psi(-h)=0$, $e_+^{-h}\psi \in W^{1,q}(-\infty,0;\mathbb{R}^\ell)$. Change the variable θ to $s=t+\theta$ and extend the bounds $t-h$ and t to $-h$ and T:

$$\begin{aligned} E &= -\int_0^T dt \int_{t-h}^t ds (e_+^{-h}\psi)(s-t)\cdot u(s)\frac{d\varphi}{dt}(t) \\ &= -\int_0^T dt \int_{-h}^T ds (e_+^{-h}\psi)(s-t)\cdot u(s)\frac{d\varphi}{dt}(t). \end{aligned}$$

Now change the order of integration and integrate by parts with respect to t:

$$E = \int_{-h}^T ds \int_0^T dt \frac{d}{dt}(e_+^{-h}\psi)(s-t)\cdot u(s)\varphi(t).$$

Then change the order of integration once again and change the variable s back to $\theta = s-t$

$$\begin{aligned} E = \int_0^T dt \int_{-h-t}^{T-t} d\theta \left[-\frac{d}{d\theta}(e_+^{-h}\psi)(\theta)\right]\cdot u(t+\theta)\varphi(t) \\ = -\int_0^T dt \int_{-h}^0 d\theta \frac{d\psi}{d\theta}(\theta)\cdot u_t(\theta)\varphi(t). \end{aligned}$$

This establishes (4.21).

(ii) If $u \in W^{1,p}(-h,T;\mathbb{R}^\ell)$, then the derivative $\dot u \in L^p(-h,T;\mathbb{R}^\ell)$ and from (i) the function $t \mapsto (\dot u)_t$ belongs to $C\big(0,T;L^p(-h,0;\mathbb{R}^\ell)\big)$. So it is sufficient to establish that $Du_t = (\dot u)_t$ to conclude that $u_\bullet \in C^1\big(0,T;L^p(-h,0;\mathbb{R}^\ell)\big)$. We use identity (4.21)

$$\begin{aligned} -\langle D_\theta\,\psi, u_t\rangle = -\int_{-h}^0 \frac{d\psi}{d\theta}(\theta)\cdot u(t+\theta)\,d\theta = \int_{-h}^0 \psi(\theta)\cdot\frac{d}{d\theta}u(t+\theta)\,d\theta \\ = \int_{-h}^0 \psi(\theta)\cdot \dot u(t+\theta)\,d\theta = \langle \psi, (\dot u)_t\rangle \end{aligned}$$

and necessarily $Du_t = (\dot{u})_t$. So (4.22) implies (4.23). To show that (4.23) implies (4.24) again use identity (4.21): For all t in $[0,T]$ and ψ in $W_0^{1,q}(-h,0;\mathbb{R}^\ell)$

$$\langle \psi, Du_t \rangle = D_t \langle \psi, u_t \rangle = -\langle D_\theta \, \psi, u_t \rangle.$$

Therefore

$$Du_t = D_\theta \, u_t, \quad 0 \le t \le T \tag{4.27}$$

and $t \mapsto D_\theta \, u_t$ belongs to $C\big(0,T;L^p(-h,0;\mathbb{R}^\ell)\big)$. But we already know from part (i) that $u_t \in C\big(0,T;L^p(-h,0;\mathbb{R}^\ell)\big)$ and necessarily

$$u_\bullet \in C\big(0,T;W^{1,p}(-h,0;\mathbb{R}^\ell)\big).$$

Finally we show that (4.24) implies (4.22). For $h = +\infty$,

$$u_T \in W^{1,p}(-\infty,0;\mathbb{R}^\ell) \Longrightarrow u \in W^{1,p}(-\infty,T;\mathbb{R}^\ell).$$

For h finite

$$\forall t \in [0,T], \quad u_t \in W^{1,p}(-h,0;\mathbb{R}^\ell) \Longrightarrow u|_{[t-h,t]} \in W^{1,p}(t-h,t;\mathbb{R}^\ell),$$

and because the interval $[-h,T]$ is finite, $u \in W^{1,p}(-h,T;\mathbb{R}^\ell)$. In the process we have established (4.26). This completes the proof of the lemma. □

Proof of Theorem 4.2. (i) This is a direct consequence of Theorem 4.1.

(ii) By definition of $D(A)$, for each ϕ in $D(A)$ and $T > 0$, the function

$$t \mapsto S(t)\phi = (x(t), x_t)$$

belongs to $C^1(0,T;M^p)$. In particular $x \in C(0,T;\mathbb{R}^n)$ and

$$x_\bullet \in C^1\big(0,T;L^p(-h,0;\mathbb{R}^n)\big).$$

Now by Lemma 4.1 (ii) $x \in W^{1,p}(-h,T;\mathbb{R}^n)$ and

$$\phi^0 = x(0) = \phi^1(0) \quad \text{and} \quad \phi^1 \in W^{1,p}(-h,0;\mathbb{R}^n).$$

Conversely if $\phi^0 = \phi^1(0)$ and $\phi^1 \in W^{1,p}(-h,0;\mathbb{R}^n)$, then by Theorem 4.1 (ii) for all $T > 0$, the solution x to (4.11) belongs to $W^{1,p}(-h,T;\mathbb{R}^n)$. By Lemma 4.1 (ii),

$$x_\bullet \in C^1\big(0,T;L^p(-h,0;\mathbb{R}^n)\big) \quad \text{and} \quad D_t x_t = D_\theta \, x_t$$

and by continuity of $D_t x_t$

$$\lim_{t \searrow 0} D_t x_t = \lim_{t \searrow 0} D_\theta \, x_t = D_\theta \, \phi^1 \quad \text{in } L^p(-h,0;\mathbb{R}^n).$$

As for the $\mathbb{R}^n$-component we use identity (3.48) in Theorem 3.3 (iv): For all x in $W^{1,p}(-h,T;\mathbb{R}^n)$, $\mathcal{L}\dot{x}$ belongs to $L^p(0,T;\mathbb{R}^n)$ and

$$Lx_t = Lx_0 + \int_0^t (\mathcal{L}\dot{x})(s)\, ds.$$

Therefore

$$\lim_{t \searrow 0} \dot{x}(t) = \lim_{t \searrow 0} Lx_t = Lx_0 = L\phi^1.$$

We have shown that

$$\lim_{t \searrow 0} D_t \tilde{x}(t) = \lim_{t \searrow 0} D_t(x(t), x_t) = (L\phi^1, D_\theta\, \phi^1) \quad \text{in } M^p.$$

So $\phi \in D(A)$, which is characterized by (4.14). Moreover we have also established identity (4.13) and this completes the proof. □

Proof of Theorem 4.3. (i) By linearity it is sufficient to establish (4.15) for $\phi = 0$. In that case $\phi = 0$ verifies assumption (4.8) in Theorem 4.1 (ii) and the solution x of (4.6) is such that for all $T > 0$

$$x \in W^{1,p}(0, T; \mathbb{R}^n) \quad \text{and} \quad x(0) = 0.$$

Hence $e_+^0 x \in W^{1,p}(-h, T; \mathbb{R}^n)$ and by Theorem 4.2 (ii)

$$\tilde{x}(t) = (x(t), (e_+^0 x)_t) \in D(A), \quad \forall t \geq 0.$$

In particular

$$\frac{dx}{dt}(t) = L(e_+^0 x)_t + f(t),$$

and from Lemma 4.1 (ii), $e_+^0 x \in C\big(0, T; W^{1,p}(-h, 0; \mathbb{R}^n)\big)$ and

$$\frac{d}{dt}(e_+^0 x)_t = D_\theta (e_+^0 x)_t, \quad t > 0.$$

Hence

$$\frac{d}{dt}\tilde{x}(t) = A\tilde{x}(t) + (f(t), 0), \quad \text{a.e. in } [0, \infty[, \quad \tilde{x}(0) = 0.$$

Now from semigroup theory (cf. Chapter 1, Proposition 3.1), we conclude that (4.15) is verified.

(ii) From part (i) we know that $\tilde{x} \in C_{\text{loc}}(0, \infty; M^p)$ and that M^p is reflexive. Therefore $\{S(t)\}$ has an adjoint semigroup $\{S^*(t)\}$ on $(M^p)'$ with a densely defined infinitesimal generator A^* with domain $D(A^*)$. So for all ψ in $D(A^*)$, we rewrite (4.15) in the following weak form:

$$\langle \psi, \tilde{x}(t) \rangle = \langle S^*(t)\psi, \phi \rangle + \int_0^t \langle S^*(t-s)\psi, (f(s), 0) \rangle\, ds$$

and

$$\begin{aligned}\frac{d}{dt}\langle \psi, \tilde{x}(t) \rangle &= \langle S(t)^* A^* \psi, \phi \rangle + \int_0^t \langle S^*(t-s)A^*\psi, (f(s), 0) \rangle\, ds + \langle \psi, (f(t), 0) \rangle \\ &= \langle A^*\psi, \tilde{x}(t) \rangle + \langle \psi, (f(t), 0) \rangle.\end{aligned}$$

This gives the weak expression (4.16).

(iii) Rewrite (4.16) with the canonical injection and take the duals of i and A^* to get (4.18), and this gives the characterization of $\tilde{x}$ as an element of $\mathcal{V}(0,T;M^p;D(A^*)')$ in (4.17). The uniqueness is obvious. □

4.3 Transposed and adjoint systems

In the previous sections we considered the system

$$\dot{x}(t) = Lx_t, \quad t > 0, \qquad (x(0), x_0) = \phi \in M^p \tag{4.28}$$

associated with the continuous linear map $L\colon K(-h,0;\mathbb{R}^n) \to \mathbb{R}^n$ and constructed the semigroup $\{S(t)\}$. In Theorem 3.3 in §3.3, we have associated with L a representation in terms of an $n \times n$ matrix of regular Borel measures

$$L\phi = \int_{-h}^{0} d_\theta\, \eta\phi(\theta). \tag{4.29}$$

If we denote by $\eta^\top$ the transposed of the matrix η, then we can introduce a new continuous linear map

$$L^\top\colon K(-h,0;\mathbb{R}^n) \to \mathbb{R}^n, \quad L^\top\psi = \int_{-h}^{0} d_\theta\, \eta^\top\psi(\theta), \tag{4.30}$$

and for each $T > 0$ and q, $1 \le q < \infty$, the continuous linear map

$$\begin{cases} \mathcal{L}^\top : L^q(-h,T;\mathbb{R}^n) \to L^q(0,T;\mathbb{R}^n), \\ (\mathcal{L}^\top z)(t) = L^\top z_t, \quad \forall t \ge 0,\ \forall z \in C_c(-h,T;\mathbb{R}^n). \end{cases} \tag{4.31}$$

For $L^\top$ and $\mathcal{L}^\top$ we have the analog of Theorem 3.3 in §3.3.

Definition 4.2. Let q, $1 \le q < \infty$, be a real number:

(i) For ψ in M^q and g in $L^q_{\mathrm{loc}}(0,\infty;\mathbb{R}^n)$ the *transposed system* is defined as

$$\dot{z} = \mathcal{L}^\top z + g, \quad (z(0), z_0) = (\psi^0, \psi^1) \in M^q. \tag{4.32}$$

(ii) The *transposed semigroup* $\{S^\top(t)\}$ is defined as

$$S^\top(t)\psi = (z(t), z_t), \quad t \ge 0,\ \psi \in M^q, \tag{4.33}$$

where z is the solution of (4.32) with $g = 0$. □

The system (4.32) and the semigroup (4.33) associated with $L^\top$ have the same properties as system (4.6) and the semigroup (4.12) associated with L. So the results in the previous section apply with $L^\top$, $S^\top$, and q in place of L, S, and p. In particular the infinitesimal generator $A^\top$ of the semigroup $\{S^\top(t)\}$ is given by

$$D(A^\top) = \left\{ (\psi^0, \psi^1) \in M^q : \begin{array}{l} \psi^1 \in W^{1,q}(-h, 0; \mathbb{R}^n) \\ \psi^0 = \psi^1(0) \end{array} \right\}, \tag{4.34}$$

$$A^\top(\psi^0, \psi^1) = (L^\top \psi^1, D\psi^1).$$

Notation 4.2. It will be convenient to use the notation ψ for both the element ψ of $W^{1,q}(-h,0;\mathbb{R}^n)$ and the element $\psi = (\psi(0), \psi)$ of $D(A^\top)$. The canonical injection of $D(A^\top)$ into M^q will be denoted by j

$$\psi = (\psi(0), \psi) \mapsto j\psi = (\psi(0), \psi) \colon D(A^\top) \to M^q.$$

□

The transposed semigroup $\{S^\top(t)\}$ is however not equal to the topological adjoint semigroup $\{S^*(t)\}$ of $\{S(t)\}$. This is a very fundamental aspect of the theory of delay systems with deep implications for Control Theory. Working with (4.6) for x will yield an "adjoint system" that is equivalent to the transposed system (4.32) with a change of variable from t to $T - t$. However working with the state equation (4.18) will yield an "adjoint system" in M^q characterized by the adjoint semigroup $\{S^*(t)\}$. Of course there is a connection between the two approaches that is a consequence of the "intertwining theorem" between $S^\top(t)$ and $S^*(t)$ by the structural operator F, which will be introduced in the subsequent sections.

Terminology 2. The pair $(z(t), z_t)$ associated with the solution z of system (4.32) will be called the *transposed state* of system (4.32) and denoted $\tilde{z}(t)$. The corresponding semigroup $\{S(t)^\top\}$ defined by (4.33) will be referred to as the *transposed semigroup.*

The first technical result is an "integration by parts" formula that relates systems (4.6) and (4.32).

Lemma 4.2. *Let* $T > 0$, p, $1 < p < \infty$, *and* q, $q^{-1} + p^{-1} = 1$, *be real numbers. Then for all* x *in* $W^{1,p}(0,T;\mathbb{R}^n)$ *and* z *in* $W^{1,q}(0,T;\mathbb{R}^n)$

$$\begin{cases} \displaystyle\int_0^T z(T-t) \cdot [\dot{x} - \mathcal{L} e^0_+ x](t)\, dt + z(T) \cdot x(0), \\ \displaystyle = \int_0^T [\dot{z} - \mathcal{L}^\top e^0_+ z](T-t) \cdot x(t)\, dt + z(0) \cdot x(T). \end{cases} \tag{4.35}$$

Proof. It is sufficient to look at the term

$$\int_0^T z(T-t) \cdot (\mathcal{L} e^0_+ x)(t)\, dt.$$

For x and z in $C_c(0,T;\mathbb{R}^n), x(0) = 0$ and $z(0) = 0$, $e^0_+ x$ and $e^0_+ z$ belong to $C_c(-h,T;\mathbb{R}^n)$, and

$$
\begin{aligned}
\int_0^T z(T-t)\cdot(\mathcal{L}e_+^0 x)(t)\,dt &= \int_0^T z(T-t)\cdot L(e_+^0 x)_t\,dt \\
&= \int_0^T z(T-t)\cdot \int_{-h}^0 d_\theta\,\eta(e_+^0 x)(t+\theta)\,dt \\
&= \int_0^T \int_{-h}^0 d_\theta\,\eta^\top z(T-t)\cdot(e_+^0 x)(t+\theta)\,dt.
\end{aligned}
$$

After changing the order of integration

$$
\begin{aligned}
&= \int_{-h}^0 \int_0^T d_\theta\,\eta^\top z(T-t)\cdot(e_+^0 x)(t+\theta)\,dt \\
&= \int_{-h}^0 \int_\theta^{T+\theta} d_\theta\,\eta^\top z(T-s+\theta)\cdot(e_+^0 x)(s)\,ds \\
&= \int_{-h}^0 \int_0^T d_\theta\,\eta^\top (e_+^0 z)(T-s+\theta)\cdot x(s)\,ds.
\end{aligned}
$$

Again change the order of integration

$$
\begin{aligned}
&= \int_0^T L^\top (e_+^0 z)_{T-s}\cdot x(s)\,ds \\
&= \int_0^T (\mathcal{L}^\top e_+^0 z)(T-s)\cdot x(s)\,ds.
\end{aligned}
$$

By density of $C_c(0,T;\mathbb{R}^n)$ in $L^p(0,T;\mathbb{R}^n)$ and $L^q(0,T;\mathbb{R}^n)$ for all z in $L^q(0,T;\mathbb{R}^n)$ and x in $L^p(0,T;\mathbb{R}^n)$

$$
\int_0^T z(T-t)\cdot(\mathcal{L}e_+^0 x)(t)\,dt = \int_0^T (\mathcal{L}^\top e_+^0 z)(T-s)\cdot x(s)\,ds.
$$

In particular it is true for z in $W^{1,q}(0,T;\mathbb{R}^n)$ and x in $W^{1,p}(0,T;\mathbb{R}^n)$. □

To complete this section we relate the solution z of the transposed system to the classical backward adjoint system. For simplicity we assume that

$$
\psi^1 \in W^{1,q}(-h,0;\mathbb{R}^n) \quad \text{and} \quad \psi^0 = \psi^1(0).
$$

Under that condition system (4.32) can be rewritten as

$$
\dot z(t) = L^\top z_t + g(t) \quad \text{on } [0,T]. \tag{4.36}
$$

Now we introduce the variable

$$
p(t) = z(T-t), \quad 0 \le t \le T+h. \tag{4.37}
$$

Then

$$
\frac{d}{dt}p(t) = -\dot z(T-t)
$$

and

$$L^\top z_{T-t} = \int_{-h}^{0} d_\theta\, \eta^\top z(T-t+\theta) = \int_{-h}^{0} d_\theta\, \eta^\top p(t-\theta) = L^\top p^t,$$

where

$$p^t(\theta) = p(t-\theta). \tag{4.38}$$

Finally we obtain the *backward adjoint system*

$$\begin{cases} -\dfrac{dp}{dt}(t) = L^\top p^t + g(T-t), \quad 0 < t < T, \\ (p(T), p^T) = (\psi^0, \psi^1). \end{cases} \tag{4.39}$$

In general for time-varying systems (4.39) will be the system to work with. The transposed system can only be obtained in the time-invariant case. Keeping this in mind and the fact that the remainder of this chapter is devoted to time-invariant linear systems, we have decided to avoid the backward adjoint system and work with the transposed system.

4.4 Structural operators

Structural operators are as fundamental to delay systems as Sobolev spaces to elliptic problems. They capture the fundamental structure of a delay system and play a key role in the characterization of the properties of stability, stabilizability, controllability, and observability. They were first announced in December 1976 for a finite number of delays by A. Manitius [5] at the CDC and by M. C. Delfour and A. Manitius [1] at the INRIA. A complete treatment was later given by M. C. Delfour and A. Manitius [2, 3] for arbitrary linear delay functionals L continuous on the space of continuous functions.

There are many ways to introduce structural operators. Intuitively they describe the way the system combines and transforms initial conditions over the initial time interval $I(0,h)$. Go back to the linear homogeneous system

$$\dot{x} = \mathcal{L}x, \quad (x(0), x_0) = (\psi^0, \psi^1) \in M^p. \tag{4.40}$$

Then separate the solution x in $W^{1,p}(0,T;\mathbb{R}^n)$ from its initial function ϕ^1 in $L^p(-h,0;\mathbb{R}^n)$

$$\dot{x} = \mathcal{L}e^0_+ x + \mathcal{L}e^0_- \phi^1, \quad x(0) = \phi^0. \tag{4.41}$$

System (4.40) does not directly use ϕ^1 but only its image $\mathcal{L}e^0_-\phi^1$. When h is finite

$$(\mathcal{L}e^0_-\phi^1)(t) = 0, \quad t > h, \tag{4.42}$$

and it is sufficient to consider $\mathcal{L}e^0_-\phi^1$ on the interval $[0,h]$.

This suggests to associate with the function ϕ^1 on $I(-h,0)$ another function $\bar{L}\phi^1$ on $I(-h,0)$ defined in the following way:

$$(\bar{L}\phi^1)(\alpha) = (\mathcal{L}e_-^0\phi^1)(-\alpha), \quad \alpha \in I(-h,0). \tag{4.43}$$

In view of Theorem 3.3 (iii) in §3.3

$$\bar{L}\colon L^p(-h,0;\mathbb{R}^n) \to L^p(-h,0;\mathbb{R}^n) \tag{4.44}$$

is a continuous linear transformation and we can rewrite (4.41) to emphasize the role of the pair $(\phi^0, \bar{L}\phi^1) \in M^p$

$$\begin{cases} \dot{x}(t) = (\mathcal{L}e_+^0 x)(t) + (e_+^{-h}\bar{L}\phi^1)(-t), \quad t > 0, \\ x(0) = \phi^0. \end{cases} \tag{4.45}$$

It is clear that the "real initial data" is the pair

$$F(\phi^0, \phi^1) = (\phi^0, \bar{L}\phi^1) \in M^p, \tag{4.46}$$

which defines a continuous linear transformation of M^p.

Definition 4.3. Let p, $1 \le p < \infty$, and h, $0 < h \le +\infty$, be real numbers:

(i) The operator F defined in (4.46) will be referred to as the *structural operator associated with* L.
(ii) The *structural operator associated with* $L^\top$ is defined as

$$F^\top(\psi^0, \psi^1) = (\psi^0, \bar{L}^\top\psi^1), \tag{4.47}$$

where

$$(\bar{L}^\top\psi^1)(\alpha) = (\mathcal{L}^\top e_-^0\psi^1)(-\alpha), \quad \alpha \in I(-h,0). \tag{4.48}$$

□

The following lemma gives further insight into the structure of F and its relationship to $F^\top$.

Lemma 4.3. (i) *For all* ϕ *in* $K(-h,0;\mathbb{R}^n)$ *such that* $\phi(0) = 0$,

$$\begin{cases} (\bar{L}\phi)(\alpha) = \displaystyle\int_{-h}^{\alpha} d_\theta\, \eta\phi(\theta - \alpha), \quad \forall\alpha \in I(-h,0), \\ \bar{L}\phi \in K(-h,0;\mathbb{R}^n), \\ (\bar{L}\phi)(-h) = 0, \quad (\bar{L}\phi)(0) = L\phi, \\ (\bar{L}^T\phi)(\alpha) = \displaystyle\int_{-h}^{\alpha} d_\theta\, \eta^\top\phi(\theta - \alpha), \quad \forall\alpha \in I(-h,0), \\ \bar{L}^\top\phi \in K(-h,0;\mathbb{R}^n), \\ (\bar{L}^\top\phi)(-h) = 0, \quad (\bar{L}^\top\phi)(0) = L^\top\phi. \end{cases} \tag{4.49}$$

(ii) *For any* p, $1 < p < \infty$, *and its conjugate* $q, q^{-1} + p^{-1} = 1$,

$$\bar{L}^* = \bar{L}^\top \quad \textit{and} \quad F^* = F^\top, \tag{4.50}$$

where $\bar{L}^*$ *and* F^* *are the dual operators of* $\bar{L}$ *and* F.

Proof. (i) It is sufficient to establish (4.49) for L. The proof for $L^\top$ is exactly the same. Given any ϕ in $K(-h,0;\mathbb{R}^n)$, $e^0_-\phi \in K(-\infty,0;\mathbb{R}^n)$ is uniformly continuous and the function

$$\alpha \mapsto (e^0_-\phi)_{-\alpha} \colon I(-h,0) \to K(-h,0;\mathbb{R}^n)$$

is also continuous. As L is continuous on $K(-h,0;\mathbb{R}^n)$, the function

$$\alpha \mapsto (\bar{L}\phi)(\alpha) = (\mathcal{L}e^0_-\phi)(-\alpha) = L(e^0_-\phi)_{-\alpha} \colon I(-h,0) \to \mathbb{R}^n$$

is also continuous and $(\bar{L}\phi)(0) = L(e^0_-\phi)_0 = L\phi$. For h finite, $(\bar{L}\phi)(\alpha) = 0$ for $\alpha < -h$ and by continuity $(\bar{L}\phi)(-h) = 0$. For h infinite, $(\bar{L}\phi)(\alpha) \to 0$ as $\alpha \to -h$ because $e^0_-\phi \in K(-\infty,0;\mathbb{R}^n)$ and $(e^0_-\phi)(\alpha) \to 0$ as $\alpha \to -\infty$. Finally the last two identities (4.49) follow by definition of $\bar{L}$

$$(\bar{L}\phi)(\alpha) = L(e^0_-\phi)_{-\alpha} = \int_{-h}^{0} d_\theta\, \eta(e^0_-\phi)(\theta-\alpha) = \int_{-h}^{\alpha} d_\theta\, \eta\phi(\theta-\alpha)$$

because

$$(e^0_-\phi)(\theta-\alpha) = 0 \quad \text{for } \theta-\alpha \geq 0.$$

(ii) We compute the dual of $\bar{L}$. For ϕ and ψ in $C_c(-h,0;\mathbb{R}^n)$

$$\begin{aligned}
\langle \psi, \bar{L}\phi\rangle &= \int_{-h}^{0} \psi(\alpha)\cdot(\bar{L}\phi)(\alpha)\, d\alpha \\
&= \int_{-h}^{0} \psi(\alpha)\cdot \int_{-h}^{0} d_\theta\, \eta(e^0_-\phi)(\theta-\alpha)\, d\alpha.
\end{aligned}$$

Change the order of integration and the variable α to $\zeta = \theta-\alpha$

$$\begin{aligned}
\langle \psi, \bar{L}\phi\rangle &= \int_{-h}^{0}\int_{\theta}^{\theta+h} d_\theta\, \eta^\top\psi(\theta-\zeta)\cdot(e^0_-\phi)(\zeta)\, d\zeta \\
&= \int_{-h}^{0}\int_{-h}^{0} d_\theta\, \eta^\top(e^0_-\psi)(\theta-\zeta)\cdot\phi(\zeta)\, d\zeta \\
&= \int_{-h}^{0} d\zeta \int_{-h}^{0} d_\theta\, \eta^\top(e^0_-\psi)(\theta-\zeta)\cdot\phi(\zeta) \\
&= \langle \bar{L}^\top\psi, \phi\rangle.
\end{aligned}$$

If the dual $(L^p)^*$ of L^p is identified with L^q, then we obtain the first identity (4.50). The second one follows immediately from the first. □

4.5 Adjoint semigroup $\{S^{\top *}(t)\}$ and intertwining theorems

The next theorem gives a complete characterization of the dual semigroup $\{S^{\top *}(t)\}$ of $S^\top(t)\}$ in terms of the structural operator F. This is the first step toward the so-called intertwining theorem.

Theorem 4.4. *Fix the real number p, $1 < p < \infty$, and its conjugate q, $q^{-1} + p^{-1} = 1$, and identify the elements of $(M^p)^*$ with those of M^q:*

(i) *For all $\xi = (\xi^0, \xi^1)$ in M^p and all $t \geq 0$,*

$$S^{\top *}(t)\xi = F(x(t), (e_+^0 x)_t) + (0, \tau(t)\xi^1), \tag{4.51}$$

where $x \in W^{1,p}_{\mathrm{loc}}(0, \infty; \mathbb{R}^n)$ is the solution of the system

$$\dot{x}(t) = (\mathcal{L} e_+^0 x)(t) + (e_+^{-h}\xi^1)(-t), \quad t > 0, \; x(0) = \xi^0, \tag{4.52}$$

and $\tau(t)$ is the right translation operator

$$[\tau(t)u](\theta) = (e_+^{-h} u)(\theta - t), \theta \in [-h, 0] \tag{4.53}$$

for any arbitrary function u defined on $I(-h, 0)$.

(ii) *When $\xi = F\phi$ for some $\phi \in M^p$, then*

$$S^{\top *}(t)F\phi = F(x(t), x_t), \quad t \geq 0. \tag{4.54}$$

Remark 4.4. We shall see in §5 that it is possible to construct a state $\hat{x}(t)$ and a state equation similar to (4.16) and (4.18) in Theorem 4.3 for $\tilde{x}(t)$ in the nonhomogeneous case. We choose to do it later in the general case with the control term instead of doing the simple case here and repeating the argument for the control case. □

Corollary 4.1. *Fix the real number p, $1 < p < \infty$, and its conjugate $q, q^{-1} + p^{-1} = 1$, and identify the elements of $(M^p)^*$ with those of M^q:*

(i) *For all $\zeta = (\zeta^0, \zeta^1)$ in M^q and all $t \geq 0$,*

$$S^*(t)\zeta = F^*(z(t), (e_+^0 z)_t) + (0, \tau(t)\zeta^1), \tag{4.55}$$

where $z \in W^{1,q}_{\mathrm{loc}}(0, \infty; \mathbb{R}^n)$ is the solution of the system

$$\dot{z}(t) = (\mathcal{L}^\top e_+^0 z)(t) + (e_+^{-h}\zeta^1)(-t), \quad t > 0, \; z(0) = \zeta^0, \tag{4.56}$$

and $\tau(t)$ is the right translation operator defined in (4.53).

(ii) *When $\zeta = F^*\psi$ for some $\psi \in M^q$, then*

$$S^*(t)F^*\psi = F^*(z(t), z_t), \quad t \geq 0. \tag{4.57}$$

To see the complete picture we immediately give the intertwining theorem of $S^{\top *}$ and S with respect to F and the dual result as a corollary. The proofs will be given at the end of the section. The general version of the theorem is due to M. C. Delfour and A. Manitius [2, 3], for arbitrary linear delay functionals L continuous on the space of continuous functions (see also C. Bernier and A. Manitius [1] for a finite number of delays).

Theorem 4.5 (Intertwining theorem). *Fix the real number* p, $1 < p < \infty$, *and its conjugate* q, $q^{-1} + p^{-1} = 1$. *Denote by* $A^{\top *}$ *the infinitesimal generator of the semigroup* $\{S^{\top *}(t)\}$ *on* M^p. *Then the following properties hold and are equivalent*:

(i) $S^{\top *}(t)F\phi = FS(t)\phi$, $\forall \phi \in M^p$, $\forall t \geq 0$.
(ii) $FD(A) \subset D(A^{\top *})$ *and* $A^{\top *}F\phi = FA\phi$, $\forall \phi \in D(A)$.
(iii) *For all* ζ *in the resolvent set of* A

$$R(\zeta, A^{\top *})F\phi = FR(\zeta, A)\phi, \quad \forall \phi \in M^p, \tag{4.58}$$

where $R(\zeta, A) = [\zeta I - A]^{-1}$ *and* $R(\zeta, A^{\top *}) = [\zeta I - A^{\top *}]^{-1}$.

Corollary 4.2. *Fix* p *and* q *as in Theorem* 4.5. *If* $\{S^*(t)\}$ *is the dual semigroup of* $\{S(t)\}$ *with infinitesimal generator* A^*, *the following properties hold and are equivalent*:

(i) $S^*(t)F^*\phi = F^*S^{\top}(t)\phi$, $\forall \phi \in M^q$.
(ii) $F^*D(A^{\top}) \subset D(A^*)$ *and* $A^*F^*\phi = F^*A^{\top}\phi$, $\forall \phi \in D(A^{\top})$.
(iii) *For all* ζ *in the resolvent set of* $A^{\top}$,

$$R(\zeta, A^*)F^*\phi = F^*R(\zeta, A^{\top})\phi, \quad \forall \phi \in M^q. \tag{4.59}$$

Proof of Theorem 4.4. (i) Let z be the solution of the transposed system

$$\dot{z} = \mathcal{L}^{\top} z, \quad (z(0), z_0) = (\psi^0, \psi^1) \in M^q.$$

Let $T > 0$ be an arbitrary time. As x and z belong to $W^{1,p}(0,T;\mathbb{R}^n)$ and $W^{1,q}(0,T;\mathbb{R}^n)$, respectively, we can use identity (4.35) in Lemma 4.2 and the fact that

$$\mathcal{L}^{\top} z = \mathcal{L}^{\top} e_+^0 z + \mathcal{L}^{\top} e_-^0 \psi^1.$$

Then

$$\begin{aligned}(1) \overset{\text{def}}{=} \int_0^T z(T-t) \cdot (e_+^{-h}\xi^1)(-t)\, dt + z(T) \cdot \xi^0 \\ = \int_0^T (\mathcal{L}^{\top} e_-^0 \psi^1)(T-t) \cdot x(t)\, dt + \psi^0 \cdot x(T) \overset{\text{def}}{=} (2).\end{aligned}$$

Notice the two terms in ξ on the left-hand side (1) and the two terms in ψ on the right-hand side of (2). Change the variable t to $\theta = -t$ in (1):

$$\begin{aligned}(1) &= \int_{-T}^0 z(T+\theta) \cdot (e_+^{-h}\xi^1)(\theta)\, d\theta + z(T) \cdot \xi^0 \\ &= \int_{-h}^0 (e_+^0 z)_T(\theta) \cdot \xi^1(\theta)\, d\theta + z(T) \cdot \xi^0 \\ &= \langle z_T, \xi^1 \rangle - \langle (e_-^0 \psi^1)_T, \xi^1 \rangle + z(T) \cdot \xi^0 \\ &= \langle \tilde{z}(T), \xi \rangle_{M^q \times M^p} - \langle (e_-^0 \psi^1)_T, \xi^1 \rangle_{L^q \times L^p}.\end{aligned}$$

By a simple change of variable

$$\int_{-h}^{0} (e_-^0 \psi^1)_T(\alpha) \cdot \xi^1(\alpha)\, d\alpha = \int_{-h}^{0} \psi^1(\theta) \cdot (e_+^{-h}\xi^1)(\theta - T)\, d\theta$$

and finally by definition of $S^\top$ and τ

$$(1) = \langle S^\top(T)\psi, \xi\rangle - \big\langle \psi, (0, \tau(T)\xi^1)\big\rangle.$$

For the right-hand side we change t to $\alpha = t - T$ in (2)

$$\begin{aligned}(2) &= \int_{-T}^{0} (\mathcal{L}^\top e_-^0 \psi^1)(-\alpha) \cdot x(T+\alpha)\, d\alpha + \psi^0 \cdot x(T)\\ &= \int_{-h}^{0} (\bar{L}^\top \psi^1)(\alpha) \cdot (e_+^0 x)_T(\alpha)\, d\alpha + \psi^0 \cdot x(T)\\ &= \langle F^*\psi, (x(T), (e_+^0 x)_T)\rangle M^q \times M^p.\end{aligned}$$

By combining the above results for (1) and (2) we obtain (4.51).

(ii) We need the following lemma, which will be proved at the end. □

Lemma 4.4. *Let p, $1 \le p < \infty$, be a real number and ψ a function in $L^p(-h, 0; \mathbb{R}^n)$; then*

$$\forall t \ge 0, \quad \tau(t)(\bar{L}\psi) = \bar{L}(e_-^0 \psi)_t. \tag{4.60}$$

When $\xi = F\phi$, $\xi^1 = \bar{L}\phi^1$, from (4.60), $\tau(T)(\bar{L}\phi^1) = \bar{L}(e_-^0 \phi^1)_T$, and hence

$$\big(0, \tau(T)(\bar{L}\phi^1)\big) = F(0, (e_-^0 \phi^1)_T)$$

and

$$S^{\top *}(T)\xi = F(x(T), (e_+^0 x)_T + (e_-^0 \phi^1)_T) = F(x(T), x_T).$$

Proof. Again we prove (4.60) for functions ϕ in $C_c(-h, 0; \mathbb{R}^n)$ and extend it to $L^p(-h, 0; \mathbb{R}^n)$ by density and continuity. By assumptions $e_-^0 \phi \in C_c(-\infty, 0; \mathbb{R}^n)$ and for α in $I(-h, 0)$

$$\begin{aligned}[\tau(t)(\bar{L}\phi)](\alpha) &= [e_+^{-h}(\bar{L}\phi)](\alpha - t)\\ &= \begin{cases} L(e_-^0 \phi)_{t-\alpha}, & t - \alpha \le h,\\ 0, & t - \alpha > h.\end{cases}\end{aligned}$$

For $t - \alpha \le h$

$$L(e_-^0 \phi)_{t-\alpha} = \int_{-h}^{0} d_\theta\, \eta(e_-^0 \phi)(t - \alpha + \theta) = \int_{-h}^{\alpha} d_\theta\, \eta(e_-^0 \phi)_t(\theta - \alpha)$$

because

$$\alpha \leq \theta \leq 0 \Longrightarrow t < t - \alpha + \theta < t - \alpha \Longrightarrow (e_-^0 \phi)(t - \alpha + \theta) = 0.$$

Hence for $t - \alpha \leq h$

$$[\tau(t)(\bar{L}\phi)](\alpha) = [\bar{L}(e_-^0 \phi)_t](\alpha).$$

When $t - \alpha > h$

$$-h \leq \theta \leq \alpha \Longrightarrow 0 < t - \alpha - h \leq t - \alpha + \theta \Longrightarrow (e_-^0 \phi)(t - \alpha + \theta) = 0$$

and

$$[\bar{L}(e_-^0 \phi)_t](\alpha) = 0.$$

By regrouping the two cases we obtain (4.60). □

Proof of Theorem 4.5. (i) is a direct consequence of identity (4.54) in Theorem 4.4 (ii). The equivalence of (iii) and (i) is obtained by using the integral formulas between the resolvent and the semigroup. Finally (ii) is equivalent to (i) by the following lemma, which can be found in C. Bernier and A. Manitius [1]. □

Lemma 4.5. *Let Y be a Banach space and K a continuous linear transformation of Y. Given two strongly continuous semigroups $\{T_1(t)\}$ and $\{T_2(t)\}$ of bounded linear transformations of Y and their respective infinitesimal generators B_1 and B_2, the following statements are equivalent*:

(i) $T_2(t)K = KT_1(t)$, $t \geq 0$.
(ii) $KD(B_1) \subset D(B_2)$ *and* $B_2K = KB_1$ *on* $D(B_1)$.

4.6 Infinitesimal generators $A^{\top *}$ and A^*

Theorem 4.4 has provided a characterization of the adjoint semigroups $\{S^{\top *}(t)\}$ and $\{S^*(t)\}$, which has a more complex structure than $\{S^\top(t)\}$ and $\{S(t)\}$. So the next step is to characterize their infinitesimal generators $A^{\top *}$ and A^*. Of course they will inherit of the complexity of the adjoint semigroups. This question was initially studied by R. B. Vinter [2], but its importance was not fully appreciated at that time. This characterization was not easy to obtain because its final form was not known. It is one area where the structural operator has provided simplification both in the characterization and in the proof of the result. One source of difficulties was the search of an explicit definition of the infinitesimal generator of the adjoint semigroup. In the following theorem, we give an *implicit characterization or definition*, which turns out to be a more general and flexible result that applies to systems with finite or infinite memory.

Theorem 4.6. *Assume that the hypotheses of Theorem* 4.4 *are verified*:

(i) *The infinitesimal generator* $A^{\top *}$ *of* $\{S^{\top *}(t)\}$ *is characterized as follows*:

$$D(A^{\top *}) = \left\{ \xi \colon \begin{array}{c} \xi = F\phi + (0,\zeta), \phi \in D(A), \\ \zeta \in W^{1,p}(-h,0;\mathbb{R}^n), \zeta(-h) = 0 \end{array} \right\} \tag{4.61}$$

and the map

$$\xi = F\phi + (0,\zeta) \mapsto A^{\top *}\xi = FA\phi + (\zeta(0), -D\zeta) \colon D(A^{\top *}) \to M^p \tag{4.62}$$

is independent of the choice of the representation of ξ *in terms of* (ϕ,ζ).

(ii) *The infinitesimal generator* A^* *of* $\{S^*(t)\}$ *is characterized as follows*:

$$D(A^*) = \left\{ \xi \colon \begin{array}{c} \xi = F^*\phi + (0,\zeta), \ \phi \in D(A^\top), \\ \zeta \in W^{1,p}(-h,0;\mathbb{R}^n), \ \zeta(-h) = 0 \end{array} \right\} \tag{4.63}$$

and the map

$$\xi = F^*\phi + (0,\zeta) \mapsto A^*\zeta = F^*A^\top\phi + (\zeta(0), -D\zeta) \colon D(A^*) \to M^q \tag{4.64}$$

is independent of the choice of the representation of ξ *in terms of* (ϕ,ζ).

Remark 4.5. For h finite we can choose a representation for $D(A^{\top *})$ of the form

$$\xi = (\xi^0, \bar{L}\bar{\xi}^0 + \zeta), A^{\top *}\xi = (L\bar{\xi}^0 + \zeta(0), -D\zeta),$$

where $\bar{\xi}^0$ denotes the constant function equal to ξ^0 on $[-h,0]$. This choice can be found in the early papers on the topic but does not extend to $h = +\infty$. □

Proof. It is sufficient to prove (i). The proof of (ii) is identical up to a change in the superscript $^\top$. We have already established in Theorem 4.5 (ii) that

$$FD(A) \subset D(A^{\top *}), \quad A^{\top *}F\phi = FA\phi, \qquad \forall \phi \in D(A).$$

Hence for all ξ of the form $F\phi$, $\phi \in D(A)$,

$$A^{\top *}\xi = FA\phi = F(L\phi, D\phi) = (L\phi, \bar{L}D\phi).$$

Now we can always associate with an arbitrary ξ in $D(A^{\top *})$ an element $\phi^1 \in W^{1,p}(-h,0;\mathbb{R}^n)$ such that $\phi^1(0) = \xi^0$ and decompose ξ in two terms

$$\xi = (\xi^0, \xi^1) = (\xi^0, \bar{L}\phi^1) + (0, \xi^1 - \bar{L}\phi^1) = F\phi + (0,\zeta),$$

where by construction

$$(\xi^0, \bar{L}\phi^1) = (\phi^1(0), \bar{L}\phi^1) = F(\phi^1(0), \phi^1)$$

and $(\phi^1(0), \phi^1) \in D(A)$. In view of the previous remarks it is now sufficient to characterize the elements of $D(A^{\top *})$ of the form $(0,\zeta)$.

Given $(0, \zeta) \in D(A^{\top *})$, let x and y in $W^{1,p}_{\mathrm{loc}}(0, \infty; \mathbb{R}^n)$ be the solutions of the systems

$$\begin{cases} \dot{x}(t) = (\mathcal{L}e^0_+ x)(t) + (e^{-h}_+ \zeta)(-t), \\ x(0) = 0, \end{cases} \tag{4.65}$$

$$\begin{cases} \dot{y}(t) = (\mathcal{L}e^0_+ y)(t) + (e^{-h}_+ [A^{\top *}(0, \zeta)]^1)(-t), \\ y(0) = [A^{\top *}(0, \zeta)]^0. \end{cases} \tag{4.66}$$

Notice that

$$\forall T > 0, \quad e^0_+ x \in W^{1,p}(-h, T; \mathbb{R}^n).$$

Define

$$\hat{x}(t) = S^{\top *}(t)(0, \zeta), \quad \hat{y}(t) = S^{\top *}(t) A^{\top *}(0, \zeta).$$

By Theorem 4.4 (i)

$$\begin{cases} \hat{x}(t) = F(x(t), (e^0_+ x)_t) + (0, \tau(t)\zeta), \\ \hat{y}(t) = F(y(t), (e^0_+ y)_t) + (0, \tau(t)[A^{\top *}(0, \zeta)]^1) \end{cases} \tag{4.67}$$

and for $(0, \zeta) \in D(A^{\top *})$

$$\frac{d\hat{x}}{dt}(t) = \hat{y}(t), \quad \forall t \geq 0.$$

In particular

$$\frac{dx}{dt}(t) = y(t) \Longrightarrow x \in W^{2,p}_{\mathrm{loc}}(0, \infty; \mathbb{R}^n).$$

Define for $t \geq 0$

$$X(t) = \int_0^t y(s)\, ds, \quad t \geq 0 \quad \text{and} \quad X(t) = 0, \qquad t \in I(-h, 0)$$

and notice that

$$X = e^0_+ x, \quad \dot{X} = e^0_+ \dot{x} = e^0_+ y. \tag{4.68}$$

Integrate (4.66) from 0 to t

$$\dot{X}(t) = [A^{\top *}(0, \zeta)]^0 + \int_0^t (\mathcal{L}e^0_+ y)(s)\, ds + \int_0^t (e^{-h}_+ [A^{\top *}(0, \zeta)]^1)(-s)\, ds.$$

In view of Theorem 3.3 (iv) and (4.68)

$$\int_0^t (\mathcal{L}e^0_+ y)(s)\, ds = \int_0^t (\mathcal{L}\dot{X})(s)\, ds = LX_t - LX_0 = LX_t$$

and

$$\dot{X}(t) = LX_t + [A^{\top *}(0,\zeta)]^0 + \int_0^t (e_+^{-h}[A^{\top *}(0,\zeta)]^1)(-s)\, ds. \tag{4.69}$$

As $\dot{x}(t) = \dot{X}(t), t \geq 0$, the right-hand sides of (4.65) and (4.69) are equal and in view of the fact that $X = e_+^0 x$

$$(e_+^{-h}\zeta)(-t) = [A^{\top *}(0,\zeta)]^0 + \int_0^t (e_+^{-h}[A^{\top *}(0,\zeta)]^1)(-s)\, ds.$$

Introducing the new variable $\alpha = -t$ for $-t \in I(-h,0)$, we obtain

$$\zeta(\alpha) = [A^{\top *}(0,\zeta)]^0 + \int_\alpha^0 [A^{\top *}(0,\zeta)]^1(\theta)\, d\theta, \quad \alpha \in I(-h,0). \tag{4.70}$$

Hence

$$\begin{gathered} \zeta \in W^{1,p}(-h,0;\mathbb{R}^n), \\ \zeta(0) = [A^{\top *}(0,\zeta)]^0, \quad D_\alpha \zeta = -[A^{\top *}(0,\zeta)]^1. \end{gathered} \tag{4.71}$$

When $h = \infty$, we know by Lemma 3.5 that

$$\zeta \in W^{1,p}(-\infty,0;\mathbb{R}^n) \Longrightarrow \zeta \in C_0(-\infty,0;\mathbb{R}^n) \Longrightarrow \zeta(-\infty) = 0.$$

For h finite we go back to (4.65) and

$$\begin{gathered} \dot{x} = y \in W^{1,p}_{\text{loc}}(0,\infty;\mathbb{R}^n) \Longrightarrow \dot{x} \in C_{\text{loc}}(0,\infty;\mathbb{R}^n) \\ \forall T > 0, \quad e_+^0 x \in W^{1,p}(-h,T;\mathbb{R}^n) \Longrightarrow \mathcal{L}e_+^0 x \in C(0,T;\mathbb{R}^n). \end{gathered}$$

So

$$(e_+^{-h}\zeta)(-t) = \dot{x}(t) - (\mathcal{L}e_+^0 x)(t)$$

is continuous for $t \geq 0$ and in particular continuous at $t = h$. But

$$t > h \Longrightarrow (e_+^{-h}\zeta)(-t) = 0 \Longrightarrow \zeta(-h) = 0.$$

So we have established that

$$\zeta \in W^{1,p}(-h,0;\mathbb{R}^n), \quad \zeta(-h) = 0 \tag{4.72}$$

and that

$$A^{\top *}(0,\zeta) = (\zeta(0), -D_\alpha \zeta). \tag{4.73}$$

Conversely given ζ verifying (4.72) and x the solution of (4.65), we want to prove that $(0,\zeta) \in D(A^{\top *})$ or equivalently that for all $t \geq 0$

$$\frac{d\hat{x}}{dt}(t) \quad \text{exists in } M^p,$$

where $\hat{x}$ is given by (4.67). Specifically we shall show that

$$\frac{d\hat{x}}{dt}(t) = F\left(\dot{x}(t), \frac{d}{dt}(e_+^0 x)_t\right) + \left(0, \frac{d}{dt}\tau(t)\zeta\right) \tag{4.74}$$

and that

$$t \mapsto \left(\dot{x}(t), \frac{d}{dt}(e_+^0 x)_t, \frac{d}{dt}\tau(t)\zeta\right)$$

is continuous. By Theorem 4.1 (ii) $e_+^0 x \in W^{1,p}(-h, T; \mathbb{R}^n)$ for all $T > 0$ and by Lemma 4.1 (ii)

$$(e_+^0 x)_\cdot \in C^1\big(0, T; L^p(-h, 0; \mathbb{R}^n)\big)$$

and the function

$$t \mapsto \frac{d}{dt}(e_+^0 x)_t \colon [0, \infty[\to L^p(-h, 0; \mathbb{R}^n)$$

is continuous. By assumption (4.72) on ζ,

$$e_+^{-h}\zeta \in W^{1,p}(-\infty, 0; \mathbb{R}^n)$$

and because the function $\tau(t)\zeta$ is a shift of the function $e_+^{-h}\zeta$ the function

$$t \mapsto \tau(t)\zeta \colon [0 \infty[\to L^p(-h, 0; \mathbb{R}^n)$$

is C^1 and

$$t \mapsto \frac{d}{dt}\tau(t)\zeta = -\tau(t)\, D\zeta$$

is continuous. To show that $\dot{x}$ is continuous we prove that the right-hand side of (4.65) is continuous. The continuity of the term $(e_+^{-h}\zeta)(-t)$ for $t \geq 0$ follows from the fact that $e_+^{-h}\zeta$ belongs to $W^{1,p}(-\infty, 0; \mathbb{R}^n)$. Similarly by Lemma 4.1 (ii),

$$e_+^0 x \in W^{1,p}(-h, T; \mathbb{R}^n)$$

implies that

$$(e_+^0 x)_\cdot \in C\big(0, T; W^{1,p}(-h, 0; \mathbb{R}^n)\big)$$

and necessarily

$$t \mapsto (\mathcal{L} e_+^0 x)(t) = L(e_+^0 x)_t$$

is continuous. So by definition of $\hat{x}$ and continuity of F we obtain (4.74), the existence of $\frac{d\hat{x}}{dt}(t)$ as a continuous function of t and therefore $(0, \zeta) \in D(A^{\top *})$.

We have established that any element ξ of $D(A^{\top *})$ is of the form

$$\begin{gathered} \xi = F\phi + (0, \zeta), \\ \phi \in D(A), \\ \zeta \in W^{1,p}(-h, 0; \mathbb{R}^n), \quad \zeta(-h) = 0, \end{gathered} \tag{4.75}$$

and that

$$A^{\top *}\xi = FA\phi + (\zeta(0), -D\zeta) = (L\phi + \zeta(0), \bar{L}D\phi - D\zeta).$$

To complete the proof we have to show that $A^{\top *}\xi$ is independent of the representation of ξ in terms of ϕ and ζ. By linearity it is sufficient to establish that for all ϕ in $D(A)$ and ζ verifying (4.72)

$$\xi = F\phi + (0, \zeta) = 0 \Longrightarrow A^{\top *}\xi = FA\phi + (\zeta(0), -D\zeta) = 0$$

or for ζ, $\phi \in W^{1,p}(-h, 0; \mathbb{R}^n)$ such that $\zeta(-h) = 0$

$$\begin{cases} \phi(0) = 0 \\ \bar{L}\phi + \zeta = 0 \end{cases} \Longrightarrow \begin{cases} L\phi + \zeta(0) = 0, \\ \bar{L}D\phi - D\zeta = 0. \end{cases} \tag{4.76}$$

This will require the following lemma.

Lemma 4.6. *Let p, $1 \le p < \infty$, be a real number*:

(i) *For all ϕ in $W^{1,p}(-h, 0; \mathbb{R}^n)$, the map*

$$F\phi = (\phi(0), \bar{L}\phi) \mapsto L\phi \tag{4.77}$$

is well defined, linear, and there exists a constant $c > 0$ such that for all $\phi \in W^{1,p}(-h, 0; \mathbb{R}^n)$,

$$|L\phi| \le c[|\phi(0)| + \|\bar{L}\phi\|_p + \|\bar{L}D\phi\|_p]. \tag{4.78}$$

(ii) *For all ϕ in $W^{1,p}(-h, 0; \mathbb{R}^n)$ such that $\phi(0) = 0$*

$$\begin{gathered} \bar{L}\phi \in W^{1,p}(-h, 0; \mathbb{R}^n), \quad (\bar{L}\phi)(-h) = 0, \\ (\bar{L}\phi)(0) = L\phi, \quad D_\theta\, \bar{L}\phi = -\bar{L}D_\theta\, \phi. \end{gathered} \tag{4.79}$$

Going back to the proof of the theorem, ϕ satisfies the conditions of Lemma 4.6 (ii) and from (4.76)

$$\zeta = -\bar{L}\phi \Longrightarrow \begin{cases} \zeta(0) = -(\bar{L}\phi)(0) = -L\phi, \\ D_\theta\, \zeta = \bar{L}D_\theta\, \phi. \end{cases}$$

This completes the proof of the theorem. □

Proof of Lemma 4.6. (i) By linearity the map (4.77) is well defined if

$$(\phi(0), \bar{L}\phi) = 0 \Longrightarrow L\phi = 0.$$

But by Lemma 3.5, $W^{1,p}(-h, 0; \mathbb{R}^n) \subset K(-h, 0; \mathbb{R}^n)$ and by Lemma 4.3

$$\phi \in K(-h, 0; \mathbb{R}^n), \quad \phi(0) = 0 \Longrightarrow (\bar{L}\phi)(0) = L\phi$$

and necessarily $\bar{L}\phi = 0 \Longrightarrow L\phi = (\bar{L}\phi)(0) = 0$. For the continuity we associate with $\phi \in W^{1,p}(-h, 0; \mathbb{R}^n)$, the function

$$\tilde{\phi}(t) = \begin{cases} \phi(t), & t \in I(-h,0), \\ \phi(0), & t \geq 0. \end{cases}$$

For all $T > 0$, $\tilde{\phi} \in W^{1,p}(-h,T;\mathbb{R}^n)$ and

$$\dot{\tilde{\phi}} = D_t \tilde{\phi} = e^0_- D_\theta \, \phi.$$

Then by Theorem 3.3 (iv), for all $\alpha \in I(-h,0)$

$$\int_\alpha^0 (\bar{L} D_\theta \, \phi)(\theta) \, d\theta = \int_\alpha^0 (\mathcal{L} e^0_- D_\theta \, \phi)(-\theta) \, d\theta = L\tilde{\phi}_{-\alpha} - L\phi.$$

Let $\tau = h$ if h is finite and $\tau = 1$ if $h = \infty$. Then taking the L^p norm on $[0,\tau]$

$$\tau^{1/p} |L\phi| \leq c \|\bar{L} D_\theta \, \phi\|_{L^p(-h,0)} + \|\mathcal{L}\tilde{\phi}\|_{L^p(0,\tau)}.$$

But

$$\mathcal{L}\tilde{\phi} = \mathcal{L} e^0_+ \tilde{\phi} + \mathcal{L} e^0_- \tilde{\phi} = \mathcal{L} e^0_+ \phi(0) + \mathcal{L} e^0_- \phi$$

and

$$\|\mathcal{L}\tilde{\phi}\|_{L^p(0,\tau)} \leq c \|e^0_+ \phi(0)\|_{L^p(-h,\tau)} + \|\bar{L}\phi\|_{L^p(-h,0)} \leq c\tau^{1/p} |\phi(0)| + \|\bar{L}\phi\|_{L^p(-h,0)}.$$

Combining the last inequalities we get (4.78).

(ii) When $\phi(0) = 0$, $\tilde{\phi} = e^0_- \phi$ and for all α in $I(-h,0)$

$$(\bar{L}\phi)(\alpha) = (\mathcal{L} e^0_- \phi)(-\alpha) = L\tilde{\phi}_{-\alpha} = L\phi + \int_\alpha^0 (\bar{L} D_\theta \, \phi)(\theta) \, d\theta.$$

Hence $\bar{L}\phi \in W^{1,p}(-h,0;\mathbb{R}^n)$ and $D\bar{L}\phi = -\bar{L} \, D\phi$. Moreover by Lemma 3.5, $W^{1,p}(-h,0;\mathbb{R}^n) \subset K(-h,0;\mathbb{R}^n)$ and from Lemma 4.3 $\phi \in K(-h,0;\mathbb{R}^n)$ and $\phi(0) = 0$ imply that $(\bar{L}\phi)(0) = L\phi$ and $(\bar{L}\phi)(-h) = 0$. $\square$

4.7 The companion structural operator G of F

We have seen in Theorem 4.3 how the adjoint semigroups $\{S^{\top *}(t)\}$ and $\{S^*(t)\}$ are related to the solution of systems

$$\dot{x}(t) = (\mathcal{L} e^0_+ x)(t) + (e^{-h}_+ \xi^1)(-t), \quad t > 0, \; x(0) = \xi^0, \tag{4.80}$$

$$\dot{z}(t) = (\mathcal{L}^\top e^0_+ z)(t) + (e^{-h}_+ \zeta^1)(-t), \quad t > 0, \; z(0) = \zeta^0. \tag{4.81}$$

More precisely

$$S^{\top *}(t)\xi = F(x(t), (e^0_+ x)_t) + (0, \tau(t)\xi^1), \tag{4.82}$$

$$S^*(t)\zeta = F * (z(t), (e^0_+ z)_t) + (0, \tau(t)\zeta^1). \tag{4.83}$$

When h is finite and $t \geq h$, the second term in (4.82) and (4.83) is zero and by the intertwining theorem

$$\begin{aligned} S^{\top *}(t)\xi = S^{\top *}(t-h)S^{\top *}(h)\xi &= S^{\top *}(t-h)F(x(h), x_h) \\ &= FS(t-h)(x(h), x_h), \qquad (4.84) \\ S^*(t)\zeta = S^*(t-h)S^*(h)\zeta &= S^*(t-h)F^*(z(h), z_h) \\ &= F^* S^{\top}(t-h)(z(h), z_h). \qquad (4.85) \end{aligned}$$

For $t \geq h$ the structure of $S^{\top *}(t)$ and $S^*(t)$ considerably simplifies provided we have a knowledge of the pairs

$$G\xi = (x(h), x_h), \quad G^{\top}\zeta = (z(h), z_h), \tag{4.86}$$

which define continuous linear operators on M^p and M^q, respectively. They are related to F and F^* because for $\xi = F\phi$ or $\zeta = F^*\psi$, we obtain

$$\begin{cases} S(h)\phi = (x(h), x_h) = GF\phi, \\ S^{\top}(h)\psi = (z(h), z_h) = G^{\top}F^*\psi, \end{cases} \tag{4.87}$$

and

$$S(h) = GF, \quad S^{\top}(h) = G^{\top}F^*. \tag{4.88}$$

We shall see that $G^* = G^{\top}$ and that G and $G^{\top}$ are intertwining operators similar to F and F^*. They will play a complementary role to F and F^* in the characterization of the F-controllability and the F-observability for delay systems.

Definition 4.4. The operators G and $G^{\top}$ will be referred to as the *companion structural operators* of F and F^*, respectively. □

The operator G was introduced by A. Manitius [6]. It will not be used in this chapter, but it has played an important role in conjunction with the operator F especially for the characterization of the notions of controllability and observability.

Theorem 4.7. *Let h, $0 < h < \infty$, p, $1 \leq p < \infty$, and q, $1 \leq q < \infty$, be real numbers:*

(i) *The operators*

$$\xi \mapsto G\xi = (x(h), x_h) \colon M^p \to M^p, \tag{4.89}$$

$$\zeta \mapsto G^{\top}\zeta = (z(h), z_h) \colon M^q \to M^q \tag{4.90}$$

are linear and continuous and

$$GF = S(h), \quad G^{\top}F^* = S^{\top}(h). \tag{4.91}$$

(ii) *The operators G and $G^\top$ define isomorphisms*

$$G\colon M^p \to D(A), \quad G^\top : M^q \to D(A^\top) \tag{4.92}$$

when $D(A)$ and $D(A^T)$ are endowed with their respective graph norm topologies.

(iii) *For p, $1 < p < \infty$, and its conjugate q, $q^{-1} + p^{-1} = 1$,*

$$G^* = G^\top \tag{4.93}$$

when the elements of $(M^p)^*$ are identified with those of M^q.

Proof. (i) By construction and the remarks preceding the theorem.

(ii) We only prove this result for G. The map G is injective because $G\xi = 0$ implies that

$$(x(h), x_h) = 0 \Longrightarrow x(t) = 0, \quad 0 \le t \le h$$

and by substituting in (4.80)

$$\xi^1(-t) = 0, \quad 0 \le t \le h,\ \xi^0 = 0 \Longrightarrow \xi = 0.$$

It is surjective because for any ψ in $D(A)$

$$\psi^1 \in W^{1,p}(-h, 0; \mathbb{R}^n) \quad \text{and} \quad \psi^0 = \psi^1(0)$$

and we can define the function x in $W^{1,p}(0, h; \mathbb{R}^n)$

$$x(t) = \psi(t - h), \quad 0 \le t \le h$$

and the element $\phi = (\phi^0, \phi^1)$ of M^p

$$\phi^1(-t) = \dot{x}(t) - (\mathcal{L}e^0_+ x)(t), \quad 0 \le t \le h,\ \phi^0 = x(0).$$

By definition of G

$$G\phi = (x(h), x_h) = (\psi^1(0), \psi^1) = \psi.$$

The continuity of G from M^p to $D(A)$ follows from inequality (4.7) in Theorem 4.1 (i).

(iii) To establish (4.93) we use identity (4.35) in Lemma 4.2 with $T = h$

$$\int_0^h z(h-t)\cdot(\dot{x}-\mathcal{L}e^0_+ x)(t)\,dt + z(h)\cdot x(0) = \int_0^h (\dot{x}-\mathcal{L}e^0_+ z)(h-t)\cdot x(t)\,dt + z(0)\cdot x(h).$$

By substitution of (4.80) and (4.81)

$$\int_0^h z(h-t)\cdot(e^{-h}_+ \xi^1)(-t)\,dt + z(h)\cdot \xi^0 = \int_0^h (e^{-h}_+ \zeta^1)(-t)\cdot x(h-t)\,dt + \zeta^0 \cdot x(h)$$

and by changing the variable t to $\alpha = -t$

$$\int_{-h}^0 z_h(\alpha)\cdot \xi^1(\alpha)\,d\alpha + z(h)\cdot \xi^0 = \int_{-h}^0 \zeta^1(\alpha)\cdot x_h(\alpha)\,d\alpha + \zeta^1 \cdot x(h).$$

This completes the proof of the theorem. □

The last theorem is the intertwining theorem for $G, S^{\top *}$ and S.

Theorem 4.8 (Intertwining). *Let h, $0 < h < \infty$, p, $1 < p < \infty$, q, $p^{-1} + q^{-1} = 1$, be real numbers. Then the following properties hold and are equivalent*:

(i) $S(t)G = GS^{\top *}(t)$, $\forall t \geq 0$.
(ii) $GD(A^{\top *}) \subset D(A)$ *and* $GA^{\top *}\xi = AG\xi$, $\forall \xi \in D(A^{\top *})$.
(iii) *For all λ in the resolvent set of $A^{\top *}$*

$$R(\lambda, A)G\xi = GR(\lambda, A^{\top *})\xi, \quad \forall \xi \in M^p. \tag{4.94}$$

Proof. It is sufficient to establish (i). The equivalence of (i), (ii), and (iii) is a consequence of Lemma 4.4. Notice that for $t \geq 0$

$$(x(t+h), x_{t+h}) = S(t)(x(h), x_h) = S(t)G\xi$$

and

$$(z(t+h), z_{t+h}) = S^{\top}(t)(z(h), z_h) = S^{\top}(t)G^{\top}\zeta.$$

Now substitute in identity (4.35) in Lemma 4.2 the solutions x and z of (4.80) and (4.81) for $T = t + h$:

$$\int_0^{t+h} z(t+h-s) \cdot (e_+^{-h}\xi^1)(-s)ds + z(t+h) \cdot x(0)$$
$$= \int_0^{t+h} (e_+^{-h}\zeta^1)(-s) \cdot x(t+h-s)\, ds + z(0) \cdot x(t+h).$$

Notice that the integrands are zero for $s > h$ and change the variable s to $\alpha = -s$:

$$\int_{-h}^{0} z(t+h+\alpha) \cdot \xi^1(\alpha)\, d\alpha + z(t+h) \cdot \xi^0$$
$$= \int_{-h}^{0} \zeta^1(\alpha) \cdot x(t+h+\alpha)\, d\alpha + \zeta^0 \cdot x(t+h).$$

Therefore

$$\langle S^{\top}(t)G^{\top}\zeta, \xi\rangle = \langle \zeta, S(t)G\xi\rangle$$

and (i) is verified. □

Corollary 4.3. *Assume that the hypotheses of Theorem* 4.8 *are verified. Then the following properties are verified and are equivalent*:

(i) $S^{\top}(t)G^{\top} = G^{\top}S^*(t)$, $\quad t \geq 0$.
(ii) $G^{\top}D(A^*) \subset D(A^{\top})$ *and* $G^{\top}A^*\xi = A^{\top}G^{\top}\xi$, $\forall \xi \in D(A^*)$.
(iii) *For all λ in the resolvent set of A^**

$$R(\lambda, A^{\top})G^{\top}\xi = G^{\top}R(\lambda, A^*)\xi, \quad \forall \xi \in M^q. \tag{4.95}$$

Remark 4.6. Both G and $G^\top$ have continuous linear inverse $G^{-1}\colon D(A) \to M^p$ and $(G^\top)^{-1}\colon D(A^\top) \to M^q$. So in view of Theorem 4.8 and its corollary

$$S^{\top\star}(t) = G^{-1}S(t)G, \quad S^\star(t) = (G^\top)^{-1}S^\top(t)G^\top$$

and the adjoint semigroups are completely characterized in terms of $\{S(t)\}$ and $\{S^\top(t)\}$. □

5 State space theory of linear control systems

The basic concepts and techniques to deal with linear delay systems have been introduced in §3 and §4. In this section a control vector $u(t) \in \mathbb{R}^m$ ($m \geq 1$, an integer) is introduced in the system dynamics through a delay operator B, which is a continuous linear map

$$B\colon K(-h, 0; \mathbb{R}^m) \to \mathbb{R}^n \tag{5.1}$$

similar to the continuous linear map

$$L\colon K(-h, 0; \mathbb{R}^n) \to \mathbb{R}^n \tag{5.2}$$

of §3 and §4.

Consider the linear control system

$$\begin{cases} \dot{x}(t) = Lx_t + Bu_t + f(t), \quad t > 0, \\ (x(0), x_0, u_0) = (\phi^0, \phi^1, w) \in M^p \times L^p(-h, 0; \mathbb{R}^m) \end{cases} \tag{5.3}$$

for the real number p, $1 \leq p < \infty$, and $f \in L^p_{\text{loc}}(0, \infty; \mathbb{R}^n)$. The objective is to construct a state and give a state space formulation of equations (5.3) where delays in both variables x and u disappear. The motivation behind this objective is to put (5.3) in the form of an evolution equation in an infinite dimensional Banach space without delays to bring such systems in line with the standard Control Theory of infinite dimensional evolution systems.

The two constructions that will be considered here were originally introduced by R. B. Vinter and R. H. Kwong [1] and A. Ichikawa [3]. In the light of §3.4 they correspond to the states

$$\hat{x}(t) = F(x(t), x_t) \tag{5.4}$$

and

$$\tilde{x}(t) = (x(t), x_t) \tag{5.5}$$

as defined in (4.51) (Theorem 4.4 and (4.12)). They will require the introduction of new semigroups that are extensions or generalizations of the semigroups $\{S^{\top *}(t)\}$ and $\{S(t)\}$ of §3.4, and the analogs for B of the structural operators $\bar{L}$ and $\bar{L}^\top$ for L.

All results established for L (Theorem 3.3, Lemmas 4.2, 4.3, 4.4, 4.5, and 4.6) extend to B. So they will not be repeated here. We simply introduce the appropriate notation associated with B. There exists an $n \times m$ matrix β of regular Borel measures such that

$$Bw = \int_{-h}^{0} d_\theta \, \beta w(\theta), \quad \forall w \in K\,(-h, 0; \mathbb{R}^m). \tag{5.6}$$

For each $T > 0$, the continuous linear map

$$\begin{cases} \mathcal{B} \colon L^p(-h, T; \mathbb{R}^m) \to L^p(0, T; \mathbb{R}^n), \\ (\mathcal{B}v)(t) = Bv_t, \quad \forall t \geq 0, \ \forall v \in C_c(-h, T; \mathbb{R}^m) \end{cases} \tag{5.7}$$

has the same properties as $\mathcal{L}$. Denote by $\beta^\top$ the transposed of the matrix β and introduce the continuous linear map

$$\phi \mapsto B^\top \phi = \int_{-h}^{0} d_\theta \, \beta^\top \phi(\theta) \colon K(-h, 0; \mathbb{R}^n) \to \mathbb{R}^m. \tag{5.8}$$

For q, $1 \leq q < \infty$, and all $T > 0$, we also have the continuous linear map

$$\begin{cases} \mathcal{B}^\top \colon L^q(-h, T; \mathbb{R}^n) \to L^q(0, T; \mathbb{R}^m), \\ (\mathcal{B}^\top z)(t) = B^\top z_t, \quad \forall t \geq 0, \ \forall z \in C_c(-h, T; \mathbb{R}^n). \end{cases} \tag{5.9}$$

Finally we associate with B and $B^\top$ the structural operators

$$\begin{cases} \bar{B} \colon L^p(-h, 0; \mathbb{R}^m) \to L^p(-h, 0; \mathbb{R}^n), \\ (\bar{B}w)(\alpha) = (\mathcal{B}e^0_- w)(-\alpha), \quad \alpha \in I(-h, 0), \end{cases} \tag{5.10}$$

$$\begin{cases} \bar{B}^\top \colon L^q(-h, 0; \mathbb{R}^n) \to L^q(-h, 0; \mathbb{R}^m), \\ (\bar{B}^\top \psi)(\alpha) = (\mathcal{B}^\top e^0_- \psi)(-\alpha), \quad \alpha \in I(-h, 0). \end{cases} \tag{5.11}$$

When $1 < p < \infty$ and q is the conjugate of p, $p^{-1} + q^{-1} = 1$

$$\bar{B}^* = \bar{B}^\top, \tag{5.12}$$

where the elements of $(L^p)'$ are identified with those of L^q.

5.1 The structural state

The first step is to rewrite system (5.3) in the more accurate form

$$\begin{gathered} \dot{x} = \mathcal{L}x + \mathcal{B}u + f, \\ (x(0), x_0, u_0) = (\phi^0, \phi^1, w) \in M^p \times L^p(-h, 0; \mathbb{R}^m). \end{gathered} \tag{5.13}$$

Then separate the solution $x(t)$, $t \geq 0$ and the control $u(t)$, $t \geq 0$, from the pieces of initial functions ϕ^1 and w:

$$\begin{gathered} \dot{x} = \mathcal{L}e_+^0 x + \mathcal{B}e_+^0 u + \mathcal{L}e_-^0 \phi^1 + \mathcal{B}e_-^0 w + f, \\ x(0) = \phi^0. \end{gathered} \tag{5.14}$$

As in §4.4 notice that system (5.14) does not directly use the initial functions ϕ^1 and w but only the sum of their images $\mathcal{L}e_-^0\phi^1 + \mathcal{B}e_-^0 w$. When h is finite

$$(\mathcal{L}e_-^0\phi^1)(t) + (\mathcal{B}e_-^0 w)(t) = 0, \quad t > h \tag{5.15}$$

and it is sufficient to consider the effect of the initial functions on the interval $[0, h]$. We recognize the structural operators $\bar{L}$ and $\bar{B}$ extended by 0 for $t > h$:

$$(\mathcal{L}e_-^0\phi^1)(t) + (\mathcal{B}e_-^0 w)(t) = \big(e_+^{-h}(\bar{L}\phi^1 + \bar{B}w)\big)(-t), \quad t \geq 0. \tag{5.16}$$

So the true initial condition to system (5.13) is the pair

$$(\phi^0, \bar{L}\phi^1 + \bar{B}w) = F\phi + (0, \bar{B}w) \tag{5.17}$$

and this suggests the introduction of the following state:

$$\hat{x}(t) = (x(t), \bar{L}x_t + \bar{B}u_t) = F(x(t), x_t) + (0, \bar{B}u_t) \tag{5.18}$$

at time $t \geq 0$. Compare (5.18) to (4.51) in Theorem 4.3 or to the intertwining identity in Theorem 4.4 between $\{S^{\top *}(t)\}$ and $\{S(t)\}$. It is clear that the state (5.18) is related to a nonhomogeneous version of the evolution equation associated with the semigroup $\{S^{\top *}(t)\}$. So we are led to the embedding of the initial system (5.14) into the following larger family of systems:

$$\begin{cases} \dot{x}(t) = (\mathcal{L}e_+^0 x)(t) + (\mathcal{B}e_+^0 u)(t) + (e_+^{-h}\xi^1)(-t) + f(t), \quad t > 0, \\ x(0) = \xi^0, \quad \xi = (\xi^0, \xi^1) \in M^p, \\ f \in L^p_{\text{loc}}(0, \infty; \mathbb{R}^n), \quad u \in L^p_{\text{loc}}(0, \infty; \mathbb{R}^m). \end{cases} \tag{5.19}$$

Definition 5.1. The *structural state* $\hat{x}(t)$ at time $t \geq 0$ is defined by

$$\hat{x}(t) = (x(t), \bar{L}(e_+^0 x)_t + \bar{B}(e_+^0 u)_t + \tau(t)\xi^1), \tag{5.20}$$

where $\tau(t)$ is the right translation operator defined in (4.53) of Theorem 4.3

$$[\tau(t)u](\theta) = (e_+^{-h}u)(\theta - t), \quad \theta \in I(-h, 0) \tag{5.21}$$

for any arbitrary function u defined on $I(-h, 0)$. If $\xi^1 = \bar{L}\phi^1 + \bar{B}w$, then the definition (5.20) reduces to (5.18). □

Other terminologies have been used: the *Vinter-Kwong state* in M. C. Delfour [14] or the *forcing function state* in G. Grippenberg, S. O. Londen, and O. Staffans [1].

It turns out that the state $\hat{x}(t)$ is the solution of a nonhomogeneous differential equation that need to be interpreted in an appropriate weak sense. In the process all delays will disappear and the equation will take the same form as its finite dimensional analog.

Notation 5.1. The restriction of the continuous linear operator

$$B^\top : K(-h,0;\mathbb{R}^n) \to \mathbb{R}^m$$

to $D(A^\top)$ is well defined because $W^{1,q}(-h,0;\mathbb{R}^n)$ is continuously embedded into $K(-h,0;\mathbb{R}^n)$ (cf. Lemma 3.5. Its restriction to $D(A^\top)$

$$\psi = (\psi(0),\psi) \mapsto B^\top\psi \colon D(A^\top) \to \mathbb{R}^m \tag{5.22}$$

will also be denoted $B^\top$. □

Theorem 5.1. *Let the real number p, $1 < p < \infty$, and its conjugate q, $q^{-1} + p^{-1} = 1$, be given. Assume that x in $W^{1,p}_{\mathrm{loc}}(0,\infty;\mathbb{R}^n)$ is the solution of system* (5.19) *for $\xi \in M^p$, $f \in L^p_{\mathrm{loc}}(0,\infty;\mathbb{R}^n)$ and $u \in L^p_{\mathrm{loc}}(0,\infty;\mathbb{R}^m)$, and let $\hat{x}(t)$ be the structural state constructed from x in* (5.20)–(5.21):

(i) *For all ψ in $D(A^\top)$ and $t \geq 0$*

$$\begin{aligned}\langle \psi, \hat{x}(t)\rangle = \langle S^\top(t)\psi,\xi\rangle + \int_0^t &[B^\top S^\top(t-r)\psi \cdot u(r) \\ &+ \langle S^\top(t-s)\psi,(f(s),0)\rangle]\,dr.\end{aligned} \tag{5.23}$$

(ii) *The structural state $\hat{x}(t)$ is a solution of the weak equation: For all $t > 0$ and all $\psi \in D(A^\top)$*

$$\begin{cases} \dfrac{d}{dt}\langle \psi, \hat{x}(t)\rangle = \langle A^\top\psi, \hat{x}(t)\rangle + B^\top\psi \cdot u(t) + \psi(0)\cdot f(t), \\ \hat{x}(0) = \xi. \end{cases} \tag{5.24}$$

(iii) *For each $T > 0$, the state $\hat{x}$ is the unique solution in*

$$\begin{aligned}&\mathcal{V}(0,T;M^p,D(A^\top)') \\ &\qquad = \left\{ z \in C(0,T;M^p) \colon \frac{d}{dt} j^* z \in L^p(0,T;D(A^\top)') \right\}\end{aligned} \tag{5.25}$$

to the following equation:

$$\begin{cases} \dfrac{d}{dt} j^*\hat{x}(t) = (A^\top)^*\hat{x}(t) + (B^\top)^* u(t) + j^*(f(t),0), \quad t > 0, \\ \hat{x}(0) = \xi, \end{cases} \tag{5.26}$$

where $j^*, (A^\top)^*$, and $(B^\top)^*$ are the dual maps of the continuous linear operators

$$j \colon D(A^\top) \to M^q,\ A^\top \colon D(A^\top) \to M^q,\ \text{and } B^\top \colon D(A^\top) \to \mathbb{R}^m \tag{5.27}$$

defined on $D(A^\top)$ (cf. Notation 4.2, *Definition* 4.2, *Notation* 5.1).

Remark 5.1. In their original work, R. B. VINTER and R. H. KWONG [1] considered a delay operator B of the form

$$Bw = B_0 w(0) + \int_{-h}^{0} B_1(\theta) w(\theta)\, d\theta \tag{5.28}$$

for $n \times m$ matrices B_0 and $B_1(\theta)$, where the elements of B_1 belong to $L^p(-h,0;\mathbb{R})$ and a finite memory h. In this case

$$B^\top \psi = B_0^\top \psi(0) + \int_{-h}^{0} B_1(\theta)^\top \psi(\theta)\, d\theta \tag{5.29}$$

and both B and $B^\top$ can be considered as linear maps on $\mathbb{R}^m \times L^p(-h,0;\mathbb{R}^m)$ and $M^q = \mathbb{R}^n \times L^q(-h,0;\mathbb{R}^n)$

$$\begin{cases} B(w_0,w_1) = B_0 w_0 + \displaystyle\int_{-h}^{0} B_1(\theta) w_1(\theta)\, d\theta, \\ B^\top(\psi^0,\psi^1) = B_0^\top \psi^0 + \displaystyle\int_{-h}^{0} B_1^\top(\theta)\psi^1(\theta)\, d\theta. \end{cases} \tag{5.30}$$

This of course is the limit case where everything stays in the state space M^q and equation (5.23) reduces to

$$\hat{x}(t) = S^{\top *}(t)^* \xi + \int_0^t S^{\top *}(t-r)[(B^\top)^* u(r) + (f(r),0)]\, dr \tag{5.31}$$

because $B^{\top *} : \mathbb{R}^m \to M^p$ is now linear and continuous. The weak formulations (5.23), (5.24), and (5.26) were introduced by M. C. DELFOUR [15] to study the corresponding Riccati equation. They were absolutely necessary to handle control operators with pure delays. □

Remark 5.2. Notice that the underlying semigroup is the adjoint of the transposed semigroup associated with L. A careful interpretation of the proof of Theorem 5.1 indicates that the structural state is really the resulting product of transposition techniques applied to the differential equation (5.19). The elements entering into the structural state are obtained by isolating on one side the transposed semigroup associated with the transposed system. □

Proof of Theorem 5.1. In view of the linearity of system (5.19) and of the definition of the semigroup $\{S^{\top *}(t)\}$ in Theorem 4.3, it is sufficient to establish identity (5.23) for $\xi = 0$. Again the key tool is the integration by part formula (4.35) in Lemma 4.2 for $T = t$

$$\begin{aligned} \int_0^t z(t-s)\cdot[\dot{x} - \mathcal{L} e_+^0 x](s)\, ds + z(t)\cdot x(0) \\ = \int_0^t [\dot{z} - \mathcal{L}^\top e_+^0 z](s)\cdot x(t-s)\, ds + z(0)\cdot x(t), \end{aligned} \tag{5.32}$$

where x is the solution of

$$\dot{x} = \mathcal{L}e_+^0 x + \mathcal{B}e_+^0 u + f, \quad x(0) = 0 \tag{5.33}$$

and z is the solution of

$$\dot{z} = \mathcal{L}^\top e_+^0 z + \mathcal{L}^\top e_-^0 \psi, \quad z(0) = \psi(0), \quad \psi \in D(A^\top). \tag{5.34}$$

By techniques analogous to the ones used in the proof of Lemma 4.2 we can also show that for u in $L^p_{\text{loc}}(0,\infty;\mathbb{R}^m)$

$$\int_0^t z(t-s) \cdot (\mathcal{B}e_+^0 u)(s)\, ds = \int_0^t (\mathcal{B}^\top e_+^0 z)(t-s) \cdot u(s)\, ds. \tag{5.35}$$

Substract (5.35) from (5.32) and use (5.33) and (5.34):

$$\begin{aligned}
&\int_0^t z(t-s) \cdot f(s)\, ds \\
&\quad = \int_0^t [(\mathcal{L}^\top e_-^0 \psi)(s) \cdot x(t-s) - (\mathcal{B}^\top e_+^0 z)(t-s) \cdot u(s)]\, ds + \psi(0) \cdot x(t)
\end{aligned}$$

and the right-hand side becomes

$$\begin{aligned}
&\int_0^t [(\mathcal{L}^\top e_-^0 \psi)(s) \cdot x(t-s) + (\mathcal{B}^\top e_-^0 \psi)(s) \cdot u(t-s)]\, ds + \psi(0) \cdot x(t) \\
&\quad = \int_0^t [z(t-s) \cdot f(s) + (\mathcal{B}^\top e_+^0 z)(t-s) \cdot u(s) + (\mathcal{B}^\top e_-^0 \psi)(s) \cdot u(t-s)]\, ds.
\end{aligned}$$

But this can be rewritten as

$$\begin{aligned}
&\int_{-h}^0 [(\bar{L}^\top \psi)(\alpha) \cdot (e_+^0 x)_t(\alpha) + (\bar{B}^\top \psi)(\alpha) \cdot (e_+^0 u)_t(\alpha)]\, d\alpha \\
&\qquad\qquad + \psi(0) \cdot x(t) = \int_0^t z(t-s) \cdot f(s) + (\mathcal{B}^\top z)(t-s) \cdot u(s)\, ds,
\end{aligned}$$

and as

$$S^\top(r)\psi = (z(r), z_r) \in D(A^\top),$$

we have

$$(\mathcal{B}^\top z)(t-s) = B^\top z_{t-s} = B^\top S^\top(t-s)\psi,$$

when $B^\top$ is viewed as a continuous linear operator from $D(A^\top)$ into $\mathbb{R}^m$. So finally as $\bar{L}^\top = \bar{L}^*$, $(\bar{L}^\top)^* = \bar{L}$ and $\bar{B}^\top = \bar{B}^*$ and $(\bar{B}^\top)^* = \bar{B}$

$$\begin{aligned}
&\langle(\psi(0), \psi, (x(t), \bar{L}(e_+^0 x)_t + \bar{B}(e_+^0 u)_t)\rangle \\
&\qquad = \int_0^t \left[\left\langle S^\top(t-s)\psi, \big(0, f(s)\big)\right\rangle + \bar{B}^\top S^\top(t-s)\psi \cdot u(s)\right] ds.
\end{aligned}$$

The left-hand side of the above expression is precisely $\langle \psi, \hat{x}(t) \rangle$ for $\xi = 0$ and this completes the proof of part (i).

(ii) The differential equation (5.24) is obtained by differentiating (5.23) using the fact that for all $T > 0$

$$z(\cdot) = S^\top(\cdot)\psi \in C^0\big(0, T; D(A^T)\big) \cap C^1(0, T; M^q).$$

For instance if $\varphi \in \mathcal{D}(]0, T[$, $T > 0$, the distributional derivative of the term

$$F(t) = \int_0^t z(t-s) \cdot f(s)\, ds$$

is given by the expression

$$\begin{aligned}
-\int_0^T \frac{\partial \varphi}{\partial t}(t) \int_0^t z(t-s) \cdot f(s)\, ds\, dt \\
&= -\int_0^T ds f(s) \cdot \int_s^T dt \frac{\partial \varphi}{\partial t}(t) z(t-s) \\
&= \int_0^T ds f(s) \cdot \left[\int_s^T dt \varphi(t) \frac{dz}{dt}(t-s) + \varphi(s) z(0)\right] \\
&= \int_0^T \varphi(t) \left[\int_0^t ds \frac{dz}{dt}(t-s) \cdot f(s) + z(0) \cdot f(t)\right] dt
\end{aligned}$$

and

$$\begin{aligned}
\frac{dF}{dt}(t) = \int_0^t \frac{dz}{dt}(t-s) \cdot f(s)\, ds + z(0) \cdot f(t) \\
&= \int_0^t \langle S^\top(t-s) A^\top \psi, (f(s), 0) \rangle\, ds + \psi(0) \cdot f(t).
\end{aligned}$$

We repeat the above computation for the term in u but with $\psi \in D\big((A^\top)^2\big)$, and the sum of the derivatives of the right-hand side of (5.23) for ψ in $D\big((A^\top)^2\big)$ is equal to

$$\begin{aligned}
\langle S^\top(t) A^\top \psi, \xi \rangle + \int_0^t \langle S^\top(t-s) A^\top \psi, (f(s), 0) \rangle\, ds \\
+ \psi(0) \cdot f(t) + \int_0^t B^\top S^\top(t-s) A^\top \psi \cdot u(s)\, ds + B^\top \psi \cdot u(t) \\
= \langle A^\top \psi, \hat{x}(t) \rangle + B^\top \psi \cdot u(t) + \psi(0) \cdot f(t).
\end{aligned}$$

So we obtain (5.24) for $\psi \in D\big((A^\top)^2\big)$ and this identity extends by linearity, continuity, and density of $D\big((A^\top)^2\big)$ in $D(A^\top)$ to all elements ψ of $D(A^\top)$.

(iii) From (5.23) $\hat{x} \in C(0, T; M^p)$ and from (5.24) we obtain expression (5.26) and $\frac{d}{dt} j^* z \in L^p(0, T; D(A^\top)')$. The last element to complete the proof is the uniqueness. By linearity it is sufficient to prove that for $\xi = 0$, $f = 0$, and $u = 0$ the solution $\hat{x}$ in $\mathcal{V}(0,T; M^p, D(A^\top)')$ to

$$\frac{d}{dt} j^* \hat{x}(t) = (A^\top)^* \hat{x}(t), \quad \hat{x}(0) = 0,$$

is zero. Fix t', $0 < t' \le T$, and construct the function

$$g(t) = \langle S^\top(t'-t)\psi, \hat{x}(t)\rangle, \quad 0 \le t \le t',$$

for $\psi \in D(A^\top)$. Then $S^\top(\cdot)\psi \in C^1(0,t'; M^q) \cap C\big(0,t'; D(A^\top)\big)$ and hence $g \in W^{1,p}(0,t'; \mathbb{R})$. Moreover $g(0) = 0$ and

$$\begin{aligned}\frac{dg}{dt}(t) = &-\langle A^\top S^\top(t'-t)\psi, \hat{x}(t)\rangle_{M^q \times M^p} \\ &+ \langle S^\top(t'-t)\psi, (A^\top)^* \hat{x}(t)\rangle_{D(A^\top)\times D(A^\top)'} = 0.\end{aligned}$$

Then

$$0 = g(0) = g(t') = \langle \psi, \hat{x}(t')\rangle$$

and because this is true for all ψ in $D(A^\top)$, $\hat{x}(t') = 0$, $\forall t' > 0$. This completes the proof of the theorem. □

5.2 The extended state

In this section we come back to system (5.13) and extend the definition of the first state $\tilde{x}(t) = (x(t), x_t)$ introduced in §4.2 to

$$\begin{gathered}\tilde{x}(t) = (x(t), x_t, u_t) \in Z^p, \\ Z^p \stackrel{\text{def}}{=} \mathbb{R}^n \times L^p(-h,0;\mathbb{R}^n) \times L^p(-h,0;\mathbb{R}^m).\end{gathered} \tag{5.36}$$

So we increase the size of the state space by adding a third component. The following presentation is new and extends to general delay operators the original work of A. Ichikawa [1, 3].

Terminology 1. The *extended state* associated with L and B will be defined by (5.36) and denoted by $\tilde{x}(t)$.

5.2.1 The extended semigroup $\{\tilde{S}(t)\}$

Because of this added feature we have to somehow extend the definition of the semigroup $\{S(t)\}$. For this purpose we introduce the homogeneous system

$$\begin{cases}\dot{x}(t) = (\mathcal{L}e^0_+ x)(t) + (\mathcal{L}e^0_- \phi^1)(t) + (\mathcal{B}e^0_- w)(t), \\ x(0) = \phi^0, \quad (\phi^0, \phi^1, w) \in Z^p\end{cases} \tag{5.37}$$

and the following linear transformations $\{\tilde{S}(t) \colon t \ge 0\}$ of Z^p:

$$\tilde{S}(t)(\phi^0, \phi^1, w) = (x(t), x_t, (e^0_- w)_t), \quad t \ge 0. \tag{5.38}$$

Terminology 2. The semigroup $\tilde{S}(t)$ will be called the *extended semigroup* associated with L and B.

This terminology is a little bit more descriptive than the *Ichikawa semigroup* used in M. C. Delfour [14]. We shall use in § 6 the extended semigroup associated with $L^\top$ and $C^\top$, where C is an appropriate delay observation functional. This is a natural extension of the transposed semigroup of §4.3.

Theorem 5.2. *Let p, $1 \le p < \infty$, be a real number, let x be the solution of* (5.37) *in $W^{1,p}_{\text{loc}}(0,\infty;\mathbb{R}^n)$ corresponding to $(\phi^0,\phi^1,w) \in Z^p$, and let $\tilde{S}(t)(\phi^0,\phi^1,w)$ be given by* (5.38):

(i) *The family $\{\tilde{S}(t) \colon t \ge 0\}$ forms a strongly continuous semigroup of continuous linear transformations of Z^p.*

(ii) *The infinitesimal generator $\tilde{A}$ of $\{\tilde{S}(t)\}$ is given by*

$$\tilde{A}(\phi^0,\phi^1,w) = (L\phi^1 + Bw, D\phi^1, Dw), \tag{5.39}$$

where

$$D(\tilde{A}) = \left\{ (\phi^0,\phi^1,w) \in Z^p \colon \begin{array}{c} (\phi^0,\phi^1) \in D(A) \\ w \in W^{1,p}(-h,0;\mathbb{R}^m) \\ w(0) = 0 \end{array} \right\}. \tag{5.40}$$

(iii) *For p, $1 < p < \infty$, and its conjugate q, $q^{-1} + p^{-1} = 1$, the adjoint semigroup $\{\tilde{S}^*(t)\}$ is characterized by*

$$\tilde{S}^*(t)(\xi^0,\xi^1,\xi^2) = (z(t), \bar{L}^\top (e^0_+ z)_t + \tau(t)\xi^1, \bar{B}^\top (e^0_+ z)_t + \tau(t)\xi^2), \tag{5.41}$$

where z is the solution of system

$$\begin{cases} \dot{z}(t) = (\mathcal{L}^\top e^0_+ z)(t) + (e^{-h}_+ \xi^1)(-t), \\ z(0) = \xi^0. \end{cases} \tag{5.42}$$

Its infinitesimal generator is given by the map

$$\begin{aligned} &\xi = (\psi(0), \bar{L}^\top \psi + \zeta, \bar{B}^\top \psi + \lambda) \\ &\qquad \mapsto \tilde{A}^*\xi = (L^\top \psi + \zeta(0), \bar{L}^\top D\psi - D\zeta, \bar{B}^\top D\psi - D\lambda), \end{aligned} \tag{5.43}$$

which is independent of the choice of the representation of the element ξ in the domain of A^*,

$$D(\tilde{A}^*) = \left\{ \xi \colon \begin{array}{l} \xi = (\psi(0),\ \bar{L}^\top \psi + \zeta,\ \bar{B}^\top \psi + \lambda) \\ (\psi(0), \psi) \in D(A^\top) \\ \zeta \in W^{1,q}(-h,0;\mathbb{R}^n),\ \zeta(-h) = 0 \\ \lambda \in W^{1,q}(-h,0;\mathbb{R}^m),\ \lambda(-h) = 0 \end{array} \right\} \tag{5.44}$$

in terms of (ψ,ζ,λ).

Proof. (i) The state $\tilde{x}(t) = \tilde{S}(t)(\phi^0, \phi^1, w)$ is well defined because

$$(\mathcal{L}e^0_- \phi^1)(t) = (e^{-h}_+ \bar{L}\phi^1)(-t), (\mathcal{B}e^0_- w)(t) = (e^{-h}_+ \bar{B}w)(-t),$$

and by Theorem 4.1 (i) there exists a constant $c(T) > 0$ such that for all $T > 0$

$$\|x\|_{W^{1,p}(0,T;\mathbb{R}^n)} \leq C(T)[|\phi^0| + \|\bar{L}\phi^2\|_{L^p} + \|\bar{B}w\|_{L^p}] \tag{5.45}$$

and $t \mapsto (e^0_- w)_t$ is the shift operator for the function $e^0_- w$. It is linear and continuous by inequality (5.45). The semigroup property is obvious and the continuity with respect to t follows by continuity of x for $t \geq 0$ and the continuity of the shift operator $t \mapsto x_t$, $t \mapsto (e^0_- w)_t$ for L^p-functions.

(ii) Assume that $(\phi^0, \phi^1, w) \in D(\tilde{A})$. Then for all $T > 0$

$$\tilde{x} \in C^1(0,T; M^p) \cap C\big(0,T; D(\tilde{A})\big).$$

In particular

$$x \in C^1(0,T;\mathbb{R}^n), \quad x_\bullet \in C^1\big(0,T; L^p(-h,0;\mathbb{R}^n)\big),$$
$$(e^0_- w)_\bullet \in L^p(-h,0;\mathbb{R}^m).$$

By Lemma 4.1 (ii)

$$x \in W^{1,p}(-h,T;\mathbb{R}^n), \quad (e^0_- w) \in W^{1,p}(-h,T;\mathbb{R}^m) \tag{5.46}$$

and

$$D_t u_t = D_\theta\, u_t, \quad D_t (e^0_- w)_t = D_\theta\, (e^0_- w)_t. \tag{5.47}$$

As a result

$$\phi^1 = x|_{I(-h,0)} \in W^{1,p}(-h,0;\mathbb{R}^n),$$
$$\phi^1(0) = x(0) = \phi^0 \Longrightarrow (\phi^0, \phi^1) \in D(A^\top).$$

Therefore we have established that any $(\phi^0, \phi^1, w) \in D(\tilde{A})$ verifies

$$(\phi^0, \phi^1) \in D(A^\top), \quad w \in W^{1,p}(-h,0;\mathbb{R}^m), \quad w(0) = 0. \tag{5.48}$$

Finally because (5.46) is true, by Lemma 4.2 (ii)

$$x_\bullet \in C\big(0,T; W^{1,p}(-h,0;\mathbb{R}^n)\big), \quad (e^0_- w)_\bullet \in C\big(0,T; W^{1,p}(-h,0;\mathbb{R}^m)\big)$$

and

$$\frac{dx}{dt}(t) = Lx_t + B(e^0_- w)_t$$

has a continuous right-hand side. The same is true of the right-hand sides in (5.47) and

$$\begin{aligned}\tilde{A}(\phi^0,\phi^1,w) &= \lim_{t\searrow 0}\left(\frac{dx}{dt}(t),\frac{d}{dt}x_t,\frac{d}{dt}(e_-^0 w)_t\right)\\ &= (L\phi^1+Bw, D\phi^1, Dw).\end{aligned}$$

Conversely if (ϕ^0,ϕ^1,w) verify (5.48), then, for all $T>0$, $e_-^0 w \in W^{1,p}(-h,T;\mathbb{R}^m)$ and by Theorem 4.1 (ii), $x \in W^{1,p}(-h,T;\mathbb{R}^n)$. By Lemma 4.1 (ii)

$$x_\bullet \in C\big(0,T;W^{1,p}(-h,0;\mathbb{R}^n)\big)$$

and

$$(e_-^0 w)_\bullet \in C\big(0,T;W^{1,p}(-h,0;\mathbb{R}^m)\big).$$

Thus

$$\dot{x}(t) = Lx_t + B(e_-^0 w)_t$$

has a continuous right-hand side and $\dot{x} \in C(0,T;\mathbb{R}^n)$. Always by Lemma 4.1 (ii)

$$x_\bullet \in C^1\big(0,T;L^p(-h,0;\mathbb{R}^n)\big), \quad (e_-^0 w)_\bullet \in C^1\big(0,T;L^p(-h,0;\mathbb{R}^m)\big)$$

and necessarily

$$\frac{d}{dt}\hat{x}(t) = \left(\frac{dx}{dt}(t),\frac{d}{dt}x_t,\frac{d}{dt}(e_-^0 w)_t\right)$$

exists and is continuous for $t \geq 0$. So the limit as t goes to zero exists and by definition $(\phi^0,\phi^1,w) \in D(\tilde{A})$.

(iii) Again we use the integration by parts formula (4.35) in Lemma 4.2 for the solution x of (5.37) and the solution z of (5.42) with $T=t$:

$$\begin{aligned}&\int_0^t z(t-s)\cdot[\dot{x}-\mathcal{L}e_+^0 x](s)\,ds + z(t)\cdot x(0)\\ &\qquad\qquad = \int_0^t (\dot{z}-\mathcal{L}^T e_+^0 z)(t)\cdot x(t-s)\,ds + z(0)\cdot x(t).\end{aligned}$$

By substitution

$$\begin{aligned}&\int_0^t z(t-s)\cdot[(\mathcal{L}e_-^0\phi^1)(-s)+(\mathcal{B}e_-^0 w)(-s)]\,ds + z(t)\cdot\phi^0\\ &\qquad\qquad = \int_0^t (e_+^{-h}\xi^1)(-s)\cdot x(t-s)\,ds + \xi^0\cdot x(t)\end{aligned}$$

and

$$\int_{-h}^0 (e_+^0 z)_t(\alpha)\cdot[\bar{L}\phi^1+\bar{B}w](\alpha)\,d\alpha + z(t)\cdot\phi^0 = \int_{-h}^0 \xi^1(\alpha)\cdot(e_+^0 x)_t(\alpha)\,d\alpha + \xi^0\cdot x(t)$$

or in a more compact form

$$\langle(\xi^0,\xi^1),(x(t),(e_+^0 x)_t)\rangle = z(t)\cdot\phi^0 + \langle\bar{L}^\top(e_+^0 z)_t,\phi^1\rangle + \langle\bar{B}^\top(e_+^0 z)_t, w\rangle. \quad (5.49)$$

To complete the above identity, we add the terms

$$\langle \xi^2, (e_-^0 w)_t \rangle \quad \text{and} \quad \langle \xi^1, (e_-^0 \phi^1)_t \rangle$$

and use the identity

$$\langle \xi^2, (e_-^0 w)_t \rangle = \int_{-h}^{0} \xi^2(\alpha) \bullet (e_-^0 w)(t+\alpha)\, d\alpha = \int_{t-h}^{t} \xi^2(\theta - t) \cdot (e_-^0 w)(\theta)\, d\theta,$$

$$\langle \xi^2, (e_-^0 w)_t \rangle = \int_{-h}^{0} (e_+^{-h} \xi^2)(\theta - t) \cdot w(\theta)\, d\theta = \langle \tau(t)\xi^2, w \rangle, \tag{5.50}$$

and similarly

$$\langle \xi^1, (e_-^0 \phi^1)_t \rangle = \langle \tau(t)\xi^1, \phi^1 \rangle. \tag{5.51}$$

Combining (5.49), (5.50), and (5.51) we obtain

$$\begin{aligned} &\langle (\xi^0, \xi^1, \xi^2), (x(t), x_t, (e_-^0 w)_t) \rangle \\ &\qquad = \langle (z(t), \bar{L}(e_+^0 z)_t + \tau(t)\xi^1, \bar{B}^\top (e_+^0 z)_t + \tau(t)\xi^2), (\phi^0, \phi^1, w) \rangle \end{aligned}$$

and henceforth the adjoint semigroup expression (5.41).

We now turn to the characterization of the infinitesimal generator $\tilde{A}^*$. It is readily seen that $\tilde{S}^*$ has a special structure

$$\tilde{S}^*(t)\xi = (S^*(t)(\xi^0, \xi^1), \bar{B}(e_+^0 z)_t + \tau(t)\xi^2), \tag{5.52}$$

$$\tilde{S}^*(t)\xi = (S^*(t)(\xi^0, \xi^1), 0) + \big((0,0), \bar{B}(e_+^0 z)_t + \tau(t)\xi^2\big), \tag{5.53}$$

where the first two components that are related to S^* can be separated from the third component. So if $\xi \in D(\tilde{A}^*)$, then $(\xi^0, \xi^1) \in D(A^*)$ and by Theorem 4.5

$$\exists \psi \in D(A^\top), \quad (\text{that is } \psi = (\psi(0), \psi), \psi \in W^{1,q}(-h, 0; \mathbb{R}^n))$$

and

$$\exists \zeta \in W^{1,q}(-h, 0; \mathbb{R}^n)$$

such that

$$(\xi^0, \xi^1) = F^* \psi + (0, \zeta) = (\psi(0), \bar{L}^\top \psi + \zeta) \tag{5.54}$$

and

$$\begin{aligned} \lim_{t \searrow 0} \frac{d}{dt} S^*(t)(\xi^0, \xi^1) &= A^*[F^*\psi + (0,\zeta)] = F^* A^\top \psi + (\zeta(0), -D\zeta) \\ &= (L^\top \psi + \zeta(0), \bar{L}^\top D\psi - D\zeta). \end{aligned} \tag{5.55}$$

But in view of the decomposition (5.54), equation (5.42) reduces to

$$\begin{cases} \dot{z}(t) &= (\mathcal{L}^\top e_+^0 z)(t) + (e_+^{-h}[\bar{L}^\top \psi + \zeta])(-t), \\ &= (\mathcal{L}^\top z)(t) + (e_+^{-h} \zeta)(-t), \\ z(0) &= \psi(0), \quad z_0 = \psi. \end{cases} \tag{5.56}$$

By Theorem 4.1 (ii), for all $T > 0$, $z \in W^{1,q}(-h, T; \mathbb{R}^n)$ and by Lemma 4.1 (ii)

$$z_\bullet \in C\big(0, T; W^{1,q}(-h, 0; \mathbb{R}^n)\big) \cap C^1\big(0, T; L^q(-h, 0; \mathbb{R}^n)\big) \tag{5.57}$$

and

$$D_t \bar{B}^\top z_t = \bar{B}^\top D_\theta\, z_t. \tag{5.58}$$

Now go back to the third component of $\tilde{S}^*(t)\xi$ in (5.53)

$$F(t) = \bar{B}^\top (e_+^0 z)_t + \tau(t)\xi^2 = \bar{B}^\top z_t + \tau(t)\xi^2 - \bar{B}^\top (e_-^0 \psi)_t$$

and recall that by Lemma 4.4 applied to $B^\top$

$$\bar{B}^\top (e_-^0 \psi)_t = \tau(t)[\bar{B}^\top \psi].$$

Then

$$F(t) = \bar{B}^\top z_t + \tau(t)\lambda, \quad \lambda = \xi^2 - \bar{B}^\top \psi.$$

From (5.57) $t \mapsto \bar{B}^\top z_t$ belong to $C^1\big(0, T; L^q(-h, 0; \mathbb{R}^n)\big)$ and from (5.57)–(5.58),

$$\bar{B}^\top z_\bullet \in C\big(0, T; W^{1,q}(-h, 0; \mathbb{R}^n)\big) \cap C^1\big(0, T; L^q(-h, 0; \mathbb{R}^n)\big) \tag{5.59}$$

and

$$\frac{d}{dt} \bar{B}^\top z_t = \bar{B}^\top D_\theta\, z_t. \tag{5.60}$$

So the assumption $\xi \in D(\tilde{A}^*)$ implies that

$$\tau(\cdot)\lambda \in C^1\big(0, T; L^q(-h, 0; \mathbb{R}^m)\big). \tag{5.61}$$

But this is the shift operator for the function $e_0^{-h}\lambda$ in $L^q(-\infty, 0; \mathbb{R}^m)$ and by the analog of Lemma 4.1 (ii)

$$e_0^{-h}\lambda \in W^{1,q}(-\infty, 0; \mathbb{R}^m) \Longrightarrow \lambda \in W^{1,q}(-h, 0; \mathbb{R}^m), \quad \lambda(-h) = 0,$$

and

$$\frac{d}{dt} \tau(t)\lambda = -\tau(t) D_\theta\, \lambda \tag{5.62}$$

as in the proof of Theorem 4.5. Finally

$$\frac{dF}{dt}(t) = \bar{B}^T D_\theta\, z_t - \tau(t) D_\theta\, \lambda. \tag{5.63}$$

To summarize we have shown that if $\xi = (\xi^0, \xi^1, \xi^2) \in D(\tilde{A}^*)$

$$\begin{cases} \xi^0 = \psi(0), \quad \xi^1 = \bar{L}^\top \psi + \zeta, \quad \xi^2 = \bar{B}^\top \psi + \lambda, \\ \psi, \zeta \in W^{1,q}(-h, 0; \mathbb{R}^n), \quad \zeta(-h) = 0, \\ \lambda \in W^{1,q}(-h, 0; \mathbb{R}^m), \quad \lambda(-h) = 0. \end{cases} \tag{5.64}$$

Moreover from (5.55), (5.60), and (5.62) as t goes to zero

$$\tilde{A}^*\xi = (L^\top\psi + \zeta(0), \bar{L}^\top D\psi - D\zeta, \bar{B}^\top D\psi - D\lambda). \tag{5.65}$$

Conversely if (5.64) is verified, then by Theorem 4.5 (i), $(\xi^0, \xi^1) \in D(A^*)$ and $S^*(\cdot)(\xi^0, \xi^1)$ belongs to $C^1(0, T; M^q)$. Moreover (5.42) reduces to (5.56) and we have (5.57) and (5.59). Finally condition (5.64) on λ implies that $e_+^{-h}\lambda \in W^{1,q}(-\infty, 0; \mathbb{R}^m)$ and we have (5.61). Therefore $\tilde{S}(\cdot)\xi$ belong to $C^1(0, T; Z^q)$ and necessarily $\xi \in D(\tilde{A}^*)$.

To complete the proof we must show that expression (5.65) for $\tilde{A}^*\xi$ is independent of the representation of ξ in terms of (ψ, ζ, λ). By linearity this is equivalent to show that

$$\begin{aligned}\xi = (\psi(0), \bar{L}^\top\psi + \zeta, \bar{B}^\top\psi + \lambda) = 0 \\ \implies \tilde{A}^*\xi = (L^\top\psi + \zeta(0),\ \bar{L}^\top D\psi - D\zeta, \bar{B}^\top D\psi - D\lambda) = 0.\end{aligned}$$

So $\xi = 0$ is equivalent to

$$\begin{aligned}\psi(0) = 0, \quad \psi \in W^{1,q}(-h, 0; \mathbb{R}^n),\\ \bar{L}^\top\psi + \zeta = 0, \quad \zeta \in W^{1,q}(-h, 0; \mathbb{R}^n), \quad \zeta(-h) = 0,\\ \bar{B}^\top\psi + \lambda = 0, \quad \lambda \in W^{1,q}(-h, 0; \mathbb{R}^m), \quad \lambda(-h) = 0.\end{aligned}$$

By Lemma 4.6 extended to $L^\top$ and $B^\top$

$$\begin{aligned}0 = D[\bar{L}^\top\psi + \zeta] = -\bar{L}^\top D\psi + D\zeta,\\ 0 = D[\bar{B}^\top\psi + \lambda] = -\bar{B}^\top D\psi + D\lambda,\\ 0 = [\bar{L}\psi + \zeta](0) = L^\top\psi + \zeta(0),\end{aligned}$$

and $\tilde{A}^*\xi = 0$. □

5.2.2 The nonhomogeneous case with control

Go back to system (5.13), which can be rewritten as

$$\begin{cases}\dot{x} = \mathcal{L}e_+^0 x + \mathcal{B}e_+^0 u + \mathcal{L}e_-^0\phi^1 + \mathcal{B}e_-^0 w + f,\\ x(0) = \phi^0.\end{cases} \tag{5.66}$$

Associate with the solution x of (5.66) the extended state

$$\tilde{x}(t) = (x(t), x_t, u_t), \quad t \geq 0. \tag{5.67}$$

The objective is to obtain the analog of Theorem 5.1 for $\tilde{x}(t)$. Here $\{\tilde{S}^*(t)\}$ will play the same role as $\{S^\top(t)\}$ and it will be necessary to construct a control operator $\hat{B}^\top : D(\tilde{A}^*) \to \mathbb{R}^m$, which will play for $\tilde{x}(t)$ the same role as $B^\top : D(A^\top) \to \mathbb{R}^m$ for $\hat{x}(t)$. The results are presented in the next theorem followed by a short discussion and the proof.

Theorem 5.3. *Let the real number* $p, 1 < p < \infty$, *and its conjugate* $q, q^{-1} + p^{-1} = 1$, *be given. Denote by* x *in* $W^{1,p}_{\text{loc}}(0,\infty;\mathbb{R}^n)$ *the solution of system* (5.66) *for* $(\phi^0, \phi^1, w) \in Z^p$, $f \in L^p_{\text{loc}}(0,\infty;\mathbb{R}^n)$, *and* $u \in L^p_{\text{loc}}(0,\infty;\mathbb{R}^m)$, *and let* $\tilde{x}(t)$ *be the extended state constructed from* x *in* (5.67):

(i) *For all* ρ *in* $D(\tilde{A}^*)$ *and* $t \geq 0$

$$\langle \rho, \tilde{x}(t)\rangle = \langle \tilde{S}^*(t)\rho, (\phi^0, \phi^1, w)\rangle + \int_0^t [\hat{B}^\top \tilde{S}^*(t-s)\rho \cdot u(s) + \tilde{S}^*(t-s)\rho \cdot (f(s), 0, 0)]\, ds, \quad (5.68)$$

where

$$\rho = (\psi(0), \bar{L}^\top \psi + \zeta, \bar{B}^\top \psi \ + \lambda) \mapsto \hat{B}^\top \rho = B^\top \psi + \lambda(0) : D(\tilde{A}^*) \to \mathbb{R}^m \quad (5.69)$$

is a well-defined continuous linear operator on $D(\tilde{A}^*)$ *endowed with its graph norm topology* (*cf.* (5.43) *in Theorem* 5.2 *for the representation of elements of* $D(\tilde{A}^*)$).

(ii) *The extended state* $\tilde{x}(t)$ *is the solution of the weak equation*: *For all* $t > 0$ *and all* $\rho \in D(\tilde{A}^*)$

$$\begin{cases} \dfrac{d}{dt}\langle \rho, \tilde{x}(t)\rangle = \langle \tilde{A}^*\rho, \tilde{x}(t)\rangle + \hat{B}^\top \rho \cdot u(t) + \rho^0 \cdot f(t), \\ \tilde{x}(0) = (\phi^0, \phi^1, w). \end{cases} \quad (5.70)$$

(iii) *For each* $T > 0$, *the state* $\tilde{x}$ *is the unique solution in*

$$\mathcal{V}(0,T;Z^p, D(\tilde{A}^*)') = \left\{ y \in C(0,T;Z^p) \colon \frac{d}{dt}\tilde{\imath}^* y \in L^p(0,T;D(\tilde{A}^*)') \right\} \quad (5.71)$$

to the following equation in $D(\tilde{A}^*)'$:

$$\begin{cases} \dfrac{d}{dt}\tilde{\imath}^*\tilde{x}(t) \\ = (\tilde{A}^*)^*\tilde{x}(t) + (\hat{B}^\top)^* u(t) + \tilde{\imath}^*(f(t), 0, 0), \quad t > 0, \\ \tilde{x}(0) = (\phi^0, \phi^1, w), \end{cases} \quad (5.72)$$

where $\tilde{\imath} \colon D(\tilde{A}^*) \to M^q$ *is the canonical injection of* $D(\tilde{A}^*)$ *into* M^q *and* $\tilde{\imath}^*, (\tilde{A}^*)^*$ *and* $(\hat{B}^\top)^*$ *are the dual maps of the continuous linear operators*

$$\tilde{\imath} \colon D(\tilde{A}^*) \to Z^q, \quad \tilde{A}^* \colon D(\tilde{A}^*) \to Z^q, \quad \hat{B}^\top \colon D(\tilde{A}^*) \to \mathbb{R}^m \quad (5.73)$$

defined on $D(\tilde{A}^*)$ *endowed with its graph norm topology.*

Remark 5.3. The key element of this theorem is the introduction of the operator $\hat{B}^\top$, which makes it possible to complete and extend the work of A. ICHIKAWA [1, 3] who considered operators B of the form

$$Bw = \sum_{i=0}^{N} B_i w(\theta_i) + \int_{-h}^{0} B_{01}(\theta) w(\theta)\, d\theta, \tag{5.74}$$

where $N \geq 0$ is an integer,

$$-\infty < -h = \theta_N < \cdots > \theta_{i+1} < \theta_i < \cdots > \theta_0 = 0 \tag{5.75}$$

are real, $B_0, B_1, \ldots, B_N$ and $B_{01}(\theta)$ are $n \times m$ matrices, and the element of $B_{01}(\theta)$ are L^q-functions on $I(-h, 0)$. In that case

$$\hat{B}^\top \rho = \sum_{i=0}^{N} B_i^\top \psi(\theta_i) + \int_{-h}^{0} B_{01}(\theta)^\top \psi(\theta)\, d\theta + \lambda(0)$$

with the notation of (5.69). But the following limit exists:

$$\lim_{\alpha \nearrow 0} [\bar{B}^\top \psi](\alpha) = B^\top \psi - B_0^\top \psi(0)$$

and

$$\hat{B}^\top \rho = B_0^\top \psi(0) + [\bar{B}^\top \psi + \lambda](0), \tag{5.76}$$

where we recognize the third component $\bar{B}^\top \psi + \lambda$ of

$$\rho = (\psi(0), \bar{L}^\top \psi + \zeta, \bar{B}^\top \psi + \lambda)$$

evaluated at 0. This construction and identity (5.76) can be extended to operators B of the form

$$Bw = B_0 w(0) + \int_{-h}^{0} d_\theta\, \beta_1 w(\theta), \quad w \in K(-h, 0; \mathbb{R}^m), \tag{5.77}$$

where B_0 is an $n \times m$ matrix and β_1 is an $n \times m$ matrix of regular Borel measures such that

$$\lim_{\alpha \nearrow 0} \int_{\alpha}^{0} d_\theta\, |\beta_1| = 0. \tag{5.78}$$

However this is a special case that does not generalize to arbitrary continuous linear maps $B\colon K(-h, 0; \mathbb{R}^m) \to R^n$ because there is a priori no reason that the third component $\bar{B}^\top \psi + \lambda$ of $\rho \in D(\tilde{A}^*)$ be defined at the point 0. The difficulties encountered in the construction (5.69) of operator $\hat{B}^\top$ are analogous to the ones encountered in the characterization of the infinitesimal generators A^* of the adjoint semigroups $\{S^*(t)\}$ in Theorem 4.5. $\square$

Proof of Theorem 5.3. (i) Let $\rho \in D(\tilde{A}^*)$ be of the form

$$\begin{cases} \rho = (\psi(0), \bar{L}^\top \psi + \zeta, \bar{B}^\top \psi + \lambda), \\ \zeta, \psi \in W^{1,q}(-h, 0; \mathbb{R}^n), \quad \zeta(-h) = 0, \\ \lambda \in W^{1,q}(-h, 0; \mathbb{R}^m), \quad \lambda(-h) = 0, \end{cases} \tag{5.79}$$

and let z be the solution of the system

$$\begin{cases} \dot{z}(t) = (\mathcal{L}^\top e_+^0 z)(t) + (\mathcal{L}^\top e_-^0 \psi)(t) + (e_+^{-h} \zeta)(-t), \\ z(0) = \psi(0). \end{cases} \tag{5.80}$$

This is the final form of (5.42) when $\rho \in D(\tilde{A}^*)$ is given by (5.79). By construction and the characterization of $D(\tilde{A}^*)$ given in Theorem 5.2

$$\tilde{S}^*(t)\rho = (z(t), \bar{L}^\top z_t + \tau(t)\zeta, \bar{B}^\top z_t + \tau(t)\lambda). \tag{5.81}$$

In view of the linearity of system (5.66) and of the definition of the semigroup $\{\tilde{S}(t)\}$ in Theorem 5.2, it is sufficient to establish identity (5.68) for $(\phi^0, \phi^1, w) = 0$. The first step is the use of the integration by parts formula (4.35) in Lemma 4.2 for $T = t$

$$\begin{aligned} &\int_0^t z(t-s) \cdot [\dot{x} - \mathcal{L} e_+^0 x](s)\, ds + z(t) \cdot x(0) \\ &\qquad = \int_0^t [\dot{z} - \mathcal{L}^\top e_+^0 z](s) \cdot x(t-s)\, ds + z(0) \cdot x(t), \end{aligned} \tag{5.82}$$

where x is the solution of

$$\dot{x} = \mathcal{L} e_+^0 x + \mathcal{B} e_+^0 u + f, \quad x(0) = 0. \tag{5.83}$$

Moreover as in (4.35)

$$\int_0^t z(t-s) \cdot (\mathcal{B} e_+^0 u)(s)\, ds = \int_0^t (\mathcal{B}^\top e_+^0 z)(t-s) \cdot u(s)\, ds. \tag{5.84}$$

Substract (5.84) from (5.82) and use (5.80) and (5.83)

$$\begin{aligned} &\int_0^t \Big[z(t-s) \cdot f(s)\, ds \\ &\quad = \int_0^t [(\mathcal{L}^\top e_-^0 \psi)(s) + (e_+^{-h} \zeta)(-s)] \cdot x(t-s) + (\mathcal{B}^\top e_-^0 \psi)(s) \cdot\, u(t-s) \Big]\, ds \\ &\qquad - \int_0^t (\mathcal{B}^\top z)(s) \cdot u(t-s)\, ds + \psi(0) \cdot x(t). \end{aligned}$$

But this can be reorganized and written as follows:

$$\psi(0) \cdot x(t) + \langle \bar{L}^\top \psi + \zeta, (e_+^0 x)_t \rangle + \langle \bar{B}^\top \psi, (e_+^0 u)_t \rangle$$
$$= \int_0^t [z(t-s) \cdot f(s) + B^\top z_{t-s} \cdot u(s)] \, ds. \quad (5.85)$$

To complete the identity we add the term

$$\begin{aligned} \langle \lambda, (e_+^0 u)_t \rangle &= \int_{-h}^0 \lambda(\alpha) \cdot (e_+^0 u)(t+\alpha) \, d\alpha \\ &= \int_{t-h}^t (e_+^{-h} \lambda)(s-t) \cdot (e_+^0 u)(s) \, ds \\ &= \int_0^t (e_+^{-h} \lambda)(s-t) \cdot u(s) \, ds. \end{aligned} \quad (5.86)$$

Therefore from (5.79), (5.85), and (5.86)

$$\langle \rho, \tilde{x}(t) \rangle = \int_0^t \{ z(t-s) \cdot f(s) + [B^\top z_{t-s} + (e_+^{-h} \lambda)(s-t)] \cdot u(s) \} \, ds.$$

But $\rho \in D(\tilde{A}^*)$ and hence $\tilde{S}^*(t-s)\rho \in D(\tilde{A}^*)$ has the representation given in (5.81) with $t-s$ in place of t.

To complete the proof of the theorem, we need the following lemma, which will be proved at the end. □

Lemma 5.1. *The linear operator*

$$\rho = (\psi(0), \bar{L}^\top \psi + \zeta, \bar{B}^\top \psi + \lambda) \mapsto \hat{B}^\top \rho = B^\top \psi + \lambda(0) \colon D(\tilde{A}^*) \to \mathbb{R}^m \quad (5.87)$$

is well defined, linear, and continuous on $D(\tilde{A}^*)$ endowed with its graph norm topology (cf. (5.43) *in Theorem* 5.2 *for the representation of the elements of $D(\tilde{A}^*)$.*

So by definition of $\hat{B}^\top$

$$\begin{aligned} \hat{B}^\top \tilde{S}^*(t-s)\rho &= \hat{B}^\top (z(t-s), \bar{L}^\top z_{t-s} + \tau(t-s)\zeta, \bar{B}^\top z_{t-s} + \tau(t-s)\lambda) \\ &= B^\top z_{t-s} + [\tau(t-s)\lambda](0) = B^\top z_{t-s} + (e_+^{-h}\lambda)(s-t) \end{aligned}$$

and

$$\langle \rho, \tilde{x}(t) \rangle = \int_0^t [\langle \tilde{S}^*(t-s)\rho, (f(s), 0, 0) \rangle + \hat{B}^\top \tilde{S}^*(t-s)\rho \cdot u(s)] \, ds.$$

(ii) and (iii) now follow by the same techniques as in the proof of Theorem 5.1 (ii) and (iii).

Remark 5.4. Notice that $\hat{B}^\top \rho$ does not depend on the second component of ρ. So it readily extends to

$$\left\{ (\psi(0), \rho_2, \bar{B}^\top \psi + \lambda) \colon \begin{array}{ll} \lambda \in W^{1,p}(-h, 0; \mathbb{R}^m), & \lambda(-h) = 0 \\ \psi \in W^{1,p}(-h, 0; \mathbb{R}^n), & \rho_2 \in L^q(-h, 0; \mathbb{R}^n) \end{array} \right\}.$$

□

Proof of Lemma 5.1. The proof is very similar to the proof of Lemma 4.6 with $B^\top$ in place of L. By linearity the map (5.87) is well defined if

$$(\psi(0), \bar{L}^\top \psi + \zeta, \bar{B}^\top \psi + \lambda) = 0 \Longrightarrow B^\top \psi + \lambda(0) = 0.$$

But from Lemma 4.6 (ii)

$$\psi \in W^{1,q}(-h, 0; \mathbb{R}^n) \quad \text{and} \quad \psi(0) = 0 \Longrightarrow (\bar{B}^\top \psi)(0) = B^\top \psi.$$

As a result $\bar{B}^\top \psi + \lambda \in W^{1,q}(-h, 0; \mathbb{R}^n)$ and

$$\bar{B}^\top \psi + \lambda = 0 \Longrightarrow 0 = [\bar{B}^\top \psi + \lambda](0) = B^\top \psi + \lambda(0).$$

For the continuity we use the same techniques as in Lemma 4.6 (i):

$$B^\top \psi + \lambda(0) = B^\top \tilde{\psi}_{-\alpha} + \lambda(\alpha) - \int_\alpha^0 [\bar{B}^\top D_\theta \, \psi - D_\theta \, \lambda](\theta) \, d\theta$$

and use the fact that

$$\begin{aligned} B^\top \tilde{\psi}_{-\alpha} + \lambda(\alpha) &= (\mathcal{B}^\top e^0_- \psi)(-\alpha) + \lambda(\alpha) + \big(\mathcal{B}^\top e^0_+ \psi(0)\big)(-\alpha) \\ &= [\bar{B}^\top \psi + \lambda](\alpha) + \big(\mathcal{B}^\top e^0_+ \psi(0)\big)(-\alpha). \end{aligned}$$

Then by making the same estimates as in Lemma 4.6 (i), there exists $c > 0$ such that

$$|B^\top \psi + \lambda(0)| \leq c[|\psi(0)| + \|\bar{B}^\top \psi + \lambda\|_q + \|\bar{B}^\top D\psi - D\lambda\|_q].$$

But for $\rho = (\psi(0), \bar{L}^\top \psi + \zeta, \bar{B}^\top \psi + \lambda)$ in $D(\tilde{A}^*)$ (cf. (5.43) in Theorem 5.2 (iii),

$$\begin{aligned} \|\rho\|_{D(\tilde{A}^*)} = \|\rho\|_q + \|\tilde{A}^* \rho\|_q = |\psi(0)| + \|\bar{L}^\top \psi + \zeta\|_q + \|\bar{B}^\top \psi + \lambda\|_q \\ + |L^\top \psi + \zeta(0)| + \|\bar{L}^\top D\psi - D\zeta\|_q + \|\bar{B}^\top D\psi - D\lambda\|_q \end{aligned}$$

and hence

$$|\hat{B}^\top \rho| = |B^\top \psi + \lambda(0)| \leq c \, \|\rho\|_{D(\tilde{A}^*)}.$$

□

6 State space theory of linear control systems with observation

In this last section we add to the linear control system

$$\begin{cases} \dot{x} = Lx_t + B_0 u_t + f(t), \quad t > 0, \\ (x(0), x_0, u_0) = (\phi^0, \phi^1, w) \end{cases} \tag{6.1}$$

the observation equation

$$y(t) = Cx_t + B_1 u_t, \quad t > 0 \tag{6.2}$$

for the continuous linear maps

$$\begin{cases} L\colon K(-h,0;\mathbb{R}^n) \to \mathbb{R}^n, & B_0 : K(-h,0;\mathbb{R}^m) \to \mathbb{R}^n, \\ C\colon K(-h,0;\mathbb{R}^n) \to \mathbb{R}^k, & B_1 : K(-h,0;\mathbb{R}^m) \to \mathbb{R}^k \end{cases} \tag{6.3}$$

and integers $n \geq 1$, $m \geq 1$, and $k \geq 1$.

The objective is to construct a state, a state equation, and an unbounded observation operator for the state. As in §3.5 two parallel approaches will be considered. The first one is based on the use of the extended state introduced in §5.2

$$\tilde{x}(t) = (x(t), x_t, u_t), \quad t \geq 0, \tag{6.4}$$

and the observation operator

$$(\phi(0), \phi, w) \mapsto C\phi + B_1 w\colon D(\tilde{A}) \to \mathbb{R}^k. \tag{6.5}$$

The second one is due to D. Salamon [2] who added a third component to the structural state

$$\hat{x}(t) = (x(t), \bar{L}x_t + \bar{B}_0 u_t, \bar{C}x_t + \bar{B}_1 u_t), \tag{6.6}$$

and an appropriate observation operator can be defined on the domain of the natural semigroup associated with the state $\hat{x}(t)$ and $u(t)$. This will be referred to as the *extended structural state* in §6.2.

In this section we give a presentation that naturally fits within the general framework adopted in this chapter. Moreover we show that the two states are not unrelated. They turn out to be also intertwined with respect to a general structural operator F, which contains all operators that characterize the delay structure of the system and the control and observation operators. This result is new, but the essential ideas and constructions were already contained in the general model developed by M. C. Delfour and J. Karrakchou [1, 2]. All this can obviously be extended to more general delay structures.

Formally in the first case the natural semigroup will be the extended semigroup $\{\tilde{S}(t)\}$ associated with L and B_0. The extended state $\tilde{x}$ will be the solution of the state equation

$$\begin{cases} \dfrac{d}{dt}\tilde{\imath}^* \tilde{x}(t) = (\tilde{A}^*)^* \tilde{x}(t) + (\hat{B}^\top)^* u(t), \\ \tilde{x}(0) = (\phi^0, \phi^1, w), \end{cases} \tag{6.7}$$

and the observation will be given by

$$y(t) = \tilde{C}\tilde{x}(t), \tag{6.8}$$

where

$$\hat{B}^\top : D(\tilde{A}^*) \to \mathbb{R}^m, \quad \hat{B}^\top(\psi(0), \bar{L}^\top\psi + \zeta, \bar{B}_0^\top\psi + \lambda) = B_0^\top\psi + \lambda(0), \tag{6.9}$$

$$\tilde{C} : D(\tilde{A}) \to \mathbb{R}^k, \quad \tilde{C}(\phi(0), \phi, w) = C\phi + B_1 w. \tag{6.10}$$

In the second case the semigroup will be the adjoint $\{\tilde{S}^{\top *}(t)\}$ of the extended semigroup $\{\tilde{S}(t)^\top\}$ associated with $L^\top$ and $C^\top$. It is the analog of the extended semigroup associated with L and B_0 in Terminology 2. The extended structural state $\hat{x}$ will be the solution of

$$\begin{cases} \dfrac{d}{dt}\tilde{\jmath}^*\hat{x}(t) = (\tilde{A}^\top)^*\hat{x}(t) + (\tilde{B}^\top)^* u(t), \\ \hat{x}(0) = (\phi(0), \bar{L}\phi^1 + \bar{B}_0 w, \bar{C}\phi^1 + \bar{B}_1 w). \end{cases} \tag{6.11}$$

When $B_1 = 0$

$$y(t) = \hat{C}\hat{x}(t), \tag{6.12}$$

where

$$\begin{aligned} &\tilde{B}^\top : D(\tilde{A}^\top) \to \mathbb{R}^m, \\ &\tilde{B}^\top(\psi(0), \psi, v) = B_0^\top\psi + B_1^\top v, \end{aligned} \tag{6.13}$$

$$\begin{aligned} &\hat{C} : D(\tilde{A}^{\top *}) \to \mathbb{R}^k, \\ &\hat{C}(\phi(0), \bar{L}\phi + \zeta, \bar{C}\phi + \lambda) = C\phi + \lambda(0). \end{aligned} \tag{6.14}$$

When $B_1 \neq 0$, the situation is slightly more complex and $y(t)$ depends on $\hat{x}(t)$ and $u(t)$.

Of course everything will be made more precise as we proceed, but we would like to already draw some general conclusions to guide the reader in the subsequent developments of this theory. In §3.5 the use of the structural state $\hat{x}(t)$ with two components had a clear advantage over the extended state $\tilde{x}(t)$ with three components. Here both states have three components. The operators $\hat{B}^\top$ and $\tilde{C}$ have the same structure as the operators $\hat{C}$ and $\tilde{B}^\top$. So there is a flavor of duality or intertwining between the two approaches, but no clear technical advantage in choosing one over the other. In both cases we come up with a classical evolution system and unbounded control and observation operators ... and most important no more delays.

However from the System Theoretic viewpoint, the first approach leads to the observation $y(t)$ as a linear function of the extended state $\tilde{x}(t)$, whereas in the second approach it is a linear function of $\big(\hat{x}(t), u(t)\big)$. In other words in the first approach the state contains the control segment u_t and the control does not explicitly appear in the observation equation, whereas in the second approach the observation equation explicitly contains the control $u(t)$ at time t.

6.1 The extended state

This case is easy because we have already studied $\tilde{x}$ in Theorem 5.3 for the system

$$\begin{cases} \dot{x} = \mathcal{L}e^0_+ x + \mathcal{B}_0 e^0_+ u + \mathcal{L}e^0_- \phi^1 + \mathcal{B}_0 e^0_- w + f, \\ x(0) = \phi^0. \end{cases} \tag{6.15}$$

The observation (6.2) makes sense because

$$y = \mathcal{C}x + \mathcal{B}_1 u, \tag{6.16}$$

where, as in the case of L and $\mathcal{L}$ in Theorem 3.3, for all $T > 0$, we can associate with C the linear map

$$\begin{gathered} \mathcal{C}\colon L^p(-h,T;\mathbb{R}^n) \to L^p(0,T;\mathbb{R}^k) \\ (\mathcal{C}x)(t) = Cx_t, \quad \forall x \in C_c(]-h,T[;\mathbb{R}^n),\ t \in [0,T], \end{gathered} \tag{6.17}$$

and linear maps $\mathcal{B}_0$ and $\mathcal{B}_1$ are similarly associated with B_0 and B_1, respectively. The operator

$$\begin{aligned} (\phi^0,\phi^1,w) \mapsto \tilde{C}(\phi^0,\phi^1,w) &= C\phi^1 + B_1 w \\ &: \mathbb{R}^n \times K(-h,0;\mathbb{R}^n) \times K(-h,0;\mathbb{R}^m) \to \mathbb{R}^k \end{aligned} \tag{6.18}$$

is linear and continuous. For initial conditions of the form

$$(\phi(0),\phi,w), \quad \phi \in K(-h,0;\mathbb{R}^n), \quad w \in K(-h,0;\mathbb{R}^m) \tag{6.19}$$

and

$$f \in L^p_{\mathrm{loc}}(0,\infty;\mathbb{R}^n), \quad u \in C_{\mathrm{loc}}(0,\infty;\mathbb{R}^m),\ u(0) = w(0), \tag{6.20}$$

we have a continuous observation y

$$y(t) = \tilde{C}\tilde{x}(t) = Cx_t + B_1 u_t, \quad t \geq 0. \tag{6.21}$$

The operator $\tilde{C}$ introduced in (6.18) can also be viewed as a continuous linear operator

$$\xi = (\phi(0),\phi,w) \mapsto \tilde{C}\xi = C\phi + B_1 w\colon D(\tilde{A}) \to \mathbb{R}^k, \tag{6.22}$$

when $D(\tilde{A})$ is endowed with the graph norm topology. When $u(t) = 0$, $t \geq 0$, and $\big(\phi(0),\phi,w)\big) \in D(\tilde{A})$,

$$\tilde{C}\tilde{x}(t) = Cx_t + B_1(e^0_- w)_t. \tag{6.23}$$

6.2 The extended structural state

This section is not necessarily more difficult than the other ones, but certainly heavier in notation. Yet the constructions and the proofs are very much the same as in the previous sections. Always as in Theorem 3.3 where we have associated with L, a matrix of regular Borel measures η, a new operator $L^\top$, maps $\mathcal{L}$ and $\mathcal{L}^\top$, and structural operators $\bar{L}$ and $\bar{L}^\top$, we associate with C, B_0, B_1, the same items

$$\begin{cases} L & \mathcal{L} & \eta & L^\top & \mathcal{L}^\top & \bar{L} & \bar{L}^\top \\ C & \mathcal{C} & \gamma & C^\top & \mathcal{C}^\top & \bar{C} & \bar{C}^\top \\ B_0 & \mathcal{B}_0 & \beta_0 & B_0^\top & \mathcal{B}_0^\top & \bar{B}_0 & \bar{B}_0^\top \\ B_1 & \mathcal{B}_1 & \beta_1 & B_1^\top & \mathcal{B}_1^\top & \bar{B}_1 & \bar{B}_1^\top . \end{cases} \tag{6.24}$$

Of course the previous results for L will apply to C, B_0, and B_1. Recall that the structural state was defined for the linear control system

$$\begin{cases} \dot{x}(t) = (\mathcal{L}e_+^0 x)(t) + (\mathcal{B}_0 e_+^0 u)(t) + (e_+^{-h}\xi^1)(-t) + f(t), \quad t > 0, \\ x(0) = \xi^0 \end{cases} \tag{6.25}$$

and that (6.15) is a special case of (6.25). The new state will be an extension of (5.20) by adding a third component.

Definition 6.1. The *extended structural state* is denoted by $\hat{x}(t)$ and defined as

$$\hat{x}(t) = (x(t), \bar{L}(e_+^0 x)_t + \bar{B}_0(e_+^0 u)_t + \tau(t)\xi^1, \bar{C}(e_+^0 x)_t \\ + \bar{B}_1(e_+^0 u)_t + \tau(t)\xi^2). \tag{6.26}$$

□

The *observation equation* will be for $t \geq 0$

$$y(t) = (\mathcal{C}e_+^0 x)(t) + (\mathcal{B}_1 e_+^0 u)(t) + (e_+^{-h}\xi^2)(-t). \tag{6.27}$$

When

$$\xi = (\xi^0, \xi^1, \xi^2) = (\phi^0, \bar{L}\phi^1 + \bar{B}_0 w, \bar{C}\phi^1 + \bar{B}_1 w), \tag{6.28}$$

we exactly recover system (6.15)–(6.16).

6.2.1 The extended semigroup $\{\tilde{S}^\top(t)\}$ associated

Given the real number q, $1 \leq q < \infty$, consider the system

$$\begin{cases} \dot{z}(t) = (\mathcal{L}^\top z)(t) + (\mathcal{C}^\top e_-^0 v)(t), \quad t \geq 0, \\ (z(0), z_0, v_0) = (\psi^0, \psi^1, v) \in Z^q \end{cases} \tag{6.29}$$

and define the family of linear transformations

$$\tilde{S}^\top(t)(\psi^0, \psi^1, v) = (z(t), z_t, (e_-^0 v)_t), \quad t \geq 0. \tag{6.30}$$

It is clear from §5.2.1 that $\{\tilde{S}^\top(t)\}$ associated with $L^\top$ and $C^\top$ has the same structure as $\{\tilde{S}(t)\}$ defined in (5.38) and that Theorem 5.2 applies with obvious substitutions. The family $\{\tilde{S}^\top(t)\}$ is a strongly continuous semigroup of continuous linear transformations of $Z^q = \mathbb{R}^n \times L^q(-h, 0; \mathbb{R}^n) \times L^q(-h, 0; \mathbb{R}^k)$. Its infinitesimal generator $\tilde{A}^\top$ is given by

$$\tilde{A}^\top(\psi(0), \psi, v) = (L^\top \psi + C^\top v, D\psi, Dv), \tag{6.31}$$

where

$$D(\tilde{A}^\top) = \left\{ (\psi(0), \psi, v) \in Z^q : \begin{array}{l} (\psi(0), \psi) \in D(A^\top) \\ v \in W^{1,q}(-h, 0; \mathbb{R}^k) \\ v(-h) = 0 \end{array} \right\}. \tag{6.32}$$

The canonical injection of $D(\tilde{A}^\top)$ into Z^q will be denoted by $\tilde{\jmath}$. For p, $1 < p < \infty$, and its conjugate $q, q^{-1} + p^{-1} = 1$, the adjoint semigroup $\{\tilde{S}^{\top *}(t)\}$ is characterized by

$$\tilde{S}^{\top *}(t)(\xi^0, \xi^1, \xi^2) = (x(t), \bar{L}(e_+^0 x)_t + \tau(t)\xi^1, \bar{C}(e_+^0 x)_t + \tau(t)\xi^2), \tag{6.33}$$

where x is the solution of system

$$\begin{cases} \dot{x}(t) = (\mathcal{L} e_+^0 x)(t) + (e_+^{-h} \xi^1)(-t), \quad t > 0, \\ x(0) = \xi^0. \end{cases} \tag{6.34}$$

6.2.2 The case $B_1 = 0$

When $B_1 = 0$, the observation equation (6.27) reduces to

$$y(t) = (\mathcal{C} e_+^0 x)(t) + (e_+^{-h} \xi^2)(-t), \quad t \geq 0. \tag{6.35}$$

For $\xi \in D(A^{\top *})$,

$$\begin{cases} \xi = (\phi(0), \bar{L}\phi + \zeta, \bar{C}\phi + \lambda), \\ \phi, \zeta \in W^{1,p}(-h, 0; \mathbb{R}^n), \quad \zeta(-h) = 0, \\ \lambda \in W^{1,p}(-h, 0; \mathbb{R}^k), \quad \lambda(-h) = 0, \end{cases} \tag{6.36}$$

and (6.25) reduces to

$$\begin{cases} \dot{x}(t) = L x_t + (\mathcal{B}_0 e_+^0 u)(t) + (e_+^{-h} \zeta)(-t) + f(t), \\ (x(0), x_0) = (\phi(0), \phi), \end{cases} \tag{6.37}$$

$$y(t) = C x_t + (e_+^{-h} \lambda)(-t), \quad t > 0. \tag{6.38}$$

Always for $\xi \in D(\tilde{A}^{\top *})$ of the form (6.36)

$$\hat{x}(t) = (x(t), \bar{L} x_t + \bar{B}_0 (e_+^0 u)_t + \tau(t)\zeta, \bar{C} x_t + \tau(t)\lambda) \tag{6.39}$$

and because $e_+^{-h}\lambda \in W^{1,p}(-\infty, 0; \mathbb{R}^k)$, $[\tau(t)\lambda](0) = (e_+^{-h}\lambda)(-t)$ and

$$\begin{aligned} y(t) &= C x_t + (e_+^{-h}\lambda)(-t) \\ &= C x_t + [\tau(t)\lambda](0) = \hat{C}\hat{x}(t). \end{aligned} \tag{6.40}$$

We recognize the operator

$$\xi = (\phi(0), \bar{L}\phi + \zeta, \bar{C}\phi + \lambda) \mapsto \hat{C}\xi = C\phi + \lambda(0) \colon D(\tilde{A}^{\top *}) \to \mathbb{R}^k, \tag{6.41}$$

which is the analog of the operator $\hat{B}^{\top}$ in Lemma 5.1 with L and C in place of $L^{\top}$ and $B^{\top}$. So it is linear and continuous when $D(\tilde{A}^{\top *})$ is endowed with the graph norm topology. However recall from Remark 5.4 that it is independent of the second component of ξ, which can be arbitrarily chosen in $L^2(-h, 0; \mathbb{R}^N)$.

6.2.3 The case $B_1 \neq 0$

Comparing (6.26) and (6.33) we recognize that they coincide for $u(t) = 0$, $t \geq 0$. The next theorem is the analog of Theorem 5.1.

Theorem 6.1. *Let the real number p, $1 < p < \infty$, and its conjugate q, $q^{-1} + p^{-1} = 1$, be given. Assume that x in $W^{1,p}_{\text{loc}}(0, \infty; \mathbb{R}^n)$ is the solution of system* (6.25) *for $\xi \in M^p$, $f \in L^p_{\text{loc}}(0, \infty; \mathbb{R}^n)$ and $u \in L^p_{\text{loc}}(0, \infty; \mathbb{R}^m)$, and let $\hat{x}(t)$ be the extended structural state constructed from x in* (6.26):

(i) *For all ρ in $D(\tilde{A}^{\top})$ and all $t \geq 0$*

$$\begin{aligned}\langle \rho, \hat{x}(t) \rangle = \langle \tilde{S}^{\top}(t)\rho, \xi \rangle + \int_0^t & [\tilde{B}^{\top} \tilde{S}^{\top}(t - r)\rho \cdot u(r) \\ & + \langle \tilde{S}^{\top}(t - r)\rho, (f(r), 0, 0) \rangle] \, dr, \end{aligned} \tag{6.42}$$

where

$$\rho = (\psi(0), \psi v) \mapsto \tilde{B}^{\top}\rho = B_0^{\top}\psi + B_1^{\top} v \colon D(\tilde{A}^{\top}) \to \mathbb{R}^m \tag{6.43}$$

is linear and continuous on $D(\tilde{A}^{\top})$ endowed with its graph norm topology.

(ii) *The state $\hat{x}(t)$ is the solution of the weak equation*

$$\begin{cases} \dfrac{d}{dt}\langle \rho, \hat{x}(t) \rangle = \langle \tilde{A}^{\top}\rho, \hat{x}(t) \rangle + \tilde{B}^{\top}\rho \cdot u(t) + \rho^0 \cdot f(t), \quad t > 0, \\ \hat{x}(0) = \xi, \quad \forall \rho \in D(\tilde{A}^{\top}). \end{cases} \tag{6.44}$$

(iii) *For each $T > 0$, the state $\hat{x}$ is the unique solution in*

$$\begin{aligned} &\mathcal{V}(0, T; M^p; D(\tilde{A}^{\top})') \\ &\quad = \left\{ z \in C(0, T; M^p) \colon \frac{d}{dt}\tilde{\jmath}^* z \in L^p(0, T; D(\tilde{A}^{\top})') \right\} \end{aligned} \tag{6.45}$$

to the following equation in $D(\tilde{A}^{\top})'$:

$$\begin{cases} \dfrac{d}{dt}\tilde{\jmath}^* \hat{x}(t) = (\tilde{A}^{\top})^* \hat{x}(t) + (\tilde{B}^{\top})^* u(t) + \tilde{\jmath}^*(f(t), 0, 0), \quad t > 0, \\ \hat{x}(0) = \xi, \end{cases} \tag{6.46}$$

where $\tilde{\jmath}$ is the canonical injection of $D(\tilde{A}^\top)$ into Z^q and $\tilde{\jmath}^*$, and $(\tilde{A}^\top)^*$ and $(\tilde{B}^\top)^*$ are the dual maps of the continuous linear operators

$$\tilde{\jmath}\colon D(\tilde{A}^\top) \to Z^q, \quad \tilde{A}^\top : D(\tilde{A}^\top) \to Z^q, \quad \tilde{B}^\top : D(\tilde{A}^\top) \to \mathbb{R}^m \tag{6.47}$$

defined on $D(\tilde{A}^\top)$ endowed with the graph norm topology.

Remark 6.1. The operator $\tilde{B}^\top$ is similar to the operator $\tilde{C}$ introduced in (6.22). It is the restriction to $D(\tilde{A}^\top)$ of the continuous linear operator

$$\begin{aligned}(\psi^0, \psi^1, v) \mapsto \tilde{B}^\top(\psi^0, \psi^1, v) &= B_0^\top \psi^1 + B_1^\top v \\ &: \mathbb{R}^n \times K(-h, 0; \mathbb{R}^n) \times K(-h, 0; \mathbb{R}^k) \to \mathbb{R}^m.\end{aligned} \tag{6.48}$$

□

The proof of this theorem will be given at the end of this section. Before stating the next theorem, which describes the structure of the observation equation for smooth data, it is useful to consider the following simple example.

Example 6.1. Consider the linear system without delays

$$\dot{x}(t) = Ax(t) + B_{00}u(t), \quad x(0) = x^0, \ t \geq 0, \tag{6.49}$$

$$y(t) = C_0 x(t) + B_{10}u(t), \quad t \geq 0. \tag{6.50}$$

Then

$$L\phi = A\phi(0), \quad B_0 w = B_{00}w(0), \quad C\phi = C_0\phi(0), \quad B_1 w = B_{10}w(0),$$

and it is readily seen that

$$\bar{L} = 0, \quad \bar{B}_0 = 0, \quad \bar{C} = 0, \quad \text{and} \quad \bar{B}_1 = 0.$$

For initial conditions of the form

$$\xi = (\phi(0), \bar{L}\phi + \bar{B}_0 w, \bar{C}\phi + \bar{B}_1 w) = (\phi(0), 0, 0)$$

the state reduces to

$$\hat{x}(t) = (x(t), 0, 0)$$

and the observation $y(t)$ given by (6.50) necessitates a knowledge of both $\hat{x}(t)$ and $u(t)$. □

This example illustrates how the structural operators construct a state $\hat{x}(t)$ that is "minimal" in the sense that any artificial delay structure is removed or filtered out. It also clearly indicates that the observation $y(t)$ at time $t \geq 0$ necessitates a knowledge of both $\hat{x}(t)$ and $u(t)$. This fact is well known for finite dimensional systems of the type (6.49)–(6.50) without delays.

Consider initial conditions of the form

$$\xi + (0, \bar{B}_0 w, \bar{B}_1 w), \tag{6.51}$$

where

$$\xi \in D(\tilde{A}^{\top *}) \quad \text{and} \quad w \in W^{1,p}(-h, 0; \mathbb{R}^m); \tag{6.52}$$

that is, there exist

$$\begin{aligned} &\phi, \zeta \in W^{1,p}(-h, 0; \mathbb{R}^n), \quad \zeta(-h) = 0, \\ &\lambda \in W^{1,p}(-h, 0; \mathbb{R}^k), \quad \lambda(-h) = 0 \end{aligned} \tag{6.53}$$

such that

$$\xi = (\phi(0), \bar{L}\phi + \zeta, \bar{C}\phi + \lambda). \tag{6.54}$$

For initial data of the form (6.51) and

$$\begin{aligned} &f \in L^p_{\text{loc}}(0, \infty; \mathbb{R}^n), \\ &u \in W^{1,p}_{\text{loc}}(0, \infty; \mathbb{R}^m), \quad u(0) = w(0), \end{aligned} \tag{6.55}$$

equation (6.25) reduces to

$$\begin{cases} \dot{x}(t) = Lx_t + B_0 u_t + (e_+^{-h}\zeta)(-t) + f(t), \quad t \geq 0, \\ (x(0), x_0, u_0) = (\phi(0), \phi, w), \end{cases} \tag{6.56}$$

$$y(t) = Cx_t + B_1 u_t + (e_+^{-h}\lambda)(-t), \quad t \geq 0, \tag{6.57}$$

and the state $\hat{x}(t)$ is given by

$$(x(t), \bar{L}x_t + \bar{B}_0 u_t + \tau(t)\zeta, \bar{C}x_t + \bar{B}_1 u_t + \tau(t)\lambda). \tag{6.58}$$

By assumptions (6.51) to (6.55) the term

$$(x(t), \bar{L}x_t + \tau(t)\zeta, \bar{C}x_t + \tau(t)\lambda) \in D(\tilde{A}^{\top *}), \tag{6.59}$$

and by definition (6.41) of the observation operator $\hat{C}$ for all $t \geq 0$

$$\hat{C}(x(t), \bar{L}x_t + \tau(t)\zeta, \bar{C}x_t + \tau(t)\lambda) = Cx_t + (e_+^{-h}\lambda)(-t). \tag{6.60}$$

So to relate $y(t)$ and $\hat{x}(t)$ we concentrate on the term

$$(0, \bar{B}_0 u_t, \bar{B}_1 u_t). \tag{6.61}$$

As shown in Example 6.1 this term is not rich enough to completely reconstruct the observation $y(t)$ and deal with nondelayed terms in the variable u. To fully recover the observation $y(t)$ we need the variable $u(t)$. The situation is analogous to the one in (6.60) for the variable x, which makes sense because of the presence of the term $x(t)$. The technical result behind this construction is Lemma 4.6 applied to B_1: For all $w \in W^{1,p}\,(-h, 0; \mathbb{R}^m)$, the map

$$(w(0), \bar{B}_1 w) \mapsto B_1 w \tag{6.62}$$

is well defined, linear, and

$$\begin{aligned}&\exists c>0, \quad \text{such that } \forall w\in W^{1,p}(-h,0;\mathbb{R}^m),\\ &|B_1 w| \le c[|w(0)| + \|\bar{B}_1 w\|_p + \|\bar{B}_1 Dw\|_p].\end{aligned} \tag{6.63}$$

So Lemma 4.6 has the following natural extension here.

Lemma 6.1. *Let p, $1\le p<\infty$, be a real number. For all ξ in $D(\tilde{A}^{\top *})$ of the form* (6.54)–(6.53) *and w in $W^{1,p}(-h,0;\mathbb{R}^m)$ the map*

$$\begin{aligned}\big(\xi+(0,\bar{B}_0 w,\bar{B}_1 w),w(0)\big) &= \big((\phi(0),\bar{L}\phi+\bar{B}_0 w+\zeta,\bar{C}\phi+\bar{B}_1 w+\lambda),w(0)\big)\\ &\mapsto \hat{C}_{\text{ext}}\big(\xi+(0,\bar{B}_0 w,\bar{B}_1 w),w(0)\big) = C\phi+B_1 w+\lambda(0)\end{aligned} \tag{6.64}$$

is well defined, linear, and there exists $c>0$ such that for all ξ in $D(\tilde{A}^{\top *})$ and w in $W^{1,p}(-h,0;\mathbb{R}^m)$

$$\begin{aligned}|\hat{C}_{\text{ext}}(\xi+(0,\bar{B}_0 w,\bar{B}_1 w),w(0)| \le c[&|\phi(0)|+\|\bar{L}\phi+\bar{B}_0 w+\zeta\|_p+\|\bar{C}\phi+\bar{B}_1 w+\lambda\|_p\\ &+|L\phi+B_0 w+\zeta(0)|+\|\bar{L}\,D\phi+\bar{B}_0 Dw-D\zeta\|_p\\ &+\|\bar{C}D\phi+\bar{B}_1 Dw-D\lambda\|_p+|w(0)|].\end{aligned} \tag{6.65}$$

Proof. To verify that the map (6.64) is well defined, it is sufficient to show that

$$\big((\phi(0),\bar{L}\phi+\bar{B}_0 w+\zeta,\bar{C}\phi+\bar{B}_1 w+\lambda,w(0)\big)=0$$

implies that

$$C\phi+B_1 w+\lambda(0)=0.$$

In particular

$$\begin{aligned}&\phi(0)=0 \text{ and } \phi\in W^{1,p}(-h,0;\mathbb{R}^n) \Longrightarrow (\bar{L}\phi)(0)=L\phi \text{ and } (\bar{C}\phi)(0)=C\phi,\\ &w(0)=0 \text{ and } w\in W^{1,p}(-h,0;\mathbb{R}^m) \Longrightarrow (\bar{B}_0 w)(0)=B_0 w \text{ and } (\bar{B}_1 w)(0)=B_1 w.\end{aligned}$$

As a result

$$\bar{C}\phi+\bar{B}_1 w+\lambda=0 \Longrightarrow 0=[\bar{C}\phi+\bar{B}_1 w+\lambda](0)=C\phi+B_1 w+\lambda(0).$$

Inequality (6.65) follows by the same techniques as the ones used in the proof of Lemma 4.6. □

The direct consequence of Lemma 6.1 is that for initial conditions

$$\xi\in D(\tilde{A}^{\top *}) \quad \text{and} \quad w\in W^{1,p}(-h,0;\mathbb{R}^m)$$

and functions f and u verifying (6.55),

$$\hat{C}_{\text{ext}}\big(\hat{x}(t),u(t)\big)=Cx_t+B_1 u_t+(e_+^{-h}\lambda)(-t)=y(t) \tag{6.66}$$

as can be easily verified. This is the extension of (6.40) to the case where $B_1\neq 0$ and the equivalent of (6.21) for the state $\tilde{x}(t)$.

Proof of Theorem 6.1. (i) By linearity of system (6.25) and the definition (6.33) of the adjoint semigroup $\{\tilde{S}^{\top *}(t)\}$, it is sufficient to establish identity (6.42) for $\xi = 0$. We use the integration by parts formula (4.35) in Lemma 4.2 for $T = t$

$$\begin{cases} \int_0^t z(t-s) \cdot [\dot{x} - \mathcal{L} e_+^0 x](s)\, ds + z(t) \cdot x(0), \\ = \int_0^t [\dot{z} - \mathcal{L}^\top e_+^0 z](s) \cdot x(t-s)\, ds + z(0) \cdot x(t), \end{cases} \tag{6.67}$$

where x is the solution of

$$\begin{cases} \dot{x}(t) = [\mathcal{L} e_+^0 x + \mathcal{B}_0 e_+^0 u](t), \quad t \geq 0, \\ x(0) = 0, \end{cases} \tag{6.68}$$

$$y(t) = [\mathcal{C} e_+^0 x + \mathcal{B}_1 e_+^0 u](t), \quad t \geq 0, \tag{6.69}$$

and z is the solution of

$$\begin{cases} \dot{z}(t) = [\mathcal{L}^\top z + \mathcal{C}^\top (e_-^0 v)](t), \quad t \geq 0, \\ (z(0), z_0, (e_-^0 v)_0) = (\psi(0), \psi, v) \in D(\tilde{A}^\top). \end{cases} \tag{6.70}$$

The substitution of (6.68) and (6.70) in (6.67) yields

$$\begin{aligned} \int_0^t z(t-s) \cdot [\mathcal{B}_0 e_+^0 u](s)\, ds &= \int_0^t [\mathcal{L}^\top e_-^0 \psi + \mathcal{C}^\top e_-^0 v](s) \cdot x(t-s)\, ds + \psi(0) \cdot x(t) \\ &= \int_{-h}^0 [\bar{L}^\top \psi + \bar{C}^\top v](\alpha) \cdot (e_+^0 x)_t(\alpha)\, d\alpha + \psi(0) \cdot x(t), \end{aligned}$$

and because $\bar{L}^\top = \bar{L}^*$ and $\bar{C}^\top = \bar{C}^*$

$$\langle (\psi(0), \psi, v), (x(t), \bar{L}(e_+^0 x)_t, \bar{C}(e_+^0 x)_t) \rangle = \int_0^t (\mathcal{B}_0^\top e_+^0 z)(t-s) \cdot u(s)\, ds. \tag{6.71}$$

But

$$\mathcal{B}_0^\top e_+^0 z = \mathcal{B}_0^\top z - \mathcal{B}_0^\top e_-^0 z = \mathcal{B}_0^\top z - \mathcal{B}_0^\top e_-^0 \psi$$

and

$$\begin{aligned} \int_0^t (\mathcal{B}_0^\top e_-^0 \psi)(t-s) \cdot u(s)\, ds &= \int_0^t (\mathcal{B}_0^\top e_-^0 \psi)(s) \cdot u(t-s)\, ds \\ &= \int_{-h}^0 (\bar{B}_0^\top \psi)(\alpha) \cdot (e_+^0 u)(t+\alpha)\, d\alpha. \end{aligned}$$

Substituting in (6.71) we obtain

$$\begin{aligned} &\langle (\psi(0), \psi, v), (x(t), \bar{L}(e_+^0 x)_t + \bar{B}_0 (e_+^0 u)_t, \bar{C}(e_+^0 x)_t) \rangle \\ &\qquad = \int_0^t (\mathcal{B}_0^\top z)(t-s) \cdot u(s)\, ds = \int_0^t B_0^\top z_{t-s} \cdot u(s)\, ds. \end{aligned} \tag{6.72}$$

The missing term to obtain the state $\hat{x}(t)$ in (6.72) is

$$\begin{aligned}\langle v, \bar{B}_1(e_+^0 u)_t\rangle = \langle \bar{B}_1^\top v, (e_+^0 u)_t\rangle &= \int_{-h}^0 (\mathcal{B}_1^\top e_-^0 v)(\alpha)\cdot(e_+^0 u)(t+\alpha)\,d\alpha\\ &= \int_{t-h}^t (\mathcal{B}_1^\top e_-^0 v)(t-s)\cdot(e_+^0 u)(s)\,ds\\ &= \int_{t-h}^t B_1^\top(e_-^0 v)_{t-s}\cdot(e_+^0 u)(s)\,ds\\ &= \int_0^t B_1^\top(e_-^0 v)_{t-s}\cdot u(s)\,ds.\end{aligned}$$

Finally

$$\begin{aligned}&\langle(\psi(0),\psi,v),(x(t),\bar{L}(e_+^0 x)_t+\bar{B}_0(e_+^0 u)_t,\bar{C}(e_+^0 x)_t+\bar{B}_1(e_+^0 u)_t)\rangle\\ &\qquad = \int_0^t [B_0^\top z_{t-s}+B_1^\top(e_-^0 v)_{t-s}]\cdot u(s)\,ds. \qquad (6.73)\end{aligned}$$

But this is precisely identity (6.42) for $\xi = 0$ and $\tilde{B}^\top$ given by (6.43).

(ii) and (iii) now follow by the same techniques as in the proof of Theorem 5.1 (ii) and (iii). □

6.3 Intertwining property of the two extended states

In fact the two states are not completely unrelated: They are intertwined with respect to the structural operator

$$F\colon \mathbb{R}^n \times L^2(-h,0;\mathbb{R}^n)\times L^2(-h,0;\mathbb{R}^m) \to \mathbb{R}^n \times L^2(-h,0;\mathbb{R}^n)\times L^2(-h,0;\mathbb{R}^k)$$

defined by the following matrix of operators:

$$F = \begin{bmatrix} I & 0 & 0\\ 0 & \bar{L} & \bar{B}_0\\ 0 & \bar{C} & \bar{B}_1\end{bmatrix}. \qquad (6.74)$$

Theorem 6.2. (i) *Given initial conditions ξ of the form*

$$\xi = (\phi^0, \bar{L}\phi^1+\bar{B}_0 w, \bar{C}\phi^1+\bar{B}_1 w) = F(\phi^0,\phi^1,w) \qquad (6.75)$$

for

$$(\phi^0,\phi^1,w)\in\mathbb{R}^n\times L^2(-h,0;\mathbb{R}^n)\times L^2(-h,0;\mathbb{R}^m), \qquad (6.76)$$

then for all u in $L^2_{\mathrm{loc}}(0,\infty;\mathbb{R}^m)$

$$\hat{x}(t;F(\phi^0,\phi^1,w),u) = F\tilde{x}(t;(\phi^0,\phi^1,w),u), \qquad (6.77)$$

where $\hat{x}$ is given by (6.26) and $\tilde{x}$ by (6.4).

(ii) *The following intertwining identities hold*:

$$\tilde{S}^{\top *}(t)F = F\tilde{S}(t), \quad \forall t \geq 0, \tag{6.78}$$

$$\hat{C}F = \tilde{C} \quad \text{on } D(\tilde{A}), \tag{6.79}$$

$$\hat{B}^{\top}F^{\top} = \tilde{B}^{\top} \quad \text{on } D(\tilde{A}^{\top}), \tag{6.80}$$

where

$$F^* = F^{\top} = \begin{bmatrix} I & 0 & 0 \\ 0 & \bar{L}^{\top} & \bar{C}^{\top} \\ 0 & \bar{B}_0^{\top} & \bar{B}_1^{\top} \end{bmatrix}. \tag{6.81}$$

Proof. (i) By definition (6.26)

$$\hat{x}(t) = (x(t), \bar{L}(e_+^0 x)_t + \bar{B}_0(e_+^0 u)_t + \tau(t)\xi^1, \bar{C}(e_+^0 x)_t + \bar{B}_1(e_+^0 u)_t + \tau(t)\xi^2).$$

By Lemma 4.4

$$\tau(t)[\bar{L}\phi^1 + \bar{B}_0 w] = \bar{L}(e_-^0 \phi^1)_t + \bar{B}_0(e_-^0 w)_t$$

and

$$\tau(t)[\bar{C}\phi^1 + \bar{B}_1 w] = \bar{C}(e_-^0 \phi^1)_t + \bar{B}_1(e_-^0 w)_t.$$

But

$$x_t = (e_+^0 x)_t + (e_-^0 \phi^1)_t, \quad u_t = (e_+^0 u)_t + (e_-^0 w)_t$$

and finally

$$\begin{aligned} \hat{x}(t) &= (x(t), \bar{L}x_t + \bar{B}_0 u_t, \bar{C}x_t + \bar{B}_1 u_t) \\ &= F(x(t), x_t, u_t) = F\tilde{x}(t). \end{aligned}$$

(ii) This is a special case of (i) with $u(t) = 0$, $t \geq 0$. From (6.33)

$$\begin{aligned} \tilde{S}^{\top *}(t)\xi &= (x(t), \bar{L}(e_+^0 x)_t + \tau(t)\xi^1, \bar{C}(e_+^0 x)_t + \tau(t)\xi^2) \\ &= (x(t), \bar{L}x_t + \tau(t)\bar{B}_0 w, \bar{C}x_t + \tau(t)\bar{B}_1 w) \\ &= (x(t), \bar{L}x_t + \bar{B}_0(e_-^0 w)_t, \bar{C}x_t + \bar{B}_1(e_-^0 w)_t) \\ &= F(x(t), x_t, (e_-^0 w)_t) = F\tilde{S}(t)(\phi_0, \phi_1, w) \end{aligned}$$

by definition (5.38) of the extended semigroup of Ichikawa.

To prove identity (6.79) we make use of the intertwining property (6.78) and Lemma 4.5, which says that the corresponding infinitesimal generators are also intertwined: For all $(\phi(0), \phi, w) \in D(\tilde{A})$,

$$FD(\tilde{A}) \subset D(\tilde{A}^{\top *}), \quad \tilde{A}^{\top *}F(\phi(0), \phi, w) = F\tilde{A}(\phi(0), \phi, w).$$

Hence

$$\begin{aligned} \hat{C}F(\phi(0), \phi, w) &= \hat{C}(\phi(0), \bar{L}\phi + \bar{B}_0 w, \bar{C}\phi + \bar{B}_1 w) \\ &= C\phi + (\bar{B}_1 w)(0). \end{aligned}$$

Recall that an element of $D(\tilde{A})$ is of the form $(\phi(0), \phi, w)$, where

$$w \in W^{1,2}(-h, 0; \mathbb{R}^m), \quad w(0) = 0.$$

By Lemma 4.6 applied to B_1 instead of L

$$(\bar{B}_1 w)(0) = B_1 w.$$

Finally for all $(\phi(0), \phi, w) \in D(\tilde{A})$

$$\hat{C}F(\phi(0), \phi, w) = C\phi + B_1 w = \tilde{C}(\phi(0), \phi, w)$$

for the operator $\tilde{C}$ defined by (6.10):

$$\hat{C}F = \tilde{C} \quad \text{on } D(\tilde{A}).$$

For the control operator we use the dual intertwining identity

$$F^\top \tilde{S}^\top(t) = \tilde{S}^*(t) F^\top$$

and for all $(\psi(0), \psi, v) \in D(\tilde{A}^\top)$

$$F^\top D(\tilde{A}^\top) \subset D(\tilde{A}^*), \quad \tilde{A}^* F^\top = F^\top \tilde{A}^\top.$$

By repeating the previous arguments

$$\begin{aligned} \hat{B}^\top F^\top(\psi(0), \psi, v) &= \hat{B}^\top(\psi(0), \bar{L}^\top \psi + \bar{C}^\top v, \bar{B}_0^\top \psi + \bar{B}_1^\top v) \\ &= B_0^\top \psi + (\bar{B}_1^\top v)(0) \end{aligned}$$

and as

$$(\psi(0), \psi, v) \in D(\tilde{A}^\top) \Longrightarrow v \in W^{1,2}(-h, 0; \mathbb{R}^k) \quad \text{and} \quad v(0) = 0,$$

we conclude from identity (4.79) in Lemma 4.6 applied to $B_1^\top$ that

$$(\bar{B}_1^\top v)(0) = B_1^\top v$$

and

$$\hat{B}^\top F^\top = B_0^\top \psi + B_1^\top v = \tilde{B}^\top \quad \text{on } D(\tilde{A}^\top). \qquad \square$$

Remark 6.2. The intertwining properties of Theorem 6.2 are new in the general case. Special cases were already considered in the case without delays in the observation (cf. M. C. Delfour [15]) and in the case without delays in the control and observation variables (cf. M. C. Delfour, E. B. Lee, and A. Manitius [1]). In the latter case, it was shown that the solutions of the Riccati equations for the states $\tilde{x}$ and $\hat{x}$ are also intertwined. $\square$

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Part III

Qualitative Properties of Infinite Dimensional Linear Control Dynamical Systems

1

Controllability and Observability for a Class of Infinite Dimensional Systems

1 Introduction

In §2.1 and §2.2 of Chapter 1 of Part I, we have discussed criteria for controllability, and observability for finite dimensional systems and have also shown that when the system is controllable we can transfer the state $z_0 \in H$ at time t_0 to the state $z_1 \in H$ at time t_1 using minimum energy controls. These results were obtained by considering the controllability operator

$$\begin{aligned} L_T \colon L^2(0,T;U) &\to H \\ \colon u &\longmapsto \int_0^T e^{(T-s)A} Bu(s)\, ds, \end{aligned}$$

and its adjoint

$$\begin{aligned} L_T^* \colon H &\to L^2(0,T;U) \\ \colon y &\longmapsto B^* e^{(T-\cdot)A^*} y, \end{aligned}$$

and studying the relation between the ranges and null spaces of these two operators and by showing that controllability is equivalent to invertibility of $L_T L_T^*$. As we have remarked (see Remark 2.1, Chapter 1 of Part I) in some sense the same ideas can be used to obtain characterizations of controllability when the spaces U and X are infinite dimensional Hilbert spaces, but at the expense of using much elaborate technical machinery. In this chapter we discuss questions of controllability for parabolic and second-order hyperbolic equations, the plate equation, and Maxwell's equations. We first deal with controllability of the abstract linear dynamical system

$$\begin{cases} z'(t) = Az(t) + Bu(t), \\ z(0) = 0, \end{cases} \tag{1.1}$$

evolving in a Hilbert space H and where A is the infinitesimal generator of a strongly continuous semi-group e^{tA} on H and $B \in \mathcal{L}(U, H)$, where U is a

Hilbert space, the control space. In an infinite dimensional setting there are at least two concepts of controllability: approximate controllability and exact controllability. These concepts are introduced in Definitions 2.1 and 2.2, and criteria of approximate controllability and exact controllability for the above abstract linear dynamical system are presented in part (b) of Proposition 3.1 in §3.1 and in (3.4) of §3.2. Approximate controllability and null controllability for parabolic equations (distributed, boundary, and pointwise control), are studied in §5 by converting it to the abstract model (1.1) in appropriate spaces. The exact controllability problem for hyperbolic equations (Neumann and Dirichlet boundary control), the plate equation, and Maxwell's equation is studied by first studying exact controllability of the model (1.1) by specializing A to the case of skew-symmetric operators (§6 and §7) and then converting the various partial differential equations to this abstract model in appropriate spaces where the assumptions of the abstract results are verified. The method of proof for the abstract model makes essential use of the eigenvalue–eigenfunction structure of an appropriate partial differential operator related to the generator A and hence assumptions need to be made on the control operator B in relation to these eigenfunctions (assumptions that clearly have to be verified in concrete cases), so that lower estimates on boundary operators can be obtained.

For partial differential equations, the study of controllability has some intrinsic interest. Indeed, it appears that the most natural approach to the exact controllability problem for hyperbolic equations is obtained by using the theory of pseudo-differential operators and micro-local analysis as developed by Melrose and Sjöstrand (see C. Bardos, G. Lebeau, and J. Rauch [1]). For our purposes the theory of controllability (in particular exact controllability in suitably defined spaces) provides a criterion for stabilizability, a requirement for infinite time quadratic control problems to be well posed (see Part V). The interplay between the choice of state and the control spaces in the formulation of infinite time quadratic control problems, regularity properties for solutions of partial differential equations and verifying exact controllability properties in these spaces (leading to stabilizability and uniform stabilizability) is subtle and has only been worked out in the eighties (see, in particular the recent work of I. Lasiecka and R. Triggiani [11], J. L. Lions [5], and Chapter 2 and 3 of Part V). A general methodological approach to these issues is via the Hilbert Uniqueness Method (HUM) of J. L. Lions.

The scheme of development adopted in this chapter, although not necessarily leading to the sharpest results, has considerable appeal (especially from an engineering viewpoint) and provides a bridge between the finite dimensional and infinite dimensional theory. It uses the controllability operator and minimum energy transfer between states in an essential way. It is therefore of some interest in placing it both in historical context and also in the context of most recent developments in the subject. For this purpose we cite three sources on which the remainder of this section is based (see I. Lasiecka and R. Triggiani [11], J. L. Lions [5], and D. L. Russell [1]). For finite dimensional systems, the concept of controllability first arose as a technical condition related to normality and abnormality of Calculus of Variations problems and in obtaining structural results on the reachable set (see J. P. LaSalle [1]). In the same way, the concept of approximate controllability and its characterization first arose in the work of Yu. V. Egorov [1] again as a technical requirement in the study of time-optimal control problems for parabolic systems. An important early contribution to controllability is H. O. Fattorini [1]. There is a close relation between approximate controllability and observability (see Definition 3.1 and Proposition 3.1). In concrete situations involving parabolic partial differential equations, approximate controllability is verified via observability using uniqueness and unique continuation theorems for the adjoint system (for an early use of this idea see J. L. Lions [3] who used a uniqueness theorem of S. Mizohata [1] to achieve the desired objective). Our results in §5, especially Theorem 5.2 and Lemma 5.1, implicitly use uniqueness and unique continuation ideas. For other work related to §5, see H. O. Fattorini [1, 2, 4]; R. C. MacCamy, V. J. Mizel, and T. I. Seidman [1]; E. J. P. G. Schmidt and N. Weck [1]; and §3.3, Chapter 2, Part V.

A deeper issue in the context of parabolic systems is the concept of null controllability (exact controllability to the origin), that is, the ability of transferring an arbitrary state $z_0 \in D(A)$ to the zero state in time $T > 0$ using admissible controls. This question and its observability counterpart have been investigated by several authors (see, for example, H. O. Fattorini and D. L. Russell [1], V. J. Mizel and T. I. Seidman [1, 2], T. I. Seidman [1], and W. Littman [1]). Most of the results known here are for parabolic equations in one-dimension (the work of Seidman being an exception) and are based on deep results in Harmonic Analysis (excepting the work of Littman, which uses different ideas) related to the independence of exponentials $e^{-\lambda_k t}$, where λ_k are eigenvalues of appropriate partial differential operators entering the dynamical system. The work in the early seventies started in one-dimension, but both V. J. Mizel and T. I. Seidman [1, 2] and H. O. Fattorini and D. L. Russell [1] proved the result for balls, and it was well known that this gave the result for general domains by extension to a ball containing the domain and then restriction to the domain and its boundary. Null controllability is not discussed here, but we conjecture that the methods that we adopt would lead to results for parabolic equations.[1]

The focus of this chapter is on exact controllability (which is often a mean of verifying stabilizability) for second-order hyperbolic equations, plate equations, and Maxwell's equations. The reason for this is that there is an essential difference between the problem of stabilizability (uniform stabilizability) for parabolic systems and for hyperbolic systems. In the parabolic case, we are generally dealing with a semigroup that is analytic with a compact resolvent

[1] Since 1993, this topic has received more attention. The reader is referred to the book of A. V. Fursikov and O. Yu. Imanuvilov [1] in 1996 that studies null controllability via Carleman estimates.

and hence A (the generator) has only finitely many unstable eigenvalues with finite multiplicity. In this case we have to check whether the projection of A on the unstable subspace is controllable, and this can be done via the Pole-Assignment Theorem (see Theorem 2.4, Chapter 1, Part I). These issues are dealt with in §3.3 of Chapter 1 of Part V and §3.3 of Chapter 2 of Part V. Concrete situations, such as the heat equation with control exercised through Dirichlet or Neumann boundary conditions, can be handled using these ideas (see Remark 3.1 of Chapter 2 of Part V for bibliographical references).

For second-order hyperbolic equations (and for plate and Maxwell's equation) the stabilizability question is far more difficult because these systems have an infinite dimensional unstable (marginally stable) part. It is here that the study of exact controllability is most important and this has been carried out in J. L. Lions [4, 5] and I. Lasiecka and R. Triggiani [9, 10, 13] and others (see the bibliographical references cited in the above-mentioned works). We remark here that the key to these results are estimates on boundary operators.

Let us examine this in the context of the wave equation with Dirichlet control (see Example 2.1 in Chapter 3 of Part V).

In this case we denote by Λ the operator

$$\Lambda h = -\Delta h, \quad \mathcal{D}(\Lambda) = H^2(\Omega) \cap H_0^1(\Omega),$$

and by D the Dirichlet map

$$Dv = y \Longleftrightarrow \{\Delta y = 0 \quad \text{in } \Omega \quad \text{and} \quad y_{\partial\Omega} = v\}.$$

and the state space $H = L^2(\Omega) \times H^{-1}(\Omega)$ and the control space $U = L^2(\partial\Omega)$. Then the abstract model is

$$A = \begin{bmatrix} 0 & I \\ -\Lambda & 0 \end{bmatrix}, \quad Bu = \begin{bmatrix} 0 \\ \Lambda D u \end{bmatrix}.$$

We may then compute

$$B^* \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = D^* y_2 = -\frac{\partial}{\partial \nu} \Lambda^{-1} y_2 \quad \text{since } D^* \Lambda = -\frac{\partial}{\partial \nu}.$$

Hence

$$B^* e^{A^* t} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{\partial \varphi}{\partial \nu}(t), \quad (y_1, y_2) \in L^2(\Omega) \times H^{-1}(\Omega),$$

and φ is the solution of the homogeneous equation

$$\begin{cases} \varphi_{tt} = \Delta\varphi \quad \text{in } \Omega \times]0,T[\, \varphi = 0 \quad \text{in } \partial\Omega \times]0,T[\\ \varphi(\cdot,0) = \varphi_0 & \text{in } \Omega. \\ \varphi_t(\cdot,0) = \varphi_1 \end{cases}$$

Now

$$\varphi_0 = \Lambda^{-1} y_2 \in \mathcal{D}(\Lambda^{1/2}) = H_0^1(\Omega), \quad \varphi_1 = y_1 \in L^2(\Omega).$$

The two relevant estimates are (see Example 4.3 in Chapter 3 of Part IV and Example 2.1 in Chapter 3 of Part V):

$$\text{(i)} \qquad \int_\Sigma \left| \frac{\partial \varphi}{\partial \nu} \right|^2 d\Sigma \leq c(T) \| \{\varphi_0, \varphi_1\} \|^2_{H_0^1(\Omega) \times L^2(\Omega)},$$

$$\text{(ii)} \qquad \int_\Sigma \left| \frac{\partial \varphi}{\partial \nu} \right|^2 d\Sigma \geq c(T) \| \{\varphi_0, \varphi_1\} \|^2_{H_0^1(\Omega) \times L^2(\Omega)}$$

at least for large T, where $\Sigma = \partial\Omega \times]0, T[$. By virtue of the fact that

$$B^* e^{A^* t} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{\partial \varphi}{\partial \nu}(t),$$

the inequality (ii) above is equivalent to the invertibility of the controllability operator

$$\int_0^T e^{tA} B B^* e^{tA^*} \, dt,$$

and this is the criterion for exact controllability.

In the case of control through Neumann Boundary conditions it is difficult to identify the space on which the above estimates on boundary operators hold.

For a discussion of the above examples in the abstract setting developed in this chapter, see Theorem 6.1 and §8.1 and §8.2.

2 Main definitions

2.1 Notation

Let H be a complex Hilbert space, identified with its dual. More specifically H is the complexified version of a real Hilbert space, also denoted by H to save the notation. This means that an element of H complexified is of the form

$$h = h_1 + i h_2, \quad \text{with } h_1, h_2 \in H.$$

The norm in H real is denoted by $|\cdot|$ and the scalar product by $(\cdot, \cdot)$. In H (complexified) the scalar product of h and h' is denoted by $(h, \bar{h}')$ where $\bar{h}'$ is the conjugate of h. The norm of h is $|h| = (h, \bar{h})^{1/2}$.

We consider a linear operator $A \colon D(A) \to H$ and $B \in \mathcal{L}(U; H)$, where U is also identified with its dual. We call H the state space and U the control space. The norm and scalar product in U is denoted in the same way as for H, and possibly with a subscript U when there is a risk of confusion. A and B are defined when H and U use scalars that are real valued, and are real valued. They are extended to the complexified versions of H and U by setting

$$Ah = Ah_1 + iAh_2$$

and similar formula for B. In the sequel, the complexified versions are needed because of eigenvalues (in fact only starting with §6). Note that the operator

$$A^*h = A^*h_1 + iA^*h_2$$

satisfies the condition

$$\forall h, h', \quad (A^*h, \bar{h}') = (h, A\bar{h}')$$

and is the adjoint of A considered as a linear operator in H complexified.

We consider the linear dynamical system

$$\begin{cases} z'(t) = Az(t) + Bv(t), & t \in (0,T), \\ z(0) = 0. \end{cases} \tag{2.1}$$

We assume that A is the infinitesimal generator of a strongly continuous semigroup e^{At} in H. The control function $v(\cdot)$ belongs to $L^2(0,T;U)$ (U is the ordinary real version). The solution of (2.1) is denoted by $z(t;v)$ and naturally

$$z(t;v) = \int_0^t e^{(t-s)A} Bv(s)\, ds. \tag{2.2}$$

2.2 Definitions

We set

$$F_T = \{z(T;v) \colon \forall v \in L^2(0,T;U)\}.$$

We can provide F_T with the structure of a Hilbert space thanks to the following.

Proposition 2.1. *The space F_T can be structured as a Hilbert space, with continuous injection in H.*

Proof. Consider the closed subspace of $L^2(0,T;U)$ made of $v(\cdot)$ such that $z(T;v) = 0$. Let N_T be this subspace, and $L^2(0,T;U)/N_T$ be the Hilbert quotient space of $L^2(0,T;U)$ by N_T. If $u(\cdot)^\circ$ is an element of $L^2(0,T;U)/N_T$, and $u(\cdot)$ is a representative of $u(\cdot)^\circ$, then the quotient norm is given by

$$|u^\circ|^2 = \min\{J(v) \colon v \in L^2(0,T;U), v - u \in N_T\}, \tag{2.3}$$

where we have set

$$J(v) = \int_0^T |v(t)|^2\, dt.$$

If $\hat{u}$ realizes the minimum in (2.3), then the scalar product in

$$L^2(0,T;U)/N_T$$

is defined by

$$(u^\circ, v^\circ) = \int_0^T \big(\hat{u}(t), \hat{v}(t)\big)\, dt.$$

We define the map $\varPsi_T$ from $L^2(0,T;U)/N_T$ to F_T by setting

$$\varPsi_T(u^\circ) = z(T;u)$$

and $\varPsi_T$ is clearly a bijection. We define on F_T a structure of Hilbert space by setting

$$\big((\xi,\eta)\big) = (\varPsi_T^{-1}\xi, \varPsi_T^{-1}\eta). \tag{2.4}$$

By construction $\varPsi_T$ is an isometry. It remains to show that the injection of F_T into H is continuous. We show that

$$\frac{\|\xi\|}{|\xi|} \geq c, \quad \forall \xi \in F_T. \tag{2.5}$$

Indeed if (2.5) does not hold, then there exists a sequence ξ_n in F_T, such that $|\xi_n| = 1$, and $\xi_n \to 0$ in F_T. Let $u_n^\circ = \varPsi_T^{-1}\xi_n$, and $\hat{u}_n$ be the representative of u_n° of minimum norm; then $\hat{u}_n \to 0$ in $L^2(0,T;U)$ and $\xi_n = z(T;\hat{u}_n)$; therefore $\xi_n \to 0$ in H, which contradicts the fact that $|\xi_n| = 1$. The desired result has been proved. □

Let $\varPsi_T^*$ be the adjoint of $\varPsi_T$. It is an isometry from F_T' to $L^2(0,T;U)/N_T$. Therefore the map $\varPsi_T\varPsi_T^*$ is an isometry from F_T' to F_T. If $\xi \in F_T$, defining $\zeta_* \in F_T'$ by the equation

$$\varPsi_T\varPsi_T^*\zeta_* = \xi \tag{2.6}$$

and considering the class $\varPsi_T^*\zeta_* \in L^2(0,T;U)/N_T$, then any representative of this class will realize ξ. The canonical isomorphism from F_T into F_T' is $(\varPsi_T\varPsi_T^*)^{-1} = (\varPsi_T^*)^{-1}(\varPsi_T)^{-1}$. The norm product in F_T' is $\|\zeta_*\| = |\varPsi_T^*\zeta_*|$.

Consider now the operator

$$\varGamma_T = \int_0^T e^{tA}BB^*e^{tA^*}\, dt,$$

which belongs to $\mathcal{L}(H;H)$. We call $\varGamma_T$ the *controllability operator*. We can decompose $\varGamma_T$ as a product of an operator with its transpose. For that, introduce π the injection of F_T in H and π^* its transpose. Note that π^*H is dense in F_T'. We can then give an interpretation of $\psi_T^*\pi^*$. We have

$$\psi_T^*\pi^*h(t) = \text{equivalence class of } B^*e^{(T-t)A^*}h. \tag{2.7}$$

Let $u^0 \in L^2(0,T;U)/N_T$. We have

$$(\psi_T^*\pi^*h, u^0) = \big(h, z(T;\hat{u})\big),$$

where $\hat{u}$ is the representative of minimum norm of u^0. Hence,

$$(\psi_T^* \pi^* h, u^0) = \int_0^T \big(B^* e^{(T-t)A^*} h, \hat{u}(t)\big)\, dt.$$

Therefore what remains to be done is to show that $B^* e^{(T-t)A^*} h$ minimizes the norm of elements that belong to its equivalence class. According to (2.3) we look for the element $\hat{v}$ that minimizes

$$J(v) = \int_0^T |v(t)|^2\, dt$$

among all v such that

$$\int_0^T e^{(T-t)A} B(v(t) - B^* e^{(T-t)A^*} h)\, dt = 0. \tag{2.8}$$

The optimal $\hat{v}$ satisfies (2.8) and

$$\int_0^T \big(\hat{v}(t), v(t)\big)\, dt = 0$$

for all v such that

$$\int_0^T e^{(T-t)A} B v(t)\, dt = 0.$$

It is clear that $\hat{v}(t) = B^* e^{(T-t)A^*} h$ satisfies this necessary condition, and thus the property (2.7) has been demonstrated.

From (2.7) we deduce at once the decomposition

$$\mathit{\Gamma}_T = \pi \psi_T \psi_T^* \pi^*. \tag{2.9}$$

We now introduce the following.

Definition 2.1. The pair (A, B) is *approximately controllable at time* T whenever F_T is *densely* embedded in H. □

If the pair (A, B) is approximately controllable, then the map π^* is injective. In that case we can identify an element h of H, with its image $\pi^* h$, and $\pi^* H$ with H. In that case we have

$$F_T \subset H \subset F_T' \tag{2.10}$$

with continuous and dense embedding of each space in the following one.

Moreover from (2.9) we have

$$\mathit{\Gamma}_T = \psi_T \psi_T^* \tag{2.11}$$

and $\mathit{\Gamma}_T$ extends as an isometry from F_T' to F_T. If $h \in H$, then

$$\|h\|_{F_T'} = (\mathit{\Gamma}_T h, h)^{1/2} \tag{2.12}$$

and F_T' appears as the completion of H with respect to the norm (2.12).

Suppose that the pair (A, B) is approximately controllable; then $D(A)$ and F_T are two dense subspaces of H. An important question is the comparison between these two spaces. This justifies the following definition.

Definition 2.2. The pair (A, B) is *exactly controllable at time* T whenever

$$D(A) \subset F_T$$

with continuous and dense injection. □

One may wonder why we do not use as a definition of exact controllability at time T, the property $F_T = H$. The reason is that as soon as H is infinite dimensional, this property is not generally verified, except for special families such as the hyperbolic systems. To partially support this assertion, we can state the following

Proposition 2.2. R. TRIGGIANI[1] *If H is infinite dimensional and B is compact, the property $F_T = H$ cannot hold for finite time T.*

Proof. The operator B is compact, there exists a non unique sequence of orthonormal vectors w_k of U and positive numbers α_k, which tends monotonically to 0 as k tends to infinity, such that

$$w'_k = \frac{1}{\alpha_k} B w_k \text{ is an orthonormal sequence of } H, \tag{2.13}$$

$$Bv = \sum_{k=0}^{\infty} \alpha_k (v, w_k) w'_k. \tag{2.14}$$

Let $\{v_m\}$ be a sequence in $L^2(0, T; U)$, which tends to 0 weakly and $z_m(T) = z(T; v_m)$. We shall show that

$$z_m(T) \to 0 \text{ strongly in } H. \tag{2.15}$$

If (2.15) is proved, then defining $F_T^n =$ subset of F_T obtained with v restricted to the ball of center 0 and radius n, it is clear that F_T^n has a compact closure in H. As H is infinite dimensional, $\bar{F_T^n}$ has an empty interior. It is nowhere dense.

Now

$$F_T \subset \bigcup_{n=1}^{\infty} \bar{F_T^n}.$$

Therefore, according to Baire Category theorem, F_T cannot coincide with H. Similarly

$$\overline{\bigcup_{T>0} F_T} = \overline{\bigcup_{m>0} F_m} \subset \bigcup_{n,m=1}^{\infty} \bar{F_m^n}$$

cannot coincide either with H.

It remains to prove (2.15). Using (2.13) one has

$$z_m(T) = \sum_{k=0}^{\infty} \alpha_k \int_0^T e^{(T-t)A} w'_k (v_m(t), w_k)\, dt. \tag{2.16}$$

Let

$$z_{m,N}(T) = \sum_{k=0}^{N} \alpha_k \int_0^T e^{(T-t)A} w_k'(v_m(t), w_k)\, dt.$$

As v_m tends to 0 weakly in $L^2(0,T;U)$, its norm remains bounded. Then it is easy to check that

$$|z_m(T) - z_{m,N}(T)| \leq C\alpha_N$$

independently of m. As α_N tends to 0 as N tends to infinity, to prove (2.15) it is sufficient to prove that $z_{m,N}(T)$ tends to 0 as m tends to infinity, for fixed N. For this it is enough to prove that

$$\int_0^T e^{(T-t)A} h f_m(t)\, dt \to 0 \quad \text{in } H$$

whenever f_m tends to 0 weakly in $L^2(0,T)$ and h is a fixed element of H. This is an easy consequence of the fact that $t \mapsto e^{(T-t)A}h$ is a continuous function on $[0,T]$ with values in H. The proof has been completed. □

In the sequel, we shall be interested in obtaining criteria for approximate and exact controllability.

3 Criteria for approximate and exact controllability

3.1 Criterion for approximate controllability

We introduce the following definition.

Definition 3.1. Given $T > 0$, we say that the pair (A^*, B^*) is observable on $(0,T)$ when

$$B^* e^{tA^*} h = 0 \quad \text{on } (0,T) \implies h = 0. \tag{3.1}$$

□

We can state the following result.

Proposition 3.1. *The following statements are equivalent*:

(a) *The pair* A^*, B^* *is observable on* $(0,T)$,
(b) $(\Gamma_T h, h)^{1/2}$ *is a norm on* H,
(c) *the pair* (A,B) *is approximately controllable at* Γ.

Proof. If the observability property holds, then $(\Gamma_T h, h) \geq 0$ and $(\Gamma_T h, h) = 0$ implies $B^* e^{tA^*} h = 0$ on $(0,T)$; therefore, $h = 0$.

If $(\Gamma_T h, h)^{1/2}$ is a norm on H, then if h satisfies $B^* e^{tA^*} h = 0$ on $(0,T)$, $(\Gamma_T h, h) = 0$ hence $h = 0$. Hence (a) and (b) are equivalent.

Now approximate controllability at T is equivalent to

$$(z(T;v),h)=0, \quad \forall v \implies h=0. \tag{3.2}$$

But

$$(z(T;v),h)=\left(\int_0^T e^{(T-t)A}Bv(t)\,dt, h\right)=\int_0^T (v(t), B^* e^{(T-t)A^*}h)\,dt.$$

Therefore

$$\forall v, \quad (z(T;v),h)=0 \iff B^* e^{(T-t)A^*}h=0$$

and (3.2) is thus the same thing as observability. Hence (a) and (c). □

3.2 Criteria for exact controllability and continuous observability

From our definition of exact controllability we deduce the sequence

$$D(A)\subset F_T \subset H \subset F_T' \subset \big(D(A)\big)', \tag{3.3}$$

each space being densely and continuously embedded in the following one. From (2.12) we deduce

$$(\Gamma_T h,h)\geq c_T\|h\|^2_{(D(A))'} \tag{3.4}$$

for any $h\in H$.

Conversely, if (3.4) holds, then (3.3) holds. Indeed necessarily

$$F_T' \subset \big(D(A)\big)'. \tag{3.5}$$

To prove (3.5) consider the vector space $F_T'\cap\big(D(A)\big)'$ provided with the norm

$$\|\xi\|_{F_T'\cap(D(A))'}=\max\{\|\xi\|_{F_T'}, \|\xi\|_{(D(A))'}\}.$$

Let $\xi\in F_T'$ and consider a sequence $\{\xi_n\}$ in H, $\xi_n \implies \xi$ in F_T'. Note that $\{\xi_n\}$ is a Cauchy sequence in F_T' and from (3.4) a Cauchy sequence in $\big(D(A)\big)'$. Therefore ξ_n is a Cauchy sequence in $F_T'\cap\big(D(A)\big)'$. It follows that $\xi\in F_T'\cap\big(D(A)\big)'$, in particular to $\big(D(A)\big)'$. Hence (3.5). From (3.4) the injection is clearly continuous. As H is dense in $\big(D(A)\big)'$, F_T' is also densely embedded in $\big(D(A)\big)'$. By duality we deduce (3.3). Therefore, the estimate (3.4) is a *criterion of exact controllability.*

Remark 3.1. There is an analog of Proposition 3.1 for exact controllability. For this purpose, define the dual system as follows: First introduce the observation equation for the system (2.1)

$$y(t)=Cz(t),$$

where $C\in\mathcal{L}(H;Y)$, and the observation space Y is a Hilbert space identified with its dual. Then introduce the dual system (backward equation)

$$\begin{cases} p'(t) = -A^*p(t) + C^*y(t), \\ p(T) = 0, \end{cases}$$

and the dual observation equation

$$u(t) = B^*p(t).$$

We then say that the pair (A, C) is continuously observable if and only if the pair (A^*, C^*) is exactly controllable at time 0 (starting at time T). Therefore we obtain a criterion analogous to (3.4) for continuous observability. $\square$

3.3 Approximation

Let h be an element of H. There exists a sequence $\{\xi_n\}$ of elements of F_T which converges to h in H. Solving the equation

$$\Psi_T \Psi_T^* \zeta_n^* = \xi_n$$

defines a unique ζ_n^* in F_T'. Any control of the equivalence class $\Psi_T^*\zeta_n^*$ realizes ξ_n. In this way we can construct a sequence of controls whose corresponding state at time T is as close as possible to h. But this procedure is not very constructive. A constructive approach is obtained in the following way: Define

$$J_\varepsilon(v) = J(v) + \frac{1}{\varepsilon}|z(T; v) - h|^2 \tag{3.6}$$

with

$$z(T; v) = \int_0^T e^{(T-t)A} Bv(t)\, dt \tag{3.7}$$

and minimize $J_\varepsilon(v)$ over all $v \in L^2(0, T; U)$. This problem has a unique solution u_ε. We have

$$|z(T; u_\varepsilon) - h|^2 \le |z(T; v) - h|^2 + \varepsilon J(v), \quad \forall v,$$

and thus

$$\limsup_{\varepsilon \to 0} |z(T; u_\varepsilon) - h|^2 \le \inf |z(T; v) - h|^2 = 0.$$

Therefore the sequence $z(T; u_\varepsilon)$ belongs to F_T and converges to h in H as ε tends to 0. Now we may write the necessary conditions of optimality for u_ε. It is easy to check that if p_ε is the adjoint state defined by

$$-p_\varepsilon' = A^*p_\varepsilon, \quad p_\varepsilon(T) = \frac{1}{\varepsilon}(x_\varepsilon(T) - h), \tag{3.8}$$

then

$$u_\varepsilon(t) = -B^*p_\varepsilon(t) = -B^*e^{(T-t)A^*}p_\varepsilon(T) = -\Psi_T^* p_\varepsilon(T). \tag{3.9}$$

Therefore

$$z(T; u_\varepsilon) = -\Gamma_T p_\varepsilon(T). \tag{3.10}$$

Since $p_\varepsilon(T) \in H$ and not just to F'_T, we have an even better result than that explained at the beginning of the paragraph.

If h belongs to F_T, then we can say something more. Indeed let $\hat{u}$ be such that

$$\Psi_T \hat{u} = h$$

and $\hat{u}$ has minimum L^2 norm. As

$$J(u_\varepsilon) \leq J(\hat{u}), \tag{3.11}$$

it follows that u_ε remains bounded in L^2. From the formula (3.9), it follows that $p_\varepsilon(T)$ remains in a bounded subset of F'_T. We pick a subsequence, still denoted by $p_\varepsilon(T)$, which converges weakly to ζ_* in F'_T. Going to the limit in (3.10) yields

$$h = -\Gamma_T \zeta_*. \tag{3.12}$$

From (3.11) and the minimality of $\hat{u}$, it is easy to check that u_ε tends to $\hat{u}$ in $L^2(0, T; U)$. From (3.9) we deduce also

$$\hat{u} = -\Psi_T^* \zeta_*, \tag{3.13}$$

and thus we have an approximation procedure for the element ζ_*.

We refer to A. Bel Fekih [1] for related topics.

4 Finite dimensional control space

We shall consider in this section the case when U is finite dimensional, $U = \mathbb{R}^m$, and make precise the conditions of controllability (clearly approximate and exact controllability coincide in that case).

4.1 Finite dimensional case

Assume here that $H = \mathbb{R}^n$ and $U = \mathbb{R}^m$, $A \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n)$, $B \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n)$. We have the following classical result due to R. E. Kalman [1], for which see Theorems 2.1 and 2.2, Part I, Chapter 1.

Proposition 4.1. *The pair (A, B) is controllable at any time $T > 0$ if and only if the matrix $[B\ AB \ldots A^{n-1}B] \in \mathcal{L}(\mathbb{R}^{nm}; \mathbb{R}^n)$ has* rank n.

Consider now the case of diagonalizable operators

$$A = \sum_{j=1}^{J} \lambda_j P_j, \tag{4.1}$$

where λ_j is complex valued, P_j is a projector, satisfying

$$P_j = P_j^*, \quad P_j P_k = \delta_{jk}, \quad \sum_{j=1}^{J} P_j = I.$$

Note that we have complexified the set up, so P_j^* means

$$(P_j^* h, k) = (h, \bar{P_j} k) = (P_j h, k);$$

hence

$$A^* = \sum_{j=1}^{J} \bar{\lambda}_j P_j.$$

We then state the following proposition.

Proposition 4.2. *Assume* (4.1)*; then the pair* (A, B) *is controllable at any time* $T > 0$*, if and only if for any* j*, the operator* $P_j B B^* P_j$ *has full rank, in the subspace* $P_j H$*.*

Proof.

- The condition is necessary.

Suppose otherwise that there exists some h, with $P_j h \neq 0$, and $B^* P_j h = 0$; then

$$\begin{bmatrix} B^* \\ B^* A^* \\ \cdots \\ \cdots \\ B^* A^{*n-1} \end{bmatrix} P_j h = \begin{bmatrix} B^* P_j h \\ \bar{\lambda}_j B^* P_j h \\ \cdots \\ \cdots \\ \bar{\lambda}_j^{\,n-1} B^* P_j h \end{bmatrix} = 0,$$

which contradicts the fact that the pair (A, B) is controllable.

- The condition is sufficient.

Consider an element h such that $(\Gamma_T h, h) = 0$; hence $B^* e^{A^* t} h = 0$; i.e.,

$$\sum_{j=1}^{J} \exp(\bar{\lambda}_j t) B^* P_j H = 0, \quad t \in (0, T). \tag{4.2}$$

From the analyticity of the function to the left of (4.2), it follows that (4.2) holds for any $t \in (-\infty, +\infty)$. It first follows that $B^* P_j h = 0$, for any j such that Re $\lambda_j \neq 0$. Indeed, suppose λ_1 is such that $|\operatorname{Re} \lambda_1| > |\operatorname{Re} \lambda_j|$, $\forall j \neq 1$. We can write

$$B^* P_1 h + \sum_{j \neq 1} \exp\big((\bar{\lambda}_j - \bar{\lambda}_1) t\big) B^* P_j h = 0.$$

Taking the limit as t tends to $+$ or $-$ infinity, according to the fact that $\operatorname{Re} \lambda_1$ is positive or negative, we deduce $B^* P_1 h = 0$. More generally, there may be a set J_0 of indices j such that $\operatorname{Re} \lambda_j = \operatorname{Re} \lambda_1$, for $j \in J_0$, and $|\operatorname{Re} \lambda_1| > |\operatorname{Re} \lambda_j|$, $\forall j \notin J_0$. Dividing (4.2) by $\exp(\operatorname{Re} \lambda_1 t)$ we deduce

$$\sum_{j\in J_0} e^{-i\gamma jt} B^* P_j h + \sum_{j\notin J_0} \exp(\bar{\lambda_j} - \operatorname{Re}\lambda_1)tB^* P_j h = 0, \tag{4.3}$$

where $\gamma_j = \operatorname{Im}\lambda_j$. Suppose to fix the ideas that $\operatorname{Re}\lambda_1 > 0$. We derive from (4.3)

$$0 = \int_0^T \left| \sum_{j\in J_0} e^{-i\gamma_j t} B^* P_j h \right|^2 dt \\ + \int_0^T \left(\sum_{j\in J_0} e^{+i\gamma_j t} B^* \bar{P}_j h, \sum_{k\notin J_0} \exp(\bar{\lambda_k} - \operatorname{Re}\lambda_1)tB^* P_k h \right) dt;$$

hence

$$0 = T\sum_{j\in J_0} (P_j BB^* P_j h, h) + \sum_{j\in J_0} \sum_{\substack{k\in J_0 \\ k\neq j}} \frac{\exp(i(\gamma_k - \gamma_j)T) - 1}{i(\gamma_k - \gamma_j)} (P_k BB^* P_j h, h) \\ + \sum_{j\in J_0} \sum_{k\neq J_0} \frac{\exp(\bar{\lambda}_k - \operatorname{Re}\lambda_1 + i\gamma_j)T - 1}{\bar{\lambda}_k - \operatorname{Re}\lambda_1 + i\gamma_j} (P_j BB^* P_k h, h).$$

Dividing by T, and letting T tend to ∞, we deduce

$$\sum_{j\in J_0} (P_j BB^* P_j h, h) = 0,$$

or $B^* P_j h = 0$, $\forall j \in J_0$. If $\operatorname{Re}\lambda_1 < 0$, one should consider a similar integral between $-T$ and 0, to deduce a similar result.

Successively, we can treat in the same way all j such that $\operatorname{Re}\lambda_j \neq 0$. So we have

$$B^* P_j h = 0, \quad \forall j \quad \text{with } \operatorname{Re}\lambda_j \neq 0.$$

There remains the case when (4.2) reduces to

$$\sum_{j\in J_0} e^{-i\gamma_j t} B^* P_j h = 0, \quad \forall t \in (-\infty, +\infty),$$

where $\lambda_j = i\gamma_j$, γ_j real, and J_0 is a subset of $\{1, \ldots, J\}$. We make a calculation similar to that done after (4.3) to prove that

$$\sum_{j\in J_0} (P_j BB^* P_j h, h) = 0,$$

or $B^* P_j h = 0$, $\forall j \in J_0$.

Hence, we have proved that $B^* P_j h = 0$, $\forall j$, and this implies $h = 0$, from the assumption. $\square$

4.2 General state space

We now turn to the case of a general state space, with finite dimensional control space $U = R^m$. We have

$$Bu = \sum_{i=1}^{m} b_i u_i, \quad \text{with } b_1, b_2, \ldots, b_m \in H, \quad u \in \mathbb{R}^m. \tag{4.4}$$

We state (see L. Markus [1]) the following theorem.

Theorem 4.1. *Assume that*

$$b_1, b_2, \ldots, b_m \in D_\infty(A) = \bigcap_{k>0} D(A^k).$$

If the linear set generated by the vectors

$$A^k b_j, \quad j = 1, \ldots, m, \qquad k = 0, 1, \ldots$$

is dense in H, then the pair (A, B) is approximately controllable at any time $T > 0$. Conversely, assume that $b_j = e^{t_j A}\bar{b_j}$, with $\bar{b_j} \in D_\infty(A)$, $t_j > 0$, and the trajectories $(e^{tA}\bar{b_j}, z)$ for any $z \in H$ are real analytic; then if the pair (A, B) is approximately controllable at some $T > 0$, necessarily the linear set generated by the vectors $\{A^k b_j, j = 1, \ldots, m, k = 0, 1, \ldots\}$ is dense in H.

Proof. If A, B is not approximately controllable at $T > 0$, then A^*, B^* is not observable on $(0, T)$ (see Proposition 3.1). Hence

$$\exists h \neq 0, \quad \text{such that } \forall t \in (0, T);\ B^* e^{tA^*} h = 0;$$

hence also

$$\forall u \in \mathbb{R}^m, \qquad (e^{tA} Bu, h) = 0 \quad \text{in } (0, T).$$

This implies

$$\forall j = 1, \ldots, m, \qquad (e^{tA} b_j, h) = 0 \quad \text{in } (0, T). \tag{4.5}$$

From the assumption on the b_j's, we can differentiate in t as many times as we wish and set $t = 0$. We deduce

$$\forall k > 0, \qquad j = 1, \ldots, m, \quad (A^k b_j, h) = 0. \tag{4.6}$$

As $h \neq 0$, this contradicts the fact that the set generated by the vectors $\{A^k b_j, j = 1, \ldots, m, k = 0, 1, \ldots\}$. is dense in H.

Let us now prove the second part of the statement. Assume that the set generated by the vectors $\{A^k b_j, j = 1, \ldots m, k = 0, 1, \ldots\}$ is not dense in H. There exists $h \neq 0$ such that (4.6) holds. This means also

$$(A^k e^{t_j A}\bar{b_j}, h) = 0. \tag{4.7}$$

Set

$$f_j(t) = (e^{tA}\bar{b}_j, h),$$

which is real analytic. Then (4.7) means that $f_j(t_j) = 0$, $f_j^{(k)}(t_j) = 0$, $\forall k > 0$. Therefore, $f_j(t) = 0$, $\forall t > 0$. Writing $f_j(t + t_j) = 0$, we have also (4.5). Therefore the pair (A^*, B^*) is not observable on $(0, T)$, which implies that the pair cannot be approximately controllable at T. $\square$

Let us give a variant of Theorem 4.1 in the case when $-A$ has a discrete spectrum $\lambda_1 < \lambda_2 < \cdots \lambda_k \cdots$ with $\lambda_k \uparrow \infty$ and λ_k has multiplicity r_k. It bears similarity with Proposition 4.2. Using the projector P_k on the finite dimensional eigenspace corresponding to λ_k, the semigroup e^{tA} is then represented by the expansion

$$e^{tA}h = \sum_k e^{-\lambda_k t} P_k h. \tag{4.8}$$

We consider P_k as an element of $\mathcal{L}(H; H)$. It is self-adjoint and $P_k^2 = P_k$. We have

$$z(T; u) = \sum_k \int_0^T e^{-(T-t)\lambda_k} P_k B u(t)\, dt, \tag{4.9}$$

$$(\varGamma_T h, h) = \int_0^T \left| \sum_k e^{-\lambda_k t} B^* P_k h \right|^2 dt. \tag{4.10}$$

We shall prove the following result.

Theorem 4.2. *Assume that* (4.8) *is verified. Then the pair* (A, B) *is approximately controllable at* T *if and only if* $P_k B B^* P_k$ *is full rank, for any* k.

Proof. From Proposition 3.1 we know that the pair (A, B), is approximately controllable at T if and only if $(\varGamma_T h, h)$ is a norm on H. From the expression (4.10), we deduce that $(\varGamma_T h, h)$ is a norm on H if and only if

$$\sum_k e^{-\lambda_k t} B^* P_k h = 0 \quad \text{on } (0, T) \implies h = 0. \tag{4.11}$$

The function

$$t \mapsto \sum_k e^{-\lambda_k t} B^* P_k h$$

being analytic, the left part of the statement (4.11) implies also

$$\sum_k e^{-\lambda_k t} B^* P_k h = 0, \quad \forall t \geq 0.$$

Multiplying by $e^{\lambda_1 t}$ and letting t tend to infinity, we deduce $B^* P_1 h = 0$, and successively $B^* P_k h = 0$. Therefore (4.11) means also

$$B^* P_k h = 0, \quad \forall k \geq 0 \implies h = 0. \tag{4.12}$$

It is easy to check that (4.12) is equivalent to the property that the matrix $P_k B B^* P_k$ be full rank for any k. $\square$

Remark 4.1. The restriction of the system to E_k (eigenspace corresponding to λ_k) is given by the dynamics

$$z_k' = -\lambda_k z_k + B_k u, \quad z_k(0) = 0, \tag{4.13}$$

where $B_k = P_k B$. The state z_k lies in the finite dimensional space E_k. The pair $(-\lambda_k I, B_k)$ is controllable. □

Remark 4.2. Note that the condition (4.12) holds also when U is not finite dimensional. Note also that from the formula (4.10) it follows that there exists a positive constant c such that

$$(\mathit{\Gamma}_T h, h) \leq c \sum_k \frac{(1 - e^{-2\lambda_k T})|P_k h|^2}{2\lambda_k}. \tag{4.14}$$

□

5 Controllability for the heat equation

5.1 Distributed control

We consider a bounded domain of $\mathbb{R}^n$, denoted Ω, assumed to be smooth to simplify a little. Let $\mathit{\Gamma}$ be the boundary of Ω and $\mathit{\Sigma} = \mathit{\Gamma} \times (0, T)$. Let O be a subdomain of Ω. We denote by χ_O the characteristic function of the subdomain O. We consider the dynamic system

$$\begin{cases} z' - \mathit{\Delta} z = v(x,t)\chi_O, \\ z\big|_{\mathit{\Sigma}} = 0, \\ z(x,0) = 0, \end{cases} \tag{5.1}$$

with v the control belongs to $L^2\big(O \times (0,T)\big)$. In fact, (5.1) can be written under the general framework (2.1). Indeed, let $H = L^2(\Omega)$ and $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. We take $A = \mathit{\Delta}$, and $D(A)$ is equipped with the norm $|Az|, z \in D(A)$. Note that A is an isometry from $D(A)$ to H. Let next $U = L^2(O)$, and if u belongs to U,

$$Bu \equiv \begin{cases} u(x), & \text{if } x \in O, \\ 0, & \text{if } x \notin O. \end{cases}$$

It is well known that there exists a sequence $0 < \lambda_1 < \lambda_2 < \cdots \lambda_k \uparrow \infty$ of eigenvalues and w_{kj}, $j = 1, \ldots, r_k$, corresponding eigenvectors where r_k is the multiplicity of λ_k, with

$$\begin{cases} -\mathit{\Delta} w_{kj} = \lambda_k w_{kj}, \\ w_{kj} \in H_0^1(\Omega), \quad |w_{kj}|_{L^2} = 1. \end{cases} \tag{5.2}$$

Moreover (4.5) hold true. We can then state the

Theorem 5.1. *The pair (A, B) defined above is approximately controllable at any time $T > 0$.*

Proof. From Remark 4.2 it is sufficient to check that the condition (4.12) holds. Note that $B^*h = h|_O$, and thus since $B_k^* = B^* P_k$, condition (4.12) means

$$\sum_{j=1}^{r_k} (w_{kj}, h)\, w_{kj}|_O = 0 \quad \forall k \implies h = 0. \tag{5.3}$$

The function

$$\sum_{j=1}^{r_k} (w_{kj}, h)\, w_{kj}$$

is analytic in Ω. From (5.3) it follows that it is identically 0. Hence the coefficients $(w_{kj}, h) = 0$. As this holds for any k and $j = 1, \ldots, r_k$, necessarily $h = 0$. The desired result is thus proved. □

Remark 5.1. From (4.14) we have

$$\begin{aligned}(\varGamma_T h, h) &\leq C \sum_k \frac{1 - e^{-2\lambda_k T}}{2\lambda_k} \sum_{j=1}^{r_k} (w_{kj}, h)^2 \\ &\leq C \sum_k \frac{1}{\lambda_k} \sum_{j=1}^{r_k} (w_{kj}, h)^2 = \|h\|^2_{H^{-1}}.\end{aligned}$$

This means that

$$H^{-1}(\Omega) \subset F_T'$$

with dense and continuous injection. By duality, it follows that

$$F_T \subset H_0^1(\Omega)$$

with dense and continuous injection. Clearly F_T increases with the Lebesgue measure of O, and $F_T = H_0^1(\Omega)$, when $O = \Omega$. In that case, exact controllability holds true. □

5.2 Boundary control

Consider the following situation, using a formal write-up

$$\zeta' - \varDelta\zeta = 0, \quad \zeta\big|_\varGamma = v, \quad \zeta(z; 0) = 0, \tag{5.4}$$

where $v(t) \in U = L^2(\varGamma)$. We need to clarify in what space $\zeta(T)$ lies and to what extent we can reduce (5.4) to the general framework (2.1).

In fact, we shall show that $\zeta(T; v)$ lies in $(H^2 \cap H_0^1)'$, and $\zeta \in C(0; T; (H^2 \cap H_0^1)')$. One defines $\zeta(T)$ by the transposition method of J. L. Lions and E. Magenes [1]. Let h be an element of $L^2(\Omega) = H$; then $(-\varDelta)^{-1}h \in H^2 \cap H_0^1$. Consider the solution ψ of

$$\begin{cases} -\psi' - \Delta\psi = 0, \\ \psi\big|_\Gamma = 0, \\ \psi(x;T) = -(-\Delta)^{-1}h = A^{-1}h \\ \text{(recalling the notation } A = \Delta\text{)}, \end{cases} \tag{5.5}$$

and we note that $\psi = -(-\Delta)^{-1}\phi$, where

$$\begin{cases} -\phi' - \Delta\phi = 0, \\ \phi\big|_\Gamma = 0, \\ \phi(x;T) = h. \end{cases} \tag{5.6}$$

Clearly $\phi \in C(0;T;H)$ and $\psi \in C(0;T;H^2 \cap H_0^1)$. If we perform a formal integration by parts between (5.4) and (5.5), we obtain

$$\langle \zeta(T), \psi(T)\rangle = -\int_0^T \int_\Gamma v \frac{\partial\psi}{\partial\nu}\, d\Gamma\, dt, \tag{5.7}$$

and this constitutes the definition of $\zeta(T)$ as an element of $(H^2 \cap H_0^1)' = \big(D(A)\big)'$.

Consider now the operator $n \in \mathcal{L}\big(H^2; L^2(\Gamma)\big)$ defined by

$$n\phi = \frac{\partial\phi}{\partial\nu}. \tag{5.8}$$

We set

$$B = (-\Delta)^{-1}n^* = -A^{-1}n^*, \tag{5.9}$$

which belongs to $\mathcal{L}(L^2(\Gamma); H)$. We consider

$$\begin{cases} z' = Az + Bv, \\ z(0) = 0. \end{cases} \tag{5.10}$$

Noting that (5.6) is equivalent to

$$\begin{cases} -\phi' = A\phi, \\ \phi(T) = h, \end{cases} \tag{5.11}$$

then from (5.10) and (5.11) we deduce

$$\begin{aligned} (z(T), h) &= \int_0^T (\phi, Bv)\, dt = \int_0^T (B^*\phi, v)\, dt \\ &= -\int_0^T (nA^{-1}\phi, v)\, dt = -\int_0^T \left(\frac{\partial\psi}{\partial\nu}, v\right) dt, \end{aligned} \tag{5.12}$$

which is exactly the right-hand side of (5.7). Hence

$$(z(T), h) = (\zeta(T), -(-\Delta)^{-1}h). \tag{5.13}$$

Therefore

$$z(T) = -(-\Delta)^{-1}\zeta(T). \tag{5.14}$$

Similarly for any t, $z(t) = -(-\Delta)^{-1}\zeta(t)$, and as $z \in C(0, T; H)$,[2] we deduce that $\zeta \in C(0; T; (H^2 \cap H_0^1)')$. Now (5.10) is exactly equivalent to (2.1).

We shall prove the following theorem.

Theorem 5.2. *The pair (A, B) is approximately controllable at any time $T > 0$. The range of $\zeta(T; v)$ is dense in $(H^2 \cap H_0^1)'$.*

Proof. It is clear that it is sufficient to prove that the pair (A, B) is approximately controllable at any time $T > 0$. For that we must check the condition (4.12); i.e.,

$$\sum_{j=1}^{r_k} (h, w_{kj}) B^* w_{kj} = 0, \quad \forall k > 0 \implies h = 0. \tag{5.15}$$

This means also

$$\sum_{j=1}^{r_k} (h, w_{kj}) \frac{\partial w_{kj}}{\partial \nu} = 0 \implies (h, w_{kj}) = 0, \quad \forall j = 1, \ldots, r_k. \tag{5.16}$$

But from the following lemma we have

$$\sum_{j=1}^{r_k} (h, w_{kj}) w_{kj} = 0.$$

Hence the desired result. □

Lemma 5.1. *Let w be an eigenfunction of the Laplace operator corresponding to the eigenvalue λ; i.e.,*

$$-\Delta w = \lambda w, \quad w\big|_{\Gamma} = 0; \tag{5.17}$$

then one has

$$\int_{\Gamma} \left(\frac{\partial w}{\partial \nu}\right)^2 m \cdot \nu \, d\Gamma = 2\lambda \int_{\Omega} w^2 \, dx, \tag{5.18}$$

where $m(x) = x - x_0$.

Proof. We multiply (5.17) by

$$m_\alpha \frac{\partial w}{\partial x_\alpha}$$

(using summation convention) and perform integration by parts. The desired result follows. □

[2] Given a Banach space X, the notation $C(0, T; X)$ stands for $C([0, T]; X)$. See also footnote 2 on page 3 in the Introduction to the book.

Remark 5.2. Let Γ_0 be a part of Γ. The control is exerted on Γ_0 only, which means that it is 0 on $\Gamma_1 = \Gamma - \Gamma_0$. To treat this case, let us define

$$n_0\phi = \frac{\partial \phi}{\partial \nu}\bigg|_{\Gamma_0}.$$

Take $B = -A^{-1}n_0^*$. The result of Theorem 5.2 extends to the pair (A, B) provided that for an eigenfunction w as in (5.17)

$$\frac{\partial w}{\partial \nu}\bigg|_{\Gamma_0} = 0$$

implies $w = 0$. From the formula (5.18), it is sufficient that $m \cdot \nu \geq 0$ on Γ_0. □

Remark 5.3. From (4.14) again we deduce

$$(\Gamma_T h, h) \leq C \sum_k \frac{1 - e^{-2\lambda_k T}}{2\lambda_k} \sum_{j=1}^{r_k} (w_{kj}, h)^2 \leq C(\|h\|_{H^{-1}})^2. \tag{5.19}$$

Therefore $H^{-1} \subset F_T'$, with continuous and dense injection. Hence also $F_T \subset H_0^1(\Omega)$, with continuous and dense injection. It follows that the range of $\zeta(T; v)$, when v lies in $L^2\big(0, T; L^2(\Gamma)\big)$ is dense in H^{-1}. □

5.3 Neumann boundary control

We introduce some notation. We take

$$H = L^2(\Omega), \quad V = H^1(\Omega), \quad A = \Delta - I,$$

associated with the Neumann boundary condition. Hence

$$D(A) = \left\{ z \in H^2(\Omega) \colon \frac{\partial z}{\partial \nu} = 0 \right\}.$$

Again $D(A)$ is equipped with the norm $|Az|, z \in D(A)$. The eigenvectors are defined by

$$-\Delta w + w = \lambda w, \quad \frac{\partial w}{\partial \nu} = 0. \tag{5.20}$$

The first eigenvalue is 1. It is isolated; hence, $1 < \lambda_2 \leq \lambda_3 \cdots$ and $w_1 = 1/\sqrt{|\Omega|}$. We set $m(x) = x - x_0$ and define

$$\Gamma_0 = \{x \in \Gamma \colon m \cdot v > 0\}, \quad \Gamma_1 = \Gamma - \Gamma_0. \tag{5.21}$$

Let γ be the trace operator on Γ. Note the $\gamma \in \mathcal{L}\big(V; L^2(\Gamma)\big) \cap \mathcal{L}\big(H^2(\Omega); H^1(\Gamma)\big)$. Let also σ_j be the tangential operators on $\Gamma \in \mathcal{L}\big(H^1(\Gamma); L^2(\Gamma)\big)$, such that

$$\gamma \frac{\partial \phi}{\partial x_j}\bigg|_{\Gamma} = \nu_j \frac{\partial \phi}{\partial \nu} + \sigma_j \gamma \phi = \nu_j n\phi + \sigma_j \gamma\phi, \quad \forall \phi \in H^2(\Omega). \tag{5.22}$$

Note that $n\phi = \nu \cdot \gamma D\phi$, and $\nu \cdot \sigma = 0$.

Similarly, we define by $\gamma_0, \sigma_{j,0}\,(\gamma_1, \sigma_{j,1})$ the respective restrictions of γ, σ_j to Γ_0 and Γ_1. We also use the notation

$$D_\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathcal{L}\big(H^1(\Gamma); \big(L^2(\Gamma)\big)^n\big).$$

The operators $\gamma, \sigma_1\gamma, \sigma_2\gamma, \ldots \sigma_n\gamma \in \mathcal{L}\big(D(A); L^2(\Gamma)\big)$ and not identically 0, whereas $n = 0$ on $D(A)$. This should be compared with the situation in the Dirichlet case, where γ, $\sigma_1\gamma, \sigma_2\gamma, \ldots \sigma_n\gamma$ are 0 on $D(A)$ and n is not 0. This explains why the analogy with the Dirichlet case implies the control to be an element of $U = \big(L^2(\Gamma)\big)^{n+1}$, instead of just $L^2(\Gamma)$. The operator B is defined by

$$\begin{aligned} B &= A^{-1}(\gamma^* \ \gamma^*\sigma_1^* \ \gamma^*\sigma_2^* \ldots \gamma^*\sigma_n^*) \\ &= A^{-1}(\gamma^* \ \gamma^* D_\sigma^*) \in \mathcal{L}(U; H). \end{aligned} \tag{5.23}$$

We can also view U as $\big(L^2(\Gamma_0)\big)^{n+1} \times \big(L^2(\Gamma_1)\big)^{n+1}$. Some components might be taken equal to 0; in which case, they are omitted and we use γ_0^* or γ_1^* instead of γ^* (or $\sigma_{1,0}^*$, $\sigma_{1,1}^* \ldots; D_{\sigma,0}^*, D_{\sigma,1}^*$). We are now interested in the pair A, B. Let us first interpret the dynamic system (5.10), corresponding to this pair,

$$z' = Az + Bv, \quad z(0) = 0. \tag{5.24}$$

Consider also (5.11), i.e.

$$-\phi' = A\phi, \quad \phi(T) = h, \tag{5.25}$$

then we have

$$(z(T), h) = \int_0^T (B^*\phi, v)\,dt = \int_0^T \left[(\gamma\psi, v_0) + \sum_{i=1}^n (\sigma_i \gamma\psi, v_i)\right] dt, \tag{5.26}$$

where $\psi = A^{-1}\phi$ and $v = (v_0, v_1, \ldots, v_n)$. As $n\psi = 0$, we can also write the above relation as

$$(z(T), h) = \int_0^T \int_\Gamma \left(\psi v_0 + \sum_{i=1}^n \frac{\partial \psi}{\partial x_i} v_i\right) d\Gamma, \tag{5.27}$$

where we omit to indicate explicitly the operator γ.

Note that $z(\cdot) \in C(0, T; H)$. Let $\zeta(t) = Az(t) \in C\big(0, T; \big(D(A)\big)'\big)$. Then from (5.27) we deduce

$$\langle \zeta(T), \psi(T)\rangle = \int_0^T \int_\Gamma \left(\psi v_0 + \sum_{i=1}^n \frac{\partial \psi}{\partial x_i} v_i\right) d\Gamma. \tag{5.28}$$

Note that if $v \in \big(H^1(\Gamma)\big)'$ and $\psi \in D(A)$ (hence $\gamma\psi \in H^1(\Gamma)$), we can write

$$\langle v, \gamma\psi\rangle = \int_\Gamma \left(\psi v_0 + \sum_{i=1}^n \frac{\partial \psi}{\partial x_i} v_i\right) d\Gamma,$$

where $(v_0, v_1, \ldots, v_n)$ is a representation of v.

Therefore (5.28) reads also

$$\langle \zeta(T), \psi(T)\rangle = \int_0^T \langle v(t), \gamma\psi(t)\rangle \, dt. \tag{5.29}$$

This is exactly the formula that follows from the definition of the solution of

$$\begin{cases} \zeta' - \Delta\zeta = 0, \\ \dfrac{\partial \zeta}{\partial \nu} = v, \\ \zeta(x,0) = 0 \end{cases} \tag{5.30}$$

obtained by the Method of Transposition of J. L. Lions and E. Magenes [1] (see also Chapter 2 of Part II), using the dual equation

$$\begin{cases} -\psi' - \Delta\psi = 0, \\ \dfrac{\partial \psi}{\partial \nu} = 0, \\ \psi(T) \in D(A). \end{cases} \tag{5.31}$$

We are interested in the density of the range of $\zeta(T; v)$, solution of (5.30) at some time T, in the space $\big(D(A)\big)'$. This is equivalent to the density of $z(T; v)$ in H, where z is the solution of (5.24). We are in the framework of the model (2.1), with a semigroup satisfying the condition (4.5), and thus the problem is that of the approximate controllability of the pair (A, B) at some time T. The general theory implies that the pair (A, B) is approximately controllable at any positive time T, if and only if the condition (4.12) holds.

We write the condition (4.12)

$$\sum_{j=1}^{r_k} (h, w_{kj}) B^* w_{kj} = 0, \quad \forall k > 0 \implies h = 0. \tag{5.32}$$

This implies that

$$w = \sum_{j=1}^{r_k} (h, w_{kj}) w_{kj}$$

satisfies $\gamma w = 0$, $D_\sigma w = 0$. Clearly $w = 0$; hence the coefficients $(h, w_{kj}) = 0$, for $j = 1, \ldots, r_k$. Hence $h = 0$.

It is clear from the above that if we control the whole boundary, we can take $v(t)$ in $L^2(\Gamma)$ instead of $\big(H^1(\Gamma)\big)'$; i.e., $v(t) = v_0(t)$ and $v_1(t) = \cdots = v_n(t) = 0$.

On the other hand, we need $\left(H^1(\Gamma)\right)'$ if we control only a part of the boundary. Let us make this precise. We first state a lemma.

Lemma 5.2. *The eigenvector w solution of* (5.20) *satisfies the relation*

$$\int_\Gamma m\cdot\nu|D_\sigma w|^2\,d\Gamma + 2(\lambda-1)\int_\Omega w^2\,dx = (\lambda-1)\int_\Gamma m\cdot\nu w^2\,d\Gamma. \tag{5.33}$$

Proof. As usual, we multiply (5.20) by

$$m_\alpha\frac{\partial w}{\partial x_\alpha}$$

and perform integration by parts. Details are left to the reader. □

We then state the following theorem.

Theorem 5.3. *Let $A=\Delta-I$ associated with the Neumann boundary condition, and the two following cases for U, B. Either $U=L^2(\Gamma)$ and $B=A^{-1}\gamma^*$ or $U=L^2(\Gamma_0)\times\left(L^2(\Gamma_1)\right)^n$ and $B=A^{-1}(\gamma_0^*\ \gamma_1^*D_{\sigma,1}^*)$. Then the pair (A,B) is approximately controllable at any $T>0$. The range of $\zeta(T;v)$ solution of* (5.40) *is dense in $\left(D(A)\right)'$.*

Proof. The first case has already been discussed. Consider the second case. We have to prove (5.32). Note first that for $k=1$, $r_k=1$, and w_1 is a constant. We necessarily have $(h,w_1)=0$. Starting with $k=2$, we have $\lambda_k>1$. From the hypothesis of the statement (5.32) we have, writing

$$\begin{aligned} w &= \sum_{j=1}^{r_k}(h,w_{kj})w_{kj},\\ \gamma_0 w &= 0,\quad D_{\sigma,1}w=0. \end{aligned} \tag{5.34}$$

Splitting the surface integrals in (5.33) in two parts Γ_0 and Γ_1 and using the sign properties of $m.\nu$ on Γ_0,Γ_1 and the fact that $\lambda>1$ we deduce that

$$\int_\Omega w^2\,dx=0.$$

Therefore the coefficients $(h,w_{kj})=0$, for $j=1,\dots,r_k$. The desired result has been proved. □

Remark 5.4. Apparently, it is not sufficient to take $U=L^2(\Gamma_0)$; unlike the Dirichlet case, see Remark 5.2. □

Remark 5.5. Consider again $(\Gamma_T h,h)$. As usual from (4.14), we can prove that $(\Gamma_T h,h)\le C(\|h\|_{V'})^2$, recalling that $V=H^1$. Hence $V'\subset F_T'$ and $F_T\subset V$, with continuous and dense injection. In the case $U=L^2(\Gamma)$ and $B=A^{-1}\gamma^*$, one can obtain an additional result, namely $\left(D(A)\right)'\subset F_T'$; therefore $F_T\subset D(A)$. It follows that the range of $\zeta(T;v)$, solution of (5.30), is dense in H as $v(\cdot)$ varies in $L^2\left(0,T;L^2(\Gamma)\right)$. □

5.4 Pointwise control

We consider the system

$$\begin{cases} \zeta' - \Delta\zeta = \sum_{i=1}^m v_i(t)\delta(x-\alpha_i), \\ \zeta\big|_\Gamma = 0, \\ \zeta(x;0) = 0, \end{cases} \tag{5.35}$$

where the points α_i belong to Ω and represent actuators. We assume that the dimension of the space is $n \leq 3$. We shall reduce the formal model (5.35) to our general set up.

We take $A = \Delta$ with Dirichlet conditions. We define $b_i \in H$ by the equation

$$(b_i, h) = (A^{-1}h)(\alpha_i). \tag{5.36}$$

This makes sense because for $h \in H = L^2$, $A^{-1}h$ belongs to $H^2 \subset C(\bar{\Omega})$ with continuous injection. Hence the right-hand side of (5.36) is defined and represents a linear continuous functional on H. Consider the dynamic system

$$z' = Az + \sum_{i=1}^m v_i(t)b_i, \quad z(0) = 0. \tag{5.37}$$

Let us check that

$$\zeta(T) = Az(T) \tag{5.38}$$

coincides with the definition of (5.35) through the Method of Transposition. Indeed consider

$$-\phi' = A\phi, \quad \phi(T) = h. \tag{5.39}$$

Then from (5.37) and (5.39), it follows taking into account the definition of b_i

$$(z(T), h) = \sum_{i=1}^m \int_0^T v_i(t)\psi(\alpha_i, t)\, dt, \tag{5.40}$$

where $\psi = A^{-1}\phi$. But from the Method of Transposition we define $\zeta(T)$ by

$$\big(\zeta(T), \psi(T)\big) = \sum_{i=1}^m \int_0^T v_i(t)\psi(\alpha_i, t)\, dt. \tag{5.41}$$

From (5.40) and (5.41), using the fact that $\psi(T) = A^{-1}h$, we deduce that (5.38) holds. Now the system (5.37) has already been studied in §4.2. Consider the matrices

$$F_k, \quad \text{with } F_{k,ji} = w_{kj}(\alpha_i), \quad j = 1, \dots, r_k, i = 1, \dots, m; \tag{5.42}$$

then we can state the following theorem.

Theorem 5.4. *The pair* (A, B), $B = (b_1, b_2, \ldots, b_m)$ *is approximately controllable at any* $T > 0$, *if and only if* $\operatorname{rank} F_k = r_k$ *for any* k.

Remark 5.6. We refer to A. El Jai and A. J. Pritchard [2] for general results of controllability in the case of pointwise control. □

Remark 5.7. We assume that the assumption of Theorem 5.4 is satisfied. Using condition (4.14), we obtain again that $F_T \subset H_0^1$ with dense and continuous injection. Hence the range of $\zeta(T; v)$ is dense in H^{-1}. □

6 Controllability for skew-symmetric operators

6.1 Notation and general comments

We shall need to work here with a complex spectrum. We assume that A has a purely imaginary spectrum of the form

$$\left\{ i\sqrt{\lambda_j}, -i\sqrt{\lambda_j} : j = 1, \ldots \right\},$$

where

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots, \quad \lambda_j \uparrow \infty,$$

with only a finite number of successive eigenvalues possibly equal. If ϕ_j is the eigenvector corresponding to $i\sqrt{\lambda_j}$, then $\bar{\phi_j}$ is the eigenvector corresponding to $-i\sqrt{\lambda_j}$. We assume that

$$\phi_j, \bar{\phi_j} \tag{6.1}$$

form an orthonormal basis of H (complexified).

This implies with the notation of the scalar product in H complexified

$$(\phi_j, \bar{\phi_k}) = \delta_{jk}, \quad (\phi_j, \phi_k) = 0. \tag{6.2}$$

By definition

$$A\phi_j = i\sqrt{\lambda_j}\phi_j, \quad A\bar{\phi_j} = -i\sqrt{\lambda_j}\bar{\phi_j}. \tag{6.3}$$

If ϕ is an element of H (real valued), then it can be represented with the expansion

$$\phi = \sum_j (c_j \bar{\phi_j} + \bar{c_j}\phi_j), \tag{6.4}$$

with

$$c_j = (\phi, \phi_j), \quad \bar{c_j} = (\phi, \bar{\phi_j}), \quad |\phi|^2 = 2\sum_j |c_j|^2. \tag{6.5}$$

It is easy to check that the operator A is skew adjoint,

$$A^* = -A. \tag{6.6}$$

In fact we can even characterize the type of operators we are considering, thanks to the following proposition.

Proposition 6.1. *Assume that A is a linear operator from a dense subspace $D(A)$ of H into H, satisfies* (6.6) *and*

$$\textit{the null space of } A = \{0\}, \tag{6.7}$$

$$\begin{cases} \textit{the mapping } A+I \textit{ from } D(A) \textit{ to } H \textit{ is onto} \\ (\textit{and thus invertible since already one to one}), \end{cases} \tag{6.8}$$

$$\textit{and the map } T = (A+I)^{-1} \textit{ is compact.} \tag{6.9}$$

Then the spectrum of A is of the form $\pm i\sqrt{\lambda_j}$, with

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots, \quad \lambda_j \uparrow \infty,$$

and the corresponding eigenvectors ϕ_j and $\bar{\phi}_j$ form an orthonormal basis for H. Conversely if A has these spectral properties, then it satisfies (6.6) *to* (6.9)*.*

Proof. Let us assume (6.6); then if $z \in D(A)$, we can write

$$(Az, \bar{z}) = -(A^* z, \bar{z}) = -(z, \overline{Az}) = -\overline{(Az, \bar{z})},$$

and thus $(Az, \bar{z})$ is purely imaginary. Hence the eigenvalues are necessarily purely imaginary. As -1 cannot be an eigenvalue, the map $A+I$ is necessarily one to one. Moreover if λ is an eigenvalue corresponding to the eigenvector ϕ, then $\bar{\phi}$ is an eigenvector corresponding to the eigenvalue $\bar{\lambda}$.

Assume now (6.7), (6.8), and (6.9). Let us check the T is a normal operator on H; i.e.,

$$T^*T = TT^*. \tag{6.10}$$

Note that $T^* = (A^* + I)^{-1}$. Let $f \in H$ and define

$$\begin{aligned} \phi &= \tfrac{1}{2}\{(A+I)^{-1} f + (-A+I)^{-1} f\}, \\ \psi &= \tfrac{1}{2}\{-(A+I)^{-1} f + (-A+I)^{-1} f\}; \end{aligned}$$

then clearly $\psi = A\phi$; thus

$$A\phi + \phi = (A^* + I)^{-1} f, \quad A^*\phi + \phi = (A+I)^{-1} f,$$

which proves (6.10).

From the spectral theory of normal compact operators, see T. Kato [4, p. 260], we can state that T has a discrete spectrum $\{\mu_1, \mu_2, \dots\}$, where $|\mu_1| \geq |\mu_2| \geq \cdots \geq |\mu_n| \geq \cdots$, $|\mu_n|$ tends to 0 as n tends to infinity. Each of these eigenvalues has a finite multiplicity. Moreover, as the null space of T is 0, the eigenvectors form an orthonormal basis of H. But eigenvectors of T are eigenvectors of A, with eigenvalues obtained by the transformation

$$\lambda = -1 + 1/\mu.$$

The spectral properties of A are easily deduced.

Let us check the reverse. The set $D(A)$ is made of elements

$$\phi = \sum_j (c_j \bar{\phi_j} + \bar{c_j} \phi_j)$$

such that

$$\sum_j |c_j|^2 \lambda_j < \infty.$$

It is clear that $D(A)$ is dense in H. If $\phi \in D(A)$, then

$$A\phi = \sum_j (-ic_j \sqrt{\lambda_j} \bar{\phi_j} + i\bar{c_j} \sqrt{\lambda_j} \phi_j).$$

Therefore we can solve the equation

$$A\phi + \phi = f \tag{6.11}$$

as follows:

$$c_j = \frac{f_j}{(1 - i\sqrt{\lambda_j})},$$

where f_j denotes the components of f. Therefore, (6.11) has one and only one solution. To prove (6.9) consider a sequence $\{f^n\}$ that tends to 0 in H weakly, and let ϕ^n be the corresponding solution of (6.11); we must prove that ϕ^n tends to 0 in H strongly. Let f_j^n be the components of f^n and

$$c_j^n = \frac{f_j^n}{(1 - i\sqrt{\lambda_j})}$$

be the corresponding components of ϕ^n.

We have

$$\sum_j |c_j^n|^2 = \sum_j \frac{|f_j^n|^2}{1 + \lambda_j} \leq \sum_{j=1,\dots,N} |f_j^n|^2 + \frac{|f^n|^2}{\lambda_N}.$$

As f^n tends to 0 weakly, $|f^n|^2$ remains bounded and f_j^n tends to 0 for any fixed j. Moreover λ_n tends to 0 as N tends to infinity. It follows from the above inequality that

$$\sum_j |c_j^n|^2$$

tends to 0 as n tends to infinity. This means that ϕ^n tends to 0 in H. □

The solution of

$$z' = Az, \quad z(0) = \phi \tag{6.12}$$

is given by

$$z(t) = e^{tA}\phi = \sum_j (c_j e^{-i\sqrt{\lambda_j} t} \bar{\phi_j} + \bar{c_j} e^{+i\sqrt{\lambda_j} t} \phi_j), \tag{6.13}$$

where c_j are the components of ϕ.

In the sequel we shall use the notation

$$\phi = \operatorname{Re}\phi + i \operatorname{Im}\phi$$

to express the decomposition of an element of H complexified.

Note also that if $\phi \in D(A)$, then

$$\|\phi\|^2_{D(A)} = |A\phi|^2 = 2\sum_j \lambda_j |c_j|^2$$

and

$$\frac{\phi_j}{\sqrt{\lambda_j}}, \quad \frac{\bar{\phi}_j}{\sqrt{\lambda_j}}$$

form an orthonormal basis of $D(A)$. Similarly $\phi_j\sqrt{\lambda_j}$, $\bar{\phi}_j\sqrt{\lambda_j}$ form an orthonormal basis of $\big(D(A)\big)'$.

6.2 Dynamical system

Consider $B \in \mathcal{L}(U;H)$. We shall make several assumptions on B in relation with the eigenvectors ϕ_j.

There exist N integer ≥ 0 and operators $M \in \mathcal{L}(H;H)$, $\Lambda \in \mathcal{L}(H;D(A)')$ such that

$$|B^* z|^2 \geq (Mz, z), \quad \forall z \in H, \tag{6.14}$$

for all $j, k \geq N+1$,

$$(M \operatorname{Re}\phi_j, \operatorname{Im}\phi_k) = 0, \tag{6.15}$$

for all $j, k \geq N+1$, $j \neq k$

$$(M \operatorname{Re}\phi_j, \operatorname{Re}\phi_k) = \frac{(\Lambda \operatorname{Re}\phi_j, \operatorname{Re}\phi_k)}{\lambda_k} + \frac{(\Lambda \operatorname{Re}\phi_k, \operatorname{Re}\phi_j)}{\lambda_j} \tag{6.16}$$

for all $j, k \geq N+1$, $j \neq k$

$$(\Lambda \operatorname{Re}\phi_j, \operatorname{Re}\phi_k) + (\Lambda \operatorname{Re}\phi_k, \operatorname{Re}\phi_j) + \sqrt{\lambda_j\lambda_k}(M \operatorname{Im}\phi_j, \operatorname{Im}\phi_k) = 0. \tag{6.17}$$

There exists $c_0 > 0$ such that for all $j \geq N+1$,

$$(M \operatorname{Re}\phi_j, \operatorname{Re}\phi_j) + (M \operatorname{Im}\phi_j, \operatorname{Im}\phi_j) \geq \frac{c_0}{\lambda_j}, \tag{6.18}$$

for all $j \geq N+1$

$$\left|(M \operatorname{Re}\phi_j, \operatorname{Re}\phi_j) - (M \operatorname{Im}\phi_j, \operatorname{Im}\phi_j) - 4\frac{(\Lambda \operatorname{Re}\phi_j, \operatorname{Re}\phi_j)}{\lambda_j}\right| \leq \frac{K}{\sqrt{\lambda_j}} \tag{6.19}$$

and

$$\lambda_{N+1} > \lambda_N.$$

For $j \le N$, we consider only the λ_j having different values and call $P_j(\bar{P}_j)$ the projector on the finite dimensional eigensubspace of H corresponding to the conjugate pair of eigenvalues $\{i\sqrt{\lambda_j}, -i\sqrt{\lambda_j}\}$; then

$$|B^* P_j z|^2 \ge c_1 \frac{|P_j z|^2}{\lambda_j}, \quad \forall z \in H,\ j \le N. \tag{6.20}$$

With our convention we have

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_N < \lambda_{N+1} \le \lambda_{N+2} \le \cdots .$$

An element ϕ of H is represented by the expansion

$$\phi = \sum_{j=1}^{N} (P_j\phi + \bar{P}_j\phi) + \sum_{j \ge N+1} (c_j \bar{\phi}_j + \bar{c}_j \phi_j). \tag{6.21}$$

Naturally if $N = 0$, the condition (6.20) is void.

We now consider the dynamic system corresponding to the pair (A, B)

$$z' = Az + Bv, \quad z(0) = 0. \tag{6.22}$$

Our main result is as follows.

Theorem 6.1. *We assume that A satisfies* (6.6) *to* (6.9) *and B satisfies* (6.14) *to* (6.20)*; then the pair (A, B) is exactly controllable at T, for T sufficiently large.*

Proof. We shall consider the controllability operator

$$\Gamma_T = \int_0^T e^{tA} B B^* e^{tA^*}\, dt$$

and prove that

$$\forall h \in H, \quad (\Gamma_T h, h) \ge c_T \|h\|^2_{(D(A))'} \tag{6.23}$$

for T sufficiently large.

According to §3.2, this implies the desired result. Consider

$$h = \sum_{j=1}^{N} (P_j h + \bar{P}_j h) + \sum_{j \ge N+1} (c_j \bar{\phi}_j + \bar{c}_j \phi_j)$$

with $c_j = (h, \phi_j)$; then

$$\begin{aligned}
(\Gamma_T h, h) &= \int_0^T |B^* e^{tA^*} h|^2\, dt \\
&= 4 \int_0^T \left| B^* \left(\sum_{j=1}^{N} \mathrm{Re}(e^{ti\sqrt{\lambda_j}} P_j h) + \sum_{j \ge N+1} \mathrm{Re}(e^{-ti\sqrt{\lambda_j}} \bar{c}_j \phi_j) \right) \right|^2 dt \qquad (6.24) \\
&= X_1 + X_2 + X_3,
\end{aligned}$$

where

$$X_1 = 4\int_0^T \left| B^* \left[\sum_{j=1}^N \mathrm{Re}(e^{-ti\sqrt{\lambda_j}} P_j h) \right] \right|^2 dt,$$

$$X_2 = 8\int_0^T B^* \left[\sum_{j=1}^N \mathrm{Re}(e^{-ti\sqrt{\lambda_j}} P_j H) \right], \left(B^* \left[\sum_{j\geq N+1} \mathrm{Re}(e^{-ti\sqrt{\lambda_j}} \bar{c}_j \phi_j) \right] \right) dt,$$

$$X_3 = 4\int_0^T \left| B^* \left[\sum_{j\geq N+1} \mathrm{Re}(e^{-ti\sqrt{\lambda_j}} \bar{c}_j \phi_j) \right] \right|^2 dt.$$

By an easy computation

$$X_1 = 2T \sum_{j=1}^N |B^* P_j h|^2 + Y_1$$

with

$$\begin{aligned} Y_1 = 2\,\mathrm{Re} \sum_{j,k=1}^N (B^* P_j h, B^* P_k h) \frac{1 - e^{-iT(\sqrt{\lambda_j}+\sqrt{\lambda_k})}}{i(\sqrt{\lambda_j} + \sqrt{\lambda_k})} \\ + 2\,\mathrm{Re} \sum_{j=1}^N \sum_{\substack{k=1 \\ k\neq j}}^N (B^* P_j h, B^* \bar{P}_k h) \frac{1 - e^{-iT(\sqrt{\lambda_j}-\sqrt{\lambda_k})}}{i(\sqrt{\lambda_j} - \sqrt{\lambda_k})}, \end{aligned}$$

$$\begin{aligned} X_2 = 2\,\mathrm{Re} \sum_{j=1}^N \sum_{k\geq N+1} (B^* P_j h, B^* \phi_k \bar{c}_k) \frac{1 - e^{-iT(\sqrt{\lambda_j}+\sqrt{\lambda_k})}}{i(\sqrt{\lambda_j} + \sqrt{\lambda_k})} \\ + 2\,\mathrm{Re} \sum_{j=1}^N \sum_{k\geq N+1} (B^* P_j h, B^* \bar{\phi}_k c_k) \frac{1 - e^{-iT(\sqrt{\lambda_j}-\sqrt{\lambda_k})}}{i(\sqrt{\lambda_j} - \sqrt{\lambda_k})}. \end{aligned}$$

Now according to (6.14)

$$X_3 \geq 4\int_0^T \left(M \left(\sum_{j\geq N+1} \mathrm{Re}(e^{-ti\sqrt{\lambda_j}} c_j \bar{\phi}_j) \right), \sum_{j\geq N+1} \mathrm{Re}(e^{-ti\sqrt{\lambda_j}} \bar{c}_j \phi_j) \right) dt.$$

At this stage it is convenient to introduce the following notation, which will help reduce the length of the following equations and estimates:

$$m_{ij} = (M\ \mathrm{Im}\,\phi_i, \mathrm{Im}\,\phi_j), \quad l_{ij} = (\Lambda\,\mathrm{Re}\,\phi_i, \mathrm{Re}\,\phi_j).$$

From (6.15)

$$X_3 \geq 4 \sum_{j,k\geq N+1} \int_0^T (M \operatorname{Re}\phi_j, \operatorname{Re}\phi_k) \operatorname{Re}(e^{-ti\sqrt{\lambda_j}}\bar{c_j}) \operatorname{Re}(e^{-ti\sqrt{\lambda_k}}\bar{c_k})\, dt$$
$$+4 \sum_{j,k\geq N+1} \int_0^T m_{jk}(e^{-ti\sqrt{\lambda_j}}\bar{c_j}) \operatorname{Im}(e^{-ti\sqrt{\lambda_k}}\bar{c_k})\, dt,$$

and from (6.16),

$$X_3 \geq Y_3 + Z_3,$$

where

$$Y_3 = 4 \sum_{j\geq N+1} \int_0^T (M \operatorname{Re}\phi_j, \operatorname{Re}\phi_j) |\operatorname{Re}(e^{-ti\sqrt{\lambda_j}}\bar{c_j})|^2\, dt$$
$$+ 4 \sum_{j\geq N+1} \int_0^T (M \operatorname{Im}\phi_j, \operatorname{Im}\phi_j) |\operatorname{Im}(e^{-ti\sqrt{\lambda_j}}\bar{c_j})|^2\, dt,$$
$$Z_3 = 4 \sum_{\substack{k\geq N+1\\ j\neq k}} \int_0^T \left[\frac{l_{jk}}{\lambda_k} + \frac{l_{kj}}{\lambda_j}\right] \operatorname{Re}(e^{-ti\sqrt{\lambda_j}}\bar{c_j}) \operatorname{Re}(e^{-ti\sqrt{\lambda_k}}\bar{c_k})\, dt$$
$$+ 4 \sum_{\substack{k\geq N+1\\ j\neq k}} \int_0^T m_{jk} \operatorname{Im}(e^{-ti\sqrt{\lambda_j}}\bar{c_j}) \operatorname{Im}(e^{-ti\sqrt{\lambda_k}}\bar{c_k})\, dt.$$

Set

$$a_j = \frac{c_j}{\sqrt{\lambda_j}}.$$

Then we can write

$$Z_3/4 = \sum_{\substack{k\geq N+1\\ j\neq k}} \int_0^T \left[\frac{l_{jk}\sqrt{\lambda_j}}{\sqrt{\lambda_k}} + \frac{l_{kj}\sqrt{\lambda_k}}{\sqrt{\lambda_j}}\right] \operatorname{Re}(e^{-ti\sqrt{\lambda_j}}\,\bar{a_j})\ \operatorname{Re}(e^{-ti\sqrt{\lambda_k}}\,\bar{a_k})\, dt$$
$$+ \sum_{\substack{k\geq N+1\\ j\neq k}} \int_0^T \sqrt{\lambda_j\lambda_k} m_{jk}\ \operatorname{Im}(e^{-ti\sqrt{\lambda_j}}\bar{a_j}) \operatorname{Im}(e^{-ti\sqrt{\lambda_k}}\bar{a_k})\, dt.$$

Then

$$Z_3 = -4 \sum_{\substack{k\geq N+1\\ j\neq k}} \int_0^T \frac{l_{jk}}{\sqrt{\lambda_k}} \frac{d}{dt} \operatorname{Im}(e^{-ti\sqrt{\lambda_j}}\bar{a_j}) \operatorname{Re}(e^{-ti\sqrt{\lambda_k}}\bar{a_k})\, dt$$
$$- 4 \sum_{\substack{k\geq N+1\\ j\neq k}} \int_0^T \frac{l_{kj}}{\sqrt{\lambda_j}} \operatorname{Re}(e^{-ti\sqrt{\lambda_j}}\bar{a_j}) \frac{d}{dt} \operatorname{Im}(e^{-ti\sqrt{\lambda_k}}\bar{a_k})\, dt$$
$$+ 4 \sum_{\substack{k\geq N+1\\ j\neq k}} \int_0^T \sqrt{\lambda_j\lambda_k}\, m_{jk} \operatorname{Im}(e^{-ti\sqrt{\lambda_j}}\bar{a_j}) \operatorname{Im}(e^{-ti\sqrt{\lambda_k}}\bar{a_k})\, dt.$$

Integrating by parts and using (6.17) obtains

$$\begin{aligned} Z_3 = &-4 \sum_{\substack{k\geq N+1\\ j\neq k}} \frac{(\Lambda \operatorname{Re}\phi_j, \operatorname{Re}\phi_k)}{\sqrt{\lambda_k}} \big(\operatorname{Im}(e^{iT\sqrt{\lambda_j}}\bar{a_j}) \cdot \operatorname{Re}(e^{iT\sqrt{\lambda_k}}\bar{a_k}) \\ &\qquad - \operatorname{Im}(\bar{a_j}) \operatorname{Re}(\bar{a_k})\big) \\ &-4 \sum_{\substack{k\geq N+1\\ j\neq k}} \frac{(\Lambda \operatorname{Re}\phi_k, \operatorname{Re}\phi_j)}{\sqrt{\lambda_j}} \big(\operatorname{Im}(e^{iT\sqrt{\lambda_k}}\bar{a_k}) \cdot \operatorname{Re}(e^{iT\sqrt{\lambda_j}}\bar{a_j}) \\ &\qquad - \operatorname{Im}(\bar{a_k}) \operatorname{Re}(\bar{a_j})\big). \end{aligned}$$

Hence also

$$\begin{aligned} Z_3 = &-8 \left(\Lambda \left(\sum_{j\geq N+1} \operatorname{Re}\phi_j \operatorname{Im}(e^{-iT\sqrt{\lambda_j}}\bar{a_j}) \right), \sum_{k\geq N+1} \operatorname{Re}\phi_k \frac{\operatorname{Re}(e^{-iT\sqrt{\lambda_k}}\bar{a_k})}{\sqrt{\lambda_k}} \right) \\ &+ 8 \left(\Lambda \left(\sum_{j\geq N+1} \operatorname{Re}\phi_j \operatorname{Im}(\bar{a_j}) \right), \sum_{k\geq N+1} \operatorname{Re}\phi_k \frac{\operatorname{Re}(\bar{a_k})}{\sqrt{\lambda_k}} \right) \\ &+ 8 \sum_{j\geq N+1} (\Lambda \operatorname{Re}\phi_j, \operatorname{Re}\phi_j) \operatorname{Im}(e^{-iT\sqrt{\lambda_j}}\bar{a_j}) \frac{\operatorname{Re}(e^{-iT\sqrt{\lambda_j}}\bar{a_j})}{\sqrt{\lambda_j}} \\ &- 8 \sum_{j\geq N+1} (\Lambda \operatorname{Re}\phi_j, \operatorname{Re}\phi_j) \operatorname{Im}(\bar{a_j}) \frac{\operatorname{Re}(\bar{a_j})}{\sqrt{\lambda_j}}. \end{aligned}$$

Next as easily seen

$$Y_3 = 2T \sum_{j\geq N+1} |c_j|^2 \big((M \operatorname{Re}\phi_j, \operatorname{Re}\phi_j) + (M \operatorname{Im}\phi_j, \operatorname{Im}\phi_j) \big) + Y_3',$$

with

$$Y_3' = \sum_{j\geq N+1} \{ (M \operatorname{Re}\phi_j, \operatorname{Re}\phi_j) - m_{jj} \} \frac{\operatorname{Im}(c_j^2(e^{2iT\sqrt{\lambda_j}} - 1))}{\sqrt{\lambda_j}}.$$

Collecting results we deduce that

$$\begin{aligned} (\Gamma_T h, h) \geq 2T \sum_{j=1}^{N} |B^* P_j h|^2 + 2T \sum_{j\geq N+1} |c_j|^2 [(M \operatorname{Re}\phi_j, \operatorname{Re}\phi_j) + m_{jj}] \\ + Y_1 + X_2 + Z_3 + Y_3'. \quad (6.25) \end{aligned}$$

Using the assumptions (6.18) and (6.19) we get, setting $c = \min\{c_0, c_1\}$:

$$\begin{aligned} (\Gamma_T h, h) &\geq cT \left\{ \sum_{j=1}^{N} \frac{|P_j h|^2}{\lambda_j} + \sum_{j\geq N+1} |a_j|^2 \right\} + Y_1 + X_2 + Z_3 + Y_3' \\ &\geq cT \|h\|^2_{(D(A))'} + Y_1 + X_2 + Z_3 + Y_3'. \quad (6.26) \end{aligned}$$

We now estimate the remainder $+Y_1+X_2+Z_3+Y_3'$.

We first check

$$
\begin{aligned}
&Y_1+X_2\\
&=2\operatorname{Re}\left(\sum_{j=1}^{N}BB^*P_jh,\sum_{\substack{k=1\\k\neq j}}^{N}\left[P_kh\frac{1-e^{iT(\sqrt{\lambda_j}+\sqrt{\lambda_k})}}{i(\sqrt{\lambda_j}+\sqrt{\lambda_k})}+\bar{P}_kh\frac{1-e^{-iT(\sqrt{\lambda_j}-\sqrt{\lambda_k})}}{i(\sqrt{\lambda_j}-\sqrt{\lambda_k})}\right]\right.\\
&\qquad\left.+\sum_{k\geq N+1}\left[\phi_k\bar{c}_k\frac{1-e^{-iT(\sqrt{\lambda_j}-\sqrt{\lambda_k})}}{i(\sqrt{\lambda_j}+\sqrt{\lambda_k})}+\bar{\phi}_kc_k\frac{1-e^{-iT(\sqrt{\lambda_j}-\sqrt{\lambda_k})}}{i(\sqrt{\lambda_j}-\sqrt{\lambda_k})}\right]\right). \quad (6.27)
\end{aligned}
$$

Hence

$$
|Y_1+X_2|\leq\beta_N\|h\|^2_{(D(A))'}, \quad (6.28)
$$

where

$$
\beta_N=\sqrt{2}\sup_h\sum_{j=1}^{N}\sqrt{\lambda_j}\Lambda_j\frac{|BB^*P_jh|}{|P_jh|}
$$

and

$$
\Lambda_j=\max\left\{\frac{\lambda_{j+1}^{1/2}(\lambda_j+\lambda_{j+1})^{1/2}}{\lambda_{j+1}-\lambda_j},\frac{\lambda_{j-1}^{1/2}(\lambda_j+\lambda_{j-1})^{1/2}}{\lambda_j-\lambda_{j-1}}\right\}.
$$

We interpret $\lambda_{j-1}=0$, if $j=1$. Now

$$
Z_3+Y_3'=T_3+T_3',
$$

where

$$
\begin{aligned}
T_3=&-8\left(\Lambda\left(\sum_{j\geq N+1}\operatorname{Re}\phi_j\operatorname{Im}(e^{-iT\sqrt{\lambda_j}}\bar{a_j})\right),\sum_{k\geq N+1}\operatorname{Re}\phi_k\frac{\operatorname{Re}(e^{-iT\sqrt{\lambda_k}}\bar{a_k})}{\sqrt{\lambda_k}}\right)\\
&+8\left(\Lambda\left(\sum_{j\geq N+1}\operatorname{Re}\phi_j\operatorname{Im}(\bar{a_j})\right),\sum_{k\geq N+1}\operatorname{Re}\phi_k\frac{\operatorname{Re}(\bar{a_k})}{\sqrt{\lambda_k}}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
T_3'=\sum_{j\geq N+1}&\left\{(M\operatorname{Re}\phi_j,\operatorname{Re}\phi_j)-(M\operatorname{Im}\phi_j,\operatorname{Im}\phi_j)-4\frac{(\Lambda\operatorname{Re}\phi_j\operatorname{Re}\phi_j)}{\lambda_j}\right\}\\
&\frac{\operatorname{Im}(c_j^2(e^{2iT\sqrt{\lambda_j}}-1))}{\sqrt{\lambda_j}}.
\end{aligned}
$$

As $\Lambda\in\mathcal{L}\big(H;\big(D(A)\big)'\big)$ and using the fact that

$$
\frac{\phi_j}{\sqrt{\lambda_j}},\quad\frac{\bar{\phi}_j}{\sqrt{\lambda_j}}
$$

are orthonormal in $D(A)$, we can write the estimate

$$|T_3| \leq 4\|\Lambda\| \, \|h\|^2_{(D(A))'}. \tag{6.29}$$

Similarly thanks to (6.19)

$$|T_3'| \leq K\|h\|^2_{(D(A))'}. \tag{6.30}$$

From (6.26), (6.28), (6.29), and (6.30), we deduce the desired result. □

Remark 6.1. We have proved that

$$(\Gamma_T h, h) \geq c(T - T_0)\|h\|^2_{(D(A))'}$$

with

$$T_0 = \frac{\beta_N + K + 4\|\Lambda\|}{c},$$

where $c = \min(c_0, c_1)$ ($= c_0$ if $N = 0$). Therefore there is exact controllability if $T > T_0$. Moreover this estimate implies the sequence

$$D(A) \subset F_T \subset H \subset F_T' \subset \big(D(A)\big)' \tag{6.31}$$

with continuous and dense injection, as seen in §3.2. □

Remark 6.2. In finite dimension, Theorem 6.1 reduces to Proposition 4.2. □

There is some flexibility in the type of assumption we can make to achieve exact controllability. We give next a variant of Theorem 6.1.

Theorem 6.2. *We make the assumptions* (6.6) *to* (6.9), (6.14), (6.15), (6.20), *and*

$$BB^* \in \mathcal{L}\big(\big(D(A)\big)'; D(A)\big) \tag{6.32}$$

and $\forall j, k \geq N+1$, $j \neq k$

$$\lambda_j \lambda_k (M \operatorname{Re}\phi_j, \operatorname{Re}\phi_k) = (\Lambda \operatorname{Re}\phi_j, \operatorname{Re}\phi_k) + (\Lambda \operatorname{Re}\phi_k, \operatorname{Re}\phi_j). \tag{6.33}$$

There exists $Q \in \mathcal{L}(H; H) \geq 0$, *self-adjoint such that* $\forall j, k \geq N+1$, $j \neq k$

$$\frac{(\Lambda \operatorname{Re}\phi_j, \operatorname{Re}\phi_k)}{\lambda_k} + \frac{(\Lambda \operatorname{Re}\phi_k, \operatorname{Re}\phi_j)}{\lambda_j} + \sqrt{\lambda_j \lambda_k}(M \operatorname{Im}\phi_j, \operatorname{Im}\phi_k) = (Q \operatorname{Re}\phi_j, \operatorname{Re}\phi_k). \tag{6.34}$$

There exists $c_0 >$ *such that* $\forall j \geq N+1$

$$(M \operatorname{Re}\phi_j, \operatorname{Re}\phi_j) + (M \operatorname{Im}\phi_j, \operatorname{Im}\phi_j) - \frac{(Q \operatorname{Re}\phi_j, \operatorname{Re}\phi_j)}{\lambda_j} \geq \frac{c_0}{\lambda_j^2}, \tag{6.35}$$

and $\forall j, k \geq N+1$,

$$\Big|(M\operatorname{Re}\phi_j,\operatorname{Re}\phi_j)-(M\operatorname{Im}\phi_j,\operatorname{Im}\phi_j) \\ +\frac{(Q\operatorname{Re}\phi_j,\operatorname{Re}\phi_j)}{\lambda_j}-4\frac{\Lambda\operatorname{Re}\phi_j,\operatorname{Re}\phi_j)}{\lambda_j^2}\Big|\leq\frac{K}{\lambda_j^{3/2}}. \quad (6.36)$$

Then the pair (A, B) is exactly controllable for T sufficiently large, and $D(A^2) \subset F_T$, with dense and continuous injection.

Proof. One proves that

$$(\Gamma_T h, h) \geq c_T \|h\|^2_{(D(A^2))'}, \quad \forall h \in H. \quad (6.37)$$

Of course this estimate is not as good as (6.23), but it is sufficient to establish that $(\Gamma_T h, h)$ is a norm on H and the rest of the statement. We again write

$$(\Gamma_T h, h) = X_1 + X_2 + X_3,$$

where

$$X_1 = 4\int_0^T \left|B^*\left(\sum_{j=1}^N \operatorname{Re}(e^{-ti\sqrt{\lambda_j}}P_j h)\right)\right|^2 dt,$$

$$X_2 = 8\int_0^T \left(B^*\left(\sum_{j=1}^N \operatorname{Re}(e^{-ti\sqrt{\lambda_j}}P_j h)\right), B^*\left(\sum_{j\geq N+1} \operatorname{Re}(e^{-ti\sqrt{\lambda_j}}\bar{c}_j\phi_j)\right)\right) dt,$$

$$X_3 = 4\int_O^T \left|B^*\left(\sum_{j\geq N+1} \operatorname{Re}(e^{-ti\sqrt{\lambda_j}}\bar{c}_j\phi_j)\right)\right|^2 dt.$$

As in Theorem 6.1

$$X_1 = 2T\sum_{j=1}^N |B^* P_j h|^2 + Y_1.$$

We can combine

$$X_2 + Y_1 = \text{ right-hand side of (6.27)}. \quad (6.38)$$

We can state the estimate

$$|X_2 + Y_1| \leq \beta'_N \|h\|^2_{(D(A^2))'}, \quad (6.39)$$

where

$$\beta'_N = \sqrt{2}\sup_h \sum_{j=1}^N \sqrt{\lambda_j}\Lambda_j \frac{\|BB^* P_j h\|_{D(A)}}{\|P_j h\|_{(D(A))'}} \quad (6.40)$$

and

$$\Lambda_j = \max\left\{\frac{\lambda_{j+1}^{1/2}(\lambda_j+\lambda_{j+1})^{1/2}}{\lambda_{j+1}-\lambda_j}, \frac{\lambda_{j-1}^{1/2}(\lambda_j+\lambda_{j-1})^{1/2}}{\lambda_j-\lambda_{j-1}}\right\}.$$

Moreover again as in the proof of Theorem 6.1.

$$\begin{aligned} X_3 \geq 4 \sum_{j,k\geq N+1} \int_0^T (M \operatorname{Re}\phi_j, \operatorname{Re}\phi_k) \operatorname{Re}(e^{-ti\sqrt{\lambda_j}}\bar{c_j}) \cdot \operatorname{Re}(e^{-ti\sqrt{\lambda_k}}\bar{c_k})\,dt \\ + 4 \sum_{j,k\geq N=1} \int_0^T (M \operatorname{Im}\phi_j, \operatorname{Im}\phi_k) \operatorname{Im}(e^{-ti\sqrt{\lambda_j}}\bar{c_j}) \cdot \operatorname{Im}(e^{-ti\sqrt{\lambda_k}}\bar{c_k})\,dt. \end{aligned}$$

Let us write $b_j = c_j/\lambda_j$; then from (6.33)

$$X_3 \geq Y_3 + Z_3,$$

where

$$\begin{aligned} Y_3 = 4 \sum_{j\geq N+1} \int_0^T \lambda_j^2 (M \operatorname{Re}\phi_j, \operatorname{Re}\phi_j) |\operatorname{Re}(e^{-ti\sqrt{\lambda_j}}\bar{b_j})|^2\,dt \\ + 4 \sum_{j\geq N+1} \int_0^T \lambda_j^2 (M \operatorname{Im}\phi_j, \operatorname{Im}\phi_j) |\operatorname{Im}(e^{-ti\sqrt{\lambda_j}}\bar{b_j})|^2\,dt, \end{aligned} \tag{6.41}$$

$$\begin{aligned} Z_3 = 4 \sum_{\substack{k\geq N+1\\ j\neq k}} \int_0^T \big((\Lambda \operatorname{Re}\phi_j, \operatorname{Re}\phi_k) + (\Lambda \operatorname{Re}\phi_k, \operatorname{Re}\phi_j)\big) \\ \cdot \operatorname{Re}(e^{-ti\sqrt{\lambda_j}}\bar{b_j}) \operatorname{Re}(e^{-ti\sqrt{\lambda_k}}\bar{b_k})\,dt \\ +4 \sum_{\substack{k\geq N+1\\ j\neq k}} \int_0^T \lambda_j\lambda_k (M \operatorname{Im}\phi_j, \operatorname{Im}\phi_k) \\ \cdot \operatorname{Im}(e^{-ti\sqrt{\lambda_j}}\bar{b_j}) \operatorname{Im}(e^{-ti\sqrt{\lambda_k}}\bar{b_k})\,dt. \end{aligned} \tag{6.42}$$

We can write

$$\begin{aligned} &\int_0^T \big((\Lambda \operatorname{Re}\phi_j, \operatorname{Re}\phi_k) + (\Lambda \operatorname{Re}\phi_k, \operatorname{Re}\phi_j)\big) \cdot \operatorname{Re}(e^{-ti\sqrt{\lambda_j}}\bar{b_j}) \operatorname{Re}(e^{-ti\sqrt{\lambda_k}}\bar{b_k})\,dt \\ &= -\left(\Lambda \operatorname{Re}\phi_j, \operatorname{Re}\frac{\phi_k}{\sqrt{\lambda_k}}\right)\big(\operatorname{Re}(e^{-Ti\sqrt{\lambda_j}}\bar{b_j}) \cdot \operatorname{Im}(e^{-Ti\sqrt{\lambda_k}}\bar{b_k}) - \operatorname{Re}\bar{b_j} \operatorname{Im}\bar{b_k}\big) \\ &\quad -\left(\Lambda \operatorname{Re}\phi_k, \operatorname{Re}\frac{\phi_j}{\sqrt{\lambda_j}}\right)\big(\operatorname{Re}(e^{-Ti\sqrt{\lambda_k}}\bar{b_k}) \cdot \operatorname{Im}(e^{-Ti\sqrt{\lambda_j}}\bar{b_j}) - \operatorname{Re}\bar{b_k} \operatorname{Im}\bar{b_j}\big) \\ &\qquad + \int_0^T \left(\frac{(\Lambda \operatorname{Re}\phi_j, \operatorname{Re}\phi_k)\sqrt{\lambda_j}}{\sqrt{\lambda_k}} + \frac{(\Lambda \operatorname{Re}\phi_k, \operatorname{Re}\phi_j)\sqrt{\lambda_k}}{\sqrt{\lambda_j}}\right) \\ &\qquad \cdot \operatorname{Im}(e^{-ti\sqrt{\lambda_j}}\bar{b_j}) \operatorname{Im}(e^{-ti\sqrt{\lambda_k}}\bar{b_k})\,dt \end{aligned} \tag{6.43}$$

and thus from the assumption (6.34)

$$Z_3 = -8\left(\Lambda\left[\sum_{j\geq N+1} \operatorname{Re}\phi_j \operatorname{Re}(e^{-Ti\sqrt{\lambda_j}}\bar{b_j})\right], \sum_{k\geq N+1} \operatorname{Re}\frac{\phi_k}{\sqrt{\lambda_k}} \operatorname{Im}(e^{-Ti\sqrt{\lambda_k}}\bar{b_k})\right)$$

$$+ 8\left(\Lambda\left[\sum_{j\geq N+1}\operatorname{Re}\phi_j\operatorname{Re}(\bar{b_j})\right], \sum_{k\geq N+1}\operatorname{Re}\frac{\phi_k}{\sqrt{\lambda_k}}\operatorname{Im}(\bar{b_k})\right)$$

$$+ 8\sum_{j\geq N+1}\left(\Lambda\operatorname{Re}\phi_j, \operatorname{Re}\frac{\phi_j}{\sqrt{\lambda_j}}\right)\big(\operatorname{Re}(e^{-Ti\sqrt{\lambda_j}}\bar{b_j})\operatorname{Im}(e^{-Ti\sqrt{\lambda_j}}\bar{b_j})$$

$$- \operatorname{Re}\big(\bar{b_j}\big)\operatorname{Im}(\bar{b_j})\big)$$

$$+ 4\int_0^T\Bigg(Q\sum_{j\geq N+1}\sqrt{\lambda_j}\operatorname{Re}\phi_j\operatorname{Im}(e^{-Ti\sqrt{\lambda_j}}\bar{b_j}),$$

$$\sum_{k\geq N+1}\sqrt{\lambda_k}\operatorname{Re}\phi_k\operatorname{Im}(e^{-Ti\sqrt{\lambda_k}}\bar{b_k})\Bigg)$$

$$- 4\int_0^T\sum_{j\geq N+1}\lambda_j(Q\operatorname{Re}\phi_j, \operatorname{Re}\phi_j)|\operatorname{Im}(e^{-ti\sqrt{\lambda_j}}\bar{b_j})|^2\,dt. \tag{6.44}$$

Since $Q\geq 0$, we can assert that

$$X_1 + X_2 + X_3$$

$$\geq 2T\left\{\sum_{j=1}^{N}|B^*P_jh|^2 + \sum_{j\geq N+1}|b_j|^2[\lambda_j^2\big((M\operatorname{Re}\phi_j, \operatorname{Re}\phi_j) + (M\operatorname{Im}\phi_j, \operatorname{Im}\phi_j)\big)\right.$$

$$\left. - \lambda_j(Q\operatorname{Re}\phi_j, \operatorname{Re}\phi_j)]\right\} + Y_1 + X_2 + \Delta_1 + \Delta_2, \tag{6.45}$$

with

$$\Delta_1 = \sum_{j\geq N+1}\left\{\lambda_j^{3/2}[(M\operatorname{Re}\phi_j, \operatorname{Re}\phi_j) - (M\operatorname{Im}\phi_j, \operatorname{Im}\phi_j)]\right.$$

$$\left. + \lambda_j^{1/2}(Q\operatorname{Re}\phi_j, \operatorname{Re}\phi_j) - 4\frac{(\Lambda\operatorname{Re}\phi_j, \operatorname{Re}\phi_j)}{\lambda_j^{1/2}}\right\}\operatorname{Im}\big(b_j^2(e^{2Ti\sqrt{\lambda_j}} - 1)\big),$$

$$\Delta_2 = -8\left(\Lambda\sum_{j\geq N+1}\operatorname{Re}\phi_j\operatorname{Re}(e^{-Ti\sqrt{\lambda_j}}\bar{b_j})\right), \sum_{k\geq N+1}\operatorname{Re}\frac{\phi_k}{\sqrt{\lambda_k}}\operatorname{Im}(e^{-Ti\sqrt{\lambda_k}}\bar{b_k})\big)$$

$$+ 8\left(\Lambda\left[\sum_{j\geq N+1}\operatorname{Re}\phi_j\operatorname{Re}(\bar{b_j})\right]\right), \sum_{k\geq N+1}\operatorname{Re}\frac{\phi_k}{\sqrt{\lambda_k}}\operatorname{Im}(\bar{b_k}).$$

Then, according to the assumption (6.36)

$$|\Delta_1| \leq K\|h\|^2_{(D(A^2))'}$$

and

$$|\Delta_2| \leq 4\|\Lambda\| \, \|h\|^2_{(D(A^2))'}.$$

Then we deduce from the assumptions (6.20) and (6.35)

$$(\Gamma_T h, h) \geq c(T - T_0)\|h\|^2_{(D(A^2))'}, \tag{6.46}$$

with $c = \min\{c_0, c_1\}$ (c_0 if $N = 0$) and

$$T_0 = \frac{\beta'_N + K + 4\|\Lambda\|}{c},$$

and the desired result has been proved. □

6.3 Approximation

We shall study the properties of Γ_T, assuming only (6.6) to (6.9) and thus nothing on B except naturally $B \in \mathcal{L}(U; H)$.

We now keep only the eigenvalues with different values denoted $0 < \lambda_1 < \lambda_2 \cdots < \lambda_j < \cdots$ and call P_j the projector, which has been defined in (6.20) (but now j is any integer). An element ϕ of H is represented as

$$\phi = \sum_j (P_j\phi + \bar{P}_j\phi) \tag{6.47}$$

and

$$e^{At}\phi = \sum_j (e^{i\sqrt{\lambda_j}t} P_j\phi + e^{-i\sqrt{\lambda_j}t}\bar{P}_j\phi). \tag{6.48}$$

Call

$$\Pi^N\phi = \sum_{j=1,\ldots,N} (P_j\phi + \bar{P}_j\phi). \tag{6.49}$$

We also define the operator Γ by

$$\Gamma = 2\,\mathrm{Re} \sum_j P_j BB^* P_j. \tag{6.50}$$

We can state the following proposition.

Proposition 6.2. *Assume* (6.6) *to* (6.9)*; then*

$$\frac{\Gamma_T h}{T} \Longrightarrow \Gamma h \quad \text{in } H, \qquad \forall h \in H, \quad \text{as } T \Longrightarrow \infty. \tag{6.51}$$

Proof. We can write, picking η in H

$$(\Gamma_T h, \eta) = (\Gamma_T(h - \Pi^N h), \eta) + 2T\,\mathrm{Re}\sum_{j=1}^{N}(P_j BB^* P_j h, \eta) + X_{N,T} + Y_{N,T} \tag{6.52}$$

with

$$X_{N,T} = 2\,\mathrm{Re}\left(\sum_{j=1}^{N} BB^* P_j h, \sum_{k\neq j}\left\{P_k\eta\frac{1-e^{-iT(\sqrt{\lambda_j}+\sqrt{\lambda_k})}}{i(\sqrt{\lambda_j}+\sqrt{\lambda_k})} + \bar{P}_k\eta\frac{1-e^{-iT(\sqrt{\lambda_j}-\sqrt{\lambda_k})}}{i(\sqrt{\lambda_j}-\sqrt{\lambda_k})}\right\}\right)$$

and

$$Y_{N,T} = \mathrm{Re}\sum_{j=1}^{N}(BB^* P_j h, P_j\eta)\frac{(1-e^{2iT\sqrt{\lambda_j}})}{i\sqrt{\lambda_j}}.$$

We can check that

$$|X_{N,T}| \leq \beta_N \|h\|_{(D(A))'}\|\eta\|_{(D(A))'}, \tag{6.53}$$

where β_N has been defined in the proof of Theorem 6.1.

Similarly

$$|Y_{N,T}| \leq \sqrt{2}\sum_{j=1}^{N}|BB^* P_j h|\colon \|\eta\|_{(D(A))'}. \tag{6.54}$$

Now

$$2T\,\mathrm{Re}\sum_{j=1}^{N}(P_j BB^* P_j h, \eta) = 2T(\Gamma h, \eta) - 2T(\Gamma(h - \Pi^N h), \eta).$$

Therefore, collecting results, we deduce noticing that $\|\Gamma_T\| \leq T\|BB^*\|$ and $\|\Gamma\| \leq \|BB^*\|$

$$\left|\left(\frac{\Gamma_T}{T} - \Gamma\right)h\right| \leq 2\|BB^*\|\colon |h - \Pi^N h| + \frac{\beta_N\|h\|_{(D(A))'} + \sqrt{2}\sum_{j=1}^{N}|BB^* P_j h|}{T\lambda_1},$$

and the desired result follows. □

We next prove the following estimate.

Lemma 6.1.

$$|(\Gamma_T h, \eta)| \leq \|BB^*\|(2|h| + T\|h\|_{D(A)})\|\eta\|_{(D(A))'}. \tag{6.55}$$

Proof. From the definition of Γ_T we have

$$(\Gamma_T h, \eta) = 4\int_0^T \left(BB^*\sum_j \mathrm{Re}(e^{it\sqrt{\lambda_j}} P_j h), \frac{d}{dt}\sum_k \mathrm{Im}\left(e^{it\sqrt{\lambda_k}}\frac{P_k\eta}{\sqrt{\lambda_k}}\right)\right)\,dt,$$

and by integrating by parts

$$
\begin{aligned}
&= 4\left(BB^* \sum_j \mathrm{Re}(e^{iT\sqrt{\lambda_j}} P_j h), \sum_k \mathrm{Im}\left(e^{iT\sqrt{\lambda_k}} \frac{P_k \eta}{\sqrt{\lambda_k}}\right)\right) \\
&\quad -4\left(BB^* \sum_j \mathrm{Re}(P_j h), \sum_k \mathrm{Im}\left(\frac{P_k \eta}{\sqrt{\lambda_k}}\right)\right) \\
&\quad +4\int_0^T \left(BB^* \sum_j \mathrm{Im}(\sqrt{\lambda_j} e^{it\sqrt{\lambda_j}} P_j h), \sum_k \mathrm{Im}\left(e^{it\sqrt{\lambda_k}} \frac{P_k \eta}{\sqrt{\lambda_k}}\right)\right) dt.
\end{aligned}
$$

Thus the result (6.55) is easily deduced. □

One deduces from (6.55) that

$$
\Gamma_T \in \mathcal{L}\big(D(A); D(A)\big). \tag{6.56}
$$

Remark 6.3. The formula used in the proof of Lemma 6.1 shows also that that if $BB^* \in \mathcal{L}\big(D(A); D(A)\big)$, then

$$
\Gamma_T \in \mathcal{L}\big(D(A^2); D(A^2)\big).
$$

□

Similarly we check the estimate

$$
|(\Gamma h, \eta)| \leq \|BB^*\| \colon \|h\|_{D(A)} \|\eta\|_{(D(A))'}, \tag{6.57}
$$

and then we can assert the following proposition.

Proposition 6.3. *One has*

$$
\frac{\Gamma_T}{T} h \implies \Gamma h \quad \text{in } D(A), \qquad \forall h \in D(A), \qquad \text{as } T \implies \infty. \tag{6.58}
$$

Proof. Using the formulas of Proposition 6.2 and Lemma 6.1, we obtain easily the estimate

$$
\begin{aligned}
\left\| \left(\frac{\Gamma_T}{T} - \Gamma\right) h \right\|_{D(A)} &\leq 2\|BB^*\| \left\{ \|h - \Pi^N h\|_{D(A)} + \frac{|h - \Pi^N h|}{T} \right\} \\
&\quad + \frac{\beta_N \|h\|_{(D(A))'}}{T} b + \sqrt{2} \sum_{j=1}^N |BB^* P_j h|,
\end{aligned} \tag{6.59}
$$

and thus (6.59) follows. □

We now make the assumptions of Theorem 6.1. We first notice that $(\Gamma h, h)$ is also a norm on H, thanks to the following proposition.

Proposition 6.4. *We make the assumptions of Theorem* 6.1*; then*

$$
(\Gamma h, h) \geq c\|h\|^2_{(D(A))'} \quad \forall h \in H. \tag{6.60}
$$

Proof. We have

$$(\Gamma h, h) = 2\sum_j (|B^* \operatorname{Re} P_j h|^2 + |B^* \operatorname{Im} P_j h|^2),$$

and from (6.14) and (6.20)

$$\geq 2 \sum_{j \geq N+1} \big((M \operatorname{Re} P_j h, \operatorname{Re} P_j h) + (M \operatorname{Im} P_j h, \operatorname{Im} P_j h)\big) + 2c_1 \sum_{j=1}^{N} \frac{|P_j h|^2}{\lambda_j}.$$

Now (6.15) to (6.17) imply that if

$$P_j h = \sum_{k=1}^{rj} \bar{c_{jk}} \phi_{jk},$$

then

$$\sum_{j \geq N+1} \big((M \operatorname{Re} P_j h, \operatorname{Re} P_j h) + (M \operatorname{Im} P_j h, \operatorname{Im} P_j h)\big)$$

$$= \sum_{j \geq N+1} \sum_{k=1}^{r_j} |\bar{c_{jk}}|^2 \big((M \operatorname{Re} \phi_{jk}, \operatorname{Re} \phi_{jk}) + (M \ \operatorname{Im} \phi_{jk}, \operatorname{Im} \phi_{jk})\big),$$

and from (6.18)

$$\geq c_0 \sum_{j \geq N+1} \sum_{k=1}^{r_j} \frac{|\bar{c_{jk}}|^2}{\lambda_j} = c_0 \sum_{j \geq N+1} \frac{|P_j h|^2}{\lambda_j}.$$

Therefore collecting results, the desired result follows. □

Let us define the space F' completing H with the norm (6.60). Clearly one has as for (6.31)

$$D(A) \subset F \subset H \subset F' \subset \big(D(A)\big)'. \tag{6.61}$$

As Γ_T is invertible from F'_T to F_T and Γ from F' to F, we can consider $(\Gamma_T)^{-1}$ from F_T to F'_T and Γ^{-1} from F to F', hence also from $D(A)$ to $\big(D(A)\big)'$ for both.

In fact, it will be useful to notice that something more can be said for Γ^{-1}, namely the following lemma.

Lemma 6.2.

$$\Gamma^{-1} \in \mathcal{L}\big(D(A^2); H\big).$$

Proof. Setting $\phi = \Gamma^{-1} h$; then from the definition of Γ

$$2 \operatorname{Re} \sum_j P_j (BB^* P_j \phi) = 2 \operatorname{Re} \sum_j P_j h;$$

hence

$$P_j(BB^*P_j\phi) = P_jh, \quad \forall j.$$

Therefore

$$|B^*P_j\phi|^2 = (P_jh, \bar{P}_j\phi), \quad \forall j.$$

But from the proof of Proposition 6.4, it follows that

$$|B^*P_j\phi|^2 \geq c\frac{|P_j\phi|^2}{\lambda_j}.$$

Therefore

$$|P_j\phi| \leq \frac{\lambda_j}{c}|P_jh|.$$

Hence

$$|\phi|^2 = 2\sum_j |P_j\phi|^2 \leq \frac{2}{c}\sum_j \lambda_j|P_jh|^2 \leq \frac{1}{c}\|h\|^2_{D(A^2)},$$

which is the desired result. □

We can then state the the following theorem.

Theorem 6.3. *We make the assumptions of Theorem* 6.1*; then one has*

$$T(\Gamma_T)^{-1}h \implies \Gamma^{-1}h \quad \textit{in } \big(D(A)\big)', \qquad \forall h \in D(A). \tag{6.62}$$

Proof. Let us set $\rho_T = T(\Gamma_T)^{-1}h$ and $\rho = \Gamma^{-1}h$.

We begin by proving the weak convergence in $\big(D(A)\big)'$. But

$$(h, \rho_T) = \left(\frac{\Gamma_T}{T}\rho_T, \rho_T\right) \geq c\frac{T - T_0}{T}\|\rho_T\|^2_{(D(A))'}$$

and thus

$$\|\rho_T\|_{(D(A))'} \leq \frac{2}{c}\|h\|_{D(A)}$$

as soon as $T > 2T_0$.

Therefore we can extract a subsequence converging weakly to some σ in $\big(D(A)\big)'$. But we can write for any η in $D(A)$

$$\left(\frac{\Gamma_T}{T}\eta, \rho_T\right) = (h, \eta),$$

and using Proposition 6.3 we get

$$(\Gamma\eta, \sigma) = (h, \eta).$$

Therefore

$$(\Gamma(\sigma - \rho), \eta) = 0$$

for any η in $D(A)$. Hence $\Gamma(\sigma-\rho)=0$. But Γ is invertible from $\big(D(A)\big)'$ to $D(A)$; hence $\rho=\sigma$. Note that by the uniqueness of the limit, we can assert that the full subsequence converges. To prove the strong convergence, we first assume that $h\in D(A^2)$; in which case, we know that ρ belongs to H. But

$$\left(\frac{\Gamma_T}{T}(\rho_T-\rho),\rho_T-\rho\right)=(h,\rho_T)-2(h,\rho)+\left(\frac{\Gamma_T}{T}\rho,\rho\right),$$

and therefore the right-hand side tends to 0. It easily follows that ρ_T tends to ρ in $\big(D(A)\big)'$. In the general case, we pick a sequence h_μ that belongs to $D(A^2)$ and converges to h in $D(A)$. Let ρ_μ and $\rho_{T\mu}$ correspond to h_μ. We first notice that

$$\|\rho_{T\mu}-\rho_T\|\le\frac{2}{c}\|h-h_\mu\|_{D(A)}. \tag{6.63}$$

Now

$$\|\rho_T-\rho\|\le\|\rho_{T\mu}-\rho_T\|+\|\rho_{T\mu}-\rho_\mu\|+\|\rho_\mu-\rho\|.$$

From the uniform estimate (6.63) and from the fact that $\|\rho_{T\mu}-\rho_\mu\|$ tends to 0 as T tends to infinity and μ is fixed, we deduce that ρ_T tends to ρ in $\big(D(A)\big)'$ as T tends to infinity. □

6.4 Exact controllability for T arbitrarily small

In this section, we consider additional assumptions, which will guarantee that the pair (A,B) is not only exactly controllable for large T, but in fact for arbitrarily small T. We work in the framework of Theorem 6.1. We replace (6.19) by

$$\left|(M\operatorname{Re}\phi_j,\operatorname{Re}\phi_j)-(M\operatorname{Im}\phi_j,\operatorname{Im}\phi_j)-4\frac{(\Lambda\operatorname{Re}\phi_j,\operatorname{Re}\phi_j)}{\lambda_j}\right| \le\frac{K}{\lambda_j},\quad\forall j\ge N+1. \tag{6.64}$$

In fact, we shall need (6.64) for j sufficiently large, but modifying the constant, we can always assume that it is satisfied for $j\ge N+1$, without loss of generality. We also assume that

$$\begin{aligned}&\text{there exists a Hilbert space } W \text{ such that } D(A)\subset W\subset H,\\ &\qquad\text{the injection of } D(A) \text{ in } W \text{ is compact, and } \Gamma\in\mathcal{L}(H;W').\end{aligned} \tag{6.65}$$

Our objective is to prove the following.

Theorem 6.4. *We make the assumptions of Theorem* 6.1*, except* (6.19)*, which is replaced by* (6.64)*, and we assume* (6.65)*. Then the pair A, B is exactly controllable for any time $T>0$.*

Proof. We begin by proving that for any $T > 0$, then

$$(\Gamma_T h, h) = 0 \implies h = 0. \tag{6.66}$$

This will imply that $(\Gamma_T h, h)^{1/2}$ is a norm on H, for $T > 0$, arbitrary. Of course, we already know that it is true for $T > T_0$. Naturally (6.66) is equivalent to

$$\Gamma_T h = 0 \implies h = 0. \tag{6.67}$$

We denote by Π^J the projector on the subspace of H, generated by the eigenvalues $\pm i\sqrt{\lambda_j}$, with $j \leq J$. We shall need to consider only J large, so in particular $J > N$; hence for any element h of H, one has

$$\Pi^J h = \sum_{j=1}^{N} (P_j h + \bar{P}_j h) + \sum_{j=N+1}^{J} (\bar{c_j}\phi_j + c_j\bar{\phi_j}). \tag{6.68}$$

We shall prove that for any $T > 0$, there exists J depending on T, sufficiently large and a constant β^J, such that

$$(\Gamma_T h, h) + \beta^J |\Pi^J h|^2 \geq c_T \|h\|^2_{(D(A))'}, \quad \forall h \in H. \tag{6.69}$$

Assume for a while that this is proved.

We shall then prove that

$$\Gamma_T h = 0 \quad \text{implies } h \in D(A). \tag{6.70}$$

Indeed, from the assumption we have

$$B^* e^{A^* t} h = 0, \quad t \in (0, T). \tag{6.71}$$

Let τ such that $T - 2\tau > 0$. Consider a sequence of C^∞ functions on R, with compact support on $(0, \tau)$, denoted by ρ_j, which converges to $\delta(0)$ in the distribution sense. Define successively $h_{1,j}$ and $h_{2,j}$ by the formulas

$$h_{1,j} = \int_0^\tau e^{A^* s} h \rho_j(s)\, ds, \tag{6.72}$$

$$h_{2j} = \int_0^\tau e^{A^* s} h_{1,j} \rho_j(s)\, ds; \tag{6.73}$$

then $h_{1,j} \in D(A)$, $h_{2,j} \in D(A^2)$.

Indeed

$$Ah_{1,j} = \int_0^\tau e^{A^* s} h \rho_j'(s)\, ds,$$

$$Ah_{2,j} = \int_0^\tau e^{A^* s} h_{1,j} \rho_j'(s)\, ds.$$

From (6.71), we deduce easily

$$\begin{aligned} B^* e^{A^* t} h_{1,j} &= 0, \quad t \in (0, T - \tau), \\ B^* e^{A^* t} h_{2,j} &= 0, \quad t \in (0, T - 2\tau). \end{aligned} \tag{6.74}$$

As $h_{2,j} \in D(A^2)$, we deduce by differentiating twice the second relation that

$$B^* e^{A^* t} A^2 h_{2,j} = 0, \quad t \in (0, T - 2\tau).$$

Moreover $A^2 h_{2,j} \in H$. Therefore, we may apply (6.65) with $h = A^2 h_{2,j}$, and T changed into $T - 2\tau$. As $\mathit{\Gamma}_{T-2\tau} A^2 h_{2j} = 0$, we deduce that $A^2 h_{2,j}$ remains in a bounded subset of $\big(D(A)\big)'$; hence $h_{2,j}$ remains in a bounded subset of $D(A)$. On the other hand, h_{2j} converges to h in $\big(D(A)\big)'$ weakly. This follows from the fact that if $\phi \in D(A)$, then

$$\langle \phi, h_{2,j} \rangle \left([A + I] \int_0^\tau e^{As} \phi \rho_j(s)\, ds, [A + I]^{-1} \int_0^\tau e^{A^* s} h \rho_j(s)\, ds \right)$$

and

$$[A + I]^{-1} \int_0^\tau e^{A^* s} h \rho_j(s)\, ds$$

tends to h in H strongly, and

$$[A + I] \int_0^\tau e^{As} \phi \rho_j(s)\, ds$$

tends to ϕ in H weakly. Therefore $h \in D(A)$.

Let Y be the subspace of H of elements such that $h \in H$, $\mathit{\Gamma}_T h = 0$. Our objective is to prove that $Y = \{0\}$. We first prove that Y is finite dimensional. But if $h \in Y$, then one can write

$$\mathit{\Gamma}_T h + \beta^J \mathit{\Pi}^J h = \beta^J \mathit{\Pi}^J h;$$

hence

$$h = \beta^J (\mathit{\Gamma}_T + \beta^J \mathit{\Pi}^J)^{-1} \mathit{\Pi}^J h$$

and

$$\mathit{\Pi}^J h = \beta^J \mathit{\Pi}^J (\mathit{\Gamma}_T + \beta^J \mathit{\Pi}^J)^{-1} \mathit{\Pi}^J h. \tag{6.75}$$

These formulas prove that Y is a vector subspace of the span of $(\mathit{\Gamma}_T + \beta^J \mathit{\Pi}^J)^{-1} \phi_j$, $(\mathit{\Gamma}_T + \beta^J \mathit{\Pi}^J)^{-1} \bar{\phi_j}, j = 1, \ldots, J$, where we have written for simplicity $\mathit{\Pi}^J$ in (6.75) as

$$\mathit{\Pi}^J h = \sum_{j=1}^J (\bar{c_j} \phi_j + c_j \bar{\phi_j}),$$

which is always possible by a convenient renumbering.

Now, thanks to (6.70), $Ah \in Y$. Furthermore, if $h \in Y$, $A^2 h \in Y$ and thus

$$\Gamma_T A^2 h = \beta^J \Pi^J A^2 h = \beta^J \Pi^J A^2 h$$

and also

$$(\Gamma_T A^2 h, A^2 h) + \beta^J (\Pi^J A^2 h, A^2 h) = \beta^J (\Pi^J A^2 h, A^2 h).$$

Therefore, by (6.69), we deduce

$$c_T \|A^2 h\|^2_{(D(A))'} \leq \beta^J (\lambda^J)^2 |h|^2$$

or

$$c_T |Ah|^2_H \leq \beta^J (\lambda^J)^2 |h|^2,$$

which proves that A is linear continuous on Y. But then Y can be spanned by the eigenvectors of A. If ϕ_j is an eigenvector of A belonging to Y, then

$$B^* \phi_j e^{it\sqrt{\lambda_j}} = 0, \quad \forall t \in (0, T).$$

By the analyticity property, it follows that

$$B^* \phi_j e^{it\sqrt{\lambda_j}} = 0, \quad \forall t \in (-\infty, +\infty).$$

In particular $(\Gamma_T \phi_j, \phi_j) = 0$, $\forall T > 0$. This is impossible, because we know that for T sufficiently large $(\Gamma_T h, h)^{1/2}$ is a norm on H. Therefore $Y = \{0\}$, and (6.67) is proved. It is easy to check then, as a consequence of (6.67) and (6.69), one has also

$$(\Gamma_T h, h) \geq c'_T \|h\|^2_{(D(A))'}, \quad \forall h \in H,\ \forall T > 0. \tag{6.76}$$

Suppose that (6.76) is not true; then there exists a sequence $\{h_n\}$, $\|h_n\|_{(D(A))'} = 1$, such that $(\Gamma_T h_n, h_n)$ tends to 0. As $(\Gamma_T h, h)^{1/2}$ is a norm on H, we have $h_n \to 0$ in F_T (completed of H with the norm $(\Gamma_T h, h)^{1/2}$). As h_n is bounded in $\big(D(A)\big)'$, it converges also weakly to 0 in $\big(D(A)\big)'$; hence, $\Pi^j h_n \to 0$. But then from (6.69), $\|h_n\|_{(D(A))'} \to 0$, which is impossible.

It remains to prove (6.69). It is sufficient to prove that

$$\forall T > 0,\ \exists J = J_T \quad \text{and} \quad c_T \text{ such that}$$
$$(\Gamma_T(h - \Pi^J h), h - \Pi^J h) \geq c_T \|h - \Pi^J h\|^2_{(D(A))'}. \tag{6.77}$$

Now

$$(\Gamma_T(h - \Pi^J h), h - \Pi^J h) = 4 \int_0^T \left| B^* \left(\sum_{j \geq J+1} \operatorname{Re}(e^{-ti\sqrt{\lambda_j}} \bar{c}_j \phi_j) \right) \right|^2 dt$$

and with calculations already done

$$\geq 2T \sum_{j \geq J+1} \|c_j\|^2 \big((M \operatorname{Re} \phi_j, \operatorname{Re} \phi_j) + (M \operatorname{Im} \phi_j, \operatorname{Im} \phi_j) \big) + Z_3 + Y'_3,$$

with (recalling that $a_j = c_j/\sqrt{\lambda_j}$)

$$Z_3 = -8\left(\varLambda\left[\sum_{j\geq J+1}\operatorname{Re}\phi_j\operatorname{Im}(e^{-iT\sqrt{\lambda_j}}\bar{a_j})\right], \sum_{k\geq J+1}\operatorname{Re}\phi_k\frac{\operatorname{Re}(e^{iT\sqrt{\lambda_k}}\bar{a_k})}{\sqrt{\lambda_k}}\right)$$
$$+8\left(\varLambda\left[\sum_{j\geq J+1}\operatorname{Re}\phi_j\operatorname{Im}(\bar{a_j})\right], \sum_{k\geq J+1}\operatorname{Re}\phi_k\frac{\operatorname{Re}(\bar{a_k})}{\sqrt{\lambda_k}}\right)$$
$$+8\sum_{j\geq J+1}(\varLambda\operatorname{Re}\phi_j,\operatorname{Re}\phi_j)\operatorname{Im}(e^{-iT\sqrt{\lambda_j}}\bar{a_j})\frac{\operatorname{Re}(e^{-iT\sqrt{\lambda_j}}\bar{a_j})}{\sqrt{\lambda_j}}$$
$$-8\sum_{j\geq J+1}(\varLambda\operatorname{Re}\phi_j,\operatorname{Re}\phi_j)\operatorname{Im}(\bar{a_j})\frac{\operatorname{Re}(\bar{a_j})}{\sqrt{\lambda_j}}$$

and

$$Y_3' = \sum_{j\geq J+1}\left[(M\operatorname{Re}\phi_j,\operatorname{Re}\phi_j)-(M\operatorname{Im}\phi_j,\operatorname{Im}\phi_j)\right]\frac{\operatorname{Im}(c_j^2(e^{2iT\sqrt{\lambda_j}}-1))}{\sqrt{\lambda_j}}.$$

Again we can write

$$Z_3 + Y_3' = T_3 + T_3',$$

where

$$T_3 = -8\left(\varLambda\left[\sum_{j\geq N+1}\operatorname{Re}\phi_j\operatorname{Im}(e^{-iT\sqrt{\lambda_j}}\bar{a_j})\right], \sum_{k\geq N+1}\operatorname{Re}\phi_k\frac{\operatorname{Re}(e^{-iT\sqrt{\lambda_k}}\bar{a_k})}{\sqrt{\lambda_k}}\right)$$
$$+8\left(\varLambda\left[\sum_{j\geq N+1}\operatorname{Re}\phi_j\operatorname{Im}(\bar{a_j})\right], \sum_{k\geq N+1}\operatorname{Re}\phi_k\frac{\operatorname{Re}(\bar{a_k})}{\sqrt{\lambda_k}}\right)$$

and

$$T_3' = \sum_{j\geq N+1}\Bigg[(M\operatorname{Re}\phi_j,\operatorname{Re}\phi_j)-(M\operatorname{Im}\phi_j,\operatorname{Im}\phi_j)$$
$$-4\frac{(\varLambda\operatorname{Re}\phi_j,\operatorname{Re}\phi_j)}{\lambda_j}\Bigg]\frac{\operatorname{Im}(c_j^2(e^{2iT\sqrt{\lambda_j}}-1))}{\sqrt{\lambda_j}}.$$

We use (6.64) to assert that

$$|T_3'| \leq 2K\sum_{j\geq J+1}\frac{|c_j|^2}{\lambda_j^{3/2}} = 2K\sum_{j\geq J+1}\frac{|a_j|^2}{\lambda_j^{1/2}}.$$

Next, from (6.65) we can assert that, for any ε, there exists $C(\varepsilon)$ such that

$$\|h\|_W \leq \varepsilon\|h\|_{D(A)} + C(\varepsilon)|h|$$

(see J. L. LIONS [1]). Therefore, as easily seen

$$|T_3| \leq 8\|\Lambda\|_{\mathcal{L}(H;W')} \cdot \left[\varepsilon \sum_{j \geq J+1} |\bar{a}_j|^2 + C(\varepsilon) \left(\sum_{j \geq J+1} |\bar{a}_j|^2 \right)^{1/2} \left(\sum_{j \geq J+1} \frac{|\bar{a}_j|^2}{\lambda_j} \right)^{1/2} \right].$$

Therefore, collecting results we have

$$(\Gamma_T(h - \Pi^J h), h - \Pi^J h) \\ \geq \|h - \Pi^J h\|^2_{(D(A))'} \left\{ c_0 T - \frac{K}{\sqrt{\lambda_J}} - 4\|\Lambda\|\varepsilon - \frac{C(\varepsilon)}{\sqrt{\lambda_j}} \right\}. \quad (6.78)$$

Therefore for any $T > 0$, choose

$$\varepsilon = \frac{c_0 T}{8\|\Lambda\|},$$

and J sufficiently large so that

$$\frac{c_0 T}{2} > \frac{K + C(\varepsilon)}{\sqrt{\lambda_J}}.$$

This concludes (6.77). □

Remark 6.4. The idea of the regularizing functions ρ_j has been given to us by J. L. LIONS [6]. □

7 General framework: skew-symmetric operators

7.1 Operator A

Let L be a self-adjoint operator in a Hilbert space H_L (which is identified with its dual). Let $D(L)$ be the domain of L. We assume that there exists a Hilbert space V_L such that

$$V_L \subset H_L \subset V_L', \quad (7.1)$$

continuously and densily embedded.

We assume that

$$\langle L z_1, z_2 \rangle = ((z_1, z_2)), \quad \forall z_1, z_2 \in V_L, \quad (7.2)$$

where ((,)) denotes the scalar product in V_L.

We furthermore assume that

$$\text{the injection of } V_L \text{ into } H_L \text{ is compact.} \quad (7.3)$$

In that case $L^{-1} \in \mathcal{L}(H_L; H_L)$ is compact and thus there exists an orthonormal base w_j of H_L, of eigenvectors of L; i.e.,

$$Lw_j = \lambda_j w_j, \quad |w_j|_{H_L} = 1,$$

with $0 < \lambda_1 \leq \lambda_2 \cdots \leq \lambda_j \cdots \uparrow \infty$.

Besides $D(L)$ we shall need

$$\varDelta(L) = \{z \in V_L \colon Lz \in V_L\}$$

and we have the sequence of spaces, each of them being densily and continuously embedded in the next one

$$\varDelta_L \subset D(L) \subset V_L \subset H_L \subset V_L' \subset \big(D(L)\big)' \subset \varDelta_L'.$$

We can associate with L a skew-symmetric operator A, satisfying the assumptions (6.6) to (6.9). Take indeed

$$H = H_L \times V_L', \quad A = \begin{bmatrix} 0 & I \\ -L & 0 \end{bmatrix}, \quad D(A) = V_L \times H_L. \tag{7.4}$$

We have to pay attention to the fact that because H is identified with its dual, V_L' is in the second component the pivot space, identified with its dual. In that framework the dual of H_L is $\big(D(L)\big)'$, with the duality

$$\langle z, \zeta \rangle = (z, L^{-1}\zeta)_{H_L}, \quad \forall z \in H_L,\ \zeta \in \big(D(L)\big)'.$$

Note that the scalar product in H is expressed as follows:

$$(h, h')_H = (h_1, h_1') + \langle L^{-1}h_2, h_2' \rangle.$$

Moreover $\big(D(A)\big)' = V_L' \times \big(D(L)\big)'$.

The eigenvectors of A are given by

$$\phi_j = \frac{1}{\sqrt{2}} \begin{bmatrix} w_j \\ i\sqrt{\lambda_j} w_j \end{bmatrix}.$$

7.2 Operator B

We next define the operator B. We shall consider a control space $U = U_1 \times U_2$ where U_1, U_2 are Hilbert spaces, identified with their duals. Let q_1, q_2 be operators such that

$$q_1 \in \mathcal{L}(D(L); U_1), \quad q_2 \in \mathcal{L}(V_L; U_2). \tag{7.5}$$

We set

$$Bv = -\begin{bmatrix} L^{-1}q_1^* v_1 \\ q_2^* v_2 \end{bmatrix} \tag{7.6}$$

and $B \in \mathcal{L}(U; H)$. Note that

$$B^* h = -\begin{bmatrix} q_1 L^{-1} h_1 \\ q_2 L^{-1} h_2 \end{bmatrix}. \tag{7.7}$$

The presence of L^{-1} in the second component of B^* arises from the fact that the pivot space is V_L'.

7.3 Dynamical system

Consider the dynamic system

$$\begin{gathered} z' = Az + Bv, \quad z(0) = 0, \\ z \in C([0,T];H), \quad z' \in L^2\big(0,T;\big(D(A)\big)'\big); \end{gathered} \tag{7.8}$$

we deduce

$$\begin{aligned} z_1' &= z_2 - L^{-1}q_1^* v_1, \qquad & z_1(0) = 0, \\ z_2' &= -Lz_1 - q_2^* v_2, \qquad & z_2(0) = 0. \end{aligned} \tag{7.9}$$

Hence $z_1 \in C([0,T];H_L)$, $z_2 \in C([0,T];V_L')$; $z_1' \in L^2(0,T;V_L')$, $z_2' \in L^2\big(0,T;\big(D(L)\big)'\big)$.

We can associate with (7.9) a second order (in time) equation as follows. Set $\eta = z_2$; then we also write

$$\begin{gathered} \eta'' + L\eta = q_1^* v_1 - q_2^* v_2', \\ \eta(0) = 0, \quad \eta'(0) + q_2^* v_2(0) = 0, \\ \eta \in C([0,T];V_L'), \quad \eta' + q_2^* v_2 \in C\big([0,T];\big(D(L)\big)'\big), \\ \eta' \in L^2\big(0,T;\big(D(L)\big)'\big), \\ \eta'' \in C([0,T];\Delta_L') \oplus L^2\big(0,T;\big(D(L)\big)'\big) \oplus H^{-1}(0,T;V_L'). \end{gathered} \tag{7.10}$$

Note that all terms in (7.10) make perfect sense. However, it is important to notice that the value of η' is not defined at each point, and is not 0 in any sense in general. It is useful to reinterpret (7.10) in terms of the Method of Transposition of J. L. Lions and E. Magenes [1]. Note that for any $\psi_0 \in D(L), \psi_1 \in V_L$, there exists one and only one solution ψ such that

$$\begin{gathered} \psi'' + L\psi = f, \\ \psi(T) = \psi_0, \quad \psi'(T) = \psi_1, \\ \psi \in C\big([0,T];D(L)\big), \quad \psi' \in C([0,T];V_L). \end{gathered} \tag{7.11}$$

By making appropriate integration by parts between (7.10) and (7.11), we deduce the formula

$$\langle \psi_1, \eta(T)\rangle - \langle \psi_0, \eta'(T) + q_2^* v_2(T)\rangle = \int_0^T [(v_1, q_1\psi) + (v_2, q_2\psi')]\, dt, \tag{7.12}$$

which can be considered as the definition of $\eta(T)$ and $\eta'(T) + q_2^* v_2(T)$.

There is a formal but mnemonic way of writing (7.10). Introduce the operator

$$J_0 \in \mathcal{L}\big(H^1(0,T;U_2);L^2(0,T;U_2)\big)$$

defined by

$$J_0 u = u'.$$

Its transpose $J_0^* \in \mathcal{L}\big(L^2(0,T;U_2);\big(H^1(0,T;U_2)\big)'\big)$. One should not mix up $J_0^* u$, which belongs to $\big(H^1(0,T;U_2)\big)'$, with $-u'$, which belongs to $H^{-1}(0,T;U_2)$. They coincide only, when applied to $H_0^1(0,T;U_2)$. Next, consider q_2 as a linear bounded operator from $H^1(0,T;V_L)$ to $H^1(0,T;U_2)$, and thus q_2^* as an element of $\mathcal{L}\big(\big(H^1(0,T;U_2)\big)';\big(H^1(0,T;V_L)\big)'\big)$. Therefore $q_2^* J_0^* \in \mathcal{L}\big(L^2(0,T;U_2);\big(H^1(0,T;V_L)\big)'\big)$. We write (7.10) as

$$\eta'' + L\eta = q_1^* v_1 + q_2^* J_0^* v_2, \qquad \eta(0) = 0, \quad \eta'(0) = 0. \tag{7.13}$$

We know that

$$\eta \in C([0,T];V_L') \cap L^2(0,T;H_L), \quad \eta' \in L^2\big(0,T;\big(D(L)\big)'\big).$$

From (7.13) we read

$$\eta'' \in C([0,T];\Delta_L') \oplus L^2\big(0,T;\big(D(L)\big)'\big) \oplus \big(H^1(0,T;V_L)\big)'.$$

The writing is formal as far as $\eta'(0)$ (which is not defined) and η'' are concerned. What it means is that for any

$$\psi \in H^1\big(0,T;D(L)\big) \cap L^2(0,T;\Delta_L)$$

with $\psi(T) = 0$, then one has

$$-\int_0^T (\eta', \psi')\,dt + \int_0^T (\eta, L\psi)\,dt = \int_0^T [(v_1, q_1\psi) + (v_2, q_2\psi')]\,dt, \tag{7.14}$$

where all terms make sense. This equation defines a unique η because it implies in particular (7.12). The value of η' at any given time t (in particular 0) is not the value of the derivative, but a short for $\eta' + q_2^* v_2$.

7.4 Exact controllability

Let us first express the operator $\Gamma_T \in \mathcal{L}(H;H)$. By the definition

$$(\Gamma_T h, k) = \int_0^T (B^* e^{A^* t} h, B^* e^{A^* t} k)\,dt. \tag{7.15}$$

If we write

$$h = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}, \quad k = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

and define ϕ, ψ by

$$\begin{aligned} \phi'' + L\phi = 0, &\quad \psi'' + L\psi = 0, \\ \phi(0) = L^{-1} h_1, &\quad \psi(0) = L^{-1} k_1, \\ \phi'(0) = -L^{-1} h_2, &\quad \psi'(0) = -L^{-1} k_2, \end{aligned} \tag{7.16}$$

we deduce easily from (7.7) that

$$B^* e^{A^* t} h = \begin{bmatrix} -q_1 \phi(t) \\ q_2 \phi'(t) \end{bmatrix}, \tag{7.17}$$

$$(\Gamma_T h, k) = \int_0^T [(q_1\phi, q_1\psi) + (q_2\phi', q_2\psi')]\, dt. \tag{7.18}$$

Therefore the estimate

$$(\Gamma_T h, k) \geq c(T - T_0)\|h\|^2_{(D(A))'}$$

is clearly equivalent to

$$\int_0^T (|q_1\phi|^2 + |q_2\phi'|^2)\, dt \geq c(T - T_0)(\|\phi(0)\|^2_{V_L} + |\phi'(0)|^2_{H_L}).$$

Moreover

$$\Gamma_T h = \begin{bmatrix} -L^{-1}\big(\eta'(T) + q_2^* q_2 \phi'(T)\big) \\ \eta(T) \end{bmatrix}, \tag{7.19}$$

where η is the solution of (7.10) with

$$v_1(t) = -q_1\phi(t), \quad v_2(t) = q_2\phi'(t).$$

If we use the formulation (7.13) for the η equation, then we may write

$$\Gamma_T h = \begin{bmatrix} -L^{-1}\eta'(T) \\ \eta(T) \end{bmatrix}, \tag{7.20}$$

but $\eta'(T)$ is not the value of η' at T, which is not meaningful because η' is only an L^2 function, but a mnemonic to denote an element of $\big(D(L)\big)'$ defined by the relation

$$\begin{aligned} \big(\eta'(T), \psi(T)\big) - \int_0^T (\eta', \psi')\, dt + \int_0^T (\eta, L\psi)\, dt \\ = \int_0^T [(v_1, q_1\psi) + (v_2, q_2\psi')]\, dt \end{aligned} \tag{7.21}$$

for any $\psi \in H^1\big(0, T; D(L)\big) \cap L^2(0, T; \Delta_L)$.

Let us now try to check how we can satisfy the sufficient assumptions of exact controllability as stated in Theorem 6.1 and find the operators M, Λ satisfying (6.14) and (6.19). We shall suppose that there exist operators π_1, π_2 and a form $b(\xi, \xi')$ such that

$$\begin{gathered} \pi_1 \in \mathcal{L}\big(D(L); D(L)'\big), \quad \pi_2 \in \mathcal{L}(V_L; V_L'), \\ b \text{ is bilinear continuous on } H_L \times V_L, \\ |q_1\zeta|^2 \geq (\pi_1\zeta, \zeta), \quad \forall \zeta \in D(L), \\ |q_2\zeta|^2 \geq (\pi_2\zeta, \zeta), \quad \forall \zeta \in V_L, \end{gathered} \tag{7.22}$$

$$(\pi_1 w_j, w_k) = \lambda_j b(w_j, w_k) + \lambda_k b(w_k, w_j), \quad \forall j \neq k \geq N+1, \tag{7.23}$$

$$b(w_j, w_k) + b(w_k, w_j) + (\pi_2 w_j, w_k) = 0, \quad \forall j \neq k \geq N+1, \tag{7.24}$$

$$\frac{1}{\lambda_j}(\pi_1 w_j, w_j) + (\pi_2 w_j, w_j) \geq 2c_0, \quad \forall j \geq N+1, \tag{7.25}$$

$$\left| \frac{(\pi_1 w_j, w_j)}{\lambda_j^{\frac{3}{2}}} - \frac{(\pi_2 w_j, w_j)}{\lambda_j^{1/2}} - 4\frac{b(w_j, w_j)}{\lambda_j^{1/2}} \right| \leq 2K, \quad \forall j \geq N+1. \tag{7.26}$$

We then define

$$(M\xi, \eta) = (\pi_1 L^{-1}\xi_1, L^{-1}\eta_1) + (\pi_2 L^{-1}\xi_2, L^{-1}\eta_2), \quad \forall \xi, \eta \in H, \tag{7.27}$$

$$(\Lambda\xi, \eta) = b(\xi_1, \eta_1), \quad \forall \xi \in H,\ \eta \in D(A). \tag{7.28}$$

It is easy to check that all assumptions (6.14)–(6.19) are satisfied.

As a consequence of Theorem 6.1 we can state the following theorem.

Theorem 7.1. *Consider the pair* (A, B) *defined in* §7.1 *and* §7.2*. Assume* (7.22) *to* (7.26) *and* (6.20)*. Then the pair* (A, B) *is exactly controllable for* T *sufficiently large.*

Let us check what must be added to obtain exact controllability at any $T > 0$. We replace (7.26) by

$$\left| \frac{(\pi_1 w_j, w_j)}{\lambda_j} - (\pi_2 w_j, w_j) - 4b(w_j, w_j) \right| \leq 2K, \quad \forall j \geq N+1. \tag{7.29}$$

We also assume that there exists a Hilbert space W_L such that

$$\begin{gathered} V_L \subset W_L \subset H_L, \text{ the injection of } V_L \text{ in } W_L \text{ is compact} \\ b \text{ is bilinear continuous on } H_L \times W_L. \end{gathered} \tag{7.30}$$

Then (7.29) implies (6.64) and (7.30) implies (6.65).

We can then state the following theorem.

Theorem 7.2. *We make the assumptions of Theorem* 7.1*, except* (7.26)*, which is replaced by* (7.29)*, and we assume* (7.30)*; then the pair* (A, B) *is exactly controllable for any* $T > 0$*.*

8 Exact controllability of hyperbolic equations

We shall apply in this section the results of §6 and §7 to the exact controllability of the wave equation, Maxwell equations, and the plate equation with boundary control. The domain Ω considered in the sequel is smooth and bounded.

8.1 Wave equation with Dirichlet boundary control

We take $L = -\mathit{\Delta}$, with

$$H_L = L^2(\Omega), \quad V_L = H_0^1(\Omega),$$
$$D(L) = H^2(\Omega) \cap H_0^1(\Omega),$$
$$\mathit{\Delta}_L = \{z \in D(L)\colon \mathit{\Delta} z \in H_0^1(\Omega)\}.$$

We take $U_1 = U_2 = L^2(\mathit{\Gamma})$ and $q_1 = \partial/\partial\nu, q_2 = 0$. Let

$$m(x) = x - x_0, \quad R(x_0) = \sup_{x \in \mathit{\Gamma}} |m(x)|.$$

We define

$$(\pi_1 \zeta, \zeta') = \frac{1}{R(x_0)} \int_{\mathit{\Gamma}} m \cdot \nu \frac{\partial \zeta}{\partial \nu} \frac{\partial \zeta'}{\partial \nu} \, d\mathit{\Gamma}, \quad \pi_2 = 0, \tag{8.1}$$

$$b(z, \zeta) = -\frac{1}{R(x_0)} \int_{\Omega} \sum_{\alpha} m_\alpha z \frac{\partial \zeta}{\partial x_\alpha} \, dx. \tag{8.2}$$

Consider the eigenvectors w_j defined by

$$-\mathit{\Delta} w_j = \lambda_j w_j, \quad w_j\big|_{\mathit{\Gamma}} = 0; \tag{8.3}$$

then we have the relation

$$\begin{aligned} &\int_{\mathit{\Gamma}} m \cdot \nu \frac{\partial w_j}{\partial \nu} \frac{\partial w_k}{\partial \nu} \, d\mathit{\Gamma} \\ &\quad = (2 - n)\sqrt{\lambda_j \lambda_k}\, \delta_{jk} - \int_{\Omega} \sum_{\alpha} m_\alpha \left(\lambda_j w_j \frac{\partial w_k}{\partial x_\alpha} + \lambda_k w_k \frac{\partial w_j}{\partial x_\alpha} \right) dx \end{aligned} \tag{8.4}$$

for any j, k. Then taking $N = 0$, all assumptions of Theorem 7.1 are satisfied. The system to be controlled is, according to (7.12)

$$\langle \psi_1, \eta(T) \rangle - \langle \psi_0, \eta'(T) \rangle = \int_0^T \left(v_1, \frac{\partial \psi}{\partial \nu} \right) dt, \tag{8.5}$$

where ψ satisfies

$$\begin{cases} \psi'' - \mathit{\Delta} \psi = 0, \\ \psi\big|_{\mathit{\Gamma}} = 0, \\ \psi(T) = \psi_0, \quad \psi'(T) = \psi_1, \quad \psi_0 \in H^2 \cap H_0^1, \quad \psi_1 \in H_0^1. \end{cases} \tag{8.6}$$

The relation (8.6) is the "Method of Transposition" definition of

$$\begin{cases} \eta'' - \mathit{\Delta} \eta = 0, \\ \eta\big|_{\mathit{\Gamma}} = -v_1, \\ \eta(0) = \eta'(0) = 0. \end{cases} \tag{8.7}$$

The exact controllability property is expressed as follows: Given $y_0 \in L^2$, $y_1 \in H^{-1}$, for T sufficiently large, the system

$$\begin{cases} \eta'' - \Delta\eta = 0, \quad \phi'' - \Delta\phi = 0, \\ \eta\big|_\Gamma = \dfrac{\partial\phi}{\partial\nu}, \quad \phi_\Gamma = 0, \\ \eta(0) = \eta'(0) = 0, \quad \eta(T) = y_0, \quad \eta'(T) = y_1 \end{cases} \tag{8.8}$$

has a solution with $\phi(0) \in H_0^1$, and $\phi'(0) \in L^2$.

8.2 Wave equation with Neumann boundary control

We take now

$$\begin{gathered} L = -\Delta + I, \quad H_L = L^2(\Omega), \quad V_L = H^1(\Omega), \\ D(L) = \left\{ z \in H^2(\Omega) \colon \frac{\partial z}{\partial\nu} = 0 \right\}, \\ \Delta_L = \left\{ z \in H^3(\Omega) \colon \frac{\partial z}{\partial\nu} = 0 \right\}. \end{gathered}$$

The eigenvectors are defined by

$$\begin{cases} -\Delta w_j + w_j = \lambda_j w_j, \\ \dfrac{\partial w_j}{\partial\nu} = 0. \end{cases} \tag{8.9}$$

Note that $\lambda_1 = 1$ and $w_1 = 1/|\Omega|$.

The eigenspace corresponding to λ_1 is one dimensional. We next take $U_1 = \big(L^2(\Gamma_1)\big)^{n+1}$, $U_2 = L^2(\Gamma_0)$ (see (5.21) for the definition of Γ_0 and Γ_1), and

$$q_1 z = \begin{bmatrix} \gamma_1 z \\ \gamma_1 Dz \end{bmatrix} \quad \text{for } z \in D(L), \qquad q_2 z = \gamma_0 z, \quad \text{for } z \in H^1. \tag{8.10}$$

Note that for

$$\forall z \in D(L), \quad \gamma_1 Dz = D_{\sigma,1}\gamma_1 z.$$

See §5.3 for the notation. Define then

$$(\pi_1\zeta, \zeta') = -\frac{1}{R(x_0)} \int_\Gamma m \cdot \nu(\zeta\zeta' + D\zeta \cdot D\zeta')\, d\Gamma, \quad \forall \zeta, \zeta' \in D(L), \tag{8.11}$$

$$(\pi_2\zeta, \zeta') = \frac{1}{R(x_0)} \int_\Gamma m \cdot \nu\zeta\zeta'\, d\Gamma, \quad \forall \zeta, \zeta' \in H^1. \tag{8.12}$$

Similarly define

$$b(z, \zeta) = -\frac{1}{R(x_0)} \int_\Omega \sum_\alpha m_\alpha z \frac{\partial\zeta}{\partial x_\alpha}\, dx. \tag{8.13}$$

We use the following relation among eigenvalues:

$$\int_\Gamma m\cdot\nu(w_jw_k+Dw_jDw_k)\,d\Gamma=(n-2)\sqrt{\lambda_j\lambda_k}\delta_{jk}+2\delta_{jk}$$
$$+\int_\Omega\sum_\alpha m_\alpha\left(\lambda_jw_j\frac{\partial w_k}{\partial x_\alpha}+\lambda_kw_k\frac{\partial w_j}{\partial x_\alpha}\right)dx. \quad (8.14)$$

Pick $N=1$. Consider $j\neq k\geq 2$. It is easily checked that the properties (7.22) to (7.26) are satisfied. Let us check (6.20). It follows from the fact that

$$|B^*\phi_1|^2=\frac{|\Gamma|}{2|\Omega|}.$$

Therefore all assumptions of Theorem 7.1 are satisfied.

The system to be controlled is, according to (7.12),

$$\langle\psi_1,\eta(T)\rangle-\langle\psi_0,\eta'(T)+q_2^*v_2(T)\rangle$$
$$=\int_0^T\int_{\Gamma_1}(v_{10}\psi+\sum_{i=1}^n v_{1i}\frac{\partial\psi}{\partial x_i})\,d\Gamma+\int_0^T\int_{\Gamma_0}v_2\psi'\,d\Gamma, \quad (8.15)$$

where

$$v_1=\begin{bmatrix}v_{10}\\ v_{12}\\ \cdots\\ \cdots\\ v_{1n}\end{bmatrix}$$

and ψ is the solution of

$$\begin{cases}\psi''-\Delta\psi+\psi=f,\\ \dfrac{\partial\psi}{\partial\nu}=0,\\ \psi(T)=\psi'(T)=0.\end{cases} \quad (8.16)$$

We can consider v_1 as an element of $\big(H^1(\Gamma_1)\big)'$, by the following duality formula (see also (5.29)):

$$\langle v_1,\theta\rangle=\int_{\Gamma_1}\left(v_{10}\gamma_1\theta+\sum_{i=1}^n v_{1i}\gamma_1\frac{\partial\theta}{\partial x_i}\right)d\Gamma. \quad (8.17)$$

Consider the special case

$$v_{10}=\gamma_1\chi,\quad v_{1i}=\gamma_1\frac{\partial\chi}{\partial x_i}\quad\text{with }\frac{\partial\chi}{\partial\nu}=0. \quad (8.18)$$

Then

$$\begin{aligned}\langle v_1, \theta\rangle &= \int_{\Gamma_1} \left\{ \gamma_1\chi\gamma_1\theta + \sum_{i=1}^n \gamma_1 \frac{\partial \chi}{\partial x_i} \gamma_1 \frac{\partial \theta}{\partial x_i} \right\} d\Gamma \\ &= \int_{\Gamma_1} \left\{ \gamma_1\chi\gamma_1\theta + \sum_{i=1}^n \sigma_{i,1}\gamma_1\chi\sigma_{i,1}\gamma_1\theta \right\} d\Gamma \\ &= \int_{\Gamma_1} \{\gamma_1\chi - \Delta_{\sigma 1}\gamma_1\chi\}\, \gamma_1\theta \, d\Gamma,\end{aligned} \tag{8.19}$$

where

$$-\Delta_{\sigma 1} = \sum_i \sigma_{i,1}^* \sigma_{i,1}$$

represents the tangential Laplacian on Γ_1, which is an element of $\mathcal{L}\big(H^1(\Gamma_1); \big(H^1(\Gamma_1)\big)'\big)$.

So we can write (8.15) as

$$\begin{cases} \eta'' - \Delta\eta + \eta = 0, \\ \left.\dfrac{\partial \eta}{\partial \nu}\right|_{\Gamma_0} = -J_0^* v_2, \quad \left.\dfrac{\partial \eta}{\partial \nu}\right|_{\Gamma_1} = v_1, \\ \eta(0) = \eta'(0) = 0, \end{cases} \tag{8.20}$$

with the interpretation of the operator J_0 already discussed in §7.3.

Let us express the property of exact controllability. Let $y_0 \in L^2$, $y_1 \in \big(H^1(\Omega)\big)'$; then for T sufficiently large, the system

$$\begin{cases} \eta'' - \Delta\eta + \eta = 0, \quad \phi'' - \Delta\phi + \phi = 0, \\ \left.\dfrac{\partial \eta}{\partial \nu}\right|_{\Gamma_0} = -J_0^* \phi', \quad \dfrac{\partial \phi}{\partial \nu} = 0, \quad \left.\dfrac{\partial \eta}{\partial \nu}\right|_{\Gamma_1} = \Delta_{\sigma 1}\phi - \phi, \\ \eta(T) = y_0, \quad \eta'(T) = y_1, \\ \eta(0) = 0, \quad \eta'(0) = 0 \end{cases} \tag{8.21}$$

has a solution with $\phi(0) \in H^1$, $\phi'(0) \in L^2$. We recall that $J_0^*\phi'$ belongs to $\big(H^1\big(0,T;L^2(\Gamma_0)\big)'\big)$ and $-\Delta_{\sigma 1}\phi + \phi$ to $L^2\big(0,T;\big(H^1(\Gamma_1)\big)'\big)$.

8.3 Maxwell equations

Let us introduce some notations. Let $\mathbf{L}^2(\Omega) = \big(L^2(\Omega)\big)^3$, $\mathbf{L}^2(\Gamma) = \big(L^2(\Gamma)\big)^3$ and similarly, $\mathbf{H}^k(\Omega) = \big(H^k(\Omega)\big)^3$. If $v \in \mathbf{H}^1(\Omega)$, we set $\operatorname{curl} v = D \times v$, where $\times$ symbolizes the external product. We recall the relations

$$\operatorname{curl} \operatorname{grad} v = 0, \quad \operatorname{div} \operatorname{curl} v = 0.$$

Furthermore, the set Ω being assumed smooth and singly connected, we have

$$\operatorname{curl} u = 0 \implies u = Dp, \qquad \operatorname{div} u = 0 \implies u = \operatorname{curl} v.$$

Define

$$\begin{aligned}
J &= \text{closure in } \mathbf{L}^2(\Omega) \text{ of functions belonging to}\\
&\qquad \left(C^\infty(\bar{\Omega})\right)^3 \text{ with divergence } 0,\\
K &= \text{closure in } \mathbf{L}^2(\Omega) \text{ of functions belonging to}\\
&\qquad \left(C_0^\infty(\Omega)\right)^3 \text{ with divergence } 0,\\
J_1 &= \{v \in \mathbf{H}^1(\Omega)\colon\ \operatorname{div} v = 0, v \times \nu = 0\},\\
K_1 &= \{v \in \mathbf{H}^1(\Omega)\colon\ \operatorname{div} v = 0, v \cdot \nu = 0\},
\end{aligned}$$

where ν represents the external unit normal. We provide J_1 and K_1 with the scalar product

$$\big((u, v)\big) = (\operatorname{curl} u, \operatorname{curl} v) \tag{8.22}$$

for which they become Hilbert spaces. Indeed if $v \in J_1$, curl $v = 0$, then

$$v = \operatorname{curl} \chi, \quad \operatorname{curl}\operatorname{curl} \chi = 0.$$

Using the integration by parts formula

$$(\operatorname{curl} \phi, \psi) = (\phi, \operatorname{curl} \psi) + \int_\Gamma \phi \times \nu \cdot \psi \, d\Gamma, \tag{8.23}$$

we deduce easily that curl $\chi = 0$; hence $v = 0$. Similarly if $v \in K_1$, curl $v = 0$, then $v = Dp$ and $\Delta p = 0$, with $\partial p/\partial \nu = 0$; hence p is a constant, which implies again $v = 0$. It can be proved (see (8)) that

$$J_1 \subset J, \quad K_1 \subset K \tag{8.24}$$

with continuous and dense embedding. Moreover curl is an isometry from J_1 to K, and K_1 to J. The norm thus defined on J_1 and K_1 is equivalent to that induced by $\mathbf{H}^1(\Omega)$. We shall further need the spaces

$$\begin{aligned}
J_2 &= \left\{v \in \mathbf{H}^2(\Omega)\colon\ \operatorname{div} v = 0,\ v \times \nu = 0,\ \nu \cdot \operatorname{curl} v = 0\right\},\\
J_3 &= \left\{v \in \mathbf{H}^3(\Omega)\colon\ \begin{matrix} \operatorname{div} v = 0, & v \times \nu = 0, \\ \nu \cdot \operatorname{curl} v = 0, & \nu \times \operatorname{curl}\operatorname{curl} v = 0 \end{matrix}\right\}
\end{aligned} \tag{8.25}$$

and

$$\begin{aligned}
K_2 &= \left\{v \in \mathbf{H}^2(\Omega)\colon\ \operatorname{div} v = 0,\ v \cdot \nu = 0,\ \nu \times \operatorname{curl} v = 0\right\}\\
K_3 &= \left\{v \in \mathbf{H}^3(\Omega)\colon\ \begin{matrix} \operatorname{div} v = 0, & \nu \cdot v = 0, \\ \nu \times \operatorname{curl} v = 0, & \nu \cdot \operatorname{curl}\operatorname{curl} v = 0 \end{matrix}\right\}.
\end{aligned} \tag{8.26}$$

The spaces J_2 and K_2 are provided with the norms

$$\|v\| = |\operatorname{curl}\operatorname{curl} v|$$

and J_3, K_3 with the norms

$$\|v\| = |\operatorname{curl}\operatorname{curl}\operatorname{curl} v| \,.$$

Then curl is an isometry from J_2 to K_1, J_3 to K_2, K_2 to J_1, and K_3 to J_2.

Note that from (8.23) we have

$$(\operatorname{curl}\phi, \psi) = (\operatorname{curl}\psi, \phi) \tag{8.27}$$

with $\phi \in J_1$, $\psi \in K_1$, or $\phi \in K_1$, $\psi \in J_1$. It follows that curl can be extended as an isometry from J to $(K_1)'$ and K to $(J_1)'$. Similarly it extends as an isometry from $(K_1)'$ to $(J_2)'$, $(J_1)'$ to $(K_2)'$, $(J_2)'$ to $(K_3)'$, and $(K_2)'$ to $(J_3)'$.

Summarizing we have the sequences

$$\begin{gathered} J_3 \subset J_2 \subset J_1 \subset J \subset (J_1)' \subset (J_2)' \subset (J_3)', \\ K_3 \subset K_2 \subset K_1 \subset K \subset (K_1)' \subset (K_2)' \subset (K_3)', \end{gathered} \tag{8.28}$$

with each space being continuously and densely embedded into the next one. Moreover curl curl is an isometry from J_3 to J_1, J_2 to J, J_1 to $(J_1)'$, J to $(J_2)'$, and $(J_1)'$ to $(J_3)'$, and a similar result with J changed into K.

Define

$$H_L = K, \quad V_L = K_1, \quad D(L) = K_2, \quad \Delta(L) = K_3, \quad L = \operatorname{curl}\operatorname{curl}. \tag{8.29}$$

The eigenvectors w_j are given by the relations

$$\begin{cases} \operatorname{curl}\operatorname{curl} w_j = \lambda_j w_j, \quad \operatorname{div} w_j = 0, \\ \nu \cdot w_j = 0, \quad \nu \times \operatorname{curl} w_j = 0 \quad \text{on } \Gamma, \\ |w_j| = 1. \end{cases} \tag{8.30}$$

We shall use the following relations among eigenvectors:

$$\begin{aligned} \int_\Gamma m \cdot \nu \operatorname{curl} w_j \cdot \operatorname{curl} w_k \, d\Gamma &= (n-2)\sqrt{\lambda_j \lambda_k}\, \delta_{jk} \\ &\quad + \int_\Omega \sum_\alpha m_\alpha \left(\lambda_j w_j \cdot \frac{\partial w_k}{\partial x_\alpha} + \lambda_k w_k \cdot \frac{\partial w_j}{\partial x_\alpha} \right) dx \end{aligned} \tag{8.31}$$

with again $m(x) = x - x_0$.

Define next

$$\begin{gathered} U_2 = \mathbf{L}^2(\Gamma_0), \quad U_1 = \{v \in \mathbf{L}^2(\Gamma_1) \colon \nu \times v = 0\}, \\ q_1 = -\gamma_1 \operatorname{curl}, \quad q_2 = -\gamma_0. \end{gathered}$$

We can take

$$(\pi_1\zeta,\zeta') = -\frac{1}{R(x_0)}\int_\Gamma m\cdot\nu\,\mathrm{curl}\,\zeta\cdot\mathrm{curl}\,\zeta'\,d\Gamma,\quad \forall\zeta,\zeta'\in K_2, \tag{8.32}$$

$$(\pi_2\zeta,\zeta') = \frac{1}{R(x_0)}\int_\Gamma m\cdot\nu\zeta\cdot\zeta'\,d\Gamma,\quad \forall\zeta,\zeta'\in K_1, \tag{8.33}$$

$$b(z,\zeta) = -\frac{1}{R(x_0)}\int_\Omega\sum_\alpha m_\alpha z\cdot\frac{\partial\zeta}{\partial x_\alpha}\,dx,\quad \forall z\in K,\zeta\in K_1. \tag{8.34}$$

Then all assumptions of Theorem 7.1 are satisfied, with $N=0$.

We next interpret the system to be controlled. Consider (7.11) and (7.12); we get for $\psi_0\in K_2$, $\psi_1\in K_j$

$$\begin{cases}\psi''+\mathrm{curl}\,\mathrm{curl}\,\psi=0,\\ \mathrm{div}\,\psi=0,\\ \nu\cdot\psi=0,\quad \nu\times\mathrm{curl}\,\psi=0\quad\text{on }\Gamma,\\ \psi(T)=\psi_0,\quad \psi'(T)=\psi_1,\end{cases} \tag{8.35}$$

$$\begin{aligned}&\langle\psi_1,\eta(T)\rangle-\langle\psi_0,\eta'(T)+q_2^*v_2(T)\rangle\\ &\qquad = -\int_0^T\left[\int_{\Gamma_1}v_1\cdot\gamma_1\,\mathrm{curl}\,\psi\,d\Gamma+\int_{\Gamma_0}v_2\cdot\gamma_0\psi'\,d\Gamma\right]dt.\end{aligned} \tag{8.36}$$

Now, we use the formula (see (5.22))

$$\gamma\,\mathrm{curl}\,\psi=\nu\times\frac{\partial\psi}{\partial\nu}+D_\sigma\times\gamma\psi\quad\text{on }\Gamma \tag{8.37}$$

and similar relations on Γ_0 and Γ_1. Using the fact that v_1 belongs to U_1, we have

$$\int_{\Gamma_1}v_1\cdot\gamma_1\,\mathrm{curl}\,\psi\,d\Gamma=-\langle D_{\sigma 1}^*\times v_1,\gamma_1\psi\rangle,$$

$$\int_{\Gamma_0}v_2\cdot\gamma_0\psi'\,d\Gamma=\langle J_0^*v_2,\gamma_0\psi\rangle.$$

We then can write (8.36) as follows (using the Method of Transposition):

$$\begin{cases}\eta''+\mathrm{curl}\,\mathrm{curl}\,\eta=0,\\ \mathrm{div}\,\eta=0,\\ \eta(0)=\eta'(0)=0,\\ \nu\times\mathrm{curl}\,\eta\big|_{\Gamma_1}=D_{\sigma 1}^*\times v_1,\\ \nu\times\mathrm{curl}\,\eta\big|_{\Gamma_0}=-J_0^*v_2.\end{cases} \tag{8.38}$$

We see that $\nu\times\eta\big|_{\Gamma_1}$ belongs to $L^2(0,T;H^1(\Gamma_1)')$ and $\nu\times\eta\big|_{\Gamma_0}$ belongs to $\big(H^1(0,T;L^2(\Gamma_0))\big)'$. Let us then express the condition of exact controllability. Given y_0 in K and y_1 in $(K_1)'$, we consider the system

$$\begin{cases} \eta'' + \operatorname{curl}\operatorname{curl}\eta = 0, \\ \operatorname{div}\eta = 0, \\ \eta(0) = \eta'(0) = 0, \\ \nu \times \operatorname{curl}\eta\big|_{\Gamma_1} = D_{\sigma 1}^* \times \gamma_1 \operatorname{curl}\phi, \\ \nu \times \operatorname{curl}\eta\big|_{\Gamma_0} = J_0^*\phi', \end{cases} \qquad \begin{cases} \phi'' + \operatorname{curl}\operatorname{curl}\phi = 0, \\ \operatorname{div}\phi = 0, \\ \eta(T) = y_0, \quad \eta'(T) = y_1, \\ \phi\cdot\nu = 0, \\ \nu \times \operatorname{curl}\phi = 0, \end{cases} \tag{8.39}$$

which has a solution with $\phi(0) \in K_1$ and $\phi'(0) \in K$, at least for T sufficiently large. Furthermore we know that $\eta \in C([0,T];K_1')$ and $\eta' \in L^2([0,T];K_2')$. Reducing (8.39) to Maxwell equations by setting

$$H = \operatorname{curl}\eta, \quad E = -\eta', \quad \psi = -\operatorname{curl}^{-1}\phi',$$

we deduce the relations

$$\begin{cases} E' = \operatorname{curl} H, \\ H' + \operatorname{curl} E = 0, \\ \operatorname{div} E = \operatorname{div} H = 0, \\ \nu \times H\big|_{\Gamma_1} = D_{\sigma 1}^* \times \gamma_1\psi', \\ \nu \times H\big|_{\Gamma_0} = J_0^*\phi', \\ E(0) = H(0) = 0, \end{cases} \qquad \begin{cases} \phi' + \operatorname{curl}\psi = 0, \\ \psi' = \operatorname{curl}\phi, \\ \operatorname{div}\phi = \operatorname{div}\psi = 0, \\ E(T) = E_T, \quad H(T) = H_T, \\ \phi\cdot\nu = 0, \quad \nu\times\psi = 0, \end{cases} \tag{8.40}$$

with

$$\begin{aligned} &H \in C([0,T];J_2'), && E \in L^2([0,T];K_2'), \\ &H' \in L^2([0,T];J_3'), && E' \in L^2([0,T];K_3'), \\ &\phi \in C([0,T];K_1), && \psi \in C([0,T];J_1), \\ &\phi' \in C([0,T];K), && \psi' \in C([0,T];J). \end{aligned}$$

The values E_T and H_T are any elements of K_1' and J_1', respectively. The system (8.40) has a solution for T sufficiently large.

We now turn to another situation where we shall be able to use Theorem 6.2 instead of Theorem 6.1 (which permitted to state Theorem 7.1). However we assume the geometric condition

$$m \cdot \nu \geq 0. \tag{8.41}$$

Consider again (8.29). Take $U_1 = U_2 = \mathbf{L}^2(\Gamma)$ and $q_1 = -\gamma$, $q_2 = 0$. We define A, B as in §7.1 and §7.2. We note that $B \in D(A)$ and thus (6.31) is satisfied. We then take

$$(M\xi, \eta) = \frac{1}{R(x_0)} \int_\Gamma m\cdot\nu L^{-1}\xi_1 \cdot L^{-1}\eta_1 \, d\Gamma, \tag{8.42}$$

$$(\varLambda\xi, \eta) = \frac{1}{R(x_0)} \int_\Omega \sum_\alpha m_\alpha \xi_1 \cdot \frac{\partial \eta_1}{\partial x_\alpha} \, dx, \tag{8.43}$$

$$(Q\xi, \eta) = \frac{1}{R(x_0)} \int_\Gamma m\cdot\nu \operatorname{curl} L^{-1}\xi_1 \cdot \operatorname{curl} L^{-1}\eta_1 \, d\Gamma. \tag{8.44}$$

Let us check the assumptions of Theorem 6.2, with $N = 0$. Thanks to (8.41) the assumption (6.14) is satisfied and (6.15) is trivial. The property (6.33) is clear from the definition of ϕ_j and that of M, Λ. From the definition of Q and the relation (8.31) the property (6.34) is easily checked. It remains to check (6.35) and (6.36). But

$$
\begin{aligned}
(M \operatorname{Re}\phi_j, \operatorname{Re}\phi_j) &= \frac{1}{2R(x_0)\lambda_j^2}\int_\Gamma m\cdot\nu|w_j|^2\,d\Gamma,\\
(Q \operatorname{Re}\phi_j, \operatorname{Re}\phi_j) &= \frac{1}{2R(x_0)\lambda_j^2}\int_\Gamma m\cdot\nu|\operatorname{curl} w_j|^2\,d\Gamma
\end{aligned}
$$

and from (8.31)

$$
(M \operatorname{Re}\phi_j, \operatorname{Re}\phi_j) - \frac{1}{\lambda_j}(Q \operatorname{Re}\phi_j, \operatorname{Re}\phi_j) = \frac{1}{R(x_0)\lambda_j^2};
$$

therefore (6.35) is satisfied with

$$
c_0 = \frac{1}{R(x_0)}.
$$

Now

$$
(M \operatorname{Re}\phi_j, \operatorname{Re}\phi_j) + \frac{1}{\lambda_j}(Q \operatorname{Re}\phi_j, \operatorname{Re}\phi_j) = \frac{1}{R(x_0)\lambda_j^2}\left\{\int_\Gamma m\cdot\nu|w_j|^2\,d\Gamma - 1\right\}
$$

and

$$
\begin{aligned}
\int_\Gamma m\cdot\nu|w_j|^2\,d\Gamma &= 2\int_\Omega \sum_\alpha m_\alpha w_j\cdot\frac{\partial w_j}{\partial x_\alpha}\,dx + n\\
&= 4R(x_0)(\Lambda \operatorname{Re}\phi_j, \operatorname{Re}\phi_j) + n,
\end{aligned}
$$

and in particular the property (6.36) follows. The dynamic system is given by

$$
\begin{cases}
\eta'' + \operatorname{curl}\operatorname{curl}\eta = 0,\\
\operatorname{div}\eta = 0,\\
\eta(0) = \eta'(0) = 0,\\
\nu\times\operatorname{curl}\eta\big|_\Gamma = -v.
\end{cases}
\tag{8.45}
$$

We express the property of exact controllability as follows.

The system

$$
\begin{cases}
\eta'' + \operatorname{curl}\operatorname{curl}\eta = 0,\\
\operatorname{div}\eta = 0,\\
\eta(0) = \eta'(0) = 0,\\
\nu\times\operatorname{curl}\eta\big|_\Gamma = \gamma\phi,
\end{cases}
\qquad
\begin{cases}
\phi'' + \operatorname{curl}\operatorname{curl}\phi = 0,\\
\operatorname{div}\phi = 0,\\
\eta(T) = y_0, \quad \eta'(T) = y_1,\\
\phi\cdot\nu = 0, \quad \nu\times\operatorname{curl}\phi = 0
\end{cases}
\tag{8.46}
$$

has a solution for T sufficiently large, with $y_0 \in K_1$, $y_1 \in K$. Moreover $\phi(0) \in K$, $\phi'(0) \in (K_1)'$. We have $\eta \in C([0,T];K)$, $\eta' \in C([0,T];(K_1)')$. Reducing to Maxwell equations as above we obtain the system

$$\begin{cases} E' = \operatorname{curl} H, \\ H' + \operatorname{curl} E = 0, \\ \operatorname{div} E = \operatorname{div} H = 0, \\ \nu \times H\big|_{\Gamma} = \gamma\phi, \\ E(0) = H(0) = 0, \end{cases} \qquad \begin{cases} \phi' + \operatorname{curl} \psi = 0, \\ \psi' = \operatorname{curl} \phi, \\ \operatorname{div} \phi = \operatorname{div} \psi = 0, \\ E(T) = E_T, \quad H(T) = H_T, \\ \phi \cdot \nu = 0, \quad \nu \times \psi = 0. \end{cases} \tag{8.47}$$

For any given E_T and H_T in K and J, respectively, and for sufficiently large T, the system (8.47) has a solution such that

$$\begin{aligned} &H \in C([0,T];J_1'), && E \in C([0,T];K_1'), \\ &H' \in C([0,T];J_2'), && E' \in C([0,T];K_2'), \\ &\phi \in C([0,T];K), && \psi \in C([0,T];J), \\ &\phi' \in C\big([0,T];(K_1)'\big), && \psi' \in C\big([0,T];(J_1)'\big), \end{aligned}$$

and $\gamma\phi$ belongs to $L^2(0,T;\mathbf{L}^2(\Gamma)\big)$.

8.4 Plate equation

We begin with Neumann control. We take $H_L = L^2(\Omega)$, $V_L = H_0^2(\Omega)$, $D(L) = H^4(\Omega) \cap H_0^2(\Omega)$, and

$$L = \Delta^2, \quad \Delta(L) = \{z \in H_0^2(\Omega) \colon \Delta^2 z \in H_0^2(\Omega)\}.$$

The eigenvalues related to L are defined by

$$\begin{cases} \Delta^2 w_j = \lambda_j w_j, \\ w_{j|\Gamma} = \dfrac{\partial w_j}{\partial \nu}\bigg|_{\Gamma} = 0, \quad |w_j| = 1. \end{cases} \tag{8.48}$$

Considering again the multiplier $m(x) = x - x_0$, we can check the relation

$$\begin{aligned} &\int_\Gamma m \cdot \nu \Delta w_j \Delta w_k \, d\Gamma \\ &\qquad = (4-n)\sqrt{\lambda_j \lambda_k}\delta_{jk} - \int_\Omega \sum_\alpha m_\alpha \left(\lambda_j w_j \frac{\partial w_k}{\partial x_\alpha} + \lambda_k w_k \frac{\partial w_j}{\partial x_\alpha} \right) dx. \end{aligned} \tag{8.49}$$

We take $U_1 = U_2 = L^2(\Gamma_0)$, where we recall that Γ_0 is the part of the boundary on which $m \cdot \nu \geq 0$. We pick $q_1 = -\gamma_0 \Delta$ and $q_2 = 0$.

It is useful to introduce the space $H^3(\Omega) \cap H_0^2(\Omega)$, intermediary between V_L and $D(L)$, equipped with the norm $\|z\| = |D(\Delta z)|$. Note that L is an isomorphism between $H^3 \cap H_0^2$ and H^{-1}. We shall check the property of approximate controllability for any $T > 0$, by checking the assumptions of Theorem 7.2. We take $N = 0$, and we define

$$(\pi_1 \zeta, \zeta') = \frac{1}{R(x_0)} \int_\Gamma m \cdot \nu \Delta\zeta \Delta\zeta' \, d\Gamma, \quad \pi_2 = 0, \tag{8.50}$$

$$b(z, \zeta) = -\frac{1}{R(x_0)} \int_\Omega \sum_\alpha m_\alpha z \frac{\partial \zeta}{\partial x_\alpha} \, dx. \tag{8.51}$$

The properties (7.22), (7.23), and (7.24) are easily verified. As

$$(\pi_1 w_j, w_j) = \frac{4\lambda_j}{R(x_0)} \text{ and } b(w_j, w_j) = \frac{n}{2R(x_0)}, \tag{8.52}$$

the properties (7.25) and (7.29) are clear. It remains to check (7.30). We define $W_L = H_0^1$ and the assumption (7.30) is satisfied. Therefore, all the assumptions of Theorem 7.2 are now satisfied. Thus the pair (A, B) is approximately controllable for arbitrary positive T. Let us interpret this result. The dynamical system is described by

$$\langle \psi_1, \eta(T) \rangle - \langle \psi_0, \eta'(T) + q_2^* v_2(T) \rangle = -\int_0^T \int_{\Gamma_0} v \Delta\psi \, dx \, dt \tag{8.53}$$

for $\psi_0 \in H^4 \cap H_0^2$ and $\psi_1 \in H_0^2$ and ψ being the solution of

$$\begin{cases} \psi'' + \Delta^2 \psi = f, \\ \psi(T) = \psi'(T) = 0, \\ \psi\big|_\Gamma = \dfrac{\partial \psi}{\partial \nu}\Big|_\Gamma = 0, \\ \psi \in C([0,T]; H^4 \cap H_0^2), \quad \psi' \in C([0,T]; H_0^2). \end{cases} \tag{8.54}$$

We interpret (8.53) as follows:

$$\begin{cases} \eta'' + \Delta^2 \eta = 0, \\ \eta(0) = \eta'(0) = 0, \\ \eta\big|_\Gamma = 0, \quad \dfrac{\partial \eta}{\partial \nu}\Big|_{\Gamma_1} = 0, \\ \dfrac{\partial \eta}{\partial \nu}\Big|_{\Gamma_0} = v, \\ \eta \in C([0,T]; V_L'), \quad \eta' \in C\big([0,T]; \big(D(L)\big)'\big). \end{cases} \tag{8.55}$$

Let us express the controllability property. Given $y_0 \in L^2(\Omega)$, $y_1 \in H^{-2}(\Omega)$, the system

$$\begin{cases} \eta'' + \Delta^2\eta = 0, \quad \phi'' + \Delta^2\phi = 0, \\ \eta(0) = \eta'(0) = 0, \quad \phi\big|_\Gamma = \dfrac{\partial\phi}{\partial\nu}\Big|_\Gamma = 0, \\ \eta\big|_\Gamma = 0, \quad \dfrac{\partial\eta}{\partial\nu}\Big|_{\Gamma_1} = 0 \quad \eta(T) = y_0, \quad \eta'(T) = y_1, \\ \dfrac{\partial\eta}{\partial\nu} = \Delta\phi\big|_{\Gamma_0}, \\ \phi \in C([0,T]; H_0^2), \quad \phi' \in C([0,T]; L^2), \\ \Delta\phi\big|_{\Gamma_0} \in L^2\big(0,T; L^2(\Gamma_0)\big) \end{cases}$$

has a solution for any $T > 0$.

We next consider a more elaborate case, where we control y and Δy on the boundary. We need some notation. Let J_n be such that

$$J_n = \big\{z \in H^n \colon z\big|_\Gamma = 0, \Delta z\big|_\Gamma = 0, \ldots, \Delta^{n-1/2} z\big|_\Gamma = 0\big\}.$$

Hence $J_1 = H_0^1$, $J_2 = H^2 \cap H_0^1 \ldots$. We provide J_n with the norm

$$\|z\|_{J_n}^2 = \begin{cases} |\Delta^{n/2} z|^2, & \text{if } n \text{ is even,} \\ |D\Delta^{n/2} z|^2, & \text{if } n \text{ is odd.} \end{cases}$$

We set $J_0 = L^2(\Omega)$. The operator $-\Delta$ is an isometry from J_n to J_{n-2}, $n \geq 2$.

Denote J_n the dual of J_n, when J_0 is the pivot space. Clearly $-\Delta$ extends as an isometry between J_n and J_{-n-2}, and we have the sequence

$$J_n \subset J_{n-1} \subset J_{n-2} \subset \cdots \subset J_1 \subset J_0 \subset J_{-1} \cdots \subset J_{-n} \cdots .$$

In the above sequence, suppose that we pick J_1 to be the pivot space, identified with its dual. Then the dual of J_2 is J_0, the dual of J_3 is J_{-1}, and the dual of J_n is J_{-n+2}.

The duality pairing is in the first case (J_0 pivot)

$$\begin{cases} \big((-\Delta)^{n/2} u, (-\Delta)^{-n/2} v\big), & \text{if } n \text{ is even,} \\ (D(-\Delta)^{n-1/2} u, D(-\Delta)^{-n+1/2} v), & \text{if } n \text{ is odd} \end{cases} \tag{8.56}$$

with $u \in J_n$, $v \in J_{-n}$.

Now in the case when J_1 is chosen as the pivot space, then the duality pairing is given by

$$\begin{cases} \big((-\Delta)^{n/2} u, (-\Delta)^{-n/2-1} v\big), & \text{if } n \text{ is even,} \\ (D(-\Delta)^{n-1/2} u, D(-\Delta)^{-n-1/2} v), & \text{if } n \text{ is odd} \end{cases} \tag{8.57}$$

with $u \in J_n$, $v \in J_{-n+2}$.

We choose in the sequel J_1 to be the pivot space. Let $H_L = J_1$, $V_L = J_3$, $D(L) = J_5$, and $\Delta(L) = J_7$. Then $V_L' = J_{-1}$, $\big(D(L)\big)' = J_{-3}$.

We pick $L = \Delta^2$. The eigenvectors are defined by

$$\begin{cases} \Delta^2 w_j = \lambda_j w_j, \\ w_j\big|_\Gamma = \Delta w_j\big|_\Gamma = 0, \quad \|w_j\| = 1, \end{cases} \tag{8.58}$$

and the w_j form an orthonormal system in J_1 the pivot space. We shall use the relations

$$\begin{aligned} &\int_\Gamma m\cdot\nu \frac{\partial}{\partial\nu}\Delta w_j \frac{\partial}{\partial\nu}\Delta w_k \, d\Gamma \\ &\quad = (2+n)\sqrt{\lambda_j\lambda_k}\delta_{jk} - \int_\Omega \sum_\alpha m_\alpha \left(\lambda_j \Delta w_k \frac{\partial w_j}{\partial x_\alpha} + \lambda_k \Delta w_j \frac{\partial w_k}{\partial x_\alpha}\right) dx, \end{aligned} \tag{8.59}$$

$$\int_\Gamma m\cdot\nu \left(\frac{\partial}{\partial\nu}\Delta w_j\right)^2 d\Gamma + \lambda_j \int_\Gamma m\cdot\nu \left(\frac{\partial}{\partial\nu} w_j\right)^2 d\Gamma = 4\lambda_j. \tag{8.60}$$

We take $U_1 = U_2 = L^2(\Gamma_0)$. We recall that

$$n_0 = \frac{\partial}{\partial\nu}$$

on Γ_0. We define $q_1 = n_0\Delta$, $q_2 = n_0$. We then take

$$(\pi_1\zeta, \zeta') = \frac{1}{R(x_0)} \int_\Gamma m\cdot\nu \frac{\partial}{\partial\nu}\Delta\zeta \frac{\partial}{\partial\nu}\Delta\zeta' \, d\Gamma, \tag{8.61}$$

$$(\pi_2\zeta, \zeta') = \frac{1}{R(x_0)} \int_\gamma m\cdot\nu \frac{\partial}{\partial\nu}\zeta \frac{\partial}{\partial\nu}\zeta' \, d\Gamma, \tag{8.62}$$

$$b(z,\zeta) = -\frac{1}{R(x_0)} \int_\Omega \sum_\alpha m_\alpha \Delta\zeta \frac{\partial z}{\partial x_\alpha} \, dx. \tag{8.63}$$

Let us check that the assumptions of Theorem 7.2 are satisfied, with $N = 0$. The properties (7.22) and (7.23) are clearly satisfied. The relation (7.24) can also be checked, taking into account the fact that the w_j are orthogonal in H_0^1. Let us check (7.25). We have

$$\begin{aligned} &\frac{1}{\lambda_j}(\pi_1 w_j, w_j) + (\pi_2 w_j, w_j) \\ &\quad = \frac{1}{R(x_0)}\left[\frac{1}{\lambda_j}\int_\Gamma m\cdot\nu \left|\frac{\partial}{\partial\nu}\Delta w_j\right|^2 d\Gamma + \int_\Gamma m\cdot\nu \left|\frac{\partial}{\partial\nu} w_j\right|^2 d\Gamma\right] = \frac{4}{R(x_0)} \end{aligned}$$

from (8.60). Hence (7.25) is verified. Let us check (7.29). We have

$$b(w_j, w_j) = -\frac{(\pi_2 w_j, w_j)}{2} + \frac{2-n}{2R(x_0)};$$

hence

$$\frac{1}{\lambda_j}(\pi_1 w_j, w_j) - (\pi_2 w_j, w_j) - 4b(w_j, w_j) = \frac{2n}{R(x_0)}$$

and (7.29) is satisfied. Let us finally check (7.30). We take $W_L = J_2$ and note that

$$|b(z,\zeta)| \leq \frac{\mu(x_0)}{R(x_0)} \|z\|_{J_1} \|\zeta\|_{J_2}.$$

As the injection of J_3 into J_2 is compact the assumption (7.30) is also satisfied. Therefore the assumptions of Theorem 7.2 are now satisfied. Thus the pair (A, B) is exactly controllable for arbitrary positive T.

The dynamic system is given by the relation

$$\langle \psi_1, \eta(T) \rangle - \langle \psi_0, \eta'(T) + q_2^* v_2(T) \rangle = \int_0^T \int_{\Gamma_0} \left(v_1 \frac{\partial}{\partial \nu} \Delta \psi + v_2 \frac{\partial}{\partial \nu} \psi' \right) d\Gamma \, dt, \tag{8.64}$$

where $\psi_0 \in J_5$, $\psi_1 \in J_3$, and ψ is the solution of

$$\begin{cases} \psi'' + \Delta^2 \psi = 0, \\ \psi(T) = \psi_0, \quad \psi'(T) = \psi_1, \\ \psi\big|_\Gamma = \Delta\psi\big|_\Gamma = \Delta^2\psi\big|_\Gamma = 0, \\ \psi \in C([0,T]; J_5), \quad \psi' \in C([0,T]; J_3). \end{cases} \tag{8.65}$$

Writing $\zeta = -\Delta\eta$, we interpret (8.64) as follows:

$$\begin{cases} \zeta'' + \Delta^2 \zeta = 0, \\ \zeta(0) = \zeta'(0) = 0, \\ \zeta\big|_{\Gamma_1} = 0, \quad \Delta\zeta\big|_{\Gamma_1} = 0, \\ \zeta\big|_{\Gamma_0} = v_1, \quad \Delta\zeta\big|_{\Gamma_0} = J_0^* v_2, \\ \zeta \in L^2(0,T; J_{-1}) \cap C([0,T]; J_{-3}), \quad \zeta' \in L^2([0,T]; J_{-5}). \end{cases} \tag{8.66}$$

Let us express the controllability property. Given y_0 in J_1 and y_1 in J_{-3}, the system

$$\begin{cases} \zeta'' + \Delta^2 \zeta = 0, \quad \phi'' + \Delta^2 \phi = 0, \\ \zeta(0) = \zeta'(0) = 0, \quad \phi\big|_\Gamma = \Delta\phi\big|_\Gamma = 0, \\ \zeta\big|_{\Gamma_1} = 0, \quad \Delta\zeta\big|_{\Gamma_1} = 0, \quad \zeta(T) = y_0, \\ \zeta\big|_{\Gamma_0} = -\frac{\partial}{\partial \nu}\Delta\phi, \quad \Delta\zeta\big|_{\Gamma_0} = J_0^* \frac{\partial}{\partial \nu}\phi', \quad \zeta'(T) = y_1, \\ \zeta \in L^2(0,T; J_{-1}) \cap C([0,T]; J_{-3}), \quad \phi \in C([0,T]; J_3), \\ \zeta' \in L^2([0,T]; J_{-5}), \quad \phi' \in C([0,T]; J_1) \end{cases} \tag{8.67}$$

has a solution for $T > 0$ arbitrary.

Remark 8.1. For other examples and results of exact controllability see J. L. Lions [1, 5] where the HUM method is presented in full generality, I. Lasiecka and R. Triggiani [9, 10, 11, 13], R. Triggiani [4], J. E. Lagnese [1], and J. E. Lagnese and J. L. Lions [1]. The Maxwell equation is treated in J. E. Lagnese [2]. The result of exact controllability for the plate equation is given in E. Zuazua [1] and I. Lasiecka and R. Triggiani [13]. For the use of microlocal analysis in the context of exact controllability, we refer to C. Bardos, G. Lebeau, and J. Rauch [1] □

Part IV

Quadratic Optimal Control: Finite Time Horizon

1

Bounded Control Operators: Control Inside the Domain

1 Introduction and setting of the problem

In this chapter we consider the dynamical system governed by the equation

$$\begin{cases} x'(t) = Ax(t) + Bu(t), \quad t \geq 0, \\ x(0) = x_0 \in H, \end{cases} \tag{1.1}$$

where $A\colon D(A) \subset H \to H$, $B\colon U \to H$ are linear operators defined on the Hilbert spaces H (*state space*) and U (*control space*), respectively. x is the *state* and u the *control* of the system. We shall also consider another Hilbert space Y, the space of *observations*. The inner product and norm in H, U, and Y will be denoted by $(\cdot,\cdot)$ and $|\cdot|$. Whenever confusion is possible a subscript H, U or Y will be added.

Given $T > 0$, we want to minimize the *cost function*

$$J(u) = \int_0^T \left\{ |Cx(s)|^2 + |u(s)|^2 \right\} ds + (P_0 x(T), x(T)) \tag{1.2}$$

over all controls $u \in L^2(0,T;U)$ subject to the differential equation constraint (1.1). Concerning the operators A, B, C, and P_0 we shall assume that

$$(\mathcal{H}) \begin{cases} \text{(i)} & A \text{ generates a strongly continuous semigroup } e^{tA} \text{ on } H, \\ \text{(ii)} & B \in \mathcal{L}(U;H), \\ \text{(iii)} & P_0 \in \mathcal{L}(H) \text{ is hermitian and non-negative}, \\ \text{(iv)} & C \in \mathcal{L}(H;Y). \end{cases}$$

We shall also say that (A,B,C,P_0) verifies assumptions $(\mathcal{H})$. Assumption $(\mathcal{H})$–(iv) means that the observation operator is bounded. However in §4 below we shall also consider the more general situation when C is unbounded. Under assumptions $(\mathcal{H})$–(i)–(ii), problem (1.1) has a unique mild solution

$x \in C([0,T];H)$ for any $x_0 \in H$ and any $u \in L^2(0,T;U)$ (see Part II, Chapter 1, §3.1). Moreover x is given by the expression

$$x(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A}Bu(s)\,ds.$$

A function $u^\star \in L^2(0,T;U)$ is called an *optimal control* if

$$J(u^\star) \le J(u), \quad \forall u \in L^2(0,T;U).$$

In this case the corresponding solution of (1.1) is called an *optimal state* and the pair $(u^\star, x^\star)$ an *optimal pair*. Under hypotheses $(\mathcal{H})$ it is easy to see that there exists a unique optimal control (because the quadratic form $J(u)$ is coercive on $L^2(0,T;U)$). However we are interested in showing that the optimal control can be obtained as a *feedback control* (*synthesis problem*). For this purpose we shall describe the *Dynamic Programming* approach, which consists, essentially, in the following two steps:

Step 1.

We solve the Riccati equation

$$\begin{cases} P' = A^\star P + PA - PBB^\star P + C^\star C, \\ P(0) = P_0. \end{cases}$$

Step 2.

We prove that the optimal control $u^\star$ is related to the optimal state $x^\star$ by the feedback formula

$$u^\star(t) = -B^\star P(T-t)x^\star(t), \quad t \in [0,T],$$

and moreover that $x^\star$ is the solution of the *closed loop equation*

$$\begin{cases} x'(t) = [A - BB^\star P(T-t)]x(t), \quad t \in [0,T], \\ x(0) = x_0. \end{cases}$$

2 Solution of the Riccati equation

2.1 Notation and preliminaries

We first introduce some notation. Let H be a complex Hilbert space; define

$$\Sigma(H) = \{T \in \mathcal{L}(H) \colon T \text{ is hermitian}\},$$
$$\Sigma^+(H) = \{T \in \Sigma(H) \colon (Tx,x) \ge 0, \forall x \in H\}.$$

The space $\Sigma(H)$, endowed with the norm of $\mathcal{L}(H)$, is a real Banach space and $\Sigma^+(H)$ is a cone in $\Sigma(H)$. For any interval I in $\mathbb{R}$, we shall denote by $C\big(I;\Sigma(H)\big)$ the set of all continuous mappings from I to $\Sigma(H)$. We next consider the set of all strongly continuous mappings $F\colon I \to \Sigma(H)$ (that is, such that $F(\cdot)x$ is continuous for any $x \in H$). We shall mainly use the following concept of convergence, called *strong convergence*. A sequence $\{F_n\}$ is strongly convergent to F if

$$\forall x \in H, \qquad \lim_{n\to\infty} F_n(\cdot)x = F(\cdot)x \quad \text{in } C(I;H),$$

where $C(I;H)$ is endowed with the topology of uniform convergence on compact subsets of I. This topological space will be denoted by $C_s\big(I;\Sigma(H)\big)$. If I is compact we can also consider a stronger topology. Indeed, if F is strongly continuous, the number

$$\|F\| = \sup_{t\in I} \|F(t)\| \tag{2.1}$$

is finite by virtue of the Uniform Boundedness Theorem. The space of all strongly continuous mappings $F\colon I \to \Sigma(H)$, endowed with the norm (2.1), is a Banach space and will be referred to as $C_u\big(I;\Sigma(H)\big)$, where the subscript u stands for uniform convergence. Note that the spaces $C_u\big(I;\Sigma(H)\big)$ and $C_s\big(I;\Sigma(H)\big)$ are equal as sets, but their topologies are different. In particular $C\big(I;\Sigma(H)\big)$ is a proper closed subspace of $C_u\big(I;\Sigma(H)\big)$, but it is a dense subspace of $C_s\big(I;\Sigma(H)\big)$ if H is separable. (The fact that H is separable will not be used in the sequel.) If $F \in C_s\big(I;\Sigma(H)\big)$, then it is easy to check that $\|F(\cdot)\|$ is Lebesgue measurable (see N. Dunford and R. S. Schwartz [1], Lemma 3, p. 616).

Remark 2.1. The above terminology is both natural and convenient, but it may deserve some extra general comments. Given a compact interval I in $\mathbb{R}$, there are two underlying spaces $C\big(I;\mathcal{L}(H)\big)$ and $\mathcal{L}\big(H;C(I;H)\big)$ and the space $\mathcal{L}\big(H;C(I;H)\big)$ has two different topologies. The notation $C_u\big(I;\mathcal{L}(H)\big)$ applies to the space $\mathcal{L}\big(H;C(I;H)\big)$ endowed with its natural Banach space topology; $C_s\big(I;\mathcal{L}(H)\big)$ denotes the same space endowed with the weaker non-Banach topology $\mathcal{T}_s$ of *pointwise convergence*:

$$a_n \to a \Leftrightarrow \forall x \in H, \qquad a_n(x) \to a(x) \quad \text{in } C(I;H).$$

The space $\mathcal{L}\big(H;C(I;H)\big)$ coincides with the space of all strongly continuous mappings $A\colon I \to \mathcal{L}(H)$. The mapping A induces a map $a\colon H \to C(I;H)$ defined by $a(x)(t) = A(t)x$ and because

$$\forall x \in H, \quad \sup_{t\in I} |A(t)x| < \infty,$$

we conclude from the Uniform Boundedness Theorem that

$$\sup_{t\in I} \|A(t)\|_{\mathcal{L}(H)} < \infty.$$

By interchanging the order of the sup's

$$\sup_{|x|=1}\sup_{t\in I}|A(t)x| = \sup_{t\in I}\sup_{|x|=1}|A(t)x| = \sup_{t\in I}\|A(t)\|_{\mathcal{L}(H)},$$

we conclude that a belongs to $\mathcal{L}\big(H;C(I;H)\big)$. Using this construction the space $C\big(I;\mathcal{L}(H)\big)$ of all uniformly continuous mappings can be identified with a closed but smaller subspace of $\mathcal{L}\big(H;C(I;H)\big)$ because the norm on $\mathcal{L}\big(H;C(I;H)\big)$ coincides with the norm on $C\big(I;\mathcal{L}(H)\big)$ by interchanging the sup's. In particular $\mathcal{L}\big(H;C(I;H)\big)$ contains all strongly continuous semigroups of class C_0, whereas $C\big(I;\mathcal{L}(H)\big)$ only contains uniformly continuous semigroups. The reader can easily check that the space $\mathcal{L}\big(H;C(I;H)\big)$, endowed with the weaker topology $\mathcal{T}_s$, remains closed, but the space $C\big(I;\mathcal{L}(H)\big)$ is now dense when H is separable. □

We define $C_s^1\big(I;\Sigma(H)\big)$ as the set of all mappings $F\in C_s\big(I;\Sigma(H)\big)$, which are strongly differentiable (that is, such that $F(\cdot)x$ is differentiable for any $x\in H$) and such that their derivative belongs to $C_s\big(I;\Sigma(H)\big)$. We set

$$C_s^1\big(I;\Sigma^+(H)\big) = \big\{F\in C_s^1\big(I;\Sigma(H)\big)\colon F(t)\in\Sigma^+(H),\forall t\in I\big\}.$$

Finally we denote by $C^\alpha\big(I;\Sigma(H)\big)$ the set of all Hölder continuous mappings from I to $\Sigma(H)$ with exponent $\alpha\in\,]0,1[$.

We are given a linear operator A that is the infinitesimal generator of a strongly continuous semigroup e^{tA} in H. By the Hille–Yosida theorem, there exist $M\geq 1$ and $\omega\in\mathbb{R}$ such that

$$\|e^{tA}\|\leq Me^{\omega t} \quad\text{and}\quad \|e^{tA_n}\|\leq Me^{\omega t}, \qquad t>0, \tag{2.2}$$

where the A_n's are the Yosida approximations of A. For any $T>0$ we set

$$M_T = \sup\{\|e^{tA_n}\|\colon t\in[0,T], n\in\mathbb{N}\}.$$

Finally we denote by $A^\star$ the adjoint operator of A.

In the sequel, we shall need a generalization of the Contraction Mapping Principle. Let $T>0$, and let $\{\gamma_n\}$ be a sequence of mappings from $C_u\big([0,T];\Sigma(H)\big)$ into itself such that

$$\|\gamma_n(P)-\gamma_n(Q)\|\leq\alpha\|P-Q\|,$$

for all $P,Q\in C_u\big([0,T];\Sigma(H)\big)$ and all $n\in\mathbb{N}$, where $\alpha\in[0,1[$.

Moreover assume that there exists a mapping γ from the space $C_u\big([0,T];\Sigma(H)\big)$ into itself such that

$$\lim_{n\to\infty}\gamma_n^m(P) = \gamma^m(P) \quad\text{in } C_s\big([0,T];\Sigma(H)\big), \tag{2.3}$$

for all $P\in C_u\big([0,T];\Sigma(H)\big)$ and all $m\in\mathbb{N}$, where γ^m and γ_n^m are defined by recurrence as

$$\gamma^1 = \gamma, \quad \gamma^{m+1}(P) = \gamma\big(\gamma^m(P)\big),$$
$$\gamma_n^1 = \gamma_n, \quad \gamma_n^{m+1}(P) = \gamma_n\big(\gamma_n^m(P)\big),$$

for $m = 2, 3, \ldots$, and $P \in C_s\big([0,T]; \Sigma(H)\big)$. It is easy to check that

$$\|\gamma(P) - \gamma(Q)\| \le \alpha \|P - Q\|, \quad \forall P, Q \in C_u\big([0,T]; \Sigma(H)\big).$$

Then, by the Contraction Mapping Principle, there exist unique P_n and P in $C_u\big([0,T]; \Sigma(H)\big)$ such that

$$\gamma_n(P_n) = P_n \quad \text{and} \quad \gamma(P) = P.$$

However, as we do not assume that

$$\gamma_n(P) \to \gamma(P) \quad \text{in } C_u\big([0,T]; \Sigma(H)\big),$$

we cannot conclude that $P_n \to P$ in $C_u\big([0,T]; \Sigma(H)\big)$, but a weaker result holds.

Lemma 2.1. *Under the previous hypotheses on the sequence of mappings* $\{\gamma_n\}$*,*

$$P_n \to P \quad \textit{in } C_s\big([0,T]; \Sigma(H)\big).$$

Proof. Set

$$P^0 = 0, \quad P_n^0 = 0,$$

and define

$$P^m = \gamma^m(P^0), \quad P_n^m = \gamma_n^m(P^0), \qquad m = 1, 2, \ldots.$$

By the Contraction Mapping Principle, we have

$$\lim_{m\to\infty} P^m = P, \quad \lim_{m\to\infty} P_n^m = P_n \qquad \text{in } C_u\big([0,T]; \Sigma(H)\big), \qquad n = 1, 2, \ldots.$$

Moreover

$$\|P - P^m\| \le \sum_{k=m}^{\infty} \alpha^k \|\gamma(P^0)\|, \quad \|P_n - P_n^m\| \le \sum_{k=m}^{\infty} \alpha^k \|\gamma_n(P^0)\|.$$

Now fix x in H; then for all t in $[0,T]$

$$|P(t)x - P_n(t)x| \le |P(t)x - P^m(t)x| + |P^m(t)x - P_n^m(t)x| + |P_n^m(t)x - P_n(t)x|. \quad (2.4)$$

Given $\varepsilon > 0$, there exists $m_\varepsilon \in \mathbb{N}$ such that

$$\sum_{k=m}^{\infty} \alpha^k \{\|\gamma(P^0)\| + \|\gamma_n(P^0)\|\} \le \frac{\varepsilon}{2}, \quad (2.5)$$

for all $m \ge m_\varepsilon$ and all $n \in \mathbb{N}$. By (2.4) and (2.5) it follows that

$$|P(t)x - P_n(t)x| \le \frac{\varepsilon}{2} + |P^{m_\varepsilon}(t)x - P_n^{m_\varepsilon}(t)x|, \quad \forall t \in [0,T]. \quad (2.6)$$

Now (2.3) yields the conclusion. □

2.2 Riccati equation

Let A, B, C, and P_0 be given linear operators such that assumptions $(\mathcal{H})$ are verified. Consider the following Riccati equation:

$$\begin{cases} P' = A^\star P + PA - PBB^\star P + C^\star C, \\ P(0) = P_0. \end{cases} \tag{2.7}$$

Problem (2.7) was studied by several authors when H, U, and Y are finite dimensional. We recall the pioneering work in the early papers by R. E. Kalman [2] and W. M. Wonham [2]. The first systematic approach in the infinite dimensional case is due to J. L. Lions [3] who solved (2.7) by variational methods. Here we follow the direct approach by G. Da Prato [1]. For other results see D. L. Lukes and D. L. Russell [1], R. Temam [1], L. Tartar [1], and R. F. Curtain and A. J. Pritchard [1].

As the operator A is unbounded it is not clear a priori what a solution of (2.7) means. We shall define now two kinds of solutions: *mild solutions* and *weak solutions*. It is also possible to define a notion of *strict solution* and *classical solution*. These definitions are more technical and will be given in §3. We first notice that if $A \in \mathcal{L}(H)$, then, as is easily checked, problem (2.7) is equivalent to the following integral equation:

$$\begin{aligned} P(t)x = e^{tA^\star} P_0 e^{tA} x + \int_0^t e^{sA^\star} C^\star C e^{sA} x \, ds \\ - \int_0^t e^{(t-s)A^\star} P(s) BB^\star P(s) e^{(t-s)A} x \, ds, \quad x \in H. \end{aligned} \tag{2.8}$$

Now (2.8) is meaningful for any A, which satisfies assumptions $(\mathcal{H})$. In particular the right-hand side of (2.8) belongs to the function space $C_s\big([0,T]; \Sigma(H)\big)$ and it is natural to seek solutions in that space.

Definition 2.1.

(i) A *mild solution* of problem (2.7) in the interval $[0,T]$ is a function $P \in C_s\big([0,T]; \Sigma(H)\big)$ that verifies the integral equation (2.8).

(ii) A *weak solution* of problem (2.7) in the interval $[0,T]$ is a function $P \in C_s\big([0,T]; \Sigma(H)\big)$ such that, $P(0) = P_0$ and, for any $x, y \in D(A)$, $(P(\cdot)x, y)$ is differentiable in $[0,T]$ and verifies the equation

$$\begin{aligned} \frac{d}{dt}(P(t)x, y) = (P(t)x, Ay) + (P(t)Ax, y) \\ - \big(B^\star P(t)x, B^\star P(t)y\big) + (Cx, Cy). \end{aligned} \tag{2.9}$$

□

Note that the choice of the space $C_s\big([0,T]; \Sigma(H)\big)$ is natural because the mapping $t \mapsto e^{tA}$ is in general strongly continuous but not uniformly continuous.

Proposition 2.1. *Let $P \in C_s\big([0,T];\Sigma(H)\big)$. Then P is a mild solution of problem* (2.7) *if and only if P is a weak solution of* (2.7).

Proof. If P is a mild solution of (2.7), then for any $x, y \in D(A)$ we have

$$\big(P(t)x,y\big) = (P_0e^{tA}x, e^{tA}y) + \int_0^t ([C^\star C - P(s)BB^\star P(s)]e^{(t-s)A}x, e^{(t-s)A}y)\, ds.$$

It follows that $\big(P(t)x,y\big)$ is differentiable with respect to t and, by a simple computation, that (2.9) holds. Conversely if P is a weak solution, then it is easy to check that for all $x, y \in D(A)$

$$\begin{aligned}\frac{d}{ds}&\big(P(s)e^{(t-s)A}x, e^{(t-s)A}y\big)\\ &= (Ce^{(t-s)A}x, Ce^{(t-s)A}y) - \big(B^\star P(t)e^{(t-s)A}x, B^\star P(t)e^{(t-s)A}y\big).\end{aligned}$$

Integrating from 0 to t we obtain

$$\big(P(t)x,y\big) = (e^{tA^\star}P_0e^{tA}x, y) + \int_0^t \big(e^{(t-s)A^\star}[C^\star C - P(s)BB^\star P(s)]e^{(t-s)A}x, y\big)\, ds$$

for all $x, y \in D(A)$. As $D(A)$ is dense in H, (2.8) follows. □

It is useful to introduce the approximating problem:

$$\begin{cases} P_n' = A_n^\star P_n + P_nA_n - P_nBB^\star P_n + C^\star C, \\ P_n(0) = P_0, \end{cases} \tag{2.10}$$

where $A_n = n^2R(n,A) - nI$ is the Yosida approximation of A and $R(n,A)$ is the resolvent of A. Problem (2.10) is obviously equivalent to the following integral equation:

$$\begin{aligned}P_n(t) = e^{tA_n^\star}P_0e^{tA_n} &+ \int_0^t e^{sA_n^\star}C^\star Ce^{sA_n}\, ds\\ &- \int_0^t e^{(t-s)A_n^\star}P_n(s)BB^\star P_n(s)e^{(t-s)A_n}\, ds.\end{aligned}$$

We now solve problem (2.7). We first prove the local existence of a solution.

Lemma 2.2. *Assume that* $(\mathcal{H})$ *is verified, fix $T > 0$, set*

$$r = 2M_T^2\|P_0\|, \tag{2.11}$$

and let τ be such that

$$\tau \in\,]0,T], \quad \tau(\|C\|^2 + r^2\|B\|^2) \le \|P_0\|, \quad 2rM_T^2\tau \le \tfrac{1}{2}, \tag{2.12}$$

where M_T has been previously defined just below (2.2). *Then problems* (2.7) *and* (2.10) *have unique mild solutions P and P_n in the ball*

$$B_{r,\tau} = \left\{F \in C_u\big([0,\tau];\Sigma(H)\big) \colon \|F\| \le r\right\}.$$

Moreover

$$\lim_{n\to\infty} P_n = P \quad \textit{in } C_s\big([0,\tau];\Sigma(H)\big). \tag{2.13}$$

Proof. Equation (2.8) (resp. the integral version of (2.10)) can be written in the form

$$P = \gamma(P) \quad \big(\text{resp. } P_n = \gamma_n\big(P_n)\big),$$

where for $x \in H$

$$\gamma(P)(t)x = e^{tA^\star} P_0 e^{tA} x + \int_0^t e^{(t-s)A^\star}[C^\star C - P(s)BB^\star P(s)]e^{(t-s)A} x\, ds$$

and

$$\gamma_n(P)(t)x = e^{tA_n^\star} P_0 e^{tA_n} x + \int_0^t e^{(t-s)A_n^\star}[C^\star C - P(s)BB^\star P(s)]e^{(t-s)A_n} x\, ds.$$

Choose now r and τ such that (2.11) and (2.12) hold. We show that γ and γ_n are $\frac{1}{2}$-contractions on the ball $B_{r,\tau}$ of $C_u\big([0,\tau];\Sigma(H)\big)$. Let in fact $P \in B_{r,\tau}$. Then, recalling (2.2), we have

$$\begin{aligned} |\gamma(P)(t)x| &\le M_T^2\{\|P_0\| + \tau[\|C\|^2 + r^2\|B\|^2]\}|x| \\ &\le 2M_T^2\|P_0\|\,|x|, \end{aligned}$$

and analogously

$$|\gamma_n(P)(t)x| \le 2M_T^2\|P_0\|\,|x|.$$

It follows that

$$\|\gamma(P)(t)\| \le r, \quad \|\gamma_n(P)(t)\| \le r, \qquad \forall t \in [0,\tau],\ n \in \mathbb{N},\ P \in B_{r,\tau}$$

so that γ and γ_n map $B_{r,\tau}$ into $B_{r,\tau}$.

For $P, Q \in B_{r,\tau}$ we have

$$\gamma(P)(t)x - \gamma(Q)(t)x = \int_0^t e^{(t-s)A^\star}[PBB^\star(Q-P)+(Q-P)BB^\star Q](s)e^{(t-s)A} x\, ds,$$

and a similar formula holds for $\gamma_n(P)(t)x - \gamma_n(Q)(t)x$. It follows that

$$\begin{aligned} \|\gamma(P)(t) - \gamma(Q)(t)\| &\le 2rM_T^2\tau\|B\|^2\|P - Q\| \le \tfrac{1}{2}\|P - Q\|, \\ \|\gamma_n(P)(t) - \gamma_n(Q)(t)\| &\le 2rM_T^2\tau\|B\|^2\|P - Q\| \le \tfrac{1}{2}\|P - Q\|. \end{aligned}$$

Thus γ and γ_n are $\frac{1}{2}$-contractions in $B_{r,\tau}$ and there exist unique mild solutions P and P_n in $B_{r,\tau}$. Finally (2.13) follows from Lemma 2.1. □

We now prove global uniqueness.

Lemma 2.3. *Assume that* $(\mathcal{H})$ *is verified, let* $T > 0$*, and let* P*,* Q *be two mild solutions of problem* (2.7) *in* $[0, T]$*. Then* $P = Q$*.*

Proof. Set

$$\alpha = \sup_{t \in [0,T]} \max\{\|P(t)\|, \|Q(t)\|\};$$

α is finite by the Uniform Boundedness Theorem. Choose $r > 0$ and $\tau \in [0, T]$ such that

$$r = 2M_T^2 \alpha, \quad \tau(\|C\|^2 + r^2\|B\|^2) \leq \alpha, \quad 2rM_T^2\tau \leq \tfrac{1}{2}.$$

By Lemma 2.2 it follows that $P(t) = Q(t)$ for any $t \in [0, \tau]$. It is now sufficient to repeat this argument in the interval $[\tau, 2\tau]$ and so on. □

The main result of this section is the following theorem.

Theorem 2.1. *Assume that* $(\mathcal{H})$ *is verified. Then problem* (2.7) *has a unique mild solution* $P \in C_s\big([0, \infty[; \Sigma^+(H)\big)$*. Moreover, for each* $n \in \mathbb{N}$*, problem* (2.10) *has a unique solution* $P_n \in C\big([0, \infty[; \Sigma^+(H)\big)$ *and*

$$\lim_{n \to \infty} P_n = P \quad \textit{in } C_s\big([0, T]; \Sigma(H)\big),$$

for any $T > 0$*.*

Proof. Fix $T > 0$, set $\beta = M_T^2(\|P_0\| + T\|C\|^2)$, and choose $r > 0$ and $\tau \in]0, T]$ such that

$$r = 2\beta M_T^2, \quad \tau(\|C\|^2 + r^2\|B\|^2) \leq \beta, \quad 2r\tau M_T^2 \leq \tfrac{1}{2}.$$

By Lemma 2.3 there exists a unique solution P (resp. P_n) of (2.7) (resp. (2.10)) in $[0, \tau]$, and $P_n \to P$ in $C_s\big([0, \tau]; \Sigma(H)\big)$. We now prove that

$$P_n(t) \geq 0, \quad \forall t \in [0, \tau]. \tag{2.14}$$

This will imply that

$$P(t) \geq 0, \quad \forall t \in [0, \tau]. \tag{2.15}$$

To this end we notice that P_n is the solution of the following linear problem in $[0, \tau]$:

$$P_n' = L_n^\star P_n + P_n L_n + C^\star C, \quad P_n(0) = P_0,$$

where $L_n = A_n - \frac{1}{2}BB^\star P_n$. Denote by $U_n(t, s)$, $0 \leq s \leq t \leq \tau$, the evolution operator associated with $L_n^\star$ (see Part II, Chapter 1 §3.5). Then we can write the solution $P_n(t)$ as

$$P_n(t) = U_n(t, 0)P_0U_n^\star(t, 0) + \int_0^t U_n(t, s)C^\star C\, U_n^\star(t, s)\, ds.$$

Thus (2.14) and (2.15) follow immediately.

We now prove that, for n large enough, we have

$$P(t) \le \beta I, \quad P_n(t) \le \beta I, \qquad \forall t \in [0,\tau]. \tag{2.16}$$

These inequalities will allow us to repeat the previous argument in the interval $[\tau, 2\tau]$ and so on. In this way the theorem will be proved. We have in fact

$$\big(P_n(t)x, x\big) = (P_0 e^{tA_n}x, e^{tA_n}x) + \int_0^t |Ce^{sA_n}x|^2\,ds \\ - \int_0^t |B^\star P_n(s)e^{(t-s)A_n}x|^2\,ds \le \beta|x|^2.$$

As $P_n(t) \ge 0$ this implies (2.16). The proof is complete. □

We now prove *continuous dependence with respect to the data.* Consider a sequence of Riccati equations

$$\begin{cases} (P^k)' = A_k^\star P^k + P^k A_k - P^k B_k B_k^\star P^k + C_k^\star C_k, \\ P^k(0) = P_0^k, \end{cases} \tag{2.17}$$

under the following hypotheses:

$$\begin{cases} \text{(i)} & \text{for any } k \in \mathbb{N},\ (A_k, B_k, C_k, P_0^k) \quad \text{fulfill } (\mathcal{H}), \\ \text{(ii)} & \text{for all } T > 0 \text{ and all } x \in H, \\ & \lim_{k\to\infty} e^{tA_k}x = e^{tA}x \text{ uniformly in } [0,T], \\ \text{(iii)} & \text{for all } T > 0 \text{ and all } x \in H, \\ & \lim_{k\to\infty} e^{tA_k^\star}x = e^{tA^\star}x \text{ uniformly in } [0,T], \\ \text{(iv)} & \text{the sequences } \{B_k\}, \{B_k^\star\}, \{C_k\}, \{C_k^\star\}, \{P_0^k\} \text{ are} \\ & \text{strongly convergent to } B, B^\star, C, C^\star, P_0, \\ & \text{respectively.} \end{cases} \tag{2.18}$$

Theorem 2.2. *Assume that* $(\mathcal{H})$ *and* (2.18) *hold. Let* P *and* P_k *be the respective mild solutions to* (2.7) *and* (2.17). *Then, for any* $T > 0$, *we have*

$$\lim_{k\to\infty} P_k = P \quad \text{in } C_s([0,T]; H). \tag{2.19}$$

Proof. Fix $T > 0$. By the Uniform Boundedness Theorem there exist positive numbers p, b, and c such that

$$\|P_0^k\| \le p, \quad \|C_k^\star C_k\| \le c, \quad \|B_k B_k^\star\| \le b, \qquad \forall k \in \mathbb{N}.$$

Set $\beta = M_T^2(p + cT)$ and choose r and $\tau \in]0,T]$ such that

$$r = 2M_T^2\beta, \quad \tau(c + r^2 b) \le \beta, \quad 2M_T^2\tau \le \tfrac{1}{2}.$$

Then, arguing as we did in the proof of Lemma 2.2 we can show that $P^k(\cdot)x \to P(\cdot)x$ in $C([0,\tau]; H)$ for any x in H. Finally, proceeding as in the proof of Theorem 2.1, we prove that this argument can be iterated in the interval $[\tau, 2\tau]$ and so on. □

We conclude this section by proving an important *monotonicity property* of the solutions of the Riccati equation (2.7).

Proposition 2.2. *Consider the Riccati equations*:

$$\begin{cases} P_i' = A^\star P_i + P_i A - P_i B_i B_i^\star P_i + C_i^\star C_i, \\ P_i(0) = P_{i,0}, \quad i = 1,2. \end{cases} \tag{2.20}$$

Assume that $(A_i, B_i, C_i, P_{i,0})$, *verify* $(\mathcal{H})$ *for* $i = 1,2$ *and in addition that*

$$P_{1,0} \le P_{2,0}, \quad C_1^\star C_1 \le C_2^\star C_2, \quad B_2 B_2^\star \le B_1 B_1^\star.$$

Then we have

$$P_1(t) \le P_2(t), \quad t \ge 0. \tag{2.21}$$

Proof. Due to Theorem 2.1 it is sufficient to prove (2.21) when A is bounded (because we can approximate A with the sequence of bounded operators A_n). Set $Z = P_2 - P_1$; then, as easily checked, Z is the solution to the linear problem:

$$\begin{cases} Z' = X^\star Z + ZX - P_2[B_2 B_2^\star - B_1 B_1^\star]P_2 + C_2^\star C_2 - C_1^\star C_1, \\ Z(0) = P_{2,0} - P_{1,0}, \end{cases}$$

where

$$X = A - \tfrac{1}{2} B_1 B_1^\star (P_1 + P_2).$$

Let $V(t,s)$ be the evolution operator associated with $X^\star$. Then we have

$$\begin{aligned} Z(t) = V(t,0)(P_{2,0} - P_{1,0})V^\star(t,0) &+ \int_0^t V(t,s)\{C_2^\star C_2 - C_1^\star C_1\}V^\star(t,s)\,ds \\ &+ \int_0^t V(t,s)P_1(s)[B_1 B_1^\star - B_2 B_2^\star]P_1(s)V^\star(t,s)\,ds \end{aligned}$$

so that $Z(t) \ge 0$ and the conclusion follows. □

2.3 Representation formulas for the solution of the Riccati equation

We want here to give an explicit formula for the solution of the Riccati equation (2.7). In fact several variants are possible; see A. V. Balakrishnan [4] and I. Lasiecka and R. Triggiani [1, 3].

We assume that $(\mathcal{H})$ is verified and introduce the operator

$$K_t \in \mathcal{L}\big(L^2(0,t;H); L^2(0,t;H)\big)$$

defined by

$$(K_t x)(s) = \int_s^t e^{(\rho - s)A} x(\rho)\, d\rho, \quad x \in L^2(0,t;H), \qquad s \in [0,t].$$

Its adjoint $K_t^\star \in \mathcal{L}\big(L^2(0,t;H); L^2(0,t;H)\big)$ is given by

$$(K_t^\star x)(s) = \int_0^s e^{(s-\sigma)A^\star} x(\sigma)\, d\sigma, \quad s \in [0,t].$$

In fact $K_t, K_t^\star \in \mathcal{L}\big(L^2(0,t;H); C([0,t];H)\big)$ (with $K_t^\star$ not representing the dual for these functional spaces). Therefore we can define $K_t^0 \in \mathcal{L}\big(L^2(0,t;H); H\big)$ as the operator

$$K_t^0 x = (K_t x)(0) = \int_0^t e^{\rho A} x(\rho)\, d\rho, \quad x \in L^2(0,t;H),$$

and its dual $(K_t^0)^\star \in \mathcal{L}\big(H; L^2(0,t;H)\big)$ is given by

$$[(K_t^0)^\star h](s) = e^{sA^\star} h, \quad h \in H.$$

We can assert the following proposition.

Proposition 2.3. *Assume that* $(\mathcal{H})$ *is verified, and let* P *be the mild solution of* (2.7). *Then one has the formula*

$$P(t) = \big[I + \big(K_t^\star C^\star C K_t + (K_t^0)^\star P_0 K_t\big) BB^\star\big]^{-1} \times [K_t^\star C^\star C e^{(t-\cdot)A} + (K_t^0)^\star P_0 e^{tA}](t). \quad (2.22)$$

Proof. We first clarify the meaning of the right-hand side of (2.22). For fixed t, $e^{(t-\cdot)A}$ is the operator in $\mathcal{L}\big(H; L^2(0,t;H)\big)$ defined by $h \mapsto e^{(t-s)A}h$, $s \in]0,t[$. Next $C \in \mathcal{L}(H;Y)$ is identified with an element of $\mathcal{L}\big(L^2(0,t;H); L^2(0,t;Y)\big)$ defined by

$$Cx(\cdot)(s) = Cx(s).$$

Similarly $B \in \mathcal{L}\big(L^2(0,t;U); L^2(0,t;H)\big)$ and analogous considerations hold for $C^\star$, $B^\star$. Then the operator

$$\Lambda_t = I + \big(K_t^\star C^\star C K_t + (K_t^0)^\star P_0 K_t\big) BB^\star$$

is an element of $\mathcal{L}\big(L^2(0,t;H); L^2(0,t;H)\big)$, which even belongs to $\mathcal{L}\big(L^2(0,t;H); C([0,t];H)\big)$. According to Proposition 1.1 in Appendix A, it is invertible and

$$\Lambda_t^{-1} \in \mathcal{L}\big(L^2(0,t;H); C([0,t];H)\big)$$

and finally

$$\Lambda_t^{-1}[K_t^\star C^\star C e^{(t-\cdot)A} + (K_t^0)^\star P_0 e^{tA}] \in \mathcal{L}\big(H; C([0,t];H)\big),$$

and it makes sense to take its value at time t, defining in this way an element of $\mathcal{L}(H)$, and therefore (2.22) has a meaning.

We shall proceed with the formal derivation of (2.22). The rigorous derivation is first done for the approximation (2.10), where the formal computations are valid, because A_n is bounded, and then the final result is obtained by going to the limit. Define $X(t,s)$ to be the solution of the problem

$$\begin{cases} -\dfrac{\partial X(t,s)}{\partial s} = \big(A - BB^\star P(s)\big)X(t,s), \quad 0 < t < s, \\ X(t,t) = I. \end{cases} \tag{2.23}$$

Define next

$$Z(t,s) = P(s)X(t,s).$$

An easy computation (from the Riccati equation and (2.23)) shows that the pair $X(t,s)$, $Z(t,s)$ is the solution of the Hamiltonian system

$$\begin{cases} -\dfrac{\partial X(t,s)}{\partial s} = AX(t,s) - BB^\star Z(t,s), \\ \dfrac{\partial Z(t,s)}{\partial s} = A^\star Z(t,s) + C^\star C X(t,s), \\ Z(t,0) = P_0 X(t,0), \quad X(t,t) = I. \end{cases} \tag{2.24}$$

Therefore we have

$$\begin{cases} Z(t,s) = e^{sA^\star} P_0 X(t,0) + \displaystyle\int_0^s e^{(s-\sigma)A^\star} C^\star C X(t,\sigma)\, d\sigma, \\ X(t,s) = e^{(t-s)A} - \displaystyle\int_s^t e^{(\rho-s)A} BB^\star Z(t,\rho)\, d\rho, \end{cases}$$

and combining the two relations we obtain

$$\begin{aligned} Z(t,s) = e^{sA^\star} P_0 &\left\{ e^{tA} - \int_0^t e^{\rho A} BB^\star Z(t,\rho)\, d\rho \right\} \\ &+ \int_0^s e^{(s-\sigma)A^\star} C^\star C \left\{ e^{(t-\sigma)A} - \int_\sigma^t e^{(\rho-\sigma)A} BB^\star Z(t,\rho)\, d\rho \right\} d\sigma. \end{aligned}$$

Setting $Z_t(s) = Z(t,s) \in \mathcal{L}\big(L^2(0,t;H); L^2(0,t;H)\big)$, this relation reads as

$$Z_t + (K_t^0)^\star P_0 K_t^0 BB^\star Z_t + K_t^\star C^\star C K_t BB^\star Z_t = K_t^\star C^\star C e^{(t-\cdot)A} + (K_t^0)^\star P_0 e^{tA}.$$

As $P(t) = Z_t(t) = Z(t,t)$, we obtain formula (2.22). □

3 Strict and classical solutions of the Riccati equation

In this section we are interested in strict and classical solutions of the Riccati equation. The section is divided into three subsections, namely the general case, the case when e^{tA} is an analytic semigroup, and finally the case when A is a variational operator. In §3.1 and §3.2 we follow G. Da Prato [1].

3.1 The general case

In order to define strict solutions to (2.7), we need to give a precise meaning to the linear mapping $\mathcal{A}$: $T \mapsto \mathcal{A}(T) = A^\star T + TA$. For any $T \in \Sigma(H)$ we set

$$\varphi_T(x, y) = (Tx, Ay) + (Ax, Ty), \quad x, y \in D(A) \tag{3.1}$$

and define

$$D(\mathcal{A}) = \{T \in \Sigma(H) \colon \varphi_T \text{ is continuous in } H \times H\}. \tag{3.2}$$

If $T \in D(\mathcal{A})$ then φ_T has a unique extension (which we still denote by φ_T) as a continuous sesquilinear form in $H \times H$. In this case there exists a linear operator that we denote by $\mathcal{A}(T) \in \Sigma(H)$ such that

$$\big(\mathcal{A}(T)x, y\big) = \varphi_T(x, y), \quad x, y \in D(A),\ T \in D(\mathcal{A}). \tag{3.3}$$

Thus we have defined a linear operator from $D(\mathcal{A}) \subset \Sigma(H) \to \Sigma(H)$. It is easy to check that $\mathcal{A}$ is closed in $\Sigma(H)$. However $D(\mathcal{A})$ is not dense in $\Sigma(H)$ in general. The following proposition shows the relationship between $\mathcal{A}(T)$ and $A^\star T + TA$.

Proposition 3.1. *Let $T \in D(\mathcal{A})$. Then for any $x \in D(A)$ we have $Tx \in D(A^\star)$ and*

$$\mathcal{A}(T)x = A^\star Tx + TAx. \tag{3.4}$$

Proof. For any $x, y \in D(A)$ we have

$$(Tx, Ay) = \varphi_T(x, y) - (Ax, Ty).$$

It follows that, for $x \in D(A)$, the linear mapping

$$y \mapsto (Tx, Ay) \colon D(A) \to \mathbb{C},$$

is continuous for H. This implies that $Tx \in D(A^\star)$ and for all $y \in H$

$$\big(\mathcal{A}(T)x, y\big) = \varphi_T(x, y) = (A^\star Tx, y) + (TAx, y),$$

which proves (3.4). □

Remark 3.1. We define a semigroup on $\Sigma(H)$

$$e^{t\mathcal{A}}(T) = e^{tA^\star} T e^{tA}, \quad t \geq 0,\ T \in \Sigma(H). \tag{3.5}$$

Notice that $e^{t\mathcal{A}}$ is not strongly continuous in general (this should imply that $e^{t\mathcal{A}}(T) \to T$ as $t \to 0$ in $\mathcal{L}(H)$). However it is easy to see that

$$e^{t\mathcal{A}}(T)x \to Tx \quad \text{as } t \to 0, \qquad \forall x \in H, \quad \forall T \in \Sigma(H). \qquad \square$$

We now return to the Riccati equation (2.7), which we write in the form

$$P' = \mathcal{A}(P) - PBB^\star P + C^\star C, \quad P(0) = P_0. \tag{3.6}$$

We say that P is a *strict solution* to the Riccati equation (2.7) (or (3.6)) if

$$\begin{cases} \text{(i)} & P \in C_s^1\big([0,\infty[;\Sigma(H)\big), \\ \text{(ii)} & P(t) \in D(\mathcal{A}),\ \forall t \geq 0, \\ \text{(iii)} & \mathcal{A}(P) \in C_s\big([0,\infty[\,;\Sigma(H)\big) \quad \text{and} \quad (3.6) \text{ holds.} \end{cases} \tag{3.7}$$

Remark 3.2. Assume that P is a strict solution of problem (2.7). By Proposition 3.1 it follows that for any $x \in D(A)$, $P(t)x \in D(A^\star)$ and the following equation holds:

$$P'(t)x = A^\star P(t)x + P(t)Ax - P(t)BB^\star P(t)x + C^\star Cx. \tag{3.8}$$

By (3.8) one can easily check that P is a weak solution of problem (2.7). Thus, by Proposition 2.2, a strict solution is a mild solution of (2.7). □

Proposition 3.2. *Assume that* ($\mathcal{H}$) *is verified and that* $P_0 \in D(\mathcal{A})$. *Then the Riccati equation* (2.7) *has a unique strict solution.*

Proof. Uniqueness follows from Theorem 2.1. Let us prove existence. Fix $T > 0$ and let P (resp. P_n) be the mild solution of (2.7) (resp. (2.10)). Set

$$\mathcal{A}_n(S) = A_n^\star S + SA_n, \quad n > \omega,\ S \in \Sigma(H),$$

where the A_n's are the Yosida approximations of A and ω is given in (2.2) (note that the $\mathcal{A}_n$'s are not the Yosida approximations of $\mathcal{A}$). We have

$$\begin{cases} P_n' = \mathcal{A}_n(P_n) - P_nBB^\star P_n + C^\star C, \\ P_n(0) = P_0. \end{cases}$$

Moreover $V_n = P_n'$ is the solution of the problem

$$\begin{cases} V_n' = \mathcal{A}_n(V_n) - P_nBB^\star V_n - V_nBB^\star P_n, \\ V_n(0) = \mathcal{A}_n(P_0) - P_0BB^\star P_0 + C^\star C. \end{cases}$$

We want to show that $V_n(\cdot)x$ is convergent, as $n \to \infty$, to $V(\cdot)x$ in $C([0,T];H)$, where V is the solution to the following linear integral equation: For all x in H

$$V(t)x = e^{tA^\star}V(0)e^{tA}x - \int_0^t e^{(t-s)A^\star}[P(s)BB^\star V(s) + V(s)BB^\star P(s)]e^{(t-s)A}x\,ds,$$

which can be easily solved by successive approximations. In fact, by using similar arguments as in the proof of Lemma 2.2, it follows that

$$P_n \to P, \qquad V_n \to V \quad \text{in } C_s([0,T];H).$$

As $V_n = P_n'$, we have that $P \in C_s^1([0,T];H)$. It remains to show that $P(t) \in D(\mathcal{A})$ and that (3.6) holds. Fix $t \geq 0$ and let $x, y \in D(A)$. We have

$$\begin{aligned}
\varphi_{P(t)}(x,y) &= \big(P(t)x, Ay\big) + \big(Ax, P(t)y\big) \\
&= \lim_{n\to\infty} \big[\big(P_n(t)x, A_n y\big) + \big(A_n x, P_n(t)y\big)\big] \\
&= \lim_{n\to\infty} \big(\mathcal{A}_n\big(P_n(t)\big)x, y\big) \\
&= \lim_{n\to\infty} \big[\big(V_n(t)x, y\big) + \big(P_n(t)BB^\star P_n(t)x, y\big) - (C^\star Cx, y)\big] \\
&= \big(V(t)x, y\big) + \big(P(t)BB^\star P(t)x, y\big) - (C^\star Cx, y).
\end{aligned}$$

It follows that $P(t) \in D(\mathcal{A})$ and

$$\mathcal{A}\big(P(t)\big) = P'(t) + P(t)BB^\star P(t) - C^\star C. \qquad \square$$

3.2 The analytic case

We assume here that the semigroup e^{tA} is analytic. Then there exists $c > 0$ and $\gamma > 0$ such that

$$\|e^{tA_n}\| \leq ce^{t\gamma}, \quad \|A_n e^{tA_n}\| \leq \frac{c}{t} e^{t\gamma}, \qquad 0 < t < 1. \tag{3.9}$$

As $e^{t\mathcal{A}_n}(T) = e^{tA_n^\star} T e^{tA_n}$, $\forall T \in \mathcal{L}(H)$, it follows that

$$\|e^{t\mathcal{A}_n}\| \leq c^2 e^{2t\gamma}, \quad \|\mathcal{A}_n e^{t\mathcal{A}_n}\| \leq \frac{2c^2}{t} e^{2t\gamma}, \qquad 0 < t < 1. \tag{3.10}$$

Moreover the semigroup $e^{t\mathcal{A}}$, defined by (3.5), is clearly analytic for $t > 0$.

We now introduce the notion of classical solution. We say that $P \in C_s\big([0,\infty[\,; \Sigma(H)\big)$ is a *classical solution* of the Riccati equation (2.7) if

$$\begin{cases}
\text{(i)} \quad \text{for any } t > 0,\ P(t) \text{ is differentiable, } P(t) \in D(\mathcal{A}) \quad \text{and} \\
\qquad\ P'(t) = \mathcal{A}\big(P(t)\big) - P(t)BB^\star P(t) + C^\star C, \quad t > 0, \\
\text{(ii)} \quad P(0) = P_0.
\end{cases}$$

Proposition 3.3. *Assume that* $(\mathcal{H})$ *is verified and that* e^{tA} *is an analytic semigroup. If* $P \in C_s\big([0,\infty[\,; \Sigma(H)\big)$ *is a classical solution of* (2.7)*, then* P *is a weak solution and a mild solution of* (2.7).

Proof. Let P be a classical solution, let $x, y \in D(A)$, and let $t > 0$. By Proposition 3.1 we have $P(t)x \in D(A^\star)$ and

$$\mathcal{A}\big(P(t)\big)x = A^\star P(t)x + P(t)Ax.$$

It follows that

$$\frac{d}{dt}\big(P(t)x,y\big) = \big(P(t)x,y\big) + (P(t)Ax,y) - \big(B^\star P(t)x, B^\star P(t)y\big) + (Cx,Cy)$$

for all $t > 0$. By integrating this identity between $\varepsilon > 0$ and t, and letting ε tend to 0, we find that P is a weak solution and then a mild solution of (2.7). □

Remark 3.3. By Proposition 3.3 it follows that there exists at most one classical solution of (2.7). □

Theorem 3.1. *Assume* $(\mathcal{H})$ *and that* e^{tA} *is an analytic semigroup. If* $P \in C_s\big([0,\infty[\,;\Sigma(H)\big)$ *is the mild solution of* (2.7)*, then* P *is a classical solution and belongs to* $C^\infty\big([\varepsilon,\infty[;\Sigma(H)\big)$*,* $\forall\varepsilon > 0$.

In order to prove Theorem 3.1 we need the following two lemmas.

Lemma 3.1. *Assume that* e^{tA} *is an analytic semigroup. Let* $E \in C_s\big([0,T];$ $\Sigma(H)\big)$ *for some* $T > 0$*, and let* F *be defined by*

$$F(t)x = \int_0^t e^{(t-s)A^\star} E(s) e^{(t-s)A} x\,ds, \quad x \in H.$$

Then for any $\alpha \in\,]0,1[$*, we have* $F \in C^\alpha\big([0,T];\Sigma(H)\big)$ *and there exists a constant* $C_T > 0$ *such that*

$$|F(t)x - F(r)x| \le C_T |t-r|^\alpha \|E\|_{C([0,T];\Sigma(H))} |x|,$$

for all $t, r > 0$.

Proof. Let $x \in H$ and $0 < r < t \le T$. We have

$$F(t)x - F(r)x = \int_r^t e^{(t-s)\mathcal{A}}\big(E(s)\big)x\,ds + \int_0^r \int_{r-s}^{t-s} (\mathcal{A}e^{\sigma\mathcal{A}})\big(E(s)\big)x\,d\sigma.$$

Taking into account (3.10) we have

$$\|F(t) - F(r)\| \le c^2 e^{2t\gamma}\left\{|t-r| + \int_0^r ds \int_{r-s}^{t-s} \frac{d\sigma}{\sigma}\right\}\|E\|.$$

As

$$\int_0^r ds \int_{r-s}^{t-s}\frac{d\sigma}{\sigma} \le \int_0^r (r-s)^{-\alpha}\,ds \int_{r-s}^{t-s} \sigma^{\alpha-1}\,d\sigma \le \frac{1}{\alpha}|t-r|^\alpha \int_0^r (r-s)^{-\alpha}\,ds,$$

the conclusion follows. □

Lemma 3.2. *Assume that* e^{tA} *is an analytic semigroup. Let* $\alpha \in\,]0,1[$*,* $M \in C^\alpha\big([0,T];\Sigma(H)\big)$*, and let* G *be defined by*

$$G(t)x = \int_0^t e^{(t-s)A^\star} M(s) e^{(t-s)A} x\,ds, \quad x \in H. \tag{3.11}$$

Then $G \in C^1\big([0,T];\Sigma(H)\big) \cap C\big([0,T];D(\mathcal{A})\big)$ *and*

$$G'(t) = \mathcal{A}\big(G(t)\big) + M(t).$$

Proof. Set

$$G_n(t)x = \int_0^t e^{(t-s)\mathcal{A}_n}\big(M(s)\big)x\,ds.$$

Then we have

$$G_n'(t) = \int_0^t \mathcal{A}_n e^{(t-s)\mathcal{A}_n}\big(M(s) - M(t)\big)\,ds + e^{t\mathcal{A}_n}\big(M(t)\big). \tag{3.12}$$

By using (3.10) we have

$$\|\mathcal{A}_n e^{(t-s)\mathcal{A}_n}\big(M(s) - M(t)\big)\| \le 2c^2 e^{2t\gamma}|t-s|^{\alpha-1}\|M\|_\alpha, \tag{3.13}$$

where

$$\|M\|_\alpha = \sup_{0\le t<s\le T} \frac{\|M(t)-M(s)\|}{|t-s|^\alpha}.$$

By (3.12) and (3.13), it follows that $G \in C^1\big([0,T];\Sigma(H)\big)$ and

$$G'(t)x = \int_0^t \mathcal{A}e^{(t-s)\mathcal{A}}\big(M(s)-M(t)\big)x\,ds + e^{t\mathcal{A}}\big(M(t)\big)x.$$

Finally, arguing as we did in the proof of Proposition 3.2, we find that $G \in C\big([0,T];D(\mathcal{A})\big)$ and that (3.11) holds. □

Proof. Proof of Theorem 3.1. Let P be the mild solution of (2.7), which we write in the form

$$P(t)x = e^{tA^\star}P_0e^{tA}x + \int_0^t e^{(t-s)A^\star}E(s)e^{(t-s)A}x\,ds, \tag{3.14}$$

where $x \in H$ and $E = C^\star C - PBB^\star P$. As $e^{tA^\star}P_0e^{tA}$ is analytic in t for $t > 0$, from Lemma 3.1 we have $P \in C^\alpha\big([\varepsilon,T];\Sigma(H)\big)$ for $0 < \varepsilon < T$, $\alpha \in]0,1]$. Moreover for all x in H and $t \ge \varepsilon$

$$P(t)x = e^{(t-\varepsilon)A^\star}P(\varepsilon)e^{(t-\varepsilon)A}x + \int_\varepsilon^t e^{(t-s)A^\star}E(s)e^{(t-s)A}x\,ds.$$

From Lemma 3.2, it follows that

$$P \in C^1\big([2\varepsilon,T];\Sigma(H)\big) \cap C([2\varepsilon,T];D\big(\mathcal{A})\big), \quad 0 < 2\varepsilon < T,$$

which implies $E \in C^1\big([2\varepsilon,T];\Sigma(H)\big)$. Moreover, by the identity

$$P(t)x = e^{(t-2\varepsilon)A^\star}P(2\varepsilon)e^{(t-2\varepsilon)A}x + \int_{2\varepsilon}^t e^{\sigma A^\star}E(t-\sigma)e^{\sigma A}x\,d\sigma, \quad x \in H,$$

with $t \ge 2\varepsilon$, it follows that $P \in C^2\big([3\varepsilon,T];\Sigma(H)\big)$, $0 < 3\varepsilon < T$. By repeating this argument several times we find $P \in C^\infty\left([\varepsilon,T];\Sigma(H)\right)$, for any $\varepsilon > 0$, as required. □

The next corollary follows easily from (3.9).

Corollary 3.1. *Assume that* $(\mathcal{H})$ *is verified and let* e^{tA} *be an analytic semigroup. Let* $P \in C_s([0,\infty[\,;H)$ *be the classical solution to* (2.7)*. Then* P *belongs to* $C\big([\varepsilon,\infty[\,;\mathcal{L}\big(H;D\big((-A^\star)^{1-\varepsilon}\big)\big)\big)$ *for all* $\varepsilon \in\,]0,1[$.

Theorem 3.2. *Assume that* $(\mathcal{H})$ *is verified and that the semigroup* e^{tA} *is analytic in the sector* $S_\theta = \{\lambda \in \mathbb{C}\colon |\arg\lambda| < \theta\}$ *for some* $\theta \in\,]0,\pi/2[$*. Let* P *be the classical solution of problem* (2.7)*. Then* P *has an analytical extension, as a function with values in* $\Sigma(H)$*, on the sector* S_θ*.*

Proof. Define r and τ as in (2.12). Then by Lemma 2.2 we have

$$P(\cdot)x = \lim_{m\to\infty} P^{(m)}(\cdot)x \quad \text{in } C([0,\tau];H), \qquad x \in H,$$

where $P^{(m)}$ are defined by recurrence as

$$\begin{aligned}
P^{(0)}(t)x &= e^{tA^\star}P_0e^{tA}x\\
P^{(m+1)}(t)x &= e^{tA^\star}P_0e^{tA}x + \int_0^t e^{(t-s)A^\star}C^\star Ce^{(t-s)A}x\,ds\\
&\qquad\qquad - \int_0^t e^{(t-s)A^\star}P^{(m)}(s)BB^\star P^{(m)}(s)e^{(t-s)A}x\,ds.
\end{aligned}$$

Clearly $P^{(m)}(\cdot)x$ are analytic in S_θ. Thus by a classical result, $P(\cdot)x$ is analytic in S_θ for any $x \in H$. This yields the conclusion (see for instance A. E. Taylor [1], Theorem 4.4.F). □

3.3 The variational case

We consider here the situation described in Chapter 2 of Part II. We assume that there exists a Hilbert space V such that $V \subset H$, algebraically and topologically, and V is dense in H. Moreover, H is identified with its dual, and $V \subset H \subset V'$, where V' is the dual of V. We denote by $\langle\cdot,\cdot\rangle$ the duality pairing between V and V' and by $\|\cdot\|$ the norm on V (recall that $|\cdot|$ and $(\cdot,\cdot)$ represent the norm and the scalar product in H). Let $a\colon V\times V \to \mathbb{R}$ be a V–H coercive continuous bilinear form; that is

$$\exists\alpha > 0, \quad \exists\lambda \in \mathbb{R}, \qquad \forall x \in V, \quad a(v,v) + \lambda|v|^2 \geq \alpha\|v\|^2.$$

Let $-A\colon D(A) \subset H \to H$ be the operator generated by a

$$-(Av,v) = a(v,v), \quad \forall v \in D(A),$$

and denote by A_λ the operator $A - \lambda I$. The operator A_λ, as well as A, is the generator of an analytic semigroup on H (cf. Part II, Chapter 1, §2.7, Theorem 2.12). Moreover we have seen that when

$$D_A\left(\tfrac{1}{2},2\right) = D_{A^\star}\left(\tfrac{1}{2},2\right),$$

then

$$D_A\left(\tfrac{1}{2},2\right) = D\big((-A_\lambda)^{1/2}\big) = V = D\big((-A_\lambda^\star)^{1/2}\big) = D_{A^\star}\left(\tfrac{1}{2},2\right)$$

(cf. Part II, Chapter 2, §1.5, Proposition 1.1).

We shall need the following result due to F. FLANDOLI [6], which is true for arbitrary variational operators.

Lemma 3.3. *Let A be a variational operator in H. Then for any $T>0$, there exists $K_T>0$ such that*

$$\int_0^T |(-A)^{1/2}e^{tA}x|^2 dt \le K_T|x|^2, \quad \forall x\in H. \tag{3.15}$$

Proof. For variational operators we know from the result of A. YAGI [1] (cf. Part II, Chapter 1, Theorem 6.1) that $D_A(\frac{1}{2},2)$ and $D\big((-A)^{1/2}\big)$ are isomorphic. It is sufficient to prove (3.15) for all $x\in D(A)$; if $x\in D(A)$, we have

$$\int_0^T |(-A)^{1/2}e^{tA}x|^2\,dt = \int_0^T |Ae^{tA}(-A)^{-1/2}x|^2\,dt \le |(-A)^{-1/2}x|^2_{D_A(1/2,2)}.$$

As $D_A(\frac{1}{2},2)$ is isomorphic to $D\big((-A)^{1/2}\big)$, the conclusion follows. □

In this case we have the following additional regularity result.

Theorem 3.3. *Assume that $(\mathcal{H})$ is verified, and let A be a variational operator. Let $P\in C_s([0,\infty[\,;H)$ be the classical solution to (2.7). Then for any $t>0$ and $x\in D\big((-A)^{1/2}\big)$ we have $P(t)x\in D\big((-A^\star)^{1/2}\big)$ and the mapping $(-A^\star)^{1/2}P(\cdot)(-A)^{1/2}x$ is continuous for $t>0$.*

Proof. Let $x\in D\big((-A)^{1/2}\big)$ and let P be given by (3.14); then we have

$$\begin{aligned}\big(P(t)(-A)^{1/2}x,(-A)^{1/2}x\big) &= \big(P_0(-A)^{1/2}e^{tA}x,(-A)^{1/2}e^{tA}x\big)\\ &\quad+\int_0^t \big(E(s)(-A)^{1/2}e^{(t-s)A}x,(-A)^{1/2}e^{(t-s)A}x\big)\,ds\\ &\le \|P_0\|\,\|(-A)^{1/2}\|^2|e^{tA}x|^2+\|E\|\int_0^t|(-A)^{1/2}e^{sA}x|^2\,ds.\end{aligned}$$

It follows, recalling (3.15), that

$$\big|\big(P(t)(-A)^{1/2}x,(-A)^{1/2}x\big)\big| \le \frac{c^2}{t^2}e^{2\gamma T}\|(-A)^{1/2}\|^2|x|^2 + \|E\|K_T|x|^2,$$

which yields the conclusion. □

So when A is a variational operator generated by a continuous and coercive bilinear form a such that

$$D\big((-A)^{1/2}\big) = D\big((-A^\star)^{1/2}\big),$$

Theorem 3.3 says that for all $t>0$ the operator $P(t)$ belongs to $\mathcal{L}(V';V)$.

4 The case of the unbounded observation

In this section we consider the case when the observation C is unbounded. More precisely we make the following assumption:

$$(H)\text{–(v)}\begin{cases} \text{(i)} \quad C \in \mathcal{L}\big(D(A); Y\big) \\ \text{(ii)} \quad \exists \text{ a real continuous function } K \text{ such that} \\ \qquad \displaystyle\int_0^t |Ce^{sA}x|^2\, ds \le K(t)|x|^2, \quad \forall x \in D(A). \end{cases}$$

Clearly, if (H)–(v) holds, then for any $t \ge 0$, there exists a linear operator $F_C(t) \in \Sigma^+(H)$ such that

$$\big(F_C(t)x, y\big) = \int_0^t (Ce^{sA}x, Ce^{sA}y)\, ds, \quad \forall x, y \in D(A). \tag{4.1}$$

Lemma 4.1. *Assume that (H)–(v) is verified, and let $F_C(\cdot)$ be defined by (4.1). Then $F_C \in C_s\big([0, +\infty[\,; \Sigma^+(H)\big)$. Moreover, setting*

$$F_{C,n}(t)x = \int_0^t e^{sA^\star}(CJ_n)^\star CJ_n e^{sA}x\, ds, \quad x \in H,$$

where $J_n = nR(n, A)$, one has for all $T > 0$

$$\lim_{n\to\infty} F_{C,n}(\cdot) = F_C(\cdot),$$

in $C_s\big([0, T]; \Sigma(H)\big)$.

Proof. Let $t_0 \in [0, +\infty[$, $t > t_0$, and $x \in D(A)$; then we have

$$\left|\sqrt{\big(F(t) - F(t_0)\big)}x\right|^2 = \big(F(t)x - F(t_0)x, x\big) = \int_{t_0}^t |Ce^{sA}x|^2\, ds \to 0, \quad \text{as } t \to t_0.$$

As $D(A)$ is dense in H we have

$$\lim_{t\to t_0} \sqrt{\big(F(t) - F(t_0)\big)}\; x = 0$$

for all $x \in H$. Consequently, the first part of the lemma follows easily. The last part follows immediately because

$$F_{C,n}(t) = J_n^\star F_C J_n. \qquad \square$$

We consider now the Riccati equation in the integral form

$$P(t)x = e^{tA^\star}P_0 e^{tA}x + F_C(t)x - \int_0^t e^{(t-s)A^\star}P(s)BB^\star P(s)e^{(t-s)A}x\, ds \tag{4.2}$$

and the approximating equation

$$P_n(t)x = e^{tA_n^\star} P_0 e^{tA_n} x + F_{C,n}(t)x - \int_0^t e^{(t-s)A_n^\star} P_n(s)BB^\star P_n(s)e^{(t-s)A_n} x\,ds, \quad n \in \mathbb{N}. \quad (4.3)$$

Equation (4.3) has clearly a unique solution $P_n \in C([0,\infty[\,;\Sigma^+(H))$, which is also the solution of the problem

$$\begin{cases} P_n' = A_n^\star P_n + P_n A_n - P_n BB^\star P_n + (CJ_n)^\star CJ_n, \\ P_n(0) = P_0. \end{cases}$$

We now prove the following theorem.

Theorem 4.1. *Assume that* $(\mathcal{H})$–(i)–(ii)–(iii)–(v) *are verified. Then* (4.2) *has a unique solution* $P \in C_s\big([0,\infty[\,;\Sigma^+(H)\big)$ *and*

$$\lim_{n\to\infty} P_n(\cdot) = P(\cdot) \quad \textit{in } C_s\big([0,T];\Sigma(H)\big), \qquad \forall T > 0. \quad (4.4)$$

Proof. The proof is similar to the one of Theorem 2.1, with minor differences. Fix $T > 0$, set

$$L_T = \sup_{t\in[0,T],n\in\mathbb{N}} \|F_{C,n}(t)\|, \quad r = 2(M_T^2\|P_0\| + L_T),$$

and choose $\tau \in [0,T]$ such that

$$\tau r^2 M_T^2 \|B\|^2 \le M_T^2\|P_0\| + L_T, \quad 2\tau r M_T^2 \le \tfrac{1}{2}.$$

Now set

$$\gamma(P)(t)x = e^{tA^\star} P_0 e^{tA} x + F_C(t)x - \int_0^t e^{(t-s)A^\star} P(s)BB^\star P(s)e^{(t-s)A} x\,ds,$$

and let $\gamma_n(P)$ be defined in a similar way. Then γ and γ_n are $\frac{1}{2}$-contractions on the ball

$$\{P \in C_s([0,\tau];H) \colon \|P(t)\| \le r, \forall t \in [0,\tau]\}.$$

Thus there exists P (resp. P_n) such that $\gamma(P) = P$ (resp. $\gamma_n(P_n) = P_n$) and P (resp. P_n) is the unique solution of (4.2) (resp. (4.3)) in $[0,\tau]$. Now, by Lemma 2.1, it follows that

$$\lim_{n\to\infty} P_n(\cdot) = P(\cdot) \quad \text{in } C_s\big([0,\tau];\Sigma(H)\big).$$

This yields $P(t) \ge 0$, $\forall t \in [0,\tau]$. From now on the proof is completely similar to the one of Theorem 2.1. □

4.1 The analytic case

We choose once and for all $\lambda_0 > \omega$ and set $A_{\lambda_0} = \lambda_0 - A$. We assume besides $(\mathcal{H})$–(i)–(ii)–(iii)

$(\mathcal{H})$–(vi) *A is the infinitesimal generator of an analytic semigroup,*
$(\mathcal{H})$-(vii) *there exists $\alpha \in [0, \frac{1}{2}[$ such that $C \in \mathcal{L}\big(D(A_{\lambda_0}^{\alpha}); Y\big)$.*

We remark that $(\mathcal{H})$–(vi)–(vii) imply that there exists a constant $K_\alpha > 0$ such that (see Part II, Chapter 1, §5, Theorem 5.2)

$$\|Ce^{tA}\| \le K_\alpha e^{\omega t} t^{-\alpha}, \quad t \ge 0.$$

It follows that (H)–(v) holds true and so we have the following straightforward generalization of Theorem 2.1.

Proposition 4.1. *Assume that* $(\mathcal{H})$–(i)–(ii)–(iii)–(vi)–(vii) *are verified. Then the Riccati equation* (2.7) *has a unique mild solution $P \in C_s\big([0, \infty[\,; \Sigma^+(H)\big)$.*

4.2 The variational case

As in §3.3, we are given a variational operator A, which is V–H coercive for some $\alpha > 0$ and $\lambda \in \mathbb{R}$. We assume that V is isomorphic to $D\big((-A)^{1/2}\big)$ and to $D\big((-A^\star)^{1/2}\big)$. We assume in addition that

- $(\mathcal{H})$–(viii) $C \in \mathcal{L}(V; Y)$.

In fact this framework corresponds to the limit case of the situation considered in §4.1, with $\alpha = \frac{1}{2}$. If $(\mathcal{H})$–(viii) holds, then by Lemma 3.3, it follows that (H)–(v) is also fulfilled. Thus we have the final result.

Proposition 4.2. *Assume that* $(\mathcal{H})$–(i)–(ii)–(iii)–(viii) *are verified. Then the Riccati equation* (2.7) *has a unique mild solution $P \in C_s\big([0, \infty[\,; \Sigma^+(H)\big)$.*

5 The case when A generates a group

We shall denote by $\mathcal{L}_r(H)$ the set of elements in $\mathcal{L}(H)$ that have a continuous inverse. It is well known that $\mathcal{L}_r(H)$ is open in $\mathcal{L}(H)$.

In this section we assume $(\mathcal{H})$ and in addition that A is the infinitesimal generator of a strongly continuous group of operators. This is equivalent to say that A fulfills $(\mathcal{H})$–(i) and $-A$ generates a C_0 semigroup e^{-tA}. Besides Riccati equation (2.7) we shall consider the following:

$$\begin{cases} Q' = -AQ - QA^\star - QC^\star CQ + BB^\star, \\ Q(0) = Q_0. \end{cases} \tag{5.1}$$

We shall show that, as proved in F. Flandoli [2], when $Q_0 = P_0^{-1}$, then we have $Q(t) = P^{-1}(t)$. We first consider the case when A is bounded.

Proposition 5.1. *Assume that* $(\mathcal{H})$ *is verified, that* $A \in \mathcal{L}(H)$*, and that* $P_0 \in \mathcal{L}_r(H)$*. Let* $P(\cdot)$ *be the mild solution of* (2.7) *and* $Q(\cdot)$ *the mild solution of* (5.1) *with* $Q_0 = P_0^{-1}$*. Then* $P(t) \in \mathcal{L}_r(H)$ *for all* $t > 0$ *and we have*

$$P(t)^{-1} = Q(t), \quad \forall t \geq 0.$$

Proof. Set $\Lambda = PQ - I$; as easily checked we have

$$\begin{cases} \Lambda' = (A^* - PBB^*)\Lambda - \Lambda(A^* - C^*CQ) =: L(\Lambda), \\ \Lambda(0) = 0. \end{cases}$$

As L is a linear bounded operator in $\mathcal{L}(H)$ we have $\Lambda(t) = 0$, $\forall t \geq 0$ and so $P(t)Q(t) = I$, $\forall t \geq 0$. In a similar way one shows that $Q(t)P(t) = I$, $\forall t \geq 0$. □

We prove now the main result of this section.

Theorem 5.1. *Assume that* $(\mathcal{H})$ *is verified, that* A *generates a strongly continuous group in* H*, and that* $P_0 \in \mathcal{L}_r(H)$*. Let* $P(\cdot)$ *be the mild solution of* (2.7) *and* $Q(\cdot)$ *the mild solution of* (5.1) *with* $Q_0 = P_0^{-1}$*. Then* $P(t) \in \mathcal{L}_r(H)$ *for all* $t > 0$ *and we have*

$$P(t)^{-1} = Q(t), \quad \forall t \geq 0.$$

Proof. Denote by P_n and Q_n the solutions of problems (2.7) and (5.1) with A replaced by the Yosida approximants A_n, $n \in \mathbb{N}$. By Theorem 2.1 we have

$$\lim_{n\to\infty} P_n = P, \quad \lim_{n\to\infty} Q_n = Q$$

in $C_s\big([0,T];\Sigma^+(H)\big)$ for all $T > 0$. By the previous proposition it follows that

$$P_n^{-1}(t) = Q_n(t), \quad \forall t \geq 0.$$

Letting n tend to infinity on the equality

$$P_n(t)Q_n(t)x = Q_n(t)P_n(t)x = x, \quad x \in H,$$

the conclusion follows. □

6 The linear quadratic control problem with finite horizon

6.1 The main result

In this section we consider the control problem (1.1)–(1.2). We assume that $(\mathcal{H})$ is verified and denote by $P \in C_s\big([0,\infty[\,;\Sigma^+(H)\big)$ the mild solution of the Riccati equation (2.7). We first consider the closed loop equation

$$\begin{cases} x'(t) = Ax(t) - BB^\star P(T-t)x(t), \quad t \in [0,T], \\ x(0) = x_0 \in H. \end{cases} \tag{6.1}$$

Proposition 6.1. *Assume that* $(\mathcal{H})$ *is verified, and let* $x_0 \in H$*. Then* (6.1) *has a unique mild solution* $x \in C([0,T];H)$*.*

Proof. It follows from Proposition 3.6 (Part II, Chapter 1, §3). □

We now prove a basic identity.

Proposition 6.2. *Assume that* $(\mathcal{H})$ *is verified, and let* $u \in L^2(0,T;U)$, $x_0 \in H$*. Let* x *be the mild solution of the state equation* (1.1)*, and let* P *be the mild solution of Riccati equation* (2.7)*. Then the following identity holds*:

$$J(u) = \int_0^T |u(s) + B^\star P(T-s)x(s)|^2 ds + \big(P(T)x_0, x_0\big). \tag{6.2}$$

Proof. Let P_n be the solution to (2.10), and let x_n be the solution to the problem

$$\begin{cases} x_n'(t) = A_n x_n(t) + Bu(t), \quad t \in [0,T], \\ x_n(0) = x_0 \in H, \end{cases}$$

where the A_n's are the Yosida approximations of A. We follow here a classical argument; see for instance R. W. Brockett [1]. By computing the derivative

$$\frac{d}{ds}\big(P_n(T-s)x_n(s), x_n(s)\big)$$

and completing the squares, we obtain the identity

$$\frac{d}{ds}\big(P_n(T-s)x_n(s), x_n(s)\big) = |u(s)+B^\star P_n(T-s)x_n(s)|^2 - |Cx_n(s)|^2 - |u(s)|^2.$$

Integrating from 0 to T and letting n tend to infinity, we obtain (6.2). □

We are now ready to prove the following result.

Theorem 6.1. *Assume that* $(\mathcal{H})$ *is verified, and let* $x_0 \in H$*. Then there exists a unique optimal pair* $(u^\star, x^\star)$*. Moreover the following statements hold:*

(i) $x^\star \in C([0,T];H)$ *is the mild solution to the closed loop equation* (6.1)*,*
(ii) $u^\star \in C([0,T];U)$ *is given by the feedback formula*

$$u^\star(t) = -B^\star P(T-t)x^\star(t), \quad t \in [0,T], \tag{6.3}$$

(iii) *The optimal cost* $J(u^\star)$ *is given by*

$$J(u^\star) = (P(T)x_0, x_0). \tag{6.4}$$

Proof. We first remark that by identity (6.2) it follows that

$$J(u) \geq \big(P(T)x_0, x_0\big), \tag{6.5}$$

for any control $u \in L^2(0,T;U)$. Let now $x^\star$ be the mild solution to (6.1), and let $u^\star$ be given by (6.3). Setting in (6.2) $u = u^\star$ and taking into account (6.5), it follows that $(u^\star, x^\star)$ is an optimal pair and that (6.4) holds. It remains to prove uniqueness. Let $(\bar{u}, \bar{x})$ be another optimal pair. Setting in (6.2), $u = \bar{u}$ and $x = \bar{x}$, we obtain

$$\int_0^T |\bar{u}(s) + B^\star P(T-s)\bar{x}(s)|^2\, ds = 0,$$

so that $\bar{u}(s) = -B^\star P(T-s)\bar{x}(s)$ for almost every s in $[0,T]$. But this implies that $\bar{x}$ is a mild solution of (6.1) so that $\bar{x} = x^\star$ and, consequently, $\bar{u} = u^\star$. □

6.2 The case of unbounded observation

We now consider the case of unbounded observation, assuming that $(\mathcal{H})$–(i)–(ii)–(iii)–(v) are verified. It is convenient to introduce the linear operator:

$$\xi \mapsto L_T(\xi)\colon D(A) \to L^2(0,T;H),$$

where $(L_T\xi)(t) = Ce^{tA}\xi$. As, by (H)–(v)

$$\|L_T\xi\|^2_{L^2(0,T;H)} \le K(T)|\xi|^2,$$

L_T has an extension to the whole space H, which will still be denoted by L_T.

Let $x_0 \in H$, $u \in L^2(0,T;U)$, and let x be the corresponding solution of (1.1). We want to define the cost functional $J(u)$; this is not a priori defined because $x(t)$ does not necessarily belong to $D(C)$ and the term $\int_0^T |Cx(t)|^2\, dt$ is not well defined. We set

$$Cx(t) = (L_T x_0)(t) + \int_0^t L_T\big(Bu(s)\big)(t-s)\, ds.$$

This definition is meaningful in virtue of the following lemma.

Lemma 6.1. *Let $z \in L^2(0,T;H)$, and set*

$$w(t) = \int_0^t L_T\big(z(s)\big)(t-s)\, ds.$$

Then $w \in L^2(0,T;H)$ and the following estimate holds:

$$\|w\|^2_{L^2(0,T;H)} \le K(T)\|z\|^2_{L^2(0,T;H)}.$$

Proof. It is sufficient to verify the estimate for $z \in L^2\big(0,T;D(A)\big)$; in this case we have

$$w(t) = \int_0^t Ce^{(t-s)A}z(s)\, ds.$$

Let $\varphi \in L^2(0,T;H)$; then we have

$$\int_0^T \big(w(t),\varphi(t)\big)\,dt = \int_0^T ds \int_s^T \big(Ce^{(t-s)A}z(s),\varphi(t)\big)\,dt.$$

It follows that

$$\left|\int_0^T \big(w(t),\varphi(t)\big)\,dt\right| \le \int_0^T ds \left[\int_s^T |Ce^{(t-s)A}z(s)|^2\,dt\right]^{1/2} \left[\int_s^T |\varphi(t)|^2 dt\right]^{1/2}$$
$$\le K(T)\|\varphi\|^2_{L^2(0,T;H)}\|z\|^2_{L^2(0,T;H)}.$$

As this is true for any φ, we obtain the estimate. □

Now the following result is proved as Theorem 6.1.

Theorem 6.2. *Assume that* $(\mathcal{H})$–(i)–(ii)–(iii)–(v) *are verified, and let* $x_0 \in H$*. Then there exists a unique optimal pair* $(u^\star, x^\star)$*. Moreover the following statements hold*:

(i) $x^\star \in C([0,T];H)$ *is the mild solution to the closed loop equation* (6.1)*,*
(ii) $u^\star \in C([0,T];U)$ *is given by the feedback formula* (6.3)*,*
(iii) *the optimal cost* $J(u^\star)$ *is given by* (6.4)*.*

6.3 Regularity properties of the optimal control

We give here some regularity results for the optimal pair $(u^\star, x^\star)$.

Proposition 6.3. *Assume that* $(\mathcal{H})$ *is verified. Let* $P_0 \in D(\mathcal{A})$ *and* $x \in D(A)$*, where* $\mathcal{A}$ *is the linear operator defined by* (3.3)*. Then*

$$x^\star \in C^1([0,T];H) \cap C\big([0,T];D(A)\big), \quad u^\star \in C^1([0,T];H).$$

Proof. By Proposition 3.2 we know that the mild solution P to the Riccati equation (2.7) is a strict solution, so that

$$P \in C^1_s\big([0,T];\Sigma(H)\big) \cap C_s\big([0,T];D(\mathcal{A})\big).$$

Now the assertion concerning $x^\star$ follows from Proposition 3.3 (Part II, Chapter 1). □

Proposition 6.4. *Assume that* $(\mathcal{H})$ *is verified and that* e^{tA} *is an analytic semigroup. Then the following statements hold*:

(i) *for any* $\varepsilon \in \,]0,T/2[$*,* $x^\star$ *(resp.* $u^\star$*) belongs to* $C^\infty(]\varepsilon,T-\varepsilon[\,;H)$ *(resp.* $C^\infty(]\varepsilon,T-\varepsilon[\,;U)$*).*
(ii) $u^\star$ *and* $x^\star$ *are analytic in* $]0,T[$*.*

Proof. (i) We first remark that by (6.3) $u^* \in C([0,T];U)$. Moreover, as $(x^\star)' = Ax^\star + Bu^\star$ and $x^\star(0) = x_0$, we have, by Proposition 3.10 (Part II, Chapter 1), $x^\star \in C^\alpha([\varepsilon,T];H)$ (for any $\alpha \in \,]0,1[$) and $\varepsilon \in [0,T]$. By (6.3) and Theorem 3.1 we have $u^\star \in C^\alpha([\varepsilon,T-\varepsilon];U)$. Then, by Proposition 3.9 (Part II, Chapter 1), $x^\star$ belongs to $C^1([\varepsilon,T-\varepsilon];H)$. By iterating this argument the assertion (i) follows. Finally (ii) is a consequence of Theorem 3.2. □

6.4 Hamiltonian systems

Assume that $(\mathcal{H})$ is verified, let P be the mild solution of (2.7), and let $(u^\star, x^\star)$ be an optimal pair. In (6.1) it is useful to introduce the following function:

$$p^\star(t) = P(T-t)x^\star(t)$$

called the *adjoint variable*. If we differentiate $p^\star(t)$, assuming for a while that the conditions of Proposition 6.3 hold, we easily deduce from (2.7) and (6.1) that $p^\star(t)$ satisfies the equations

$$\begin{cases} -p'(t) = A^\star p(t) + C^\star Cx(t), \\ P(T) = P_0 x(T) \end{cases} \tag{6.6}$$

and thus considering (6.6) together with

$$\begin{cases} x'(t) = Ax(t) - BB^\star p(t), \\ x(0) = x_0, \end{cases} \tag{6.7}$$

we obtain a system (the two point boundary value problem) whose solution is $(x^\star, p^\star)$. The system (6.6)–(6.7) is called *Hamiltonian system*. Once we have established (6.6)–(6.7) for regular x_0, P_0, it is easy to extend it to the general case.

Remark 6.1. One can notice the analogy between (6.6)–(6.7) and (2.24), which was the source of the explicit formula (2.22) for $P(t)$. □

7 Some generalizations and complements

7.1 Nonhomogeneous state equation

Consider the following optimal control problem: To minimize

$$J(u) = \int_0^T \{|Cx(t)|^2 + |u(t)|^2\}\,dt + \big(P_0 x(T), x(T)\big) \tag{7.1}$$

over all controls $u \in L^2(0,T;U)$ subject to the differential equation constraint

$$\begin{cases} x'(t) = Ax(t) + f(t) + Bu(t), \quad t \in [0,T], \\ x(0) = x_0 \in H. \end{cases} \tag{7.2}$$

We assume that hypothesis $(\mathcal{H})$ holds and that $f \in L^2(0,T;H)$. Moreover we denote by P the mild solution of the Riccati equation (2.7). We want to show that it is possible to generalize the Dynamic Programming approach to this more general situation. The main difference is the introduction of a dual variable r, defined by the following backward Cauchy problem: For all t in $[0,T]$

$$\begin{cases} r'(t) + \big(A^\star - P(T-t)BB^\star\big)r(t) + P(T-t)f(t) = 0, \\ r(T) = 0. \end{cases} \tag{7.3}$$

Notice that, by the change of variable $t \to T-t$, problem (7.3) reduces to an initial value problem, which has a unique mild solution.

The following identity generalizes identity (6.2).

Lemma 7.1. *Let $x_0 \in H$, $f \in L^2(0,T;H)$, and $u \in L^2(0,T;U)$. Then we have*

$$\begin{aligned} J(u) = (P(T)x_0, x_0) + 2(r(0), x_0) + \int_0^T \left\{2\big(r(s), f(s)\big) - |B^\star r(s)|^2\right\} ds \\ + \int_0^T |u(s) + B^\star r(s) + B^\star P(T-s)x(s)|^2\, ds, \end{aligned} \tag{7.4}$$

where x and r are, respectively, the mild solutions of (7.2) *and* (7.3)*.*

Proof. Let P_n be the solution of (2.10), and let x_n and r_n be the solutions of the problems

$$\begin{cases} x_n'(t) = A_n x_n(t) + f(t) + Bu(t), \\ x_n(0) = x_0, \end{cases}$$

$$\begin{cases} r_n'(t) = -\big(A_n^\star - P_n(T-t)BB^\star\big)r_n(t) - P_n(T-t)f(t), \\ r_n(T) = 0. \end{cases}$$

Then, by integrating the identity

$$\begin{aligned} &\frac{d}{ds}\left\{\big(P_n(T-s)x_n(s), x_n(s)\big) + 2\big(r_n(s), x_n(s)\big)\right\} \\ &\quad = |B^\star P_n(T-s)x_n(s) + B^\star r_n(s) + u(s)|^2 + 2\big(r_n(s), f(s)\big) - |B^\star r_n(s)|^2 \\ &\qquad - |Cx_n(s)|^2 - |u(s)|^2 \end{aligned}$$

between 0 and T and by letting n tend to infinity, we obtain (7.4). □

By using identity (7.4) we can easily generalize Theorem 6.1. The proof is similar and will be omitted.

Theorem 7.1. *Assume that $(\mathcal{H})$ is verified, let $x_0 \in H$, and let $f \in L^2(0,T;H)$. Then there exists a unique optimal pair $(u^\star, x^\star)$ for problem* (7.1)–(7.2)*. Moreover the following statements hold:*

(i) *$x^\star$ is the mild solution to the closed loop equation*

$$x'(t) = \big(A - BB^\star P(T-t)\big)x(t) - BB^\star r(t) + f(t), \quad t \in [0,T],\ x(0) = x_0,$$

(ii) $u^\star$ *is given by the feedback formula*

$$u^\star(t) = -B^\star[P(T-t)x^\star(t) + r(t)],$$

where r *is the mild solution to* (7.3)*,*

(iii) *the optimal cost is given by*

$$J(u^\star) = \big(P(T)x_0, x_0\big) + 2\big(r(0), x_0\big) + \int_0^T \left[\big(2r(s), f(s)\big) - |B^\star r(s)|^2\right] ds.$$

Remark 7.1. (Tracking Problem). Consider the following optimal control problem: To minimize

$$\begin{aligned} K(u) = \int_0^T &\{|Cx(t) - \xi(t)|^2 + |u(t)|^2\}\, dt \\ &+ \Big(P_0\big(x(T) - \xi(T)\big), \big(x(T) - \xi(T)\big)\Big) \end{aligned} \tag{7.5}$$

over all controls $u \in L^2(0,T;U)$ subject to the differential equation constraint

$$\begin{cases} z'(t) = Az(t) + Bu(t), \quad t \in [0,T], \\ z(0) = z_0 \in H. \end{cases} \tag{7.6}$$

Here $\xi \in W^{1,2}([0,T];H) \cap L^2\big(0,T;D(A)\big)$ is a given function. Now, by setting

$$x = z - \xi, \quad f = A\xi - \xi', \quad x_0 = z_0 - x(0),$$

equation (7.6) reduces to equation (7.2) and problem (7.5)–(7.6) to problem (7.1)–(7.2). □

7.2 Time-dependent state equation and cost function

Consider the system

$$\begin{cases} x'(t) = A(t)x(t) + B(t)u(t), \\ x(0) = x_0, \end{cases} \tag{7.7}$$

where $A(t)\colon D\big(A(t)\big) \subset H \to H$ and $B(t) \in \mathcal{L}(U;H)$ are linear operators.

We make the following assumptions on the families $\{A(t)\}_{t\in[0,T]}$ and $\{B(t)\}_{t\in[0,T]}$:

$$\begin{cases} \text{(i)} & A(t)\colon D\big(A(t)\big) \subset H \to H \text{ generates a } C_0 \text{ semigroup in } H \\ & \text{for all } t \in [0.T], \\ \text{(ii)} & \text{there exists a strongly continuous mapping} \\ & U_A(\cdot,\cdot)\colon \{(t,s) \in \mathbb{R}^2\colon t \geq s\} \to \mathcal{L}(H) \\ & \text{such that } U_A^*(\cdot,\cdot) \text{ is also strongly continuous and} \\ & \dfrac{\partial}{\partial t} U_A(t,s)x = A(t)U_A(t,s)x, U_A(s,s)x = x, \\ & \forall x \in D\big(A(t)\big), 0 \leq s \leq t \leq T, \\ \text{(iii)} & \text{we have } \lim\limits_{n\to\infty} U_{A_n}(t,s)x = U_A(t,s)x, \forall x \in H \text{ uniformly on} \\ & \text{the bounded sets of } \{(t,s) \in \mathbb{R}^2\colon t \geq s\}, \text{ where} \\ & U_{A_n}(t,s) \text{ is the evolution operator generated by the Yosida} \\ & \text{approximations of } A(t), \\ \text{(iv)} & B(\cdot)u \text{ is continuous for all } u \in U. \end{cases} \tag{7.8}$$

Under these assumptions problem (7.7) has a unique mild solution given by

$$x(t) = U(t,0)x_0 + \int_0^t U(t,s)Bu(s)\,ds.$$

Assumptions (7.8) are verified in many problems both parabolic and hyperbolic (see for instance P. Acquistapace and B. Terreni [1], A. Lunardi [3], A. Pazy [2], and H. Tanabe [1]).

We want to minimize the *cost function*

$$J(u) = \int_0^T \big\{|C(s)x(s)|^2 + |u(s)|^2\big\}\,ds + \big(P_0x(T), x(T)\big)$$

over all controls $u \in L^2(0,T;U)$ subject to the differential equation constraint (7.7). Concerning the operators B, C, and P_0, we shall assume that

$$\begin{gathered} P_0 \in \Sigma^+(H), \quad C \in C_s\big([0,T];\mathcal{L}(H;Y)\big), \\ B \in C_s\big(\big[0,T];\mathcal{L}(U;H)\big). \end{gathered} \tag{7.9}$$

It is convenient to consider the backward Riccati equation

$$\begin{cases} Q' + A^\star Q + QA - QBB^\star Q + C^\star C = 0, \\ Q(T) = P_0. \end{cases} \tag{7.10}$$

Its corresponding mild form is

$$\begin{aligned} Q(t)x = U^\star(T,t)P_0U(T,t)x &+ \int_t^T U^\star(s,t)C^\star(s)C(s)U(s,t)x\,ds \\ &- \int_t^T U^\star(s,t)Q(s)B(s)B^\star(s)Q(s)U(s,t)x\,ds, \quad x \in H, \end{aligned}$$

where $U^\star(s,t)$ is the adjoint of $U(s,t)$.

The following theorem is proved as Theorem 2.1.

Theorem 7.2. *Assume* (7.8) *and* (7.9). *Then problem* (7.10) *has a unique mild solution* $Q \in C_s\big([0,\infty[\,;\Sigma^+(H)\big)$ *and*

$$\lim_{n\to\infty} Q_n = Q \quad \textit{in } C_s\big([0,\infty[\,;\Sigma(H)\big),$$

for any $T>0$, *where* Q_n *is the solution to problem* (7.10) *with* $A(t)$ *replaced by* $A_n(t)$.

The following monotonicity property of the solutions of Riccati equations is proved as Proposition 2.2.

Proposition 7.1. *Consider the Riccati equations*

$$\begin{cases} Q_i' + A_i^\star Q_i + Q_i A_i - Q_i B_i B_i^\star Q_i + C_i^\star C_i = 0, \\ Q_i(T) = Q_{i,0}, \quad i=1,2. \end{cases} \tag{7.11}$$

Assume that $(A_i, B_i, C_i, P_{i,0})$ *verify* (7.8) *and* (7.9) *for* $i=1,2$ *and that, in addition,*

$$\begin{gathered} P_{1,0} \le P_{2,0}, \\ C_1^\star(t)C_1(t) \le C_2^\star(t)C_2(t), \quad B_2(t)B_2^\star(t) \le B_1(t)B_1^\star(t), \qquad t\in[0,T]. \end{gathered}$$

Then we have

$$P_1(t) \le P_2(t), \quad t \ge 0. \tag{7.12}$$

By using Theorem 7.2, we can easily generalize Theorem 6.1.

Theorem 7.3. *Assume* (7.8) *and* (7.9), *and let* $x_0 \in H$. *Then there exists a unique optimal pair* $(u^\star, x^\star)$ *and* $u^\star \in C([0,T];U)$ *is related to* $x^\star$ *by the feedback formula*

$$u^\star(t) = -B^\star(t)Q(t)x^\star(t), \quad t\in[0,T].$$

Finally,the optimal cost $J(u^\star)$ *is given by*

$$J(u^\star) = \big(Q(0)x_0, x_0\big).$$

Remark 7.2. For the time-varying variational case the reader is referred to the book of J. L. Lions [3]. □

7.3 Dual Riccati equation

We assume here that $(\mathcal{H})$ is verified and consider the Riccati equation

$$\begin{cases} P' = A^*P + PA - PBB^*P + C^*C, \quad t\ge 0, \\ P(0) = P_0 \end{cases} \tag{7.13}$$

and the dual equation

$$\begin{cases} Q' = AQ + QA^* - QC^*CQ + BB^*, \quad t \geq 0, \\ Q(0) = Q_0, \end{cases} \tag{7.14}$$

where $Q_0 \in \Sigma^+(H)$. Clearly the set (A^*, C^*, B^*, Q_0) also verifies assumption $(\mathcal{H})$; thus the equations (7.13) and (7.14) have unique solutions $P, Q \in C_s\big([0, +\infty[\,; \Sigma^+(H)\big)$, respectively.

We want to present here a simple formula that gives P in terms of Q. This formula will turn out to be useful in Chapter 3 below, and it was proved in V. Barbu and G. Da Prato [1], by variational methods.

Fix $T > 0$, and set

$$\tilde{Q}_T(s) = Q(T - s), \quad s \in [0, T], \tag{7.15}$$

and

$$G_T(s) = A^* - C^*CQ(T - s), \quad s \in [0, T]; \tag{7.16}$$

we shall denote by U_{G_T} the evolution operator associated with G_T.

We first prove a lemma.

Lemma 7.2. *Fix $T > 0$, and set*

$$Z_T(t) = [I + P(t)\tilde{Q}_T(t)]^{-1}P(t) = P(t)[I + \tilde{Q}_T(t)P(t)]^{-1}.$$

Then for any $x \in H$ we have

$$Z_T(t)x = U_{G_T}(t,0)Z_T(0)U^*_{G_T}(t,0)x + \int_0^t U_{G_T}(t,s)C^*CU^*_{G_T}(t,s)x\,ds. \tag{7.17}$$

Proof. We assume that A is bounded; otherwise we replace A by its Yosida approximations A_n and then we let n tend to infinity. We first remark that the definition of Z_T is meaningful by Proposition 1.1 in Appendix A. Setting for simplicity $Z = Z_T$ and $\tilde{Q} = \tilde{Q}_T$ we have

$$(I + P\tilde{Q})Z = P$$

and so

$$(P'\tilde{Q} + P\tilde{Q}')Z + (I + P\tilde{Q})Z' = P',$$

which implies

$$\begin{aligned} (I + P\tilde{Q})Z' &= P' - (P'\tilde{Q} + P\tilde{Q}')[I + P\tilde{Q}]^{-1}P \\ &= P' - (P'\tilde{Q} + P\tilde{Q}')P[I + \tilde{Q}P]^{-1} \\ &= [P'(I + \tilde{Q}P) - (P'\tilde{Q} + P\tilde{Q}')P][I + \tilde{Q}P]^{-1} \\ &= [P' - P\tilde{Q}'P][I + \tilde{Q}P]^{-1}. \end{aligned}$$

It follows that

$$(I + P\tilde{Q})Z'[I + \tilde{Q}P] = P' - P\tilde{Q}'P.$$

Then, by substituting P' and Q' with the expressions given by (7.13) and (7.14), we have

$$\begin{aligned}[I + P\tilde{Q}]Z'[I + \tilde{Q}P] &= A^*P + PA - PBB^*P + C^*C \\ &\qquad + P[A\tilde{Q} + \tilde{Q}A^* - \tilde{Q}C^*\,C\tilde{Q} + BB^*]P \\ &= [I + P\tilde{Q}](A^* - C^*C\tilde{Q})P + P(A - \tilde{Q}C^*C)[I + \tilde{Q}P] \\ &\qquad + [I + P\tilde{Q}]C^*C[I + \tilde{Q}P].\end{aligned}$$

Therefore

$$Z' = (A^* - C^*C\tilde{Q})Z + Z(A - \tilde{Q}C^*C) + C^*C,$$

and the conclusion follows. □

Theorem 7.4. *Assume that* $(\mathcal{H})$ *is verified, and let* P *and* Q *be the respective solutions in* $C_s\big([0, +\infty[\,; \Sigma^+(H)\big)$ *of* (7.13) *and* (7.14) *with* $Q_0 = 0$. *Let* $T > 0$ *and set*

$$G_T(t) = A^* - C^*C\tilde{Q}_T(t) = A^* - C^*CQ(T - t), \quad t \in [0, T].$$

Then we have

$$\begin{aligned}P(T) = U_{G_T}(T, 0)[I + P_0Q(T)]^{-1}P_0U^*_{G_T}(T, 0) \\ + \int_0^T U_{G_T}(T, s)C^*CU^*_{G_T}(T, s)\,ds.\end{aligned} \tag{7.18}$$

Proof. As

$$Z_T(0) = [I + P_0Q(T)]^{-1}P_0$$

and $Z_T(T) = P(T)$, the conclusion follows from (7.17). □

8 Examples of controlled systems

8.1 Parabolic equations

Let Ω be an open bounded set of $\mathbb{R}^n$ with regular boundary $\partial\Omega$. Consider the state equation

$$\begin{cases} \dfrac{\partial x}{\partial t}(t, \xi) = \big(\Delta_\xi + c)x(t, \xi) + (Bu(t, \cdot)\big)(\xi), & \text{in }]0, T] \times \Omega, \\ x(t, \xi) = 0 \quad \text{on }]0, T] \times \partial\Omega, \\ x(0, \xi) = x_0(\xi) \quad \text{in } \Omega, \end{cases} \tag{8.1}$$

where $c \in \mathbb{R}$ and $B \in \mathcal{L}\big(L^2(\Omega)\big)$. We choose $H = U = Y = L^2(\Omega)$ as space of states, controls, and observations. The control u is said to be a *distributed control*. We denote by A the linear self-adjoint operator in H:

$$\begin{cases} Ax = \Delta_\xi x + cx, \quad \forall x \in D(A), \\ D(A) = H^2(\Omega) \cap H_0^1(\Omega). \end{cases}$$

Setting $x(t) = x(t, \cdot), u(t) = u(t, \cdot)$ we can write problem (8.1) in the abstract form (1.1). By Proposition 2.11 (Part II, Chapter 1), A is the infinitesimal generator of an analytic semigroup in H. Let C and P_0 be non-negative linear bounded operators in $L^2(\Omega)$. Consider the following problem: To minimize

$$J(u) = \int_0^T \int_\Omega \left\{ \left|\big(Cx(t,\cdot)\big)(\xi)\right|^2 + |u(t,\xi)|^2 \right\} dt\, d\xi + \int_\Omega \big(P_0 x(T,\cdot)(\xi) x(T,\xi)\big)\, d\xi$$

subject to state equation (8.1). We remark that assumption $(\mathcal{H})$ is fulfilled. Due to Theorem 2.1, the Riccati equation (2.7) has a unique solution $P \in C_s\big([0,T]; \Sigma^+(H)\big)$. Then, by Theorem 6.1, there exists a unique optimal pair $(u^\star, x^\star)$, where $x^\star \in C\big([0,T]; L^2(\Omega)\big)$ is the classical solution in $]0,T[$ to the closed loop equation

$$\begin{cases} \dfrac{\partial x}{\partial t}(t,\xi) = (\Delta_\xi + c)x(t,\xi) - (BB^\star\big(P(T-t)x(t,\cdot)\big)(\xi), \\ \text{in }]0,T] \times \Omega, \\ x(t,\xi) = 0 \quad \text{on }]0,T] \times \partial\Omega, \\ x(0,\xi) = x_0(\xi) \quad \text{in } \Omega, \end{cases}$$

that is,

$$\forall \varepsilon \in]0,T[, \quad \frac{\partial x}{\partial t} \quad \text{and} \quad \frac{\partial^2 x}{\partial \xi^2} \in L^2([\varepsilon, T-\varepsilon] \times \Omega).$$

Moreover $u^\star$ is given by the formula

$$u^\star(t,\xi) = -\big(B^\star\big(P(T-t)x^\star(t,\cdot)\big)(\xi)\big).$$

Consider now a problem with unbounded observation: To minimize

$$J(u) = \int_0^T \int_\Omega \{|(\nabla_\xi x(t,\xi)|^2 + |u(t,\xi)|^2\}\, dt\, d\xi + \int_\Omega |x(T,\xi)|^2\, d\xi$$

subject to state equation (8.1). In this case the linear operator C is given by

$$Cx = -\sqrt{\Delta_\xi}\, x,$$

and Proposition 4.2 can be applied.

8.2 Wave equation

Let Ω be as in the previous example and consider the problem:

$$\begin{cases} \dfrac{\partial^2 x}{\partial t^2}(t,\xi) = \varDelta_\xi x(t,\xi) + \big(Bu(t,\cdot)\big)(\xi) \quad \text{in }]0,T] \times \Omega, \\ x(t,\xi) = 0 \quad \text{on }]0,T] \times \partial\Omega, \\ x(0,\xi) = x_0(\xi), \quad \text{in } \Omega, \\ \dfrac{\partial x}{\partial t}(0,\xi) = x_1(\xi) \quad \text{in } \Omega. \end{cases} \tag{8.2}$$

Set

$$H = Y = H_0^1(\Omega) \oplus L^2(\Omega), \quad U = L^2(\Omega) \tag{8.3}$$

and denote by $X = \left[\begin{smallmatrix} x^0 \\ x^1 \end{smallmatrix}\right]$, the generic element of H. The inner product in H is defined by

$$\left(\begin{bmatrix} x^0 \\ x^1 \end{bmatrix}, \begin{bmatrix} z^0 \\ z^1 \end{bmatrix} \right) = \int_\Omega (\nabla_\xi x^0 \cdot \nabla_\xi z^0 + x^1 z^1)\, d\xi. \tag{8.4}$$

Let $\varLambda$ be the self-adjoint positive operator on $L^2(\Omega)$ defined by

$$D(\varLambda) = H^2(\Omega) \cap H_0^1(\Omega), \quad \varLambda x = -\varDelta_\xi x. \tag{8.5}$$

Then we have

$$(X,Z) = (\sqrt{\varLambda} x^0, \sqrt{\varLambda} z^0) + (x^1, z^1).$$

Define the linear operator A on H:

$$\begin{cases} AX = \begin{bmatrix} 0 & 1 \\ -\varLambda & 0 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \end{bmatrix}, \quad \forall X \in D(A), \\ D(A) = H^2(\Omega) \cap H_0^1(\Omega) \oplus H_0^1(\Omega). \end{cases} \tag{8.6}$$

We have $A^\star = -A$ and by Proposition 2.12 (Part II, Chapter 1) we know that A is the infinitesimal generator of a contraction group in H. Let finally $B \in \mathcal{L}(U;H)$ be defined as

$$Bu = \begin{bmatrix} 0 \\ u \end{bmatrix}, \quad u \in U;$$

then $B^\star \in \mathcal{L}(H,U)$ is given by

$$B^\star \begin{bmatrix} x^0 \\ x^1 \end{bmatrix} = x^1, \quad \begin{bmatrix} x^0 \\ x^1 \end{bmatrix} \in H,$$

and we have

$$BB^\star \begin{bmatrix} x^0 \\ x^1 \end{bmatrix} = \begin{bmatrix} 0 \\ x^1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \end{bmatrix}.$$

Setting

$$x^0(t) = x(t,\cdot), \quad x^1(t) = \frac{\partial x}{\partial t}(t,\cdot), \quad u(t) = u(t,\cdot),$$

we can write (8.2) in the abstract form

$$Y' = AY + Bu, \quad Y(0) = Y_0, \tag{8.7}$$

where

$$Y(t) = \begin{bmatrix} x^0(t) \\ x^1(t) \end{bmatrix}, \quad Y_0 = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}.$$

We are interested in the following optimal control problem: To minimize

$$\begin{aligned} J(u) = \int_0^T \int_\Omega \left(|\nabla_\xi x(t,\xi)|^2 + \left| \frac{\partial x}{\partial t}(t,\xi) \right|^2 + |u(t,\xi)|^2 \right) dt\, d\xi \\ + \int_\Omega \left(|\nabla_\xi x(T,\xi)|^2 + \left| \frac{\partial x}{\partial t}(T,\xi) \right|^2 \right) d\xi, \end{aligned} \tag{8.8}$$

over all $u \in L^2([0,\infty] \times \Omega)$ subject to (8.2). The cost function $J(u)$ can be written as

$$J(u) = \int_0^T \{|Y(t)|_H^2 + |u(t)|_U^2\}\, dt + |Y(T)|_H^2.$$

The Riccati equation is

$$P' = \begin{bmatrix} 0 & -1 \\ \Lambda & 0 \end{bmatrix} P + P \begin{bmatrix} 0 & 1 \\ -\Lambda & 0 \end{bmatrix} - P \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} P + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$P(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We can represent P as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix},$$

where

$$\begin{aligned} &P_{11} \in \mathcal{L}\big(H_0^1(\Omega)\big), && P_{12} \in \mathcal{L}\big(L^2(\Omega); H_0^1(\Omega)\big), \\ &P_{21} \in \mathcal{L}\big(H_0^1(\Omega); L^2(\Omega)\big), && P_{22} \in \mathcal{L}\big(L^2(\Omega)\big), \end{aligned}$$

and the following identities hold:

$$P_{11}^\star = \Lambda P_{11} \Lambda^{-1}, \quad P_{12}^\star = P_{21} \Lambda^{-1}, \quad P_{21}^\star = \Lambda P_{12}, \quad P_{22}^\star = P_{22}.$$

Setting $C = I$, we can apply Theorem 2.1 and conclude that the Riccati equation has a unique mild solution $P \in C_s\big([0,\infty[\ ; \Sigma^+(H)\big)$. Moreover by Theorem 6.1, there exists a unique optimal pair $(u^\star, x^\star)$ and

$$u^\star(t,\xi) = -\big(P_{21}(T-t) x^\star(t,\cdot)\big)(\xi) - \left(P_{22}(T-t) \frac{\partial x^\star}{\partial t}(t,\cdot) \right)(\xi).$$

The closed loop equation is

$$\begin{cases} \dfrac{\partial^2 x}{\partial t^2}(t,\xi) = \Delta_\xi x(t,\xi) - \big(P_{21}(T-t)x(t,\cdot)\big)(\xi) \\ \qquad\qquad\qquad -\left(P_{22}(T-t)\dfrac{\partial x}{\partial t}(t,\cdot)\right)(\xi) \quad \text{in }]0,T[\times\Omega, \\ x(t,\xi) = 0 \quad \text{on }]0,T[\times\partial\Omega, \\ x(0,\xi) = x_0(\xi), \\ \dfrac{\partial x}{\partial t}(t,\xi) = x_1(\xi) \quad \text{in } \Omega. \end{cases}$$

Remark 8.1. Similar results can be obtained if one replaces in problem (8.2) the Dirichlet by the Neumann boundary condition.

$$\begin{cases} \dfrac{\partial^2 x}{\partial t^2}(t,\xi) = \Delta_\xi x(t,\xi) + \big(Bu(t,\cdot)\big)(\xi) \quad \text{in }]0,T]\times\Omega, \\ \dfrac{\partial x}{\partial \nu}(t,\xi) = 0 \quad \text{on }]0,T]\times\partial\Omega, \\ x(0,\xi) = x_0(\xi) \quad \text{in } \Omega, \\ \dfrac{\partial x}{\partial t}(0,\xi) = x_1(\xi) \quad \text{in } \Omega, \end{cases} \tag{8.9}$$

where ν is the outward normal to $\partial\Omega$. In this case we set

$$H = Y = H^1(\Omega) \oplus L^2(\Omega), \quad U = L^2(\Omega), \tag{8.10}$$

and define the scalar product in H by

$$\left(\begin{bmatrix} x^0 \\ x^1 \end{bmatrix}, \begin{bmatrix} z^0 \\ z^1 \end{bmatrix} \right) = \int_\Omega (\nabla_\xi x^0 \cdot \nabla_\xi z^0 + x^1 z^1)\, d\xi + \int_\Omega x^0 z^0\, d\xi. \tag{8.11}$$

Moreover we define a self-adjoint positive operator Λ_1 on $L^2(\Omega)$ by

$$\begin{cases} D(\Lambda_1) = \left\{ x \in H^2(\Omega) \colon \dfrac{\partial x}{\partial \nu} = 0 \quad \text{on } \partial\Omega \right\}, \\ \Lambda_1 x = -\Delta_\xi x, \end{cases} \tag{8.12}$$

and the linear operator A_1 on H:

$$\begin{cases} A_1 X = \begin{bmatrix} 0 & 1 \\ -\Lambda_1 & 0 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \end{bmatrix}, \quad \forall X \in D(A_1), \\ D(A_1) = \left\{ x \in H^2(\Omega) \colon \dfrac{\partial x}{\partial \nu} = 0 \quad \text{on } \partial\Omega \right\} \oplus H_0^1(\Omega). \end{cases} \tag{8.13}$$

Now all previous considerations can be easily generalized. □

8.3 Delay equations

We start with a simple example given in §2.10 (Part II, Chapter 1). A more elaborate treatment of differential delay systems with delays in the control and observation variables will be given after using the general state space frameworks developed in Chapter 4 of Part II.

Let D, $E \in \mathcal{L}(\mathbb{C}^n)$, $r > 0$, and consider the problem:

$$\begin{cases} z'(t) = Dz(t) + Ez(t-r) + u(t), \quad t \geq 0, \\ z(0) = h_0 \in \mathbb{C}^n, \\ z(\theta) = h_1(\theta), \quad a.e.\ \theta \in [-r, 0], \end{cases} \tag{8.14}$$

where $h_1 \in L^2(-r, 0; \mathbb{C}^n)$ and $u \in L^2(0, T; \mathbb{C}^n)$. We shall use here the notation of §2.10 (Chapter 1 in Part II). We set

$$H = L^2(-r, 0; \mathbb{C}^n), \quad U = \mathbb{C}^n, \quad \tilde{x}(t) = \big(z(t), z_t\big).$$

Then problem (8.14) is equivalent to

$$\tilde{x}' = A\tilde{x} + Bu, \quad \tilde{x}(0) = (h_0, h_1),$$

where A is defined by (2.104) in Chapter 1 of Part II and B is the linear operator

$$u \mapsto Bu = (u, 0) \colon \mathbb{C}^n \to H.$$

We remark that the adjoint operator $B^\star$ is given by

$$(x_0, x_1) \mapsto B^\star(x_0, x_1) = x_0 \colon H \to \mathbb{C}^n.$$

Consider now the problem: To minimize

$$J(u) = \int_0^T \{|z(t)|^2 + |u(t)|^2\}\, dt + |z(T)|^2$$

over all $u \in L^2(0, T; \mathbb{C}^n)$ subject to (8.14). $J(u)$ can be written as

$$J(u) = \int_0^T \{|Cx(t)|^2 + |u(t)|^2\}\, dt + \big(P_0 z(T), z(T)\big)_{\mathbb{C}^n \times L^2(-r,0;\mathbb{C}^n)},$$

where

$$C(x_0, x_1) = (x_0, 0) \quad \text{and} \quad P_0(x_0, x_1) = (x_0, 0).$$

The Riccati equation is

$$P' = A^\star P + PA - P^2 + C^\star C, \quad P(0) = P_0,$$

because, as easily seen, $BB^\star = \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right]$.

We can write P as

$$P = \begin{bmatrix} P_{01} & P_{01} \\ P_{10} & P_{11} \end{bmatrix},$$

where

$$\begin{aligned}
&P_{00} \in \mathcal{L}(\mathbb{C}^n), \quad P_{01} \in \mathcal{L}\big(\big(L^2(0,T;\mathbb{C}^n);\mathbb{C}^n\big)\big), \\
&P_{10} \in \mathcal{L}\big(\mathbb{C}^n; \big(L^2(0,T;\mathbb{C}^n)\big), P_{11} \in \mathcal{L}\big(L^2(0,T;\mathbb{C}^n)\big).
\end{aligned}$$

Then the optimal control is given by

$$u^\star(t) = -P_{00}(T-t)z^\star(t) - P_{01}(T-t)z^\star_t.$$

We now consider the more general situation. The abstract theory developed in Chapter 4 of Part II (Theorem 4.1) for unbounded observation operators applies to a class of delay systems with delayed observations where the control operator of the state equation is bounded. Following the notation and definitions of §5 (Chapter 4, Part II), a delay differential system with controls and observations is given by

$$\begin{cases} \dot{x}(t) = Lx_t + Bu_t, \quad t > 0, \\ (x(0), x_0, u_0) = (\varphi^0, \varphi^1, w) \in M^2 \times L^2(-h,0;\mathbb{R}^m), \end{cases} \tag{8.15}$$

$$y(t) = Cx_t, \quad t > 0. \tag{8.16}$$

The spaces of controls and observation are $U = \mathbb{R}^m$ and $Y = \mathbb{R}^k$, respectively. By introducing the *extended structural state* $\hat{x}(t)$ in $H = M^2 \times L^2(-h,0;Y)$ (cf. Part II, Chapter 4, §6.2, Theorem 6.1, equations (6.11) to (6.14)), system (8.15)–(8.16) can be transformed in the form

$$\begin{cases} \dfrac{d}{dt}\hat{\jmath}^*\hat{x}(t) = (\tilde{A}^\top)^*\hat{x}(t) + (\tilde{B}^\top)^*u(t), \quad t > 0, \\ \hat{x}(0) = \xi, \end{cases} \tag{8.17}$$

where $\tilde{B}^\top \colon D(\tilde{A}^\top) \to U$ and $\hat{C} \colon D\big((\tilde{A}^\top)^*\big) \to Y$ are continuous linear maps when the domains are endowed with their respective graph norm topologies. The operator $\tilde{B}^\top$ is bounded and continuous on H when it is of the form

$$\tilde{B}^\top(\psi^0, \psi^1, v) = B_0^\top \psi^0 + \int_{-h}^0 B_1^\top(\theta)\psi^1(\theta)\, d\theta \tag{8.18}$$

for some matrices B_0 and $B_1(\theta)$, $\theta \in I(-h,0)$, such that the elements of $B_1(\cdot)$ belong to $L^2(-h,0)$. Then

$$(\tilde{B}^\top)^* u = \big(B_0 u, B_1(\cdot)u, 0\big). \tag{8.19}$$

So we specialize to systems (8.15) of the form

$$\begin{cases} \dot{x}(t) = Lx_t + B_0 u(t) + \displaystyle\int_{-h}^{0} B_1(\theta)\psi^1(\theta)u(t+\theta)\,d\theta, & t > 0, \\ (x(0), x_0, u_0) = (\varphi^0, \varphi^1, w). \end{cases} \tag{8.20}$$

The operator $\hat{C}$ is of the general form (cf. Part II, Chapter 4, §6.2.2)

$$\xi = (\varphi_0, \bar{L}\varphi_1 + \zeta, \bar{C}\varphi_1 + \lambda) \mapsto \hat{C}\xi = C\varphi_1 + \lambda(0) \colon D\big((\tilde{A}^\top)^*\big) \to Y, \tag{8.21}$$

where $(\varphi, \zeta, \lambda)$ is a representation of the elements of $D\big((\tilde{A}^\top)^*\big)$.

So we only need to check part (ii) of condition (H)–(v)

$$\int_0^t |\hat{C}e^{s(\tilde{A}^\top)^*}\xi|_Y^2\,ds \leq K(t)|\xi|_H^2, \quad \forall \xi \in D\big((\tilde{A}^\top)^*\big).$$

In the notation of Part II, Chapter 4, $e^{t(\tilde{A}^\top)^*}$ is $(\tilde{S}^\top)^*(t)$ and

$$\begin{aligned} \hat{C}(\tilde{S}^\top)^*(t)\xi = C\hat{x}(t) &= Cx_t + [\tau(t)\lambda](0) \\ &= (Ce_+^0 x)(t) + (e_+^{-h}\xi^2)(-t), \end{aligned}$$

which is continuous as an element of $L^2(0,T;Y)$ with respect to ξ and $(e_+^{-h}\xi^2)(-t)$ is the shift of an L^2–function. This holds for each finite $T > 0$.

As a result we have the following proposition.

Proposition 8.1. *When the mapping B is of the form* (8.18)*, the Riccati equation*

$$\begin{cases} P' = \tilde{A}^\top P + P(\tilde{A}^\top)^* - P(\tilde{B}^\top)^*\tilde{B}^\top P + \hat{C}^*\hat{C}, \\ P(0) = P_0, \end{cases} \tag{8.22}$$

has a unique mild solution P in $C_s\big([0,\infty[; \Sigma^+(H)\big)$, where $H = M^2 \times L^2(-h,0;Y)$.

The Riccati equation (8.22) on $[0,T]$, $T > 0$, is associated with the minimization of the cost function

$$J(u) = \int_0^T [|Cx_t|^2 + |u(t)|^2]\,dt + Fx(T)\cdot x(T) \tag{8.23}$$

over all u in $L^2(0,T;Y)$, where

$$P_0(\xi^0, \xi^1, \xi^2) = (F\xi^0, 0, 0), \quad \forall \xi = (\xi^0, \xi^1, \xi^2).$$

The optimal control u^* can be expressed in feedback form

$$u^*(t) = -\tilde{B}^\top P(T-t)\hat{x}(t), \quad t > 0,$$

where for the initial conditions in (8.15)

$$\hat{x}(t) = \big(x(t), \bar{L}x_t + \bar{B}u_t^*, \bar{C}x_t\big).$$

As $P(t)$ can be decomposed into a 3×3 matrix of operators on the state space $H = \mathbb{R}^n \times L^2(-h,0,\mathbb{R}^n) \times L^2(-h,0,Y)$

$$P(t) = \begin{bmatrix} P_{00}(t) & P_{01}(t) & P_{02}(t) \\ P_{10}(t) & P_{11}(t) & P_{12}(t) \\ P_{20}(t) & P_{21}(t) & P_{22}(t) \end{bmatrix},$$

and $\tilde{B}^\top$ has the special form (8.18)

$$\begin{aligned} \tilde{B}^\top P(T-t)\hat{x}(t) = B_0^\top \{ P_{00}(T-t)x(t) + P_{01}(T-t)x(t)[\bar{L}x_t + \bar{B}u_t^*] \\ + P_{02}(T-t)\bar{C}x_t\} + \int_{-h}^{0} B_1^\top(\theta)\{P_{10}(T-t)x(t) \\ + P_{11}(T-t)x(t)[\bar{L}x_t + \bar{B}u_t^*] + P_{12}(T-t)\bar{C}x_t\}(\theta)\, d\theta. \end{aligned}$$

It is important to notice that the feedback is through the structural operators $\bar{L}$, $\bar{B}$, and $\bar{C}$ on x_t and u_t^*. This state captures the minimal information necessary for the feedback synthesis. Without this notion of state, determining directly expressions as above is almost hopeless in the general case. We leave it to the reader to specify $\bar{L}$, $\bar{B}$ and $\bar{C}$ in special cases!

The same problem can also be formulated by using the *extended state* $\tilde{x}(t)$ in $H = M^2 \times L^2(-h,0;U)$ (cf. Part II, Chapter 4, §§5.2 and 6.1, eqs. (6.7) to (6.10)) which verifies the state equation

$$\begin{cases} \dfrac{d}{dt}\tilde{\imath}^*\tilde{x}(t) = (\tilde{A}^*)^*\tilde{x}(t) + (\hat{B}^\top)^* u(t), \\ \tilde{x}(0) = (\varphi^0 \varphi^1, w), \end{cases} \tag{8.24}$$

with the observation equation

$$y(t) = \tilde{C}\tilde{x}(t) \tag{8.25}$$

(cf. Chapter 4, §6, equations (6.7) and (6.8)), where $\hat{B}^\top : D(\tilde{A}^*) \to U$ and $\tilde{C} \colon D(\tilde{A}) \to Y$ are continuous linear maps when the domains are endowed with their respective graph norm topologies. When the control operator is given as in (8.20), the operator $\hat{B}^\top$ is equal to

$$\hat{B}^\top(\xi) = B_0^\top \psi(0) + \int_{-h}^{0} B_1^\top(\theta)\psi(\theta)\, d\theta + \lambda(0), \tag{8.26}$$

where ψ, ζ, λ is a representation of an arbitrary element ξ of $D(\tilde{A}^*)$

$$\xi = \big(\psi(0), \bar{L}^\top\psi + \zeta, \bar{B}^\top\psi + \lambda\big).$$

The operator $\hat{B}^\top$ is unfortunately not bounded on H. As for $\tilde{C}$ it takes the simple form

$$\tilde{C}(\varphi(0), \varphi, w) = C\varphi, \tag{8.27}$$

and part (ii) of condition (H)–(v)

$$\int_0^t |\tilde{C} e^{s\tilde{A}} \xi|_Y^2 \, ds \le K(t) |\xi|_H^2, \quad \forall \xi \in D(\tilde{A}),$$

is verified. In the notation of Part II, Chapter 4, $e^{t\tilde{A}}$ is $\tilde{S}(t)$ and

$$\tilde{C}\tilde{S}(t)\xi = \tilde{C}(x(t), x_t, (e_-^0 w)_t) = (\mathcal{C}x)(t),$$

which is continuous as an element of $L^2(0,T;Y)$ with respect to ξ in $M^2 \times L^2(0,T;U)$. However $\hat{B}^\top$ is not bounded on H because it contains a delta function. So we cannot use the general theory of §4.

However we have seen in §6.3 of Chapter 4 in Part II that $\hat{x}$ and $\tilde{x}$ are intertwined via the structural operator

$$F = \begin{bmatrix} I & 0 & 0 \\ 0 & \bar{L} & \bar{B}_0 \\ 0 & \bar{C} & \bar{B}_1 \end{bmatrix} \tag{8.28}$$

(cf. (6.74) and (6.77) to (6.80) in Theorem 6.2). Then for the optimal control u^*

$$\hat{x}\big(t; F(\phi^0, \phi^1, w), u^*\big) = F\tilde{x}\big(t; (\phi^0, \phi^1, w), u^*\big).$$

So it is not too difficult to guess that for the formulation (8.23)–(8.24)–(8.25), there will be a Riccati equation and a solution $\Pi(t)$ related to $P(t)$ through the identity

$$\Pi(t) = F^\top P(t) F.$$

To see that, multiply the Riccati equation (8.22) by $F^\top$ on the left and F on the right and use the intertwining identities

$$\hat{C}F = \tilde{C}, \quad \hat{B}^\top F^\top = \tilde{B}^\top, \quad F^\top \tilde{A}^\top = \tilde{A}^* F^\top, \quad \tilde{A}^{\top *} F = F\tilde{A}$$

(cf. Part II, Chapter 4, (6.79), (6.80) and from (6.78)) to obtain

$$\begin{cases} \Pi' = \tilde{A}^* \Pi + \Pi \tilde{A} - \Pi (\hat{B}^\top)^* \hat{B}^\top \Pi + \tilde{C}^* \tilde{C}, \\ \Pi(0) = F^\top P_0 F. \end{cases} \tag{8.29}$$

It corresponds to the control of an infinite dimensional system with both unbounded control and observation operators. We know what the Riccati equation associated with the state $\tilde{x}(t)$ will look like, but it remains to give a precise meaning to such an equation and establish the properties of its solution. The necessary techniques that would allow us to do it are different than the ones that will be developed in the next two chapters.

The last equation is associated with the same problem (8.23) with

$$\Pi_0(\xi^0, \xi^1, \xi^2) = (F\xi^0, 0, 0), \quad \forall \xi = (\xi^0, \xi^1, \xi^2). \tag{8.30}$$

The optimal control u^* can be expressed in feedback form

$$u^*(t) = -\hat{B}^\top \Pi(T-t)\tilde{x}(t), \quad t > 0,$$

where for the initial conditions in (8.15)

$$\tilde{x}(t) = \big(x(t), x_t, u_t\big).$$

However in view of (8.26) the interpretation of $\hat{B}^\top \Pi(T-t)$ is not clear unless some additional properties of $\Pi(t)$ are obtained. In that respect the *extended state* is simpler to define but because it does not incorporate the delay structure of the problem, the feedback law is difficult to explicit. For a simple example of this property, the reader is referred to M. C. Delfour, E. B. Lee, and A. Manitius [1] and M. C. Delfour [15]. For a more detailed bibliography on the control of delay systems, the reader is referred to the references in Chapter 4 of Part II. Although the general case with delays in control and observation has a unique solution, it was not possible here to detail all special cases and the history of the problems

8.4 Evolution equations in noncylindrical domains

Consider the state equation

$$\begin{cases} \dfrac{\partial x}{\partial t}(t,\xi) = \Delta_\xi x(t,\xi) + u(t,\xi), \quad t \in \,]0,T], \qquad \xi \in \Omega_t, \\ x(t,\xi) = 0, \quad t \in \,]0,T], \qquad \xi \in \partial\Omega_t, \\ x(0,\xi) = x_0(\xi), \quad \xi \in \Omega_0, \end{cases} \tag{8.31}$$

where for any $t \in [0,T]$, Ω_t is a bounded set in $\mathbb{R}^n$ with a regular boundary $\partial\Omega_t$.

We want to minimize the cost

$$J(u) = \int_0^T dt \int_{\Omega_t} \{|x(t,\xi)|^2 + |u(t,\xi)|^2\}\, d\xi + \int_{\Omega_T} |x(T,\xi)|^2\, d\xi \tag{8.32}$$

over all controls $u \in L^2([0,T] \times \Omega_t)$ subject to state equation (8.31). By $L^2([0,T] \times \Omega_t)$ we mean the set of all functions u

$$(t,\xi) \mapsto u(t,\xi) \colon \{(t,\xi) \colon t \in [0,T], \xi \in \Omega_t\} \to \mathbb{R},$$

such that $t \mapsto \int_{\Omega_t} |u(t,\xi)|^2\, d\xi$ is measurable and

$$\int_0^T dt \int_{\Omega_t} |u(t,\xi)|^2\, d\xi < +\infty.$$

We first reduce problem (8.31) to an evolution equation in a cylindrical domain, following G. Da Prato and J. P. Zolésio [1]. To this end we introduce, as in J. P. Zolésio [1], a suitable change of variables:

$$x(t,\xi) = z\big(t, T_t(\xi)\big), \quad u(t,\xi) = v\big(t, T_t(\xi)\big),$$

where T_t is a regular mapping such that

$$T_t(\Omega_0) = \Omega_t, \quad t \in [0,T].$$

Then problem (8.31) reduces to

$$\begin{cases} z_t = J_t^{-1} \operatorname{div} \left[J_t\big((DT_t)^{-1}\big)^\star (DT_t)^{-1} \nabla z\right] \\ \qquad\qquad + \big((DT_t)^{-1}\big)^\star \nabla z \cdot \dfrac{d}{dt} T_t + v, \qquad t \in [0,T],\ \xi \in \Omega_0, \\ z(t,\xi) = 0, \quad t \in [0,T], \qquad \xi \in \partial\Omega_0, \\ z(0,\xi) = x_0(\xi), \quad \xi \in \Omega_0, \end{cases} \tag{8.33}$$

where J_t is the determinant of the Jacobian matrix DT_t of T_t.

We set $H = U = L^2(\Omega_0)$, $z(t,\cdot) = z(t)$, $v(t,\cdot) = v(t)$ and write (8.33) as an abstract evolution equation in H

$$z'(t) = A(t)z(t) + v(t), \quad z(0) = x_0,$$

where $A(t)$ is defined by

$$\begin{cases} A(t)w = J_t^{-1} \operatorname{div} \left[J_t\big((DT_t)^{-1}\big)^\star (DT_t)^{-1} \nabla w\right] + \big((DT_t)^{-1}\big)^\star \nabla w \cdot \dfrac{d}{dt} T_t, \\ D\big(A(t)\big) = H^2(\Omega_0) \cap H_0^1(\Omega_0). \end{cases}$$

$A(t)$ is the realization of an elliptic operator under Dirichlet boundary conditions, and consequently it is the infinitesimal generator of an analytic semigroup in H. Moreover we have $A \in C^1\big([0,T]; \mathcal{L}(D;H)\big)$ so that by H. Tanabe [1], assumptions (7.8) hold. The cost function (8.32) becomes

$$J(v) = \int_0^T dt \int_{\Omega_0} J_t\big(T_t^{-1}(x)\big)\big(|z(t,\xi)|^2 + |v(t,\xi)|^2\big)\, d\xi + \int_{\Omega_0} |z(T,\xi)|^2\, d\xi,$$

and we can study the problem as in §7.2.

Remark 8.2. It is also possible to study problem (8.31) without changing variables. By proceeding as in P. Cannarsa, G. Da Prato, and J. P. Zolésio [1], we assume that there exists a bounded open set D that contains all sets Ω_t, $t \in [0,T]$, and introduce unbounded operators $\{A(t)\}$, $t \in [0,T]$ with domain depending on time, but in the fixed space $H = L^2(D)$

$$\begin{cases} D\big(A(t)\big) = \{y \in L^2(D) \colon y\big|_{\Omega_t} \in H^2(\Omega_t), y\big|_{D\backslash\Omega_t} \in H^2(D\backslash\Omega_t), \\ \qquad\qquad\qquad\qquad\qquad\qquad y = 0 \quad \text{on } \partial D, \quad y = 0 \quad \text{on } \partial\Omega_t\} \\ A(t)u = \Delta u, \forall u \in D\big(A(t)\big). \end{cases}$$

Then we write problem (8.31) in the abstract form (7.7) and prove that (7.8) hold. □

Remark 8.3. An optimal control problem for the wave equation in moving domain has been studied with similar techniques in G. Da Prato and J. P. Zolésio [2]. □

2

Unbounded Control Operators: Parabolic Equations With Control on the Boundary

1 Introduction

As in the previous chapter, we shall denote by H, U, and Y the Hilbert spaces of states, controls, and observations, respectively. We consider a dynamical system, whose state $x(t)$ is subject to the following equation:

$$x'(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \in H,$$

where $u \in L^2(0, T; U)$ and $A\colon D(A) \subset H \to H$ generates an analytic semigroup in H. However, in the current case, the linear operator B is not supposed to be bounded from U into H. This situation has been discussed at length in Chapters 1 and 2 (Part II). However some key constructions will be repeated here as needed. In that case many possibilities could be considered. However, in practice, it will be natural to consider situations where B maps U into the dual space $\big(D(A^\star)\big)'$ of $D(A^\star)$. This will be apparent in the following Examples 1.1 and 1.2. Equivalently, B is supposed to be of the form $B = (\lambda_0 - A)D$, where $D \in \mathcal{L}(U; H)$ and λ_0 is an element in $\rho(A)$. Under these assumptions we write the state equation as

$$x'(t) = Ax(t) + (\lambda_0 - A)Du(t), \quad x(0) = x_0,$$

or in the mild form as

$$x(t) = e^{tA}x_0 + (\lambda_0 - A)\int_0^t e^{(t-s)A} Du(s)\, ds. \tag{1.1}$$

Remark that formula (1.1) is meaningful and $x \in L^2(0, T; H)$; see Chapters 1 to 3 of Part II. This formula will represent the state of our system.

The above representation of the operator B is also discussed in Chapter 2 of Part II. The differential equation for the state still makes sense for a control operator $B \in \mathcal{L}(U; D(A^*)')$, and a representation formula can be obtained for $x(t)$ (cf. Theorem 1.1 in Chapter 3 of Part II). These results can be sharpened when A is the generator of an analytic semigroup (cf. Theorems 2.2 and 2.3 in Chapter 3 of Part II) and the natural operator $B \in \mathcal{L}(U, D(A^*)')$ is $D = [\lambda_0 - A]^{-1}B$ through formula (1.1).

We shall assume that

$$(\mathcal{HP})_1 \begin{cases} \text{(i)} & A \text{ generates an analytic semigroup } e^{tA} \text{ of type } \omega_0 \\ & \text{and} \quad \lambda_0 \text{ is a real number in } \rho(A) \text{ such that } \omega_0 < \lambda_0, \\ \text{(ii)} & \exists \alpha \in \,]0,1[\, \text{ such that } D \in \mathcal{L}\big(U, D(A^\alpha)\big), \end{cases}$$

where $D(A^\alpha) = D([\lambda_0 - A]^\alpha)$ is the domain of the fractional power $[\lambda_0 - A]^\alpha$ of the operator $\lambda_0 - A$, as defined in §5 of Chapter 1 in Part II. For more details on the above techniques, the reader is referred to Chapter 1 (Part II).

Assumption $(\mathcal{HP})_1$ is equivalent to say that

$$B = [\lambda_0 - A]D = [\lambda_0 - A]^{1-\alpha}[\lambda_0 - A]^\alpha D \tag{1.2}$$

belongs to $\mathcal{L}\big(U, D\big((A^*)^{1-\alpha}\big)'\big)$. In the variational case

$$D\big((A^*)^{1-\alpha}\big) = [D(A^*), H]_\alpha \implies B \in \mathcal{L}(U, [D(A^*), H]'_\alpha)$$

and the regularity results of Theorem 2.2 (Part II, Chapter 3) would apply:

$$\begin{aligned} &\forall u \in L^2(0,T;U) \text{ and } \forall x(0) \in \big[[D(A), H]_\alpha, [D(A^*), H]'_{1-\alpha}\big]_{1/2} \\ &\qquad \implies x \in W([0,T; [D(A), H]_\alpha, [D(A^*), H]'_{1-\alpha}]). \end{aligned} \tag{1.3}$$

Remark 1.1. Assume that $(\mathcal{HP})_1$ is verified, and let x be defined by (1.1). By the Closed Graph Theorem, $[\lambda_0 - A]^\alpha D$ is a bounded operator. Moreover, because

$$(\lambda_0 - A)e^{(t-s)A}Du(s) = [\lambda_0 - A]^{1-\alpha}e^{(t-s)A}[\lambda_0 - A]^\alpha Du(s),$$

there exists a constant $k_\alpha > 0$ such that

$$|(\lambda_0 - A)e^{(t-s)A}Du(s)| \le k_\alpha (t-s)^{\alpha-1}|u(s)|, \quad s \in [0,t].$$

Thus the state $x(\cdot)$ can also be written as

$$x(t) = e^{tA}x_0 + \int_0^t (\lambda_0 - A)e^{(t-s)A}Du(s)\,ds,$$

and the following estimate holds:

$$|x(t)| \le |e^{tA}x_0| + k_\alpha \int_0^t (t-s)^{\alpha-1}|u(s)|\,ds.$$

If $\alpha > \frac{1}{2}$, from the Hölder estimate, it follows that $x \in L^\infty(0,T;H)$; this implies that $x \in C([0,T];H)$ by a standard density argument. In fact setting

$$x_k(t) = e^{tA}x_0 + \int_0^t (\lambda_0 - A)e^{(t-s)A}kR(k,A)Du(s)\,ds,$$

we clearly have $x_k \in C([0,T];H)$ and it is easy to check that $x_k(t) \to x(t)$ uniformly in t.

If $\alpha \le 1/2$, then $x \notin C([0,T];H)$ in general; however, arguing as before, we can easily check that $[\lambda_0 - A]^{-\beta}x \in C([0,T];H)$ for all $\beta > 1/2 - \alpha$. In the sequel, we shall discuss separately the cases $\alpha > 1/2$ and $\alpha \le 1/2$; as we shall see, the second case is much more difficult to deal with. □

Consider the following *optimal control problem*: To minimize the cost function

$$J(u) = \int_0^T \{|Cx(t)|^2 + |u(t)|^2\}\,dt + \big(P_0x(T), x(T)\big), \tag{1.4}$$

over all $u \in L^2(0,T;U)$, subject to the differential equation constraint (1.1). We assume, besides $(\mathcal{HP})_1$,

$$(\mathcal{HP})_2 \begin{cases} \text{(i)} & \text{that } C \in \mathcal{L}(H;Y), P_0 \in \Sigma^+(H),\\ \text{(ii)} & \text{and that if } \alpha \le 1/2 \text{ there exists } \beta \in [1/2-\alpha, (1-\alpha)/2]\\ & \text{such that the following holds}\\ & x \in D([\lambda_0 - A]^\beta) \implies P_0[\lambda_0 - A]^\beta x \in D([\lambda_0 - A^\star]^\beta),\\ & \text{and} \quad [\lambda_0 - A^\star]^\beta P_0 [\lambda_0 - A]^\beta \text{ is bounded.} \end{cases}$$

If assumption $(\mathcal{HP})_2$ holds, we shall use the notation

$$P_\beta = \text{closure } [\lambda_0 - A^\star]^\beta P_0 [\lambda_0 - A]^\beta,$$

so that $P_\beta \in \Sigma^+(H)$ and

$$P_0 = [\lambda_0 - A^\star]^{-\beta} P_\beta [\lambda_0 - A]^{-\beta}.$$

We remark that if $\alpha > 1/2$, then we do not require additional conditions on P_0, except that $P_0 \in \Sigma^+(H)$, whereas if $\alpha \le 1/2$, we need the assumption $(\mathcal{H}P)_2$–(ii) to make sense of the term

$$\big(P_0x(T), x(T)\big) = \big(P_\beta[\lambda_0 - A]^{-\beta}x(T), [\lambda_0 - A]^{-\beta}x(T)\big),$$

in the cost functional. In fact, if $\beta > 1/2 - \alpha$, then $[\lambda_0 - A]^{-\beta}x \in C([0,T];H)$ (see Remark 2.1), and the trace value $[\lambda_0 - A]^{-\beta}x(T)$ is meaningful.

By Assumption $(\mathcal{HP})$ we mean the set of assumptions $(\mathcal{HP})_1$ and $(\mathcal{HP})_2$. We shall also say that (A, D, C, P_0) fulfill $(\mathcal{HP})$. The definitions of *optimal control, optimal state*, and *optimal pair* are as given in Chapter 1.

We shall study the optimization problem (1.1)–(1.4), by using Dynamic Programming and by proceeding into the usual two steps.

First step.

We consider the Riccati equation that formally reads as follows:

$$\begin{cases} P' = A^\star P + PA - P(\lambda_0 - A)D\big((\lambda_0 - A)D\big)^\star P + C^\star C, \\ P(0) = P_0. \end{cases} \tag{1.5}$$

We set

$$E = [\lambda_0 - A]^\alpha D, \quad V = [\lambda_0 - A^\star]^{1-\alpha} P.$$

Then $E \in \mathcal{L}(U; H)$ by virtue of assumption $(\mathcal{HP})_1$–(ii) . Thus the meaningless term $P(\lambda_0 - A)D\big((\lambda_0 - A)D\big)^\star P$ can be written as $V^\star EE^\star V$ and the Riccati equation rewritten as follows:

$$\begin{cases} P' = A^\star P + PA - V^\star EE^\star V + C^\star C, \\ P(0) = P_0, \end{cases} \tag{1.6}$$

or in the integral form

$$\begin{aligned} P(t)x = e^{tA^\star} P_0 e^{tA} x &+ \int_0^t e^{(t-s)A^\star} C^\star C e^{(t-s)A} x\, ds \\ &- \int_0^t e^{(t-s)A^\star} V^\star(s) EE^\star V(s) e^{(t-s)A} x\, ds, \quad x \in H. \end{aligned} \tag{1.7}$$

We look for a solution $P(\cdot)$ of (1.7) such that $V(t) \in \mathcal{L}(H)$ for all $t > 0$.

Second step.

We shall show that the optimal control $u^\star$ is related to the optimal state $y^\star$ by the feedback formula

$$u^\star(t) = -D^\star(\lambda_0 - A^\star)P(T-t)x^\star(t) = -E^\star V(T-t)x^\star(t). \tag{1.8}$$

Moreover $x^\star(t)$ is the solution of the *closed loop equation*

$$x(t) = e^{tA}x_0 + \int_0^t [\lambda_0 - A]^{1-\alpha} e^{(t-s)A} EE^\star V(T-s)x(s)\, ds, \tag{1.9}$$

which can be solved by a fixed point argument.

We now give some bibliographical comments. Early papers treating boundary control problems are due to A. V. Balakrishnan [4, 5] and D. C. Washburn [1], who developed further an old idea in H. O. Fattorini [3]. Simplifications and refinements through domains of fractional powers were introduced in R. Triggiani [3]. Another approach is due to A. Chojnowska-Michalik [1] and J. Zabczyk [4] (see also Chapter 3 in Part II for an alternate interpretation of their constructions). The Riccati equation (1.7) was first studied by A. V. Balakrishnan [5] when $P_0 = 0$ and by I. Lasiecka and R. TRIGGIANI [3] when P_0 is a positive multiple of the identity. They were able to build an explicit solution of the Riccati equation by generalizing the representation formula (2.22) of Chapter 1; we remark, however, that this method does not give uniqueness. A direct approach to solve (1.7) was first used by F. FLANDOLI [1] under the assumption that $V_0 = [\lambda_0 - A^\star]^{1-\alpha} P_0$ is bounded; he showed that, in this case, there exists a unique solution $P(t)$ such that $[\lambda_0 - A^\star]^{1-\alpha} P(t)$ is bounded for any t. Moreover the closed loop (1.9) can be solved directly and the feedback formula (1.8) holds true. The same direct approach was used by G. DA PRATO and A. ICHIKAWA [1], which proved that (1.7) has a unique solution for any $P_0 \in \Sigma^+(H)$ if $\alpha > 1/2$, whereas if $\alpha \leq 1/2$ they need in addition that the linear operator $[\lambda_0 - A^\star]^\gamma P_0$ is bounded for some $\gamma > 1 - 2\alpha$. For some other results in this direction, see M. C. DELFOUR and M. SORINE [1] and M. SORINE [2] using J. L. Lions' direct method and A. J. PRITCHARD and D. SALAMON [1].

In this chapter we assume $(\mathcal{HP})$ following the recent paper of F. FLANDOLI [7], and show the existence and uniqueness of the Riccati equation (see §2.2 below). Then, we solve the control problem in §2.3.

We shall not study the Riccati equation (1.7) when $\alpha \leq 1/2$ and P_0 only belongs to $\Sigma^+(H)$. In this case the cost functional (1.4) is not defined on the whole $L^2(0, T; U)$, (see Remark 1.1), and the Dynamic Programming approach, based on the direct solution of (1.7), does not work. Concerning the existence (but not the uniqueness) of a solution to (1.7) we mention the following results:

(i) If P_0 is a positive multiple of the identity, there exists a solution of (1.7) and one can also solve the corresponding control problem (1.1)–(1.4) (see I. LASIECKA and R. TRIGGIANI [3], and F. FLANDOLI [5]).
(ii) There exists $P_0 \in \Sigma^+(H)$ such that (1.7) does not have a solution (see F. FLANDOLI [5]).
(iii) A sharp sufficient condition, which is "almost" necessary, for the existence of a solution of (1.7) is given in I. LASIECKA and R. TRIGGIANI [14].

It is also possible to study generalizations as in §2.7, §3, and §4 of Chapter 1 in Part I to

(i) nonhomogeneous state equation,
(ii) tracking problem,
(iii) time-dependent state equation and cost function.

The generalizations (i) and (ii) are straightforward, and they will be left to the reader. For the point (iii), which is much more technical, see the papers by P. ACQUISTAPACE, F. FLANDOLI, and B. TERRENI [1]; P. ACQUISTAPACE and B. TERRENI [3]; and P. ACQUISTAPACE [1].

We end this section by giving two examples concerning the *heat equation*. For a result concerning the *strongly damped wave equation*, see F. BUCCI [1].

Several other examples can be found in the lecture notes by I. LASIECKA and R. TRIGGIANI [11].

Another approach to boundary control problems using a quadratic cost function not necessarily coercive can be found in L. PANDOLFI [1].

Example 1.1 *(Dirichlet boundary condition).* Let Ω be a bounded open set in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$. Consider the optimal control problem: To minimize

$$J(u) = \int_0^T \int_\Omega |x(t,\xi)|^2 \, d\xi + \int_0^T \int_{\partial\Omega} |u(t,\xi)|^2 \, d\sigma + \int_\Omega |\Gamma x(T,\cdot)(\xi)|^2 \, d\xi,$$

over all controls $u \in L^2([0,T] \times \partial\Omega)$ subject to the state constraints

$$\begin{cases} \dfrac{\partial x}{\partial t}(t,\xi) = \Delta_\xi x(t,\xi), \quad (t,\xi) \in \,]0,T[\times \Omega, \\ x(0,\xi) = x_0(\xi), \quad \xi \in \Omega, \\ x(t,\xi) = u(t,\xi), \quad (t,\xi) \in \,]0,T[\times \partial\Omega, \end{cases} \tag{1.10}$$

where $x_0 \in L^2(\Omega)$,

$$\Delta_\xi = \sum_{i=1}^N \frac{\partial^2 x}{\partial \xi_i^2}$$

is the Laplace operator and $\Gamma \in \mathcal{L}\big(L^2(\partial\Omega)\big)$.

We set $H = Y = L^2(\Omega)$, $U = L^2(\partial\Omega)$, and introduce the Dirichlet realization A of the Laplace operator

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega), \quad Ax = \Delta_\xi x, \qquad \forall x \in D(A).$$

As is well known (see for instance S. AGMON [2]), A is a strictly negative self-adjoint operator in $L^2(\Omega)$, so that assumption $(\mathcal{HP})_1$-(i) holds with $\lambda_0 = 0$. Moreover we have

$$D\big((-A)^\alpha\big) = \begin{cases} H^{2\alpha}(\Omega), & \text{if } \alpha \in \,]0,1/4[, \\ \{u \in H^{2\alpha}(\Omega) \colon u = 0 \quad \text{on } \partial\Omega\}, & \text{if } \alpha \in \,]1/4,1[. \end{cases}$$

Let us introduce now the *Dirichlet mapping*

$$v \mapsto Dv = w \colon L^2(\partial\Omega) \to L^2(\Omega),$$

where

$$\Delta_\xi w = 0 \quad \text{in } \Omega, \qquad w(\xi) = v(\xi) \quad \text{in } \partial\Omega. \tag{1.11}$$

As proved in J. L. LIONS and E. MAGENES [1], we have $D \in \mathcal{L}\big(L^2(\partial\Omega); H^{1/2}(\Omega)\big)$; therefore $D \in \mathcal{L}\big(L^2(\partial\Omega); D\big((-A)^\alpha\big)\big)$ for any $\alpha \in \,]0,1/4[$. Thus Assumptions $(\mathcal{HP})$ are verified provided that $\alpha \in \,]0,1/4[$, $\beta \in \,]1/2-\alpha, 1/2[$, and $\Gamma \in \mathcal{L}\big(L^2(\partial\Omega); D\big((-A)^\beta\big)\big)$, as noted in R. TRIGGIANI [1]. We now show, following A. V. Balakrishnan [4], that problem (1.10) can be set in the form (1.1).

First we assume that $u \in W^{1,2}\big(0,T;L^2(\partial\Omega)\big)$ and introduce a new variable by setting

$$y(t,\xi) = x(t,\xi) - \big(Du(t,\cdot)\big)(\xi).$$

We obtain

$$\begin{cases} \dfrac{\partial y}{\partial t}(t,\xi) = \Delta_\xi y(t,\xi) - \dfrac{\partial Du}{\partial t}(t,\xi), \quad (t,\xi) \in \,]0,T[\, \times \Omega, \\ y(0,\xi) = x_0(\xi) - \big(Du(0,\cdot)\big)(\xi), \quad \xi \in \Omega, \\ y(t,\xi) = 0, \quad (t,\xi) \in \,]0,T[\, \times \partial\Omega. \end{cases}$$

Then, we set $y(t) = y(t,\cdot)$, $u(t) = u(t,\cdot)$, and write the above problem as

$$y(t) = e^{tA}\big(x_0 - Du(0)\big) - \int_0^t e^{(t-s)A} Du'(s)\, ds.$$

By performing an integration by parts, it is not difficult to check that x fulfills (1.1). Finally, the hypothesis $u \in W^{1,2}\big([0,T];L^2(\partial\Omega)\big)$ can be easily removed by regularization. $\square$

Example 1.2 *(Neumann boundary condition).* Let Ω be a bounded open set in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$. Consider the optimal control problem: To minimize

$$J(u) = \int_0^T \int_\Omega |x(t,\xi)|^2\, d\xi + \int_0^T \int_{\partial\Omega} |u(t,\xi)|^2\, d\sigma + \int_\Omega |\Gamma x(T,\cdot)(\xi)|^2\, d\xi,$$

over all controls $u \in L^2([0,T] \times \partial\Omega)$, subject to the state constraints

$$\begin{cases} \dfrac{\partial x}{\partial t}(t,\xi) = \Delta_\xi x(t,\xi), \quad (t,\xi) \in \,]0,T[\, \times \Omega, \\ x(0,\xi) = x_0(\xi), \quad \xi \in \Omega, \\ \dfrac{\partial x}{\partial \nu}(t,\xi) = u(t,\xi), \quad (t,\xi) \in \,]0,T[\, \times \partial\Omega, \end{cases} \tag{1.12}$$

where $x_0 \in L^2(\Omega)$, $\Gamma \in \mathcal{L}\big(L^2(\Omega)\big)$, and ν represents the outward normal to $\partial\Omega$.

We set $H = Y = L^2(\Omega)$, $U = L^2(\partial\Omega)$, and introduce the Neumann realization A of the Laplace operator

$$\begin{cases} D(A) = \left\{ x \in H^2(\Omega) \colon \dfrac{\partial x}{\partial \nu} = 0 \right\}, \\ Ax = \Delta_\xi x, \quad \forall x \in D(A). \end{cases}$$

As is well known (see for instance S. Agmon [2]), A is a nonpositive self-adjoint operator in $L^2(\Omega)$, so that Assumption $(\mathcal{HP})_1$–(i) holds with any $\lambda_0 > 0$; we choose $\lambda_0 = 1$. We have

$$D([\lambda_0 - A]^\alpha) = \begin{cases} H^{2\alpha}(\Omega), & \text{if } \alpha \in]0, 3/4[, \\ \left\{ x \in H^{2\alpha}(\Omega) \colon \dfrac{\partial x}{\partial \nu} = 0 \right\}, & \text{if } \alpha \in]3/4, 1[. \end{cases}$$

Introduce the *Neumann mapping*

$$v \mapsto Nv = w \colon L^2(\partial\Omega) \to L^2(\Omega),$$

where

$$\varDelta_\xi w - w = 0 \quad \text{in } \Omega, \qquad \frac{\partial w}{\partial \nu}(\xi) = v(\xi) \quad \text{in } \partial\Omega.$$

As proved in J. L. Lions and E. Magenes [1], we have

$$N \in \mathcal{L}(L^2\big(\partial\Omega); H^{3/2}(\Omega)\big);$$

thus $N \in \mathcal{L}\big(L^2(\partial\Omega); D([\lambda_0 - A]^\alpha)\big)$ for any $\alpha \in]0, 3/4[$; and Assumptions $(\mathcal{HP})$ are verified. Moreover, by proceeding as in Example 1.1, we see that problem (1.12) is equivalent to problem (1.1) with $D = N$ and $\lambda_0 = 1$. □

Remark 1.2. One obtains similar results for an elliptic operator with general boundary conditions (see I. Lasiecka [1]). □

2 Riccati equation

2.1 Notation

We use here the notation introduced in §2.1 of Chapter 1, and we assume that Assumptions $(\mathcal{HP})$ are verified (that is both $(\mathcal{HP})_1$ and $(\mathcal{HP})_2$) hold. If $P \in C_s\big([a, b]; \varSigma^+(H)\big)$, we set

$$V_P(t) = [\lambda_0 - A^\star]^{1-\alpha} P(t), \quad t \in [a, b].$$

When no confusion may arise, we will drop index P, writing $V_P = V$.

We are going to study the Riccati equation

$$\begin{cases} P' = A^\star P + PA - V^\star E E^\star V + C^\star C, \\ P(0) = P_0, \end{cases} \tag{2.1}$$

where

$$E = [\lambda_0 - A]^\alpha D.$$

We also consider the integral form

$$P(t)x = e^{tA^\star} P_0 e^{tA} x + \int_0^t e^{(t-s)A^\star} C^\star C e^{(t-s)A} x \, ds - \int_0^t e^{(t-s)A^\star} V^\star(s) E E^\star V(s) e^{(t-s)A} x \, ds, \tag{2.2}$$

where $x \in H$.

It is useful to introduce the approximating problem for $k > \lambda_0$:

$$\begin{cases} P_k' = A^\star P_k + P_k A - V_k^\star E_k E_k^\star V_k + C^\star C, \\ P_k(0) = P_0, \end{cases} \tag{2.3}$$

where

$$E_k = kR(k,A)E, \quad V_k = V_{P_k} = [\lambda_0 - A^\star]^{1-\alpha} P_k$$

and $R(k,A) = (k-A)^{-1}$. Problem (2.3) can also be written in mild form as follows:

$$\begin{aligned} P_k(t)x = e^{tA^\star} P_0 e^{tA} x &+ \int_0^t e^{(t-s)A^\star} C^\star C e^{(t-s)A} x\, ds \\ &- \int_0^t e^{(t-s)A^\star} V_k^\star(s) E_k E_k^\star V_k(s) e^{(t-s)A} x\, ds, \end{aligned} \tag{2.4}$$

where $x \in H$.

Remark 2.1. The operators (A, B_k, C, P_0),

$$B_k = [\lambda_0 - A^\star]^{1-\alpha} kR(k,A) [\lambda_0 - A^\star]^\alpha D,$$

fulfill assumptions $(\mathcal{H})$ of Chapter 1. So, by Theorem 2.1 of Chapter 1, equation (2.4) has a unique mild solution $P_k \in C_s\big([0,\infty[; \Sigma^+(H)\big)$. □

Finally, in order to make simpler some estimates, we introduce a constant L such that

$$\begin{cases} \text{(i)} & L \geq 1, \quad \|(\lambda_0 - A)^{-1}\| \leq L, \|(\lambda_0 - A)^{-\gamma}\| \leq L, \\ & \forall \gamma \in]0,1[, \|E_k E_k^\star\| \leq L, \quad \forall k \in \mathbb{N}. \\ \text{(ii)} & \|e^{t(A-\lambda_0)}\| \leq L, t\|(A-\lambda_0)e^{t(A-\lambda_0)}\| \\ & \qquad \leq L, t^\gamma \|(\lambda_0 - A)^{-\gamma} e^{t(A-\lambda_0)}\| \leq L, \\ & \forall \gamma \in]0,1[, \quad \forall t > 0. \end{cases}$$

2.2 Riccati equation for $\alpha > 1/2$

Assume that Assumptions $(\mathcal{HP})$ are verified with $\alpha > \frac{1}{2}$, and set, for the sake of simplicity, $\lambda_0 = 0$. We shall look for a solution of the Riccati equation (2.1) in the following functional space. For any interval $[a,b]$ we denote by $C_{s,\alpha}\big([a,b]; \Sigma(H)\big)$ the set of all $P \in C_s\big([a,b]; \Sigma(H)\big)$ such that

$$\begin{cases} \text{(i)} & P(t)x \in D\big((-A^\star)^{1-\alpha}\big), \quad \forall x \in H, \ \forall t \in]a,b], \\ \text{(ii)} & (-A^\star)^{1-\alpha} P \in C\big(]a,b]; \mathcal{L}(H)\big), \\ \text{(iii)} & \lim_{t \to a} (t-a)^{1-\alpha} (-A^\star)^{1-\alpha} P(t)x = 0, \quad \forall x \in H. \end{cases} \tag{2.5}$$

Define

$$\|P\|_1 = \sup_{t\in]a,b]} \|(t-a)^{1-\alpha}(-A^\star)^{1-\alpha}P(t)\|.$$

$C_{s,\alpha}\big([a,b];\Sigma(H)\big)$ endowed with the norm

$$\|P\|_\alpha = \|P\| + \|P\|_1$$

is a Banach space. We set

$$C_{s,\alpha}\big([a,b];\Sigma^+(H)\big) = \big\{P \in C_{s,\alpha}\big([a,b];\Sigma(H)\big) \colon P(t) \geq 0, \forall t \in]a,b]\big\}$$

and denote by $C_{s,\alpha}\big([0,\infty];\Sigma(H)\big)$ the set of all $P \in C_s\big([0,\infty];\Sigma(H)\big)$ such that $P \in C_{s,\alpha}\big([0,T];\Sigma(H)\big)$ for all $T > 0$.

Definition 2.1. A *mild solution* of problem (2.1) in the interval $[0,T]$ is an operator valued function $P \in C_{s,\alpha}\big([0,T];\Sigma(H)\big)$ that verifies the integral equation

$$\begin{aligned} P(t)x = e^{tA^\star}P_0e^{tA}x + \int_0^t e^{(t-s)A^\star}C^\star Ce^{(t-s)A}x\,ds \\ - \int_0^t e^{(t-s)A^\star}V^\star(s)EE^\star V(s)e^{(t-s)A}x\,ds, \end{aligned} \tag{2.6}$$

where $x \in H$. A *weak solution* of problem (2.1) in $[0,T]$ is an operator valued function $P \in C_{s,\alpha}\big([0,T];\Sigma(H)\big)$ such that, for any $x,y \in D(A)$, $\big(P(\cdot)x,y\big)$ is differentiable in $[0,T]$ and verifies

$$\begin{cases} \dfrac{d}{dt}\big(P(t)x,y\big) = \big(P(t)x,Ay\big) + \big(P(t)Ax,y\big) + (Cx,Cy) \\ \qquad\qquad\qquad\qquad\qquad -\big(E^\star V(t)x,E^\star V(t)y\big), \\ P(0) = P_0. \end{cases} \tag{2.7}$$

□

Proposition 2.1. *Let $P \in C_{s,\alpha}\big([0,T];\Sigma(H)\big)$. Then P is a mild solution of problem* (2.1) *if and only if P is a weak solution.*

Proof. It is completely similar to the proof of Proposition 2.3 of Chapter 1, and so, it will be omitted. □

We need, as in Chapter 1, a generalization of the Contraction Mapping Principle. Let $T > 0$, and let $\{\gamma_k\}$ be a sequence of mappings from $C_{s,\alpha}\big([0,T];\Sigma(H)\big)$ into itself, such that

$$\|\gamma_k(P) - \gamma_k(Q)\|_\alpha \leq a\|P - Q\|_\alpha$$

for some $a \in [0,1[$ and all $P,Q \in C_{s,\alpha}\big([0,T];\Sigma(H)\big)$. Moreover assume that there exists a mapping γ from $C_{s,\alpha}\big([0,T];\Sigma(H)\big)$ into itself such that $\forall P \in C_{s,\alpha}\big([0,T];\Sigma(H)\big)$, $\forall m \in \mathbb{N}$, $\forall x \in H$

$$\begin{cases} \lim\limits_{k\to\infty} \gamma_k^m(P)x = \gamma^m(P)x, \quad \text{in } C([0,T];H), \\ \lim\limits_{k\to\infty} t^{1-\alpha}(-A^\star)^{1-\alpha}\gamma_k^m(P)x = t^{1-\alpha}(-A^\star)^{1-\alpha}\gamma^m(P)x, \\ \qquad\qquad\qquad\qquad \text{in } C([0,T];H), \end{cases} \tag{2.8}$$

Then, by the Contraction Mapping Principle, there exist unique P_k and P in $C_{s,\alpha}\big([0,T];\Sigma(H)\big)$ such that

$$\gamma_k(P_k) = P_k, \quad \gamma(P) = P.$$

The following result can be proved as Lemma 2.1 of Chapter 1.

Lemma 2.1. *Under the previous assumptions on the sequence of mappings* $\{\gamma_k\}$

$$\begin{cases} \lim\limits_{k\to\infty} P_k(\cdot)x = P(\cdot)x \quad \textit{in } C([0,T];H), \\ \lim\limits_{k\to\infty} t^{1-\alpha}(-A^\star)^{1-\alpha}P_k(\cdot)x = t^{1-\alpha}(-A^\star)^{1-\alpha}P(\cdot)x, \\ \qquad\qquad\qquad \textit{in } C([0,T];H), \qquad \forall x\in H,\ T>0. \end{cases} \tag{2.9}$$

The main result of this section is as follows.

Theorem 2.1. *Assume that Assumptions* $(\mathcal{HP})$ *are verified with* $\alpha > 1/2$. *Then, problem* (2.1) *has a unique mild solution* $P \in C_{s,\alpha}\big([0,T];\Sigma(H)\big)$. *Moreover, the solution* P_k *of problem* (2.4) *also belongs to* $C_{s,\alpha}\big([0,T];\Sigma(H)\big)$ *for all* $k\in\mathbb{N}$ *and* (2.9) *holds for all* $x \in H$ *and* $T > 0$.

Proof. We write problem (2.1) as

$$P = \gamma(P) = F + H - \varphi(P),$$

where

$$F(t) = e^{tA^\star}P_0e^{tA},$$

$$H(t) = \int_0^t e^{(t-s)A^\star}C^\star Ce^{(t-s)A}ds,$$

$$\big(\varphi(P)\big)(t) = \int_0^t e^{(t-s)A^\star}V^\star(s)EE^\star V(s)e^{(t-s)A}\,ds.$$

Analogously, we write problem (2.4) as

$$P_k = \gamma_k(P_k) = F + H - \varphi_k(P_k),$$

where

$$\big(\varphi_k(P)\big)(t)x = \int_0^t e^{(t-s)A^\star}V_k^\star(s)E_kE_k^\star V_k(s)e^{(t-s)A}x\,ds, \quad x\in H$$

and $V_k(s) = V_{P_k}(s)$.

In the sequel of the proof $T > 0$, $\alpha \in\,]0,1[$ and $L \geq 1$ are fixed. We proceed in several steps.

Step 1.

$F, H \in C_{s,\alpha}\big([0,T];\Sigma(H)\big)$ and

$$\|F\|_\alpha \le 2L^2\|P_0\|, \quad \|H\|_\alpha \le \left(1+\frac{1}{\alpha}\right)L^2\|C\|^2T. \tag{2.10}$$

In fact, we clearly have $\|F\| \le L^2\|P_0\|$. Moreover

$$\|t^{1-\alpha}(-A^\star)^{1-\alpha}F(t)\| = \|t^{1-\alpha}(-A^\star)^{1-\alpha}e^{tA^\star}P_0e^{tA}\| \le L^2\|P_0\|$$

so that $\|F\|_1 \le L^2\|P_0\|$ and the first inequality in (2.10) follows. Concerning the second, it suffices to remark that $\|H(t)\| \le L^2\|C\|^2T$ and, for all $x \in H$,

$$\begin{aligned}|t^{1-\alpha}(-A^\star)^{1-\alpha}H(t)| &= \left|t^{1-\alpha}(-A^\star)^{1-\alpha}\int_0^t e^{(t-s)A^\star}C^\star Ce^{(t-s)A}\,ds\right| \\ &\le L^2\|C\|^2t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}\,ds\,|x| \le \frac{1}{\alpha}L^2\|C\|^2T.\end{aligned}$$

Step 2.

φ and φ_k map $C_{s,\alpha}([0,T];\Sigma(H))$ into itself. Moreover there exists a constant $C_{1,T}>0$ such that, for all $k \in \mathbb{N}$ and $P \in C_{s,\alpha}([0,T];\Sigma(H))$

$$\|\varphi(P)\|_\alpha + \|\varphi_k(P)\|_\alpha \le C_{1,T}\|P\|_\alpha^2, \tag{2.11}$$

where

$$C_{1,T} = 2\left[\frac{1}{2\alpha-1} + \beta(2\alpha-1,\alpha)\right]T^{2\alpha-1}L^3,$$

and β represents the Euler function,

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt.$$

We only estimate $\|\varphi(P)\|_\alpha$. The estimate for $\|\varphi_k(P)\|_\alpha$ is similar. Let $P \in C_{s,\alpha}\big([0,T];\Sigma(H)\big)$ and $x \in H$; then $\varphi(P) \in C_s\big([0,T];\Sigma(H)\big)$ and we have

$$\big|\big(\varphi(P)\big)(t)\big| \le L^3\int_0^t s^{2\alpha-2}\,ds\,\|P\|_1^2 \le \frac{1}{2\alpha-1}L^3\|P\|_1^2T^{2\alpha-1},$$

which implies

$$\|\varphi(P)\| \le \frac{1}{2\alpha-1}L^3\|P\|_1^2T^{2\alpha-1}. \tag{2.12}$$

By arguing as in Lemma 3.1 of Chapter 1, it is not difficult to show that $\varphi(P) \in C_{s,\alpha}\big([0,T];\Sigma(H)\big)$. Moreover we have

$$
\begin{aligned}
|t^{1-\alpha}(-A^\star)^{1-\alpha}\big(\varphi(P)\big)(t)| &\le L^3 t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}s^{2\alpha-2}\,ds\,\|P\|_1^2\\
&\le L^3\|P\|_1^2 T^{2\alpha-1}\int_0^1 (1-\sigma)^{\alpha-1}\sigma^{2\alpha-2}\,d\sigma\,|x|\\
&= L^3\|P\|_1^2 T^{2\alpha-1}\beta(2\alpha-1,\alpha),
\end{aligned}
$$

which yields

$$
\|\varphi(P)\|_1 \le T^{2\alpha-1}L^3\beta(2\alpha-1,\alpha)\|P\|_1^2. \tag{2.13}
$$

Now, (2.11) follows from (2.12) and (2.13).

Step 3.

There exists a constant $C_{2,T}>0$ such that, for all $P,Q\in C_{s,\alpha}\big([0,T];\Sigma(H)\big)$, we have

$$
\|\varphi(P)-\varphi(Q)\|_\alpha+\|\varphi_k(P)-\varphi_k(Q)\|_\alpha \le C_{2,T}(\|P\|_\alpha+\|Q\|_\alpha)\|P-Q\|_\alpha, \tag{2.14}
$$

where

$$
C_{2,T} = 2\left[\frac{1}{2\alpha-1}+\beta(2\alpha-1,\alpha)\right]T^{2\alpha-1}L^3.
$$

In fact, let $P,Q\in C_{s,\alpha}\big([0,T];\Sigma(H)\big)$, $V=V_P$, $Z=V_Q$, and $x\in H$; we have

$$
\begin{aligned}
&\big|t^{1-\alpha}\big(\varphi(P)\big)(t)x - t^{1-\alpha}\big(\varphi(Q)\big)(t)\big|\\
&\qquad\le \left|t^{1-\alpha}\int_0^t e^{(t-s)A^\star}\big[\big(V(s)-Z(s)\big)^\star EE^\star V(s)\big]e^{(t-s)A}\,ds\right|\\
&\qquad\quad + \left|t^{1-\alpha}\int_0^t e^{(t-s)A^\star}\big[Z^\star(s)EE^\star\big(V(s)-Z(s)\big)\big]e^{(t-s)A}\,ds\right|,
\end{aligned}
$$

and the conclusion follows by arguing as in the second step.

We denote by $B(r,t)$ the ball

$$
B(r,t) = \big\{P\in C_{s,\alpha}\big([0,t];\Sigma(H)\big)\colon \|P\|_\alpha\le r\big\}.
$$

Step 4.

For all $p>0$, there exist $\tau=\tau(p)>0$ and $r=r(p)>0$, such that

$$
\|P_0\|\le p \implies \gamma\big(B(r,\tau)\big)\cup\gamma_k\big(B(r,\tau)\big)\subset B(r,\tau) \tag{2.15}
$$

and

$$
\|\gamma(P)-\gamma(Q)\|_\alpha\le \tfrac12\|P-Q\|_\alpha,\quad \|\gamma_k(P)-\gamma_k(Q)\|_\alpha\le\tfrac12\|P-Q\|_\alpha, \tag{2.16}
$$

for all P, $Q\in\gamma\big(B(r,\tau)\big)$. This follows easily from the inequalities (2.10), (2.11), and (2.14).

Step 5.

Existence and positivity. Choose $p > \|P_0\|$, and let $\tau = \tau(p)$. By (2.15), (2.16), and the Contraction Mapping Principle, it follows that there exist unique P and P_k in $C_{s,\alpha}\big([0,T];\Sigma(H)\big)$ such that

$$\gamma(P) = P, \quad \gamma_k(P_k) = P_k.$$

So problems (2.1) and (2.3) have unique solutions P and P_k, respectively, in $[0,\tau]$. Moreover, by Lemma 2.1 it follows that (2.9) holds for all $x \in H$ and $T > 0$. Since, by Theorem 2.1 of Chapter 1, $P_k(t) \geq 0$ for all $t \in [0,\tau]$, we finally find

$$P(t) \geq 0, \quad \text{for all } t \in [0,\tau].$$

Step 6.

Global existence and uniqueness. We first remark that we have shown existence in the interval $[0,\tau]$, which only depends on the norm of P_0. Thus, in order to prove global existence we only have to estimate the norm of $P(t)$, $t \in [0,\tau[$. Let $P \in C_{s,\alpha}\big([0,\tau];\Sigma(H)\big)$ be the solution of (2.1) in $[0,\tau]$. As $P(t) \geq 0$, we have

$$\begin{aligned}\big(P(t)x,x\big) &\leq (P_0 e^{tA}x, e^{tA}x) + \int_0^t |Ce^{(t-s)A}x|^2\, ds \\ &\leq L^2\{\|P_0\| + T\|C\|^2\}|x|^2,\end{aligned}$$

which implies that

$$\|P(t)\| \leq \|L\|^2\{\|P_0\| + T\|C\|^2\}, \quad \text{for all } t \in [0,\tau].$$

Set now $p_1 = L^2\{\|P_0\| + T\|C\|^2\}$ and $\tau_1 = \tau(p_1)$. Let P be the solution to the Riccati equation (2.1) in $[0,\tau_1]$. Proceeding as above, we can solve the Riccati equation

$$\begin{aligned}Q(t) = e^{(t-\tau_1+\varepsilon)A^\star} P(\tau_1-\varepsilon) e^{(t-\tau_1+\varepsilon)A} &+ \int_{\tau_1-\varepsilon}^t e^{(t-s)A^\star} C^\star C e^{(t-s)A}\, ds \\ &- \int_{\tau_1-\varepsilon}^t e^{(t-s)A^\star} V^\star_{Q(s)} E^\star E V_{Q(s)} e^{(t-s)A}\, ds,\end{aligned}$$

in the interval $[\tau_1-\varepsilon, 2\tau_1-\varepsilon]$ (here ε is a small positive number to be chosen later), and we find a solution $Q \in C_{s,\alpha}\big([\tau_1-\varepsilon, 2\tau_1-\varepsilon];\Sigma(H)\big)$ such that

$$\|Q(t)\| \leq L^2\{\|P_0\| + T\|C\|^2\}, \quad \forall t \in [\tau_1-\varepsilon, 2\tau_1-\varepsilon].$$

Now, it is easy to check that setting

$$\bar{P}(t) = \begin{cases} P(t), & \text{if } t \in [0, \tau_1], \\ Q(t), & \text{if } t \in [\tau_1, 2\tau_1 - \varepsilon]; \end{cases}$$

then $\bar{P}$ is the unique solution of (2.1) in $[0, 2\tau_1 - \varepsilon]$; moreover $\bar{P}(t) \geq 0$ in $[0, 2\tau_1 - \varepsilon]$ and

$$\|\bar{P}(t)\| \leq L^2\{\|P_0\| + T\|C\|^2\}, \quad t \in [0, 2\tau_1 - \varepsilon].$$

Thus we can repeat this argument successively in the intervals $[2\tau_1 - 2\varepsilon, 3\tau_1 - 2\varepsilon]$, $[3\tau_1 - 3\varepsilon, 4\tau_1 - 3\varepsilon]$, and so on. So we get the conclusions in N steps, provided N and ε are chosen such that $N\tau_1 > T$ and

$$\varepsilon \leq \frac{N\tau_1 - T}{N - 1}.$$

The proof is complete. □

We now prove the *continuous dependence* with respect to P_0 and C of the solutions of (2.1). Consider a sequence of Riccati equations:

$$\begin{cases} \dfrac{dP^h}{dt} = A^\star P^h + P^h A - (V^h)^\star E E^\star V^h + (C^h)^\star C^h, \\ P^h(0) = P_0^h, \end{cases} \tag{2.17}$$

where $V^h = V^{P^h}$, $h \in \mathbb{N}$. Assume that

$$\begin{cases} \text{(i)} & \text{For any } h \in \mathbb{N}, (A, D, C^h, P_0^h) \text{ fulfill } (\mathcal{HP}) \text{ with } \alpha > \frac{1}{2}, \\ \text{(ii)} & \lim\limits_{h\to\infty} (C^h)^\star C^h x = C^\star C x, \quad \text{for all } x \in H, \\ \text{(iii)} & \lim\limits_{h\to\infty} P_0^h x = P_0 x \quad \text{for all } x \in H, \end{cases} \tag{2.18}$$

and prove the following result.

Proposition 2.2. *Assume that Assumptions* $(\mathcal{HP})$ *and* (2.18) *are verified with* $\alpha > \frac{1}{2}$. *Let* P *and* P^h *be the mild solutions to* (2.1) *and* (2.17), *respectively. Then, for any* $x \in H$ *and any* $T > 0$, *we have*

$$\begin{cases} \lim\limits_{h\to\infty} P^h(\cdot)x = P(\cdot)x, \quad \text{in } C([0,T]; H), \\ \lim\limits_{h\to\infty} t^{1-\alpha} V^h(\cdot)x = t^{1-\alpha} V(\cdot)x \quad \text{in } C([0,T]; H). \end{cases} \tag{2.19}$$

Proof. Fix $T > 0$. By the Uniform Boundedness Theorem, there exist positive numbers q and c such that

$$\|P_0^h\| \leq q, \quad \|(C^h)^\star C^h\| \leq c, \quad \forall h \in \mathbb{N}.$$

Set $p = L^2(q + Tc)$, $\tau = \tau(p)$; then, arguing as in the proof of Theorem 2.2 of Chapter 1, we first show that (2.19) holds in $[0, \tau]$ and then we prove that this argument can be iterated in the interval $[\tau, 2\tau]$ and so on. □

We finally show a monotonicity property of the solutions of the Riccati equation

Proposition 2.3. *Assume that* $(A, D, C_i, P_{i,0})$ *fulfill* $(\mathcal{HP})$ *for* $i = 1, 2$*, with* $\alpha > 1/2$ *and, in addition, that*

$$P_{1,0} \le P_{2,0}, \quad C_1^\star C_1 \le C_2^\star C_2. \tag{2.20}$$

Let P_i*,* $i = 1, 2$ *be the mild solution of the Riccati equations*

$$\begin{cases} \dfrac{d}{dt} P_i = A^\star P_i + P_i A - V_i^\star E E^\star V_i + (C_i)^\star C_i, \\ P_i(0) = P_{i,0}, \end{cases} \tag{2.21}$$

where $V_i = V_{P_i}$*. Then we have*

$$P_1(t) \le P_2(t), \quad t \ge 0.$$

Proof. For any $k \in \mathbb{N}$, $i = 1, 2$, let $P_{i,k}$ be the mild solution to the Riccati equation

$$\begin{cases} \dfrac{d}{dt} P_{i,k} = A^\star P_{i,k} + P_{i,k} A - V_{i,k}^\star E E^\star V_{i,k} + (C_i)^\star C_i, \\ P_{i,k}(0) = P_{i,0}, \end{cases}$$

where $V_{i,k} = (-A_k^\star)^{1-\alpha} P_{i,k}$. Then, by Proposition 2.2 of Chapter 1,

$$P_{1,k}(t) \le P_{2,k}(t), \quad t \ge 0, k \in \mathbb{N}.$$

The conclusion now follows from Theorem 2.1. □

2.3 Solution of the Riccati equation for $\alpha \le 1/2$

We assume here that Assumptions $(\mathcal{HP})$ are verified with $\alpha \le 1/2$ and some $\beta \in]1/2 - \alpha, (1-\alpha)/2[$. We set again $\lambda_0 = 0$ and denote by $P_\beta \in \Sigma^+(H)$ the closure of $(-A^\star)^\beta P_0 (-A)^\beta$, so that

$$P_0 = (-A^\star)^{-\beta} P_\beta (-A)^{-\beta}.$$

One can see very easily that the proof of Theorem 2.1 cannot be repeated, because $2 - 2\alpha \ge 1$ and some integrals would be divergent. Thus we introduce, following F. Flandoli [7], a more complicated functional space and some additional notations. If $[a, b]$ is any interval, we set

$$\mu_{[a,b]}(t) = (t-a)^{1-\alpha-\beta}, \quad t \in [a, b],$$

and

$$\nu_{[a,b]}(t,s) = \frac{(s-a)^{1-\alpha-\beta}(t-s)^{\beta}}{(t-a)^{\beta}}, \quad a \le s \le t \le b.$$

If $P \in C_s\big([a,b];\Sigma^+(H)\big)$, we set

$$V_P(t) = (-A^\star)^{1-\alpha}P(t), \quad t \in [a,b],$$
$$W_P(t,s) = (-A^\star)^{1-\alpha}P(s)(-A)^{\beta}e^{(t-s)A}, \quad a \le s \le t \le b.$$

When no confusion may arise, we will drop indexes $[a,b]$ and P, in the formulas above.

For any interval $[a,b]$ we denote by $C_{s,\alpha,\beta}\big([a,b];\Sigma(H)\big)$ the set of all $P \in C_s\big([a,b];\Sigma(H)\big)$ such that (2.5)–(i)–(ii) hold and moreover

$$\begin{cases} \text{(i)} & P(t)x \in D\big((-A^\star)^{1-\alpha}\big), \quad \forall x \in H,\ \forall t \in \,]a,b], \\ \text{(ii)} & V_P \in C\big(]a,b];\mathcal{L}(H)\big), \\ \text{(iii)} & \lim\limits_{t\to a} \mu_{[a,b]}(t)V_P(t)x = 0, \quad \forall x \in H, \\ \text{(iv)} & \lim\limits_{(t,s)\to(a,a)} \nu_{[a,b]}(t,s)W_P(t,s)x = 0, \quad \forall x \in H. \end{cases} \tag{2.22}$$

Define

$$\|P\|_2 = \sup_{t\in]a,b]} \|\mu_{[a,b]}(t)V_P(t)\|$$

and

$$\|P\|_3 = \sup_{a\le s<t\le b} \|\nu_{[a,b]}(t,s)W_P(t,s)\|.$$

$C_{s,\alpha,\beta}\big([a,b];\Sigma(H)\big)$, endowed with the norm

$$\|P\|_{\alpha,\beta} = \|P\| + \|P\|_2 + \|P\|_3,$$

is a Banach space. We set

$$C_{s,\alpha,\beta}\big([a,b];\Sigma^+(H)\big) = \big\{P \in C_{s,\alpha,\beta}\big([a,b];\Sigma(H)\big) \colon P(t) \ge 0, \forall t \in \,]a,b]\big\},$$

and let $C_{s,\alpha,\beta}\big([0,\infty[;\Sigma(H)\big)$ be the set of all $P \in C_s\big([0,\infty[;\Sigma(H)\big)$ such that $P \in C_{s,\alpha,\beta}\big([0,T];\Sigma(H)\big)$ for all $T > 0$.

Definition 2.2. A *mild solution* of problem (2.1) in the interval $[0,T]$ is an operator valued function $P \in C_{s,\alpha,\beta}\big([0,T];\Sigma(H)\big)$, which verifies the integral equation (2.6). A *weak solution* of problem (2.1) in $[0,T]$ is an operator valued function $P \in C_{s,\alpha,\beta}\big([0,T];\Sigma(H)\big)$ such that, for any $x,y \in D(A)$, $\big(P(\cdot)x,y\big)$ is differentiable in $[0,T]$ and verifies (2.7). □

Proposition 2.4. *Let $P \in C_{s,\alpha,\beta}\big([0,T];\Sigma(H)\big)$. Then P is a mild solution of problem* (2.1) *if and only if P is a weak solution.*

Proof. Completely similar to the proof of Proposition 2.3 of Chapter 1. □

We shall need, as in §2, a generalization of the Contraction Mapping Principle. Let $T > 0$, and let $\{\gamma_k\}$ be a sequence of mappings from $C_{s,\alpha,\beta}\big([0,T];\Sigma(H)\big)$ into itself, such that

$$\|\gamma_k(P) - \gamma_k(Q)\|_\alpha \leq a\|P - Q\|_\alpha,$$

for some $a \in [0,1[$ and all $P, Q \in C_{s,\alpha,\beta}\big([0,T];\Sigma(H)\big)$. Assume moreover that there exists a mapping γ from $C_{s,\alpha,\beta}\big([0,T];\Sigma(H)\big)$ into itself such that for all $m \in \mathbb{N}$ and all $x \in H$:

$$\begin{cases} \lim\limits_{k\to\infty} \gamma_k^m(P)x = \gamma^m(P)x \quad \text{in } C([0,T];H), \\ \lim\limits_{k\to\infty} \mu(\cdot)V_{\gamma_k^m(P)}x = \mu(\cdot)V_{\gamma^m(P)}x \quad \text{in } C([0,T];H), \\ \lim\limits_{k\to\infty} \nu(\cdot,\cdot)W_{\gamma_k^m(P)}x = \nu(\cdot,\cdot)W_{\gamma^m(P)}x \quad \text{in } C(\varDelta_T;H), \end{cases}$$

where $\varDelta_T = \{(t,s)\colon 0 \leq s \leq t \leq T\}$, $\forall P \in C_{s,\alpha,\beta}\big([0,T];\Sigma(H)\big)$. Then, by the Contraction Mapping Principle, there exist unique P_k and P in $C_{s,\alpha,\beta}\big([0,T];\Sigma(H)\big)$ such that

$$\gamma_k(P_k) = P_k, \quad \gamma(P) = P.$$

The following result can be proved as Lemma 2.1 of Chapter 1.

Lemma 2.2. *Under the previous assumptions on the sequence of mappings* $\{\gamma_k\}$*, for all* $x \in H$

$$\begin{cases} \lim\limits_{k\to\infty} P_k(\cdot)x = P(\cdot)x \quad \textit{in } C([0,T];H), \\ \lim\limits_{k\to\infty} \mu(\cdot)V_{P_k}(\cdot)x = \mu(\cdot)V_P(\cdot)x \quad \textit{in } C([0,T];H), \\ \lim\limits_{k\to\infty} \nu(\cdot,\cdot)W_{P_k}(\cdot,\cdot)x = \nu(\cdot,\cdot)W_P(\cdot,\cdot)x \quad \textit{in } C(\varDelta_T;H). \end{cases} \tag{2.23}$$

Prior to state the main result of this section, we need another lemma.

Lemma 2.3. *If* $P \in C_{s,\alpha,\beta}\big([0,T];\Sigma(H)\big)$*, we have*

$$\|(-A^\star)^{1-\alpha}P(s)(-A)^{1-\alpha}e^{(t-s)A}\| \leq 2^{1-\alpha}L\frac{t^\beta}{s^{1-\alpha-\beta}(t-s)^{(1-\alpha)}}\|P\|_3.$$

Proof. Let $t \in]0,T]$, $s \in [0,t[$. We have

$$\begin{aligned} &(-A^\star)^{1-\alpha}P(s)(-A)^{1-\alpha}e^{(t-s)A} \\ &\qquad = (-A^\star)^{1-\alpha}P(s)(-A)^\beta e^{(t+s/2-s)A}(-A)^{1-\alpha-\beta}e^{(t+s/2-s)A}. \end{aligned}$$

It follows that

$$\begin{aligned} &\|(-A^\star)^{1-\alpha}P(s)(-A)^{1-\alpha}e^{(t-s)A}\| \\ &\qquad \leq L(t+s)^\beta s^{\alpha+\beta-1}(t-s)^{-\beta}\left(\frac{t-s}{2}\right)^{\alpha+\beta-1}\|P\|_3, \end{aligned}$$

which implies the conclusion. □

The main result of this section as follows.

Theorem 2.2. *Assume that Assumptions* $(\mathcal{HP})$ *are verified for* $\alpha \le 1/2$ *and some* $\beta \in\,]1/2-\alpha, 1-\alpha/2[$. *Then the Riccati equation* (2.1) *has a unique mild solution* $P \in C_{s,\alpha,\beta}\big([0,\infty[;\Sigma^+(H)\big)$. *Moreover, the solution* P_k *of problem* (2.4) *also belongs to* $C_{s,\alpha,\beta}\big([0,\infty[;\Sigma(H)\big)$ *for all* $k \in \mathbb{N}$ *and* (2.23) *holds* $\forall x \in H$.

Proof. We write problems (2.1) and (2.4) as

$$P = \gamma(P) = F + H - \varphi(P), \quad P_k = \gamma_k(P_k) = F + H - \varphi_k(P_k), \qquad k \in \mathbb{N},$$

where F, H, φ, and φ_k are defined as in the proof of Theorem 2.1. We fix $T > 0$ and proceed in several steps.

Step 1.

$F \in C_{s,\alpha,\beta}\big([0,T];\Sigma(H)\big)$ and

$$\|F\|_{\alpha,\beta} \le L^2\|P_0\| + 2L^3\|P_\beta\|. \tag{2.24}$$

We have indeed

$$\|t^{1-\alpha-\beta}(-A^\star)^{1-\alpha}F(t)\| = \|t^{1-\alpha-\beta}(-A^\star)^{1-\alpha-\beta}e^{tA^\star}P_\beta(-A)^{-\beta}e^{tA}\| \le L^3\|P_\beta\|,$$

which implies

$$\|F\|_2 \le L^3\|P_\beta\|. \tag{2.25}$$

Moreover, if $t \in\,]0,T]$ and $s \in\,]0,t]$, we have

$$\begin{aligned}\|\nu(t,s)(-A^\star)^{1-\alpha}F(s)(-A)^\beta e^{(t-s)A}\| &= \|\nu(t,s)(-A^\star)^{1-\alpha-\beta}e^{sA^\star}P_\beta e^{tA}\| \\ &\le L^2[(t-s)/t]^\beta\|P_\beta\| \le L^3\|P_\beta\|\end{aligned}$$

because $(t-s)/t \le 1$. It follows that

$$\|F\|_3 \le L^3\|P_\beta\|. \tag{2.26}$$

Now (2.24) follows from (2.25) and (2.26) and the estimate $\|F\| \le L^2\|P_0\|$.

Step 2.

$H \in C_{s,\alpha,\beta}\big([0,T];\Sigma(H)\big)$ and

$$\begin{aligned}\|H\|_{\alpha,\beta} \le L^2\|C\|^2T + \frac{1}{\alpha}L^2\|C\|^2T^{1-\beta}& \\ +L^2\|C\|^2T^{1-2\beta+2\varepsilon}\int_0^{+\infty}\rho^{\varepsilon-1}(1+\rho)^\beta\,d\rho,&\end{aligned} \tag{2.27}$$

where $\varepsilon = \frac{1}{2}\min\{\alpha,\beta\}$.

Let $x \in H$; then

$$|t^{1-\alpha-\beta}(-A^\star)^{1-\alpha}H(t)x| = \left|t^{1-\alpha-\beta}(-A^\star)^{1-\alpha}\int_0^t e^{(t-s)A^\star}C^\star Ce^{(t-s)A}x\,ds\right|$$
$$\leq L^2\|C\|^2 t^{1-\alpha-\beta}\int_0^t (t-s)^{\alpha-1}\,ds\,|x| \leq \frac{1}{\alpha}L^2\|C\|^2T^{1-\beta}|x|.$$

It follows that

$$\|H\|_2 \leq \frac{1}{\alpha}L^2\|C\|^2T^{1-\beta}. \tag{2.28}$$

Moreover

$$\begin{aligned}
&|\nu(t,s)(-A^\star)^{1-\alpha}H(s)(-A)^\beta e^{(t-s)A}x|\\
&\qquad = \left|\nu(t,s)(-A^\star)^{1-\alpha}\int_0^s e^{(s-\sigma)A^\star}C^\star C(-A)^\beta e^{(t-\sigma)A}\,d\sigma\,x\right|\\
&\qquad \leq L^2\|C\|^2\nu(t,s)\int_0^s (s-\sigma)^{\alpha-1}(t-\sigma)^{-\beta}\,d\sigma\,|x|\\
&\qquad = L^2\|C\|^2\nu(t,s)\int_0^{s/(t-s)}(\rho)^{\alpha-1}(1+\rho)^{-\beta}\,d\rho\,|x|.
\end{aligned}$$

As $\rho^{\alpha-\varepsilon} \leq [s/(t-s)]^{\alpha-\varepsilon}$, we have

$$\begin{aligned}
&|\nu(t,s)(-A^\star)^{1-\alpha}H(s)(-A)^\beta e^{(t-s)A}x|\\
&\qquad \leq L^2\|C\|^2\frac{(t-s)^\varepsilon s^{1-\beta-\varepsilon}}{t^\beta}\int_0^{s/(t-s)}\frac{d\rho}{\rho^{1+\varepsilon}(1+\rho)^\beta}|x|\\
&\qquad \leq L^2\|C\|^2T^{1-2\beta+2\varepsilon}\int_0^{+\infty}\frac{d\rho}{\rho^{1+\varepsilon}(1+\rho)^\beta}|x|,
\end{aligned}$$

which implies

$$\|H\|_3 \leq L^2\|C\|^2T^{1-2\beta}\int_0^{+\infty}\frac{d\rho}{\rho^{1+\varepsilon}(1+\rho)^\beta}. \tag{2.29}$$

As $\|H\| \leq L^2\|C\|^2T$, the conclusion follows from (2.28) and (2.29).

Step 3.

φ and φ_k map $C_{s,\alpha,\beta}\big([0,T];\Sigma(H)\big)$ into itself. Moreover there exists two constants $C_1 > 0$ and $C_2 > 0$ such that for all $P \in C_{s,\alpha,\beta}\big([0,T];\Sigma(H)\big)$

$$\begin{aligned}
&\|\varphi(P)\|_{\alpha,\beta} + \|\varphi(P_k)\|_{\alpha,\beta}\\
&\qquad \leq T^{2\alpha+2\beta-1}C_1\|P\|^2_{\alpha,\beta} + C_2(\|P\|_2\|P\|_3 + \|P\|_3^2).
\end{aligned} \tag{2.30}$$

Let $P \in C_{s,\alpha,\beta}\big([0,T];\Sigma(H)\big)$; we have

$$\|\varphi(P)\| \le L^3\|P\|_2^2 \int_0^t (t-s)^{2\alpha+2\beta-2}\, ds = \frac{1}{2\alpha+2\beta-1} L^3\|P\|_2^2 T^{2\alpha+2\beta-1}, \quad (2.31)$$

because $2\alpha+2\beta-1>0$. Moreover for all $x\in H$ we have

$$\begin{aligned}|t^{1-\alpha-\beta}(\varphi(P))(t)x| &= \left|t^{1-\alpha-\beta}\int_0^t e^{(t-s)A^\star} V_P^\star(s)EE^\star V_P(s)e^{(t-s)A}x\, ds\right| \\ &= \left|t^{1-\alpha-\beta}\int_0^t [(-A^\star)^{1-\alpha}P(s)(-A)^{1-\alpha}e^{(t-s)A}]^\star\right. \\ &\qquad\qquad \left. EE^\star[(-A^\star)^{1-\alpha}P(s)]e^{(t-s)A}x\, ds\right|.\end{aligned}$$

By using Lemma 2.3, we obtain

$$\begin{aligned}|t^{1-\alpha-\beta}\big(\varphi(P)\big)(t)x| &\le 2^{1-\alpha}L^3t^{1-\alpha}\int_0^t s^{2\alpha+2\beta-2}(t-s)^{\alpha-1}\, ds\|P\|_2\|P\|_3|x| \\ &= 2^{1-\alpha}L^3t^{2\alpha+2\beta-1}\int_0^1 \sigma^{2\alpha+2\beta-2}(1-\sigma)^{\alpha-1}d\sigma\|P\|_2\|P\|_3|x|.\end{aligned}$$

Hence

$$\|\varphi(P)\|_2 \le 2^{1-\alpha}L^3T^{2\alpha+2\beta-1}\int_0^1 \sigma^{2\alpha+2\beta-2}(1-\sigma)^{\alpha-1}\, d\sigma\, \|P\|_2\|P\|_3. \quad (2.32)$$

Finally, using once again Lemma 2.3, we find

$$\begin{aligned}&\left|\nu(t,s)(-A^\star)^{1-\alpha}\big(\varphi(P)\big)(s)(-A)^\beta e^{(t-s)A}x\right| \\ &\quad = \left|\nu(t,s)\int_0^s [(-A^\star)^{1-\alpha}P(\sigma)(-A)^{1-\alpha}e^{(s-\sigma)A}]^\star\right. \\ &\qquad\qquad \left. EE^\star(-A^\star)^{1-\alpha}P(\sigma)(-A)^\beta e^{(t-\sigma)A}]x\, d\sigma\right| \\ &\quad \le 2^{1-\alpha}L^3s^{1-\alpha}(t-s)^\beta\int_0^s \sigma^{2\alpha+2\beta-2}(t-\sigma)^{-\beta}(s-\sigma)^{1-\alpha}\, d\sigma\, \|P\|_3^2|x|.\end{aligned}$$

As $(t-s)^\beta \le (t-\sigma)^\beta$, we obtain

$$\begin{aligned}&\left|\nu(t,s)(-A^\star)^{1-\alpha}\big(\gamma(P)\big)(s)(-A)^\beta e^{(t-s)A}x\right| \\ &\qquad \le 2^{1-\alpha}L^3s^{1-\alpha}\int_0^s \sigma^{2\alpha+2\beta-2}(s-\sigma)^{1-\alpha}\, d\sigma\, \|P\|_3^2|x| \\ &\qquad \le 2^{1-\alpha}L^3s^{2\alpha+2\beta-1}\int_0^1 \rho^{2\alpha+2\beta-2}(1-\rho)^{\alpha-1}\, d\rho\, \|P\|_3^2|x|\end{aligned}$$

so that

$$\|\gamma(P)\|_3 \le 2^{1-\alpha}L^3s^{2\alpha+2\beta-1}\int_0^1 \rho^{2\alpha+2\beta-2}(1-\rho)^{1-\alpha}d\rho\, \|P\|_3^2 \quad (2.33)$$

and (2.30) follows from (2.31), (2.32), and (2.33).

Step 4.

There exists a constant $C_3 > 0$ such that for all $P, Q \in C_{s,\alpha,\beta}\big([0,T]; \Sigma(H)\big)$ we have

$$\begin{aligned}\|\gamma(P) - \gamma(Q)\|_{\alpha,\beta} + \|\gamma(P_k) - \gamma(Q_k)\|_{\alpha,\beta} \\ \leq C_3 T^{2\alpha+2\beta-1}(\|P\|_{\alpha,\beta} + \|Q\|_{\alpha,\beta})\|P - Q\|_{\alpha,\beta}.\end{aligned} \tag{2.34}$$

The proof is similar to the previous one in Step 3.

We denote now by $B(r,t)$ the ball

$$B(r,t) = \big\{P \in C_{s,\alpha,\beta}\big([0,t]; \Sigma(H)\big) \colon \|P\|_{\alpha,\beta} \leq r\big\}.$$

Step 5.

For all $p > 0$, there exist $\tau = \tau(p) > 0$ and $r = r(p) > 0$, such that

$$\|P_\beta\| \leq p \implies \gamma\big(B(r,\tau)\big) \cup \gamma_k\big(B(r,\tau)\big) \subset B(r,\tau), \tag{2.35}$$

and for all $P, Q \in B(r,\tau)$

$$\begin{cases} \|\gamma(P) - \gamma(Q)\|_{\alpha,\beta} \leq \frac{1}{2}\|P - Q\|_{\alpha,\beta}, \\ \|\gamma_k(P) - \gamma_k(Q)\|_{\alpha,\beta} \leq \frac{1}{2}\|P - Q\|_{\alpha,\beta}. \end{cases} \tag{2.36}$$

This follows easily from the inequalities (2.24), (2.27), (2.34), and (2.36).

Step 6. *Local existence, convergence, and positivity.*

Choose $p > \|P_\beta\|$ and $\tau = \tau(p)$. Then, by (2.35), (2.36), and the Contractions Mapping Principle, there exist unique P and P_k in $C_{s,\alpha,\beta}\big([0,\tau]; \Sigma(H)\big)$ such that

$$\gamma(P) = P, \quad \gamma_k(P_k) = P_k, \tag{2.37}$$

and problems (2.1) and (2.4) have unique solutions P and P_k, respectively, in $[0,\tau]$.

Moreover, by Lemma 2.2, (2.23) follows. Finally, by Theorem 2.1 of Chapter 1, we have $P_k(t) \geq 0$ for all $t \in [0,\tau]$; this yields

$$P(t) \geq 0, \quad \forall t \in [0,\tau].$$

Step 7. *Conclusion.*

As the interval $[0,\tau]$ depends only on the norm of P_β, in order to prove global existence, we have to find an estimate for the norm of $(P(t))_\beta$, $t \in [0,\tau[$. Let $P \in C_{s,\alpha,\beta}\big([0,\tau]; \Sigma(H)\big)$ be the solution of (2.1) in $[0,\tau]$. Let $x \in D\big((-A)^\beta\big)$, as $P(t) \geq 0$, we have

$$\begin{aligned}\big(P(t)(-A)^\beta x,(-A)^\beta x\big) &\leq \big(P_0(-A)^\beta e^{tA}x,(-A)^\beta e^{tA}x\big)+\int_0^t |C(-A)^\beta e^{sA}x|^2\,ds\\ &\leq L^2\left\{\|P_\beta\|+\frac{1}{\beta}T^\beta\|C\|^2\right\}|x|^2,\end{aligned}$$

which implies

$$\|(-A^\star)^\beta P(t)(-A)^\beta\| \leq L^2\left\{\|P_\beta\|+\frac{1}{\beta}T^\beta\|C\|^2\right\}|x|^2, \tag{2.38}$$

for all $t\in[0,\tau]$. Set now

$$p=L^2\left\{\|P_\beta\|+\frac{1}{\beta}T^\beta\|C\|^2\right\}|x|^2, \quad \tau_1=\tau(p);$$

and let P be the solution to the Riccati equation (2.1) in $t\in[0,\tau_1]$. Proceeding as in Step 6 of the proof of Theorem 2.1, we prove that there exists a unique solution of (2.1) in $[0,2\tau_1-\varepsilon]$ and so on. The proof is complete. □

We now prove the *continuous dependence* with respect to P_β and C of the solutions of (2.1). Consider the sequence (2.17) of the Riccati equations. We assume that

$$\begin{cases}\text{(i)} & \text{for any } h\in\mathbb{N}, (A,D,C^h,P_0^h) \text{ fulfill } (\mathcal{HP}) \text{ with } \alpha\leq 1/2,\\ \text{(ii)} & \lim\limits_{n\to\infty}(C^h)^\star C^h x = C^\star C x, \quad \text{for all } x\in H,\\ \text{(iii)} & \lim\limits_{n\to\infty} P_\beta^h x = P_\beta x \quad \text{for all } x\in H,\end{cases} \tag{2.39}$$

and we prove the result

Proposition 2.5. *Assume that Assumptions* $(\mathcal{HP})$ *and* (2.39) *are verified for* $\alpha\leq 1/2$. *Let P and P^h be the mild solutions to* (2.1) *and* (2.17), *respectively. Then, for any $x\in H$ and any $T>0$, we have the following properties*: *For all $x\in H$.*

$$\begin{cases}\lim\limits_{h\to\infty} P^h(\cdot)x = P(\cdot)x \quad \text{in } C([0,T];H),\\ \lim\limits_{h\to\infty} \mu(\cdot)V_{P^h}(\cdot)x = \mu(\cdot)V_P(\cdot)x \quad \text{in } C([0,T];H),\\ \lim\limits_{h\to\infty} \nu(\cdot,\cdot)W_{P^k}(\cdot,\cdot)x = \nu(\cdot,\cdot)W_P(\cdot,\cdot)x \quad \text{in } C(\varDelta_T;H).\end{cases} \tag{2.40}$$

Proof. Fix $T>0$. By the Uniform Boundedness Theorem, there exist positive numbers q_β and c such that

$$\forall h\in\mathbb{N}, \qquad \|P_\beta^h\|\leq q_\beta, \quad \|(C^h)^\star C^h\|\leq c. \tag{2.41}$$

Set

$$p=L^2\left(q_\beta+\frac{1}{\beta}T^\beta c\right), \quad \tau=\tau(p).$$

Then, arguing as in the proof of Theorem 2.2, we first show that $P_k\to P$ in $C_s\big([0,\tau];\Sigma(H)\big)$ and then we show that this argument can be iterated in the interval $[\tau,2\tau]$ and so on. □

Finally, the following result can be proved as Proposition 2.3.

Proposition 2.6. *Assume that* $(A, D, C_i, P_{i,0})$ *fulfill Assumptions* $(\mathcal{HP})$ *for* $i = 1, 2$*, with* $\alpha \le 1/2$*, and, in addition, that*

$$P_{1,0} \le P_{2,0}, \quad C_1^\star C_1 \le C_2^\star C_2,$$

Let P_i*,* $i = 1, 2$*, be the mild solution of the Riccati equations* (2.21)*. Then we have*

$$P_1(t) \le P_2(t), \quad t \ge 0.$$

3 Dynamic programming

In this section, we are going to solve minimization problem (1.4) associated with the state equation (1.1). In the sequel, this will be referred to as problem (1.1)–(1.4). We assume that Assumptions $(\mathcal{HP})$ are verified with $\lambda_0 = 0$ (recall that under Assumption $(\mathcal{HP})_1$ properties (1.2) and (1.3) are verified) and denote by P and P_k the solutions of the Riccati equations (2.1) and (2.4), respectively, given by Theorems 2.1 and 2.2. We set also $V = V_P$, $V_k = V_{P_k}$. We shall extend the arguments of §6 of Chapter 1 based on the fundamental identity (3.1) and the solution of the closed loop equation (3.2) below.

Proposition 3.1. *Let* $u \in L^2(0, T; U)$ *and let* x *be the solution to state equation* (1.1)*. Then the following identity holds*:

$$\begin{aligned}\big(P(t)x_0, x_0\big) &+ \int_0^t |u(s) + E^\star V(t-s)x(s)|^2\, ds \\ &= \int_0^t \{|Cx(s)|^2 + |u(s)|^2\}\, ds + \big(P_0 x(t), x(t)\big). \qquad (3.1)\end{aligned}$$

Proof. Let $u \in L^2(0, T; U)$ and let x be the solution to (1.1). Let $\{u_k\}$ be a sequence in $W^{1,2}(0, T; U)$ such that $u_k \to u$ in $L^2(0, T; U)$, and let x_k be the solution to (1.1) corresponding to u_k. We have, as can be easily checked,

$$x_k(t) = e^{tA}x_0 + Du_k(t) - \int_0^t e^{(t-s)A} Du_k'(s)\, ds.$$

By Proposition 3.8 in Part II, Chapter 1, it follows that $x_k(t)$ is differentiable for any $t \in \,]0, T]$ and

$$x_k'(t) = Ae^{tA}x_0 - A\int_0^t e^{(t-s)A} Du_k'(s)\, ds.$$

Now we compute the derivative

$$\frac{d}{ds}\big(P_k(T-s)x_k(s), x_k(s)\big),$$

and integrating from 0 and T and letting k tend to infinity, we get (3.1). □

Let us consider now the *closed loop equation*

$$x(t) = e^{tA}x_0 - \int_0^t (-A)^{1-\alpha} e^{(t-s)A} EE^\star V(T-s)x(s)\, ds. \tag{3.2}$$

Remark that the right-hand side of (3.2) is meaningful for any $t \in [0,T[$ because

$$\begin{aligned}\|(-A)^{1-\alpha} e^{(t-s)A} EE^\star V(T-s)\| \\ \le \begin{cases} L^2\|P\|_\alpha[(T-s)(t-s)]^{\alpha-1} & \text{if } \alpha > 1/2, \\ L^2\|P\|_{\alpha,\beta}(T-s)^{\alpha+\beta-1}(t-s)^{\alpha-1} & \text{if } \alpha \le 1/2. \end{cases}\end{aligned} \tag{3.3}$$

Proposition 3.2. *Assume that Assumptions* $(\mathcal{HP})$ *hold. Then the following statements hold true*:

(i) *If* $\alpha > 1/2$*, there exists a unique solution of* (3.2)*,* $x \in C([0,T];H)$*.*
(ii) *If* $\alpha \le 1/2$*, there exists a unique solution of* (3.2)*,* $x \in C([0,T[;H)$ *and a constant* $C > 0$ *such that*

$$|x(t)| \le C(T-t)^{-\beta}, \quad t \in [0,T[. \tag{3.4}$$

Proof. For any $y \in C([0,T];H)$, we set

$$\lambda(y)(t) = \int_0^t (-A)^{1-\alpha} e^{(t-s)A} EE^\star V(T-s)x(s)\, ds, \quad t \in [0,T].$$

We first prove (i). Assume $\alpha > 1/2$; then by (3.3) using the Hölder estimate, we have

$$\begin{aligned} |(\lambda(y))(t)| &\le L^2\|P\|_\alpha \int_0^t (t-s)^{\alpha-1}(T-s)^{\alpha-1}\, ds\, \|y\|_{C([0,t];H)} \\ &\le L^2\|P\|_\alpha \left\{ \int_0^t (t-s)^{2\alpha-2} ds \int_0^t (T-s)^{2\alpha-2}\, ds \right\}^{1/2} \|y\|_{C([0,t];H)} \\ &\le \frac{1}{2\alpha-1} L^2\|P\|_\alpha t^{2\alpha-1}\|y\|_{C([0,t];H)}, \end{aligned}$$

so that $\lambda \in \mathcal{L}\big(C([0,t];H)\big)$ for any $t \in]0,T]$ and

$$\|\lambda\|_{\mathcal{L}(C([0,t];H))} \le \frac{1}{2\alpha-1} L^2\|P\|_\alpha t^{2\alpha-1}.$$

Thus, if t is small, λ is a contraction and (3.2) has a unique solution in $C([0,t];H)$. Now this argument can be repeated in $[t,2t]$ and so on, giving the conclusion. We now prove (ii). Fix $T_1 \in]0,T[$, and let $\alpha \le 1/2$ and $\beta \in]1/2-\alpha, 1-\alpha/2[$. By (3.3) we have

$$\begin{aligned}|(\lambda(y))(t)| &\le L^2\|P\|_{\alpha,\beta}\int_0^t (t-s)^{\alpha-1}(T-s)^{\alpha+\beta-1}\,ds\,\|y\|_{C([0,t];H)}\\ &\le L^2\|P\|_{\alpha,\beta}(T-t)^{\alpha+\beta-1}\int_0^t s^{\alpha-1}\,ds\,\|y\|_{C([0,t];H)}\\ &\le \frac{1}{\alpha}L^2\|P\|_{\alpha,\beta}(T-T_1)^{\alpha+\beta-1}t^{\alpha}\|y\|_{C([0,t];H)}, \quad t\in[0,T],\end{aligned}$$

so that $\lambda \in \mathcal{L}\big(C([0,t];H)\big)$ for any $t\in\,]0,T_1]$ and

$$\|\lambda\|_{\mathcal{L}(C([0,t];H))} \le \frac{1}{\alpha}L^2\|P\|_{\alpha,\beta}(T-T_1)^{\alpha+\beta-1}t^{\alpha}.$$

Thus, if t is small, λ is a contraction and (3.2) has a unique solution in $C([0,t];H)$, and proceeding by iteration, we prove the first part of (ii).

It remains to show (3.4). Following F. Flandoli [7], we set

$$z(t) = V(T-t)x(t), \quad t\in[0,T[. \tag{3.5}$$

Then z verifies the equation

$$z(t) = V(T-t)e^{tA}x_0 - \int_0^t V(T-t)(-A)^{1-\alpha}e^{(t-s)A}EE^{\star}z(s)\,ds.$$

Taking into account Lemma 2.3, it follows that

$$(T-t)^{1-\alpha-\beta}|z(t)| \le L|x_0|\|P\|_{\alpha,\beta}+2^{1-\alpha}L^2\|P\|_{\alpha,\beta}\int_0^t (T-s)^{\beta}(t-s)^{\alpha-1}|z(s)|\,ds.$$

Finally set $w(t) = (T-t)^{1-\alpha-\beta}z(t)$; then we have

$$\begin{aligned}|w(t)| &\le L|x_0|\;\|P\|_{\alpha,\beta}\\ &\qquad + 2^{1-\alpha}L^2\|P\|_{\alpha,\beta}\int_0^t (T-s)^{\alpha+2\beta-1}(t-s)^{\alpha-1}|w(s)|\,ds\\ &\le L|x_0|\;\|P\|_{\alpha,\beta}\\ &\qquad + 2^{1-\alpha}L^2\|P\|_{\alpha,\beta}\int_0^t (t-s)^{2\alpha+2\beta-2}|w(s)|\,ds.\end{aligned} \tag{3.6}$$

As $2-2\alpha-2\beta<1$, by the Gronwall's lemma, we have $|w(t)|\le K$, for some $K>0$ and all $t\in[0,T[$. It follows that

$$|z(t)| \le K(T-t)^{\alpha+\beta-1}, \quad t\in[0,T[. \tag{3.7}$$

Now, from (3.2), we have

$$\begin{aligned}|x(t)| &\le L|x_0| + L^2K\int_0^t (t-s)^{\alpha-1}(T-s)^{\alpha+\beta-1}\,ds\\ &= L|x_0| + L^2K(T-t)^{2\alpha+\beta-1}\int_0^{t/T-t}\sigma^{\alpha-1}(1+\sigma)^{\alpha+\beta-1}\,d\sigma.\end{aligned}$$

As $\sigma^{2\alpha+2\beta-1} \leq (t/T-t)^{2\alpha+2\beta-1}$, it follows that

$$|x(t)| \leq L|x_0| + L^2 K T^{2\alpha+2\beta-1}(T-t)^{-\beta} \int_0^\infty \sigma^{\alpha+2\beta}(1+\sigma)^{\alpha+\beta-1} d\sigma,$$

and the proof is complete. □

We now prove the main result of this section.

Theorem 3.1. *Assume that Assumptions* $(\mathcal{HP})$ *are verified and let* $x_0 \in H$. *Then there exists a unique optimal pair* $(u^\star, x^\star)$. *Moreover the following statements hold*:

(i) $x^\star$ *is the mild solution to the closed loop equation* (3.2) *and* $x^\star \in C([0,T];H)$ *if* $\alpha \in]\frac{1}{2},1[$, *whereas* $x^\star \in C(]0,T];H) \cap L^2(0,T;H)$ *if* $\alpha \in]0,\frac{1}{2}[$.

(ii) $u^\star \in C([0,T];U)$ *is given by the feedback formula:*

$$u^\star(t) = -E^\star V(T-t)x^\star(t). \tag{3.8}$$

(iii) *the optimal cost* $J(u^\star)$ *is given by*

$$J(u^\star) = \big(P(T)x_0, x_0\big). \tag{3.9}$$

Proof. Let $u \in L^2(0,T;U)$ and let y be the solution to state equation (1.1). Then, by (3.1), we have

$$\big(P(T)x_0, x_0\big) + \int_0^T |u(s) + E^\star E V(T-s)x(s)|^2\, ds = J(u). \tag{3.10}$$

It follows that

$$\big(P(T)x_0, x_0\big) \leq J(u)$$

for all $u \in L^2(0,T;U)$. Now, let $x^\star$ be the solution of the closed loop equation (3.2). By Proposition 3.2, $x^\star$ belongs to $C([0,T];H)$ if $\alpha < \frac{1}{2}$ and to $L^2(0,T;H)$ if $\alpha > \frac{1}{2}$ (due to the estimate (3.4)). Now, let $u^\star$ be defined by (3.8) and set $u = u^\star$, $x = x^\star$ in (3.10). We find $\big(P(T)x_0, x_0\big) = J(u^\star)$, so that $u^\star$ is optimal. Finally, the uniqueness of the optimal control can be proved as in the proof of Theorem 6.1 in Chapter 1. □

3

Unbounded Control Operators: Hyperbolic Equations With Control on the Boundary

1 Introduction

As before, we shall denote by H, U, and Y the Hilbert spaces of states, controls, and observations, respectively, and consider a dynamical system, whose state $x(\cdot)$ is the solution of the following equation:

$$\begin{cases} x'(t) = Ax(t) + Bu(t), & t \in [0,T], \\ x(0) = x_0 \in H, \end{cases}$$

where $u \in L^2(0,T;U)$ and $A\colon D(A) \subset H \to H$ generates a strongly continuous group on H. We identify the elements of H' with those of H so that the linear operator $(A^*)^*\colon H \to D(A^*)'$ is a linear extension of the linear operator $A\colon D(A) \to H$. As in the previous chapter, the linear operator B is not supposed to be bounded from U into H, but it belongs to $\mathcal{L}\big(U; D(A^\star)'\big)$, or equivalently, B is of the form $B = (\lambda_0 - A)E$, where $E \in \mathcal{L}(U;H)$ and λ_0 is an element in $\rho(A)$. More precisely, following I. Lasiecka and R. Triggiani [1, 2, 11], we shall assume that

$$(\mathcal{H}\mathcal{H})_1 \begin{cases} \text{(i)} & A \text{ generates a strongly continuous group } e^{tA} \text{ in } H \text{ of type } \omega_0 \\ & \text{and} \quad \lambda_0 \text{ is a real number in } \rho(A) \text{ such that } \omega_0 < \lambda_0, \\ \text{(ii)} & E \in \mathcal{L}(U;H), \\ \text{(iii)} & \exists K > 0 \text{ such that } \displaystyle\int_0^T |E^\star A^\star e^{sA^\star} x|^2\, ds \le K^2 |x|^2,\ \forall x \in D(A^\star). \end{cases}$$

If assumptions $(\mathcal{H}\mathcal{H})_1$ hold, then we can give a precise meaning to the state equation. We have in fact the following result due to I. Lasiecka and R. Triggiani [1, 2, 11].

Proposition 1.1. *Assume that $(\mathcal{H}\mathcal{H})_1$ is verified, and for each control function $u \in L^2(0,T;U)$, define*

$$z(t) = \int_0^t e^{(t-s)A} Eu(s)\, ds, \quad t \in [0,T]. \tag{1.1}$$

Then $z(t) \in D(A)$ *for all* $t \in [0,T]$ *and* $Az \in C([0,T];H)$. *Moreover the following inequality holds*:

$$|Az(t)| \leq K\|u\|_{L^2(0,T;U)}, \quad t \in [0,T]. \tag{1.2}$$

Proof. We divide the proof into two steps.

Step 1.

$z(t) \in D(A)$ for all $t \in [0,T]$.
If $x \in D(A^\star)$ we have

$$\big(z(t), A^\star x\big) = \int_0^t \big(u(s), E^\star A^\star e^{(t-s)A^\star} x\big)\, ds.$$

It follows that

$$\begin{aligned} |(z(t), A^\star x)|^2 &\leq \|u\|^2_{L^2(0,T;U)} \int_0^t |E^\star A^\star e^{(t-s)A^\star} x|^2\, ds \\ &\leq K^2 \|u\|^2_{L^2(0,T;U)} |x|^2. \end{aligned}$$

If we identify elements of H' with those of H, then this inequality implies that the linear form $x \mapsto \big(z(t), A^\star x\big)$ on H, is continuous, and hence that $z(t) \in D(A)$ and that (1.2) holds.

Step 2.

$Az \in C([0,T];H)$.
Let $\{u_n\}$ be a sequence in $W^{1,2}(0,T;U)$ such that u_n converges to u in $L^2(0,T;U)$. Set

$$z_n = \int_0^t e^{(t-s)A} Eu_n(s)\, ds.$$

It follows that $z_n \in C\big([0,T];D(A)\big)$ (cf. Part II, Chapter 1, Proposition 3.3). Now setting $w_n(t) = Az_n(t)$, $w_n \in C([0,T];H)$ and we have

$$|w(t) - w_n(t)|^2 = |Az(t) - Az_n(t)|^2 \leq K\|u - u_n\|^2_{L^2(0,T;U)} \to 0$$

as $n \to \infty$ uniformly in $[0,T]$. The proof is complete. □

In the sequel we shall write the solution of the state equation as follows:

$$\begin{aligned} x(t) &= e^{tA} x_0 + G(u)(t) \\ &= e^{tA} x_0 + (\lambda_0 - A) \int_0^t e^{(t-s)A} Eu(s)\, ds. \end{aligned} \tag{1.3}$$

By Proposition 1.1, $G \in \mathcal{L}\big(L^2(0,T;U);C([0,T];H)\big)$.

We now consider the following optimal control problem: To minimize

$$J(u) = \int_0^T \{|Cx(t)|^2 + |u(t)|^2\}\,dt + \big(P_0x(T),x(T)\big), \tag{1.4}$$

over all $u \in L^2(0,T;U)$, subject to the state equation constraint (1.3). In addition to the assumptions $(\mathcal{HH})_1$, we assume that $(\mathcal{HH})_2$ $C \in \mathcal{L}(H;Y)$, $P_0 \in \Sigma^+(H)$.
In the sequel $(\mathcal{HH})$ will mean that both assumptions $(\mathcal{HH})_1$ and $(\mathcal{HH})_2$ are verified. We shall also say that (A,E,C,P_0) fulfill $(\mathcal{HH})$. From now on we assume for simplicity that $\lambda_0 = 0$.

The definitions of *optimal control, optimal state,* and *optimal pair* are as in the previous chapters.

We again study the optimization problem (1.3)–(1.4) by using Dynamic Programming. However, in the current case, new technical difficulties arise. Consider in fact the Riccati equation that we formally write as follows:

$$\begin{cases} P' = A^\star P + PA - P(E^\star A^\star)^\star E^\star A^\star P + C^\star C, \\ P(0) = P_0. \end{cases} \tag{1.5}$$

If the data C and P_0 are regular or more precisely if the linear operators $\sqrt{P_0}A$ and CA admit bounded extensions, $\overline{\sqrt{P_0}A}$ and $\overline{CA}$, then problem (1.5) can be easily solved. In fact by setting $R = A^\star PA$ it reduces to

$$\begin{cases} R' = A^\star R + RA - REE^\star R + (\overline{CA})^\star \overline{CA}, \\ R(0) = (\overline{\sqrt{P_0}A})^\star \overline{\sqrt{P_0}A}, \end{cases}$$

which has a unique solution by Theorem 2.1 of Chapter 1.

However these conditions on C and P_0 are not natural, so we prefer to proceed in the following different way:

(i) We consider the approximating problem

$$\begin{cases} P_n' = A^\star P_n + P_nA - P_n(E^\star A_n^\star)^\star E^\star A_n^\star P_n + C^\star C, \\ P(0) = P_0, \end{cases} \tag{1.6}$$

where $A_n = nAR(n,A)$ are the Yosida approximations of A, and prove the convergence in $C_s\big([0,T];\Sigma(H)\big)$ of the sequence $\{P_n\}$ to an element P.

(ii) For any $n \in \mathbb{N}$ we consider the control problem: To minimize

$$J_n(u) = \int_0^T \{|Cx_n(t)|^2 + |u(t)|^2\}\,dt + \big(P_0x_n(T),x_n(T)\big), \tag{1.7}$$

over all $u \in L^2(0,T;U)$, subject to the constraint

$$\begin{cases} x_n'(t) = Ax_n(t) - A_nEu(t), \quad t \in [0,T], \\ x_n(0) = x_0. \end{cases} \tag{1.8}$$

Denote by $(u_n^\star, x_n^\star)$ the corresponding optimal pair given by Theorem 6.1 of Chapter 1. Then we prove that $(u_n^\star, x_n^\star)$ converges to the optimal pair of problem (1.4) with state equation (1.3).

Notice that we avoid the difficulty of giving a general definition of a solution of the Riccati equation (1.5); for us the solution will namely be the limit of the sequence $\{P_n\}$. The proof of the existence of this limit is given in §2, whereas in §3 we shall see how to use this result to study the initial optimal control problem.

For an existence and uniqueness result, under suitable regularity assumptions on C and P_0 (weaker than the ones discussed before), see G. Da Prato, I. Lasiecka, and R. Triggiani [1]. Finally we notice that in the previously quoted papers by I. Lasiecka and R. Triggiani, explicit formulas for the solutions to the Riccati equation and the optimal pair (even under more general assumptions) are constructed by generalizing the representation formula (2.22) of Chapter 1.

In §5 we shall consider, following V. Barbu and G. Da Prato [1], the case when e^{tA} is a general semigroup, and a condition similar to $(\mathcal{HH})$ holds. In this case we are able to define a solution of the Riccati equation by using the dual Riccati equation introduced in §7.3 of Chapter 1.

2 Riccati equation

We assume here that $(\mathcal{HH})$ is verified and consider the Riccati equation (1.6), whose mild solution is denoted by P_n. For the sake of simplicity, we assume that $P_0 \in \mathcal{L}_r(H)$, that is, that P_0 is invertible with a bounded inverse (cf. Chapter 1, §5); at the end of this section, we shall explain how to handle the general case.

Following F. Flandoli [2], we consider the Riccati equation

$$\begin{cases} Q' = -AQ - QA^\star - QC^\star CQ + (E^\star A^\star)^\star E^\star A^\star, \\ Q(0) = P_0^{-1}. \end{cases} \tag{2.1}$$

and

$$\begin{cases} Q_n' = -AQ_n - Q_nA^\star - Q_nC^\star CQ_n + (E^\star A_n^\star)^\star E^\star A_n^\star, \\ Q_n(0) = P_0^{-1}, \end{cases} \tag{2.2}$$

where $A_n = nAR(n,A)$ are the Yosida approximations of A. As the linear operator $E^\star A^\star$ fulfills the assumptions of Theorem 4.1 of Chapter 1, problem (2.1) has a unique mild solution Q that belongs to $C_s\big([0,T];\Sigma(H)\big)$ and we have

$$\lim_{n\to\infty} Q_n = Q \quad \text{in } C_s\big([0,T];\Sigma(H)\big) \tag{2.3}$$

and

$$Q_n(t) = P_n^{-1}(t), \quad \forall t \geq 0. \tag{2.4}$$

Theorem 2.1. *Assume that* $(\mathcal{H}\mathcal{H})$ *is verified and that* $P_0 \in \mathcal{L}_r(H)$*. Let* P_n *be the solution to the Riccati equation* (1.6)*. Then we have*

$$\lim_{n\to\infty} P_n = P \quad \text{in } C_s\big([0,T];\Sigma(H)\big), \tag{2.5}$$

where $P(t) = Q^{-1}(t)$*,* $t \in [0,T]$ *and* Q *is the mild solution of* (2.1)*.*

We shall call P the *mild solution* of (1.5).
Proof. It is sufficient to prove that there exists a constant $C_1 > 0$ such that

$$\|P_n(t)\| \leq C_1, \quad \forall t \in [0,T],\ \forall n \in \mathbb{N}. \tag{2.6}$$

In fact, if (2.6) holds, then we have

$$\|Q_n^{-1}(t)\| \leq C_1, \quad \forall t \in [0,T],\ \forall n \in \mathbb{N}.$$

Thus $Q(t) \in \mathcal{L}_r(H)$, $\forall t \in [0,T]$. Moreover from the identity

$$Q^{-1}(t) - P_n(t) = P_n(t)[Q_n(t) - Q(t)]Q^{-1}(t),$$

one can prove the existence of the limit (2.5). Finally to prove (2.6), it is sufficient to observe that

$$P_n(t) \leq e^{tA^\star} P_0 e^{tA} + \int_0^t e^{(t-s)A^\star} C^\star C e^{(t-s)A}\, ds.$$

The proof is complete. □

Remark 2.1. Assume that $(\mathcal{H}\mathcal{H})$ is verified and that P_0, $\bar{P}_0 \in \mathcal{L}_r(H)$ with $P_0 \leq \bar{P}_0$. Let P and $\bar{P}$ be the corresponding mild solutions of (1.5). Then we have $P(t) \leq \bar{P}(t)$, $\forall t \geq 0$. □

Remark 2.2. Assume that P_0 does not have a bounded inverse, and denote by P_ε the mild solution of (1.5) with P_0 replaced by $P_0 + \varepsilon I$. Then by the previous remark P_ε is decreasing and so the limit of P_ε exists in $C_s\big([0,T];\Sigma(H)\big)$ as ε goes to zero. □

3 Dynamic programming

Here we assume that $(\mathcal{H}\mathcal{H})$ is verified. We denote by P and P_n the mild solutions of (1.5) and (1.6), and by $(u^\star, x^\star)$ (resp. $(u_n^\star, x_n^\star)$) the optimal pair for the control problem (1.3)–(1.4) (resp. (1.7)–(1.8)).

We first consider the case when $P_0 \in \mathcal{L}_r(H)$.

Theorem 3.1. *Assume that $(\mathcal{HH})$ is verified and that $P_0 \in \mathcal{L}_r(H) \cap \Sigma^+(H)$. Then we have*

(i) $\lim_{n\to\infty} u_n^\star = u^\star$ *in* $L^2(0,T;U)$,
(ii) $\lim_{n\to\infty} x_n^\star = x^\star$ *in* $C([0,T];H)$.

Proof. Let $u \in L^2(0,T;U)$ and let x be given by (1.3) and x_n by (1.8). We remark that, as $x_n(t) = e^{tA}x_0 + n(nI - A)^{-1}G(u)$, we have $x_n \to x$ in $C([0,T];H)$. Now by formula (6.2) in Chapter 1, it follows that

$$
\begin{aligned}
(P_n(T)x_0, x_0) &+ \int_0^T |u(s) + E^\star A_n^\star P_n(T-s)x_n(s)|^2\, ds \\
&= \int_0^T [|Cx_n(s)|^2 + |u(s)|^2]\, ds + \big(P_0 x_n(T), x_n(T)\big). \quad (3.1)
\end{aligned}
$$

Setting $u = u_n^\star$ in (3.1), we find

$$
(P_n(T)x_0, x_0) = \int_0^T [|Cx_n^\star(s)|^2 + |u_n^\star(s)|^2]\, ds + \big(P_0 x_n^\star(T), x_n^\star(T)\big). \quad (3.2)
$$

Thus we have

$$
\big(P(T)x_0, x_0\big) \le J(u), \quad \forall u \in L^2(0,T;U). \quad (3.3)
$$

Moreover from identities (3.1) and (3.2) the sequence $\{u_n^*\}$ is bounded in $L^2(0,T;U)$ and so there exists a subsequence $\{u_{n_k}^*\}$ that weakly converges to an element $\widetilde{u}$ of $L^2(0,T;U)$. As

$$
u_{n_k}^*(\cdot) = S(\cdot)x_0 - n_k R(n_k, A) G(u_{n_k}^*)
$$

and

$$
G(u_{n_k}^*) \rightharpoonup G(\widetilde{u}) \quad \text{in } L^2(0,T;H),
$$

we have

$$
x_{n_k}^* \rightharpoonup \widetilde{x} \quad \text{in } L^2(0,T;H),
$$

where

$$
\widetilde{x} = S(\cdot)x_0 - G(\widetilde{u}),
$$

and the symbol $\rightharpoonup$ denotes the weak convergence.

By letting n go to infinity in (3.2), we have

$$
\big(P(T)x_0, x_0\big) \ge J_T(\widetilde{u}),
$$

which, along with (3.3), yields $\widetilde{u} = u^*$, $\widetilde{x} = x^*$ and

$$
\big(P(T)x_0, x_0\big) = J_T(u^*).
$$

Thus the sequence $\{u_n^*\}$ weakly converges to u^* and it only remains to prove that $u_n^* \to u^*$. By (3.2) we have

$$\lim_{n\to\infty} \left[\|u_n^*\|^2_{L^2(0,T;U)} + \|x_n^*\|^2_{L^2(0,T;H)} + |\sqrt{P_0}x_n^*(T)|^2\right]$$
$$\leq J(u^*) = \|u^*\|^2_{L^2(0,T;U)} + \|x^*\|^2_{L^2(0,T;H)} + |\sqrt{P_0}x^*(T)|^2.$$

As

$$\begin{aligned} u_n^* &\rightharpoonup u^* \quad \text{in } L^2(0,T;U),\\ x_n^* &\rightharpoonup x^* \quad \text{in } L^2(0,T;H),\\ \sqrt{P_0}x_n^*(T) &\rightharpoonup \sqrt{P_0}x^*(T) \quad \text{in } H, \end{aligned}$$

as $n \to \infty$, it follows that the above convergences are strong. The proof is complete. □

We now consider the case where $P_0 \in \Sigma^+(H)$ is not invertible. Define for any $\varepsilon > 0$, the cost

$$J_\varepsilon(u) = \int_0^T \{|Cx(t)|^2 + |u(t)|^2\}\, dt + \big(P_0x(T), x(T)\big) + \varepsilon|x(T)|^2,$$

and let $(u_\varepsilon^\star, x_\varepsilon^\star)$ be an optimal pair for the corresponding control problem. Let moreover P_ε be the mild solution of the Riccati equation (1.5) with P_0 replaced by $P_0+\varepsilon I$. As we know (see Remark 3.1), P_ε is convergent in $C_s\big([0,T];\Sigma(H)\big)$ to an element $P \in C_s\big([0,T];\Sigma(H)\big)$, which has been defined as the mild solution of (1.5).

Theorem 3.2. *Assume that $(\mathcal{HH})$ is verified and that $P_0 \in \Sigma^+(H)$. Then there exists a unique optimal pair $(u^\star, x^\star)$ for the control problem* (1.4) *subject to the state equation* (1.3). *Moreover*

(i) $\lim_{\varepsilon\to 0} u_\varepsilon^\star = u^\star$ *in* $L^2(0,T;U)$,
(ii) $\lim_{\varepsilon\to 0} x_\varepsilon^\star = x^\star$ *in* $C([0,T];H)$.

Proof. We have

$$\big(P_\varepsilon(T)x_0, x_0\big) \leq J_\varepsilon(u), \quad \forall u \in L^2(0,T;U), \tag{3.4}$$

$$\begin{aligned}\big(P_\varepsilon(T)x_0, x_0\big) = \int_0^T &\{|Cx_\varepsilon^\star(t)|^2 + |u_\varepsilon^\star(t)|^2\}dt\\ &+ \big(P_0x(T), x(T)\big) + \varepsilon|x_\varepsilon^\star(T)|^2.\end{aligned} \tag{3.5}$$

We notice that the sequence $\{u_\varepsilon^\star\}$ is bounded in $L^2(0,T;U)$; we have in fact

$$\int_0^T |u_\varepsilon^\star(s)|^2\, ds \leq J_\varepsilon(u_\varepsilon^\star) \leq J_\varepsilon(0),$$

where

$$J_\varepsilon(0) = \int_0^T |Ce^{sA}x_0|^2\, ds + (P_0Ce^{TA}x_0, Ce^{TA}x_0) + \varepsilon|e^{TA}x_0|^2.$$

It follows that there exists a sequence $\{\varepsilon_k\} \downarrow 0$, and elements $\widetilde{u} \in L^2(0,T;U)$, $\widetilde{x} \in C([0,T];H)$ such that

(i) $u_{\varepsilon_k} \rightharpoonup \widetilde{u}$ in $L^2(0,T;U)$,
(ii) $x_{\varepsilon_k} \rightharpoonup \widetilde{x}$ in $L^2(0,T;H)$,
(iii) $x_{\varepsilon_k}(T) \rightharpoonup \widetilde{x}(T)$ in H.

Moreover

$$\widetilde{x}(t) = e^{tA}x_0 - A\int_0^t e^{(t-s)A}E\widetilde{u}(s)\,ds, \quad t \in [0,T].$$

Now from (3.4) it follows that

$$\big(P_\varepsilon(T)x_0, x_0\big) \le J_\varepsilon(\widetilde{u}),$$

and by (3.5)

$$\big(P_\varepsilon(T)x_0, x_0\big) \ge J_\varepsilon(\widetilde{u}).$$

The conclusion follows arguing as in the proof of Theorem 3.1. □

Remark 3.1. We shall denote by $P_{\min}$ the mild solution of the Riccati equation corresponding to $P_0 = 0$. By Remark 2.1 it follows that $P_{\min}$ is the minimal nonnegative solution of the Riccati equation. □

4 Examples of controlled hyperbolic systems

Let $A\colon D(A) \subset H \to H$ be a linear operator, infinitesimal generator of a strongly continuous group e^{tA} and let $E\colon U \to H$ be a linear bounded operator. In this section we shall discuss several examples such that the key condition $(\mathcal{HH})$–(iii) is fulfilled. We start with a very simple example.

Example 4.1 *(first order problem).*
Consider a system

$$\begin{cases} \dfrac{\partial x}{\partial t}(t,\xi) = \dfrac{\partial x}{\partial \xi}(t,\xi), \quad t \in [0,T],\ \xi \in [0,2\pi], \\ x(0,\xi) = x_0(\xi), \quad \xi \in [0,2\pi], \\ x(t,2\pi) = x(t,0) + u(t), \quad t \in [0,T]. \end{cases}$$

We choose here $H = L^2_{\#}(\mathbb{R})$, the set of all 2π-periodic functions that belong to $L^2(0,2\pi;\mathbb{R})$, and $U = \mathbb{R}$. We consider the linear operator A in H

$$\begin{cases} D(A) = W^{1,2}_{\#}(\mathbb{R}) = \{x \in L^2_{\#}(\mathbb{R})\colon x' \in L^2_{\#}(\mathbb{R}), x(0) = x(2\pi)\}, \\ Ax = x', \quad \forall x \in D(A). \end{cases}$$

As it is well known, A is the infinitesimal generator of the strongly continuous group of translations

$$e^{tA}x(\xi) = x(t+\xi), \quad t \in \mathbb{R},\ \xi \in \mathbb{R}.$$

We proceed here as in the Example 1.1 of Chapter 2. First we assume that u is differentiable and introduce a new variable y by setting

$$y(t,\xi) = x(t,\xi) - \frac{e^{\xi}}{e^{2\pi}-1}u(t),$$

so that state equation becomes

$$\begin{cases} \dfrac{\partial y}{\partial t}(t,\xi) = \dfrac{\partial y}{\partial \xi}(t,\xi) + \dfrac{e^{\xi}}{e^{2\pi}-1}\big(u(t) - u'(t)\big), & t \in [0,T],\ \xi \in [0,2\pi], \\ y(0,\xi) = x_0(\xi) - \dfrac{e^{\xi}}{e^{2\pi}-1}u(0) = y_0, & \xi \in [0,2\pi], \\ y(t,2\pi) = y(t,0), & t \in [0,T]. \end{cases}$$

Now we can write this equation in abstract form, namely

$$y(t,\cdot) = e^{tA}y_0 + \int_0^t e^{(t-s)A}z_0\big(u(s) - u'(s)\big)\,ds,$$

where z_0 is the element of $L^2_{\#}(\mathbb{R})$ defined by

$$z_0(\xi) = \frac{e^{\xi}}{e^{2\pi}-1}, \quad \xi \in [0,2\pi].$$

By integration by parts we eventually find that the state equation can be written in the required form

$$x'(t) = Ax(t) + (1-A)Eu(t), \quad x(0) = x_0,$$

where the linear operator $E \in \mathcal{L}(U;H)$ is defined as

$$E\alpha = \alpha z_0, \quad \forall \alpha \in U = \mathbb{R}.$$

Observe that the adjoint operator $E^\star \in \mathcal{L}(H;U)$ is defined as

$$E^\star x = (x, z_0), \quad \forall x \in H.$$

We now check condition $(\mathcal{HH})$–(iii). Let $t \in [0,2\pi]$ and $x \in W^{1,2}_{\#}(\mathbb{R})$. If $s \in [0,2\pi]$ we have

$$\begin{aligned} E^\star A^\star e^{sA^\star}x = (A^\star x, e^{sA}z_0) \\ &= \frac{1}{e^{2\pi}-1}\left\{\int_0^{2\pi-s} x'(\xi)e^{s+\xi}\,d\xi + \int_{2\pi-s}^{2\pi} x'(\xi)e^{s+\xi-2\pi}\,d\xi\right\} \\ &= x(2\pi - s) - (x, e^{sA}z_0). \end{aligned}$$

It follows that

$$\begin{aligned} \int_0^t |E^\star A^\star e^{sA^\star}x|^2\,ds &\le 2\int_0^t |x(2\pi - s)|^2\,ds + 2t\int_0^t |(x, e^{sA}z_0)|^2\,ds \\ &\le 2(1 + t^2|z_0|^2)|x|^2. \end{aligned}$$

Thus condition $(\mathcal{HH})$–(iii) is fulfilled. □

Example 4.2 *(Abstract wave equation).*

Let Z, U be Hilbert spaces, $\Lambda \colon D(\Lambda) \subset Z \to Z$ a linear self-adjoint strictly positive operator, D a linear operator in $\mathcal{L}(U; Z)$, and $\lambda_0 \geq 0$. Here we are dealing with the system

$$\begin{cases} y''(t) + \Lambda y(t) + (\lambda_0^2 + \Lambda) D u(t) = 0, \quad t \geq 0, \\ y(0) = y_0, \quad y'(0) = y_1. \end{cases} \tag{4.1}$$

In order to write problem (4.1) in the form (1.3), we have to introduce the space $D(\Lambda^{-1/2})$ defined as the completion of Z with respect to the norm $\|x\|_{D(\Lambda^{-1/2})} = \|\Lambda^{-1/2} x\|$. Then we choose as space of states and observations $H = Z \oplus D(\Lambda^{-1/2})$, endowed with the inner product:

$$\left(\begin{bmatrix} x^0 \\ x^1 \end{bmatrix}, \begin{bmatrix} \bar{x}^0 \\ \bar{x}^1 \end{bmatrix} \right) = (x^0, \bar{x}^0)_Z + (\Lambda^{-1/2} x^1, \Lambda^{-1/2} \bar{x}^1)_Z,$$

and define the linear operator A on H

$$\begin{cases} AX = \begin{bmatrix} 0 & 1 \\ -\Lambda & 0 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \end{bmatrix}, \quad \forall X \in D(A), \\ D(A) = D(\Lambda^{1/2}) \oplus Z. \end{cases}$$

A is the infinitesimal generator of a strongly continuous group of contractions, given by the formula

$$e^{tA} = \begin{bmatrix} \cos(\Lambda^{1/2} t) & \Lambda^{-1/2} \sin(\Lambda^{1/2} t) \\ -\Lambda^{1/2} \sin(\Lambda^{1/2} t) & \cos(\Lambda^{1/2} t) \end{bmatrix}, \quad t \in \mathbb{R}.$$

Setting now

$$X(t) = \begin{bmatrix} y(t, \cdot) \\ y'(t, \cdot) \end{bmatrix}, \quad X_0 = \begin{bmatrix} y_0 \\ y_0' \end{bmatrix},$$

we can write (4.1) as

$$\begin{cases} X'(t) = AX(t) + (\lambda_0 - A) E u(t), \quad t \geq 0, \\ X(0) = X_0, \end{cases} \tag{4.2}$$

where $E \colon U \to H$ is defined by

$$E\eta = -\begin{bmatrix} D\eta \\ \lambda_0 D\eta \end{bmatrix}, \quad \eta \in U.$$

We remark that the adjoint of E, $E^\star \colon H \to U$ is given by

$$E^\star \begin{bmatrix} x^0 \\ x^1 \end{bmatrix} = -D^\star (x^0 - \lambda_0 \Lambda^{-1}) x^1.$$

It follows that

$$E^\star A^\star e^{tA^\star}\begin{bmatrix} x^0 \\ x^1 \end{bmatrix} = -D^\star \Lambda^{1/2}\sin(\Lambda^{1/2}t)x^0 + D^\star\cos(\Lambda^{1/2}t)x^1$$
$$+\lambda_0 D^\star \Lambda^{-1}\cos(\Lambda^{1/2}t)x^0 + \lambda_0 D^\star \Lambda^{1/2}\sin(\Lambda^{1/2}t)x^1.$$

Now it follows easily that condition $(\mathcal{H}\mathcal{H})$–(iii) is fulfilled if and only if there exists $K_1 > 0$ such that

$$\begin{aligned} &\int_0^t |D^\star \Lambda^{1/2}\sin(\Lambda^{1/2}s)z|_U^2\,ds \le K_1(t)|z|_Z^2, \quad \forall z\in D(\Lambda^{1/2}),\\ &\int_0^t |D^\star \Lambda^{1/2}\cos(\Lambda^{1/2}s)z|_U^2\,ds \le K_1(t)|z|_Z^2, \quad \forall z\in D(\Lambda^{1/2}). \end{aligned} \tag{4.3}$$

□

Example 4.3 *(Wave equation).*

Let Ω be a bounded set in $\mathbb{R}^n$ with a regular boundary $\partial\Omega$. Consider the problem

$$\begin{cases} y_{tt}(t,\xi) = \Delta_\xi y(t,\xi), & t\ge 0,\ \xi\in\Omega,\\ y(t,\xi) = u(t,\xi), & t\ge 0,\ \xi\in\partial\Omega,\\ y(0,\xi) = y_0(\xi), & y_t(0,\xi) = y_1(\xi),\ \xi\in\Omega. \end{cases} \tag{4.4}$$

We set $Z = L^2(\Omega)$ and $U = L^2(\partial\Omega)$ and denote by $-\Lambda$ the Laplace operator with Dirichlet boundary conditions in Ω. Then, setting $y(t) = y(t,\cdot)$ and $u(t) = u(t,\cdot)$ and denoting by D the Dirichlet mapping defined in Example 1.1 of Chapter 2, we see that problem (4.4) is equivalent to problem (4.1) with $\lambda_0 = 0$.

We now prove $(\mathcal{H}\mathcal{H})$–(iii); for this it is enough to show the first inequality (4.3), because the second one can be proven similarly. Thus we fix $z\in D(\Lambda)$ and set

$$w(t) = D^\star \Lambda^{1/2}\sin(\Lambda^{1/2}t)z, \quad v(t) = \Lambda^{-1/2}\sin(\Lambda^{1/2}t)z;$$

we remark that

$$w(t) = \frac{\partial v}{\partial \nu}(t),$$

where ν is the the outward normal to $\partial\Omega$ (because $D^\star\Lambda$ coincides with the linear operator $\partial/\partial\nu$). Moreover $v(t,\cdot) = v(t)$ is the classical solution to the following problem:

$$\begin{cases} v_{tt}(t,\xi) = \Delta_\xi v(t,\xi), \quad t\ge 0,\ \xi\in\Omega,\\ v(t,\xi) = 0, \quad t\ge 0,\ \xi\in\partial\Omega,\\ v(0,\xi) = 0, \quad v_t(0,\xi) = x_0(\xi),\ \xi\in\Omega. \end{cases} \tag{4.5}$$

Fix $T > 0$, to prove the first inequality in (4.3), it suffices to prove that there exists a constant $C_1 > 0$ such that

$$\int_0^T (T-t)\left|\frac{\partial v}{\partial \nu}(t,\cdot)\right|^2_{L^2(\partial\Omega)} dt \le C_1 |x_0|^2_{L^2(\Omega)}. \tag{4.6}$$

For this purpose, we follow the proof in I. LASIECKA, J. L. LIONS, and R. TRIGGIANI [1]. We consider a regular vector field h in Ω that extends the outward normal ν; then we multiply the first equation in (4.5) by $(T-t)h\cdot\nabla_\xi v$ and set

$$I = \int_Q (T-t)v_{tt} h\cdot\nabla_\xi v\,dt\,d\xi, \quad J = \int_Q (T-t)(\Delta_\xi v)h\cdot\nabla_\xi v\,dt\,d\xi,$$

where $Q = [0,T]\times\Omega$ and " $\cdot$ " denotes the scalar product in $\mathbb{R}^n$.

Step 1. *Estimate of I.*

We have

$$\begin{aligned} I = \int_\Omega [(T-t)(h\cdot\nabla_\xi v)v_t]\Big|_{t=0}^{t=T} d\xi + \int_Q (h\cdot\nabla_\xi v)v_t\,dt\,d\xi \\ - \int_Q (T-t)(h\cdot\nabla_\xi v_t)v_t\,dt\,d\xi = I_1 + I_2 + I_3. \end{aligned}$$

Clearly, $I_1 = 0$; moreover, by the usual energy estimates, there exists $C_2 > 0$ such that

$$I_2 \le C_2|x_0|^2_{L^2(\Omega)}. \tag{4.7}$$

Concerning I_3 we have

$$\begin{aligned} I_3 &= -\frac{1}{2}\int_Q (T-t)\operatorname{div}_\xi(hv_t^2)\,dt\,d\xi + \frac{1}{2}\int_Q (T-t)v_t^2 \operatorname{div}_\xi h\,dt\,d\xi \\ &= \frac{1}{2}\int_{[0,T]\times\partial\Omega} (T-t)v_t^2 h\cdot\nu\,dt\,d\sigma + \frac{1}{2}\int_Q (T-t)v_t^2\operatorname{div}_\xi h\,dt\,d\xi. \end{aligned}$$

The first term is equal to 0 because $v = 0$ on $\partial\Omega$; moreover the second one can be estimated as I_2. Thus, there exists a constant $C_3 > 0$ such that

$$|I| \le C_2|x_0|^2_{L^2(\Omega)}. \tag{4.8}$$

Step 2. *Estimate of J.*

Using Gauss–Green formulas for the term J, we obtain the boundary term in (4.6) minus the term J_1 given by the expression

$$\begin{aligned} J_1 &= \sum_{i,j=1}^n \int_Q (T-t)\partial_{\xi_i} v\partial_{\xi_i}(h_j\partial_{\xi_j} v)\,dt\,d\xi \\ &= \sum_{i,j=1}^n \int_Q (T-t)\partial_{\xi_i} v\partial_{\xi_i} h_j\partial_{\xi_j} v\,dt\,d\xi + \sum_{i,j=1}^n \int_Q (T-t)h_j\partial_{\xi_i\xi_j} v\,\partial_{\xi_i} v\,dt\,d\xi, \end{aligned}$$

where ∂_{ξ_i} denotes the partial derivative with respect to ξ_i and $\partial_{ij} = \partial_i(\partial_j)$. Then

$$J_1 = \sum_{i,j=1}^{n} \int_Q (T-t)\partial_{\xi_i} h_j \partial_{\xi_j} v \partial_{\xi_i} v \, dt \, d\xi + \frac{1}{2} \sum_{i,j=1}^{n} \int_Q (T-t) h_j \partial_{\xi_j} (\partial_{\xi_i} v)^2 \, dt \, d\xi$$

$$\begin{aligned} J_1 = \sum_{i,j=1}^{n} \int_Q (T-t)\partial_{\xi_i} h_j \partial_{\xi_j} v \partial_{\xi_i} v \, dt \, d\xi &+ \frac{1}{2} \sum_{i,j=1}^{n} \int_Q (T-t) \partial_{\xi_j} [h_j |\partial_{\xi_i} v|^2] \, dt \, d\xi \\ &- \frac{1}{2} \sum_{i,j=1}^{n} \int_Q (T-t) \partial_{\xi_j} h_j |\partial_{\xi_i} v|^2 \, dt \, d\xi \\ = K_1 + \frac{1}{2} \int_{[0,T]\times\partial\Omega} (T-t) |\nabla_\xi v|^2 \, dt \, d\sigma \\ = K_1 + \frac{1}{2} \int_{[0,T]\times\partial\Omega} (T-t) \left|\frac{\partial v}{\partial \nu}\right|^2 \, dt \, d\sigma, \end{aligned}$$

because $v = 0$ on $\partial\Omega$ and

$$h \cdot \nabla_\xi = \frac{\partial v}{\partial \nu} \quad \text{on } \partial\Omega.$$

Now

$$|K_1| \le C_4 |x_0|^2_{L^2(\Omega)} \tag{4.9}$$

for some constant $C_4 > 0$. We have

$$J = -K_1 + \frac{1}{2} \int_{[0,T]\times\partial\Omega} (T-t) \left|\frac{\partial v}{\partial \nu}\right|^2 dt \, d\sigma. \tag{4.10}$$

Finally as $I = J$

$$\int_0^T (T-t) \left|\frac{\partial v}{\partial \nu}\right|^2_{L^2(\partial\Omega)} dt = I + K_1, \tag{4.11}$$

so that the required estimate (4.6) follows from (4.9), (4.10), and (4.11). □

Remark 4.1. One can consider different situations such as the wave equations with Neumann Boundary conditions, or the plate equations with several boundary conditions. As for the verification of the condition $(\mathcal{HH})$–(i), the reader is referred to the review paper by I. Lasiecka and R. Triggiani [11]. □

5 Some result for general semigroups

In this section we do not assume that A generates a strongly continuous group but that a condition weaker than $(\mathcal{HH})_1$ holds.

$$(\mathcal{HH})'_1 \begin{cases} \text{(i)} & A \text{ generates a strongly continuous semigroup } e^{tA} \text{ in } H \text{ of} \\ & \text{type } \omega_0 \text{ and } \lambda_0 \text{ is a real number in } \rho(A) \text{ such that } \omega_0 < \lambda_0, \\ \text{(ii)} & E \in \mathcal{L}(U;H), \\ \text{(iii)} & \exists K>0 \text{ such that } \displaystyle\int_0^T |E^\star A^\star e^{sA^\star} x|^2\, ds \le K^2|x|^2, \ \forall x \in D(A^\star). \end{cases}$$

If we replace $(\mathcal{HH})_1$ with $(\mathcal{HH})'_1$ Proposition 1.1 still holds true as can easily be seen and the state equation (1.3) is meaningful. We consider again the cost function (1.4) and assume that $(\mathcal{HH})_2$ is verified. We denote by $(\mathcal{HH})'$ the set of all hypotheses $(\mathcal{HH})'_1$ and $(\mathcal{HH})_2$. Besides the Riccati equation (1.5) and the approximating equation (1.6) we consider, as in V. Barbu and G. Da Prato [1], the dual Riccati equation

$$\begin{cases} Q' = AQ + QA^\star - QC^\star CQ + (E^\star A^\star)^\star E^\star A^\star, \\ Q(0) = 0, \end{cases} \tag{5.1}$$

as well as the approximating problem

$$\begin{cases} Q_n' = AQ_n + Q_nA^\star - Q_nC^\star CQ_n + (E^\star A_n^\star)^\star E^\star A_n^\star, \\ Q_n(0) = 0, \end{cases} \tag{5.2}$$

where $A_n = nAR(n,A)$ are the Yosida approximations of A.

Proposition 5.1. *Assume* $(\mathcal{HH})'$ *and let* $P_n \in C_s\big([0,T];\Sigma(H)\big)$ *be the mild solution of* (1.6)*. Then there is* $P \in C_s\big([0,T];\Sigma(H)\big)$ *such that the following limit exists*:

$$\lim_{n\to\infty} P_n = P, \quad \text{in } C_s\big([0,T];\Sigma(H)\big).$$

In the sequel we shall call P the *mild solution* of (1.5).

Proof. We first remark that the hypotheses of Theorem 4.1 in Chapter 1 are verified, so that problems (5.1) and (5.2) have mild solutions, and moreover the following limit exists:

$$\lim_{n\to\infty} Q_n = Q \quad \text{in } C_s\big([0,T];\Sigma(H)\big).$$

Denote by C_T a positive constant such that

$$\|Q_n(t)\| \le C_T, \quad \forall t \in [0,T],\ n \in \mathbb{N}. \tag{5.3}$$

Fix now $t \in [0,T]$. Then by Theorem 7.4 in Chapter 1 we have

$$\begin{aligned} P_n(t) = U_{G_{n,t}}(t,0)[I + P_0Q_n(t)]^{-1}P_0U^*_{G_{n,t}}(t,0) \\ + \int_0^t U_{G_{n,t}}(t,s)C^*CU^*_{G_{n,t}}(t,s)\,ds, \end{aligned} \tag{5.4}$$

where

$$G_{n,t}(s) = A^* - C^*CQ_n(t-s), \quad s \in [0,t].$$

Therefore $U_{G_{n,t}}$ is the evolution operator with respect to $G_{n,t}$.

Step 1.

There exists $C_1 > 0$ such that

$$\|U_{G_{n,t}}(\tau, s)\| \le C_1, \quad \forall (s,t),\ 0 \le s \le \tau \le t, \quad \text{and} \quad \forall n \in \mathbb{N}. \tag{5.5}$$

In fact let $x \in H$ and set $\varphi_n(\tau) = U_{G_{n,t}}(\tau, s)x$. Then we have

$$\varphi_n(\tau) = e^{(\tau - s)A^*}x - \int_s^\tau e^{(\tau-\sigma)A^*} C^* C Q_n(t-\sigma)\varphi_n(\sigma)\, d\sigma.$$

Let $M_T > 0$ such that

$$\|e^{sA}\| \le M_T, \quad \forall s \in [0, T].$$

Then, recalling (5.3), we obtain

$$|\varphi_n(\tau)| \le M_T|x| + M_T C \int_s^\tau |\varphi_n(\sigma)|\, d\sigma,$$

and the conclusion follows from Gronwall's lemma.

Step 2.

We have

$$\lim_{n\to\infty} U_{G_{n,t}}(\tau, s)x = U_{G_t}(\tau, s)x, \quad \forall x \in H,\ 0 \le s \le \tau \le t. \tag{5.6}$$

Let $x \in H$ and set $\varphi_n(\tau) = U_{G_{n,t}}(\tau, s)x$ and $\varphi(\tau) = U_{G_n}(\tau, s)x$. Setting $\zeta_n(\cdot) = \varphi(\cdot) - \varphi_n(\cdot)$; then ζ_n is the mild solution of the following problem:

$$\begin{cases} \zeta_n'(\tau) & = A^*\zeta_n(\tau) - C^*CQ_n(t-\tau)\zeta_n(\tau) + C^*C\big(Q_n(t-\tau) - Q(t-\tau)\big)\varphi(\tau), \\ \zeta_n(0) & = 0, \end{cases}$$

that is to the integral equation

$$\zeta_n(\tau) = \int_s^\tau U_{G_{n,t}}(\tau - \sigma)C^*C\big(Q_n(t-\sigma) - Q(t-\sigma)\big)\varphi(\sigma)\, d\sigma.$$

Now (5.6) follows from (5.5).

Step 3.

Conclusion. It is sufficient to let n tend to infinity in (5.4). □

We can now proceed as in §3. Let $(u^\star, x^\star)$ (resp. $(u_n^\star, x_n^\star)$) be the optimal pair for the control problem (1.3)–(1.4) (resp. (1.7)–(1.8)).

Then the following result is proved as Theorem 3.1.

Theorem 5.1. *Assume that* $(\mathcal{H}\mathcal{H}')$ *is verified and that* $P_0 \in \Sigma^+(H)$. *Then we have*

(i) $\lim_{n\to\infty} u_n^\star = u^\star$ *in* $L^2(0,T;U)$,
(ii) $\lim_{n\to\infty} x_n^\star = x^\star$ *in* $C([0,T];H)$.

Example 5.1 *(Age-dependent equations).*

Consider a dynamical system describing the evolution of a certain population (see for instance G. F. Webb [5]), governed by the equations

$$\begin{cases} p_t(t,a) + p_a(t,a) + \mu(a)p(t,a) = 0, \quad a \in [0,a_+],\ t \geq 0, \\ p(0,a) = p_0(a), \quad a \in [0,a_+], \\ p(t,0) = \displaystyle\int_0^{a_+} \beta(b)p(t,b)\,db + u(t), \quad t \geq 0. \end{cases} \tag{5.7}$$

Here $p(t,a)$ is the density of the population of age a at time t, μ is the mortality rate, β is the birth rate, a_+ is the maximal age, and u is the control. We shall assume that $\mu \geq 0$, $\beta \geq 0$, $\mu, \beta \in C^1([0,a_+])$ and

$$\int_0^{a_+} \mu(b)\,db = +\infty, \quad \int_0^{a_+} \beta(b) \exp\left\{-\int_0^{\beta} \mu(c)dc\right\} db \neq 1. \tag{5.8}$$

We want to minimize a quadratic cost function of the form

$$J(u) = \int_0^T dt \left[\int_0^{a_+} p^2(t,a)\,da + u^2(t)\right], \tag{5.9}$$

over all controls $u \in L^2(0,T)$ subject to state equation (5.7).

As shown in G. Da Prato and M. Iannelli [1], problem (5.7) can be written on the abstract form

$$p(t) = e^{tA}p_0 - \alpha A \int_0^t e^{(t-s)A} Eu(s)\,ds, \quad t \in [0,T], \tag{5.10}$$

where A is the infinitesimal generator of a C_0 semigroup on $H = L^2(0,a_+)$ and D is a mapping from $\mathbb{R}$ into H. More precisely the operator A is given by

$$\begin{aligned} &A\varphi \stackrel{\text{def}}{=} -\varphi' - \mu\varphi, \\ &D(A) = \left\{\varphi \in H \colon \varphi' + \mu\varphi \in H \text{ and } \varphi(0) = \int_0^{a_+} \beta(b)\varphi(b)\,db\right\}, \end{aligned} \tag{5.11}$$

and E is defined by

$$x \mapsto E(x) = \pi(x) \colon \mathbb{R} \to H, \tag{5.12}$$

where π is the element of H given by

$$\pi(a) = e^{-\int_0^a \mu(b)\,db} \quad a \in [0,a_+]. \tag{5.13}$$

Moreover the semigroup generated by A is given by

$$e^{tA}\varphi = \begin{cases} B_\varphi(t-a)\pi(a) & \text{if } a \in [0,t[\\ \varphi(a-t)\pi(t) & \text{if } a \in [t,a_+] \end{cases}, \quad \varphi \in L^2(0,a_+), \tag{5.14}$$

where B is the solution to the integral equation

$$B(a) = F_\varphi(a) + \int_0^a K(a-b)B(b)\,db, \quad a \in [0,a_+], \tag{5.15}$$

and

$$F_\varphi(a) = \int_a^{a_+} \varphi(b-a)K(b)\frac{\pi(a)}{\pi(b)}\,db, \quad a \in [0,a_+]. \tag{5.16}$$

We now check hypotheses $(\mathcal{HH})'$. It is sufficient to show that $(\mathcal{HH})_1$–(iii) holds. If $\varphi \in D(A^*)$ we have in fact

$$\begin{aligned}\int_0^{a_+} |E^*A^*e^{tA^*}\varphi|^2\,dt = \int_0^{a_+} |\langle A^*e^{tA^*}\varphi,\pi\rangle|^2 dt \\ = \int_0^{a_+} \left|\frac{d}{dt}\langle\varphi, e^{tA}\pi\rangle\right|^2 dt.\end{aligned} \tag{5.17}$$

Moreover, recalling (5.14), we have

$$\begin{aligned}\frac{d}{dt}\langle\varphi, e^{tA}\pi\rangle &= \frac{d}{dt}\left\{\int_0^t \varphi(a)B_\pi(t-a)\pi(a)\,da + \int_t^{a_+} \varphi(a-t)\pi(t)\,da\right\} \\ &= \varphi(t)B_\pi(0)\pi(b) + \int_0^t \varphi(a)B'_\pi(t-a)\pi(a)\,da \\ &\quad - \varphi(a_+-t)\pi(t) + \int_0^{a_+-t} \varphi(a)\pi'(t)\,da.\end{aligned} \tag{5.18}$$

Now, by plugging (5.18) into (5.17) the conclusion follows. □

Part V

Quadratic Optimal Control: Infinite Time Horizon

1

Bounded Control Operators: Control Inside the Domain

1 Introduction and setting of the problem

As in Chapter 1 (Part IV) we consider a dynamical system governed by the following state equation:

$$\begin{cases} x'(t) = Ax(t) + Bu(t), \quad t \geq 0, \\ x(0) = x_0 \in H, \end{cases} \tag{1.1}$$

and we use the notation introduced in §1 of that chapter. We assume that

$$(\mathcal{H})_\infty \begin{cases} \text{(i)} & A \text{ generates a } C_0 \text{ semigroup } e^{tA} \quad \text{in } H, \\ \text{(ii)} & B \in \mathcal{L}(U; H), \\ \text{(iii)} & C \in \mathcal{L}(H; Y). \end{cases}$$

Clearly, if the assumptions $(\mathcal{H})_\infty$ hold, then the assumptions $(\mathcal{H})$ of §1 in Chapter 1 (Part IV) are verified with $P_0 = 0$.

We want to minimize the *cost function*:

$$J_\infty(u) = \int_0^\infty \{|Cx(s)|^2 + |u(s)|^2\}\, ds, \tag{1.2}$$

over all controls $u \in L^2(0, \infty; U)$ subject to the differential equation constraint (1.1). We say that the control $u \in L^2(0, \infty; U)$ is *admissible* if $J_\infty(u) < \infty$. An admissible control $u^\star \in L^2(0, \infty; U)$ is called an *optimal control* if

$$J_\infty(u^\star) \leq J_\infty(u), \quad \forall u \in L^2(0, \infty; U).$$

In this case the corresponding solution of (1.1) is called an *optimal state* and the pair $(u^\star, x^\star)$ an *optimal pair*.

Simple examples in finite dimension show that admissible controls can fail to exist. When, for any $x_0 \in H$, an admissible control exists, we say that (A, B) is *stabilizable with respect to the observation operator* C, or, for brevity, that (A, B) is C-*stabilizable*. In this case it is still possible to solve problem (1.1)–(1.2) by using Dynamic Programming. In fact when there exists an admissible control, one can show that the *minimal solution* $P_{\min}(t)$ to the Riccati equation (2.7) in §1 of Chapter 1 in Part IV, that is the solution corresponding to the initial condition $P_0 = 0$, converges, as $t \to \infty$ to a solution $P_{\min}^{\infty}$ of the *algebraic Riccati equation*:

$$A^\star X + XA - XBB^\star X + C^\star C = 0. \tag{1.3}$$

Moreover (1.3) has a nonnegative solution if and only if (A, B) is C-stabilizable. The *minimal nonnegative solution* of (1.3) $P_{\min}^{\infty}$ is the main tool in solving problem (1.1)–(1.2). In fact one can show (see §3) that if (A, B) is C-stabilizable, then the optimal control $u^\star$ is given by

$$u^\star(t) = -B^\star P_{\min}^{\infty} x^\star(t), \tag{1.4}$$

where $x^\star$ is the solution to the *closed loop equation*:

$$\begin{cases} x'(t) = (A - BB^\star P_{\min}^{\infty})x(t), \quad t \in [0, T], \\ x(0) = x_0 \in H. \end{cases} \tag{1.5}$$

The operator

$$F = A - BB^\star P_{\min}^{\infty} \tag{1.6}$$

is called the *feedback operator*. It is important for the applications to know whether F is exponentially stable or not. A sufficient condition for this is that (A, C) be *detectable*, i.e. that $(A^\star, C^\star)$ be I-stabilizable. This condition is also necessary if H is finite dimensional or in special situations (see §3). In §4 we study some qualitative properties of the solutions of (1.3). In particular we find that if F is exponentially stable, then $P_{\min}^{\infty}$ is globally attractive (among all nonnegative solutions of Riccati equations), and so it is the unique solution of (1.3). We also study the existence of a maximal solution $P_{\mathrm{Max}}^{\infty}$ of (1.3). We prove that a maximal solution $P_{\mathrm{Max}}^{\infty}$ exists if (A, B) is I-stabilizable. In this case one can show that the infimum of $J_\infty(u)$ over all $u \in L^2(0, \infty; U)$ such that $x \in L^2(0, \infty; H)$, subject to the differential equation constraint (1.1), is precisely $(P_{\mathrm{Max}}^{\infty} x_0, x_0)$.

We end §4 by studying periodic solutions of Riccati equations. Finally §5 is devoted to examples and §6 to complements.

2 The algebraic Riccati equation

We assume here that $(\mathcal{H})_\infty$ is verified and consider the system (1.1).

Definition 2.1. (i) (A, B) is said to be stabilizable with respect to the observation C, or C-stabilizable, if for any $x_0 \in H$ there exist $u \in L^2(0, \infty; U)$ such that $Cx \in L^2(0, \infty; Y)$, where x is the corresponding solution to (1.1).

(ii) (A, B) is called feedback stabilizable with respect to the observation C, if there exists $K \in \mathcal{L}(H; U)$, $N > 0$, $\omega > 0$ such that

$$\|Ce^{t(A-BK)}\| \leq Ne^{-\omega t}, \quad \forall t \geq 0. \qquad \square$$

We consider the Riccati equation

$$P' = A^\star P + PA - PBB^\star P + C^\star C \tag{2.1}$$

and the corresponding stationary equation

$$A^\star X + XA - XBB^\star X + C^\star C = 0. \tag{2.2}$$

In the sequel we shall consider only nonnegative solutions of (2.1) and (2.2).

Definition 2.2. (i) We say that $X \in \Sigma^+(H)$ is a *weak solution* of (2.2) if

$$(Xx, Ay) + (Ax, Xy) - (B^\star Xx, B^\star Xy) + (Cx, Cy) = 0 \tag{2.3}$$

for all $x, y \in D(A)$.

(ii) X is called a *strict solution* of (2.2) if $X \in D(\mathcal{A})$ (the operator defined by (3.3) in Chapter 1 (Part IV) and

$$\mathcal{A}(X) - XBB^\star X + C^\star C = 0. \tag{2.4}$$

$\square$

We remark that if X is a strict solution of (2.2), then by Proposition 3.1 (Chapter 1, Part IV), we have

$$A^\star Xx + XAx - XBB^\star Xx + C^\star Cx = 0 \tag{2.5}$$

for any $x \in D(A)$ (because if $x \in D(A)$ then $Xx \in D(A^\star)$).

Finally we introduce the following definition.

Definition 2.3. We say that $X \in \Sigma^+(H)$ is a *stationary solution* of (2.1) if it coincides with the mild solution of (2.1) with initial condition $P(0) = X$. $\square$

Proposition 2.1. *Let $X \in \Sigma^+(H)$, then the following statements are equivalent:*

(i) *X is a weak solution of* (2.2).
(ii) *X is a strict solution of* (2.2).
(iii) *X is a stationary solution of* (2.1).

Proof. (i) $\Longrightarrow$ (ii). Assume that (2.3) holds for any $x, y \in D(A)$. Then we have

$$\varphi_X(x, y) = (B^\star Xx, B^\star Xy) - (Cx, Cy), \qquad \forall x, y \in D(A),$$

where φ_X is the bilinear form defined by (3.1) in Chapter 1 of Part IV. Clearly φ_X is continuous in $H \times H$ so that $X \in D(\mathcal{A})$ and (2.4) holds.

(ii) $\Longrightarrow$ (i). Let $X \in D(\mathcal{A})$ be a strict solution of (2.1) and let $x \in D(A)$; then $Xx \in D(A^\star)$ and (2.5) holds true. Now, if $x, y \in D(A)$, (2.3) follows and X is a weak solution.

(ii) $\Longrightarrow$ (iii). If X is a strict solution of (2.2), then $P(t) = X$ is a strict solution. Therefore it is a stationary solution, of (2.1).

(iii) $\Longrightarrow$ (i). Let X be a stationary solution of (2.1), and set $P(t) = X$. Then by Proposition 2.1 in Chapter 1 of Part IV, P is a weak solution of (2.1) so that (2.3) holds and X is a weak solution of (2.2). □

Due to Proposition 2.1, all kinds of solutions are the same; in the sequel we shall call a *solution* both a weak and a strict solution of (2.2).

We are going to study existence of a solution of the algebraic Riccati equation. We follow here A. J. Pritchard and J. Zabczyk [1]. It is useful to consider the solution of the Riccati equation with initial condition 0,

$$\begin{cases} P' = A^\star P + PA - PBB^\star P + C^\star C, \\ P(0) = 0. \end{cases} \tag{2.6}$$

Its solution will be denoted by $P_{\min}(\cdot)$.

Remark 2.1. $P_{\min}(\cdot)$ is the *minimal nonnegative solution* of the Riccati equation. In fact if $P_0 \in \Sigma^+(H)$ and P is the mild solution of (2.1) such that $P(0) = P_0$, then by Proposition 2.2 in Chapter 1 of Part IV, we have

$$P_{\min}(t) \le P(t), \quad \forall t \ge 0.$$

It follows that if X is a solution of (2.2), then

$$P_{\min}(t) \le X, \quad \forall t \ge 0.$$ □

We now prove the following properties.

Proposition 2.2. *The following statements hold*:

(i) *for any* $x \in H, (P_{\min}(\cdot)x, x)$ *is non decreasing,*
(ii) *assume that, for some* $R \in \Sigma^+(H)$*, we have*

$$P_{\min}(t) \le R, \quad \forall t \ge 0.$$

Then for all $x \in H$ *the limit*

$$P_{\min}^\infty x = \lim_{t\to\infty} P_{\min}(t)x, \tag{2.7}$$

exists, and $P_{\min}^\infty$ *is a solution of equation* (2.2).

Proof. Let $\varepsilon > 0$, $t \ge 0$ and let P be the solution of (2.1) such that $P(0) = P_{\min}(\varepsilon)$. By Proposition 2.2 in Chapter 1 of Part IV, we have

$$P_{\min}(t+\varepsilon) = P(t) \ge P_{\min}(t)$$

and (i) is proved. Assume now $P_{\min}(t) \le R$; as $P_{\min}(t)$ is nondecreasing and bounded we can set

$$\gamma(x) = \lim_{t\to\infty} (P_{\min}(t)x, x), \quad \forall x \in H.$$

For $x, y \in H$ we have

$$2\,\mathrm{Re}(P_{\min}(t)x, y) = (P_{\min}(t)(x+y), x+y) - (P_{\min}(t)x, x) - (P_{\min}(t)y, y),$$
$$2\,\mathrm{Im}(P_{\min}(t)x, y) = i(P_{\min}(t)(x+iy), x+iy) - (P_{\min}(t)x, x) - (P_{\min}(t)(iy), iy).$$

So the limit

$$\Gamma(x, y) = \lim_{t\to\infty} (P_{\min}(t)x, y), \quad \forall x, y \in H,$$

exists and the following operator $P_{\min}^{\infty} \in \Sigma^+(H)$ can be defined:

$$\lim_{t\to\infty} (P_{\min}(t)x, y) = (P_{\min}^{\infty}(t)x, y), \quad \forall x, y \in H.$$

It follows that

$$\lim_{t\to\infty} ([P_{\min}^{\infty} - P_{\min}(t)]x, x) = 0, \quad x \in H,$$

which is equivalent to

$$\lim_{t\to\infty} \left(P_{\min}^{\infty} - P_{\min}(t)\right)^{1/2} x = 0, \quad \forall x \in H.$$

This implies that

$$\lim_{t\to\infty} \left(P_{\min}^{\infty} - P_{\min}(t)\right) x = 0, \quad \forall x \in H,$$

so that (2.7) holds. It remains to show that $P_{\min}^{\infty}$ is a solution of (2.2). For this we denote by P_h the solution of (2.1) for which $P_h(0) = P_{\min}(h)$, i.e. $P_h(t) = P_{\min}(h+t)$. As

$$\lim_{h\to\infty} P_{\min}(h)x = P_{\min}^{\infty}x, \quad \forall x \in H,$$

by Theorem 2.2 in Chapter 1 of Part IV, we have

$$\lim_{h\to\infty} P_h(\cdot)x = P_{\min}^{\infty}x \quad \text{in } C([0,T]; H), \qquad \forall x \in H,\ \forall T > 0.$$

Moreover $P_{\min}^{\infty}$ is a solution of (2.1) (hence stationary).

Remark 2.2. Assume that there exists a solution $X \in \Sigma^+(H)$ of (2.2). Then, by Proposition 2.2 and Remark 2.1, the solution $P_{\min}^{\infty}$ defined by (2.7) exists. By the above proposition it follows that

$$P_{\min}^{\infty} \le X,$$

for all solution $X \in \Sigma^+(H)$ of (2.2). Thus $P_{\min}^{\infty}$ is the *minimal solution* of the algebraic Riccati equation (2.2). □

We now prove that a nonnegative solution of the algebraic Riccati equation exists if and only if (A, B) is C-stabilizable.

Proposition 2.3. *Assume that $(\mathcal{H})_\infty$ is verified and that (A, B) is C-stabilizable. Then there exists a minimal solution $P_{\min}^\infty$ of (2.2).*

Proof. We first recall that by (6.2) in Part IV, Chapter 1, we have

$$(P_{\min}(t)x_0, x_0) + \int_0^t |u(s) + B^\star P_{\min}(t-s)x(s)|^2\, ds \\ = \int_0^t \{|Cx(s)|^2 + |u(s)|^2\}\, ds, \quad (2.8)$$

for any $x_0 \in H$ and any $u \in L^2(0, \infty; U)$, where x is the solution of (1.1). Let u be a control in $L^2(0, \infty; U)$ such that the corresponding solution x of (1.1) is such that Cx belongs to $L^2(0, \infty; Y)$. By (2.8) it follows that

$$\sup_{t \geq 0}(P_{\min}(t)x_0, x_0) \leq \int_0^\infty \{|Cx(s)|^2 + |u(s)|^2\}\, ds < \infty$$

for any $x_0 \in H$. By the Uniform Boundedness Theorem it follows that $P_{\min}(t)$ is bounded, so that, by Proposition 2.2, there exists a solution of (2.2). $\square$

In order to prove the converse result it is useful to introduce, for any $t > 0$, the following auxiliary optimal control problems over the finite time horizon $[0, t]$: To minimize

$$J_t(u) = \int_0^t \{|Cx(s)|^2 + |u(s)|^2\}\, ds, \quad (2.9)$$

over all controls $u \in L^2(0, t; U)$ subject to the differential equation constraint (1.1). By Theorem 6.1 in Part IV, Chapter 1, we know that there exists a unique optimal pair (x_t, u_t) for problem (2.9), where x_t is the mild solution to the closed loop equation:

$$\begin{cases} x_t'(s) = [A - BB^\star P_{\min}(t-s)]x_t(s), \quad 0 \leq s \leq t, \\ x_t(0) = x_0, \end{cases}$$

and u_t is given by the feedback formula

$$u_t(s) = -B^\star P_{\min}(t-s)x_t(s), \quad 0 \leq s \leq t.$$

Moreover the optimal cost is given by

$$(P_{\min}(t)x_0, x_0) = \int_0^t \{|Cx_t(s)|^2 + |u_t(s)|^2\}\, ds. \quad (2.10)$$

Lemma 2.1. *Assume that the minimal solution $P_{\min}^{\infty}$ of* (2.2) *exists. Denote by x_∞ the corresponding mild solution to the equation*

$$\begin{cases} x_\infty'(s) = [A - BB^\star P_{\min}^\infty] x_\infty(s), \quad s \ge 0, \\ x_\infty(0) = x_0, \end{cases}$$

and set

$$u_\infty(s) = -B^\star P_{\min}^\infty x_\infty(s), \quad s \ge 0. \tag{2.11}$$

Then we have

$$\lim_{t\to\infty} x_t(s) = x_\infty(s), \quad s \ge 0, \tag{2.12}$$

$$\lim_{t\to\infty} u_t(s) = u_\infty(s), \quad s \ge 0. \tag{2.13}$$

Proof. Fix $T > t$ and set $z_t = x_t - x_\infty$; then for all $0 \le s \le T$, z_t is the mild solution of the problem:

$$\begin{cases} z_t'(s) = [A - BB^\star P_{\min}(t-s)] z_t(s) + BB^\star [P_{\min}(t-s) - P_{\min}^\infty] x_\infty(s), \\ z_t(0) = 0. \end{cases} \tag{2.14}$$

Denote by $U(r,s)$ the evolution operator corresponding to $A - BB^\star P_{\min}(t-\cdot)$; then for $x \in H$

$$\begin{cases} U(r,\sigma)x = e^{(r-\sigma)A}x - \displaystyle\int_\sigma^r e^{(r-\rho)A} BB^\star P_{\min}(t-\rho) U(\rho,\sigma)x \, d\rho, \\ U(\sigma,\sigma) = I. \end{cases}$$

It follows that

$$\|U(r,\sigma)\| \le M e^{(r-\sigma)\omega} + M\|B\|^2 \|P_{\min}^\infty\| \int_\sigma^r e^{(r-\rho)\omega} \|U(\rho,\sigma)\| \, d\rho.$$

By Gronwall's Lemma we have

$$\|U(r,\sigma)\| \le M e^{(r-\sigma)[\omega + M\|B\|^2 \|P_{\min}^\infty\|]}, \quad 0 \le \sigma \le r \le T. \tag{2.15}$$

We now return to problem (2.14), which we write in the form

$$z_t(s) = \int_0^s U(s,\sigma) BB^\star [P_{\min}(t-\sigma) - P_{\min}^\infty] x_\infty(\sigma) \, d\sigma.$$

By (2.15) and by the Dominated Convergence Theorem we obtain $z_t(s) \to 0$ as $t \to \infty$. So (2.12) and then (2.13) follow. □

We can now prove the following proposition.

Proposition 2.4. *Assume that there exists a solution of* (2.2). *Then (A, B) is C-stabilizable.*

Proof. Let x_t and u_t be defined as in the lemma. By (2.10) we have for $t \geq T$

$$(P^{\infty}_{\min} x_0, x_0) \geq \int_0^T \{|Cx_t(s)|^2 + |u_t(s)|^2\}\, ds \tag{2.16}$$

and, as $t \to \infty$,

$$(P^{\infty}_{\min} x_0, x_0) \geq \int_0^T \{|Cx_{\infty}(s)|^2 + |u_{\infty}(s)|^2\}\, ds. \tag{2.17}$$

But as T is arbitrary we find

$$(P^{\infty}_{\min} x_0, x_0) \geq \int_0^{\infty} \{|Cx_{\infty}(s)|^2 + |u_{\infty}(s)|^2\}\, ds \tag{2.18}$$

and thus $u_{\infty} \in L^2(0, \infty; U)$ is an admissible control. □

3 Solution of the control problem

We now consider the control problem (1.1)–(1.2) and prove the central result.

Theorem 3.1. *Assume that the conditions* $(\mathcal{H})_{\infty}$ *are verified and that* (A, B) *is* C*-stabilizable. Then there exists a unique optimal pair* $(u^{\star}, x^{\star})$ *for the optimal control problem* (1.1)–(1.2)*. Moreover the following statements hold*:

(i) $x^{\star} \in C([0, \infty[; H)$ *is the mild solution to the closed loop equation* (1.5)*.*
(ii) $u^{\star} \in C([0, \infty[; U)$ *is given by the feedback formula*

$$u^{\star}(t) = -B^{\star} P^{\infty}_{\min} x^{\star}(t), \quad t \in [0, T], \tag{3.1}$$

where $P^{\infty}_{\min}$ *represents the minimal solution of* (2.2)*.*
(iii) *The optimal cost* $J_{\infty}(u^{\star})$ *is given by*

$$J_{\infty}(u^{\star}) = (P^{\infty}_{\min} x_0, x_0). \tag{3.2}$$

Proof. Let $u \in L^2(0, \infty; U)$ and let x be the corresponding solution of the state equation (1.1). By the identity (2.8) we have

$$(P_{\min}(t) x_0, x_0) \leq \int_0^t \{|Cx(s)|^2 + |u(s)|^2\}\, ds \leq J_{\infty}(u).$$

It follows that

$$J_{\infty}(u) \geq (P^{\infty}_{\min}(t) x_0, x_0), \quad \forall u \in L^2(0, \infty; U).$$

Let now u_{∞} be defined by(2.11); by (2.18) we have

$$(P^{\infty}_{\min}(t) x_0, x_0) \geq J_{\infty}(u_{\infty}),$$

so that u_∞ is optimal. Formula (3.1) with $u^\star = u_\infty$, $x^\star = x_\infty$ follows from (2.12)–(2.13). It remains to show uniqueness. Let $(\hat{u}, \hat{x})$ be another optimal pair; then $J_\infty(\hat{u}) = (P_{\min}^\infty x_0, x_0)$. Fix $T > 0$. By applying (2.8) with $t \geq T$ we obtain

$$\begin{aligned}\int_0^T |\hat{u}(s) + B^\star P_{\min}(t-s)\hat{x}(s)|^2\, ds &\leq J_\infty(\hat{u}) - (P_{\min}(t)x_0, x_0)\\ &\leq ([P_{\min}^\infty - P_{\min}(T)]x_0, x_0).\end{aligned}$$

As $t \to \infty$ we have

$$\int_0^T |\hat{u}(s) + B^\star P_{\min}\hat{x}(s)|^2\, ds \leq ([P_{\min}^\infty - P_{\min}(T)]x_0, x_0),$$

and letting T tend to ∞, necessarily $\hat{u}(s) = -B^\star P_{\min}^\infty \hat{x}(s)$. Consequently $\hat{x}$ is also a solution of the closed loop equation that necessarily coincides with x_∞. Then $\hat{u} = u_\infty$ and the proof is complete. □

We now give a regularity result for the optimal pair $(u^\star, x^\star)$.

Proposition 3.1. *Assume that the conditions of Theorem* 3.1 *are verified and let* $(u^\star, x^\star)$ *be the optimal pair for the control problem* (1.1)–(1.2). *Then the following statements hold*:

(i) *If* $x_0 \in D(A)$, *then* $x^\star$ *is a strict solution of* (1.5) *and belongs to*

$$C^1([0,\infty[; H) \cap C\big([0,\infty[; D(A)\big).$$

(ii) *If* e^{tA} *is an analytic semigroup, then* $x^\star$ *is a classical solution of* (1.5).

Proof. (i) follows from Proposition 3.3 in Part II, Chapter 1, and (ii) from Proposition 3.9 in Part II, Chapter 1. □

3.1 Feedback operator and detectability

We assume here that the conditions $(\mathcal{H})$ are verified and that (A, B) is C-stabilizable. We denote by $P_{\min}^\infty$ the minimal solution of the algebraic Riccati equation (2.2). Under these hypotheses, by Theorem 3.1, there exists a unique optimal pair $(u^\star, x^\star)$, and

$$\begin{cases} x^\star(t) = e^{tF}x_0, \quad t \geq 0,\\ u^\star(t) = -B^\star P_{\min}^\infty x^\star(t) = -B^\star P_{\min}^\infty e^{tF}x_0, \quad t \geq 0, \end{cases}$$

where

$$F = A - BB^\star P_{\min}^\infty$$

is called the *closed loop operator*. Moreover, by (3.2) we have

$$(P_{\min}^\infty x_0, x_0) = \int_0^\infty \{|Ce^{tF}x_0|^2 + |B^\star P_{\min}^\infty e^{tF}x_0|^2|\}\, ds. \tag{3.3}$$

Remark 3.1. When $C = I$ (or if $C^{-1} \in \mathcal{L}(H)$), by (3.3) and Datko's Theorem, it follows that F is exponentially stable. In particular, if (A, B) is stabilizable with respect to the identity I then it is *feedback stabilizable.* □

It is important for the applications to give conditions under which F is stable (even when C is not invertible). A sufficient condition is given by the *detectability* property. In order to define detectability we introduce the *dual system*

$$\begin{cases} y'(t) = A^\star y(t) + C^\star v(t), \quad t \geq 0, \\ y(0) = y_0 \in H. \end{cases}$$

For this system the space of states, controls, and observations are H, Y, and H, respectively.

Definition 3.1. We say that the pair (A, C) is detectable if the pair $(A^\star, C^\star)$ is I-stabilizable. □

Remark 3.2. By Remark 3.1 it follows that (A, C) is detectable if and only if there exists $K \in \mathcal{L}(Y; H)$ such that $A - KC$ is exponentially stable. □

The next result is due to W. M. Wonham [2] in the finite dimensional case and to J. Zabczyk [3] in the general case.

Proposition 3.2. *Assume that (A, C) is detectable. Then F is exponentially stable.*

Proof. Let $x_0 \in H$, $x(t) = e^{tF}x_0$; by Remark 3.2 there exists $K \in \mathcal{L}(Y; H)$ such that $A - KC$ is exponentially stable. As

$$F = (A - KC) + (KC - BB^\star P_{\min}^\infty),$$

we have that x is the mild solution to the problem

$$\begin{cases} x' = (A - KC)x + z, \\ x(0) = x_0, \end{cases}$$

where $z = (KC - BB^\star P_{\min}^\infty)x$. It follows that

$$x(t) = e^{t(A-KC)}x_0 + \int_0^t e^{(t-s)(A-KC)}[KC - BB^\star P_{\min}^\infty]x(s)\, ds.$$

By (3.3) we have that Cx and $BB^\star P_{\min}^\infty x$ belong to $L^2(0, \infty; H)$; it follows that $x \in L^2(0, \infty; H)$ because $A - KC$ is exponentially stable. By Datko's Theorem this implies the conclusion. □

Remark 3.3. The assumption that (A, C) is detectable is not a necessary condition that $F = A - BB^\star P_{\min}^\infty$ be exponentially stable, as Example 3.1 below shows. Notice however that this condition is necessary and sufficient in some special cases (see §3.2). □

The following example was discussed in the reference of G. Da Prato and M. C. Delfour [1, 2].

Example 3.1. Let $H = U = Y = \ell^2$, the Hilbert space of all the sequences $\{x_n\}$ of complex numbers such that $\sum_{n=1}^{\infty} |x_n|^2 < \infty$. We set

$$A\{x_n\} = \left\{\frac{n}{n+1}x_n\right\}, \quad B = I, \quad C\{x_n\} = \left\{\frac{\sqrt{2n+1}}{n+1}x_n\right\}.$$

Then (2.2) reduces to

$$2AX - X^2 + C^2 = 0,$$

and we have

$$\begin{aligned} P_{\min}^{\infty} &= A + \sqrt{A^2 + C^2} = A + I, \\ F &= A - P_{\min}^{\infty} = -I \end{aligned}$$

Thus F is exponentially stable. We now show that (A, C) is not detectable. In fact assume, by contradiction, that (A, C) is detectable. Then the algebraic Riccati equation

$$2AX - C^2X^2 + I = 0$$

has a positive solution X and we have

$$X\{x_n\} = \{(n+1)x_n\}.$$

However this is not a bounded operator in H. □

3.2 Stabilizability and stability of the closed loop operator F in the point spectrum case

We assume here $(\mathcal{H})_\infty$–(i)–(ii); we are interested in the stabilizability of the system (1.1) under suitable conditions on the spectrum of A. We denote by $\sigma^-(A)$, $\sigma^+(A)$, and $\sigma^0(A)$ those elements of the spectrum of A that have a negative, positive, and zero real part, respectively. The elements of $\sigma^+(A) \cup \sigma^0(A)$ are called *unstable* points of $\sigma(A)$. Obviously, if $\sigma^+(A) \cup \sigma^0(A)$ is empty and if A verifies the spectral determining condition (see Part II, Chapter 1, §2.9 §2.9), then (A, B) is stabilizable. We now assume that

$$(\mathcal{P}) \begin{cases} \text{(i)} & \text{the set } \sigma^+(A) \cup \sigma^0(A) \text{ consists of a finite set of} \\ & \text{eigenvalues of finite algebraic multiplicity,} \\ \text{(ii)} & \text{there exists } \varepsilon > 0,\ N_A > 0 \text{ such that} \\ & \displaystyle\sup_{\lambda \in \sigma^-(A)} \operatorname{Re}\lambda < -\varepsilon, \quad \|e^{tA}\Pi_A^-\| \le N_A e^{-\varepsilon t}, \quad \forall t \ge 0. \end{cases}$$

Here Π_A^- represents the projector on $\sigma^-(A)$ defined by

$$\Pi_A^- = \frac{1}{2\pi i}\int_{\gamma^-} R(\lambda, A)\, d\lambda$$

and γ^- is a simple Jordan curve around $\sigma^-(A)$. The projectors Π_A^+ and Π_A^0 are defined analogously. We set

$$H_A^- = \Pi_A^-(H), \quad H_A^+ = \Pi_A^+(H), \quad H_A^0 = \Pi_A^0(H).$$

Then H_A^-, H_A^+, H_A^0 are invariant subspace for the semigroup e^{tA}.

Remark 3.4. Assume that A fulfills $(\mathcal{P})$. Then by hypothesis (ii) H_A^+ is finite dimensional and there exists $\eta > 0$ such that

$$\operatorname{Re}\lambda > \eta, \quad \forall \lambda \in \sigma^+(A). \tag{3.4}$$

Moreover $e^{tA}H_A^+$ can be extended for $t < 0$ by the formula

$$e^{tA}\Pi_A^+ = \frac{1}{2\pi i}\int_{\gamma^+} e^{t\lambda} R(\lambda, A)\, d\lambda, \quad t \in \mathbb{R}.$$

Finally, by (3.4) there exists $N_A' > 0$ such that

$$\|e^{tA}\Pi_A^+\| \le N_A', \quad \forall t \le 0. \tag{3.5}$$

□

Remark 3.5. Assumptions $(\mathcal{P})$ are verified in each of the following cases:

(i) H is finite dimensional.
(ii) e^{tA} is compact for any $t > 0$. □

3.3 Stabilizability

In this section we want to give a necessary and sufficient condition in order that (A, B) be stabilizable with respect to the identity I. We first prove a lemma on the existence of solutions of (1.1) in $[0, +\infty[$.

Lemma 3.1. *Assume that conditions* $(\mathcal{H})_\infty$–(i)–(ii) *are verified and moreover that A fulfills* $(\mathcal{P})$ *with* $\sigma^0(A) = \varnothing$. *Then the following statements are equivalent*:

(i) *for all* $x \in H$, *there exists* $u \in L^2([0, +\infty[\,; U)$ *such that* $x \in L^2([0, +\infty[\,; H)$, x *being the solution to* (1.1) *corresponding to* u.
(ii) *Range* $\gamma \supset \Pi_A^+(H)$, *where* γ *is defined by*

$$u \mapsto \gamma u = \int_0^\infty e^{-sA}\Pi_A^+ Bu(s)\, ds \colon L^2([0, +\infty[\,; U) \to H,$$

(iii) *the mapping*

$$\xi \mapsto \gamma^\star\xi = B^\star(\Pi_A^+)^\star e^{-\cdot A^{STAR}}\xi\colon H \to L^2([0,+\infty[\,;U),$$

is one-to-one,

(iv) *for all* $\lambda \in \sigma^+(A^\star)$, $\quad$ *Ker* $(B^\star) \cap \mathrm{Ker}(\lambda - A^\star) = \{0\}$.

Proof. Let $u \in L^2([0,+\infty[\,;U)$, and let x be the solution to (1.1). Then we have

$$x(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A}Bu(s)\,ds = x_-(t) + x_+(t),$$

where

$$x_-(t) = e^{tA}\Pi_A^- x_0 + \int_0^t e^{(t-s)A}\Pi_A^- Bu(s)\,ds - \int_t^{+\infty} e^{(t-s)A}\Pi_A^+ Bu(s)\,ds,$$

$$x_+(t) = e^{tA}\left\{\Pi_A^+ x_0 + \int_0^{+\infty} e^{-sA}\Pi_A^+ Bu(s)\,ds\right\}.$$

Remark that $x_- \in L^2(0,\infty;H)$ by $(\mathcal{P})$–(ii) and (3.5), whereas $x_+ \in L^2(0,\infty;H)$ if and only if

$$\Pi_A^+ x_0 + \int_0^{+\infty} e^{-sA}\Pi_A^+ Bu(s)\,ds = 0.$$

It follows that (i) $\Longleftrightarrow$ (ii). Moreover as the space $\Pi_A^+(H)$ is finite dimensional, (ii) $\Longleftrightarrow$ (iii) by the Alternative Principle. Thus it remains to show that (iii) $\Longleftrightarrow$ (iv). Assume (iii) and, by contradiction, that (iv) does not hold. Then there exist $\lambda \in \sigma^+(A^\star)$ and $\xi \in \left(\Pi_A^+(H)\right)^\star$ different from 0 such that $B^\star\xi = 0$, $\lambda\xi = A^\star\xi = 0$. Then $e^{-sA^\star}\xi = e^{-\lambda s}\xi$ and we have

$$B^\star e^{-sA^\star}(\Pi_A^+)^\star\xi = 0, \quad s \geq 0, \tag{3.6}$$

which is a contradiction.

(iv) $\Longrightarrow$ (iii). Assume (iv) and, by contradiction, that (iii) does not hold. Then there exists $\xi \in \left(\Pi_A^+(H)\right)^\star$ different from 0 such that (3.6) holds. By differentiating (3.6) several times with respect to s and by setting $s = 0$ we see that

$$p(A^\star)\xi = 0$$

for all polynomials p. Choose p as the minimal degree polynomial such that $p(A^\star)\xi = 0$. Let λ_0 be a root of p; then clearly $\lambda_0 \in \sigma^+(A^\star)$. Set

$$q(\lambda) = \frac{p(\lambda)}{\lambda - \lambda_0} \quad \text{and} \quad \psi = q(A^\star)\xi;$$

then by (3.6) it follows that $\psi \in \mathrm{Ker}(B^\star) \cap \mathrm{Ker}(\lambda I - A^\star)$, which is a contradiction. □

The following result is a straightforward generalization of a result due to M. L. J. Hautus [2].

Proposition 3.3. *Assume that conditions* $(\mathcal{H})_\infty$–(i)–(ii) *and* $(\mathcal{P})$ *are verified. Then the following statements are equivalent*:

(i) (A,B) *is stabilizable with respect to* I,
(ii) $\mathrm{Ker}(\lambda - A^\star) \cap \mathrm{Ker}(B^\star) = \{0\}, \quad \forall\lambda \in \sigma^0(A) \cup \sigma^+(A)$.

Proof. We can clearly choose $\varepsilon > 0$ such that, setting $A_\varepsilon = A + \varepsilon I$, then $\sigma^0(A_\varepsilon)$ is empty, so that we can apply Lemma 3.1. Setting $v(t) = e^{\varepsilon t}u(t)$, $z(t) = e^{\varepsilon t}x(t)$, problem (1.1) becomes

$$z'(t) = A_\varepsilon z(t) + Bv(t), \quad z(0) = x.$$

Now assume that (ii) holds. By Lemma 3.1, for all $x_0 \in H$, there exists $v \in L^2([0,\infty[;U)$ such that $z \in L^2(0,\infty;H)$; thus (A,B) is I-stabilizable and (i) holds.

It remains to prove that (i) $\Longrightarrow$ (ii). In fact assume (i) and that, by contradiction, (ii) does not hold. Then there exists $\lambda \in \sigma^0(A^\star) \cup \sigma^+(A^\star)$ and $\xi \in D(A^\star)$ different from 0 such that

$$B^\star\xi = 0, \quad \lambda\xi - A^\star\xi = 0. \tag{3.7}$$

Choose now $x \in H$ such that $(x_0, \xi) = 1$; by the hypothesis (i) there exists $u \in L^2(0,\infty;U)$ such that $x \in L^2(0,\infty;H)$. As

$$(x_t, \xi) = (x_0, e^{tA^\star}\xi) + \int_0^t (u(s), B^\star e^{tA^\star}\xi)\,ds,$$

by (3.7) it follows that $(x_t,\xi) = (x_0, e^{tA^\star}\xi) = e^{\lambda t}$, which is a contradiction, because $\mathrm{Re}\,\lambda \ge 0$. The proof is complete. □

3.4 Exponential stability of F

We assume here that (A,B) is C-stabilizable and denote by $F = A - BB^\star P^\infty_{\min}$ the closed loop operator. We prove the following result.

Proposition 3.4. *Assume that* (A,B) *is* C*-stabilizable and that* A *and* F *fulfill* $(\mathcal{P})$. *Then the following statements are equivalent*:

(i) F *is exponentially stable,*
(ii) *for any* $\lambda \in \sigma^+(A) \cup \sigma^0(A)$, *we have*

$$\mathrm{Ker}(A - \lambda I) \cap \mathrm{Ker}(C) = \{0\}. \tag{3.8}$$

Proof. (i) $\Longrightarrow$ (ii). Assume that F is exponentially stable and, by contradiction, that (ii) does not hold. Then there exists $x_0 \neq 0$ and $\lambda_0 \in \sigma^+(A)$ such that

$$\mathrm{Re}\,\lambda_0 \ge 0, \quad Ax_0 = \lambda_0 x_0, \quad Cx_0 = 0. \tag{3.9}$$

It follows that

$$\frac{d}{dt}(P_{\min}(t)x_0, x_0) = 2\operatorname{Re}\lambda_0(P_{\min}(t)x_0, x_0) - |B^\star P_{\min}(t)x_0|^2$$
$$\leq 2\operatorname{Re}\lambda_0(P_{\min}(t)x_0, x_0),$$

where $P_{\min}$ is the minimal nonnegative solution of (2.1). As $P_{\min}(0) = 0$ by (3.9), it follows that $P_{\min}(t)x_0 = 0$ and then $P_{\min}^\infty x_0 = 0$. Thus

$$Fx_0 = Ax_0 - BB^\star P_{\min}^\infty x_0 = Ax_0 = \lambda x_0,$$

which contradicts the fact that F be exponentially stable.

(ii) $\Longrightarrow$ (i) Assume now that (3.8) holds and, by contradiction, that F is not exponentially stable. Then there exists $\lambda_0 \in \mathbb{C}$ and $x_0 \neq 0$ in H such that

$$Fx_0 = \lambda_0 x_0, \quad \operatorname{Re}\lambda_0 \geq 0.$$

Because, as easily checked, $P_{\min}^\infty$ verifies

$$F^\star P_{\min}^\infty + P_{\min}^\infty F + P_{\min}^\infty BB^\star P_{\min}^\infty + C^\star C = 0,$$

we have

$$2\operatorname{Re}\lambda_0(P_{\min}^\infty x_0, x_0) + |B^\star P_{\min}^\infty x_0|^2 + |Cx_0|^2 = 0,$$

which implies $B^\star P_{\min}^\infty x_0 = 0$, $Ax_0 = \lambda_0 x_0$, and $Cx_0 = 0$, and this fact contradicts (ii). □

We now give a characterization for the stability of F.

Proposition 3.5. *Assume that the conditions of Proposition* 3.4 *are verified. Then the following statements are equivalent*:

(i) F *is exponentially stable,*
(ii) (A, C) *is detectable.*

Proof. (i) $\Longrightarrow$ (ii). If F is exponentially stable, then, by Proposition 3.4, (3.8) holds. But this implies that $(A^\star, C^\star)$ is stabilizable with respect to I (by Proposition 3.3). Thus (ii) is proved. The implication (i) $\Longrightarrow$ (ii) was proved in Proposition 3.4. □

4 Qualitative properties of the solutions of the Riccati equation

In this section we assume that conditions $(\mathcal{H})_\infty$ are verified and that (A, B) is stabilizable with respect to C. We again consider the equations

$$P' = A^\star P + PA - PBB^\star P + C^\star C, \tag{4.1}$$
$$A^\star X + XA - XBB^\star X + C^\star C = 0, \tag{4.2}$$

and denote by $P_{\min}$ and $P_{\min}^\infty$ the minimal solutions of (4.1) and (4.2), respectively. We say that a solution $X \in \Sigma^+(H)$ of (4.2) is *maximal* if

$$\forall \text{ solution } Y \in \Sigma^+(H) \quad \text{of (4.2)} \implies Y \le X.$$

If a maximal solution exists, it is clearly unique; however a maximal solution does not exist in general (see Remark 4.2 below). The existence of a maximal solution under suitable assumptions will be proved in §4.3.

4.1 Local stability results

We study here the exponential stability of any positive solution X of (4.2). This property of X is naturally related to the exponential stability of the linear operator $A - BB^\star X$.

Proposition 4.1. *Let $X \in \Sigma^+(H)$ be a solution of* (4.2). *Assume that $A - BB^\star X = K$ is exponentially stable, that is, that there exist $a > 0$, $N > 0$ such that*

$$\|e^{tK}\| \le Ne^{-at}, \quad t \ge 0. \tag{4.3}$$

Then there exist $r > 0$ and $a > 0$ such that, if $P_0 \in \Sigma^+(H)$ and $\|P_0 - X\| < 1/2r\,N^2$, we have

$$\|P(t) - X\| \le re^{-at}, \quad t \ge 0,$$

where P is the mild solution to (4.1) *such that $P(0) = P_0$.*

Proof. Set $Y(t) = P(t) - X$; then Y is the mild solution to the problem

$$Y' = K^\star Y + YK - YBB^\star Y, \quad Y(0) = P_0 - X = Y_0, \tag{4.4}$$

which is equivalent to the equation $\gamma(Y) = Y$, where γ is defined by

$$\gamma(Y)(t)x = e^{tK^\star} Y_0 e^{tK} x - \int_0^t e^{(t-s)K^\star} Y(s) e^{(t-s)K} x\, ds, \tag{4.5}$$

for any $x \in H$. For any $a \ge 0$ we introduce the Banach space

$$C_a\big([0,\infty[\,; \Sigma(H)\big) = \big\{Y \in C_s\big([0,\infty[\,;\Sigma(H)\big) \colon \sup_{t\ge 0} \|e^{ta}Y(t)\| < \infty\big\},$$

and, for any $r > 0$ we set

$$B_r = \big\{Y \in C_a\big([0,\infty[\,;\Sigma(H)\big) \colon \|Y\|_a \le r\big\}.$$

We now want to solve (4.4) by proving that γ has a fixed point in B_r, for a suitable r. If $Y, Z \in B_r$ we have, taking into account (4.3),

$$\|\gamma(Y)\|_a \le N^2\|Y_0\| + \frac{N^2r^2}{ea}\|B\|^2$$

$$\|\gamma(Y) - \gamma(Z)\|_a \le N^2\|Y_0\| + 2\frac{rN^2}{ea}\|B\|^2\|Y - Z\|.$$

Now choose r such that

$$2\frac{rN^2}{ea}\|B\|^2 \le \frac{1}{2}, \quad \frac{N^2r^2}{ea}\|B\|^2 \le \frac{1}{2}$$

and then Y_0 such that $\|Y_0\| \le \rho/2N^2$. Then γ is a contraction on B_r and the conclusion holds. □

4.2 Attractivity properties of a stationary solution

We first show that the minimal solution $P_{\min}^\infty$ is globally attractive *from below.*

Proposition 4.2. *Let $P_0 \in \Sigma^+(H)$ such that $P_0 \le P_{\min}^\infty$ and let P be the mild solution of* (4.1) *with $P(0) = P_0$. Then we have*

$$\lim_{t\to\infty} P(t)x = P_{\min}^\infty x, \quad \forall x \in H.$$

Proof. By Proposition 2.2 we have

$$P_{\min}(t) \le P(t) \le P_{\min}^\infty,$$

so that

$$0 \le (P_{\min}^\infty x - P(t)x, x) \le (P_{\min}^\infty x - P_{\min}(t)x, x), \quad x \in H,$$

and by (2.7)

$$\lim_{t\to\infty} (P_{\min}^\infty x - P(t)x, x) = 0, \quad x \in H,$$

which implies the conclusion. □

We now consider a general positive solution X of (4.2) and set $K = A - BB^\star X$. We first show that if K is exponentially stable, then X is globally attractive *from above.*

Proposition 4.3. *Let $X \in \Sigma^+(H)$ be a solution of* (4.2), *and assume that $K = A - BB^\star X$ is exponentially stable and that* (4.3) *holds. Then for any $P_0 \ge X$ we have*

$$\|P(t) - X\| \le N^2 e^{-2at} \|P_0 - X\|, \quad t \ge 0, \tag{4.6}$$

where P is the solution to (4.1) *such that $P(0) = P_0$.*

Proof. Set $Y = P(t) - X$; as $P_0 \ge X$, we have, by Proposition 2.2 in Part IV, Chapter 1, $Y(t) \ge 0$. Moreover Y is the mild solution to problem (4.4). For any $x \in D(A)$ we have (using the fact that Y is a strict solution of (4.4))

$$\frac{d}{ds}(Y(t-s)e^{sK}x, e^{sK}x) = |B^\star Y(t-s)e^{sK}x|^2, \quad \forall x \in D(A)$$

so that (because $D(A)$ is dense in H)

$$(Y(t)x, x) + \int_0^t |BY(t-s)e^{sK}x|^2\, ds = (Y(0)e^{sK}x, e^{sK}x), \quad \forall x \in H.$$

It follows that

$$(P(t)x - Xx, x) \le (Y(0)e^{sK}x, e^{sK}x), \quad \forall x \in H,$$

which yields (4.6). □

The following result shows that if X is a positive solution of (4.2) such that $K = A - BB^\star X$ is exponentially stable, then X is *maximal.*

Proposition 4.4. *Let* $X, Y \in \Sigma^+(H)$ *be solutions of* (4.2). *Assume that* $K = A - BB^\star X$ *is exponentially stable. Then* $X \geq Y$.

Proof. Set $Z = X - Y$; by (4.4) it follows that

$$(Zx, Kx) + (ZKx, x) + (ZBB^\star Zx, x) = 0, \quad \forall x \in D(A),$$

which implies that

$$\frac{d}{dt}(Ze^{sK}x, e^{sK}x) = -|B^{STAR}Ze^{sK}x|^2, \quad \forall x \in D(A).$$

As $D(A)$ is dense in H, by integrating this between 0 and t, we obtain

$$\begin{aligned}(Zx, x) &= (Ze^{tK}x, e^{tK}x) + \int_0^t |B^\star Ze^{sK}x|^2\, ds \\ &\geq (Ze^{tK}x, e^{tK}x), \quad \forall x \in H.\end{aligned}$$

As $t \to \infty$ we have $(Zx, x) \geq 0$, $\forall x \in H$ so that $X \geq Y$. □

Corollary 4.1. *Equation* (4.2) *has at most one positive solution* X *such that* $A - BB^\star X$ *is exponentially stable.*

Proof. Let $X, Y \in \Sigma^+(H)$ be two solutions of (4.2) such that $A - BB^\star X$ and $A - BB^\star Y$ are exponentially stable. By Proposition 4.4 it follows that $X \geq Y$ and $Y \geq X$ so that $X = Y$. □

Concerning the minimal solution $P_{\min}^\infty$ we have a uniqueness result.

Corollary 4.2. *If* $A - BB^\star P_{\min}^\infty$ *is exponentially stable, then* $P_{\min}^\infty$ *is the unique positive solution of* (4.2).

Proof. It is sufficient to observe that $P_{\min}^\infty$ is both the maximal and the minimal solution of (4.2). □

We show now that if $F = A - BB^\star P_{\min}^\infty$ is exponentially stable, then $P_{\min}^\infty$ is *globally attractive* among all the positive solutions of (4.2). In finite dimension this result is well known; see W. M. Wonham [2].

Proposition 4.5. *If* $F = A - BB^\star P_{\min}^\infty$ *is exponentially stable, then for any* $P_0 \in \Sigma^+(H)$ *we have*

$$\lim_{t\to\infty} P(t)x = P_{\min}^\infty x, \quad \forall x \in H, \tag{4.7}$$

where P *is the mild solution of* (4.1) *such that* $P(0) = P_0$.

Proof. Choose $n \in \mathbb{N}$ such that $P_0 \le nI$, $P^\infty_{\min} \le nI$. Then by Proposition 2.2 in Part IV, Chapter 1, we have

$$P_{\min}(t) \le P(t) \le Q(t), \tag{4.8}$$

where Q is the mild solution of (4.1) such that $Q(0) = nI$. Now, as $t \to \infty$, $P_{\min}(t)x \to P^\infty_{\min}x$ and, by (4.6), $Q(t)x \to P^\infty_{\min}x$; this, along with (4.8), implies (4.7). □

Remark 4.1. By Proposition 3.5 we know that if (A, C) is detectable, then the feedback operator F is exponentially stable. Thus, by Corollary 4.2, $P^\infty_{\min}$ is the unique nonnegative solution to (4.2). However, detectability of (A, C) is not necessary for the uniqueness, as the following example shows. Let $H = U = Y = \mathbb{R}^2$,

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = I, \quad C = 0;$$

as easily seen $X = 0$ is the unique solution of (4.2), but the feedback operator $F = A$ is not stable and (A, C) is not detectable.

When H is finite dimensional, M. Sorine [2] has shown that the following conditions are sufficient for the uniqueness of the solution:

(i) (A, B) is C-stabilizable,
(ii) $(A^\star, C^\star)$ is $B^\star$-stabilizable,
(iii) no eigenvalue of the closed loop operator F lies on the imaginary axis.

This result can be easily generalized to any Hilbert space H provided that A and F fulfill assumptions $(\mathcal{P})$ of §3.2. □

4.3 Maximal solutions

We assume here, beside $(\mathcal{H})_\infty$, that (A, B) is I-stabilizable (which, obviously, is stronger than assuming that (A, B) is C-stabilizable). We follow here A. Bensoussan [1].

For any $\varepsilon > 0$, we consider the problem: To minimize

$$J_\varepsilon(u) = \int_0^\infty \{|Cx(s)|^2 + \varepsilon|x(s)|^2 + |u(s)|^2\}\, ds, \tag{4.9}$$

over all controls $u \in L^2(0, \infty; U)$ subject to the differential equation constraint (1.1). Consider the observation $D_\varepsilon = \sqrt{C^\star C + \varepsilon I}$ (the observation space being H). Clearly (A, B) is D_ε-stabilizable, so that, by Theorem 3.1, there exists a unique optimal pair $(u_\varepsilon, x_\varepsilon)$ and a solution P_ε of the algebraic Riccati equation

$$A^\star X + XA - XBB^\star X + C^\star C + \varepsilon I = 0, \tag{4.10}$$

such that

$$\begin{aligned} x_\varepsilon' &= (A - BB^\star P_\varepsilon)x,\\ x_\varepsilon(0) &= x_0, \\ u_\varepsilon &= -B^\star P_\varepsilon x_\varepsilon. \end{aligned} \tag{4.11}$$

Moreover (A, D_ε) is detectable (because D_ε is invertible and $D_\varepsilon^{-1} \in \mathcal{L}(H)$). Thus by Corollary 4.2, P_ε is the unique solution in $\Sigma^+(H)$ of (4.10). Moreover, by Proposition 4.5, given $Q \in \Sigma^+(H)$ we have

$$\lim_{t\to\infty} P_Q(t) = P_\varepsilon x, \quad \forall x \in H, \tag{4.12}$$

where P_Q is the mild solution of the Riccati equation

$$P' = A^\star P + PA - PBB^\star P + C^\star C + \varepsilon I, \quad P(0) = Q;$$

that is, P_ε is globally attractive.

Proposition 4.6. *Assume that conditions* $(\mathcal{H})_\infty$ *are verified and that* (A, B) *is I-stabilizable. Then* (4.2) *has a maximal solution* P_{Max}^∞. *Moreover*

$$\lim_{\varepsilon\to 0} P_\varepsilon x = P_{\mathrm{Max}}^\infty x, \quad \forall x \in H. \tag{4.13}$$

Proof. We first remark that, as $\{P_\varepsilon\}$ is nonincreasing (by Proposition 2.2 in Part II, Chapter 1) and bounded below by 0, the limit

$$\lim_{\varepsilon\to 0} P_\varepsilon x = P_{\mathrm{Max}}^\infty x, \quad \forall x \in H, \tag{4.14}$$

exists. We have to show that P_{Max}^∞ is the required maximal solution.

Step 1.

P_{Max}^∞ is a solution to(4.2).
In fact for all $x, y \in D(A)$, we have

$$(P_\varepsilon Ax, y) + (P_\varepsilon x, Ay) - (B^\star P_\varepsilon x, B^\star P_\varepsilon x) + (Cx, Cy) + \varepsilon(x, y) = 0;$$

letting ε tend to 0 and using (4.14) we see that P_{Max}^∞ is a solution of (4.2).

Step 2.

P_{Max}^∞ is maximal.
Let Q be any solution of (4.2); then, by Proposition 2.2 in Part II, Chapter 1, we have $P_Q(t) \geq Q$ (where P_Q is the solution to (4.9)). Thus, by (4.12) and (4.14) it follows that $P_{\mathrm{Max}}^\infty \geq Q$, so that P_{Max}^∞ is maximal. □

We will now give the variational interpretation of the maximal solution.

Theorem 4.1. *Assume* $(\mathcal{H})_\infty$ *and that* (A,B) *is I-stabilizable. Let* P_{Max}^∞ *be the maximal solution of* (4.2) *and let* $x_0 \in H$*. Then*

$$(P_{\text{Max}}^\infty x_0, x_0) = \inf\{J_\infty(u)\colon u \in L^2(0,\infty;U), x \in L^2(0,\infty;H)\}, \tag{4.15}$$

where (x,u) *are subject to the differential equation constraint* (1.1)*. Moreover*

$$(P_{\text{Max}}^\infty x_0, x_0) \geq J_\infty(\hat{u}), \tag{4.16}$$

where $\hat{u} = \lim_{\varepsilon\to 0} u_\varepsilon$ *and* u_ε *is defined by* (4.11)*.*

Proof. Set

$$U_{ad} = \inf\{J_\infty(u)\colon u \in L^2(0,\infty;U), x \in L^2(0,\infty;H)\},$$

where x is the solution of (1.1).

We first remark that, recalling (4.13) and arguing as we did in the proof of Lemma 2.1, we can show that the following limits:

$$\hat{x}(t) = \lim_{\varepsilon\to 0} x_\varepsilon(t), \quad \hat{u}(t) = \lim_{\varepsilon\to 0} u_\varepsilon(t), \qquad t \geq 0 \tag{4.17}$$

exist and are uniform on the bounded subsets of $[0,\infty[$. Let now $u \in U_{ad}$. By identity (6.2) in Chapter 1 of Part IV, we have

$$\begin{aligned}(P_{\text{Max}}^\infty x_0, x_0) &+ \int_0^t \{|u(s) + B^\star P_{\text{Max}}^\infty x(s)|^2\}\, ds \\ &= \int_0^t \{|Cx(s)|^2 + |u(s)|^2\}\, ds + \big(P_{\text{Max}}^\infty x(t), x(t)\big).\end{aligned}$$

As $x \in L^2(0,\infty;H)$, there exists a sequence $t_n \nearrow \infty$ such that $x(t_n) \to 0$. Therefore

$$(P_{\text{Max}}^\infty x_0, x_0) + \int_0^t |u(s) + B^\star P_{\text{Max}}^\infty x(s)|^2\, ds = J_\infty(u), \tag{4.18}$$

which implies that

$$(P_{\text{Max}}^\infty x_0, x_0) \leq J_\infty(u), \quad \forall u \in U_{ad}.$$

Letting ε tend to zero in the equality

$$(P_\varepsilon x_0, x_0) = J_\infty(u_\varepsilon) + \varepsilon \|x_\varepsilon\|^2_{L^2(0,\infty;H)},$$

one obtains (4.16); also, as $u_\varepsilon \in U_{ad}$, the conclusion follows. □

Proposition 4.7. *Under the conditions of Theorem* 4.1*, the following statements are equivalent.*

(i) *The operator* $F_M = A - BB^\star P_{\text{Max}}^\infty$ *is exponentially stable.*

(ii) $(P^{\infty}_{\text{Max}}x_0, x_0) = \min\{J_\infty(u)\colon u \in L^2(0,\infty;U), x \in L^2(0,\infty;H)\}$, *where x is given by* (1.1).

Proof. (i) $\Longrightarrow$ (ii). Let $\hat{x}(t)$ and $\hat{u}(t)$ be defined by (4.17). If F_M is exponentially stable, we have $\hat{x} \in L^2(0,\infty;H)$ so that $\hat{u} \in U_{ad}$ and (ii) holds.

(ii) $\Longrightarrow$ (i). Let $u^\star \in L^2(0,\infty;U)$ such that the corresponding solution $x^\star$ of (1.1) belongs to $L^2(0,\infty;H)$. Setting $u = u^\star$ in (4.18), we obtain

$$\int_0^\infty |u^\star(s) + B^\star P^{\infty}_{\text{Max}} x^\star(s)|^2\, ds = 0$$

which implies that

$$x^\star(t) = e^{tF_M}x_0, \quad u^\star(t) = -B^\star P^{\infty}_{\text{Max}} e^{tF_M}x_0,$$

and the conclusion follows from Datko's Theorem. □

Corollary 4.3. *Assume that the conditions $(\mathcal{H})_\infty$ are verified and that (A,B) is I-stabilizable. Let $P^{\infty}_{\min}$ and P^{∞}_{Max} be the minimal and maximal solutions of the algebraic Riccati equation. Then*

(i) $P^{\infty}_{\min} \le P^{\infty}_{\text{Max}}$,
(ii) $P^{\infty}_{\min}$ *is globally attractive from below,*
(iii) P^{∞}_{Max} *is globally attractive from above.*

Proof. (i) is clear, (ii) follows from Proposition 4.2, and (iii) follows from Theorem 4.1 and Proposition 4.3. □

Remark 4.2.

(i) The maximal solution does not exist in general. Let in fact

$$H = U = Y = \mathbb{R}^2, \quad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = 0, \quad C = 0.$$

Then (4.2) reduces to $A^\star X + XA = 0$ and $X = \lambda I$ is a solution for all $\lambda \ge 0$.

(ii) In (4.15) the infimum is not a minimum in general. In fact, let $A = 0$, $U = H$, $B = I$, and $C = 0$. Then the algebraic Riccati equation reduces to $P^2 = 0$ and one has $P^{\infty}_{\min} = P^{\infty}_{\text{Max}} = 0$. Thus the infimum in (4.15) is 0; however the control $u = 0$ does not belong to U_{ad} if $x_0 \ne 0$. Remark also that F_M is not exponentially stable. □

Remark 4.3. For the existence of the maximal solution it is not necessary that (A,B) be I-stabilizable. It is sufficient to assume that there exists $D \ge C^\star C$ such that

(i) (A,B) is $D^{1/2}$-*stabilizable.*
(ii) $(A,D^{1/2})$ is *detectable.*

In fact, under these assumptions, one can easily repeat the previous arguments. □

4.4 Continuous dependence of stationary solutions with respect to the data

We consider here a sequence of Riccati equations

$$\begin{aligned} P_k' &= A_k^\star P_k + P_k A_k - P_k B_k B_k^\star P_k + C_k^\star C_k, \\ A_k^\star X_k &+ X_k A_k - X_k B_k B_k^\star X_k + C_k^\star C_k = 0. \end{aligned} \tag{4.19}$$

Assume that hypotheses (2.18) in Chapter 1 of Part IV are verified and set

$$J_\infty^k(u) = \int_0^\infty \{|C_k x_k(s)|^2 + |u(s)|^2\}\, ds, \quad u \in L^2(0,\infty;U),$$

where x_k is the mild solution of the system

$$x_k' = A_k x_k + B_k u, \quad x_k(0) = x_{k0}. \tag{4.20}$$

We say that (A_k, B_k) is *stabilizable with respect to C_k uniformly in k*, if for any $x_0 \in H$, there exists $u \in L^2(0,\infty;U)$ such that

$$J_\infty^k(u) < \infty, \quad \forall k \in \mathbb{N}. \tag{4.21}$$

If this assumption is verified, then, according to Theorem 3.1, the minimal solution of (2.2) exists. We denote it by $P_{k,\min}^\infty$. Finally we set

$$F_k = A_k - B_k B_k^\star P_{k,\min}^\infty.$$

We now prove the following theorem (see also J. S. Gibson [1]).

Theorem 4.2. *Assume that assumptions* (2.18) *in Chapter* 1 *of Part* IV *are verified and that (A_k, B_k) is stabilizable with respect to C_k uniformly in k. If, in addition, there exist $N > 0$ and $a > 0$ such that*

$$\|e^{tF_k}\| \le N e^{-at}, \quad t \ge 0, \tag{4.22}$$

then we have

$$\lim_{k\to\infty} P_{k,\min}^\infty x = P_{\min}^\infty x, \quad \forall x \in H.$$

Proof. We have

$$(P_{k,\min}^\infty x_0, x_0) \le \int_0^\infty \{|x_k(s)|^2 + |u(s)|^2\}\, ds,$$

where x_k is the solution of (4.20). Choose u such that (4.21) holds; then, by using the Uniform Boundedness Theorem, it is easy to show that there exists $c > 0$ such that

$$P_{k,\min}^\infty \le cI, \quad \forall k \in \mathbb{N}. \tag{4.23}$$

Now set $V_k = Q_k - P_{k,\min}^\infty$ (resp. $V = Q - P_k^\infty$) where Q_k (resp. Q) is the solution of (4.19) (resp. (4.1)) such that $Q_k(0) = cI$ (resp. $Q(0) = cI$). Then V_k is the mild solution of the problem

$$V_k' = F_k^\star V_k + V_k F_k - V_k B_k B_k^\star V_k, \quad V_k(0) = cI - P_{k,\min}^\infty.$$

Remark that $V_k(0) \geq 0$ in virtue of (4.23). By (4.22) it follows that

$$\|V_k(t)\| \leq N^2 e^{-2at}, \quad t \geq 0.$$

Moreover

$$\begin{aligned} |P_{\min}^\infty x - P_{k,\min}^\infty x| &\leq |P_{\min}^\infty x - Q(t)x| + |Q(t)x - Q_k(t)x| + |P_{k,\min}^\infty x - Q_k(t)x| \\ &\leq 2cN^2 e^{-2at}|x| + |Q(t)x - Q_k(t)x|, \end{aligned}$$

and the result follows from Theorem 2.2 of Chapter 1 (Part IV). □

4.5 Periodic solutions of the Riccati equation

In this section we study periodic solutions of Riccati equations. For the finite dimensional case, see M. A. Shayman [1]. Here we follow G. Da Prato [2,3].

Let $T > 0$ be fixed. We say that $P \in C_s\big(\mathbb{R}; \Sigma(H)\big)$ is a T-*periodic solution* of the Riccati equation

$$P' = A^\star P + PA - PBB^\star P + C^\star C, \tag{4.24}$$

if

(i) $P(t+T) = P(t)$, $\forall t \in \mathbb{R}$.
(ii) *P is a mild solution of* (4.24); that is, for any $t, s \in \mathbb{R}$ with $t < s$ we have

$$\begin{aligned} P(t)x = e^{(t-s)A^\star} P(s) e^{(t-s)A} x &+ \int_t^s e^{(t-\tau)A^\star} C^\star C e^{(t-\tau)A} x \, d\tau \\ &- \int_t^s e^{(t-\tau)A^\star} P(\tau) BB^\star P(\tau) e^{(t-\tau)A} x d\tau. \end{aligned}$$

If, in addition, P is not constant we say that P is a *nontrivial periodic solution* of (4.24).

Proposition 4.8. *Assume that $(\mathcal{H})_\infty$ is verified, then the following statements are equivalent*:

(i) (A, B) *is C-stabilizable,*
(ii) *there exists a positive T-periodic solution* (*possibly trivial*) *of* (4.24).

Proof. (i) $\Longrightarrow$ (ii) follows from Proposition 2.3 because any stationary solution is also periodic.

(ii) $\Longrightarrow$ (i). Assume, by contradiction, that there exists a T-periodic solution of (4.24) whereas (i) does not hold. Then, by Proposition 2.2, there exists $x_0 \in H$ such that $(P_{\min}(t)x_0, x_0) \to \infty$ as $t \to \infty$, where $P_{\min}(\cdot)$ is the minimal positive solution of (4.24). As $P(t) \geq P_{\min}(t)$ we have $(P(t)x_0, x_0) \to \infty$ as $t \to \infty$, which is a contradiction because P is periodic. □

Lemma 4.1. *Assume that (A, B) is C-stabilizable and let P be a positive T-periodic solution of (4.24). Then we have $P(t) \geq P^{\infty}_{\min}$, where $P^{\infty}_{\min}$ is the minimal solution of (2.2). If, in addition, (A, B) is I-stabilizable we have $P(t) \leq P^{\infty}_{\mathrm{Max}}$.*

Proof. For the first statement, it is sufficient to let n tend to infinity in the inequality

$$P(t) = P(t + nT) \geq P_{\min}(t + nT),$$

whereas the second one follows from Corollary 4.3. □

Let now P be a positive T-periodic solution of (4.24) and set $Q = P - P^{\infty}_{\min}$ and $F = A - BB^{\star}P^{\infty}_{\min}$. By Lemma 4.1 we have $Q \geq 0$; moreover, as easily checked, Q is the mild solution of (4.24) such that $Q(0) = P(0) - P^{\infty}_{\min} = Q_0$; that is

$$\begin{cases} Q' = F^{\star}Q + QF - QBB^{\star}Q, \\ Q(0) = P(0) - P^{\infty}_{\min} = Q_0. \end{cases} \tag{4.25}$$

Problem (4.25) can be explicitly solved; we could use formula (2.22) in Part IV, Chapter 1, but we prefer to give a simpler proof in the following lemma.

Lemma 4.2. *The solution of problem* (4.25) *is given by the formula*

$$Q(t) = e^{tF^{\star}}Q_0(I + \Omega(t)Q_0)^{-1}e^{tF}, \tag{4.26}$$

where

$$\Omega(t)x = \int_0^t e^{sF}BB^{\star}e^{sF^{\star}}x\, ds, \quad x \in H.$$

Proof. First of all we remark that formula (4.26) is meaningful because $\Omega(t) \in \Sigma(H)$ so that the inverse of $I + \Omega(t)Q_0$ belongs to $\mathcal{L}(H)$ (see Proposition 1.1 in Appendix A). Now let Q_n be the solution to the approximating problem

$$Q_n' = F_n^{\star}Q_n + Q_nF_n - Q_nBB^{\star}Q_n, \quad Q_n(0) = Q_0 + \frac{1}{n}I,$$

where F_n are the Yosida approximation of F. Setting $V_n = Q_n^{-1}$, this problem reduces to the following linear one:

$$V_n' = -F_nV_n - V_nF_n^{\star} + BB^{\star} \quad V_n(0) = \left(Q_0 + \frac{1}{n}I\right)^{-1}.$$

Thus we have

$$\begin{aligned} V_n(t) &= e^{-tF_n}\left(Q_0 + \frac{1}{n}I\right)^{-1}e^{-tF_n^{\star}} + \int_0^t e^{(t-s)F_n}BB^{\star}e^{(t-s)F_n^{\star}}\, ds \\ &= e^{-tF_n}\left\{I + \int_0^t e^{sF_n}BB^{\star}e^{sF_n^{\star}}\, ds\, Q_0 + \frac{1}{n}I\right\}\left(Q_0 + \frac{1}{n}I\right)^{-1}e^{-tF_n^{\star}}, \end{aligned}$$

which implies that

$$Q_n(t) = e^{tF_n^\star}\left(Q_0 + \frac{1}{n}I\right)\left\{I + \int_0^t e^{sF_n}BB^\star e^{sF_n^\star}\,dsQ_0 + \frac{1}{n}I\right\}^{-1} e^{tF_n}.$$

Now the conclusion follows from Theorem 2.1 of Chapter 1 in Part IV. $\square$

From Lemma 4.1 we have the following result.

Proposition 4.9. *Assume that the conditions $(\mathcal{H})_\infty$ are verified and that (A, B) is stabilizable with respect to C. Let $P_0 \geq P_{\min}^\infty$ and let P be the solution of (4.24) such that $P(0) = P_0$. Then P is T-periodic if and only if $X = P_0 - P_{\min}^\infty$ verifies the equation*

$$e^{TF^\star}X(I + \Omega(T)X)^{-1}e^{TF} = X, \tag{4.27}$$

where

$$\Omega(T)x = \int_0^T e^{sF}BB^\star e^{sF^\star}x\,ds, \quad x \in H,$$

and

$$F = A - BB^\star P_{\min}^\infty.$$

We will now study a special case of (4.27). We remark that it is also important to decide whether a given solution of (4.27) is a nontrivial periodic solution. Assume that

$$\begin{cases} \text{(i)} \quad A = D + \dfrac{\alpha}{2}I, \quad C = 0, \quad \alpha > 0, \\ \text{(ii)} \quad D + D^\star = 0, \quad e^{2\pi D} = I, \quad \|e^{tD}\| \leq 1, \\ \text{(iii)} \quad \text{there exists } \delta > 0 \text{ such that} \\ \qquad \displaystyle\int_0^{2\pi} e^{\alpha s}e^{sD}BB^\star e^{sD^\star}\,ds \geq \delta. \end{cases} \tag{4.28}$$

Then (4.27) is equivalent to

$$e^{2\pi\alpha}X(I + \Omega X)^{-1} = X, \tag{4.29}$$

where

$$\Omega = \Omega(2\pi) = \int_0^{2\pi} e^{\alpha s}e^{sD}BB^\star e^{sD^\star}\,ds$$

and the Riccati equation (4.24) becomes

$$P' = D^\star P + PD + \alpha P - PBB^\star P. \tag{4.30}$$

Proposition 4.10. *Assume that the hypotheses of Proposition 4.9 hold and that, in addition, the conditions (4.28) are verified. Then all solutions of (4.29) are given by the formula*

$$X = (e^{2\pi\alpha} - 1)\Omega^{-1/2}\Xi\Omega^{-1/2},$$

where Ξ is any hermitian projector operator in $\Sigma^+(H)$.

Proof. Equation (4.29) is equivalent to $(e^{2\pi\alpha} - 1)X = X\Omega X$ and setting $Y = \Omega^{1/2}X\Omega^{1/2}$ to $(e^{2\pi\alpha} - 1)Y = Y^2$; thus the conclusion follows. □

Let X be a solution of (4.29) and let P be the solution of (4.30) such that $P(0) = X$; P is 2π-periodic. We now want to see if P is nontrivial. Let us consider the special case when $B = I$. In this case we have $X = \alpha\Xi$ and moreover

$$P(t) = \alpha e^{\alpha t} e^{tD^\star} \Xi [I + (e^{\alpha t} - 1)\Xi]^{-1} e^{tD}.$$

As Ξ is a projector we have $\Xi(I + b\Xi)^{-1} = (1+b)^{-1}\Xi$ for any $b > 0$, so that

$$P(t) = \alpha e^{tD^\star} \Xi e^{tD}.$$

Thus, if $\Xi = 0$ or I, $P(t)$ is constant, whereas if Ξ is different from 0 or I, $P(t)$ is not trivial. □

Remark 4.4 *(Unbounded observation operator).* Assume here that $(\mathcal{H}_\infty)$–(i)–(ii) and $(\mathcal{H}_\infty)$–(v) are verified and consider the optimal control (1.1)–(1.2). Obviously, in the definition of the cost functional J_∞, Cx must be defined as in §6 of Chapter 1 of Part IV. The definitions of optimal pair, stabilizability, and weak and stationary solutions (but not strict solutions) of the Riccati equation are now the same as before. However several of the previous results can be generalized, with the exception of strict solutions of the algebraic Riccati equation and of the detectability, because it is related to the dual system, which has an unbounded control operator. □

5 Some generalizations and complements

5.1 Nonhomogeneous state equation

We consider, as in §7 of Chapter 1 in Part IV, a system governed by a nonhomogeneous state equation and the following infinite horizon problem: To minimize

$$J_\infty(u) = \int_0^\infty \{|Cx(s)|^2 + |u(s)|^2\}\, ds, \tag{5.1}$$

over all controls $u \in L^2(0, \infty; U)$ subject to the differential equation constraint

$$x'(t) = Ax(t) + f(t) + Bu(t), \quad t \geq 0, \qquad x(0) = x_0. \tag{5.2}$$

We assume that

$$\begin{cases} \text{(i)} & f \in L^2(0, \infty; H), \\ \text{(ii)} & \text{hypotheses } (\mathcal{H}) \text{ hold}, \\ \text{(iii)} & (A, B) \text{ is } C\text{-stabilizable}, \\ \text{(vi)} & F = A - BB^\star P_{\min}^\infty \text{ is exponentially stable.} \end{cases} \tag{5.3}$$

Under conditions (5.3) it is easy to check that the following problem:

$$r'(t) + F^\star r(t) + P^\infty_{\min} f(t) = 0, \quad t \geq 0, \qquad r(\infty) = 0$$

has a unique mild solution given by

$$r(t) = \int_t^\infty e^{(s-t)F^\star} P^\infty_{\min} f(s)\, ds.$$

We can now prove the following result.

Lemma 5.1. *Assume* (5.3) *and let $x_0 \in H$ and $u \in L^2(0,\infty;U)$; then we have*

$$J_\infty(u) = (P^\infty_{\min} x_0, x_0) + 2(r(0), x_0) + \int_0^\infty \left\{2\big(r(s), f(s)\big) - |B^\star r(s)|^2\right\} ds$$
$$+ \int_0^\infty |u(s) + B^\star r(s) + B^\star P^\infty_{\min} x(s)|^2\, ds.$$

Proof. It is sufficient to set $P = P_{\min}$ (the minimal solution of the Riccati equation) in (7.4) in Chapter 1 of Part IV, and to let T tend to infinity. □

Now the following result is easily proved.

Theorem 5.1. *Assume* (5.3) *and let $x_0 \in H$. Then there exists a unique optimal pair $(u^\star, x^\star)$ for problem* (5.1)–(5.2)*. Moreover the following statements hold.*

(i) *$x^\star$ is the mild solution to the closed loop equation*

$$\begin{cases} x'(t) = [A - BB^\star P^\infty_{\min}]x(t) - BB^\star r(t) + f(t), \quad t \geq 0,\\ x(0) = x_0. \end{cases}$$

(ii) *$u^\star$ is given by the feedback formula*

$$u^\star(t) = -B^\star[P^\infty_{\min} x^\star(t) + r(t)].$$

(iii) *The optimal cost is given by*

$$J_\infty(u) = (P^\infty_{\min} x_0, x_0) + 2(r(0), x_0) + \int_0^\infty \left\{2\big(r(s), f(s)\big) - |B^\star r(s)|^2\right\} ds.$$

Remark 5.1. If $f \notin L^2(0,\infty;H)$ it can happen that no admissible control exists. Consider in fact the following example. Let $H = U = Y = \mathbb{C}$, $A = 0$, $B = C = 1$, and $f = 1$. It is readily seen that

$$J_\infty(u) = \int_0^\infty \left\{\left|t + \int_0^t u(s)\, ds\right|^2 + |u(t)|^2\right\} dt = \infty.$$

In this case the following cost function seems more appropriate (see G. Da Prato and A. Ichikawa [4]):

$$J_\infty(u) = \limsup_{T\to\infty} \frac{1}{T}\int_0^T \{|Cx(s)|^2 + |u(s)|^2\}\, ds. \qquad □$$

5.2 Time-dependent state equation and cost function

We consider here the system

$$\begin{cases} x'(t) = A(t)x(t) + B(t)u(t), \quad t \geq 0, \\ x(0) = x_0, \end{cases} \tag{5.4}$$

where $A(t)\colon D\big(A(t)\big) \subset H \to H$, $t \geq 0$ and $B(t) \in \mathcal{L}(U;H)$, $t \geq 0$, are linear operators. We assume that condition (7.8) in Chapter 1 of Part IV is verified. We want to minimize the cost function

$$J_\infty(u) = \int_0^\infty \{|C(s)x(s)|^2 + |u(s)|^2\}\, ds$$

over all controls $u \in L^2(0,\infty;U)$ subject to the differential equation constraint (5.4).

The definitions of admissible controls and optimal pair are the same as before. Also when, for any $x_0 \in H$, an admissible control exists, we say that (A,B) is *C-stabilizable* (cf. Definition 2.1). In the autonomous case this problem was studied with the tool of the minimal positive solution of the algebraic Riccati equation. Now the same role is played by the *minimal positive bounded solution* of the Riccati equation

$$Q' + A^\star Q + QA - QBB^\star Q + C^\star C = 0 \tag{5.5}$$

in $[0,+\infty[$. In the remaining of this subsection we follow G. Da Prato and A. Ichikawa [4].

Definition 5.1. We say that $Q \in C_s\big([0,+\infty[\,;\Sigma^+(H)\big)$ is a (mild) solution to (5.5), if for any $a > 0$, $t \in [0,a]$ and $x \in H$, we have

$$\begin{aligned} Q(t)x = U^*(a,t)Q(a)U(a,t)x + \int_t^a U^*(s,t)C^*(s)C(s)U(s,t)x\, ds \\ - \int_t^a U^*(s,t)Q(s)B(s)B^*(s)Q(s)U(s,t)x\, ds. \quad \Box \end{aligned}$$

Theorem 5.2. *Assume that conditions* (7.8) *and* (7.9) *in Chapter* 1 *of Part* IV *are verified, and that* (A,B) *is* C*-stabilizable. Then* (5.5) *has a nonnegative bounded solution* Q*. This solution is minimal among all nonnegative bounded solutions of* (5.5)*.*

Proof. For any $\lambda > 0$ and $n > 0$ sufficiently large, we introduce the Riccati equations

$$\begin{cases} \dfrac{dQ^\lambda}{dt} + A^\star Q^\lambda + Q^\lambda A - Q^\lambda BB^\star Q^\lambda + C^\star C = 0, \\ Q^\lambda(\lambda) = 0, \quad t \in [0,\lambda], \end{cases} \tag{5.6}$$

$$\begin{cases} \dfrac{dQ_n^\lambda}{dt} + A_n^\star Q_n^\lambda + Q_n^\lambda A_n - Q_n^\lambda BB^\star Q_n^\lambda + C^\star C = 0, \\ Q_n^\lambda(\lambda) = 0, \quad t \in [0,\lambda], \end{cases} \tag{5.7}$$

where A_n are the Yosida approximations of A. By Theorem 7.2 of Chapter 1 of Part IV, problems (5.6) and (5.7) have unique solutions Q^λ and Q_n^λ, respectively, and moreover for any $x \in H$

$$\lim_{n\to\infty} Q_n^\lambda(t)x = Q^\lambda(t)x, \tag{5.8}$$

uniformly in $t \in [0,\lambda]$. Also, by Proposition 7.1 of Chapter 1 of Part IV,

$$Q_n^\lambda(t) \le Q_n^\mu(t), \quad t\in[0,\lambda] \quad \text{if } \mu > \lambda. \tag{5.9}$$

We now prove that there exists $C_1 > 0$ such that

$$\|Q^\lambda(t)\| \le C_1, \quad \forall \lambda > 0,\ \forall t \in [0,\lambda]. \tag{5.10}$$

For this it is sufficient to prove that

$$\|Q_n^\lambda(t)\| \le C_1, \quad \forall \lambda > 0,\ \forall t \in [0,\lambda],\ \forall n > 0. \tag{5.11}$$

Let $x \in H$ and let $u \in L^2(0,\infty;U)$ such that $J_\infty(u) < +\infty$. Let moreover x_n be the solution of the initial value problem:

$$x_n' = A_n x_n + Bu, \quad x_n(0) = x.$$

Then we have

$$\frac{d}{dt}\big(Q_n^\lambda(t)x_n(t), x_n(t)\big) = |u(t) + B^\star(t)Q_n^\lambda(t)x_n(t)|^2 - |C(t)x_n(t)|^2 - |u(t)|^2.$$

Integrating this from t and λ, we find

$$\langle Q_n^\lambda(t)x, x\rangle \le \int_t^\lambda \{|C(s)x_n(s)|^2 + |u(s)|^2\}\, ds \le \text{Const}\,|x|^2,$$

and the inequalities (5.10) and (5.11) follow.

From the estimates (5.9) and (5.10) the following limit exists:

$$Q_\infty(t)x = \lim_{\lambda\to\infty} Q_n^\lambda(t)x, \quad \forall x \in H.$$

By letting λ tend to infinity in the equality

$$\begin{aligned} Q^\lambda(t)x = U^*(a,t)Q^\lambda(a)U(a,t)x &+ \int_t^a U^*(s,t)C^*(s)C(s)U(s,t)x\,ds \\ &- \int_t^a U^*(s,t)Q^\lambda(s)B(s)B^*(s)Q^\lambda(s)U(s,t)x\,ds, \end{aligned}$$

we see that Q_∞ is a bounded solution of (5.5). It remains to prove minimality. Let R be a bounded nonnegative solution of (5.5). Then by Proposition 7.1 of Chapter 1 in Part IV and the fact that $Q^\lambda(\lambda) = 0$,

$$R(t) \ge Q^\lambda(t), \quad \forall t \in [0,\lambda],$$

and so $R \ge Q_\infty$. □

Corollary 5.1. *Assume that the conditions of Theorem* 5.2 *are verified and that, in addition, A, B, and C are τ-periodic functions for some $\tau > 0$. Then* (5.5) *has a nonnegative τ-periodic solution Q. This solution is minimal among all nonnegative τ-periodic solutions of* (5.5).

Proof. Let Q_∞ be the minimal nonnegative bounded solution of the Riccati equation. Then we have

$$Q^\lambda(t+\tau) = Q^{\lambda-\tau}(t), \quad \forall t \in [0, \lambda - \tau];$$

as $\lambda \to \infty$, we find that Q_∞ is τ-periodic. □

Remark 5.2. This result was proved in G. Da Prato and A. Ichikawa [3]. If A, B, C are almost periodic and H is a finite dimensional space, then Q_∞ is almost periodic (see T. Morozan [1]). □

5.3 Periodic control problems

Consider a dynamical system governed by a linear equation

$$x'(t) = A(t)x(t) + f(t) + B(t)u(t) \tag{5.12}$$

with periodic coefficients $f(\cdot)$, $A(\cdot)$, $B(\cdot)$ of period 2π.

Then it is natural to consider 2π-periodic controls u. For any Banach space E we denote by $L^2_\#(E)$ the space of all 2π-periodic functions $u\colon \mathbb{R} \to E$ that belong to $L^2(0, 2\pi; E)$.

We assume that

$$\begin{cases} \text{(i)} & \text{Conditions (7.8) and (7.9) in Chapter 1 of Part IV} \\ & \text{hold.} \\ \text{(ii)} & A, B, C \text{ are } 2\pi \text{ periodic.} \\ \text{(iii)} & f \in L^2_\#(H). \end{cases} \tag{5.13}$$

Obviously (5.12) does not necessarily have a 2π-periodic solution for any control $u \in L^2_\#(U)$. This happens when no Floquet exponent of A is equal to 1. In such a case we say that A is *nonresonant.* As it is well known, this is equivalent to require that 1 belongs to the resolvent set of $U_A(2\pi, 0)$, where $U_A(t, s)$ is the evolution operator relative to A. In this case (5.12) has a unique mild solution given by the formula

$$\begin{aligned} x(t) = U_A(t,0)[I - U_A(2\pi,0)]^{-1} \int_0^{2\pi} U_A(2\pi, s)[Bu(s) + f(s)]\, ds \\ + \int_0^t U(t,s)[Bu(s) + f(s)]\, ds. \end{aligned}$$

We are interested in the general case, that is, when A is possibly resonant. For any $u \in L^2_\#(U)$ we set

$$\Lambda(u) = \{y \in L^2_\#(H)\colon y \text{ fulfills } (5.12)\}. \tag{5.14}$$

The control u is said to be *admissible* if $\Lambda(u) \neq \varnothing$. The set of all admissible controls will be denoted by U_{ad}.

We want to minimize the *cost function*

$$J(u,y) = \int_0^{2\pi} \{|C(t)y(t)|^2 + |u(t)|^2\}\, dt \tag{5.15}$$

over all $u \in U_{ad}$ and $y \in \Lambda(u)$.

If there exist $u^\star \in U_{ad}$ and $x^\star \in \Lambda(u^\star)$ such that

$$J(u^\star, x^\star) \le J(u,x), \quad \forall u \in U_{ad},\ \forall x \in \Lambda(u)$$

the function $u^\star$ is called an *optimal control* and the associated state $x^\star$ is called the *optimal state*. The pair $(u^\star, x^\star)$ is called an *optimal pair*.

We will study this minimization problem by using again the Dynamic Programming approach and by proceeding in the following steps.

(i) We consider a periodic solution to the Riccati equation

$$Q' + A^\star Q + QA - QBB^\star Q + C^\star C = 0. \tag{5.16}$$

(ii) We look for a periodic solution to the dual equation

$$r'(t) + (A - BB^\star Q)^\star r(t) + Qf(t) = 0 \tag{5.17}$$

and to the closed loop equation

$$x'(t) = (A - BB^\star Q)x(t) - BB^\star r(t) + f(t). \tag{5.18}$$

Then, we show that x is an optimal state and that u, given by

$$u(t) = -B^\star\big(Qx(t) + r(t)\big), \tag{5.19}$$

is an optimal control.

Concerning the point (i), the existence of a nonnegative periodic solution of (5.16) was proved in Corollary 5.1 when (A,B) is C-stabilizable. The approach described below was introduced in G. Da Prato and A. Ichikawa [3] where the existence of an optimal pair was proved with the additional condition that (A,C) be detectable. In fact in this case one can show that the Floquet exponents of the closed loop operator $F = A - BB^\star Q$ have all modulus less than 1 and then F is nonresonant and (5.17) and (5.18) have periodic solutions.

Here we do not assume detectability, but we give, following G. Da Prato [3], a characterization of the Floquet exponents of F with modulus greater than or equal to one. In the following theorem U_F represents the evolution operator associated with F (U_F is well defined because F is a bounded perturbation of A) and Q_∞ is the minimal nonnegative periodic solution of the Riccati equation (5.16).

Theorem 5.3. *Assume that condition* (5.13) *is verified and that* (A, B) *is* C-*stabilizable. Let* Q_∞ *be the minimal nonnegative* 2π-*periodic solution of Riccati equation* (5.16) *and* $F = A - BB^*Q_\infty$. *Let* $\mu \in \mathbb{C}$ *such that* $|\mu| \geq 1$ *and let* $x_0 \in H$ *be different from* 0. *Then the following statements are equivalent.*

(i) $U_F(2\pi, 0)x_0 = \mu x_0$.
(ii) $U_A(2\pi, 0)x_0 = \mu x_0$ *and* $C(t)U_A(t, 0)x_0 = 0,$ *for all* $t \geq 0$.

Moreover if either (i) *or* (ii) *holds true, we have*

$$U_F(t, 0)x_0 = U_A(t, 0)x_0, \quad \forall t \geq 0. \tag{5.20}$$

Proof. (i) $\Longrightarrow$ (ii). Let $x_0 \in H$ and let x be the mild solution of the problem

$$x' = Fx, \quad t \geq 0, \quad x(0) = x_0,$$

then the following identity holds:

$$\begin{aligned}\big(Q_\infty(t)x(t), x(t)\big) &- (Q_\infty(0)x_0, x_0)\\ &+ \int_0^t \{|B^*(s)Q_\infty(s)x(s)|^2 + |C(s)x(s)^2|\, ds\} = 0.\end{aligned} \tag{5.21}$$

If $x_0 \in D(A)$ this follows easily by integrating the identity

$$\frac{d}{dt}\big(Q_\infty(t)x(t), x(t)\big) = -|B^\star(t)Q_\infty(t)x(t)|^2 - |C(t)x(t)|^2$$

between 0 and t. For general x_0, (5.21) follows by density.

Let now $x_0 \in H$ be such that $U_F(2\pi, 0)x_0 = x(2\pi) = \mu x_0$ with $|\mu| \geq 1$. Then, setting $t = 2\pi$ in (5.21) we find

$$(|\mu|^2 - 1)(Q_\infty(0)x_0, x_0) + \int_0^{2\pi} \{|B^*Q_\infty x|^2 + |Cx^2|\, ds\} = 0,$$

which yields

$$B^\star(t)Q_\infty(t)x(t) = 0, C(t)x(t) = 0, \quad \forall t \geq 0.$$

Consequently $x' = Ax$, and

$$x(t) = U_F(t, 0)x_0 = U_A(t, 0)x_0, \quad t \geq 0,$$

which implies (ii).

(ii) $\Longrightarrow$ (i). Given $x_0 \in H$, let y be the mild solution of the system $y' = Ay$, $y(0) = x_0$. Moreover let Q^λ be the mild solution of the Riccati equation

$$Q' + A^*Q + QA - QBB^*Q + C^*C = 0, \quad Q(2\pi\lambda) = 0.$$

Then the following identity holds:

$$(Q^\lambda(0)x_0, x_0) + \int_0^{2\pi\lambda} |B^*(s)Q^\lambda(s)x(s)|^2\, ds = \int_0^{2\pi\lambda} |C(s)x(s)^2|\, ds. \quad (5.22)$$

In fact if $x_0 \in D(A)$ this result readily follows by integrating the identity

$$\frac{d}{dt}\big(Q^\lambda(t)y(t), y(t)\big) = |B^\star(t)Q^\lambda(t)y(t)|^2 - |C(t)y(t)|^2$$

between 0 and $2\pi\lambda$. For general x_0, (5.22) follows by density.

Let now x_0 be such that

$$y(2\pi) = U_A(2\pi, 0)x_0 = \mu x_0, \quad C(t)y(t) = 0, \quad t \geq 0.$$

Then, by letting λ tend to infinity in (5.22) we find

$$(Q_\infty(0)x_0, x_0) + \int_0^\infty |B^*(t)Q_\infty(t)x(t)|^2\, dt = 0,$$

which implies $B^*(t)Q_\infty(t) = 0$, $t \geq 0$, so that $y' = Fy$ and $U_A(t,0)x_0 = U_F(t,0)x_0$ and (i) holds. $\square$

Remark 5.3. Assume that the spectra of $U_A(2\pi, 0)$ and $U_F(2\pi, 0)$ consist only of eigenvalues (this is for instance the case for parabolic state equations in bounded domains; see Example 6.1 below). By the above theorem it follows that F is nonresonant if one of the following conditions holds

(i) A is nonresonant.
(ii) A is resonant but the following implication holds:

$$x_0 \in H,\ x_0 \neq 0, \quad U_A(2\pi, 0)x_0 = x_0 \implies C(t_0)U_A(t_0, 0)x_0 \neq 0,$$

for at least one $t_0 \geq 0$.

For a case where the eigenvalues of $U_A(2\pi, 0)$ and $U_F(2\pi, 0)$ have limit points, see G. Da Prato [7]. $\square$

We can prove now the result.

Theorem 5.4. *Assume that condition* (5.13) *is verified, that A, B is C-stabilizable, and that $F = A - BB^*Q_\infty$ is nonresonant, where Q_∞ is the minimal nonnegative 2π-periodic solution of the Riccati equation* (5.16)*. Then there exists a unique optimal pair $(u^\star, x^\star)$ for problem* (5.12)–(5.15) *and the following conditions are satisfied*:

(i) $x^\star$ *is the unique periodic solution to the closed loop equation* (5.18),
(ii) $u^\star$ *is given by the feedback formula*

$$u^\star(t) = -B^\star[Q_\infty(t)x^\star(t) + r^\star(t)], \tag{5.23}$$

where r *is the unique periodic solution of* (5.17),
(iii) *the optimal cost* $J(u^\star, x^\star)$ *is given by*

$$J(u^\star, x^\star) = \int_0^{2\pi} \left\{\big(r(t), f(t)\big) - |B^\star(t)r(t)|^2\right\} dt. \tag{5.24}$$

Proof. Let $u \in U_{ad}$ and $y \in \Lambda(u)$. By computing

$$\frac{d}{dt}\left\{\big(Q_\infty(t)y(t), y(t)\big) + 2\big(r(t), f(t)\big)\right\}$$

and by integrating between 0 and 2π, we find the identity

$$J(u, y) = J^\star + \int_0^{2\pi} \|R(t)\|^2 dt, \tag{5.25}$$

where

$$J^\star = \int_0^{2\pi} \left\{\big(r(t), f(t)\big) - |B^\star(t)r(t)|^2\right\} dt$$

and

$$R(t) = B^*(t)[Q_\infty(t)y(t + r(t)] + u(t).$$

We remark that the computation can be made rigorous by approximating $A(t)$ by their Yosida approximations. It follows that

$$J(u^\star, y^\star) \geq J^\star, \quad \forall u \in U_{ad},\ \forall y \in \Lambda(u).$$

Now, let x^* be the solution of (5.18) and let u^* be given by (5.23). Setting in (5.25), $u = u^\star$ and $y = y^\star$ we obtain $J(u^\star, y^\star) = J^\star$ so that the pair $u^\star$, $y^\star$ is optimal. Finally uniqueness of the optimal pair is proved as in Theorem 6.1 of Chapter 1 in Part IV. □

Remark 5.4. It is also possible to study almost periodic control problems (see G. Da Prato and A. Ichikawa [2]). □

6 Examples of controlled systems

6.1 Parabolic equations

We shall continue here the example of §8.1 in Chapter 1 of Part IV. Moreover we denote by $\{\varphi_k\}$ a complete set of eigenvectors of A and by $\{\lambda_k\}$ the corresponding sequence of eigenvalues. We assume that $\{\lambda_k\}$ is nonincreasing. We consider the infinite horizon control problem: To minimize

$$J_\infty(u) = \int_0^\infty \int_\Omega \left\{\left|\big(Cx(t,\cdot)\big)(\xi)\right|^2 + |u(t,\xi)|^2\right\} dt\, d\xi, \tag{6.1}$$

over all $u \in L^2([0,\infty] \times \Omega)$ subject to condition (1.48) of Chapter 1 in Part II.

We have to discuss the existence of admissible controls. We consider two cases.

First case. $\lambda_1 < 0$.

In this case A is exponentially stable so that (A,B) is C-stabilizable. Thus by Theorem 3.1 and Corollary 4.2, the algebraic Riccati equation

$$AP + PA - PBB^\star P + C^\star C = 0, \tag{6.2}$$

has a unique solution $P^\infty_{\min} = P^\infty_{\mathrm{Max}}$ and the feedback operator $F = A - BB^\star P^\infty_{\min}$ is exponentially stable.

Second case. $\lambda_1 \geq 0,\ \lambda_2 < 0$.

In this case we have $\sigma^+(A^\star) \cup \sigma^0(A^\star) = \{\lambda_1\}$; thus, by Proposition 3.3, (A,B) is I-stabilizable if and only if $B^\star\varphi_1 \neq 0$. Under this assumption (6.2) has a minimal and a maximal nonnegative solution. Moreover by Proposition 3.4, F is exponentially stable if and only if $C\varphi_1$ is not identically zero. It is easy to generalize the previous discussion when $\lambda_m \geq 0$ and $\lambda_{m+1} < 0$ for some $m \in \mathbb{N}$.

6.2 Wave equation

We continue here the example of §8.2 in Chapter 1 of Part IV. Consider the infinite horizon problem: To minimize

$$J_\infty(u) = \int_0^\infty \int_\Omega \left\{\left|\nabla_\xi x(t,\xi)\big)\right|^2 + \left|\frac{\partial}{\partial t}(t,\xi)\right|^2 + |u(t,\xi)\|^2\right\} dt\, d\xi, \tag{6.3}$$

over all $u \in L^2([0,\infty] \times \Omega)$ subject to (8.2) in Chapter 1 of Part IV. In this case the algebraic Riccati equation reads as follows:

$$\begin{bmatrix} 0 & -1 \\ \Lambda & 0 \end{bmatrix} X + X \begin{bmatrix} 0 & 1 \\ -\Lambda & 0 \end{bmatrix} - X \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} X + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0.$$

We prove now that (A,B) is I-stabilizable; for this it is sufficient to show that $A - 2\alpha BB^\star$ is exponentially stable if $0 < \alpha^2 < \lambda_0$ where λ_0 is the principal eigenvalue of the Laplace operator in Ω, with Dirichlet boundary conditions. By a direct computation we find

$$e^{t(A-2\alpha BB^\star)} = e^{-\alpha t}\begin{bmatrix} \cos(Et) + \dfrac{\alpha}{E}\sin(Et) & \dfrac{1}{E}\sin(Et) \\ \dfrac{\alpha^2+E^2}{E}\sin(Et) & -\dfrac{\alpha}{E}\sin(Et) + \cos(Et)\end{bmatrix},$$

where $E = \sqrt{\Lambda - \alpha^2 I}$. As $(A, C) = (A, I)$ is detectable, the Riccati equation has a unique nonnegative solution

$$P_{\min}^\infty = P_{\mathrm{Max}}^\infty = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22}\end{bmatrix}.$$

Then, by Theorem 3.1, there exists a unique optimal pair $(u^\star, x^\star)$ with

$$u^\star(t,\xi) = -\big(P_{21}x^\star(t,\cdot)\big)(\xi) - \left(P_{22}\frac{\partial x^\star}{\partial t}\right)(t,\cdot)(\xi).$$

Remark 6.1. Similar consideration apply to the wave equation with Neumann boundary conditions. See Remark 8.1 in Chapter 1 of Part IV. □

Remark 6.2. For a situation in which there exist periodic nontrivial solutions (see G. Da Prato [6]). □

6.3 Strongly damped wave equation

Let Ω be an open bounded set of $\mathbb{R}^n$ with regular boundary $\partial\Omega$. Consider the equation

$$\begin{cases} \dfrac{\partial^2 x}{\partial t^2}x(t,\xi) = \Delta_\xi(t,\xi) + \rho\Delta_\xi\dfrac{\partial x}{\partial t}(t,\xi) \\ \qquad\qquad\qquad + \big(Bu(t,\cdot)\big)(\xi) \quad \text{in }]0,T] \times \Omega, \\ x(t,\xi) = 0 \quad \text{on }]0,T] \times \partial\Omega, \\ x(0,\xi) = x_0(\xi), \quad \dfrac{\partial x}{\partial t}(0,\xi) = x_1(\xi) \quad \text{in } \Omega, \end{cases} \tag{6.4}$$

where ν is the outward normal to $\partial\Omega$ and ρ is a given positive number.

As in §8.2 of Chapter 1 of Part IV, we choose the spaces H, Y, U as in (8.3), the scalar product in H as in (8.4), and consider the positive self-adjoint operator Λ defined by (8.5). As Ω is bounded there exists a complete orthonormal system in $L^2(\Omega)$, $\{e_k\}$, and a sequence of real numbers

$$0 < \mu_1 \le \mu_2 \le \mu_2 \le \cdots \to +\infty$$

such that

$$\Lambda e_k = \mu_k e_k, \quad k = 1, 2, \ldots.$$

Define the linear operator A_1 on H

$$\begin{cases} A_1 X = \begin{bmatrix} 0 & 1 \\ -\Lambda & -\rho\Lambda \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \end{bmatrix}, \quad \forall X \in D(A_1), \\ D(A_1) = H^2(\Omega) \cap H_0^1(\Omega) \oplus H_0^1(\Omega). \end{cases} \tag{6.5}$$

It is an easy exercise to prove that the spectrum of A_1 is given by

$$\sigma(A_1) = \{-1/\rho\} \cup \Big\{ -\tfrac{1}{2}(\sigma\mu_k \pm \sqrt{(\rho^2\mu_k^2 - 4\mu_k}) \colon k = 1, 2, \dots \Big\}$$

and the resolvent by

$$R(\lambda, A_1) = \begin{bmatrix} \lambda + \rho\Lambda & 1 \\ -\Lambda & \lambda \end{bmatrix} [\lambda^2 + \rho\Lambda\lambda + \Lambda]^{-1}, \quad \forall \lambda \in \sigma(A_1).$$

From the above formula, it is not difficult to prove that A_1 is the infinitesimal generator of an analytic semigroup in H.

Now set

$$Bu = \begin{bmatrix} 0 \\ u \end{bmatrix}, \quad u \in U.$$

Condition (ii) of Proposition 3.3 (Hautus condition) is fulfilled because A_1 only has a spectrum with strictly negative real parts, so that (A_1, B) is stabilizable with respect to the observation I.

We can now consider the cost functional (6.3) and solve the corresponding minimization problem.

Remark 6.3. The strongly damped wave equation with time-dependent periodic coefficients has been studied in G. Da Prato and A. Lunardi [1]. □

2

Unbounded Control Operators: Parabolic Equations With Control on the Boundary

1 Introduction and setting of the problem

As in Chapter 2 of Part IV we consider a dynamical system governed by the following equation:

$$\begin{cases} x'(t) = Ax(t) + (\lambda_0 - A)Du(t), \quad t \geq 0, \\ x(0) = x_0, \end{cases}$$

or equivalently

$$x(t) = e^{tA}x_0 + (\lambda_0 - A)\int_0^t Du(s)\,ds, \tag{1.1}$$

where $x_0 \in H$ and $u \in L^2(0,\infty;U)$. We assume that

$$(\mathcal{HP})_\infty \begin{cases} \text{(i)} & A \text{ generates an analytic semigroup } e^{tA} \text{ of type } \omega_0 \\ & \text{ and } \lambda_0 \text{ is a real number in } \rho(A) \text{ such that } \omega_0 < \lambda_0, \\ \text{(ii)} & \exists \alpha \in \,]0,1[\text{ such that } D \in \mathcal{L}\big(U; D([\lambda_0 - A]^\alpha)\big), \\ \text{(iii)} & C \in \mathcal{L}(H;Y). \end{cases}$$

Clearly, if hypotheses $(\mathcal{HP})_\infty$ hold, then the hypotheses $(\mathcal{HP})$ of Chapter 2 of Part IV are fulfilled with $P_0 = 0$. If $\alpha \leq 1/2$, we will choose once and for all a number β belonging to $]1-\alpha/2, 1-\alpha/2[$. We want to minimize the cost function:

$$J_\infty(u) = \int_0^\infty \{|Cx(s)|^2 + |u(s)|^2\}\,ds \tag{1.2}$$

over all controls $u \in L^2(0,\infty;U)$ subject to the differential equation constraint (1.1). We say that the control $u \in L^2(0,\infty;U)$ is *admissible* if $J_\infty(u) < \infty$. The definitions of optimal control, optimal state, and optimal pair are the same as in Chapter 1. When, for any $x_0 \in H$, an admissible control exists, we say that (A, AD) is C-stabilizable.

In this chapter, we want to generalize the results of Chapter 1. Several of these generalizations are straightforward and the corresponding proofs will only be sketched. We will start by proving the existence of a minimal solution $P_{\min}^{\infty}$ of the algebraic Riccati equation

$$\begin{aligned} A^{\star}X + XA - Y^{\star}EE^{\star}Y + C^{\star}C = 0, \\ Y = [\lambda_0 - A^{\star}]^{1-\alpha}X, \end{aligned} \tag{1.3}$$

where $E = (\lambda_0 - A)^{\alpha}D$, under the hypothesis that (A, AD) is C-stabilizable. We remark that in order that (1.3) be meaningful, we need a regularity property of X, namely

$$X \in \Sigma^{+}(H) \cap \mathcal{L}\big(H; D\big([\lambda_0 - A^{\star}]^{1-\alpha}\big)\big).$$

We shall set $V_{\min}^{\infty} = [\lambda_0 - A^{\star}]^{1-\alpha}P_{\min}^{\infty}$. The above requirement will make the proof of existence much more involved than in Chapter 1 (see §2 below). Once the existence is proved, one can show, quite easily (see §3), that the optimal control $u^{\star}$ exists and is given by the feedback formula

$$u^{\star}(t) = -E^{\star}V_{\min}^{\infty}x^{\star}(t),$$

where $x^{\star}$ is the solution to the closed loop equation

$$x(t) = e^{tA}x_0 + \int_0^t [\lambda_0 - A]^{1-\alpha}e^{(t-s)A}EE^{\star}V_{\min}^{\infty}x(s)\,ds. \tag{1.4}$$

Another new difficulty arises in the study of the closed loop operator F (see §3), formally defined by

$$F = A - EE^{\star}V_{\min}^{\infty}.$$

In §3 we prove a characterization of F that enables us to generalize all results concerning detectability and Hautus conditions of Chapter 1. Also we leave the reader to extend all results on the qualitative behavior of the solutions of Riccati equations proved in Chapter 1 .

2 The algebraic Riccati equation

We assume here that the assumptions $(\mathcal{HP})_{\infty}$ are verified and consider system (1.1) and the Riccati equation

$$P' = A^{\star}P + PA - V^{\star}EE^{\star}V + C^{\star}C, \tag{2.1}$$

along with the algebraic equation (1.3). We say that

$$P \in \begin{cases} C_{s,\alpha}\big([0,\infty[\,;\Sigma^{+}(H)\big), & \text{if } \alpha > 1/2, \\ C_{s,\alpha,\beta}([0,\infty[\,;\Sigma^{+}(H)), & \text{if } \alpha \le 1/2 \end{cases}$$

is a *mild solution* of (2.1), if

$$P(t)x = e^{tA^\star} P(0) e^{tA} x + \int_0^t e^{(t-s)A^\star} C^\star C e^{(t-s)A} x\, ds \\ + \int_0^t e^{(t-s)A^\star} V^\star(s) E E^\star V(s) e^{(t-s)A} x\, ds. \quad (2.2)$$

Moreover, X is said to be a *solution* of (1.3) if

(i) $X \in \Sigma^+(H) \cap \mathcal{L}\big(H; D([\lambda_0 - A^\star]^{1-\alpha})\big)$,
(ii) $(Xx, Ay) + (Ax, Xy) - (E^\star Y x, E^\star Y y) + (Cx, Cy) = 0$

for all $x, y \in D(A)$, where $Y = [\lambda_0 - A^\star]^{1-\alpha} X$.

As in the previous chapter, we shall denote by $P_{\min}$ the minimal nonnegative solution of Riccati equation (2.1); that is, the solution of (2.1) such that $P(0) = 0$ (see Propositions 2.3 and 2.4 of Chapter 2 of Part IV).

We start by proving existence in the easier case $\alpha > 1/2$.

Proposition 2.1. *Assume $(\mathcal{HP})_\infty$ with $\alpha > 1/2$ and that (A, AD) is C-stabilizable. Then there exists*

$$P_{\min}^\infty \in \Sigma^+(H) \cap \mathcal{L}\big(H; D([\lambda_0 - A^\star]^{1-\alpha})\big)$$

such that

(i) *for each $x \in H$ the following limit exists:*

$$\lim_{t\to\infty} P_{\min}(t)x = P_{\min}^\infty x, \quad (2.3)$$

(ii) $P_{\min}^\infty \in \Sigma^+(H) \cap \mathcal{L}\big(H; D([\lambda_0 - A^\star]^{1-\alpha})\big)$,
(iii) $P_{\min}^\infty$ *is a solution of* (1.3).

Proof. As (A, AD) is C-stabilizable, we can prove, by repeating the proof of Proposition 1.2.3, that there exists $K > 0$ such that

$$\|P_{\min}(t)\| \le K, \quad t \ge 0. \quad (2.4)$$

Now, arguing as we did in the proof of Proposition 2.2 in Chapter 1, we see that there exists $P_{\min}^\infty$ such that (2.3) holds true. Let Q and Q_n, $n = 0, 1, \dots$, be the mild solutions (granted by Theorem 2.1 in Chapter 2 of Part IV of the Riccati equations

$$\begin{cases} Q' = A^\star Q + QA - V_Q^\star E E^\star V_Q + C^\star C, \\ Q(0) = P_{\min}^\infty, \end{cases} \quad (2.5)$$

and

$$\begin{cases} Q_n' = A^\star Q_n + Q_n A - V_{Q,n}^\star E E^\star V_{Q,n} + C^\star C, \\ Q_n(0) = P_{\min}(n), \end{cases} \quad (2.6)$$

where $V_Q = [\lambda_0 - A^\star]^{1-\alpha} Q$ and $V_{Q,n} = [\lambda_0 - A^\star]^{1-\alpha} Q_n$. We clearly have

$$Q_n(t) = P_{\min}(t+n), \quad t > 0.$$

Moreover, by Proposition 2.2 in Chapter 2 of Part IV, because

$$Q_n(0)x \to Q(0)x, \quad \forall x \in H,$$

we have

$$Q(t)x = \lim_{n\to\infty} Q_n(t)x = \lim_{n\to\infty} P_{\min}(t+n)x = P_{\min}^\infty x, \quad \forall t > 0,\ x \in H.$$

Thus $Q(t)$ is constant and coincides with $P_{\min}^\infty$. As, by Theorem 2.1 in Chapter 2 of Part IV, $Q \in C_{s,\alpha}\big([0,\infty[\ ; \Sigma^+(H)\big)$, we see that (ii) holds true. Finally, (iii) follows from Proposition 2.1 in Chapter 2 of Part IV, because if $x, y \in D(A)$, we have

$$0 = \frac{d}{dt}(Q(t)x, y) = (P_{\min}^\infty x, Ay) + (P_{\min}^\infty Ax, y) + (Cx, Cy) - (E^\star V_{\min}^\infty x, E^\star V_{\min}^\infty y).$$

The proof is complete. □

Next, we consider the case $\alpha \le \frac{1}{2}$. We first recall that, by Theorem 2.2 in Chapter 2 of Part IV, for any $x \in D([\lambda_0 - A]^\beta)$ we have $P_{\min}(t)[\lambda_0 - A]^\beta x \in D([\lambda_0 - A^\star]^\beta)$, and the linear operator $[\lambda_0 - A^\star]^\beta P_{\min}(t)[\lambda_0 - A]^\beta$ is closable. We denote by $\big(P_{\min}(t)\big)_\beta$ its closure.

Proposition 2.2. *Assume $(\mathcal{HP})_\infty$ with $\alpha \le \frac{1}{2}$ and that (A, AD) is C-stabilizable. Then there exists*

$$P_{\min}^\infty \in \Sigma^+(H) \cap \mathcal{L}\big(H; D([\lambda_0 - A^\star]^{1-\alpha})\big)$$

such that for all $x \in H$

(i) *the limits*

$$\lim_{t\to\infty} P_{\min}(t)x = P_{\min}^\infty x, \qquad \forall x \in H, \tag{2.7}$$

$$\lim_{t\to\infty} \big(P_{\min}(t)\big)_\beta x = R_{\min}^\infty x, \quad \forall x \in H, \tag{2.8}$$

exist, where $R_{\min}^\infty = (P_{\min}^\infty)_\beta$ and

$$(P_{\min}^\infty)_\beta = \text{ closure of } [\lambda_0 - A^\star]^\beta P_{\min}^\infty [\lambda_0 - A^\star]^\beta,$$

(ii) $P_{\min}^\infty \in \Sigma^+(H) \cap \mathcal{L}\big(H; D([\lambda_0 - A^\star]^{1-\alpha})\big)$,
(iii) $P_{\min}^\infty$ *is a solution of* (1.3).

Proof. By proceeding as in the proof of Proposition 2.1, we can prove (2.7) and that there exists a constant L_1 such that

$$\left\|\left(P_{\min}(t)\right)_\beta\right\| \le L_1, \quad t \ge 0. \tag{2.9}$$

We follows here an argument of F. FLANDOLI [7]; as the proof of (2.9) is easier for $\lambda_0 < 0$, we introduce a shifting $L = A - \lambda_0 I$ of A. Clearly L is of negative type and, as easily checked, $P(t) = P_{\min}(t)$ is the solution to Riccati equation

$$\begin{cases} P' = L^\star P + PL + \Delta^2 - V^\star E E^\star V, \\ P(0) = 0, \end{cases} \tag{2.10}$$

where

$$\Delta^2 = C^\star C + (2\lambda_0 + 2)P.$$

Now, if $x \in D([\lambda_0 - A]^\beta) = D([I - L]^\beta)$, we have

$$\begin{aligned}(P(t)(\lambda_0 - A)^\beta x, (\lambda_0 - A)^\beta x) &= (P(t)(I - L)^\beta x, (I - L)^\beta x) \\ &\le \int_0^t |\Delta(I - L)^\beta e^{(t-s)L} x|^2 ds \\ &\le L^2[\|C\|^2 + 2K(\lambda_0 + 1] \int_0^\infty e^{-\lambda_0 s} s^{-2\beta}\, ds\, |x|^2,\end{aligned}$$

which implies (2.9). As the family $\left\{\left(P_{\min}(t)\right)_\beta\right\}$ is bounded and nondecreasing in β, (2.8) follows.

We now prove (ii). Let Q and Q_n be the mild solutions (established in Theorem 2.2 in Chapter 2 of Part IV of the Riccati equations (2.5) and (2.6)). We clearly have

$$Q_n(t) = P_{\min}(t + n), \quad t > 0.$$

Moreover, by Proposition 2.2 in Chapter 2 of Part IV, because $Q_n(0)x \to Q(0)x$, $\forall x \in H$, we have

$$\begin{aligned}Q(t) &= \lim_{n\to\infty} Q_n(t)x \\ &= \lim_{n\to\infty} P_{\min}(t + n)x = P_{\min}^\infty x, \quad \forall t > 0,\ x \in H.\end{aligned}$$

Thus $Q(t)$ is constant and coincides with $P_{\min}^\infty$. As, by Theorem 2.2, in Chapter 2 of Part IV, $Q \in C_{s,\alpha,\beta}\big([0,\infty[\ ; \Sigma^+(H)\big)$, we see that (ii) holds true. Finally, the proof of (iii) is similar to that of the previous proposition. So it will be omitted. The proof is complete. □

3 Dynamic programming

3.1 Existence and uniqueness of the optimal control

In this section, we consider the control problem (1.1)–(1.2). We assume $(\mathcal{HP})_\infty$ and that (A, AD) is C-stabilizable, and denote by

$$P_{\min}^{\infty} \in \Sigma^{+}(H) \cap \mathcal{L}\big(H; D([\lambda_0 - A^{\star}]^{1-\alpha})\big)$$

the minimal solution of the algebraic equation (1.3) and we set $V_{\min}^{\infty} = [\lambda_0 - A^{\star}]^{1-\alpha} P_{\min}^{\infty}$.

We first recall that, in virtue of (3.1) in Chapter 1, we have

$$(P_{\min}(t)x_0, x_0) + \int_0^t |u(s) + E^{\star} V_{\min}(t-s)x(s)|^2\, ds = \int_0^t \{|Cx(s)|^2 + |u(s)|^2\}\, ds, \quad (3.1)$$

for any $x_0 \in H$ and any $u \in L^2(0,\infty;U)$, where x is the solution of (1.1). Now we study the closed loop equation (1.4) which is meaningful because $P_{\min}^{\infty} \in \Sigma^{+}(H) \cap \mathcal{L}\big(H; D([\lambda_0 - A^{\star}]^{1-\alpha})\big)$ and $\|[\lambda_0 - A]^{1-\alpha} e^{t(A-\lambda_0)}\| \le L(t-s)^{\alpha-1}$.

Proposition 3.1. *There exists a unique solution $x \in C([0,\infty[\,;H)$ of* (1.4).

Proof. For any $x \in C([0,\infty[;H)$ we set

$$\big(\lambda(y)\big)(t) = \int_0^t [\lambda_0 - A]^{1-\alpha} e^{(t-s)A} E E^{\star} V_{\min}^{\infty} x(s)\, ds,$$

for all $t \in [0,T]$. Then we have

$$\begin{aligned}\|(\lambda(y))(t)\| &\le L^2 \|V_{\min}^{\infty}\| \int_0^t (t-s)^{\alpha-1}\, ds\, \|y\|_{C([0,t];H)} \\ &\le \frac{L^2}{2\alpha - 1} \|V_{\min}^{\infty}\| t^{\alpha} \|y\|_{C([0,t];H)},\end{aligned}$$

so that $\lambda \in \mathcal{L}\big(C([0,t];H)\big)$ for any $t \in\,]0,T]$ and

$$\|\lambda\|_{\mathcal{L}(C([0,t];H))} \le \frac{L^2}{2\alpha - 1} \|V_{\min}^{\infty}\|\, t^{\alpha}.$$

Thus, if t is small, λ is a contraction and (1.4) has a unique solution in $C([0,t];H)$. Now this argument can be repeated in the interval $[t, 2t]$ and so on, giving the conclusion. □

We prove now the main result of this section.

Theorem 3.1. *Assume that assumption $(\mathcal{HP})_{\infty}$ is verified and that (A,B) is C-stabilizable. Then there exists a unique optimal pair $(u^{\star}, x^{\star})$ for the optimal control problem* (1.1)–(1.2)*. Moreover the following statements hold*:

(i) *$x^{\star} \in C([0,\infty[\,;H)$ is the mild solution to the closed loop equation* (1.4).
(ii) *$u^{\star} \in C([0,\infty[\,;U)$ is given by the feedback formula*

$$u^{\star}(t) = -E^{\star} V_{\min}^{\infty} x^{\star}(t), \quad (3.2)$$

where $P_{\min}^{\infty}$ represents the minimal solution of the algebraic Riccati equation (1.3).

(iii) *the optimal cost* $J_\infty(u^\star)$ *is given by*

$$J_\infty(u^\star) = (P^\infty_{\min} x_0, x_0). \tag{3.3}$$

Proof. In view of Proposition 3.1, the proof is similar to that of Theorem 3.1 in Chapter 1. So it will be omitted. □

3.2 Feedback operator and detectability

Assume that the hypotheses of Theorem 3.1 hold and let $(u^\star, x^\star)$ be the optimal pair for problem (1.1)–(1.2). We want here to construct, following G. Da Prato and A. Ichikawa [1], a closed loop operator F for the system (3.1); that is a linear operator, infinitesimal generator of a strongly continuous semigroup e^{tF}, such that

$$x^\star(t) = e^{tF} x_0, \quad t \ge 0. \tag{3.4}$$

Proposition 3.2. *Assume* $(\mathcal{HP}_\infty)$ *and that* (A, B) *is* C*-stabilizable. For any* $x_0 \in H$ *set*

$$S(t)x = x^\star(t),$$

where $x^\star(t)$ *is the optimal state corresponding to* x_0*. Then* $S(\cdot)$ *is an analytic semigroup in* H *and its infinitesimal generator* F *is given by*

$$\begin{aligned} Fx &= (\lambda_0 - A)(AR(\lambda_0, A)x + DE^\star V^\infty_{\min} x),\\ D(F) &= \{x \in H \colon AR(\lambda_0, A)x + DE^\star V^\infty_{\min} x \in D(A)\}. \end{aligned} \tag{3.5}$$

Finally, if the resolvent of A *is compact, so is the resolvent of* F*.*

Proof. We first remark that for any $x_0 \in H$, $x(t) = S(t)x_0$ is precisely the solution of the closed loop equation (1.4). Thus, x is continuous, by Proposition 3.1, and $S(t)$ is a strongly continuous semigroup in H. Let $F \colon D(F) \subset H \to H$ be its infinitesimal generator; we want to prove that F is given by (3.5). To this aim, let $x_0 \in D(F)$, $x(t) = S(t)x_0$. We have

$$R(\lambda_0, A)x(t) = e^{tA} R(\lambda_0, A)x_0 + \int_0^t e^{(t-s)A} DE^\star V^\infty_{\min} x(s)\, ds.$$

As $x(t)$ is continuously differentiable, we have

$$R(\lambda_0, A)x'(t) = AR(\lambda_0, A)x(t) + DE^\star V^\infty_{\min} x(t).$$

Setting $t = 0$, it follows that

$$R(\lambda_0, A)Fx_0 = AR(\lambda_0, A)x_0 + DE^\star V^\infty_{\min} x_0, \tag{3.6}$$

so $AR(\lambda_0, A)x_0 + DE^\star V^\infty_{\min} x_0 \in D(A)$ and

$$Fx_0 = (\lambda_0 - A)(AR(\lambda_0, A)x_0 + DE^\star V_{\min}^\infty x_0). \tag{3.7}$$

Conversely assume that

$$AR(\lambda_0, A)x_0 + DE^\star V_{\min}^\infty x_0 \in D(A). \tag{3.8}$$

Then, by (3.6) and the density of $D(F)$, the linear operator $R(\lambda_0, A)F$ is closable and its closure $N = \overline{R(\lambda_0, A)F}$ is bounded. Now, if (3.8) holds, we have $Nx_0 \in D(A)$ and it is not difficult to show that this implies $x_0 \in D(F)$ and $(\lambda_0 - A)Nx_0 = Fx_0$.

It remains to show that e^{tF} is an analytic semigroup. Let

$$S_{\omega,\theta_0} = \{\lambda \in \mathbb{C}\colon |\arg(\lambda - \omega)| \le \theta_0\}$$

be a sector contained in the resolvent set $\rho(A)$ and assume that

$$\|R(\lambda, A)\| \le \frac{M(\theta)}{|\lambda - \omega|}, \quad \lambda \in S_{\omega,\theta_0}, \tag{3.9}$$

for some constants ω and $M(\theta) = M(-\theta)$ (see §2.7 of Chapter 1 of Part II). Let $\lambda \in S_{\omega,\theta_0}$, $\eta \in H$; consider the equation $\lambda\xi - F\xi = \eta$, which is equivalent to

$$\lambda\xi - [\lambda_0 - A]\{AR(\lambda_0, A)\xi + DE^\star V_{\min}^\infty \xi\} = \eta. \tag{3.10}$$

Multiplying (3.10) by $R(\lambda, A)$, gives

$$\xi - [\lambda_0 - A]^{1-\alpha} R(\lambda, A)EE^\star V_{\min}^\infty \xi] = R(\lambda, A)\eta. \tag{3.11}$$

As $[\lambda_0 - A]^\alpha D$ is bounded and $\|[\lambda_0 - A]^{1-\alpha} R(\lambda, A)\| \le \text{const.}\, |\lambda|^{-\alpha}$, there exists $\rho > 0$ such that, for any $\lambda \in \{\mu \in S_{\omega,\theta_0}\colon |\mu| \ge \rho\}$, (3.11) has a unique solution $\xi = R(\lambda, F)\eta$, where

$$R(\lambda, F) = \{I - [\lambda_0 - A]^{1-\alpha} R(\lambda, A)([\lambda_0 - A]^\alpha D)DE^\star V_{\min}^\infty\}^{-1} R(\lambda, A). \tag{3.12}$$

From (3.9) and (3.12) it follows that the semigroup generated by F is analytic and that if $R(\lambda, A)$ is compact so is $R(\lambda, F)$. □

The operator F is said to be the *closed loop* operator for the optimal control problem (1.1)–(1.2). Remark that, by (3.3), we know that the function $Cx^\star(t) = Ce^{tF}x_0$ belongs to $L^2(0, \infty; H)$. Thus, when $C = I$ (or if $C^{-1} \in \mathcal{L}(H)$), by Datko's Theorem, it follows that F is exponentially stable. Thus, in this case, (A, B) is feedback stabilizable.

We give now a generalization of Proposition 3.4 of Chapter 1.

Proposition 3.3. *Assume that (A, C) is detectable. Then F is exponentially stable.*

Proof. Let $x_0 \in H$, and let $(u^\star, x^\star)$ be the optimal pair corresponding to x_0. By Remark 3.2 in Chapter 1, there exists $K \in \mathcal{L}(Y; H)$ such that $A - KC$ is exponentially stable. Now it is easy to check that

$$x^*(t) = e^{t(A-KC)}x_0 + \int_0^t (\lambda_0 - A)e^{(t-s)(A-KC)}Du(s)\,ds + \int_0^t e^{(t-s)(A-KC)}KCx(s)\,ds.$$

As $Cx^\star$, $u^\star \in L^2(0,\infty;U)$ and $A-KC$ is exponentially stable it follows that $x^\star \in L^2(0,\infty;H)$. Datko's Theorem yields the conclusion. □

3.3 Stabilizability and stability of F in the point spectrum case

We consider here the system (1.1) under hypothesis $(\mathcal{HP}_\infty)$ and we assume that A verifies the *point spectrum* hypotheses $(\mathcal{P})$ (see §3.2 in Chapter 1). We want to give a necessary and sufficient condition in order that (A, AD) be stabilizable with respect to the identity I. We need a lemma whose proof is similar to that of Lemma 3.1 (Chapter 1).

Lemma 3.1. *Assume $(\mathcal{HP}_\infty)$ and $(\mathcal{P})$ with $\sigma^0(A) = \varnothing$. Then the following statements are equivalent*:

(i) *For all $x \in H$, there exists $u \in L^2(0,\infty;U)$ such that the solution x of* (1.1) *belongs to $L^2(0,\infty;H)$.*
(ii) *The mapping*

$$u \mapsto \gamma u = (\lambda_0 - A)\int_0^\infty e^{-sA}\Pi_A^+ Du(s)\,ds\colon L^2(0,\infty;U) \to H$$

is onto.
(iii) *The mapping*

$$\gamma^\star\colon H \to L^2(0,\infty;U),$$
$$(\gamma^\star\xi)(s) = D^\star[\lambda_0 - A^\star](\Pi_A^+)^\star e^{-sA^\star}\xi, \quad \xi \in H$$

is one-to-one.
(iv) Ker $\big(D^\star(\lambda - A^\star)\big) \cap \mathrm{Ker}(\lambda - A^\star) = \{0\}$, *for all $\lambda \in \sigma^+(A^\star)$.*

Proof. It is completely similar to the proof of Lemma 3.1 in Chapter 1. □

We now generalize Propositions 3.3, 3.4, and 3.5 of Chapter 1.

Proposition 3.4. *Assume $(\mathcal{HP})_\infty$ and $(\mathcal{P})$. Then the following statements are equivalent*:

(i) *(A, AD) is stabilizable with respect to I.*
(ii) *For any $\lambda \in \sigma^0(A^\star) \cup \sigma^+(A^\star)$,*

$$\mathrm{Ker}\,\big(D^\star(\lambda - A^\star) \cap \mathrm{Ker}(\lambda - A^\star)\big) = \{0\}.$$

Proposition 3.5. *Assume that (A, AD) is C-stabilizable and that A and F fulfill $(\mathcal{P})$. Then the following statements are equivalent*:

(i) *F is exponentially stable.*
(ii) *For any $\lambda \in \sigma^+(A) \cup \sigma^0(A)$, we have*

$$\operatorname{Ker}(A - \lambda) \cap \operatorname{Ker}(C) = \{0\}. \tag{3.13}$$

Proposition 3.6. *Assume that the hypotheses of Proposition* 3.5 *hold. Then the following statements are equivalent*:

(i) *F is exponentially stable.*
(ii) *(A, C) is detectable.*

Example 3.1. Let Ω be an open bounded set of $\mathbb{R}^n$ with regular boundary $\partial\Omega$. Consider the state equation

$$\begin{cases} \dfrac{\partial x}{\partial t}(t, \xi) = (\Delta_\xi + c)x(t, \xi) \quad \text{in }]0, T[\times \Omega, \\ x(t, \xi) = u(t, \xi) \quad \text{on }]0, T[\times \partial\Omega, \\ x(0, \xi) = x_0(\xi) \quad \text{in } \Omega, \end{cases} \tag{3.14}$$

where $c > 0$, $x_0 \in L^2(\Omega)$, and $u \in L^2(\partial\Omega)$. We choose $H = Y = L^2(\Omega)$ as space of states and observations and $U = L^2(\partial\Omega)$ as space of controls. We denote by A the linear self-adjoint operator in H:

$$\begin{cases} Ax = \Delta_\xi x + cx, \quad \forall x \in D(A), \\ D(A) = H^2(\Omega) \cap H_0^1(\Omega). \end{cases}$$

We consider the following problem: To minimize

$$J(u) = \int_0^\infty \int_\Omega |x(t, \xi)|^2 \, dt \, d\xi + \int_0^\infty \int_{\partial\Omega} |u(t, \xi)|^2 \, dt \, d\xi.$$

Let $D\colon L^2(\partial\Omega) \to L^2(\Omega)$ the Dirichlet mapping introduced in Example 1.1 of Chapter 2 in Part IV; then problem (3.14) can be written as

$$x(t) = e^{tA}x_0 + (cI - A)\int_0^t e^{(t-s)A} Du(s) \, ds,$$

and hypotheses $(\mathcal{HP}_\infty)$ hold with $\lambda_0 = c$. We denote by $\{\varphi_k\}$ a complete set of eigenvectors of A and by $\{-\lambda_k\}$ the corresponding sequence of eigenvalues. We assume that $\{\lambda_k\}$ is a nondecreasing sequence and that

$$0 < \lambda_1 < c, \quad \lambda_k > c, \quad k = 2, 3, \ldots,$$

and we prove that (A, AD) is stabilizable. Recalling Proposition 3.4 it is enough to observe that

$$D^\star(A^\star - c)\varphi_1 = \frac{\partial \varphi_1}{\partial \nu}$$

is not identically zero; this in fact is a consequence of a result that can be found in E. J. P. G. Schmidt and N. Weck [1]. □

Remark 3.1. Stabilizability for parabolic boundary control problems was studied by several authors, see T. Nanbu [1], I. Lasiecka and R. Triggiani [4, 5], and H. Amann [1].

These results have been generalized by A. Lunardi [2, 4] for non-autonomous state equations with periodic coefficients (see also P. K. Medina [1]). □

3

Unbounded Control Operators: Hyperbolic Equations With Control on the Boundary

1 Introduction and setting of the problem

We use here the notation of Chapter 3 in Part IV. We assume that

$$(\mathcal{H}\mathcal{H})_\infty \begin{cases} \text{(i)} & A \text{ generates a strongly continuous group } e^{tA} \text{ in } H \\ & \text{of type } \omega_0 \text{ and } \lambda_0 \text{ is a real number in } \rho(A) \text{ such} \\ & \text{that } \omega_0 < \lambda_0, \\ \text{(ii)} & E \in \mathcal{L}(U;H), \\ \text{(iii)} & \forall T > 0,\ \exists K_T > 0 \text{ such that} \\ & \displaystyle\int_0^t |E^\star A^\star e^{sA^\star} x|^2\, ds \le K_T^2 |x|^2, \quad \forall x \in D(A^\star),\ t \ge 0, \\ \text{(iv)} & C \in \mathcal{L}(H;Y). \end{cases}$$

Clearly, if $(\mathcal{H}\mathcal{H})_\infty$ hold, then the hypotheses $(\mathcal{H}\mathcal{H})$ of Chapter 3 in Part IV are fulfilled with $P_0 = 0$. We want to minimize the cost function:

$$J_\infty(u) = \int_0^\infty \{|Cx(s)|^2 + |u(s)|^2\} ds, \tag{1.1}$$

over all controls $u \in L^2(0,\infty;U)$ subject to the equation constraint

$$\begin{aligned} x(t) &= e^{tA}x_0 + G(u)(s), \\ G(u)(s) &= (\lambda_0 - A)\int_0^t e^{(t-s)A} Eu(s)\, ds. \end{aligned} \tag{1.2}$$

Moreover, $x_0 \in H$ and $u \in L^2(0,\infty;U)$. We recall that by Proposition 3.1 in Chapter 1 in Part II, $x \in C([0,T];H)$ for all $\in L^2(0,T;U)$; more precisely

$$G \in \mathcal{L}\big(L^2(0,T;U); C([0,T];H)\big), \quad \forall T > 0.$$

We say that the control $u \in L^2(0,\infty;U)$ is *admissible* if $J_\infty(u) < \infty$. The definitions of optimal control, optimal state, and optimal pair are the same as in Chapters 1 and 2. When, for any $x_0 \in H$, an admissible control exists, we say that (A, AE) is C-stabilizable.

In order to solve the control problem (1.1)–(1.2), we consider the optimization problems: to minimize

$$J_t(u) = \int_0^t \{|Cx(s)|^2 + |u(s)|^2\}\, ds, \tag{1.3}$$

over all controls $u \in L^2(0, t; U)$ subject to the equation constraint (1.1), and denote by $(u_t^\star, x_t^\star)$ the corresponding optimal pair, then we prove the convergence of $(u_t^\star, x_t^\star)$ to an optimal pair for the problem (1.1)–(1.2).

For a different approach to the study of the algebraic Riccati equation and more results, see the paper by F. Flandoli, I. Lasiecka, and R. Triggiani [1], and the Lecture Notes by I. Lasiecka and R. Triggiani [11]. The situation where assumption $(\mathcal{HH})_\infty$–(iii) is not fulfilled is considered in I. Lasiecka and R. Triggiani [12], where specific examples are provided.

2 Main results

The main result of this chapter is as follows.

Theorem 2.1. *Assume $(\mathcal{HH})_\infty$ and that (A, AE) is C-stabilizable. Let $(u_t^\star, x_t^\star)$ be the optimal pair corresponding to problem* (1.2)–(1.3)*; then there exist $u^\star \in L^2(0, \infty; U)$ and $x^\star \in L^2(0, \infty; H)$ such that*

(i) $\lim_{t \to \infty} u_t^\star = u^\star$ *in $L^2(0, T; U)$, for all $T > 0$.*
(ii) $\lim_{t \to \infty} x_t^\star = x^\star$ *in $L^2(0, T; H)$, for all $T > 0$.*
(iii) $(u^\star, x^\star)$ *is an optimal pair for the problem* (1.1)–(1.2).

Proof. Denote by $P_{\min}(\cdot)$ the mild solution to (1.5) of Part IV, Chapter 3, with $P_0 = 0$. Let $x_0 \in H$, $u \in L^2(0, \infty; U)$ and let x be the corresponding solution to (1.1); we have

$$\begin{aligned}(P_{\min}(t)x_0, x_0) \le \int_0^t \{|Cx(s)|^2 + |u(s)|^2\}\, ds& \\ \le J_\infty(u), \quad \forall u \in L^2(0, \infty; U).& \quad (2.1)\end{aligned}$$

As (A, AE) is C-stabilizable, there exists $M(x_0) > 0$ such that

$$(P_{\min}(t)x_0, x_0) \le M(x_0), \quad \forall t > 0.$$

By the Uniform Boundedness Theorem, it follows that there exists $M > 0$ such that

$$\|P_{\min}(t)\| \le M, \quad \forall t \ge 0. \tag{2.2}$$

Then there exists the limit

$$P_{\min}^{\infty}x_0 = \lim_{t\to\infty} P_{\min}(t)x_0, \quad \forall x_0 \in H.$$

From (2.1) it follows that

$$(P_{\min}^{\infty}x_0, x_0) \le J_\infty(u), \quad \forall u \in L^2(0,\infty;U). \tag{2.3}$$

On the other hand we have

$$(P_{\min}(t)x_0, x_0) = \int_0^t \{|Cx_t^*(s)|^2 + |u_t^*(s)|^2\}\, ds, \quad \forall t > 0, \tag{2.4}$$

and taking into account (2.2), we see easily that the set $\{\hat{u}_t\}_{t\ge 0}$, where

$$\hat{u}_t(s) = \begin{cases} u_t^\star(s), & \text{if } s \in [0,t], \\ 0, & \text{if } s \ge t, \end{cases}$$

is bounded in $L^2(0,\infty;U)$. Thus there exists a sequence $t_n \uparrow \infty$ and a function $\hat{u} \in L^2(0,\infty;U)$ such that

$$\hat{u}_{t_n} \rightharpoonup \hat{u}, \quad \text{in } L^2(0,\infty;U) \text{ as } n \to \infty. \tag{2.5}$$

Set

$$\hat{x}(t) = e^{tA}x_0 + G(u)(t).$$

Fix now $T > 0$. Obviously

$$u_{t_n}^* \rightharpoonup \hat{u}, \quad \text{in } L^2(0,T;U) \text{ as } n \to \infty,$$

and consequently,

$$x_{t_n}^* \rightharpoonup \hat{x}, \quad \text{in } L^2(0,T;U) \text{ as } n \to \infty.$$

From (2.4) it follows that

$$(P_{\min}^{\infty}x_0, x_0) \ge \int_0^T \{|\hat{x}(s)|^2 + |\hat{u}(s)|^2\} ds, \quad \forall t > 0.$$

As T is arbitrary we can conclude that $C\hat{x} \in L^2(0,\infty;H)$ and

$$(P_{\min}^{\infty}x_0, x_0) \ge J_\infty(\hat{u}), \tag{2.6}$$

which, along with (2.3), implies that the pair $(\hat{u}, \hat{x})$ is optimal. It remains to prove (i) and (ii). We first remark that, by the Lebesgue dominated convergence theorem

$$Cx_{t_n}^* \rightharpoonup \hat{x}, \quad \text{in } L^2(0,T;\infty) \text{ as } n \to \infty.$$

Letting n tend to infinity in the equality

$$\int_0^{T_n} \{|Cx_{T_n}^*(s)|^2 + |u_{T_n}^*(s)|^2\}\, ds = (P_{\min}(T_n)x_0, x_0),$$

we find

$$\lim_{n\to\infty}\int_0^\infty \{|Cx^*_{T_n}(s)|^2 + |u^*_T(s)|^2\}\,ds = (P^\infty_{\min}x_0, x_0).$$

This implies that the convergences of $\{u^*_T(s)\}$ to $\hat{u}$ and of $\{Cx^*_T(s)\}$ to $C\hat{x}$ are strong. As the optimal pair is unique, by the strict convexity of the cost J_∞, (i) and (ii) follows. □

Example 2.1. We consider here the system described in Example 4.3 of Chapter 3 of Part IV, but with $T = +\infty$. In order to apply Theorem 2.1 it is enough to check that (A, AE) is exactly controllable. As remarked at the end of §2 in Chapter 2 of Part II, the relevant estimate to prove is that, for T sufficiently large, there exists a constant $C(T) > 0$ such that

$$\int_0^T \left|\frac{\partial v}{\partial \nu}\right|^2_{L^2(\partial\Omega)} dt \geq C(T)E(0), \tag{2.7}$$

where v is the solution to the problem

$$\begin{cases} v_{tt}(t,\xi) = \Delta_\xi v(t,\xi), \quad t \geq 0,\ \xi \in \Omega, \\ v(t,\xi) = 0, \quad t \geq 0,\ \xi \in \partial\Omega, \\ v(0,\xi) = v_0 \in H^1_0(\Omega), \\ v_t(0,\xi) = v_1 \in L^2(\Omega), \end{cases} \tag{2.8}$$

and

$$E(t) = \frac{1}{2}\int_\Omega [|\nabla v(t,\xi)|^2 + |v_t(t,\xi)|^2]\,d\xi\,dt.$$

To prove (2.7) we follows J. L. Lions [4]. We introduce the following notation:

$$X(t) = (v_t, \xi\cdot\nabla v), \quad Y(t) = (v_t, v),$$

where $(\cdot,\cdot)$ denotes the scalar product and $|\cdot|$ the norm in $L^2(\Omega)$. We first prove the identity

$$Y(T) - Y(0) = \int_{\Omega\times[0,T]} [v_t^2 - |\nabla v|^2]\,d\xi\,dt. \tag{2.9}$$

In fact, multiplying both sides of the first equation in (2.8) by v and taking into account that

$$(v^2)_{tt} = 2v_t^2 + 2vv_{tt},$$

we find

$$vv_{tt} = \tfrac{1}{2}(v^2)_{tt} - v_t^2 = v\Delta v.$$

Integrating on Ω

$$\frac{1}{2}\int_\Omega (v^2)_{tt}\,d\xi - \int_\Omega v_t^2\,d\xi = \int_\Omega v\Delta v\,d\xi = -\int_\Omega |\nabla v|^2\,d\xi,$$

and so

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}(v^2)_t\,d\xi = \int_{\Omega}[v_t^2 - |\nabla v|^2]\,d\xi,$$

which implies (2.9).

Now we multiply both sides of the first equation in (2.8) by $\xi \cdot \nabla v$ and integrate in $\Omega \times [0,T]$. We find

$$I = J,$$

where

$$I = \int_{\Omega\times[0,T]}(\xi\cdot\nabla v)v_{tt}\,d\xi\,dt$$

and

$$J = \int_{\Omega\times[0,T]}(\xi\cdot\nabla v)\Delta v\,d\xi\,dt.$$

We proceed now in three steps.

Step 1. *Estimate of* I.

By integrating by parts successively in t and in ξ, we find

$$\begin{aligned} I &= X(T) - X(0) - \int_{\Omega\times[0,T]}(\xi\cdot\nabla v_t)v_t\,d\xi\,dt \\ &= X(T) - X(0) - \frac{1}{2}\int_{\Omega\times[0,T]}\xi\cdot\nabla(v_t^2)\,d\xi\,dt \\ &= X(T) - X(0) + \frac{n}{2}\int_{\Omega\times[0,T]}v_t^2\,d\xi\,dt. \end{aligned}$$

Step 2. *Estimate of* J.

By integrating by parts in ξ, we find

$$\begin{aligned} J &= \sum_{h,k=1}^{n}\int_{\Omega\times[0,T]}\xi_h\frac{\partial v}{\partial\xi_h}\frac{\partial^2 v}{\partial\xi_k^2}\,d\xi\,dt \\ &= \sum_{h,k=1}^{n}\int_{\partial\Omega\times[0,T]}\xi_h\nu_k\frac{\partial v}{\partial x_h}\frac{\partial v}{\partial\xi_k}\,d\sigma\,dt - \sum_{h,k=1}^{n}\int_{\Omega\times[0,T]}\frac{\partial}{\partial\xi_k}\left(\xi_h\frac{\partial v}{\partial\xi_h}\right)\frac{\partial v}{\partial\xi_k}\,d\xi\,dt \\ &= \int_{\partial\Omega\times[0,T]}(\xi\cdot\nabla v)\left|\frac{\partial v}{\partial\nu}\right|^2 d\sigma\,dt - \frac{1}{2}\sum_{h,k=1}^{n}\int_{\Omega\times[0,T]}\xi_h\frac{\partial}{\partial\xi_k}\left|\frac{\partial v}{\partial\xi_h}\right|^2 d\xi\,dt \\ &\qquad - \int_{\Omega\times[0,T]}|\nabla v|^2\,d\xi\,dt. \end{aligned}$$

We observe now that, as $v = 0$ on $\partial\Omega$, we have $\nabla v = \frac{\partial v}{\partial\nu}\nu$. It follows that

$$J = \int_{\partial\Omega\times[0,T]} (\xi \cdot \nabla v)\left|\frac{\partial v}{\partial \nu}\right|^2 d\sigma\, dt - \frac{1}{2}\int_{\Omega\times[0,T]} \xi \cdot \nabla(|\nabla v|^2)\, dx\, dt$$
$$- \int_{\Omega\times[0,T]} |\nabla v|^2\, d\xi\, dt$$
$$= \int_{\partial\Omega\times[0,T]} (\xi \cdot v)\left|\frac{\partial v}{\partial \nu}\right|^2 d\sigma\, dt + \left(\frac{n}{2} - 1\right)\int_{\Omega\times[0,T]} |\nabla v|^2\, d\xi\, dt.$$

Step 3. *Conclusion.*

As $I = J$, we find

$$\int_{\partial\Omega\times[0,T]} (\xi \cdot v)\left|\frac{\partial v}{\partial \nu}\right|^2 d\sigma$$
$$= X(T) - X(0) + \frac{n}{2}\int_{\Omega\times[0,T]} v_t^2\, d\xi\, dt - \left(\frac{n}{2} - 1\right)\int_{\Omega\times[0,T]} |\nabla v|^2\, dx\, dt$$
$$= X(T) - X(0) + \frac{n}{2}\int_{\Omega\times[0,T]} [v_t^2 - |\nabla v|^2]\, d\xi\, dt + \int_{\Omega\times[0,T]} |\nabla v|^2\, d\xi\, dt.$$

In conclusion

$$\int_{\partial\Omega\times[0,T]} (\xi \cdot v)\left|\frac{\partial v}{\partial \nu}\right|^2 d\sigma\, dt = \left[X + \frac{n-1}{2} Y\right]_0^T + TE(0).$$

Now, let $R > 0$ such that Ω is included in the ball $B(0, R)$ and let $\alpha_0 > 0$ such that $|v| \leq \alpha_0 |\nabla v|$; then

$$|X| \leq R|v_t|\, |\nabla v| \leq RE(0)$$

and

$$|Y| \leq |v|\, |v_t| \leq \alpha_0 |v_t|\, |\nabla v| \leq \alpha_0 E(0),$$

and so

$$\int_{\partial\Omega\times[0,T]} (\xi \cdot v)\left|\frac{\partial v}{\partial \nu}\right|^2 d\sigma\, dt \geq E(0)[T - 2R - 3\alpha_0],$$

and the required estimate (2.7) holds for T large. □

3 Some result for general semigroups

In this section we do not assume that A generates a strongly continuous group but

$$
(\mathcal{HH})'_\infty \begin{cases} \text{(i)} & A \text{ generates a strongly continuous semigroup } e^{tA} \\ & \text{in } H \text{ of type } < \lambda_0, \text{ for some real number } \lambda_0, \\ \text{(ii)} & E \in \mathcal{L}(U;H), \\ \text{(iii)} & \forall T > 0,\ \exists K_T > 0 \text{ such that} \\ & \displaystyle\int_0^t |E^\star A^\star e^{sA^\star} x|^2\, ds \le K_T^2 |x|^2, \quad \forall x \in D(A^\star), t \ge 0, \\ \text{(iv)} & C \in \mathcal{L}(H;Y). \end{cases}
$$

By the results of §5 in Chapter 3 of Part IV, we are able to show the existence of an optimal pair (u_t^*, x_t^*) of the control problem (1.2)–(1.3) under hypotheses $(\mathcal{HH})'_\infty$. Then the following theorem is proved as Theorem 2.1.

Theorem 3.1. *Assume* $(\mathcal{HH})'_\infty$ *and that* (A, AE) *is* C*-stabilizable. Let* $(u_t^\star, x_t^\star)$ *be the optimal pair corresponding to problem* (1.2)–(1.3)*. Then there exist* $u^\star \in L^2(0,\infty;U)$ *and* $x^\star \in L^2(0,\infty;H)$ *such that*

(i) $\lim_{t\to\infty} u_t^\star = u^\star$ *in* $L^2(0,T;U)$*, for all* $T > 0$*.*
(ii) $\lim_{t\to\infty} x_t^\star = x^\star$ *in* $L^2(0,T;H)$*, for all* $T > 0$*.*
(iii) $(u^\star, x^\star)$ *is an optimal pair for the problem* (1.1)–(1.2)*.*

Example 3.1 *(Age-dependent equations).* We continue here Example 5.1 of Chapter 3 of Part IV, but taking $T = +\infty$. In order to apply Theorem 2.2 we have only to check stabilizability. Set

$$
\begin{aligned}
\sigma_1(A) &= \{\lambda \in \sigma(A) \colon\ \operatorname{Re}\lambda < 0\}, \\
\sigma_2(A) &= \{\lambda \in \sigma(A) \colon\ \operatorname{Re}\lambda \ge 0\},
\end{aligned}
$$

and denote by Q_1 and Q_2 the spectral projectors on $\sigma_1(A)$ and $\sigma_2(A)$, respectively. As is well known Q_2 is a finite dimensional projector; moreover, because the semigroup $S(\cdot)$ is differentiable for $t > a_+$ the spectral determining condition holds true, and there exist $M > 0$ and $r > 0$ such that

$$
\|S(t)Q_1\| \le Me^{-rt}, \quad t \ge 0. \tag{3.1}
$$

We set $p_1(t) = Q_1 p(t)$, $t \ge 0$ and $p_2(t) = Q_2 p(t)$, $t \ge 0$ and denote by A_1 and A_2 the restrictions of A to $Q_1(H)$ and $Q_2(H)$, respectively. By (3.1) it follows easily that $p_1 \in L^2(0,\infty;H)$ for any $u \in L^2(0,\infty)$. So, we have only to check that, for any $p_0 \in H$ there exists $u \in L^2(0,\infty)$ such that $p_2 \in L^2\big(0,\infty;Q_2(H)\big)$. Now p_2 is the solution of the finite dimensional Cauchy problem

$$
\begin{cases} p_2'(t) = A_2 p_2(t) - \alpha A_2 D u(t), \\ p_2(0) = Q_2 p_0. \end{cases} \tag{3.2}
$$

By a well-known result due to Hautus (see Part I, Chapter 1, §2.6), the system (3.2) is stabilizable if and only if

$$\operatorname{Ker}(\lambda - A_2^*) \cap \operatorname{Ker}(D^* A_2^*) = \{0\}, \quad \forall \lambda \in \sigma_2(A). \tag{3.3}$$

Now we can easily check that the spectrum $\sigma(A)$ of consists of simple eigenvalues λ that are exactly all the solutions to the equation

$$1 = \int_0^{a_+} e^{-\lambda b} K(b)\, db. \tag{3.4}$$

If λ is an eigenvalue of A, then a corresponding eigenvector is given by

$$\varphi_\lambda(a) = e^{-\lambda a} \pi(a), \quad a \in [0, a_+]. \tag{3.5}$$

Moreover the adjoint A^* of A is given by

$$\begin{cases} D(A^*) = \{\psi \in H^1(0, a_+) \colon \psi(T) = 0\}, \\ A^* \psi = \psi' - \mu \psi + \beta \psi(0). \end{cases} \tag{3.6}$$

Finally $\sigma(A^*) = \sigma(A)$, and if $\lambda \in \sigma(A^*)$, then a corresponding eigenvector is given by

$$\psi_\lambda(a) = \frac{1}{\pi(a)} \int_a^{a_+} e^{\lambda(a-b)} K(b)\, db, \quad a \in [0, a_+]. \tag{3.7}$$

Let now $\lambda \in \sigma_2(A)$ and $\psi \in \operatorname{Ker}(D^* A_2^*)$; then we have

$$1 = \int_0^{a_+} e^{-\lambda b} K(b)\, db, \tag{3.8}$$

and a corresponding eigenvector to λ is given by

$$\psi_\lambda(a) = \frac{1}{\pi(a)} \int_a^{a_+} e^{\lambda(a-b)} K(b)\, db, \quad a \in [0, a_+]. \tag{3.9}$$

It follows, recalling hypotheses (5.8) of Part II, Chapter 3,

$$D^* A_2^* \psi = \langle \psi, \pi \rangle = \frac{1}{\lambda} \left[\int_0^{a_+} e^{-\lambda b} K(b)\, db - 1 \right] \neq 0.$$

Thus (A, AE) is I-stabilizable. $\square$

A

An Isomorphism Result

We present two proofs of an isomorphism result that plays a key role in the solution of the linear quadratic optimal control problem. Proposition 1.1 gives a direct proof, whereas Proposition 1.2 gives a proof based on the variational characterization of the minimum of a linear quadratic cost function under a linear state equation constraint. This second proof also yields a general result on the invertibility of a 2×2 matrix of operators, which corresponds to the usual optimality system obtained from the coupled system in the state and adjoint state variables. In this appendix X represents a Hilbert space, with inner product $(\cdot, \cdot)$ and norm $|\cdot|$ and T and S linear bounded operators in X.

Proposition 1.1. *Assume that T and S are symmetric and nonnegative; then $I + TS$ is one-to-one and onto. Moreover*

$$\|S(I+TS)^{-1}\| \leq \|S\|$$

and

$$\|(I+TS)^{-1}\| \leq 1 + \|T\| \, \|S\|.$$

Proof. *Step* 1. $I + TS$ is one-to-one. Let $x_0 \in X$ such that $x_0 + TSx_0 = 0$. Then we have $Sx_0 + STSx_0 = 0$, and so

$$(Sx_0, x_0) + (TSx_0, Sx_0) = 0,$$

which implies $(Sx_0, x_0) = 0$, and consequently $Sx_0 = 0$ and finally $0 = x_0 + TSx_0 = x_0$.

Step 2.

$R := (I+TS)(X)$ is dense in X. Let $y_0 \in X$ be such that

$$\big((I+TS)x, y_0\big) = 0, \quad \forall x \in X.$$

Then

$$(x, (I+ST)y_0) = 0, \quad \forall x \in X,$$

which implies $y_0 = 0$ by Step 1 (exchanging T and S).

Step 3.

$0 \le (S(I+TS)^{-1}x, x) \le (Sx, x)$, $\forall x \in R$. Let $x \in R$ and let $z = (I+TS)^{-1}x$. We have, recalling that $T \ge 0$, $S \ge 0$,

$$\begin{aligned}(Sx, x) &= (S(I+TS)z, (I+TS)z) \\ &= (Sz, z) + (Sz, TSz) + (STSz, z) + (STSz, TSz) \\ &\ge (Sz, z) + (Sz, TSz) = (Sz, (I+TS)z) \\ &= (S(I+TS)^{-1}x, x) \ge 0.\end{aligned}$$

Step 4.

$(S(I+TS)^{-1}x, y) = (x, S(I+TS)^{-1}y)$, $\forall x, y \in R$. Let $x, y \in R$ and let $z = (I+TS)^{-1}x$, $w = (I+TS)^{-1}y$. We have

$$\begin{aligned}(S(I+TS)^{-1}x, y) &= (Sz, w + TSw) = \big((I+TS)z, Sw\big) \\ &= (x, S(I+TS)^{-1}y).\end{aligned}$$

Step 5.

Conclusion. By steps 2, 3, and 4 the symmetric operator $S(I+TS)^{-1}$ has a unique extension to a symmetric bounded operator that we denote again by $S(I+TS)^{-1}$. Now for any $x \in R$ we have

$$(I+TS)^{-1}x = x - TS(I+TS)^{-1}x;$$

thus, by Step 3

$$|(I+TS)^{-1}x| \le |x| + \|T\|\, |Sx|.$$

This implies that R is closed; because it is dense by Step 2, it is onto. □

The above result was first proved in an early version of this book via an optimal control argument. We give it below because this type of result is useful in many contexts.

Consider the real Hilbert spaces U and H, the continuous linear operators

$$B \in \mathcal{L}(U; H), \quad Q \in \mathcal{L}(H; H)$$

and assume that

$$\begin{aligned}&\forall x \in H, &&(Qx, x) \ge 0, \\ &\forall x,\ y \in H, &&(Qx, y) = (Qy, x).\end{aligned}$$

Define the following optimal control problem: To find $u \in U$ such that

$$\inf_{u \in U} J(u),$$

where

$$J(u) = \tfrac{1}{2}(Qx + 2q, x) + \tfrac{1}{2}|u|^2$$

and

$$x = Bu + f$$

for some given pair $(f, g) \in H \times H$.

This problem has a unique solution $u \in U$ that is completely characterized by the following optimality condition:

$$dJ(u; v) = 0, \quad \forall v \in U,$$

where

$$dJ(u; v) = (Qx + q, y) + (u, v)$$

and

$$x = Bu + f, \quad y = Bv.$$

Introduce the adjoint variable $p \in H$

$$p = Qx + q$$

and rewrite the optimality condition

$$(p, Bv) + (u, v) = 0, \quad \forall v \in U \implies u + B^*p = 0.$$

Using this characterization we can now write the coupled system in the form

$$\begin{bmatrix} I & BB^* \\ -Q & I \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} f \\ q \end{bmatrix}.$$

We conclude that for any given pair (f, q), there exists a pair (x, p) for which the above identity is verified. Moreover when $f = g = 0$,

$$\begin{aligned} x + BB^*p = 0 &\implies (x, p) + |B^*p|^2 = 0, \\ -Qx + p = 0 &\implies -(Qx, x) + (p, x) = 0 \end{aligned}$$

and $x = p = 0$. Therefore the matrix of operators is invertible.

Finally set $q = 0$ and eliminate p in the first equation of the coupled system. Then

$$x + BB^*Qx = f \implies [I + BB^*Q]x = f.$$

Again for any f in H, there exists an x in H for which the above identity is verified. For $f = 0$

$$\begin{aligned} x + BB^*Qx = 0 &\implies (Qx, x) + |B^*Qx|^2 = 0 \\ &\implies Q^{1/2}x = 0 \implies x = 0. \end{aligned}$$

Hence $I + BB^*Q$ is invertible.

Proposition 1.2. *Given two real Hilbert spaces U and H and two linear bounded operators*

$$B \in \mathcal{L}(U, H) \quad \textit{and} \quad Q \in \mathcal{L}(H, H)$$

such that

$$\begin{aligned} &\forall x \in H, && (Qx, x) \geq 0, \\ &\forall x,\ y \in H, && (Qx, y) = (Qy, x), \end{aligned}$$

then the operators

$$\begin{bmatrix} I & BB^* \\ -Q & I \end{bmatrix} \quad \textit{and} \quad [I + BB^*Q]$$

are (algebraic and topological) isomorphisms.

This last proposition was also proved by M. Sorine [1].

Proposition 1.1 is now obtained with $X = H = U$, $S = Q$, and $B = T^{1/2}$.

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