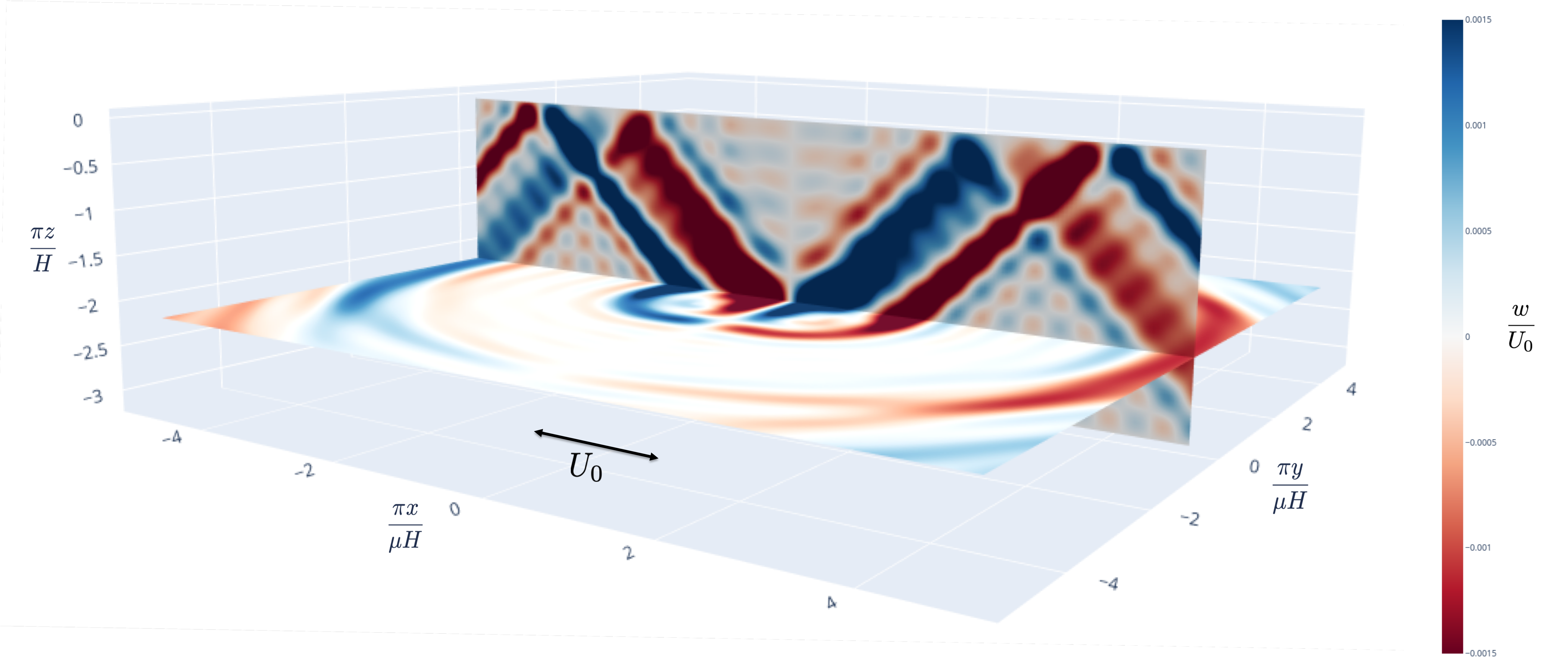
3D modelling of internal tides generation : a Green's function approach for any localized topographies.

Internal tides are internal waves generated by the interaction of tidal currents with underwater topography. They contribute to **energy transfer** and mixing within the ocean and can thus influence ocean circulation, climate variability and ecosystems behavior. We present here a semi-analytical method using Green's functions for 3D modelling of internal tides generation, **not restricted to weak topographies** and with enhanced efficiency allowing for fine-scale resolution.



Non-dimensional vertical velocity for a small symmetric gaussian topography and unidirectional barotropic tide : test case discussed in "Numerical strategy"

The problem to solve

We assume **linear, non-viscous waves**. The perturbations of velocity, pressure and density verify the Boussinesq approximation.

$$\begin{cases} \partial_x u + \partial_y v + \partial_z w = 0 & (1a) \\ \partial_t u + fv + \partial_x P/\rho_* = 0 & (1b) \\ \partial_t v - fu + \partial_y P/\rho_* = 0 & (1c) \\ \partial_t w - b + \partial_z P/\rho_* = 0 & (1d) \\ \partial_t b + N^2 w = 0 & (1e) \end{cases} \quad \text{with} \quad \begin{cases} b = -\frac{\rho' g}{\rho_*} \\ N = \sqrt{-\frac{g}{\rho_*}\frac{d\rho_0}{dz}} \end{cases}$$

The response to the tidal forcing of frequency ω is assumed monochromatic, which leads to the following wave equation :

$$\Rightarrow \quad \mu^2(z)\nabla_h^2 w - w_{zz} = 0 \qquad (2)$$

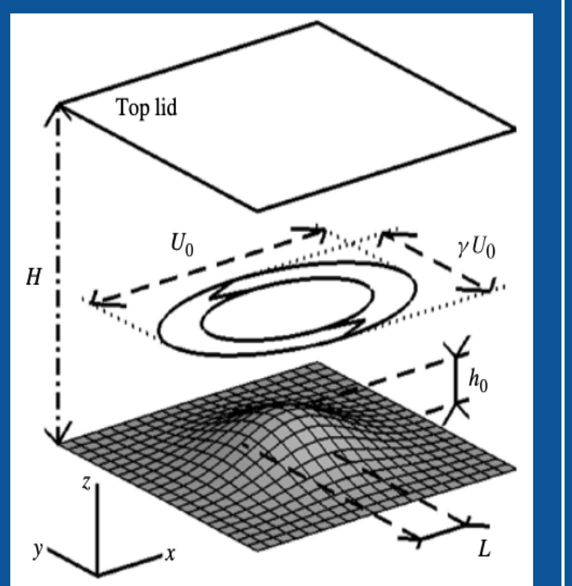
with $\mu^{-1}(z) = \sqrt{\frac{\omega^2 - f^2}{N(z)^2 - \omega^2}}$ the local slope of the wave beams.

Forcing : The M2 component of the barotropic tides is assumed elliptic and uniform.

$$\vec{U}_0 = U_0 e^{j\omega t}(\vec{e}_x + j\gamma\vec{e}_y) \qquad (3)$$

Boundary conditions :

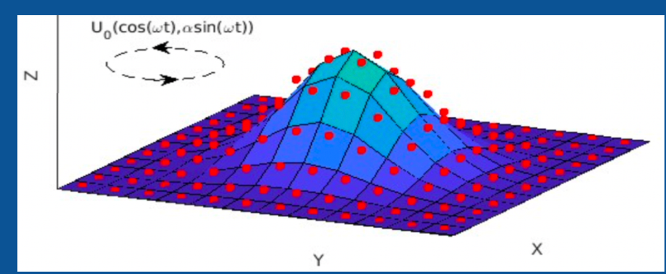
$$\begin{cases} w_{|z=H} = 0 & (4a) \\ w_{|z=h} = (\vec{U}_0 + \vec{u}_h).\vec{\nabla}_h h & (4b) \\ \text{Waves radiated outward as } r \to \infty & (4c) \end{cases}$$



Green's function approach

The underwater topography is associated with a distribution $S(x,y)$ of source points, and the problem $(2)/(4)$ is recast in the form :

$$(5) \quad \begin{cases} \mu^2(z)\nabla_h^2 w - w_{zz} = S(\vec{r}) \\ w(\vec{r}, z = H) = 0 \\ w(\vec{r}, z = 0) = 0 \end{cases}$$



We look for the solution using the **decomposition into vertical modes** :

$$\begin{aligned} [w, f^2N^{-2}b]\,(\vec{r}, z) &= \textstyle\sum_{n=1}^{\infty} [w_n, b_n]\,(\vec{r})a_n(z) \\ [u, v, p]\,(\vec{r}, z) &= \textstyle\sum_{n=1}^{\infty} [u_n, v_n, p_n]\,(\vec{r})a_n'(z) \end{aligned} \qquad (6)$$

where $\frac{d^2 a_n}{dz^2} + \mu^2(z)k_n^2 a_n = 0$ and $\int_0^H \mu^2(z)a_m(z)a_n(z)dz = \Gamma_m\delta_{mn}$

The Green's function associated with the problem is given by :

$$G(\vec{r}, z\,;\vec{r'}, h(\vec{r'})) = \sum_{m=1}^{\infty} \frac{a_m(h(\vec{r'}))a_m(z)}{4i\Gamma_m} H_0^{(1)}(k_m||\vec{r} - \vec{r'}||) \qquad (7)$$

where $H_0^{(1)}$ is an Hankel function of the first kind. The solution of the problem (5) is then : $\boxed{w = G * S}$ (8)

Using the decomposition and the equation (1), we can express $[u, v, p]$ as convolution products with S, which is the **only unknown**.

$$\begin{cases} u_n = k_n^{-2}(w_{n,x} + i\frac{f}{\omega}w_{n,y}) \\ v_n = k_n^{-2}(w_{n,y} - i\frac{f}{\omega}w_{n,x}) \\ p_n = -\frac{\omega^2 - f^2}{i\omega}k_n^{-2}w_n \end{cases} \Rightarrow \begin{cases} u = G_u * S \\ v = G_v * S \\ p = G_p * S \end{cases} \qquad (9)$$

Numerical strategy

The source distribution has to verify the boundary condition at the topography.

$$\begin{aligned} w_{|\vec{r}, h(\vec{r})} &= (\vec{U}_0 + \vec{u}_h).\vec{\nabla}_h h|\vec{r} \\ \Rightarrow \; U_0 h_x(\vec{r}) + i\gamma U_0 h_y(\vec{r}) &= w_{|\vec{r}, h(\vec{r})} - u_{|\vec{r}, h(\vec{r})}h_x(\vec{r}) - v_{|\vec{r}, h(\vec{r})}h_y(\vec{r}) \\ &= (G - h_x(\vec{r})G_u - h_y(\vec{r})G_v) * S \end{aligned}$$

We discretize the topography and assume the distribution of sources to be constant on each cell. We then solve the above matrix equation and compute the wavefield and the energy flux based on $(8)/(9)$.

Test case : The figure above represents the ratio of the vertical velocity by the tide velocity for a small Gaussian topography with the following parameters : $h_{max}/H_0 \sim 0.02$; criticality $\epsilon = \mu \max(\partial_x h) \sim 0.1$; unidirectional barotropic tide $\gamma = 0$; solved with 60 modes ; reconstructed with 10 modes.

Conclusion

Advantages :

- Not restricted to weak topographies or constant stratification
- Allow fine-scale resolution
- Better understanding of the underlying physics

Limitations and open issues

- Slow modal convergence
- Dense matrix : large storage required
- No direct resolution of energy dissipation
- Comparison with realistic reliefs

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