Project 1.1

Study a pendulum and a harmonic oscillator using Euler-Cromer, velocity Verlet and Runge-Kutta. Assume $\sqrt{g/l} = 2\ \mathrm{s}^{-1}$ and $m = 1$ kg. Compare the different methods with each other and with the exact solution of the harmonic oscillator. Choose a good integrator and plot $\theta(t)$, $\dot{\theta}(t)$ and E. Study the dependence on the time step, try values between 0.01 and 0.2 s. Consider initial conditions $\theta(0)/\pi = 0.1$ and 0.5; $\dot{\theta}(0) = 0$.

Although Runge-Kutta has a higher order accuracy than Verlet, the former is not good for many physical simulations. Can you see why? (hint: check the integrators' behavior at sufficiently long times).

Project 1.2

Determine the period time T as a function of the initial position $\theta(0)$. Which system (harmonic osc./pendulum) has a larger period? Explain! Compare the pendulum with the perturbation series:

$$T = 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{1}{16}\theta^2(0) + \frac{11}{3072}\theta^4(0) + \frac{173}{737280}\theta^6(0) + \ldots\right)$$

Project 1.3

Study the damped harmonic oscillator equation:

$$F/m = \ddot{x} = -\omega_0^2\, x - \gamma\, \dot{x}$$

Take $\omega_0 = 2$, $\gamma = 0.5,\ 3$, $x(0) = 1$, $\dot{x}(0) = 0$. Plot $x(t)$, $v(t)$, $E(t)$. Discuss the features of these plots. Estimate the relaxation time τ, i.e. the time for the amplitude to be reduced to $1/e \approx 0.37$ of the initial amplitude (note that you should look at the "envelope", rather than at the exact value of the solution). Study the dependence of τ on γ. Find the smallest γ such that the oscillator does not pass $x = 0$. This is called the critical damping, γ_c and for $\gamma > \gamma_c$ the systems is called overdamped.

Project 1.4

Consider a damped pendulum with damping given by $-\gamma\,\dot{\theta}$. Take $\gamma = 1$, $\sqrt{g/l} = 2$, $\theta(0) = \pi/2$, $\dot{\theta}(0) = 0$. Determine the phase space portrait, i.e. plot $\dot{\theta}$ vs θ. Discuss.

Project 1.5

Similar to Problem 6.21 in the book. Double pendulum

a. Use the fourth-order Runge-Kutta algorithm (with $\Delta t = 0.003$) to simulate the double pendulum. Choose $m = 1$, $L = 1$, and $g = 9.8$. The input parameter is the total energy E. The initial values of q_1 and q_2 can be either chosen randomly within the interval $|q_i| < \pi$ or by the user. Then set the initial $p_1 = 0$, and solve for p_2 using equation (6.52) from the book:

$$H = \frac{1}{2mL^2}\frac{p_1^2 + 2p_2^2 - 2p_1p_2\cos(q_1 - q_2)}{1 + \sin^2(q_1 - q_2)} + mgL(3 - 2\cos q_1 - \cos q_2)$$

with $H = E$. First explore the pendulum's behavior by plotting the generalized coordinates and momenta as a function of time in four windows. Consider the energies $E = 1$, 15, and 40. Try a few initial conditions for each value of E. Visually determine whether the steady state behavior is regular or appears to be chaotic. Are there some values of E for which all the trajectories appear regular? Are there values of E for which all trajectories appear chaotic? Are there values of E for which both types of trajectories occur?

Tip: checking energy conservation is one good check for your code

b. Repeat part (a), but plot the phase space diagrams p_1 versus q_1 and p_2 versus q_2. Are these plots more useful for determining the nature of the trajectories than those drawn in part (a)?

c. Draw the Poincaré plot with p_1 plotted versus q_1 only when $q_2 = 0$ and $p_2 > 0$. Overlay trajectories from different initial conditions, but with the same total energy on the same plot. Duplicate the plot shown in Figure 6.13. Then produce Poincaré plots for the values of E given in part (a), with at least five different initial conditions for each energy. Describe the different types of behavior.

d. Is there a critical value of the total energy at which some chaotic trajectories first occur?