Supplementary Materials to *Network Formation Dynamics in Asymmetric Coordination Games*

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1 Intermediate impact case: $\alpha = 1.75$

Figure 1 illustrates a qualitative pattern that closely resembles Figure 6 in the main manuscript when $\alpha = 1.25$. However, all dynamics in this scenario are shifted to the right. This shift implies that the activation of the preferential attachment term on average occurs in groups with a more balanced composition of L and H agents, as opposed to groups composed of only seven L agents.

The observed shift can be explained as follows. Suppose agent i prefers yellow and has already established 3 connections. Now, consider agent j with a preference for yellow and $d_j^* = 4$. Forming its first connection with agent i would yield a utility of $2 - 1.75 \times 3 + 3\beta$. Under the condition $\alpha = 1.25$, a $\beta > 0.58$ would be sufficient to induce j to form ij. However, with $\alpha = 1.75$, a higher β is necessary, specifically $\beta > 1.08$. This also explains why preferential attachment often fails to activate in most groups when $\beta = 0.35$.

An alternative to requiring a very high β is for agent j to connect with an agent possessing a higher degree than agent i. Since initial degree differences are mainly driven by baseline connectivity—which is influenced by entry costs and coordination payoffs, it is more likely that agent j will find the necessary connectivity in groups with a larger number of agents facing lower entry costs. As a result, this shift to the right in activation dynamics arises from these nuanced interactions.

When considering the probability of unanimity, we notice a qualitative pattern akin to the case when $\alpha = 1.25$, albeit shifted to the right (Figure 2). As illustrated in the diverse scenarios outlined in the main manuscript, the average color prevalence attains its peak values when there is a higher likelihood of achieving color unanimity. Conversely, it is lower when the probability of unanimity is low. See Figure 3.

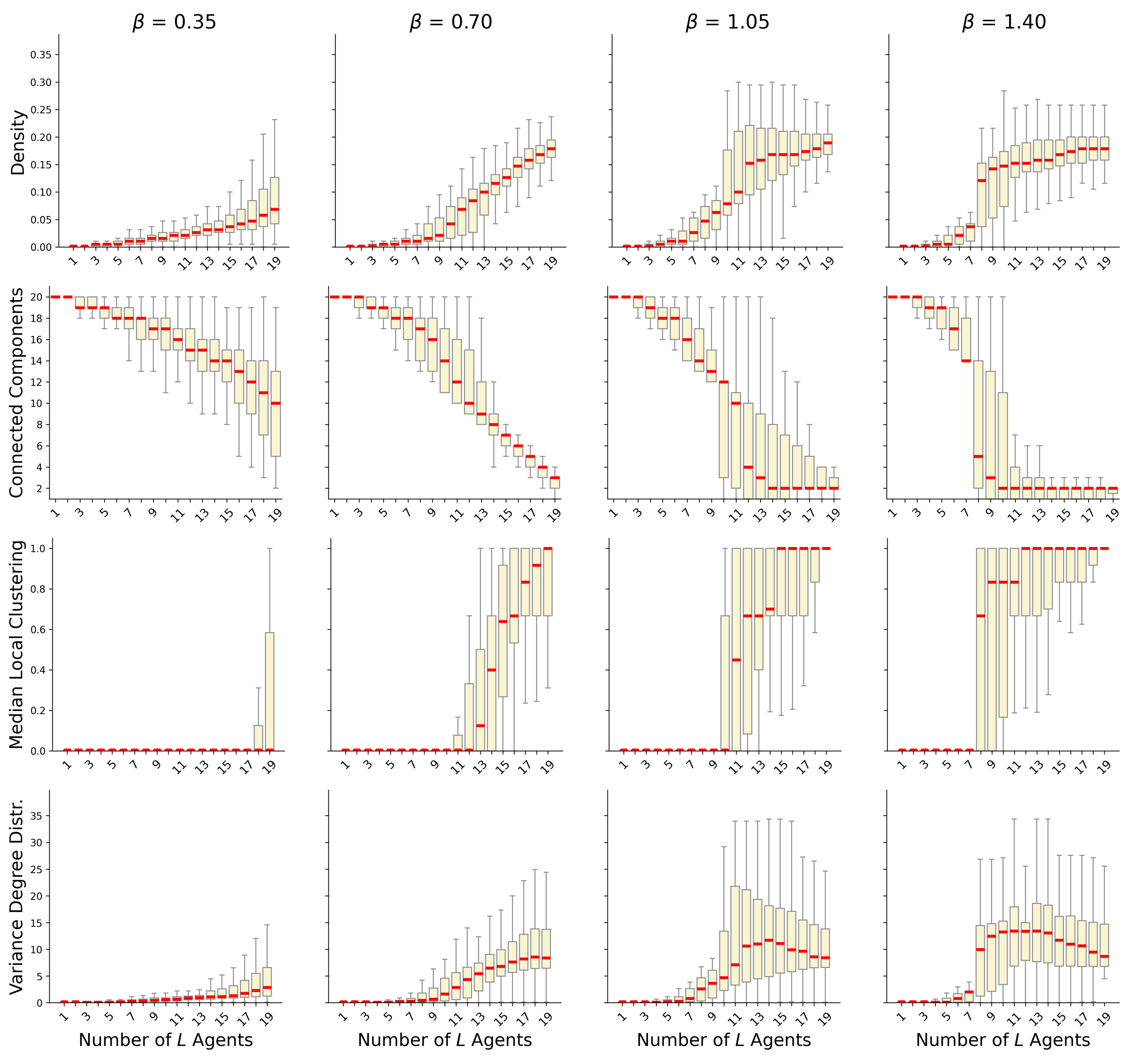

Figure 1: Density, number of connected components, median of the local clustering coefficients distribution and variance of the degree distribution over 1000 simulations with fixed $\alpha = 1.75$, while β and group composition by L agents were varied. In red: median values.

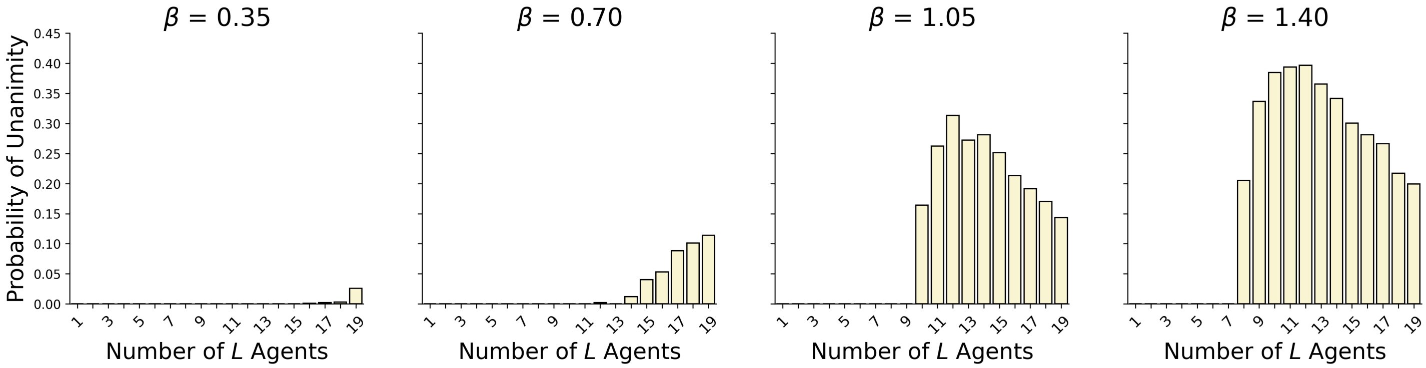

Figure 2: Probability of unanimity i.e., every agent playing either yellow or blue over 1000 simulations with fixed $\alpha = 1.75$, while β and group composition by L agents were varied.

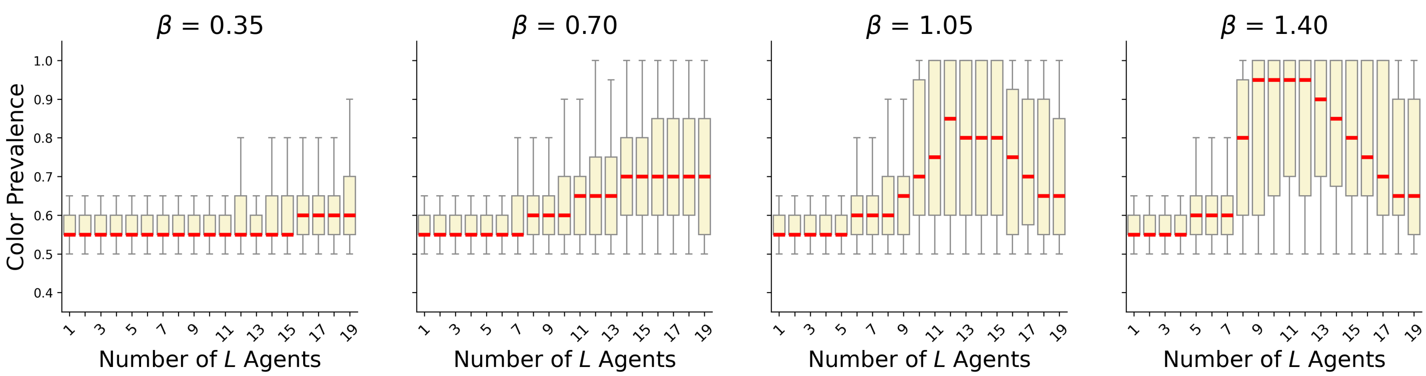

Figure 3: Color prevalence over 1000 simulations with fixed $\alpha = 1.75$, while β and group composition by L agents were varied. In red: median values.

Qualitatively consistent patterns also emerge in terms of segregation by equilibrium color choices and exogenous color preferences l_i (Figures 4 and 5). Segregation begins to manifest in groups predominantly composed of L agents, with color segregation generally being more pronounced than segregation based on l_i.

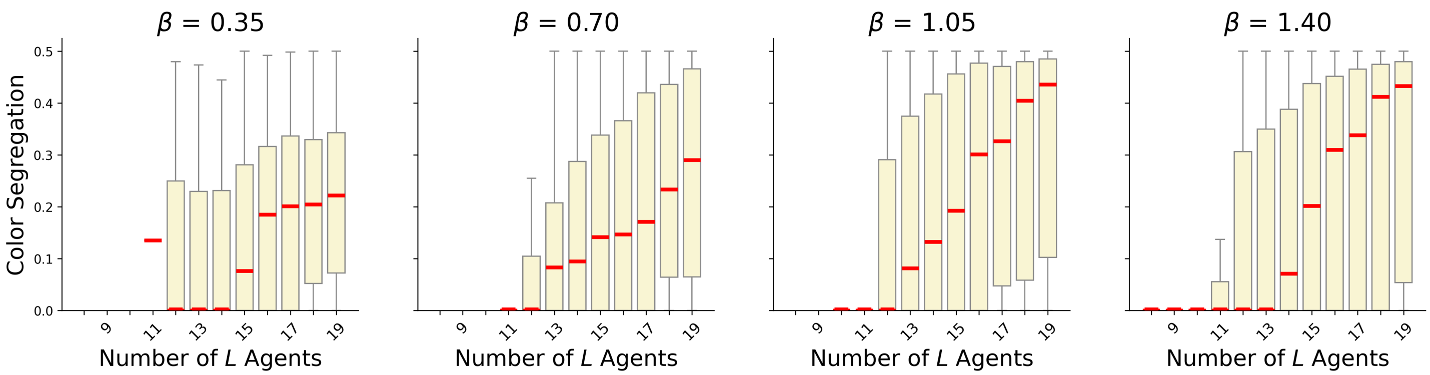

Figure 4: Color segregation over 1000 simulations with fixed $\alpha = 1.75$, while β and group composition by L agents were varied. In red: median values.

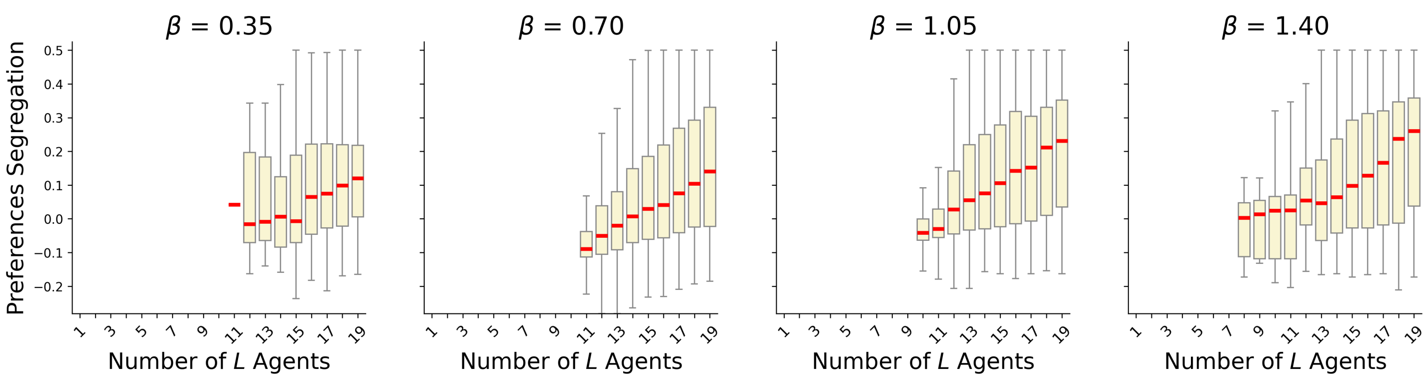

Figure 5: Segregation by initial color preferences l_i over 1000 simulations with fixed $\alpha = 1.75$, while β and group composition by L agents were varied. In red: median values.

2 Intermediate impact case: $\alpha = 1.50$

In the scenario where $\alpha = 1.50$, we observe similar qualitative patterns to those seen when $\alpha = 1.25$. These patterns are shown in Figures 6, 7, 8, 9, and 10. Although there are some minor quantitative differences – such as a slightly lower probability of unanimity in Figure 7 compared to Figure 7 in the main manuscript, or an overall lower density when $\beta = 0.30$ – these distinctions are not of substantive interest.

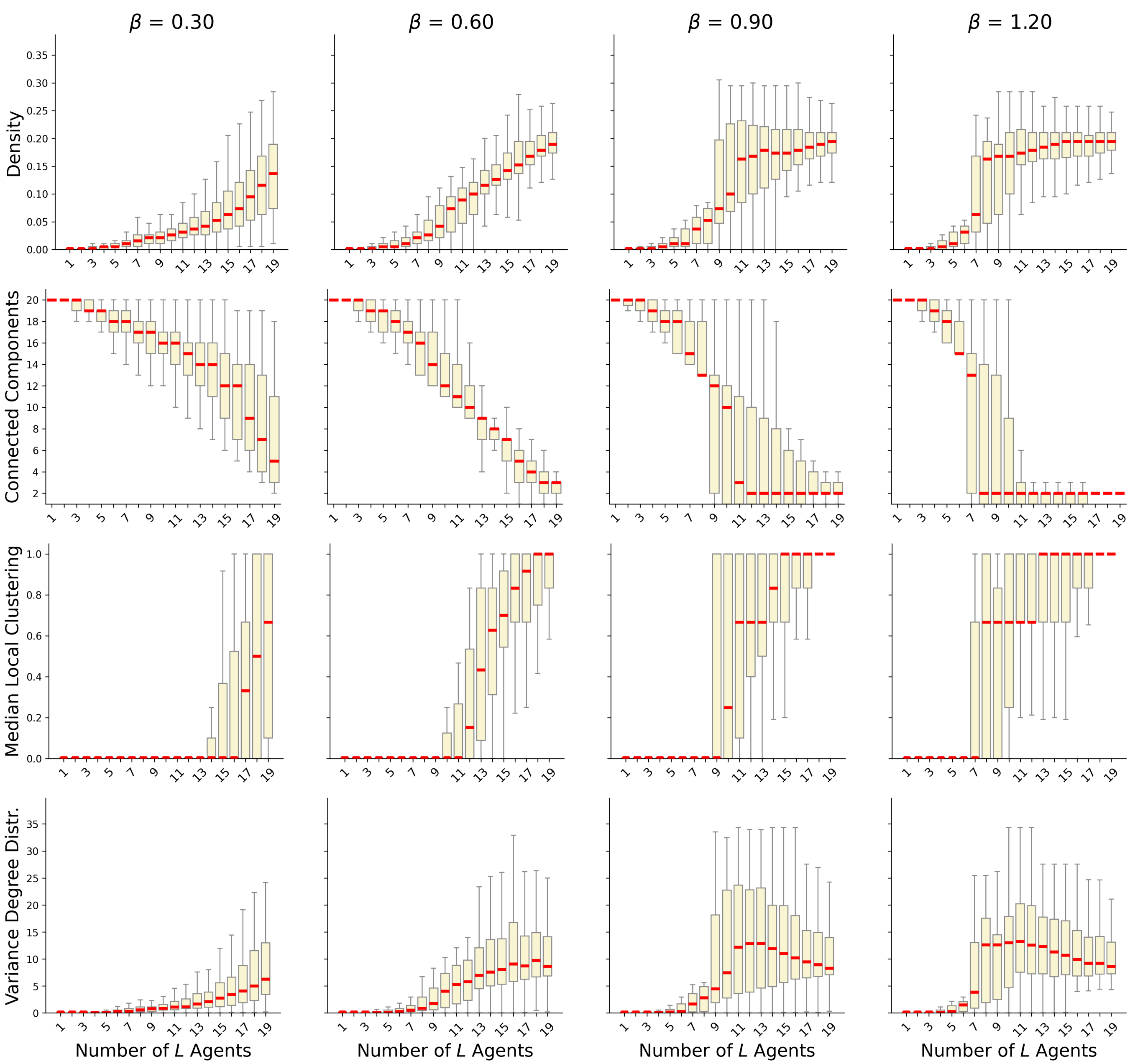

Figure 6: Density, number of connected components, median of the local clustering coefficients distribution and variance of the degree distribution over 1000 simulations with fixed $\alpha = 1.50$, while β and group composition by L agents were varied. In red: median values.

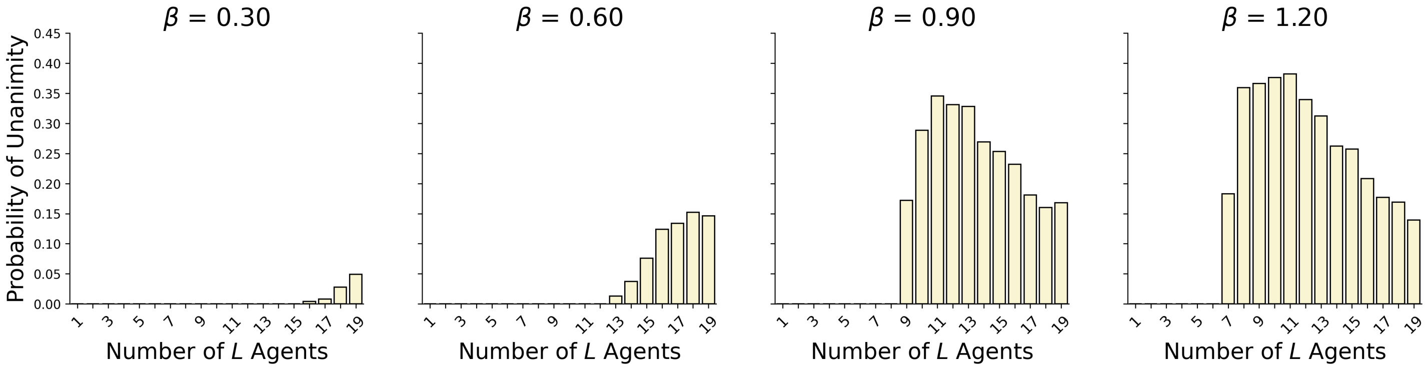

Figure 7: Probability of unanimity i.e., every agent playing either yellow or blue over 1000 simulations with fixed $\alpha = 1.50$, while β and group composition by L agents were varied.

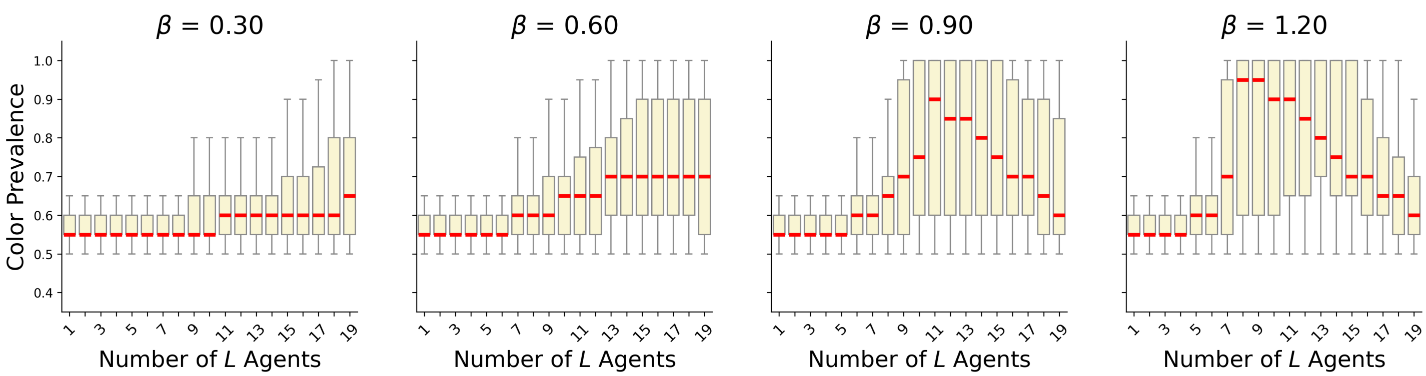

Figure 8: Color prevalence over 1000 simulations with fixed $\alpha = 1.50$, while β and group composition by L agents were varied. In red: median values.

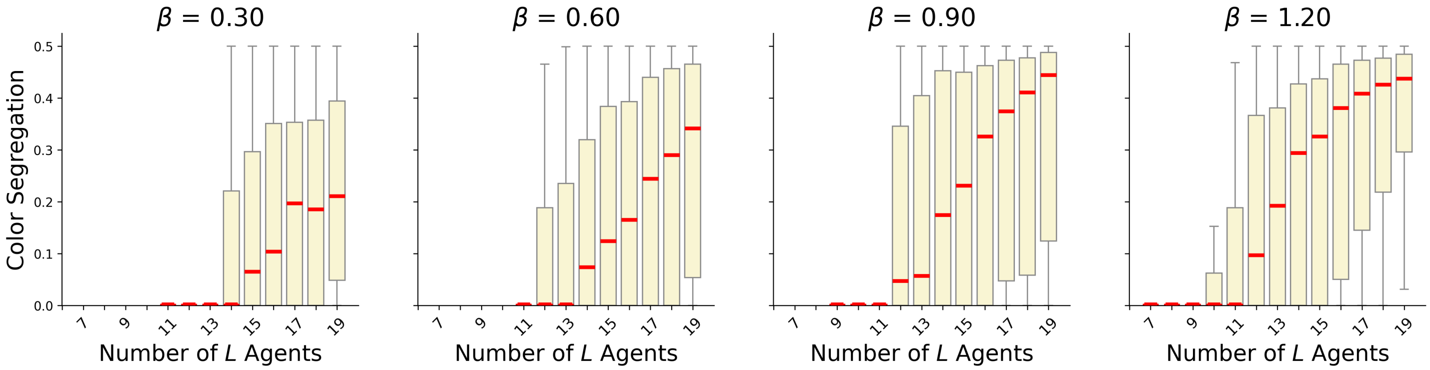

Figure 9: Color segregation over 1000 simulations with fixed $\alpha = 1.50$, while β and group composition by L agents were varied. In red: median values.

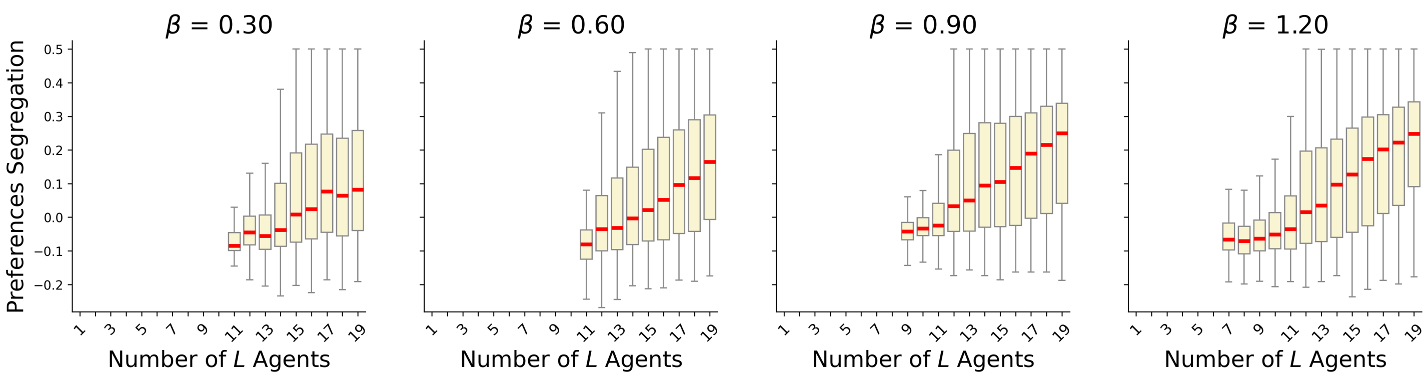

Figure 10: Segregation by initial color preferences l_i over 1000 simulations with fixed $\alpha = 1.50$, while β and group composition by L agents were varied. In red: median values.

3 Low impact case: $\alpha = 0.50$

Consistent with the findings for $\alpha = 0.75$ in the main manuscript, the earlier onset of the preferential attachment dynamics becomes even more evident with $\alpha = 0.50$. In this scenario, an even greater number of agents are willing to form their initial connections, as the impact of entry costs on utility is further reduced. This sets the stage for the preferential attachment mechanism to activate even earlier, occurring within groups with as few as four L agents. Figure 11 illustrates the continuation of the pattern observed in Figure 11 in the main manuscript. When density begins to rise in groups with four L agents, the initial rate of increase becomes more pronounced compared to the initial rate observed when $\alpha = 0.75$ in groups with six L agents (see Figure 11 in the main manuscript). This trend is further reflected in the degree variance, showcasing an accelerated dynamic.

The trends observed with $\alpha = 0.75$ persist regarding the probability of unanimity and color prevalence, with even more pronounced effects. Specifically, in highly centralized networks i.e., those composed of 4 to 8 L agents, where centralization levels by degree peak, the probability of coordination reaches its maximum values compared to all scenarios examined in both this analysis and the main manuscript. For detailed visual representations, please refer to Figures 12, 13, 14, and 15.

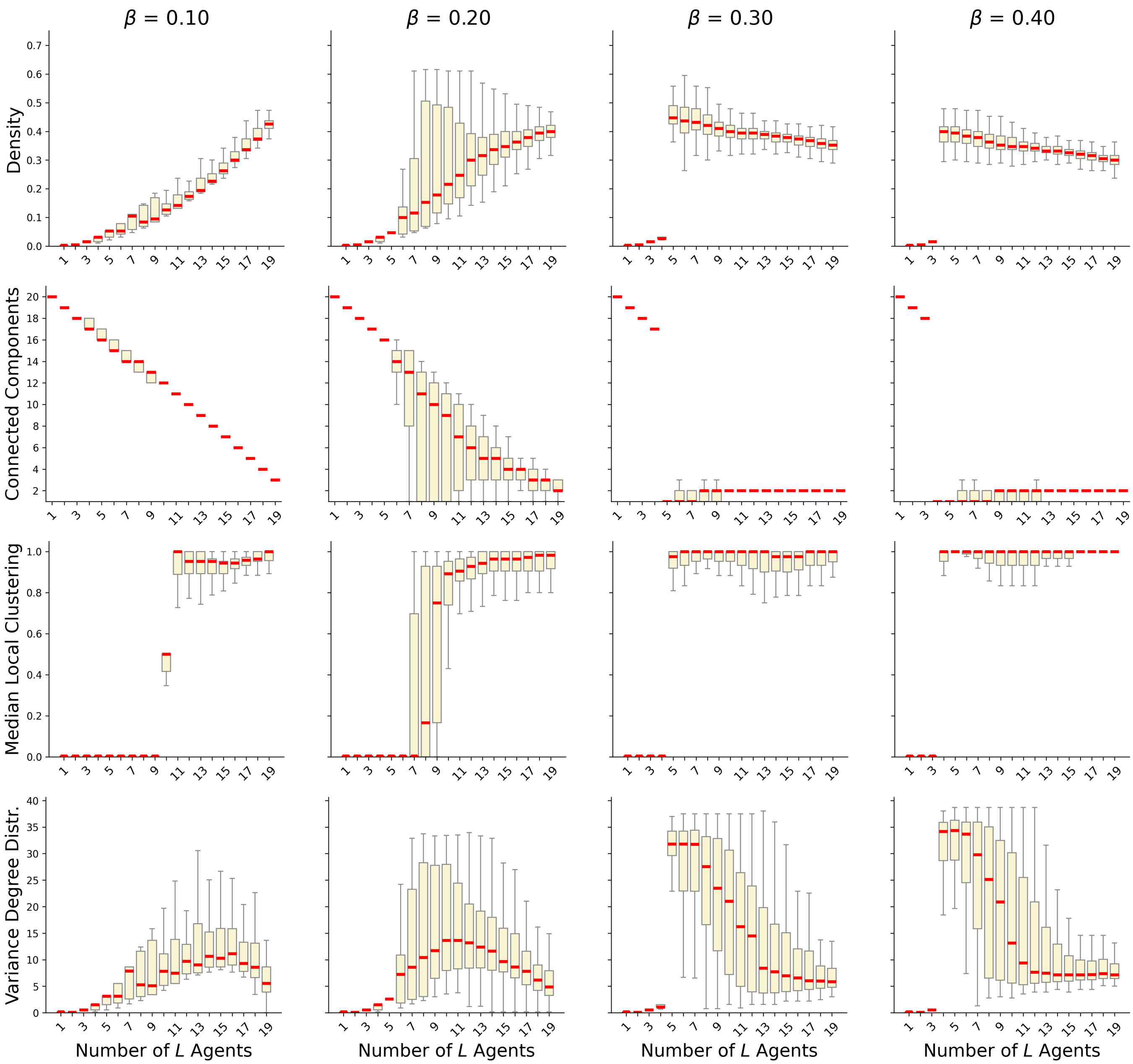

Figure 11: Density, number of connected components, median of the local clustering coefficients distribution and variance of the degree distribution over 1000 simulations with fixed $\alpha = 0.50$, while β and group composition by L agents were varied. In red: median values.

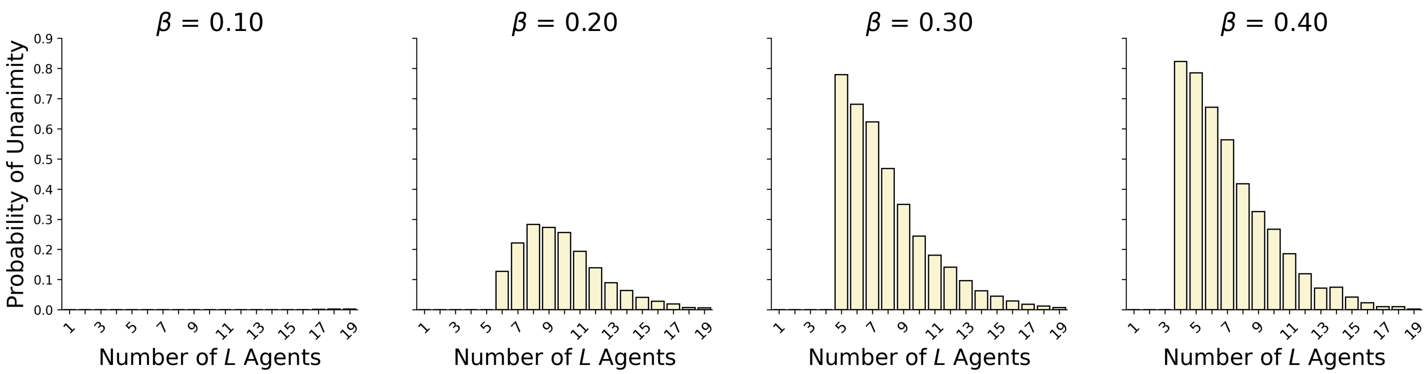

Figure 12: Probability of unanimity i.e., every agent playing either yellow or blue over 1000 simulations with fixed $\alpha = 0.50$, while β and group composition by L agents were varied.

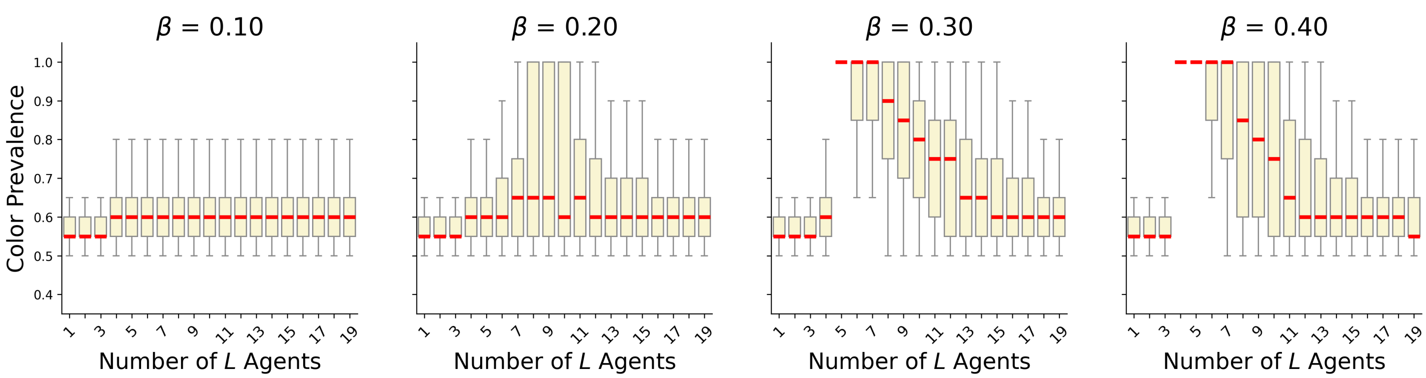

Figure 13: Color prevalence over 1000 simulations with fixed $\alpha = 0.50$, while β and group composition by L agents were varied. In red: median values.

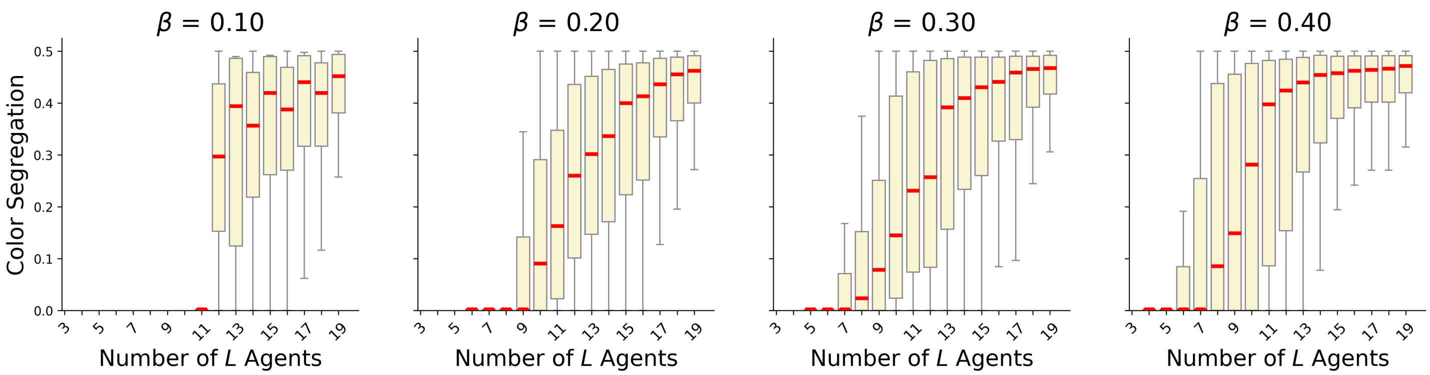

Figure 14: Color segregation over 1000 simulations with fixed $\alpha = 0.50$, while β and group composition by L agents were varied. In red: median values.

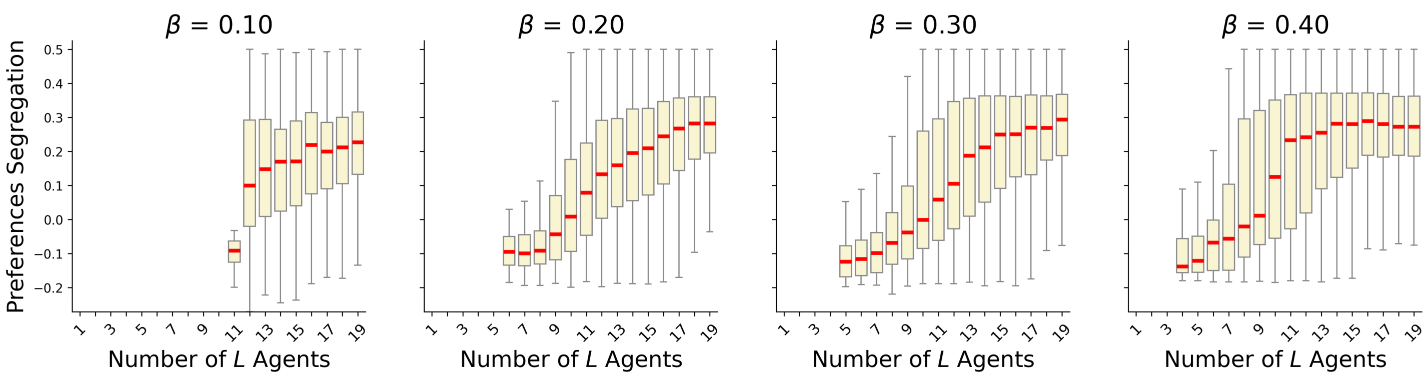

Figure 15: Segregation by initial color preferences l_i over 1000 simulations with fixed $\alpha = 0.50$, while β and group composition by L agents were varied. In red: median values.

4 Low impact case: $\alpha = 0.25$

As anticipated in the main manuscript, when α approaches zero (e.g., $\alpha = 0.25$), everyone, including agents with high entry costs, finds it advantageous to form connections. Consequently, network formation decisions are primarily driven by coordination payoffs and color preferences, resulting in highly segregated networks in equilibrium. This is evident from Figures 16 and 19 (see also Figure 16 in the main manuscript).

In Figure 16, we observe that, regardless of the group compositions by L and H agents and the values of β, the number of connected components consistently remains 2. The emerging components are almost completely connected within themselves but remain disconnected across each other (see Figure 16 in the main manuscript). Since, at $t = 0$, there are exactly $\frac{N}{2}$ agents preferring and playing blue and $\frac{N}{2}$ agents preferring and playing yellow in every scenario, the resulting densities are always close to 50%. This observed pattern is also evident in Figure 19, where color segregation consistently approaches its theoretical maximum of 0.5 with two colors.

As networks exhibit high segregation into two components, where one component consists of agents exclusively playing yellow and the other of agents exclusively playing blue in equilibrium (refer to Figure 16 in the main manuscript), the probability of unanimity is consistently low across all cases, displaying no discernible pattern with the values of β, as otherwise observed in all previous scenarios. Color prevalence remains almost always very close to the initial $50 - 50$ color distribution at $t = 0$.

Segregation by l_i is consistently high, approaching its theoretical maximum of 0.5 (refer to Figure 20). However, even in this case, it is generally lower than color segregation (refer to Figure 19). This discrepancy arises because, when $\alpha = 0.25$, there is still a possibility that someone preferring blue might find itself in the yellow component or vice versa, especially in groups composed predominantly by agents facing lower entry costs.

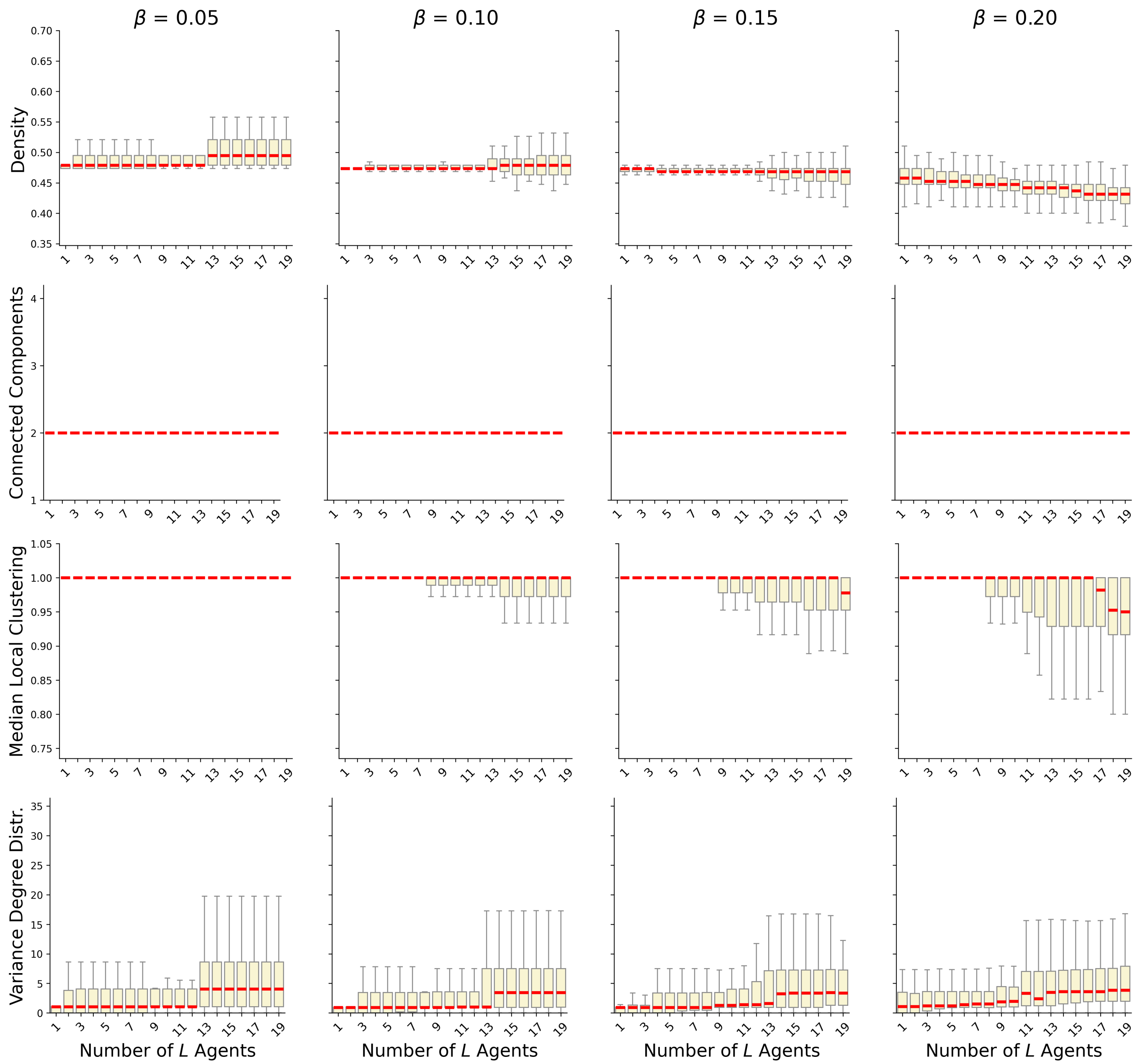

Figure 16: Density, number of connected components, median of the local clustering coefficients distribution and variance of the degree distribution over 1000 simulations with fixed $\alpha = 0.25$, while β and group composition by L agents were varied. In red: median values.

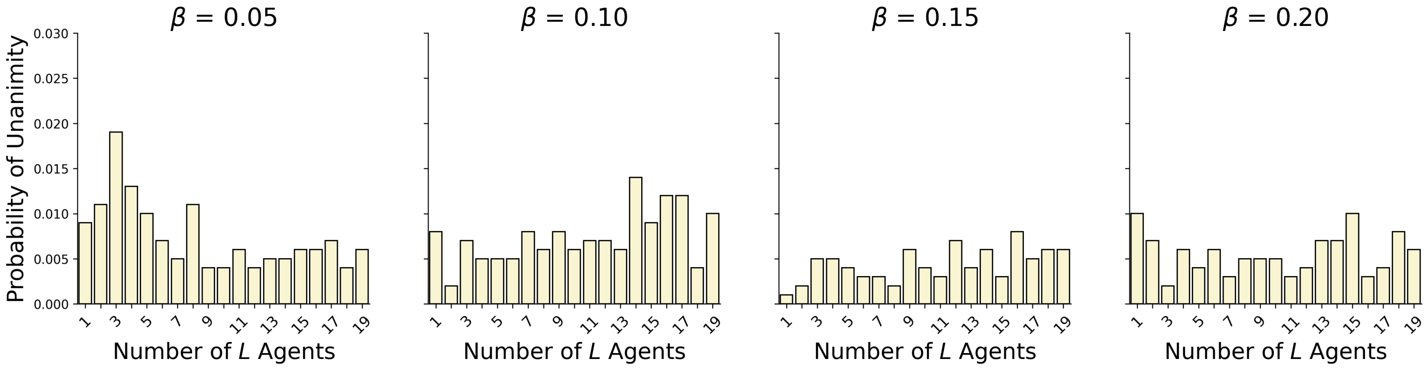

Figure 17: Probability of unanimity i.e., every agent playing either yellow or blue over 1000 simulations with fixed $\alpha = 0.25$, while β and group composition by L agents were varied.

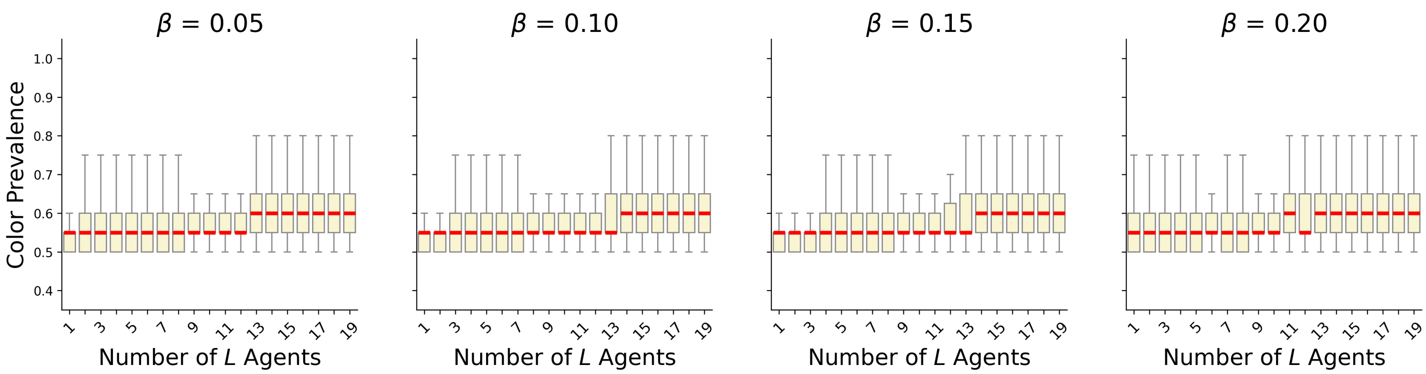

Figure 18: Color prevalence over 1000 simulations with fixed $\alpha = 0.25$, while β and group composition by L agents were varied. In red: median values.

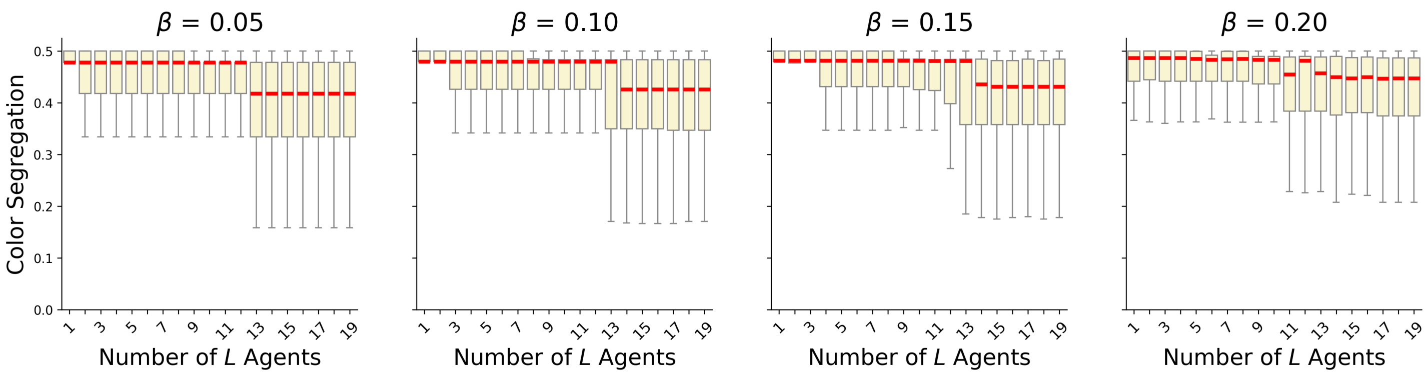

Figure 19: Color segregation over 1000 simulations with fixed $\alpha = 0.25$, while β and group composition by L agents were varied. In red: median values.

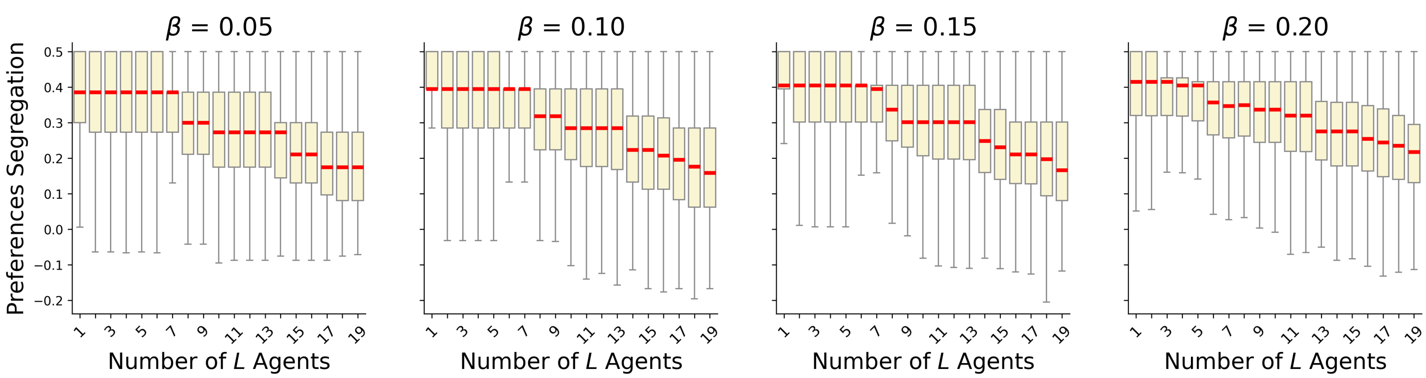

Figure 20: Segregation by initial color preferences l_i over 1000 simulations with fixed $\alpha = 0.25$, while β and group composition by L agents were varied. In red: median values.

5 $\epsilon = 10\%$

Here, we present results for the scenario where $\alpha = 1.25$ and ϵ is reduced to 10% from 20%. However, we observe no significant differences in the various outcomes.

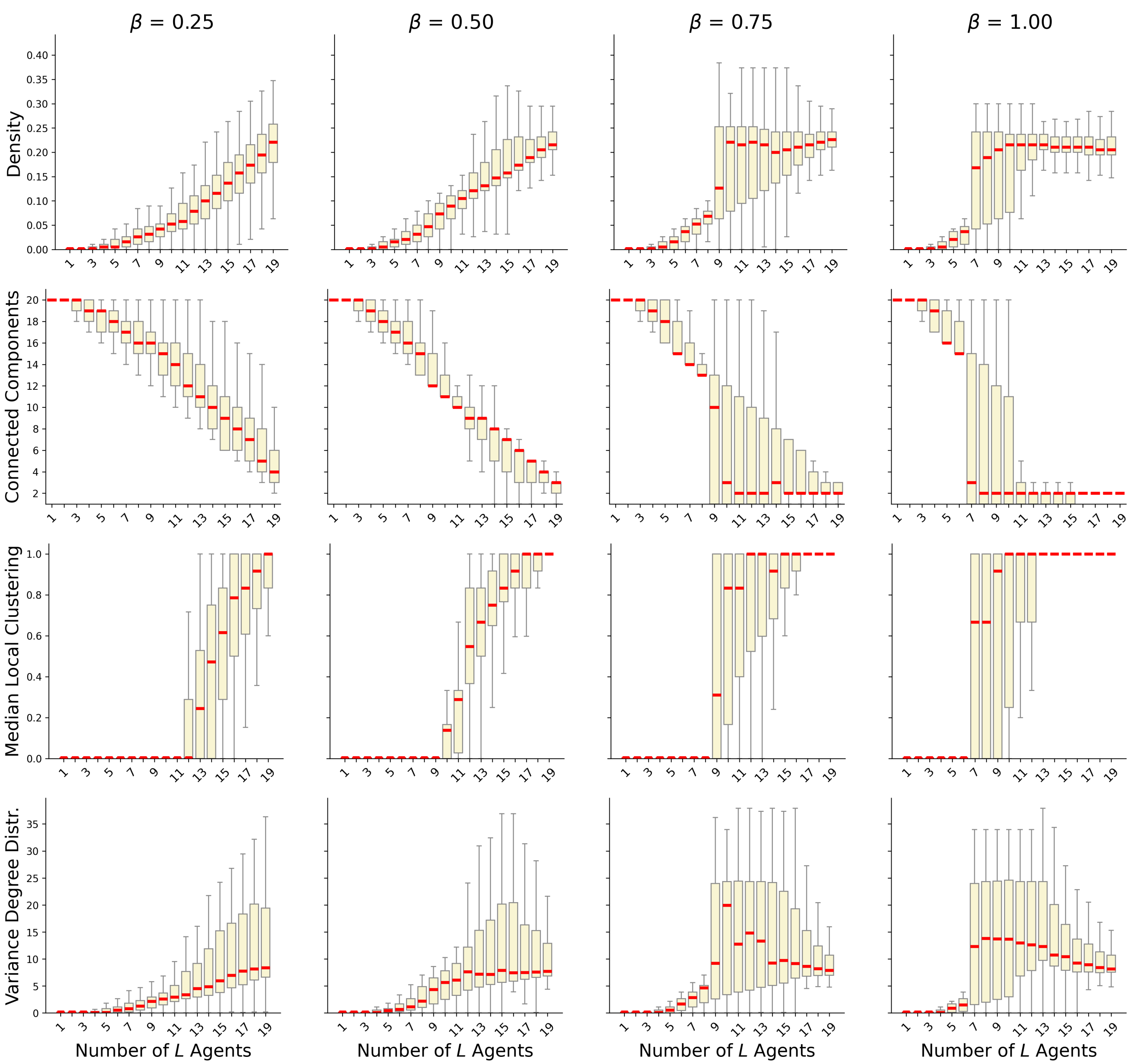

Figure 21: Density, number of connected components, median of the local clustering coefficients distribution and variance of the degree distribution over 1000 simulations with fixed $\alpha = 1.25$, $\epsilon = 10\%$, while β and group composition by L agents were varied. In red: median values.

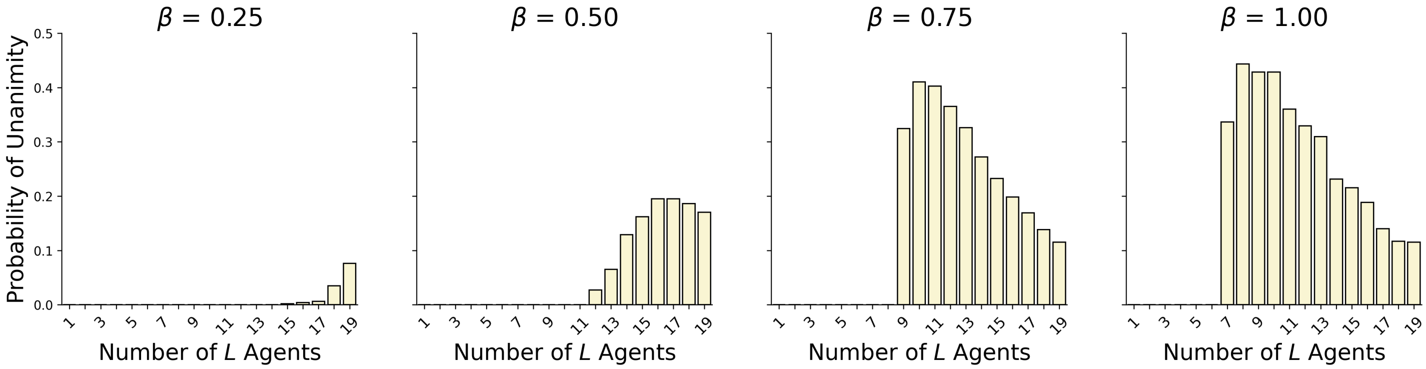

Figure 22: Probability of unanimity i.e., every agent playing either yellow or blue over 1000 simulations with fixed $\alpha = 1.25$, $\epsilon = 10\%$, while β and group composition by L agents were varied.

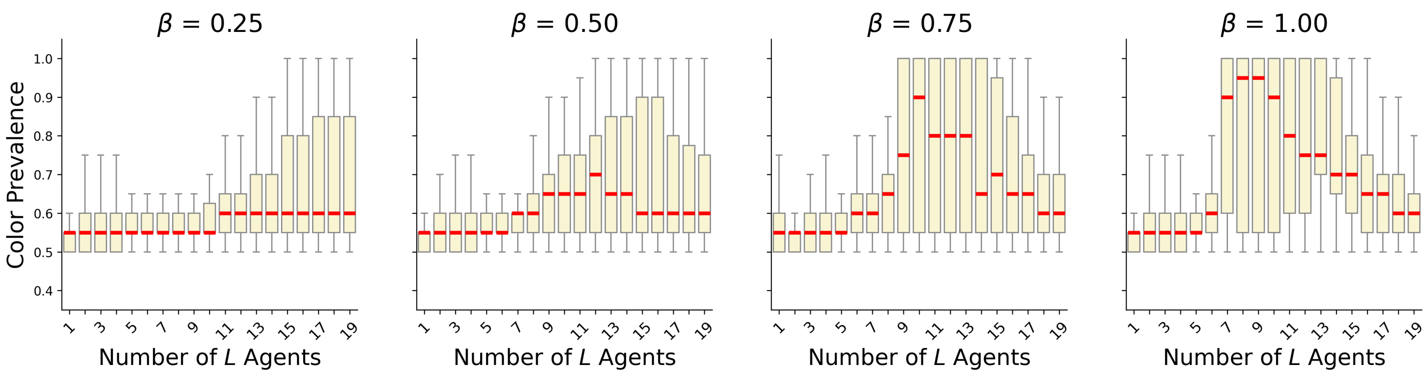

Figure 23: Color prevalence over 1000 simulations with fixed $\alpha = 1.25$, $\epsilon = 10\%$, while β and group composition by L agents were varied. In red: median values.

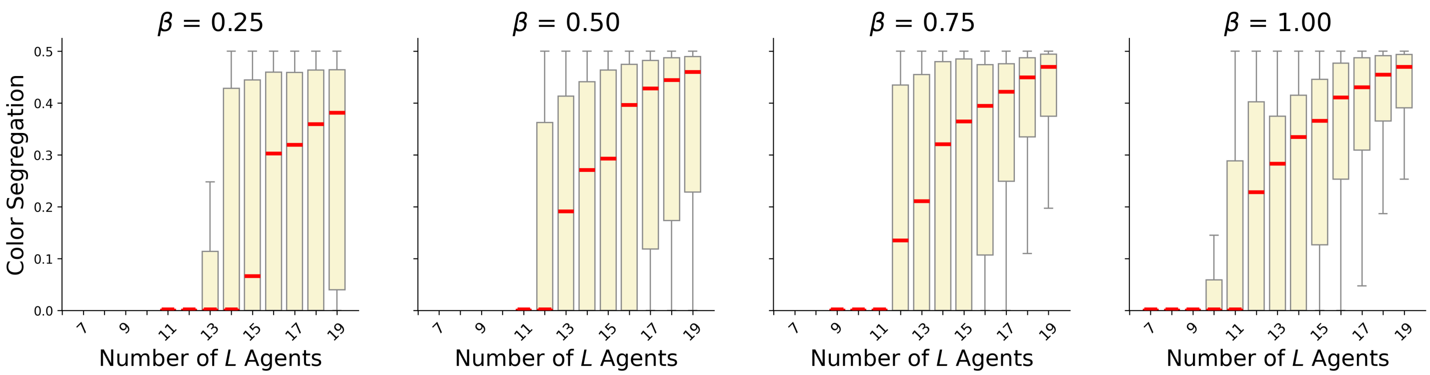

Figure 24: Color segregation over 1000 simulations with fixed $\alpha = 1.25$, $\epsilon = 10\%$, while β and group composition by L agents were varied. In red: median values.

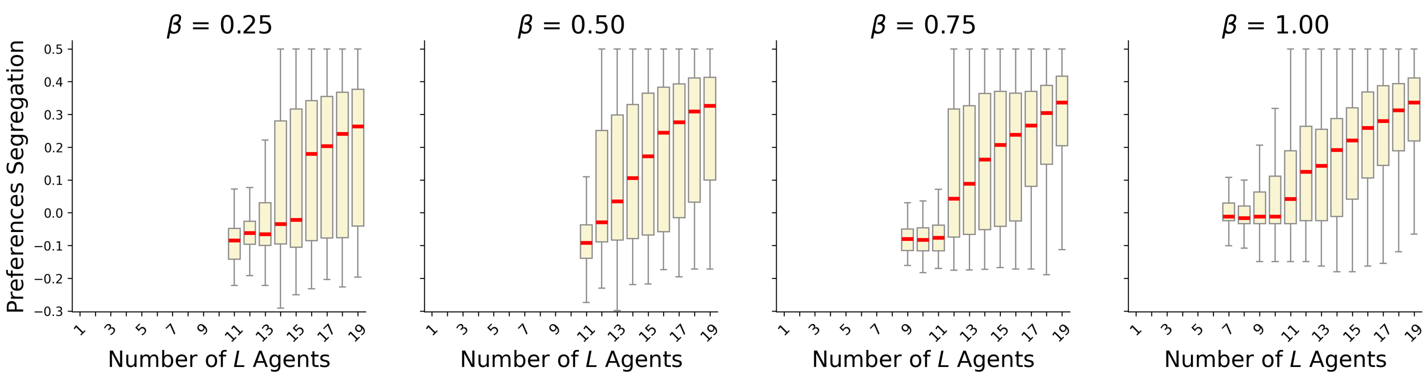

Figure 25: Segregation by initial color preferences l_i over 1000 simulations with fixed $\alpha = 1.25$, $\epsilon = 10\%$, while β and group composition by L agents were varied. In red: median values.

6 $N = 30$

We explored groups consisting of $N = 30$ agents in the case of $\alpha = 0.75$. In this setup, 15 agents had a preference for blue, and the remaining 15 favored yellow at $t = 0$. While maintaining consistent values for β, we adjusted the bounds for the d_i^* distribution for both L and H agents. Importantly, we maintained the same proportions as in the $N = 20$ case. Specifically, d_i^* for L was drawn from a uniform distribution with 1 and 7 as bounds, while d_i^* for H was between 10 and 15. We can see that the same qualitative patterns are obtained even when $N = 30$.

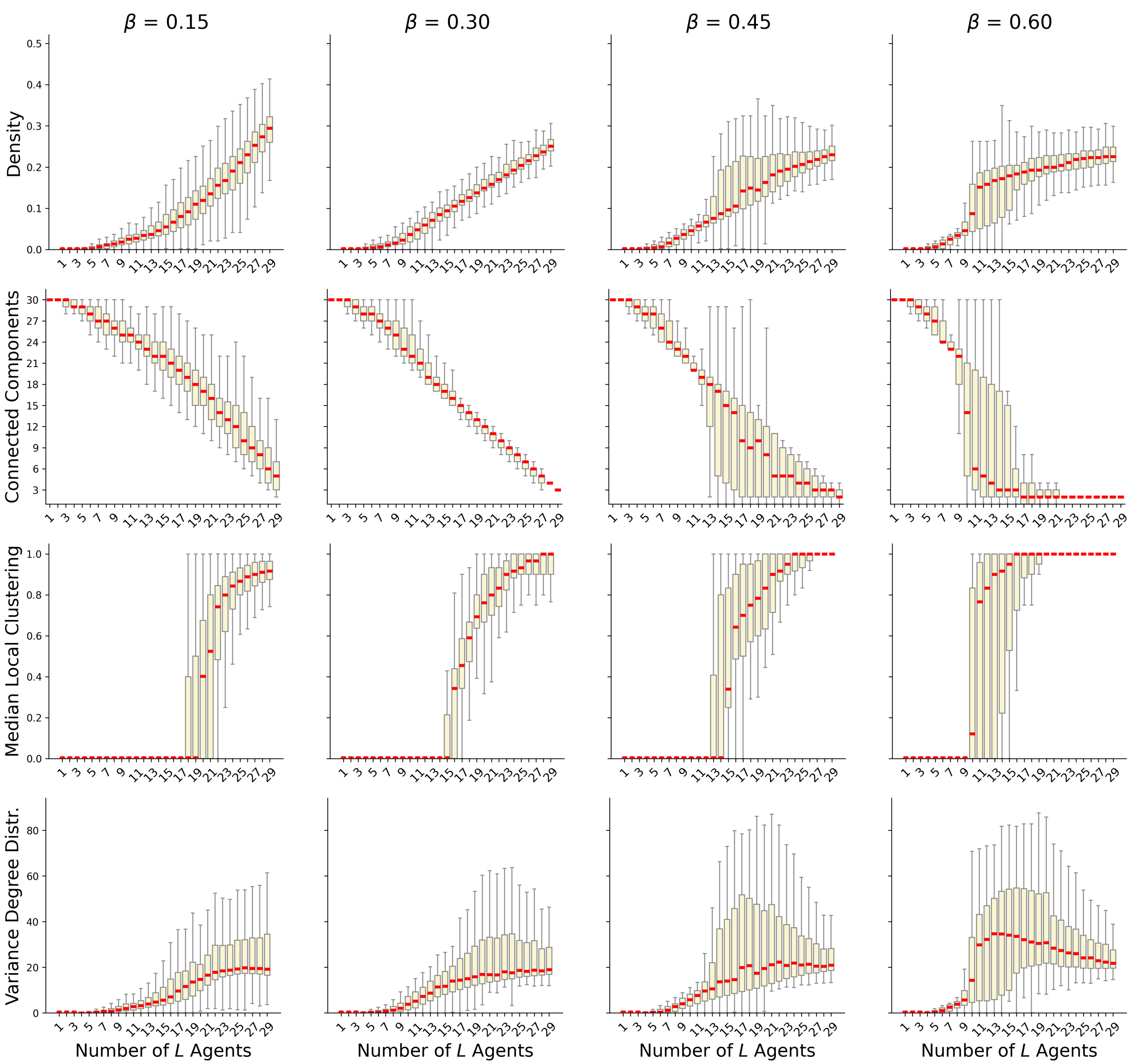

Figure 26: Density, number of connected components, median of the local clustering coefficients distribution and variance of the degree distribution over 1000 simulations with fixed $\alpha = 0.75$, $N = 30$, while β and group composition by L agents were varied. In red: median values.

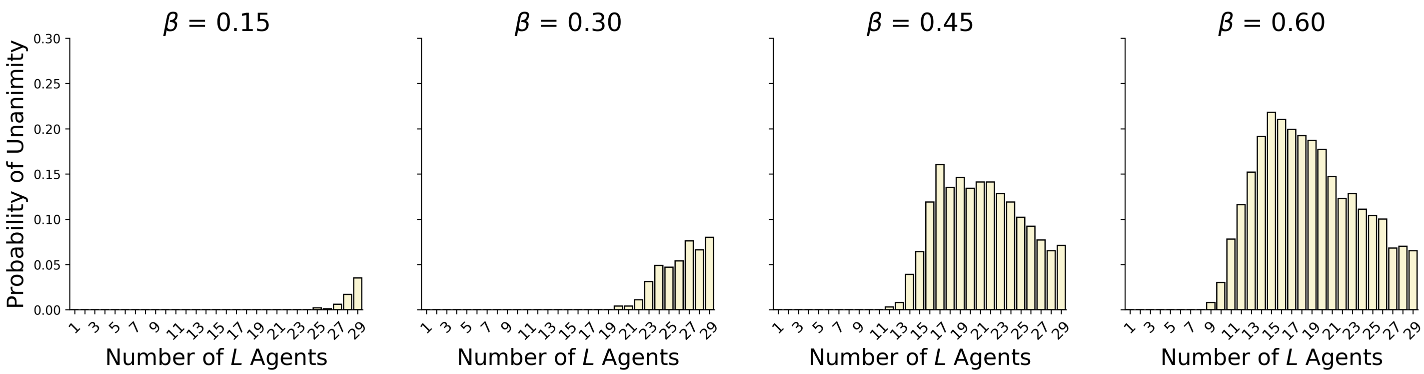

Figure 27: Probability of unanimity i.e., every agent playing either yellow or blue over 1000 simulations with fixed $\alpha = 0.75$, $N = 30$, while β and group composition by L agents were varied.

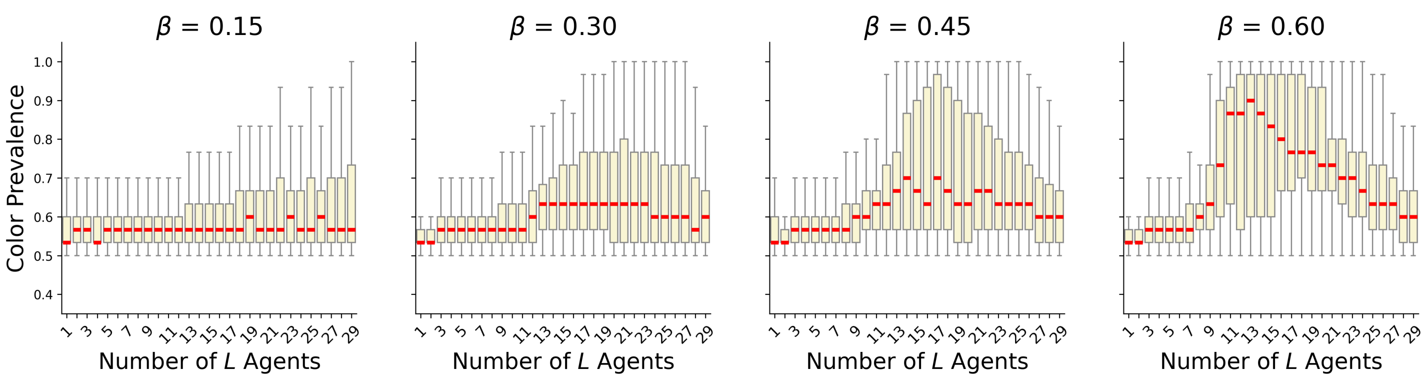

Figure 28: Color prevalence over 1000 simulations with fixed $\alpha = 0.75$, $N = 30$, while β and group composition by L agents were varied. In red: median values.

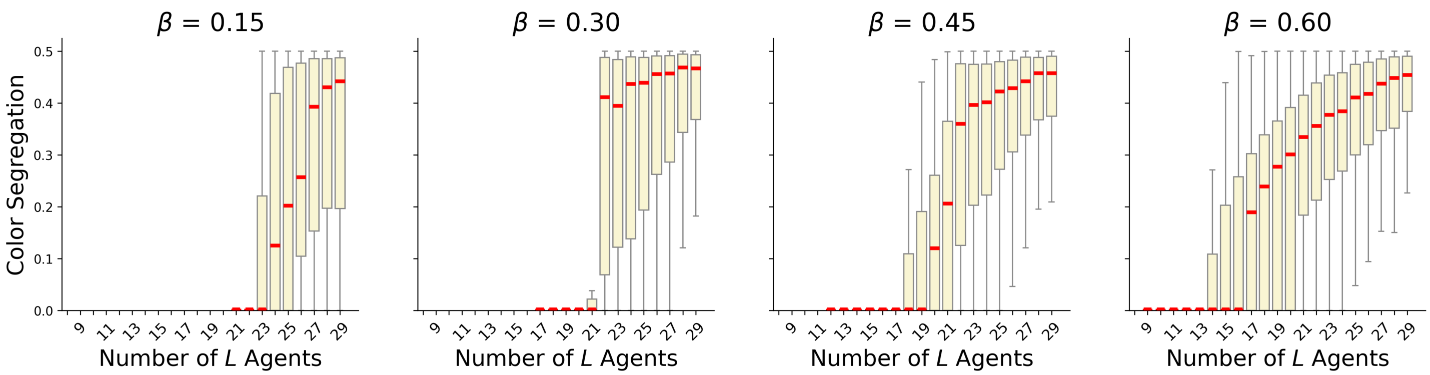

Figure 29: Color segregation over 1000 simulations with fixed $\alpha = 0.75$, $N = 30$, while β and group composition by L agents were varied. In red: median values.

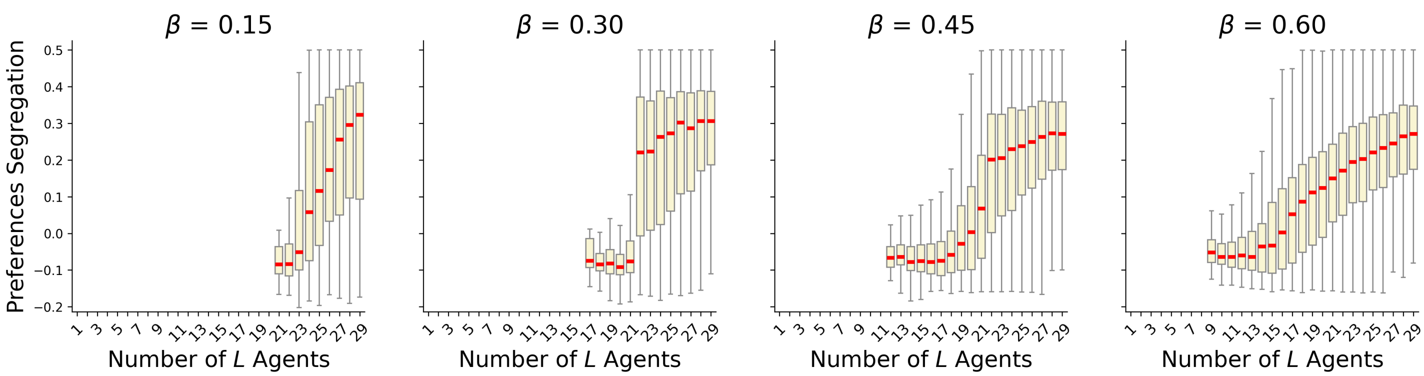

Figure 30: Segregation by initial color preferences l_i over 1000 simulations with fixed $\alpha = 0.75$, $N = 30$, while β and group composition by L agents were varied. In red: median values.

7 A Preliminary Test with Linear Costs

Here, we present results from a preliminary test in which we relaxed the assumption that network formation costs follow a V-shaped function of the number of ties formed. Instead, we adopted a linear cost function with increasing costs for larger personal networks. Heterogeneity was retained by setting the slope of the cost function proportional to d_i^*, using a proportionality factor of 0.35 to align average entry costs with the scenario of $\alpha = 1.25$ from the main manuscript. As a result, H agents continued to encounter higher entry costs, necessitating additional utility derived from network preferences to participate in the game, while L agents primarily formed their initial ties based on coordination payoffs alone. For these preliminary simulations, we fixed $\alpha = 1.25$ and varied β across $\{0.25, 1\}$. For simplicity, the analysis focuses on the probability of unanimity (Figure 31), color segregation (Figure 32), and the variance of the degree distribution (Figure 33).

We observe that the probability of unanimity increases with higher β, reaching its maximum when networks are more centralized by degree. It is either zero or very low when groups are predominantly composed of H or L agents. In the former case, the preferential attachment mechanism is not activated, leading to very sparse networks, while in the latter, color segregation dominates. Additionally, a higher β activates preferential attachment in groups with more H agents compared to a lower β. These findings align with the discussion in the main manuscript regarding the effects of changing β when $\alpha = 1.25$ on consensus and emerging network structures. However, further and more systematic tests are needed to confirm these results.

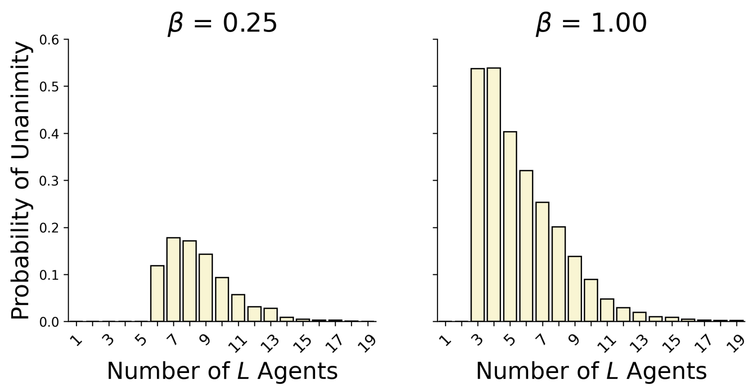

Figure 31: Color segregation over 1000 simulations with fixed $\alpha = 0.75$, $N = 30$, while β and group composition by L agents were varied. In red: median values.

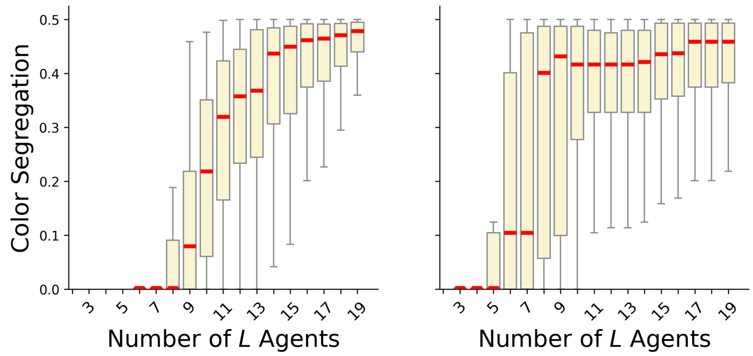

Figure 32: Color segregation over 1000 simulations with fixed $\alpha = 0.75$, $N = 30$, while β and group composition by L agents were varied. In red: median values.

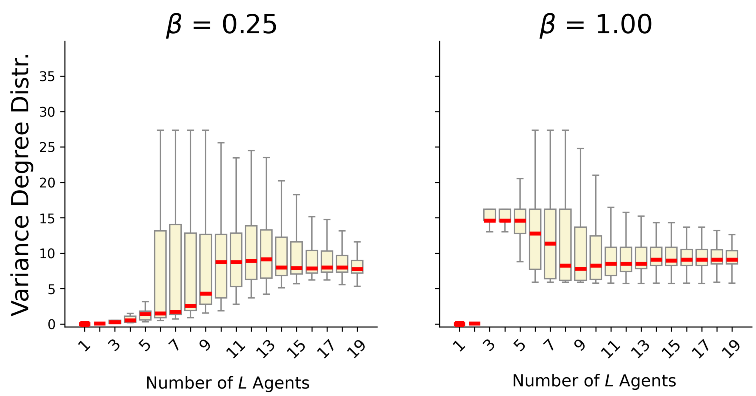

Figure 33: Color segregation over 1000 simulations with fixed $\alpha = 0.75$, $N = 30$, while β and group composition by L agents were varied. In red: median values.