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Analyzing DAX Volatility: An Application of GARCH and TARCH Models

Applied Econometrics

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Chapter 1

Introduction

It is common knowledge that the stock market has become much more volatile, especially in the last several years. In the subject of financial econometrics, modelling and forecasting volatility has drawn a lot of attention, due to the volatility increase. In the fields of option pricing and trading, risk management and volatility measurement are critical. Since stocks are highly volatile assets, our ability to forecast and anticipate changes in advance can help us minimise losses and prepare more effectively for scenarios that may induce either higher or lower volatility. Hedging, diversification, and portfolio risk management all rely upon precisely estimating stock volatility. (Kim & Won 2018)

Volatility, which refers to the degree of variation in the price of a stock over time, in the financial markets, is characterized by periods of high and low fluctuations, which is a common feature of financial time series. Stock volatility can be influenced by several factors, including economic indicators, company news, geopolitical events, changes in interest rates, or investor behaviour. Higher volatility may lead to larger returns, however, it can also lead to significant losses. Analysing the volatility of the stocks, gives the ability to adjust portfolios, depending on the current situation in the market.

Germany's major blue-chip stock market index, the DAX (Deutscher Aktienindex), is an important indicator of the country's economic health and investor attitude. It represents the top 40 largest and most liquid companies traded on the Frankfurt Stock Exchange. The DAX is considered the primary benchmark for the German stock market and a key indicator of the health of the German economy. It greatly affects international financial markets due to the strength that each of the companies that make up the index have. Industries represented in the DAX include companies from various sectors, such as automotive (BMW), chemicals (BASF), financial services (Allianz), healthcare (Bayer), and technology (SAP). As Germany is one of the largest economies in the world, the DAX is a crucial index to consider in the global financial space.

In the past, the DAX has shown stable growth over the long term, mainly reflecting the performance of Germany's top companies. It has increased from 30 companies to 40 in 2021, the reason being to enhance the index's presentation of the German economy, leading to a more precise estimation of the German market. In recent years, the German stock market has experienced times of higher volatility due to economic and geopolitical events, such as the global financial crisis, the European sovereign debt crisis and the COVID-19 pandemic. Nevertheless, it has grown throughout the years to an all-time high, currently standing at about 18 500 EUR [1].

Investors, policymakers, and economists should all understand the DAX index's volatility dynamics because they provide insight into market risk, portfolio management techniques, and larger economic trends, not only concerning the German market.

Using GARCH and TARCH models, which are used to understand these dynamics, this research aims to investigate the volatility dynamics of the DAX index. Through the use of these econometric techniques, our goal is to understand the complex patterns of volatility present in the DAX index and investigate whether using TARCH, an extended and more complex version of GARCH, could bring us a more precise understanding. This would indicate that the DAX index responds asymmetrically to positive and negative shocks.

[1] Observed in May 2024 at `https://www.investing.com/indices/germany-30`

Chapter 2

Literature review

2.1 Theoretical Background

Using Generalized Autoregressive Conditional Heteroskedasticity (GARCH), which has become a standard tool for analyzing financial time series characterized by volatility clustering, and Threshold Autoregressive Conditional Heteroskedasticity (TARCH), many researchers have examined the volatility of stock market indices. Engle & Bollerslev (1986) created the GARCH framework, which extends the Autoregressive Conditional Heteroskedasticity (ARCH) model, from Engle (1982) and showed how well it captured the time-varying volatility of financial assets. Zakoian (1994), followed their steps and laid the foundation for TARCH models that take into account non-linearities in volatility dynamics by extending the GARCH model to include asymmetry in volatility shocks.

In addition to the widely used ARCH and GARCH models, there exists a variety of advanced models designed to capture more complexities in financial time series. One notable extension is the Exponential GARCH (EGARCH) model, proposed by Nelson (1991). It models the logarithm of the conditional variance, allowing for asymmetries in the response of volatility to past errors and eliminating the need for non-negativity constraints on the variance. Furthermore, another widely used model, the Fractionally Integrated GARCH (FIGARCH) addresses the long memory properties and was also created by Baillie *et al.* (1996) and allows shock effects on volatility to last indefinitely.

2.2 DAX Index

Focusing more on the DAX index, Stapf & Werner (2003) found that after 1997, not only was the volatility of the DAX higher, but volatility persistence also went up. This implies that there is a higher chance that days with high volatility will be followed by other days with high volatility.

On the other hand, Claessen & Mittnik (2002) have focused in their work on evaluating whether the German DAX-index options market is informationally efficient and have looked into the applicability of several forecasting methodologies for the volatility of the returns of the index. They found using return data from the German DAX index, that historical returns do not provide any further insight into volatility beyond what option prices already factor in. In the case of the DAX-index options market, this validates the efficient market theory.

Bartlmae & Rauscher (2000) present performance metrics that assess two distinct aspects of the models' volatility forecasts that are crucial for risk mitigation. The DAX is used to test the framework daily. The neural network volatility mixture technique performs better than GARCH models, according to the results.

This research tackles the volatility of the DAX index. The German stock market's changing risk profile is highlighted by the market's heightened volatility and persistence after 1997. Option prices are a reliable source of volatility information, according to the DAX-index options market's support of the efficient market hypothesis.

While previous research offers insightful analyses of the volatility dynamics of the DAX index, this study aims to further advance the field by doing a thorough analysis utilising the GARCH and TARCH models.

Chapter 3

Data and Methodology

In this chapter, we would like to outline our motivation, subsequently, briefly describe the used data, and then go through the econometric procedures we used.

3.1 Motivation

As described above, in this study we focus on modelling the volatility of returns of the **DAX** index, which is an index of 40 major German blue-chip companies. In particular, we question whether we can estimate DAX volatility better using **GARCH** or **TARCH** model. In our modelling process, we first try to eliminate linear dependencies using a suitable **ARMA** model and then fit a suitably parameterized GARCH and TARCH model. Suitable models are selected on the basis of information criteria, namely the Akaike Information Criterion (**AIC**) and Bayesian Information Criterion (**BIC**). All of these methods are further explained in section 3.3.

3.2 Data

For the analysis, we use daily data from 5 January 2010 to 6 March 2023 of the DAX index closing prices. From the closing prices we calculate the logarithmic returns as follows:

$$r_t = log(\frac{P_t}{P_{t-1}})$$

where $t \in \{1, ..., T\}$ is a time index and P_t is a closing price at time t

Figure 3.1: DAX Index Price

3.3 Estimation methods

3.3.1 ARMA process

Before we proceed to GARCH and TARCH models, we should first swiftly introduce **ARMA** process. The ARMA process consists of two processes - The Autoregressive Process (**AR**) and the Moving Average Process (**MA**).

We define the autoregressive process of order p as follows:

$$r_t = \rho_0 + \sum_{i=1}^{p} \rho_i r_{t-i} + \epsilon_t$$

where ϵ_t follows White Noise with zero mean μ and constant variance σ^2.

Similarly, we define the moving average process of order q:

$$r_t = \theta_0 + \sum_{i=1}^{q} \theta_i \epsilon_{t-i} + \epsilon_t$$

where ϵ_t same as in case of $AR(p)$ follows White Noise.

Finally, the $ARMA(p, q)$ process could be perceived as a combination of both mentioned processes, mathematically:

$$r_t = \rho_0 + \sum_{i=1}^{p} \rho_i r_{t-i} + \sum_{j=1}^{q} \theta_j \epsilon_{t-j} + \epsilon_t$$

where the residuals are again White Noise.

As can be seen from the model, the ARMA model tries to describe the returns using their historical values and also historical residuals. Thus the ARMA approach is autoregressive and mean-reverting (Box 2013).

3.3.2 GARCH

The Generalized Autoregressive Conditional Heteroscedasticity model (GARCH), introduced by Engle & Bollerslev (1986), which extends the ARCH model as mentioned above, by allowing past conditional variances to affect present variances. As the name suggests the model does not assume constant variance throughout the examined timeframe (homoscedasticity). The GARCH model provides a more robust framework for modelling the time-varying volatility commonly observed in financial markets. Mathematically is GARCH(p, q) defined as follows:

we define returns as the following process:

$$r_t = \mu_t + a_t$$

$$a_t = \sigma_t \epsilon_t$$

which we use as an input into the variance volatility:

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i a_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2$$

3.3.3 TARCH

The Threshold Autoregressive Conditional Heteroscedasticity model was proposed by Zakoian (1994) as a possible correction of the GARCH model. The problem with GARCH might be that it assigns equivalent effects for positive and negative shocks, which TARCH deals with by allowing for asymmetric effects. We define TARCH(p, q) as follows:

We define the underlying process of returns in the same manner as in the case of GARCH:

$$r_t = \mu_t + a_t$$

$$a_t = \sigma_t \epsilon_t$$

In the case of the variance equation, we use the Indicative function $I(.)$ which returns 1 if $a_{t-i} < 0$ and 0 otherwise.

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i a_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2 + \sum_{k=1}^{p} \gamma_k a_{t-k}^2 I(a_{t-k} < 0)$$

As we can see, the TARCH model accounts for different effects of negative and positive shocks.

3.3.4 Akaike & Bayesian Information Criteria

When selecting the appropriate model for time series analysis, it might be crucial to balance model fit and complexity. Often used metrics to evaluate several models on the same dataset are the Akaike Information Criterion (AIC) and the Bayesian Information Criterion (BIC).

Both criteria are derived from the Maximal Likelihood Principle, which is weighted by the complexity of the model:

$$AIC = 2k - 2log(\hat{L})$$

where k is number of parameters of the model and $\hat{L}$ is a maximal value of the Likelihood function.

$$BIC = klog(N) - 2log(\hat{L})$$

where N is number of observations in the dataset.

Thus, this implies that the better value of the likelihood function does not necessarily mean a better model, and if we have two models with the same value of the likelihood function, we should choose the one with fewer parameters - In the case of GARCH/ARCH models the model with smaller sum $p + q$ (Akaike 1974; Schwarz 1978).

Chapter 4

Results

4.1 ARMA Model

This chapter goes through the whole procedure in detail. Firstly we go through the modeling process and after that, we interpret the results, which have come up in our analysis.

As mentioned in the methodology section, first we obtain log returns from our financial time series of the DAX index. Subsequently, we test logarithmic returns using the Augmented Dickey-Fuller test, to check for a possible unit root (Fuller 2009). The null hypothesis of the ADF test is that the time series follows a random walk. For our purposes, it is essential to reject the null hypothesis. From the resulting statistics of the test, we reject the null hypothesis at a 0.05 significance level, indicating, that the logarithmic returns series does not contain a unit root.

Following the unit root testing, we select an appropriate ARMA model, which helps us remove linear dependencies for the GARCH model. To choose which ARMA model is suitable, we looked at the Autocorrelated Function and Partial Autocorrelated Function plots.

Looking at the graphs, there can be observed a significance for lags 4 and 7, for both functions. These observations do not give us a robust indication of which ARMA model should be used to model the data.

Therefore, we decided to select the appropriate ARMA(p, q) lags based on comparing the lowest combined information criteria (combination of AIC and BIC). We take into account all combinations $(p, q) \in [0, 1, 2, 4, 7]^2$ [1]. The lowest

[1] We list the smallest lags combined with lags which turned out to be significant in ACF/-PACF

Figure 4.1: Autocorrelated Function

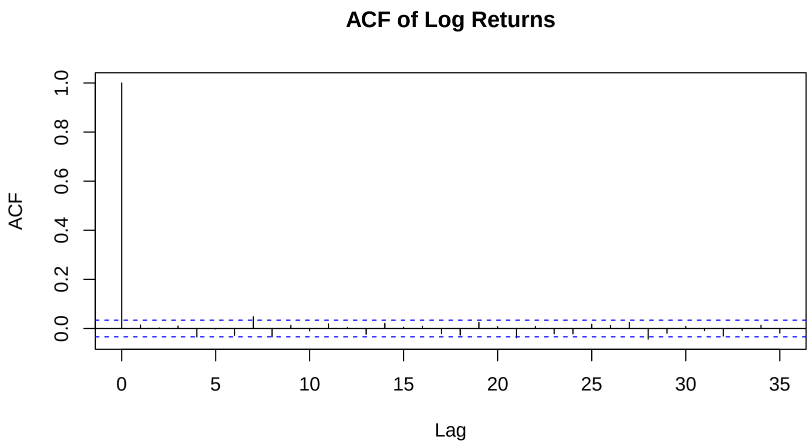
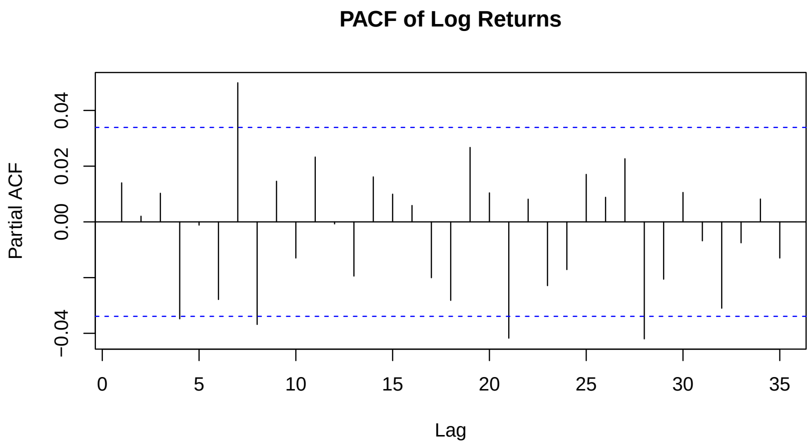


Figure 4.2: Partial Autocorrelated Function



AIC result was calculated for lags (p, q) = (7,7). The resulting statistic was -19 651.56. On the contrary, the lowest BIC result was calculated for lag (p, q) = (0,0), with the resulting statistic -19 626.76. Upon looking at the calculations, we decided to use the lag, which minimized the sum of AIC and BIC, therefore the lag we decided to further use in our modelling process was selected to be (p, q) = (0,0).

Table 4.1: Sum of AIC and BIC

	0	1	2	4	7
0	**-39265.75**	-39256.94	-39246.85	-39236.75	-39228.65
1	-39256.94	-39246.83	-39236.74	-39226.63	-39227.07
2	-39246.86	-39236.74	-39226.62	-39231.84	-39217.02
4	-39235.42	-39225.30	-39231.79	-39222.23	-39195.66
7	-39226.96	-39228.85	-39218.87	-39216.48	-39205.30

Note: The table shows combination (p,q), where rows represent lag p and columns represent lag q.

By arriving at the result, that ARMA (0,0) is our preferred model, we do not have to correct for any linear dependencies.

4.2 GARCH and TARCH Models

The goal of this analysis is to model the DAX index, by employing GARCH and TARCH models. The appropriate ARMA model selection is explained above. The next step in modelling the volatility is to select the best GARCH and TARCH parametrizations.

Similarly to the ARMA model selection, we now select the best parametrization for the GARCH and TARCH models. We selected the models analogously, based on the information criteria, which minimizes the sum, where we took into account lags 1 and 2.

Table 4.2: GARCH Estimators

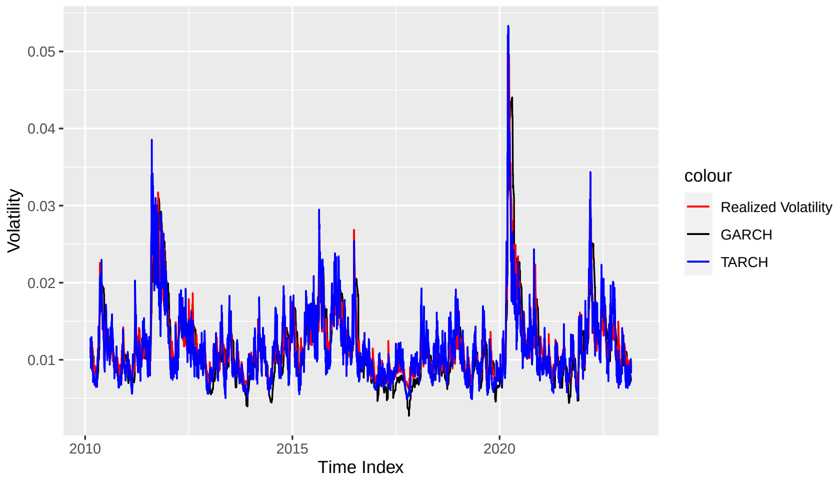
Order (p,q)	AIC	BIC	LogLikelihood
(1,1)	**-6.129635**	**-6.122317**	**10246.62**
(1,2)	-6.129001	-6.119853	10246.56
(2,1)	-6.129549	-6.120401	10247.48
(2,2)	-6.128950	-6.117973	10247.48

Table 4.3: TARCH Estimators

Order (p,q)	AIC	BIC	LogLikelihood
1,1	-6.184334	-6.175186	10339.02
1,2	-6.184094	-6.173117	10339.62
2,1	-6.185423	-6.172616	10342.84
2,2	**-6.188346**	**-6.173710**	**10348.73**

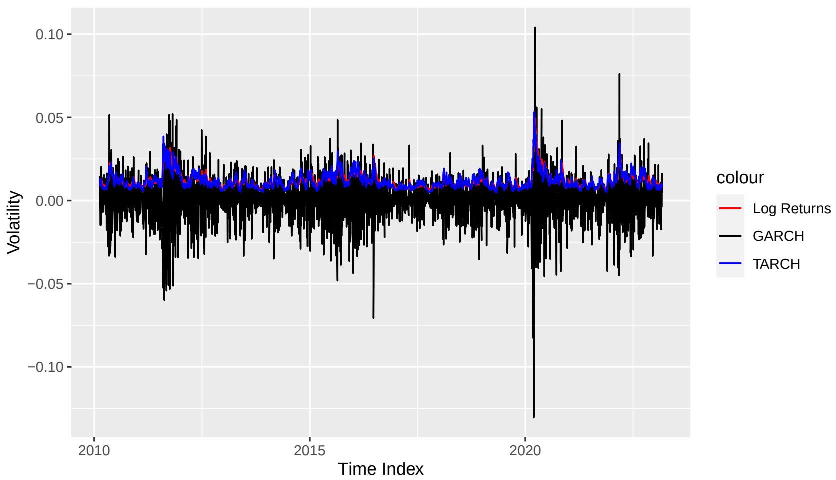
The information criteria analysis resulted in the application of GARCH (1,1) and TARCH(2,2). Furthermore, the TARCH model information criteria are better than the GARCH model criteria for all chosen orders. Therefore TARCH seems to outperform GARCH on DAX index data, which might indicate asymmetry between negative and positive returns to volatility. On the examined time frame the TARCH$(2, 2)$ model turns out to describe the volatility with the greatest accuracy.

Figure 4.3: Estimated Volatility Compared With Realized Volatility



We have compared the selected TARCH and GARCH models with the realized volatility of the DAX, which was computed as a 30-day rolling standard deviation. We can observe that estimated volatility seems to describe the realized volatility quite well.

Figure 4.4: Estimated Volatility of Logarithmic Returns



We have plotted the volatility estimated by GARCH$(1,1)$ and TARCH$(2,2)$. From our perspective, the resulting volatility representation seems to match the DAX index returns with solid precision.

Chapter 5

Conclusion

We have analyzed and applied GARCH and TARCH models on the daily data of the DAX index from 5 January 2010 to 6 March 2023. Results provide a thorough examination of the index and the estimated volatility appears to be consistent with realized volatility and also with log returns. Based on AIC and BIC the TARCH models seem to outperform GARCH models. We examined several GARCH orders (p,q) and the best model turned out to be TARCH(2,2). Also, all TARCH(1,1), TARCH(1,2), and TARCH(2,1) seem to outperform the best GARCH model, which is GARCH(1,1). This is probably due to the presence of the asymmetric effects of negative and positive shocks on volatility, which TARCH models account for.

Bibliography

Akaike, H. (1974): "A new look at the statistical model identification." *IEEE transactions on automatic control* **19(6)**: pp. 716–723.

Baillie, R. T., T. Bollerslev, & H. O. Mikkelsen (1996): "Fractionally integrated generalized autoregressive conditional heteroskedasticity." *Journal of econometrics* **74(1)**: pp. 3–30.

Bartlmae, K. & F. A. Rauscher (2000): "Measuring dax market risk: a neural network volatility mixture approach." In "Presentation at the FFM2000 Conference, London," volume 31.

Box, G. (2013): "Box and jenkins: time series analysis, forecasting and control." In "A Very British Affair: Six Britons and the Development of Time Series Analysis During the 20th Century," pp. 161–215. Springer.

Claessen, H. & S. Mittnik (2002): "Forecasting stock market volatility and the informational efficiency of the dax-index options market." *The European Journal of Finance* **8(3)**: pp. 302–321.

Engle, R. F. (1982): "Autoregressive conditional heteroscedasticity with estimates of the variance of united kingdom inflation." *Econometrica: Journal of the econometric society* pp. 987–1007.

Engle, R. F. & T. Bollerslev (1986): "Modelling the persistence of conditional variances." *Econometric reviews* **5(1)**: pp. 1–50.

Fuller, W. A. (2009): *Introduction to statistical time series.* John Wiley & Sons.

Kim, H. Y. & C. H. Won (2018): "Forecasting the volatility of stock price index: A hybrid model integrating lstm with multiple garch-type models." *Expert Systems with Applications* **103**: pp. 25–37.

NELSON, D. B. (1991): "Conditional heteroskedasticity in asset returns: A new approach." *Econometrica: Journal of the econometric society* pp. 347–370.

SCHWARZ, G. (1978): "Estimating the dimension of a model." *The annals of statistics* pp. 461–464.

STAPF, J. & T. WERNER (2003): "How wacky is the dax? the changing structure of german stock market volatility." .

ZAKOIAN, J.-M. (1994): "Threshold heteroskedastic models." *Journal of Economic Dynamics and control* **18(5)**: pp. 931–955.

Appendix A

Additional materials

All the necessary material to replicate the work, i.e. data and R script, is publicly available and easily accessible at `https://github.com/cedav12/Analyzing-DAX-Volatility-An-Application-of-GARCH-and-TARCH-models`