1) Notion of representation

- time-frequency
- inverse problem

2) Inverse problems

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60% exams
40% labs

Inverse problem

Recording problem → Parameters

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Use cases :
- Restoration
- Inpainting
- Geometrical transformation
- Super-resolution
- Classification
- Diagnosis (quantitative measurement)
- Texture synthesis

Image processing → Machine Learning

↓ create from "true" data

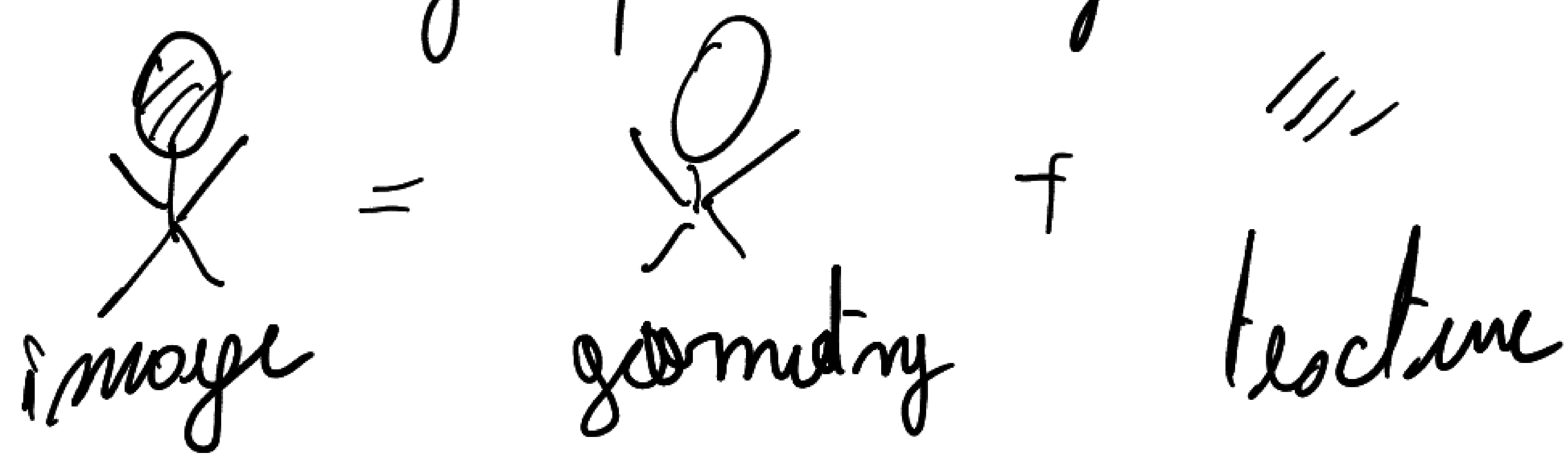
↓ predicte based on observation

Representation: How to access useful information?

Inverse problem

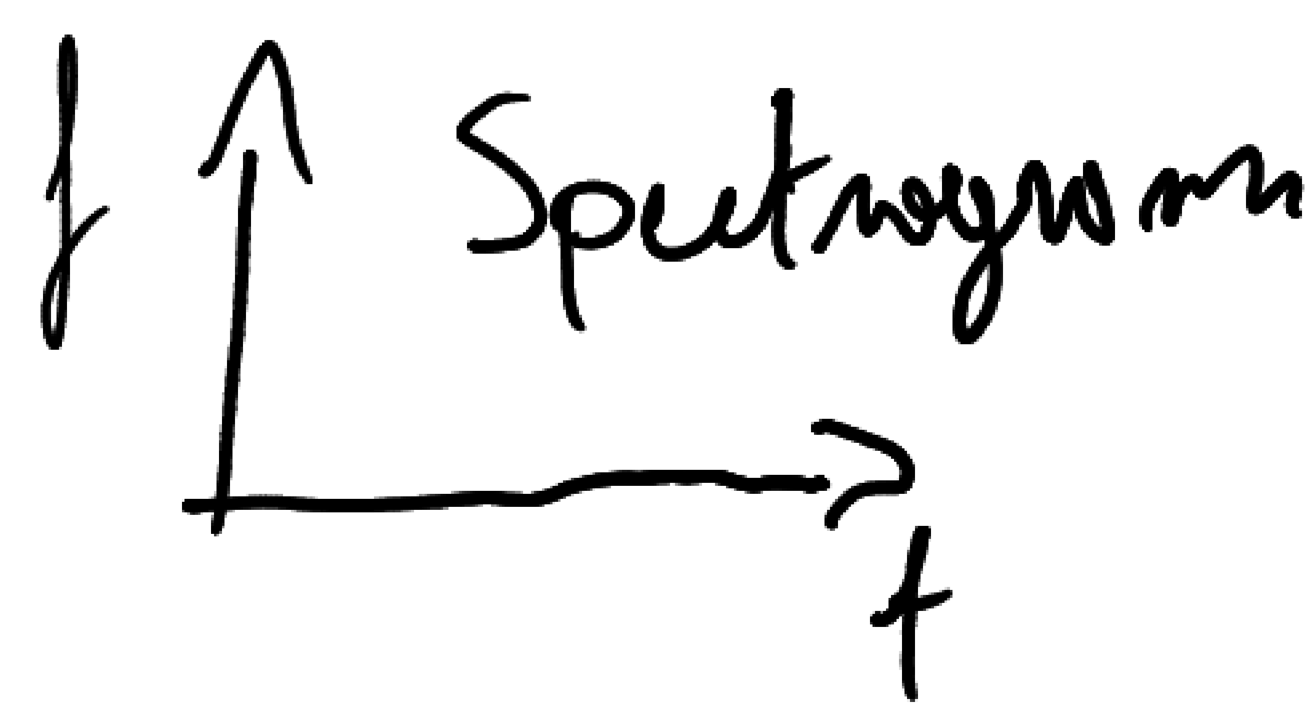
get parameters from data on a priori knowledge

Natural image processing



Which frequency? at which moment?

(TF does describe a signal by frequency but not when frequency exists)



Representation of signals and inverse problem

Part 1: Representation of signals

time-frequency representation
wavelets (time-scale representation)
and application

Chapter 1: Introduction representation of signals

1) Notion of representation

Known example: system of coordinates
(cartesian, cylindric, spherical)
↪ Choose the good representation

→ Choice of an adapted representation
(depending on the signal of interest)

⟹ better description of the signal
easier characterization of its
specific feature ("information")

(better = more compact)

- Large volume of data to analyse
  ⟹ sparse representation are required
  (concise
  parsimonious)
  [ie, composed only of a few non-zero
  coefficients]

to simplify / accelerate signal processing
(via automated numerical procedure algorithms)

⇒ Representation in orthogonal bases of functions

(minimum size description encoded via a dictionary)

So that the energy of the signal is concentrated on a few coefficients in the representation

2) Fourier transform: an appropriate representation for stationary (homogeneous) systems

- Complex exponential functions are eigen-functions of linear stationary (=time-invariant) operators

$$\underset{\uparrow}{L}\left[e^{2i\pi\nu t}\right] = \underbrace{H(\nu)}_{\text{eigenvalue}} e^{2i\pi\nu t}$$

linear time-invariant (LTI) operator

eigenvalue associated with $t \mapsto e^{2i\pi\nu t}$

ex: (i) $\frac{d}{dt}\left(e^{2i\pi\nu t}\right) = (2i\pi\nu)\, e^{2i\pi\nu t}$

(ii) linear filtering

$$y(t) = \underset{\substack{\uparrow \\ \text{impulse response} \\ \text{of a LTI system}}}{h} * x(t)$$

—

$$L\left[e^{2i\pi\nu t}\right] = \int_{\mathbb{R}} h(u)\, e^{2i\pi\nu(t-u)}\, du$$

$$= \underset{\text{eigenfunction}}{e^{2i\pi\nu t}} \underbrace{\int h(u)\, e^{-2i\pi\nu u}\, du}_{\hat{h}(\nu)}$$

eigen value

Brief refresher on Fourier transform

(i) Defined for functions $v \in L^1(\mathbb{R})$

$L^1(\mathbb{R}) \cap L^2(\mathbb{R})$

$$\mathcal{S}(\mathbb{R}) = \left\{ \phi \in \mathcal{C}^\infty(\mathbb{R}) : \sup_{t \in \mathbb{R}} |x^\alpha \phi^{(\beta)}(t)| < \infty \text{ for any } \alpha, \beta \right\}$$

$$\hat{x}(\nu) = \int_{\mathbb{R}} x(t) e^{-2i\pi\nu t} dt$$

$\mathcal{F} \updownarrow$

$$x(t) = \int_{\mathbb{R}} \hat{x}(\nu) e^{2i\pi\nu t} d\nu$$

Remarque FT is a non-local transform

ie for each t all freq ν are needed and conversely

Parseval-Plancherel theorem

For any $x, y \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ we have:

$$\langle x, y \rangle = \int_{\mathbb{R}} x(t)\, y^*(t)\, dt$$

(Plancherel)

$$= \int_{\mathbb{R}} \hat{x}(\nu)\, \hat{y}^*(\nu)\, d\nu = \langle \hat{x}, \hat{y} \rangle$$

Specific case (energy conservation)

$$\|x\|_2^2 = \int_{\mathbb{R}} |x(t)|^2\, dt = \int_{\mathbb{R}} |\hat{x}(\nu)|^2\, d\nu = \|\hat{x}\|_2^2$$

(Parseval)

2 main property

→ one representation to another

→ energy conservation

These definition and results are extended to $L^2(\mathbb{R})$ using density of $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$

We will also consider FT of tempered distributions $S'(\mathbb{R})$

Gabor - Heisenberg Let $x \in L^2(\mathbb{R})$ such that the following quantities are well defined

(i) time average $u = \frac{1}{\|x\|^2} \int_{\mathbb{R}} t |x(t)|^2 dt$

frequency average $\xi = \frac{1}{\|x\|_2^2} \int_{\mathbb{R}} \nu |\hat{x}(\nu)|^2 d\nu$

average of energy distribution in time

$$t \mapsto \frac{|x(t)|^2}{\|x\|_2^2}$$

(ii) time variance

$$\sigma_t = \left(\frac{1}{\|x\|_2^2}\int_{\mathbb{R}} (t-u)^2\,|x(t)|^2\,dt\right)^{1/2}$$

freq variance

$$\sigma_\nu = \left(\frac{1}{\|x\|_2^2}\int_{\mathbb{R}} (\nu-\xi)^2\,|x(\nu)|^2\,d\nu\right)^{1/2}$$

then $\boxed{\sigma_t\,\sigma_\nu \geq \frac{1}{4\pi}}$ $(*)$

Remark: equality reached in (*)
if and only if (iff)
there exists $(u, \xi, a, b) \in \mathbb{R}^2 \times \mathbb{C}^2$
such that

$$x(t) = a e^{-b(t-u)^2} e^{2i\pi \xi t}$$

Gaussian window translated
in time by u and in frequency by ξ

Remarks

(i) compact support signal x
(in time)

$\Longleftrightarrow$ $\hat{x}$ has an infinite size support

Compromise in localization between the time and the frequency domain

localized in time $\overset{\mathcal{F}}{\Longleftrightarrow}$ diffused in frequency

(and conversely)
$\Rightarrow$ compromise

(ii) For all representations, we need an inverse transform (to move back from representation to the initial domain

* In general, energy conservation is a desirable property (no distortion in the info conveyed by the energy via the change of representation

In 2D (images), similar construction

a) $f \in L^2(\mathbb{R}^2)$,

$$\hat{f}(\nu_1, \nu_2) = \iint_{\mathbb{R}^2} f(x_1, x_2) e^{-2i\pi \langle \underline{x}, \underline{\nu} \rangle} dx_1 dx_2$$

$$x = \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} \text{ and } \nu = \begin{vmatrix} \nu_1 \\ \nu_2 \end{vmatrix}$$

b) $\hat{f} \in L^1(\mathbb{R}^2)$, $f = \mathcal{F}^{-1}[\hat{f}]$ almost everywhere (a.e.)

c) $f \in L^1(\mathbb{R}^2)$, $h \in L^1(\mathbb{R}^2)$

$$\mathcal{F}[f * h] = \hat{f}\,\hat{h}$$

d) Parseval-Plancherel still holds

Discrete Fourier transform (DFT)

Let $f = (f[n])_{0 \leqslant n \leqslant N-1}$ be a sequence of N samples from a continuous-time signal f, $N \geqslant 1$

(i) DFT of f, named $\hat{f}$ by

$$\forall k \in [\![0, N-1]\!] , \quad \hat{f}[k] = \sum_{n=0}^{N-1} f[n] e^{-\frac{2i\pi kn}{N}}$$

(ii) inverse DFT of $\hat{f}$

$$\forall n \in [\![0, N-1]\!] , \quad f[n] = \frac{1}{N} \sum_{k=0}^{N-1} \hat{f}[k] e^{\frac{2i\pi kn}{N}}$$

discrete index $\longrightarrow$ continuous domain

$$n \longrightarrow t = n\,T_s \;,\; T_s \text{ sampling period}$$

$$k \longrightarrow \nu = \frac{k}{N} F_s \;,\; F_s = \frac{1}{T_s} \text{ sampling frequency}$$

($\rightsquigarrow$ there is an equivalent of the convolution theorism, see part 2)

3. Geometry in space of functions: bases (dictionaries) and atoms (an element of a dictionary)

a) Similarities between geometry notions in an Euclidean and in spaces of functions

Euclidean space ($\mathbb{R}^3$)	Functional space (H an Hilbert space)
$\mathcal{B} = (\underline{e}_1, \underline{e}_2, \underline{e}_3)$	$\mathcal{B} = \{g_n, n \in \mathbb{Z}\}$
	or $\{g_\alpha, \alpha \in \mathbb{R}\}$

atom

dictionary

Euclidean space

$\underline{v} = v_1 e_1 + v_2 e_2 + v_3 e_3$

$v_j = \langle v, e_j \rangle$

$\langle v, u \rangle = \sum_{i=1}^{3} v_i u_i$

Functional space

$f = \sum_{m \in \mathbb{Z}} c_f[m] g_m$

$c_f[m] = \langle f, g_m \rangle$

esc (i) $H = L^2(\mathbb{R})$

$\langle f, g \rangle = \int_{\mathbb{R}} f(t) g^*(t) dt$

(ii) $H = L^2(\mathbb{R})$: FT

$\hat{x}(\nu) = \langle x, e^{2i\pi \cdot} \rangle$

STFT

WFT