II) Continuous wavelets transform

Definition (wavelet)

A wavelet is a function $\psi \in L^*(\mathbb{R})$ s.t.:

(i) $\|\psi\|_2 = 1$

(ii) has 0-average

(iii) ψ is centred around 0

(iv) ψ satisfies the following admissibility conditions

$$0 < C_\psi = \int_0^{+\infty} \frac{|\hat{\psi}(\nu)|^2 \, d\nu}{\nu} < +\infty$$

$\Rightarrow$ We define for $(u, s) \in \mathbb{R} \times \mathbb{R}_+^*$, the family of functions $\left\{ \psi_{s,u} = \frac{1}{\sqrt{s}} \psi\left(\frac{\cdot - u}{s}\right) \right\}$

obtained by translating (by u) and dilating (by s) the function ψ

Definition (continuous wavelet transform)

The continuous wavelet transform (CWT) of a signal $f \in L^2(\mathbb{R})$ at time $u \in \mathbb{R}$ and scale $s > 0$ is defined as

$$Wf(u,s) = \langle f, \psi_{u,s} \rangle \quad (*)$$

$$= \int_{\mathbb{R}} f(t) \frac{1}{\sqrt{s}} \psi^*\left(\frac{t-u}{s}\right) dt$$

Remark

(i) Wf can be interpreted as the result of a linear filtering operation

Introducing $\tilde{\psi}_s = \frac{1}{\sqrt{s}} \psi^*\left(\frac{\cdot}{s}\right)$

then $(*)$ can be rewritten as

$$Wf(u,s) = \int_{\mathbb{R}} f(t)\, \tilde{\psi}_s(u-t)\, dt = f * \tilde{\psi}_s(u)$$

Since $\hat{\psi}(0) = \int_{\mathbb{R}} \psi(t)\,dt = 0$ (by definition)

and

$$\mathcal{F}[\tilde{\psi}_s](\nu) = \sqrt{s}\,\hat{\psi}^*(s\nu)$$

$\tilde{\psi}_s$ can be interpreted as (the impulse response of) a band-pass filter

2 types (i) analytic wavelets (complex-valued)
aimed at defining / capturing "instantaneous frequencies" $\rightarrow$ not covered here
(ii) real wavelets to detect sharp transitions, singularities
and characterize textures
(local regularity)

2) Real wavelets ($\psi(t) \in \mathbb{R}$)

Examples

(i) Haar wavelets (appropriate for the description of piece-wise constant signals) $\longrightarrow$ sparse representation

fonction continue par morceau

(ii) Mexican hat $\longrightarrow$ see slides

$\Rightarrow$ band-pass filter, capturing details/textures information from an input signal

Remark

the decay of wavelet coefficient as $s \to 0$ characterizes the regularity of a signal around the time-instant u

2.1 Inversion an energy conservation

Theorem (Calderon (1964), Grossman, Morlet (1984))

Let $\psi \in L^2(\mathbb{R})$ be real wavelet, then for any $f \in L^2(\mathbb{R})$,

inversion formula

$$t \in \mathbb{R}, \quad f(t) = \frac{1}{C_\psi} \int_0^{+\infty} \int_{\mathbb{R}} Wf(u,s) \frac{1}{\sqrt{s}} \psi\left(\frac{t-u}{s}\right) du \frac{ds}{s^2}$$

and

energy conservation

$$\|f\|_2^2 = \frac{1}{C_\psi} \int_0^{+\infty} \int_{\mathbb{R}} |Wf(u,s)|^2 du \frac{ds}{s^2}$$

↳ do not need to retain the expressions

2.2. Scaling function

When $Wf(u,s)$ is known only for $s < s_0$, the one also needs to known $Wf(u,s)$ for $s > s_0$ to apply the reconstruction formula

- This is obtained by introducing the so-called scaling function ϕ that is an aggregation of wavelets at scales larger than s_0. The scaling function is such that:

$$|\hat{\phi}(\omega)|^2 = \int_1^{+\infty} |\hat{\psi}(s\omega)|^2 \frac{ds}{s} = \int_\omega^{+\infty} \frac{|\hat{\psi}(\xi)|^2}{\xi} d\xi$$

and it's phase can be defined arbitrarily

- To garantee that $\hat{\phi}(0)$ is finite, the admissibility condition ensures that $\lim_{\omega \to 0} |\hat{\phi}(\omega)|^2 = C_\psi < +\infty$

- The scaling function can be associated with the impulse response of a low-pass filter, complemented the band-pass filter associated with ψ

Summary: $\begin{cases} \psi \text{ (mother wavelet)} \to \text{band-pass filter} \\ \phi \text{ (scaling function)} \to \text{low-pass filter} \\ \text{"father wavelet"} \end{cases}$

$\tilde{\phi}_s = \frac{1}{\sqrt{s}} \phi^*\left(\frac{-\cdot}{s}\right)$ a low-frequency approximation of $f \in L^2(\mathbb{R})$ at a scale s, around u is:

$$Lf(u,s) = \left\langle f, \frac{1}{\sqrt{s}} \phi\left(\frac{\cdot - u}{s}\right)\right\rangle = f * \tilde{\phi}_s(u)$$

Revisiting the reconstruction formula leads to

$$f(t) = \underbrace{\frac{1}{C_\psi} \int_0^{s_0} Wf(\cdot, s) * \psi_s(u) \frac{ds}{s^2}}_{\text{details}} + \underbrace{\frac{1}{C_\psi s_0} Lf(\cdot, s_0) * \phi_{s_0}(t)}_{\text{low-frequency approximation}}$$

$$\begin{cases} \psi_s = \dfrac{1}{\sqrt{s}}\,\psi\left(\dfrac{\cdot}{s}\right) \\ \phi_s = \dfrac{1}{\sqrt{s}}\,\phi\left(\dfrac{\cdot}{s}\right) \end{cases}$$

Summary of representation covered

Time-frequency representation

- Short-time Fourier transform (STFT)

$$Sf(u,\xi) = \langle f, g_{u,\xi} \rangle = \int_{\mathbb{R}} f(t) \underbrace{g(t-u)\, e^{-2i\pi u \xi}}_{g^*_{u,\xi}(t)} dt$$

- Objectives
  - time-frequency analysis
  - fixed-size tessellation of the time frequency plane (Heisenberg boxes)

Time-scale representation

- Continuous wavelet transform (CWT)

$$Wf(u,s) = \langle f, \psi_{u,s} \rangle = \int_{\mathbb{R}} f(t) \frac{1}{\sqrt{s}} \psi^*\left(\frac{t-u}{s}\right) ds$$

- Objectives

(i) real wavelets

- detect sharp transition local regularity
- Wavelet ψ $\Longrightarrow$ band-pass filter
  $\Longrightarrow$ captures details

- Scaling function ϕ $\longrightarrow$ low-pass filter
  $\longrightarrow$ smooth approximation at scale s

(ii) analytic wavelets

- characterize the temporal evolution of the frequency content of a signal using scale-adaptive Heisenberg boxes

IV Orthogonal wavelet bases

1) Practical interest and objectives

Objectives obtain a representation of a signal composed of a limited number of non zero coefficients (sparse representation)

a) Approximation

Let $f \in L^2(\mathbb{R})$ and $\{e_n\}_{n \in \mathbb{N}}$ an orthonormal basis of $L^2(\mathbb{R})$.

- Linear approximation error

$$\mathcal{E}_L(N) = \| f - f_N \|_2^2 = \sum_{n \geq N} | \langle f, e_n \rangle |^2 \text{ with}$$

$$f_N = \sum_{n=0}^{N-1} \langle f, e_n \rangle e_n$$

• Non Linear approximation error

Let I_N denote the subspace of indices corresponding to the N largest values of $|\langle f, e_n\rangle|$

Then: $$\varepsilon_{NL}(N) = \| f - \underbrace{\sum_{n \in I_N} \langle f, e_n\rangle e_n}_{f_N} \|_2^2$$

$$= \sum_{n \notin I_N} |\langle f, e_n\rangle|^2$$

→ equivalent to using a fine approximation around features of interest (singularities for the wavelet transform)

ii) Denoising

In general, the noise affects all the coefficients of a signal in the original domain

→ a representation adapted to the content of a signal concentrates the energy of the signal onto a few coefficients, corresponding to those with the highest magnitude

<u>Example</u> Donoho & Johnstone (1994) have shown that a simple thresholding operation of the wavelet coeffs is efficient for denoising

$$y[n] = x[n] + w[n]$$

$$y[n] \xrightarrow{DWT} c_y[n] \underset{\text{thresholding operation}}{\longrightarrow} \begin{cases} 0 \text{ if } c_y[n] < \gamma & \gamma > 0 \text{ fixed} \\ c_y[n] \text{ otherwise} \end{cases} \xrightarrow{DWT} \tilde{x}[n]$$

⇝ a difficult problem in the original can be addressed with a basic procedure

c) Compression : Keep only N most significant decomposition coefficient

JPEG ← DCT (discrete cosine transform)