2. Multi-resolution analysis

<u>Idea principle</u> project the signal of interest onto consecutive space of approximations V_j (j scale coefficient)

a) one can build wavelets ψ such that

$$\left\{ \psi_{j,k}(t) = \frac{1}{\sqrt{2^j}} \psi\left(\frac{t-2^j k}{2^j}\right) \right\}$$

is an o.n.b. of $L^2(\mathbb{R})$

b) equivalence between wavelet bases and conjugate mirror filters

→ Efficient and fast numerical procedures to compute a wavelet transform (relying on filter banks, complexity in $O(N)$)

Definition : (multi-resolution analysis)

A multi-resolution analysis is a sequence of closed subspaces $(V_j)_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ such that

(i) $\forall (j,k)\in\mathbb{Z}^2,\ f\in V_j \iff f(\cdot-2^j k)\in V_j$

(translated property)

(ii) $\forall j\in\mathbb{Z},\ V_{j+1}\subset V_j$ [nested subspace]

(iii) $\forall j\in\mathbb{Z},\ f\in V_j \iff f\left(\frac{\cdot}{2}\right)\in V_{j+1}$

(iv) $\displaystyle\lim_{j\to+\infty} V_j = \bigcap_{j\in\mathbb{Z}} V_j = \{0\}$

(v) $\displaystyle\lim_{j\to-\infty} V_j = cl\Big(\bigcup_{j\in\mathbb{Z}} V_j\Big) = L^2(\mathbb{R})$ (density)

"closure" = adherence

(vi) There exist a function $\phi \in V_0$ (called scaling function) such that $\{\phi(t-k)\}_{k \in \mathbb{Z}}$ is an orthonormal basis of V_0

Remarks / interpretations

(i) V_j invariant by translation proportional to the scale 2^j (grid of scale given by 2^j, leading to an approximation at the resolution 2^{-j})

(ii) Causality : the approximation at resolution 2^{-j} (i.e, at scale 2^j) contains all the necessary information to compute the approximation at a coarser resolution 2^{-j-1} (scale 2^{j+1}) → plus grossier

(iii) Dilating functions on V_j by a factor 2 dilates the details by a factor 2 and (iii) ensures these definition approximation at the worser resolution 2^{-j-1}

(iv) As the resolution $2^{-j} \to 0$ all the details on f are lost: $\lim_{j \to +\infty} \| P_{V_j} f \| = 0$

(v) As $2^{-j} \to +\infty$, the approximation $P_{V_j} f$ converges to f: $\lim_{j \to -\infty} \| P_{V_j} f - f \| = 0$

(vi) The existence of a basis for V_0 gives a discretization theorem: allowing numerical computation

For any j, the function $\{ 2^{-j/2} \phi(2^{-j} t - k) \}_{k \in \mathbb{Z}}$ form an orthonormal bases of V_j

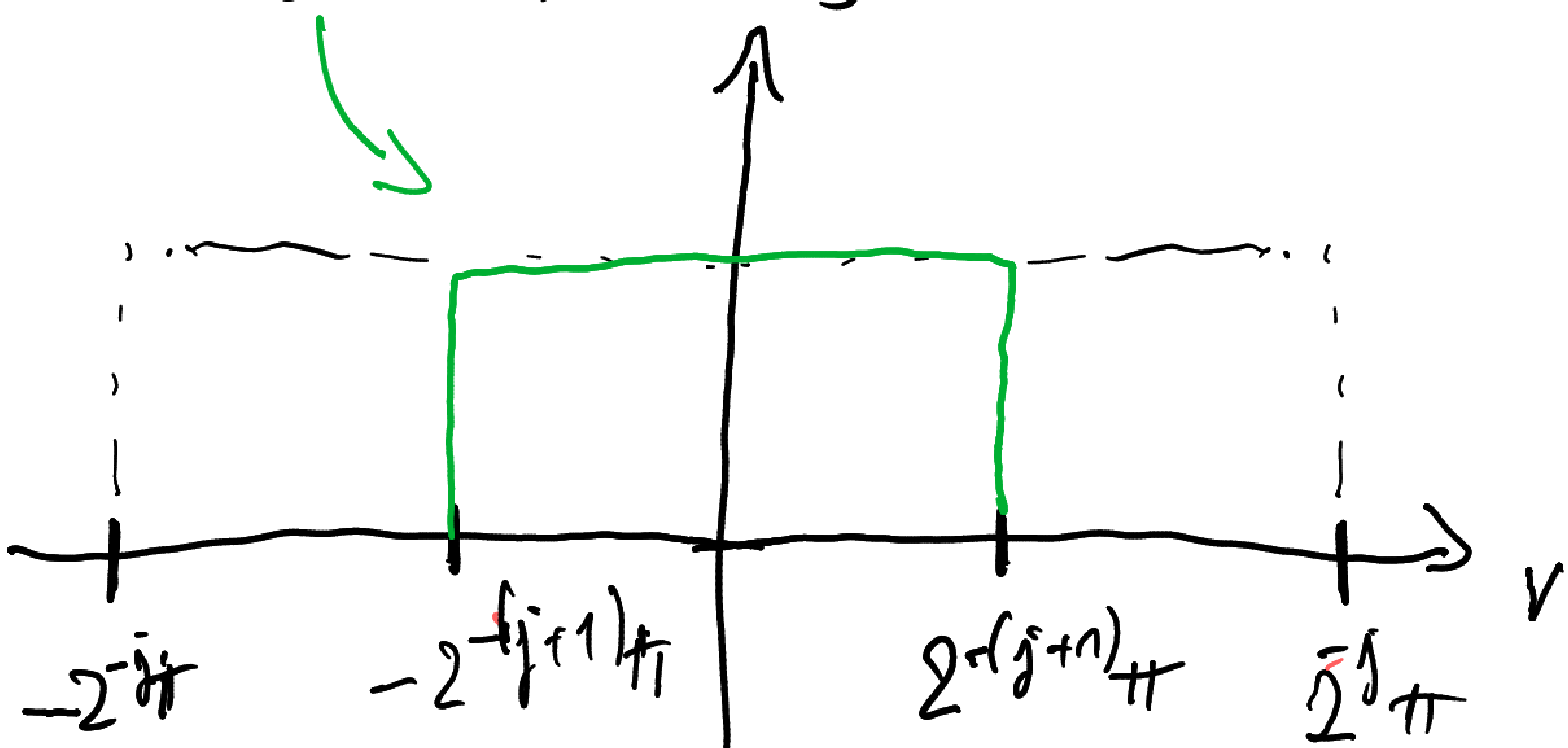
Example

1) $V_j = \{ f \in L^2(\mathbb{R}),\ f \text{ is constant on } [k2^{-j}, (k+1)2^{-j}[\}$

is a multi-resolution analysis for $\phi(t) = 1_{[0,1[}(t)$

2) $\phi(t) = \text{sinc}(\pi t)$ leads to the MRA

$V_j = \{ f \in L^2(\mathbb{R}) : f \text{ is a band-limited in } [-2^{-j}\pi, 2^{-j}\pi] \}$



$V_{j+1} \subset V_j$

By construction $V_j \subset V_{j-1}$ (V_j approximation space) One introduces the space of details W_j defined by : $V_{j-1} = V_j \overset{\perp}{\oplus} W_j$

Properties of W_j

(i) $\forall j \neq l \quad W_j \perp W_l$

Indeed $\begin{cases} j < l & W_l \subset V_{l-1} \subset \dots \subset V_j \\ W_j \perp V_j & \end{cases} \Rightarrow W_j \perp W_l \subset V_j$

(ii) $L^2(\mathbb{R})$ is the direct sum of the W_j's

$\Leftrightarrow \forall f \in L^2(\mathbb{R}) \qquad f = \sum_{j \in \mathbb{Z}} P_{W_j} f$

where P_{W_j} is the projector onto W_j

$\rightarrow$ One can build a function ψ such that (a wavelet)
$\{ \psi_{j,k}(t) = 2^{-j/2} \psi\left(\frac{t}{2^j} - k\right) \}_{k \in \mathbb{Z}}$ is an o.n.b. of W_j

Summary : $V_j \overset{\perp}{\oplus} W_j = V_{j-1}$

space of approximation — space of details

ϕ scaling function — ψ wavelet function

o.n.b on V_j — o.n.b on W_j

Theorem (orthogonal wavelet basis)

The family $\left\{ \psi_{j,k}(t) = 2^{-j/2} \psi\left(\frac{t}{2^j} - k\right) \right\}$

is an o.n.b of $L^2(\mathbb{R})$, and

$t \in \mathbb{R}$, $f(t) = \sum_{j,k} \underbrace{\langle f, \psi_{j,k} \rangle}_{d_{j,k}\text{ : detail coefficients}} \psi_{j,k}$

Stopping at scale index J

$$(t \in \mathbb{R}) \quad f(t) = \sum_k \underbrace{a_J[k]}_{\text{approximation at scale } J} \phi_{J,k}(t) + \sum_{j \leq J} \sum_k \underbrace{d_j[k]}_{\text{detail coefficient}} \psi_{j,k}(t)$$

Low-pass filter (see next section)

3 Wavelets and filter banks

Scale equations

$$V_1 \subset V_0 \Rightarrow \underbrace{\frac{1}{\sqrt{2}} \phi\left(\frac{t}{2}\right)}_{\in V_1} = \sum_{l \in \mathbb{Z}} h[l] \underbrace{\phi(t-l)}_{\text{o.n.b. of } V_0}$$

$$h[l] = \left\langle \frac{1}{\sqrt{2}} \phi\left(\frac{\cdot}{2}\right), \phi(\cdot - l) \right\rangle$$

Similarly $V_{j+1} \subset V_j$

$\forall k \in \mathbb{Z}$

$$\frac{1}{2^{(j+1)/2}} \phi\left(\frac{t}{2^{j+1}} - k\right) = \sum_{l \in \mathbb{Z}} h_{j,k}[l]\, 2^{-j/2} \phi\left(\frac{t}{2^j} - l\right)$$

with $h_{j,k}[l] = \left\langle \frac{1}{2^{\frac{j+1}{2}}} \phi\left(\frac{\cdot}{2^{j+1}} - k\right), 2^{-j/2} \phi\left(\frac{\cdot}{2^j} - l\right)\right\rangle$

$$= \frac{1}{2^{j+\frac{1}{2}}} \int_{\mathbb{R}} \phi\left(\frac{t}{2^{j+1}} - k\right) \phi^*\left(\frac{t}{2^j} - l\right) dt$$

$$\underset{u = \frac{t}{2^j} - 2k}{=} \frac{1}{2^{1/2}} \int_{\mathbb{R}} \phi\left(\frac{u}{2}\right) \phi^*(u + 2k - l)\, du$$

$$= h[l - 2k]$$

$\Rightarrow$ The coefficients $\underline{(h[l]})_{l \in \mathbb{Z}}$ are interpreted as the impulse response of a discrete filter, with frequency response

$$\hat{h}(\nu) = \sum_{l \in \mathbb{Z}} h[l] e^{-2i\pi\nu l}$$

(1-periodic function) with

$$\begin{cases} \hat{h}(0) = \sqrt{2} \\ \forall \nu \in [0, 1[, \ |\hat{h}(\nu)|^2 + |\hat{h}(\nu + \frac{1}{2})|^2 = 2 \end{cases}$$

h is called a conjugate mirror filter

Doing the same computation with the space of details

$$W_{j+1} \subset V_j \Rightarrow \forall k \in \mathbb{Z}$$

$$\frac{1}{2^{\frac{j+1}{2}}} \psi\left(\frac{t}{2^{j+1}} - k\right) = \sum_{l \in \mathbb{Z}} g[l-2k] \frac{1}{2^{\frac{j}{2}}} \phi\left(\frac{t}{2^j} - l\right)$$

Similarly: g is interpreted as the impulse response of a filter whose Fourier transform $\hat{g}$ verifies:

$$\begin{cases} (i)\ \hat{g}(0)=0 \\ (ii)\ \forall \nu \in [0,1[,\ |\hat{g}(\nu)|^2+|\hat{g}(\nu+\frac{1}{2})|^2=2 \end{cases}$$

- For j fixed, the orthogonality of $\phi_{j,k}$ and $\psi_{j,k}$ implies that: (*)

$$\forall \nu \in [0,1[,\ \hat{h}(\nu)\hat{g}^*(\nu)+\hat{h}(\nu+\tfrac{1}{2})\hat{g}^*(\nu+\tfrac{1}{2})=0$$

$\Rightarrow$ combined with the properties of h and g this suggest that

$$\begin{cases} \underline{h \text{ is a low-pass filter}} \rightarrow \underline{\text{associated with } \phi} \text{ scaling fct} \\ \underline{g \text{ is a high-pass filter}} \rightarrow \underline{\text{associated with } \psi} \text{ wavelet fct} \end{cases}$$

From ⊛ $\forall \nu \in \mathbb{R}$, $\hat{\psi}(\nu) = \frac{1}{\sqrt{2}} \hat{g}\left(\frac{\nu}{2}\right) \hat{\phi}\left(\frac{\nu}{2}\right)$,

with $\hat{g}(\nu) = e^{-2i\pi\nu} \hat{h}^*\left(\nu + \frac{1}{2}\right)$

→ choosing $\phi \to h \to \overset{\psi}{g}$

→ The wavelet ψ (and the filter g associated to it) are directly deduced from the scaling function ϕ

4. À-trous algorithm from Mallat

a) Decomposition = analysis

Let $x \in L^2(\mathbb{R})$. At scale $j \in \mathbb{Z}$

→ $P_{V_j} x(t) = \sum_k a_j[k] \phi_{j,k}(t)$

→ $P_{W_j} x(t) = \sum_k d_j[k] \psi_{j,k}(t)$

As a result from 3:

$$\underbrace{a_j[k] = \sum_{l \in \mathbb{Z}} a_{j-1}[l]\, h[l-2k]}_{\text{convolution}}$$

$$\underbrace{d_j[k] = \sum_{l \in \mathbb{Z}} a_{j-1}[l]\, g[l-2k]}_{\text{convolution}}$$

- decimation by factor 2

$h \rightarrow$ low pass filter
$g \rightarrow$ high pass filter

$\rightsquigarrow$ from the approximation and details coefficient at scale $j-1$, get approximation coeffs at scale j

$$V_j \subset V_{j-1} \Rightarrow \phi_{j,p} = \sum_m \langle \phi_{j,p}, \phi_{j-1,m} \rangle \phi_{j-1,m}$$

$$\Rightarrow a_j[k] = \langle x, \phi_{j,k} \rangle$$

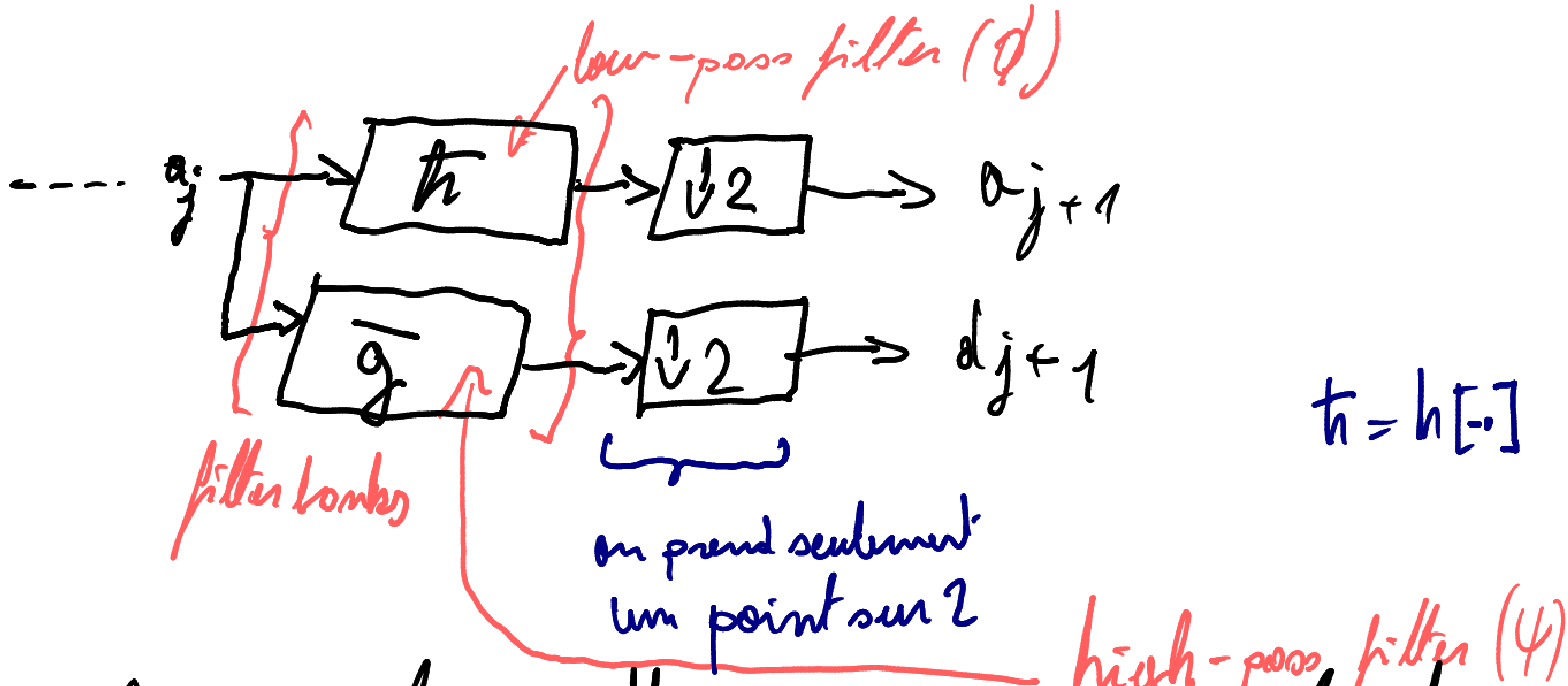
$$= \langle x, \sum_l h[l-2k]\, \phi_{j-1,k} \rangle$$

$$= \sum_l h[l-2k] \langle x, \phi_{j-1,k} \rangle$$

$$= \sum_l h[l-2k]\, a_{j-1}[l]$$

$$= (a_{j-1} * \bar{h})[2k]$$

with $\bar{h} = h[-\cdot]$



$\bar{h} = h[-\cdot]$

⇝ Approximation coeffs a_j are progressively refined (in the sense details are progressively extracted from it)

⇝ Hierarchical description of the signal across scales

f) Reconstruction (synthesis) [inverse discrete wavelet transform]

Since $V_j = V_{j+1} \oplus W_{j+1}$, $\phi_{j,p}$ can be decomposed onto $\{\phi_{j+1,k}\}_k$ and $\{\psi_{j+1,k}\}_k$

$$\phi_{j,p}(t) = \underbrace{\sum_k \langle \phi_{j,p}, \phi_{j+1,k} \rangle \phi_{j+1,k}(t)}_{P_{V_{j+1}} \phi_{j,p}(t)}$$

$$+ \underbrace{\sum_k \langle \phi_{j,p}, \psi_{j+1,k} \rangle \psi_{j+1,k}(t)}_{P_{W_{j+1}} \phi_{j,p}(t)}$$

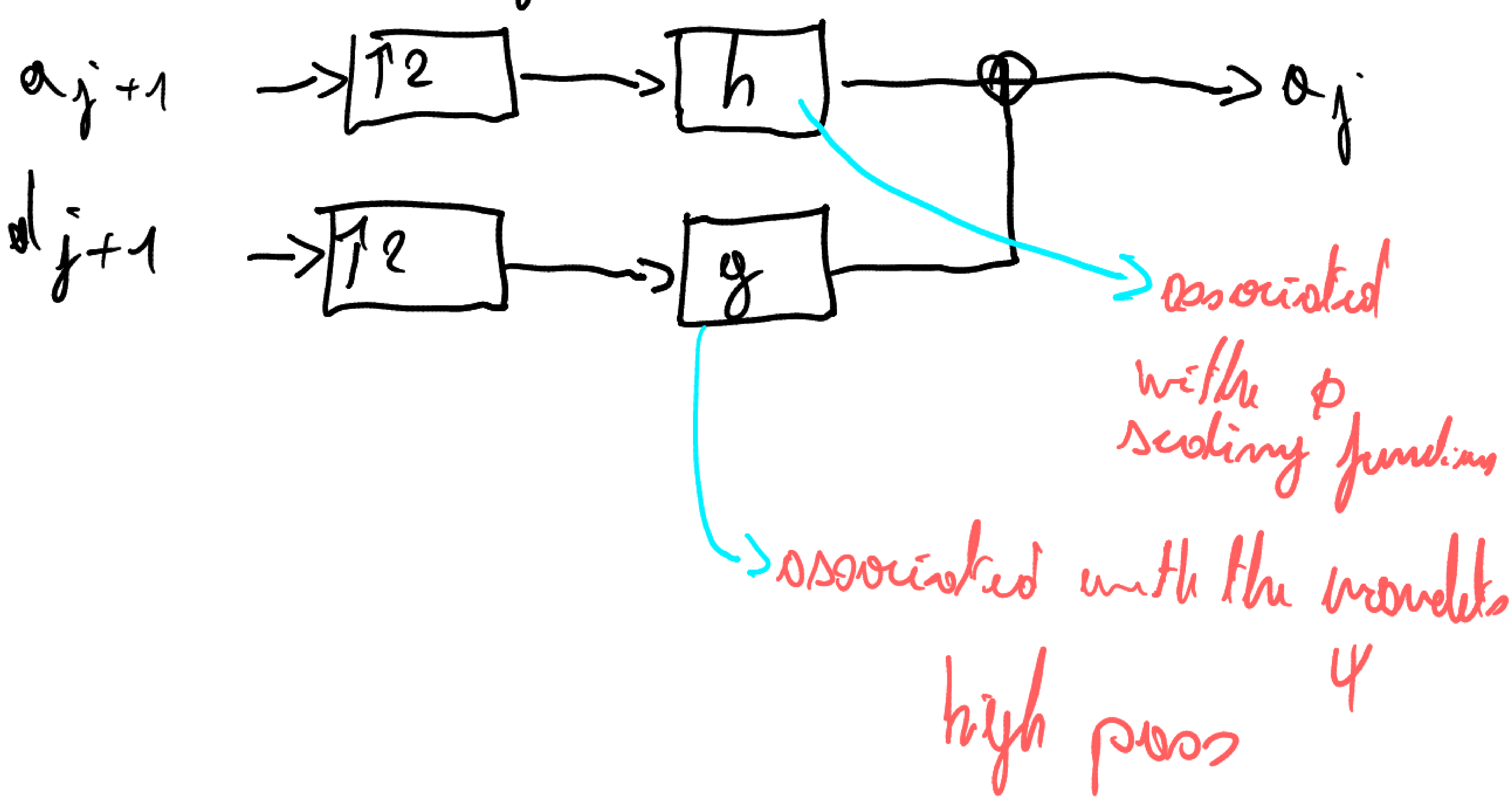
$$= \sum_k h[p-2k]\, \phi_{j+1,k}(t) + \sum_k g[p-2k]\, \psi_{j+1,k}(t)$$

To reconstruct the signal, one needs to up-sample by a factor 2 (insert 0s) then apply some filter to the results and combine the resulting contributions (ie up-sampled and filtered versions of both details and approximation coeffs)

$$a_j[p] = \langle x, \phi_{j,p} \rangle = \sum_k h[p-2k] \underbrace{\langle x, \phi_{j+1,k} \rangle}_{a_{j+1}[k]}$$

$$+ \sum_k g[p-2k] \underbrace{\langle x, \psi_{j+1,k} \rangle}_{d_{j+1}[k]}$$

Similarity to a), the reconstruction algorithm can be described by



5) Properties of wavelets

Proposition

If an MRA is r-regular

(ie, for $\phi \in \mathcal{C}^r$, and $\phi, \phi', \dots \phi^{(r)}$ have a fast decay

$\forall l \in [\![0,r]\!]\ \forall m \in \mathbb{N}, \exists c > 0$ such that $|\phi^{(l)}(x)| \leq \frac{c}{(1+|x|)^m}$)

Then ψ is a class $\mathcal{C}^r$ and admits r null moments:

$$\forall l \in \{0, \dots, r\}, \int_{\mathbb{R}} x^l \psi(x)\, dx = 0$$

$$\langle x^l, \psi \rangle = 0$$

The decomposition focus on less regular term

Remark/interpretations

ψ is orthogonal to any polynomial of order smaller or equal to $n-1$

$\Rightarrow$ appropriate tool to detect singularities (irregular contributions to the signal)

(i.e only singularities have a non-zero contribution in the decomposition)

⚠ Limitation **Theorem (Daubechies)**

Let ψ a wavelet and ϕ a scaling function that generate an orthogonal basis. If ψ has n null moments, then its support is of size $\geq 2n-1$

$\rightsquigarrow$ compromise between singularity detection and accurate description of time-instants at which singularities occur

6. A trous algorithm in 2D

MRA analysis in 2D is defined using 1D MRA

The following functions will be required

$$\phi(x) = \phi_1(x_1)\phi_2(x_2) \rightsquigarrow \text{approximation coeffs}$$

space coordinate $x=(x_1,x_2)$

vertical details $\rightsquigarrow$ $\Psi^{(1)}(x) = \phi_1(x_1)\psi_2(x_2)$

horizontal details $\rightsquigarrow$ $\Psi^{(2)}(x) = \psi_1(x_1)\phi_2(x_2)$

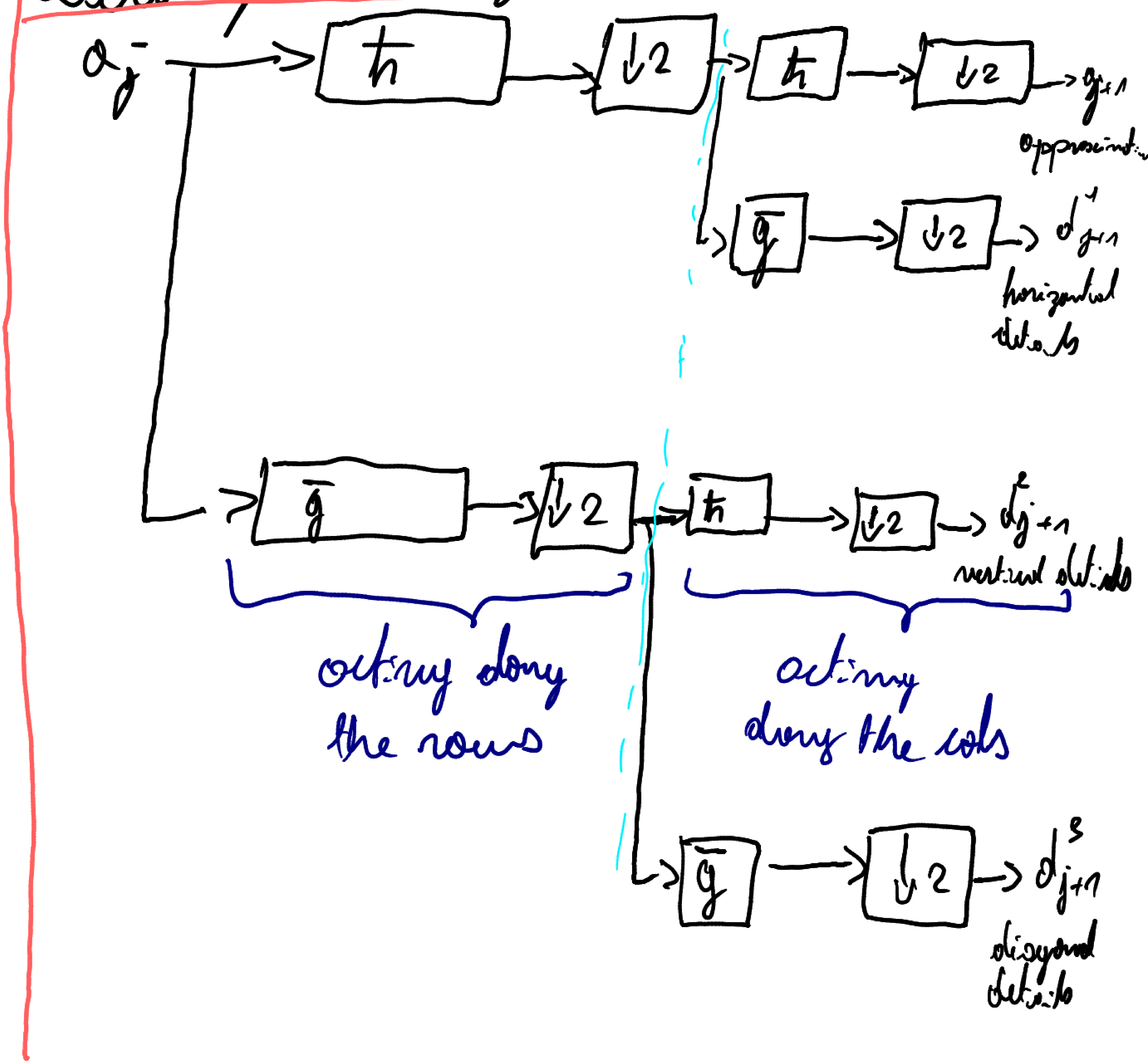
diagonal details $\rightsquigarrow$ $\Psi^{(3)}(x) = \psi_1(x_1)\psi_2(x_2)$

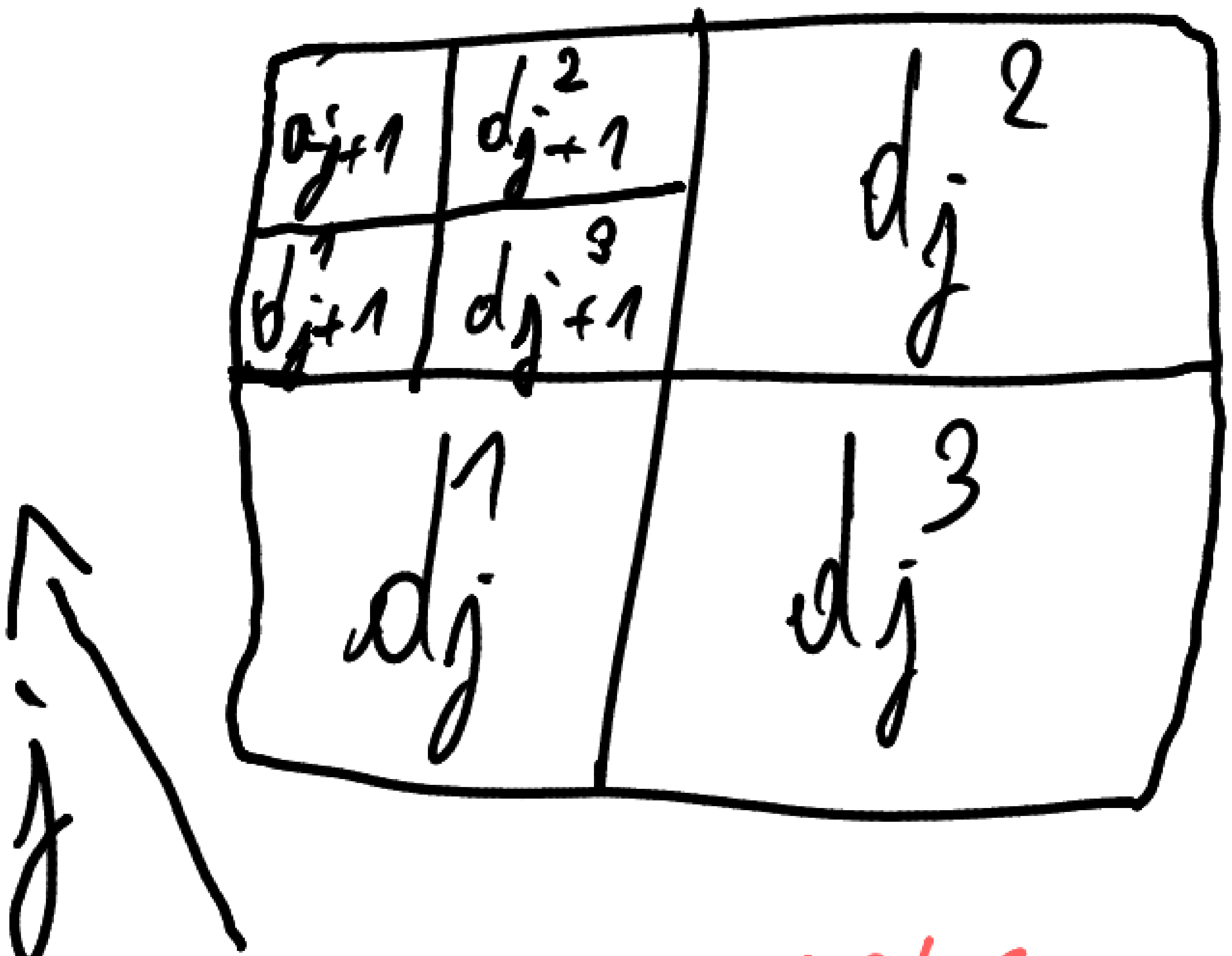
$$x = \sum_{k\in\mathbb{Z}^2} \langle x, \Phi_{J,k}\rangle \Phi_{J,k}$$

$$+ \sum_{i=1}^{3}\sum_{j=1}^{J}\sum_{k\in\mathbb{Z}^2} \langle x, \Psi^{(i)}_{j,k}\rangle \Psi^{(i)}_{j,k}$$

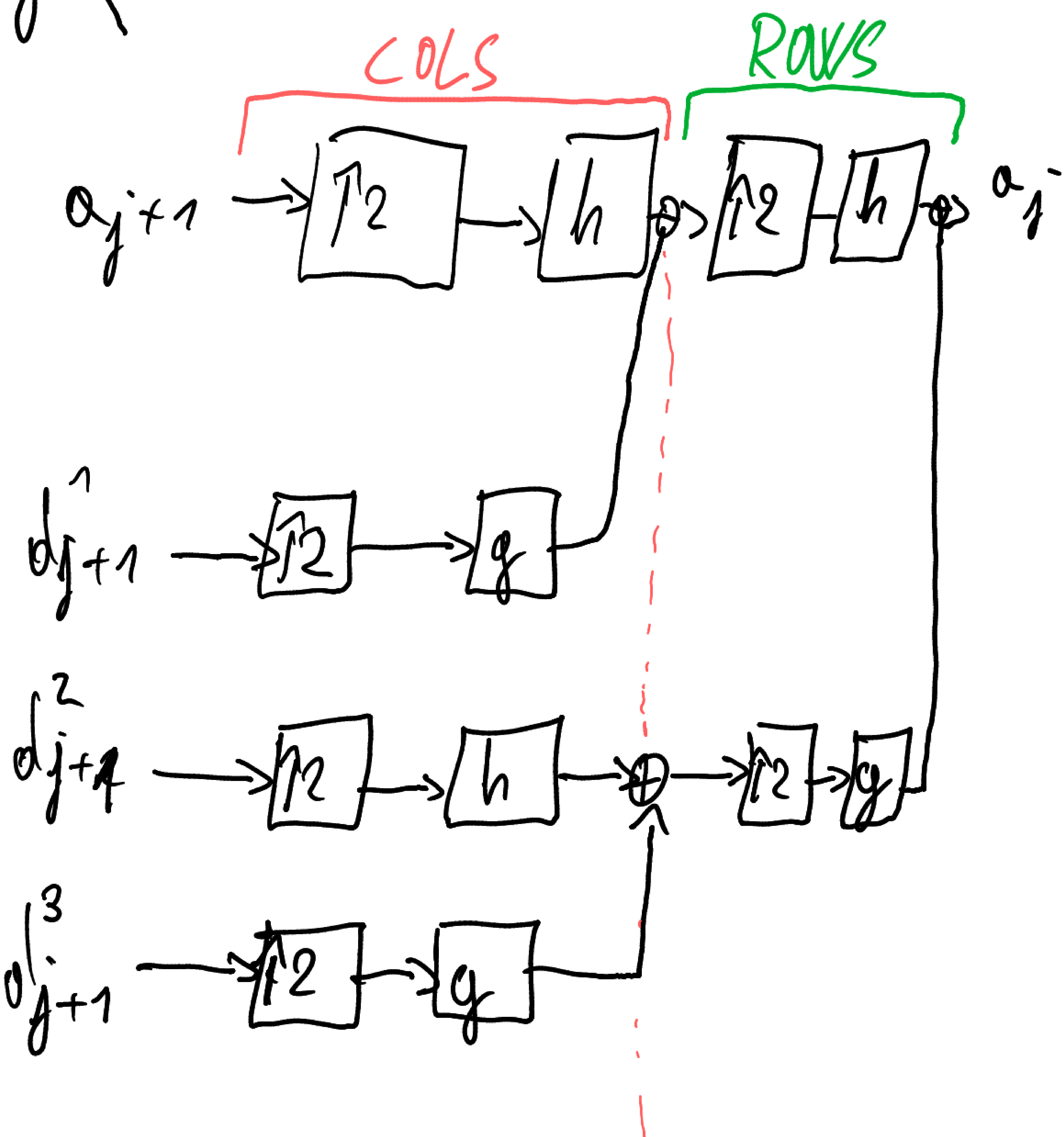
Filters applied successively to the rows and columns of the image considered

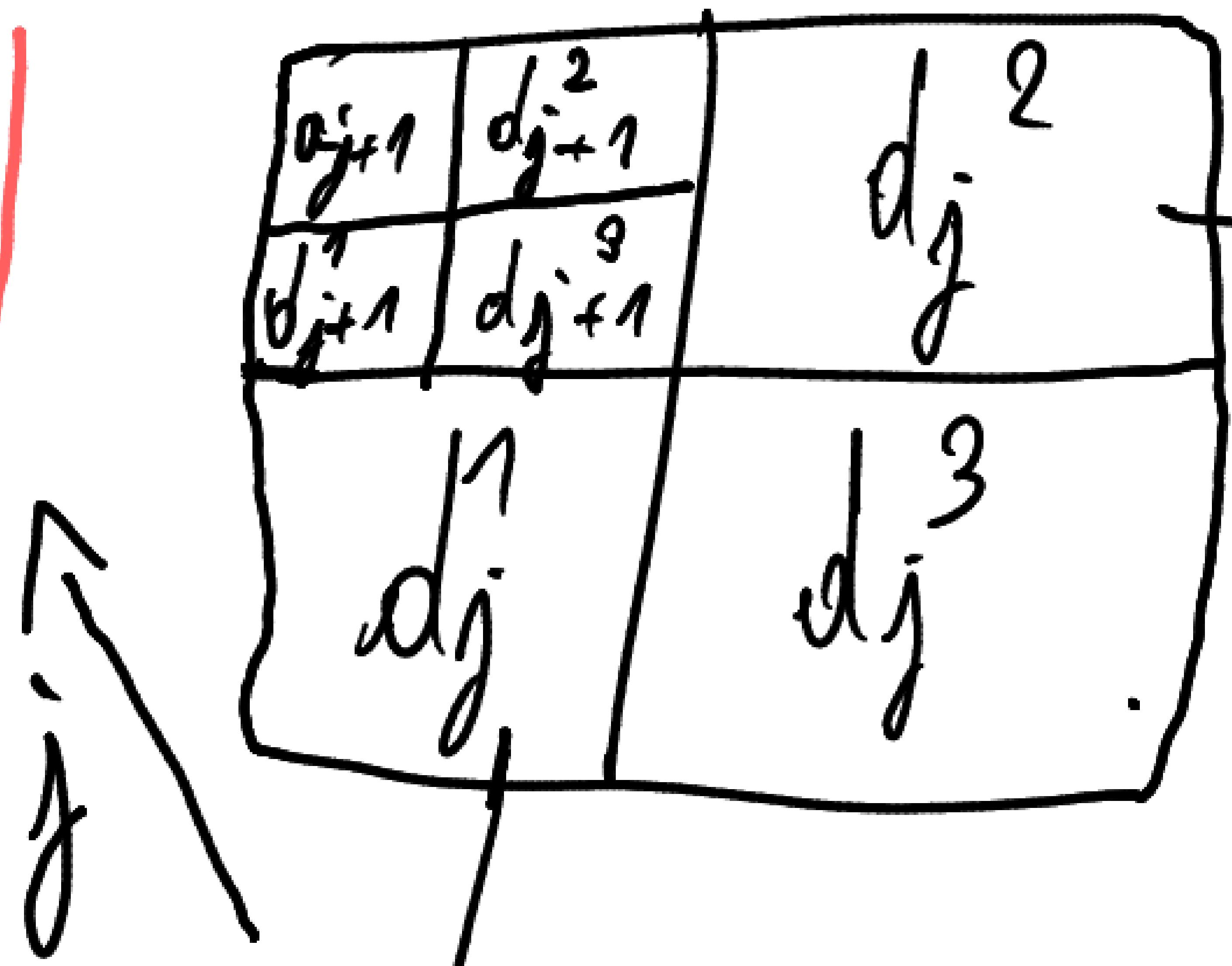
Decomposition algorithm





(2D representation of wavelet coefficients)





horizontal
high frequencies
(vertical edges)

vertical
high frequencies
(horizontal edges)