Hermitian

$\forall \nu \in \mathbb{R} \qquad X(-\nu) = X^*(\nu)$

3) a)

b) Example in an Euclidian space: the DFT

The space of discrete signals of period N, denoted H in this section, is a Euclidean space of dimension N, with inner product

$$\forall f, g \in H, \langle f, g \rangle = \sum_{n=0}^{N-1} f[n] g^*[n]$$

<u>Theorem</u>: The family $\left\{ e_k[n] = e^{\left(\frac{2i\pi nk}{N}\right)} \right\}_{0 \leq k \leq N-1}$ of N vectors is an orthogonal family basis of H

Since H is of dim N, any orthogonal family of N vectors is an orthogonal basis implying

$$f = \sum_{k=0}^{N-1} \frac{\langle f, e_k \rangle}{\| e_k \|_2^2} e_k \qquad *$$

By definition of the DFT $\hat{f}[k] = \langle f, e_k \rangle$

Since $\| e_k \|_2^2 = N$, $*$ directly gives the expression for the iDFT:

$$\forall n \in [\![0, N-1]\!] , \quad f[n] = \frac{1}{N} \sum_{k=0}^{N-1} \hat{f}[k] \, e^{\frac{2i\pi k n}{N}}$$

$$\| f \|_2^2 = \sum_{n=0}^{N-1} | f[n] |^2 = \sum_{k=0}^{N-1} | \hat{f}[k] |^2 = \| \hat{f} \|_2^2$$

(Parseval formula in H)

4) Refreshers: analysis in Hilbert spaces

a) Norm and convergence

Def (norm): Let H be a complex vector space

A norm $\|\cdot\| : H \to \mathbb{R}_+$ is a function such that

$\forall f \in H, g \in H, \lambda \in \mathbb{C}$

(i) $\|f\| \geq 0$

(ii) $\|f\| = 0 \Rightarrow f = 0$

(iii) $\|\lambda f\| = |\lambda| \|f\|$

(iv) $\|f+g\| \leq \|f\| + \|g\|$

Examples usual norm on $L^1(\mathbb{R})$, $\ell^1(\mathbb{Z})$

on $\mathbb{N}$

$$\|f\|_1 = \int_{\mathbb{R}} |f|$$

$$\sum_{n \in \mathbb{Z}} |f[n]|$$

Def (convergence in H):

Let H be a metric space, with norm $\|\cdot\|$.
The sequence $(f_n)_{n\in\mathbb{N}} \in H^{\mathbb{N}}$ is said to converge to $f \in H$ iff $\|f - f_n\| \underset{n\to+\infty}{\longrightarrow} 0$

in this case $f = \lim\limits_{n\to+\infty} f_n$

Def (complete metric space = Banach space)

A metric space H is said to be complete iff any Cauchy sequence of elements in H converge in H

$(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence iff

$$\forall \varepsilon>0, \exists N \in \mathbb{N} \quad \left.\begin{matrix}\forall n \geq N \\ \forall p \geq N\end{matrix}\right\} \Rightarrow \|f_n - f_p\| < \varepsilon$$

b) Inner product and Hilbert spaces

Def (inner product) voir cours prepa $\langle \cdot , \cdot \rangle$

Def (Hilbert space) A Hilbert space H is a Banach space endowed with an inner product (Distance usually considered : $\|f\| = \langle f, f \rangle^{1/2}$)

c) Orthonormal bases of an Hilbert space

Definition . A family $\{e_n\}_{n \in \mathbb{N}}$ of elements of a Hilbert space H is orthogonal iff (by def)

$\langle e_n, e_k \rangle = 0$ when $n \neq k$

If, for any $f \in H$ there exists a sequence $(a_n)_{n \in \mathbb{N}}$ of elements in $\mathbb{C}$ such that

$$\lim_{N \to +\infty} \left\| f - \sum_{n=0}^{N-1} a_n e_n \right\| = 0$$

(generative family of functions)

then the orthogonal family $\{e_k\}_{k \in \mathbb{N}}$ is said to be an orthogonal bases of H

The orthogonality implies $a = \frac{\langle f, e_n \rangle}{\|e_n\|}$

hence $f = \sum_{n=0}^{+\infty} \frac{\langle f, e_n \rangle}{\|e_n\|^2} e_n$

A Hilbert space which admits an orthogonal basis is said to be separable

$\{e_n\}_{n \in \mathbb{N}}$ is said to be orthonormal if $\|e_n\| = 1$

Theorem (Parseval - Plancherel):

Let H be a separable Hilbert space and $\{e_n\}_{n \in \mathbb{N}}$ be an orthonormal basis

Then, for any $f, g \in H$

(Plancherel) $\langle f, g \rangle = \sum_{n=0}^{+\infty} \langle f, e_n \rangle \langle g, e_n \rangle^*$

(Parseval) $\|f\|^2 = \sum_{n=0}^{+\infty} |\langle f, e_n \rangle|^2$

$\Rightarrow$ "Energy conservation"

Example: $\ell^2(\mathbb{Z})$ is separable and $\{e_n[k] = \delta[n-k]\}_{n\in\mathbb{Z}}$ is an orthogonal basis of $\ell^2(\mathbb{Z})$

Example 2: $L^2(\mathbb{R})$ is separable $\to$ next chapter

Chapter 2 (part 1) Beyond the Fourier representation when time meets frequency

Observation: listening to song, one perceives.
the succession (temporal variations) of notes
(characterized by a fundamental frequency harmonics depending on the music instrument with which the note was played

⟹ Fourier analysis gives no information about the order in which the notes are played, but precisely indicates all the frequency components activated

Fourier gives a perfectly localized info in frequency

Objective: describe a signal simultaneously in both the time and the frequency domain

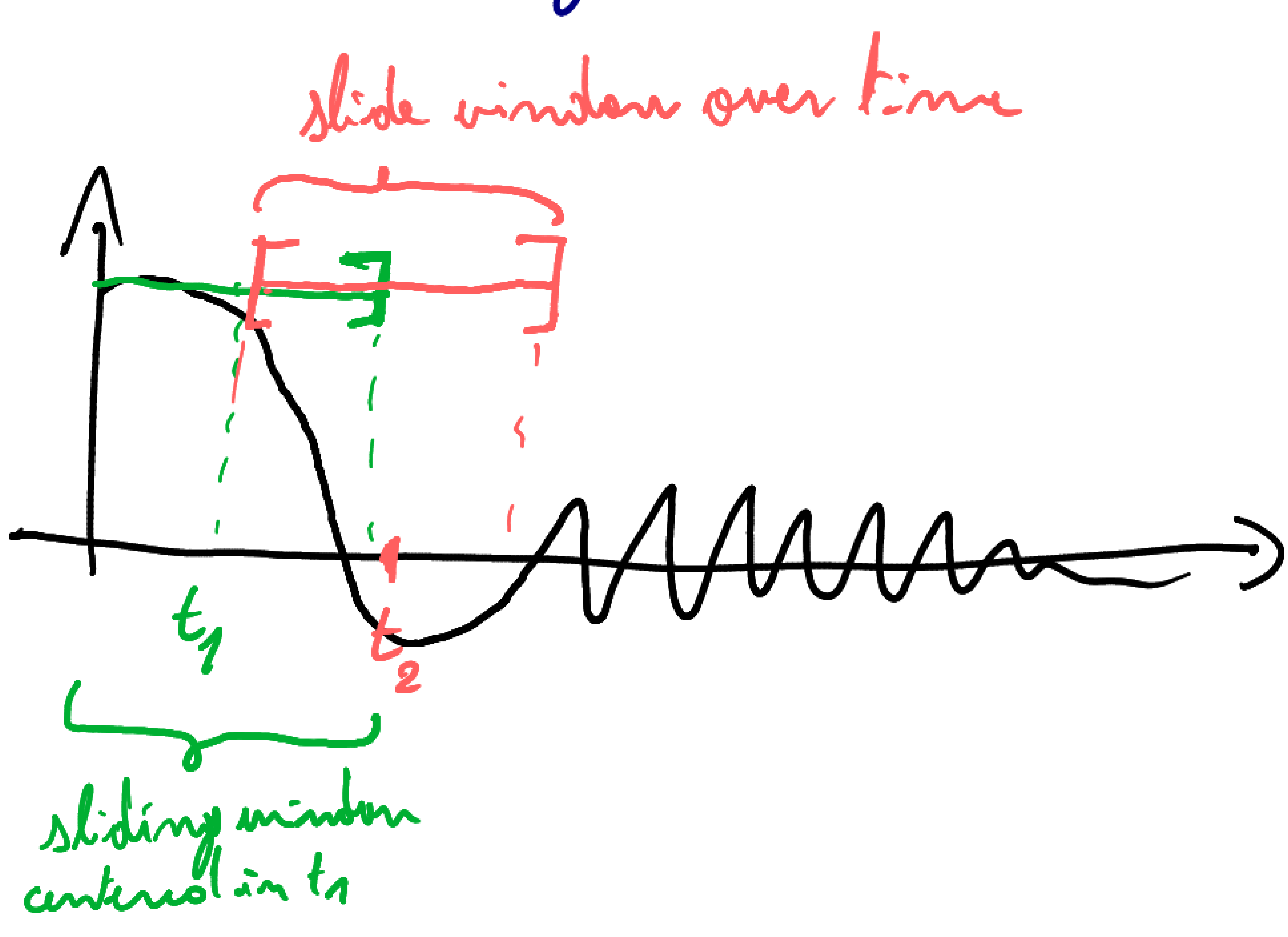
⟶ It will rely on Fourier

Idea: decompose the signal onto family of functions whose elements are well localized both in time and in frequency

⚠ "Function well localized in time"
the energy of the function is well localized in time
→ Compromise to be found due to the limitation inherent to Fourier described by the Heisenberg-Gabor theorem
→ Divide the time-frequency depending on the target application

1) Introduction to time-frequency analysis

A "local" spectral analysis of a signal can be obtain with a sliding window

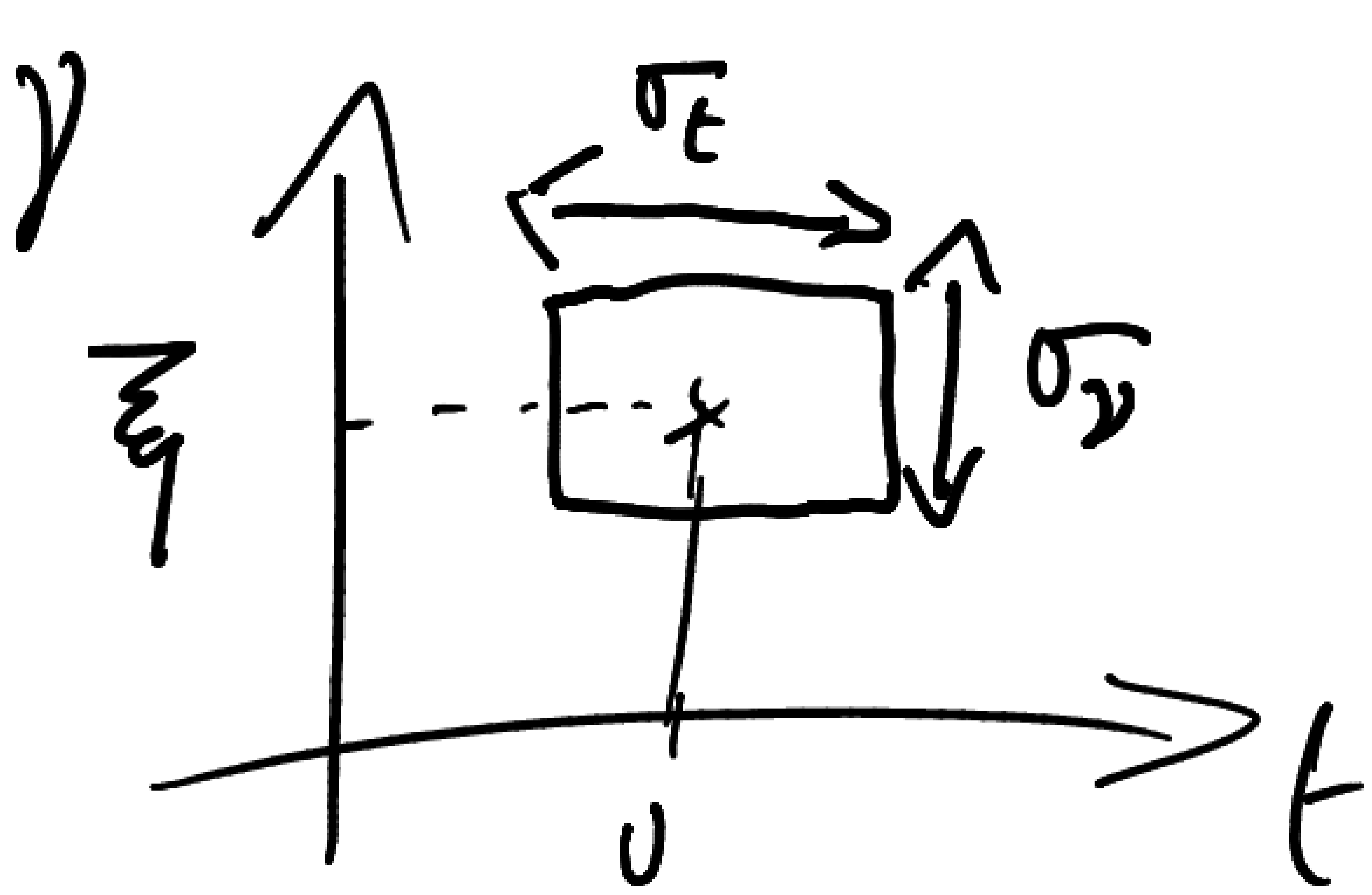


→ Decompose the signal over a basis of time-frequency atom, whose energy is well localized

- Heisenberg uncertainty principle : implies that a signal cannot be perfectly both in time and in frequency

⇒ time-frequency boxes, whose size depends on the time/frequency uncertainity of the observations

→ Heisenberg boxes of surface $\sigma_t \sigma_\nu$



A first approach towards a time-frequency representation: starting from Fourier, re-localize the information in time and in frequency using Fourier transforms on a time-windowed version of the signal, the window being progressively translated over time.

$\Rightarrow$ principle of the short-time Fourier transform (STFT)

2) Time-frequency atoms

Let $(\phi_\gamma)_{\gamma\in\Gamma}$ be a family of functions (dictionary) called atoms, such that:

$$\phi_\gamma \in L^2(\mathbb{R}),\ \|\phi_\gamma\| \text{ with FT } \hat{\phi}_\gamma .$$

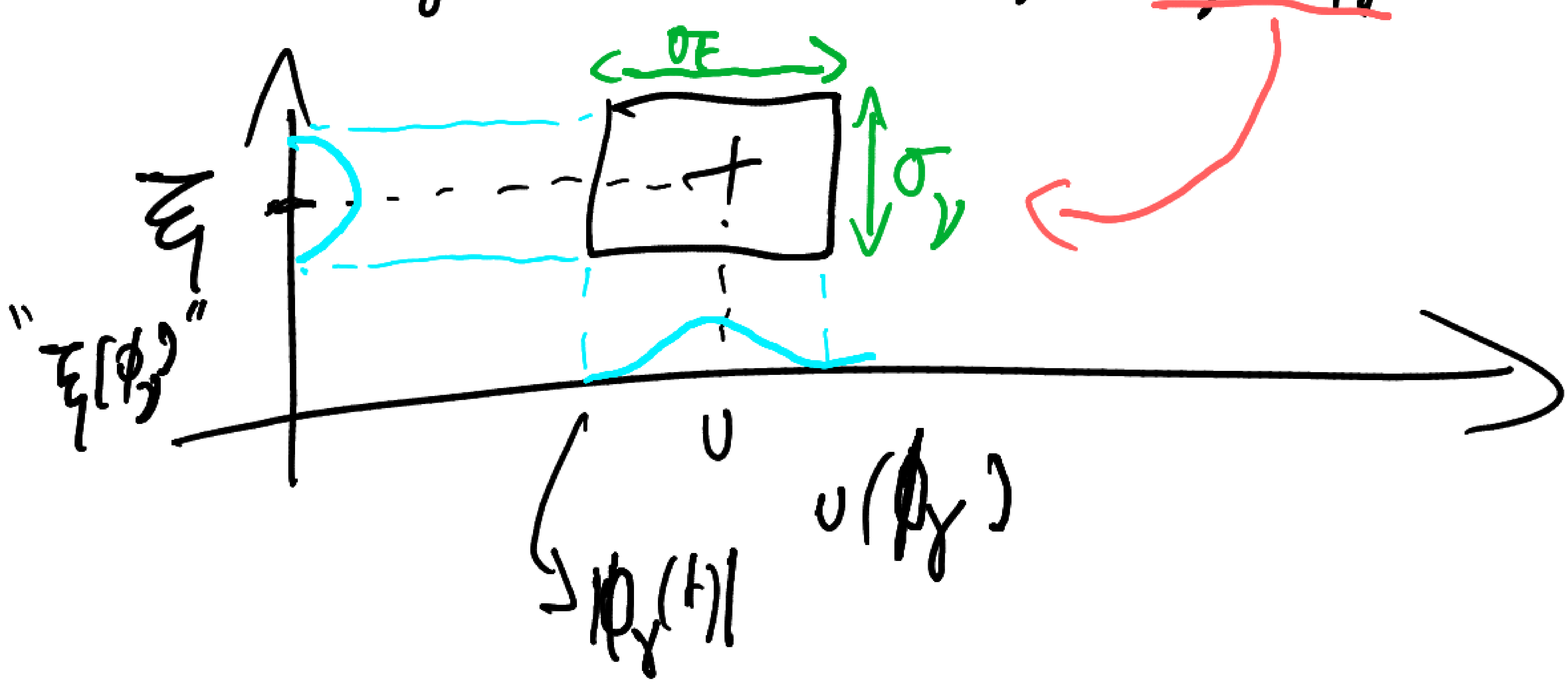
The corresponding time frequency representation in application of chap 1.3, is given by the following coefficients

$$c_f(\gamma) = \langle f, \phi_\gamma \rangle = \int_{\mathbb{R}} f \, \phi_\gamma^*$$

$$= \int_{\mathbb{R}} \hat{f} \, \hat{\phi}_\gamma^*$$

Then $c_f(\gamma)$ gives a well localized info (i.e. concentrated in time and freq)

iff $\begin{cases} \phi_\gamma \text{ is well localized } (\sigma_t(\phi_\gamma) \text{ "small"}) \\ \hat{\phi}_\gamma \text{ is well localized } (\sigma_\nu(\phi_\gamma) \text{ "small"}) \end{cases}$



3) The short-time Fourier transform (STFT)

a) Context and definition

In 1946, windowed Fourier transforms introduced by Gabor to describe temporal variations of the frequency content of sounds

Definition (family of time-freq atoms, STFT)

Consider the family of functions $(g_{u,\xi})_{(u,\xi)\in\mathbb{R}^2}$ generated from time window g (ie $\|g\|_2 = 1$, g real-valued, symmetric, non negative)

↳ centred

as follows

$$\forall (u,\xi)\in\mathbb{R}^2, \forall t\in\mathbb{R}, g_{u,\xi}(t) = \underbrace{g(t-u)}_{\text{translation in time } g}\ \underbrace{e^{2i\pi\xi t}}_{\text{translation in freq } g}$$

$$x(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-2i\pi t_0 \nu}\, \hat{x}(\nu)$$

$$e^{2i\pi \nu_0 t} x(t) \longleftrightarrow \hat{x}(\nu - \nu_0)$$

Given $(g_{u,\xi})_{(u,\xi)\in\mathbb{R}^2}$ a family of time-freq atoms, the STFT of $f \in L^2(\mathbb{R})$ is given by

$$\forall (u,\xi) \in \mathbb{R}^2, \quad Sf(u,\xi) = \langle f, g_{u,\xi} \rangle$$

$$= \int_{\mathbb{R}} f(t)\, g(t-u) e^{-2i\pi \xi t}$$

Definition (spectrogram): The spectrogram defines the density of energy corresponding to the STFT time-frequency representation

$$\forall (u,\xi)\in\mathbb{R}^2, \quad P_S f(u,\xi) = |Sf(u,\xi)|^2$$

→ quantifies the of f in the neighbourhood time-frequency point (u,ξ). The size of the neighbourhood is specified by the Heisenberg box associated with $g_{u,\xi}$

Remark: the expression "short-term" in the name of STFT comes from the fact that $g(\cdot - u)$ localises the Fourier integral around u

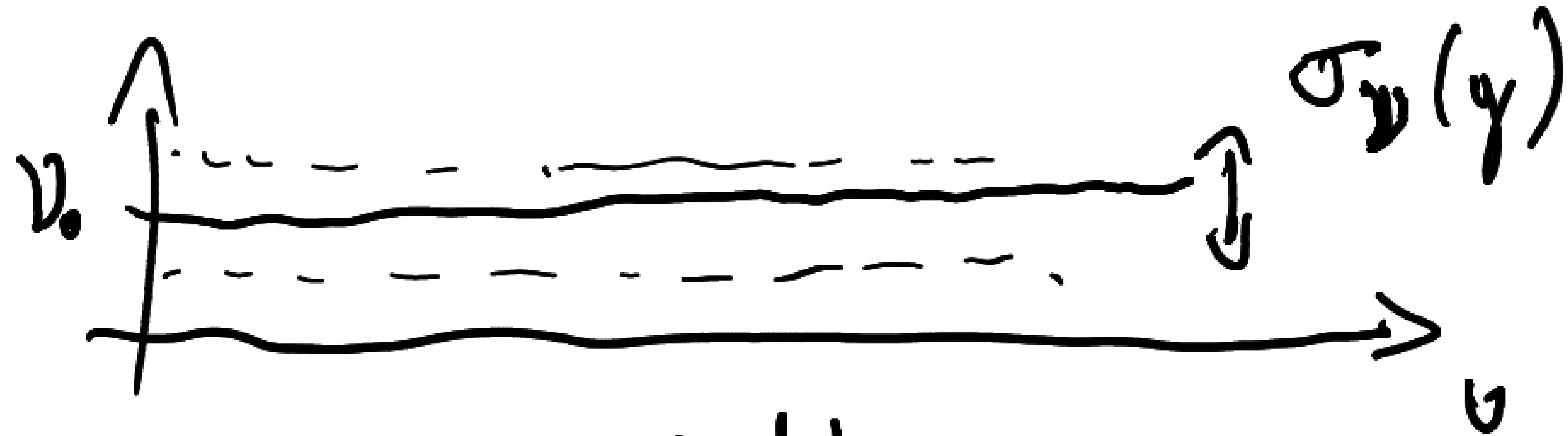
$$Sf(u,\xi) = \mathcal{F}[f g(\cdot - u)](\xi)$$

Example: (i) complex exponential

$$x(t) = e^{2i\pi \nu_0 t}$$

$$\hat{x}(\nu) = \delta(\nu - \nu_0)$$

$$S_x(u, \xi) = \hat{g}(\xi - \nu_0)\, e^{-2i\pi(\xi - \nu_0) u}$$



$\Rightarrow$ energy concentrated in the region of $\mathbb{R}^2$ given by $\xi \in \left[\nu_0 - \frac{\sigma_\nu(g)}{2}, \nu_0 + \frac{\sigma_\nu(g)}{2}\right]$

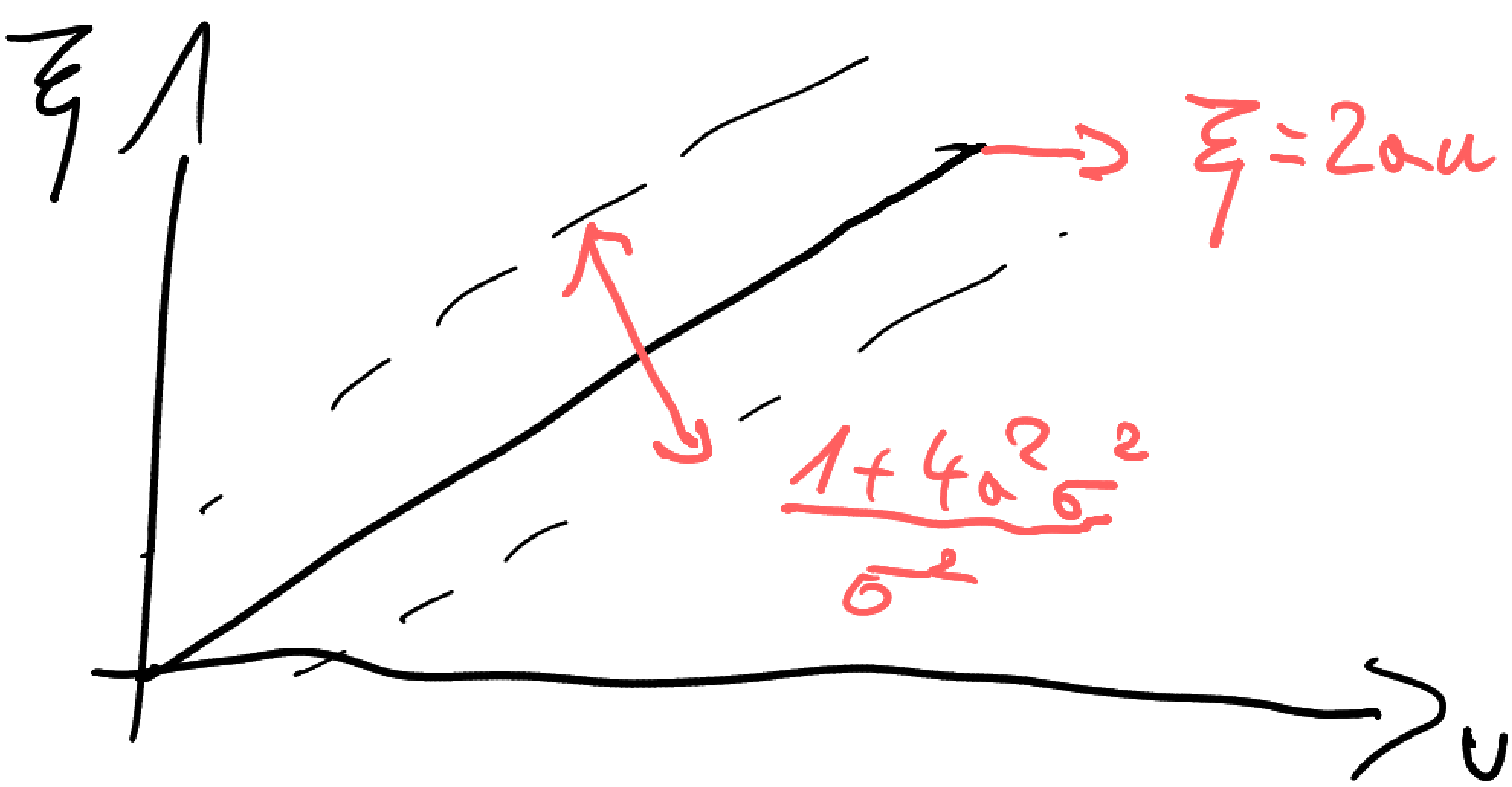
(ii) Linear chirp $x(t) = e^{at^2}$

Consider a Gaussian window $g(t) = \frac{1}{(\pi\sigma^2)^{1/2}} \exp\left(\frac{-t^2}{2\sigma^2}\right)$,

with $\underbrace{\hat{x}(\omega)}_{2\pi\nu} = \int_{\mathbb{R}} x(t) e^{-i\omega t} dt$, one can show

$$P_S f(u, \xi) \propto \exp\left(-\frac{\sigma^2(\xi - 2au)^2}{1 + 4a^2\sigma^2}\right)$$

max in $\xi = 2au$



b) Choice of the window g

· Optimal time - frequency localization

obtained when $\sigma_t(g)\sigma_\nu(g) = \frac{1}{4\pi}$

$\Leftrightarrow \exists a, b \in \mathbb{R}$ such that $g_{u,\xi}(t) = \underbrace{a e^{-b(t-u)^2}}_{g \text{ Gaussian window}} e^{2i\pi\xi t}$

→ such atoms are called Gabor time-frequency atoms

Playing with the scale of the window

dilate / shrink g:

$$\forall s \in \mathbb{R}_+^* \quad g_s(t) = \frac{1}{\sqrt{s}}\, g\left(\frac{t}{s}\right)$$

$$\xleftrightarrow{\mathcal{F}} \quad \hat{g}_s(\nu) = \sqrt{s}\,\hat{g}(s\nu)$$

$\sigma_t(g_s) = s\,\sigma_t(g)$, $\sigma_\nu(g_s) = \frac{\sigma_\nu(g)}{s}$

→ constant surface Heisenberg box through dilatation ($\sigma_t \uparrow$ when $\sigma_\nu \downarrow$) a compromise can be found between $\sigma_t(g_s)$ and $\sigma_\nu(g_s)$ by adjusting the scale s depending on the target application

Considerations about the choice of a window

→ numerical implementation

a) has the window a compact support in frequency/time ?

b) frequency spread and secondary lobes of g quantified in terms of

(i) $\Delta\nu(g)$, the root mean square bandwidth, that is, the width of the primary lobe of $\hat{g}$ at half-height
$$\frac{\left|\hat{g}\left(\frac{\Delta\nu(g)}{2}\right)\right|^2}{|\hat{g}(0)|^2} = \frac{1}{2}$$

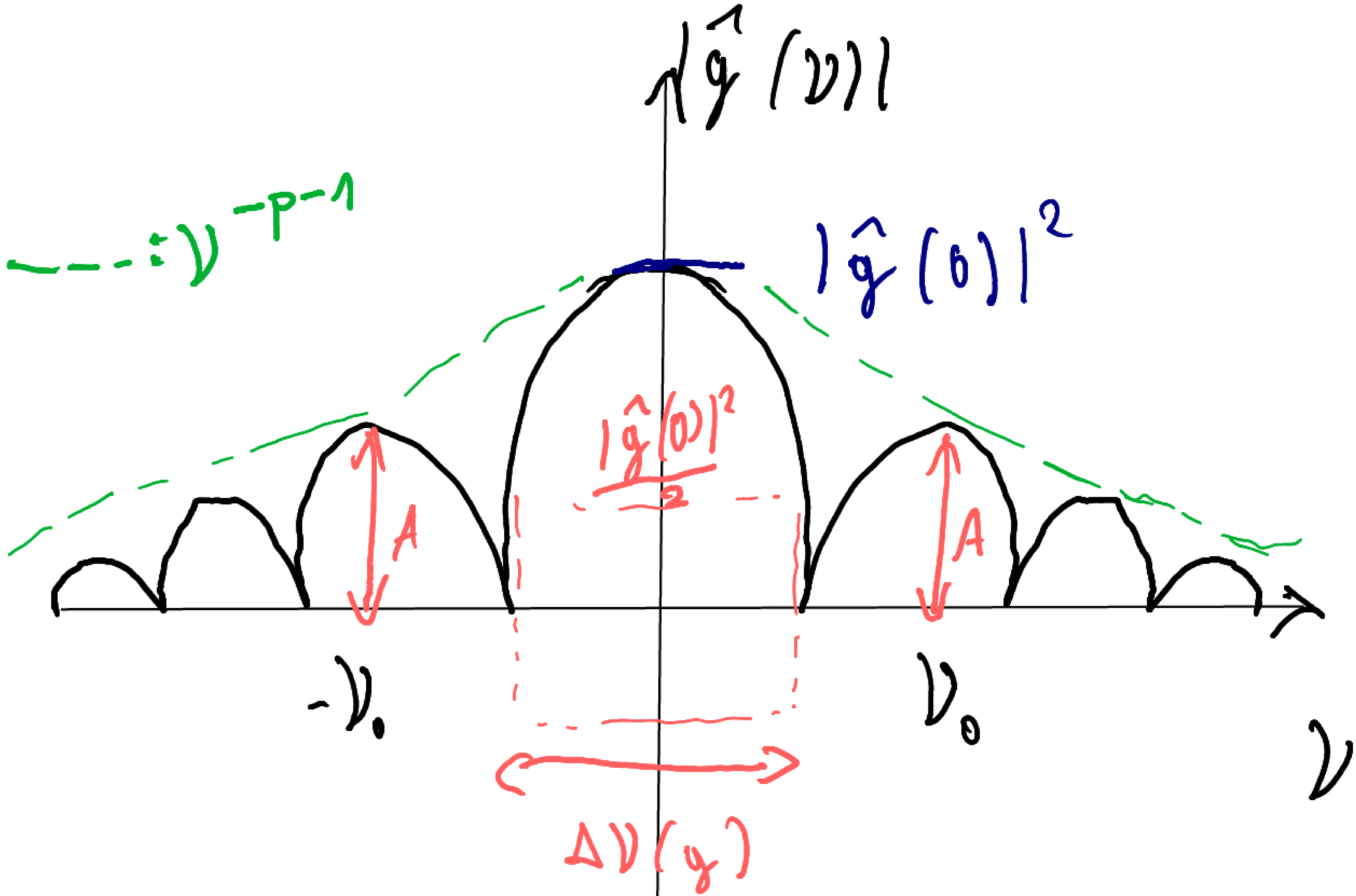
(ii) maximum amplitude of the first secondary lobe of $\hat{g}$, denoted by A, located in $\pm \nu_0$ expressed in dB (the lower the better)

$$A = 10 \log_{10}\left(\frac{|\hat{g}(\nu)|^2}{|\hat{g}(0)|^2}\right)$$

(iii) the polynomial exponent p which indicates the asymptotic speed of decay of $|\hat{g}|$

$$|\hat{g}(\nu)| = O(\nu^{-p-1})$$

(the higher the better)



(even since $g(t) \in \mathbb{R}$)

__Remark__ the rectangular window $\mathbb{1}_{[-T/2, T/2]}(t)$
has the thinnest primary but
has a hight amplitude for the first
secondary lobe, and a slow decay
for the amplitude of sec lobes

→ compromise between b) (i)-(ii) to be found

c) Energy conservation and inversion formula

Theorem (pas besoin de l'apprendre en détail)

Let $f \in L^2(\mathbb{R})$, and $g \in L^2(\mathbb{R})$ a time window. Then

(i) $$f(t) = \iint Sf(u,\xi)\, g(t-u)\, e^{2i\pi \xi t}\, d\xi\, du$$

(inversion formula for the STFT)

(ii) $$\int_{\mathbb{R}} |f(t)|^2\, dt = \iint_{\mathbb{R}^2} \underbrace{|Sf(u,\xi)|^2}_{P_S f,\ \text{time frequency energy density}}\, du\, d\xi$$

(energy conservation)

d) Discrete STFT

Definition: • Space considered: space of discrete-time N-periodic signal, with $N \in \mathbb{N}^*$

• g is real-valued symetric discrete signal of period N with $\|g\|_2 = 1$

• Atoms: $g_{m,l}[n] = g[n-m]\, e^{\frac{2i\pi l n}{N}}$

$\mathcal{F}$ $\hookrightarrow \hat{g}_{m,l}[k] = \hat{g}[k-l]\, e^{-\frac{2i\pi m(k-l)}{N}}$

• $\forall (m,l) \in [\![0, N-1]\!]^2$, $\boxed{Sf[m,l] = \langle f, g_{m,l} \rangle}$

$$= \sum_{n=0}^{N-1} f[n]\, g_{m,l}[n]$$

$$= \sum_{n=0}^{N-1} f[n]\, g[n-m]\, e^{\frac{2i\pi l n}{N}}$$

Remark

For each m, $Sf[m, l]$ computed for $l \in [\![0, N-1]\!]$ with a DFT of $f[n]g[n-m]$

The DFT admits a fast numerical implementation: FFT with complexity in $O(N \log_2 N)$

$\Rightarrow$ total complexity $O(N^2 \log_2 N)$

Summary

* the STFT answers all the requirements of time-frequency analysis (energy conservation, inversion formula, description of time variations of the freq. content + efficient) though limited in time and frequency resolution (due to Heisenberg-Gabor)
* for other signal, essential information is conveyed through their regularity (images, texture, transitory phases of an ECG, ...) $\Rightarrow$ wavelet transform

Chapter 3: Continuous wavelet transforms

Context: * Introduced in the 60s (1964)
by Morlet for geological applications
(probing geological layers)

↪ need to send short-times impulses at right frequency to discriminate contiguous layers

⇒ need for new time-frequency tools

* Grossman: studied similar notion in quantum physics

1) Definition (wavelet, wavelet transform)

A wavelet is a function $\psi \in L^2(\mathbb{R})$ st

(i) ψ has a 0 average $\int_{\mathbb{R}} \psi(t)\,dt = 0$

(ii) $\|\psi\|_2 = 1$

(iii) Support is centred around 0

(iv) ψ is admissible $0 < C_\psi = \int \frac{|\hat{\psi}(\nu)|^2}{\nu} d\nu < \infty$