Raytracing model

This is a numerical model for the propagation and refraction of internal gravity waves. It integrates the six-dimensional energy transfer equation in time. The waves propagate under the influence of variable stratification and mean flow.

Energy transfer equation

The energy transfer equation for internal gravity waves with wavenumber vector $\boldsymbol{k}$ and intrinsic frequency ω_i given by

$$\omega_i = \sqrt{(N^2(\ell^2+k^2)+f^2m^2)/(k^2+\ell^2+m^2)} \quad , \quad \boldsymbol{k} = (k,\ell,m) = (k_1,k_2,k_3) \tag{1}$$

which are propagating in a variable stratification given by $N(x,y,z)$ and horizontal mean flow $\boldsymbol{U}(x,y,z) = (U,V,0)$ is given by

$$\partial_t \mathcal{E} + \partial_{\boldsymbol{x}}(\dot{\boldsymbol{x}}\,\mathcal{E}) + \partial_{\boldsymbol{k}}(\dot{\boldsymbol{k}}\,\mathcal{E}) = \omega_i^{-1}\mathcal{E}\dot{\omega}_i \tag{2}$$

with group velocities $\dot{\boldsymbol{x}}$ and refraction parameter $\dot{\boldsymbol{k}}$ given by

$$\dot{\boldsymbol{x}} = \partial_{\boldsymbol{k}}\omega_e = (\partial_k\omega_e, \partial_\ell\omega_e, \partial_m\omega_e) \quad , \quad \dot{\boldsymbol{k}} = -\partial_{\boldsymbol{x}}\omega_e = -(\partial_x\omega_e, \partial_y\omega_e, \partial_z\omega_e) \tag{3}$$

and extrinsic (Doppler shifted) frequency of encounter

$$\omega_e = \omega_i + \boldsymbol{k}\cdot\boldsymbol{U} \tag{4}$$

The energy exchange with the mean flow is given by the right hand side of the energy transfer equation with

$$\begin{aligned}
\dot{\omega}_i &= -\boldsymbol{k}\cdot(\boldsymbol{\nabla}_{\boldsymbol{k}}\omega_i\cdot\boldsymbol{\nabla U}) - \dot{z}\boldsymbol{k}\cdot\partial_z\boldsymbol{U} = -\boldsymbol{\nabla}_{\boldsymbol{k}}\omega_i\cdot(\boldsymbol{k}\cdot\boldsymbol{\nabla U}) - \dot{z}\boldsymbol{k}\cdot\partial_z\boldsymbol{U} && (5)\\
&= -k_i(\partial_{k_j}\omega_i)\partial_{x_j}U_i - \dot{z}k_i\partial_z U_i && (6)\\
&= -k\boldsymbol{\nabla}_{\boldsymbol{k}}\omega_i\cdot\boldsymbol{\nabla}U - \ell\boldsymbol{\nabla}_{\boldsymbol{k}}\omega_i\cdot\boldsymbol{\nabla}V - \dot{z}(k\partial_z U + \ell\partial_Z V) && (7)
\end{aligned}$$

since $\boldsymbol{\nabla}_{\boldsymbol{k}}\omega_i||\boldsymbol{k}$.

Numerical grid

The model is discretised on a staggered C-grid in all six dimensions. Energy E is centered in a tracer grid box. On the eastern, northern and upper sides of these boxes, the zonal, meridional and vertical velocities, $\dot{x}$, $\dot{y}$, $\dot{z}$, respectively, are located, and similar for directions k, ℓ, and m and the corresponding velocities. N, U, and V are given on the same grid position in $\boldsymbol{x}$ as E – i.e. on tracer points.

All indices for the six dimensions are counted positive in the respective direction. This also holds for the vertical coordinate. Energy is given at points zt_k, $k = 1, ..., K$, where K is the number of grid points in the vertical. Above zt_k ends the grid cell at the level

zu_k which is given by $zu_k = (zt_{k+1} + zt_k)/2$. The grid box size for the tracer is thus $\Delta zt_k = zu_k - zu_{k-1}$ and we also define $\Delta zu_k = zt_{k+1} - zt_k$. The vertical velocity $\dot{z}$ is defined at zu_k and is thus in the center of its box, while the tracer point zt_k is displaced relative to the center of its box in case of a non-equidistant grid. This is sometimes called a u-centered grid. The grid in the other directions follows equivalent rules and naming convention. The sea surface is at $zu_K = 0\,\mathrm{m}$, where K is the number of vertical levels. The number of grid points in k, ℓ, and m direction must be even (or just one), such that $kt_{K/2} = -kt_{K/2-1}$ represent the smallest wavenumber when K is the number of grid points in k direktion, and such that $ku_{K/2} = 0$. Corresponding rules apply for ℓ and m.

Discretisation

The velocities are calculated from gradients of ω_e, which is given on the same grid position as E. The velocities $\dot{x}$ and $\dot{k}$ are e.g. given by

$$\dot{x} = \delta^{-}_{kt} \overline{\overline{\omega_e}^{x+}}^{k+} \ , \ \dot{k} = -\delta^{-}_{xt} \overline{\overline{\omega_e}^{x+}}^{k+} \tag{8}$$

with the finite differencing operators

$$\delta^{+}_{x} h_i = (h_{i+1} - h_i)/\Delta xt_i \quad , \quad \delta^{-}_{x} h_i = (h_i - h_{i-1})/\Delta xt_i \tag{9}$$

and with the finite averaging operators

$$\overline{h_i}^{x+} = (h_i + h_{i+1})/2 \quad , \quad \overline{h_i}^{x-} = (h_i + h_{i-1})/2 \tag{10}$$

and similar for the other dimensions y, z, k, ℓ, m. This satisfies on the discrete grid the condition $\partial_x \dot{x} + \partial_k \dot{k} = \partial_x \partial_k \omega_e - \partial_k \partial_x \omega_e = 0$ and similar for the other dimensions.

The fluxes in the energy transfer terms are calculated using a second order advection scheme with superbee flux limiter. This scheme introduces a certain/necessary amount of numerical diffusion, and ensures positive definite energies. Time stepping is then simply forward in time.

Boundary conditions

Boundary conditions in z and m are reflection boundary conditions, i.e. the upward energy flux at the surface $z = 0, k = -|k^*|$ is converted to a downward energy flux at $z = 0, k = |k^*|$, and similar for the bottom $z = -h$. For all other boundaries energy can flow out if the velocity is directed such, while there is no inflow of energy. The energy flux at k, ℓ, and m direction is integrated and stored in an output file.

Parallelisation

The model domain is decomposed into sub domains for each processor. The number of subdomains for each direction in $\boldsymbol{x}$ and $\boldsymbol{k}$ is specified during run time.

Dimension subsets

Since integrating the model in six dimensions is computionally very demanding it is also possible to use just one grid point in certain dimensions. E.g. for the case of horizontal homogenuity, i.e. $\boldsymbol{U} = \boldsymbol{U}(z)$ and $N = N(z)$, the grid points in x and y direction can be set to one. All wavenumber dimensions can also be set to one, a special model variable for the choice of wavenumber in this dimensions then needs to be specified to calculate wave frequencies, etc.