Documentation for pseudo-spectral 3D ocean model (ps3D)

Continuous equations

The Boussinesq equations are

$$\begin{aligned}
\partial_t u &= -\partial_x p + \delta u \quad , \quad \delta u = +fv - \boldsymbol{u}\cdot\boldsymbol{\nabla} u + A_h \boldsymbol{\nabla}_h^2 u + A_v \partial_{zz} u \\
\partial_t v &= -\partial_y p + \delta v \quad , \quad \delta v = -fu - \boldsymbol{u}\cdot\boldsymbol{\nabla} v + A_h \boldsymbol{\nabla}_h^2 v + A_v \partial_{zz} v \\
\partial_t w &= -\partial_z p + \delta w \quad , \quad \delta w = b - \boldsymbol{u}\cdot\boldsymbol{\nabla} w + A_h \boldsymbol{\nabla}_h^2 w + A_v \partial_{zz} w
\end{aligned}$$

plus frictional terms in the δ terms. We solve them on a triple-periodic domain. Note that a factor ρ_0 is absorbed in the pressure. The continuity equation is given by

$$\partial_x u + \partial_y v + \partial_z w = 0$$

We use a conservation equation for buoyancy b

$$\partial_t b = -wN^2 + K_h \boldsymbol{\nabla}^2 b + K_v \partial_{zz} b - \boldsymbol{u}\cdot\boldsymbol{\nabla} b$$

The constant stability frequency N results from the stratification of the mean state. Kinetic and potential energy are $(u^2+v^2+w^2)/2$ and $b^2/2N^2$.

Fully non-hydrostatic pressure

For fully non-hydrostatic conditions, we solve for the full pressure p. Taking the divergence of the momentum equation yields

$$\boldsymbol{\nabla}^2 p = \boldsymbol{\nabla}_h \cdot \delta \boldsymbol{u}_h + \partial_z \delta w$$

which needs to be solved at each time step. Discrete version

$$(\delta_x^- \delta_x^+ + ...) p_{i,j,k} = f_{i,j,k}$$

with the finite differencing operators

$$\begin{aligned}
\delta_x^+ h_{i,j} = (h_{i+1,j} - h_{i,j})/\Delta_x \ , \ \delta_y^+ h_{i,j} = (h_{i,j+1} - h_{i,j})/\Delta_y \\
\delta_x^- h_{i,j} = (h_{i,j} - h_{i-1,j})/\Delta_x \ , \ \delta_y^- h_{i,j} = (h_{i,j} - h_{i,j-1})/\Delta_y
\end{aligned}$$

The Fourier transform of the discrete system becomes simple with the definitions

$$\begin{aligned}
i\hat{k}_x^+(k_x) = \frac{e^{ik_x\Delta x} - 1}{\Delta x} \overset{\Delta x\to 0}{=} ik_x \quad , \quad i\hat{k}_x^-(k_x) = \frac{1 - e^{-ik_x\Delta x}}{\Delta x} \overset{\Delta x\to 0}{=} ik_x \\
\hat{1}_x^+(k_x) = \frac{e^{ik_x\Delta x} + 1}{2} \overset{\Delta x\to 0}{=} 1 \quad , \quad \hat{1}_x^-(k_x) = \frac{e^{-ik_x\Delta x} + 1}{2} \overset{\Delta x\to 0}{=} 1
\end{aligned}$$

and similar for $\hat{k}_y^+$, $\hat{k}_y^-$, etc, with $(\hat{1}_x^+)^* = \hat{1}_x^-$, $(\hat{k}_x^+)^* = \hat{k}_x^-$, $\hat{k}_x^+(-k_x) = -\hat{k}_x^-(k_x)$ and $\hat{1}_x^+(-k_x) = \hat{1}_x^-(k_x)$.

$$-(\hat{k}_x^+(k_x)\hat{k}_x^-(k_x) + ...)\hat{p}_{i,j,k} = \hat{f}_{i,j,k}$$

Scaled equations

Introducing L, H for vertical and horizontal scales, using $T = 1/f$ as time scale which is appropriate for gravity waves, using for the scaling of buoyancy and pressure the hydrostatic and geostrophic balance, and the continuity equation for the scaling of w, we obtain

$$\partial_t \boldsymbol{u} + f \underline{\boldsymbol{u}}\,\lrcorner + \boldsymbol{\nabla} p = -Ro\,(\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} + w\partial_z \boldsymbol{u}) \quad , \quad \partial_t b + N^2 w = -Ro\,(\boldsymbol{u}\cdot\boldsymbol{\nabla} b + w\partial_z b)$$

$$\delta^2 \partial_t w + \partial_z p - b = -Ro\delta^2\,(\boldsymbol{u}\cdot\boldsymbol{\nabla} w + w\partial_z w)$$

with the diagnostic relation $\boldsymbol{\nabla}\cdot\boldsymbol{u} + \partial_z w = 0$, and with the Rossby number $Ro = U/(L\Omega)$ and $N = Ro/Fr = L_r/L$ where Fr denotes the Froude number $Fr = U/(\tilde{N}H)$ and $L_r = \tilde{N}H/\Omega$ the Rossby radius. $f = 1$ was kept for reference, and with the small aspect ratio $\delta = H/L$. The scaled background stratification $\bar{b}(z)$ is given by $N^2 z/Ro$. Kinetic and potential energy are $\boldsymbol{u}^2/2 + \delta^2 w^2/2$ and $b^2/2N^2$.

To solve for the pressure we first we write as

$$\partial_t \boldsymbol{u} = -\boldsymbol{\nabla} p + \dot{\boldsymbol{u}}\ ,\ \delta^2 \partial_t w = -\partial_z p + \dot{w}\ \rightarrow\ \boldsymbol{\nabla}^2 p + \partial_{zz} p/\delta^2 = \boldsymbol{\nabla}\cdot\dot{\boldsymbol{u}} + \partial_z \dot{w}/\delta^2$$

Then we first time step without pressure gradient

$$\boldsymbol{u}^* = \boldsymbol{u}^{n-1} + \Delta t\dot{\boldsymbol{u}}\ ,\ w^* = w^{n-1} + \Delta t/\delta^2 \dot{w}$$

Then we take divergence for pressure equation

$$\boldsymbol{\nabla}\cdot\boldsymbol{u}^* + \partial_z w^* = \Delta t\boldsymbol{\nabla}\cdot\dot{\boldsymbol{u}} + \Delta t/\delta^2 \partial_z \dot{w} = \Delta t(\boldsymbol{\nabla}^2 p + \partial_{zz} p/\delta^2)$$

Spectral transform yields

$$-k^2 \hat{p} - m^2/\delta^2 \hat{p} = \hat{div}$$

Pressure with rigid lid

Taking the divergence of the momentum equation yields

$$(\delta_x^- \delta_x^+ + \delta_y^- \delta_y^+) p_{i,j,k} + \delta_z^- \delta_z^+ p_{i,j,k}/\delta^2 \quad = \quad f_{i,j,k}$$

Horizontal Fourier transform yields

$$-(\hat{k}_x^+(k_x)\hat{k}_x^-(k_x) + \hat{k}_y^+(k_y)\hat{k}_y^-(k_y))\hat{p}_{i,j,k} + \delta_z^- \delta_z^+ p_{i,j,k}/\delta^2 \quad = \quad \hat{f}_{i,j,k}$$

For $k = N$

$$\delta_z^- \delta_z^+ p_{i,j,k} = ((p_{k+1} - p_k) - (p_k - p_{k-1}))/\Delta z^2 = (-p_k + p_{k-1})/\Delta z^2$$

$$-(\hat{k}_x^+(k_x)\hat{k}_x^-(k_x) + \hat{k}_y^+(k_y)\hat{k}_y^-(k_y) + 1/\Delta z^2/\delta^2)\hat{p}_{i,j,k} + p_{k-1}/\Delta z^2/\delta^2 \quad = \quad \hat{f}_{i,j,k}$$

For interior k

$$\hat{p}_{k+1}/\Delta z^2/\delta^2 - (\hat{k}_x^+(k_x)\hat{k}_x^-(k_x) + \hat{k}_y^+(k_y)\hat{k}_y^-(k_y) + 2/\Delta_z^2/\delta^2)\hat{p}_{i,j,k} + \hat{p}_{k-1}/\Delta z^2/\delta^2 \quad = \quad \hat{f}_{i,j,k}$$

For $k = 1$

$$\delta_z^- \delta_z^+ p_{i,j,k} = ((p_{k+1} - p_k) - (p_k - p_{k-1}))/\Delta z^2 = (p_{k+1} - p_k)/\Delta z^2$$
$$p_{k+1}/\Delta z^2/\delta^2 - (\hat{k}_x^+(k_x)\hat{k}_x^-(k_x) + \hat{k}_y^+(k_y)\hat{k}_y^-(k_y) + 1/\Delta z^2/\delta^2)\hat{p}_{i,j,k} \;\; = \;\; \hat{f}_{i,j,k}$$

Set for horizontal mean $p_{1,0} = 0$. Then for $k = 1$

$$\delta_z^- \delta_z^+ p_{i,j,k} = ((p_{k+1} - p_k) - (p_k - p_{k-1}))/\Delta z^2 = p_{k+1}/\Delta z^2 \;\rightarrow\; p_2 = f_1 \Delta z^2 \delta^2$$

For $k = 2$

$$\delta_z^- \delta_z^+ p_{i,j,k} = ((p_{k+1} - p_k) - (p_k - p_{k-1}))/\Delta z^2 = (p_{k+1} - 2p_k)/\Delta z^2 \;\rightarrow\; p_3 = 2p_2 + f_2 \Delta z^2 \delta^2$$

For k

$$\delta_z^- \delta_z^+ p_{i,j,k} = ((p_{k+1} - p_k) - (p_k - p_{k-1}))/\Delta z^2 = f_k \delta^2 \;\rightarrow\; p_{k+1} = 2p_k - p_{k-1} + f_k \Delta z^2 \delta^2$$

Biharmonic friction

Biharmonic friction reads as

$$\partial_t u = ... - A_h \boldsymbol{\nabla}^4 u - A_v \partial_z^4 u$$

Scaling yields with $T = 1/\Omega$, $W/U = H/L = \delta$ and $A_h \sim \Delta x^4 \Omega$ and $A_v \sim \Delta z^4 \Omega$

$$\partial_t u = ... - \Delta x^4/L^4 \boldsymbol{\nabla}^4 u - \Delta z^4/H^4 \partial_z^4 u$$

On the grid scale Δx friction and advection terms should scale equally.

$$RoU^2/\Delta x \sim A_h U/\Delta x^4 \;\rightarrow\; A_h \sim URo\Delta x^3 \;,\; A_v \sim URo\Delta z^3$$

Waves and modes for rigid lid

Neglecting the non-linear terms, mixing and friction

$$\partial_t \boldsymbol{u} + f\underset{\neg}{\underline{\boldsymbol{u}}} + \boldsymbol{\nabla} p = 0 \;\;,\;\; \delta^2 \partial_t w + \partial_z p - b = 0 \;\;,\;\; \partial_t b + N^2 w = 0 \;\;,\;\; \partial_z w + \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$$

Divergence and vorticity for $f = const$

$$\partial_t \boldsymbol{\nabla} \cdot \boldsymbol{u} - f\underset{\neg}{\underline{\boldsymbol{\nabla}}} \cdot \boldsymbol{u} + \boldsymbol{\nabla}^2 p = 0 \;\;,\;\; \partial_t \underset{\neg}{\underline{\boldsymbol{\nabla}}} \cdot \boldsymbol{u} + f \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$$

Time derivative of divergence and vertical momentum

$$\partial_{tt} \boldsymbol{\nabla} \cdot \boldsymbol{u} + f^2 \boldsymbol{\nabla} \cdot \boldsymbol{u} + \boldsymbol{\nabla}^2 \partial_t p = 0 \;\;,\;\; \delta^2 \partial_{tt} w + \partial_z \partial_t p + N^2 w = 0$$

Combination yields the wave equation for w

$$\partial_{zz}\left(\partial_{tt} w + f^2 w\right) + \boldsymbol{\nabla}^2(\delta^2 \partial_{tt} w + N^2 w) = 0$$

Free wave ansatz $w = \exp i(\boldsymbol{k}\cdot\boldsymbol{x} + mz - \omega t)$

$$(im)^2\left((-i\omega)^2 + f^2\right) + (i\boldsymbol{k})^2(\delta^2(-i\omega)^2 + N^2) = 0 \;\rightarrow\; \omega^2 = \frac{m^2 f^2 + k^2 N^2}{\delta^2 k^2 + m^2}$$

Wave polarisation from $u = U_0 \exp i(\boldsymbol{k}\cdot\boldsymbol{x} + mz - \omega t)$, etc, which yields for instance

$$R_k = \frac{w^2}{u^2+v^2} = \frac{m^2\omega^2(\omega^2-f^2)^2}{(\delta^2\omega^2-N^2)^2k^2(\omega^2+f^2)} = \frac{\omega^2(\omega^2-f^2)}{(N^2-\delta^2\omega^2)(\omega^2+f^2)}$$

or

$$\omega^2 = \frac{\sqrt{4R_kN^2f^2(1+R_k\delta^2) + (f^2 - R_k\delta^2 f^2 + R_kN^2)^2}}{2(1+R_k\delta^2)} + \frac{(f^2 - R_k\delta^2 f^2 + R_kN^2)}{2(1+R_k\delta^2)}$$

With constant mean flow $\boldsymbol{U}$

$$(\omega - Ro\,\boldsymbol{U}\cdot\boldsymbol{k}_h)^2 = \frac{m^2 f^2 + k^2 N^2}{\delta^2 k^2 + m^2}$$

Wave ansatz with $w = \exp i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)\partial_z\phi(z)$ for rigid lid

$$\partial_{zzz}\phi + k^2\frac{\delta^2\omega^2 - N^2}{f^2-\omega^2}\partial_z\phi = 0\ ,\ \partial_z\phi = \sin mz\ ,\ m = n\pi/h$$

Thus we have rigid lid modes $\phi = \cos mz$ for $\boldsymbol{u}$ and p and $\partial_z\phi = \sin(mz)$ for w and b.

Potential vorticity

Scaling unscaled PV yields

$$\begin{aligned} Q &= f + \frac{f Ro}{N^2}\partial_z b + Ro(\partial_x v - \partial_y u) + Ro^2/N^2\left((\partial_x v - \partial_y u)\partial_z b - \partial_z v\partial_x b + \partial_z u\partial_y b\right) \\ &\quad + Ro^2\delta^2/N^2(\partial_y w\partial_x b - \partial_x w\partial_y b) \end{aligned}$$

The last term vanishes in primitive equations.

Scaled eigenvectors

The scaled Boussinesq equations are

$$\begin{aligned} \partial_t\boldsymbol{u} + f\boldsymbol{u}^{\neg} + \boldsymbol{\nabla}p &= -Ro\,(\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} + w\partial_z\boldsymbol{u}) \\ \partial_t w + (\partial_z p - b)/\delta^2 &= -Ro\,(\boldsymbol{u}\cdot\boldsymbol{\nabla}w + w\partial_z w) \\ \partial_t b + N^2 w &= -Ro\,(\boldsymbol{u}\cdot\boldsymbol{\nabla}b + w\partial_z b) \end{aligned}$$

with Rossby number Ro and aspect ratio δ. We add for the moment the sound wave equation

$$\partial_t p = -c_s^2(\boldsymbol{\nabla}\cdot\boldsymbol{u} + \partial_z w)$$

with the scaled sound speed $c_s = Ro/Ma$ and with the Mach number $Ma = U/\tilde{c}_s \ll Ro$. Assuming a triple-periodic domain and applying the Fourier ansatz

$$\boldsymbol{u}(\boldsymbol{x},t) = \int_{-\infty}^{\infty} d\boldsymbol{K}\, \hat{\boldsymbol{u}}_n(\boldsymbol{K},t) e^{i\boldsymbol{K}\cdot\boldsymbol{x}}$$

and similar for w and b with wavenumber vector $\boldsymbol{K} = (k_x, k_y, k_z)$, yields after multiplication with $e^{-i\boldsymbol{K}\cdot\boldsymbol{x}}$ and integration over $\boldsymbol{x}$

$$\partial_t \boldsymbol{z} = i\boldsymbol{A}\cdot\boldsymbol{z} - Ro\,\hat{\boldsymbol{n}}$$

with the linear system matrix $\boldsymbol{A}(\boldsymbol{K})$, the state vector $\boldsymbol{z}(\boldsymbol{K},t)$, and the vector function $\boldsymbol{N}$ given by

$$\boldsymbol{A} = \begin{pmatrix} 0 & -if & 0 & 0 & -k_x \\ if & 0 & 0 & 0 & -k_y \\ 0 & 0 & 0 & -i/\delta^2 & -m/\delta^2 \\ 0 & 0 & iN^2 & 0 & 0 \\ -c_s^2 k_x & -c_s^2 k_y & -c_s^2 m & 0 & 0 \end{pmatrix} \quad , \quad \boldsymbol{z} = \begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{w} \\ \hat{b} \\ \hat{p} \end{pmatrix}$$

For the system without the auxiliary sound wave equation we then have the eigenvectors

$$\boldsymbol{q}^{\pm} = \left(\frac{if\underset{\neg}{\boldsymbol{k}} + \boldsymbol{k}\omega}{f^2 - \omega^2} \, , \, \frac{m\omega}{N^2 - \delta^2\omega^2} \, , \, \frac{imN^2}{N^2 - \delta^2\omega^2} \right)^T$$

$$\boldsymbol{p}^{\pm} = p_{\pm} \left(\frac{-if\underset{\neg}{\boldsymbol{k}} + \boldsymbol{k}\omega}{f^2 - \omega^2} \, , \, \frac{m\omega\delta^2}{N^2 - \delta^2\omega^2} \, , \, -\frac{im}{N^2 - \delta^2\omega^2} \right)$$

Eigenvalues ω are given by

$$\omega = 0 \ , \ k^2N^2 + f^2m^2 = (k^2\delta^2 + m^2)\omega^2$$

Orthonormalisation is given by

$$p_0 = \frac{f^2N^2}{k^2N^2 + f^2m^2} \ , \ p_{\pm} = \frac{(N^2 - \delta^2\omega^2)(\omega^2 - f^2)}{2(m^2f^2 + k^2N^2)}$$

The following relation holds

$$\boldsymbol{q} = (q_u, q_v, q_w, q_b, q_p)^T \ , \ \boldsymbol{p} = p_s(q_u^*, q_u^*, q_w\delta^2, q_b^*/N^2)$$

Rigid lid vertical mode eigenvectors

We have the linear system

$$\partial_t \boldsymbol{u} = -f\underset{\neg}{\boldsymbol{u}} - \boldsymbol{\nabla} p \ , \ \partial_t w = -(\partial_z p - b)/\delta^2 \ , \ \partial_t b = -N^2 w \ , \ \partial_t p = -c_s^2(\boldsymbol{\nabla}\cdot\boldsymbol{u} + \partial_z w)$$

Assume a double periodic domain and use

$$\boldsymbol{u}(\boldsymbol{x}, z, t) = \int_{-\infty}^{\infty} d\boldsymbol{k}\, \hat{\boldsymbol{u}}(\boldsymbol{k}, z, t) e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

and similar for w and b with wavenumber vector $\boldsymbol{k} = (k_x, k_y)$, yields after multiplication with $e^{-i\boldsymbol{k}'\cdot\boldsymbol{x}}$ and integration over $\boldsymbol{x}$

$$\partial_t \hat{\boldsymbol{u}} = -f \underset{\neg}{\hat{\boldsymbol{u}}} - i\boldsymbol{k}\hat{p} \ , \ \partial_t \hat{w} = -(\partial_z \hat{p} - \hat{b})/\delta^2 \ , \ \partial_t \hat{b} = -N^2 \hat{w} \ , \ \partial_t \hat{p} = -c_s^2 (i\boldsymbol{k}\hat{\boldsymbol{u}} + \partial_z \hat{w})$$

The wave equation for $c_s \to \infty$ becomes

$$-\partial_{tt}\partial_{zz} w + \delta^2 k^2 \partial_{tt}\hat{w} - f^2 \partial_{zz} w + N^2 k^2 \hat{w} = 0 \ \to \ \partial_{zz}\psi + \lambda_n^2 \psi = 0 \ , \ \lambda_n^2 = k^2 \frac{N^2 - \delta^2\omega^2}{\omega^2 - f^2} = (n\pi/h)^2$$

with separation ansatz $w \sim \exp(-\omega t)\psi(z)$. Solution is $\psi_n = \sqrt{2/h}\sin n\pi z/h$ with $\int \psi_n \psi_n dz = \delta_{n,m}$.

$$\int \psi_m \partial_{zz}\psi_n dz = \psi_m \partial_z \psi_n| - \int (\partial_z \psi_m)\partial_z \psi_n dz \ \to \ \int (\partial_z \psi_m)\partial_z \psi_n dz = \lambda_n^2 \delta_{n,m}$$

Modes for $\boldsymbol{u}$ and p are $\phi_n = \sqrt{2/h}\cos n\pi z/h$, with $\int \phi_n \phi_n dz = \delta_{n,m}$ and $\partial_z \psi_n = \lambda_n \phi_n$. Multiplication by ϕ_n or ψ_n, integration over depth, and using the modal representation yields

$$\partial_t \hat{\boldsymbol{u}}_n = -f \underset{\neg}{\hat{\boldsymbol{u}}_n} - i\boldsymbol{k}\hat{p}_n \ , \ \delta^2 \partial_t \hat{w}_n = \hat{p}_n \lambda_n^2/(n\pi/h) + \hat{b}_n \ , \ \partial_t \hat{b}_n = -N^2 \hat{w}_n \ , \ \partial_t \hat{p}_n = -c_s^2 (i\boldsymbol{k}\cdot\hat{\boldsymbol{u}}_n + (n\pi/h)\hat{w}_n)$$

The system matrix in $\partial_t \boldsymbol{z} = i\boldsymbol{A}\cdot\boldsymbol{z}$ is then

$$\partial_t \hat{\boldsymbol{u}}_n = i(f \underset{\neg}{\hat{\boldsymbol{u}}_n} - \boldsymbol{k}\hat{p}_n) \ , \ \partial_t \hat{w}_n = i(-i\hat{p}_n \lambda_n^2/(n\pi/h) - i\hat{b}_n)/\delta^2 \ , \ \partial_t \hat{b}_n = i^2 N^2 \hat{w}_n \ , \ \partial_t \hat{p}_n = ic_s^2(-\boldsymbol{k}\cdot\hat{\boldsymbol{u}}_n + i(n\pi/h$$

$$\boldsymbol{A} = \begin{pmatrix} 0 & -if & 0 & 0 & -k_x \\ if & 0 & 0 & 0 & -k_y \\ 0 & 0 & 0 & -i/\delta^2 & -i\lambda_n/\delta^2 \\ 0 & 0 & iN^2 & 0 & 0 \\ -c_s^2 k_x & -c_s^2 k_y & c_s^2 i\lambda_n & 0 & 0 \end{pmatrix} \ , \ \boldsymbol{z} = \begin{pmatrix} \hat{u}_n \\ \hat{v}_n \\ \hat{w}_n \\ \hat{b}_n \\ \hat{p}_n \end{pmatrix}$$

Difference to before in red: $m \to \pm i\lambda_n$. The right eigenvectors $\omega\boldsymbol{q} = \boldsymbol{A}\cdot\boldsymbol{q}$ and the left ones $\omega\boldsymbol{p} = \boldsymbol{p}\cdot\boldsymbol{A}$ of the linear system are given by the relations

$$\boldsymbol{q}^{\pm} = \left(\frac{if\underset{\neg}{\boldsymbol{k}} + \boldsymbol{k}\omega}{f^2 - \omega^2} \ , \ \frac{i\lambda_n \omega}{N^2 - \delta^2\omega^2} \ , \ \frac{-\lambda_n N^2}{N^2 - \delta^2\omega^2} \right)^T$$

$$\boldsymbol{p}^{\pm} = p_{\pm} \left(\frac{-if\underset{\neg}{\boldsymbol{k}} + \boldsymbol{k}\omega}{f^2 - \omega^2} \ , \ \frac{-i\lambda_n \omega\delta^2}{N^2 - \delta^2\omega^2} \ , \ \frac{-\lambda_n}{N^2 - \delta^2\omega^2} \right)$$

Only the w and b component are different to before. It holds that $\boldsymbol{p} = p_s(q_u^*, q_v^*, q_w^*\delta^2, q_b/N^2)^T$. For $\boldsymbol{q}$, $m \to i\lambda_n$ and for $\boldsymbol{p}$, $m \to -i\lambda_n$. Orthonormalisation is given by

$$p_0 = \frac{f^2 N^2}{k^2 N^2 + f^2 \lambda_n^2} \ , \ p_{\pm} = \frac{(N^2 - \delta^2\omega^2)(\omega^2 - f^2)}{2(\lambda_n^2 f^2 + k^2 N^2)}$$

Discrete vertical mode eigenvectors

Discrete version for vertical after horizontal Fourier transform

$$\partial_t \hat{\boldsymbol{u}} = -f \underset{\neg}{\hat{\boldsymbol{u}}} - i\boldsymbol{k}\hat{p} \ , \ \partial_t \hat{w} = -(\delta_z^+ \hat{p} - \overline{\hat{b}}^{k+})/\delta^2 \ , \ \partial_t \hat{b} = -N^2 \overline{\hat{w}}^{k-} \ , \ \boldsymbol{k} \cdot \hat{\boldsymbol{u}} = i\delta_k^- \hat{w}$$

The discrete wave equation is given by

$$\left(-\partial_{tt} - f^2\right) \delta_k^+ \delta_k^- \hat{w} + k^2 \left(N^2 \overline{\overline{\hat{w}}^{k-}}^{k+} + \delta^2 \partial_{tt} \hat{w} \right) = 0$$

Now apply separation ansatz $\hat{w}_k = \hat{w}_0 \psi_k \exp(-i\omega t)$

$$\delta_k^+ \delta_k^- \psi + \frac{k^2}{\omega^2 - f^2} \left(N^2 \overline{\overline{\psi}^{k-}}^{k+} - \delta^2 \omega^2 \psi \right) = 0 \ , \ \psi_0 = \psi_N = 0 \ , \ k = 1, N-1$$

with solution

$$\psi_k^n = a \sin(nk\pi/N) \ , \ 1 < n < N \ , \ a = \sqrt{2/h} \ , \ \Delta z \sum_n \psi_k^n \psi_k^m = \delta_{n,m}$$

and the (semi) discrete dispersion relation

$$\omega_n^2 = \frac{f^2 \lambda_n^2 + k^2 N^2 \cos^2 n\pi/(2N)}{\lambda_n^2 + k^2 \delta^2}$$

with $\lambda_n^2 = \left(2 \sin n\pi/(2N)\right)^2 / \Delta z^2$. For $N \to \infty$ and $\Delta z = h/N \to 0$ we find $\lambda_n^2 \to n^2\pi^2/h^2$ and $\cos^2 n\pi/(2N) \to 1$ so convergence is satisfied. With $k \to \hat{k}_x^+ \hat{k}_x^- + \hat{k}_y^+ \hat{k}_y^-$ and $f^2 \to f^2 1_x^- 1_y^+ 1_x^+ 1_y^-$ we obtain the fully discrete version.

The orthonormal vertical modes for $\hat{\boldsymbol{u}}_k = \sum_n \hat{\boldsymbol{u}}_n \phi_k^n$, $\hat{p}_k = \sum_n \hat{p}_n \phi_k^n$ and $\hat{b}_k = \sum_n \hat{b}_n \tilde{\psi}_k^n$ are

$$\phi_k^n = \frac{\psi_k^n - \psi_{k-1}^n}{2 \sin n\pi/(2N)} = \frac{\Delta z \delta_k^- \psi_k^n}{2 \sin n\pi \Delta z/(2h)} \ , \ \tilde{\psi}_k^n = \frac{\psi_k^n + \psi_{k-1}^n}{2 \cos n\pi/(2N)}$$

with $\Delta z \sum_n \phi_k^n \phi_k^m = \delta_{n,m}$ and $\Delta z \sum_n \tilde{\psi}_k^n \tilde{\psi}_k^m = \delta_{n,m}$. Using the modal representation we obtain

$$\partial_t \hat{u} = -i^2 f 1_x^+ 1_y^- v - i\hat{k}_x^+ p \quad , \quad \partial_t \hat{v} = i^2 f 1_x^- 1_y^+ u - i\hat{k}_y^+ p$$

$$\partial_t \hat{b}_n = i^2 N^2 \hat{w}_n \cos n\pi/(2N) \ , \ \partial_t p = -c_s i \left(\hat{k}_x^- \hat{u}_n + \hat{k}_y^- \hat{v}_n - i\hat{w}_n \lambda_n \right)$$

$$\delta^2 \partial_t \hat{w}_n = -i^2 \hat{p}_n \lambda_n - i^2 \hat{b}_n \cos n\pi/(2N)$$

with $\lambda_n = \frac{2 \sin n\pi/(2N)}{\Delta z}$, and so for $\partial_t \boldsymbol{z} = i\boldsymbol{A} \cdot \boldsymbol{z}$ (note that eigenvalues ω will have opposite sign)

$$\boldsymbol{A} = \begin{pmatrix} 0 & -if 1_x^+ 1_y^- & 0 & 0 & -\hat{k}_x^+ \\ if 1_x^- 1_y^+ & 0 & 0 & 0 & -\hat{k}_y^+ \\ 0 & 0 & 0 & -(i/\delta^2) \cos n\pi/(2N) & -i\lambda_n/\delta^2 \\ 0 & 0 & iN^2 \cos n\pi/(2N) & 0 & 0 \\ -c_s^2 \hat{k}_x^- & -c_s^2 \hat{k}_y^- & c_s^2 i \lambda_n & 0 & 0 \end{pmatrix} \ , \ \boldsymbol{z} = \begin{pmatrix} \hat{u}_n \\ \hat{v}_n \\ \hat{w}_n \\ \hat{b}_n \\ \hat{p}_n \end{pmatrix}$$

Difference to vertically propagating version below is that $\hat{1}_z^{\pm} \to \cos n\pi/(2N)$ and $\hat{k}_z^{\pm} \to \pm i\lambda_n$. Right eigenvectors are given by

$$\boldsymbol{q} = \left(\frac{-if 1_x^+ 1_y^- \hat{k}_y^+ + \hat{k}_x^+ \omega}{f^2 1_x^+ 1_y^- 1_x^- 1_y^+ - \omega^2} \;,\; \frac{if 1_x^- 1_y^+ \hat{k}_x^+ + \hat{k}_y^+ \omega}{f^2 1_x^+ 1_y^- 1_x^- 1_y^+ - \omega^2} \;,\; \frac{i\lambda_n \omega}{N^2 \cos^2 n\pi/(2N) - \delta^2\omega^2} \;,\; \frac{-N^2 \lambda_n \cos n\pi/(2N)}{N^2 \cos^2 n\pi/(2N) - \delta^2\omega^2} \right)^T$$

The left eigenvectors are given by

$$\boldsymbol{p} = p_s \left(\frac{if 1_x^- 1_y^+ \hat{k}_y^- + \hat{k}_x^- \omega}{f^2 1_x^+ 1_y^- 1_x^- 1_y^+ - \omega^2} \;,\; \frac{-if 1_x^+ 1_y^- \hat{k}_x^- + \hat{k}_y^- \omega}{f^2 1_x^- 1_y^+ 1_x^+ 1_y^- - \omega^2} \;,\; \frac{-i\lambda_n \omega \delta^2}{N^2 \cos^2 n\pi/(2N) - \delta^2\omega^2} \;,\; \frac{-\lambda_n \cos n\pi/(2N)}{N^2 \cos^2 n\pi/(2N) - \delta^2\omega^2} \right)$$

It holds that $\boldsymbol{p} = p_s(q_u^*, q_v^*, q_w^* \delta^2, q_b/N^2)^T$. Orthonormalisation with

$$\begin{aligned} p_0 &= \frac{f^2 1_x^- 1_y^+ 1_x^+ 1_y^- N^2 \cos^2 n\pi/(2N)}{\lambda_n^2 f^2 1_x^- 1_y^+ 1_x^+ 1_y^- + (\hat{k}_y^+ \hat{k}_y^- + \hat{k}_x^+ \hat{k}_x^-) N^2 \cos^2 n\pi/(2N)} \\ p_\pm &= \frac{1}{2} \frac{(N^2 \cos^2 n\pi/(2N) - \delta^2\omega^2)(\omega^2 - f^2 1_x^+ 1_y^- 1_y^+ 1_x^-)}{\lambda_n^2 f^2 1_x^+ 1_y^- 1_y^+ 1_x^- + (k_y^+ k_y^- + k_x^+ k_x^-) N^2 \cos^2 n\pi/(2N)} \end{aligned}$$

PV and eigenvectors

Linear PV is $QN^2/Ro = f\partial_z b + N^2\zeta$. From

$$\boldsymbol{q}^{\pm} = \left(\frac{if \underset{\neg}{\boldsymbol{k}} + \boldsymbol{k}\omega}{f^2 - \omega^2} \;,\; \frac{m\omega}{N^2 - \delta^2\omega^2} \;,\; \frac{imN^2}{N^2 - \delta^2\omega^2} \right)^T$$

it follows that

$$\begin{aligned} \zeta &= \frac{ifk_x^2 + k_x k_y \omega}{f^2 - \omega^2} - \frac{-ifk_y^2 + k_y k_x \omega}{f^2 - \omega^2} = if \frac{k_x^2 + k_y^2}{f^2 - \omega^2} \\ QN^2/Ro &= ifN^2 \left(\frac{m^2}{N^2 - \delta^2\omega^2} + \frac{k_x^2 + k_y^2}{f^2 - \omega^2} \right) \\ &= ifN^2 \frac{m^2(f^2 - \omega^2) + k^2(N^2 - \delta^2\omega^2)}{(N^2 - \delta^2\omega^2)(f^2 - \omega^2)} \\ &= ifN^2 \frac{k^2\delta^2\omega^2 + m^2\omega^2 - m^2\omega^2 - \delta^2 k^2 \omega^2}{(N^2 - \delta^2\omega^2)(f^2 - \omega^2)} = 0 \end{aligned}$$

with the dispersion relation $k^2N^2 + f^2m^2 = (k^2\delta^2 + m^2)\omega^2$.

Eigenvectors again

For $c_s \to \infty$ we have $\boldsymbol{\nabla} \cdot \boldsymbol{u} + \partial_z w = 0$ and

$$\delta^2 \boldsymbol{\nabla}^2 p + \partial_{zz} p = \partial_z b - \delta^2 f \boldsymbol{\nabla} \cdot \underset{\neg}{\boldsymbol{u}} = \partial_z b + \delta^2 f(\partial_x v - \partial_y u) \;\to\; \hat{p} = -i \frac{m\hat{b} + \delta^2 f \underset{\neg}{\boldsymbol{k}} \cdot \boldsymbol{u}}{\delta^2 k^2 + m^2}$$

and so

$$\boldsymbol{A} = \begin{pmatrix} -ik_x \frac{\delta^2 f k_y}{\delta^2 k^2+m^2} & -if + if\frac{\delta^2 k_x^2}{\delta^2 k^2+m^2} & 0 & \frac{ik_x m}{\delta^2 k^2+m^2} \\ if - if\frac{\delta^2 k_y^2}{\delta^2 k^2+m^2} & ik_y \frac{\delta^2 f k_x}{\delta^2 k^2+m^2} & 0 & \frac{ik_y m}{\delta^2 k^2+m^2} \\ -im\frac{f k_y}{\delta^2 k^2+m^2} & +im\frac{f k_x}{\delta^2 k^2+m^2} & 0 & -i\frac{k^2}{\delta^2 k^2+m^2} \\ 0 & 0 & iN^2 & 0 \end{pmatrix} \quad , \quad \boldsymbol{z} = \begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{w} \\ \hat{b} \end{pmatrix}$$

Eigenvalues of $\boldsymbol{A}$ are

$$\omega^2 = 0 \ , \ 0 = \omega^2 K^2 - k^2 N^2 - f^2 m^2$$

with $K^2 = \delta^2 k^2 + m^2$. Note that there is an additional zero eigenvalue. Right eigenvectors are

$$\boldsymbol{q} = \left(\frac{(f k_y + \omega i k_x)\, m}{N^2 k^2} \ , \ \frac{(-f k_x + \omega i k_y)\, m}{N^2 k^2} \ , \ -i\omega/N^2 \ , \ 1 \right)^T$$

Multiply with $\frac{imN^2}{N^2-\delta^2\omega^2}$ to obtain previous ones

$$\frac{(f k_y + \omega i k_x)\, m}{N^2 k^2} \frac{imN^2}{N^2 - \delta^2\omega^2} = (ifk_y - \omega k_x)\frac{m^2}{k^2(N^2-\delta^2\omega^2)} = \frac{(ifk_y - \omega k_x)}{(\omega^2 - f^2)}$$

with $(\omega^2 - f^2)m^2 = (N^2 - \omega^2\delta^2)k^2$, which is the same as the first component of previous ones. Left eigenvector follow then from symmetry relation

$$\boldsymbol{q} = (q_u, q_v, q_w, q_b, q_p)^T \ , \ \boldsymbol{p} = p_s(q_u^*, q_u^*, q_w\delta^2, q_b^*/N^2)$$

Discrete eigenvalues

The discrete system matrix becomes

$$\boldsymbol{A} = \begin{pmatrix} 0 & -if 1_x^+ 1_y^- & 0 & 0 & -\hat{k}_x^+ \\ if 1_x^- 1_y^+ & 0 & 0 & 0 & -\hat{k}_y^+ \\ 0 & 0 & 0 & -i\hat{1}_z^+/\delta^2 & -\hat{k}_z^+/\delta^2 \\ 0 & 0 & iN^2 1_z^- & 0 & 0 \\ -c_s^2\hat{k}_x^- & -c_s^2\hat{k}_y^- & -c_s^2\hat{k}_z^- & 0 & 0 \end{pmatrix} \quad , \quad \boldsymbol{z} = \begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{w} \\ \hat{b} \\ \hat{p} \end{pmatrix}$$

which is similar to the continuous case. Discrete eigenvalues for $c \to \infty$ are given by

$$\omega = 0 \ , \ f^2 1_x^+ 1_y^- 1_x^- 1_y^+ \hat{k}_z^+ \hat{k}_z^- + N^2 1_z^+ 1_z^- (\hat{k}_y^+ \hat{k}_y^- + \hat{k}_x^+ \hat{k}_x^-) = \omega^2(\hat{k}_z^+ \hat{k}_z^- + \delta^2 \hat{k}_y^+ \hat{k}_y^- + \delta^2 \hat{k}_x^+ \hat{k}_x^-)$$

Right eigenvectors are given by

$$\boldsymbol{q} = \left(\frac{-if 1_x^+ 1_y^- \hat{k}_y^+ + \hat{k}_x^+ \omega}{f^2 1_x^+ 1_y^- 1_x^- 1_y^+ - \omega^2} \ , \ \frac{if 1_x^- 1_y^+ \hat{k}_x^+ + \hat{k}_y^+ \omega}{f^2 1_x^+ 1_y^- 1_x^- 1_y^+ - \omega^2} \ , \ \frac{\hat{k}_z^+ \omega}{N^2 1_z^+ 1_z^- - \delta^2\omega^2} \ , \ \frac{iN^2 1_z^- \hat{k}_z^+}{N^2 1_z^- 1_z^+ - \delta^2\omega^2} \right)^T$$

which is analogous to the analytical version. The left eigenvectors are given by

$$\boldsymbol{p} = p_s \left(\frac{if 1_x^- 1_y^+ \hat{k}_y^- + \hat{k}_x^- \omega}{f^2 1_x^+ 1_y^- 1_x^- 1_y^+ - \omega^2} \, , \, \frac{-if 1_x^+ 1_y^- \hat{k}_x^- + \hat{k}_y^- \omega}{f^2 1_x^- 1_y^+ 1_x^+ 1_y^- - \omega^2} \, , \, \frac{\hat{k}_z^- \omega \delta^2}{N^2 1_z^- 1_z^+ - \delta^2 \omega^2} \, , \, \frac{-i \hat{1}_z^+ \hat{k}_z^-}{N^2 1_z^- 1_z^+ - \delta^2 \omega^2} \right)$$

Orthonormalisation with

$$\begin{aligned} p_0 &= \frac{f^2 1_x^- 1_y^+ 1_x^+ 1_y^- N^2 1_z^+ 1_z^-}{\hat{k}_z^+ \hat{k}_z^- f^2 1_x^- 1_y^+ 1_x^+ 1_y^- + (\hat{k}_y^+ \hat{k}_y^- + \hat{k}_x^+ \hat{k}_x^-) N^2 1_z^+ 1_z^-} \\ p_\pm &= \frac{1}{2} \frac{(N^2 1_z^+ 1_z^- - \delta^2 \omega^2)(\omega^2 - f^2 1_x^+ 1_y^- 1_y^+ 1_x^-)}{\hat{k}_z^+ \hat{k}_z^- f^2 1_x^+ 1_y^- 1_y^+ 1_x^- + (k_y^+ k_y^- + k_x^+ k_x^-) 1_z^+ 1_z^- N^2} \end{aligned}$$

Balancing

The system is given by

$$\partial_t \hat{\boldsymbol{z}} - i \boldsymbol{A} \cdot \hat{\boldsymbol{z}} = Ro \, \hat{\boldsymbol{n}}$$

with the state vector $\hat{\boldsymbol{z}} = (\hat{\boldsymbol{u}}, \hat{w}, \hat{b})^T$, the vector of the non-linearities $\boldsymbol{n} = (\boldsymbol{n}_u, n_p)^T$, and its Fourier transform $\hat{\boldsymbol{n}} = (2\pi)^{-2} \int d\boldsymbol{x} \, \boldsymbol{n} e^{-i\boldsymbol{k} \cdot \boldsymbol{x}}$. The matrix $\boldsymbol{A}$ has eigenvalues ω^s and right and left eigenvectors $\boldsymbol{q}^s$ and $\boldsymbol{p}^s$, respectively.

It is useful to project on the fast or slow modes using the eigenvectors of $\boldsymbol{A}$. The fast or slow mode amplitude $g^s = \boldsymbol{p}^s \cdot \hat{\boldsymbol{z}}$ is then governed by

$$\partial_t g^s - i \omega^s g^s = Ro \, \boldsymbol{p}^s \cdot \hat{\boldsymbol{n}} = -iRo \, I^s(g^0, g^\pm) \quad , \quad s = 0, \pm$$

where the case $s = 0$ corresponds to the slow geostrophic mode with amplitude g^0, and $s = \pm$ to the (two) fast gravity wave mode with g^+ and g^-. I^s is the interaction integral with

$$I^s(g^0, g^\pm) = i \boldsymbol{p}^s \cdot \hat{\boldsymbol{n}}$$

and accounts for the quadratic non-linearities, which can also be written as

$$I^s(g^0, g^\pm) \;\; = \;\; I^s(g^0, 0) + I^s(0, g^\pm) + K^s(g^0, g^\pm)$$

Expand the gravity wave amplitudes as

$$g^\pm = Ro f_1^\pm + Ro^2 f_2^\pm + Ro^3 f_3^\pm + \ldots$$

and introduce a fast and slow time scale with $T = Ro \, t^*$ and $\partial_t = Ro \, \partial_T + \partial_{t^*}$ and $g^0 = g^0(T)$ and $g^\pm = g^\pm(T, t^*)$. For $s = 0$ the governing equation becomes

$$Ro \, \partial_T g^0 = -iRo I^0(g^0, g^\pm) = -iRo \left(I^s(g^0, 0) + I^s(0, \sum Ro^n f_n^\pm) + K^s(g^0, \sum Ro^n f_n^\pm) \right)$$

The first order

$$\partial_T g^0 = -iI^0(g^0, 0)$$

and the second order

$$\begin{aligned} 0 = K^0(g^0, f_1^\pm) = I^0(g^0, f_1^\pm) - I^0(g^0, 0) + I^s(0, f_1^\pm) \\ \partial_T g^0 = -iI^0(g^0, f_1^\pm) + iI^0(0, f_1^\pm) \end{aligned}$$

$f_1^\pm$ is the first order slaved mode. For $s = \pm$ the governing equation becomes

$$\begin{aligned} Ro\partial_T \sum Ro^n f_n^\pm + \partial_{t^*} \sum Ro^n f_n^\pm - i\omega^\pm \sum Ro^n f_n^\pm &= -iRo\, I^\pm(g^0, \sum Ro^n f_n^\pm) \\ &= -iRo\left(I^\pm(g^0, 0) + I^\pm(0, \sum Ro^n f_n^\pm) + K^\pm(g^0, \sum Ro^n f_n^\pm)\right) \end{aligned}$$

First order

$$\partial_{t^*} f_1^\pm - i\omega^\pm f_1^\pm = -i\, I^\pm(g^0, 0)$$

Eliminate $\partial_{t^*} f_1^\pm$ such that the slaved mode becomes

$$f_1^\pm = I^\pm(g^0, 0)/\omega^\pm$$

Periodic forcing

Consider a periodic wavemaker in the equations

$$\begin{aligned} \partial_t u - fv + \partial_x p &= -Ro\ (\boldsymbol{u} \cdot \boldsymbol{\nabla} u + w\partial_z u) + \cos(\omega_f t) f(\boldsymbol{x}) \\ \partial_t v + fu + \partial_y p &= -Ro\ (\boldsymbol{u} \cdot \boldsymbol{\nabla} v + w\partial_z v) \\ \partial_t w + (\partial_z p - b)/\delta^2 &= -Ro\ (\boldsymbol{u} \cdot \boldsymbol{\nabla} w + w\partial_z w) \\ \partial_t b + N^2 w &= -Ro\ (\boldsymbol{u} \cdot \boldsymbol{\nabla} b + w\partial_z b) \end{aligned}$$

Assuming again a triple-periodic domain and applying the Fourier ansatz

$$\boldsymbol{u}(\boldsymbol{x}, t) = \int_{-\infty}^{\infty} d\boldsymbol{K}\, \hat{\boldsymbol{u}}(\boldsymbol{K}, t) e^{i\boldsymbol{K}\cdot\boldsymbol{x}}$$

and similar for w and b with wavenumber vector $\boldsymbol{K} = (k_x, k_y, k_z)$, yields after multiplication with $e^{-i\boldsymbol{K}\cdot\boldsymbol{x}}$ and integration over $\boldsymbol{x}$

$$\partial_t \boldsymbol{z} = i\boldsymbol{A} \cdot \boldsymbol{z} + Ro\, \hat{\boldsymbol{n}} + \cos(\omega_f t)\hat{\boldsymbol{f}}\ ,\ \ \hat{\boldsymbol{f}} = (\hat{f}, 0, ...)^T$$

with $f(\boldsymbol{x}) = \int d\boldsymbol{K} \hat{f}(\boldsymbol{K}) e^{i\boldsymbol{K}\cdot\boldsymbol{x}}$ and with the linear same system matrix $\boldsymbol{A}(\boldsymbol{K})$, state vector $\boldsymbol{z}(\boldsymbol{K}, t)$, and vector function $\boldsymbol{N}$.

Expanding $\boldsymbol{z} = \sum_s g^s(t)\boldsymbol{q}^s$ and $\hat{\boldsymbol{f}} = \sum_s f^s \boldsymbol{q}^s$ and projecting on the left eigenvectors $\omega^s \boldsymbol{p}^s = \boldsymbol{p}^s \cdot \boldsymbol{A}$ yields

$$\partial_t g^n = i g^n \omega^n + Ro\, \boldsymbol{p}^n \cdot \hat{\boldsymbol{n}} + \cos(\omega_f t) f^n$$

Now set $g^n = e^{i\omega^n t} a(t)$

$$\partial_t a^n = e^{-i\omega^n t} Ro\, \boldsymbol{p}^n \cdot \hat{\boldsymbol{n}} + e^{-i\omega^n t} \cos(\omega_f t) f^n$$

For $\omega^n = \omega_f$

$$\partial_t a^n = e^{-i\omega^n t} Ro\, \boldsymbol{p}^n \cdot \hat{\boldsymbol{n}} + \left(\cos^2(\omega^n t) - i \sin(\omega^n t)\cos(\omega^n t)\right) f^n$$

First part of the bracket is always positive, second part is oscillatory.

Discrete frequency with Doppler shift

The scaled Boussinesq equations are

$$\begin{aligned}
\partial_t \boldsymbol{u} + f \underset{\neg}{\boldsymbol{u}} + \boldsymbol{\nabla} p &= -Ro\, (\boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} + w \partial_z \boldsymbol{u}) \\
\partial_t w + (\partial_z p - b)/\delta^2 &= -Ro\, (\boldsymbol{u} \cdot \boldsymbol{\nabla} w + w \partial_z w) \\
\partial_t b + N^2 w &= -Ro\, (\boldsymbol{u} \cdot \boldsymbol{\nabla} b + w \partial_z b)
\end{aligned}$$

A basic state $\boldsymbol{U} = (U = const, 0, 0)$ and $B = const$ and $P(y,z) = fUy + Bz$ is balanced if

$$fU = -\partial_y P \ , \ \partial_z P = B$$

Then with $\boldsymbol{u} = \boldsymbol{u}' + U$, ... and dropping all primes

$$\begin{aligned}
(\partial_t + Ro\, U \partial_x)\, \boldsymbol{u} + f \underset{\neg}{\boldsymbol{u}} + \boldsymbol{\nabla} p &= -Ro\, (\boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} + w \partial_z \boldsymbol{u}) \\
(\partial_t + Ro\, U \partial_x)\, w + (\partial_z p - b)/\delta^2 &= -Ro\, (\boldsymbol{u} \cdot \boldsymbol{\nabla} w + w \partial_z w) \\
(\partial_t + Ro\, U \partial_x)\, b + N^2 w &= -Ro\, (\boldsymbol{u} \cdot \boldsymbol{\nabla} b + w \partial_z b)
\end{aligned}$$

Fourier transform yields

$$\begin{aligned}
(\partial_t + Ro\, U i k_x)\, \hat{\boldsymbol{u}} &= i \left(i f \underset{\neg}{\hat{\boldsymbol{u}}} - \boldsymbol{k} \hat{p} \right) - Ro\, \hat{F}\, (\boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} + w \partial_z \boldsymbol{u}) \\
(\partial_t + Ro\, U i k_x)\, \hat{w} &= i \left(-k_z \hat{p}/\delta^2 - i \hat{b}/\delta^2 \right) - Ro\, \hat{F}\, (\boldsymbol{u} \cdot \boldsymbol{\nabla} w + w \partial_z w) \\
(\partial_t + Ro\, U i k_x)\, \hat{b} &= i \left(i N^2 \hat{w} \right) - Ro\, \hat{F}\, (\boldsymbol{u} \cdot \boldsymbol{\nabla} b + w \partial_z b)
\end{aligned}$$

With the pressure equation this can be written as

$$(\partial_t + RoiUk_x)\boldsymbol{z} = i\boldsymbol{A} \cdot \boldsymbol{z} - Ro\, \hat{\boldsymbol{n}}$$

with the linear system matrix $\boldsymbol{A}(\boldsymbol{K})$, the state vector $\boldsymbol{z}(\boldsymbol{K},t)$, and the vector function $\boldsymbol{N}$ given by

$$\boldsymbol{A} = \begin{pmatrix} 0 & -if & 0 & 0 & -k_x \\ if & 0 & 0 & 0 & -k_y \\ 0 & 0 & 0 & -i/\delta^2 & -m/\delta^2 \\ 0 & 0 & iN^2 & 0 & 0 \\ -c_s^2 k_x & -c_s^2 k_y & -c_s^2 m & 0 & 0 \end{pmatrix} \quad , \quad \boldsymbol{z} = \begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{w} \\ \hat{b} \\ \hat{p} \end{pmatrix}$$

With ansatz $\boldsymbol{z} = \boldsymbol{z}_0 \exp -\omega t$

$$(\omega - Ro U k_x)\boldsymbol{z}_0 = -\boldsymbol{A}\cdot\boldsymbol{z}_0 = -\omega_A \boldsymbol{z}_0 \ , \ \omega_A = -\omega + Ro U k_x$$

with the eigenvalues ω_A of the system matrix $\boldsymbol{A}$

$$\omega_A^2 = 0 \ , \ 0 = \omega_A^2 K^2 - k^2 N^2 - f^2 m^2$$

with $K^2 = \delta^2 k^2 + m^2$. The intrinsic frequency is ω_A and the one which figures in the eigenvector definitions. ω is the frequency of encounter and vanishes for the lee waves. Any state $\boldsymbol{z}$ can be decomposed with the left and right eigenvectors of $\boldsymbol{A}$

$$\begin{aligned} \boldsymbol{z} = \sum g^s \boldsymbol{q}^s \ , \ \boldsymbol{p}^s \cdot \boldsymbol{z} = \sum g^t \boldsymbol{p}^s \cdot \boldsymbol{q}^t = g^s \\ (\omega - Ro U k_x) \sum g^s \boldsymbol{q}^s = -\boldsymbol{A}\cdot \sum g^s \boldsymbol{q}^s = -\sum g^s \omega_A^s \boldsymbol{q}^s \\ (\omega - Ro U k_x) g^s = -g^s \omega_A^s \end{aligned}$$

Discrete linear system is

$$\partial_t b_{ijk} + N^2 1_z^-(w_{ijk}) \quad = \quad -Ro\, U \overline{\delta_x^+(b_{ijk})}^{x-}$$

and so on, such that Fourier transform becomes

$$\begin{aligned} -i\omega\hat{u} - f\,\underset{\neg}{\hat{v}} + i\hat{k}_x^+ \hat{p} &= -iRo\,U \hat{1}_x^- \hat{k}_x^+ \hat{u} \\ -i\omega\hat{v} + f\,\underset{\neg}{\hat{u}} + i\hat{k}_y^+ \hat{p} &= -iRo\,U \hat{1}_x^- \hat{k}_x^+ \hat{v} \\ -i\omega\hat{w} + (i\hat{k}_z^+ p - \hat{1}_z^+ b)/\delta^2 &= -iRo\,U \hat{1}_x^- \hat{k}_x^+ \hat{w} \\ -i\omega\hat{b} + N^2 \hat{1}_z^- \hat{w} &= -iRo\,U \hat{1}_x^- \hat{k}_x^+ \hat{b} \end{aligned}$$

and so Doppler shift is $\omega \to \omega - Ro\,U \hat{1}_x^- \hat{k}_x^+$

$$(\omega - Ro\,U \hat{1}_x^- \hat{k}_x^+)\hat{b} \quad = \quad -iN^2 \hat{1}_z^- \hat{w}$$

with $\hat{k}_x^+ = -i(e^{ik_x\Delta_x} - 1)/\Delta_x$ and $\hat{1}_x^- = \left(1 + e^{-ik_x\Delta x}\right)/2$

$$\begin{aligned} 2\Delta_x \hat{1}_x^- \hat{k}_x^+ &= -i(1 + e^{-ik_x\Delta x})(e^{ik_x\Delta_x} - 1) = -i\left(e^{ik_x\Delta_x} + e^{-ik_x\Delta x} e^{ik_x\Delta_x} - 1 - e^{-ik_x\Delta x}\right) \\ &= -i\left(e^{ik_x\Delta_x} - e^{-ik_x\Delta x}\right) = 2\sin k_x \Delta x \ , \ \hat{1}_x^- \hat{k}_x^+ = \sin(k_x\Delta x)/\Delta_x \end{aligned}$$

The scaled GM spectrum

We want to have the scaled GM spectrum.

$$\begin{aligned} E(k,\omega,\phi,\boldsymbol{x}) &= \frac{E_0(\boldsymbol{x})}{4\pi}\frac{\tilde{A}(k/k_\star)}{k_\star}B(\omega) \\ E(k_1,k_2,m) &= \frac{E_0(\boldsymbol{x})}{4\pi}\frac{\tilde{A}(m/m_\star)}{m_\star}B(\omega)\frac{(N^2-\delta^2\omega^2)^2}{m^2\omega(N^2-\delta^2 f^2)} \end{aligned}$$

where $\tilde{A}$ and B are dimensionless functions given by

$$\tilde{A}(\lambda) = \frac{n_a}{(1+\lambda^2)^t}\ ,\ B(\omega) = \frac{n_b|f|}{\omega\sqrt{\omega^2-f^2}}$$

with the parameter $c_\star = 1/(j_\star\pi)\int N dz$, the modal bandwidth $j_\star = 3,..15$, the exponent $t = 1$, and the normalisations $n_a = n_b = 2/\pi$, and

$$k_\star = \frac{\sqrt{\omega^2-f^2}}{c_\star}\ ,\ m_\star = \frac{\sqrt{N^2-\delta^2\omega^2}}{c_\star}$$