

The simple Primitive equation model

Consider the Boussinesq equations in hydrostatic approximation

$$\partial_t \mathbf{u} + \underline{\mathbf{f}} \mathbf{u} + \nabla p = -\mathbf{u} \cdot \nabla \mathbf{u} - w \partial_z \mathbf{u} + \mathbf{F}(\mathbf{u}) \quad , \quad \partial_t b = -\mathbf{u} \cdot \nabla b - w \partial_z b + Mb \quad (1)$$

with the diagnostic relations $\partial_z p = b$ and $\nabla \cdot \mathbf{u} + \partial_z w = 0$. Vectors are two-dimensional and $\underline{\mathbf{u}}$ denotes anticlockwise rotation by 90° . Pressure is scaled with background density ρ_0 and thermodynamics have been simplified to a buoyancy equation. \mathbf{F} and M are the friction and mixing operators. At the free surface $z = \eta$, we also have the condition

$$\partial_t p_s = -g \nabla \cdot \int_{-H}^{\eta} \mathbf{u} dz \quad (2)$$

with $p_s = p|_{z=\eta} = g\eta$. Alternatively, we use a rigid lid with $w = 0$ at $z = 0$ and $p_s = p|_{z=0}$. We split full pressure p into hydrostatic part and surface pressure.

$$p = p_h(z) + p_s \quad , \quad p_h = \int_{-h}^z b dz' \quad (3)$$

where p_s is to be determined by the external mode solvers.

Simple implicit external solver for rigid lid

Velocity and buoyancy and pressure are all co-located in time in this scheme. The buoyancy equation is integrated as

$$b^{n+1} - b^n = \Delta t \left(\overline{\delta b}^{n+1/2} + M(b^n) \right) \quad (4)$$

with the advective buoyancy tendency

$$\delta b^n = \nabla \cdot \mathbf{u}^n b^n + \partial_z w^n b^n \quad , \quad \overline{\delta b}^{n+1/2} = A \delta b^n + B \delta b^{n-1} + C \delta b^{n-2} \quad (5)$$

interpolated with the Adam-Bashforth (AB3) scheme to time level $n + 1/2$. Mixing is treated with an Euler forward step, or maybe implicit in case of vertical mixing.

An intermediate velocity \mathbf{u}^* is calculated from

$$\mathbf{u}^* - \mathbf{u}^n = \Delta t \left(\overline{\delta \mathbf{u}}^{n+1/2} + \mathbf{F}(\mathbf{u}^n) \right) \quad (6)$$

with the momentum tendency $\delta \mathbf{u}$ excluding the surface pressure gradient and friction

$$\delta \mathbf{u}^n = -\underline{\mathbf{f}} \mathbf{u}^n - \nabla p_h^n - \mathbf{u}^n \cdot \nabla \mathbf{u}^n - w^n \partial_z \mathbf{u}^n \quad (7)$$

These tendencies are extrapolated with AB3 to time level $n + 1/2$

$$\overline{\delta \mathbf{u}}^{n+1/2} = A \delta \mathbf{u}^n + B \delta \mathbf{u}^{n-1} + C \delta \mathbf{u}^{n-2} \quad (8)$$

Friction is treated with an Euler forward step. From the vertically integrated continuity equation follows the condition

$$\nabla \cdot \int_{-h}^0 \mathbf{u}^* = \Delta t \nabla \cdot h \nabla p_s^{n+1} \quad (9)$$

which is solved with some iterative method for p_s^{n+1} . The intermediate velocity is then corrected as

$$\mathbf{u}^{n+1} = \mathbf{u}^* - \Delta t \nabla p_s^{n+1} \quad (10)$$

It is possible to include an implicit free surface in the time stepping. Vertical velocity is calculated from continuity

$$w^{n+1} = - \int_{-h}^z \nabla \cdot \mathbf{u}^{n+1} \quad (11)$$

This scheme is also used as default in the MITgcm.

Simple split-explicit external mode solver

Velocity is known in this scheme at $t = n + 1/2$, while b and p at level n . The momentum equation is discretised as

$$\mathbf{u}^* - \mathbf{u}^{n-1/2} = \Delta t \left(-f \underline{\mathbf{u}}^n + \overline{\delta \mathbf{u}}^n - \nabla p_h^n - \nabla p_s^n + \mathbf{F}(\mathbf{u}^{n-1/2}) \right) \quad (12)$$

with the intermediate velocity \mathbf{u}^* and AB3 extrapolation of Coriolis force and advection terms tendencies

$$\underline{\mathbf{u}}^n = A \underline{\mathbf{u}}^{n-1/2} + B \underline{\mathbf{u}}^{n-3/2} + C \underline{\mathbf{u}}^{n-5/2} \quad , \quad \overline{\delta \mathbf{u}}^n = A \delta \mathbf{u}^{n-1/2} + B \delta \mathbf{u}^{n-3/2} + C \delta \mathbf{u}^{n-5/2} \quad (13)$$

and

$$\delta \mathbf{u}^{n-1/2} = -\mathbf{u}^{n-1/2} \cdot \nabla \mathbf{u}^{n-1/2} - w^{n-1/2} \partial_z \mathbf{u}^{n-1/2} \quad (14)$$

For the hydrostatic and surface pressure $p_h + p_s$ no extrapolation is made, since they are at level n anyways, and friction is treated with Euler forward.

Now the momentum tendencies are vertically integrated but without surface pressure and Coriolis contributions.

$$\mathbf{R}^n = \int_{-h}^0 \left(\overline{\delta \mathbf{u}}^n - \nabla p_h^n + \mathbf{F}(\mathbf{u}^{n-1/2}) \right) dz \quad (15)$$

The external mode is integrated with sub-cycling over $m = 1, \dots, M$ from n to $n + 1$ with a simple, dissipative scheme proposed by Demange et al (2019) using the forcing \mathbf{R}^n

$$\mathbf{U}^{n+(m+1)/M} - \mathbf{U}^{n+m/M} = \Delta t/M \left(-f \underline{\mathbf{U}}^{n+(m+1/2)/M} - h \nabla p_s^{n+m/M} + \mathbf{R}^n \right) \quad (16)$$

$$p_s^{n+(m+1)/M} - p_s^{n+m/M} = -g \Delta t/M \left((1 + \theta) \nabla \cdot \mathbf{U}^{n+(m+1)/M} - \theta \nabla \cdot \mathbf{U}^{n+m/M} \right) \quad (17)$$

with the transport $\mathbf{U}^n = \int_{-h}^0 \mathbf{u}^n dz$, and $\overline{\mathbf{U}}^{n+(m+1/2)/M}$ interpolated with AB2. Demange et al (2019) found $\theta = 0.14$ to be the best choice in terms of phase error and damping. The new external mode is given by

$$\mathbf{U}^{n+1} = \frac{1}{M} \sum_{m=1}^M \mathbf{U}^{n+m/M} + \frac{\theta}{M} (\mathbf{U}^{n+1} - \mathbf{U}^n) \quad (18)$$

After the sub-cycling, the barotropic part of the intermediate velocity is corrected

$$\mathbf{u}^{n+1/2} = \mathbf{u}^* - 1/h \left(\int_{-h}^0 \mathbf{u}^* dz - \mathbf{U}^{n+1} \right) \quad (19)$$

The buoyancy equation is integrated using $\mathbf{u}^{n+1/2}$ and $w^{n+1/2} = - \int_{-h}^z \nabla \cdot \mathbf{u}^{n+1/2}$ as

$$b^{n+1} - b^n = \Delta t (\delta b^{n+1/2} + M(b^n)) \quad (20)$$

with the buoyancy tendency

$$\delta b^n = \nabla \cdot \mathbf{u}^{n+1/2} b^n + \partial_z w^{n+1/2} b^n \quad (21)$$

interpolated with AB3 to time level $n + 1/2$. This scheme is one option in FESOM2.