The Saturn model

Consider the non-linear barotropic or reduced gravity model

$$\partial_t \boldsymbol{u} + f\underset{\neg}{\boldsymbol{u}} + \boldsymbol{\nabla} h = -\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} \quad , \quad \partial_t h + c^2 \boldsymbol{\nabla}\cdot\boldsymbol{u} = -\boldsymbol{\nabla}\cdot h\boldsymbol{u} \tag{1}$$

with the Coriolis parameter f, the gravity wave speed c^2, and the layer velocity $\boldsymbol{u}$. The layer thickness perturbation η was re-scaled to $h = g\eta$ with reduced gravity g such that $c^2 = g\bar{\eta}$, with the mean thickness $\bar{\eta}$. The vector $\underset{\neg}{\boldsymbol{u}}$ denotes anticlockwise rotation of the vector $\boldsymbol{u}$ by $\pi/2$, i.e. $\underset{\neg}{\boldsymbol{u}} = (-v, u)$ for $\boldsymbol{u} = (u, v)$.

Scaling

Using the beta plane $f(y) = f(0) + f'y$, the scaling $h \sim fUL$ from geostrophy, $x, y \sim L$, and the wave scaling $t \sim 1/f$, a scaled version of Eq. (1) becomes

$$\partial_t \boldsymbol{u} + (\tilde{f} + \beta y)\underset{\neg}{\boldsymbol{u}} + \boldsymbol{\nabla} h = -Ro\,\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} \quad , \quad \partial_t h + \tilde{c}^2 \boldsymbol{\nabla}\cdot\boldsymbol{u} = -Ro\,\boldsymbol{\nabla}\cdot h\boldsymbol{u} \tag{2}$$

with the Rossby number $Ro = U/(fL)$ and $\tilde{c} = L_r/L = Ro/F$ where $L_r = c/f$ denotes the Rossby radius, with the Froude number $F = U/c$, and $\beta = L/a \ll 1$, where a denotes the Earth radius. $\tilde{f} = 1$ is kept for reference. For $Ro = 1$ and $f = \tilde{f}$, $\beta = f'$, $c = \tilde{c}$ this becomes the dimensional version again and thus we drop the tilde for f and c from now.

Total, kinetic, and potential energy and potential voticity

Using the relation $\boldsymbol{\nabla}\boldsymbol{u}^2/2 + \underset{\neg}{\boldsymbol{u}}\underset{\neg}{\boldsymbol{\nabla}}\cdot\boldsymbol{u} = \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}$, the system Eq. (2) can be written as

$$\partial_t \boldsymbol{u} + q\underset{\neg}{\boldsymbol{U}} + \boldsymbol{\nabla}(h + Ro\,K) = 0 \quad , \quad \partial_t h + \boldsymbol{\nabla}\cdot\boldsymbol{U} = 0 \tag{3}$$

with total thickness $H = c^2 + Ro\,h$, volume transport $\boldsymbol{U} = H\boldsymbol{u}$, kinetic energy $K = \boldsymbol{u}^2/2$ and potential vorticity $q = (f + Ro\,\underset{\neg}{\boldsymbol{\nabla}}\cdot\boldsymbol{u})/H$. Kinetic energy K is given by

$$\partial_t K + \boldsymbol{u}\cdot\boldsymbol{\nabla} h = -Ro\,\boldsymbol{u}\cdot\boldsymbol{\nabla} K \tag{4}$$

the term $\boldsymbol{u}\cdot\boldsymbol{\nabla} h$ is the exchange with potential energy. Total energy T is given by

$$T = Ro^2\,HK + H^2/2 \tag{5}$$

obtained by adding $Ro^2 H$ times the kinetic energy equation and ϵH times the thickness equation which yields

$$\partial_t T + Ro\,\boldsymbol{\nabla}\cdot(H + Ro^2 K)\boldsymbol{U} = 0 \tag{6}$$

Total energy is conserved since it holds that

$$\int dA\partial_t T = Ro\int dA\left(Ro\,\boldsymbol{U}\cdot\partial_t\boldsymbol{u} + (H + Ro^2 K)\partial_t h\right) = 0 \tag{7}$$

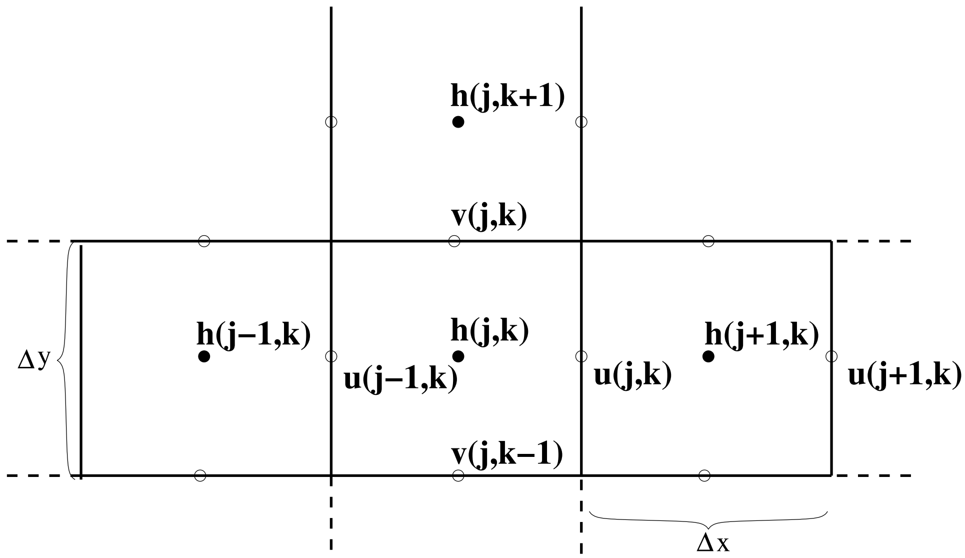

Figure 1: The staggered grid arrangement.

Linear discrete equations

Omitting the non-linear terms by setting $Ro = 0$ for the moment, rewrite Eq. (2) as

$$\partial_t u = fv - \partial_x h \quad , \quad \partial_t v = -fu - \partial_y h \quad , \quad \partial_t h = -c^2 \left(\partial_x u + \partial_y v\right) \tag{8}$$

For discretisation we use the C-grid arrangement shown in Fig. 1 which yields

$$\frac{du_{j,k}}{dt} = f\overline{\overline{v_{j,k}}^{j+}}^{k-} - \delta_x^+ h_{j,k} \ , \ \frac{dv_{j,k}}{dt} = -f\overline{\overline{u_{j,k}}^{j-}}^{k+} - \delta_y^+ h_{j,k} \ , \ \frac{dh_{j,k}}{dt} = -c^2 \left(\delta_x^- u_{j,k} + \delta_y^- v_{j,k}\right) \tag{9}$$

with the finite differencing operators

$$\delta_x^+ h_{j,k} = (h_{j+1,k} - h_{j,k})/\Delta_x \quad , \quad \delta_x^- h_{j,k} = (h_{j,k} - h_{j-1,k})/\Delta_x \tag{10}$$

$$\delta_y^+ h_{j,k} = (h_{j,k+1} - h_{j,k})/\Delta_y \quad , \quad \delta_y^- h_{j,k} = (h_{j,k} - h_{j,k-1})/\Delta_y \tag{11}$$

and with the finite averaging operators

$$\overline{h_{j,k}}^{j+} = (h_{j,k} + h_{j+1,k})/2 \quad , \quad \overline{h_{j,k}}^{j-} = (h_{j,k} + h_{j-1,k})/2 \tag{12}$$

$$\overline{h_{j,k}}^{k+} = (h_{j,k} + h_{j,k+1})/2 \quad , \quad \overline{h_{j,k}}^{k-} = (h_{j,k} + h_{j,k-1})/2 \tag{13}$$

Non-linear discrete equations

For the discrete non-linear system of Eq. (2), we use the momentum equation in the form

$$\partial_t \boldsymbol{u} + q\underset{\neg}{\boldsymbol{U}} = -\boldsymbol{\nabla}(h + Ro\, K) \quad , \quad \partial_t h + \boldsymbol{\nabla} \cdot \boldsymbol{U} = 0 \tag{14}$$

and discretise it using the energy conserving scheme by Sadourny (1975). The volume transport $\boldsymbol{U} = (U, V) = H\boldsymbol{u}$ with total thickness $H = c^2 + Ro\, h$ is defined at $u_{j,k}$ and $v_{j,k}$ points

$$U_{j,k} = u_{j,k}\overline{H}^{j+} \quad , \quad V = v_{j,k}\overline{H}^{k+} \quad \rightarrow \quad \frac{dh_{j,k}}{dt} = -\delta_x^- U_{j,k} - \delta_y^- V_{j,k} \tag{15}$$

Potential vorticity q is defined at grid corners as

$$q_{j,k} = \left(f + \delta_x^+ v_{j,k} - \delta_y^+ u_{j,k}\right) / \overline{\overline{H}^{j+}}^{k+} \tag{16}$$

The gradient force in momentum equation is given by

$$p_{j,k} = (h + Ro\, K)_{j,k} = h_{j,k} + Ro\,(\overline{u_{j,k}^2}^{j-} + \overline{v_{j,k}^2}^{k-})/2 \tag{17}$$

and the momentum equation is then discretized as

$$\frac{du_{j,k}}{dt} = \overline{q_{j,k}\overline{V_{j,k}}^{j+}}^{k-} - \delta_x^+ p_{j,k} \quad , \quad \frac{dv_{j,k}}{dt} = -\overline{q_{j,k}\overline{U_{j,k}}^{k+}}^{j-} - \delta_y^+ p_{j,k} \tag{18}$$

It can be shown for the discrete equations that total energy T is indeed conserved by this scheme. However, it is much easier to check this in the code.