2.2.1 - Questions to Investigate

1).
$$Ax^* = \vec{b}$$
$$\alpha A x^* = \alpha \vec{b}$$
$$(I + -I + \alpha A)x^* = \alpha \vec{b}$$
$$(I - (I - \alpha A))x^* = \alpha \vec{b}$$
$$\underline{x^* = (I - \alpha A)x^* + \alpha \vec{b}}$$

2)
$$\|x^* - x_{k+1}\| = \|(I - \alpha A)x^* + \alpha \vec{b} - x_{k+1}\| \quad (1)$$
$$= \|(I - \alpha A)x^* + \cancel{\alpha \vec{b}} - (I - \alpha A)x_k - \cancel{\alpha \vec{b}}\|$$
$$= \|(I - \alpha A)(x^* - x_k)\|$$
$$\leq \|I - \alpha A\| \|x^* - x_k\|$$
$$\hookrightarrow \frac{\|x^* - x_{k+1}\|}{\|x^* - x_k\|} \leq \|I - \alpha A\|$$

$\therefore$ $\|I - \alpha A\| < 1$ for convergence.

We also want to minimize $\|I - \alpha A\|$ for faster convergence which depends on choice of α.

Considering A as a symmetric matrix, we can define the spectral norm of A ($\|A\|$) as $\|A\| = \max_i |\lambda_i|$ w/ λ_i an eigenvalue of A.

define $\lambda_1 \leq \dots \leq \lambda_n$ as eigenvalues of $A \in \mathbb{R}^{n \times n}$

Note as A is semi-positive definite, $\lambda_i \geq 0$ $\forall i \in \{1, \dots, n\}$

$\therefore$ $\|I - \alpha A\| = \max_i |1 - \alpha \lambda_i| = \max\{|1 - \alpha \lambda_1|, |1 - \alpha \lambda_n|\}$ as $1 - \alpha\lambda_1 \geq 1 - \alpha\lambda_2 \geq \dots \geq 1 - \alpha\lambda_n$ which is minimized taking $\alpha = \frac{2}{\lambda_1 + \lambda_n}$ and guarantees convergence when A is positive definite st all λ_i are non-zero

There are no conditions for $\vec{x}_0$ as convergence depends only on $\|I - \alpha A\|$. Thus we could simply say $\vec{x}_0 = \vec{0}$.