APPM 4600 — HOMEWORK # 4

1. In laying water mains, utilities must be concerned with the possibility of freezing. Although soil and weather conditions are complicated, reasonable approximations can be made on the basis of the assumption that soil is uniform in all directions. In that case the temperature in degrees Celsius $T(x,t)$ at a distance x (in meters) below the surface, t seconds after the beginning of a cold snap, approximately satisfies

$$\frac{T(x,t) - T_s}{T_i - T_s} = \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right),$$

where T_s is the constant temperature during a cold period, T_i is the initial soil temperature before the cold snap, α is the thermal conductivity (in meters^2 per second), and

$$\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t \exp(-s^2) ds$$

Assume that $T_i = 20$ [degrees C], $T_s = -15$[degrees C], $\alpha = 0.138 \cdot 10^{-6}$ [meters^2 per second]. *It is convenient to use* **`scipy`** *to evaluate the erf function.*

For parts (b) and (c), run your experiments with a tolerance of $\epsilon = 10^{-13}$.

   (a) We want to determine how deep a water main should be buried so that it will only freeze after 60 *days* exposure at this constant surface temperature.
   Formulate the problem as a root finding problem $f(x) = 0$. What is f and what is f'?
   Plot the function f on $[0, \bar{x}]$, where $\bar{x}$ is chosen so that $f(\bar{x}) > 0$.

   (b) Compute an approximate depth using the Bisection Method with starting values $a_0 =$ 0[meters] and $b_0 = \bar{x}$ [meters].

   (c) Compute an approximate depth using Newton's Method with starting value $x_0 = 0.01$[meters]. What happens if you start with $x_0 = \bar{x}$?

2. Let $f(x)$ denote a function with root α of multiplicity m.

   (a) Write down a formal mathematical definition of what it means for α to be a root of multiplicity m of $f(x)$.

   (b) Show that Newton's method applied to $f(x)$ only converges linearly to the root α.

   (c) Show that the fixed point iteration applied to $g(x) = x - m\frac{f(x)}{f'(x)}$ is second order convergent.

   (d) What does part (c) provide for Newton's method in the case of roots with multiplicity greater than 1?

3. Beginning with the definition of order of convergence of a sequence $\{x_k\}_{k=1}^{\infty}$ that converges to α, derive a relationship between the $\log(|x_{k+1} - \alpha|)$ and $\log(|x_k - \alpha|)$. What is the order p in this relationship?

4. There are two ways of improving the convergence of Newton's method when a root has multiplicity greater than 1: Problem 2 c and apply Newton's method to $g(x) = \frac{f(x)}{f'(x)}$.

   In this problem consider finding the root of the function $f(x) = e^{3x} - 27x^6 + 27x^4e^x - 9x^2e^{2x}$ in the interval $(3, 5)$.

   Explore the order of convergence when applying (i) Newton's method, (ii) the modified Newton's method from class and (iii) the modified Newton's method in Problem 2. Which method do you perfer and why?

5. Use Newton and Secant method to approximate the largest root of

$$f(x) = x^6 - x - 1.$$

   Start Newton's method with $x_0 = 2$. Start Secant method with $x_0 = 2$ and $x_1 = 1$.

   (a) Create at table of the error for each step in the iteration. Does the error decrease as you expect?

   (b) Plot $|x_{k+1} - \alpha|$ vs $|x_k - \alpha|$ on log-log axes where α is the exact root for both methods. What are the slopes of the lines that result from this plot? How does this relate to the order?