APPM 4600 - HW 9 Gustav Cedergrund

1) Find least squares solution for linear system

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

by Thm in class, for $\mathbb{R}$ matrices, vector $\bar{x}$ that minimizes $\|Ax-b\|_2$ is solution of $A^TAx = A^Tb$.

Thus $\rightarrow$ $A^T \cdot A \quad x = A^T \quad b$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\hookrightarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 2 & 1 \end{array}\right] \Rightarrow \boxed{\bar{x} = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}} \leftarrow \text{Least Squares solution}$$

2) Find vector that min. $E^2 = b_1^2 + 4b_2^2 + 25b_3^2 + 9b_4^2$ when

$$\begin{bmatrix} 1 & 3 \\ 6 & -1 \\ 4 & 0 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \Rightarrow \begin{aligned} x_1 + 3x_2 - 1 &= b_1 \\ 6x_1 - x_2 - 2 &= b_2 \\ 4x_1 \quad - 3 &= b_3 \\ 2x_1 + 7x_2 - 4 &= b_4 \end{aligned}$$

s.t. $E^2 = (x_1 + 3x_2 - 1)^2 + (12x_1 - 2x_2 - 4)^2 + (20x_1 - 15)^2 + (6x_1 + 21x_2 - 12)^2$

which is least squares of

$$\begin{bmatrix} 1 & 3 \\ 12 & -2 \\ 20 & 0 \\ 6 & 21 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 15 \\ 12 \end{bmatrix}$$

so

$$\vec{x} = (A^TA)^{-1}A^T\vec{b}$$

$$= \begin{bmatrix} 581 & 105 \\ 105 & 454 \end{bmatrix}^{-1} \begin{bmatrix} 421 \\ 247 \end{bmatrix}$$

$$\boxed{\vec{x} \approx \begin{bmatrix} 0.65360 \\ 0.39288 \end{bmatrix}} \text{ to min. } E^2$$

3)

a) Prove that $\{1, x, x^2, \dots, x^n\}$ are linearly independent.

AFSOC that functions are not linearly independent. Then $\exists a_i \neq 0$ s.t

$$a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n = 0.$$

Using observation from (3), if we take the derivative of both sides n times, we have
$$\frac{d^n}{dx^n}(a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n) = \frac{d^n}{dx^n}(0)$$
$$a_n(n!) = 0$$

$\therefore$ $a_n = 0$.

The same argument can be conveyed inductively, taking one less derivative every time and seeing that $a_k \neq 0$ b/c $a_{k+1}, a_{k+2}, \dots, a_n = 0$ $\forall k \in \{0, 1, \dots, n-1\}$.

This is a contradiction as then $a_i = 0$ $\forall i$,

$\therefore$ $\{1, x, x^2, \dots, x^n\}$ are linearly independent.

b) Show function set $\{1, \cos(x), \cos(2x), \dots, \cos(nx), \sin(x), \dots, \sin(nx)\}$ is LI

We proceed in same fashion as with (a) w/ derivatives.

AFSOC that functions not LI s.t $\exists a_i \neq 0$ s.t

$$a_0 + a_1\cos(x) + a_2\cos(2x) + \dots + a_n\cos(nx) + a_{n+1}\sin(x) + \dots + a_{2n}\sin(nx) = 0$$

$f' \rightarrow$ $-a_1\sin(x) - 2a_2\sin(2x) - \dots - n a_n \sin(nx) + a_{n+1}\cos(x) + \dots + n a_{2n}\cos(nx) = 0$

$f'' \rightarrow$ $-a_1\cos(x) - 2^2 a_2\cos(2x) - \dots - n^2 a_n\cos(nx) - a_{n+1}\sin(x) - \dots - n^2 a_{2n}\sin(nx) = 0$

$f^{(3)} \rightarrow$ $a_1\sin(x) + 2^3 a_2\sin(2x) + \dots + n^3 a_n\sin(nx) - a_{n+1}\cos(x) - \dots - n^3 a_{2n}\cos(nx) = 0$

and so on. If we evaluate each derivative at $x=0$, we get

$a_0 + a_1 + a_2 + \dots + a_n = 0$, $\quad a_{n+1} + 2a_{n+2} + \dots + n \cdot a_{2n} = 0$

$a_1 + 2^2 a_2 + \dots + n^2 a_n = 0$, $\quad a_{n+1} + 2^3 a_{n+2} + \dots + n^3 \cdot a_{2n} = 0$ and so on.

Therefore, we can build 2 systems

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 2^2 & \cdots & n^2 \\ 0 & 1 & 2^4 & \cdots & n^4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 2^n & \cdots & n^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 & \cdots & n \\ 1 & 2^3 & \cdots & 2^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 2^n & \cdots & n^n \end{bmatrix} \begin{bmatrix} a_{n+1} \\ a_{n+2} \\ \vdots \\ a_{2n} \end{bmatrix} = \vec{0}$$

which can both be written into the Vandermonde matrix via transformation.

As Vandermonde matrix is invertible by thm., there is only 1 solution s.t $a_1 - a_n = 0$, $a_{n+1} - a_{2n} = 0$, and by translation, $a_0 = 0$ also. $\therefore$ contradiction and functions are LI.

4) Prove 3 term recursion for orthogonal polynomials:

$$\phi_k(x) = (x - b_k)\phi_{k-1}(x) - c_k\phi_{k-2}(x)$$

where $b_k = \frac{\langle x\phi_{k-1}, \phi_{k-1}\rangle}{\langle \phi_{k-1}, \phi_{k-1}\rangle}$, $c_k = \frac{\langle x\phi_{k-1}, \phi_{k-2}\rangle}{\langle \phi_{k-2}, \phi_{k-2}\rangle}$

Proof: As $\phi_k(x)$ is polynomial of degree k, we know ϕ_k can be written as

$$\phi_k(x) = (x - b_k)\phi_{k-1}(x) - c_k\phi_{k-2}(x) - (a_{k-3}\phi_{k-3}(x) + \dots + a_0\phi_0(x))$$

where $a_0, \dots, a_{k-3}, b_k, c_k \in \mathbb{R}$ and each ϕ_i & ϕ_j are orthogonal if $i \neq j$

thus, for $j \leq k-3$

$$\langle \phi_k, \phi_j\rangle = 0 = \langle x\phi_{k-1}, \phi_j\rangle - b_k\langle \phi_{k-1}, \phi_j\rangle - c_k\langle \phi_{k-2}, \phi_j\rangle - a_{k-3}\langle \phi_{k-3}, \phi_j\rangle - \dots - a_0\langle \phi_0, \phi_j\rangle$$

$$= 0 + \dots + 0 + a_j\langle \phi_j, \phi_j\rangle + 0 + \dots + 0 \quad \text{as } \langle \phi_i, \phi_j\rangle = 0 \text{ for } i \neq j$$

$$0 = a_j\langle \phi_j, \phi_j\rangle \quad \text{so} \quad a_j = 0 \quad \text{b/c} \quad \langle \phi_j, \phi_j\rangle \neq 0$$

so $a_j = 0$ $\forall j \leq k-3$. Note for $j \in \{k-2, k-1\}$, $\langle x\phi_{k-1}, \phi_j\rangle \neq 0$ so more is needed.

Now for $j = k-2$,

$$\langle \phi_k, \phi_{k-2}\rangle = 0 = \langle x\phi_{k-1}, \phi_{k-2}\rangle - b_k\cancelto{0}{\langle \phi_{k-1}, \phi_{k-2}\rangle} - c_k\langle \phi_{k-2}, \phi_{k-2}\rangle$$

$$c_k\langle \phi_{k-2}, \phi_{k-2}\rangle = \langle x\phi_{k-1}, \phi_{k-2}\rangle$$

$$c_k = \frac{\langle x\phi_{k-1}, \phi_{k-2}\rangle}{\langle \phi_{k-2}, \phi_{k-2}\rangle}$$

And for $j = k-1$,

$$\langle \phi_k, \phi_{k-1}\rangle = 0 = \langle x\phi_{k-1}, \phi_{k-1}\rangle - b_k\langle \phi_{k-1}, \phi_{k-1}\rangle - c_k\cancelto{0}{\langle \phi_{k-2}, \phi_{k-1}\rangle}$$

$$b_k\langle \phi_{k-1}, \phi_{k-1}\rangle = \langle x\phi_{k-1}, \phi_{k-1}\rangle$$

$$b_k = \frac{\langle x\phi_{k-1}, \phi_{k-1}\rangle}{\langle \phi_{k-1}, \phi_{k-1}\rangle}$$

$\therefore$ $\phi_k(x) = (x - b_k)\phi_{k-1}(x) - c_k\phi_{k-2}(x)$

w/ b_k and c_k as defined. ∎

5) Proof that $T_n(x) = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right)$ w/ $x = \frac{1}{2}\left(z + \frac{1}{z}\right)$ accurately generates Chebychev Polynomials:

T_0: $T_0(x) = \frac{1}{2}\left(z^0 + \frac{1}{z^0}\right) = \frac{1}{2}(1+1) = 1$ ✓ correct as $T_0(x) = 1$

T_1: $T_1(x) = \frac{1}{2}\left(z + \frac{1}{z}\right) = x$ ✓ correct as $T_1(x) = x$

Satisfies 3-term recursion by induction

BC: $\underline{n=2}$ $T_2(x) = 2xT_1(x) - T_0(x)$

$= 2x(x) - 1$

$= 2\left(\frac{1}{4}\left(z + \frac{1}{z}\right)^2\right) - 1$

$= \frac{1}{2}\left(z^2 + 2 + \frac{1}{z^2}\right) - 1 = \frac{1}{2}\left(z^2 + \frac{1}{z^2}\right)$ ✓

$\underline{IH}$: Assume $T_n(x) = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right)$ $\forall n \leq k$. Now consider case '$k+1$'

$T_{k+1}(x) = 2x\,T_k(x) - T_{k-1}(x)$

$= 2\left(\frac{1}{2}\left(z + \frac{1}{z}\right)\right)\left(z^k + \frac{1}{z^k}\right)\frac{1}{2} - \frac{1}{2}\left(z^{k-1} + \frac{1}{z^{k-1}}\right)$ (by IH)

$= \frac{1}{2}\left(z^{k+1} + \frac{1}{z^{k+1}} + \cancel{z^{k-1}} + \cancel{\frac{1}{z^{k-1}}} - \cancel{z^{k-1}} - \cancel{\frac{1}{z^{k-1}}}\right)$

$= \frac{1}{2}\left(z^{k+1} + \frac{1}{z^{k+1}}\right)$ ✓

$\therefore$ $T_n(x) = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right)$ accurately generates Chebychev Polynomials $\forall n \in \mathbb{N}$. ∎