

$$1) P(x) = (x-2)^9 = x^9 - 18x^8 + 144x^7 - 672x^6 + 2016x^5 - 4032x^4 + 5376x^3 - 4608x^2 + 2304x - 512$$

(i)-(ii): Plots attached on next page

(iii) Difference b/w the plots is that the first is much more jagged in its graph while the second is a bit rounder and smoother, like the plot of a polynomial should be. The discrepancy is most likely caused by round-off error. In the second graph, there is only one "addition" operation and then one exponential; however, in the first, a large # of multiplication and addition operations of numbers w/ many decimal places causes round-off error, which is only propagated the more operations are completed. Therefore, I would say the second graph is more accurate, as it resembles more what a polynomial "should" look like (smooth/rounded/differentiable).

2) How would you perform floating calculations to avoid cancellations?

i) Evaluate  $\sqrt{x+1} - 1$  for  $x \approx 0$

To avoid loss of cancelling significant digits in subtraction of nearly equal #s, I would rewrite  $\sqrt{x+1} - 1$  to  $\downarrow$

$$\frac{\sqrt{x+1} - 1}{1} \cdot \frac{(\sqrt{x+1} + 1)}{(\sqrt{x+1} + 1)} \rightarrow \frac{(x+1) - 1}{\sqrt{x+1} + 1} = \frac{x}{\sqrt{x+1} + 1}$$

now we have less cancellation as we are adding not subtracting. Less rel. error  
Evaluate that instead

ii) Evaluate  $\sin(x) - \sin(y)$  for  $x \approx y$

I would follow same process for same reasons as in (i)

$$\frac{\sin(x) - \sin(y)}{1} \cdot \frac{(\sin(x) + \sin(y))}{(\sin(x) + \sin(y))} \Rightarrow \frac{\sin^2(x) - \sin^2(y)}{\sin(x) + \sin(y)} = \frac{\sin(x+y)\sin(x-y)}{\sin(x) + \sin(y)}$$

which would produce answers w/ less rel. error when evaluated

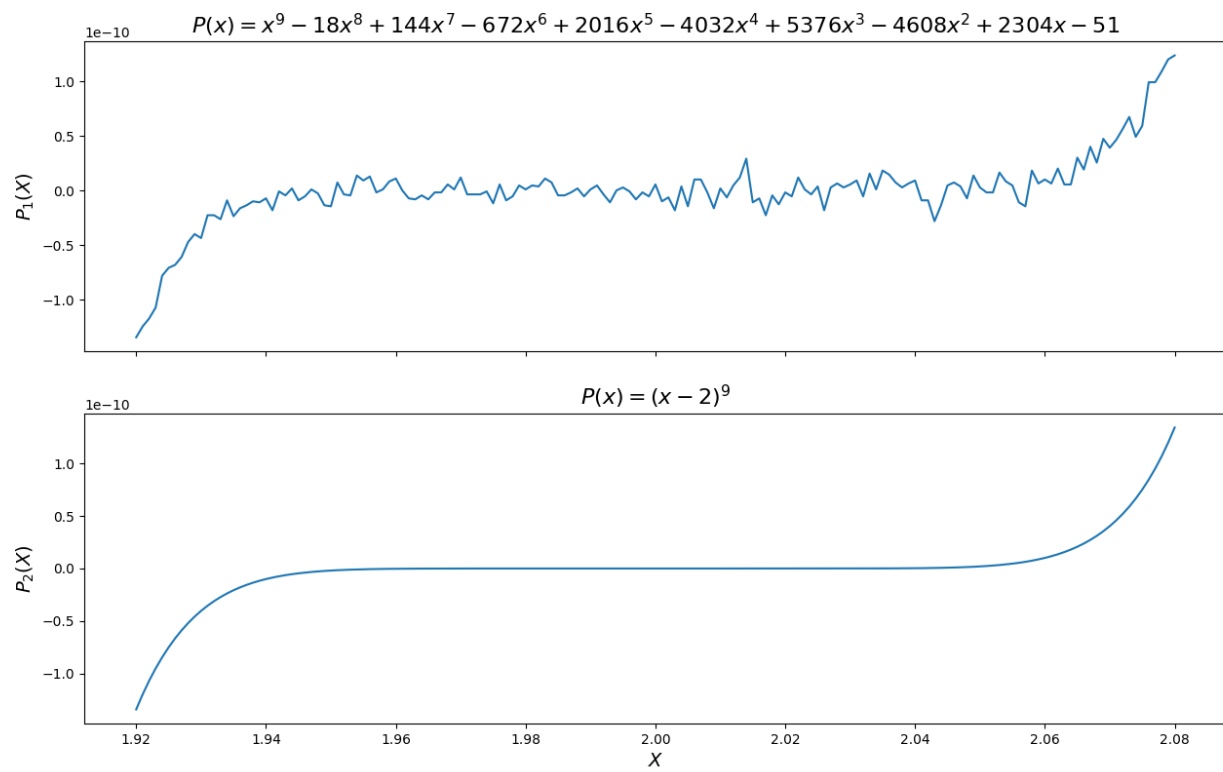
iii) Evaluate  $\frac{1 - \cos(x)}{\sin(x)}$  for  $x \approx 0$

rewrite to:

$$\frac{1 - \cos(x)}{\sin(x)} \cdot \frac{(1 + \cos(x))}{(1 + \cos(x))} = \frac{1 - \cos^2(x)}{\sin(x)(1 + \cos(x))} = \frac{\sin^2(x)}{\sin(x)(1 + \cos(x))} = \frac{\sin(x)}{1 + \cos(x)}$$

and evaluate that instead for more accurate output.

## Plot for Problem 1



$$3) f(x) = (1+x+x^3)\cos(x)$$

↓ 2nd degree Taylor Polynomial of  $f(x)$  about  $x_0 = 0$

$$P_2(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2}(x)^2$$

$$f(x) = (1+x+x^3)\cos(x) \Rightarrow f(0) = (1+0+0)1 = 1$$

$$f'(x) = (3x^2+1)\cos(x) - (x^3+x+1)\sin(x) \Rightarrow f'(0) = 1 \cdot 1 - 0 = 1$$

$$f''(x) = -2(3x^2+1)\sin(x) - (x^3+x+1)\cos(x) + 6x\cos(x) \Rightarrow f''(0) = -1 \cdot 1 = -1$$

$$f'''(x) = (-x^3-17x+1)\sin(x) + (3-9x^2)\cos(x) \Rightarrow f'''(0) = 3$$

$$P_2(x) = 1 + x - \frac{1}{2}x^2$$

$$a) P_2(0.5) = 1 + 0.5 - \frac{1}{2}(0.5)^2 = 1.5 - \frac{1}{2}\left(\frac{1}{4}\right) = \boxed{\frac{11}{8} \approx f(0.5)} \leftarrow \text{approximation}$$

Error bound:

$$|f(0.5) - P_2(0.5)| \leq \frac{f'''(0.5)(0.5)^3}{6} = \frac{3}{6}(0.5)^3 = \frac{1}{16}$$

Actual error:

$$|f(0.5) - P_2(0.5)| = |1.426 - \frac{11}{8}| = 0.05107 < 0.0625$$

$$b) |f(x) - P_2(x)| \leq \frac{f'''(x)x^3}{6} \leq \boxed{\frac{(x^3-17x+1)\sin(x) + (3-9x^2)\cos(x)}{6} x^3} \leftarrow \text{error bound for } P_2(x)$$

$$c) \int_0^1 f(x) dx \approx \int_0^1 (1+x-\frac{1}{2}x^2) dx$$

$$\approx x + \frac{1}{2}x^2 - \frac{1}{6}x^3 \Big|_0^1$$

$$\approx \boxed{\frac{4}{3}}$$

$$d) \left| \int_0^1 f(x) dx - \int_0^1 P_2(x) dx \right| \leq \left| \int_0^1 \frac{(x^3-17x+1)\sin(x) + (3-9x^2)\cos(x)}{6} x^3 \right|$$

$$\leq \boxed{0.4428381}$$



4) Consider quadratic equation  $x^2 + 56x + 1 = 0$

a) Assuming we can calculate square root w/ 3 correct decimals,

$$\begin{aligned} r_1^* &= \frac{56 + \sqrt{(-56)^2 - 4(1)(1)}}{2(1)} \\ &= \frac{56 + \sqrt{3132}}{2} = \frac{56 + 55.964}{2} \\ &= \frac{111.964}{2} = \boxed{55.982} \end{aligned}$$

$$\begin{aligned} r_2^* &= \frac{56 - \sqrt{(-56)^2 - 4(1)(1)}}{2(1)} \\ &= \frac{56 - \sqrt{3132}}{2} = \frac{56 - 55.964}{2} \\ &= \frac{0.036}{2} = \boxed{0.018} \end{aligned}$$

Actual roots  $\Rightarrow r_1 = 28 + 3\sqrt{87}, r_2 = 28 - 3\sqrt{87}$

s.t. rel. error:  $\downarrow$

$$\begin{aligned} \text{root 1: } \frac{|r_1 - r_1^*|}{|r_1|} &= \frac{|28 + 3\sqrt{87} - 55.982|}{|28 + 3\sqrt{87}|} & \text{root 2: } \frac{|r_2 - r_2^*|}{|r_2|} &= \frac{|28 - 3\sqrt{87} - 0.018|}{|28 - 3\sqrt{87}|} \\ &= \underline{2.45005 \cdot 10^{-6}} & &= \underline{7.67846 \cdot 10^{-3}} \end{aligned}$$

$\leftarrow$  bad root  $\rightarrow$

b) two relations that better approximate "bad" root

relation 1: (given)  $(x - r_1)(x - r_2) = 0 = x^2 - 56x + 1$

$$x^2 - r_2x - r_1x + r_1r_2 = x^2 - 56x + 1$$

s.t.  $r_2 + r_1 = 56$  and  $r_1r_2 = 1$

now, using the "good" root  $r_1 = 55.982$ , we can find better approx. for  $r_2$  by "plugging in"  $r_1$ . If we use the 2nd equation  $r_1r_2 = 1$  b/c the first would involve subtracting 2 figs that are very similar, which we established was less accurate in problem (2) of this hw.

$\therefore r_2 = 1/r_1 = 1/55.982 = \boxed{0.017862} \leftarrow$  new  $r_2$

relative error: for new  $r_2$

$$\hookrightarrow \frac{|28 - 3\sqrt{87} - 0.017862|}{|28 - 3\sqrt{87}|} = \boxed{2.450058 \cdot 10^{-6}} \leftarrow \text{new } r_2 \text{ rel. error}$$

way better!

relation 2: re-writing quadratic formula, as done on pg. 26 of textbook (and in lecture)

into  $r_2 = \frac{-2c}{b + \sqrt{b^2 - 4ac}}$

solving, we get  $r_2 = \frac{-2}{-56 + \sqrt{56^2 - 4}} = \frac{-2}{-111.964} = \boxed{0.0178628}$

Continued  $\rightarrow$

4) continued

b) continued

$$\frac{\text{relative error for new } r_2}{\text{new}} = \frac{|28 - 3\sqrt{87} - 0.0178628|}{|28 - 3\sqrt{87}|} = \boxed{2.450060 \cdot 10^{-6}}$$

which is also more accurate than original.

$\therefore$  both relations can be used to similar effectiveness in finding a more accurate root. turning "bad"  $\rightarrow$  "good #2"

5) Cancellation of Terms:

a) Find upper bounds on the absolute error  $|\Delta y|$  and relative error  $\frac{|\Delta y|}{|y|}$

As  $\Delta y = \Delta x_1 - \Delta x_2$ , upper bound on absolute error can be given by

$$|\Delta y| = |\Delta x_1 - \Delta x_2| \leq |\Delta x_1| + |\Delta x_2|$$

We can continue this line of thought w/ relative error

$$\frac{|\Delta y|}{|y|} = \frac{|\Delta y|}{|x_1 - x_2|} \leq \frac{|\Delta x_1| + |\Delta x_2|}{|x_1 - x_2|}$$

relative error then becomes large when  $x_1, x_2$  are really close

b) Manipulation of  $\cos(x+\delta) - \cos(x)$  to expression w/o subtraction

$$\begin{aligned} \cos(x+\delta) - \cos(x) &= -2\sin\left(\frac{x+\delta+x}{2}\right)\sin\left(\frac{x+\delta-x}{2}\right) \leftarrow \text{by trig. identities} \\ &= -2\sin\left(x+\frac{\delta}{2}\right)\sin\left(\frac{\delta}{2}\right) \leftarrow \text{new expression} \end{aligned}$$

Tables for each expression is attached on next page.

One can see how new expression doesn't cancel to become 0 when  $\delta$  is low.

c) Using Taylor expansion  $f(x+\delta) - f(x) = \delta f'(x) + \frac{\delta^2}{2!} f''(\xi)$ ,  $\xi \in [x, x+\delta]$

We can create this algorithm for approximating  $\cos(x+\delta) - \cos(x)$

$$\left\{ \begin{aligned} \rightarrow -\delta \sin(x) - \frac{\delta^2}{2} \cos(\xi) &= \left[ -\left( \delta \sin(x) + \frac{\delta^2}{2} \cos(\xi) \right) \right] \end{aligned} \right.$$

choose algorithm b/c it uses  $\cos(x)$  as  $f(x)$  and it doesn't use subtraction so cancellation less likely

$\rightarrow$  For  $\xi$ , I plot and calculate using both bounds, w/ the real value being somewhere in b/w

$\rightarrow$  Comparing to the techniques from part (b), the approximation is very close to my expression at lower values for  $\delta$ , then diverges w/  $\delta$  at  $\pi$ , which is expected for Taylor.

## Tables for Problem 5

X = Pi					
$\delta$	Given expression (part b)	My expression (part b)	Taylor w $\xi=x$ (part c)	Taylor w $\xi=x+6$ (part c)	
1.000000e-16	0.000000e+00	-1.224647e-32	-7.246468e-33	-7.246468e-33	
1.000000e-15	0.000000e+00	3.216245e-31	3.775353e-31	3.775353e-31	
1.000000e-14	0.000000e+00	4.762517e-29	4.877535e-29	4.877535e-29	
1.000000e-13	0.000000e+00	5.005962e-27	4.987754e-27	4.987754e-27	
1.000000e-12	0.000000e+00	4.999220e-25	4.998775e-25	4.998775e-25	
1.000000e-11	0.000000e+00	4.999878e-23	4.999878e-23	4.999878e-23	
1.000000e-10	0.000000e+00	4.999988e-21	4.999988e-21	4.999988e-21	
1.000000e-09	0.000000e+00	4.999999e-19	4.999999e-19	4.999999e-19	
1.000000e-08	0.000000e+00	5.000000e-17	5.000000e-17	5.000000e-17	
1.000000e-07	4.996004e-15	5.000000e-15	5.000000e-15	5.000000e-15	
1.000000e-06	5.000445e-13	5.000000e-13	5.000000e-13	5.000000e-13	
1.000000e-05	5.000000e-11	5.000000e-11	5.000000e-11	5.000000e-11	
1.000000e-04	5.000000e-09	5.000000e-09	5.000000e-09	5.000000e-09	
1.000000e-03	5.000000e-07	5.000000e-07	5.000000e-07	4.999997e-07	
1.000000e-02	4.999958e-05	4.999958e-05	5.000000e-05	4.999750e-05	
1.000000e-01	4.995835e-03	4.995835e-03	5.000000e-03	4.975021e-03	
1.000000e+00	4.596977e-01	4.596977e-01	5.000000e-01	2.701512e-01	

X = 10e6					
$\delta$	Given expression (part b)	My expression (part b)	Taylor w $\xi=x$ (part c)	Taylor w $\xi=x+6$ (part c)	
1.000000e-16	0.000000e+00	-4.205478e-17	-4.205478e-17	-4.205478e-17	
1.000000e-15	0.000000e+00	-4.205478e-16	-4.205478e-16	-4.205478e-16	
1.000000e-14	0.000000e+00	-4.205478e-15	-4.205478e-15	-4.205478e-15	
1.000000e-13	0.000000e+00	-4.205478e-14	-4.205478e-14	-4.205478e-14	
1.000000e-12	0.000000e+00	-4.205478e-13	-4.205478e-13	-4.205478e-13	
1.000000e-11	0.000000e+00	-4.205478e-12	-4.205478e-12	-4.205478e-12	
1.000000e-10	0.000000e+00	-4.205478e-11	-4.205478e-11	-4.205478e-11	
1.000000e-09	-7.833313e-10	-4.205478e-10	-4.205478e-10	-4.205478e-10	
1.000000e-08	-3.916656e-09	-4.205478e-09	-4.205478e-09	-4.205478e-09	
1.000000e-07	-4.229989e-08	-4.205477e-08	-4.205477e-08	-4.205477e-08	
1.000000e-06	-4.206485e-07	-4.205473e-07	-4.205473e-07	-4.205473e-07	
1.000000e-05	-4.205660e-06	-4.205433e-06	-4.205433e-06	-4.205433e-06	
1.000000e-04	-4.205017e-05	-4.205024e-05	-4.205024e-05	-4.205024e-05	
1.000000e-03	-4.200942e-04	-4.200941e-04	-4.200942e-04	-4.200939e-04	
1.000000e-02	-4.160045e-03	-4.160045e-03	-4.160114e-03	-4.159906e-03	
1.000000e-01	-3.745215e-02	-3.745215e-02	-3.751843e-02	-3.733117e-02	
1.000000e+00	6.319134e-02	6.319134e-02	3.308740e-02	1.491731e-03	