APPM 4600 - HW #5 Gustav Cedergrund

1) Let

$$f(x,y) = 3x^2 - y^2 = 0, \quad g(x,y) = 3xy^2 - x^3 - 1 = 0$$

a) Iterate using scheme w/ $x_0 = y_0 = 1$

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - \begin{bmatrix} 1/6 & 1/18 \\ 0 & 1/6 \end{bmatrix} \begin{bmatrix} f(x_n, y_n) \\ g(x_n, y_n) \end{bmatrix}$$

Iteration output attached. Converges at a rate of order 1 (linearly) w/ $\lambda \approx 0.5$.

b) $\begin{bmatrix} 1/6 & 1/18 \\ 0 & 1/6 \end{bmatrix}$ is the inverse Jacobian evaluated at (x_0, y_0).
This resembles "Lazy Newton" outlined in class.

c) Iteration attached.
Converges quadratically

d) From numerical result, solution at

$$x = 0.5, \quad y = 0.8660254037844387$$

Verification

$$f(x,y) = 3(0.5)^2 - (0.86602\ldots)^2 = -1.11 \cdot 10^{-16} \approx 0$$
$$g(x,y) = 3(0.5)(0.86602\ldots)^2 - (0.5)^3 - 1 = 2.22 \cdot 10^{-16} \approx 0$$

Not entirely exact, but very close

---

2) Based on n-dimensional fixed point theorem, find region D in xy-plane where fixed point iteration $\to$ converges $\forall (x_0, y_0) \in D$

$$x_{n+1} = \frac{1}{\sqrt{2}} \sqrt{1 + (x_n + y_n)^2} - \frac{2}{3} \qquad y_{n+1} = \frac{1}{\sqrt{2}} \sqrt{1 + (x_n - y_n)^2} - \frac{2}{3}$$

As dimension $n = 2$, $\left| \frac{\partial g_i(\vec{x})}{\partial x_j} \right| \leq \frac{K}{2} \leq \frac{1}{2}$ as $K \leq 1$ $\forall i,j \in \{1,2\}$

Note partial derivatives: Let $f(x_n) = x_{n+1}$, $g(x_n) = y_{n+1}$

$$f_x = \frac{x+y}{\sqrt{2(x+y)^2 + 2}} \qquad g_x = \frac{x-y}{\sqrt{2(x-y)^2 + 2}}$$

$$f_y = \frac{x+y}{\sqrt{2(x+y)^2 + 2}} \qquad g_y = \frac{y-x}{\sqrt{2(x-y)^2 + 2}}$$

which all need to be $\in$ less than 1/2

continued

2) continued

$|f_x| \le \frac{1}{2}$

$\frac{x+y}{\sqrt{2(x+y)^2+2}} \le \frac{1}{2} \Rightarrow (x+y)^2 \le \frac{1}{4}(2(x+y)^2+2)$

$\frac{1}{2}(x+y)^2 \le \frac{1}{2} \Rightarrow \underline{(x+y)^2 \le 1}$ ← same for f_y

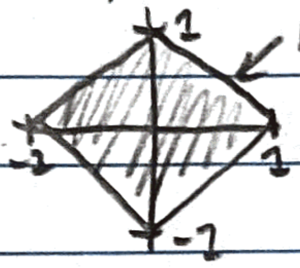
$|g_x| \le 1/2$

$\frac{x-y}{\sqrt{2(x-y)^2+2}} \le \frac{1}{2} \Rightarrow (x-y)^2 \le \frac{1}{4}(2(x-y)^2+2)$

$\frac{1}{2}(x-y)^2 \le \frac{1}{2} \Rightarrow (x-y)^2 \le 1$ ← same for g_y

Thus, D at this point defined by $(x+y)^2 \le 1$ and $(x-y)^2 \le 1$

s.t. D



Still need to check $\forall (x_0,y_0) \in D$, $f(x_0,y_0), g(x_0,y_0) \in D$.

Thus we look at equations.

$x = \frac{1}{\sqrt{2}}\sqrt{1+(x+y)^2} - \frac{2}{3}$ $\qquad y = \frac{1}{\sqrt{2}}\sqrt{1+(x-y)^2} - \frac{2}{3}$

See equations max when $(x+y)^2 = (x-y)^2 = 1$ which is in D

and min when $(x+y)^2 = (x-y)^2 = 0$ which is also in D

Max: $(x+y)^2 = (x-y)^2 = 1$	Min: $(x+y)^2 = (x-y)^2 = 0$
$x = \frac{1}{\sqrt{2}}\sqrt{1+1} - \frac{2}{3} = 1 - \frac{2}{3} = 1/3$	$x = \frac{1}{\sqrt{2}}\sqrt{1+0} - \frac{2}{3} = \frac{\sqrt{2}}{2} - 2/3$
$y = \frac{1}{\sqrt{2}}\sqrt{1+1} - \frac{2}{3} = 1 - \frac{2}{3} = 1/3$	$y = \frac{1}{\sqrt{2}}\sqrt{1+0} - 2/3 = \sqrt{2}/2 - 2/3$
s.t. $(1/3 + 1/3)^2 = 4/9 \le 1$	s.t. $((\frac{\sqrt{2}}{2} - 2/3) \cdot 2)^2 \approx 0.0065 \le 1$
$(1/3 - 1/3)^2 = 0 \le 1$	$(0)^2 = 0 \le 1$
so max $\in D$	so min $\in D$

Thus we see $\forall (x_0,y_0) \in D$, $f(x_0,y_0) \in D$ and $g(x_0,y_0) \in D$.

$$\boxed{D = \{(x,y) : (x+y)^2 \le 1, (x-y)^2 \le 1\}}$$

3)

a) Derive iteration scheme

$$\begin{cases} x_{n+1} = x_n - d f_x \\ y_{n+1} = y_n - d f_y \end{cases} \quad w/ \quad d = f/(f_x^2 + f_y^2)$$

We can derive this iteration by finding a second iterate across the line normal to the tangent at the curve defined by $f(x,y)=0$.

In this case, as all x_n, y_n are at 0-plane, we can define new iterate g as $g(x_n, y_n) = 0$

And as it is the normal, we put $g_x = f_y$ and $g_y = -f_x$

So

$$J = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \underline{\begin{bmatrix} f_x & f_y \\ f_y & -f_x \end{bmatrix}}$$

$\therefore$

$$J^{-1} = \begin{bmatrix} \frac{f_x}{f_x^2+f_y^2} & \frac{f_y}{f_x^2+f_y^2} \\ \frac{f_y}{f_x^2+f_y^2} & \frac{-f_x}{f_x^2+f_y^2} \end{bmatrix}$$

Now involving Newton's method according to formula, with $g(x_n, y_n) = 0$

$$x_{n+1} = x_n - J^{-1}[x_n] F[x_n]$$

We get

$$\underline{\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - \begin{bmatrix} f \cdot f_x / (f_x^2 + f_y^2) \\ f \cdot f_y / (f_x^2 + f_y^2) \end{bmatrix}} \quad \blacksquare$$

b) Iteration applied to 3d attached.

Here we can see that iteration is quadratically convergent.