APPM 4600 HW #3 - Gustav Cedergrund

1) Consider $2x-1=\sin(x)$

a) An interval $[a,b]$ on which equation has root r is on $[0,\frac{\pi}{2}]$ b/c

$x=0 \Rightarrow 2(0)-1-\sin(0) = 0-1-0 = \underline{-1} < 0$

$x=\frac{\pi}{2} \Rightarrow 2(\frac{\pi}{2})-1-\sin(\frac{\pi}{2}) = \pi-1-1 = \underline{\pi-2} > 0$

$\therefore$ by IVT, as equation is continuous and $-1 < 0 < \pi-2$, then $\exists$ root 'r' s.t. $\underline{2(r)-1-\sin(r)=0}$.

b) To prove that r from (a) is only root, we can look to 1st derivative,

$\frac{d}{dx}\left(2x-1-\sin(x)\right) = 2-\cos(x) > 0 \quad \forall x \in \mathbb{R}$.

$\therefore$ the equation is monotonically increasing $\forall x \in \mathbb{R}$ s.t. it continues to increase forever. Thus, 'r' is the only root b/c the equation only goes past 0 once.

c) Using bisection code from class, we can approximate r to 8 correct dec. places as

$\boxed{r = 0.88786221}$ calling script w/ # of iterations attached on next page

2) function: $f(x)=(x-5)^9$ is viable candidate for bisection

a) bisection method + results for f(x) attached

b) bisection method + results for expanded f(x) attached

c) the difference in computation + result for bisection in this example is caused by error in computation for the function. The second version of the function that is expanded becomes very inaccurate as a result of many computations being done near the root, so it is unable to check if the guess for the root at each iteration is close to the real root. As a result, it produces 5.128 rather than ~5.0 b/c the error propagation causes an input of 5.128 resulting in output under the tolerance of root, while actually, a root does not exist there.

3)

a) Using theorem 2.1, we know that bisection method approximates zero p of function f w/

$$|p_n - p| < \frac{b-a}{2^n}, \text{ when } n \geq 1$$

$\therefore$ w/ $a=1, b=4$ on $f(x) = x^3 + x - 4$, to predict w/ accuracy of 10^{-3}

$$\hookrightarrow \quad 10^{-3} < \frac{4-1}{2^n}$$

$$2^n < 3 \cdot 10^3$$

$$\log_2(3000) = 11.55 < n \quad \text{so} \quad n \geq 12$$

We need at most $\boxed{12}$ iterations to find p

b) Using bisection code (output attached), it takes 11 iterations to get root w/ this accuracy. This is less than upper bound, which checks out.

4) Evaluate convergence of following iterations

a) $x_{n+1} = -16 + 6x_n + \frac{12}{x_n}$, $x_* = 2$

Let $x_{n+1} = g(x_n) = -16 + 6x_n + \frac{12}{x_n}$

first, $x_* = 2$ is a fixed point b/c $g(x_*) = -16 + 6(2) + \frac{12}{2} = 2 = x_*$

However, as $g'(x_n) = 6 - 12x_n^{-2}$ evaluated at x_* is

$$g'(2) = 6 - \frac{12}{2^2} = 6 - 3 = 3$$

$|g'(x_*)| \not< 1$, so by Thm 2.4 x_{n+1} does not converge to x_*

b) $x_{n+1} = \frac{2}{3}x_n + \frac{1}{x_n^2}$, $x_* = 3^{1/3}$

Let $x_{n+1} = g(x_n)$. $x_* = 3^{1/3}$ is a fixed point as $g(3^{1/3}) = \frac{2}{3}(3^{1/3}) + \frac{1}{3^{2/3}} = 3^{1/3}$

And $g'(x_n) = \frac{2}{3} - 2x_n^{-3}$ s.t $g'(3^{1/3}) = \frac{2}{3} - 2(3^{1/3})^{-3} = 0 < 1$

s.t $\{x_n\}$ converges when x_0 sufficiently close to x_* by Thm 2.4.

However as $g'(3^{1/3}) = 0$ rate of convergence is not linear.

Note: $\lim_{n\to\infty} \frac{|x_{n+1} - x_*|}{|x_n - x_*|^2} = \lim_{n\to\infty} \frac{|g''(x_n)|}{2}$, and b/c $g''(x_n) = 6x_n^{-4}$

$$\frac{g''(x_*)}{2} = \frac{6}{2 \cdot 3^{4/3}} = 3^{-1/3} < 1. \quad \therefore \boxed{\text{order of convergence is 2}}$$

4) continued

c) $x_{n+1} = \frac{12}{1+x_n}$, $x_* = 3$

Let $x_{n+1} = g(x_n)$, then $g(3) = \frac{12}{1+3} = 3$ $\therefore$ x_* is fixed point,

Also, as $g'(x_n) = -12/(1+x_n)^2$ at x_* is $|g'(3)| = |-12/(1+3)^2| = 3/4$ and $0 < 3/4 < 1$. The sequence converges linearly w/ order 1 and asymptotic constant $3/4 = \lambda$.

5) Consider scalar equation $x - 4\sin(2x) - 3 = 0$

a) Plot for $f(x) = x - 4\sin(2x) - 3$ attached.
There are 5 zero crossings

b) Program output attached
Empirically only 2/5 roots can be found.
↳ 1 @ -0.5444424006, 1 @ 3.1618264865
The other roots can not be found by fixed root method, b/c, $|f'(x)|$ at fixed points are > 1 s.t Fixed Point Theorem 2.4 doesn't guarantee the fixed point method work. Simply, this is b/c successive iterations will grow away from root.