APPM 4600 — HOMEWORK # 9

1. Find the least squares solution to the overdetermined linear system

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

2. Find the vector $\boldsymbol{x}$ that minimizes the quantity $E^2 = b_1^2 + 4b_2^2 + 25b_3^2 + 9b_4^2$, when it holds that

$$\begin{bmatrix} 1 & 3 \\ 6 & -1 \\ 4 & 0 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

   Hint: When we solve a linear system of equations $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$, multiplication from the left with a nonsingular matrix will leave the solution unchanged. This is **not** the case when finding the least squares solution to an overdetermined system. Exploit this and multiply the system above with a suitable diagonal matrix, so that the problems becomes a regular least squares problem (for which we can apply the normal equation approach.)

3. If a function is identically zero over an interval, all its derivatives must also be identically zero over the same interval. Based on this observation:

   (a) Prove that $\{1, x, x^2, \ldots, x^n\}$ are linearly independent.

   (b) Show that the function set

$$\{1, \cos(x), \cos(2x), \ldots, \cos(nx), \sin(x), \ldots, \sin(nx)\}$$

   is linearly independent (also over any interval).

4. Prove the three-term recursion formula for orthogonal polynomials:

$$\phi_k(x) = (x - b_k)\phi_{k-1}(x) - c_k\phi_{k-2}(x)$$

   where

$$b_k = \frac{< x\phi_{k-1}, \phi_{k-1} >}{< \phi_{k-1}, \phi_{k-1} >} \quad c_k = \frac{< x\phi_{k-1}, \phi_{k-2} >}{< \phi_{k-2}, \phi_{k-2} >}$$

   Hint: Since $\phi_k(x)$ is a polynomial of degree k and of the form $\phi_k = x^k + \{\text{lower order terms}\}$, we can clearly select b_k and c_k so that the right hand side (RHS) of (1) matches $\phi_k(x)$ for powers x^k , x^{k-1} and x^{k-2} . We have no obvious reason to expect that the two sides will match the other lower order terms. Hence, we would expect to need to include a lot more terms in the RHS to get the two sides to become equal:

$$\phi_k(x) = (x - b_k)\phi_{k-1}(x) - c_k\phi_{k-2}(x) - \{a_{k-3}\phi_{k-3}(x) + a_{k-4}\phi_{k-4}(x) + \cdots + a_0\phi_0(x)\} \quad (1)$$

   We now need to show that all these a's are in fact are zero. To show that $a_j = 0$, $j \leq k-3$, we form the scalar product of (1) with $\phi_j(x)$ for $j = 0, \ldots, k-1$. You need to show that everything in (1) apart from $a_j < \phi_j, \phi_j >$ then vanishes, thereby showing that $a_j = 0$, $j \leq k-3$. After that, it remains to determine the values of b_k and c_k . These coefficients follow by again forming suitable scalar products.

5. One of the many formulas for computing the Chebychev polynomials $T_n(x)$ is

$$T_n(x) = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right), \tag{2}$$

where z is implicitly defined through x via $x = \frac{1}{2}\left(z + \frac{1}{z}\right)$. Confirm that the formula (2) indeed generates the same polynomials as the standard definition of the Chebychev polynomials. **Hint:** One way would be to verify that it produces the correct result for T_0 and T_1 and that it satisfies the 3 term recursion.