APPM 4600 - HW #5 Gustav Cedergrund

1) Let

$$f(x,y)=3x^2-y^2=0, \quad g(x,y)=3xy^2-x^3-1=0$$

a) Iterate using scheme w/ $x_0=y_0=1$

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - \begin{bmatrix} 1/6 & 1/18 \\ 0 & 1/6 \end{bmatrix} \begin{bmatrix} f(x_n,y_n) \\ g(x_n,y_n) \end{bmatrix}$$

Iteration output attached. Converges at a rate of order 1 (linearly) w/ $\lambda \approx 0.5$.

b) $\begin{bmatrix} 1/6 & 1/18 \\ 0 & 1/6 \end{bmatrix}$ is the inverse Jacobian evaluated at (x_0,y_0).
This resembles "Lazy Newton" outlined in class.

c) Iteration attached.
Converges quadratically

d) From numerical result, solution at

$$x=0.5, \quad y=0.8660254037844387$$

Verification

$$f(x,y)=3(0.5)^2-(0.86602...)^2 = -1.11\cdot10^{-16} \approx 0$$
$$g(x,y)=3(0.5)(0.86602...)^2-(0.5)^3-1 = 2.22\cdot10^{-16} \approx 0$$

Not entirely exact, but very close

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2) Based on n-dimensional fixed point theorem, find region D in xy-plane where fixed point iteration converges $\forall (x_0,y_0) \in D$

$$x_{n+1}=\frac{1}{\sqrt{2}}\sqrt{1+(x_n+y_n)^2}-\frac{2}{3} \qquad y_{n+1}=\frac{1}{\sqrt{2}}\sqrt{1+(x_n-y_n)^2}-\frac{2}{3}$$

As dimension $n=2$, $\left|\frac{\partial g_i(\bar{x})}{\partial x_j}\right| \leq \frac{K}{2} \leq \frac{1}{2}$ as $K\leq 1$ $\forall i,j \in \{1,2\}$

Note partial derivatives: Let $f(x_n)=x_{n+1}$, $g(x_n)=y_{n+1}$

$$f_x=\frac{x+y}{\sqrt{2(x+y)^2+2}} \qquad g_x=\frac{x-y}{\sqrt{2(x-y)^2+2}}$$

which all need to be less than 1/2

$$f_y=\frac{x+y}{\sqrt{2(x+y)^2+2}} \qquad g_y=\frac{y-x}{\sqrt{2(x-y)^2+2}}$$

continued

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(1a)
iteration with initial guess at x0=y0=1:
tolerance @ 5e-09

Found solution after 29 iterations.

the approximate root is [0.5        0.8660254]
f(root) = [-4.3136214600281164e-09, -4.714810764028243e-10]

running order of convergence for alpha = 1
[[0.22374855 0.22374855]
 [0.47528213 0.47528213]
 [0.5900429  0.5900429 ]
 [0.61478579 0.61478579]
 [0.58802571 0.58802571]
 [0.54858881 0.54858881]
 [0.50906251 0.50906251]
 [0.4728112  0.4728112 ]
 [0.44479692 0.44479692]
 [0.44300887 0.44300887]
 [0.50178519 0.50178519]
 [0.59368033 0.59368033]
 [0.62106123 0.62106123]
 [0.59409387 0.59409387]
 [0.55422616 0.55422616]
 [0.51493429 0.51493429]
 [0.47873698 0.47873698]
 [0.44891152 0.44891152]
 [0.43944687 0.43944687]
 [0.48377176 0.48377176]
 [0.5774849  0.5774849 ]
 [0.62168904 0.62168904]
 [0.60153744 0.60153744]
 [0.56161959 0.56161959]
 [0.5190582  0.5190582 ]
 [0.47378119 0.47378119]
 [0.41634855 0.41634855]
 [0.31459123 0.31459123]]
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newton's method with initial guess at x0=y0=1:
tolerance @ 5e-16

Found solution after 5 iterations.
the approximate root is [0.5        0.8660254]
f(root) = [-1.1102230246251565e-16, 2.220446049250313e-16]

running order of convergence for alpha = 1
[[2.23748544e-01 2.23748544e-01]
 [1.21023214e-01 1.21023214e-01]
 [2.93860849e-03 2.93860849e-03]
 [4.77656972e-05 4.77656972e-05]]

running order of convergence for alpha = 2
[[0.43224899 0.43224899]
 [1.04491807 1.04491807]
 [0.20964601 0.20964601]
 [1.15962953 1.15962953]]

running order of convergence for alpha = 3
[[8.35040933e-01 8.35040933e-01]
 [9.02185402e+00 9.02185402e+00]
 [1.49565517e+01 1.49565517e+01]
 [2.81528529e+04 2.81528529e+04]]
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2) continued

$|f_x| \le \frac{1}{2}$

$\frac{x+y}{\sqrt{2(x+y)^2+2}} \le \frac{1}{2} \Rightarrow (x+y)^2 \le \frac{1}{4}(2(x+y)^2+2)$

$\frac{1}{2}(x+y)^2 \le \frac{1}{2} \Rightarrow \underline{(x+y)^2 \le 1}$ ← same for f_y

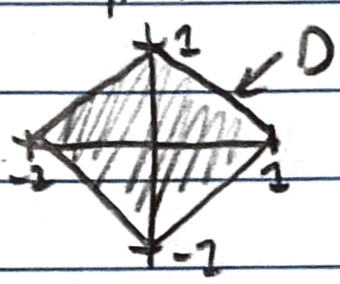
$|g_x| \le 1/2$

$\frac{x-y}{\sqrt{2(x-y)^2+2}} \le \frac{1}{2} \Rightarrow (x-y)^2 \le \frac{1}{4}(2(x-y)^2+2)$

$\frac{1}{2}(x-y)^2 \le \frac{1}{2} \Rightarrow (x-y)^2 \le 1$ ← same for g_y

Thus, D at this point defined by $(x+y)^2 \le 1$ and $(x-y)^2 \le 1$

st D



Still need to check $\forall (x_0,y_0) \in D$, $f(x_0,y_0), g(x_0,y_0) \in D$.

Thus we look at equations.

$x = \frac{1}{\sqrt{2}}\sqrt{1+(x+y)^2} - \frac{2}{3}$ $\qquad y = \frac{1}{\sqrt{2}}\sqrt{1+(x-y)^2} - \frac{2}{3}$

See equations max when $(x+y)^2 = (x-y)^2 = 1$ which is in D

and min when $(x+y)^2 = (x-y)^2 = 0$ which is also in D

Max: $(x+y)^2 = (x-y)^2 = 1$

$x = \frac{1}{\sqrt{2}}\sqrt{1+1} - \frac{2}{3} = 1 - \frac{2}{3} = 1/3$

$y = \frac{1}{\sqrt{2}}\sqrt{1+1} - \frac{2}{3} = 1 - \frac{2}{3} = 1/3$

s.t. $(1/3 + 1/3)^2 = 4/9 \le 1$

$(1/3 - 1/3)^2 = 0 \le 1$

so max $\in D$

Min: $(x+y)^2 = (x-y)^2 = 0$

$x = \frac{1}{\sqrt{2}}\sqrt{1+0} - \frac{2}{3} = \frac{\sqrt{2}}{2} - 2/3$

$y = \frac{1}{\sqrt{2}}\sqrt{1+0} - 2/3 = \sqrt{2}/2 - 2/3$

s.t. $((\frac{\sqrt{2}}{2} - 2/3) \cdot 2)^2 \approx 0.0065 \le 1$

$(0)^2 = 0 \le 1$

so min $\in D$

Thus we see $\forall (x_0,y_0) \in D$, $f(x_0,y_0) \in D$ and $g(x_0,y_0) \in D$.

$$\boxed{D = \{(x,y) : (x+y)^2 \le 1, (x-y)^2 \le 1\}}$$

3)

a) Derive iteration scheme

$$\begin{cases} x_{n+1} = x_n - d f_x \\ y_{n+1} = y_n - d f_y \end{cases} \quad w/ \quad d = f/(f_x^2 + f_y^2)$$

We can derive this iteration by finding a second iterate across the line normal to the tangent at the curve defined by $f(x,y)=0$.

In this case, as all x_n, y_n are at 0-plane, we can define new iterate g as $g(x_n, y_n) = 0$

And as it is the normal, we put $g_x = f_y$ and $g_y = -f_x$

So

$$J = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} f_x & f_y \\ f_y & -f_x \end{bmatrix}$$

$\therefore$

$$J^{-1} = \begin{bmatrix} \frac{f_x}{f_x^2+f_y^2} & \frac{f_y}{f_x^2+f_y^2} \\ \frac{f_y}{f_x^2+f_y^2} & \frac{-f_x}{f_x^2+f_y^2} \end{bmatrix}$$

Now involving Newton's method according to formula, with $g(x_n, y_n) = 0$

$$x_{n+1} = x_n - J^{-1}[x_n]\, F[x_n]$$

We get

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - \begin{bmatrix} f \cdot f_x / (f_x^2 + f_y^2) \\ f \cdot f_y / (f_x^2 + f_y^2) \end{bmatrix}$$

b) Iteration applied to 3d attached.

Here we can see that iteration is quadratically convergent.

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(3b)
iteration with initial guess at x0=y0=z0=1:
tolerance @ 5e-16

Found solution after 6 iterations.

the approximate root is:
  x= 1.3603283832230446
  y= 1.3603283832230446
  z= 1.3603283832230446
f(root) = 0.0

iterations:
[[1.         1.         1.        ]
 [1.10606061 1.42424242 1.42424242]
 [1.09392616 1.36174169 1.36174169]
 [1.09364246 1.36032911 1.36032911]
 [1.09364232 1.36032838 1.36032838]
 [1.09364232 1.36032838 1.36032838]
 [1.09364232 1.36032838 1.36032838]]
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running order of convergence for alpha = 1
[[1.76094847e-01 1.76094847e-01 1.76094847e-01]
 [2.21266952e-02 2.21266952e-02 2.21266952e-02]
 [5.11241374e-04 5.11241374e-04 5.11241374e-04]
 [2.61523416e-07 2.61523416e-07 2.61523416e-07]
 [1.16340316e-03 1.16340316e-03 1.16340316e-03]]

running order of convergence for alpha = 2
[[3.39876736e-01 3.39876736e-01 3.39876736e-01]
 [2.42518401e-01 2.42518401e-01 2.42518401e-01]
 [2.53243077e-01 2.53243077e-01 2.53243077e-01]
 [2.53393911e-01 2.53393911e-01 2.53393911e-01]
 [4.31027770e+09 4.31027770e+09 4.31027770e+09]]

running order of convergence for alpha = 3
[[6.55988505e-01 6.55988505e-01 6.55988505e-01]
 [2.65810933e+00 2.65810933e+00 2.65810933e+00]
 [1.25443791e+02 1.25443791e+02 1.25443791e+02]
 [2.45517114e+05 2.45517114e+05 2.45517114e+05]
 [1.59690936e+22 1.59690936e+22 1.59690936e+22]]
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