HW 10 - Gustav Cedergrund

1) Let $f(x)=\sin(x)$. Determine Padé approximations of degree 6 for following problems.

Note: 6th order Maclaurin for f is $T_6(x) = 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} - 0$

$$= \underline{x - \frac{x^3}{3!} + \frac{x^5}{5!}}$$

a) Both numerator and denominator are cubic,

s.t. $r(x) = \frac{p(x)}{q(x)} = \frac{p_0 + p_1 x + p_2 x^2 + p_3 x^3}{1 + q_1 x + q_2 x^2 + q_3 x^3}$

So need $p_0, p_1, p_2, p_3, q_1, q_2, q_3$ s.t. $T_6(x) = r(x)$

s.t. $\left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right)(1 + q_1 x + q_2 x^2 + q_3 x^3) = p_0 + p_1 x + p_2 x^2 + p_3 x^3$

constant	$p_0 = 0$
x	$p_1 = 1$
x^2	$p_2 = q_1$
x^3	$p_3 = q_2 - \frac{1}{6}$
x^4	$0 = q_3 - \frac{1}{6} q_1$
x^5	$0 = -\frac{1}{6} q_2 + \frac{1}{120}$
x^6	$0 = -\frac{1}{6} q_3 + \frac{1}{120} q_1$

$\therefore$ $p_0 = 0$, $p_1 = 1$

$q_2 = 6/120 = 1/20$

$p_3 = -7/60$

$$\begin{bmatrix} \frac{1}{120} & -\frac{1}{6} \\ -1/6 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow q_1 = 0,\ q_3 = 0,\ p_2 = 0$$

$\therefore$ $\boxed{P_3^3(x) = \frac{x - 7/60\, x^3}{1 + 1/20\, x^2}}$

b) Numerator is quadratic and denominator 4th degree,

s.t. $P_2^4(x) = \frac{a_0 + a_1 x + a_2 x^2}{1 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4}$

w/

$\left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right)(1 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4) = a_0 + a_1 x + a_2 x^2$

term	constant	x	x^2	x^3	x^4	x^5	x^6
eq	$a_0 = 0$	$a_1 = 1$	$a_2 = b_1$	$0 = b_2 - \frac{1}{6}$	$0 = b_3 - \frac{1}{6} b_1$	$0 = b_4 - \frac{1}{6} b_2 + \frac{1}{120}$	$0 = -\frac{1}{6} b_3 + \frac{1}{120} b_1$

$a_0 = 0$, $a_1 = 1$, $b_2 = 1/6$, $b_4 = 7/360$

so $\begin{bmatrix} 1/120 & -1/6 \\ -1/6 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow b_1 = a_2 = b_3 = 0$

$\therefore$ $\boxed{P_2^4(x) = \frac{x}{1 + 1/6\, x^2 + 7/360\, x^4}}$

c) Numerator 4th degree and denominator is quadratic

$$P_4^2(x) = \frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4}{1 + b_1 x + b_2 x^2}$$

s.t.

$$\left(x - \frac{x^3}{3!} + \frac{x^5}{120}\right)(1 + b_1 x + b_2 x^2) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

constant => $a_0 = 0$, $x \Rightarrow a_1 = 1$, $x^2 \Rightarrow a_2 = b_1$, $x^3 \Rightarrow a_3 = b_2 - 1/6$

$x^4 \Rightarrow a_4 = \frac{-1}{6} b_1$, $x^5 \Rightarrow 0 = \frac{-1}{6} b_2 + \frac{1}{120}$, $x^6 \Rightarrow 0 = b_1$

so

$a_0 = 0$, $a_1 = 1$, $b_2 = 1/20$, $a_3 = -7/60$, $b_1 = 0$, $a_2 = 0$, $a_4 = 0$

$\therefore$ $$P_4^2(x) = \frac{x - 7/60 x^3}{1 + 1/20 x^2}$$

Accuracy of approximations attached on next page. Here, we can see that Padé approximations perform better than Maclaurin and are more accurate for further along the interval.

2) Find constants x_0, x_1, c_1 s.t. quadrature formula has highest possible degree accuracy

$$\int_0^1 f(x)\,dx = \frac{1}{2} f(x_0) + c_1 f(x_1)$$

Want choices to hold for $f(x) = 1, x, x^2, \ldots$ $\therefore$

$f(x) = 1 \rightarrow \int_0^1 1\,dx = x|_0^1 = 1 = \frac{1}{2}(1) + c_1(1) \Rightarrow \underline{c_1 = 1/2}$

$f(x) = x \rightarrow \int_0^1 x\,dx = \frac{1}{2}x^2|_0^1 = 1/2 = 1/2(x_0) + \frac{1}{2}(x_1) \Rightarrow x_0 + x_1 = 1$

$f(x) = x^2 \rightarrow \int_0^1 x^2\,dx = \frac{1}{3}x^3|_0^1 = 1/3 = 1/2(x_0^2) + \frac{1}{2}(x_1^2) \Rightarrow 2 = 3x_0^2 + 3x_1^2$

$f(x)$ $\therefore$ $3x_0^2 + 3x_1^2 = 3x_0^2 + 3(1 - x_1)^2$

$\Rightarrow 6x_0^2 - 6x_0 + 3 = 2 \Rightarrow 6x_0^2 - 6x_0 + 1 = 0$

s.t. $x_0 = \frac{1}{2} \pm \frac{\sqrt{3}}{6}$ and $x_1 = \frac{1}{2} \mp \frac{\sqrt{3}}{6}$

check for $f(x) = x^3$

$\hookrightarrow \int_0^1 x^3\,dx = \frac{1}{4}x^4|_0^1 = 1/4 = 1/2 \cdot f\left(\frac{1}{2} + \frac{\sqrt{3}}{6}\right) + \frac{1}{2} f\left(\frac{1}{2} - \frac{\sqrt{3}}{6}\right)$

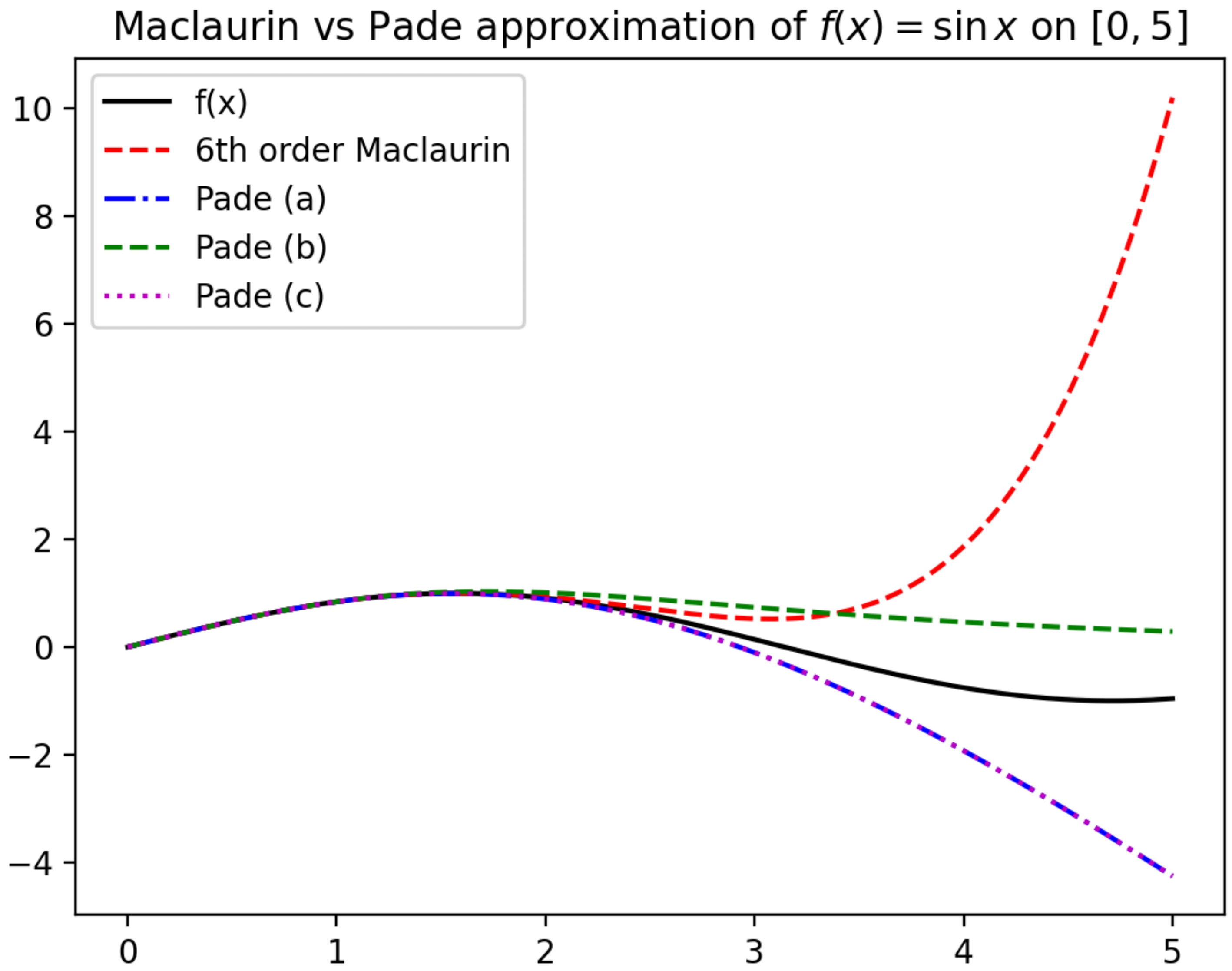
$= 1/4$

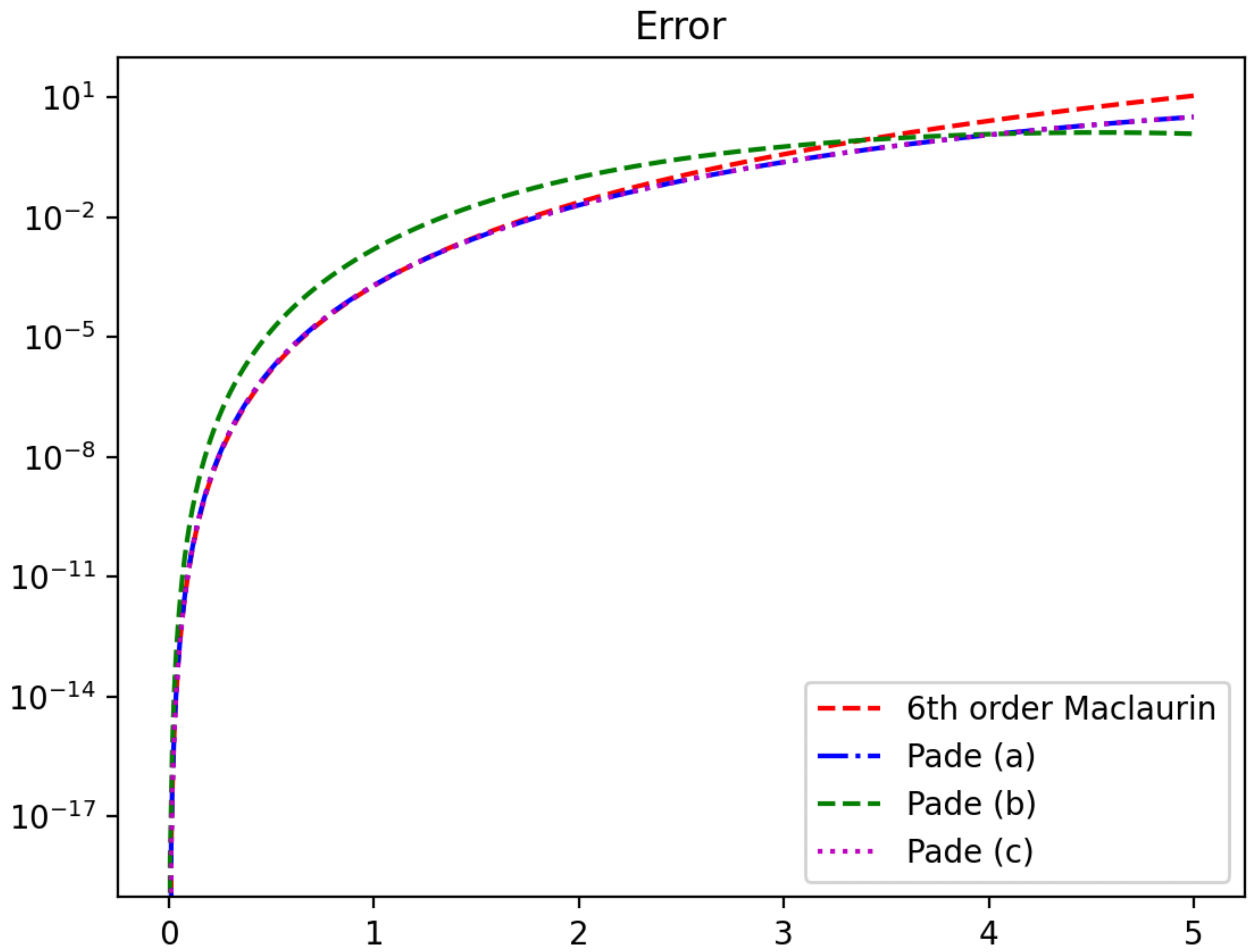
check for $f(x) = x^4$

$\hookrightarrow \int_0^1 x^4\,dx = \frac{1}{5}x^5|_0^1 = 1/5 = 1/2\, f\left(\frac{1}{2} + \frac{\sqrt{3}}{6}\right) + \frac{1}{2} f\left(\frac{1}{2} - \frac{\sqrt{3}}{6}\right)$

$\neq 7/36$

$\therefore$ $c_1 = 1/2$, $x_0 = \frac{1}{2} + \sqrt{3}/6$, $x_1 = 1/2 - \sqrt{3}/6$	is accurate to degree 3


Maclaurin vs Pade approximation of $f(x) = \sin x$ on $[0, 5]$
f(x)
6th order Maclaurin
Pade (a)
Pade (b)
Pade (c)
10
8
6
4
2
0
−2
−4
0
1
2
3
4
5


Error
10^{1}
10^{-2}
10^{-5}
10^{-8}
10^{-11}
10^{-14}
10^{-17}
6th order Maclaurin
Pade (a)
Pade (b)
Pade (c)
0
1
2
3
4
5

3)

a) Code included on following page.

b) Use error estimates derived in class to choose 'n' s.t.

$\left| \int_{-5}^{5} \frac{1}{1+s^2} ds - T_n \right| < 10^{-4}$ and $\left| \int_{-5}^{5} \frac{1}{1+s^2} ds - S_n \right| < 10^{-4}$

T_n:

Error for trapezoidal rule is $\frac{b-a}{12} h^2 f''(\xi)$ for $\xi \in (a,b)$

So for $f(x) = \frac{1}{1+x^2}$ w/ $a=-5$, $b=5$

Error is $\left| \frac{10}{12} h^2 f''(\xi) \right| = \left| \frac{5}{6} h^2 \left(\frac{2(3\xi^2-1)}{((\xi)^2+1)^3} \right) \right|$

$\leq \frac{5}{6} h^2 \left| \frac{148}{1} \right|$ ← bound numerator w/ $\xi = 5$, denominator w/ $\xi = 0$

$\leq \frac{370}{3} h^2 < 10^{-4}$

s.t. $h < \sqrt{\frac{10^{-4} \cdot 3}{370}}$

$\approx 9.004 \cdot 10^{-4}$ and $h = b-a/n = 10/n$

s.t. $n > 11105.55$

so choose $\underline{n = 11106}$ to be sure

S_n:

Error for Simpson's rule is $\frac{b-a}{180} h^4 f^{(4)}(\xi)$ w/ $\xi \in (a,b)$

S.t. Error is $\left| \frac{10}{120} h^4 f^{(4)}(\xi) \right| = \left| \frac{1}{12} h^4 \left(\frac{24(5(\xi)^4 - 10(\xi)^2 + 1)}{((\xi)^2+1)^5} \right) \right|$

$\leq \frac{1}{12} h^4 \left| \frac{2876}{1} \right|$ ← bounded by $\xi = 5$ on numerator and $\xi = 0$ on denominator

$\leq h^4 \frac{719}{3} < 10^{-4}$

s.t. $h < \left(\frac{10^{-4} \cdot 3}{719} \right)^{1/4} \approx 0.0254$

$h = b-a/n = 10/n$ s.t. $n > 393.46$

choose even # greater than that s.t. $\boxed{n = 394}$

c) Outputs attached. Will note default tol is 10^{-8} not 10^{-6} so used 10^{-8} instead.

Default tolerance # evaluations: 147

10^{-4} # of evaluations: 63

These are less than 'n' calculated, but this makes sense as calculated 'n' is an upper bound but 'quad' method returns when tol. reached exactly. I would say, however, that b/c of this upper bound not exact, that returned values for T_n and S_n are more accurate than the quad func.

3)

Code:

```python
def compositeTrapezoid(f, a, b, n):
    # define h and other sums
    h = (b - a) / n
    sum = f(a) + f(b)

    # perform trapezoid
    for i in range(1, n):
        x = a + i * h
        sum += 2 * f(x)

    # return as given by composite formula
    return sum * h / 2
```

```python
def compositeSimpsons(f, a, b, n):
    if n % 2 == 1:  # ensure n is even
        n += 1
    # define h and other sums
    h = (b - a) / n
    sum = f(a) + f(b)

    # perform composite simpsons summation
    for i in range(1, n):
        x = a + i * h
        if i % 2 == 0:
            sum += 2 * f(x)
        else:
            sum += 4 * f(x)

    # return as given by composite formula
    return sum * h / 3
```

Output:

```
Composite Trapezoidal Rule (n=11106):  2.7468015318911543
Composite Simpson's Rule (n=394):  2.746801533860976
Default quad Evaluation (n=147): 2.7468015338900327
Quad Evaluation 1e-4 tolerance (n=63): 2.746801533909586

Error from default evalation of quad():
-> Trapezoidal Rule:  1.9988783961366607e-09
-> Simpson's Rule:  2.9056757000489597e-11
```