APPM 4600 - HW #2 Gustav Cedergrund

1)

a) Show $(1+x)^n = 1+nx+o(x)$ as $x \to 0$

$$\lim_{x\to 0} \frac{(1+x)^n - 1 - nx}{x} = \lim_{x\to 0} \frac{\left(1+nx+\frac{n(n-1)}{2!}x^2+\dots\right)-1-nx}{x} \leftarrow \text{binomial formula}$$

$$= \lim_{x\to 0} \frac{\frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots \cdot \frac{1}{x}}{x \cdot \frac{1}{x}}$$

$$= \lim_{x\to 0} \frac{\frac{n(n-1)}{2!}x + \frac{n(n-1)(n-2)}{3!}x^2 + \dots}{1} = \underline{0} \quad \text{so } (1+x)^n - 1 - nx \text{ is } o(x)$$

$\therefore (1+x)^n = 1+nx+o(x)$ as $x\to 0$ ■

b) Show $x\sin(\sqrt{x}) = O(x^{3/2})$ as $x \to 0$

$$\lim_{x\to 0^+} \frac{x\sin(\sqrt{x})}{x^{3/2}} = \lim_{x\to 0^+} \frac{\sin(\sqrt{x})}{\sqrt{x}}$$

$$\overset{LH}{=} \lim_{x\to 0^+} \frac{\cos(\sqrt{x})\cdot \cancel{\frac{1}{2x^{1/2}}}}{\cancel{\frac{1}{2x^{1/2}}}} = \lim_{x\to 0^+} \cos(\sqrt{x}) = \boxed{1} \neq 0$$

$\therefore x\sin(\sqrt{x}) = O(x^{3/2})$ as $x\to 0$ ■

c) Show $e^{-t} = o\left(\frac{1}{t^2}\right)$ as $t \to \infty$

$$\lim_{t\to\infty} \frac{e^{-t}}{1/t^2} = \lim_{t\to\infty} \frac{t^2}{e^t}$$

$$\overset{LH}{=} \lim_{t\to\infty} \frac{2t}{e^t}$$

$$\overset{LH}{=} \lim_{t\to\infty} \frac{2}{e^t} = \boxed{0}$$

$\therefore e^{-t} = o\left(\frac{1}{t^2}\right)$ as $t\to\infty$ ■

d) Show $\int_0^\varepsilon e^{-x^2}dx = O(\varepsilon)$ as $\varepsilon \to 0$

$$\lim_{\varepsilon\to 0} \frac{\int_0^\varepsilon e^{-x^2}dx}{\varepsilon} \overset{LH}{=} \lim_{\varepsilon\to 0} \frac{e^{-\varepsilon^2}}{1}$$

$$= \lim_{\varepsilon\to 0} e^{-\varepsilon^2} = e^{-0^2} = e^0 = \boxed{1} \neq 0$$

$\therefore \int_0^\varepsilon e^{-x^2}dx = O(\varepsilon)$ as $\varepsilon\to 0$ ■

2) $A = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1+10^{-10} & 1-10^{-10} \end{bmatrix}$ $\quad \vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} 1-10^{10} & 10^{10} \\ 1+10^{10} & -10^{10} \end{bmatrix}$ $\quad \vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Here, we are solving $A\vec{x} = \vec{b}$ w/ some perturbation in b of $\begin{bmatrix} \Delta b_1 \\ \Delta b_2 \end{bmatrix}$

a) An exact formula for change in solution b/w exact problem and perturbed problem Δx is

$$\|\Delta x - x\| = \left\| A^{-1}\begin{bmatrix} 1+\Delta b_1 \\ 1+\Delta b_2 \end{bmatrix} - x \right\|$$

$$= \left\| \begin{bmatrix} 1-10^{10} & 10^{10} \\ 1+10^{10} & -10^{10} \end{bmatrix} \begin{bmatrix} 1+\Delta b_1 \\ 1+\Delta b_2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\|$$

$$= \left\| \begin{bmatrix} \Delta b_1(1-10^{10}) + 10^{10}\Delta b_2 + 1 - 1 \\ \Delta b_1(1+10^{10}) - 10^{10}\Delta b_2 + 1 - 1 \end{bmatrix} \right\| = \boxed{\left\| \begin{bmatrix} \Delta b_1(1-10^{10}) + 10^{10}\Delta b_2 \\ \Delta b_1(1+10^{10}) - 10^{10}\Delta b_2 \end{bmatrix} \right\|}$$

$$\boxed{= \sqrt{2\Delta b_1^2(10^{20}+1) - 4\Delta b_1 \Delta b_2 \cdot 10^{20} + 10^{20} \cdot 2\Delta b_2^2}}$$

b) Condition # of A, by definition, is given (bounded) by formula

$$K(A) = \|A^{-1}\| \cdot \|A\| = (-2 \cdot 10^{10})(-10^{-10}) = \frac{-2}{-1} \cdot \frac{10^{10}}{10^{10}}$$

$$= \boxed{2}$$

c) Let $\Delta b_1, \Delta b_2$ be of magnitude 10^{-5}

relative error is $\frac{\|\Delta x - x\|}{\|x\|} = \frac{\sqrt{2\Delta b_1^2(10^{20}+1) - 4\Delta b_1 \Delta b_2 \cdot 10^{20} + 10^{20} \cdot 2\Delta b_2^2}}{\sqrt{2}}$

$\underline{\text{w/ } \Delta b_1, \Delta b_2 \text{ different magnitude}} \rightarrow \leq 2 \cdot 10^{5}$

Relationship b/w rel. error, condition #, and the perturbation is __direct.__

If $\Delta b_1, \Delta b_2$ are same

$$\|\Delta x - x\| = \sqrt{\cancel{2\Delta b^2 \cdot 10^{20}} + 2\Delta b^2 - \cancel{4\Delta b^2 \cdot 10^{20}} + \cancel{2 \Delta b^2 \cdot 10^{20}}} = \sqrt{2\Delta b^2} = \sqrt{2}\Delta b$$

s.t. __relative error is simply equal to the magnitude of perturbation.__

I feel perturbations are more likely to be the same as the perturbations themselves are usually __patterned__/systematic as they come from the same place. Thus, they would cause perturbations that are similar, or the same. If the perturbation came from a random source, however, the perturbations would be different.

3) Let $f(x) = e^x - 1$

a) Find relative condition # $\kappa(f(x))$. By formula,

$$\kappa(f(x)) = \frac{|f'(x) \cdot x|}{|f(x)|}$$

so, $f(x) = e^x - 1$
$f'(x) = e^x$ s.t $\boxed{\kappa(f(x)) = \frac{|xe^x|}{|e^x - 1|} = \frac{xe^x}{e^x - 1} \quad \forall x \in \mathbb{R}}$

No specific values for x on which $f(x)$ is ill-conditioned, but condition # increases linearly as $x \to \infty$

b) Computing $f(x)$ using the algorithm:
1: $y = \text{math.e\^{}x}$; 2: return $y-1$
Is the algorithm stable?
Let ε be a perturbation s.t $\tilde{x} = x + \varepsilon$,
then $\tilde{y} = e^{(x+\varepsilon)} \Rightarrow \tilde{y} = e^x e^\varepsilon$ s.t $\tilde{y} - 1$ is $e^x e^\varepsilon - 1$

As algorithm produces error of around e^ε for error ε, I would say algorithm is not stable, especially when input $x \approx 0$ b/c then there would be lots of cancellation of significant digits.

c) Let $x = 9.999999995000000 \cdot 10^{-10}$ s.t $f(x) = 10^{-9}$ to 16 dec. places.
Algorithm produces $1.000000082740371 \cdot 10^{-9}$, which is correct to 8 digits (7 decimals). It is expected as when x is close to 0, we subtract two floats that are close to each other s.t. cancellation occurs.

d) Find poly. representation of $f(x)$ accurate to 16 digits for 'x' from (c)
Using Taylor,
$$f(x) = e^x - 1 = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) - 1 = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$
Need to be accurate to 16 digits (relative error), $\therefore$ we need Taylor expansion to $\frac{\text{abs. error}}{x} \leq 10^{-16} \Rightarrow \text{abs. error} \leq 9.999999995 \cdot 10^{-25}$.
$\therefore$ we need $R_n \leq 9.999999995 \cdot 10^{-25}$

continued

3) continued

d) continued

$$R_n \leq 9.99999995 \cdot 10^{-25}$$

when $n = 3$ as

$$e^x \cdot \frac{x^3}{3!} = 1.6666666658 \cdot 10^{-28} < 9.99999995 \cdot 10^{-25}$$

Thus, polynomial representation of $f(x)$ accurate to 16 digits for x is $\boxed{f(x) = x + \frac{x^2}{2!}}$

e) $f(x) = x + \frac{x^2}{2!}$ evaluated as $x = 9.9999999995000000 \cdot 10^{-16}$ is 10^{-9}.

$\therefore$ (d) is correct.

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4) Codes and outputs attached.