APPM 4600 - HW #4 Gustav Cedergrund

1) Assume temp. in C $T(x,t)$ at distance 'x' meters below surface 't' seconds after cold snap satisfies

$$\frac{T(x,t)-T_s}{T_i - T_s} = erf\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

Let $T_i = 20°C$, $T_s = -15°C$, $\alpha = 0.138 \cdot 10^{-6}\ m^2/s$

a) How deep should water main be buried so only freeze after 60 days?

Then, we want $T(x,t) = 0°C$ at $t = 60 \cdot 24 \cdot 60^2 = 5184000\ s$

$\therefore$ $\boxed{f(x) = T(x,t) = (T_i - T_s)\, erf\left(\frac{x}{2\sqrt{\alpha t}}\right) + T_s = 0}$ at $t = 5184000$

st $f(x) = 0$ is simply a root-finding problem

$\therefore$ $f'(x) = (T_i - T_s) \cdot \frac{2}{\sqrt{\pi}} \cdot e^{-\left(\frac{x}{2\sqrt{\alpha t}}\right)^2} \cdot \frac{1}{2\sqrt{\alpha t}} + 0$

$\boxed{f'(x) = (T_i - T_s) \cdot \frac{1}{\sqrt{\pi \alpha t}} \cdot e^{\frac{-x^2}{4\alpha t}}}$

Plot for $f(x)$ attached

b) Using bisection w/ starting values $a_0 = 0$ and $b_0 = \bar{x}$ ($\bar{x}$ chosen at 1) we get root approximated at $x = 0.6769618544819309\ m$ (Full output attached)

c) Using newton's method w/ starting values $x_0 = 0.01\ m$ and $x_0 = \bar{x} = 1\ m$, we get an approximate depth of

$x = 0.6769618544819365$ (...9366 for $\bar{x}$)

Both converge after 4 iterations, way faster than that for bisection method, even though it produces a smaller error.

(Full output attached)

2)

a) root 'α' has multiplicity 'm' of function $f(x)$ given that $f(x)$ can be written $f(x) = (x-\alpha)^m g(x)$

b) For $f(x)$ w/ multiplicity m at root α, note that

$f(\alpha) = f'(\alpha) = f''(\alpha) = \cdots = f^{m-1}(\alpha) = 0$, $f^{(m)}(\alpha) \neq 0$

$\therefore$ Taylor for $f(x)$ at root α

$\hookrightarrow f(x) = 0 + \cdots + 0 + \frac{f^m(x)}{m!}(x-\alpha)^m + \underbrace{\frac{f^{m+1}(\xi)}{(m+1)!}(x-\alpha)^{m+1}}_{\text{error term}}$

s.t. $f'(x) = \frac{f^m(x)}{m!} \cdot m(x-\alpha)^{m-1} + \underbrace{\frac{f^{m+1}(\xi)}{(m+1)!}(m+1)(x-\alpha)^m}_{\text{error term}}$

$\therefore$ by Newton

$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \Rightarrow x_{k+1} - \alpha = x_k - \alpha - \frac{f(x_k)}{f'(x_k)}$ ← plug in taylor and cancel common terms

$\approx x_k - \alpha - \frac{(x_k - \alpha)}{m}$

$x_{k+1} - \alpha \approx (x_k - \alpha)\left(1 - \frac{1}{m}\right)$

$\therefore \frac{x_{k+1} - \alpha}{x_k - \alpha} \approx \frac{m-1}{m} < 1$ $\therefore$ linear convergence

c) If we instead use fixed point w/ $g(x) = x - m\frac{f(x)}{f'(x)}$

we have

$x_{k+1} = g(x_k) = x_k - m\frac{f(x_k)}{f'(x_k)} \Rightarrow x_{k+1} - \alpha = x_k - \alpha - m\frac{f(x_k)}{f'(x_k)}$

plugging in the same Taylor expansions from (b) we have

$x_{k+1} - \alpha = x_k - \alpha - m\left(\frac{f^m(x)/m!\,(x-\alpha)^m + f^{m+1}(\xi_k)/(m+1)!\,(x-\alpha)^{m+1}}{f^m(x)/m! \cdot m(x-\alpha)^{m-1} + f^{m+1}(\xi_k)/(m+1)!\,(m+1)(x-\alpha)^m}\right)$

$x_{k+1} - \alpha \approx x_k - \alpha - \cancel{m}\frac{(x_k - \alpha)}{\cancel{m}} \approx x_k - x_k + \alpha - \alpha \approx 0 \neq \lambda$ so not linear

$\therefore$ greater than linear guarantees second-order convergence.

d) part (c) provides a modified method of Newton's that keeps order of convergence as quadratic even if the original function has a multiplicity.

3) Let $\{x_k\}_{k=1}^{\infty}$ be a sequence that converges to α

By definition, $\{x_k\} \to \alpha$ w/ order p given $\exists \lambda$ w/ $0 < \lambda < 1$ s.t

$$\lim_{k \to \infty} \frac{|x_{k+1} - \alpha|}{|x_k - \alpha|^p} = \lambda$$

Then when k is sufficiently large,

$$|x_{k+1} - \alpha| = \lambda |x_k - \alpha|^p$$

$$\log(|x_{k+1} - \alpha|) = \log(\lambda |x_k - \alpha|^p) = \log(\lambda) + \log(|x_k - \alpha|^p)$$

$$\boxed{\log(|x_{k+1} - \alpha|) = \log(\lambda) + p\log(|x_k - \alpha|)}$$

relating this to slope equation $y = mx + b$,

we see that the relationship b/w $\log(|x_{k+1} - \alpha|)$ and $\log(|x_k - \alpha|)$ is linear w/ order of convergence 'p' being the slope and $\log(\lambda)$ as y-intercept when k is sufficiently large.

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4) Using $f(x) = e^{3x} - 27x^6 + 27x^4 e^x - 9x^2 e^{2x}$ w/ the three methods we can see different orders of convergence. I modelled convergence using the different methods through coding (code + output attached) and saw that (i) converges linearly while (ii) and (iii) converge closer to quadratically, slightly faster for (ii).

I prefer method (iii) b/c you don't need to calculate $f''(x)$ and convergence is still very quick.

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5)

a) Plotting error by iteration (output attached) we see that error does decrease as expected, just a lot slower w/ secant method than w/ Newton's

b) As was shown in (3), the slopes of these lines are the order of convergence 'p'. Newton's method has a steeper plot than secant method, showing faster order of convergence.