APPM 4660 - HW #1 Gustav Cedergrund

1) $p(x) = (x-2)^9 = x^9 - 18x^8 + 144x^7 - 672x^6 + 2016x^5 - 4032x^4 + 5376x^3 - 4608x^2 + 2304x - 512$

(i)-(ii): Plots attached on next page

(iii) Difference b/w the plots is that the first is much more jagged in its graph while the second is a lot rounder and smoother, like the plot of a polynomial should be. The discrepancy is most likely caused by round-off error. In the second graph, there is only one "addition" operation and then one exponential; however, in the first, a large # of multiplication and addition operations of numbers w/ many decimal places causes round-off error, which is only propagated the more operations are completed. Therefore, I would say the second graph is more accurate, as it resembles more what a polynomial "should" look like (smooth/rounded/differentiable).

2) How would you perform following calculations to avoid cancellations?

i) Evaluate $\sqrt{x+1} - 1$ for $x \simeq 0$

To avoid loss of cancelling significant digits w/ subtraction of nearly equal #s, I would rewrite '$\sqrt{x+1} - 1$' to ↓

$$\frac{\sqrt{x+1} - 1}{1} \cdot \frac{(\sqrt{x+1}+1)}{(\sqrt{x+1}+1)} \Rightarrow \frac{(x+1) - 1}{\sqrt{x+1}+1} = \boxed{\frac{x}{\sqrt{x+1}+1}}$$

now we have less cancellation as we are adding not subtracting. Less rel. error

Evaluate that instead

ii) Evaluate $\sin(x) - \sin(y)$ for $x \simeq y$

I would follow same process for same reasons as in (i)

$$\frac{\sin(x) - \sin(y)}{1} \cdot \frac{(\sin(x) + \sin(y))}{(\sin(x) + \sin(y))} \Rightarrow \frac{\sin^2(x) - \sin^2(y)}{\sin(x) + \sin(y)} \underset{\text{by identity}}{=} \boxed{\frac{\sin(x+y)\sin(x-y)}{\sin(x) + \sin(y)}}$$

which would produce answers w/ less rel. error when evaluated

iii) Evaluate $\frac{1-\cos(x)}{\sin(x)}$ for $x \simeq 0$

rewrite to:

$$\frac{1-\cos(x)}{\sin(x)} \cdot \frac{(1+\cos(x))}{(1+\cos(x))} = \frac{1-\cos^2(x)}{\sin(x)(1+\cos(x))} = \frac{\sin^2(x)}{\cancel{\sin(x)}(1+\cos(x))} = \boxed{\frac{\sin(x)}{1+\cos(x)}}$$

and evaluate that instead for more accurate output.

3) $f(x) = (1 + x + x^3)\cos(x)$

↓ 2nd degree Taylor Polynomial of $f(x)$ about $x_0 = 0$

$$P_2(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2}(x)^2$$

$f(x) = (1+x+x^3)\cos(x) \Rightarrow f(0) = (1+0+0)\,1 = \underline{1}$

$f'(x) = (3x^2+1)\cos(x) - (x^3+x+1)\sin(x) \Rightarrow f'(0) = 1\cdot 1 - 0 = 1$

$f''(x) = -2(3x^2+1)\sin(x) - (x^3+x+1)\cos(x) + 6x\cos(x) \Rightarrow f''(0) = -1\cdot 1 = -1$

$f^3(x) = (-x^3 - 17x + 1)\sin(x) + (3 - 9x^2)\cos(x) \Rightarrow f^3(0) = 3$

$P_2(x) = 1 + x - \frac{1}{2}x^2$

a) $P_2(0.5) = 1 + 0.5 - \frac{1}{2}(0.5)^2 = 1.5 - \frac{1}{2}\left(\frac{1}{4}\right) = \boxed{11/8 \approx f(0.5)}$ ← approximation

Error bound:

$|f(0.5) - P_2(0.5)| \le \frac{f^3(0.5)(0.5)^3}{6} = \frac{3}{6}(0.5)^3 = 1/16$

Actual error:

$|f(0.5) - P_2(0.5)| = |1.426 - 11/8| = 0.05107 < 0.0625$

b) $|f(x) - P_2(x)| \le \frac{f^3(x)\,x^3}{6}$

$\le \boxed{\left|\frac{(x^3 - 17x + 1)\sin(x) + (3 - 9x^2)\cos(x)}{6}\right| x^3}$ ← error bound for $P_2(x)$

c) $\int_0^1 f(x)\,dx \approx \int_0^1 1 + x - \frac{1}{2}x^2\,dx$

$\approx x + \frac{1}{2}x^2 - \frac{1}{6}x^3 \Big|_0^1$

$\approx \boxed{4/3}$

d) $\int_0^1 f(x)dx - \int_0^1 P_2(x)dx \le \left|\int_0^1 \frac{(x^3 - 17x + 1)\sin(x) + (3 - 9x^2)\cos(x)}{6} x^3\right|$

$\le \boxed{0.4428381}$

4) Consider quadratic equation $x^2-56x+1=0$

a) Assuming we can calculate square root w/ 3 correct decimals,

$$r_1^* = \frac{56+\sqrt{(-56)^2-4(1)(1)}}{2(1)} = \frac{56+\sqrt{3132}}{2} = \frac{56+55.964}{2} = \frac{111.964}{2} = \boxed{55.982}$$

$$r_2^* = \frac{56-\sqrt{(-56)^2-4(1)(1)}}{2(1)} = \frac{56-\sqrt{3132}}{2} = \frac{56-55.964}{2} = \frac{0.036}{2} = \boxed{0.018}$$

Actual roots => $r_1 = 28+3\sqrt{87}$, $r_2 = 28-3\sqrt{87}$

s.t rel error:

root 1: $\frac{|r_1-r_1^*|}{|r_1|} = \frac{|28+3\sqrt{87}-55.982|}{|28+3\sqrt{87}|} = 2.45005\cdot10^{-6}$

root 2: $\frac{|r_2-r_2^*|}{|r_2|} = \frac{|(28-3\sqrt{87})-0.018|}{|28-3\sqrt{87}|} = 7.67846\cdot10^{-3}$

"bad" root

b) two relations that better approximate "bad" root

relation 1: (given) $\to (x-r_1)(x-r_2)=0=x^2-56x+1$

$$x^2-r_2x-r_1x+r_1r_2 = x^2-56x+1$$

s.t $r_2+r_1=56$ and $r_1r_2=1$

now, using the "good" root $r_1=55.982$, we can find better approx. for r_2 by "plugging in" r_2. I use the 2nd equation $r_1r_2=1$ b/c the first would involve subtracting 2 #s that are very similar, which we established was less accurate in problem (2) of this hw.

$\therefore$ $r_2 = 1/r_1 = 1/55.982 = \boxed{0.017862}$ ← new r_2

relative error for new r_2

↳ $\frac{|28-3\sqrt{87}-0.017862|}{|28-3\sqrt{87}|} = \boxed{2.450058\cdot10^{-6}}$ ← new r_2 rel. error

way better!

relation 2: re-writing quadratic formula, as done on pg. 26 of textbook (and in lecture)

into $r_2 = \frac{-2c}{b-\sqrt{b^2-4ac}}$

solving, we get $r_2 = \frac{-2}{-56-\sqrt{56^2-4}} = \frac{-2}{-111.964} = \boxed{0.0178628}$ ← new r_2

continued

4) continued

b) continued

relative error for new r_2:

$$\frac{|28-3\sqrt{87}-0.01786281|}{|28-3\sqrt{87}|} = \boxed{2.450060\cdot10^{-6}}$$

which is also more accurate than original.

$\therefore$ both relations can be used to similar effectiveness in finding a more accurate root. turning "bad" → "good #2"

5) Cancellation of Terms:

a) Find upper bounds on the absolute error $|\Delta y|$ and relative error $\frac{|\Delta y|}{|y|}$

As $\Delta y = \Delta x_1 - \Delta x_2$, upper bound on absolute error can be given by

$$|\Delta y| = |\Delta x_1 - \Delta x_2| \leq \underline{|\Delta x_1| + |\Delta x_2|}$$

We can continue this line of thought w/ relative error

$$\frac{|\Delta y|}{|y|} = \frac{|\Delta y|}{|x_1 - x_2|} \leq \underline{\frac{|\Delta x_1| + |\Delta x_2|}{|x_1 - x_2|}}$$

relative error then becomes large when x_1, x_2 are really close

b) Manipulation of $\cos(x+\delta)-\cos(x)$ to expression w/o subtraction

$$\cos(x+\delta)-\cos(x) = -2\sin\left(\frac{x+\delta+x}{2}\right)\sin\left(\frac{x+\delta-x}{2}\right) \leftarrow \text{by trig. relations}$$

$$= \underline{-2\sin(x+\delta/2)\sin(\delta/2)} \leftarrow \text{new expression}$$

Tables for each expression is attached on next page.

One can see how new expression doesn't cancel to become 0 when δ is low.

c) Using Taylor expansion $f(x+\delta)-f(x) = \delta f'(x) + \frac{\delta^2}{2!}f''(\xi)$, $\xi \in [x, x+\delta]$

We can create this algorithm ~~for approximating~~ $\cos(x+\delta)-\cos(x)$

$\hookrightarrow -\delta\sin(x) - \frac{\delta^2}{2}\cos(\xi) = \boxed{-\left(\delta\sin(x) + \frac{\delta^2}{2}\cos(\xi)\right)}$

chose algorithm b/c it uses $\cos(x)$ as $f(x)$ and it doesn't use subtraction so cancellation less likely

→ For ξ, I plot and calculate using both bounds, w/ the real value being somewhere in b/w

→ Comparing to the techniques from part (b), the approximation is very close to my expression at lower values for δ, then diverges w/ δ at 1, which is expected for Taylor.