APPM 4600 — HOMEWORK # 11

For all homeworks, you should use Python. **Do not use** symbolic software such as Maple or Mathematica.

1. Assume the error in an integration formula has the asymptotic expansion

$$I - I_n = \frac{C_1}{n\sqrt{n}} + \frac{C_2}{n^2} + \frac{C_3}{n^2\sqrt{n}} + \frac{C_4}{n^3} + \cdots$$

   Generalize the Richardson extrapolation process to obtain an estimate of I with an error of order $\frac{1}{n^2\sqrt{n}}$. Assume that three values I_n, $I_{n/2}$ and $I_{n/4}$ have been computed.

2. Use the transformation $t = x^{-1}$ and Composite Simpson's rule with 5 nodes to approximate

$$\int_1^\infty \frac{\cos(x)}{x^3} dx.$$

3. The gamma function is defined by the formula

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad x > 0.$$

   Write a program to compute the value of this function from the definition using each of the following approaches:

   (a) Truncate the infinite interval of integration and write a composite trapezoidal rule code to perform the numerical integration. You will need to do some experimentation or analysis to determine where to truncate the interval based upon the usual trade-offs between accuracy and efficiency. Please describe your reasoning for your choice of interval and step size for the Trapezoidal Rule. Compare the relative accuracy of this solution with the value given by the Python gamma function (`scipy.special.gamma`) at $x = 2, 4, 6, 8, 10$ (Recall $\Gamma(k) = (k-1)!$ for positive integer k) .

   (b) Use the Matlab adaptive quadrature routine `quad` to solve the above integral on the same interval you used for part (a). Compare the accuracy of this solution with the one you obtained in part (a) at the same values of x. Also, compare the number of function evaluations required by the two methods.

   (c) Gauss-Laguerre quadrature is designed for the interval $[0, \infty)$ and the weight function $w(t) = e^{-t}$. It is therefore ideal to use for this problem. Call the Numpy subroutine `numpy.polynomial.laguerre.laggauss` to to obtain the n weights $\mathbf{w}$ and n abscissae $\mathbf{x}$ for Gauss-Laguerre quadrature and use this to approximate $\Gamma(x)$.

4. Given the linear system

$$2x_1 - 6\alpha x_2 = 3$$
$$3\alpha x_1 - x_2 = \frac{3}{2}$$

(a) Find the value(s) of α for which the system has no solutions.

(b) Find the value(s) for α for which the system has an infinite number of solutions.

(c) Assuming a unique solution exists for a given α, find the solution.