**APPM 4600 — HOMEWORK # 1**

## General Information, applicable to all homework assignments

*Homework will be assigned on a regular basis. Generally the problems will be based on the text, but occasionally they will require you to fill in details from class discussions, or further explore a topic outside of class. Problems assigned during any given week will be due the following Friday in class. The solutions should include the following:*

- *Clear, brief restatement of the problem or question.*
- *Statement of important assumptions.*
- *Neat, detailed, step–by–step solution including sufficient comments to make the solution "read" well.*
- *When appropriate, a discussion of the accuracy (or lack thereof) of final and partial results. Is the answer consistent with the "physics of the problem"? What does the answer mean? Is it reasonable? How many digits can you trust (are significant)?*

*You are encouraged to work together on the assignments, however, the major portion should be done on your own. In all cases your submission should demonstrate that you understand the problem and its solution.*

1. Consider the polynomial
$p(x) = (x-2)^9 = x^9 - 18x^8 + 144x^7 - 672x^6 + 2016x^5 - 4032x^4 + 5376x^3 - 4608x^2 + 2304x - 512.$

   i. Plot $p(x)$ for $x = 1.920, 1.921, 1.922, \ldots, 2.080$ (i.e. $x = [1.920 : 0.001 : 2.080];$) evaluating $p$ via its coefficients.
   ii. Produce the same plot again, now evaluating $p$ via the expression $(x-2)^9$.
   iii. What is the difference? What is causing the discrepancy? Which plot is correct?

2. How would you perform the following calculations to avoid cancellation? Justify your answers.

   i. Evaluate $\sqrt{x+1} - 1$ for $x \simeq 0$.
   ii. Evaluate $\sin(x) - \sin(y)$ for $x \simeq y$.
   iii. Evaluate $\frac{1-\cos(x)}{\sin(x)}$ for $x \simeq 0$.

3. Find the second degree Taylor polynomial $P_2(x)$ for $f(x) = (1 + x + x^3)\cos(x)$ about $x_0 = 0$.

   (a) Use $P_2(0.5)$ to approximate $f(0.5)$. Find an upper bound for the error $|f(0.5) - P_2(0.5)|$ using the error formula and compare it to the actual error.
   (b) Find a bound for the error $|f(x) - P_2(x)|$ when $P_2(x)$ is used to approximate $f(x)$. This will be a function of $x$.
   (c) Approximate $\int_0^1 f(x)dx$ using $\int_0^1 P_2(x)dx$.
   (d) Estimate the error in the integral.

4. Consider the quadratic equation $ax^2 + bx + c = 0$ with $a = 1, b = -56, c = 1$.

   (a) Assume you can calculate the square root with 3 correct decimals (e.g. $\sqrt(2) \approx 1.414 \pm \frac{1}{2}10^{-3}$) and compute the relative errors for the two roots to the quadratic when computed using the standard formula

   $$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

   (b) A better approximation for the "bad" root can be found by manipulating $(x - r_1)(x - r_2) = 0$ so that $r_1$ and $r_2$ can be related to $a, b, c$. Find such relations (there are two) and see if either can be used to compute the "bad" root more accurately.

5. **Cancellation of terms**. Consider computing $y = x_1 - x_2$ with $\tilde{x}_1 = x_1 + \Delta x_1$ and $\tilde{x}_2 = x_2 + \Delta x_2$ being approximations to the exact values. If the operation $x_1 - x_2$ is carried out exactly we have $\tilde{y} = y + \underbrace{(\Delta x_1 - \Delta x_2)}_{\Delta y}$.

   *Play with different values of $x$. One really small value ($< 1$) and one large value $> 10^5$.*

   (a) Find upper bounds on the absolute error $|\Delta y|$ and the relative error $|\Delta y|/|y|$, when is the relative error large?

   (b) First manipulate $\cos(x + \delta) - \cos(x)$ into an expression without subtraction. Pick two values of $x$; say $x = \pi$ and $x = 10^6$. Then for each $x$, tabulate or plot the difference between your expression and $\cos(x + \delta) - \cos(x)$ for $\delta = 10^{-16}, 10^{-15}, \cdots, 10^{-2}, 10^{-1}, 10^0$ (note that you can use your logx command to uniformly distribute $\delta$ on the x-axis).

   (c) Taylor expansion yields $f(x + \delta) - f(x) = \delta f'(x) + \frac{\delta^2}{2!} f''(\xi), \quad \xi \in [x, x + \delta]$. Use this expression to create your own algorithm for approximating $\cos(x + \delta) - \cos(x)$. Explain why you chose the algorithm. Then compare the approximation from your algorithm with the techniques in part (b). Use the same values for $x$ and $\delta$.