Introduction to Programming and Proving in Cedille

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Plan for the tutorial

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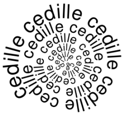
$\vdash$ *CeDilLE*

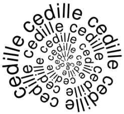
`cedille`

$\leadsto$ `cedille`$_{\texttt{core}}$

`c d ll`



Plan for the tutorial

Ç?	Motivation and background for Cedille
$\vdash$ *CeDilLE*	Syntax and semantics
`cedille`	Tooling: emacs frontend $\leftrightarrow$ backend
$\leadsto$ $\texttt{cedille}_{\texttt{core}}$	Elaboration to Cedille Core
`c d ll`	Spine-local type inference
	Future directions

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Motivation and background for Cedille

A little history

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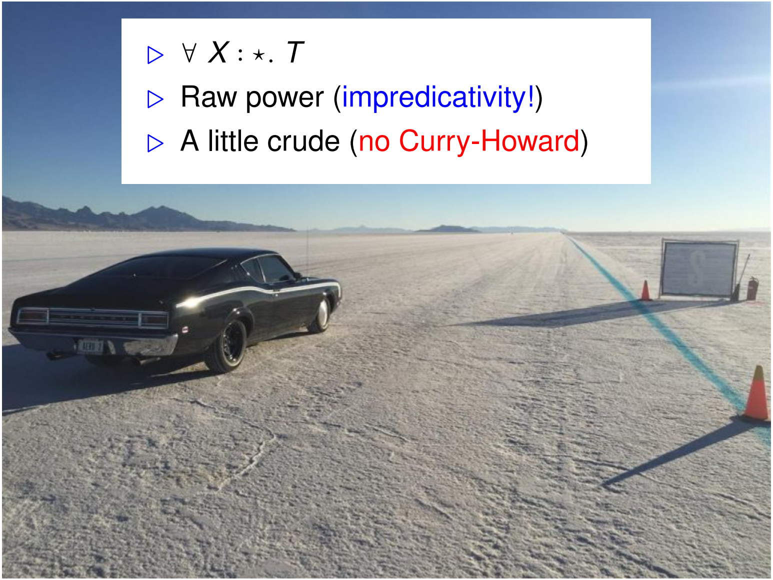
System F (Girard, Reynolds, early 1970s)

1969 Mercury Cyclone Spoiler II

System F (Girard, Reynolds, early 1970s)

- $\forall\, X : \star.\ T$
- Raw power (impredicativity!)
- A little crude (no Curry-Howard)



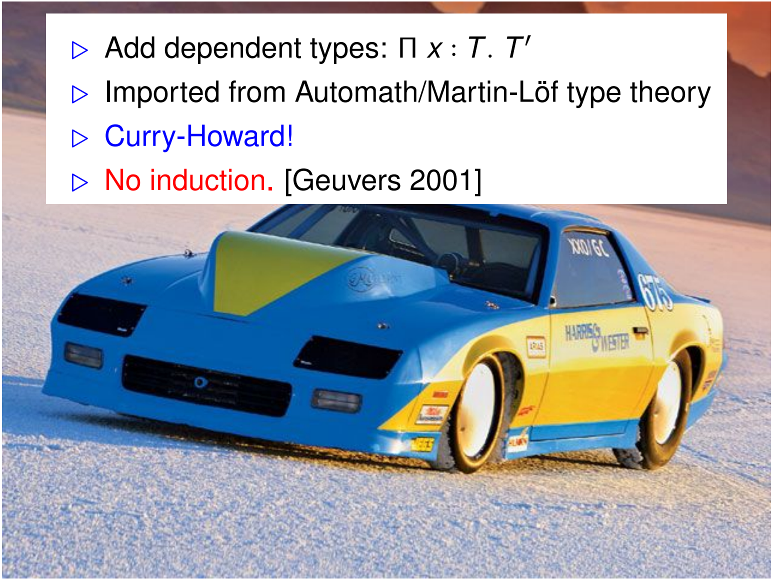
1969 Mercury Cyclone Spoiler II

Calculus of Constructions (Coquand, Huet 1988)

1988 Chevrolet Camaro

Calculus of Constructions (Coquand, Huet 1988)

- ▷ Add dependent types: $\Pi\, x : T.\; T'$
- ▷ Imported from Automath/Martin-Löf type theory
- ▷ Curry-Howard!
- ▷ No induction. [Geuvers 2001]



1988 Chevrolet Camaro

Calculus of Inductive Constructions (Werner 1994)

1992 Hoffman-Markley Streamliner

Calculus of Inductive Constructions (Werner 1994)

- ▷ Add primitive inductive types
- ▷ Finally ready for constructive mathematics!
- ▷ Basis for Coq

1992 Hoffman-Markley Streamliner

But Coq $\neq$ CIC

- Coinductive types
- Universe hierarchy (Extended CC, Luo 1990)
- Proof-irrelevant universe `Prop`
- *And we might want more:*
  - definitional proof irrelevance
  - inductive-inductive types
  - inductive-recursive types

Similarly, Agda $\neq$ MLTT.

Issues and limitations, Coq and Agda

- No formal semantics/correctness proof
  - Despite a lot of interest: TT in TT
- (Hence!) bugs and surprises
  - incompatibilities with various axioms
  - actual contradictions!
  - type soundness broken in Coq
- Commitment to a set of datatypes
  - theory of datatypes not finished...
  - e.g., higher-order abstract syntax prohibited

Have we created a monster?

Schaufelradbagger 258

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Schaufelradbagger 258

If I could turn back time...

Good-bye to:

- ▷ primitive datatypes
- ▷ (also universe hierarchy, my bias)

Hello to

- ▷ lambda-encodings of data

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Wanted: a new type theory

where

- ▷ inductive datatypes are <u>derived</u> (lambda-encoded)
- ▷ impredicativity is central
- ▷ core theory is small and verifiable

Tooling goals:

- ▷ see all typing/inference information
- ▷ predictable inference
- ▷ elaborate to core with independent checker

Introducing Cedille

CC

$\forall\, x : T.\ T'$	implicit products (Miquel)
$\iota\, x : T.\ T'$	dependent intersections (Kopylov)
$\{\ t \simeq t' \}$	untyped equality

- Small theory, formal syntax and semantics
- Core checker implemented in $<$ 1000loc Haskell
- Logically sound
- Turing complete(!)
- Supports inductive lambda-encodings

Back the truck up

Back the truck up

Not your forebear's lambda encodings

- Usual rap: inefficient accessors
- Corrected by Parigot 1988 for typed encoding
- Perfect untyped encoding Böhm et al. 1994
  - linear space
  - constant-time accessors
  - intrinsic support for iteration
- Cedille: perfect inductive (typed) encodings

What do we get from this?

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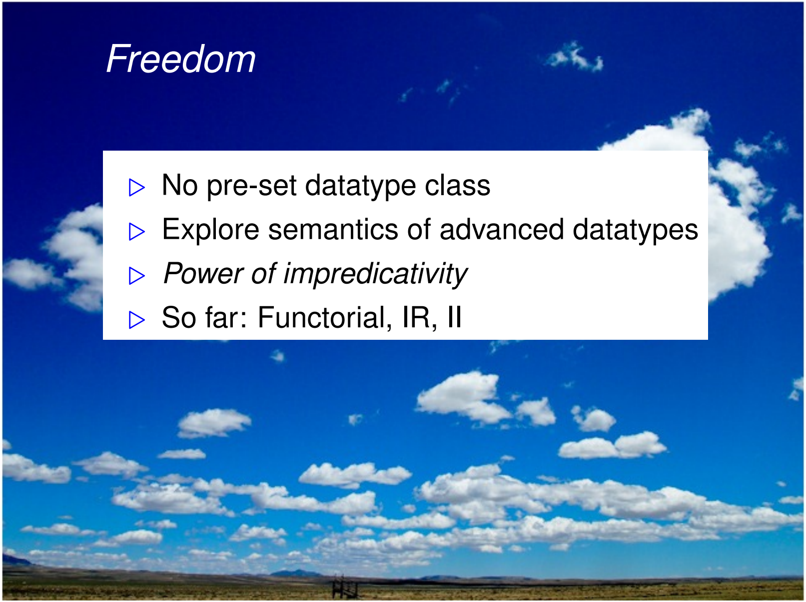

So which car are we?

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$\vdash$ *CeDilLE*

Syntax and semantics of Cedille

Types T and kinds κ

$\forall\, X : \kappa.\ T$	Impredicative quantification
$\lambda\, X : \kappa.\ T$	Type-level function with type input
$\lambda\, x : T.\ T'$	Type-level function with term input
$T\ T'$	Type-level application to a type
$T\ t$	Type-level application to a term
$\Pi\, x : T.\ T'$	Dependent function type
$\forall\, x : T.\ T'$	Implicit product
$\iota\, x : T.\ T'$	Dependent intersection
$\{\ t \simeq t'\ \}$	Untyped equality

$\star$	Kind for types
$\Pi\, x : T.\ \kappa$	Kind for predicates on terms
$\Pi\, X : \kappa.\ \kappa'$	Kind for predicates on types

Dependent intersections $\iota\, x : T_1 .\ T_2$

- Usual intersection types:

$$\frac{t : T_1 \qquad t : T_2}{t : T_1 \cap T_2}$$

- Dependent intersection:

$$\frac{t : T_1 \qquad t : [t/x]\, T_2}{t : \iota\, x : T_1 .\ T_2}$$

$\boxed{T_2 \text{ can refer to subject of typing, at weaker type } T_1}$

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Yes!

For each type

- Formation rule for the type

$$\frac{\Gamma \vdash T' : \star \qquad \Gamma, x : T' \vdash T : \star}{\Gamma \vdash \forall\ x : T'.\ T : \star}$$

- Introduction and elimination rules

$$\frac{\Gamma \vdash \forall\ x : T'.\ T : \star \qquad \Gamma, x : T' \vdash t : T}{\Gamma \vdash t : \forall\ x : T'.\ T} \qquad \frac{\Gamma \vdash t : \forall\ x : T'.\ T \qquad \Gamma \vdash t' : T'}{\Gamma \vdash t : [t'/x]T}$$

- Annotated terms
  - Introduction: $\Lambda\ x : T'.\ t$
  - Elimination: $t\ {-}t'$

- Annotated terms erase to pure lambda terms

$$\begin{array}{rcl} |\Lambda\ x : T.\ t| & = & |t| \\ |t\ \text{-}t'| & = & |t| \end{array}$$

Equality type

Formation:

$$\frac{FV(t\ t') \subseteq dom(\Gamma)}{\Gamma \vdash \{t \simeq t'\} : \star}$$

Introduction and elimination:

$$\frac{FV(t') \subseteq dom(\Gamma)}{\Gamma \vdash \color{blue}{t} : \{t' \simeq t'\}} \quad \frac{\Gamma \vdash t' : \{t_1 \simeq t_2\} \qquad \Gamma \vdash t : [t_1/x]T}{\Gamma \vdash t : [t_2/x]T}$$

Direct computation rule:

$$\frac{\Gamma \vdash t : \{t_1 \simeq t_2\} \qquad \Gamma \vdash t_1 : T}{\Gamma \vdash t_2 : T}$$

Annotations:

$$\begin{array}{rcl} |\beta\{\color{blue}{t}\}| & = & |t| \\ |\rho\ t'\ \text{-}\ t| & = & |t| \\ |\phi\ t\ \text{-}\ t_1\{t_2\}| & = & |t_2| \end{array}$$

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Dependent intersections

Formation:

$$\frac{\Gamma \vdash T : \star \qquad \Gamma, x : T \vdash T' : \star}{\Gamma \vdash \iota\, x : T.\ T'}$$

Introduction and elimination:

$$\frac{\Gamma \vdash t : T \qquad \Gamma \vdash t : [t/x]T'}{\Gamma \vdash t : \iota\, x : T.\ T'} \quad \frac{\Gamma \vdash t : \iota\, x : T.\ T'}{\Gamma \vdash t : T} \quad \frac{\Gamma \vdash t : \iota\, x : T.\ T'}{\Gamma \vdash t : [t/x]T'}$$

Annotations:

$$\begin{array}{rcl} |[t, t']| & = & |t| \\ |t.1| & = & |t| \\ |t.2| & = & |t| \end{array}$$

How are inductive datatypes defined?

- Many variations, one theme:

  The type of *d* expresses an induction principle for *d*

- For Nat:

$$n \ : \ \forall\ P : Nat \to \star.\ (\forall\ x : Nat.\ P\ x \to P\ (S\ x)) \to P\ Z \to P\ n$$

- Essentially due to Leivant 1983
- Will walk through example a little later

Another example: casts

- A cast is an identity function from A to B

```
Cast : ⋆ → ⋆ → ⋆ = λ A : ⋆ . λ B : ⋆ .
    ι cast : A → B . { cast ≃ λ x . x }.
```

- If there is a cast, you can change the type:

```
cast : ∀ A : ⋆ . ∀ B : ⋆ . Cast · A · B ⇒ A → B
  = ...
```

- Nontrivial casts exist in extrinsic type theory:

$$\lambda\, x.\, x \quad : \quad (\forall\, X{:}\star.\, X \to X) \to (Nat \to Nat)$$

- Not in intrinsic type theory:

$$\lambda\, x.\, (x \cdot Nat) \quad : \quad (\forall\, X{:}\star.\, X \to X) \to (Nat \to Nat)$$

Monotone recursive types

- Explored in works by R. Matthes [eg, in Synthese, 2002]
- Given monotone F, extend the theory with μF
- Monotonicity expressed by a term t of type

  $\forall X{:}\star.\, \forall Y{:}\star.\, (X \to Y) \to (F \cdot X \to F \cdot Y)$
- Matthes considers how to extend theory, retaining SN

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We can derive a stronger form, within our theory!

Starting point: proof of Tarski's Theorem

Consider monotone $f : \mathcal{P}(X) \to \mathcal{P}(X)$

Let $Q = \bigcap\{A \mid f(A) \subseteq A\}$

Prove $f(Q) = Q$ by both inclusions:

- For $f(Q) \subseteq Q$, show $f(Q) \subseteq A$ for all $f(A) \subseteq A$
  - Assume A s.t. $f(A) \subseteq A$.
  - $Q \subseteq A$
  - $f(Q) \subseteq f(A)$ by mono.
  - $f(A) \subseteq A$
- Now by mono., $f(f(Q)) \subseteq f(Q)$, so $f(Q) \in \{A \mid f(A) \subseteq A\}$
  - Hence $Q \subseteq f(Q)$

Translating the proof to Cedille

- "$A \subseteq B$" becomes `Cast · A · B`
- Monotonicity becomes

```
∀ X : ★ . ∀ Y : ★ .
Cast · X · Y → Cast · (F · X) · (F · Y)
```

- "$\bigcap\{A \mid f(A) \subseteq A\}$" becomes

```
Rec = ∀ X : ★ . Cast · (F · X) · X ⇒ X
```

- "$f(Q) \subseteq Q$" becomes

```
Cast · (F · Rec)) · Rec
```

- similarly for "$Q \subseteq f(Q)$":

```
Cast · Rec · (F · Rec)
```

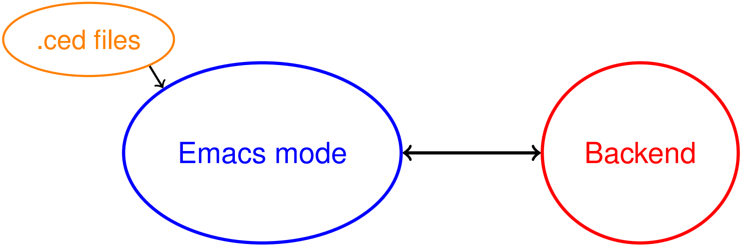
Monotone recursive types: summary

- Derive `Rec` for any montone `F`
- Casts (identity functions) between `F · Rec` and `Rec`
- Proof translates proof of Tarski fixed-point theorem
- Code is 18 lines of Cedille
- To emphasize: no addition to the theory, these are derived

`cedille`

Tooling: emacs frontend $\leftrightarrow$ backend

Architecture of Cedille



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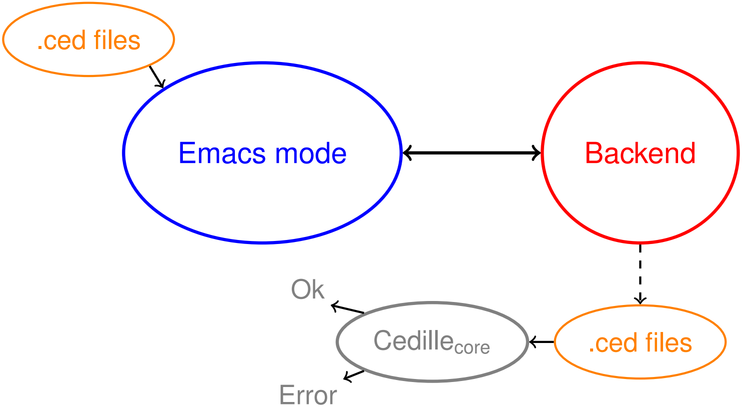
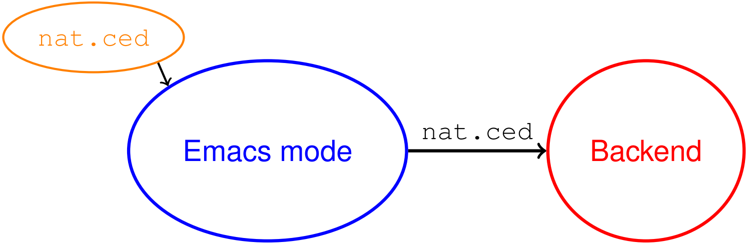


Emacs mode

- Interact with Cedille through an emacs mode



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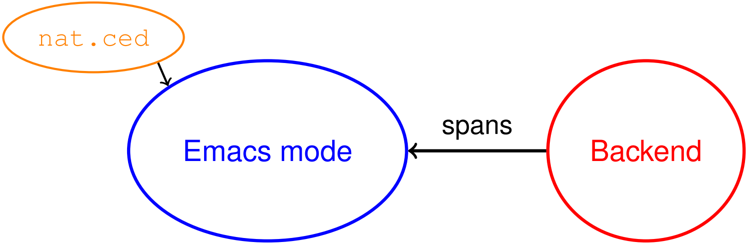

- A span is [*label*, *start-pos*, *end-pos*, *attributes*]
- Spans communicated in JSON
- Cedille sends <u>all</u> type information, in span attributes
- Monadic style for writing the backend (type checker)

Demo `cedille`