Introduction to Programming and Proving in Cedille

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Plan for the tutorial

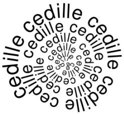
ç ?

$\vdash$ *CeDiLlE*

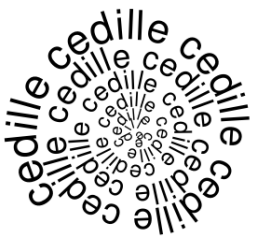
`cedille`

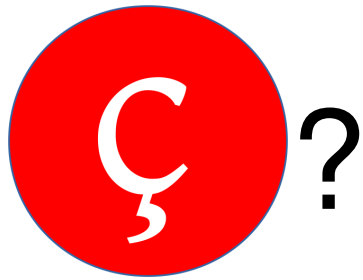
$\rightsquigarrow$ $\texttt{cedille}_{\texttt{core}}$

`c d ll`



Plan for the tutorial

Ç?	Motivation and background for Cedille
$\vdash$ *CeDiLIE*	Syntax and semantics
`cedille`	Tooling: emacs frontend $\leftrightarrow$ backend
$\rightsquigarrow$ `cedille`$_{\texttt{core}}$	Elaboration to Cedille Core
`c d ll`	Spine-local type inference
	Future directions



Motivation and background for Cedille

A little history

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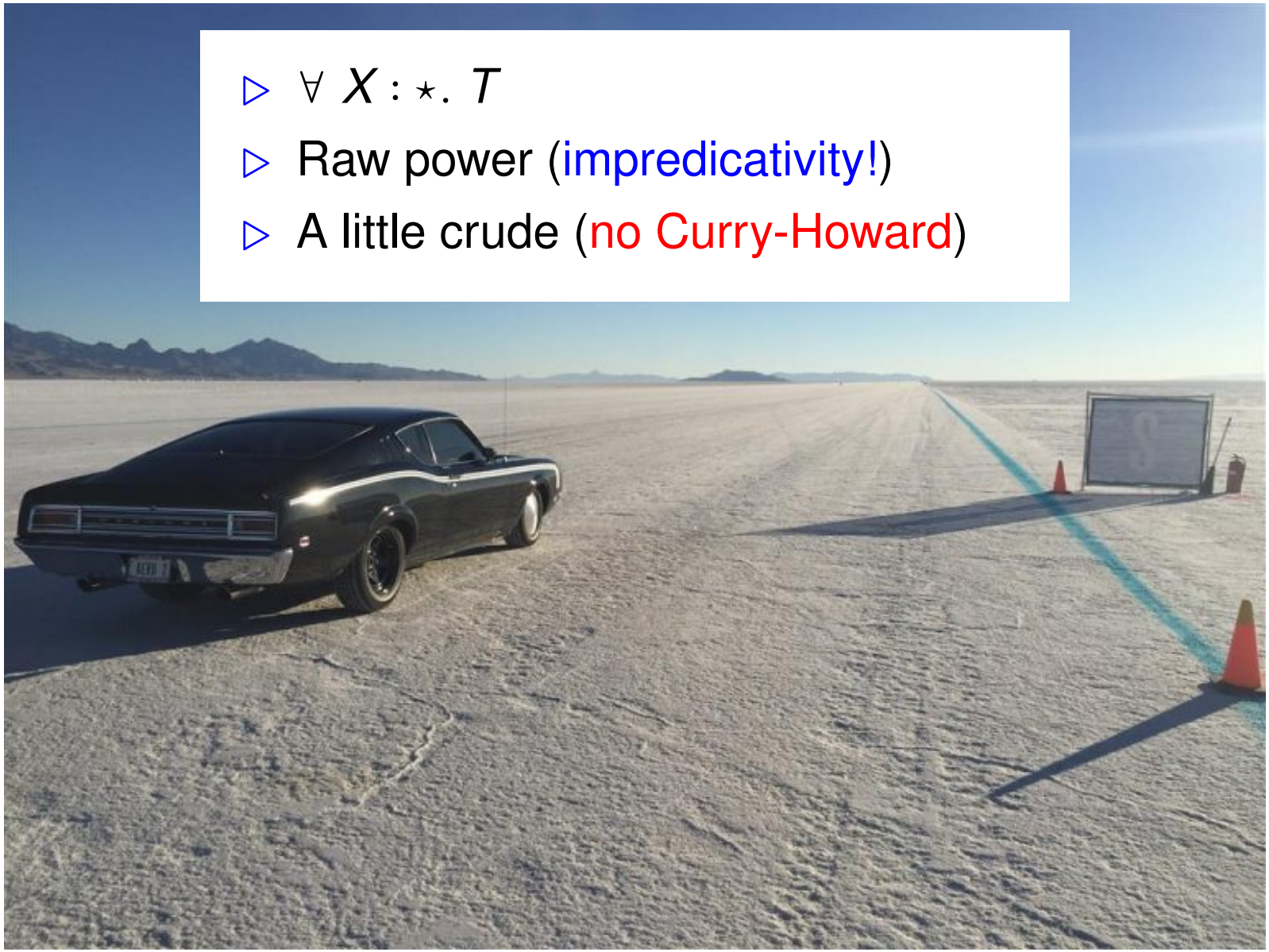
System F (Girard, Reynolds, early 1970s)

1969 Mercury Cyclone Spoiler II

System F (Girard, Reynolds, early 1970s)

- $\forall\, X : \star.\ T$
- Raw power (impredicativity!)
- A little crude (no Curry-Howard)



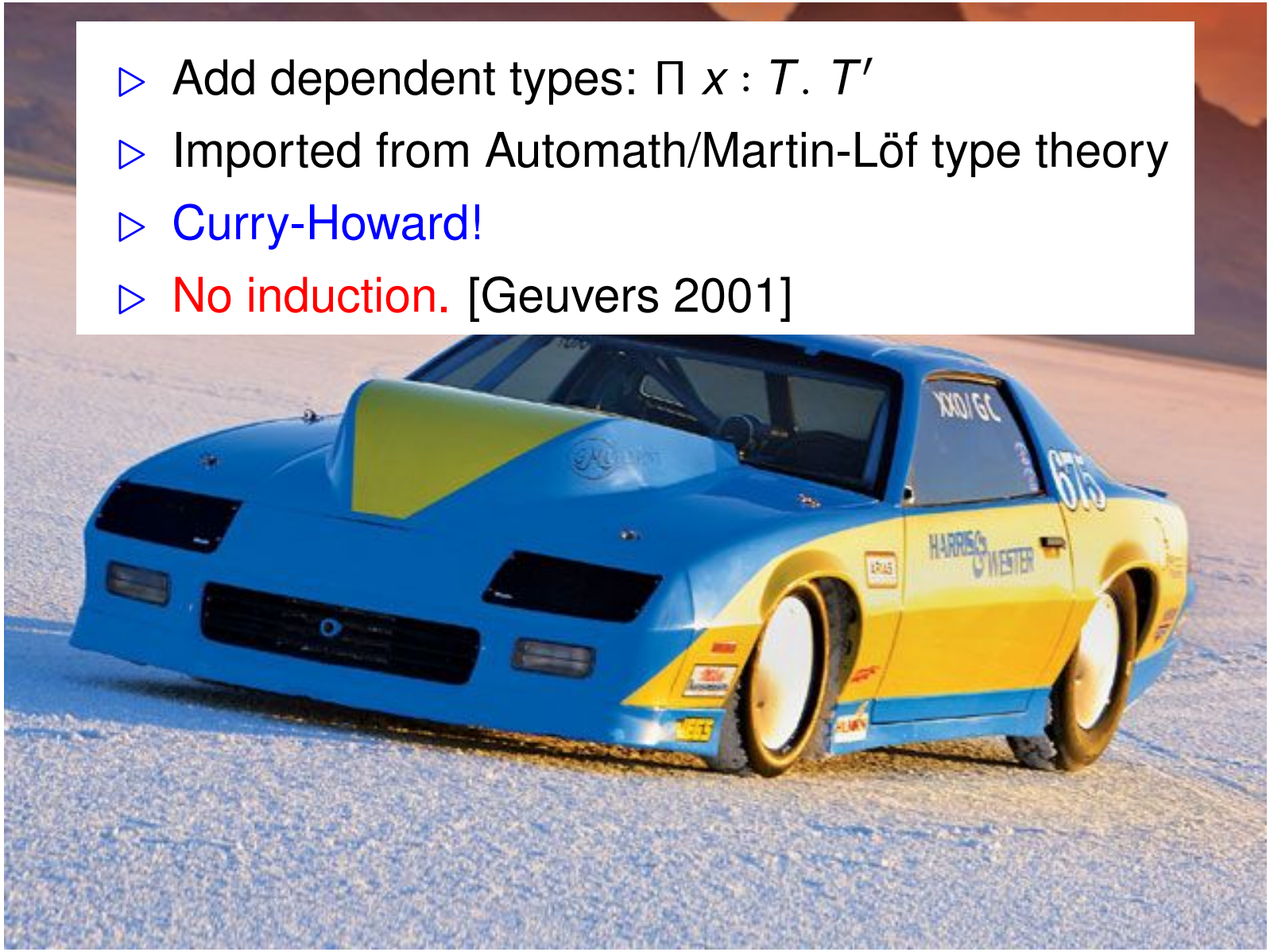
1969 Mercury Cyclone Spoiler II

Calculus of Constructions (Coquand, Huet 1988)

1988 Chevrolet Camaro

Calculus of Constructions (Coquand, Huet 1988)

- ▷ Add dependent types: $\Pi\, x : T.\ T'$
- ▷ Imported from Automath/Martin-Löf type theory
- ▷ Curry-Howard!
- ▷ No induction. [Geuvers 2001]



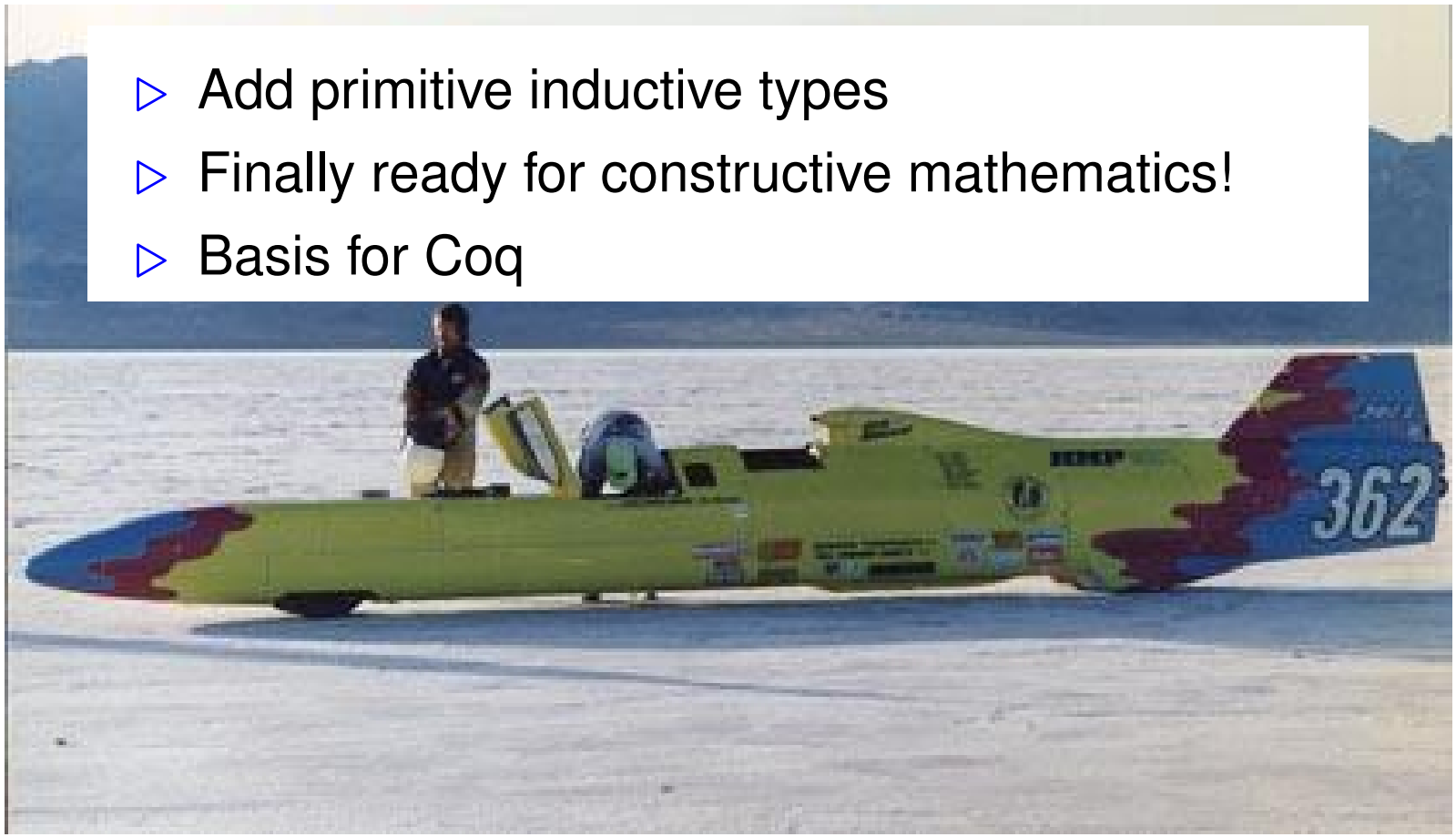
1988 Chevrolet Camaro

Calculus of Inductive Constructions (Werner 1994)

1992 Hoffman-Markley Streamliner

Calculus of Inductive Constructions (Werner 1994)

- Add primitive inductive types
- Finally ready for constructive mathematics!
- Basis for Coq



1992 Hoffman-Markley Streamliner

But Coq ≠ CIC

- Coinductive types
- Universe hierarchy (Extended CC, Luo 1990)
- Proof-irrelevant universe `Prop`
- *And we might want more:*
  - definitional proof irrelevance
  - inductive-inductive types
  - inductive-recursive types

Similarly, Agda ≠ MLTT.

Issues and limitations, Coq and Agda

- No formal semantics/correctness proof
  - Despite a lot of interest: TT in TT
- (Hence!) bugs and surprises
  - incompatibilities with various axioms
  - actual contradictions!
  - type soundness broken in Coq
- Commitment to a set of datatypes
  - theory of datatypes not finished...
  - e.g., higher-order abstract syntax prohibited

Have we created a monster?

Schaufelradbagger 258

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Schaufelradbagger 258

If I could turn back time...

Good-bye to:

- primitive datatypes
- (also universe hierarchy, my bias)

Hello to

- lambda-encodings of data

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Wanted: a new type theory

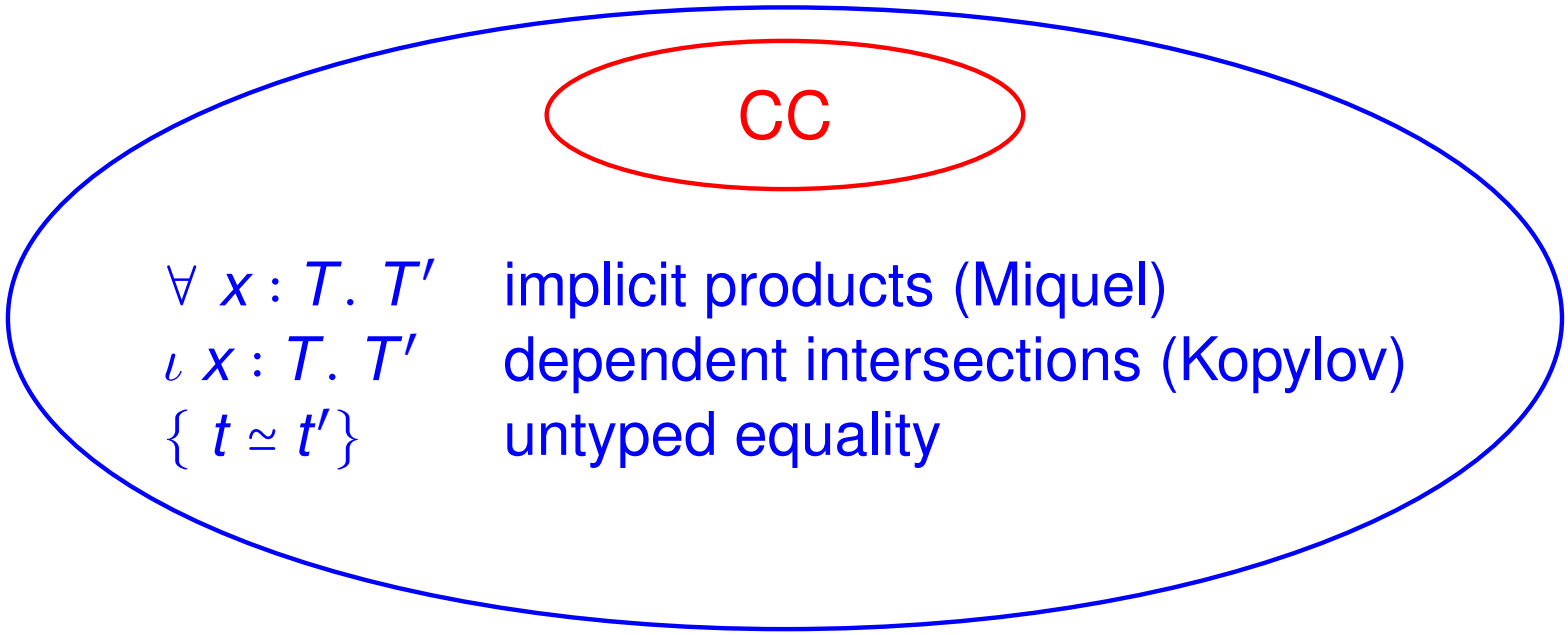
where

- inductive datatypes are derived (lambda-encoded)
- impredicativity is central
- core theory is small and verifiable

Tooling goals:

- see all typing/inference information
- predictable inference
- elaborate to core with independent checker

Introducing Cedille



- Small theory, formal syntax and semantics
- Core checker implemented in $<$ 1000loc Haskell
- Logically sound
- Turing complete(!)
- Supports inductive lambda-encodings

Back the truck up

Back the truck up

Not your forebear's lambda encodings

- Usual rap: inefficient accessors
- Corrected by Parigot 1988 for typed encoding
- Perfect untyped encoding Böhm et al. 1994
  - linear space
  - constant-time accessors
  - intrinsic support for iteration
- Cedille: perfect inductive (typed) encodings

What do we get from this?

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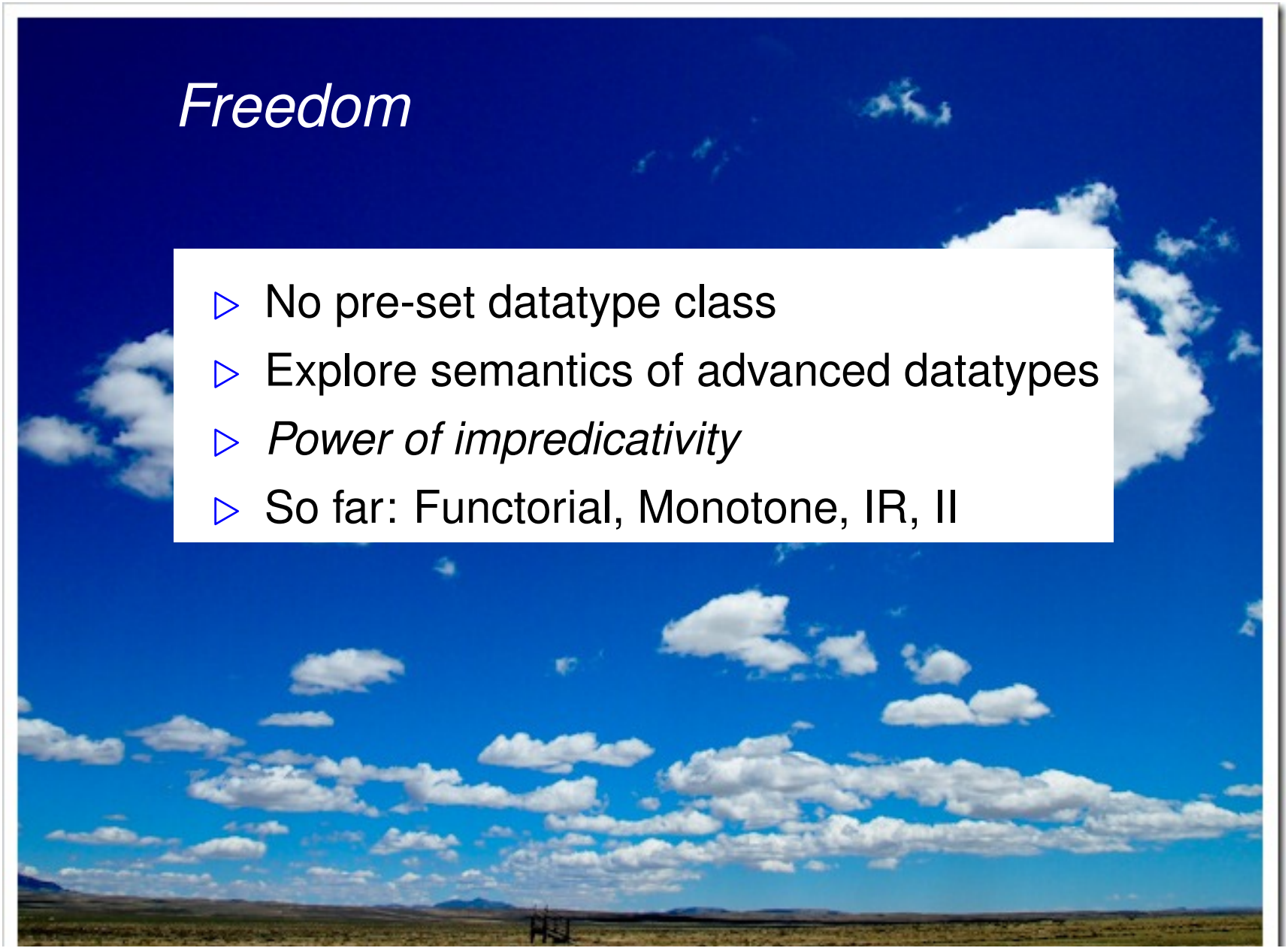


So which car are we?

So which car are we?

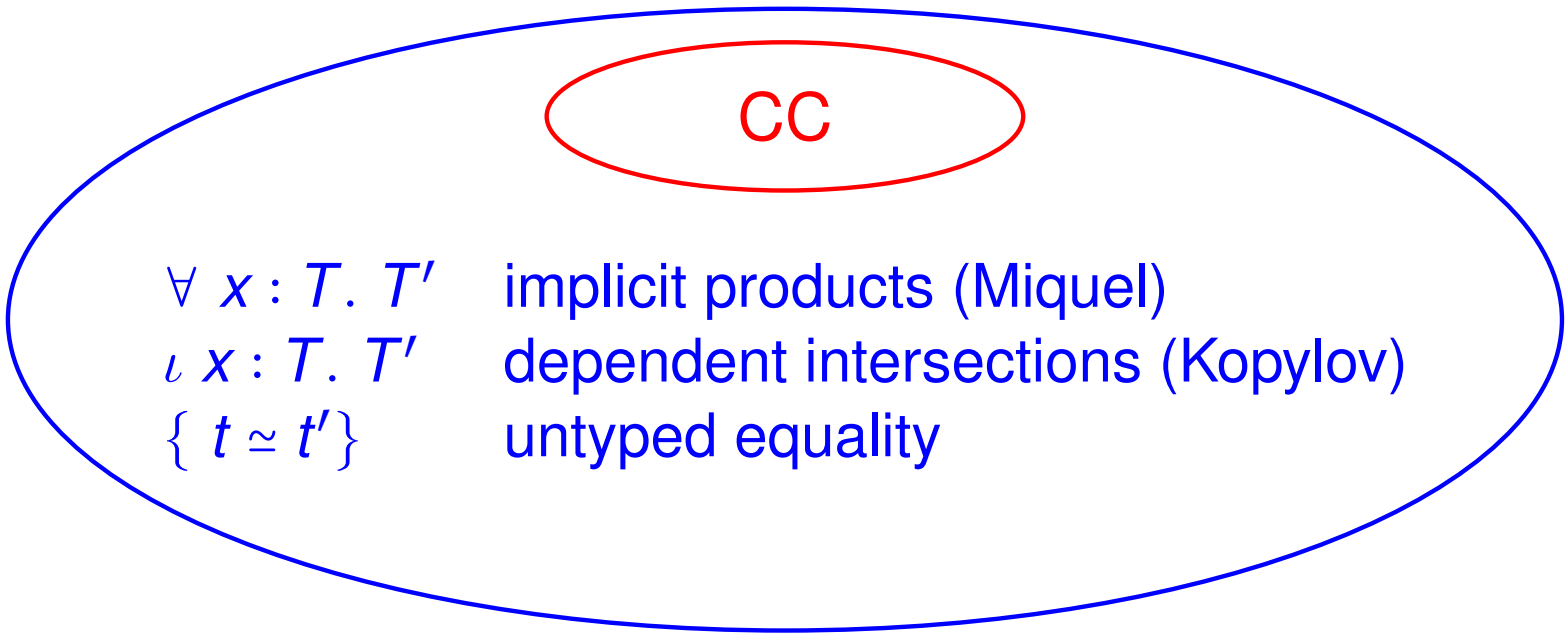
So which car are we?

$\vdash$ *CeDiLIE*

Syntax and semantics of Cedille

Calculus of Dependent Lambda Eliminations

CDLE


CC
$\forall\ x : T.\ T'$ implicit products (Miquel)
$\iota\ x : T.\ T'$ dependent intersections (Kopylov)
$\{\ t \simeq t'\}$ untyped equality

Dependent intersections $\iota\, x : T_1.\ T_2$

- Usual intersection types:

$$\frac{t : T_1 \qquad t : T_2}{t : T_1 \cap T_2}$$

- Dependent intersection:

$$\frac{t : T_1 \qquad t : [t/x]T_2}{t : \iota\, x : T_1.\ T_2}$$

T_2 can refer to subject of typing, at weaker type T_1

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Yes!

For each type (here, implicit products)

- Formation rule for the type

$$\frac{\Gamma \vdash T' : \star \qquad \Gamma, x : T' \vdash T : \star}{\Gamma \vdash \forall\ x : T'.\ T : \star}$$

- Introduction and elimination rules

$$\frac{\Gamma \vdash \forall\ x : T'.\ T : \star \qquad \Gamma, x : T' \vdash t : T}{\Gamma \vdash t : \forall\ x : T'.\ T} \qquad \frac{\Gamma \vdash t : \forall\ x : T'.\ T \qquad \Gamma \vdash t' : T'}{\Gamma \vdash t : [t'/x]T}$$

- Annotated terms
  - Introduction: $\Lambda\ x : T'.\ t$
  - Elimination: $t\ -t'$

- Annotated terms erase to pure lambda terms

$$\begin{aligned} |\Lambda\ x : T.\ t| &= |t| \\ |t\ -t'| &= |t| \end{aligned}$$

Equality type

Formation:

$$\frac{FV(t\ t') \subseteq dom(\Gamma)}{\Gamma \vdash \{t \simeq t'\} : \star}$$

Introduction and elimination:

$$\frac{FV(t') \subseteq dom(\Gamma)}{\Gamma \vdash t : \{t' \simeq t'\}} \qquad \frac{\Gamma \vdash t' : \{t_1 \simeq t_2\} \qquad \Gamma \vdash t : [t_1/x]T}{\Gamma \vdash t : [t_2/x]T}$$

Direct computation rule:

$$\frac{\Gamma \vdash t : \{t_1 \simeq t_2\} \qquad \Gamma \vdash t_1 : T}{\Gamma \vdash t_2 : T}$$

Annotations:

$$\begin{aligned} |\beta\{t\}| &= |t| \\ |\rho\ t'\ \text{-}\ t| &= |t| \\ |\phi\ t\ \text{-}\ t_1\{t_2\}| &= |t_2| \end{aligned}$$

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The Kleene trick

- Any term can be assigned a trivially true equality

$$\mathsf{t} : \{t' \simeq t'\}$$

- Restricted form of subset type, combined with *conversion*

$$\frac{\Gamma \vdash t : T' \qquad \Gamma \vdash T : \star \qquad T \cong T'}{\Gamma \vdash t : T}$$

- When $[t/x]t' =_{\beta\eta} [t/x]t''$, yields

$$t : \iota\, x : T.\, \{t' \simeq t''\}$$

- E.g.,

$$\begin{array}{rcl} \mathit{True} & = & \forall\, X : \star.\, X \to X \\ \lambda\, y.\, y\ y & : & \iota\, x : \mathit{True} \to \mathit{True}.\, \{x\ \lambda z.z \simeq \lambda z.z\} \\ \mathit{anything} & : & \{\lambda x.x \simeq \lambda x.x\} \end{array}$$

Aside on rewriting

ρ t - t'

- Suppose $t : \{t_1 \simeq t_2\}$, and T is type for t'.
- Find each subterm of T convertible to t_1 and rewrite.
- Error if no matches
- Use ρ anywhere in a term (cf. Agda)
- Other forms of ρ:
  - $\rho+$, head-normalize as you look for matches
  - $\rho\langle n_1 \cdots n_k \rangle$, rewrite occurrences $n_1, \cdots n_k$

Dependent intersections

Formation:

$$\frac{\Gamma \vdash T : \star \qquad \Gamma, x : T \vdash T' : \star}{\Gamma \vdash \iota\, x : T.\ T'}$$

Introduction and elimination:

$$\frac{\Gamma \vdash t : T \qquad \Gamma \vdash t : [t/x]T'}{\Gamma \vdash t : \iota\, x : T.\ T'} \qquad \frac{\Gamma \vdash t : \iota\, x : T.\ T'}{\Gamma \vdash t : T} \qquad \frac{\Gamma \vdash t : \iota\, x : T.\ T'}{\Gamma \vdash t : [t/x]T'}$$

Annotations:

$$\begin{aligned} |[t, t']| &= |t| \\ |t.1| &= |t| \\ |t.2| &= |t| \end{aligned}$$

How are inductive datatypes defined?

- Many variations, one theme:

  The type of d expresses an induction principle for d

- For Nat:

$$n : \forall P : Nat \to \star.\ (\forall x : Nat.\ P\ x \to P\ (S\ x)) \to P\ Z \to P\ n$$

- Essentially due to Leivant 1983
- Will walk through example a little later

Another example: casts

- A cast is an identity function from A to B

```
Cast : ⋆ → ⋆ → ⋆ = λ A : ⋆ . λ B : ⋆ .
   ι cast : A → B . { cast ≃ λ x . x }.
```

- If there is a cast, you can change the type:

```
cast : ∀ A : ⋆ . ∀ B : ⋆ . Cast · A · B ⇒ A → B
  = ...
```

- Nontrivial casts exist in extrinsic type theory:

$$\lambda\, x.\, x \;\; : \;\; (\forall\, X{:}\star.\, X \to X) \to (\mathit{Nat} \to \mathit{Nat})$$

- Not in intrinsic type theory:

$$\lambda\, x.\, (x \cdot \mathit{Nat}) \;\; : \;\; (\forall\, X{:}\star.\, X \to X) \to (\mathit{Nat} \to \mathit{Nat})$$

Monotone recursive types

- Explored in works by R. Matthes [eg, in Synthese, 2002]
- Given monotone F, extend the theory with μF
- Monotonicity expressed by a term t of type

  $$\forall\, X{:}\star.\, \forall\, Y{:}\star.\, (X \to Y) \to (F \cdot X \to F \cdot Y)$$

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We can derive a stronger form, within our theory!

Starting point: proof of Tarski's Theorem

Consider monotone $f : \mathcal{P}(X) \to \mathcal{P}(X)$

Let $Q = \bigcap\{A \mid f(A) \subseteq A\}$

Prove $f(Q) = Q$ by both inclusions.

Translating the proof to Cedille

- "$A \subseteq B$" becomes `Cast · A · B`
- Monotonicity becomes

```
∀ X : ⋆ . ∀ Y : ⋆ .
Cast · X · Y → Cast · (F · X) · (F · Y)
```

- "$\bigcap\{A \mid f(A) \subseteq A\}$" becomes

```
Rec = ∀ X : ⋆ . Cast · (F · X) · X ⇒ X
```

Monotone recursive types: summary

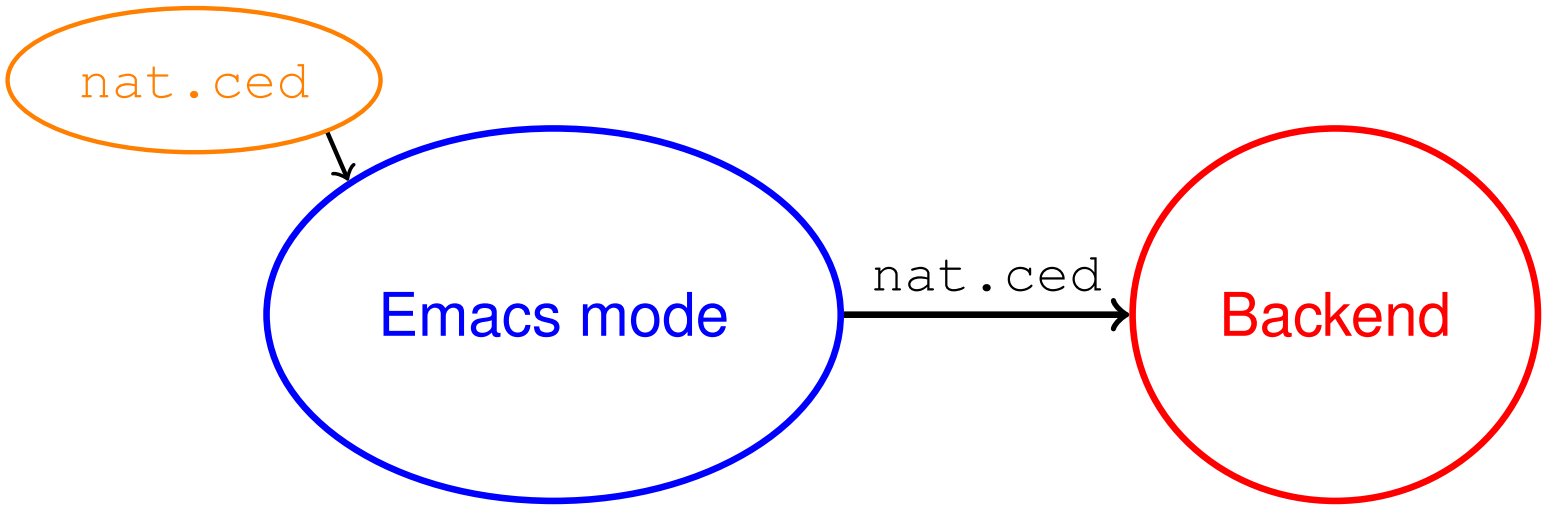
- Derive `Rec` for any monotone `F`
- Casts (identity functions) between `F · Rec` and `Rec`
- Proof translates proof of Tarski fixed-point theorem
- Code is 18 lines of Cedille
- To emphasize: no addition to the theory, these are derived

`cedille`

Tooling: emacs frontend $\leftrightarrow$ backend

Emacs mode

- Interact with Cedille through an emacs mode



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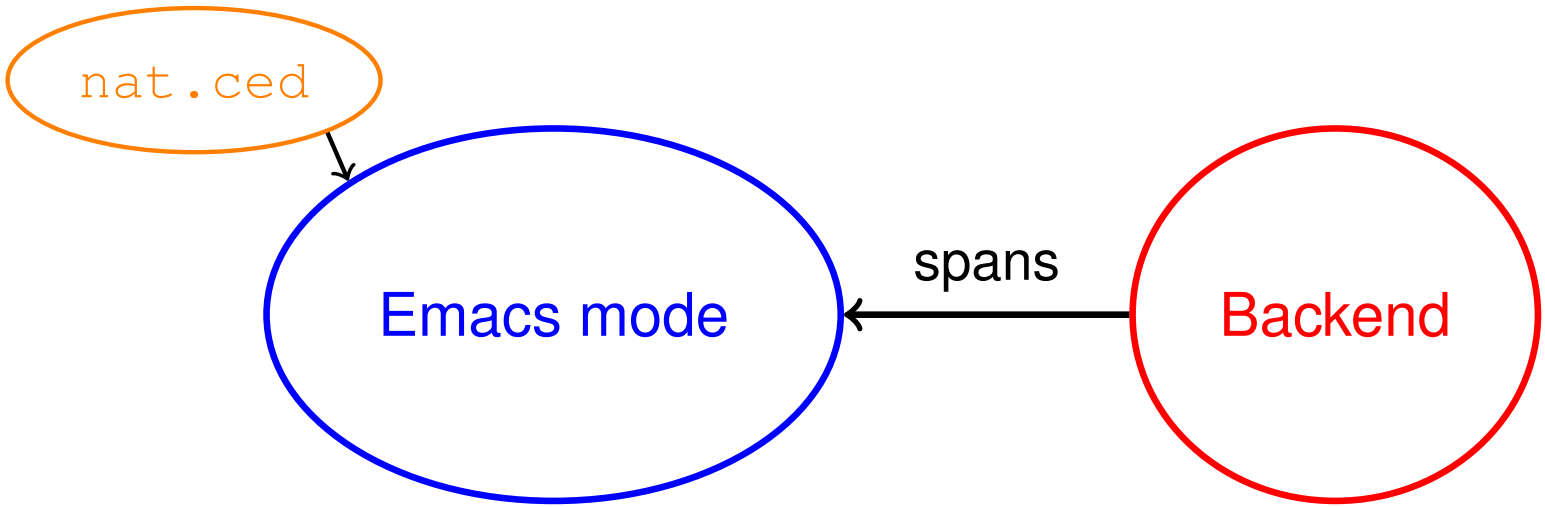

- A span is [*label*, *start-pos*, *end-pos*, *attributes*]
- Spans communicated in JSON
- Cedille sends all type information, in span attributes
- Monadic style for writing the backend (type checker)

Demo `cedille`