A. Durmus alain.durmus@polytechnique.edu

5 Classification

Exercise 5.1. Let (X, Y) be some random variables on $\mathbb{R}^d \times \{0, 1\}$ with density

$$\mathbb{P}(X \in \mathsf{A}, Y = y) = \int_{\mathsf{A}} p_y(x) \mathrm{d}x \; , \mathsf{A} \in \mathcal{B}(\mathbb{R}^d) \; , \tag{22}$$

where p_0, p_1 are densities with respect to the Lebesgue measure. We define a classifier as any functions $\mathscr{C} : \mathbb{R}^d \to \{0, 1\}$ and the $0 - 1$ risk:

$$\mathrm{Risk}(\mathscr{C}) = \mathbb{E}\left[\mathbb{1}\left\{Y, \mathscr{C}(X)\right\}\right] \; . \tag{23}$$

(1) Show that for any classifier Risk, it holds:

$$\mathrm{Risk}(\mathscr{C}^\star) \leq \mathrm{Risk}(\mathscr{C}^\star) \; , \tag{24}$$

where $\mathscr{C}^\star(x) = \mathbb{1}\left\{\eta(x) \geq 1/2\right\}$ with $\eta(x) = \mathbb{E}\left[Y | X = x\right]$.

(2) Show that if

$$\mathrm{Risk}(\mathscr{C}^\star) = \mathrm{Risk}(\mathscr{C}^\star) \tag{25}$$

then almost surely it holds

$$\mathbb{1}\left\{\eta(x) \neq 1/2\right\} \mathscr{C}^\star(X) = \mathscr{C}(X) \mathbb{1}\left\{\eta(x) \neq 1/2\right\} \; . \tag{26}$$

Exercise 5.2. We define the convex hull of a set of points $\{x_i\}_{i=1}^n \subset \mathbb{R}^d$ as

$$\mathrm{Conv}(\{x_i\}_{i=1}^n) = \left\{ \sum_{i=1}^n \varpi_i x_i \; : \; \{\varpi_i\}_{i=1}^n \in \mathrm{Simplex}_n \right\} \; . \tag{27}$$

We consider $\{x'_i\}_{i=1}^{n'} \subset \mathbb{R}^d$ another set of points.

- Recall what does it mean that $\{x_i\}_{i=1}^n$ and $\{x'_i\}_{i=1}^{n'}$ are linearly separable.
- Show that if $\{x_i\}_{i=1}^n$ and $\{x'_i\}_{i=1}^{n'}$ are linearly separable, then $\mathrm{Conv}(\{x_i\}_{i=1}^n) \cap \mathrm{Conv}(\{x'_i\}_{i=1}^{n'}) = \emptyset$.
- Conversely, show that if $\{x'_i\}_{i=1}^{n'}$ are not linearly separable then $\mathrm{Conv}(\{x_i\}_{i=1}^n) \cap \mathrm{Conv}(\{x'_i\}_{i=1}^{n'}) \neq \emptyset$.