École polytechnique
MAP 534: introduction to machine learning
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4 Conditional probabilities and gradient descent

4.1 Discrete conditional density/distribution

Exercise 4.1. Let X and Y be two random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that X is valued in $\mathbb{N}$ and Y follows a exponential distribution with parameter 1 on $\mathbb{R}$. In addition, we assume that the conditional distribution of X given Y is the Poisson distribution with parameter Y. Give the distribution of (X, Y) and the conditional distribution of Y given X.

Exercise 4.2. Let X and Y be two real random variables with distribution $\mathbf{Pn}(\lambda)$ et $\mathbf{Pn}(\mu)$ respectively. Denote $S = X + Y$.

(i) Give the distribution of S.

(ii) For any $s \in \mathbb{N}$ give the conditional distribution of X given S.

(iii) Give $\mathbb{E}[X|S]$.

(iv) Check that $\mathrm{Var}\left(\mathbb{E}\left[X|S\right]\right) \leq \mathrm{Var}\left(X\right)$.

4.2 Continuous conditional density

Exercise 4.3. Let X and Y be two independent real random variables with distribution $\mathbf{Unif}([0,1])$. Denote $D = X - Y$.

(i) Find the distribution of D.

(ii) For any $d \in \mathbb{R}$ find the conditional distribution of X given $D = d$.

(iii) Compute $\mathbb{E}[X|D]$.

(iv) Check that $\mathrm{Var}\left(\mathbb{E}\left[X|D\right]\right) \leq \mathrm{Var}\left(X\right)$.

Exercice 4.4. Let X and Y be two independent real random variables with distribution $\mathbf{Exp}(\lambda)$, $\lambda > 0$ Denote $S = X + Y$.

(i) Find the distribution of S.

(ii) For any $s \in \mathbb{R}$ find the conditional distribution of X given $S = s$.

(iii) Compute $\mathbb{E}[X|S]$.

(iv) Check that $\mathrm{Var}\left(\mathbb{E}\left[X|S\right]\right) \leq \mathrm{Var}\left(X\right)$.

Exercise 4.5. Let $Z = (X, Y)$ be a random variable on $\mathbb{R}^2$ such that:

(i) X has distribution $\mathbf{Gamma}(2, \lambda)$ (with continuous density $f_X(x) = \lambda^2 x \mathrm{e}^{-\lambda x} \mathbb{1}_{\mathbb{R}_+}(x)$),

(ii) the conditional distribution of Y given X is the uniform distribution on $[0, X]$ (i.e. the conditional density of Y given $X = x$ is $f_{Y|X=x}(y) = (1/x)\mathbb{1}_{[0,x]}(y)$).

1. Find the density of $Z = (X, Y)$ and the distribution of Y.

2. Find the conditional density of X given Y.

3. Compute the following quantities:

   (a) $\mathbb{E}[Y|X]$,

   (b) $\mathbb{E}[X|Y]$,

   (c) $\mathbb{E}[X + XY|Y]$,

   (d) $\mathbb{E}[\mathbb{E}[Y|X]]$,

   (e) $\mathbb{E}[XY]$ (on pourra utiliser le fait que $\mathbb{E}[X^2] = 6/\lambda^2$).