Symbolic Weighted Language Models and Quantitative Parsing over Infinite Alphabets

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Abstract

We propose a framework for weighted parsing over infinite alphabets. It is based on language models called Symbolic Weighted Automata (swA) at the joint between Symbolic Automata (sA) and Weighted Automata (wA), as well as Transducers (swT) and Visibly Pushdown (sw-VPA) variants. Like sA, swA deal with large or infinite input alphabets, and like wA, they output a weight value in a semiring domain. The transitions of swA are labeled by functions from an infinite alphabet into the weight domain. This is unlike sA whose transitions are guarded by boolean predicates overs symbols in an infinite alphabet and also unlike wA whose transitions are labeled by constant weight values, and who deal only with finite automata. We present some properties of swA, swT and sw-VPA models, that we use to define and solve a variant of parsing over infinite alphabets. We illustrate the models with examples taken from a motivating application, namely a parse-based approach to automated music transcription.

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1 Introduction

Parsing is the problem of structuring a linear representation on input (a finite word), according to a language model. Most of the context-free parsing approaches [15] assume a finite and reasonably small input alphabet. Such a restriction makes perfect sense in the context of NLP tasks such as constituency parsing, or of programming languages compilers or interpreters. Considering large or infinite alphabets can however be of practical interest, for instance, when dealing with large characters encodings such as UTF-16, *e.g.* for vulnerability detection in Web-applications [8], for the analysis (*e.g.* validation or filtering) of data streams or serialization of structured documents (with textual or numerical attributes) [26], or for processing timed execution traces [3].

The latter case is related to a study that motivated the present work: automated music transcription. Most representations of music are essentially linear. This is true for audio files, but also for widely used symbolic representations like MIDI. Such representations ignore the hierarchical structures that frame the conception of music, at least in the western area. These structures, on the other hand, are present, either explicitly or implicitly, in music notation [14]: music scores are partitioned in measures, measures in beats, and beats can be further recursively divided. It follows that music events do not occur at arbitrary timestamps, but respect a discrete division of the timeline incurred by these recursive divisions. The *transcription problem* takes as input a linear repreentation (audio or MIDI) and aims at re-constructing these structures by mapping input events to this hierarchical rhythmic space. It can therefore be stated as a parsing problem [12], over an infinite alphabet of timed events. Throughout the paper we will illustrate our framework with examples taken for this

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$$\begin{tikzcd}[column sep=small]
 & \begin{array}{c}\mathsf{FA} : \Sigma_{\mathsf{fin}}^* \to \mathbb{B} \\ q \xrightarrow{a} q' \quad a \in \Sigma_{\mathsf{fin}}\end{array} \arrow[dl] \arrow[dr] & \\
\begin{array}{c}\mathsf{WA} : \Sigma_{\mathsf{fin}}^* \to \mathbb{S} \\ q \xrightarrow{a,w} q' \quad a \in \Sigma_{\mathsf{fin}}, w \in \mathbb{S}\end{array} \arrow[dr] & & \begin{array}{c}\mathsf{SA} : \Sigma_{\mathsf{inf}}^* \to \mathbb{B} \\ q \xrightarrow{\phi} q' \quad \phi : \Sigma_{\mathsf{inf}} \to \mathbb{B}\end{array} \arrow[dl] \\
 & \begin{array}{c}\mathsf{SWA} : \Sigma_{\mathsf{inf}}^* \to \mathbb{S} \\ q \xrightarrow{\phi} q' \quad \phi : \Sigma_{\mathsf{inf}} \to \mathbb{S}\end{array} &
\end{tikzcd}$$

Figure 1 Classes of Symbolic/Weighted Automata. Σ_{fin} is a finite alphabet, Σ_{inf} is a countable alphabet, $\mathbb{B}$ is the Boolean algebra, $\mathbb{S}$ is a commutative semiring, $q \xrightarrow{\cdots} q'$ is a transition between states q and q'.

motivating application.

Various extensions of language models for handling infinite alphabets have been studied. For instance, some automata with memory extensions allow restricted storage and comparison of input symbols, (see [26] for a survey), with pebbles for marking positions [25], registers [18], or the possibility to compute on subsequences with the same attribute values [2]. The automata at the core of model checkers compute on input symbols represented by large bitvectors [27] (sets of assignments of Boolean variables) and in practice, each transition accepts a set of such symbols (instead of an individual symbol), represented by Boolean formula or Binary Decision Diagrams. Following a similar idea, in symbolic automata (sA) [7, 8], the transitions are guarded by predicates over infinite alphabet domains. With appropriate closure conditions on the sets of such predicates, all the good properties enjoyed by automata over finite alphabets are preserved.

register: skip refs and details, add Mikolaj recent

Other extensions of language models help in dealing with non-determinism, by the computation of weight values. With an ambiguous grammar, there may exist several derivations (*abstract syntax trees* – AST) yielding one input word. The association of one weight value to each AST permits to select a best one (or n bests). This is roughly the principle of *weighted parsing* approaches [13, 24, 23]. In *weighted language models*, like *e.g.* probabilistic context-free grammars and weighted automata (wA) [11], a weight value is associated to each transition rule, and the rule's weights can be combined with an associative product operator $\otimes$ into the weight of an AST. A second operator $\oplus$, associative and commutative, is moreover used to resolve the ambiguity raised by the existence of several (in general exponentially many) AST associated to a given input word. Typically, $\oplus$ will select the best of two weight values. The weight domain, equipped with these two operators shall be, at minima, a *semiring* where $\oplus$ can be extended to infinite sums, such as the Viterbi semiring and the tropical min-plus algebra, see Figure 2.

La figure 2 est citée avant la figure 1 mais apparait longtemps après. A corriger.

In this paper, we present a uniform framework for weighted parsing over infinite input alphabets. It is based on *symbolic weighted* finite states language models (swM), generalizing the Boolean guards of sA into functions into an arbitrary semiring, and generalizing also wA, by handling infinite alphabets, see Figure 1.

Tu fais une différence entre model et automata?

In short, a transition rule $q \xrightarrow{\phi} q'$ from state q to q' of a swM, is labeled by a function ϕ associating to every input symbol a a weight value $\phi(a)$ in a semiring domain. The models presented here are finite automata called symbolic-weighted (swA), transducers (swT), and

This sentence (symbols as variables) is not immediately clear to me. Maybe a short example or intuition?

modified

Tu veux dire: les modèles formels que tu combines?

pushdown automata with a visibly restriction [1] (sw-VPA). The latter model of automata operates on *nested words* [1], a structured form of words parenthesized with markup symbols, corresponding to a linearization of trees. In the context of parsing, they can represent (weighted) AST of CF grammars. More precisely, a sw-VPA A associates a weight value $A(t)$ to a given nested word t, which is the linearization of an AST. On the other hand, a swT can define a distance $T(s,t)$ between finite words s and t over infinite alphabets. Then, the *SW-parsing* problem aims at finding t minimizing $T(s,t) \otimes A(t)$ (*wrt* the ranking defined by $\oplus$), given an input word s. The latter value is called the distance between s and A in [21].

Like weighted-parsing methods [13, 24, 23], our approach proceeds in two steps, based on properties of the swM. The first step is an intersection (Bar-Hillel construction [15]) where, given a swT T, a sw-VPA A, and an input word s, a sw-VPA $A_{T,s}$ is built, such that for all t, $A_{T,s}(t) = T(s,t) \otimes A(t)$. In the second step, a best AST t is found by applying to $A_{T,s}$ a best search algorithm similar to the shortest distance in graphs [20, 17].

chap. intersection in [15]

The notation $A_{T,s}$ has not been introduced so far. It is not clear why T is a parameter there

The main contributions of the paper are: (*i*) the introduction of automata, swA, transducers, swT (Section 3), and visibly pushdown automata sw-VPA (Section 4), generalizing the corresponding classes of symbolic and weighted models, (*ii*) a polynomial best-search algorithm for sw-VPA, and (*iii*) a uniform framework (Section 5) for parsing over infinite alphabets, the keys to which are (*iii.a*) the swT-based definition of generic edit distances between input and output (yield) words, and (*iii.b*) the use, convenient in this context, of nested words, and sw-VPA, instead of syntax trees and grammars.

expressiveness: VPA have restricted equality test. comparable to pebble automata? → conclusion

2 Preliminary Notions

Semirings

We shall consider semirings for the weight values of our language models. A *semiring* $\langle \mathbb{S}, \oplus, \mathbb{0}, \otimes, \mathbb{1} \rangle$ is a structure with a domain $\mathbb{S}$, equipped with two associative binary operators $\oplus$ and $\otimes$, with respective neutral elements $\mathbb{0}$ and $\mathbb{1}$, and such that:

- $\oplus$ is commutative: $\langle \mathbb{S}, \oplus, \mathbb{0} \rangle$ is a commutative monoid and $\langle \mathbb{S}, \otimes, \mathbb{1} \rangle$ a monoid,
- $\otimes$ distributes over $\oplus$: $\forall x, y, z \in \mathbb{S}$, $x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$, and $(x \oplus y) \otimes z = (x \otimes z) \oplus (y \otimes z)$,
- $\mathbb{0}$ is absorbing for $\otimes$: $\forall x \in \mathbb{S}$, $\mathbb{0} \otimes x = x \otimes \mathbb{0} = \mathbb{0}$.

Intuitively, in the models presented in this paper, $\oplus$ selects an optimal value from two given values, in order to handle non-determinism, and $\otimes$ combines two values into a single value.

A semiring $\mathbb{S}$ is *commutative* if $\otimes$ is commutative. It is *idempotent* if for all $x \in \mathbb{S}$, $x \oplus x = x$. Every idempotent semiring $\mathbb{S}$ induces a partial ordering $\leq_\oplus$ called the *natural ordering* of $\mathbb{S}$ [20] defined, by: for all $x, y \in \mathbb{S}$, $x \leq_\oplus y$ iff $x \oplus y = x$. The natural ordering is sometimes defined in the opposite direction [10]; We follow here the direction that coincides with the usual ordering on the Tropical semiring *min-plus* (Figure 2). An idempotent semiring $\mathbb{S}$ is called *total* if it $\leq_\oplus$ is total *i.e.* when for all $x, y \in \mathbb{S}$, either $x \oplus y = x$ or $x \oplus y = y$.

is total necessary?

▶ **Lemma 1** (Monotony, [20]). *Let $\langle \mathbb{S}, \oplus, \mathbb{0}, \otimes, \mathbb{1} \rangle$ be an idempotent semiring. For all $x, y, z \in \mathbb{S}$, if $x \leq_\oplus y$ then $x \oplus z \leq_\oplus y \oplus z$, $x \otimes z \leq_\oplus y \otimes z$ and $z \otimes x \leq_\oplus z \otimes y$.*

To express the property of Lemma 1, we call $\mathbb{S}$ *monotonic wrt* $\leq_\oplus$. Another important semiring property in the context of optimization is superiority [16], which corresponds to the *non-negative weights* condition in shortest-path algorithms [9]. Intuitively, it means that combining elements with $\otimes$ always increase their weight. Formally, it is defined as the property (*i*) below.

	domain	$\oplus$	$\otimes$	$\mathbb{0}$	$\mathbb{1}$
Boolean	$\{\bot, \top\}$	$\vee$	$\wedge$	$\bot$	$\top$
Counting	$\mathbb{N}$	$+$	$\times$	0	1
Viterbi	$[0,1] \subset \mathbb{R}$	max	$\times$	0	1
Tropical min-plus	$\mathbb{R}_+ \cup \{\infty\}$	min	$+$	∞	0

Figure 2 Some commutative, bounded, total and complete semirings.

▶ **Lemma 2** (Superiority, Boundedness). *Let $\langle \mathbb{S}, \oplus, \mathbb{0}, \otimes, \mathbb{1} \rangle$ be an idempotent semiring. The two following statements are equivalent:*

i. for all $x, y \in \mathbb{S}$, $x \leq_\oplus x \otimes y$ and $y \leq_\oplus x \otimes y$

ii. for all $x \in \mathbb{S}$, $\mathbb{1} \oplus x = \mathbb{1}$.

Proof. $(ii) \Rightarrow (i)$: $x \oplus (x \otimes y) = x \otimes (\mathbb{1} \oplus y) = x$, by distributivity of $\otimes$ over $\oplus$. Hence $x \leq_\oplus x \otimes y$. Similarly, $y \oplus (x \otimes y) = (\mathbb{1} \oplus x) \otimes y = y$, hence $y \leq_\oplus x \otimes y$. $(i) \Rightarrow (ii)$: by the second inequality of (i), with $y = \mathbb{1}$, $\mathbb{1} \leq_\oplus x \otimes \mathbb{1} = x$, *i.e.*, by definition of $\leq_\oplus$, $\mathbb{1} \oplus x = \mathbb{1}$. ◀

In [16], when the property (i) holds, $\mathbb{S}$ is called *superior wrt* the ordering $\leq_\oplus$. We have seen in the proof of Lemma 2 that it implies that $\mathbb{1} \leq_\oplus x$ for all $x \in \mathbb{S}$. Similarly, by the first inequality of (i) with $y = \mathbb{0}$, $x \leq_\oplus x \otimes \mathbb{0} = \mathbb{0}$. Hence, in a superior semiring, it holds that for all $x \in \mathbb{S}$, $\mathbb{1} \leq_\oplus x \leq_\oplus \mathbb{0}$. Intuitively, from an optimization point of view, it means that $\mathbb{1}$ is the best value, and $\mathbb{0}$ the worst. In [20], $\mathbb{S}$ with the property (ii) of Lemma 2 is called *bounded* – we shall use this term in the rest of the paper. It implies that, when looking for a best path in a graph whose edges are weighted by values of $\mathbb{S}$, the loops can be safely avoided, because, for all $x \in \mathbb{S}$ and $n \geq 1$, $x \oplus x^n = x \otimes (\mathbb{1} \oplus x^{n-1}) = x$.

Ca j'ai pas compris

▶ **Lemma 3.** *Every bounded semiring is idempotent.*

Proof. By boundedness, $\mathbb{1} \oplus \mathbb{1} = \mathbb{1}$, and idempotency follows by multiplying both sides by x and distributing. ◀

Here the difference between $\mathbb{S}$ as a structure and as a domain is blurred.

We shall need below infinite sums with $\oplus$. A semiring $\mathbb{S}$ is called *complete* [11] if it has an operation $\bigoplus_{i \in I} x_i$ for every family $(x_i)_{i \in I}$ of elements of $dom(\mathbb{S})$ over an index set $I \subset \mathbb{N}$, such that:

$j \in \mathbb{N}$: j is en element of $\mathbb{N}$, not the same s $j \subset \mathbb{N}$

i. infinite sums extend finite sums:

$$\bigoplus_{i \in \emptyset} x_i = \mathbb{0}, \quad \forall j \in \mathbb{N}, \bigoplus_{i \in \{j\}} x_i = x_j, \forall j, k \in \mathbb{N}, j \neq k, \bigoplus_{i \in \{j,k\}} x_i = x_j \oplus x_k,$$

ii. associativity and commutativity:

for all $I \subseteq \mathbb{N}$ and all partition $(I_j)_{j \in J}$ of I, $\bigoplus_{j \in J} \bigoplus_{i \in I_j} x_i = \bigoplus_{i \in I} x_i$,

iii. distributivity of product over infinite sum:

for all $I \subseteq \mathbb{N}$, $\bigoplus_{i \in I} (x \otimes y_i) = x \otimes \bigoplus_{i \in I} y_i$, and $\bigoplus_{i \in I} (x_i \otimes y) = (\bigoplus_{i \in I} x_i) \otimes y$.

results of this paper for semirings commutative, bounded, total and complete

Label Theory

We shall now define the functions labeling the transitions of SW automata and transducers, generalizing the Boolean algebras of [7] from Boolean to other semiring domains. We consider *alphabets*, which are countable sets of symbols denoted Σ, Δ,... Given a semiring $\langle \mathbb{S}, \oplus, \mathbb{0}, \otimes, \mathbb{1} \rangle$, a *label theory* over $\mathbb{S}$ is a set $\bar{\Phi}$ of recursively enumerable sets denoted Φ_Σ,

OK, donc c'est là que les fonctions d'étiquettes prennent en argument l'input de la règle. Je ne sais pas dans quelle mesure il faut donner un peu d'explications pour faciliter la compréhension du formalisme.

containing unary functions of type $\Sigma \to \mathbb{S}$, or $\Phi_{\Sigma,\Delta}$, containing binary functions $\Sigma \times \Delta \to \mathbb{S}$, and such that:

- for all $\Phi_{\Sigma,\Delta} \in \bar{\Phi}$, we have $\Phi_\Sigma \in \bar{\Phi}$ and $\Phi_\Delta \in \bar{\Phi}$
- every $\Phi_\Sigma \in \bar{\Phi}$ contains all the constant functions from Σ into $\mathbb{S}$,
- for all $\alpha \in \mathbb{S}$ and $\phi \in \Phi_\Sigma$, $\alpha \otimes \phi : x \mapsto \alpha \otimes \phi(x)$, and $\phi \otimes \alpha : x \mapsto \phi(x) \otimes \alpha$ belong to Φ_Σ, and similarly for $\oplus$ and for $\Phi_{\Sigma,\Delta}$
- for all $\phi, \phi' \in \Phi_\Sigma$, $\phi \otimes \phi' : x \mapsto \phi(x) \otimes \phi'(x)$ belongs to Φ_Σ
- for all $\eta, \eta' \in \Phi_{\Sigma,\Delta}$ $\eta \otimes \eta' : x, y \mapsto \eta(x,y) \otimes \eta'(x,y)$ belongs to $\Phi_{\Sigma,\Delta}$
- for all $\phi \in \Phi_\Sigma$ and $\eta \in \Phi_{\Sigma,\Delta}$, $\phi \otimes_1 \eta : x, y \mapsto \phi(x) \otimes \eta(x,y)$ and $\eta \otimes_1 \phi : x, y \mapsto \eta(x,y) \otimes \phi(x)$ belong to $\Phi_{\Sigma,\Delta}$
- for all $\psi \in \Phi_\Delta$ and $\eta \in \Phi_{\Sigma,\Delta}$, $\psi \otimes_2 \eta : x, y \mapsto \psi(y) \otimes \eta(x,y)$ and $\eta \otimes_2 \psi : x, y \mapsto \eta(x,y) \otimes \psi(y)$ belong to $\Phi_{\Sigma,\Delta}$
- similar closures hold for $\oplus$.

partial application is needed?

Intuitively, the operators $\bigoplus_\Sigma$ return global minimum, *wrt* $\leq_\oplus$, of functions of Φ_Σ. When the semiring $\mathbb{S}$ is complete, we consider the following operators on the functions of $\bar{\Phi}$.

$$\begin{array}{l}
\bigoplus_\Sigma : \Phi_\Sigma \to \mathbb{S},\ \phi \mapsto \bigoplus\limits_{a \in \Sigma} \phi(a) \\
\bigoplus^1_\Sigma : \Phi_{\Sigma,\Delta} \to \Phi_\Delta,\ \eta \mapsto \big(y \mapsto \bigoplus\limits_{a \in \Sigma} \eta(a,y)\big) \quad \bigoplus^2_\Delta : \Phi_{\Sigma,\Delta} \to \Phi_\Sigma,\ \eta \mapsto \big(x \mapsto \bigoplus\limits_{b \in \Delta} \eta(x,b)\big)
\end{array}$$

In what follows, we might omit the sub- and superscripts in $\otimes_1$, $\bigoplus^1_\Sigma$..., when there is no ambiguity. We shall keep them only for the special case $\Sigma = \Delta$, *i.e.* $\eta \in \Phi_{\Sigma,\Sigma}$, in order to be able to distinguish between the first and the second argument.

▶ **Definition 4.** *A label theory $\bar{\Phi}$ is* complete *when the underlying semiring $\mathbb{S}$ is complete, and for all $\Phi_{\Sigma,\Delta} \in \bar{\Phi}$ and all $\eta \in \Phi_{\Sigma,\Delta}$, $\bigoplus^1_\Sigma \eta \in \Phi_\Delta$ and $\bigoplus^2_\Delta \eta \in \Phi_\Sigma$.*

notion of diagram of functions akin BDD for transitions in practice

The following facts are immediate.

mv appendix?

▶ **Lemma 5.** *For $\bar{\Phi}$ complete $\alpha \in \mathbb{S}$, $\phi, \phi' \in \Phi_\Sigma$, $\psi \in \Phi_\Delta$, and $\eta \in \Phi_{\Sigma,\Delta}$:*

i. $\bigoplus_\Sigma \bigoplus^2_\Delta \eta = \bigoplus_\Delta \bigoplus^1_\Sigma \eta$

ii. $\alpha \otimes \bigoplus_\Sigma \phi = \bigoplus_\Sigma (\alpha \otimes \phi)$ *and* $\big(\bigoplus_\Sigma \phi\big) \otimes \alpha = \bigoplus_\Sigma (\phi \otimes \alpha)$*, and similarly for* $\oplus$

iii. $\big(\bigoplus_\Sigma \phi\big) \oplus \big(\bigoplus_\Sigma \phi'\big) = \bigoplus_\Sigma (\phi \oplus \phi')$ *and* $\big(\bigoplus_\Sigma \phi\big) \otimes \big(\bigoplus_\Sigma \phi'\big) = \bigoplus_\Sigma (\phi \otimes \phi')$

iv. $\big(\bigoplus^2_\Delta \eta\big) \oplus \big(\bigoplus^2_\Delta \eta'\big) = \bigoplus^2_\Delta (\eta \oplus \eta')$*, and* $\big(\bigoplus^2_\Delta \eta\big) \otimes \big(\bigoplus^2_\Delta \eta'\big) = \bigoplus^2_\Delta (\eta \otimes \eta')$

v. $\phi \otimes \big(\bigoplus^2_\Delta \eta\big) = \bigoplus_\Delta (\phi \otimes_1 \eta)$*, and* $\big(\bigoplus^2_\Delta \eta\big) \otimes \phi = \bigoplus_\Delta (\eta \otimes_1 \phi)$*, and similarly for* $\oplus$

vi. $\psi \otimes \big(\bigoplus^1_\Sigma \eta\big) = \bigoplus_\Sigma (\psi \otimes_2 \eta)$*, and* $\big(\bigoplus^1_\Sigma \eta\big) \otimes \psi = \bigoplus_\Sigma (\eta \otimes_2 \psi)$*, and similarly for* $\oplus$

Je trouve qu'il y a beaucoup de notions à retenir (complete, effective) et ça devient difficile pour un lecteur non spécialiste. Est-ce que tout est nécessaire (je ne sais plus qui m'avait dit: un concept en plus, un point en moins.

A label theory is called *effective* when for all $\phi \in \Phi_\Sigma$ and $\eta \in \Phi_{\Sigma,\Delta}$, $\bigoplus_\Sigma \phi$, $\bigoplus_\Delta \bigoplus_\Sigma \eta$, and $\bigoplus_\Sigma \bigoplus_\Delta \eta$ can be effectively computed from ϕ and η.

Concretely, in one of the language models defined below, we consider a finite number of base functions ϕ, η of the underlying label theory, labelling transitions, and combine them with the above operators for construction of other models. The combinations might be represented by dags (diagrams) whose leaves are labeled by base functions and inner nodes by operators.

$\exists$ oracle returning ... in worst time complexity T.

3 SW Automata and Transducers

We follow the approach of [21] for the computation of distances, between words and languages, using weighted transducers, and extend it to infinite alphabets. The models introduced in this section generalize weighted automata and transducers [11] by labeling each transition with a weight function (instead of a simple weight value), that takes the input and output symbols as parameters. These functions are similar to the guards of symbolic automata [7, 8], but they can return values in a generic semiring, whereas the latter guards are restricted to the Boolean semiring.

Let $\mathbb{S}$ be a commutative semiring, Σ and Δ be alphabets called respectively *input* and *output*, and $\bar{\Phi}$ be a label theory over $\mathbb{S}$ containing Φ_Σ, Φ_Δ, $\Phi_{\Sigma,\Delta}$.

▶ **Definition 6.** *A* symbolic-weighted transducer *(swT) over Σ, Δ, $\mathbb{S}$ and $\bar{\Phi}$ is a tuple $T = \langle Q, \mathsf{in}, \bar{\mathsf{w}}, \mathsf{out}\rangle$, where Q is a finite set of states, $\mathsf{in} : Q \to \mathbb{S}$ (respectively $\mathsf{out} : Q \to \mathbb{S}$) are functions defining the weight for entering (respectively leaving) computation in a state, and $\bar{\mathsf{w}}$ is a triplet of transition functions $\mathsf{w}_{10} : Q \times Q \to \Phi_\Sigma$, $\mathsf{w}_{01} : Q \times Q \to \Phi_\Delta$, and $\mathsf{w}_{11} : Q \times Q \to \Phi_{\Sigma,\Delta}$.*

We call *number of transitions* of T the number of pairs of states $q, q' \in Q$ such that w_{10} or w_{01} or w_{11} is not the constant $\mathbb{0}$. For convenience, we shall sometimes present transitions as functions of $Q \times (\Sigma \cup \{\varepsilon\}) \times (\Delta \cup \{\varepsilon\}) \times Q \to \mathbb{S}$, overloading the function names, such that, for all $q, q' \in Q$, $a \in \Sigma$, $b \in \Delta$,

I missed sth: what is this ε? Intuitively clear but not defined?

$$
\begin{array}{rcll}
\mathsf{w}_{10}(q, a, \varepsilon, q') & = & \phi(a) & \text{where } \phi = \mathsf{w}_{10}(q, q') \in \Phi_\Sigma, \\
\mathsf{w}_{01}(q, \varepsilon, b, q') & = & \psi(b) & \text{where } \psi = \mathsf{w}_{01}(q, q') \in \Phi_\Delta, \\
\mathsf{w}_{11}(q, a, b, q') & = & \eta(a, b) & \text{where } \eta = \mathsf{w}_{11}(q, q') \in \Phi_{\Sigma,\Delta}.
\end{array}
$$

The swT T computes on pairs of words $\langle s, t\rangle \in \Sigma^* \times \Delta^*$, s and t, being respectively called *input* and *output* word. More precisely, T defines a mapping from $\Sigma^* \times \Delta^*$ into $\mathbb{S}$, based on an intermediate function weight_T defined recursively, for every states $q, q' \in Q$, and every pairs of strings $\langle s, t\rangle \in \Sigma^* \times \Delta^*$, where au, and bv, denote the concatenation of the symbol $a \in \Sigma$ (resp. $b \in \Delta$) with a word $u \in \Sigma^*$ (resp. $v \in \Delta^*$).

added u and v def

$$
\mathsf{weight}_T(q, \varepsilon, \varepsilon, q') = \mathbb{1} \quad \text{if } q = q' \text{ and } \mathbb{0} \text{ otherwise} \tag{1}
$$

$$
\begin{aligned}
\mathsf{weight}_T(q, s, t, q') = & \bigoplus_{\substack{q'' \in Q \\ s = au,\, a \in \Sigma}} \mathsf{w}_{10}(q, a, \varepsilon, q'') \otimes \mathsf{weight}_T(q'', u, t, q') \\
& \oplus \bigoplus_{\substack{q'' \in Q \\ t = bv,\, b \in \Delta}} \mathsf{w}_{01}(q, \varepsilon, b, q'') \otimes \mathsf{weight}_T(q'', s, v, q') \\
& \oplus \bigoplus_{\substack{q'' \in Q \\ s = au,\, t = bv}} \mathsf{w}_{11}(q, a, b, q'') \otimes \mathsf{weight}_T(q'', u, v, q')
\end{aligned}
$$

OK tout ça se lit bien :-)

We recall that, by convention (Section 2), an empty sum with $\bigoplus$ is equal to $\mathbb{0}$. Intuitively, using a transition $\mathsf{w}_{ij}(q, a, b, q')$ means for T: when reading respectively a and b at the current positions in the input and output words, increment the current position in the input word if and only if $i = 1$, and in the output word iff $j = 1$, and change state from q to q'. When $a = \varepsilon$ (resp. $b = \varepsilon$), the current symbol in the input (resp. output) is not read. Since $\mathbb{0}$

is absorbing for $\otimes$ in $\mathbb{S}$, one term $\mathsf{w}_{ij}(q, a, b, q'')$ equal to $\mathbb{0}$ in the above expression will be ignored in the sum, meaning that there is no possible transition from state q into state q' while reading a and b. This is analogous to the case of a transition's guard not satisfied by $\langle a, b\rangle$ for symbolic transducers.

The expression (1) can be seen as a stateful definition of an edit-distance between a word $s \in \Sigma^*$ and a word $t \in \Delta^*$, see also [22]. Intuitively, $\mathsf{w}_{10}(q, a, \varepsilon, r)$ is the cost of the deletion of the symbol $a \in \Sigma$ in s, $\mathsf{w}_{01}(q, \varepsilon, b, r)$ is the cost of the insertion of $b \in \Delta$ in t, and $\mathsf{w}_{11}(q, a, b, r)$ is the cost of the substitution of $a \in \Sigma$ by $b \in \Delta$. The cost of a sequence of such operations transforming s into t, is the product, with $\otimes$, of the individual costs of the operations involved; and the distance between s and t is the sum, with $\oplus$, of all possible products. Formally, the weight associated by T to $\langle s, t\rangle \in \Sigma^* \times \Delta^*$ is:

$$T(s,t) = \bigoplus_{q,q' \in Q} \mathsf{in}(q) \otimes \mathsf{weight}_T(q, s, t, q') \otimes \mathsf{out}(q') \tag{2}$$

Je crois qu'il faudrait numéroter les exemples indépendamment des définitions. Cet exemple est le premier qui donne des détails sur l'application visée. Il arrive peut-être un peu tard et est long. On pourrait introduire la motivation dans l'intro, et développer des petits exemples au fur et à mesure.

▶ **Example 7.** In Common Western Music Notation [14], several symbols may be used to represent one single sounding event. For instance, several notes can be combined with a tie, like in ♩‿♪, and one note can be augmented by half its duration with a dot like in ♩. These notations are perceived equivalent when played, as their duration is equal, yet the notation is different. We thus want to be able to compare a music score with music played by a performer. We propose a small weighted transducer model that calculates the distance bewteen an input sequence of sounding events (music "performance") to an output sequence of written events (music "score"). Let us consider the tropical (*min-plus*) semiring $\mathbb{S}$ of Figure 2 and let $\Sigma = \mathbb{R}_+$ be an input alphabet of event dates and $\Delta = \{\mathsf{e}, -\} \times \mathbb{R}_+$ be an output alphabet of symbols with timestamps. A symbol $\langle \mathsf{e}, d\rangle \in \Delta$ represents an event starting at date d, and $\langle -, d\rangle$ is a continuation of the previous event.

unique → similar

similar → single

modif.

changed end

We consider a swT with two states q_0 and q_1 whose purpose is to compare a recorded performance $s \in \Sigma^*$ with a notated music sheet $t \in \Delta^*$. One timestamp $d_i \in \Sigma$ may correspond to one notated event $\langle \mathsf{e}, d'_i\rangle \in \Delta$, in which case the weight value computed by the swT is the time distance between both (see transitions w_{11} below). If $\langle \mathsf{e}, d'_i\rangle$ is followed by continuations $\langle -, d'_{i+1}\rangle$..., they are just skipped with no cost (transitions w_{01} or weight $\mathbb{1}$).

$$\begin{array}{rclrcl} \mathsf{w}_{11}(q_0, d, \langle \mathsf{e}, d'\rangle, q_0) & = & |d' - d| & \mathsf{w}_{11}(q_1, d, \langle \mathsf{e}, d'\rangle, q_0) & = & |d' - d| \\ \mathsf{w}_{01}(q_0, \varepsilon, \langle -, d'\rangle, q_0) & = & \mathbb{1} & \mathsf{w}_{01}(q_1, \varepsilon, \langle -, d'\rangle, q_0) & = & \mathbb{1} \\ \mathsf{w}_{10}(q_0, d, \varepsilon, q_1) & = & \alpha & & & \end{array}$$

We also must be able to take performing errors into account, while still being able to compare with the score, since a performer could, for example, play an unwritten extra note. This is modelled by the transition w_{10} with an arbitrary weight value $\alpha \in \mathbb{S}$, switching from state q_0 (normal) to q_1 (error). The transitions in the second column below switch back to the normal state q_0. At last, we let q_0 be the only initial and final state, with $\mathsf{in}(q_0) = \mathsf{out}(q_0) = \mathbb{1}$, and $\mathsf{in}(q_1) = \mathsf{out}(q_1) = \mathbb{0}$.

reformulated this sentence

That way, an swT is capable of evaluating the differences between a score and a performance, all the while ensuring that performance errors are plausible.

ccl to the ex

◇

The *Symbolic Weighted Automata* are defined similarly as the transducers of Definition 6, by simply omitting the output symbols.

▶ **Definition 8.** *A* symbolic-weighted automaton *(swA) over Σ, $\mathbb{S}$ and $\bar{\Phi}$ is a tuple $A = \langle Q, \mathsf{in}, \mathsf{w}_1, \mathsf{out}\rangle$, where Q is a finite set of states, $\mathsf{in} : Q \to \mathbb{S}$ (respectively $\mathsf{out} : Q \to \mathbb{S}$) are functions defining the weight for entering (respectively leaving) computation in a state, and w_1 is a transition function from $Q \times Q$ into Φ_Σ.*

As above in the case of swT, when $\mathsf{w}_1(q, q') = \phi \in \Phi_\Sigma$, we may write $\mathsf{w}_1(q, a, q')$ for $\phi(a)$. The computation of A on words $s \in \Sigma^*$ is defined with an intermediate function weight_A, defined as follows for $q, q' \in Q$, $a \in \Sigma$, $u \in \Sigma^*$,

$$\begin{aligned}
\mathsf{weight}_A(q, \varepsilon, q) &= \mathbb{1} \\
\mathsf{weight}_A(q, \varepsilon, q') &= \mathbb{0} \quad \text{if } q \neq q' \\
\mathsf{weight}_A(q, au, q') &= \bigoplus_{q'' \in Q} \mathsf{w}_1(q, a, q'') \otimes \mathsf{weight}_A(q'', u, q')
\end{aligned} \tag{3}$$

and the weight value associated by A to $s \in \Sigma^*$ is defined as follows:

$$A(s) = \bigoplus_{q, q' \in Q} \mathsf{in}(q) \otimes \mathsf{weight}_A(q, s, q') \otimes \mathsf{out}(q') \tag{4}$$

The following property will be useful to the approach on symbolic weighted parsing presented in Section 5.

▶ **Proposition 9.** *Given a swT T over Σ, Δ, $\mathbb{S}$ commutative, bounded and complete, and $\bar{\Phi}$ effective, and a swA A over Σ, $\mathbb{S}$ and $\bar{\Phi}$, there exists an effectively constructible swA $B_{A,T}$ over Δ, $\mathbb{S}$ and $\bar{\Phi}$, such that for all $t \in \Delta^*$, $B_{A,T}(t) = \bigoplus_{s \in \Sigma^*} A(s) \otimes T(s, t)$.*

Proof. Let $T = \langle Q, \mathsf{in}_T, \bar{\mathsf{w}}, \mathsf{out}_T\rangle$, where $\bar{\mathsf{w}}$ contains w_{10}, w_{01}, and w_{11}, from $Q \times Q$ into respectively Φ_Σ, Φ_Δ, and $\Phi_{\Sigma,\Delta}$, and let $A = \langle P, \mathsf{in}_A, \mathsf{w}_1, \mathsf{out}_A\rangle$ with $\mathsf{w}_1 : Q \times Q \to \Phi_\Sigma$. The state set of $B_{A,T}$ will be $Q' = P \times Q$. The entering, leaving and transition functions of $B_{A,T}$ will simulate synchronized computations of A and T, while reading an output word of Δ^*. Its state entering functions is defined for all $p \in P$, $q \in Q$ by $\mathsf{in}'(p, q) = \mathsf{in}_A(p) \otimes \mathsf{in}_T(q)$. The transition function w'_1 will roughly perform a synchronized product of transitions defined by w_1, w_{01} (T reading in output word and not an input word) and w_{11} (T reading both an input word and an output word). Moreover, w'_1 also needs to simulate transitions defined by w_{10}: T reading in input word and not an output word. Since $B_{A,T}$ will read only in the output word, such a transition corresponds to an ε-transition of swA, but swA have been defined without ε-transitions. Therefore, in order to take care of this case, we perform an on-the-fly suppression of ε-transition in the swA in construction, following the algorithm of [19]. Initially, for all $p_1, p_2 \in P$, and $q_1, q_2 \in Q$, let

$$\mathsf{w}'_1\big(\langle p_1, q_1\rangle, \langle p_2, q_2\rangle\big) = \mathsf{w}_1(p_1, p_2) \otimes \big[\mathsf{w}_{01}(q_1, q_2) \oplus \bigoplus_{\Sigma} \mathsf{w}_{11}(q_1, q_2)\big].$$

Iterate the following for all $p_1 \in P$ and $q_1, q_2 \in Q$: for all $p_2 \in P$ and $q_3 \in Q$,

$$\mathsf{w}'_1\big(\langle p_1, q_1\rangle, \langle p_2, q_3\rangle\big) \mathrel{\oplus}= \bigoplus_{\Sigma} \mathsf{w}_{10}(q_1, q_2) \otimes \mathsf{w}'_1\big(\langle p_1, q_2\rangle, \langle p_2, q_3\rangle\big)$$

proof correctness

and $\mathsf{out}'(p_1, q_1) \mathrel{\oplus}= \bigoplus_{\Sigma} \mathsf{w}_{10}(q_1, q_2) \otimes \mathsf{out}'(p_1, q_2)$ ◀

The construction time and size for $B_{A,T}$ are $O(\|T\|^3.\|A\|^2)$, where the sizes $\|T\|$ and $\|A\|$ are their number of states.

revise with nb of tr. and states

▶ **Corollary 10.** *Given a swT T over Σ, Δ, $\mathbb{S}$ commutative, bounded and complete, and $\bar{\Phi}$ effective, and $s \in \Sigma^+$, there exists an effectively constructible swA $B_{s,T}$ over Δ, $\mathbb{S}$ and $\bar{\Phi}$, such that for all $t \in \Delta^*$, $B_{s,T}(t) = T(s,t)$.*

4 SW Visibly Pushdown Automata

The model presented in this section generalizes Symbolic VPA [6] from Boolean semirings to arbitrary semiring weight domains. It will compute on nested words over infinite alphabets, associating to every such word a weight value. Nested words are able to describe structures of labeled trees, and in the context of parsing, they will be useful to represent AST.

Let Ω be a countable alphabet that we assume partitioned into three subsets Ω_i, Ω_c, Ω_r, whose elements are respectively called *internal*, *call* and *return* symbols. Let $\langle \mathbb{S}, \oplus, \mathbb{0}, \otimes, \mathbb{1} \rangle$ be a commutative and complete semiring and let $\bar{\Phi} = \langle \Phi_\mathsf{i}, \Phi_\mathsf{c}, \Phi_\mathsf{r}, \Phi_\mathsf{ci}, \Phi_\mathsf{cc}, \Phi_\mathsf{cr} \rangle$ be a label theory over $\mathbb{S}$ where Φ_i, Φ_c, Φ_r and Φ_cx (with $\mathsf{x} \in \{\mathsf{i}, \mathsf{c}, \mathsf{r}\}$) stand respectively for Φ_{Ω_i}, Φ_{Ω_c}, Φ_{Ω_r} and $\Phi_{\Omega_\mathsf{c},\Omega_\mathsf{x}}$.

Là je crois qu'il faudrait expliquer ces Omega, je commence à fatiguer et je suis un peu largué par toutes ces définitions. J'intuite qu'il s'agit des symboles, parenthèses ouvrantes et fermantes? Pourquoi il faut un alphabet pour les parenthèses?

▶ **Definition 11.** *A* Symbolic Weighted Visibly Pushdown Automata *(sw-VPA) over $\Omega = \Omega_\mathsf{i} \uplus \Omega_\mathsf{c} \uplus \Omega_\mathsf{r}$, $\mathbb{S}$ and $\bar{\Phi}$ is a tuple $A = \langle Q, P, \mathsf{in}, \bar{\mathsf{w}}, \mathsf{out} \rangle$, where Q is a finite set of states, P is a finite set of stack symbols, $\mathsf{in} : Q \to \mathbb{S}$ (respectively $\mathsf{out} : Q \to \mathbb{S}$) are functions defining the weight for entering (respectively leaving) a state, and $\bar{\mathsf{w}}$ is a sextuplet composed of the transition functions : $\mathsf{w_i} : Q \times P \times Q \to \Phi_\mathsf{ci}$, $\mathsf{w_i^e} : Q \times Q \to \Phi_\mathsf{i}$, $\mathsf{w_c} : Q \times P \times Q \times P \to \Phi_\mathsf{cc}$, $\mathsf{w_c^e} : Q \times P \times Q \to \Phi_\mathsf{c}$, $\mathsf{w_r} : Q \times P \times Q \to \Phi_\mathsf{cr}$, $\mathsf{w_r^e} : Q \times Q \to \Phi_\mathsf{r}$.*

Est-ce que tout le monde sait ce qu'est un pushdown automata? Je suppose que c'est lié à la pile.

Similarly as in Section 3, we extend the above transition functions as follows for all $q, q' \in Q$, $p \in P$, $a \in \Omega_\mathsf{i}$, $c \in \Omega_\mathsf{c}$, $r \in \Omega_\mathsf{r}$, overloading their names:

$$
\begin{array}{lll}
\mathsf{w_i} : Q \times \Omega_\mathsf{c} \times P \times \Omega_\mathsf{i} \times Q \to \mathbb{S} & \mathsf{w_i}(q, c, p, a, q') = \eta_\mathsf{ci}(c, a) & \text{where } \eta_\mathsf{ci} = \mathsf{w_i}(q, p, q'), \\
\mathsf{w_i^e} : Q \times \Omega_\mathsf{i} \times Q \to \mathbb{S} & \mathsf{w_i^e}(q, a, q') = \phi_\mathsf{i}(a) & \text{where } \phi_\mathsf{i} = \mathsf{w_i^e}(q, q'). \\
\mathsf{w_c} : Q \times \Omega_\mathsf{c} \times P \times \Omega_\mathsf{c} \times P \times Q \to \mathbb{S} & \mathsf{w_c}(q, c, p, c', p', q') = \eta_\mathsf{cc}(c, c') & \text{where } \eta_\mathsf{cc} = \mathsf{w_c}(q, p, p', q'), \\
\mathsf{w_c^e} : Q \times \Omega_\mathsf{c} \times P \times Q \to \mathbb{S} & \mathsf{w_c^e}(q, c, p, q') = \phi_\mathsf{c}(c) & \text{where } \phi_\mathsf{c} = \mathsf{w_c^e}(q, p, q'). \\
\mathsf{w_r} : Q \times \Omega_\mathsf{c} \times P \times \Omega_\mathsf{r} \times Q \to \mathbb{S} & \mathsf{w_r}(q, c, p, r, q') = \eta_\mathsf{cr}(c, r) & \text{where } \eta_\mathsf{cr} = \mathsf{w_r}(q, p, q'), \\
\mathsf{w_r^e} : Q \times \Omega_\mathsf{r} \times Q \to \mathbb{S} & \mathsf{w_r^e}(q, r, q') = \phi_\mathsf{r}(r) & \text{where } \phi_\mathsf{r} = \mathsf{w_r^e}(q, q').
\end{array}
$$

The intuition is the following for the above transitions. $\mathsf{w_i^e}$, $\mathsf{w_c^e}$, and $\mathsf{w_r^e}$ describe the cases where the stack is empty. $\mathsf{w_i}$ and $\mathsf{w_i^e}$ both read an input internal symbol a and change state from q to q', without changing the stack. Moreover, $\mathsf{w_i}$ reads a pair made of $c \in \Omega_\mathsf{c}$ and $p \in P$ on the top of the stack (c is compared to a by the weight function $\eta_\mathsf{ci} \in \Phi_\mathsf{ci}$). $\mathsf{w_c}$ and $\mathsf{w_c^e}$ read the input call symbol c', push it to the stack along with p', and change state from q to to q'. Moreover, $\mathsf{w_c}$ reads c and p at the top of the stack (c is compared to c'). $\mathsf{w_r}$ and $\mathsf{w_r^e}$ read the input return symbol r, and change state from q to to q'. Moreover, $\mathsf{w_r}$ reads and pop from stack a pair made of c and p, (c is compared to r).

moved this to the beginning

Formally, the transitions of the automaton A are defined in term of an intermediate function weight_A, like in Section 3. A configuration, denoted by $q[\gamma]$, is here composed of a state $q \in Q$ and a stack content $\gamma \in \Gamma^*$, where $\Gamma = \Omega_\mathsf{c} \times P$. Hence, weight_A is a function from $[Q \times \Gamma^*] \times \Omega^* \times [Q \times \Gamma^*]$ into $\mathbb{S}$. The empty stack is denoted by $\perp$, and the upmost symbol is the last pushed content. The following functions illustrate each of the possible cases, being : reading $a \in \Omega_\mathsf{i}$, or $c \in \Omega_\mathsf{c}$, or $r \in \Omega_\mathsf{r}$ for each possible state of the stack (empty or not), to add to $u \in \Omega^*$.

intro to func

introduced the 6 cases

notation cp for $\langle c, p \rangle$?

$$\mathsf{weight}_A\big(q[\perp], \varepsilon, q'[\perp]\big) = \mathbb{1} \text{ if } q = q' \text{ and } \mathbb{0} \text{ otherwise} \tag{5}$$

$$\mathsf{weight}_A\big(q\begin{bmatrix}\langle c,p\rangle\\ \gamma\end{bmatrix}, a\,u, q'[\gamma']\big) = \bigoplus_{q''\in Q} \mathsf{w_i}(q,c,p,a,q'') \otimes \mathsf{weight}_A\big(q''\begin{bmatrix}\langle c,p\rangle\\ \gamma\end{bmatrix}, u, q'[\gamma']\big)$$

$$\mathsf{weight}_A\big(q[\bot], a\,u, q'[\gamma']\big) = \bigoplus_{q''\in Q} \mathsf{w_i^e}(q,a,q'') \otimes \mathsf{weight}_A\big(q''[\bot], u, q'[\gamma']\big)$$

$$\mathsf{weight}_A\big(q\begin{bmatrix}\langle c,p\rangle\\ \gamma\end{bmatrix}, c'\,u, q'[\gamma']\big) = \bigoplus_{\substack{q''\in Q\\ p'\in P}} \mathsf{w_c}\big(q,c,p,c',p',q''\big) \otimes \mathsf{weight}_A\big(q''\begin{bmatrix}\langle c',p'\rangle\\ \langle c,p\rangle\\ \gamma\end{bmatrix}, u, q'[\gamma']\big)$$

$$\mathsf{weight}_A\big(q[\bot], c\,u, q'[\gamma']\big) = \bigoplus_{\substack{q''\in Q\\ p\in P}} \mathsf{w_c^e}(q,c,p,q'') \otimes \mathsf{weight}_A\big(q''[\langle c,p\rangle], u, q'[\gamma']\big)$$

$$\mathsf{weight}_A\big(q\begin{bmatrix}\langle c,p\rangle\\ \gamma\end{bmatrix}, r\,u, q'[\gamma']\big) = \bigoplus_{q''\in Q} \mathsf{w_r}\big(q,c,p,r,q''\big) \otimes \mathsf{weight}_A\big(q''[\gamma], u, q'[\gamma']\big)$$

$$\mathsf{weight}_A\big(q[\bot], r\,u, q'[\gamma']\big) = \bigoplus_{q''\in Q} \mathsf{w_r^e}(q,r,q'') \otimes \mathsf{weight}_A\big(q''[\bot], u, q'[\gamma']\big)$$

c p to <c, p>

The weight associated by A to $s \in \Omega^*$ is defined according to empty stack semantics:

$$A(s) = \bigoplus_{q,q'\in Q} \mathsf{in}(q) \otimes \mathsf{weight}_A\big(q[\bot], s, q'[\bot]\big) \otimes \mathsf{out}(q'). \tag{6}$$

todo example VPA

▶ **Example 12.** structured words with timed symbols... intro language of music notation? (markup = time division, leaves = events etc)

Every swA $A = \langle Q, \mathsf{in}, \mathsf{w}_1, \mathsf{out}\rangle$, over Σ, $\mathbb{S}$ and $\bar{\Phi}$ is a particular case of sw-VPA $\langle Q, \emptyset, \mathsf{in}, \bar{\mathsf{w}}, \mathsf{out}\rangle$ over Ω, $\mathbb{S}$ and $\bar{\Phi}$ with $\Omega_\mathsf{i} = \Sigma$ and $\Omega_\mathsf{c} = \Omega_\mathsf{r} = \emptyset$, and computing with an always empty stack: $\mathsf{w_i^e} = \mathsf{w}_1$ and all the other functions of $\bar{\mathsf{w}}$ are the constant $\mathbb{0}$.

Like VPA and symbolic VPA, the class of sw-VPA is closed under the binary operators of the underlying semiring.

▶ **Proposition 13.** *Let A_1 and A_2 be two* sw-VPA *over the same Ω, $\mathbb{S}$ and $\bar{\Phi}$. There exists two effectively constructible* sw-VPA *$A_1 \oplus A_2$ and $A_1 \otimes A_2$, such that for all $s \in \Omega^*$, $(A_1 \oplus A_2)(s) = A_1(s) \oplus A_2(s)$ and $(A_1 \otimes A_2)(s) = A_1(s) \otimes A_2(s)$.*

Proof. The construction is essentially the same as in the case of the Boolean semiring [6]. ◀

total?

Let us assume that the semiring $\mathbb{S}$ is commutative, bounded, and complete, and that $\bar{\Phi}$ is an effective label theory. We propose a Dijkstra algorithm computing, for a sw-VPA A over Ω, $\mathbb{S}$ and $\bar{\Phi}$, the minimal weight for a word in Ω^*. We distinguish two cases : when the stack is empty, and when it is not. In the case of an empty stack, let $b_\bot : Q \times Q \to \mathbb{S}$ be such that :

introduced 2 cases for b

$$b_\bot(q, q') = \bigoplus_{s\in\Omega^*} \mathsf{weight}_A\big(q[\bot], s, q'[\bot]\big). \tag{7}$$

Since $\mathbb{S}$ is complete, the infinite sum in (7) is well defined, and, providing that $\mathbb{S}$ is total, it is the minimum in Ω^*, *wrt* $\leq_\oplus$, of the fonction $s \mapsto \mathsf{weight}_A(q[\sigma], s, q'[\sigma])$. The term $q[\bot], s, q'[\bot]$ of this sum is the central expression in the definition (6) of $A(s_0)$, for the minimum s_0 of the function weight_A.

so ?

For all $q_0, q_3 \in Q$,

$$\begin{aligned}
d_\top(q_1,p,q_3) &\oplus= d_\top(q_1,p,q_2) \otimes \bigoplus_{\Omega_\mathsf{i}} \mathsf{w_i}(q_2,p,q_3)\\
d_\bot(q_1,p,q_3) &\oplus= d_\bot(q_1,q_2) \otimes \bigoplus_{\Omega_\mathsf{i}} \mathsf{w_i^e}(q_2,q_3)\\
d_\top(q_0,p,q_3) &\oplus= \bigoplus_{\Omega_\mathsf{c}}^{2} \Big[\big(\mathsf{w_c}(q_0,p,p',q_1) \otimes_2 d_\top(q_1,p',q_2)\big) \otimes_2 \bigoplus_{\Omega_\mathsf{r}} \mathsf{w_r}(q_2,p',q_3)\Big]\\
d_\bot(q_0,q_3) &\oplus= \bigoplus_{\Omega_\mathsf{c}} \big(\mathsf{w_c^e}(q_0,p,q_1) \otimes d_\top(q_1,p,q_2) \otimes \bigoplus_{\Omega_\mathsf{r}} \mathsf{w_r}(q_2,p,q_3)\big)\\
d_\bot(q_1,q_3) &\oplus= d_\bot(q_1,q_2) \otimes \bigoplus_{\Omega_\mathsf{r}} \mathsf{w_r^e}(q_2,q_3)\\
d_\top(q_1,p,q_3) &\oplus= d_\top(q_1,p,q_2) \otimes d_\top(q_2,p,q_3), \text{if } \langle q_2,\top,q_3\rangle \notin P\\
d_\bot(q_1,q_3) &\oplus= d_\bot(q_1,q_2) \otimes d_\bot(q_2,q_3), \text{if } \langle q_2,\bot,q_3\rangle \notin P
\end{aligned}$$

Figure 3 Update $d_\bot$ with $\langle q_1,q_2\rangle$ or $d_\top$ with $\langle q_1,p,q_2\rangle$.

$b_\top$: mot bien parenthèsé c/r

If the stack is not empty, let $\top$ be a fresh stack symbol which does not belong to Γ, and let $b_\top : Q \times P \times Q \to \Phi_\mathsf{c}$ be such that, for every two states $q, q' \in Q$ and stack symbol $p \in P$:

$$b_\top(q,p,q') : c \mapsto \bigoplus_{s\in\Omega^*} \mathsf{weight}_A\Big(q \begin{bmatrix} \langle c,p\rangle \\ \top \end{bmatrix}, s, q' \begin{bmatrix} \langle c,p\rangle \\ \top \end{bmatrix}\Big) \tag{8}$$

Intuitively, the function defined in (8) associates to $c \in \Omega_\mathsf{c}$ the minimum weight of a computation of A starting in state q with a stack $\langle c,p\rangle \cdot \gamma \in \Gamma^+$ and ending in state q' with the same stack, such that the computation can not pop the pair made of c and p at the top of this stack, but may only read these symbols. Moreover, A may push another pair $\langle c',p'\rangle$ on the top of $\langle c,p\rangle \cdot \gamma$, following the third case of in the definition (5) of weight_A, and may pop $\langle c',p'\rangle$ later, following the fifth case of (5) (return symbol).

Algorithm 1 Best search for sw-VPA

initially let $\mathcal{Q} = (Q \times Q) \cup (Q \times P \times Q)$, and let $d_\bot(q_1,q_2) = d_\top(q_1,p,q_2) = \mathbb{1}$ if $q_1 = q_2$ and $d_\bot(q_1,q_2) = d_\top(q_1,p,q_2) = \mathbb{0}$ otherwise

while $\mathcal{Q} \neq \emptyset$ **do**

  extract $\langle q_1,q_2\rangle$ or $\langle q_1,p,q_2\rangle$ from $\mathcal{Q}$ such that $d_\bot(q_1,q_2)$, resp. $\bigoplus_{c\in\Omega_\mathsf{c}} d_\top(q_1,p,q_2)(c)$, is minimal in $\mathbb{S}$ *wrt* $\leq_\oplus$

  update $d_\bot$ with $\langle q_1,q_2\rangle$ or $d_\top$ with $\langle q_1,p,q_2\rangle$ (Figure 3).

Algorithm 1 constructs iteratively markings $d_\bot : Q \times Q \to \mathbb{S}$ and $d_\top : Q \times P \times Q \to \Phi_\mathsf{c}$ that converges eventually to $b_\top$ and $b_\bot$.
The infinite sums in the updates of d in Algorithm 1, Figure 3 are well defined since $\mathbb{S}$ is complete. ** effectively computable by hypothesis that the label theory is effective**
The algorithm performs $2.|Q|^2$ iterations until P is empty, and each iteration has a time complexity $O(|Q|^2.|P|)$. That gives a time complexity $O(|Q|^4.|P|)$. It can be reduced by implementing P as a priority queue, prioritized by the value returned by d.

explication Fig. 3 suivant cas de (5)

complete **

detail with nb tr. and states

The correctness of Algorithm 1 is ensured by the invariant expressed in the following lemma.

▶ **Lemma 14.** *For all $\langle q_1,q_2\rangle \notin \mathcal{Q}$, $d_\bot(q_1,q_2) = b_\bot(q_1,q_2)$/*

The proof is by contradiction, assuming a counter-example minimal in the length of the witness word.

▶ **Lemma 15.** *For all* $\langle q_1, p, q_2\rangle \notin \mathcal{Q}$, $d_\top(q_1, p, q_2) = b_\top(q_1, p, q_2)$,

For computing the minimal weight of a computation of A, we use the fact that, at the termination of Algorithm 1, $\bigoplus_{s\in\Omega^*} A(s) = \bigoplus_{q,q'\in Q} \mathsf{in}(q) \otimes d_\perp(q, q') \otimes \mathsf{out}(q')$.
In order to obtain effectively a witness (word of Ω^* with a computation of A of minimal weight), we require the additional property of convexity of weight functions.

▶ **Proposition 16.** *For a* sw-VPA A *over* Ω, $\mathbb{S}$ *commutative, bounded, total and complete, and* $\bar{\Phi}$ *effective, one can construct in PTIME a word* $t \in \Omega^*$ *such that* $A(t)$ *is minimal wrt the natural ordering for* $\mathbb{S}$.

5 Symbolic Weighted Parsing

Let us now apply the models and results of the previous sections to the problem of parsing over an infinite alphabet. Let Σ and $\Omega = \Omega_\mathsf{i} \uplus \Omega_\mathsf{c} \uplus \Omega_\mathsf{r}$ be countable input and output alphabets, let $\langle \mathbb{S}, \oplus, \mathbb{0}, \otimes, \mathbb{1}\rangle$ be a commutative, bounded, and complete semiring and let $\bar{\Phi}$ be an effective label theory over $\mathbb{S}$, containing Φ_Σ, $\Phi_{\Sigma,\Omega_\mathsf{i}}$, as well as Φ_i, Φ_c, Φ_r, Φ_cr (following the notations of Section 4). We assume given the following input:

total?

- a swT T over Σ, Ω_i, $\mathbb{S}$, and $\bar{\Phi}$, defining a measure $T : \Sigma^* \times {\Omega_\mathsf{i}}^* \to \mathbb{S}$,
- a sw-VPA A over Ω, $\mathbb{S}$, and $\bar{\Phi}$, defining a measure $A : \Omega^* \to \mathbb{S}$,
- an input word $s \in \Sigma^*$.

For all $u \in \Sigma^*$ and $t \in \Omega^*$, let $d(u, t) = T\big(u, t|_{\Omega_\mathsf{i}}\big)$, where $t|_{\Omega_\mathsf{i}} \in {\Omega_\mathsf{i}}^*$ is the projection of t onto Ω_i, obtained from t by removing all symbols in $\Omega \setminus \Omega_\mathsf{i}$. *Symbolic weighted parsing* is the problem, given the above input, to find $t \in \Omega^*$ minimizing $d(s, t) \otimes A(t)$ *wrt* $\leq_\oplus$, *i.e.* s.t.

$$d(s,t) \otimes A(t) = \bigoplus_{t'\in\mathcal{T}(\Omega)} d(s,t') \otimes A(t') \tag{9}$$

Following the terminology of [21], sw-parsing is the problem of computing the distance (9) between the input s and the output weighted language of A, and returning a witness t.

▶ **Proposition 17.** *The problem of Symbolic Weighted parsing can be solved in PTIME in the size of the input* swT T, sw-VPA A *and input word* s, *and the computation time of the functions and operators of the label theory.*

Proof. (sketch) We follow a *Bar-Hillel* construction, for parsing by intersection. Let us first extend the swT T over Σ, Ω_i into a swT T' over Σ and Ω (and the same semiring and label theory $\mathbb{S}$ and $\bar{\Phi}$), such that for all $u \in \Sigma^*$, and $t \in \Omega^*$, $T'(u, u) = T(u, t|_{\Omega_\mathsf{i}})$. The transducer T' simply skips every symbol $b \in \Omega \setminus \Omega_\mathsf{i}$, by the addition to T, of new transitions of the form $\mathsf{w}_{01}(q, \varepsilon, b, q')$. Then, using Corolary 10, we construct from the input word $s \in \Sigma^*$ and T' a swA $B_{s,T'}$, such that for all $t \in \Omega^*$, $B_{s,T'}(t) = d(s, t)$. Next, we compute the sw-VPA $B_{s,T'} \otimes A$, using Proposition 13. It remains to compute a best nested-word $t \in \Omega^*$ using the best-search procedure of Proposition 16. ◀

The sw-parsing generalizes the problem of searching the best derivation (AST) of a weighted CF-grammar that yields a given input word. The latter problem, sometimes called *weighted parsing*, (see *e.g.* [13] and [23] for general weighted parsing frameworks) corresponds to sw-parsing in the case of finite alphabets, a transducer T computing the identity and some sw-VPA A obtained from the weighted CF grammar. Indeed, the *depth-first* traversal of an AST τ yields a well-parenthesised word $\mathsf{lin}(\tau)$ over an alphabet $\Omega = \Omega_\mathsf{i} \uplus \Omega_\mathsf{c} \uplus \Omega_\mathsf{r}$, assuming *e.g.* that Ω_i contain the symbols labelling the leaves of τ (symbols of rank 0) and Ω_c and Ω_r contain respectively one left and right parenthesis $\langle_b$ and $_b\rangle$ for each symbol b labelling inner nodes of τ (symbols of rank > 0). With this representation, the projection $\mathsf{lin}(t)|_{\Omega_\mathsf{i}}$ is then the sequence of leaves of τ. We show in Appendix A how to convert a (sw) tree automaton A into a sw-VPA computing $A(\mathsf{lin}(\tau))$ for every tree τ. That also holds for the set of ASTs of a weighted CF-grammar.

Ah oui, ça aurait pu être dit avant.

2 lines Application to Automated Music Transcription: implementation $\neq$ but same principle, on-the-fly automata construction during best search, for efficiency.

Conclusion

We have introduced weighted language models (SW transducers and visibly pushdown automata) computing over infinite alphabets, and applied them to the problem of parsing with infinitely many possible input symbols (typically timed events). This approach extends conventional parsing and weighted parsing by computing a derivation tree modulo a generic distance between words, defined by a SW transducer given in input. This enables to consider finer word relationships than strict equality, opening possibilities of quantitative analysis via this method.

Ongoing and future work include

TODO future work

– The study of other theoretical properties of SW models, such as the extension of the best search algorithm from 1-best to n-best [17], and to k-*closed* semirings [20] (instead of *bounded*, which corresponds to 0-*closed*).

– ...there is room to improve the complexity bounds for the algorithms ... modular approach with oracles ...

– present here an offline algorithm for best search, semi-online implementation for AMT (bar-by-bar approach) with an on-the-fly automata construction.

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A Nested-Words and Parse-Trees

The hierarchical structure of nested-words, defined with the *call* and *return* markup symbols suggest a correspondence with trees. The lifting of this correspondence to languages, of tree automata and VPA, has been discussed in [1], and [4] for the weighted case. In this section, we describe a correspondence between the symbolic-weighted extensions of tree automata and VPA.

Let Ω be a countable ranked alphabet, such that every symbol $a \in \Omega$ has a rank $\mathsf{rk}(a) \in [0..M]$ where M is a fixed natural number. We denote by Ω_k the subset of all symbols a of Ω with $\mathsf{rk}(a) = k$, where $0 \leq k \leq M$, and $\Omega_{>0} = \Omega \setminus \Omega_0$. The free Ω-algebra of finite, ordered, Ω-labeled trees is denoted by $\mathcal{T}(\Omega)$. It is the smallest set such that $\Omega_0 \subset \mathcal{T}(\Omega)$ and for all $1 \leq k \leq M$, all $a \in \Omega_k$, and all $t_1, \ldots, t_k \in \mathcal{T}(\Omega)$, $a(t_1, \ldots, t_k) \in \mathcal{T}(\Omega)$. Let us assume a commutative semiring $\mathbb{S}$ and a label theory $\bar{\Phi}$ over $\mathbb{S}$ containing one set Φ_{Ω_k} for each $k \in [0..M]$.

▶ **Definition 18.** *A* symbolic-weighted tree automaton *(swTA) over Ω, $\mathbb{S}$, and $\bar{\Phi}$ is a triplet $A = \langle Q, \mathsf{in}, \bar{\mathsf{w}} \rangle$ where Q is a finite set of states, $\mathsf{in} : Q \to \Phi_\Omega$ is the starting weight function, and $\bar{\mathsf{w}}$ is a tuplet of transition functions containing, for each $k \in [0..M]$, the functions $\mathsf{w}_k : Q \times Q^k \to \Phi_{\Omega_{>0}, \Omega_k}$ and $\mathsf{w}^{\mathsf{e}}_k : Q \times Q^k \to \Phi_{\Omega_k}$.*

We define a transition function $\mathsf{w} : Q \times (\Omega_{>0} \cup \{\varepsilon\}) \times \Omega \times \bigcup_{k=0}^{M} Q^k \to \mathbb{S}$ by:

$$\begin{array}{rcll} \mathsf{w}(q_0, a, b, q_1 \ldots q_k) & = & \eta(a, b) & \text{where } \eta = \mathsf{w}_k(q_0, q_1 \ldots q_k) \\ \mathsf{w}(q_0, \varepsilon, b, q_1 \ldots q_k) & = & \phi(b) & \text{where } \phi = \mathsf{w}^{\mathsf{e}}_k(q_0, q_1 \ldots q_k). \end{array}$$

where $q_1 \ldots q_k$ is ε if $k = 0$. The first case deals with a strict subtree, with a parent node labeled by a, and the second case is for a root tree.

Every swTA defines a mapping from trees of $\mathcal{T}(\Omega)$ into $\mathbb{S}$, based on the following intermediate function $\mathsf{weight}_A : Q \times (\Omega \cup \{\varepsilon\}) \times \mathcal{T}(\Omega) \to \mathbb{S}$

$$\mathsf{weight}_A(q_0, a, t) = \bigoplus_{q_1 \ldots q_k \in Q^k} \mathsf{w}(q_0, a, b, q_1 \ldots q_k) \otimes \bigotimes_{i=1}^{k} \mathsf{weight}_A(q_i, b, t_i) \tag{10}$$

where $q_0 \in Q$, $a \in \Omega_{>0} \cup \{\varepsilon\}$ and $t = b(t_1, \ldots, t_k) \in \mathcal{T}(\Omega)$, $0 \leq k \leq M$.

Finally, the weight associated by A to $t \in \mathcal{T}(\Omega)$ is

$$A(t) = \bigoplus_{q \in Q} \mathsf{in}(q) \otimes \mathsf{weight}_A(q, \varepsilon, t) \tag{11}$$

Intuitively, $\mathsf{w}(q_0, a, b, q_1 \ldots q_k)$ can be seen as the weight of a production rule $q_0 \to b(q_1, \ldots, q_k)$ of a regular tree grammar [5], that replaces the non-terminal symbol q_0 by $b(q_1, \ldots, q_k)$, provided that the parent of q_0 is labeled by a (or q_0 is the root node if $a = \varepsilon$). The above production rule can also be seen as a rule of a weighted CF grammar, of the form $[a, b]\, q_0 := q_1 \ldots q_k$ if $k > 0$, and $[a]\, q_0 := b$ if $k = 0$. In the first case, b is a label of the rule, and in the second case, it is a terminal symbol. And in both cases, a is a constraint on the label of rule applied on the parent node in the derivation tree. This features of observing the parent's label are useful in the case of infinite alphabet, where it is not possible to memorize a label with the states. The weight of a labeled derivation tree t of the weighted CF grammar associated to A as above, is $\mathsf{weight}_A(q, t)$, when q is the start non-terminal. We shall now establish a correspondence between such derivation tree t and some word describing a linearization of t, in a way that $\mathsf{weight}_A(q, t)$ can be computed by a $\mathsf{sw\text{-}VPA}$.

Let $\hat{\Omega}$ be the countable (unranked) alphabet obtained from Ω by: $\hat{\Omega} = \Omega_\mathsf{i} \uplus \Omega_\mathsf{c} \uplus \Omega_\mathsf{r}$, with $\Omega_\mathsf{i} = \Omega_0$, $\Omega_\mathsf{c} = \{ \langle_a |\ a \in \Omega_{>0}\}$, $\Omega_\mathsf{r} = \{ \ _a\rangle \mid a \in \Omega_{>0}\}$.
We associate to $\hat{\Omega}$ a label theory $\hat{\Phi}$ like in Section 4, and we define a linearization of trees of $\mathcal{T}(\Omega)$ into words of $\hat{\Omega}^*$ as follows:

$\mathsf{lin}(a) = a$ for all $a \in \Omega_0$,

$\mathsf{lin}\big(b(t_1, \ldots, t_k)\big) = \langle_b\ \mathsf{lin}(t_1) \ldots \mathsf{lin}(t_k)\ _b\rangle$ when $b \in \Omega_k$ for $1 \leq k \leq M$.

▶ **Proposition 19.** *For all swTA A over Ω, $\mathbb{S}$ commutative, and $\bar{\Phi}$, there exists an effectively constructible sw-VPA A' over $\hat{\Omega}$, $\mathbb{S}$ and $\hat{\Phi}$ such that for all $t \in \mathcal{T}(\Omega)$, $A'\big(\mathsf{lin}(t)\big) = A(t)$.*

Proof. Let $A = \langle Q, \mathsf{in}, \bar{\mathsf{w}}\rangle$ where $\bar{\mathsf{w}}$ is presented as above by a function We build $A' = \langle Q', P', \mathsf{in}', \bar{\mathsf{w}}', \mathsf{out}'\rangle$, where $Q' = \bigcup_{k=0}^{M} Q^k$ is the set of sequences of state symbols of A, of length at most M, including the empty sequence denoted by ε, and where $P' = Q'$ and $\bar{\mathsf{w}}$ is defined by:

$$
\begin{array}{lcll}
\mathsf{w}_\mathsf{i}(q_0\,\bar{u}, \langle_c, \bar{p}, a, \bar{u}) & = & \mathsf{w}(q_0, c, a, \varepsilon) & \text{for all } c \in \Omega_{>0}, a \in \Omega_0 \\
\mathsf{w}^\mathsf{e}_\mathsf{i}(q_0\,\bar{u}, a, \bar{u}) & = & \mathsf{w}(q_0, \varepsilon, a, \varepsilon) & \text{for all } a \in \Omega_0 \\
\mathsf{w}_\mathsf{c}(q_0\,\bar{u}, \langle_c, \bar{p}, \langle_d, \bar{u}, \bar{q}) & = & \mathsf{w}(q_0, c, d, \bar{q}) & \text{for all } c, d \in \Omega_{>0} \\
\mathsf{w}^\mathsf{e}_\mathsf{c}(q_0\,\bar{u}, \langle_c, \bar{u}, \bar{q}) & = & \mathsf{w}(q_0, \varepsilon, c, \bar{q}) & \text{for all } c \in \Omega_{>0} \\
\mathsf{w}_\mathsf{r}(\varepsilon, \langle_c, \bar{p}, {}_c\rangle, \bar{p}) & = & \mathbb{1} & \text{for all } c \in \Omega_{>0} \\
\mathsf{w}^\mathsf{e}_\mathsf{r}(\bar{u}, {}_c\rangle, \bar{q}) & = & \mathbb{0} & \text{for all } c \in \Omega_{>0}
\end{array}
$$

All cases not matched by one of the above equations have a weight $\mathbb{0}$, for instance $\mathsf{w}_\mathsf{r}(\bar{u}, \langle_c, \bar{p}, {}_d\rangle, \bar{q}) = \mathbb{0}$ if $c \neq d$ or $\bar{u} \neq \varepsilon$ or $\bar{q} \neq \bar{p}$. ◀

Todo list