

Weighted Visibly Pushdown Automata and Automated Music Transcription

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Abstract

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Symbolic Weighted (SW) extension of symbolic automata where...

Semirings. We shall consider semiring domains for weight values. A *semiring* $\langle \mathbb{S}, \oplus, \otimes, \mathbb{0}, \mathbb{1} \rangle$ is a structure with a domain \mathbb{S} , equipped with two associative binary operators \oplus and \otimes with respective neutral elements $\mathbb{0}$ and $\mathbb{1}$ and such that: \oplus is commutative, \otimes distributes over \oplus : $\forall x, y, z \in \mathbb{S}, x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$, and $\mathbb{0}$ is absorbing for \otimes : $\forall x \in \mathbb{S}, \mathbb{0} \otimes x = x \otimes \mathbb{0} = \mathbb{0}$. In the application presented in this paper, intuitively, \oplus selects an optimal value amongst two values and \otimes combines two values into a single value.

A semiring \mathbb{S} is *commutative* if \otimes is commutative. It is *bounded* [4] if $\forall x \in \text{dom}(\mathbb{S}), \mathbb{1} \oplus x = x$, and *idempotent* if for all $x \in \mathbb{S}, x \oplus x = x$. Note that every bounded semiring is idempotent: by boundedness, $\mathbb{1} \oplus \mathbb{1} = \mathbb{1}$, and idempotency follows by multiplying both sides by x and distributing.

A semiring \mathbb{S} is *monotonic wrt* a partial ordering \leq iff for all $x, y, z \in \mathbb{S}, x \leq y$ implies $x \oplus z \leq y \oplus z$, $x \otimes z \leq y \otimes z$ and $z \otimes x \leq z \otimes y$, and it is *superior wrt* \leq iff for all $x, y \in \mathbb{S}, x \leq x \otimes y$ and $y \leq x \otimes y$ [3]. The latter property corresponds to the *non-negative weights* condition in shortest-path algorithms [1]. Intuitively, it means that combining elements always increase their weight. Note that when \mathbb{S} is superior wrt \leq , then $\mathbb{1} \leq \mathbb{0}$ and moreover, for all $x \in \mathbb{S}, \mathbb{1} \leq x \leq \mathbb{0}$.

Every idempotent semiring \mathbb{S} induces a partial ordering $\leq_{\mathbb{S}}$ called the *natural ordering* of \mathbb{S} and defined by: for all x and $y, x \leq_{\mathbb{S}} y$ iff $x \oplus y = y$. This ordering is sometimes defined in the opposite direction [2]; The above definition follows [4], and coincides than the usual ordering on the Tropical semiring (*min-plus*). It holds that \mathbb{S} is monotonic wrt $\leq_{\mathbb{S}}$. An idempotent Semiring \mathbb{S} is called *total* if it $\leq_{\mathbb{S}}$ is total *i.e.* when for all $x, y \in \mathbb{S}$, either $x \oplus y = x$ or $x \oplus y = y$.

We shall consider below infinite sums with \oplus . A semiring \mathbb{S} is called *complete* if for every family $(x_i)_{i \in I}$ of elements of $\text{dom}(\mathbb{S})$ over an index set $I \subset \mathbb{N}$, the infinite sum $\bigoplus_{i \in I} x_i$ is well-defined and in $\text{dom}(\mathbb{S})$, and the following properties hold:

- i. *infinite sums extend finite sums*: $\bigoplus_{i \in \emptyset} x_i = \mathbb{0}$, $\forall j \in \mathbb{N}, \bigoplus_{i \in \{j\}} x_i = x_j$,
- $$\forall j, k \in \mathbb{N}, j \neq k, \bigoplus_{i \in \{j, k\}} x_i = x_j \oplus x_k,$$
- ii. *associativity and commutativity*: for all $I \subseteq \mathbb{N}$ and all partition $(I_j)_{j \in J}$ of I , $\bigoplus_{j \in J} \bigoplus_{i \in I_j} x_i = \bigoplus_{i \in I} x_i$,
- iii. *distributivity of product over infinite sum*:
for all $I \subseteq \mathbb{N}$, $\bigoplus_{i \in I} (x \otimes y_i) = x \otimes \bigoplus_{i \in I} y_i$, and $\bigoplus_{i \in I} (x_i \otimes y) = (\bigoplus_{i \in I} x_i) \otimes y$.

1 SW Visibly Pushdown Automata

We follow the approach of [5] for the computation of distances...

1.1 SW Automata and Transducers

The following definition of weighted transducers over infinite alphabets generalizes weighted transducers over finite alphabets, see e.g. [5], by considering weight functions generalizing the guards of symbolic automata

Let Σ and Δ be respectively an input and output *alphabets*, which are finite or infinite sets of symbols, and let \mathbb{S} be a semiring. A *label theory* is a 4-uplet of recursively enumerable sets: Φ_0 containing constant functions valued in \mathbb{S} , Φ_Σ and Φ_Δ , containing unary functions in $\Sigma \rightarrow \mathbb{S}$, resp. $\Delta \rightarrow \mathbb{S}$, and $\Phi_{\Sigma, \Delta}$ containing binary functions in $\Sigma \times \Delta \rightarrow \mathbb{S}$. Moreover, we assume that each of these sets is closed under \oplus and \otimes , and all partial applications of functions $\Phi_{\Sigma, \Delta}$, resp. $f_a : y \mapsto f(a, y)$ for $a \in \Sigma$ and $y \in \Delta$ and $f_b : x \mapsto f(x, b)$ for $b \in \Delta$ and $x \in \Sigma$, belong resp. to Φ_Σ and Φ_Δ .

Definition 1 A *weighted transducer* T over the input and output alphabet Σ and Δ and the semiring \mathbb{S} is a tuple $T = \langle Q, \text{in}, \text{weight}, \text{out} \rangle$, where Q is a finite set of states, $\text{in} : Q \rightarrow \mathbb{S}$, respectively $\text{out} : Q \rightarrow \mathbb{S}$, is a function defining the weight for entering, respectively leaving, a state, and weight is a transition function of $Q \times Q$ into $\langle \Phi_0, \Phi_\Sigma, \Phi_\Delta, \Phi_{\Sigma, \Delta} \rangle$.

We extend the above transition function into a function from $Q \times (\Sigma \cup \{\epsilon\}) \times (\Delta \cup \{\epsilon\}) \times Q$ into \mathbb{S} , also called **weight** for simplicity, such that for all $q, q' \in Q$, $a \in \Sigma$, $b \in \Delta$, and with $\langle \phi_\epsilon, \phi_\Sigma, \phi_\Delta, \phi_{\Sigma, \Delta} \rangle = \text{weight}(q, q')$,

$$\begin{aligned} \text{weight}(q, \epsilon, \epsilon, q') &= \phi_\epsilon \\ \text{weight}(q, a, \epsilon, q') &= \phi_\Sigma(a) \\ \text{weight}(q, \epsilon, b, q') &= \phi_\Delta(b) \\ \text{weight}(q, a, b, q') &= \phi_{\Sigma, \Delta}(a, b) \end{aligned}$$

These functions ϕ act as guards for the transducer's transitions, preventing a transition when they return the absorbing $\mathbb{0}$ of \mathbb{S} .

1.2 Distance between words or languages

distance d : defined over $\Sigma^* \times \Sigma^*$ into a semiring $\mathbb{S} = (\mathbb{S}, \oplus, \mathbb{0}, \otimes, \mathbb{1})$.

Edit-Distance. ...algebraic definition of edit-distance of Mohri, in [5] Let $\Omega = \Sigma \cup \{\epsilon\} \times \Sigma \cup \{\epsilon\} \setminus \{(\epsilon, \epsilon)\}$, and let h be the morphism from Ω^* into $\Sigma^* \times \Sigma^*$ defined over the concatenation of strings of Σ^* (that removes the ϵ 's). An *alignment* between 2 strings $s, t \in \Sigma^*$ is an element $\omega \in \Omega^*$ such that $h(\omega) = (s, t)$. We assume a base cost function $\delta : \Omega \rightarrow S$, extended to Ω^* as follows (for $\omega \in \Omega^*$):

$$\delta(\omega) = \bigotimes_{0 \leq i < |\omega|} \delta(\omega_i).$$

Definition 2 For $s, t \in \Sigma^*$, the edit-distance between s and t is $d(s, t) = \bigoplus_{\omega \in \Omega^* \mid h(\omega) = (s, t)} \delta(\omega)$.

e.g. Levenstein edit-distance: S is min-plus and $\delta(a, b) = 1$ for all $(a, b) \in \Omega$.

1.3 SW Visibly Pushdown Automata

2 Application

Symbolic Automated Music Transcription

2.1 Representations

Performance.

Score.

2.2 Transducer for Distance Computation

References

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