

# Symbolic Weighted Language Models and Parsing over Infinite Alphabets

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**Abstract.** We propose language models called Symbolic Weighted Automata (SWA) at the joint between Symbolic Automata (SA) and Weighted Automata (WA), as well as Transducers (SWT) and Visibly Pushdown (SWPDA) variants. Like SA, SWA deal with large or infinite input alphabets, and like WA, they output a weight value in a semiring domain. The transitions of WA are labeled by functions from an infinite alphabet into the weight domain. This is unlike WA whose transitions are guarded by boolean predicates over symbols in an infinite alphabet and also unlike WA whose transitions are labeled by constant weight values, and who deal only with finite automata. We present some properties of WA, WT and SWPDA models, that we use to define and solve a variant of parsing over infinite alphabets. We also briefly describe the application that motivated the introduction of these models: an parse-based approach to automated music transcription.

Weighted parsing [11,21,20] consists in analyzing a word in input *wrt* a weighted language model, *e.g.* a probabilistic context-free grammar (CFG). In such models, a weight value is associated to each transition rule, and the rule's weights can be combined with a associative product operator  $\otimes$  into the weight of a derivation (or *abstract syntax tree*, *i.e.* the result of an analyze). A second operator  $\oplus$ , associative and commutative, is moreover used to handle ambiguity of the model, by summing the weights of the possibly several (in general exponentially many) syntax trees associated to a given input word. Intuitively,  $\oplus$  aims at selecting, or ranking, the syntax trees. The weight domain, equipped with these two operators is assumed, at minima, to form a *semiring*  $S$ , such as a min-plus algebra of domain  $\mathbb{R} \cup \{+\infty\}$ , where  $\oplus$  is min and  $\otimes$  is plus. Some efficient specialized parsing algorithms [13] have been proposed in this context in order to compute the  $n$  best syntax trees of a given input word without having to enumerate them all. Generally based on dynamic programming, these algorithms rely on additional algebraic properties of  $S$ . – see *e.g.* [13] for some NLP applications.

The approaches to weighted parsing usually assume a finite input alphabet. Considering infinite or large alphabets can however be of practical interest, for instance in the context of processing strings in UTF16 encodings, for vulnerability detection in Web-applications (see *e.g.* [7]), for the representation of execution traces in model checking, or when dealing with sequences of timestamped events

– we describe briefly at the end of the paper a parsing based approach to of automated music transcription, that is the conversion of a symbolic representation of a music performance into a structured music score.

We present here some weighted language models computing on words over infinite input alphabets - but with finite sets of states and transitions rules. In their transition rules, input symbols appear as variables and the weight associated to a transition rule is a function of these variables. This approach is close to the case of Symbolic Automata (SA) [6,7], except that the domain for weight values is not restricted to be Boolean, like for the guards in the rules of SA, but can be an arbitrary commutative semiring (assuming some restrictions). The models defined here are finite automata (swA), transducers (swT) and pushdown automata with a visibly restriction [1] (sw-VPA). The latter model of automata computes on *nested words* [1], a structured form of words parenthesized with markup symbols, corresponding to a linearization of trees. In expressiveness, they correspond to a strict restriction of weighted CFG. Let  $A$  be a sw-VPA, associating  $A(w) \in \mathbb{S}$  to a given a nested word  $w$  (representing a parse tree), and let a swT compute a distance  $d$ , in  $\mathbb{S}$ , between 2 strings over respectively an infinite input alphabet and the same (infinite) alphabet of  $A$ . Then, the problem of Symbolic Weighted Parsing is, given an input string  $s$ , to find a nested word  $w$  minimizing (according to the ranking defined by  $\oplus$ ) the distance  $d(s, w) \otimes A(w)$  between  $s$  and  $A$ , as defined in [18].

In Section 1 we introduce swA and swT, then sw-VPA are defined in Section 2, where a polynomial 1-best algorithm is described that can be use to solve Symbolic Weighted Parsing. Finally, , we present an application of this approach to automated music transcription that has been implemented.

## 1 SW Automata and Transducers

We follow the approach of [18] for the computation of distances between words and languages, with weighted transducers, and extending it to infinite alphabets. The models introduced in this section generalize weighted automata and transducers [10] by labeling each transition with a weight function that takes the input and output symbols as parameters, instead of a simple weight value. These functions are similar to the guards of symbolic automata [6,7], but they can return values in an arbitrary semiring, where the latter guards are restricted to the Boolean semiring.

### 1.1 Semirings

We shall consider semiring domains for the weight values of our language models. A *semiring*  $\langle \mathbb{S}, \oplus, \otimes, 0, 1 \rangle$  is a structure with a domain  $\mathbb{S}$ , equipped with two associative binary operators  $\oplus$  and  $\otimes$  with respective neutral elements  $0$  and  $1$  and such that:

- $\oplus$  is commutative – therefore  $\langle \mathbb{S}, \oplus, 0 \rangle$  is a commutative monoid and  $\langle \mathbb{S}, \otimes, 1 \rangle$  is a monoid,

- $\otimes$  distributes over  $\oplus$ :  $\forall x, y, z \in \mathbb{S}, x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$ , and  $(x \oplus y) \otimes z = (x \otimes z) \oplus (y \otimes z)$ ,
- $\mathbb{0}$  is absorbing for  $\otimes$ :  $\forall x \in \mathbb{S}, \mathbb{0} \otimes x = x \otimes \mathbb{0} = \mathbb{0}$ .

Intuitively, in the models presented in this paper,  $\oplus$  selects an optimal value from two given values, in order to handle non-determinism, and  $\otimes$  combines two values into a single value, in order to describe the chaining of several transitions.

A semiring  $\mathbb{S}$  is *commutative* if  $\otimes$  is commutative. It is *idempotent* if for each  $x \in \text{dom}(\mathbb{S})$ ,  $x \oplus x = x$ . Every idempotent semiring  $\mathbb{S}$  induces a partial ordering  $\leq_\oplus$  called the *natural ordering* of  $\mathbb{S}$  [17] and defined, by: for all  $x$  and  $y$ ,  $x \leq_\oplus y$  iff  $x \oplus y = x$ . The natural ordering is sometimes defined in the opposite direction [9]; We follow here the direction that coincides than the usual ordering on the Tropical semiring (*min-plus*).

**Lemma 1 (Monotony, [17]).** *Let  $\langle \mathbb{S}, \oplus, \mathbb{0}, \otimes, \mathbb{1} \rangle$  be an idempotent semiring. For all  $x, y, z \in \mathbb{S}$ , if  $x \leq_\oplus y$  then  $x \oplus z \leq_\oplus y \oplus z$ ,  $x \otimes z \leq_\oplus y \otimes z$  and  $z \otimes x \leq_\oplus z \otimes y$ .*

We say in this case that  $\mathbb{S}$  is *monotonic*. Another important semiring property in the context of optimization is superiority [12], which corresponds to the *non-negative weights* condition in shortest-path algorithms [8]. Intuitively, it means that combining elements with  $\otimes$  always increase their weight. Formally, it is defined as the property of the following lemma.

**Lemma 2 (Superiority, Boundedness).** *Let  $\langle \mathbb{S}, \oplus, \mathbb{0}, \otimes, \mathbb{1} \rangle$  be an idempotent semiring. The two following statements are equivalent:*

- i. for all  $x, y \in \mathbb{S}$ ,  $x \leq_\oplus x \otimes y$  and  $y \leq_\oplus x \otimes y$*
- ii. for all  $x \in \mathbb{S}$ ,  $\mathbb{1} \oplus x = \mathbb{1}$ .*

In [12], the property *i.* is called  $\mathbb{S}$  is *superior wrt* the ordering  $\leq_\oplus$ . It implies that for all  $x \in \mathbb{S}$ ,  $\mathbb{1} \leq_\oplus x \leq_\oplus \mathbb{0}$ . Indeed, by the second inequality of (*i.*) with  $y = \mathbb{1}$ ,  $\mathbb{1} \leq_\oplus \mathbb{1} \otimes x = x$ , and, by the first inequality of (*i.*) with  $y = \mathbb{0}$ ,  $x \leq_\oplus x \otimes \mathbb{0} = \mathbb{0}$ . Intuitively, from an optimization point of view, it means that  $\mathbb{1}$  is the best value, and  $\mathbb{0}$  the worst. In [17], the property of Lemma 2 (in the form *ii.*) is called *boundedness* of  $\mathbb{S}$  – we shall use this term in the rest of the paper. It implies that, when looking for a best path in a graph whose edges are weighted by values of  $\mathbb{S}$ , the loops can be safely avoided.

*Proof.* (Lemma 2) For (*ii.*)  $\Rightarrow$  (*i.*) :  $x \oplus (x \otimes y) = x \otimes (\mathbb{1} \oplus y) = x$ , by distributivity of  $\otimes$  over  $\oplus$ . Hence  $x \leq_\oplus x \otimes y$ . Similarly,  $y \oplus (x \otimes y) = (\mathbb{1} \oplus x) \otimes y = y$ , hence  $y \leq_\oplus x \otimes y$ . For (*i.*)  $\Rightarrow$  (*ii.*) : as we have seen, (*i.*) implies that  $\mathbb{1} \leq_\oplus x$  for all  $x \in \mathbb{S}$ , *i.e.*, by definition of  $\leq_\oplus$ ,  $\mathbb{1} \oplus x = \mathbb{1}$ .  $\square$

**Lemma 3.** *Every bounded semiring is idempotent.*

*Proof.* By boundedness,  $\mathbb{1} \oplus \mathbb{1} = \mathbb{1}$ , and idempotency follows by multiplying both sides by  $x$  and distributing.  $\square$

	domain	$\oplus$	$\otimes$	$\mathbb{0}$	$\mathbb{1}$
Boolean	$\{\perp, \top\}$	$\vee$	$\wedge$	$\perp$	$\top$
Counting	$\mathbb{N}$	$+$	$\times$	$0$	$1$
Viterbi	$[0, 1] \subset \mathbb{R}$	$max$	$\times$	$0$	$1$
Tropical	$\mathbb{R}_+ \cup \{+\infty\}$	$min$	$+$	$+\infty$	$0$

**Fig. 1.** Some commutative, bounded, total and complete semirings.

An idempotent semiring  $\mathbb{S}$  is called *total* if it  $\leq_\oplus$  is total *i.e.* when for all  $x, y \in \mathbb{S}$ , either  $x \oplus y = x$  or  $x \oplus y = y$ .

We shall need below infinite sums with  $\oplus$ . A semiring  $\mathbb{S}$  is called *complete* if for every family  $(x_i)_{i \in I}$  of elements of  $dom(\mathbb{S})$  over an index set  $I \subset \mathbb{N}$ , the infinite sum  $\bigoplus_{i \in I} x_i$  is well-defined and in  $dom(\mathbb{S})$ , and the following properties hold:

i. *infinite sums extend finite sums*:  $\bigoplus_{i \in \emptyset} x_i = \mathbb{0}$ ,  $\forall j \in \mathbb{N}$ ,  $\bigoplus_{i \in \{j\}} x_i = x_j$ ,  $\forall j, k \in \mathbb{N}$ ,  $j \neq k$ ,

$$\bigoplus_{i \in \{j, k\}} x_i = x_j \oplus x_k,$$

ii. *associativity and commutativity*: for all  $I \subseteq \mathbb{N}$  and all partition  $(I_j)_{j \in J}$  of  $I$ ,

$$\bigoplus_{j \in J} \bigoplus_{i \in I_j} x_i = \bigoplus_{i \in I} x_i,$$

iii. *distributivity of product over infinite sum*:

$$\text{for all } I \subseteq \mathbb{N}, \bigoplus_{i \in I} (x \otimes y_i) = x \otimes \bigoplus_{i \in I} y_i, \text{ and } \bigoplus_{i \in I} (x_i \otimes y) = \left( \bigoplus_{i \in I} x_i \right) \otimes y.$$

*Example 1.* Figure ?? presented examples of semirings interesting in practice and enjoying the above properties.

## 1.2 Label Theory

We shall now define the functions labeling the transitions of SW-automata and -transducers. Let  $\Sigma$  and  $\Delta$  be two countable sets of symbols called respectively *input* and *output alphabets*, and let  $(\mathbb{S}, \oplus, \mathbb{0}, \otimes, \mathbb{1})$  be a commutative semiring. A *label theory* over  $\Sigma$  and  $\Delta$  is made of 3 recursively enumerable sets:  $\Phi_\Sigma$  and  $\Phi_\Delta$ , containing unary functions in  $\Sigma \rightarrow \mathbb{S}$ , resp.  $\Delta \rightarrow \mathbb{S}$ , and  $\Phi_{\Sigma, \Delta}$  containing binary functions in  $\Sigma \times \Delta \rightarrow \mathbb{S}$ , such that for all  $\alpha \in \mathbb{S}$ , and  $\phi \in \Phi_\Sigma$ ,  $\alpha \otimes \phi : x \mapsto \alpha \otimes \phi(x)$ , and  $\phi \otimes \alpha : x \mapsto \phi(x) \otimes \alpha$  belong to  $\Phi_\Sigma$ , and similarly for  $\oplus$  and for  $\Phi_\Delta$  and  $\Phi_{\Sigma, \Delta}$ . Moreover, these sets are required to be closed under the following operators  $\oplus$  and  $\otimes$  of  $\mathbb{S}$ : for all  $\phi, \phi' \in \Phi_\Sigma$ , all  $\psi, \psi' \in \Phi_\Delta$ , and  $\eta, \eta' \in \Phi_{\Sigma, \Delta}$ , the function

$$\begin{aligned} \phi \otimes \phi' : x &\mapsto \phi(x) \otimes \phi'(x) \text{ belongs to } \Phi_\Sigma, \\ \psi \otimes \psi' : y &\mapsto \psi(y) \otimes \psi'(y) \text{ belongs to } \Phi_\Delta, \\ \phi \otimes \eta : x, y &\mapsto \phi(x) \otimes \eta(x, y) \text{ belongs to } \Phi_{\Sigma, \Delta}, \\ \eta \otimes \psi : x, y &\mapsto \eta(x, y) \otimes \psi(y) \text{ belongs to } \Phi_{\Sigma, \Delta}, \end{aligned}$$

$\eta \otimes \eta' : x, y \mapsto \eta(x, y) \otimes \eta'(x, y)$  belongs to  $\Phi_{\Sigma, \Delta}$ .

And the same also holds for the binary sum operator  $\oplus$ .

Finally, it is also required that all partial applications of functions  $\Phi_{\Sigma, \Delta}$ , resp.  $f_a : y \mapsto f(a, y)$  for  $a \in \Sigma$  and  $y \in \Delta$  and  $f_b : x \mapsto f(x, b)$  for  $b \in \Delta$  and  $x \in \Sigma$ , belong resp. to  $\Phi_{\Sigma}$  and  $\Phi_{\Delta}$ .

### 1.3 Definitions

**Definition 1.** A symbolic-weighted transducer (*swT*) over the input and output alphabets  $\Sigma$  and  $\Delta$  and the semiring  $\mathbb{S}$  is a tuple  $T = \langle Q, \text{in}, \vec{w}, \text{out} \rangle$ , where  $Q$  is a finite set of states,  $\text{in} : Q \rightarrow \mathbb{S}$ , respectively  $\text{out} : Q \rightarrow \mathbb{S}$ , are functions defining the weight for entering, respectively leaving, a state, and  $\vec{w}$  is a tuple made of the 4 of transition functions  $w_\epsilon$ ,  $w_\Sigma$ ,  $w_\Delta$ , and  $w_{\Sigma, \Delta}$  from  $Q \times Q$  into respectively  $\mathbb{S}$ ,  $\Phi_\Sigma$ ,  $\Phi_\Delta$ , and  $\Phi_{\Sigma, \Delta}$ .

We synthesize the above 4-uplet of transition functions into a unique function  $w$  from  $Q \times (\Sigma \cup \{\epsilon\}) \times (\Delta \cup \{\epsilon\}) \times Q$  into  $\mathbb{S}$ , such that, for all  $q, q' \in Q$ ,  $a \in \Sigma$ , and  $b \in \Delta$ ,

$$\begin{aligned} w(q, \epsilon, \epsilon, q') &= w_\epsilon(q, q'), \\ w(q, a, \epsilon, q') &= \phi_\Sigma(a) && \text{where } \phi_\Sigma = w_\Sigma(q, q'), \\ w(q, \epsilon, b, q') &= \phi_\Delta(b) && \text{where } \phi_\Delta = w_\Delta(q, q'), \\ w(q, a, b, q') &= \phi_{\Sigma, \Delta}(a, b) && \text{where } \phi_{\Sigma, \Delta} = w_{\Sigma, \Delta}(q, q'). \end{aligned}$$

The symbolic-weighted transducer  $T$  defines a mapping from the pairs of strings of  $\Sigma^* \times \Delta^*$  into the weights of  $\mathbb{S}$ , based on the following intermediate function  $\text{weight}_T$  defined recursively for every  $q, q' \in Q$ , for every strings of  $s \in \Sigma^*$ ,  $t \in \Delta^*$ :

$$\begin{aligned} \text{weight}_T(q, s, t, q') &= \bigoplus_{q'' \in Q} w(q, \epsilon, \epsilon, q'') \otimes \text{weight}_T(q'', s, t, q') \\ &\oplus \bigoplus_{q'' \in Q} w(q'', \epsilon, \epsilon, q') \otimes \text{weight}_T(q, s, t, q'') \\ &\oplus \bigoplus_{\substack{q'' \in Q \\ s=au, a \in \Sigma}} w(q, a, \epsilon, q'') \otimes \text{weight}_T(q'', u, t, q') \\ &\oplus \bigoplus_{\substack{q'' \in Q \\ t=bv, b \in \Delta}} w(q, \epsilon, b, q'') \otimes \text{weight}_T(q'', s, v, q') \\ &\oplus \bigoplus_{\substack{q'' \in Q \\ s=au, a \in \Sigma \\ t=bv, b \in \Delta}} w(q, a, b, q'') \otimes \text{weight}_T(q'', u, v, q'). \end{aligned}$$

We recall that, by convention, an empty sum with  $\oplus$  is  $\mathbb{0}$ . Since  $\mathbb{0}$  is absorbing for  $\otimes$  in  $\mathbb{S}$ , one term  $w(q, a, b, q'')$  equal to  $\mathbb{0}$  in the above expression will be

ignored in the sum, meaning that there is no possible transition from state  $q$  into state  $q'$  while reading  $a$  and writing  $b$  (including the case  $a = \epsilon$  and  $a = \epsilon$ ). This is analogous to a transition's guard not satisfied by  $\langle a, b \rangle$  in the case of symbolic transducers.

The above expression of  $\text{weight}_T$  can be seen as a stateful definition of an edit-distance between a word  $s \in \Sigma^*$  and a word  $t \in \Delta^*$ , see also [19]. Intuitively,  $w(q, a, \epsilon, q'')$ , in the third line, is the cost of the deletion of a symbol  $a \in \Sigma$  in  $s$ ,  $w(q, \epsilon, b, q'')$ , in the fourth line, is the cost of the insertion of  $b \in \Delta$  in  $s$ , and  $w(q, a, b, q'')$ , in the last line, is the cost of the substitution of  $a \in \Sigma$  by  $b \in \Delta$ . The cost of a sequence of such operations transforming  $s$  into  $t$ , is the product, with  $\otimes$ , of the individual costs of the operations involved; And the distance between  $s$  and  $t$  is the sum, with  $\oplus$ , of all such product of costs.

The weight associated by  $T$  to  $\langle s, t \rangle \in \Sigma^* \times \Delta^*$  is then defined as follows:

$$T(s, t) = \bigoplus_{q, q' \in Q} \text{in}(q) \otimes \text{weight}_T(q, s, t, q') \otimes \text{out}(q'). \quad (1)$$

*Example 2.* DTW for sequences of timestamped events \*\*

*Example 3.* (simpler) poinwise distance between two sequences of timestamped events \*\*

The *Symbolic Weighted Automata* over  $\Sigma$  and  $\mathbb{S}$  are defined similarly as the transducers of Definition 1, by simply omitting the output symbols.

**Definition 2.** A symbolic-weighted automaton (*swA*) over the input alphabet  $\Sigma$  and the semiring  $\mathbb{S}$  is a tuple  $A = \langle Q, \text{in}, \vec{w}, \text{out} \rangle$ , where  $Q$  is a finite set of states,  $\text{in} : Q \rightarrow \mathbb{S}$ , respectively  $\text{out} : Q \rightarrow \mathbb{S}$ , are functions defining the weight for entering, respectively leaving, a state, and  $\vec{w}$  is a tuple made of the 2 of transition functions  $w_\epsilon$  and  $w_\Sigma$  from  $Q \times Q$  into respectively  $\mathbb{S}$  and  $\Phi_\Sigma$ .

The above transition functions  $w_\epsilon$  and  $w_\Sigma$  are synthesized into a function  $w$  from  $Q \times (\Sigma \cup \{\epsilon\}) \times Q$  into  $\mathbb{S}$  as above. When  $w_\epsilon(q, q') = 0$  for all  $q, q' \in Q$ , the automaton  $A$  is called *without  $\epsilon$ -transitions*.

## 1.4 Properties

The two following properties will be useful to our approach on symbolic weighted parsing in Section 3.

**Proposition 1.** Given a *swT*  $T$  over alphabet  $\Sigma, \Delta$  and a commutative semiring  $\mathbb{S}$ , and  $s \in \Sigma^*$ , there exists an effectively constructible *swA*  $A_{T,s}$  over  $\Delta$  and  $\mathbb{S}$  such that for all  $t \in \Delta^*$ ,  $A_{T,s}(t) = T(s, t)$ .

*Proof.* Let  $T = \langle Q, \text{in}, \vec{w}, \text{out} \rangle$  where  $\vec{w} \langle w_\epsilon, w_\Sigma, w_\Delta, w_{\Sigma, \Delta} \rangle$  has an associated synthesized form  $w$ , and let  $s = s_1 \dots s_n$  with  $s_1, \dots, s_n \in \Sigma$ . Let  $Q' = [1..n] \times Q$  be the state set of  $A_{T,s}$ , and let, for all  $q \in Q$ ,  $\text{in}'(1, q) = \text{in}(q)$  and  $\text{in}'(i, q) = 0$  for all  $1 < i \leq n$ , and  $\text{out}'(n, q) = \text{out}(q)$  and  $\text{out}'(i, q) = 0$  for all  $1 \leq i < n$ .

We define the functions  $w'_\epsilon$  and  $w'_\Sigma$  for  $A_{T,s}$  in their synthesized form: for all  $1 \leq i, j \leq n$ ,  $q, q' \in Q$  and  $a \in \Sigma$ , and  $b \in \Delta$ ,

$$\begin{aligned} w'(\langle i, q \rangle, \epsilon, \langle i, q' \rangle) &= w(q, \epsilon, \epsilon, q'), \\ w'(\langle i, q \rangle, \epsilon, \langle i+1, q' \rangle) &= w(q, s_i, \epsilon, q') \text{ if } i < n, \\ w'(\langle i, q \rangle, \epsilon, \langle j, q' \rangle) &= 0 \text{ if } j \neq i, j \neq i+1, \\ w'(\langle i, q \rangle, b, \langle i, q' \rangle) &= w(q, \epsilon, b, q'), \\ w'(\langle i, q \rangle, b, \langle i+1, q' \rangle) &= w(q, s_i, b, q') \text{ if } i < n, \\ w'(\langle i, q \rangle, b, \langle j, q' \rangle) &= 0 \text{ if } j \neq i, j \neq i+1. \end{aligned}$$

Note that the functions defined above belong to the set  $\Phi_\Delta$  of the underlying label theory, by the hypothesis of closure under partial application. The swA wanted is  $A_{T,s} = \langle Q', \text{in}', \vec{w}', \text{out}' \rangle$ .  $\square$

The construction time and size for  $A_{T,s}$  are  $O(\|T\| \cdot |s|)$ , where the size  $\|T\|$  of  $T$  is its number of states  $|Q|$ .

**Proposition 2.** *Given a swA  $A$  over alphabet  $\Sigma$  and a commutative and bounded semiring  $\mathbb{S}$ , there exists an effectively constructible swA  $A'$  without  $\epsilon$ -transitions such that for all  $s \in \Sigma^*$ ,  $A'(s) = A(s)$ .*

*Proof.* **\*\* Tb revised \*\*** Let  $A = \langle Q, \text{in}, \langle w_\epsilon, w_\Sigma \rangle, \text{out} \rangle$ . The automaton  $A'$  is  $\langle Q, \text{in}, \langle w'_\epsilon, w'_\Sigma \rangle, \text{out} \rangle$ , where, for all  $q, q' \in Q$ ,  $w'_\epsilon(q, q') = 0$  and

$$w'_\Sigma(q, q') = \bigoplus_{q'' \in Q} w_\epsilon(q, q'') \otimes w_\Sigma(q'', q') \oplus \bigoplus_{q_1 \in Q} w_\Sigma(q, q'') \otimes w_\epsilon(q'', q')$$

In the above definition of  $w'_\Sigma$  we use the operator of product of function of  $\Phi_\Sigma$  by  $\mathbb{S}$  described in Section 1.2. By definition of  $\text{weight}_A$  and distributivity of  $\otimes$  on  $\oplus$ , **\*\* NO. TBC see [16] \*\*** it holds that  $\text{weight}_A(q, s, q') = \text{weight}_{A'}(q, s, q')$ .  $\square$

## 2 SW Visibly Pushdown Automata

The following model generalizes Symbolic VPA [5] from Boolean semirings to arbitrary semiring weight domains.

### 2.1 Definition

Let  $\Omega$  be a countable alphabet that we assume partitioned into three subsets  $\Omega_i, \Omega_c, \Omega_r$ , respectively called of *internal*, *call* and *return* symbols. Let  $\langle \mathbb{S}, \oplus, 0, \otimes, 1 \rangle$  be a commutative semiring and let  $(\Phi_\epsilon, \Phi_c, \Phi_r, \Phi_{cr})$  be a label theory over  $\mathbb{S}$  where  $\Phi_c, \Phi_r$  and  $\Phi_{cr}$  stand respectively for  $\Phi_{\Omega_c}, \Phi_{\Omega_r}$  and  $\Phi_{\Omega_c, \Omega_r}$ . Moreover, we extend this theory with a set  $\Phi_i$  of unary functions in  $\Omega_i \rightarrow \mathbb{S}$ , closed under  $\oplus$  and  $\otimes$ .

**Definition 3.** A Symbolic Weighted Visibly Pushdown Automata (*sw-VPA*)  $A$  over  $\Omega = \Omega_i \uplus \Omega_c \uplus \Omega_r$  and  $\mathbb{S}$  is a tuple  $T = \langle Q, P, \text{in}, w_i, w_c, w_r, w_e, \text{out} \rangle$ , where  $Q$  is a finite set of states,  $P$  is a finite set of stack symbols,  $\text{in} : Q \rightarrow \mathbb{S}$ , respectively  $\text{out} : Q \rightarrow \mathbb{S}$ , are functions defining the weight for entering, respectively leaving, a state, and  $w_i : Q \times Q \rightarrow \Phi_i$ ,  $w_c : Q \times Q \times P \rightarrow \Phi_c$ ,  $w_r : Q \times P \times Q \rightarrow \Phi_{cr}$ ,  $w_e : Q \times Q \rightarrow \Phi_r$ , are transition functions.

Similarly as in Section 1, we extend the above transition functions as follows for all  $q, q' \in Q$ ,  $p \in P$ ,  $a \in \Omega_i$ ,  $c \in \Omega_c$ ,  $r \in \Omega_r$ , overloading their names:

$$\begin{aligned} w_i : Q \times \Omega_i \times Q &\rightarrow \mathbb{S} & w_i(q, a, q') &= \phi_i(a) & \text{where } \phi_i &= w_i(q, q'), \\ w_c : Q \times \Omega_c \times Q \times P &\rightarrow \mathbb{S} & w_c(q, c, q', p) &= \phi_c(c) & \text{where } \phi_c &= w_c(q, q', p), \\ w_r : Q \times \Omega_c \times P \times \Sigma_r \times Q &\rightarrow \mathbb{S} & w_r(q, c, p, r, q') &= \phi_r(c, r) & \text{where } \phi_r &= w_r(q, p, q'), \\ w_e : Q \times \Omega_r \times Q &\rightarrow \mathbb{S} & w_e(q, r, q') &= \phi_e(r) & \text{where } \phi_e &= w_e(q, q'). \end{aligned}$$

The intuition is the following for the above transitions.

- $w_i$  : read the input internal symbol  $a$ , change state to  $q'$ .
- $w_c$  : read the input call symbol  $c$ , push it to the stack along with  $p$ , change state to  $q'$ .
- $w_r$  : when the stack is not empty, read and pop from stack a pair made of  $c$  and  $p$ , read the input return symbol  $r$ , change state to  $q'$ . In this case, the weight function  $\phi_r$  computes a value of matching between the call and return symbols. This value might be  $\emptyset$  in order to express that the symbols do not match.
- $w_e$  : when the stack is empty, read the input symbol  $r$ , change state to  $q'$ .

We give now a formal definition of these transitions of the automaton  $A$  in term of a weight value computed by an intermediate function  $\text{weight}_A$ . In the case of a pushdown automaton, a configuration is composed of a state  $q \in Q$  and a stack content  $\theta \in \Theta^*$ , where  $\Theta = \Omega_c \times P$ . Therefore,  $\text{weight}_A$  is a function from  $Q \times \Theta^* \times \Omega^* \times Q \times \Theta^*$  into  $\mathbb{S}$ .

$$\begin{aligned} \text{weight}_A([q], a u, [q']) &= \bigoplus_{q'' \in Q} w_i(q, a, q'') \otimes \text{weight}_A([q''], u, [q']) \\ \text{weight}_A([q], c u, [q']) &= \bigoplus_{\substack{q'' \in Q \\ p \in P}} w_c(q, c, q'', p) \otimes \text{weight}_A([c p \cdot \theta], u, [q']) \\ \text{weight}_A([c p \cdot \theta], r u, [q']) &= \bigoplus_{q'' \in Q} w_r(q, c, p, r, q'') \otimes \text{weight}_A([q''], u, [q']) \\ \text{weight}_A([\perp], r u, [q']) &= \bigoplus_{q'' \in Q} w_e(q, r, q'') \otimes \text{weight}_A([\perp], u, [q']) \end{aligned}$$

where  $\perp$  denotes the empty stack and  $c p \cdot \theta$  denotes a stack where the pair made of  $c$  and  $p$  is the top symbol and  $\theta$  is the rest of stack.

The weight associated by  $A$  to  $s \in \Omega^*$  is then defined as follows, following empty stack semantics:

$$A(s) = \bigoplus_{q, q' \in Q} \text{in}(q) \otimes \text{weight}_A([\perp], s, [q']) \otimes \text{out}(q'). \quad (2)$$

*Example 4.* structured words... intro language of music notation ?



## 2.2 Properties

The class of **sw-VPA** is closed under the binary operators of the underlying semiring.

**Proposition 3.** *Let  $A_1$  and  $A_2$  be two (sw-VPA) over the same  $\Omega$  and  $\mathbb{S}$ . There exists two sw-VPA  $A_1 \oplus A_2$  and  $A_1 \otimes A_2$ , effectively constructible, such that for all  $s \in \Omega^*$ ,  $(A_1 \oplus A_2)(s) = A_1(s) \oplus A_2(s)$  and  $(A_1 \otimes A_2)(s) = A_1(s) \otimes A_2(s)$ .*

The construction is essentially the same as the one of [5], in the case of the Boolean semiring.

## 2.3 Best-first Search

Assume a semiring  $\mathbb{S}$  commutative, complete, idempotent, superior, and total. We propose a Dijkstra algorithm computing the minimal weight by  $A$ , wrt  $\leq_\oplus$ , for a word in  $\mathbb{S}^*$ .

More precisely, let  $\top$  be a fresh stack symbol which does not belong to  $\Theta$ , and for every two states  $q, q' \in Q$  and  $\sigma \in \{\perp, \top\}$ , let

$$d_0(q, \sigma, q') = \bigoplus_{s \in \Omega^*} \text{weight}_A\left(\begin{bmatrix} q \\ \sigma \end{bmatrix}, s, \begin{bmatrix} q' \\ \sigma \end{bmatrix}\right).$$

Since  $\mathbb{S}$  is complete, this infinite sum is well defined, and since  $\leq_\oplus$  is assumed total, it is the minimum in  $\Omega^*$  of  $s \mapsto \text{weight}_A\left(\begin{bmatrix} q \\ \sigma \end{bmatrix}, s, \begin{bmatrix} q' \\ \sigma \end{bmatrix}\right)$  wrt this ordering. When  $\sigma = \perp$ ,  $d_0(q, \sigma, q')$  is the central expression in a term of the definition (2) of  $A(s_0)$  for the minimum  $s_0$  (for the above function). When  $\sigma = \top$ , intuitively, it is the minimum weight of a computation of  $A$  starting in state  $q$  with a stack  $\theta \in \Theta^*$  (possibly empty), and ending in state  $q'$  with the same stack  $\theta$ , such that moreover the computation does not touch a symbol of  $\theta$ . That means that during the computation,  $A$  may apply the first case of in the definition of  $\text{weight}_A$  (internal symbol), as well as the second case, to can push call symbols on the top of  $\theta$ , and may pop these symbols with the third case (return symbol). However, it cannot apply one of the two last cases (return symbol and empty stack) when the current stack is  $\theta$ .

The algorithm 1 constructs iteratively a a marking  $d : Q \times \{\perp, \top\} \times Q \rightarrow \mathbb{S}$  that converges eventually to  $d_0(q, \sigma, q')$ .

### Algorithm 1 (1-best for sw-VPA)

**initially** let  $P = Q \times \{\perp, \top\} \times Q$ , and  $d(q_1, \perp, q_2) = d(q_1, \top, q_2) = \mathbb{1}$  if  $q_1 = q_2$  and  $d(q_1, \perp, q_2) = d(q_1, \top, q_2) = \mathbb{0}$  otherwise.

**while**  $P$  is not empty

**extract**  $\langle q_1, \sigma, q_2 \rangle$  from  $P$  such that  $d(q_1, \sigma, q_2)$  is minimal wrt  $\leq_\oplus$ .

**for all**  $q_0, q_3 \in Q$  and  $p \in P$  do

$$\begin{aligned}
d(q_1, \sigma, q_3) &\oplus= d(q_1, \sigma, q_2) \otimes \bigoplus_{a \in \Omega_i} w_i(q_2, a, q_3) \\
\text{if } \sigma = \top d(q_0, \top, q_3) &\oplus= d(q_1, \sigma, q_2) \otimes \bigoplus_{c \in \Omega_c} \bigoplus_{r \in \Omega_r} \eta(c, r) \\
\text{and } d(q_0, \perp, q_3) &\oplus= d(q_1, \sigma, q_2) \otimes \bigoplus_{c \in \Omega_c} \bigoplus_{r \in \Omega_r} \eta(c, r) \\
&\text{where } \eta = w_c(q_0, q_1, p) \otimes w_r(q_2, p, q_3) \\
\text{if } \sigma = \perp d(q_1, \perp, q_3) &\oplus= d(q_1, \sigma, q_2) \otimes \bigoplus_{r \in \Omega_r} w_e(q_2, r, q_3) \\
d(q_1, \perp, q_3) &\oplus= d(q_1, \sigma, q_2) \otimes d(q_2, \perp, q_3), \text{ if } \langle q_2, \perp, q_3 \rangle \notin P \\
\text{if } \sigma = \top d(q_1, \top, q_3) &\oplus= d(q_1, \sigma, q_2) \otimes d(q_2, \top, q_3), \text{ if } \langle q_2, \top, q_3 \rangle \notin P
\end{aligned}$$

The infinite sums in the updates of  $d$  in Algorithm 1 are well defined since  $\mathbb{S}$  is complete. The algorithm performs  $2 \cdot |Q|^2$  iterations until  $P$  is empty, and each iteration has a time complexity  $O(|Q|^2 \cdot |P|)$ . This gives a time complexity  $O(|Q|^4 \cdot |P|)$ . It can be reduced by implementing  $P$  as a priority queue, prioritized by the value returned by  $d$  \*\*\*complete\*\*\*.

The correctness of Algorithm 1 is ensured by the invariant expressed in the following lemma.

**Lemma 4.** *For all  $\langle q_1, \sigma, q_2 \rangle \notin P$ ,  $d(q_1, \sigma, q_2) = d_0(q_1, \sigma, q_2)$ .*

The proof is by contradiction, assuming a counter-example minimal in the length of the witness word.

For computing the minimal weight of a computation of  $A$ , we use the fact that, at the termination of Algorithm 1,

$$\bigoplus_{s \in \Omega^*} A(s) = \bigoplus_{q, q' \in Q} \text{in}(q) \otimes d(q, \perp, q') \otimes \text{out}(q').$$

In order to obtain effectively a witness (word of  $\mathbb{S}^*$  with computation of  $A$  of minimal weight), we require an additional property the of weight functions.

**Definition 4.** *Let  $\Omega$  be an alphabet and  $\mathbb{S}$  a complete semiring. A function  $\phi$  from  $\Omega^n$  into  $\mathbb{S}$  is called  $k$ -convex for a natural number  $k$  iff  $\text{card}\{\mathbf{a} \in \Omega^n \mid \phi(\mathbf{a}) = \bigoplus_{\mathbf{p} \in \Omega^n} \phi(\mathbf{p})\} \leq k$ .*

A label theory is  $k$ -convex if all its functions are  $k$ -convex.

**Proposition 4.** *For a sw-VPA  $A$  over a commutative, idempotent, superior, total and complete semiring and an alphabet  $\Omega$  with a  $k$ -convex label theory, one can construct in PTIME a word  $s \in \Omega^*$  such that  $A(s)$  is minimal wrt the natural ordering for  $\mathbb{S}$ .*

### 3 Symbolic Weighted Parsing

#### 3.1 Application to Automated Music Transcription

Symbolic Automated Music Transcription and analysis of music performances

**Time Scales** Real-Time Unit (RTU) = seconds  
Musical-Time Unit (MTU) = number of measures  
conversion via tempo value

**Representation of Music Performances** We consider symbolic representations of musical performances, as finite sequences of events. It corresponds to the concrete case of a MIDI file [2] recorded from an electronic keyboard, or the output of a transcription from audio files [3]. For the sake of simplicity, we shall only consider here the case of monophonic performances, where at most one note is sounding at a time. The approach however extends to the polyphonic case.

A music performance is a finite sequence of events in a set  $\Sigma$ . Every event  $e \in \Sigma$  has attributes such from a finite domain, like a number of key for a note or a flag indicating that it is a rest (ON and OFF messages in [2]) and a velocity value (0..127 in [2]). Moreover, it contains a RTU value  $\text{ioi}(e)$  (real number) representing the time distance to the next event, or to the end of performance for the last event, also called *inter-onset interval*.

**Representation of Music Scores** Music score are represented as structured words made of timed events and parenthesized markups, akin of nested words [1].

We consider an alphabet  $\Delta$ , every symbol of which is composed of a tag, in a finite set  $\Xi$ , and an MTU (rational) IOI duration value. It is partitioned into  $\Delta = \Delta_i \uplus \Delta_c \uplus \Delta_r$ , like in Section 2. The symbols of  $\Delta_i$  represent events: with tags indicating a new note or grace-note (with null IOI), a rest or the continuation of the previous note (tie or dot). The elements of  $\Delta_c \uplus \Delta_r$  are matched markups for describing the structure of the score, *i.e.* the hierarchical grouping of events, and also, importantly the division of time in measures, tuplets... (linearization of rhythm trees [14]...). They contain additional info such as tuple number, beaming policy...

The duration values of letters of  $\Delta$ , in MTU (rational), can be computed with the markups and tags (*e.g.* grace note has duration 0).

*Example 5.* ...

**Performance/Score Distance Computation** We define a distance between performance and score representations by a swT  $T = \langle Q, \text{in}, w, \text{out} \rangle$ , over a semiring  $\mathbb{S}$ . \*\* detail the elements of  $\mathbb{S}$  ....\*\*

Every state of  $Q$  contains a tempo value in a finite domain (e.g. 30..300 bpm). This value can be fixed or recomputed by the  $T$  after reading each event, according to a perceptive/cognitive model of tempo such as [15] (also used in the context of score following [4]).

### 3.2 Transcription by Best-first Search

We assume a score language defined by a sw-VPA over the semiring  $\mathbb{S}$  of Section 3.1.

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## A Edit-Distance

...algebraic definition of edit-distance of Mohri, in [18] distance  $d$  over  $\Sigma^* \times \Sigma^*$  into a semiring  $\mathbb{S} = (\mathbb{S}, \oplus, \otimes, \mathbb{1})$ .

Let  $\Omega = \Sigma \cup \{\epsilon\} \times \Sigma \cup \{\epsilon\} \setminus \{(\epsilon, \epsilon)\}$ , and let  $h$  be the morphism from  $\Omega^*$  into  $\Sigma^* \times \Sigma^*$  defined over the concatenation of strings of  $\Sigma^*$  (that removes the  $\epsilon$ 's). An *alignment* between 2 strings  $s, t \in \Sigma^*$  is an element  $\omega \in \Omega^*$  such that  $h(\omega) = (s, t)$ . We assume a base cost function  $\Omega : \delta : \Omega \rightarrow S$ , extended to  $\Omega^*$  as follows (for  $\omega \in \Omega^*$ ):  $\delta(\omega) = \bigotimes_{0 \leq i < |\omega|} \delta(\omega_i)$ .

**Definition 5.** For  $s, t \in \Sigma^*$ , the edit-distance between  $s$  and  $t$  is  $d(s, t) = \bigoplus_{\omega \in \Omega^* \mid h(\omega) = (s, t)} \delta(\omega)$ .

e.g. Levenstein edit-distance:  $S$  is min-plus and  $\delta(a, b) = 1$  for all  $(a, b) \in \Omega$ .