

Symbolic Weighted Language Models and Parsing over Infinite Alphabets

Florent Jacquemard   

Inria & CNAM, Paris, France

Abstract

We propose a framework for weighted parsing over infinite alphabets. It is based on language models called Symbolic Weighted Automata (**swA**) at the joint between Symbolic Automata (**sA**) and Weighted Automata (**wA**), as well as Transducers (**swT**) and Visibly Pushdown (**sw-VPA**) variants. Like **sA**, **swA** deal with large or infinite input alphabets, and like **wA**, they output a weight value in a semiring domain. The transitions of **swA** are labeled by functions from an infinite alphabet into the weight domain. This is unlike **sA** whose transitions are guarded by boolean predicates over symbols in an infinite alphabet and also unlike **wA** whose transitions are labeled by constant weight values, and who deal only with finite automata. We present some properties of **swA**, **swT** and **sw-VPA** models, that we use to define and solve a variant of parsing over infinite alphabets. We also briefly describe the application that motivated the introduction of these models: a parse-based approach to automated music transcription.

2012 ACM Subject Classification Replace `ccsdsc` macro with valid one

Keywords and phrases Dummy keyword

Digital Object Identifier 10.4230/LIPIcs...

Funding Florent Jacquemard: Inria AEx Codex, ANR Collabscore, EU H2020 Polifonia

Acknowledgements I want to thank ...

1 Introduction

Parsing is the problem of structuring a linear representation in input (a finite word) according to a language model. Most of the context-free parsing approaches [15] assume a finite and reasonably small input alphabet. Such a restriction makes perfectly sense in the context of NLP tasks like constituency parsing or of programming languages compilers or interpreters. Considering large or infinite alphabets can however be of practical interest. For instance, when dealing with large characters encodings such as UTF-16, *e.g.* for vulnerability detection in Web-applications [8], for the analyse (*e.g.* validation or filtering) of data streams or serialization of structured documents (which may contain textual or numerical attributes) [26], or for processing timed execution traces [3]. The latter case is related to a case study that motivated the present work: automated music transcription. In this problem, a symbolic representation of a music performance, in the form of a sequence of timed musical events, is converted into a structured score in Common Western Music Notation. It can therefore be expressed as a parsing problem [12], over an infinite alphabet of timed events.

Various extensions of language models for handling infinite alphabets have been studied. For instance, some automata with memory extensions allow restricted storage and comparison of input symbols, (see [26] for a survey), with pebbles for marking positions [25], registers [18], or the possibility to compute on subsequences with the same attribute values [2]. The automata at the core of model checkers compute on input symbols represented by large bitvectors [27] (sets of assignments of Boolean variables) and in practice, each transition accepts a set of such symbols (instead of an individual symbol), represented by Boolean formula or Binary Decision Diagrams. Following a similar idea, in symbolic automata



© Florent Jacquemard;

licensed under Creative Commons License CC-BY 4.0

Leibniz International Proceedings in Informatics

LIPICs Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany



■ **Figure 1** Classes of Symbolic/Weighted Automata. Σ_{fin} is a finite alphabet, Σ_{inf} is a countable alphabet, \mathbb{B} is the Boolean algebra, \mathbb{S} is a commutative semiring, $q \xrightarrow{a} q'$ is a transition between states q and q' .

(sA) [7, 8], the transitions are guarded by predicates over infinite alphabet domains. With closure conditions on the sets of such predicates, all the good properties enjoyed by automata over finite alphabets are preserved.

Other extensions of language models help in dealing with non-determinism, with the computation of weight values. With an ambiguous grammar, there may exist several derivations (*abstract syntax trees* – AST), yielding one input word. The association of one weight value to each AST permits to select a best one (or n bests). This is roughly the principle of *weighted parsing* approaches [13, 24, 23]. In *weighted language models*, like *e.g.* probabilistic context-free grammars and weighted automata (wA) [11], a weight value is associated to each transition rule, and the rule's weights can be combined with an associative product operator \otimes into the weight of an AST. A second operator \oplus , associative and commutative, is moreover used to handle ambiguity of the model, by summing the weights of the possibly several (in general exponentially many) AST associated to a given input word. Typically, \oplus will select the best of two weight values. The weight domain, equipped with these two operators is assumed, at minima, to form a *semiring* where \oplus can be extended to infinite sums, such as the Viterbi semiring and the tropical min-plus algebra, see Figure 2.

In this paper, we present a uniform framework for weighted parsing over infinite input alphabets. It is based on symbolic weighted (SW) finite states language models generalizing both sA, with functions in an arbitrary semiring instead of Boolean guards, and wA, by handling infinite alphabets – Figure 1. In the transition rules of SW models, input symbols appear as variables and the weight associated to a transition rule is a function of these variables. The models presented here are finite automata called symbolic-weighted (swA), transducers (swT) and pushdown automata, with a visibly restriction [1] (sw-VPA). The latter model of automata computes on *nested words* [1], a structured form of words parenthesized with markup symbols, corresponding to a linearization of trees. In the context of parsing, they are used here to represent (weighted) AST of CF grammars. More precisely, a sw-VPA A associates a weight value $A(t)$ to a given nested word t , which is the linearization of an AST. On the other hand, a swT is used to define a distance $T(s, t)$ between two finite words s and t over an infinite alphabet, following [21]. Then, the *SW-parsing* problem aims at finding t minimizing $T(s, t) \otimes A(t)$ (wrt the ranking defined by \oplus) – this value is called the distance between s and A in [21]. Similarly to weighted-parsing methods [13, 24, 23], our

approach proceeds in two steps, based on properties of the SW models. The first step is an intersection (Bar-Hillel construction [15]) where, given a **swT** T , a **sw-VPA** A , and an input word s , a **sw-VPA** $A_{T,s}$ is built, such that for all t , $A_{T,s}(t) = T(s, t) \otimes A(t)$. In the second step, a best AST t is found by applying to $A_{T,s}$ a best search algorithm similar to shortest distance in graphs [20, 17].

chap.
in [15]

intersection

The main contributions of the paper are: (i) the introduction of automata, **swA**, transducers, **swT** (Section 3), and visibly pushdown automata **sw-VPA** (Section 4), generalizing the corresponding classes of symbolic and weighted models, (ii) a polynomial best-search algorithm for **sw-VPA**, and (iii) a uniform framework (Section 5) for parsing over infinite alphabets, the keys to which are (iii.a) the **swT**-based definition of generic edit distances between input and output words, and (iii.b) the use in this context of nested words, and **sw-VPA**, instead of syntax trees and grammars.

2 Preliminary Notions

2.1 Semirings

We shall consider semirings for the weight values of our language models. A *semiring* $\langle \mathbb{S}, \oplus, \otimes, 0, 1 \rangle$ is a structure with a domain \mathbb{S} , equipped with two associative binary operators \oplus and \otimes , with respective neutral elements 0 and 1 , and such that:

- \oplus is commutative: $\langle \mathbb{S}, \oplus, 0 \rangle$ is a commutative monoid and $\langle \mathbb{S}, \otimes, 1 \rangle$ a monoid,
- \otimes distributes over \oplus : $\forall x, y, z \in \mathbb{S}$, $x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$, and $(x \oplus y) \otimes z = (x \otimes z) \oplus (y \otimes z)$,
- 0 is absorbing for \otimes : $\forall x \in \mathbb{S}$, $0 \otimes x = x \otimes 0 = 0$.

Intuitively, in the models presented in this paper, \oplus selects an optimal value from two given values, in order to handle non-determinism, and \otimes combines two values into a single value, in a chaining of transitions.

A semiring \mathbb{S} is *commutative* if \otimes is commutative. It is *idempotent* if for each $x \in \text{dom}(\mathbb{S})$, $x \oplus x = x$. Every idempotent semiring \mathbb{S} induces a partial ordering \leq_\oplus called the *natural ordering* of \mathbb{S} [20] and defined, by: for all x and y , $x \leq_\oplus y$ iff $x \oplus y = x$. The natural ordering is sometimes defined in the opposite direction [10]; We follow here the direction that coincides with the usual ordering on the Tropical semiring *min-plus* (Figure 2). An idempotent semiring \mathbb{S} is called *total* if \leq_\oplus is total *i.e.* when for all $x, y \in \mathbb{S}$, either $x \oplus y = x$ or $x \oplus y = y$.

► **Lemma 1** (Monotony, [20]). *Let $\langle \mathbb{S}, \oplus, \otimes, 0, 1 \rangle$ be an idempotent semiring. For all $x, y, z \in \mathbb{S}$, if $x \leq_\oplus y$ then $x \oplus z \leq_\oplus y \oplus z$, $x \otimes z \leq_\oplus y \otimes z$ and $z \otimes x \leq_\oplus z \otimes y$.*

When the property of Lemma 1 holds, \mathbb{S} is called *monotonic*. Another important semiring property in the context of optimization is superiority [16], which corresponds to the *non-negative weights* condition in shortest-path algorithms [9]. Intuitively, it means that combining elements with \otimes always increase their weight. Formally, it is defined as the property (i) below.

► **Lemma 2** (Superiority, Boundedness). *Let $\langle \mathbb{S}, \oplus, \otimes, 0, 1 \rangle$ be an idempotent semiring. The two following statements are equivalent:*

- i. for all $x, y \in \mathbb{S}$, $x \leq_\oplus x \otimes y$ and $y \leq_\oplus x \otimes y$
- ii. for all $x \in \mathbb{S}$, $1 \oplus x = 1$.

	domain	\oplus	\otimes	$\mathbb{0}$	$\mathbb{1}$
Boolean	$\{\perp, \top\}$	\vee	\wedge	\perp	\top
Counting	\mathbb{N}	$+$	\times	0	1
Viterbi	$[0, 1] \subset \mathbb{R}$	\max	\times	0	1
Tropical min-plus	$\mathbb{R}_+ \cup \{\infty\}$	\min	$+$	∞	0

■ **Figure 2** Some commutative, bounded, total and complete semirings.

Proof. (ii) \Rightarrow (i) : $x \oplus (x \otimes y) = x \otimes (\mathbb{1} \oplus y) = x$, by distributivity of \otimes over \oplus . Hence $x \leq_{\oplus} x \otimes y$. Similarly, $y \oplus (x \otimes y) = (\mathbb{1} \oplus x) \otimes y = y$, hence $y \leq_{\oplus} x \otimes y$. (i) \Rightarrow (ii) : by the second inequality of (i), with $y = \mathbb{1}$, $\mathbb{1} \leq_{\oplus} x \otimes \mathbb{1} = x$, i.e., by definition of \leq_{\oplus} , $\mathbb{1} \oplus x = \mathbb{1}$. ◀ ◀

In [16], when the property (i) holds, \mathbb{S} is called *superior wrt* the ordering \leq_{\oplus} . We have seen in the proof of Lemma 2 that it implies that $\mathbb{1} \leq_{\oplus} x$ for all $x \in \mathbb{S}$. Similarly, by the first inequality of (i) with $y = \mathbb{0}$, $x \leq_{\oplus} x \otimes \mathbb{0} = \mathbb{0}$. Hence, in a superior semiring, it holds that for all $x \in \mathbb{S}$, $\mathbb{1} \leq_{\oplus} x \leq_{\oplus} \mathbb{0}$. Intuitively, from an optimization point of view, it means that $\mathbb{1}$ is the best value, and $\mathbb{0}$ the worst. In [20], \mathbb{S} with the property (ii) of Lemma 2 is called *bounded* – we shall use this term in the rest of the paper. It implies that, when looking for a best path in a graph whose edges are weighted by values of \mathbb{S} , the loops can be safely avoided (because, for all $x \in \mathbb{S}$ and $n \geq 1$, $x \oplus x^n = x \otimes (\mathbb{1} \oplus x^{n-1}) = x$).

▶ **Lemma 3.** *Every bounded semiring is idempotent.*

Proof. By boundedness, $\mathbb{1} \oplus \mathbb{1} = \mathbb{1}$, and idempotency follows by multiplying both sides by x and distributing. ◀ ◀

We shall need below infinite sums with \oplus . A semiring \mathbb{S} is called *complete* [11] if it has an operation $\bigoplus_{i \in I} x_i$ for every family $(x_i)_{i \in I}$ of elements of $\text{dom}(\mathbb{S})$ over an index set $I \subset \mathbb{N}$, such that:

i. *infinite sums extend finite sums:*

$$\bigoplus_{i \in \emptyset} x_i = \mathbb{0}, \quad \forall j \in \mathbb{N}, \bigoplus_{i \in \{j\}} x_i = x_j, \quad \forall j, k \in \mathbb{N}, j \neq k, \bigoplus_{i \in \{j, k\}} x_i = x_j \oplus x_k,$$

ii. *associativity and commutativity:*

$$\text{for all } I \subseteq \mathbb{N} \text{ and all partition } (I_j)_{j \in J} \text{ of } I, \bigoplus_{j \in J} \bigoplus_{i \in I_j} x_i = \bigoplus_{i \in I} x_i,$$

iii. *distributivity of product over infinite sum:*

$$\text{for all } I \subseteq \mathbb{N}, \bigoplus_{i \in I} (x \otimes y_i) = x \otimes \bigoplus_{i \in I} y_i, \text{ and } \bigoplus_{i \in I} (x_i \otimes y) = \left(\bigoplus_{i \in I} x_i \right) \otimes y.$$

▶ **Example 4.** Figure 2 presents examples of semirings interesting in practice and enjoying the above properties.

2.2 Label Theory

We shall now define the functions labeling the transitions of SW automata and transducers, generalizing the Boolean algebras of [7] from Boolean to other semiring domains. We consider *alphabets*, which are countable sets of symbols denoted Σ, Δ, \dots . Given a semiring $\langle \mathbb{S}, \oplus, \otimes, \mathbb{0}, \mathbb{1} \rangle$, a *label theory* over \mathbb{S} is a tuple $\bar{\Phi}$ of recursively enumerable sets denoted Φ_{Σ} , containing unary functions of type $\Sigma \rightarrow \mathbb{S}$, and $\Phi_{\Sigma, \Delta}$, containing binary functions $\Sigma \times \Delta \rightarrow \mathbb{S}$, and such that:

- 151 – for all $\Phi_{\Sigma,\Delta} \in \bar{\Phi}$, we have $\Phi_{\Sigma} \in \bar{\Phi}$ and $\Phi_{\Delta} \in \bar{\Phi}$
- 152 – every Φ_{Σ} contains all the constant functions from Σ into \mathbb{S} ,
- 153 – for all $\alpha \in \mathbb{S}$ and $\phi \in \Phi_{\Sigma}$, $\alpha \otimes \phi : x \mapsto \alpha \otimes \phi(x)$, and $\phi \otimes \alpha : x \mapsto \phi(x) \otimes \alpha$
- 154 belong to Φ_{Σ} , and similarly for \oplus and for $\Phi_{\Sigma,\Delta}$
- 155 – for all $\phi, \phi' \in \Phi_{\Sigma}$, $\phi \otimes \phi' : x \mapsto \phi(x) \otimes \phi'(x)$ belongs to Φ_{Σ}
- 156 – for all $\eta, \eta' \in \Phi_{\Sigma,\Delta}$ $\eta \otimes \eta' : x, y \mapsto \eta(x, y) \otimes \eta'(x, y)$ belongs to $\Phi_{\Sigma,\Delta}$
- 157 – for all $\phi \in \Phi_{\Sigma}$ and $\eta \in \Phi_{\Sigma,\Delta}$, $\phi \otimes_1 \eta : x, y \mapsto \phi(x) \otimes \eta(x, y)$ and
- 158 $\eta \otimes_1 \phi : x, y \mapsto \eta(x, y) \otimes \phi(x)$ belong to $\Phi_{\Sigma,\Delta}$
- 159 – for all $\psi \in \Phi_{\Delta}$ and $\eta \in \Phi_{\Sigma,\Delta}$, $\psi \otimes_2 \eta : x, y \mapsto \psi(y) \otimes \eta(x, y)$ and
- 160 $\eta \otimes_2 \psi : x, y \mapsto \eta(x, y) \otimes \psi(y)$ belong to $\Phi_{\Sigma,\Delta}$
- 161 – similar closures hold for \oplus
- 162 – the partial applications $\eta \in \Phi_{\Sigma,\Delta}$ and $\eta_a : y \mapsto \eta(a, y)$ for $a \in \Sigma$ and
- 163 $\eta_b : x \mapsto \eta(x, b)$ for $b \in \Delta$ belong respectively to Φ_{Δ} and Φ_{Σ} .

partial appli needed?

164 In what follows, we shall omit the subscripts in \otimes_1 , \otimes_2 , \oplus_1 , \oplus_2 when there is no ambiguity,
 165 and keep them only for the special case $\Sigma = \Delta$, *i.e.* $\eta \in \Phi_{\Sigma,\Sigma}$.

166 When the semiring \mathbb{S} is complete, let us consider the following operators on the functions of
 167 a label theory (we use overloading to simplify notations):

$$\begin{aligned}
 \oplus_{\Sigma} : \Phi_{\Sigma} &\rightarrow \mathbb{S}, & \phi &\mapsto \bigoplus_{a \in \Sigma} \phi(a) \\
 \oplus_{\Sigma}^1 : \Phi_{\Sigma,\Delta} &\rightarrow \Phi_{\Delta}, & \eta &\mapsto (y \mapsto \bigoplus_{a \in \Sigma} \eta(a, y)) \\
 \oplus_{\Delta}^2 : \Phi_{\Sigma,\Delta} &\rightarrow \Phi_{\Sigma}, & \eta &\mapsto (x \mapsto \bigoplus_{b \in \Delta} \eta(x, b))
 \end{aligned}$$

169 Similarly as for the notation of product and sum of functions above, the superscripts in \oplus_{Σ}^1
 170 and \oplus_{Σ}^2 shall be reserved to the ambiguous case of $\Phi_{\Sigma,\Sigma}$, in order to distinguish between
 171 the first and the second argument.

172 ► **Definition 5.** A label theory $\bar{\Phi}$ is complete when its underlying semiring \mathbb{S} is complete,
 173 and for all $\Phi_{\Sigma,\Delta} \in \bar{\Phi}$ and all $\eta \in \Phi_{\Sigma,\Delta}$, $\oplus_{\Sigma} \eta \in \Phi_{\Delta}$ and $\oplus_{\Delta} \eta \in \Phi_{\Sigma}$.

174 The following facts are immediate.

175 ► **Lemma 6.** For $\bar{\Phi}$ complete $\alpha \in \mathbb{S}$, $\phi, \phi' \in \Phi_{\Sigma}$, $\psi \in \Phi_{\Delta}$, and $\eta \in \Phi_{\Sigma,\Delta}$:

- 176 i. $\oplus_{\Sigma} \oplus_{\Delta} \eta = \oplus_{\Delta} \oplus_{\Sigma} \eta$
- 177 ii. $\alpha \otimes \oplus_{\Sigma} \phi = \oplus_{\Sigma} (\alpha \otimes \phi)$ and $(\oplus_{\Sigma} \phi) \otimes \alpha = \oplus_{\Sigma} (\phi \otimes \alpha)$, and similarly for \oplus
- 178 iii. $(\oplus_{\Sigma} \phi) \oplus (\oplus_{\Sigma} \phi') = \oplus_{\Sigma} (\phi \oplus \phi')$ and $(\oplus_{\Sigma} \phi) \otimes (\oplus_{\Sigma} \phi') = \oplus_{\Sigma} (\phi \otimes \phi')$
- 179 iv. $(\oplus_{\Delta} \eta) \oplus (\oplus_{\Delta} \eta') = \oplus_{\Delta} (\eta \oplus \eta')$, and $(\oplus_{\Delta} \eta) \otimes (\oplus_{\Delta} \eta') = \oplus_{\Delta} (\eta \otimes \eta')$
- 180 v. $\phi \otimes (\oplus_{\Delta} \eta) = \oplus_{\Delta} (\phi \otimes \eta)$, and $(\oplus_{\Delta} \eta) \otimes \phi = \oplus_{\Delta} (\eta \otimes \phi)$, and similarly for \oplus
- 181 vi. $\psi \otimes (\oplus_{\Sigma} \eta) = \oplus_{\Sigma} (\psi \otimes \eta)$, and $(\oplus_{\Sigma} \eta) \otimes \psi = \oplus_{\Sigma} (\eta \otimes \psi)$, and similarly for \oplus

182 Intuitively, the operators \oplus_{Σ} return global minimum, wrt \leq_{\oplus} , of functions of $\bar{\Phi}$. A label
 183 theory is called *effective* when for all $\phi \in \Phi_{\Sigma}$ and $\eta \in \Phi_{\Sigma,\Delta}$, $\oplus_{\Sigma} \phi$, $\oplus_{\Sigma} \eta$, and $\oplus_{\Delta} \eta$ can be
 184 effectively computed from ϕ and η .

185 ► **Definition 7.** Let Ω be an alphabet A function $\phi \in \Phi_{\Sigma}$ in a label theory over a complete
 186 semiring \mathbb{S} is called *k-convex*, for a natural number k , iff $\text{card}\{a \in \Sigma \mid \phi(a) = \oplus_{\Sigma} \phi\} \leq k$.

187 A label theory is *k-convex* if all its functions are *k-convex*.

3 SW Automata and Transducers

We follow the approach of [21] for the computation of distances, between words and languages, using weighted transducers, and extend it to infinite alphabets. The models introduced in this section generalize weighted automata and transducers [11] by labelling each transition with a weight function (instead of a simple weight value), that takes the input and output symbols as parameters. These functions are similar to the guards of symbolic automata [7, 8], but they can return values in an generic semiring, whereas the latter guards are restricted to the Boolean semiring.

Let \mathbb{S} be a commutative semiring, Σ and Δ be alphabets called respectively *input* and *output*, and $\bar{\Phi}$ be a label theory over \mathbb{S} containing $\Phi_\Sigma, \Phi_\Delta, \Phi_{\Sigma,\Delta}$.

► **Definition 8.** A symbolic-weighted transducer (*swT*) over $\Sigma, \Delta, \mathbb{S}$ and $\bar{\Phi}$ is a tuple $T = \langle Q, \text{in}, \bar{w}, \text{out} \rangle$, where Q is a finite set of states, $\text{in} : Q \rightarrow \mathbb{S}$ respectively $\text{out} : Q \rightarrow \mathbb{S}$ are functions defining the weight for entering, respectively leaving, computation in a state, and \bar{w} is a triplet of transition functions $w_{10} : Q \times Q \rightarrow \Phi_\Sigma, w_{01} : Q \times Q \rightarrow \Phi_\Delta$, and $w_{11} : Q \times Q \rightarrow \Phi_{\Sigma,\Delta}$.

For convenience, we shall sometimes present transition functions as functions of $Q \times (\Sigma \cup \{\varepsilon\}) \times (\Delta \cup \{\varepsilon\}) \times Q \rightarrow \mathbb{S}$, overloading the function names, such that, for all $q, q' \in Q, a \in \Sigma, b \in \Delta$,

$$\begin{aligned} w_{10}(q, a, \varepsilon, q') &= \phi(a) & \text{where } \phi &= w_{10}(q, q') \in \Phi_\Sigma, \\ w_{01}(q, \varepsilon, b, q') &= \psi(b) & \text{where } \psi &= w_{01}(q, q') \in \Phi_\Delta, \\ w_{11}(q, a, b, q') &= \eta(a, b) & \text{where } \eta &= w_{11}(q, q') \in \Phi_{\Sigma,\Delta}. \end{aligned}$$

The *swT* T computes on pairs of words $\langle s, t \rangle \in \Sigma^* \times \Delta^*$, s and t , being respectively called input and output word. More precisely, T defines a mapping from $\Sigma^* \times \Delta^*$ into \mathbb{S} , based on an intermediate function weight_T defined recursively for every states $q, q' \in Q$, and every strings $\langle s, t \rangle \in \Sigma^* \times \Delta^* \setminus \{\langle \varepsilon, \varepsilon \rangle\}$ by:

$$\begin{aligned} \text{weight}_T(q, \varepsilon, \varepsilon, q) &= \mathbb{1} \\ \text{weight}_T(q, \varepsilon, \varepsilon, q') &= \mathbb{0} \quad \text{if } q \neq q' \\ \text{weight}_T(q, s, t, q') &= \bigoplus_{\substack{q'' \in Q \\ s=au, a \in \Sigma}} w_{10}(q, a, \varepsilon, q'') \otimes \text{weight}_T(q'', u, t, q') \\ &\quad \oplus \bigoplus_{\substack{q'' \in Q \\ t=bv, b \in \Delta}} w_{01}(q, \varepsilon, b, q'') \otimes \text{weight}_T(q'', s, v, q') \\ &\quad \oplus \bigoplus_{\substack{q'' \in Q \\ s=au, t=bv}} w_{11}(q, a, b, q'') \otimes \text{weight}_T(q'', u, v, q') \end{aligned} \tag{1}$$



We recall that, by convention (Section 2.1), an empty sum with \bigoplus is equal to $\mathbb{0}$. Intuitively, using a transition $w_{ij}(q, a, b, q')$ means for T : when reading respectively a and b at the current positions in the input and output words, increment the current position in the input word if and only if $i = 1$, and in the output word iff $j = 1$ (otherwise, do not change it), and change state from q to q' . When $a = \varepsilon$ (resp. $b = \varepsilon$), the current symbol in the input (resp. output)

is not read. Since \emptyset is absorbing for \otimes in \mathbb{S} , one term $w_{ij}(q, a, b, q'')$ equal to \emptyset in the above expression will be ignored in the sum, meaning that there is no possible transition from state q into state q' while reading a and b . This is analogous to the case of a transition's guard not satisfied by $\langle a, b \rangle$ for symbolic transducers.

The expression (1) can be seen as a stateful definition of an edit-distance between a word $s \in \Sigma^*$ and a word $t \in \Delta^*$, see also [22]. Intuitively, $w_{10}(q, a, \varepsilon, r)$ is the cost of the deletion of the symbol $a \in \Sigma$ in s , $w_{01}(q, \varepsilon, b, r)$ is the cost of the insertion of $b \in \Delta$ in t , and $w_{11}(q, a, b, r)$ is the cost of the substitution of $a \in \Sigma$ by $b \in \Delta$. The cost of a sequence of such operations transforming s into t , is the product, with \otimes , of the individual costs of the operations involved; And the distance between s and t is the sum, with \oplus , of all possible products.

Formally, the weight associated by T to $\langle s, t \rangle \in \Sigma^* \times \Delta^*$ is:

$$T(s, t) = \bigoplus_{q, q' \in Q} \text{in}(q) \otimes \text{weight}_T(q, s, t, q') \otimes \text{out}(q') \quad (2)$$

► **Example 9.** In Common Western Music Notation [14], several symbols may be used to represent a unique sounding event. For instance, several notes can be combined with a tie, like in , and one note can be augmented by half its duration with a dot like in .

We propose a small transducer model that compares an input sequence of sounding events (music "performance") to an output sequence of written events (music "score"). Let us consider the tropical (*min-plus*) semiring \mathbb{S} of Figure 2 and let $\Sigma = \mathbb{R}_+$ be an input alphabet of event dates and $\Delta = \{\mathbf{e}, -\} \times \mathbb{R}_+$ be an output alphabet of symbols with timestamps. A symbol $\langle \mathbf{e}, d \rangle \in \Delta$ represents an event starting at date d , and $\langle -, d \rangle$ is a continuation of the previous event.

We consider a **swT** with two states q_0 and q_1 whose purpose is to compare a recorded performance $s \in \Sigma^*$ with notated music sheet $t \in \Delta^*$. One timestamp $d_i \in \Sigma$ may corresponds to one notated event $\langle \mathbf{e}, d'_i \rangle \in \Delta$, in which case the weight value computed by the **swT** is the time distance between both (see transitions w_{11} below). If $\langle \mathbf{e}, d'_i \rangle$ is followed by continuations $\langle -, d'_{i+1} \rangle \dots$, they are just skip with no cost (transitions w_{01} or weight $\mathbb{1}$).

$$\begin{aligned} w_{11}(q_0, d, \langle \mathbf{e}, d' \rangle, q_0) &= |d' - d| & w_{11}(q_1, d, \langle \mathbf{e}, d' \rangle, q_0) &= |d' - d| \\ w_{01}(q_0, \varepsilon, \langle -, d' \rangle, q_0) &= \mathbb{1} & w_{01}(q_1, \varepsilon, \langle -, d' \rangle, q_0) &= \mathbb{1} \\ w_{10}(q_0, d, \varepsilon, q_1) &= \alpha \end{aligned}$$

Moreover, it may happen that the performers plays an extra note accidentally, but only once in a row. This is modelled by the transition w_{10} with an arbitrary weight value $\alpha \in \mathbb{S}$, switching from state q_0 (normal) to q_1 (error). The transitions in the second column below switch back to the normal state q_0 . At last, we let q_0 be the only initial and final state, with $\text{in}(q_0) = \text{out}(q_0) = \mathbb{1}$, and $\text{in}(q_1) = \text{out}(q_1) = \emptyset$. \diamond

The *Symbolic Weighted Automata* are defined similarly as the transducers of Definition 8, by simply omitting the output symbols. In this case, the label theory $\bar{\Phi}$ can be reduced to a singleton $\langle \Phi_\Sigma \rangle$.

► **Definition 10.** A symbolic-weighted automaton (**swA**) over Σ , \mathbb{S} and $\bar{\Phi}$ is a tuple $A = \langle Q, \text{in}, w_1, \text{out} \rangle$, where Q is a finite set of states, $\text{in} : Q \rightarrow \mathbb{S}$, respectively $\text{out} : Q \rightarrow \mathbb{S}$ are functions defining the weight for entering, respectively leaving, computation in a state, and w_1 is a transition functions from $Q \times Q$ into Φ_Σ .

As above in the case of swT , when $w_1(q, q') = \phi \in \Phi_\Sigma$, we may write $w_1(q, a, q')$ for $\phi(a)$. The computation of A on words $s \in \Sigma^*$ is defined with an intermediate function weight_A , defined as follows for $q, q' \in Q$, $a \in \Sigma$, $u \in \Sigma^*$,

$$\begin{aligned} \text{weight}_A(q, \varepsilon, q) &= 1 \\ \text{weight}_A(q, \varepsilon, q') &= 0 \quad \text{if } q \neq q' \\ \text{weight}_A(q, au, q') &= \bigoplus_{q'' \in Q} w_1(q, a, q'') \otimes \text{weight}_A(q'', u, q') \end{aligned} \tag{3}$$

and the weight value associated by A to $s \in \Sigma^*$ is defined as follows:

$$A(s) = \bigoplus_{q, q' \in Q} \text{in}(q) \otimes \text{weight}_A(q, s, q') \otimes \text{out}(q') \tag{4}$$

The following property will be useful to the approach on symbolic weighted parsing presented in Section 5.

► **Proposition 11.** *Given a swT T over Σ , Δ , \mathbb{S} commutative, bounded and complete, and $\bar{\Phi}$ effective, and a swTA A over Σ , \mathbb{S} and $\bar{\Phi}$, there exists an effectively constructible swA $B_{T,A}$ over Δ , \mathbb{S} and $\bar{\Phi}$, such that for all $t \in \Delta^*$, $B_{T,A}(t) = \bigoplus_{s \in \Sigma^*} A(s) \otimes T(s, t)$.*

Proof. Let $T = \langle Q, \text{in}_T, \bar{w}, \text{out}_T \rangle$, where \bar{w} contains w_{10} , w_{01} , and w_{11} , from $Q \times Q$ into respectively Φ_Σ , Φ_Δ , and $\Phi_{\Sigma, \Delta}$, and let $A = \langle P, \text{in}_A, w_1, \text{out}_A \rangle$ with $w_1 : Q \times Q \rightarrow \Phi_\Sigma$. The state set of $B_{T,A}$ will be $Q' = P \times Q$. The entering, leaving and transition functions of $B_{T,A}$ will simulate synchronized computations of A and T , while reading an output word of Δ^* . Its state entering functions is defined for all $p \in P$, $q \in Q$ by $\text{in}'(p, q) = \text{in}_A(p) \otimes \text{in}_T(q)$. The transition function w'_1 will roughly perform a synchronized product of transitions defined by w_1 , w_{01} (T reading in output word and not in input word) and w_{11} (T reading in output word and input word). Moreover, w'_1 also needs to simulate transitions defined by w_{10} : T reading in input word and not in output word. Since $B_{T,A}$ will read only in the output word, such a transition corresponds to an ε -transition of swA , but swA have been defined without ε -transitions. Therefore, in order to take care of this case, we perform an on-the-fly suppression of ε -transition in the swA in construction, following the algorithm of [19]. Initially, for all $p_1, p_2 \in P$, and $q_1, q_2 \in Q$, let

$$w'_1(\langle p_1, q_1 \rangle, \langle p_2, q_2 \rangle) = w_1(p_1, p_2) \otimes [w_{01}(q_1, q_2) \oplus \bigoplus_{\Sigma} w_{11}(q_1, q_2)].$$

Iterate the following for all $p_1 \in P$ and $q_1, q_2 \in Q$: for all $p_2 \in P$ and $q_3 \in Q$,

$$w'_1(\langle p_1, q_1 \rangle, \langle p_2, q_3 \rangle) \oplus = \bigoplus_{\Sigma} w_{10}(q_1, q_2) \otimes w'_1(\langle p_1, q_2 \rangle, \langle p_2, q_3 \rangle)$$

proof correctness

$$\text{and } \text{out}'(p_1, q_1) \oplus = \bigoplus_{\Sigma} w_{10}(q_1, q_2) \otimes \text{out}'(p_1, q_2) \quad \blacktriangleleft$$

The construction time and size for $B_{T,A}$ are $O(\|T\|^3 \cdot \|A\|^2)$, where the sizes $\|T\|$ and $\|A\|$ are their number of states.

► **Corollary 12.** *Given a swT T over Σ , Δ , \mathbb{S} commutative, bounded and complete, and $\bar{\Phi}$ effective, and $s \in \Sigma^+$, there exists an effectively constructible swA $B_{T,s}$ over Δ , \mathbb{S} and $\bar{\Phi}$, such that for all $t \in \Delta^*$, $B_{T,s}(t) = T(s, t)$.*

4 SW Visibly Pushdown Automata

The model presented in this section generalizes Symbolic VPA [6] from Boolean semirings to arbitrary semiring weight domains. It will compute on nested words over infinite alphabets, associating to every such word a weight value. Nest words are able to describe structures of labeled trees, and in the context of parsing, they will be useful to represent parse trees.

4.1 Definition

Let Ω be a countable alphabet that we assume partitioned into three subsets $\Omega_i, \Omega_c, \Omega_r$, whose elements are respectively called *internal*, *call* and *return* symbols. Let $\langle \mathbb{S}, \oplus, \emptyset, \otimes, \mathbb{1} \rangle$ be a commutative and complete semiring and let $\bar{\Phi} = \langle \Phi_i, \Phi_c, \Phi_r, \Phi_{ci}, \Phi_{cc}, \Phi_{cr} \rangle$ be a label theory over \mathbb{S} where Φ_i, Φ_c, Φ_r and Φ_{cx} (with $x \in \{i, c, r\}$) stand respectively for $\Phi_{\Omega_i}, \Phi_{\Omega_c}, \Phi_{\Omega_r}$ and $\Phi_{\Omega_c, \Omega_x}$.

► **Definition 13.** A Symbolic Weighted Visibly Pushdown Automata (*sw-VPA*) over $\Omega = \Omega_i \uplus \Omega_c \uplus \Omega_r, \mathbb{S}$ and $\bar{\Phi}$ is a tuple $A = \langle Q, P, \text{in}, \bar{w}, \text{out} \rangle$, where Q is a finite set of states, P is a finite set of stack symbols, $\text{in} : Q \rightarrow \mathbb{S}$, respectively $\text{out} : Q \rightarrow \mathbb{S}$, are functions defining the weight for entering, respectively leaving, a state, and \bar{w} is a sextuplet composed of the transition functions: $w_i : Q \times P \times Q \rightarrow \Phi_{ci}$, $w_i^e : Q \times Q \rightarrow \Phi_i$, $w_c : Q \times P \times Q \times P \rightarrow \Phi_{cc}$, $w_c^e : Q \times P \times Q \rightarrow \Phi_c$, $w_r : Q \times P \times Q \rightarrow \Phi_{cr}$, $w_r^e : Q \times Q \rightarrow \Phi_r$.

Similarly as in Section 3, we extend the above transition functions as follows for all $q, q' \in Q$, $p \in P$, $a \in \Omega_i$, $c \in \Omega_c$, $r \in \Omega_r$, overloading their names:

$$\begin{array}{lll}
 w_i : Q \times \Omega_c \times P \times \Omega_i \times Q \rightarrow \mathbb{S} & w_i(q, c, p, a, q') = \eta_{ci}(c, a) & \text{where } \eta_{ci} = w_i(q, p, q'), \\
 w_i^e : Q \times \Omega_i \times Q \rightarrow \mathbb{S} & w_i^e(q, a, q') = \phi_i(a) & \text{where } \phi_i = w_i^e(q, q'), \\
 w_c : Q \times \Omega_c \times P \times \Omega_c \times P \times Q \rightarrow \mathbb{S} & w_c(q, c, p, c', p', q') = \eta_{cc}(c, c') & \text{where } \eta_{cc} = w_c(q, p, p', q'), \\
 w_c^e : Q \times \Omega_c \times P \times Q \rightarrow \mathbb{S} & w_c^e(q, c, p, q') = \phi_c(c) & \text{where } \phi_c = w_c^e(q, p, q'), \\
 w_r : Q \times \Omega_c \times P \times \Omega_r \times Q \rightarrow \mathbb{S} & w_r(q, c, p, r, q') = \eta_{cr}(c, r) & \text{where } \eta_{cr} = w_r(q, p, q'), \\
 w_r^e : Q \times \Omega_r \times Q \rightarrow \mathbb{S} & w_r^e(q, r, q') = \phi_r(r) & \text{where } \phi_r = w_r^e(q, q').
 \end{array}$$

The intuition is the following for the above transitions.

w_i and w_i^e both read an input internal symbol a and change state from q to q' , without changing the stack. Moreover, w_i reads a pair made of $c \in \Omega_c$ and $p \in P$ at the top of the stack (c is compared to a by the weight function $\eta_{ci} \in \Phi_{ci}$) and w_i^e applies if and only if the stack is empty.

w_c and w_c^e read the input call symbol c' , push it to the stack along with p' , and change state from q to q' . Moreover, w_c reads c and p at the top of the stack (c is compared to c'), and w_c^e applies iff the stack is empty.

w_r and w_r^e read the input return symbol r , and change state from q to q' . Moreover, w_r reads and pop from stack a pair made of c and p , (c is compared to r), and w_r^e applies iff the stack is empty.

Formally, the transitions of the automaton A are defined in term of an intermediate function weight_A , like in Section 3. In the case of a pushdown automaton, a configuration, denoted by $q[\gamma]$ is composed of a state $q \in Q$ and a stack content $\gamma \in \Gamma^*$, where $\Gamma = \Omega_c \times P$. Hence, weight_A is a function from $[Q \times \Gamma^*] \times \Omega^* \times [Q \times \Gamma^*]$ into \mathbb{S} (the empty stack is denoted by \perp).

$$\text{weight}_A(q[\perp], \varepsilon, q[\perp]) = \mathbb{1} \quad (5)$$

$$\text{weight}_A(q[\perp], \varepsilon, q'[\gamma]) = \emptyset \text{ if } q \neq q'$$

XX:10 Symbolic Weighted Language Models and Parsing over Infinite Alphabets

$$\text{weight}_A(q \begin{bmatrix} \langle c, p \rangle \\ \gamma \end{bmatrix}, a u, q'[\gamma']) = \bigoplus_{q'' \in Q} w_i(q, c, p, a, q'') \otimes \text{weight}_A(q'' \begin{bmatrix} \langle c, p \rangle \\ \gamma \end{bmatrix}, u, q'[\gamma'])$$

$$\text{weight}_A(q[\perp], a u, q'[\gamma']) = \bigoplus_{q'' \in Q} w_i^e(q, a, q'') \otimes \text{weight}_A(q''[\perp], u, q'[\gamma'])$$

$$\text{weight}_A(q \begin{bmatrix} \langle c, p \rangle \\ \gamma \end{bmatrix}, c' u, q'[\gamma']) = \bigoplus_{\substack{q'' \in Q \\ p' \in P}} w_c(q, c, p, c', p', q'') \otimes \text{weight}_A(q'' \begin{bmatrix} \langle c', p' \rangle \\ \langle c, p \rangle \\ \gamma \end{bmatrix}, u, q'[\gamma'])$$

$$\text{weight}_A(q[\perp], c u, q'[\gamma']) = \bigoplus_{\substack{q'' \in Q \\ p \in P}} w_c^e(q, c, p, q'') \otimes \text{weight}_A(q''[c p], u, q'[\gamma'])$$

$$\text{weight}_A(q \begin{bmatrix} \langle c, p \rangle \\ \gamma \end{bmatrix}, r u, q'[\gamma']) = \bigoplus_{q'' \in Q} w_r(q, c, p, r, q'') \otimes \text{weight}_A(q''[\gamma], u, q'[\gamma'])$$

$$\text{weight}_A(q[\perp], r u, q'[\gamma']) = \bigoplus_{q'' \in Q} w_r^e(q, r, q'') \otimes \text{weight}_A(q''[\perp], u, q'[\gamma'])$$

The weight associated by A to $s \in \Omega^*$ is defined according to empty stack semantics:

$$A(s) = \bigoplus_{q, q' \in Q} \text{in}(q) \otimes \text{weight}_A(q[\perp], s, q'[\perp]) \otimes \text{out}(q'). \quad (6)$$

► **Example 14.** structured words with timed symbols... intro language of music notation?
(markup = time division, leaves = events etc)

Every swA $A = \langle Q, \text{in}, w_1, \text{out} \rangle$, over Σ, \mathbb{S} and $\bar{\Phi}$ is a particular case of sw-VPA $\langle Q, \emptyset, \text{in}, \bar{w}, \text{out} \rangle$ over Ω, \mathbb{S} and $\bar{\Phi}$ with $\Omega_i = \Sigma$ and $\Omega_c = \Omega_r = \emptyset$, and computing with an always empty stack: $w_i^e = w_1$ and all the other functions of \bar{w} are the constant \emptyset .

4.2 Properties

Like VPA and symbolic VPA, the class of sw-VPA is closed under the binary operators of the underlying semiring.

► **Proposition 15.** Let A_1 and A_2 be two sw-VPA over the same Ω, \mathbb{S} and $\bar{\Phi}$. There exists two effectively constructible sw-VPA $A_1 \oplus A_2$ and $A_1 \otimes A_2$, such that for all $s \in \Omega^*$, $(A_1 \oplus A_2)(s) = A_1(s) \oplus A_2(s)$ and $(A_1 \otimes A_2)(s) = A_1(s) \otimes A_2(s)$.

Proof. The construction is essentially the same as in the case of the Boolean semiring [6]. ◀

4.3 Best Search

Let us assume that the semiring \mathbb{S} is commutative, bounded, and complete, and assume an effective label theory. We propose a Dijkstra algorithm computing, for a sw-VPA A over Ω, \mathbb{S} and $\bar{\Phi}$, the minimal weight for a word in Ω^* .

More precisely, let $b_\perp : Q \times Q \rightarrow \mathbb{S}$ be the function:

$$b_\perp(q, q') = \bigoplus_{s \in \Omega^*} \text{weight}_A(q[\perp], s, q'[\perp]) \quad (7)$$

Since \mathbb{S} is complete, the infinite sum in (7) is well defined, and, providing that \mathbb{S} is total, it is the minimum in Ω^* , wrt \leq_\oplus , of the function $s \mapsto \text{weight}_A(q[\perp], s, q'[\perp])$. The term

For all $q_0, q_3 \in Q$,

$$\begin{aligned}
d_{\top}(q_1, p, q_3) &\oplus= d_{\top}(q_1, p, q_2) \otimes \bigoplus_{\Omega_i} w_i(q_2, p, q_3) \\
d_{\perp}(q_1, p, q_3) &\oplus= d_{\perp}(q_1, q_2) \otimes \bigoplus_{\Omega_i} w_i^e(q_2, q_3) \\
d_{\top}(q_0, p, q_3) &\oplus= \bigoplus_{\Omega_c}^2 [(w_c(q_0, p, p', q_1) \otimes_2 d_{\top}(q_1, p', q_2)) \otimes_2 \bigoplus_{\Omega_r} w_r(q_2, p', q_3)] \\
d_{\perp}(q_0, q_3) &\oplus= \bigoplus_{\Omega_c} (w_c^e(q_0, p, q_1) \otimes d_{\top}(q_1, p, q_2) \otimes \bigoplus_{\Omega_r} w_r(q_2, p, q_3)) \\
d_{\perp}(q_1, q_3) &\oplus= d_{\perp}(q_1, q_2) \otimes \bigoplus_{\Omega_r} w_r^e(q_2, q_3) \\
d_{\top}(q_1, p, q_3) &\oplus= d_{\top}(q_1, p, q_2) \otimes d_{\top}(q_2, p, q_3), \text{ if } \langle q_2, \top, q_3 \rangle \notin P \\
d_{\perp}(q_1, q_3) &\oplus= d_{\perp}(q_1, q_2) \otimes d_{\perp}(q_2, q_3), \text{ if } \langle q_2, \perp, q_3 \rangle \notin P
\end{aligned}$$

■ **Figure 3** Update d_{\perp} with $\langle q_1, q_2 \rangle$ or d_{\top} with $\langle q_1, p, q_2 \rangle$.

365 $q[\perp], s, q'[\perp]$ of this sum is the central expression in the definition (6) of $A(s_0)$, for the
 366 minimum s_0 of the function weight_A .

367 Let \top be a fresh stack symbol which does not belong to Γ , and let $b_{\top} : Q \times P \times Q \rightarrow \Phi_c$ be
 368 such that, for every two states $q, q' \in Q$ and stack symbol $p \in P$:

$$369 \quad b_{\top}(q, p, q') : c \mapsto \bigoplus_{s \in \Omega^*} \text{weight}_A(q \left[\begin{array}{c} \langle c, p \rangle \\ \top \end{array} \right], s, q' \left[\begin{array}{c} \langle c, p \rangle \\ \top \end{array} \right]). \quad (8)$$

370 Intuitively, the function defined in (8) associates to $c \in \Omega_c$ the minimum weight of a
 371 computation of A starting in state q with a stack $cp \cdot \gamma \in \Gamma^+$ and ending in state q' with the
 372 same stack, such that the computation does pop the pair made of c and p at the top of this
 373 stack, but may read these symbols. Moreover, A may push another pair $\langle c', p' \rangle$ on the top of
 374 $cp \cdot \gamma$, following the the third case of in the definition (5) of weight_A , and may pop $\langle c', p' \rangle$
 375 later, following the fifth case of (5) (return symbol).

376 Algorithm 1 constructs iteratively markings $d_{\perp} : Q \times Q \rightarrow \mathbb{S}$ and $d_{\top} : Q \times P \times Q \rightarrow \Phi_c$
 377 that converges eventually to b_{\top} and b_{\perp} .

■ **Algorithm 1** Best search for sw-VPA

initially let $\mathcal{Q} = (Q \times Q) \cup (Q \times P \times Q)$, and let $d_{\perp}(q_1, q_2) = d_{\top}(q_1, p, q_2) = \mathbb{1}$ if
 $q_1 = q_2$ and $d_{\perp}(q_1, q_2) = d_{\top}(q_1, p, q_2) = \mathbb{0}$ otherwise
while $\mathcal{Q} \neq \emptyset$ **do**
 extract $\langle q_1, q_2 \rangle$ or $\langle q_1, p, q_2 \rangle$ from \mathcal{Q} such that $d_{\perp}(q_1, q_2)$, resp.
 $\bigoplus_{c \in \Omega_c} d_{\top}(q_1, p, q_2)(c)$, is minimal in \mathbb{S} wrt \leq_{\oplus}
 update d_{\perp} with $\langle q_1, q_2 \rangle$ or d_{\top} with $\langle q_1, p, q_2 \rangle$ (Figure 3).

378 The infinite sums in the updates of d in Algorithm 1, Figure 3 are well defined since \mathbb{S}
 379 is complete. ** effectively computable by hypothese that the label theory is effective**
 380 The algorithm performs $2 \cdot |Q|^2$ iterations until P is empty, and each iteration has a time
 381 complexity $O(|Q|^2 \cdot |P|)$. This gives a time complexity $O(|Q|^4 \cdot |P|)$. It can be reduced by
 382 implementing P as a priority queue, prioritized by the value returned by d ***complete***.

383 The correctness of Algorithm 1 is ensured by the invariant expressed in the following
 384 lemma.

XX:12 Symbolic Weighted Language Models and Parsing over Infinite Alphabets

385 ► **Lemma 16.** For all $\langle q_1, q_2 \rangle \notin Q$, $d_\perp(q_1, q_2) = b_\perp(q_1, q_2)/$

386 The proof is by contradiction, assuming a counter-example minimal in the length of the
387 witness word.

388 ► **Lemma 17.** For all $\langle q_1, p, q_2 \rangle \notin Q$, $d_\top(q_1, p, q_2) = b_\top(q_1, p, q_2)$,

389 For computing the minimal weight of a computation of A , we use the fact that, at the
390 termination of Algorithm 1,

$$391 \quad \bigoplus_{s \in \Omega^*} A(s) = \bigoplus_{q, q' \in Q} \text{in}(q) \otimes d_\perp(q, q') \otimes \text{out}(q').$$

392 In order to obtain effectively a witness (word of Ω^* with a computation of A of minimal
393 weight), we require the additional property of convexity of weight functions.

394 ► **Proposition 18.** For a *sw-VPA* A over Ω , \mathbb{S} commutative, bounded, total and complete,
395 and $\bar{\Phi}$ effective and k -convex, one can construct in *PTIME* a word $s \in \Omega^*$ such that $A(s)$ is
396 minimal wrt the natural ordering for \mathbb{S} .

397 4.4 Nested-Words and Parse-Trees

398 The hierarchical structure of nested-words, defined with the *call* and *return* markup symbols
399 suggest a correspondence with trees. The lifting of this correspondence to languages, of tree
400 automata and VPA, has been discussed in [1], and [4] for the weighted case. In this section,
401 we describe a correspondence between the symbolic-weighted extensions of tree automata
402 and VPA.

403 Let Ω be a countable ranked alphabet, such that every symbol $a \in \Omega$ has a rank
404 $\text{rk}(a) \in [0..M]$ where M is a fixed natural number. We denote by Ω_k the subset of all symbols
405 a of Ω with $\text{rk}(a) = k$, where $0 \leq k \leq M$, and $\Omega_{>0} = \Omega \setminus \Omega_0$. The free Ω -algebra of finite,
406 ordered, Ω -labeled trees is denoted by $\mathcal{T}(\Omega)$. It is the smallest set such that $\Omega_0 \subset \mathcal{T}(\Omega)$
407 and for all $1 \leq k \leq M$, all $a \in \Omega_k$, and all $t_1, \dots, t_k \in \mathcal{T}(\Omega)$, $a(t_1, \dots, t_k) \in \mathcal{T}(\Omega)$. Let us
408 assume a commutative semiring \mathbb{S} and a label theory $\bar{\Phi}$ over \mathbb{S} containing one set Φ_{Ω_k} for
409 each $k \in [0..M]$.

410 ► **Definition 19.** A symbolic-weighted tree automaton (*swTA*) over Ω , \mathbb{S} , and $\bar{\Phi}$ is a triplet
411 $A = \langle Q, \text{in}, \bar{w} \rangle$ where Q is a finite set of states, $\text{in} : Q \rightarrow \Phi_\Omega$ is the starting weight function,
412 and \bar{w} is a tuple of transition functions containing, for each $k \in [0..M]$, the functions
413 $w_k : Q \times Q^k \rightarrow \Phi_{\Omega_{>0}, \Omega_k}$ and $w_k^e : Q \times Q^k \rightarrow \Phi_{\Omega_k}$.

414 We define a transition function $w : Q \times (\Omega_{>0} \cup \{\varepsilon\}) \times \Omega \times \bigcup_{k=0}^M Q^k \rightarrow \mathbb{S}$ by:

$$415 \quad \begin{aligned} w(q_0, a, b, q_1 \dots q_k) &= \eta(a, b) & \text{where } \eta &= w_k(q_0, q_1 \dots q_k) \\ w(q_0, \varepsilon, b, q_1 \dots q_k) &= \phi(b) & \text{where } \phi &= w_k^e(q_0, q_1 \dots q_k). \end{aligned}$$

416 where $q_1 \dots q_k$ is ε if $k = 0$. The first case deals with a strict subtree, with a parent node
417 labeled by a , and the second case is for a root tree.

418 Every swTA defines a mapping from trees of $\mathcal{T}(\Omega)$ into \mathbb{S} , based on the following intermediate
419 function $\text{weight}_A : Q \times (\Omega \cup \{\varepsilon\}) \times \mathcal{T}(\Omega) \rightarrow \mathbb{S}$

$$420 \quad \text{weight}_A(q_0, a, t) = \bigoplus_{q_1 \dots q_k \in Q^k} w(q_0, a, b, q_1 \dots q_k) \otimes \bigotimes_{i=1}^k \text{weight}_A(q_i, b, t_i) \quad (9)$$

where $q_0 \in Q$, $a \in \Omega_{>0} \cup \{\varepsilon\}$ and $t = b(t_1, \dots, t_k) \in \mathcal{T}(\Omega)$, $0 \leq k \leq M$.

Finally, the weight associated by A to $t \in \mathcal{T}(\Omega)$ is

$$A(t) = \bigoplus_{q \in Q} \text{in}(q) \otimes \text{weight}_A(q, \varepsilon, t) \quad (10)$$

Intuitively, $w(q_0, a, b, q_1 \dots q_k)$ can be seen as the weight of a production rule $q_0 \rightarrow b(q_1, \dots, q_k)$ of a regular tree grammar [5], that replaces the non-terminal symbol q_0 by $b(q_1, \dots, q_k)$, provided that the parent of q_0 is labeled by a (or q_0 is the root node if $a = \varepsilon$). The above production rule can also be seen as a rule of a weighted CF grammar, of the form $[a, b] q_0 := q_1 \dots q_k$ if $k > 0$, and $[a] q_0 := b$ if $k = 0$. In the first case, b is a label of the rule, and in the second case, it is a terminal symbol. And in both cases, a is a constraint on the label of rule applied on the parent node in the derivation tree. This features of observing the parent's label are useful in the case of infinite alphabet, where it is not possible to memorize a label with the states. The weight of a labeled derivation tree t of the weighted CF grammar associated to A as above, is $\text{weight}_A(q, t)$, when q is the start non-terminal. We shall now establish a correspondence between such derivation tree t and some word describing a linearization of t , in a way that $\text{weight}_A(q, t)$ can be computed by a sw-VPA.

Let $\hat{\Omega}$ be the countable (unranked) alphabet obtained from Ω by: $\hat{\Omega} = \Omega_i \uplus \Omega_c \uplus \Omega_r$, with $\Omega_i = \Omega_0$, $\Omega_c = \{ \langle a \mid a \in \Omega_{>0} \rangle \}$, $\Omega_r = \{ \langle a \rangle \mid a \in \Omega_{>0} \}$.

We associate to $\hat{\Omega}$ a label theory $\hat{\Phi}$ like in Section 4.1, and we define a linearization of trees of $\mathcal{T}(\Omega)$ into words of $\hat{\Omega}^*$ as follows:

$$\begin{aligned} \text{lin}(a) &= a \text{ for all } a \in \Omega_0, \\ \text{lin}(b(t_1, \dots, t_k)) &= \langle_b \text{lin}(t_1) \dots \text{lin}(t_k) \rangle_b \text{ when } b \in \Omega_k \text{ for } 1 \leq k \leq M. \end{aligned}$$

► **Proposition 20.** *For all swTA A over Ω , \mathbb{S} commutative, and $\bar{\Phi}$, there exists an effectively constructible sw-VPA A' over $\hat{\Omega}$, \mathbb{S} and $\hat{\Phi}$ such that for all $t \in \mathcal{T}(\Omega)$, $A'(\text{lin}(t)) = A(t)$.*

Proof. Let $A = \langle Q, \text{in}, \bar{w} \rangle$ where \bar{w} is presented as above by a function. We build $A' = \langle Q', P', \text{in}', \bar{w}', \text{out}' \rangle$, where $Q' = \bigcup_{k=0}^M Q^k$ is the set of sequences of state symbols of A , of length at most M , including the empty sequence denoted by ε , and where $P' = Q'$ and \bar{w} is defined by:

$$\begin{aligned} w_i(q_0 \bar{u}, \langle_c \bar{p}, a, \bar{u} \rangle) &= w(q_0, c, a, \varepsilon) && \text{for all } c \in \Omega_{>0}, a \in \Omega_0 \\ w_i^e(q_0 \bar{u}, a, \bar{u}) &= w(q_0, \varepsilon, a, \varepsilon) && \text{for all } a \in \Omega_0 \\ w_c(q_0 \bar{u}, \langle_c \bar{p}, \langle_d \bar{u}, \bar{q} \rangle \rangle) &= w(q_0, c, d, \bar{q}) && \text{for all } c, d \in \Omega_{>0} \\ w_c^e(q_0 \bar{u}, \langle_c \bar{u}, \bar{q} \rangle) &= w(q_0, \varepsilon, c, \bar{q}) && \text{for all } c \in \Omega_{>0} \\ w_r(\varepsilon, \langle_c \bar{p}, c \rangle, \bar{p}) &= 1 && \text{for all } c \in \Omega_{>0} \\ w_r^e(\bar{u}, c, \bar{q}) &= 0 && \text{for all } c \in \Omega_{>0} \end{aligned}$$

All cases not matched by one of the above equations have a weight 0, for instance $w_r(\bar{u}, \langle_c \bar{p}, d \rangle, \bar{q}) = 0$ if $c \neq d$ or $\bar{u} \neq \varepsilon$ or $\bar{q} \neq \bar{p}$. ◀ ◀

5 Symbolic Weighted Parsing

Let us now apply the models and results of the previous sections to the problem of parsing over infinite alphabet. Let Σ be a countable input alphabet and Ω be a countable output ranked alphabet, with maximal rank value M , and let $\hat{\Omega} = \Omega_i \uplus \Omega_c \uplus \Omega_r$ be the alphabet with nesting symbols associated to Ω like in Section 4.4, for the linearization of trees of $\mathcal{T}(\Omega)$. Let $\langle \mathbb{S}, \oplus, \otimes, 0, \mathbb{1} \rangle$ be a commutative, bounded, and complete semiring and let $\bar{\Phi}$ be an effective

total?

XX:14 Symbolic Weighted Language Models and Parsing over Infinite Alphabets

label theory over \mathbb{S} , containing Φ_Σ , Φ_{Σ, Ω_i} , as well as Φ_i , Φ_c , Φ_r , Φ_{cr} (following the notations of Section 4.1). We assume given the following input:

- a swT T over Σ , Ω_i , \mathbb{S} , and $\bar{\Phi}$, defining a measure $T : \Sigma^* \times \Omega_i^* \rightarrow \mathbb{S}$,
- a swTA A over Ω , \mathbb{S} , and $\bar{\Phi}$, defining a measure $A : \mathcal{T}(\Omega) \rightarrow \mathbb{S}$,
- an input word $s \in \Sigma^*$.

As explained in Section 4.4, $A = \langle Q, \text{in}, \bar{w} \rangle$ can be seen as a weighted regular tree grammar, or a labeled weighted CF grammar whose derivation trees are in $\mathcal{T}(\Omega)$, with weight defined by A . The purpose of the transducer T is to measure a distance between input and output words, *i.e.* between the word s and the sequence leaves of a derivation tree, in Ω_i^* .

We extend the measure defined by T to $d : \Sigma^* \times \mathcal{T}(\Omega) \rightarrow \mathbb{S}$ as follows. Given a word $w \in \hat{\Omega}^*$, the projection of w onto Ω_i , denoted $w|_{\Omega_i}$, is the word of Ω_i^* obtained from w by removing all symbols in $\hat{\Omega} \setminus \Omega_i$. Using this notation and the tree linearization operator defined in Section 4.4, d is defined by:

$$d(s, t) = T(s, \text{lin}(t)|_{\Omega_i}) \text{ for } s \in \Sigma^*, t \in \mathcal{T}(\Omega) \quad (11)$$

Symbolic weighted parsing is the problem, given the above input, to find a tree $t \in \mathcal{T}(\Omega)$ minimizing $d(s, t) \otimes A(t)$ wrt \leq_\oplus , *i.e.* such that:

$$d(s, t) \otimes A(t) = \bigoplus_{t' \in \mathcal{T}(\Omega)} d(s, t') \otimes A(t') \quad (12)$$

The measure expressed in (12) is called the edit-distance between s and A in [21]. The problem of searching, in a WTA language, the best parse tree matching a given input, sometimes referred as *weighted parsing*, corresponds to SW parsing in the case of finite alphabets and when the transducer T characterizes identity see *e.g.* [13] and [23] for a more general weighted parsing framework.

► **Proposition 21.** *The problem of Symbolic Weighted parsing can be solved in PTIME in the size of the input swT T , swTA A and input word s , and the computation time of the functions of the label theory.*

Proof. (sketch) We follow a *Bar-Hillel* construction, also called parsing by intersection. We first extend the swT T over Σ , Ω_i , \mathbb{S} , and $\bar{\Phi}$, into a swT T' over Σ and $\hat{\Omega}$ (and the same semiring and label theory), such that for all $s \in \Sigma^*$, and $u \in \hat{\Omega}^*$, $T'(s, u) = T(s, u|_{\Omega_i})$. The transducer T' simply skips every symbol $b \in \hat{\Omega} \setminus \Omega_i$, by the addition to the transition of T , of new transitions of the form $w_{01}(q, \varepsilon, b, q')$.

Then, given an input word $s \in \Sigma^*$, we compute the swA $A_{T',s}$, using Proposition 11. This automaton is such that for all $t \in \mathcal{T}(\Omega)$,

$$A_{T',s}(\text{lin}(t)) = T'(s, \text{lin}(t)) = T'(s, \text{lin}(t)|_{\Omega_i}) = d(s, t).$$

Next, we convert the input swTA A over Ω into a sw-VPA A' over $\hat{\Omega}$, using Proposition 20, and we compute the sw-VPA $A_{T',s} \otimes A'$, using Proposition 15.

It remains to compute a best nested-word $w \in \hat{\Omega}^*$ using the best-search procedure of Proposition 18, and convert it into a best tree in $\mathcal{T}(\Omega)$ in order to solve SW parsing for T , A and s . ◀ ◀

5.0.0.1 Application to Automated Music Transcription.

...

Conclusion

We have introduced weighted language models (SW transducers and visibly pushdown automata) computing over infinite alphabets, and applied them to the problem of parsing with infinitely many possible input symbols (typically timed events). This approach extends conventional parsing and weighted parsing by computing a derivation tree modulo a generic distance between words, defined by a SW transducer given in input. This enables to consider finer word relationships than strict equality, opening possibilities of quantitative analysis via this method.

Ongoing and future work include

- The study of other theoretical properties of SW models, such as the extension of the best search algorithm from 1-best to n -best [17], and to k -closed semirings [20] (instead of *bounded*, which corresponds to 0-closed).
- ...there is room to improve the complexity bounds for the algorithms ... modular approach with oracles ...
- present here an offline algorithm for best search, semi-online implementation for AMT (bar-by-bar approach) with an on-the-fly automata construction.

References

- 1 R. Alur and P. Madhusudan. Adding nesting structure to words. *Journal of the ACM (JACM)*, 56(3):1–43, 2009.
- 2 M. Bojańczyk, C. David, A. Muscholl, T. Schwentick, and L. Segoufin. Two-variable logic on data words. *ACM Transactions on Computational Logic (TOCL)*, 12(4):1–26, 2011.
- 3 P. Bouyer, A. Petit, and D. Thérien. An algebraic approach to data languages and timed languages. *Information and Computation*, 182(2):137–162, 2003.
- 4 M. Caralp, P.-A. Reynier, and J.-M. Talbot. Visibly pushdown automata with multiplicities: finiteness and k -boundedness. In *International Conference on Developments in Language Theory*, pages 226–238. Springer, 2012.
- 5 H. Comon, M. Dauchet, R. Gilleron, F. Jacquemard, C. Löding, D. Lugiez, S. Tison, and M. Tommasi. *Tree Automata Techniques and Applications*. <http://tata.gforge.inria.fr>, 2007.
- 6 L. D’Antoni and R. Alur. Symbolic visibly pushdown automata. In *International Conference on Computer Aided Verification*, pages 209–225. Springer, 2014.
- 7 L. D’Antoni and M. Veanes. The power of symbolic automata and transducers. In *International Conference on Computer Aided Verification*, pages 47–67. Springer, 2017.
- 8 L. D’Antoni and M. Veanes. Automata modulo theories. *Communications of the ACM*, 64(5):86–95, 2021.
- 9 E. W. Dijkstra. A note on two problems in connexion with graphs. *Numerische Mathematik*, 1(1):269–271, 1959.
- 10 M. Droste and W. Kuich. Semirings and formal power series. In *Handbook of Weighted Automata*, pages 3–28. Springer, 2009.
- 11 M. Droste, W. Kuich, and H. Vogler. *Handbook of weighted automata*. Springer Science & Business Media, 2009.
- 12 F. Foscari, F. Jacquemard, P. Rigaux, and M. Sakai. A Parse-based Framework for Coupled Rhythm Quantization and Score Structuring. In *Mathematics and Computation in Music (MCM)*, volume 11502 of *Lecture Notes in Artificial Intelligence*, Madrid, Spain, 2019. Springer.
- 13 J. Goodman. Semiring parsing. *Computational Linguistics*, 25(4):573–606, 1999.
- 14 E. Gould. *Behind Bars: The Definitive Guide to Music Notation*. Faber Music, 2011.
- 15 D. Grune and C. J. Jacobs. *Parsing Techniques*. Number 2nd edition in Monographs in Computer Science. Springer, 2008.

- 545 **16** L. Huang. Advanced dynamic programming in semiring and hypergraph frameworks. In *In*
546 *COLING*, 2008.
- 547 **17** L. Huang and D. Chiang. Better k-best parsing. In *Proceedings of the Ninth International*
548 *Workshop on Parsing Technology*, Parsing '05, pages 53–64, Stroudsburg, PA, USA, 2005.
549 Association for Computational Linguistics.
- 550 **18** M. Kaminski and N. Francez. Finite-memory automata. *Theor. Comput. Sci.*, 134:329–363,
551 November 1994.
- 552 **19** S. Lombardy and J. Sakarovitch. The removal of weighted ε -transitions. In *International*
553 *Conference on Implementation and Application of Automata*, pages 345–352. Springer, 2012.
- 554 **20** M. Mohri. Semiring frameworks and algorithms for shortest-distance problems. *Journal of*
555 *Automata, Languages and Combinatorics*, 7(3):321–350, 2002.
- 556 **21** M. Mohri. Edit-distance of weighted automata: General definitions and algorithms. *Interna-*
557 *tional Journal of Foundations of Computer Science*, 14(06):957–982, 2003.
- 558 **22** M. Mohri. Edit-distance of weighted automata: General definitions and algorithms. *Interna-*
559 *tional Journal of Foundations of Computer Science*, 14(06):957–982, 2003.
- 560 **23** R. Mörbitz and H. Vogler. Weighted parsing for grammar-based language models. In *Proceedings*
561 *of the 14th International Conference on Finite-State Methods and Natural Language Processing*,
562 pages 46–55, Dresden, Germany, Sept. 2019. Association for Computational Linguistics.
- 563 **24** M.-J. Nederhof. Weighted deductive parsing and Knuth’s algorithm. *Computational Linguistics*,
564 29(1):135–143, 2003.
- 565 **25** F. Neven, T. Schwentick, and V. Vianu. Finite state machines for strings over infinite alphabets.
566 *ACM Trans. Comput. Logic*, 5(3):403–435, July 2004.
- 567 **26** L. Segoufin. Automata and logics for words and trees over an infinite alphabet. In *Computer*
568 *Science Logic*, volume 4207 of *LNCS*. Springer, 2006.
- 569 **27** M. Y. Vardi. Linear-time model checking: automata theory in practice. In *International*
570 *Conference on Implementation and Application of Automata*, pages 5–10. Springer, 2007.