MTH 9821 Numerical Methods for Finance

Fall 2017

Homework 5

Assigned: Oct 8; Due: Oct 16

This homework is to be done as a group. Each team will hand in one homework solution, and each member of the team should write at least one problem. On the cover page of the homework, please indicate the members of the team and who wrote each problem.

(1) **An alternative route to the heat equation**

Let $C(S,t)$ denote the price of a European vanilla option at time t given that the stock price is $S_t = S$.

Consider the risk-neutral dynamics of the stock:

$$S_t = S_0 \exp\left\{ \left(r - \frac{\sigma^2}{2} \right) t + \sigma W_t \right\}$$

and let $V(W,t)$ be the price of the derivative as a function of the Brownian motion ("log-return" on the stock) *expressed in units of a zero coupon bond maturing at time* T:

$$V(W_t,t) = \frac{C(S_t,t)}{B_t^T}$$

where $B_t^T = e^{r(T-t)}$. Finally, let $\tau = T - t$, be the time to maturity.

(a) Express W, τ, V as a function of S, t, C.

(b) In the Black-Scholes model, $C(S,t)$ satisfies the following equation:

$$C_t + \frac{1}{2}\sigma^2 S^2 C_{SS} + rSC_S - rC = 0$$

Rewrite this equation into W, τ, V coordinates using the transformation found in 1a.

(c) Write down the boundary condition for a put option struck at K.

(d) Compare this coordinate change with the one presented in the lecture: which one is easier to implement/remember/think about/etc.?

(2) **Model-free Gamma**

Forget about the Black-Scholes model for this question.

Let $C(S,K)$ be the price of a European vanilla call/put struck at K, given that the spot price is S.

Suppose that $C(S,K)$ is a *homogeneous function of degree one*, that is to say:

$$C(\alpha S, \alpha K) = \alpha C(S,K)$$

Intuitively, this means that the formula C is independent of units.

(a) Derive, by differentiating the above formula with respect to α, S and K an appropriate number of times, an expression for $\Gamma = C_{SS}$ in terms of S, K and C_{KK}.

(b) Assume lack of butterfly arbitrage to show that $\Gamma \geq 0$.

(c) Explain, using the previous part, how you would take advantage of the arbitrage opportunity presented by an option with negative Γ.

(3) **Black-Scholes with dividends**

Suppose that you only know the Black-Scholes formula for non-dividend paying stocks and you're trying to generalise it to a stock which pays dividends continuously at a rate q year^{-1}.

Let F_t^T be the discounted price of the (traded) forward contract on the stock, maturing at time T. Dividends are not included in the future delivery, therefore:

$$F_t^T = S_t e^{-q(T-t)}$$

(a) Write down the usual log-normal stochastic differential equation (SDE) for the dividend paying stock S. Use Itô's lemma to obtain the SDE for F_t^T.

(b) A European vanilla option on the stock can now also be treated as an option on this forward contract. Let $C(S,t)$ be the value of the option at time t given that $S_t = S$, and $V(F,t)$ be the value of the option at time t given that $F_t^T = F$. Clearly

$$C(S_t, t) = V(F_t^T, t)$$

Since F_t^T doesn't pay dividends, we see that V satisfies the Black-Scholes PDE:

$$V_t + \frac{1}{2}\sigma^2 F^2 V_{FF} + rFV_F - rV = 0$$

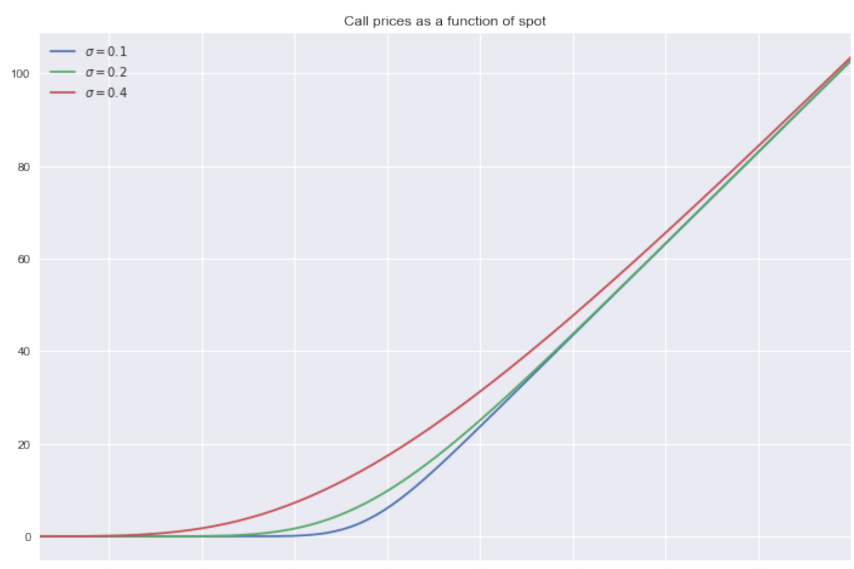
Rewrite this equation in terms of C and S and verify that you get the Black-Scholes PDE for a dividend paying asset.

(4) In this question you are asked to produce 10 plots illustrating how the value, delta and gamma of a plain vanilla option varies with spot, time to maturity and volatility.

You may use any software to produce these plots. Pay attention to the following aesthetics:

- Use 3 colours for the 3 curves in each plot and add a legend showing which colour corresponds to which volatility/maturity.
- Place "ticks" on both the x and y axis

For instance, the first plot should look similar to this:



Consider an option struck at $K = 100\,\$$ maturing in a year, on a stock with 20% yearly volatility. The risk-free rate is 5% and the stock pays dividends continuously at a rate of 1%.

All curves should be plotted with spot in the range $S_0 \in [25, 200]$ and each plot should contain 3 curves:

- Plot call prices with $\sigma = 0.1, 0.2, 0.4$
- Plot put prices with $\sigma = 0.1, 0.2, 0.4$
- Plot call prices with $T = 0.25, 1, 4$
- Plot put prices with $T = 0.25, 1, 4$
- Plot call deltas with $\sigma = 0.1, 0.2, 0.4$
- Plot put deltas with $\sigma = 0.1, 0.2, 0.4$
- Plot call deltas with $T = 0.25, 1, 4$
- Plot put deltas with $T = 0.25, 1, 4$
- Plot gammas with $\sigma = 0.1, 0.2, 0.4$
- Plot gammas with $T = 0.25, 1, 4$

All other parameters should be kept the same as above.

Place your plots in the spreadsheet provided.