# MTH 9821 Numerical Methods for Finance

Fall 2017

**Homework 11**
Assigned: November 30; Due: December 7

This homework is to be done as a group. Each team will hand in one homework solution, using the blueprint solution provided by our teaching assistant.

The homework has the following parts:

1, Pricing European Options on Assets Paying Discrete Dividends using Trees

2. Pricing European Options on Assets Paying Discrete Dividends using Monte Carlo

3. Pricing European Options on Assets Paying Discrete Dividends using Finite Differences

## Parametrizations for the tree methods

Let $\delta t = \frac{T}{N}$, where $T$ is the given maturity of an option and $N$ is the specified number of time steps in the tree model.

**Binomial trees parametrization:**

$$u \;=\; e^{\sigma\sqrt{\delta t}}; \quad d \;=\; e^{-\sigma\sqrt{\delta t}}; \quad p \;=\; \frac{e^{(r-q)\delta t}-d}{u-d}. \tag{1}$$

## Binomial Tree Methods for Discrete–Dividend–Paying Assets

Consider an asset following a lognormal process with spot price \$50, volatility 30%, and paying discrete dividends every two months, beginning two months from now. The risk free interest rate is constant at 2%. Use the binomial trees parametrization (1).

(i) A three months European put option and a three months American put option, both 10% in–the–money (i.e., with strike $K = 55.55$). The dividend is proportional, equal to 1% of the spot price. Find the value of both options, as well as the values of the Delta, Gamma, and Theta of the options (do not use an extended tree). Begin with a tree with $N = 6$ time steps and double the number of time steps until two consecutive option values are within $tol = 10^{-4}$ of each other.

Do the same for seven months put options. Begin with a tree with $N = 7$ time steps.

(ii) A three months European put option and a three months American put option, both 10% in–the–money (i.e., with strike $K = 55.55$). The dividend is fixed, equal to 50 cents. Find the value of both options, as well as the values of the Delta, Gamma, and Theta of the options. Begin with a tree with $N = 6$ time steps.

Do the same for seven months put options. Begin with a tree with $N = 7$ time steps and double the number of time steps until two consecutive option values are within $tol = 10^{-4}$ of each other.

(iii) A seven months European put option and a seven months American put option, both 10% in–the–money (i.e., with strike $K = 55.55$). The first dividend is equal to 50 cents and is paid in two months, the second dividend is 1% of the spot price in four months, and the final dividend is 75 cents and is paid in six months. Find the value of both options, as well as the values of the Delta, Gamma, and Theta of the options. Begin with a tree with $N = 7$ time steps and double the number of time steps until two consecutive option values are within $tol = 10^{-4}$ of each other.

**Generating random numbers for this homework**

Generate $N$ independent samples from the standard normal distribution as follows:

Generate $N_0$ independent samples from the uniform distribution on $[0,\ 1]$ by using the Linear Congruential Generator

$$
\begin{aligned}
x_{i+1} &= ax_i + c \pmod{k} \\
u_{i+1} &= \frac{x_{i+1}}{k},
\end{aligned}
$$

with $x_0 = 1$, $a = 39373$, $c = 0$, and $k = 2^{31} - 1$ to generate $u_1,\ u_2,\ \ldots,\ u_{N_0}$.

Use the Box–Muller Method to generate independent samples from the standard normal distribution using the Marsaglia–Bray algorithm below:

while $X > 1$
  Generate $u_1, u_2 \in U([0,1])$
  $u_1 = 2u_1 - 1$; $u_2 = 2u_2 - 1$
  $X = u_1^2 + u_2^2$
end
$Y = \sqrt{-2\frac{\ln(X)}{X}}$
$Z_1 = u_1 Y; \quad Z_2 = u_2 Y$
return $Z_1,\ Z_2$

Use the samples $u_i$, $i = 1 : N_0$, from the uniform distribution generated previously. Make sure $N_0$ is large enough in order to generate the required number $N$ independent samples of the standard normal distribution.

**Monte Carlo Methods for Discrete–Dividend–Paying Assets**

Consider a seven months European put option with strike $K = 55.55$ on an asset paying discrete dividends. The first dividend is equal to 50 cents and is paid in two months, the second dividend is 1% of the spot price in four months, and the final dividend is 75 cents and is paid in six months.

The current spot is \$50, volatility 30% and the risk free rate is constant at 2%.

We simulate the risk neutral random path of the asset on $n$ different paths, each one discretized by 4 time steps corresponding to the ex-dividend dates. To do this, $N = 4n$ independent samples of the standard normal distribution must be generated.

Each path can be simulated as follows:

$$\begin{aligned}
S_i(t_1) &= S(0)e^{(r-\sigma^2/2)t_1+\sigma\sqrt{t_1}Z_{4i+1}} - 0.5 \\
S_i(t_2) &= S_i(t_1)e^{(r-\sigma^2/2)(t_2-t_1)+\sigma\sqrt{t_2-t_1}Z_{4i+2}}(1-0.01) \\
S_i(t_3) &= S_i(t_2)e^{(r-\sigma^2/2)(t_3-t_2)+\sigma\sqrt{t_3-t_2}Z_{4i+3}} - 0.75 \\
S_i(T) &= S_i(t_3)e^{(r-\sigma^2/2)(T-t_3)+\sigma\sqrt{T-t_3}Z_{4i+4}}
\end{aligned}$$

where $t_1$, $t_2$, $t_3$ and $T$ correspond to 2, 4, 6 and 7 months respectively. Ignore the possibility (as above) of a dividend payment bankrupting the firm.

Approximate the value of the option by

$$\hat{P}(n) = \frac{1}{n}\sum_{i=0}^{n-1} P_i$$

where

$$P_i = e^{-rT}\max(K - S_i(T), 0)$$

To speed up convergence, use the same option on a non-dividend paying asset as a control variate.

$$\begin{aligned}
\widetilde{S}_i(T) &= S(0)e^{(r-\sigma^2/2)T+\sigma(\sqrt{t_1}Z_{4i+1}+\sqrt{t_2-t_1}Z_{4i+2}\sqrt{t_3-t_2}Z_{4i+3}+\sqrt{T-t_3}Z_{4i+4})} \\
\widetilde{P}_i(T) &= e^{-rT}\max(K - \tilde{S}_i(T), 0) \\
\widehat{\widetilde{P}}(n) &= \frac{1}{n}\sum_{i=0}^{n-1}\widetilde{P}_i(T)
\end{aligned}$$

Now define the new estimator $\widehat{P}_{CV}(n)$ as

$$\widehat{P}_{CV}(n) = \widehat{W}(n) = \frac{1}{n}\sum_{i=0}^{n-1} W_i$$

where

$$\begin{aligned}
W_i &= P_i - \widehat{b}(\widetilde{P}_i - \widetilde{P}_{BS}) \\
\widehat{b} &= \frac{\sum_{i=0}^{n-1}(\widetilde{P}_i - \widehat{\widetilde{P}})(P_i - \widehat{P}_i)}{\sum_{i=0}^{n-1}(\widetilde{P}_i - \widehat{\widetilde{P}})^2}
\end{aligned} \tag{2}$$

The Delta can be found as

$$\begin{aligned}
\Delta_i = \frac{dP_i}{dS_0} &= \frac{dP_i}{dS_i(T)}\frac{dS_i(T)}{dS_i(t_3)}\frac{dS_i(t_3)}{dS_i(t_2)}\frac{dS_i(t_2)}{dS_i(t_1)}\frac{dS_i(t_1)}{dS(0)} \\
&= -\mathbb{I}_{K>S_i(T)}\cdot e^{-rT}\frac{\widetilde{S}_i(T)}{S(0)}(1-0.01)
\end{aligned}$$

$$\widehat{\Delta}(n) = \frac{1}{n}\sum_{i=0}^{n-1}\Delta_i$$

Use again, as control variate, the Delta of the option on the non-dividend-paying version of the asset.

$$\widetilde{\Delta}_i = -\mathbb{I}_{K>\widetilde{S}_i(T)} \cdot e^{-rT}\frac{\widetilde{S}_i(T)}{S(0)}$$

And find the regression coefficient as in equation 2.

Carry out your calculations for $N = 4 \cdot 10,000 \cdot 2^k$ where $k = 0:8$ without variance reduction, and $k = 0:7$ with variance reduction. Comment on the convergence.

**European Options on Assets Paying Discrete Dividends**
**Finite Difference Hedging and Valuation**

We want to value a European call option on a stock with spot price $S_0 = 52$ and volatility $\sigma = 0.3$, with strike $K = 50$ and maturity one year, i.e., $T = 1$. The interest rate is assumed to be constant $r = 0.03$ over the life of the option, and the asset pays a dividend equal to $q = 2\%$ of the spot price at time $t_{div} = 5/12$, i.e., in five months.

The following change of variables transforms $x$ and $\tau$ into $S$ and $t$, respectively, and maps $V(S,t)$, the value of the call option, into $u(x,\tau)$, a solution to the heat equation:

$$V(S,t) \;=\; \exp(-ax - b\tau)u(x,\tau), \tag{3}$$

where

$$x \;=\; \ln\left(\frac{S}{K}\right); \quad \tau \;=\; \frac{(T-t)\sigma^2}{2},$$

and the constants $a$ and $b$ are given by

$$a \;=\; \frac{r}{\sigma^2} \;-\; \frac{1}{2}; \quad b \;=\; \left(\frac{r}{\sigma^2} + \frac{1}{2}\right)^2.$$

We will use finite difference methods to compute $u(x,\tau)$ approximately, as detailed below.

Let

$$\begin{aligned}\tau_{div} &= \frac{(T-t_{div})\sigma^2}{2};\\ \tau_{final} &= \frac{T\sigma^2}{2}.\end{aligned}$$

At time $t_{div}$, a dividend equal to $qS(t_{div}^-)$ will be paid. Here $S(t_{div}^-)$ is a shorthand for the left limit $\lim_{t\to t_{div}^-} S(t)$. By a no-arbitrage argument, the stock price drops by an amount equal to the paid dividend, therefore $S(t_{div}^-) \neq S(t_{div})$, and instead we have:

$$S(t_{div}) = (1-q)S(t_{div}^-) \tag{4}$$

In $\tau$ coordinates $\lim_{t\to t_{div}^-}$ corresponds to $\lim_{\tau\to\tau_{div}^+}$.

There is no change in the option value at time $t_{div}$:

$$\begin{aligned}V(S(t_{div}), t_{div}) &= V(S(t_{div}^-), t_{div}^-)\\ &= V\left(S(t_{div})/(1-q), t_{div}^-\right) \qquad \text{by (4)}\end{aligned}$$

therefore, we require that for all $s > 0$:

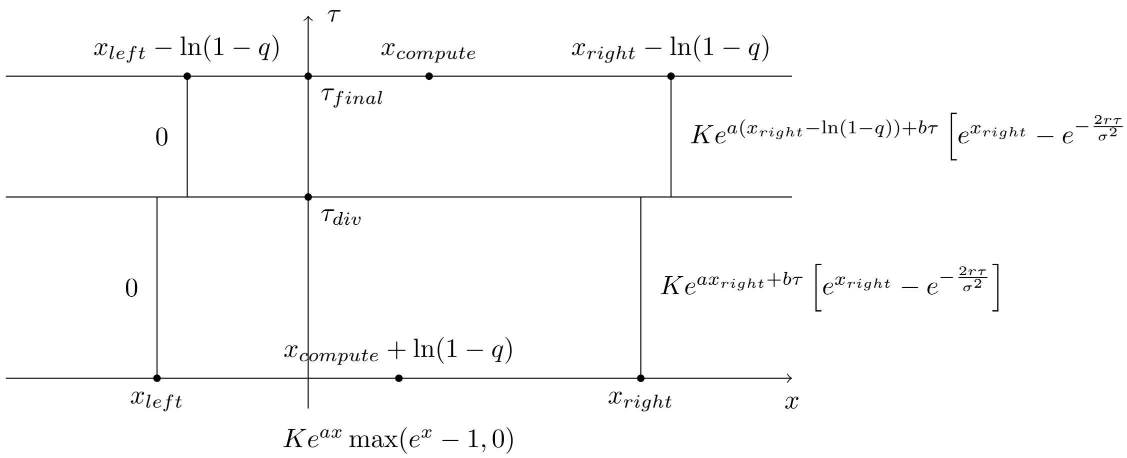
$$V(s, t_{div}^-) = V((1-q)s, t_{div}). \tag{5}$$

Converting equation (5) to $u, x, \tau$ coordinates we obtain the equivalent:

$$u(x, \tau_{div}^+) = (1-q)^{-a}u(x + \ln(1-q), \tau_{div}) \tag{6}$$

To price the call option, we solve the heat differential equation between time $\tau = 0$ and $\tau_{div}$. Then, the nodal values on the $x$-axis corresponding to time $\tau_{div}$ provide the boundary conditions at $\tau_{div}^+$ for the heat equation between $\tau_{div}^+$ and $\tau_{final}$. Note that, on the $x$-axis, the nodes are shifted to the right by $-\ln(1-q)$ when time changes from $\tau_{div}$ to $\tau_{div}^+$.

The boundary and boundary conditions are summarised in the figure below:




**Computational domain on the interval** $[0, \tau_{div}]$

One of the nodal values on the $x$–axis will be

$$\overline{x}_{compute} = \ln\left(\frac{S_0}{K}\right) + \ln(1-q).$$

The upper bound $\tau_{final}$ for $\tau$ is

$$\tau_{final} = \frac{T\sigma^2}{2}.$$

To choose the domain on the $x$-axis, fix the Courant constant $\alpha_1$. If $M_1$ is the number of time steps on the $\tau$-axis, then

$$\delta\tau_1 = \frac{\tau_{div}}{M_1}.$$

Therefore

$$\delta x = \sqrt{\frac{\delta\tau_1}{\alpha_1}},$$

since $\alpha_1 = \frac{\delta\tau_1}{(\delta x)^2}$.

Choose temporary left and right end points as follows:

$$\widetilde{x}_{left} = \ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T - 3\sigma\sqrt{T};$$
$$\widetilde{x}_{right} = \ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T + 3\sigma\sqrt{T}.$$

Let

$$N_{left} = \text{ceil}\left(\frac{\overline{x}_{compute} - \widetilde{x}_{left}}{\delta x}\right) \quad \text{and} \quad N_{right} = \text{ceil}\left(\frac{\widetilde{x}_{right} - \overline{x}_{compute}}{\delta x}\right),$$

where ceil$(y)$ is the smallest integer larger than or equal to $y$. Then

$$N = N_{left} + N_{right}$$

and

$$x_{left} = \overline{x}_{compute} - N_{left}\delta x \quad \text{and} \quad x_{right} = \overline{x}_{compute} + N_{right}\delta x.$$

The nodal points are

$$x_k = x_{left} + k\delta x, \quad \forall\ k = 0 : N. \tag{7}$$

Note that

$$x_{N_{left}} \;=\; x_{left} + N_{left}\delta x \;=\; \overline{x}_{compute} \;=\; \ln\left(\frac{S_0}{K}\right) + \ln(1-q). \tag{8}$$

**PDE to solve on the interval** $[0, \tau_{div}]$

We will solve the following heat PDE:

$$u_\tau \;=\; u_{xx}, \quad \forall\, x_{left} \;<\; x \;<\; x_{right}, \quad \forall\, 0 \;<\; \tau \;<\; \tau_{div},$$

with boundary conditions

$$\begin{aligned} u(x,0) &= K e^{ax} \max(e^x - 1, 0), \qquad \forall\, x_{left} \;\le\; x \;\le\; x_{right}; \\ u(x_{left}, \tau) &= 0, \qquad \forall\, 0 \;\le\; \tau \;\le\; \tau_{div}; \\ u(x_{right}, \tau) &= K e^{a x_{right} + b\tau}\left[e^{x_{right}} - e^{-\frac{2r\tau}{\sigma^2}}\right], \qquad \forall\, 0 \;\le\; \tau \;\le\; \tau_{div}. \end{aligned}$$

**Finite difference solvers on the interval** $[0,\ \tau_{div}]$

Use Forward Euler with $\alpha_1 = 0.4$ and Crank-Nicolson with $\alpha_1 = 4$ to solve the diffusion equation for $u(x,\tau)$. For Crank-Nicolson, use tridiagonal LU without pivoting to solve the resulting linear systems.

Choose $M_1 = 4$ to begin with. Repeat the algorithm for $M_1 \in \{4M_1, 16M_1, 64M_1\}$.

We denote by $U_1^{M_1}$ the vector of the nodal values corresponding to $u(x, \tau_{div})$. This vector will be used to calculate the boundary values at time $\tau_{div}$ of the PDE to be solved on the interval $(\tau_{div},\ \tau_{final}]$.

**Computational domain on the interval** $(\tau_{div},\ \tau_{final}]$

The number $N$ of nodes on the $x$–axis and the mesh size $\delta x$ will remain the same. However, the nodal values on the $x$–axis will shift to the right, as explained above.

Also, $\alpha$ will become slightly lower, and $M_2$ will be chosen accordingly.

$$\begin{aligned} x_{left,new} &= x_{left} - \ln(1-q) \quad \text{and} \\ x_{right,new} &= x_{right} - \ln(1-q) \end{aligned}$$

are the new end points of the computational domain on the $x$–axis corresponding to $\tau \in (\tau_{div}, \tau_{final}]$.

Note that the centre $\overline{x}_{compute}$ of the $x$-mesh on $[0, \tau_{div}]$ was chosen so that it shifts to $x_{compute} = \ln\left(\frac{S_0}{K}\right)$, thus centering the $x$-mesh on $(\tau_{div}, \tau_{final}]$ around the $x$ value corresponding to the spot price.

Denote by $\alpha_{temp}$ the Courant constant $\alpha_1$ used in the finite difference discretization of the domain corresponding to $\tau \in [0,\ \tau_{div}]$, i.e., let $\alpha_{temp} = \alpha_1$.

Then,

$$\delta\tau_{2,temp} \;=\; \alpha_{temp}(\delta x)^2$$

and[1]

$$M_2 \;=\; \text{ceil}\left(\frac{\tau_{final} - \tau_{div}}{\delta\tau_{2,temp}}\right). \tag{9}$$

[1] It is easy to see that $\delta\tau_{2,temp} = \delta\tau_1$ and

$$\frac{\tau_{final} - \tau_{div}}{\delta\tau_{2,temp}} \;=\; \frac{t_{div}}{T - t_{div}}.$$

Thus,

$$\delta\tau_2 \;=\; \frac{\tau_{final} - \tau_{div}}{M_2}$$

and

$$\alpha_2 \;=\; \frac{\delta\tau_2}{(\delta x)^2} \;<\; \alpha_{temp} \;=\; \alpha_1. \tag{10}$$

**PDE to solve on the interval** $(\tau_{div},\ \tau_{final}]$

We will solve the following heat PDE:

$$u_\tau \;=\; u_{xx},\;\; \forall\, x_{left,new} \;<\; x \;<\; x_{right,new},\;\; \forall\, \tau_{div} \;<\; \tau \;<\; \tau_{final},$$

with boundary conditions

$$\begin{aligned} u(x_{left,new}, \tau) &= 0,\;\; \forall\, \tau_{div} \;<\; \tau \;\leq\; \tau_{final}; \\ u(x_{right,new}, \tau) &= Ke^{a(x_{right} - \ln(1-q)) + b\tau}\left[e^{x_{right}} - e^{-\frac{2r\tau}{\sigma^2}}\right],\;\; \forall\, \tau_{div} \;<\; \tau \;\leq\; \tau_{final}. \end{aligned}$$

Recall that we denoted by $U_1^{M_1}$ the vector of the nodal values corresponding to $u(x, \tau_{div})$, and that by equation (6):

$$u(x - \ln(1-q), \tau_{div}^+) = (1-q)^{-a} u(x, \tau_{div})$$

We therefore choose the following boundary conditions on the interval $[x_{left,new}, x_{right,new}]$ at time $\tau_{div}^+$:

$$U_2^0(k) \;=\; (1-q)^{-a} U_1^{M_1}(k),\;\; \forall\, k = 0 : N.$$

Identify the computational domain i.e., compute and record, for each $M_1 \in \{4, 16, 64, 256\}$ and for $\alpha \in \{0.4, 4\}$ the following parameters describing the computational domain: $M_2$, $\alpha_2$, $N$, $x_{left}$, $x_{right}$, $x_{left,new}$, $x_{right,new}$, $\tau_{div}$, $\delta\tau_1$ and $\delta\tau_2$, $\delta x$.

**Finite difference solvers on the interval** $(\tau_{div},\ \tau_{final}]$

Use Forward Euler with $\alpha_1 = 0.4$ and Crank-Nicolson with $\alpha_1 = 4$ to solve the diffusion equation for $u(x, \tau)$. For Crank-Nicolson, use tridiagonal LU without pivoting to solve the resulting linear systems.

Choose $M_1 = 4$ to begin with. Repeat the algorithm for $M_1 \in \{4M_1, 16M_1, 64M_1\}$.

**Finite Difference Solvers:**

Recall that for the finite difference solution on the interval $[0, \tau_{div}]$ we used Forward Euler with $\alpha_1 = 0.4$ and Crank-Nicolson with $\alpha_1 = 4$. We chose $M_1 = 4$ to begin with and repeat the algorithm for $M_1 \in \{4M_1, 16M_1, 64M_1\}$.

For every $\alpha_1$ and $M_1$, we compute a Courant constant $\alpha_2 < \alpha_1$ and a number $M_2$ of intervals on the $(\tau_{div},\ \tau_{final}]$ interval using (10) and (9), respectively. We then use the same finite difference method on the interval $(\tau_{div},\ \tau_{final}]$ as was used on the interval $[0, \tau_{div}]$, i.e., either Forward Euler or Crank-Nicolson. (As before, use tridiagonal LU without pivoting for Crank-Nicolson to solve the resulting linear systems.)

**Pointwise Convergence and the Greeks:**

Let $U_2^{M_2}$ be the vector of length $N-1$ which gives the finite difference solution after $M_2$ time steps on the interval $(\tau_{div},\ \tau_{final}]$. Hence $U_2^{M_2}(N_{left})$ is the finite difference approximation to $u(x_{compute}, \tau_{final})$. The following change of variables computes the finite difference approximate value of the option:

$$V_{approx}(S_0, 0) \;=\; \exp(-ax_{compute} - b\tau_{final}) U_2^{M_2}(N_{left}). \tag{11}$$

Finite difference approximations for the $\Delta$, $\Gamma$, and $\Theta$ of the option can be obtained as follows: Let

$$\begin{aligned} S_{-1} &= K\exp(x_{N_{left}-1}) = K\exp(x_{compute} - \delta x); \\ S_0 &= K\exp(x_{N_{left}}) = K\exp(x_{compute}); \\ S_1 &= K\exp(x_{N_{left}+1}) = K\exp(x_{compute} + \delta x) \end{aligned}$$

be the values of $S$ corresponding to the nodes

$$\begin{aligned} x_{N_{left}-1} &= x_{compute} - \delta x; \\ x_{N_{left}} &= x_{compute}; \\ x_{N_{left}+1} &= x_{compute} + \delta x, \end{aligned}$$

respectively, and let

$$\begin{aligned} V_{-1} &= \exp(-ax_{N_{left}-1} - b\tau_{final})\, U^{M_2}(N_{left} - 1); \\ V_0 &= \exp(-ax_{N_{left}} - b\tau_{final})\, U^{M_2}(N_{left}); \\ V_1 &= \exp(-ax_{N_{left}=1} - b\tau_{final})\, U^{M_2}(N_{left} + 1) \end{aligned}$$

be the corresponding finite difference approximate values of the option.

The central difference approximations for the $\Delta$ and $\Gamma$ of the option are

$$\Delta_{central} = \frac{V_1 - V_{-1}}{S_1 - S_{-1}}; \tag{12}$$

$$\Gamma_{central} = \frac{(S_0 - S_{-1})V_1 \; - \; (S_1 - S_{-1})V_0 \; + \; (S_1 - S_0)V_{-1}}{(S_0 - S_{-1})(S_1 - S_0)((S_1 - S_{-1})/2))}. \tag{13}$$

To compute an approximation for $\Theta$, note that the next to last time step on the $\tau$–axis, $\tau_{final} - \delta\tau$ corresponds to time

$$\delta t = \frac{2\delta\tau}{\sigma^2}.$$

Let

$$V_{approx}(S_0, \delta t) = \exp(-ax_{N_{left}} - b(\tau_{final} - \delta\tau))\; u(x_{N_{left}}, \tau_{final} - \delta\tau).$$

The forward finite difference approximation of $\Theta = -\frac{\partial V}{\partial t}$ is

$$\Theta_{forward} \quad \frac{V_{approx}(S_0, 0) - V_{approx}(S_0, \delta t)}{\delta t}. \tag{14}$$

**Finite Difference Solution**

For each finite difference method, compute and record:

1. $U_2^{M_2}(N_{left})$ as "u value";
2. $V_{approx}(S_0, 0)$ given by (11) as "Option Value";
3. $\Delta_{central}$ given by (12);
4. $\Gamma_{central}$ given by (13);
5. $\Theta_{forward}$ given by (14).

To understand the numbers you provide, please include the following: for Forward Euler with $\alpha_1 = 0.4$, let $M_1 = 4$. Then $N = 15$ and $M_2 = 3$. Run your codes and record the values of the finite difference approximations at each nodes, including at the boundary nodes. You will have to fill out two tables:

- a table with five rows and 16 columns corresponding to the interval $[0, \tau_{div}]$;
- a table with four rows and 16 columns corresponding to the interval $(\tau_{div},\ \tau_{final}]$.