Holographic Entanglement Entropy as a Probe for Non-Relativistic Symmetry Breaking in the IR

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Holographic Principle

The holographic principle states that a gravity theory in $d+1$-dimension is equivalent to a d-dimensional gauge theory located at the boundary of the bulk spacetime in which the gravity theory lives.

- The extra dimension of the gravity side, the radial direction r, is related to the energy scale of the theory.
- The most important realization of the holographic principle is the AdS/CFT correspondence relating a gravity theory on bulk AdS_{d+1} to a CFT_d.

The gravity and field theory have the same symmetry. Particularly, a CFT_d is invariant under the symmetry group: $SO(d,2)$. This is consistent with the isometry of AdS_{d+1}:

$$t \to \lambda t, \quad x_i \to \lambda x_i, \quad r \to \lambda^{-1} r \tag{1}$$

Lifshitz Space

CFT are typically too symmetric to be consistent with standard model theories. They don't have a mass gap. However, one place where we do expect CFT to arise is at fixed point of the RG flow. An important class of theories arising at fixed point of many-body system is those with Lifshitz symmetry:

$$t \to \lambda^z t, \quad x_i \to \lambda x_i, \quad r \to \lambda^{-1} r \tag{2}$$

where $z \geq 1$ is some scaling constant. The energy dispersion for this theory can be obtained from this scaling symmetry as:

$$E = a\left|\vec{k}\right|^z$$

The simplest form of the geometry can also be inferred from the scaling symmetry to be:

$$ds^2_{d+1} = L^2 \left(-r^{2z} dt^2 + r^2 \delta_{ij} dx^i dx^j + \frac{dr^2}{r^2} \right) \tag{3}$$

Taking $z = 1$, we get the Poincaré patch of AdS. Therefore, we refer to $z = 1$ as the AdS and $z > 1$ as Lifshitz space.

Lifshitz Space

The Lifshitz spacetime ($z > 1$) has pp-curvature singularities (infinite tidal forces) at $r = 0$ (Horowitz, Way 2012). They are therefore geodesically incomplete. This motivates the introduction of a black hole/black brane geometry with an event horizon at $r = r_h > 0$ to shield this singularity.

Dilaton-Einstein-Maxwell Model

A model admitting a 'Lifshitz'-like solution is the Dilaton-Einstein-Maxwell model. The action is given by:

$$S = \frac{1}{16\pi G_{d+2}} \int d^{d+2}x\sqrt{-g}\left(R - 2\Lambda - 2(\nabla\phi)^2 - e^{2\alpha\phi}F^2\right) \tag{4}$$

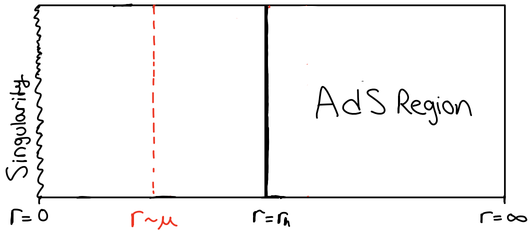
The action above contains a logarithmic running dilaton $1/(4g_{YM}^2) = e^{2\alpha\phi}$. The above action admits both pure AdS black brane (ABB) and Lifshitz black brane (LBB) solutions. The ABB is characterized by a non-propagating (constant) U(1) gauge field while in the LBB, it is propagating. Thus, the U(1) gauge field, $\mathcal{A}$, can be taken, from the perspective of the RG flow, as the relevant conformal-breaking operator.

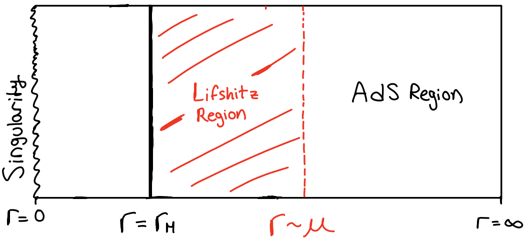
The transition between UV-AdS to IR-Lifshitz is expected to occur at $r \sim \mu$. However, we note that in order to observe a Lifshitz geometry, we must ensure that $r_h < \mu$. This can be taken as the limit:

$$\frac{\mu}{T} > 1 \tag{5}$$

Interpolating solutions: Pictorially

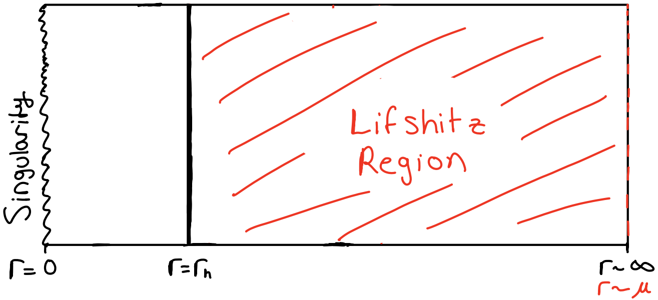
The first figure corresponds to $\mu/T < 1$ where there is no geometric interpolation. The second figure corresponds to $\mu/T > 1$ and finite where we expect a geometric interpolation to occur at $r \sim \mu$.





Interpolating solutions: Pictorially (continued)

This corresponds to taking the limit $\mu/T \to \infty$.



Dilaton-Einstein-Maxwell Model: Interpolating Solution

Following (Bertoldi et al. 2010), we guess an ansatz, which we plug-in the action to get a $1d$-Lagrangian:

$$
\begin{aligned}
ds^2 = -e^{2A(r)}dt^2 + e^{2B(r)}\delta_{ij}dx^i dx^j + e^{2C(r)}dr^2 \\
\phi = \phi(r), \quad \mathcal{A} = e^{G(r)}dt
\end{aligned}
\tag{6}
$$

The action has 4 dynamical fields and 4 global symmetries allowing us to reduce all the E.o.Ms into a first-order form. The symmetries are:

- Noether's symmetry leaving the volume term of the action, $dt\prod_{i=1}^{d} dx^i$, invariant: $(A, B, C, \phi, G) \to (A + d\delta_1, B - \delta_1, C, \phi, G + d\delta_1)$
- Global U(1) gauge symmetry associated with $e^G \to e^G + \mathrm{const}$
- Dilaton-gauge invariant coupling: $(A, B, C, \phi, G) \to (A, B, C, \phi + \delta_2, G - \alpha\delta_2)$
- Hamiltonian constraint associated with the fact that C has no derivative terms and thus acts as the Lagrange multiplier.

Equations of Motion

$$e^{A+dB-C}\left(d\partial A\partial B+\frac{d(d-1)}{2}(\partial B)^2+e^{-2A+2G+2\alpha\phi}(\partial G)^2-(\partial\phi)^2+e^{2C}\Lambda\right)=0 \tag{7}$$

$$de^{A+dB-C}\left(2\partial\phi+\alpha(\partial A-\partial B)\right)=\mathbf{D_0} \tag{8}$$

$$e^{A+dB-C}\left(\partial\phi+\alpha e^{-2A+2G+2\alpha\phi}\partial G\right)=\mathbf{P_0} \tag{9}$$

$$e^{-A+dB-C+2G+2\alpha\phi}\partial G=\mathbf{Q} \tag{10}$$

where D_0, P_0 and Q are integration constants that arised as the conserved quantities to the global symmetries. We also denote the value of the two parameters μ and T:

$$\mu=e^{G}e^{\alpha\phi} \quad T=\frac{\sqrt{\partial e^{2A(r)}\partial e^{-2C(r)}}\,|_{r=r_h}}{4\pi} \tag{11}$$

where r_h is the radius of the horizon given by the condition that $e^{2A(r_h)}=e^{-2C(r_h)}=0$ (blackening factor condition). This makes it clear theat T refers to the black brane temperature.

Numerical Analysis: Shooting Methods

We implement a shooting method by perturbing the AdS asymptotic solutions. Thus, we define:

$$\begin{aligned} A(r) &= \ln(Lr) + A_1(r), \qquad B(r) = \ln(Lr), \\ C(r) &= \ln\left(\frac{L}{r}\right) + C_1(r), \qquad G(r) = \ln(L) + G_1(r), \qquad \phi = \phi(r) \end{aligned} \tag{12}$$

The perturbed functions ("1") correspond to the deviations from pure AdS geometry arising as a result of the $U(1)$ gauge field operator. We will choose them to characterize the near-horizon data. In the equation above, we have fixed two symmetries:

- The diffeomorphism invariance to express $e^{2B(r)} = L^2/r^2$ (entropy gauge)
- The U(1) gauge symmetry via fixing the global parameter $P_0 = 0$

Seed Functions of the Near-Horizon

The perturbative functions are chosen to characterize the near-horizon data:

$$\begin{aligned} e^{A_1(r)} &= a_0 \left((r - r_h)^{1/2} + a_1 (r - r_h)^{3/2} + \ldots \right) \\ e^{C_1(r)} &= c_0 (r - r_h)^{-1/2} + c_1 (r - r_h)^{1/2} + \ldots \\ e^{G_1(r)} &= a_0 g_0 \left((r - r_h) + g_1 (r - r_h)^2 + \ldots \right) \end{aligned} \tag{13}$$

Plugging this in the equations of motion, we can extract the coefficients: a_i, c_i and g_i. We give here only the relevant parameters:

$$a_0 = \frac{2 c_0 \hat{D}_0}{\alpha d r_h^{d+2}}, \quad c_0 = c_0, \quad g_0 = \frac{d \left((d+1) c_0^2 - r_h \right) r_h^d}{2 c_0 \hat{Q}} \tag{14}$$

We only end up with one free parameter c_0. a_0 can be changed with time-rescaling and g_0 (and therefore $\hat{Q}$) can be re-scaled with the dilaton-gauge coupling symmetry, which is a transformation leaving $e^{2\alpha\phi} F^2$ invariant. We gauge fixed the above to get $e^{2\alpha\phi} |_{r \to \infty} \to 1$.

c_0 parameter

It turns out that the parameter c_0 appearing on the radial term is the only parameter controlling the 'degree of interpolation' of the black brane solution. It is useful to define a new free parameter $\hat{c}_0$ to get a better grip on the physics of this parameter:

$$c_0 = (c_0)_{AdS} + \hat{c}_0 \left[(c_0)_{Lif} - (c_0)_{AdS} \right] \tag{15}$$

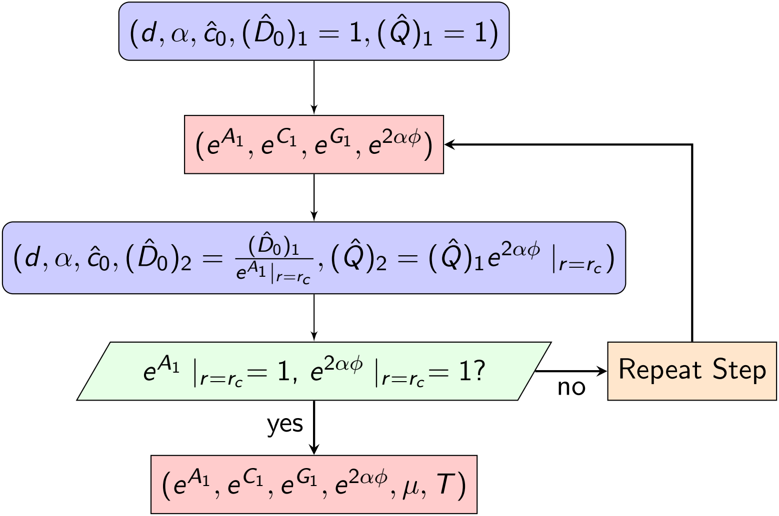
where the $(c_0)_{AdS}$ and $(c_0)_{Lif}$ can be identified by Taylor expanding the pure AdS/Lifshitz black brane solution around $r = r_h$.

Asymptopic AdS

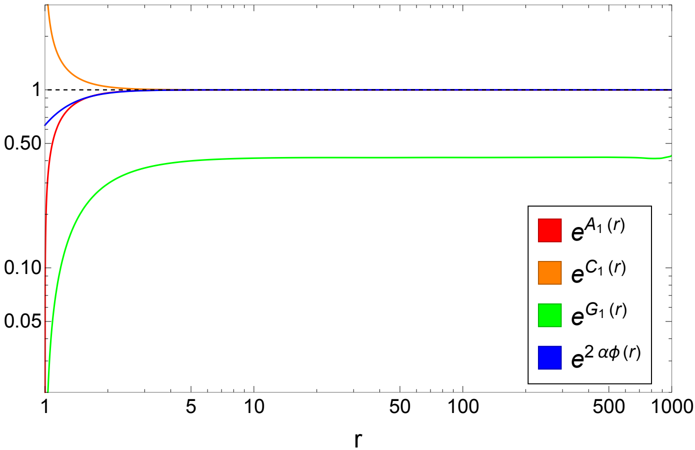
The correct AdS asymptopic are consistent with setting the perturbed functions ("1") to zero as $r \to \infty$.

$$e^{A_1} \,|_{r\to\infty} \to 1, \quad e^{C_1} \,|_{r\to\infty} \to 1, \quad e^{G_1} \,|_{r\to\infty} \to \text{const}, \quad e^{2\alpha\phi} \,|_{r\to\infty} \to 1 \tag{16}$$

We saw that it is possible to get $e^{A_1} \,|_{r\to\infty} \to 1$ by re-scaling $\hat{D}_0$ and to get $e^{2\alpha\phi} \,|_{r\to\infty} \to 1$ by re-scaling $\hat{Q}$.
We conveniently to guess a dummy solution setting $\hat{D}_0 = (\hat{D}_0)_{AdS}$ and $\hat{Q} = 1$ and then re-scaling these two parameters to get the correct AdS asymptotic.


$(d, \alpha, \hat{c}_0, (\hat{D}_0)_1 = 1, (\hat{Q})_1 = 1)$
$(e^{A_1}, e^{C_1}, e^{G_1}, e^{2\alpha\phi})$
$(d, \alpha, \hat{c}_0, (\hat{D}_0)_2 = \frac{(\hat{D}_0)_1}{e^{A_1}|_{r=r_c}}, (\hat{Q})_2 = (\hat{Q})_1 e^{2\alpha\phi}|_{r=r_c})$
$e^{A_1}|_{r=r_c} = 1,\ e^{2\alpha\phi}|_{r=r_c} = 1?$
no
Repeat Step
yes
$(e^{A_1}, e^{C_1}, e^{G_1}, e^{2\alpha\phi}, \mu, T)$

Results

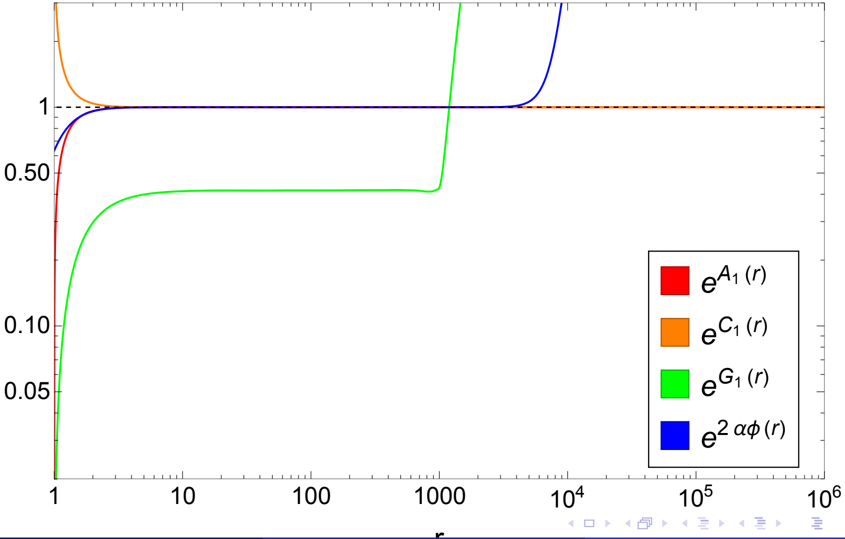


This is for the choice of parameters:

$$(d, \alpha, r_h, \hat{c}_0, \mu/T) = (3, 2, 1, 0.25, 1.52)$$

Results

But actually, if we make the numerical cutoff larger, then we see that we develop some issues with the $U(1)$ gauge field and the dilaton ϕ:



Issues at large r

This can be inferred from the equations of motion of the fields.

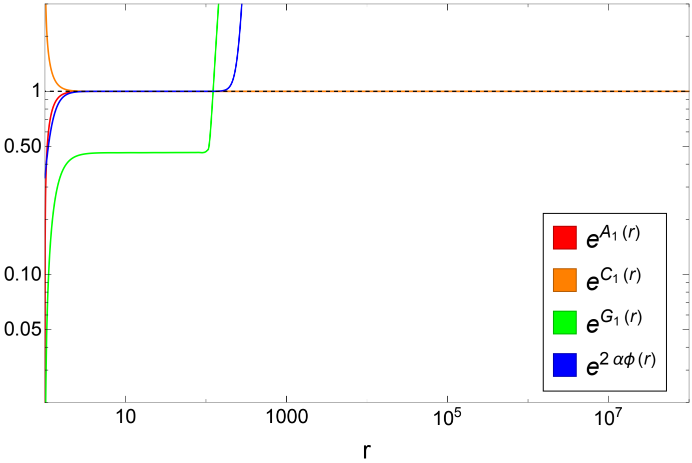
$$\partial e^{G_1} = \left(e^{2C_1} - 1\right) \lesssim r^d \tag{17}$$

$$e^{2\alpha\phi} = \frac{e^{2G_1} - r^{1+d} e^{G_1}}{r^2} \tag{18}$$

To prevent this, we need to impose a cutoff r_c, which we take to be $log_{10}(r_c) = 12/(d+1)$, which turns out to be a sufficient cutoff to use. We thought that this would not affect the computation of the entanglement entropy, because the metric fields: e^{A_1}, e^{C_1} do not share this instability, but it turns out that the cutoff r_c does affect the calculations that we will get if we are not careful.

Issue at large r

We expected that the metric components might be affected by the divergence of gauge field and the dilaton via the divergence of the stress-energy tensor, but it turns out that it is not the case.



Ryu-Takayanagi Prescription

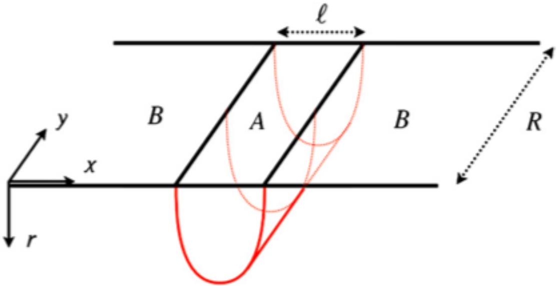
Ryu-Takayanagi prescription allows us to compute the entanglement entropy for a sub-region of a strongly CFT based purely on the geometric data of the dual theory of Einstein gravity **with AdS asymptotic** (Ryu, Takayanagi 2006). The regime where the dual gravity theory is classical corresponds to:

- Large central charge $c \gg 1$
- Large interaction parameter $\lambda \gg 1$ (or $g_{YM} \gg 1$).

The prescription states that the entanglement entropy of a region of a CFT_d characterized by a entangling region ∂A is equivalent to the extremized area of the co-dimension 2 minimal surface $\mathcal{S}_A$ extending into $d+2$ classical bulk gravity in and anchored at the boundary to ∂A.

$$S_A = \frac{\mathrm{Area}(\mathcal{S}_A)}{4 G_N^{d+1}} \tag{19}$$

Pictorial of thin-slab



Making the size of the slab larger means that you include more degrees of freedom of the CFT, which naturally increases the entanglement entropy.

Thin-Slab Example

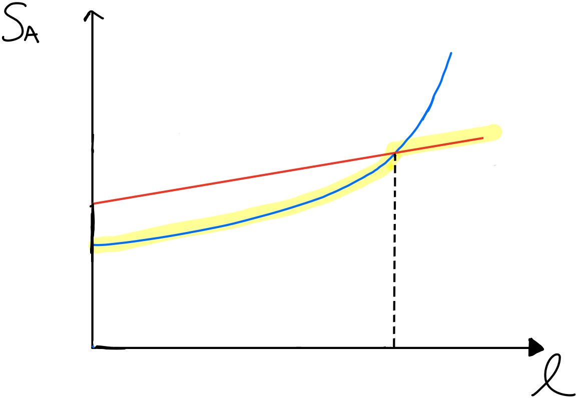
The entanglement entropy is an UV divergent quantity and to get a better control of the UV divergence, we consider the difference between the entanglement entropy of the interpolating solution (S_{inter}) and the entanglement entropy of pure AdS ($S_{pureAdS}$), both having the same maximal radial extent r_t:

$$\Delta S(r_t) = S_{inter}(r_t) - S_{pureAdS}(r_t) \tag{20}$$

An interesting fact to add is that the entropy functional depends only on one perturbed function: e^{C_1}.

Phase transition in entanglement entropy

The entanglement entropy from the RT prescription is given by extremizing an area functional. The minimal surface is consistent with the saddle point of minimal area.
The traditional expectation of the phase transition is that there will be a change of the 'minimum' saddle point at some critical length l_c.



Formally we expect that:

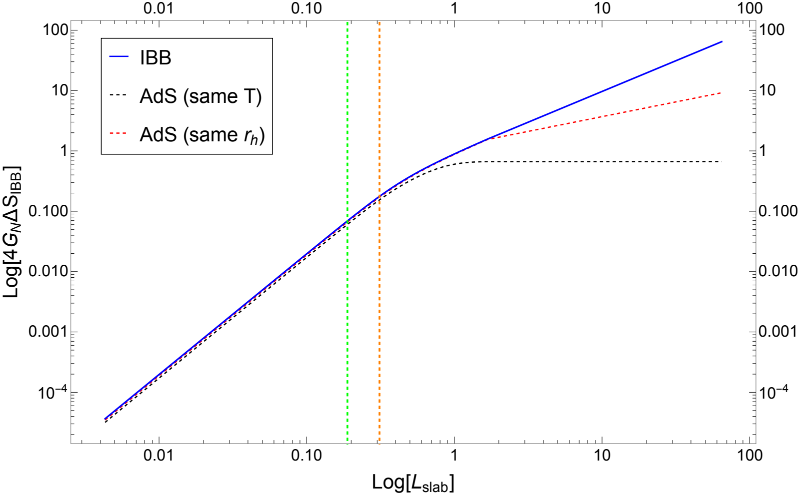
- A first-order phase transition will be evidenced by a discontinuity in the first derivative of the entanglement entropy at the critical length L_c:
$$\frac{\partial \Delta S}{\partial L} |_{L_c^+} \neq \frac{\partial \Delta S}{\partial L} |_{L_c^-}$$
- A second-order phase transition will be evidenced by a discontinuity of the second derivative of the entanglement entropy at the critical length L_c:
$$\frac{\partial^2 \Delta S}{\partial L^2} |_{L_c^+} \neq \frac{\partial^2 \Delta S}{\partial L^2} |_{L_c^-}$$

We also expect that:

- The entanglement entropy is continuous at the critical length $\Delta S |_{L_c^+} = \Delta S |_{L_c^-}$. If this is not the case, then it would suggest a lack of precision with the numerical analysis.
- The phase transition should be local and the critcial length should be related to $L_c \sim \mu$ where we expect the geometric transition to happen.

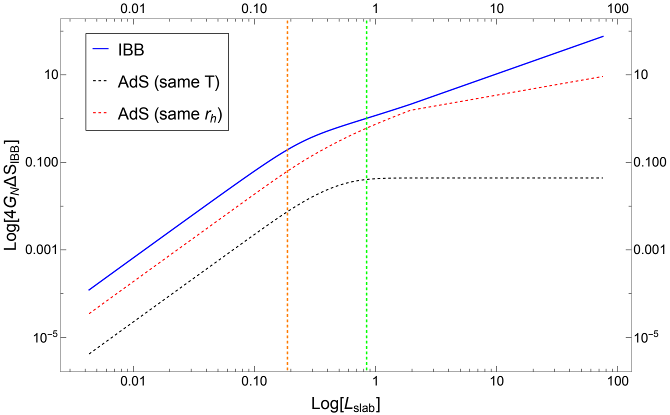
Results

We do not see any sign of discontinuity for $\mu/T < 1$ as expected. But we do see a change in profile due to the cross-over thermal behaviour.

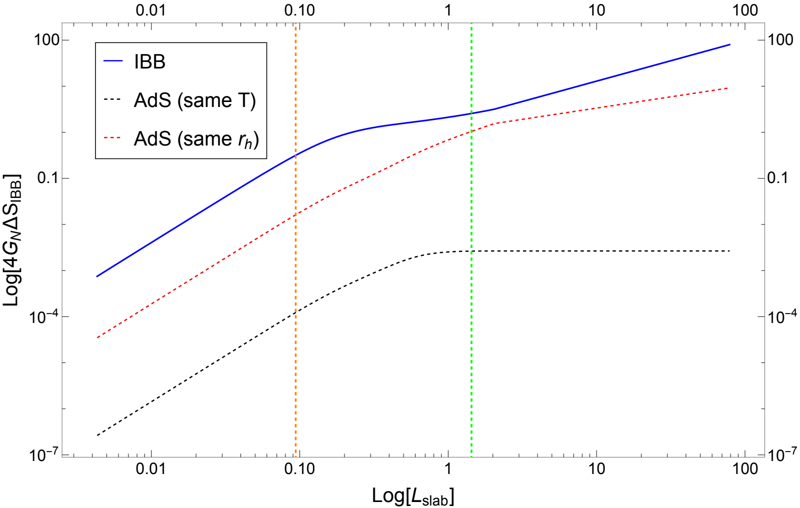


Here we plot $\mu/T \sim 0.60$ with $d = 3$ and $\alpha = 2$.

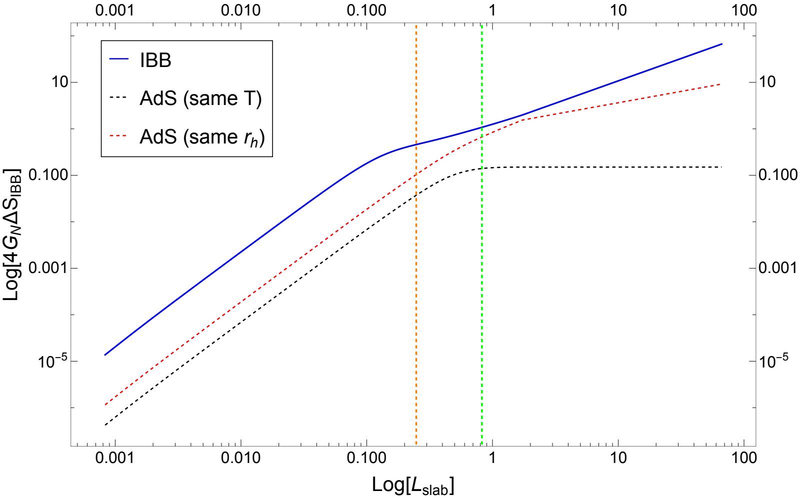
We make μ/T larger in order to separate the effects of the thermal behaviour and that of the interpolating geometry. Below we take $\mu/T = 4.49$ $(d = 3, \alpha = 2)$



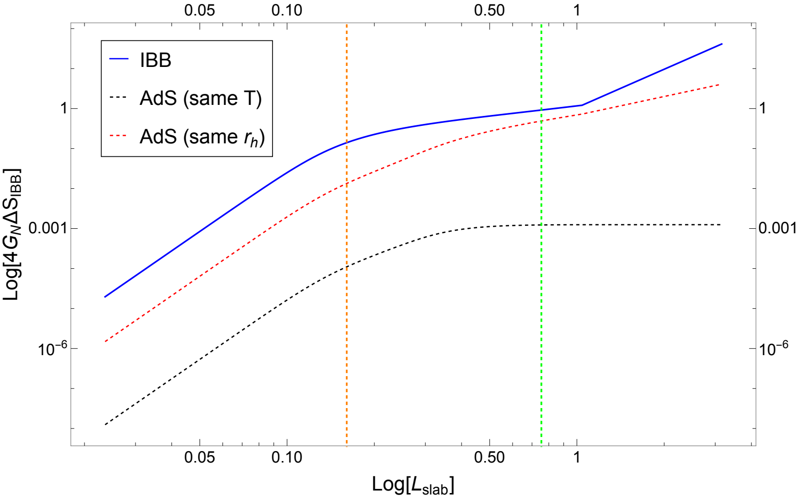
Taking $\mu/T = 15.4$ ($d = 3, \alpha = 2$).



Same for $\alpha = 2 \rightarrow \alpha = 5$ $(\mu/T = 15.4 \rightarrow 3.32)$.



Taking $d = 3 \to d = 6$ ($\mu/T = 15.4 \to \mu/T = 4.69$)



Results

- The discontinuity in the first-order derivative suggests that there is an evidence of a first-order phase transition arising as a result of the geometric interpolation.
- Another interesting fact is that the second-derivative experiences a change-of-sign. This is interesting as it might be indicative of a violation of the $\mathcal{C}$-theorem as indicated by (Myers, Singh 2012).

$\mathcal{C}$-theorem

In the course of a RG flow, we expect that the effective degrees of freedom of the theory to monotonically decrease; implying that the flow is not inversible. This is encapsulated by a $\mathcal{C}$-function, which is a function depending only on the coupling constants and the energy scale of the theory μ: via the RG flow time: $t = -\log(\mu/\Lambda)$. This function monotonically decrease along t:

$$\frac{d\mathcal{C}(\mathcal{T})}{dt} \leq 0 \tag{21}$$

Therefore, if we have an UV fixed point (small t) and an IR fixed point (large t), then we would find the condition that

$$\mathcal{C}_{UV} \geq \mathcal{C}_{IR}$$

$\mathcal{C}$-theorem: Fixed points

To build an entropic $\mathcal{C}$-function in d dimension, we use the fact that at the fixed point of the RG flow, the $\mathcal{C}$-function is typically equal to the central charge of the CFT c_{CFT}.
The entanglement entropy of a slab for a flat background spacetime is given by:

$$S_{\parallel} = \frac{4\pi c_{eff}}{d-2} R^{d-2} \left[\frac{\alpha_d}{\epsilon^{d-2}} - \frac{\beta_d}{L^{d-2}} \right] \tag{22}$$

where α_d and β_d are numerical constants, ϵ is the cutoff and R is the IR regulator for the thin-slab surface. We can isolate the c_{eff} by taking the partial derivative of $S_{\parallel}$ with L:

$$c_{eff} = \beta_d \frac{L^{d-1}}{R^{d-2}} \frac{\partial S_{\parallel}}{\partial L} \tag{23}$$

We consider the variation:

$$\frac{dc_{eff}}{dL} = \beta_d (d-1) \frac{L^{d-2}}{R^{d-2}} S_{\parallel}' + \beta_d \frac{L^{d-1}}{R^{d-2}} S_{\parallel}'' \leq 0 \tag{24}$$

where here $'$ is a partial derivative.

Future directions

It turns out to be very difficult in our case to isolate the divergences associated with taking derivatives of the entropy functional. This provides a natural next step of investigation.

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Thin-Slab Example

We pick a thin-slab boundary surface A given by:

$$A_{\text{slab}} = \left\{ \xi_0 = t = \text{fixed}, \boldsymbol{\xi}_{d+1} \in \mathbb{R}^{d+1} \mid \xi_1 \in (-a/2, a/2), \xi_i \in \mathbb{R} \text{ for } i = 2, 3, ..., d-1 \right\} \quad (25)$$

The induced metric generated by the co-dimension 2 minimal surface $\mathcal{S}_A$ extending into bulk gravity, with a radial profile: $r = r(\xi_1, a)$, is given by:

$$ds^2_{\mathcal{S}_A} = \frac{\partial X^a}{\partial \xi^i}\frac{\partial X^b}{\partial \xi^j} G_{ab}(X) = e^{2B(r)} g_{ab}(x, r) dx^a dx^b + \left(e^{2C(r)} \frac{\partial r}{\partial x_1}\frac{\partial r}{\partial x_1} + e^{2B(r)} \right) dx_1^2 \quad (26)$$

The entanglement entropy is found by extremizing the area integral of the minimal surface.

$$S_A = \frac{1}{4\pi G_N} \int d^d\boldsymbol{x} dx_1 \sqrt{\det\left(g_{\mathcal{S}_A}\right)} = \frac{R^{d-1}}{4\pi G_N} \int_{a/2}^{a/2} dx_1 \sqrt{e^{2dB(r)} + e^{2(d-1)B(r)} e^{2C(r)} \dot{r}^2} \quad (27)$$

Asymptopic AdS: Re-scaling

We can fix the e^{A_1} limit using time-rescaling: $(A, G) \to (A + \delta, G + \delta)$

$$e^A \to e^{A+\delta} = (Lr)\left[e^{\delta} a_0 \left((r - r_h)^{1/2} + a_1(r - r_h)^{3/2}\right)\right] \qquad (28)$$

and therefore a rescaling of $a_0 \to a_0 e^{\delta}$. Since $a_0 \sim \hat{D}_0$, it follows that this amounts to a rescaling of $\hat{D}_0 \to \hat{D}_0 e^{\delta}$. We choose

$$\delta = -A_1(r)\mid_{r\to\infty}\,, \quad \hat{D}_0 \to \frac{\hat{D}_0}{e^{A_1}\mid_{r\to\infty}}$$

to get the correct limit AdS asymptotic limit for e^{A_1}.
Further, the dilaton-gauge coupling amounts to a transformation leaving: $e^{2\alpha\phi}F^2$ invariant. Such a transformation is given as

$$e^{2\alpha\phi} \to e^{2\alpha\phi}e^{2\alpha\delta_2}, \quad e^G = Le^{G_1} \to Le^{G_1}e^{-\alpha\delta_2} = L\left[a_0 g_0 e^{-\alpha\delta_2}\left((r - r_h) + g_1(r\right.\right.$$

We fix: $\delta_2 = -\phi(r)\mid_{r\to\infty}$ so that we get $e^{2\alpha\phi}\mid_{r\to\infty} \to 1$. Since, $g_0 \sim 1/\hat{Q}$, it follows that this corresponds to

$$\hat{Q} \to \hat{Q}e^{\alpha\phi}\mid_{r\to\infty}$$

Zamolodchikov $\mathcal{C}$-function

The first example of such a function was discovered by Zamolodchikov for $1+1$-d QFTs. It consists of defining a c-function that respects the condition:

$$\frac{dc}{dt} \leq 0$$

It is possible to develop this function into an entanglement entropy dependent quantity. In fact, for a CFT_2, the entanglement entropy is given by:

$$S(L)\,|_{CFT} = \frac{c_{CFT}}{3} \log\left(\frac{L}{3}\right) \tag{29}$$

This tells us that the entropic c-function is given by:

$$c_E(L) = 3L\frac{\partial S(L)}{\partial L} \tag{30}$$

Zamolodchikov $\mathcal{C}$-function

We know from holographic principle that the energy scale of the CFT is related to the length of the slab: L and therefore, we impose:

$$\frac{dc_E}{dL} = 3S'(L) + 3LS''(L) \Rightarrow \frac{dc_E}{dL} \leq 0 \tag{31}$$

And therefore,

$$3\frac{\partial S}{\partial L} + 3L\frac{\partial^2 S}{\partial L^2} \leq 0 \tag{32}$$

Therefore, the change-of-sign of the first- and second-derivatives might be evidence of a violation of the $\mathcal{C}$-theorem, which might be an evidence of phase transition.

What aboud CFT$_d$?

BUT, this was only for CFT$_2$! It is unclear what the $\mathcal{C}$-theorem corresponds to in higher dimensions. For $1+2$-dimensions, the F-function has been proposed while for $1+3$-dimensions, the a-function have been proposed (Nishioka 2018). We expect that some variations of these functions will hold in larger odd and even dimensions respectively.
The important point to understand is that both these functions correspond to some version of the central charge of the CFT at critical points.