WE GENERATE A METRIC THAT TRANSITION BETWEEN ANTI DE-SITTER AND LIFSHITZ SPACE. WE COMPUTE THE ENTANGLEMENT ENTROPY FOR THIS GEOMETRY TO EXPLORE THE EFFECTS OF THE SYMMETRY BREAKING ON THE DEGREES OF FREEDOM.

HERE WE DEFINE THE NEAR-HORIZON LIFSHITZ GEOMETRY AND SHOOT TO R->INFINITY (PURE AdS). OUR CODE CONTAINS
- slnhold2 THAT SOLVES THE EQUATIONS OF MOTIONS NUMERICALLY FOR OUR FIELDS
- slngen2 THAT EXTRACTS THE ASYMPTOTICS VALUES OF THE METRIC FIELDS AND COMPUTE SOME QUANTITIES LIKE μ (CHEMICAL POTENTIAL) AND T (BRANE'S TEMPERATURE) THAT DEPENDS ON THESE ASYMPTOTIC VALUES.

In[1]:=
```
Clear[slnhold2]
slnhold2[c0h_, rh_, d_, α_, Qh_, facD0h_, RmaxFact_] :=
 Module[{E1, E2, E3, E4, E41, eAseed, eCseed, eGseed, IePhseed, ePhiseed, c0,
   D0h, a0, a1, c1, g0, g1, rIni, eOm, bc, eq, rmax, a, sol, e2aPhi, Qhlocal},
  (* EoM *)
```

```
E1 =
  D[eA1[r], r] - eC1[r] * N[(D0h + 2 * d * eG1[r] * α * Qhlocal) / (d * r^2 (r^d) * α), 50];
E2 = D[eC1[r], r] + N[eC1[r]^2 *
     (2 * Qhlocal * eG1[r] * eA1[r] * r^d * r * α * d - 2 * eC1[r] * Qhlocal^2 * eG1[r]^2 *
         α^3 + D0h * eA1[r] * r^d * r) / (r^3 * (r^d)^2 * eA1[r]^2 * d * α), 50];

E3 = D[eG1[r], r] - N[1 / (2) * 1 / (r^2 * eC1[r] * eA1[r] * Qhlocal * α * r^d) *
     (d * (r^(2 * d)) * r^2 * α * (eC1[r]^2 - 1) * (d + 1) * eA1[r]^2 -
       2 * eC1[r] * r^(d + 1) * (D0h + 2 * d * eG1[r] * α * Qhlocal) * eA1[r] +
       2 * eC1[r]^2 * eG1[r]^2 * α^3 * Qhlocal^2), 50];
(*
E3 = D[eG1[r],r]-
   N[d*r^(2*d+2)/(Qhlocal*2*r^(d+2))*((d+1)*(eA1[r]*eC1[r]-eA1[r]/eC1[r])-
        2*r^(d+1)*(D0h+2*d*α*Qhlocal*eG1[r])+
        2*α*eC1[r]/eA1[r]*α^2*Qhlocal^2*eG1[r]^2), 50];
*)
E4 = D[ePhi[r], r] -
   N[Simplify[D[1 / (2 * r^2 * Qhlocal^2 * α * eA2[r] * eC2[r]) * (d * r^(2 * d + 2) * α *
                  (d + 1) * (eA2[r] * eC2[r] - eA2[r] / eC2[r]) - 2 * r^(d + 1) * D0h -
                4 * d * r^(d + 1) * eG2[r] * α * Qhlocal + 2 * α * eC2[r] / eA2[r] * α^2 *
                 Qhlocal^2 * eG2[r]^2) /. eA2 → eA1 /. eC2 → eC1 /. eG2 → eG1 , r] /.
        eA1'[r] → eC1[r] * (D0h + 2 * d * eG1[r] * α * Qhlocal) / (d * r^2 (r^d) * α) /.
       eC1'[r] → -eC1[r]^2 * (2 * Qhlocal * eG1[r] * eA1[r] * r^d * r * α * d -
             2 * eC1[r] * Qhlocal^2 * eG1[r]^2 * α^3 + D0h * eA1[r] * r^d * r) /
          (r^3 * (r^d)^2 * eA1[r]^2 * d * α) /.
      eG1'[r] → 1 / (2) * 1 / (r^2 * eC1[r] * eA1[r] * Qhlocal * α * r^d) *
        (d * (r^(2 * d)) * r^2 * α * (eC1[r]^2 - 1) * (d + 1) * eA1[r]^2 -
          2 * eC1[r] * r^(d + 1) * (D0h + 2 * d * eG1[r] * α * Qhlocal) * eA1[r] +
          2 * eC1[r]^2 * eG1[r]^2 * α^3 * Qhlocal^2)], 50];

E41 = D[IePh[r], r] -
   N[2 * r^2 / (α * d * r^(2 * d + 2) * eA1[r] * eA1[r] * (eC1[r] - 1) * (eC1[r] + 1) * (d + 1) -
        2 * r^(d + 1) * eC1[r] * eA1[r] * (D0h + 2 * d * eG1[r] * α * Qhlocal) + 2 * eC1[r]^2 *
          eG1[r]^2 * α^3 * Qhlocal^2) * eA1[r]^2 * eC1[r]^2 * Qhlocal^2 * α, 50];

(* e^(2*α*ϕ[r]) as defined by 76 *)
(* WE WILL SEE LATER WHY WE DEFINED eA2 instead of eA1 and so on *)

(* e2aPhi[r_]:= 2*r^2/(α*d*r^(2*d+2)*eA2[r]^2*(eC2[r]-1)*(eC2[r]+1)*(d+1)-
       2*r^(d+1)*eC2[r]*eA2[r]*(D0h+2*d*eG2[r]*α*Qhlocal)+
       2*eC2[r]^2*eG2[r]^2*α^3*Qhlocal^2)*eA2[r]^2*eC2[r]^2*Qhlocal^2*α;
*)
(* Seed Functions (near-horizon boundary conditions) *)
eAseed[r_] := N[a0 * (Sqrt[r - rh] + a1 * (r - rh)^(3 / 2)), 50];
```

```
    eCseed[r_] := N[c0 / (Sqrt[r - rh]) + c1 * Sqrt[r - rh], 50];
    eGseed[r_] := N[a0 * (g0 * (r - rh) + g1 * (r - rh) ^2), 50];
    (* E^(-2*α*ϕ[r])=ePh[r]; *)
    IePhseed[r_] := N[1, 50];
    ePhiseed[r_] := 1 / (2 * r^2 * Qh^2 * α * eAseed[r] * eCseed[r]) *
      (d * r^(2 * d + 2) * α * (d + 1) * (eAseed[r] * eCseed[r] - eAseed[r] / eCseed[r]) -
        2 * r^(d + 1) * D0h - 4 * d * r^(d + 1) * eGseed[r] * α * Qh +
        2 * α * eCseed[r] / eAseed[r] * α^2 * Qh^2 * eGseed[r]^2);
    (* Coefficient of the seed functions *)
    a0 = 2 * rh^(-2 - d) * c0 * D0h / (d * α);
    a1 = (α^2 * d * (d + 1)^2 * c0^4 - 2 * d * rh * (α^2 - 2) * (d + 1) * c0^2 +
        rh^2 * α^2 * d - 4 * rh^2 - 6 * d * rh^2) / (8 * rh^3);
    c1 = c0 * (3 * α^2 * d * (d + 1)^2 * c0^4 - 2 * d * rh * (3 * α^2 + 2) * (d + 1) * c0^2 +
         3 * rh^2 * α^2 * d + 4 * rh^2 + 6 * d * rh^2) / (8 * rh^3);
    g0 = d * (c0^2 * d + c0^2 - rh) / (2 * rh^(-d) * c0 * Qhlocal);
    g1 = d^2 * (rh^(-2 + d) * α^2 * (d + 1)^3 * c0^6 -
         rh^(d - 1) * α^2 * (d + 1)^2 * c0^4 - rh^(d) * (α^2 + 2) * (d + 1) * c0^2 +
         rh^(d + 1) * α^2 + 2 * rh^(d + 1)) / (8 * Qhlocal * rh * c0);
    (*  Some gauge fixing conditions for c0,
    D0h and Qhlocal. Also, specified the  rIni,
    which is the initial radius that is slighthly away from rh,
    because the metric is divergent at this point *)
    c0 = N[(c0h * ((Sqrt[α^2 + 2] - α) / (α)) * Sqrt[rh] + Sqrt[rh]) / (Sqrt[d + 1]), 50];
    D0h = N[1 / 2 * rh^(d + 1) * (d + 1) * d * α * facD0h, 50];
    Qhlocal = N[Qh, 50];
     rIni = N[rh + rh * 1*^-10, 50];
    (* Numerical Solutions of the ODE *)
    eOm = {E1 == 0, E2 == 0, E3 == 0, (*E41==0,*) E4 == 0} (* EoM *);
    bc = N[{eA1[rIni] == eAseed[rIni], eC1[rIni] == eCseed[rIni],
       eG1[rIni] == eGseed[rIni], (*IePh[rIni]== IePhseed[rIni],*)
       ePhi[rIni] == ePhiseed[rIni] }, 50] (* BC conditions *);
    eq = Flatten[{eOm, bc}];
    a = (12 / (d + 1));
    rmax = 10^a * rh * RmaxFact; (* MAX RANGE OF THE ODE *)
    sol = NDSolve[eq, {eA1[r], eC1[r], eG1[r], ePhi[r]}, {r, rh + rh * 1*^-10, rmax},
      MaxSteps → 900 000, WorkingPrecision → MachinePrecision ];
    (* Give back eA1, eC1, eG1 and e^(2*α*ϕ[r]) *)
    List[sol[[1, 1, 2, 0]], sol[[1, 2, 2, 0]], sol[[1, 3, 2, 0]], sol[[1, 4, 2, 0]]]
   ]
 (* EXTRACT the asymptotic values of our functions and
  quantities that can be calculated from these asymptotic values,
 like the chemical potential (mu) and the brane's temperature (T).*)
 slngen2[c0h_, rh_, d_, α_, Qh_, facD0h_, RmaxFac_] :=
   Module[{sln, rmax, eA1val, eC1val, eG1val, ePval,
```

```
  mu, T, c0, D0h, Qhlocal, e2aPhi, a, scaleRinit},
 scaleRinit = 10;
sln = slnhold2[c0h, rh, d, α, Qh, facD0h, RmaxFac];
rmax =  sln〚1〛["Domain"]〚1, 2〛;
(*
(* Make the rInit larger until you get rmax = 10^(12/(d+1)) --> *)
While[ rmax<10^(12/(d+1))&& scaleRinit<1000,
 {scaleRinit,sln =slnhold2[c0h, rh, d,α,Qh,facD0h, scaleRinit],
  rmax =  sln〚1〛["Domain"]〚1,2〛}; scaleRinit =  scaleRinit*10];
rmax =   sln〚1〛["Domain"]〚1,2〛;(* 10^(12/(d+1)); *)
*)
eA1val = sln〚1〛[rmax];
eC1val = sln〚2〛[rmax];
eG1val = sln〚3〛[rmax / 2];
ePval = sln〚4〛[rmax];
(* Value of the dilaton. Note
 we need to eA2 to express as a function of eA1 *)
mu = eG1val / (eA1val * Sqrt[ePval]);

(* Some conditions on Qh and c0, D0h *)
c0 = (c0h * ((Sqrt[α^2 + 2] - α) / (α)) * Sqrt[rh] + Sqrt[rh]) / (Sqrt[d + 1]);
D0h = 1 / 2 * rh^(d + 1) * (d + 1) * d * α * facD0h;
Qhlocal = N[Qh, 50];
T = 1 / 2 * D0h / (α * d * (rh^d) * Pi * eA1val);
List[N[eA1val, 10], N[eC1val, 10], N[eG1val, 10],
 N[ePval, 10], N[mu, 10], N[T, 10], N[mu, 10] / N[T, 10] , rmax]
];
```

Out[ ]=

```
0.419273
```

THE WAY THAT WE EVALUATE OUR FUNCTION:

1) SET THE FREE PARAMETERS (d, α, rh, c0) OF OUR SOLUTIONS
>>> d = 3; α = 2; rh = 1; c0 = 0.25;
2) CALCULATE THE ASYMPTOTIC VALUES OF THE FIELD FROM THOSE FREE PARAMETERS
>>> BCval1 = slngen2[c0,rh,d,α,1,1,1]
{eA1, eC1, eG1, ePval, mu, T, mu/T , rmax} =

{1.13126,1.,0.470227,0.981598,0.419547,0.281377,1.49105,1000.}
3) RE-SCALE THE FIELDS APPROPRIATELY TO GET THE CORRECT ADS ASYMPTOTICS GIVEN BY:
eA1(r-> rmax) -> 1, eC1(r -> rmax) -> 1, e^(2α ϕ) -> 1
>>> BCval2 = slngen2[c0,rh,d,α,Sqrt[BCval1[[4]]],1/BCval1[[1]], 1]
{eA1, eC1, eG1, ePval, mu, T, mu/T , rmax} =
{1.,1.,0.41841,0.999947,0.418421,0.281377,1.48705,1000.}

```
In[18]:= Clear[BCval, sln]
d = 3; α = 2; rh = 1; c0 = 0.25; (* Set some parameters *)
BCval1 = slngen2[c0, rh, d, α, 1, 1, 1] (* We  *)
BCval2 = slngen2[c0, rh, d, α, Sqrt[BCval1⟦4⟧], 1 / BCval1⟦1⟧, 1]
sln = slnhold2[c0, rh, d, α, Sqrt[BCval1⟦4⟧], 1 / BCval1⟦1⟧, 100 000]
L = 1;
Legended[LogLogPlot[{sln⟦1⟧[r], sln⟦2⟧[r], sln⟦3⟧[r], sln⟦4⟧[r]},
  {r, rh, BCval2⟦-1⟧ * 10 000}, PlotRange → {{rh, BCval2⟦-1⟧}, {0.02, 3}},
  PlotStyle → {Red, Orange, Green, Blue}, Frame → True , FrameTicks → Automatic,
  FrameLabel → {Style["r", Black, 25], None(*Style[ "Field",Black, 25]*)},
  FrameTicksStyle → Directive[Black, 20], RotateLabel → {False, True},
  Axes → False, Mesh → False, ImageSize → 700,
  Epilog → {Black, Dashed, Line[{{0, Log[1]}, {1000, Log[1]}}]}(* ,
  GridLines→Automatic,GridLinesStyle→Directive[Gray, Dashed]*)],
 Placed[SwatchLegend[{Red, Orange, Green, Blue},
   {"e^A1 (r)", "e^C1 (r)", "e^G1 (r)" , "e^2 αϕ (r)"}, LegendLayout → "Column",
   LegendMarkerSize → 25, LabelStyle → {Black, FontSize → 25, Font → "Arial"},
   LegendFunction → Framed], {0.87, 0.25}]]

Out[20]=
{1.13126, 1., 0.470365, 0.981599, 0.419669, 0.281377, 1.49148, 1000.}

Out[21]=
{1., 1., 0.41841, 0.999947, 0.418421, 0.281377, 1.48705, 1000.}

NDSolve : Maximum number of 900000 steps reached at the point r == 1.1887134755439898` *^7.
```

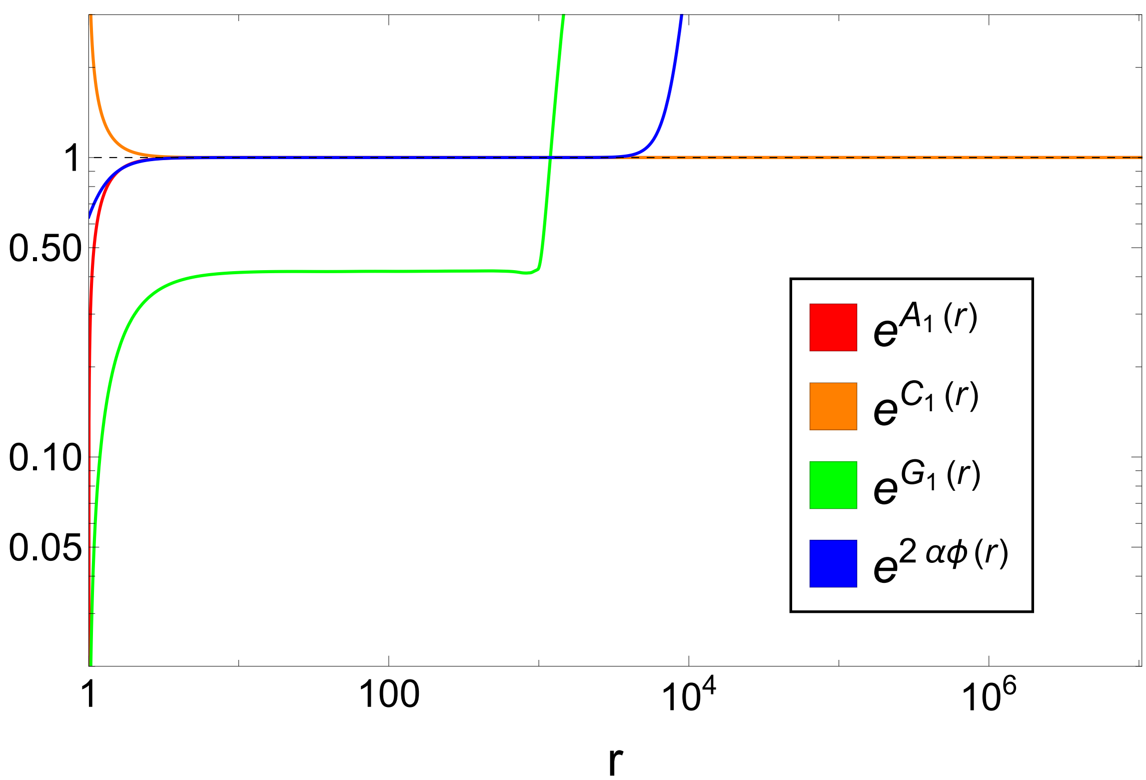
Out[22]=

{InterpolatingFunction[Domain: {{1., 1.19 ×10^7}} Output: scalar],
Data not saved. Save now

InterpolatingFunction[Domain: {{1., 1.19 ×10^7}} Output: scalar],
Data not saved. Save now

InterpolatingFunction[Domain: {{1., 1.19 ×10^7}} Output: scalar],
Data not saved. Save now

InterpolatingFunction[Domain: {{1., 1.19 ×10^7}} Output: scalar]}
Data not saved. Save now



RESULTS:

WE SEE THAT THE INTERPOLATION BREAKS DOWN BEYOND A CERTAIN VALUES OF R. THIS GIVES US A NATURAL DEFINITION OF

WHERE THE UV CUTOFF SHOULD BE.

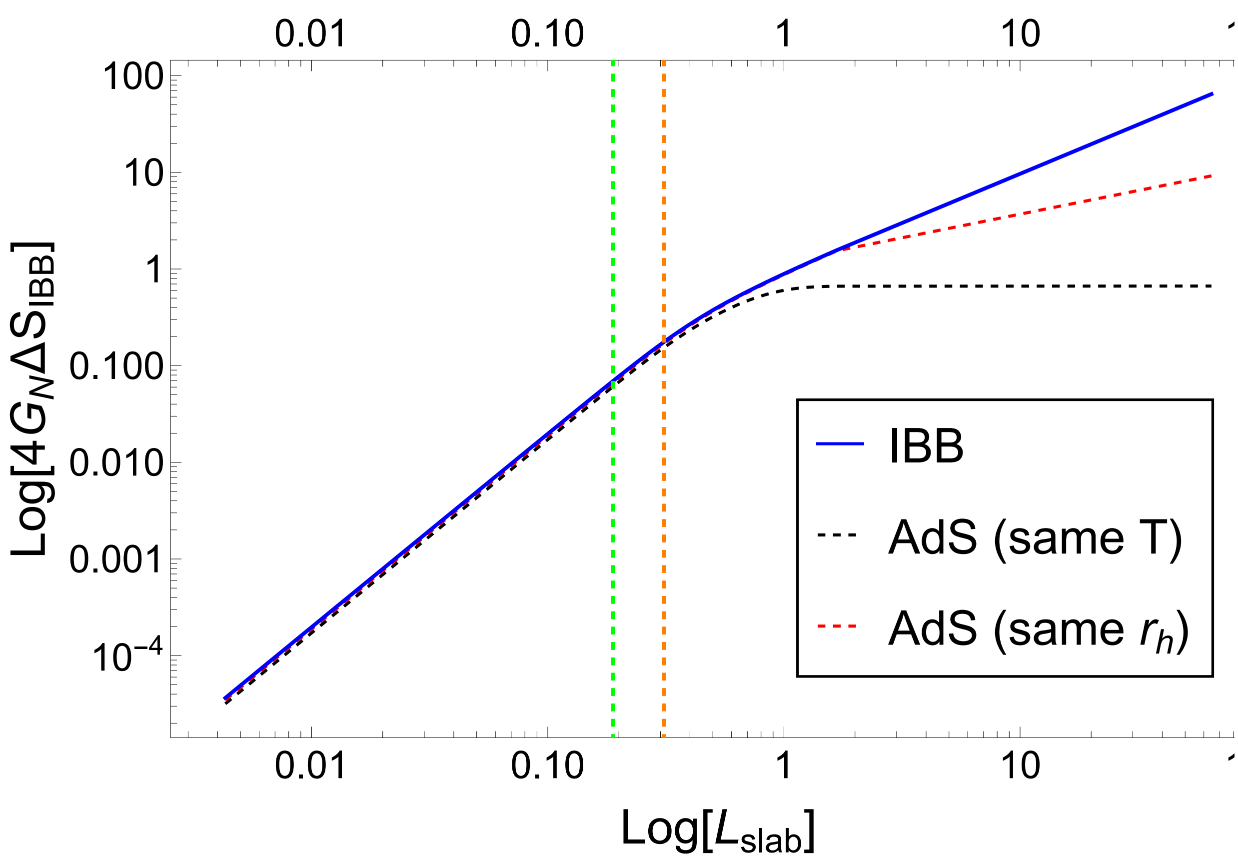
THE STRENGHT OF THE INTERPOLATION IS CONTROLED MOSTLY BY c0 WHICH IS A FREE PARAMETER. WHEN c0 = 0, WE HAVE A PURE ADS BRANE AND WHEN c0 = 1, WE HAVE A PURE LIFSHITZ BRANE.

HERE WE PICK c0 = 0.05

```
(*Parameters*)rh = 1; c0 = 0.05; d = 3; α = 2;

(*Calculate initial values*)
BCval1 = slngen2[c0, rh, d, α, 1, 1, 1]
BCval2 = slngen2[c0, rh, d, α, Sqrt[BCval1[[4]]], 1 / BCval1[[1]], 1]
sln = slnhold2[c0, rh, d, α, Sqrt[BCval1[[4]]], 1 / BCval1[[1]], 1]
L = 1;

(*Clear variables*)
Clear[Lslab, ΔSIBB, ΔSIBBr, xIBB, xIBBr, Lslabr]

(* Calculation of the entanglement entropy,
the size of the slab of the interpolating black brane (IBB) as a
 function of u and r. The reason why we integrate over u and r is to
 pick the least unstable solutions under numerical integration *)
ΔSIBB[us_, d_] := Re[NIntegrate[
   1 / u^3 * (sln[[2]][1 / u] - 1) / (Sqrt[1 - (u / us) ^ (2 * d)]), {u, 1 / BCval2[[-1]], us}]]

ΔSIBBr[rs_, d_] :=
 Re[NIntegrate[-r * (sln[[2]][r] - 1) / (Sqrt[1 - (rs / r) ^ (2 * d)]), {r, BCval2[[-1]], rs}]]

xIBB[U_, us_, d_] :=
 Re[L^2 * NIntegrate[-(sln[[2]][1 / u]) / (Sqrt[(us / u) ^ (2 * d) - 1]), {u, us, U}]]

xIBBr[U_, rs_, d_] :=
 Re[L^2 * NIntegrate[1 / r^2 * sln[[2]][r] / Sqrt[(r / rs) ^ (2 * d) - 1], {r, rs, U}]]

Lslab[us_, d_] := Re[-L^2 *
```

```
    NIntegrate[-(sln[[2]][1/u]) / (Sqrt[(us/u)^(2*d) - 1]), {u, 1/BCval2[[-1]], us}]]

Lslabr[rs_, d_] :=
  Re[L^2*NIntegrate[1/r^2*1/Sqrt[(r/rs)^(2*d) - 1], {r, rs, BCval2[[-1]]}]]



(* Calculation of the pure AdS brane *)
RSmax = 100;

If[BCval1[[5]] < rh, transPoint = BCval1[[6]], transPoint = BCval1[[5]]];
(* Statement that there is an interpolation is the value of mu < rh. Otherwise,
the interpolation occurs at the value of the temperature *)

Aads[r_, rh_, d_] := (L*r*Sqrt[1 - (rh/r)^(d+1)])^2

Cads[r_, rh_, d_] := (L/(r*Sqrt[1 - (rh/r)^(d+1)]))^2

BCval1 = slngen[c0, 1, d, α, 1, 1];

ThAds[rh_, d_] := Sqrt[
    (D[Aads[r, rh1, d], r] /. r → rh1) * (D[1/Cads[r, rh1, d], r] /. r → rh1)] / (4*Pi)

rHAds = N[Solve[ThAds[rh, 3] == BCval1[[6]], rh1][[2, 1, 2]]]

C1AdS[r_, d_, rh1_] := 1/Sqrt[1 - (rh1/r)^(d+1)]

ΔSAdS[us_, d_] :=
  Re[NIntegrate[1/u^3*(C1AdS[1/u, d, rHAds] - 1) / (Sqrt[1 - (u/us)^(2*d)]),
    {u, 1/BCval2[[-1]], us}]]

ΔSAdS1[us_, d_] :=
  Re[NIntegrate[1/u^3*(C1AdS[1/u, d, rh] - 1) / (Sqrt[1 - (u/us)^(2*d)]),
    {u, 1/BCval2[[-1]], us}]]

If[rHAds < rh, rIntAdS = rh, rIntAdS = rHAds];

(*Plot*)
Legended[
  Show[ParametricPlot[{Lslab[us, d], ΔSAdS[us, d]}, {us, 1/RSmax, 1/rIntAdS},
    ScalingFunctions → {"Log", "Log"}, Epilog → {{Green, Thick, Dashed,
        Line[{{Log[BCval1[[5]]], -100}, {Log[BCval1[[5]]], 100}}]}, {Orange, Thick,
        Dashed, Line[{{Log[BCval1[[6]]], -100}, {Log[BCval1[[6]]], 100}}]}},
    PlotStyle → {Dashed, Black}, Frame → True, FrameTicks → All,
```

```
    FrameLabel → {Style["Log[Subscript[L, slab]]", Black, 25],
      Style["Log[4Subscript[G, N] Subscript[ΔS, IBB]]", Black, 25]},
    FrameTicksStyle → Directive[Black, 20], Axes → False,
    Mesh → False, AspectRatio → 1 / GoldenRatio],
   ParametricPlot[{Lslab[us, d], ΔSAdS1[us, d]}, {us, 1 / RSmax, 1 / rh},
    ScalingFunctions → {"Log", "Log"}, Epilog → {{Green, Thick, Dashed,
       Line[{{Log[BCval1〚5〛], -100}, {Log[BCval1〚5〛], 100}}]}, {Orange, Thick,
       Dashed, Line[{{Log[BCval1〚6〛], -100}, {Log[BCval1〚6〛], 100}}]}},
    PlotStyle → {Dashed, Red}, Frame → True, FrameTicks → All,
    FrameLabel → {Style["Subscript[L, slab]", Black, 25],
      Style["4Subscript[G, N] Subscript[ΔS, IBB]", Black, 25]},
    FrameTicksStyle → Directive[Black, 20], RotateLabel → {False, True},
    Axes → False, Mesh → False, AspectRatio → 1 / GoldenRatio],
   PlotRange → All, ImageSize → 1000],
  Placed[LineLegend[{Blue, {Dashed, Black}, {Dashed, Red}},
    {"IBB", "AdS (same T)", "AdS (same \!\(\*SubscriptBox[\(r\), \(h\)]\))"},
    LegendLayout → "Column", LegendMarkerSize → 25,
    LabelStyle → {Black, FontSize → 25, Font → "Arial"},
    LegendFunction → Framed], {0.17, 0.82}]]
```



RESULTS:

WE SEE THAT THE ENTANGLEMENT ENTROPY MOVES FROM AN AREA LAW (LOG) TO A POWER LAW (VOLUME) AROUND THE PHASE TRANSITION (GREEN LINE), WHICH IS WHAT WE EXPECT TO HAPPEN. ALSO, SINCE c0 = 0.05, WE SEE THAT OUR BRANE (BLUE LINE) MIMICTS THE BEHAVIOUR OF AN ADS BRANE (RED LINE) EVEN BEYOND THE PHASE TRANSITION OCCURS (GREEN LINE).
#

WE TAKE c0 = 0.25

```
Clear[BCval, sln]
d = 3; α = 2; rh = 1; c0 = 0.25; (* Set free parameters,
we choose a larger value of c0 = a stronger interpolation *)
BCval1 = slngen2[c0, rh, d, α, 1, 1, 1]
BCval2 = slngen2[c0, rh, d, α, Sqrt[BCval1⟦4⟧], 1 / BCval1⟦1⟧, 1]
sln = slnhold2[c0, rh, d, α, Sqrt[BCval1⟦4⟧], 1 / BCval1⟦1⟧, 1]
L = 1;
(* PLOT THE VALUE OF THE FIELDS VS R *)
Legended[LogLogPlot[{sln⟦1⟧[r], sln⟦2⟧[r], sln⟦3⟧[r], sln⟦4⟧[r]},
  {r, rh, BCval2⟦-1⟧}, PlotRange → {{rh, BCval2⟦-1⟧}, {0.02, 3}},
  PlotStyle → {Red, Orange, Green, Blue}, Frame → True , FrameTicks → Automatic,
  FrameLabel → {Style["r", Black, 25], None(*Style[ "Field",Black, 25]*)},
  FrameTicksStyle → Directive[Black, 20], RotateLabel → {False, True},
  Axes → False, Mesh → False, ImageSize → 700,
  Epilog → {Black, Dashed, Line[{{0, Log[1]}, {1000, Log[1]}}]}(* ,
  GridLines→Automatic,GridLinesStyle→Directive[Gray, Dashed]*)],
 Placed[SwatchLegend[{Red, Orange, Green, Blue},
   {"e^(A1(r))", "e^(C1(r))", "e^(G1(r))" , "e^(2αϕ(r))"}, LegendLayout → "Column",
   LegendMarkerSize → 25, LabelStyle → {Black, FontSize → 25, Font → "Arial"},
   LegendFunction → Framed], {0.87, 0.25}]]


(* CALCULATE THE ENTANGLEMENT ENTROPY AND THE SIZE OF THE SLAB OF THE
  INTERPOLATING BLACK BRANES AS A FUNCTION OF u = 1/r. THE GOAL OF THIS IS
  TO PICK THE LEAST UNSTABLE SOLUTIONS UNDER NUMERICAL INTERPOLATION.  *)
Clear[Lslab, ΔSIBB, ΔSIBBr, xIBB, xIBBr, Lslab, Lslabr]
```

```
ΔSIBB[us_, d_ ] := Re[NIntegrate[
   1 / u^3 * ( sln⟦2⟧[1 / u] - 1) / (Sqrt[1 - (u / us) ^ (2 * d)]), {u, 1 / BCval2⟦-1⟧, us}]]
Lslab[us_, d_ ] := Re[-L^2 * NIntegrate[
    - ( sln⟦2⟧[1 / u]) / (Sqrt[(us / u) ^ (2 * d) - 1]), {u, 1 / BCval2⟦-1⟧, us}]]

(* CALCULATION OF THE ADS BLACK BRANE
 AT THE SAME r_h and the same temperature *)

RSmax = BCval1⟦-1⟧;

(* CALCULATION OF THE ENTROPY OF THE ADS BLACK BRANE AT
 THE SAME RADIUS OF HORIZON AS OUR INTERPOLATING BLACK BRANE *)
Cads[r_, rh_, d_] := (L / (r * Sqrt[1 - (rh / r) ^ (d + 1)])) ^2
(* PURE ADS FIELD VALUES *)
ΔSAdS1[us_, d_ ] :=
 Re[NIntegrate[1 / u^3 * (C1AdS[1 / u, d, rh] - 1) / (Sqrt[1 - (u / us) ^ (2 * d)]),
   {u, 1 / BCval2⟦-1⟧, us}]]


(* EXTRACTING THE RADIUS OF HORIZON FOR THE ADS BLACK BRANE
 AT THE SAME TEMPERATURE AS OUR INTERPOLATING BLACK BRANE *)
ThAds[rh_, d_] := Sqrt[
   (D[Aads[r, rh1, d], r] /. r → rh1) * (D[1 / Cads[r, rh1, d], r] /. r → rh1)] / (4 * Pi)
rHAds = N[Solve[ThAds[rh, 3] == BCval1⟦6⟧, rh1]⟦2, 1, 2⟧]

C1AdS[r_, d_, rh1_] := 1 / Sqrt[1 - (rh1 / r) ^ (d + 1)]
(* VALUES OF THE FIELD VALUES FOR ADS BLACK BRANE AT THE SAME TEMPERATURE *)
(* CALCULATION OF THE ENTROPY OF THE ADS BLACK BRANE
 AT THE TEMPERATURE AS OUR INTERPOLATING BLACK BRANE *)
ΔSAdS[us_, d_ ] :=
 Re[NIntegrate[1 / u^3 * (C1AdS[1 / u, d, rHAds] - 1) / (Sqrt[1 - (u / us) ^ (2 * d)]),
   {u, 1 / BCval2⟦-1⟧, us}]]

(* PLOT *)
If[rHAds < rh, rIntAdS = rh, rIntAdS = rHAds]; (* Set the range of the AdS brane,
so that we do not plot the divergence point where r = rh. *)
Legended[Show[ (* PLOT THE ENTANGLEMENT ENTROPY VS L_slab FOR THE AdS black
   branes(ABB) at the same temperature and same radius of horizon *)
  ParametricPlot[{Lslab[us, d], ΔSAdS[us, d]}, {us, 1 / RSmax, 1 / rIntAdS },
   ScalingFunctions → {"Log", "Log"}, Epilog → {{Green, Thick, Dashed,
      Line[{{Log[BCval1⟦5⟧], -100}, {Log[BCval1⟦5⟧], 100}}]}, {Orange, Thick,
      Dashed, Line[{{Log[ BCval1⟦6⟧], -100}, { Log[BCval1⟦6⟧], 100}}]}},
   PlotStyle → {Dashed, Black},  Frame → True , FrameTicks → All,
```

```
    FrameLabel → {Style["Log[L_slab]", Black, 25], Style[ "Log[4G_NΔS_IBB]", Black, 25]},
    FrameTicksStyle → Directive[Black, 20], Axes → False,
    Mesh → False, AspectRatio → 1 / GoldenRatio],
   ParametricPlot[{Lslab[us, d], ΔSAdS1[us, d]}, {us, 1 / RSmax, 1 / rh },
    ScalingFunctions → {"Log", "Log"}, Epilog → {{Green, Thick, Dashed,
       Line[{{Log[BCval1[[5]]], -100}, {Log[BCval1[[5]]], 100}}]}, {Orange, Thick,
       Dashed, Line[{{Log[ BCval1[[6]]], -100}, { Log[BCval1[[6]]], 100}}]}},
    PlotStyle → {Dashed, Red},  Frame → True , FrameTicks → All,
    FrameLabel → {Style["L_slab", Black, 25], Style[ "4G_N ΔS", Black, 25]},
    FrameTicksStyle → Directive[Black, 20], RotateLabel → {False, True},
    Axes → False, Mesh → False, AspectRatio → 1 / GoldenRatio],
   (* PLOT THE ENTANGLEMENT ENTROPY VS L_SLAB FOR
    THE INTERPOLATING BLACK BRANE (IBB) *)
   ParametricPlot[{Lslab[us, d], ΔSIBB[us, d]}, {us, 1 / RSmax, 1 / rh},
    ScalingFunctions → {"Log", "Log"}, Epilog → {{Red, Thick, Dashed,
       Line[{{Log[BCval1[[5]]], -100}, {Log[BCval1[[5]]], 100}}]}, {Orange, Thick,
       Dashed, Line[{{Log[ BCval1[[6]]], -100}, { Log[BCval1[[6]]], 100}}]}},
    PlotStyle → {Thick, Blue}, Frame → True , FrameTicks → All,
    FrameLabel → {Style["Log[L_slab]", Black, 25], Style[ "Log[4G_NΔS_IBB]", Black, 25]},
    FrameTicksStyle → Directive[Black, 20], Axes → False, Mesh → False,
    AspectRatio → 1 / GoldenRatio], PlotRange → All, ImageSize → 1000],
  Placed[LineLegend[{Blue, {Dashed, Black}, {Dashed, Red}},
    {"IBB", "AdS (same T)", "AdS (same r_h)"}, LegendLayout → "Column",
    LegendMarkerSize → 25, LabelStyle → {Black, FontSize → 25, Font → "Arial"},
    LegendFunction → Framed], {0.17, 0.82}]]
 (* ---------------- *)
```

Out[•]=

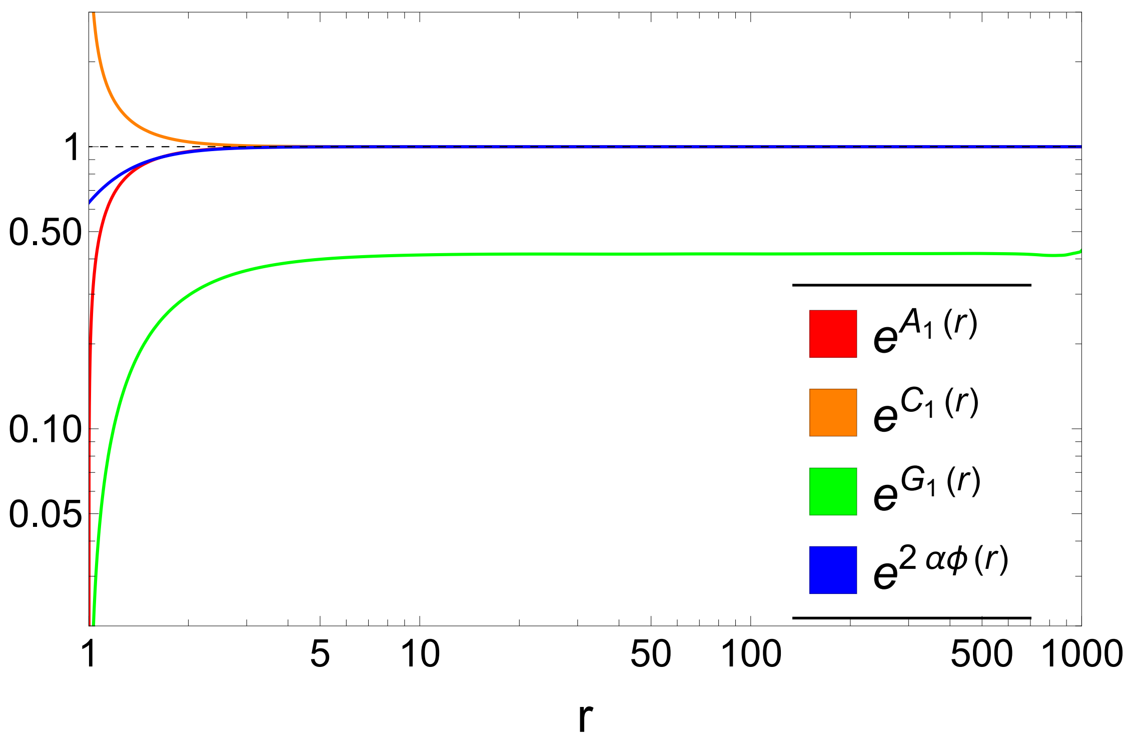
```
{1.13126, 1., 0.470227, 0.981598, 0.419547, 0.281377, 1.49105, 1000.}
```

Out[•]=

```
{1., 1., 0.418181, 0.999952, 0.418191, 0.281377, 1.48623, 1000.}
```

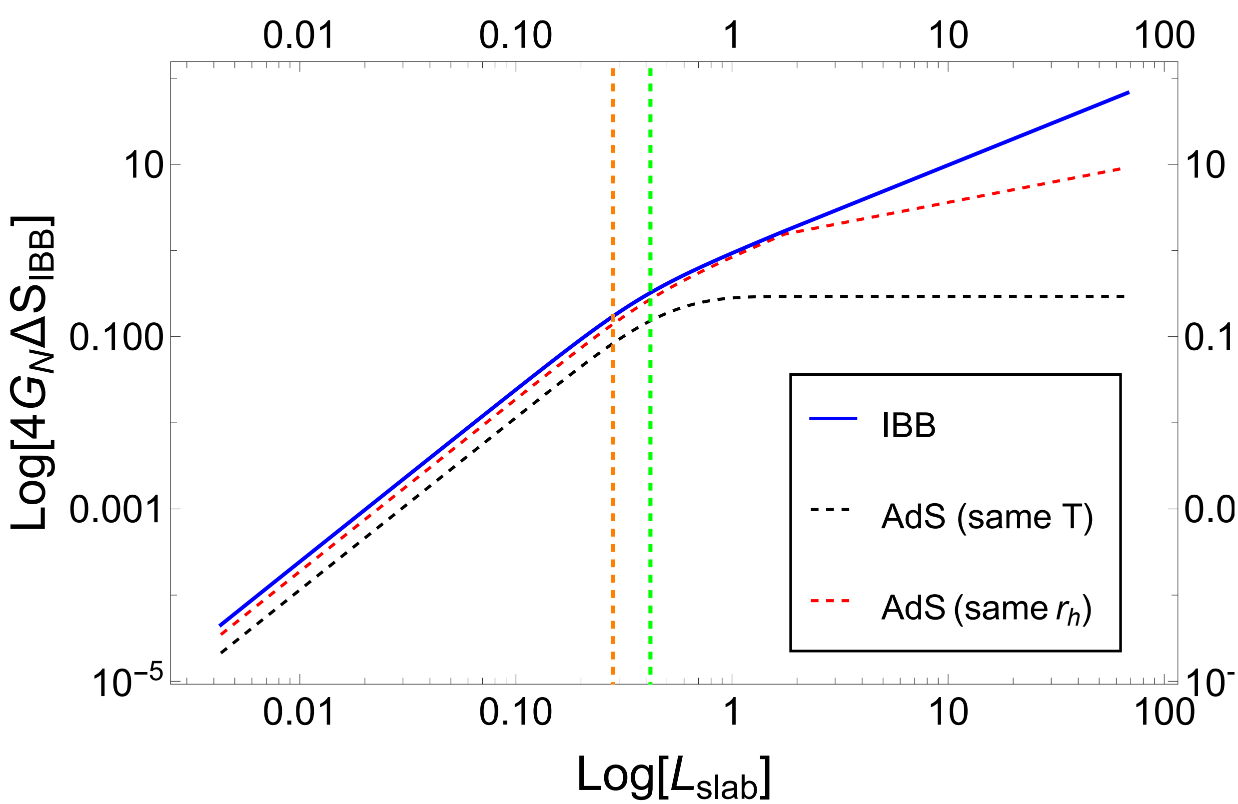
Out[•]=

```
{InterpolatingFunction[ Domain: {{1., 1.00 ×10^3}} Output: scalar ],
 InterpolatingFunction[ Domain: {{1., 1.00 ×10^3}} Output: scalar ],
 InterpolatingFunction[ Domain: {{1., 1.00 ×10^3}} Output: scalar ],
 InterpolatingFunction[ Domain: {{1., 1.00 ×10^3}} Output: scalar ]}
```



Out[ ]=

0.883973



RESULTS:

THE POINT WHERE OUR BRANE (BLUE LINE) DIVERGES FROM THE PURE ADS BRANE (RED LINE) OCCURS CLOSER TO THE POINT WHERE THE PHASE TRANSITION (GREEN LINE) OCCURS, WHICH IS AS EXPECTED FROM A LARGER VALUE OF c0.

BUT OVERALL, WE DO NOT SEE ANYTHING NOVEL ASSOCIATED WITH THE PHASE TRANSITION
#

HERE WE LOOK AT THE CASE WHERE μ = T. IT TRANSLATES INTO CHOOSING c_0 = 0.5

```
rh = 1; c0 = 0.50; d = 3; α = 2;
BCval1 = slngen2[c0, rh, d, α, 1, 1, 1]
BCval2 = slngen2[c0, rh, d, α, Sqrt[BCval1⟦4⟧], 1 / BCval1⟦1⟧, 1]
sln = slnhold2[c0, rh, d, α, Sqrt[BCval1⟦4⟧], 1 / BCval1⟦1⟧, 1]
L = 1;

(* CALCULATE THE ENTANGLEMENT ENTROPY AND THE SIZE OF THE SLAB OF THE
  INTERPOLATING BLACK BRANES AS A FUNCTION OF u = 1/r. THE GOAL OF THIS IS
  TO PICK THE LEAST UNSTABLE SOLUTIONS UNDER NUMERICAL INTERPOLATION.  *)
Clear[Lslab, ΔSIBB, ΔSIBBr, xIBB, xIBBr, Lslab, Lslabr]

ΔSIBB[us_, d_ ] := Re[NIntegrate[
   1 / u^3 * ( sln⟦2⟧[1 / u] - 1) / (Sqrt[1 - (u / us) ^ (2 * d)]), {u, 1 / BCval2⟦-1⟧, us}]]
Lslab[us_, d_ ] := Re[-L^2 * NIntegrate[
    - ( sln⟦2⟧[1 / u]) / (Sqrt[(us / u) ^ (2 * d) - 1]), {u, 1 / BCval2⟦-1⟧, us}]]

(* CALCULATION OF THE ADS BLACK BRANE
 AT THE SAME r_h and the same temperature *)

RSmax = BCval1⟦-1⟧;

(* CALCULATION OF THE ENTROPY OF THE ADS BLACK BRANE AT
 THE SAME RADIUS OF HORIZON AS OUR INTERPOLATING BLACK BRANE *)
```

```
Cads[r_, rh_, d_] := (L / (r * Sqrt[1 - (rh / r) ^ (d + 1)])) ^ 2
(* PURE ADS FIELD VALUES *)
ΔSAdS1[us_, d_ ] :=
 Re[NIntegrate[1 / u ^ 3 * (C1AdS[1 / u, d, rh] - 1) / (Sqrt[1 - (u / us) ^ (2 * d)]),
   {u, 1 / BCval2[[-1]], us}]]


(* EXTRACTING THE RADIUS OF HORIZON FOR THE ADS BLACK BRANE
 AT THE SAME TEMPERATURE AS OUR INTERPOLATING BLACK BRANE *)
ThAds[rh_, d_] := Sqrt[
   (D[Aads[r, rh1, d], r] /. r → rh1) * (D[1 / Cads[r, rh1, d], r] /. r → rh1)] / (4 * Pi)
rHAds = N[Solve[ThAds[rh, 3] == BCval1[[6]], rh1][[2, 1, 2]]]

C1AdS[r_, d_, rh1_] := 1 / Sqrt[1 - (rh1 / r) ^ (d + 1)]
(* VALUES OF THE FIELD VALUES FOR ADS BLACK BRANE AT THE SAME TEMPERATURE *)
(* CALCULATION OF THE ENTROPY OF THE ADS BLACK BRANE
 AT THE TEMPERATURE AS OUR INTERPOLATING BLACK BRANE *)
ΔSAdS[us_, d_ ] :=
 Re[NIntegrate[1 / u ^ 3 * (C1AdS[1 / u, d, rHAds] - 1) / (Sqrt[1 - (u / us) ^ (2 * d)]),
   {u, 1 / BCval2[[-1]], us}]]

(* PLOT *)
If[rHAds < rh, rIntAdS = rh, rIntAdS = rHAds]; (* Set the range of the AdS brane,
so that we do not plot the divergence point where r = rh. *)
Legended[Show[ (* PLOT THE ENTANGLEMENT ENTROPY VS L_slab FOR THE AdS black
   branes(ABB) at the same temperature and same radius of horizon *)
  ParametricPlot[{Lslab[us, d], ΔSAdS[us, d]}, {us, 1 / RSmax, 1 / rIntAdS },
   ScalingFunctions → {"Log", "Log"}, Epilog → {{Green, Thick, Dashed,
      Line[{{Log[BCval1[[5]]], -100}, {Log[BCval1[[5]]], 100}}]}, {Orange, Thick,
      Dashed, Line[{{Log[ BCval1[[6]]], -100}, { Log[BCval1[[6]]], 100}}]}},
   PlotStyle → {Dashed, Black},  Frame → True , FrameTicks → All,
   FrameLabel → {Style["Log[L_slab]", Black, 25], Style[ "Log[4G_NΔS_IBB]", Black, 25]},
   FrameTicksStyle → Directive[Black, 20], Axes → False,
   Mesh → False, AspectRatio → 1 / GoldenRatio],
  ParametricPlot[{Lslab[us, d], ΔSAdS1[us, d]}, {us, 1 / RSmax, 1 / rh },
   ScalingFunctions → {"Log", "Log"}, Epilog → {{Green, Thick, Dashed,
      Line[{{Log[BCval1[[5]]], -100}, {Log[BCval1[[5]]], 100}}]}, {Orange, Thick,
      Dashed, Line[{{Log[ BCval1[[6]]], -100}, { Log[BCval1[[6]]], 100}}]}},
   PlotStyle → {Dashed, Red},  Frame → True , FrameTicks → All,
   FrameLabel → {Style["L_slab", Black, 25], Style[ "4G_N ΔS", Black, 25]},
   FrameTicksStyle → Directive[Black, 20], RotateLabel → {False, True},
   Axes → False, Mesh → False, AspectRatio → 1 / GoldenRatio],
  (* PLOT THE ENTANGLEMENT ENTROPY VS L_SLAB FOR
   THE INTERPOLATING BLACK BRANE (IBB) *)
```

```
 ParametricPlot[{Lslab[us, d], ΔSIBB[us, d]}, {us, 1 / RSmax, 1 / rh},
  ScalingFunctions → {"Log", "Log"}, Epilog → {{Red, Thick, Dashed,
     Line[{{Log[BCval1⟦5⟧], -100}, {Log[BCval1⟦5⟧], 100}}]}, {Orange, Thick,
     Dashed, Line[{{Log[ BCval1⟦6⟧], -100}, { Log[BCval1⟦6⟧], 100}}]}},
  PlotStyle → {Thick, Blue}, Frame → True , FrameTicks → All,
  FrameLabel → {Style["Log[L_slab]", Black, 25], Style[ "Log[4G_NΔS_IBB]", Black, 25]},
  FrameTicksStyle → Directive[Black, 20], Axes → False, Mesh → False,
  AspectRatio → 1 / GoldenRatio], PlotRange → All, ImageSize → 1000],
Placed[LineLegend[{Blue, {Dashed, Black}, {Dashed, Red}},
  {"IBB", "AdS (same T)", "AdS (same r_h)"}, LegendLayout → "Column",
  LegendMarkerSize → 25, LabelStyle → {Black, FontSize → 25, Font → "Arial"},
  LegendFunction → Framed], {0.17, 0.82}]]
```

Out[ ]=

```
{1.32594, 1., 1.49548, 3.30495, 0.620402, 0.240063, 2.58432, 1000.}
```

Out[ ]=

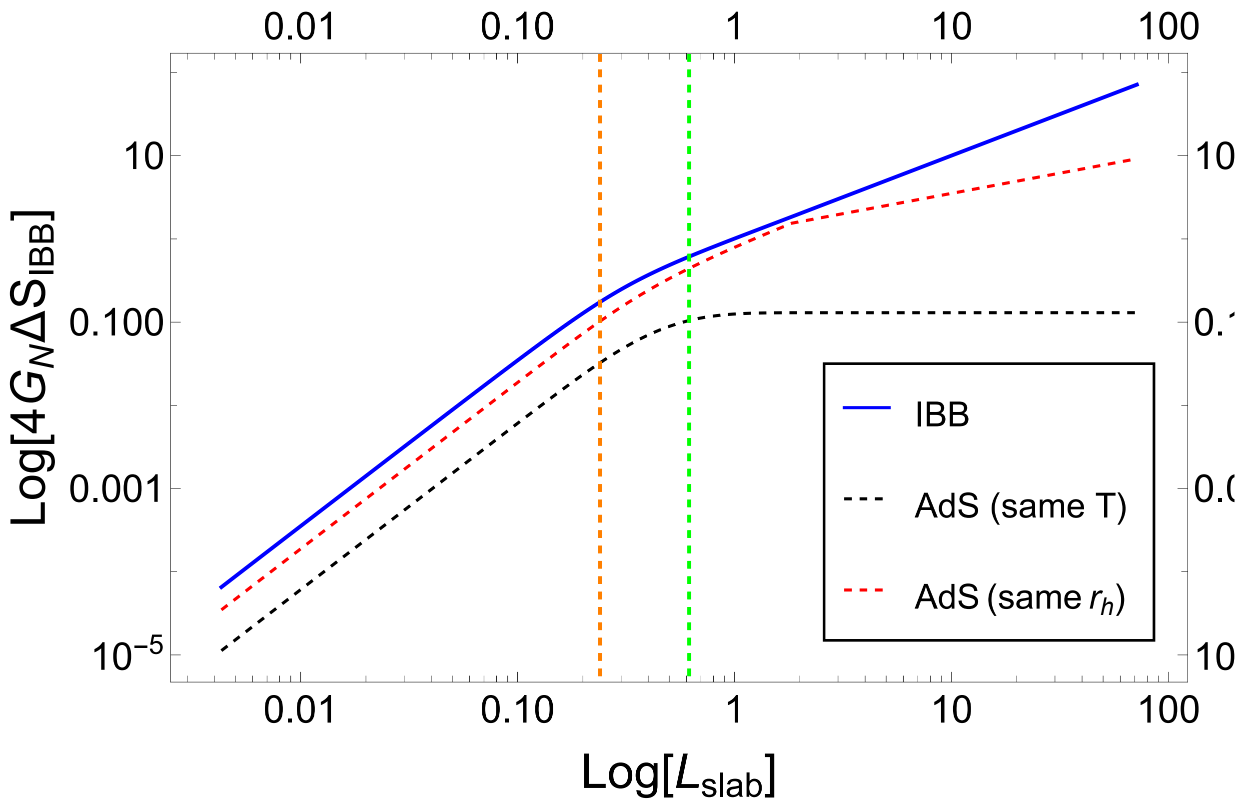
```
{1., 1., 0.619345, 0.999953, 0.61936, 0.240063, 2.57998, 1000.}
```

Out[ ]=

```
{InterpolatingFunction[ Domain: {{1., 1.00 ×10^3}} Output: scalar ],
 InterpolatingFunction[ Domain: {{1., 1.00 ×10^3}} Output: scalar ],
 InterpolatingFunction[ Domain: {{1., 1.00 ×10^3}} Output: scalar ],
 InterpolatingFunction[ Domain: {{1., 1.00 ×10^3}} Output: scalar ]}
```

Out[ ]=

```
0.754181
```



0.01
0.10
1
10
100
10
0.100
0.001
10^{-5}
$\mathrm{Log}[4 G_N \Delta S_{\mathrm{IBB}}]$
IBB
AdS (same T)
AdS (same r_h)
$\mathrm{Log}[L_{\mathrm{slab}}]$

RESULTS:
OUR BRANE (BLUE LINE) DEVIATES FROM THE PURE ADS BRANE (RED LINE) EVERYWHERE, WHICH IS AS EXPECTED FROM CHOOSING c0 = 0.5, WHICH MEANS THAT OUR BRANE HAS “AS MUCH” LIFSHITZ BEHAVIOURS AS IT HAS ADS BEHAVIOUR.

WHAT IS INTERESTING NOW IS THAT IT LOOKS LIKE OUR BRANE (BLUE LINE) STARTS TO GET A CHANGE OF CURVATURE AFTER THE PHASE TRANSITION (GREEN LINE). THIS IS SOMETHING NOVEL THAT IS NOT EXPECTED TO HAPPEN.
#

HERE WE MAKE c0 = 0.85

```
rh = 1; c0 = 0.85; d = 3; α = 2;
BCval1 = slngen2[c0, rh, d, α, 1, 1, 1]
BCval2 = slngen2[c0, rh, d, α, Sqrt[BCval1⟦4⟧], 1 / BCval1⟦1⟧, 1]
sln = slnhold2[c0, rh, d, α, Sqrt[BCval1⟦4⟧], 1 / BCval1⟦1⟧, 1]
L = 1;

(* CALCULATE THE ENTANGLEMENT ENTROPY AND THE SIZE OF THE SLAB OF THE
  INTERPOLATING BLACK BRANES AS A FUNCTION OF u = 1/r. THE GOAL OF THIS IS
  TO PICK THE LEAST UNSTABLE SOLUTIONS UNDER NUMERICAL INTERPOLATION.  *)
Clear[Lslab, ΔSIBB, ΔSIBBr, xIBB, xIBBr, Lslab, Lslabr]

ΔSIBB[us_, d_] := Re[NIntegrate[
   1 / u^3 * (sln⟦2⟧[1 / u] - 1) / (Sqrt[1 - (u / us)^(2 * d)]), {u, 1 / BCval2⟦-1⟧, us}]]
Lslab[us_, d_] := Re[-L^2 * NIntegrate[
    -(sln⟦2⟧[1 / u]) / (Sqrt[(us / u)^(2 * d) - 1]), {u, 1 / BCval2⟦-1⟧, us}]]

(* CALCULATION OF THE ADS BLACK BRANE
 AT THE SAME r_h and the same temperature *)
```

```
RSmax = BCval1[[-1]];

(* CALCULATION OF THE ENTROPY OF THE ADS BLACK BRANE AT
 THE SAME RADIUS OF HORIZON AS OUR INTERPOLATING BLACK BRANE *)
Cads[r_, rh_, d_] := (L / (r * Sqrt[1 - (rh / r) ^ (d + 1)])) ^ 2
(* PURE ADS FIELD VALUES *)
ΔSAdS1[us_, d_ ] :=
 Re[NIntegrate[1 / u ^ 3 * (C1AdS[1 / u, d, rh] - 1) / (Sqrt[1 - (u / us) ^ (2 * d)]),
   {u, 1 / BCval2[[-1]], us}]]


(* EXTRACTING THE RADIUS OF HORIZON FOR THE ADS BLACK BRANE
 AT THE SAME TEMPERATURE AS OUR INTERPOLATING BLACK BRANE *)
ThAds[rh_, d_] := Sqrt[
   (D[Aads[r, rh1, d], r] /. r → rh1) * (D[1 / Cads[r, rh1, d], r] /. r → rh1)] / (4 * Pi)
rHAds = N[Solve[ThAds[rh, 3] == BCval1[[6]], rh1][[2, 1, 2]]]

C1AdS[r_, d_, rh1_] := 1 / Sqrt[1 - (rh1 / r) ^ (d + 1)]
(* VALUES OF THE FIELD VALUES FOR ADS BLACK BRANE AT THE SAME TEMPERATURE *)
(* CALCULATION OF THE ENTROPY OF THE ADS BLACK BRANE
 AT THE TEMPERATURE AS OUR INTERPOLATING BLACK BRANE *)
ΔSAdS[us_, d_ ] :=
 Re[NIntegrate[1 / u ^ 3 * (C1AdS[1 / u, d, rHAds] - 1) / (Sqrt[1 - (u / us) ^ (2 * d)]),
   {u, 1 / BCval2[[-1]], us}]]

(* PLOT *)
If[rHAds < rh, rIntAdS = rh, rIntAdS = rHAds]; (* Set the range of the AdS brane,
so that we do not plot the divergence point where r = rh. *)
Legended[Show[ (* PLOT THE ENTANGLEMENT ENTROPY VS L_slab FOR THE AdS black
   branes(ABB) at the same temperature and same radius of horizon *)
  ParametricPlot[{Lslab[us, d], ΔSAdS[us, d]}, {us, 1 / RSmax, 1 / rIntAdS },
   ScalingFunctions → {"Log", "Log"}, Epilog → {{Green, Thick, Dashed,
      Line[{{Log[BCval1[[5]]], -100}, {Log[BCval1[[5]]], 100}}]}, {Orange, Thick,
      Dashed, Line[{{Log[ BCval1[[6]]], -100}, { Log[BCval1[[6]]], 100}}]}},
   PlotStyle → {Dashed, Black},  Frame → True , FrameTicks → All,
   FrameLabel → {Style["Log[L_slab]", Black, 25], Style[ "Log[4G_NΔS_IBB]", Black, 25]},
   FrameTicksStyle → Directive[Black, 20], Axes → False,
   Mesh → False, AspectRatio → 1 / GoldenRatio],
  ParametricPlot[{Lslab[us, d], ΔSAdS1[us, d]}, {us, 1 / RSmax, 1 / rh },
   ScalingFunctions → {"Log", "Log"}, Epilog → {{Green, Thick, Dashed,
      Line[{{Log[BCval1[[5]]], -100}, {Log[BCval1[[5]]], 100}}]}, {Orange, Thick,
      Dashed, Line[{{Log[ BCval1[[6]]], -100}, { Log[BCval1[[6]]], 100}}]}},
   PlotStyle → {Dashed, Red},  Frame → True , FrameTicks → All,
```

```
  FrameLabel → {Style["Lslab", Black, 25], Style[ "4GN ΔS", Black, 25]},
  FrameTicksStyle → Directive[Black, 20], RotateLabel → {False, True},
  Axes → False, Mesh → False, AspectRatio → 1 / GoldenRatio],
 (* PLOT THE ENTANGLEMENT ENTROPY VS L_SLAB FOR
  THE INTERPOLATING BLACK BRANE (IBB) *)
 ParametricPlot[{Lslab[us, d], ΔSIBB[us, d]}, {us, 1 / RSmax, 1 / rh},
  ScalingFunctions → {"Log", "Log"}, Epilog → {{Red, Thick, Dashed,
     Line[{{Log[BCval1⟦5⟧], -100}, {Log[BCval1⟦5⟧], 100}}]}, {Orange, Thick,
     Dashed, Line[{{Log[ BCval1⟦6⟧], -100}, { Log[BCval1⟦6⟧], 100}}]}},
  PlotStyle → {Thick, Blue}, Frame → True , FrameTicks → All,
  FrameLabel → {Style["Log[Lslab]", Black, 25], Style[ "Log[4GNΔSIBB]", Black, 25]},
  FrameTicksStyle → Directive[Black, 20], Axes → False, Mesh → False,
  AspectRatio → 1 / GoldenRatio], PlotRange → All, ImageSize → 1000],
 Placed[LineLegend[{Blue, {Dashed, Black}, {Dashed, Red}},
  {"IBB", "AdS (same T)", "AdS (same rh)"}, LegendLayout → "Column",
  LegendMarkerSize → 25, LabelStyle → {Black, FontSize → 25, Font → "Arial"},
  LegendFunction → Framed], {0.17, 0.82}]]
```

Out[ ]=

```
{2.01841, 1., 10.0432, 25.6717, 0.98206, 0.157704, 6.22725, 1000.}
```

Out[ ]=

```
{1., 1., 0.982705, 0.999994, 0.982708, 0.157704, 6.23136, 1000.}
```

Out[ ]=

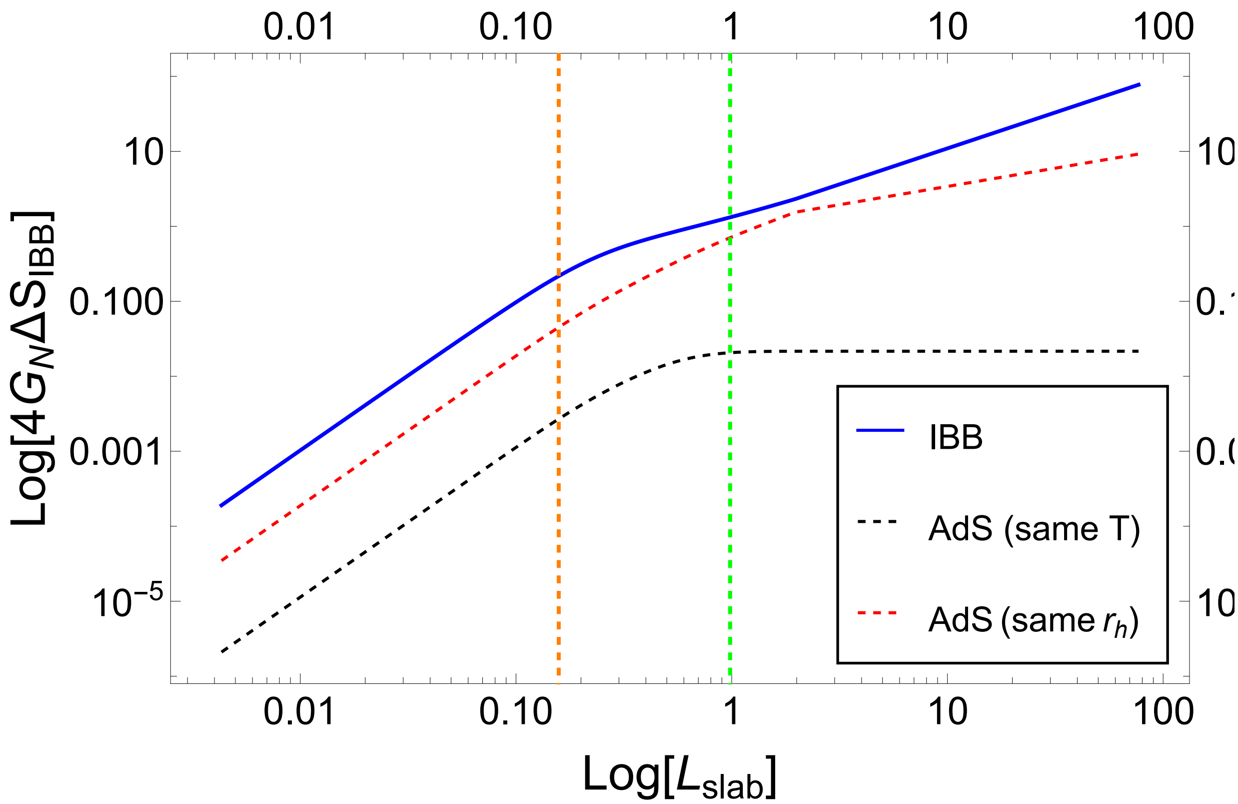
```
{InterpolatingFunction[ Domain: {{1., 1.00 ×10^3}} Output: scalar ],
 InterpolatingFunction[ Domain: {{1., 1.00 ×10^3}} Output: scalar ],
 InterpolatingFunction[ Domain: {{1., 1.00 ×10^3}} Output: scalar ],
 InterpolatingFunction[ Domain: {{1., 1.00 ×10^3}} Output: scalar ]}
```

Out[ ]=

```
0.495438
```

NIntegrate : Numerical integration converging too slowly; suspect one of the following: singularity, value of the integration is 0, highly oscillatory integrand, or WorkingPrecision too small.

NIntegrate : NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in u near {u} = {0.00104370687660094558460877750549755660358641762286424636840820312500}. NIntegrate obtained 2.1055513154500188\` \*^-6 and 3.755081109160993\` *^-12 for the integral and error estimates.



RESULTS:
THE CHANGE IN CURVATURE BECOMES MORE PRONOUNCED NOW AND OCCURS EXCTLY WHERE THE PHASE TRANSITION OCCURS. IF THE CHANGE OF CURVATURE BECOMES TOO SHARP, IT MIGHT INDICATE A VIOLATION OF THE C-THEOREM.
#

WE TAKE c0 = 0.95 (VERY CLOSE TO A PURE LIFSHITZ BRANE)

```
rh = 1; c0 = 0.95; d = 3; α = 2;
BCval1 = slngen2[c0, rh, d, α, 1, 1, 1]
BCval2 = slngen2[c0, rh, d, α, Sqrt[BCval1⟦4⟧], 1 / BCval1⟦1⟧, 1]
sln = slnhold2[c0, rh, d, α, Sqrt[BCval1⟦4⟧], 1 / BCval1⟦1⟧, 1]
L = 1;
```

```
(* CALCULATE THE ENTANGLEMENT ENTROPY AND THE SIZE OF THE SLAB OF THE
  INTERPOLATING BLACK BRANES AS A FUNCTION OF u = 1/r. THE GOAL OF THIS IS
  TO PICK THE LEAST UNSTABLE SOLUTIONS UNDER NUMERICAL INTERPOLATION.  *)
Clear[Lslab, ΔSIBB, ΔSIBBr, xIBB, xIBBr, Lslab, Lslabr]

ΔSIBB[us_, d_] := Re[NIntegrate[
   1 / u^3 * (sln[[2]][1 / u] - 1) / (Sqrt[1 - (u / us)^(2 * d)]), {u, 1 / BCval2[[-1]], us}]]
Lslab[us_, d_] := Re[-L^2 * NIntegrate[
    -(sln[[2]][1 / u]) / (Sqrt[(us / u)^(2 * d) - 1]), {u, 1 / BCval2[[-1]], us}]]

(* CALCULATION OF THE ADS BLACK BRANE
 AT THE SAME r_h and the same temperature *)

RSmax = BCval1[[-1]];

(* CALCULATION OF THE ENTROPY OF THE ADS BLACK BRANE AT
 THE SAME RADIUS OF HORIZON AS OUR INTERPOLATING BLACK BRANE *)
Cads[r_, rh_, d_] := (L / (r * Sqrt[1 - (rh / r)^(d + 1)]))^2
(* PURE ADS FIELD VALUES *)
ΔSAdS1[us_, d_] :=
 Re[NIntegrate[1 / u^3 * (C1AdS[1 / u, d, rh] - 1) / (Sqrt[1 - (u / us)^(2 * d)]),
   {u, 1 / BCval2[[-1]], us}]]


(* EXTRACTING THE RADIUS OF HORIZON FOR THE ADS BLACK BRANE
 AT THE SAME TEMPERATURE AS OUR INTERPOLATING BLACK BRANE *)
ThAds[rh_, d_] := Sqrt[
   (D[Aads[r, rh1, d], r] /. r → rh1) * (D[1 / Cads[r, rh1, d], r] /. r → rh1)] / (4 * Pi)
rHAds = N[Solve[ThAds[rh, 3] == BCval1[[6]], rh1][[2, 1, 2]]]

C1AdS[r_, d_, rh1_] := 1 / Sqrt[1 - (rh1 / r)^(d + 1)]
(* VALUES OF THE FIELD VALUES FOR ADS BLACK BRANE AT THE SAME TEMPERATURE *)
(* CALCULATION OF THE ENTROPY OF THE ADS BLACK BRANE
 AT THE TEMPERATURE AS OUR INTERPOLATING BLACK BRANE *)
ΔSAdS[us_, d_] :=
 Re[NIntegrate[1 / u^3 * (C1AdS[1 / u, d, rHAds] - 1) / (Sqrt[1 - (u / us)^(2 * d)]),
   {u, 1 / BCval2[[-1]], us}]]

(* PLOT *)
If[rHAds < rh, rIntAdS = rh, rIntAdS = rHAds]; (* Set the range of the AdS brane,
so that we do not plot the divergence point where r = rh. *)
Legended[Show[ (* PLOT THE ENTANGLEMENT ENTROPY VS L_slab FOR THE AdS black
   branes(ABB) at the same temperature and same radius of horizon *)
```

```
ParametricPlot[{Lslab[us, d], ΔSAdS[us, d]}, {us, 1 / RSmax, 1 / rIntAdS },
  ScalingFunctions → {"Log", "Log"}, Epilog → {{Green, Thick, Dashed,
     Line[{{Log[BCval1[[5]]], -100}, {Log[BCval1[[5]]], 100}}]}, {Orange, Thick,
     Dashed, Line[{{Log[ BCval1[[6]]], -100}, { Log[BCval1[[6]]], 100}}]}},
  PlotStyle → {Dashed, Black},  Frame → True , FrameTicks → All,
  FrameLabel → {Style["Log[L_slab]", Black, 25], Style[ "Log[4G_NΔS_IBB]", Black, 25]},
  FrameTicksStyle → Directive[Black, 20], Axes → False,
  Mesh → False, AspectRatio → 1 / GoldenRatio],
ParametricPlot[{Lslab[us, d], ΔSAdS1[us, d]}, {us, 1 / RSmax, 1 / rh },
  ScalingFunctions → {"Log", "Log"}, Epilog → {{Green, Thick, Dashed,
     Line[{{Log[BCval1[[5]]], -100}, {Log[BCval1[[5]]], 100}}]}, {Orange, Thick,
     Dashed, Line[{{Log[ BCval1[[6]]], -100}, { Log[BCval1[[6]]], 100}}]}},
  PlotStyle → {Dashed, Red},  Frame → True , FrameTicks → All,
  FrameLabel → {Style["L_slab", Black, 25], Style[ "4G_N ΔS", Black, 25]},
  FrameTicksStyle → Directive[Black, 20], RotateLabel → {False, True},
  Axes → False, Mesh → False, AspectRatio → 1 / GoldenRatio],
 (* PLOT THE ENTANGLEMENT ENTROPY VS L_SLAB FOR
  THE INTERPOLATING BLACK BRANE (IBB) *)
ParametricPlot[{Lslab[us, d], ΔSIBB[us, d]}, {us, 1 / RSmax, 1 / rh},
  ScalingFunctions → {"Log", "Log"}, Epilog → {{Red, Thick, Dashed,
     Line[{{Log[BCval1[[5]]], -100}, {Log[BCval1[[5]]], 100}}]}, {Orange, Thick,
     Dashed, Line[{{Log[ BCval1[[6]]], -100}, { Log[BCval1[[6]]], 100}}]}},
  PlotStyle → {Thick, Blue}, Frame → True , FrameTicks → All,
  FrameLabel → {Style["Log[L_slab]", Black, 25], Style[ "Log[4G_NΔS_IBB]", Black, 25]},
  FrameTicksStyle → Directive[Black, 20], Axes → False, Mesh → False,
  AspectRatio → 1 / GoldenRatio], PlotRange → All, ImageSize → 1000],
Placed[LineLegend[{Blue, {Dashed, Black}, {Dashed, Red}},
  {"IBB", "AdS (same T)", "AdS (same r_h)"}, LegendLayout → "Column",
  LegendMarkerSize → 25, LabelStyle → {Black, FontSize → 25, Font → "Arial"},
  LegendFunction → Framed], {0.17, 0.82}]]
```

Out[ ]=

```
{2.88215, 1., 39.373, 113.439, 1.28263, 0.110442, 11.6136, 1000.}
```

Out[ ]=

```
{1., 1., 1.28301, 1.00001, 1.283, 0.110442, 11.6169, 1000.}
```

Out[ ]=

{InterpolatingFunction[Domain: {{1., 1.00 $\times 10^3$}} Output: scalar],

InterpolatingFunction[Domain: {{1., 1.00 $\times 10^3$}} Output: scalar],

InterpolatingFunction[Domain: {{1., 1.00 $\times 10^3$}} Output: scalar],

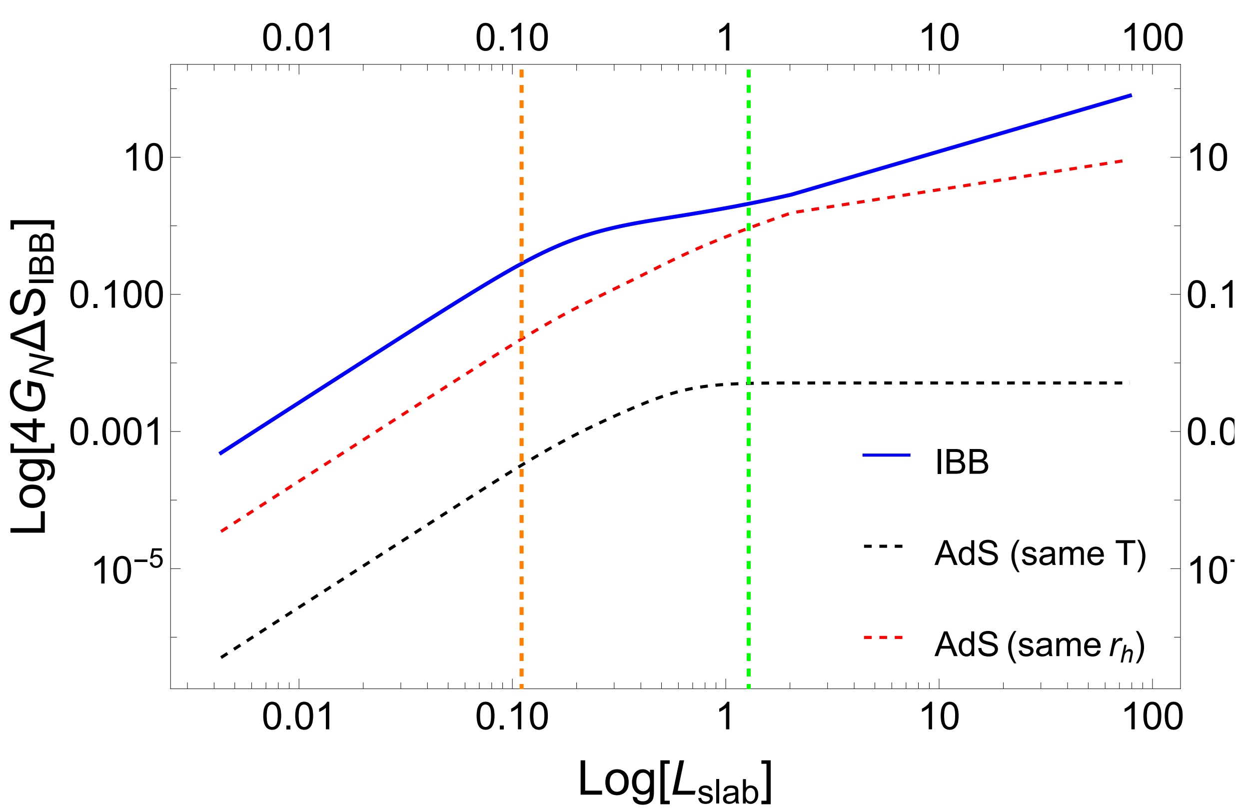
InterpolatingFunction[Domain: {{1., 1.00 $\times 10^3$}} Output: scalar]}

Out[ ]=

0.346963

NIntegrate : Numerical integration converging too slowly; suspect one of the following: singularity, value of the integration is 0, highly oscillatory integrand, or WorkingPrecision too small.

NIntegrate : NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in u near {u} = {0.00191597 }. NIntegrate obtained 5.064942805158671` *^−7 and 1.5053725736422425` *^−12 for the integral and error estimates.



RESULTS:

CHANGE OF CURVATURE IS EVEN MORE PRONOUNCED.
#

WE LOOK AT THE EFFECTS OF CHANGING OUR OTHER FREE PARAMETERS ON THE CURVATURE:
rh: 1 ---> 2

```
rh = 2; c0 = 0.95; d = 3; α = 2;
BCval1 = slngen2[c0, rh, d, α, 1, 1, 1]
BCval2 = slngen2[c0, rh, d, α, Sqrt[BCval1⟦4⟧], 1 / BCval1⟦1⟧, 1]
sln = slnhold2[c0, rh, d, α, Sqrt[BCval1⟦4⟧], 1 / BCval1⟦1⟧, 1]
L = 1;
(* CALCULATE THE ENTANGLEMENT ENTROPY AND THE SIZE OF THE SLAB OF THE
  INTERPOLATING BLACK BRANES AS A FUNCTION OF u = 1/r. THE GOAL OF THIS IS
  TO PICK THE LEAST UNSTABLE SOLUTIONS UNDER NUMERICAL INTERPOLATION.  *)
Clear[Lslab, ΔSIBB, ΔSIBBr, xIBB, xIBBr, Lslab, Lslabr]

ΔSIBB[us_, d_ ] := Re[NIntegrate[
   1 / u ^ 3 * ( sln⟦2⟧[1 / u] - 1) / (Sqrt[1 - (u / us) ^ (2 * d)]), {u, 1 / BCval2⟦-1⟧, us}]]
Lslab[us_, d_ ] := Re[-L ^ 2 * NIntegrate[
    - ( sln⟦2⟧[1 / u]) / (Sqrt[(us / u) ^ (2 * d) - 1]), {u, 1 / BCval2⟦-1⟧, us}]]

(* CALCULATION OF THE ADS BLACK BRANE
 AT THE SAME r_h and the same temperature *)

RSmax = BCval1⟦-1⟧;

(* CALCULATION OF THE ENTROPY OF THE ADS BLACK BRANE AT
 THE SAME RADIUS OF HORIZON AS OUR INTERPOLATING BLACK BRANE *)
Cads[r_, rh_, d_] := (L / (r * Sqrt[1 - (rh / r) ^ (d + 1)])) ^ 2
(* PURE ADS FIELD VALUES *)
ΔSAdS1[us_, d_ ] :=
 Re[NIntegrate[1 / u ^ 3 * (C1AdS[1 / u, d, rh] - 1) / (Sqrt[1 - (u / us) ^ (2 * d)]),
   {u, 1 / BCval2⟦-1⟧, us}]]


(* EXTRACTING THE RADIUS OF HORIZON FOR THE ADS BLACK BRANE
 AT THE SAME TEMPERATURE AS OUR INTERPOLATING BLACK BRANE *)
```

```
ThAds[rh_, d_] := Sqrt[
   (D[Aads[r, rh1, d], r] /. r → rh1) * (D[1 / Cads[r, rh1, d], r] /. r → rh1)] / (4 * Pi)
rHAds = N[Solve[ThAds[rh, 3] == BCval1⟦6⟧, rh1]⟦2, 1, 2⟧]

C1AdS[r_, d_, rh1_] := 1 / Sqrt[1 - (rh1 / r) ^ (d + 1)]
(* VALUES OF THE FIELD VALUES FOR ADS BLACK BRANE AT THE SAME TEMPERATURE *)
(* CALCULATION OF THE ENTROPY OF THE ADS BLACK BRANE
 AT THE TEMPERATURE AS OUR INTERPOLATING BLACK BRANE *)
ΔSAdS[us_, d_ ] :=
 Re[NIntegrate[1 / u ^ 3 * (C1AdS[1 / u, d, rHAds] - 1) / (Sqrt[1 - (u / us) ^ (2 * d)]),
   {u, 1 / BCval2⟦-1⟧, us}]]

(* PLOT *)
If[rHAds < rh, rIntAdS = rh, rIntAdS = rHAds]; (* Set the range of the AdS brane,
so that we do not plot the divergence point where r = rh. *)
Legended[Show[ (* PLOT THE ENTANGLEMENT ENTROPY VS L_slab FOR THE AdS black
   branes(ABB) at the same temperature and same radius of horizon *)
  ParametricPlot[{Lslab[us, d], ΔSAdS[us, d]}, {us, 1 / RSmax, 1 / rIntAdS },
   ScalingFunctions → {"Log", "Log"}, Epilog → {{Green, Thick, Dashed,
      Line[{{Log[BCval1⟦5⟧], -100}, {Log[BCval1⟦5⟧], 100}}]}, {Orange, Thick,
      Dashed, Line[{{Log[ BCval1⟦6⟧], -100}, { Log[BCval1⟦6⟧], 100}}]}},
   PlotStyle → {Dashed, Black},  Frame → True , FrameTicks → All,
   FrameLabel → {Style["Log[L_slab]", Black, 25], Style[ "Log[4G_NΔS_IBB]", Black, 25]},
   FrameTicksStyle → Directive[Black, 20], Axes → False,
   Mesh → False, AspectRatio → 1 / GoldenRatio],
  ParametricPlot[{Lslab[us, d], ΔSAdS1[us, d]}, {us, 1 / RSmax, 1 / rh },
   ScalingFunctions → {"Log", "Log"}, Epilog → {{Green, Thick, Dashed,
      Line[{{Log[BCval1⟦5⟧], -100}, {Log[BCval1⟦5⟧], 100}}]}, {Orange, Thick,
      Dashed, Line[{{Log[ BCval1⟦6⟧], -100}, { Log[BCval1⟦6⟧], 100}}]}},
   PlotStyle → {Dashed, Red},  Frame → True , FrameTicks → All,
   FrameLabel → {Style["L_slab", Black, 25], Style[ "4G_N ΔS", Black, 25]},
   FrameTicksStyle → Directive[Black, 20], RotateLabel → {False, True},
   Axes → False, Mesh → False, AspectRatio → 1 / GoldenRatio],
  (* PLOT THE ENTANGLEMENT ENTROPY VS L_SLAB FOR
   THE INTERPOLATING BLACK BRANE (IBB) *)
  ParametricPlot[{Lslab[us, d], ΔSIBB[us, d]}, {us, 1 / RSmax, 1 / rh},
   ScalingFunctions → {"Log", "Log"}, Epilog → {{Red, Thick, Dashed,
      Line[{{Log[BCval1⟦5⟧], -100}, {Log[BCval1⟦5⟧], 100}}]}, {Orange, Thick,
      Dashed, Line[{{Log[ BCval1⟦6⟧], -100}, { Log[BCval1⟦6⟧], 100}}]}},
   PlotStyle → {Thick, Blue}, Frame → True , FrameTicks → All,
   FrameLabel → {Style["Log[L_slab]", Black, 25], Style[ "Log[4G_NΔS_IBB]", Black, 25]},
   FrameTicksStyle → Directive[Black, 20], Axes → False, Mesh → False,
   AspectRatio → 1 / GoldenRatio], PlotRange → All, ImageSize → 1000],
 Placed[LineLegend[{Blue, {Dashed, Black}, {Dashed, Red}},
```

```
{"IBB", "AdS (same T)", "AdS (same r_h)"}, LegendLayout → "Column",
LegendMarkerSize → 25, LabelStyle → {Black, FontSize → 25, Font → "Arial"},
LegendFunction → Framed], {0.17, 0.82}]]
```

Out[ ]=

```
{2.88215, 1., 629.98, 7260.09, 2.5653, 0.220884, 11.6138, 2000.}
```

Out[ ]=

```
{0.999999, 1., 2.56308, 0.999982, 2.5631, 0.220884, 11.6039, 2000.}
```

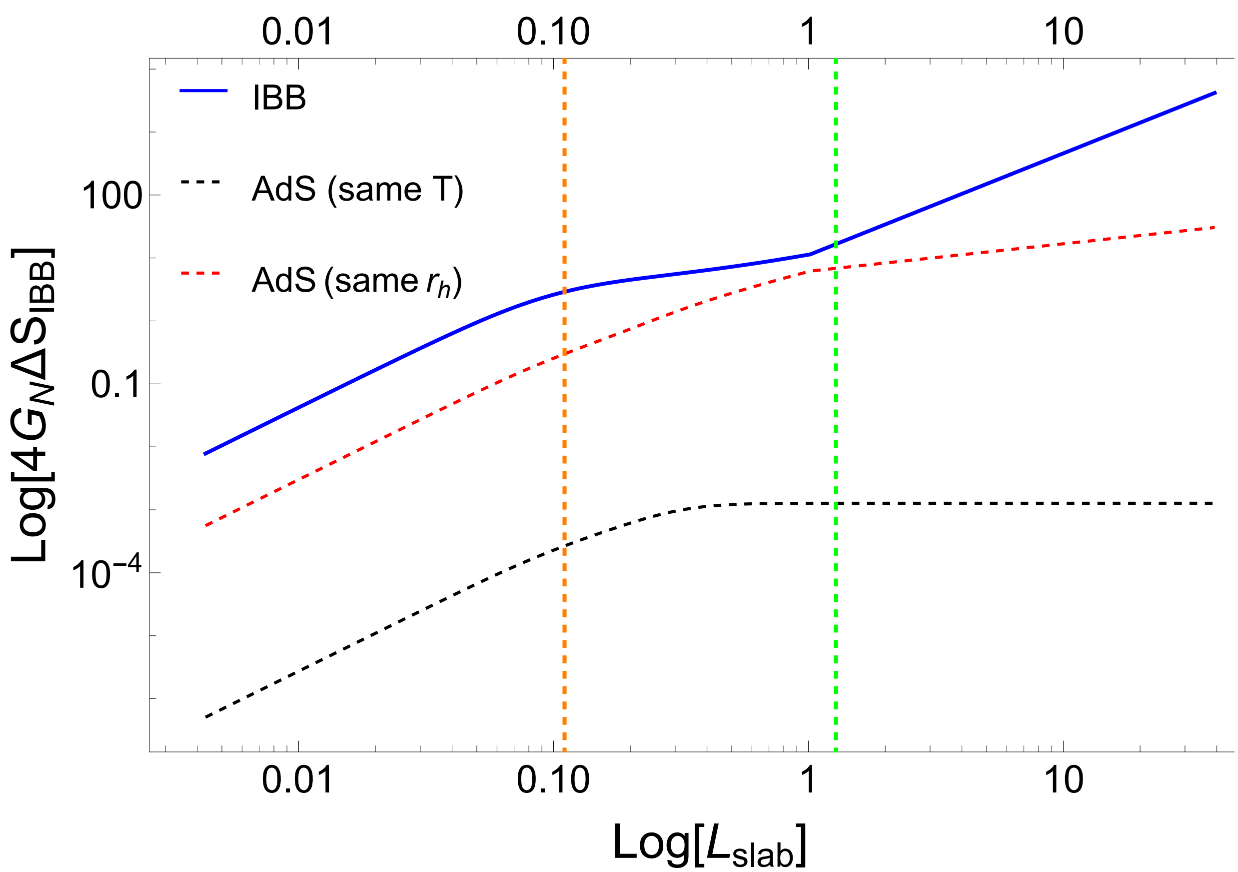
Out[ ]=

```
{InterpolatingFunction[ Domain: {{2., 2.00 ×10^3}}  Output: scalar ],
 InterpolatingFunction[ Domain: {{2., 2.00 ×10^3}}  Output: scalar ],
 InterpolatingFunction[ Domain: {{2., 2.00 ×10^3}}  Output: scalar ],
 InterpolatingFunction[ Domain: {{2., 2.00 ×10^3}}  Output: scalar ]}
```

Out[ ]=

```
0.346963
```

NIntegrate : NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in u near {u} = {0.000599876927610635019981651094855834571717423386871814727783203125000}. NIntegrate obtained 5.082856174725574` *^-7 and 7.17129358957050 44` *^-12 for the integral and error estimates.



CHANGING d : 3 -- -> 6

```
rh = 1; c0 = 0.95; d = 6; α = 2;
BCval1 = slngen2[c0, rh, d, α, 1, 1, 1]
BCval2 = slngen2[c0, rh, d, α, Sqrt[BCval1⟦4⟧], 1 / BCval1⟦1⟧, 1]
sln = slnhold2[c0, rh, d, α, Sqrt[BCval1⟦4⟧], 1 / BCval1⟦1⟧, 1]
L = 1;

(* CALCULATE THE ENTANGLEMENT ENTROPY AND THE SIZE OF THE SLAB OF THE
  INTERPOLATING BLACK BRANES AS A FUNCTION OF u = 1/r. THE GOAL OF THIS IS
  TO PICK THE LEAST UNSTABLE SOLUTIONS UNDER NUMERICAL INTERPOLATION.  *)
Clear[Lslab, ΔSIBB, ΔSIBBr, xIBB, xIBBr, Lslab, Lslabr]

ΔSIBB[us_, d_] := Re[NIntegrate[
   1 / u^3 * (sln⟦2⟧[1 / u] - 1) / (Sqrt[1 - (u / us)^(2 * d)]), {u, 1 / BCval2⟦-1⟧, us}]]
Lslab[us_, d_] := Re[-L^2 * NIntegrate[
    -(sln⟦2⟧[1 / u]) / (Sqrt[(us / u)^(2 * d) - 1]), {u, 1 / BCval2⟦-1⟧, us}]]

(* CALCULATION OF THE ADS BLACK BRANE
 AT THE SAME r_h and the same temperature *)
```

```
RSmax = BCval1[[-1]];

(* CALCULATION OF THE ENTROPY OF THE ADS BLACK BRANE AT
 THE SAME RADIUS OF HORIZON AS OUR INTERPOLATING BLACK BRANE *)
Cads[r_, rh_, d_] := (L / (r * Sqrt[1 - (rh / r) ^ (d + 1)])) ^ 2
(* PURE ADS FIELD VALUES *)
ΔSAdS1[us_, d_ ] :=
 Re[NIntegrate[1 / u ^ 3 * (C1AdS[1 / u, d, rh] - 1) / (Sqrt[1 - (u / us) ^ (2 * d)]),
   {u, 1 / BCval2[[-1]], us}]]


(* EXTRACTING THE RADIUS OF HORIZON FOR THE ADS BLACK BRANE
 AT THE SAME TEMPERATURE AS OUR INTERPOLATING BLACK BRANE *)
ThAds[rh_, d_] := Sqrt[
   (D[Aads[r, rh1, d], r] /. r → rh1) * (D[1 / Cads[r, rh1, d], r] /. r → rh1)] / (4 * Pi)
rHAds = N[Solve[ThAds[rh, 3] == BCval1[[6]], rh1][[2, 1, 2]]]

C1AdS[r_, d_, rh1_] := 1 / Sqrt[1 - (rh1 / r) ^ (d + 1)]
(* VALUES OF THE FIELD VALUES FOR ADS BLACK BRANE AT THE SAME TEMPERATURE *)
(* CALCULATION OF THE ENTROPY OF THE ADS BLACK BRANE
 AT THE TEMPERATURE AS OUR INTERPOLATING BLACK BRANE *)
ΔSAdS[us_, d_ ] :=
 Re[NIntegrate[1 / u ^ 3 * (C1AdS[1 / u, d, rHAds] - 1) / (Sqrt[1 - (u / us) ^ (2 * d)]),
   {u, 1 / BCval2[[-1]], us}]]

(* PLOT *)
If[rHAds < rh, rIntAdS = rh, rIntAdS = rHAds]; (* Set the range of the AdS brane,
so that we do not plot the divergence point where r = rh. *)
Legended[Show[ (* PLOT THE ENTANGLEMENT ENTROPY VS L_slab FOR THE AdS black
   branes(ABB) at the same temperature and same radius of horizon *)
  ParametricPlot[{Lslab[us, d], ΔSAdS[us, d]}, {us, 1 / RSmax, 1 / rIntAdS },
   ScalingFunctions → {"Log", "Log"}, Epilog → {{Green, Thick, Dashed,
      Line[{{Log[BCval1[[5]]], -100}, {Log[BCval1[[5]]], 100}}]}, {Orange, Thick,
      Dashed, Line[{{Log[ BCval1[[6]]], -100}, { Log[BCval1[[6]]], 100}}]}},
   PlotStyle → {Dashed, Black},  Frame → True , FrameTicks → All,
   FrameLabel → {Style["Log[L_slab]", Black, 25], Style[ "Log[4G_NΔS_IBB]", Black, 25]},
   FrameTicksStyle → Directive[Black, 20], Axes → False,
   Mesh → False, AspectRatio → 1 / GoldenRatio],
  ParametricPlot[{Lslab[us, d], ΔSAdS1[us, d]}, {us, 1 / RSmax, 1 / rh },
   ScalingFunctions → {"Log", "Log"}, Epilog → {{Green, Thick, Dashed,
      Line[{{Log[BCval1[[5]]], -100}, {Log[BCval1[[5]]], 100}}]}, {Orange, Thick,
      Dashed, Line[{{Log[ BCval1[[6]]], -100}, { Log[BCval1[[6]]], 100}}]}},
   PlotStyle → {Dashed, Red},  Frame → True , FrameTicks → All,
```

```
    FrameLabel → {Style["Lslab", Black, 25], Style[ "4GN ΔS", Black, 25]},
    FrameTicksStyle → Directive[Black, 20], RotateLabel → {False, True},
    Axes → False, Mesh → False, AspectRatio → 1 / GoldenRatio],
   (* PLOT THE ENTANGLEMENT ENTROPY VS L_SLAB FOR
    THE INTERPOLATING BLACK BRANE (IBB) *)
  ParametricPlot[{Lslab[us, d], ΔSIBB[us, d]}, {us, 1 / RSmax, 1 / rh},
    ScalingFunctions → {"Log", "Log"}, Epilog → {{Red, Thick, Dashed,
       Line[{{Log[BCval1⟦5⟧], -100}, {Log[BCval1⟦5⟧], 100}}]}, {Orange, Thick,
       Dashed, Line[{{Log[ BCval1⟦6⟧], -100}, { Log[BCval1⟦6⟧], 100}}]}},
    PlotStyle → {Thick, Blue}, Frame → True , FrameTicks → All,
    FrameLabel → {Style["Log[Lslab]", Black, 25], Style[ "Log[4GNΔSIBB]", Black, 25]},
    FrameTicksStyle → Directive[Black, 20], Axes → False, Mesh → False,
    AspectRatio → 1 / GoldenRatio], PlotRange → All, ImageSize → 1000],
 Placed[LineLegend[{Blue, {Dashed, Black}, {Dashed, Red}},
    {"IBB", "AdS (same T)", "AdS (same rh)"}, LegendLayout → "Column",
    LegendMarkerSize → 25, LabelStyle → {Black, FontSize → 25, Font → "Arial"},
    LegendFunction → Framed], {0.17, 0.82}]]
```

Out[ ]=

{2.9343, 1., 44.4851, 457.49, 0.708795, 0.189839, 3.73367, 51.7947}

Out[ ]=

{1., 1., 0.708695, 0.999963, 0.708709, 0.189839, 3.73322, 51.7947}

Out[ ]=

{InterpolatingFunction[Domain: {{1., 51.8 }} Output: scalar],

InterpolatingFunction[Domain: {{1., 51.8 }} Output: scalar],

InterpolatingFunction[Domain: {{1., 51.8 }} Output: scalar],

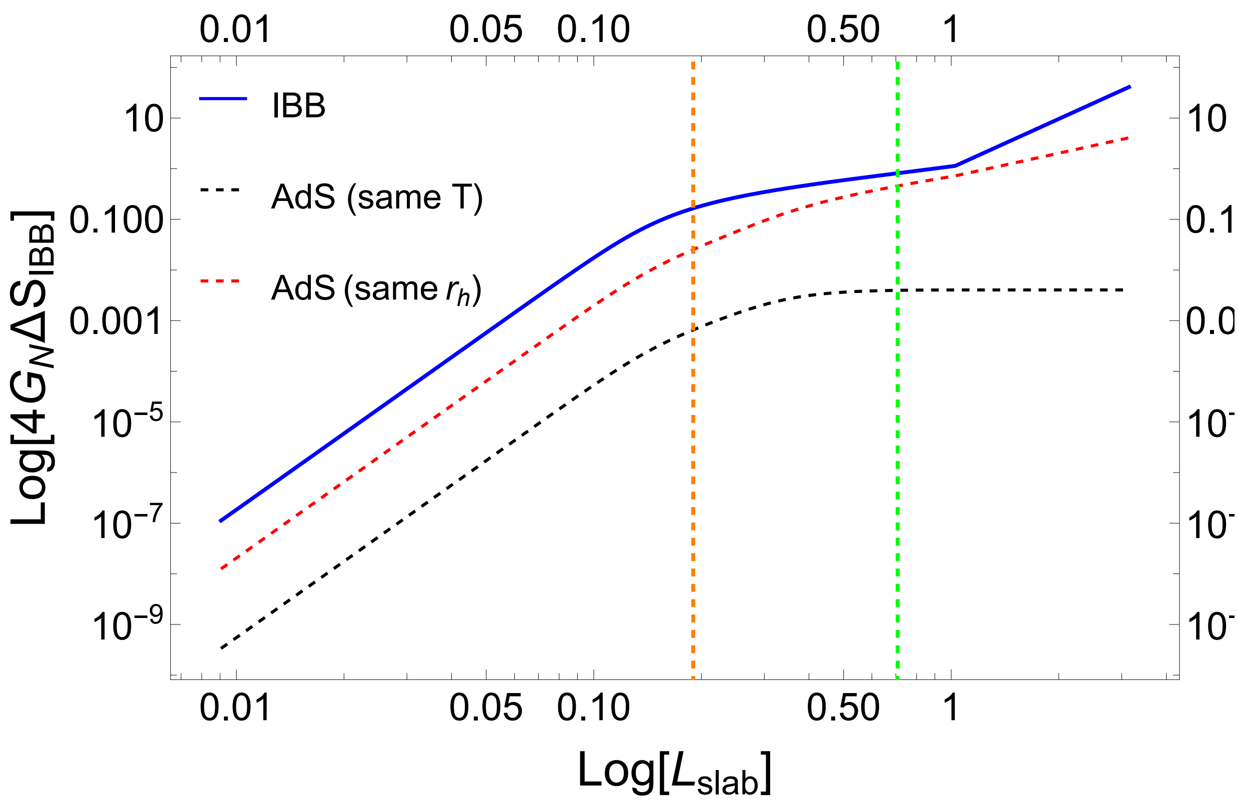
InterpolatingFunction[Domain: {{1., 51.8 }} Output: scalar]}

Out[ ]=

0.596395

NIntegrate : Numerical integration converging too slowly; suspect one of the following: singularity, value of the integration is 0, highly oscillatory integrand, or WorkingPrecision too small.

NIntegrate : NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in u near {u} = {0.0253085 }. NIntegrate obtained $3.3671 \times 10^{-10} - 2.00617 \times 10^{-19} i$ and 6.390180737112419` *^-15 for the integral and error estimates.



CHANGING α : 2 -- -> 4

```
rh = 1; c0 = 0.95; d = 2; α = 4;
BCval1 = slngen2[c0, rh, d, α, 1, 1, 1]
BCval2 = slngen2[c0, rh, d, α, Sqrt[BCval1[[4]]], 1 / BCval1[[1]], 1]
sln = slnhold2[c0, rh, d, α, Sqrt[BCval1[[4]]], 1 / BCval1[[1]], 1]
L = 1;

(* CALCULATE THE ENTANGLEMENT ENTROPY AND THE SIZE OF THE SLAB OF THE
  INTERPOLATING BLACK BRANES AS A FUNCTION OF u = 1/r. THE GOAL OF THIS IS
  TO PICK THE LEAST UNSTABLE SOLUTIONS UNDER NUMERICAL INTERPOLATION.  *)
Clear[Lslab, ΔSIBB, ΔSIBBr, xIBB, xIBBr, Lslab, Lslabr]

ΔSIBB[us_, d_] := Re[NIntegrate[
   1 / u^3 * (sln[[2]][1 / u] - 1) / (Sqrt[1 - (u / us)^(2 * d)]), {u, 1 / BCval2[[-1]], us}]]
Lslab[us_, d_] := Re[-L^2 * NIntegrate[
    -(sln[[2]][1 / u]) / (Sqrt[(us / u)^(2 * d) - 1]), {u, 1 / BCval2[[-1]], us}]]

(* CALCULATION OF THE ADS BLACK BRANE
 AT THE SAME r_h and the same temperature *)

RSmax = BCval1[[-1]];
```

```
(* CALCULATION OF THE ENTROPY OF THE ADS BLACK BRANE AT
 THE SAME RADIUS OF HORIZON AS OUR INTERPOLATING BLACK BRANE *)
Cads[r_, rh_, d_] := (L / (r * Sqrt[1 - (rh / r) ^ (d + 1)])) ^ 2
(* PURE ADS FIELD VALUES *)
ΔSAdS1[us_, d_ ] :=
 Re[NIntegrate[1 / u ^ 3 * (C1AdS[1 / u, d, rh] - 1) / (Sqrt[1 - (u / us) ^ (2 * d)]),
   {u, 1 / BCval2⟦-1⟧, us}]]


(* EXTRACTING THE RADIUS OF HORIZON FOR THE ADS BLACK BRANE
 AT THE SAME TEMPERATURE AS OUR INTERPOLATING BLACK BRANE *)
ThAds[rh_, d_] := Sqrt[
   (D[Aads[r, rh1, d], r] /. r → rh1) * (D[1 / Cads[r, rh1, d], r] /. r → rh1)] / (4 * Pi)
rHAds = N[Solve[ThAds[rh, 3] == BCval1⟦6⟧, rh1]⟦2, 1, 2⟧]

C1AdS[r_, d_, rh1_] := 1 / Sqrt[1 - (rh1 / r) ^ (d + 1)]
(* VALUES OF THE FIELD VALUES FOR ADS BLACK BRANE AT THE SAME TEMPERATURE *)
(* CALCULATION OF THE ENTROPY OF THE ADS BLACK BRANE
 AT THE TEMPERATURE AS OUR INTERPOLATING BLACK BRANE *)
ΔSAdS[us_, d_ ] :=
 Re[NIntegrate[1 / u ^ 3 * (C1AdS[1 / u, d, rHAds] - 1) / (Sqrt[1 - (u / us) ^ (2 * d)]),
   {u, 1 / BCval2⟦-1⟧, us}]]

(* PLOT *)
If[rHAds < rh, rIntAdS = rh, rIntAdS = rHAds]; (* Set the range of the AdS brane,
so that we do not plot the divergence point where r = rh. *)
Legended[Show[ (* PLOT THE ENTANGLEMENT ENTROPY VS L_slab FOR THE AdS black
   branes(ABB) at the same temperature and same radius of horizon *)
  ParametricPlot[{Lslab[us, d], ΔSAdS[us, d]}, {us, 1 / RSmax, 1 / rIntAdS },
   ScalingFunctions → {"Log", "Log"}, Epilog → {{Green, Thick, Dashed,
      Line[{{Log[BCval1⟦5⟧], -100}, {Log[BCval1⟦5⟧], 100}}]}, {Orange, Thick,
      Dashed, Line[{{Log[ BCval1⟦6⟧], -100}, { Log[BCval1⟦6⟧], 100}}]}},
   PlotStyle → {Dashed, Black},  Frame → True , FrameTicks → All,
   FrameLabel → {Style["Log[L_slab]", Black, 25], Style[ "Log[4G_NΔS_IBB]", Black, 25]},
   FrameTicksStyle → Directive[Black, 20], Axes → False,
   Mesh → False, AspectRatio → 1 / GoldenRatio],
  ParametricPlot[{Lslab[us, d], ΔSAdS1[us, d]}, {us, 1 / RSmax, 1 / rh },
   ScalingFunctions → {"Log", "Log"}, Epilog → {{Green, Thick, Dashed,
      Line[{{Log[BCval1⟦5⟧], -100}, {Log[BCval1⟦5⟧], 100}}]}, {Orange, Thick,
      Dashed, Line[{{Log[ BCval1⟦6⟧], -100}, { Log[BCval1⟦6⟧], 100}}]}},
   PlotStyle → {Dashed, Red},  Frame → True , FrameTicks → All,
   FrameLabel → {Style["L_slab", Black, 25], Style[ "4G_N ΔS", Black, 25]},
   FrameTicksStyle → Directive[Black, 20], RotateLabel → {False, True},
```

```
    Axes → False, Mesh → False, AspectRatio → 1 / GoldenRatio],
   (* PLOT THE ENTANGLEMENT ENTROPY VS L_SLAB FOR
    THE INTERPOLATING BLACK BRANE (IBB) *)
   ParametricPlot[{Lslab[us, d], ΔSIBB[us, d]}, {us, 1 / RSmax, 1 / rh},
    ScalingFunctions → {"Log", "Log"}, Epilog → {{Red, Thick, Dashed,
       Line[{{Log[BCval1[[5]]], -100}, {Log[BCval1[[5]]], 100}}]}, {Orange, Thick,
       Dashed, Line[{{Log[ BCval1[[6]]], -100}, { Log[BCval1[[6]]], 100}}]}},
    PlotStyle → {Thick, Blue}, Frame → True , FrameTicks → All,
    FrameLabel → {Style["Log[Lslab]", Black, 25], Style[ "Log[4GNΔSIBB]", Black, 25]},
    FrameTicksStyle → Directive[Black, 20], Axes → False, Mesh → False,
    AspectRatio → 1 / GoldenRatio], PlotRange → All, ImageSize → 1000],
  Placed[LineLegend[{Blue, {Dashed, Black}, {Dashed, Red}},
    {"IBB", "AdS (same T)", "AdS (same rh)"}, LegendLayout → "Column",
    LegendMarkerSize → 25, LabelStyle → {Black, FontSize → 25, Font → "Arial"},
    LegendFunction → Framed], {0.17, 0.82}]]
```

Out[ ]=

```
{1.37888, 1., 15.3783, 41.3404, 1.73457, 0.173135, 10.0186, 10000.}
```

Out[ ]=

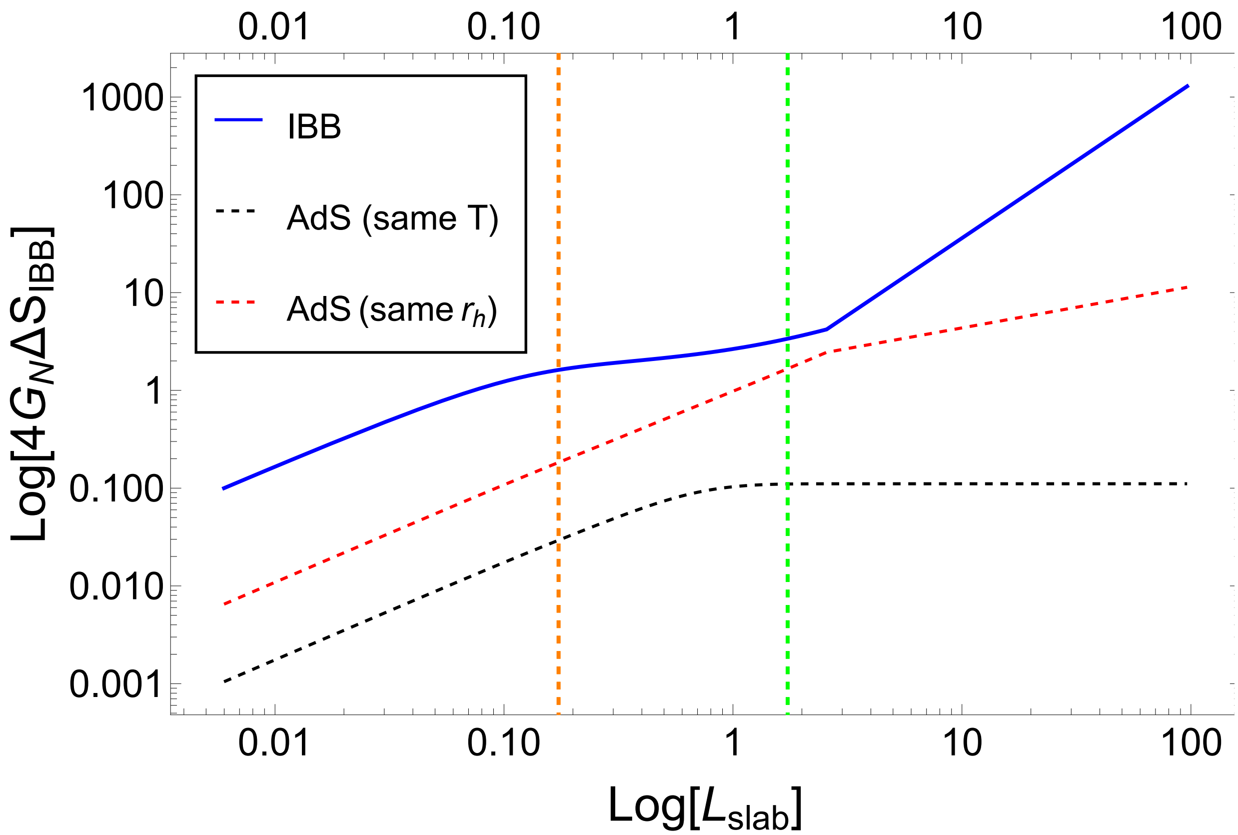
```
{0.999998, 1., 1.73324, 0.999955, 1.73328, 0.173135, 10.0111, 10000.}
```

Out[ ]=

```
{InterpolatingFunction[ Domain: {{1., 1.00 ×10^4}} Output: scalar ],
 InterpolatingFunction[ Domain: {{1., 1.00 ×10^4}} Output: scalar ],
 InterpolatingFunction[ Domain: {{1., 1.00 ×10^4}} Output: scalar ],
 InterpolatingFunction[ Domain: {{1., 1.00 ×10^4}} Output: scalar ]}
```

Out[ ]=

```
0.543918
```



SCRAP FILES

Scrap!

Circular Photon Orbits:

Get c0h:

r’’(x) and r(x) analysis at rt = 5.1 rH