

# EXPLORATION DE DONNÉES POUR L'OPTIMISATION DE TRAJECTOIRES AÉRIENNES

Cédric Rommel

Directeurs de thèse: Frédéric Bonnans, Pierre Martinon  
Encadrant Safety Line: Baptiste Gregorutti

Soutenance de thèse, 26 octobre 2018



# CONTEXT

# MOTIVATION

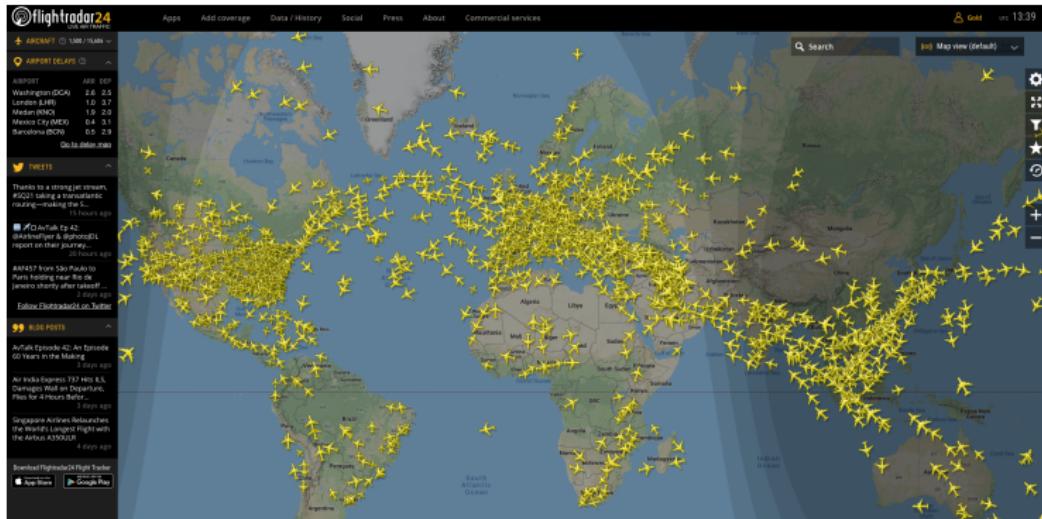


FIGURE: World air traffic - source: Flight Radar

# MOTIVATION

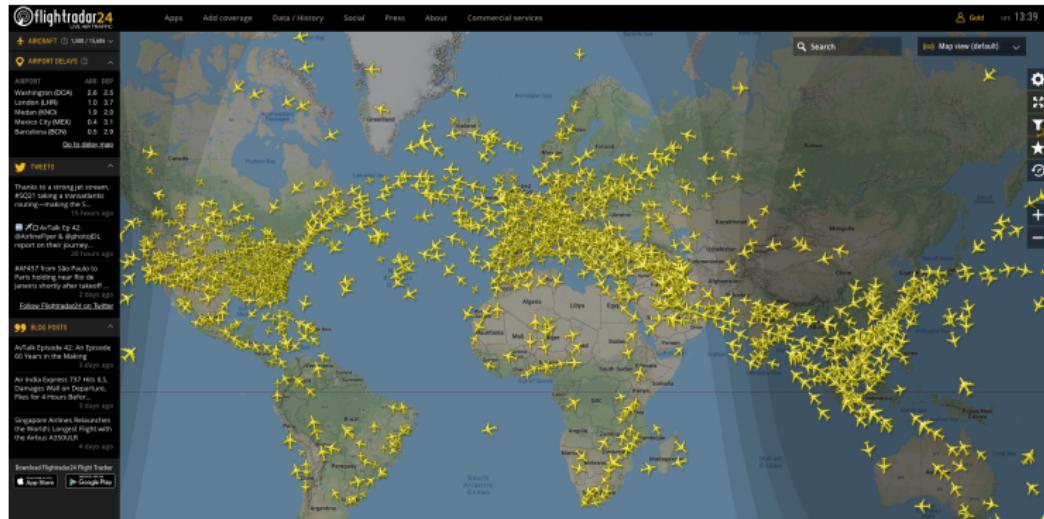


FIGURE: World air traffic - source: Flight Radar

- 20 000 airplanes — 80 000 flights per day,

# MOTIVATION

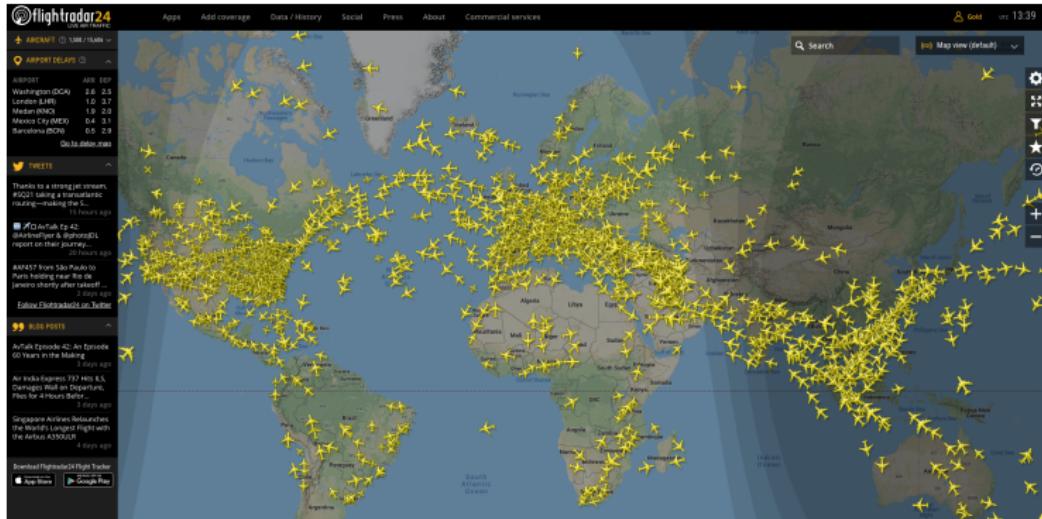
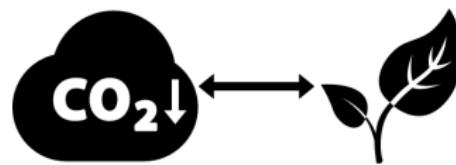


FIGURE: World air traffic - source: Flight Radar

- 20 000 airplanes — 80 000 flights per day,
- Should double until 2033,

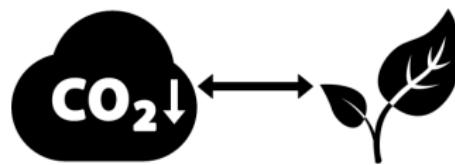
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- Most polluting means of transportation,



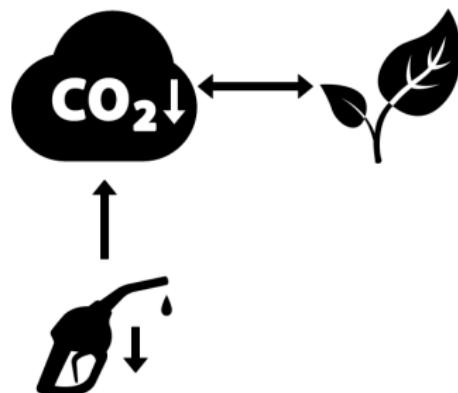
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- Most polluting means of transportation,
- Responsible for 3% of  $CO_2$  emissions,



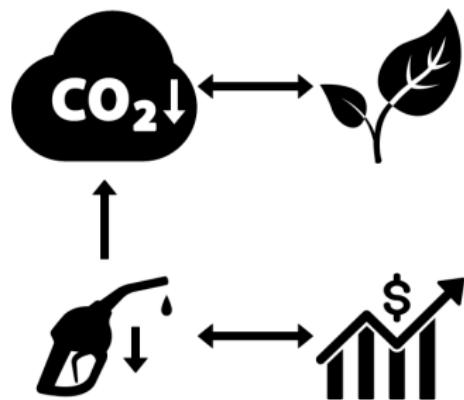
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- Most polluting means of transportation,
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- Fuel  $\simeq$  30% of an airline operational cost,



# MOTIVATION

How to tackle this problem ?

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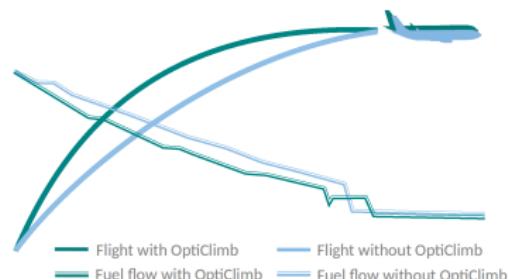
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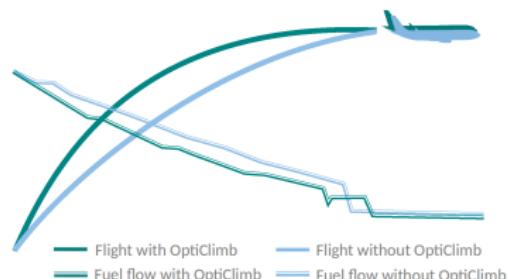
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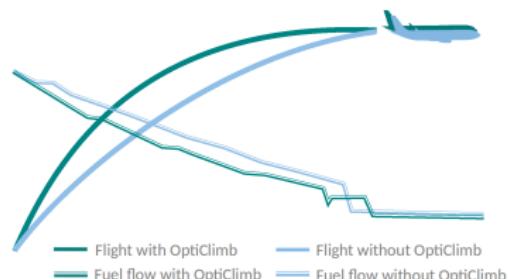
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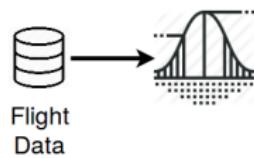
Flight  
Data

**Time**



*Many days before flight...*

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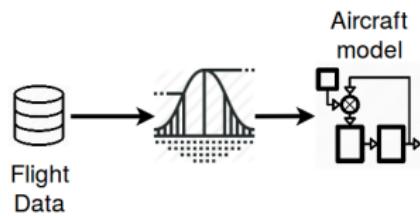


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Data

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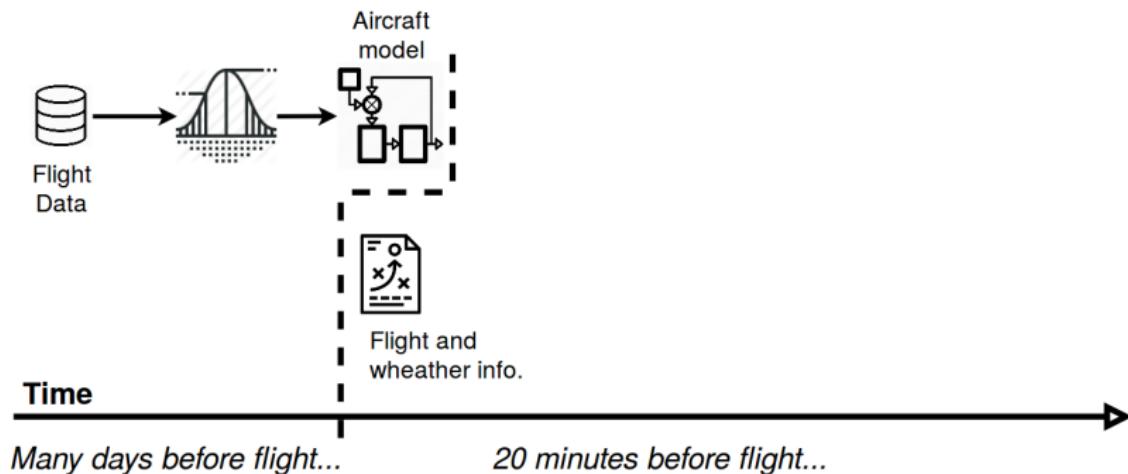
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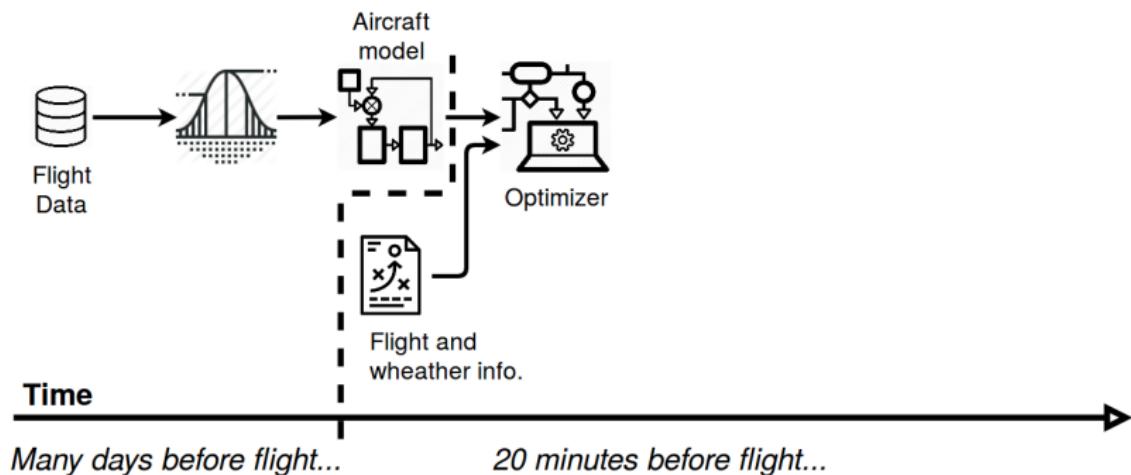
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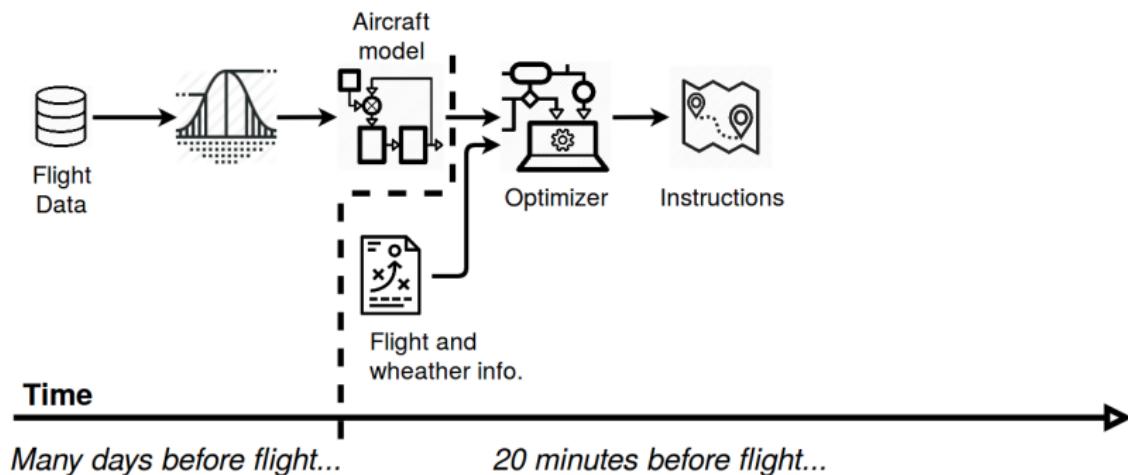
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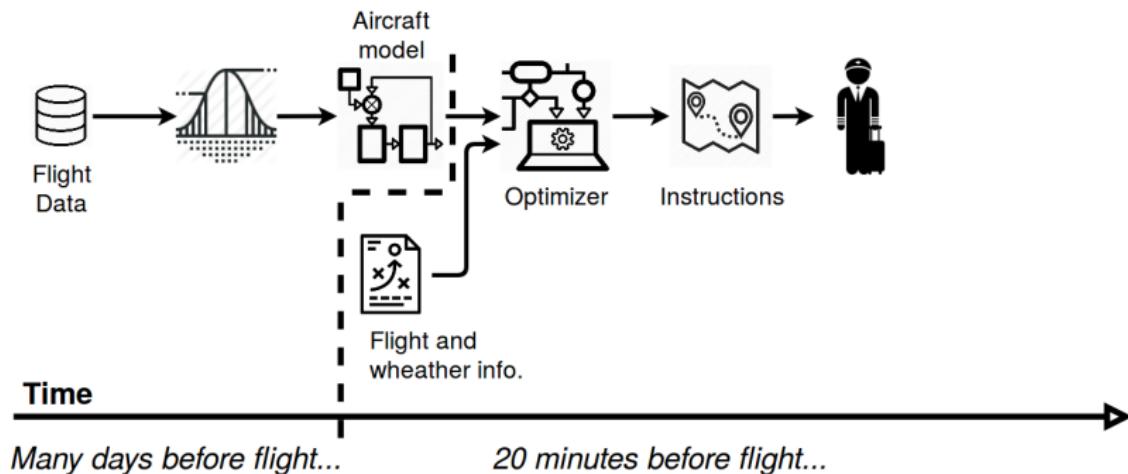
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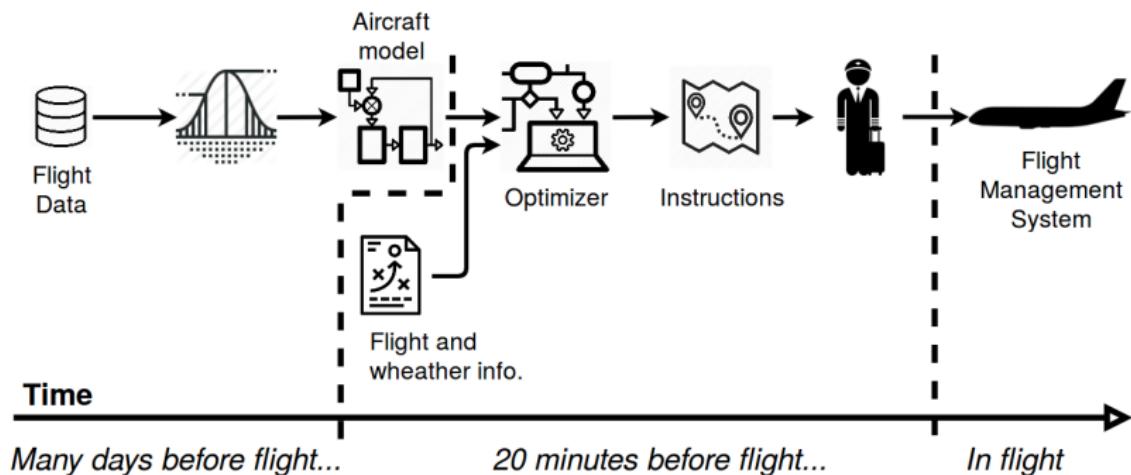
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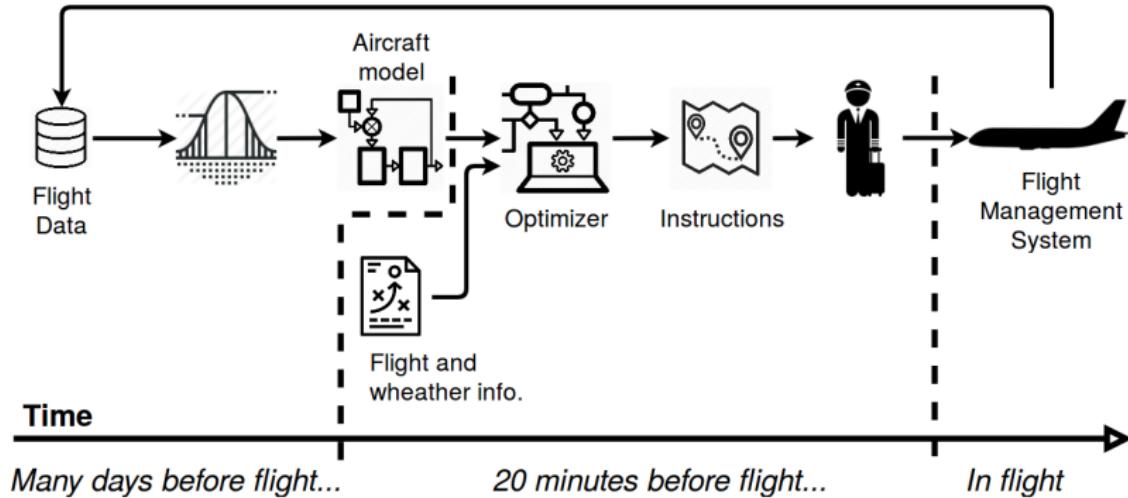
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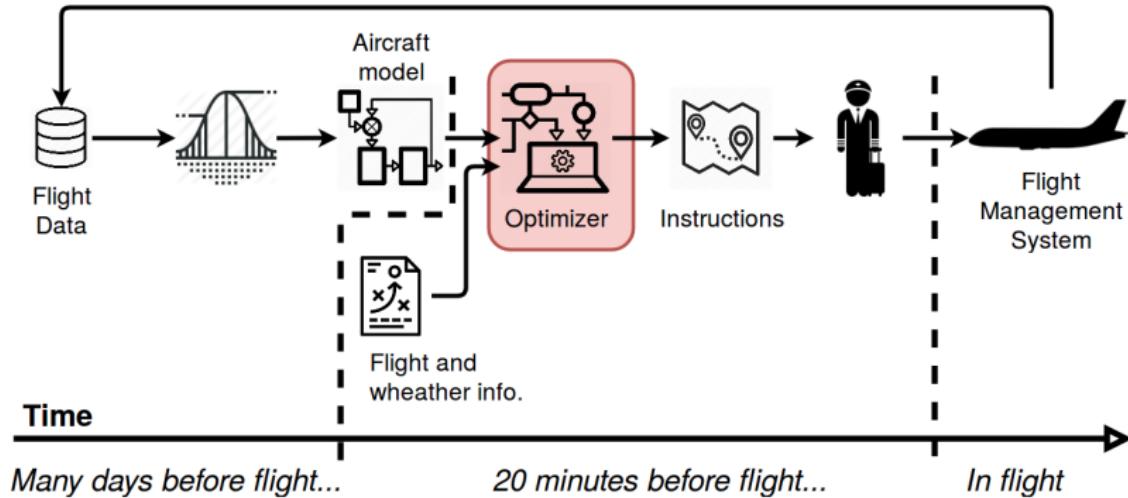
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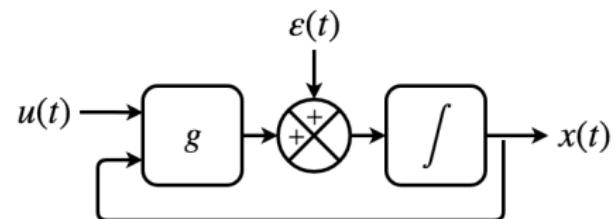


# OPTICLIMB



# TRAJECTORY OPTIMIZATION

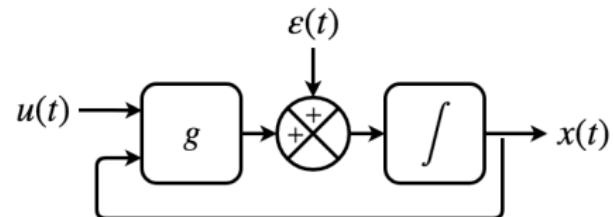
# TRAJECTORY OPTIMIZATION



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Dynamics:

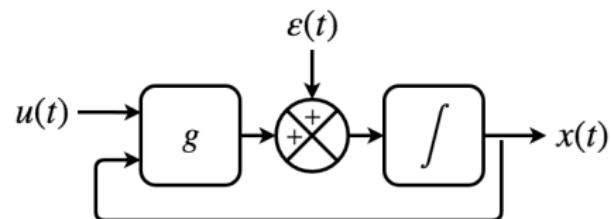
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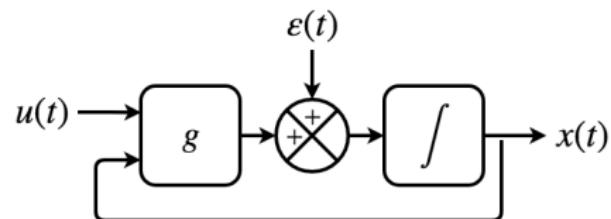


Optimization objective:  $C(\mathbf{u}(t), \mathbf{x}(t))$

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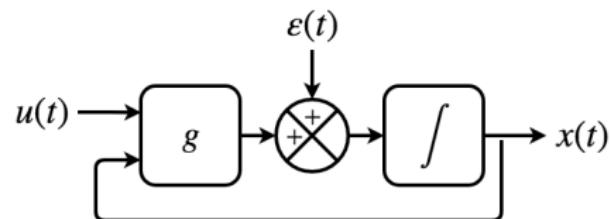


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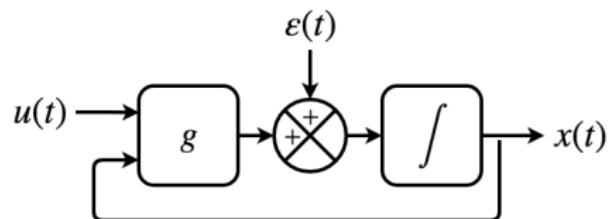


Optimization objective:  $C(\mathbf{u}(t), \mathbf{x}(t)) \Leftarrow \text{ traj. } \rightarrow \text{ goal }$

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Optimization objective:  $C(\mathbf{u}(t), \mathbf{x}(t)) \Leftarrow \text{fuel}, \text{location}$

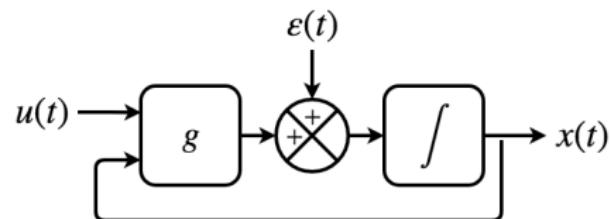
Flight constraints:

{

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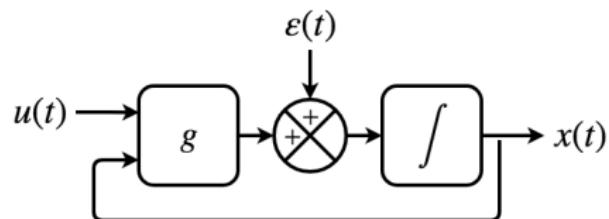
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$$\left\{ \begin{array}{l} \Phi(\mathbf{x}(0), \mathbf{x}(t_f)) \in K_\Phi \\ \end{array} \right. \quad \text{Initial and final conditions}$$

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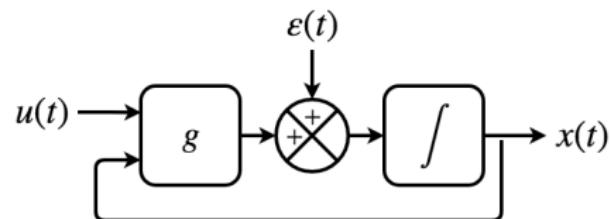
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Initial and final conditions  
Flight domain

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Optimization objective:  $C(\mathbf{u}(t), \mathbf{x}(t)) \Leftarrow \text{flight path}$

Flight constraints:

$$\begin{cases} \Phi(\mathbf{x}(0), \mathbf{x}(t_f)) \in K_\Phi \\ \mathbf{u}(t) \in U_{ad}, \quad \mathbf{x}(t) \in X_{ad}, \\ c(\mathbf{u}(t), \mathbf{x}(t)) \leq 0, \end{cases}$$

Initial and final conditions  
Flight domain  
Operational path constraints

# TRAJECTORY OPTIMIZATION

## OPTIMAL CONTROL PROBLEM

$$\begin{aligned} & \min_{(\mathbf{x}, \mathbf{u}) \in \mathbb{X} \times \mathbb{U}} \int_0^{t_f} C(\mathbf{u}(t), \mathbf{x}(t)) dt, \\ \text{s.t. } & \left\{ \begin{array}{ll} \dot{\mathbf{x}}(t) = g(\mathbf{u}(t), \mathbf{x}(t)) + \varepsilon(t), & \text{a.e. } t \in [0, t_f], \\ \Phi(\mathbf{x}(0), \mathbf{x}(t_f)) \in K_\Phi, & \\ \mathbf{u}(t) \in U_{ad}, \quad \mathbf{x}(t) \in X_{ad}, & \text{a.e. } t \in [0, t_f], \\ c(\mathbf{u}(t), \mathbf{x}(t)) \leq 0, & \text{a.e. } t \in [0, t_f]. \end{array} \right. \end{aligned} \quad (\text{OCP})$$

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## SYSTEM IDENTIFICATION



Black box

# TRAJECTORY OPTIMIZATION

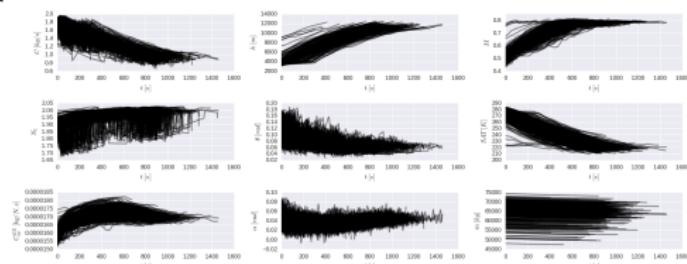
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QAR data

# TRAJECTORY OPTIMIZATION

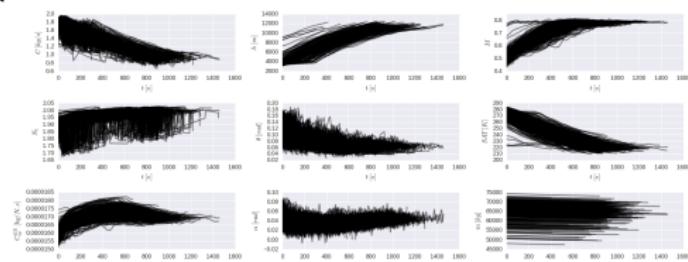
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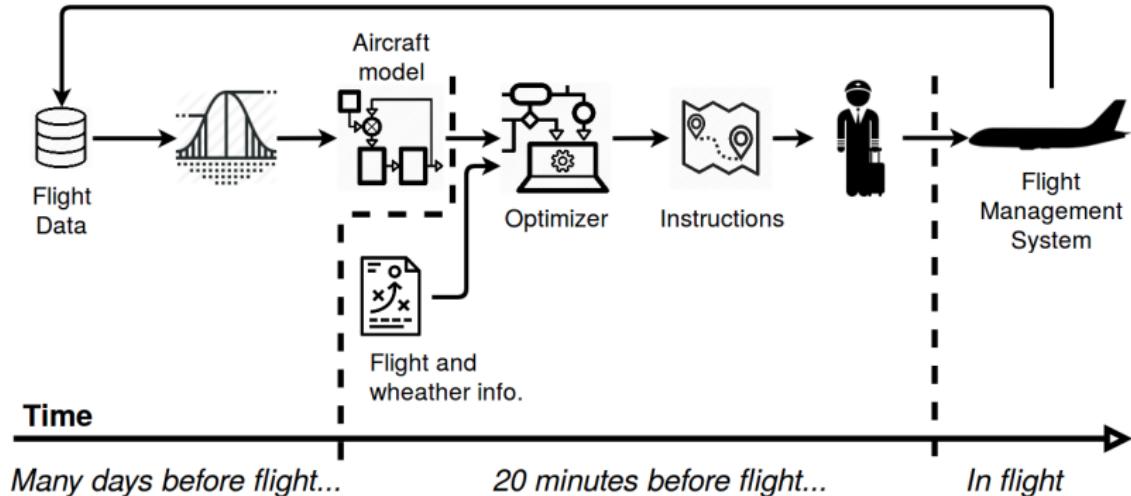


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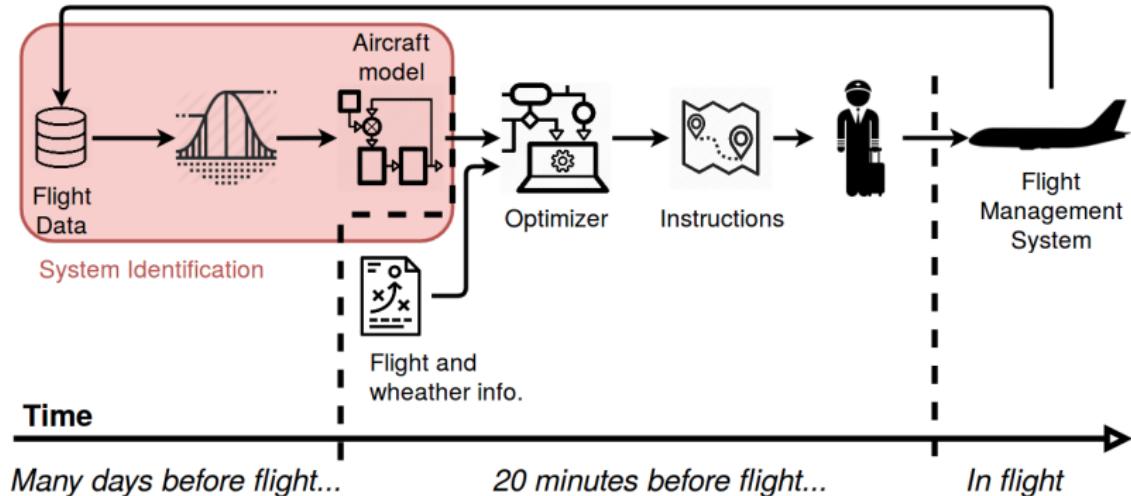


QAR data

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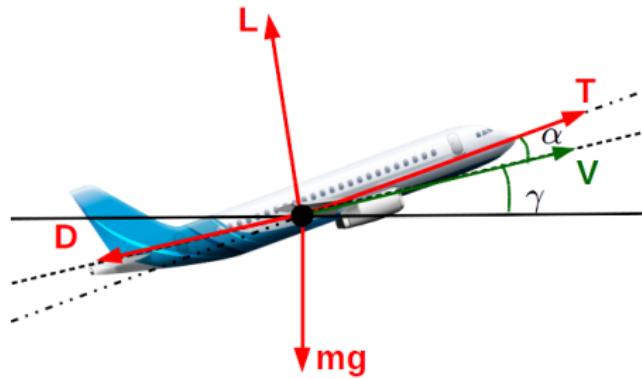


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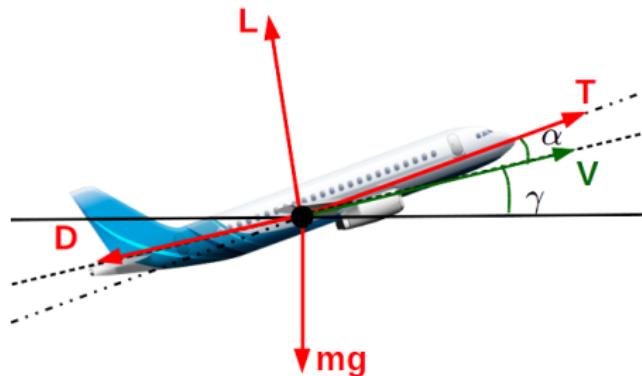


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# FLIGHT DYNAMICS

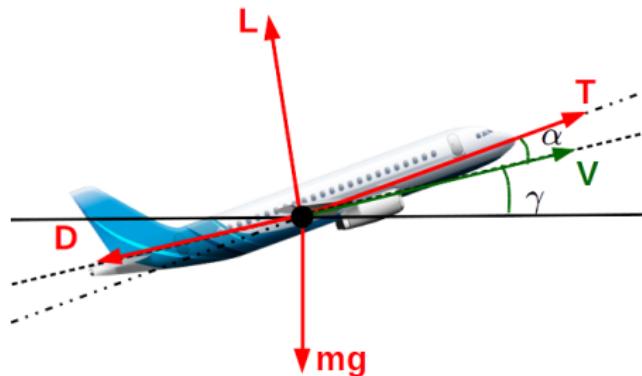


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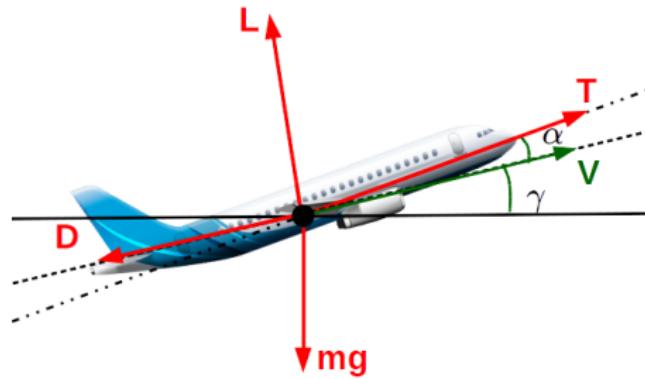
$$\left\{ \begin{array}{l} \dot{h} = V \sin \gamma \\ \dot{V} = \frac{T \cos \alpha - D - mg \sin \gamma}{m} \\ \dot{\gamma} = \frac{T \sin \alpha + L - mg \cos \gamma}{mV} \\ \dot{m} = -\frac{T}{I_{sp}} \end{array} \right.$$

# FLIGHT DYNAMICS



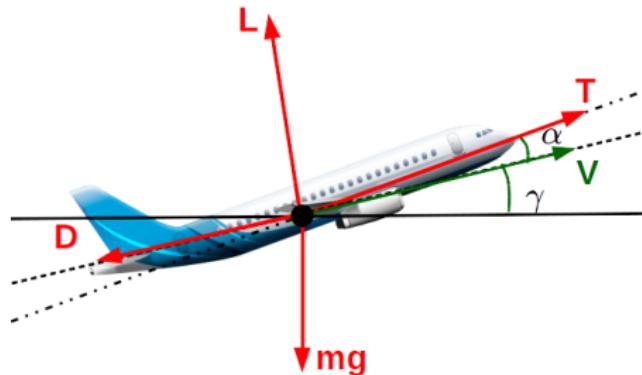
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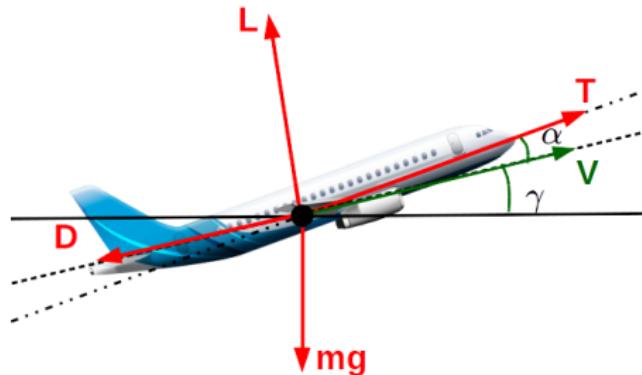
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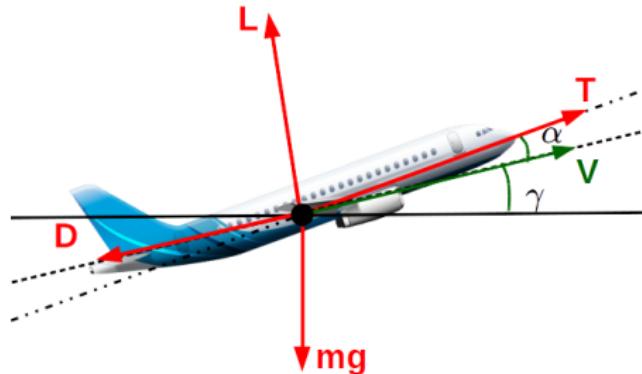
# FLIGHT DYNAMICS



States:  $x = (h, V, \gamma, m)$

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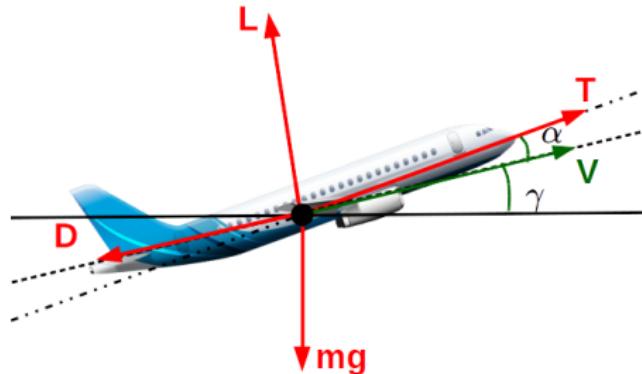


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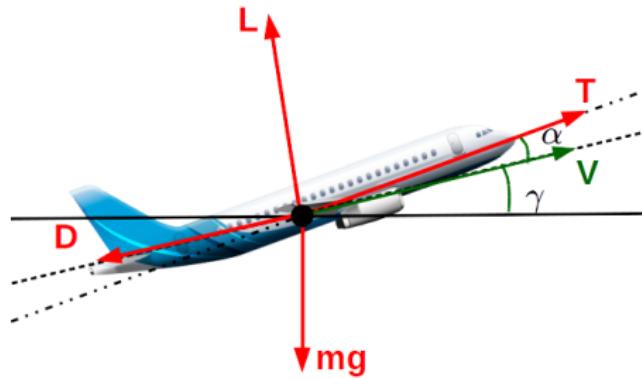
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# PHYSICAL MODELS OF NESTED FUNCTIONS

$$\begin{cases} T \text{ function of } (N_1, M, \rho) \\ D \text{ function of } (q, M, \alpha) \\ L \text{ function of } (q, M, \alpha) \\ I_{sp} \text{ function of } (SAT, M, h) \end{cases}$$

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$$\begin{cases} T \text{ function of } (N_1, M, \rho) = \varphi_T(\mathbf{x}, \mathbf{u}) \\ D \text{ function of } (q, M, \alpha) = \varphi_D(\mathbf{x}, \mathbf{u}) \\ L \text{ function of } (q, M, \alpha) = \varphi_L(\mathbf{x}, \mathbf{u}) \\ I_{sp} \text{ function of } (SAT, M, h) = \varphi_{I_{sp}}(\mathbf{x}, \mathbf{u}) \end{cases}$$

# PHYSICAL MODELS OF NESTED FUNCTIONS

$$\begin{cases} T = N_1 \times P_T(\rho, M), \\ D = q \times P_D(\alpha, M), \\ L = q \times P_L(\alpha, M), \\ I_{sp} = SAT \times P_{Isp}(h, M), \end{cases}$$

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# PHYSICAL MODELS OF NESTED FUNCTIONS

$$\begin{cases} T = X_T \cdot \theta_T, \\ D = X_D \cdot \theta_D, \\ L = X_L \cdot \theta_L, \\ I_{sp} = X_{Isp} \cdot \theta_{Isp}, \end{cases}$$

$$X_T = N_1(1, \rho, M, \rho^2, \rho M, M^2, \dots)$$

$$X_D = q(1, \alpha, M, \alpha^2, \alpha M, M^2, \dots)$$

$$X_L = q(1, \alpha, M, \alpha^2, \alpha M, M^2, \dots)$$

$$X_{Isp} = SAT(1, h, M, h^2, hM, M^2, \dots)$$

# PHYSICAL MODELS OF NESTED FUNCTIONS

$$\begin{cases} T(\mathbf{x}, \mathbf{u}, \boldsymbol{\theta}_T) = X_T \cdot \boldsymbol{\theta}_T, \\ D(\mathbf{x}, \mathbf{u}, \boldsymbol{\theta}_D) = X_D \cdot \boldsymbol{\theta}_D, \\ L(\mathbf{x}, \mathbf{u}, \boldsymbol{\theta}_L) = X_L \cdot \boldsymbol{\theta}_L, \\ I_{sp}(\mathbf{x}, \mathbf{u}, \boldsymbol{\theta}_{Isp}) = X_{Isp} \cdot \boldsymbol{\theta}_{Isp}, \end{cases}$$

$$X_T = N_1(1, \rho, M, \rho^2, \rho M, M^2, \dots)$$

$$X_D = q(1, \alpha, M, \alpha^2, \alpha M, M^2, \dots)$$

$$X_L = q(1, \alpha, M, \alpha^2, \alpha M, M^2, \dots)$$

$$X_{Isp} = SAT(1, h, M, h^2, hM, M^2, \dots)$$

# STATE-OF-THE-ART

# STATE-OF-THE-ART

- Output-Error Method

$$\{u_f\}_{f \in \mathcal{F}}$$

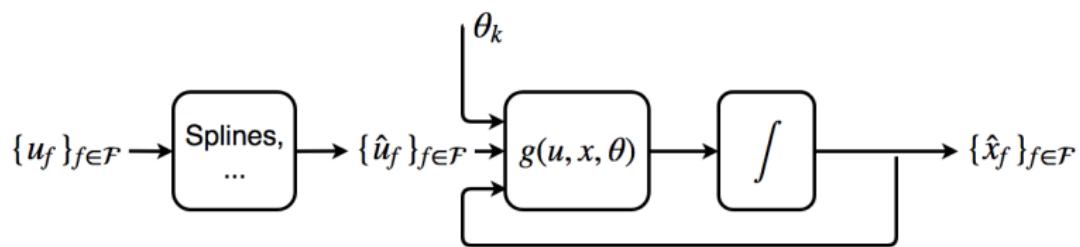
# STATE-OF-THE-ART

## ■ Output-Error Method



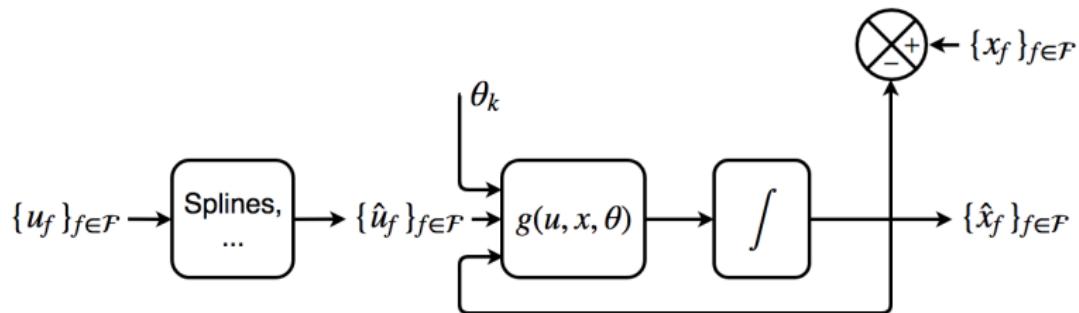
# STATE-OF-THE-ART

- Output-Error Method



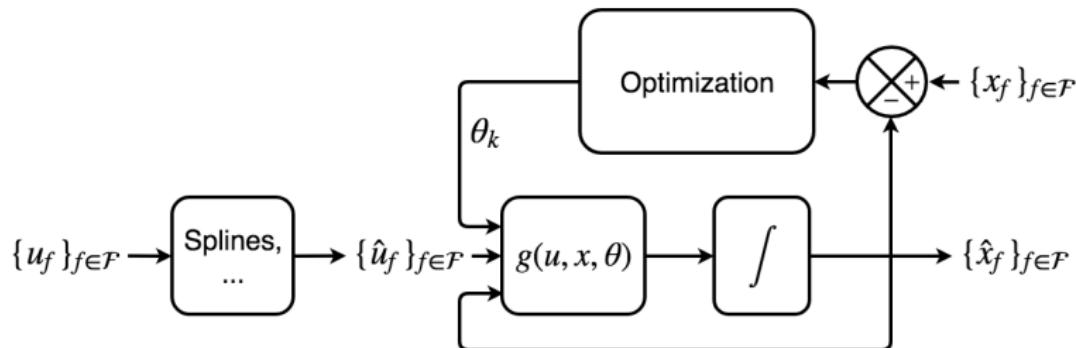
# STATE-OF-THE-ART

## ■ Output-Error Method



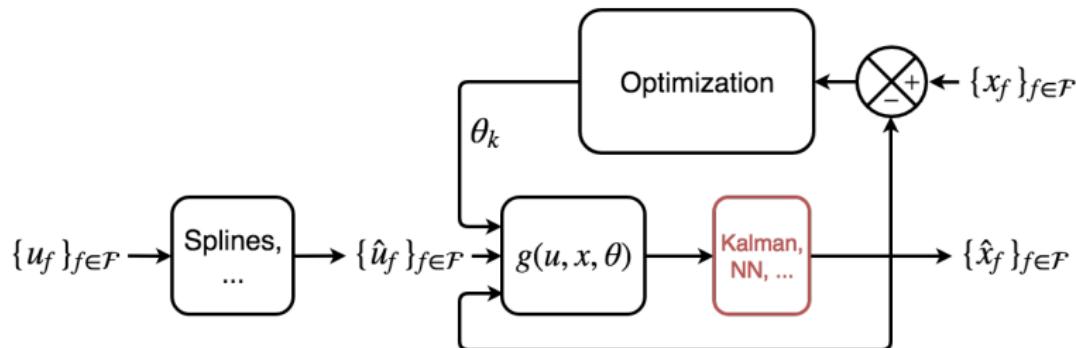
# STATE-OF-THE-ART

## ■ Output-Error Method



# STATE-OF-THE-ART

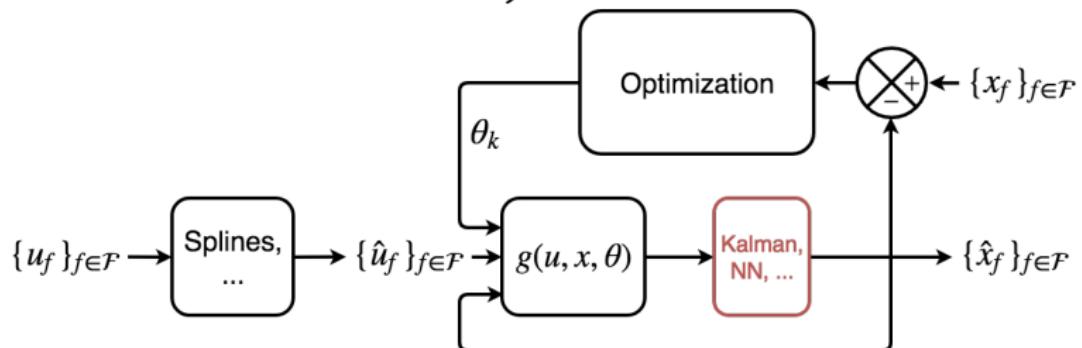
- Output-Error Method
- Filter-Error Method



# STATE-OF-THE-ART

- Output-Error Method
- Filter-Error Method

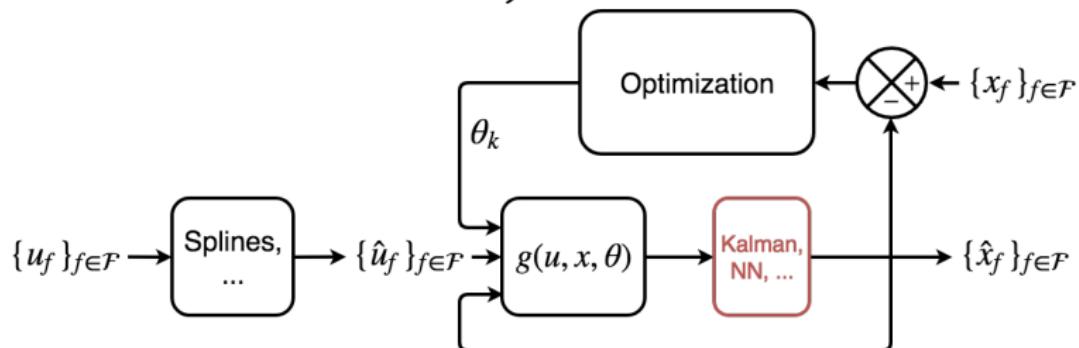
} Not scalable to many trajectories



# STATE-OF-THE-ART

- Output-Error Method
- Filter-Error Method

} Not scalable to many trajectories



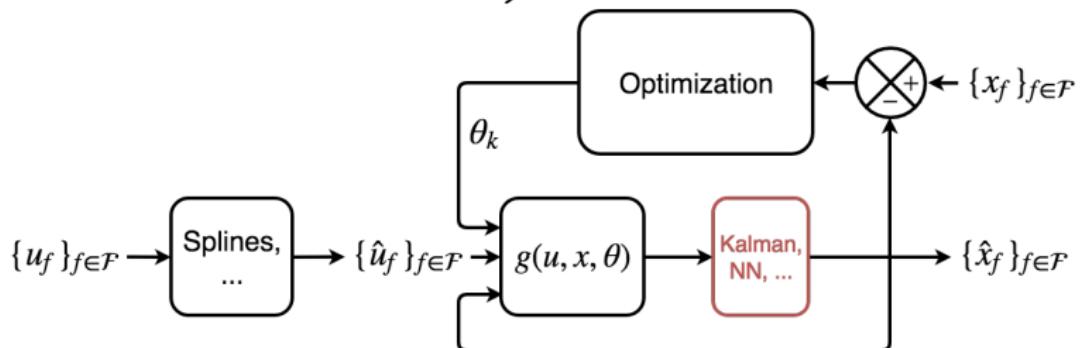
- Equation-Error Method

$$\dot{\mathbf{x}}(t) = g(\mathbf{u}(t), \mathbf{x}(t), \boldsymbol{\theta}) + \varepsilon(t), \quad t \in [0, t_f]$$

# STATE-OF-THE-ART

- Output-Error Method
- Filter-Error Method

} Not scalable to many trajectories



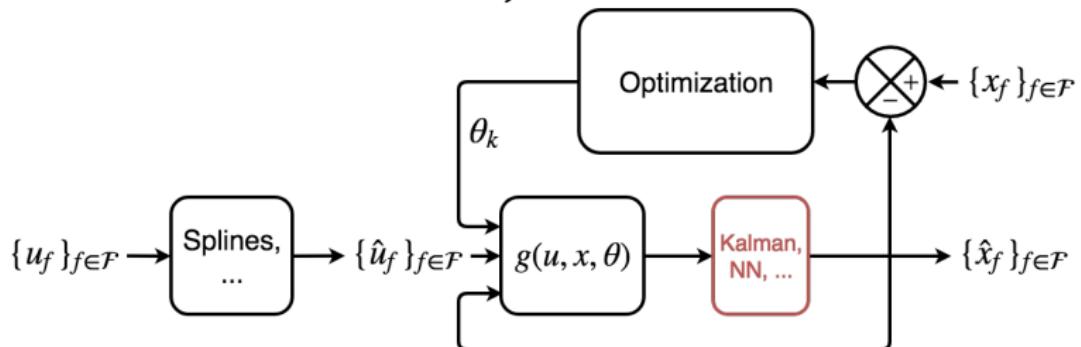
- Equation-Error Method

$$\dot{x}_i = g(\mathbf{u}_i, \mathbf{x}_i, \boldsymbol{\theta}) + \varepsilon_i, \quad i = 1, \dots, N$$

# STATE-OF-THE-ART

- Output-Error Method
- Filter-Error Method

} Not scalable to many trajectories



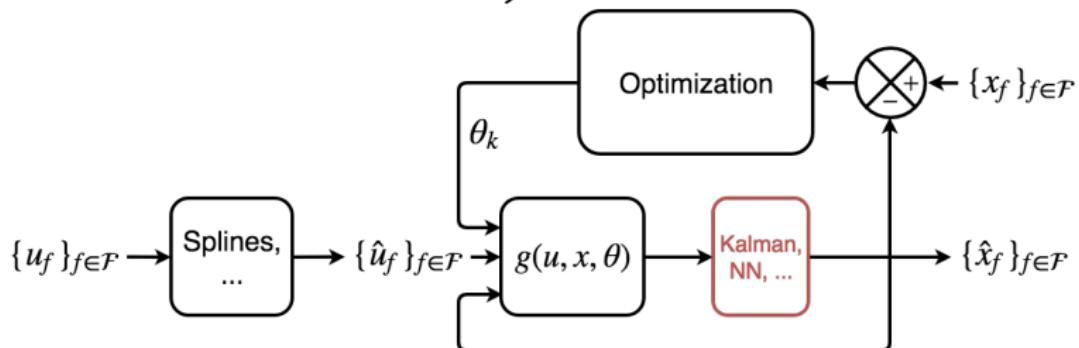
- Equation-Error Method

$$\min_{\theta} \sum_{i=1}^N \ell(\dot{x}_i, g(\mathbf{u}_i, \mathbf{x}_i, \theta))$$

# STATE-OF-THE-ART

- Output-Error Method
- Filter-Error Method

} Not scalable to many trajectories



- Equation-Error Method

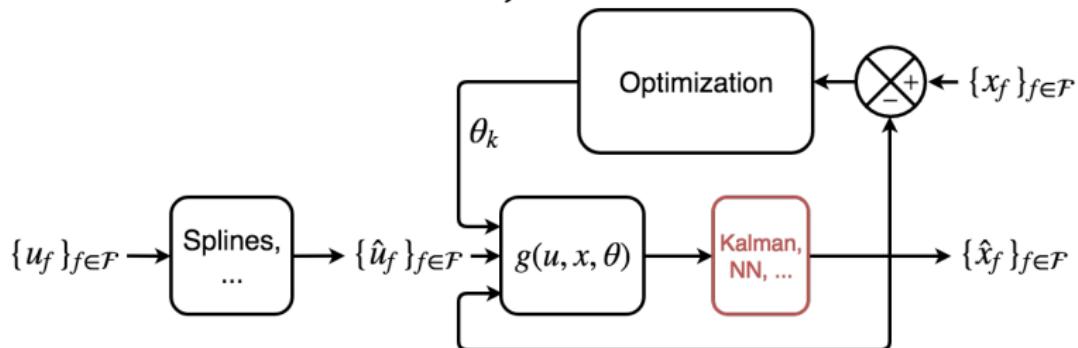
Ex: *(Nonlinear) Least-Squares*

$$\min_{\theta} \sum_{i=1}^N \left\| \dot{x}_i - g(\mathbf{u}_i, \mathbf{x}_i, \theta) \right\|_2^2$$

# STATE-OF-THE-ART

- Output-Error Method
- Filter-Error Method

} Not scalable to many trajectories



- Equation-Error Method

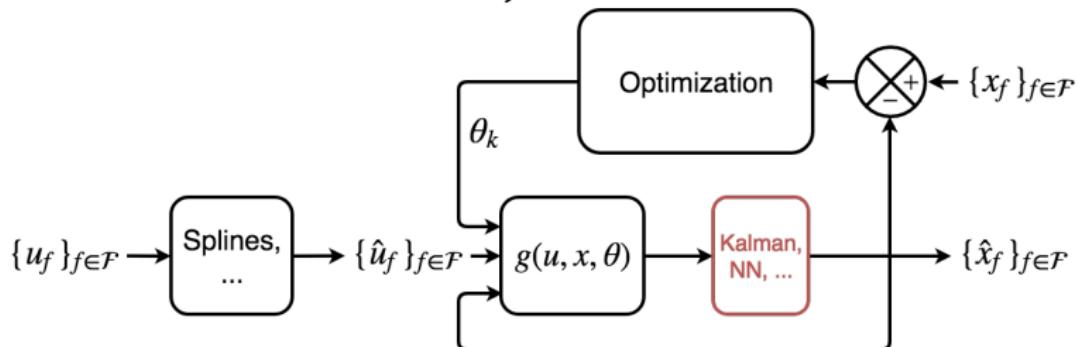
Ex: *(Nonlinear) Least-Squares*

$$\min_{\theta} \sum_{i=1}^N \left\| Y(\mathbf{u}_i, \mathbf{x}_i, \dot{\mathbf{x}}_i) - G(\mathbf{u}_i, \mathbf{x}_i, \dot{\mathbf{x}}_i, \theta) \right\|_2^2$$

# STATE-OF-THE-ART

- Output-Error Method
- Filter-Error Method

} Not scalable to many trajectories



- **Equation-Error Method**

Ex: (Nonlinear) Least-Squares

$$\min_{\theta} \sum_{i=1}^N \| Y(\mathbf{u}_i, \mathbf{x}_i, \dot{\mathbf{x}}_i) - G(\mathbf{u}_i, \mathbf{x}_i, \dot{\mathbf{x}}_i, \theta) \|_2^2$$

# LEVERAGING THE DYNAMICS STRUCTURE

$$\begin{cases} \dot{h} = V \sin \gamma \\ \dot{V} = \frac{T(\mathbf{u}, \mathbf{x}, \boldsymbol{\theta}_T) \cos \alpha - D(\mathbf{u}, \mathbf{x}, \boldsymbol{\theta}_D) - mg \sin \gamma}{m} \\ \dot{\gamma} = \frac{T(\mathbf{u}, \mathbf{x}, \boldsymbol{\theta}_T) \sin \alpha + L(\mathbf{u}, \mathbf{x}, \boldsymbol{\theta}_L) - mg \cos \gamma}{mV} \\ \dot{m} = -\frac{T(\mathbf{u}, \mathbf{x}, \boldsymbol{\theta}_T)}{I_{sp}(\mathbf{u}, \mathbf{x}, \boldsymbol{\theta}_{Isp})} \end{cases}$$

# LEVERAGING THE DYNAMICS STRUCTURE

$$\begin{cases} \dot{h} = V \sin \gamma \\ \dot{V} = \frac{T(\mathbf{u}, \mathbf{x}, \boldsymbol{\theta}_T) \cos \alpha - D(\mathbf{u}, \mathbf{x}, \boldsymbol{\theta}_D) - mg \sin \gamma}{m} \\ \dot{\gamma} = \frac{T(\mathbf{u}, \mathbf{x}, \boldsymbol{\theta}_T) \sin \alpha + L(\mathbf{u}, \mathbf{x}, \boldsymbol{\theta}_L) - mg \cos \gamma}{mV} \\ \dot{m} = -\frac{T(\mathbf{u}, \mathbf{x}, \boldsymbol{\theta}_T)}{I_{sp}(\mathbf{u}, \mathbf{x}, \boldsymbol{\theta}_{Isp})} \end{cases}$$

- Nonlinear in states and controls

# LEVERAGING THE DYNAMICS STRUCTURE

$$\begin{cases} \dot{h} = V \sin \gamma \\ \dot{V} = \frac{T(\mathbf{u}, \mathbf{x}, \boldsymbol{\theta}_T) \cos \alpha - D(\mathbf{u}, \mathbf{x}, \boldsymbol{\theta}_D) - mg \sin \gamma}{m} \\ \dot{\gamma} = \frac{T(\mathbf{u}, \mathbf{x}, \boldsymbol{\theta}_T) \sin \alpha + L(\mathbf{u}, \mathbf{x}, \boldsymbol{\theta}_L) - mg \cos \gamma}{mV} \\ \dot{m} = -\frac{T(\mathbf{u}, \mathbf{x}, \boldsymbol{\theta}_T)}{I_{sp}(\mathbf{u}, \mathbf{x}, \boldsymbol{\theta}_{Isp})} \end{cases}$$

- Nonlinear in states and controls
- Nonlinear in parameters

# LEVERAGING THE DYNAMICS STRUCTURE

$$\begin{cases} \dot{h} = V \sin \gamma \\ m\dot{V} = T(\mathbf{u}, \mathbf{x}, \boldsymbol{\theta}_T) \cos \alpha - D(\mathbf{u}, \mathbf{x}, \boldsymbol{\theta}_D) - mg \sin \gamma \\ mV\dot{\gamma} = T(\mathbf{u}, \mathbf{x}, \boldsymbol{\theta}_T) \sin \alpha + L(\mathbf{u}, \mathbf{x}, \boldsymbol{\theta}_L) - mg \cos \gamma \\ \dot{m} = -\frac{T(\mathbf{u}, \mathbf{x}, \boldsymbol{\theta}_T)}{l_{sp}(\mathbf{u}, \mathbf{x}, \boldsymbol{\theta}_{lsp})} \end{cases}$$

- Nonlinear in states and controls
- Nonlinear in parameters

# LEVERAGING THE DYNAMICS STRUCTURE

$$\begin{cases} \dot{h} = V \sin \gamma \\ m\dot{V} = T(\mathbf{u}, \mathbf{x}, \boldsymbol{\theta}_T) \cos \alpha - D(\mathbf{u}, \mathbf{x}, \boldsymbol{\theta}_D) - mg \sin \gamma \\ mV\dot{\gamma} = T(\mathbf{u}, \mathbf{x}, \boldsymbol{\theta}_T) \sin \alpha + L(\mathbf{u}, \mathbf{x}, \boldsymbol{\theta}_L) - mg \cos \gamma \\ \dot{m} = -\frac{T(\mathbf{u}, \mathbf{x}, \boldsymbol{\theta}_T)}{l_{sp}(\mathbf{u}, \mathbf{x}, \boldsymbol{\theta}_{lsp})} \end{cases}$$

- Nonlinear in states and controls
- Nonlinear in parameters

# LEVERAGING THE DYNAMICS STRUCTURE

$$\left\{ \begin{array}{l} \dot{h} = V \sin \gamma \\ m \dot{V} + mg \sin \gamma = T(\mathbf{u}, \mathbf{x}, \boldsymbol{\theta}_T) \cos \alpha - D(\mathbf{u}, \mathbf{x}, \boldsymbol{\theta}_D) \\ mV \dot{\gamma} + mg \cos \gamma = T(\mathbf{u}, \mathbf{x}, \boldsymbol{\theta}_T) \sin \alpha + L(\mathbf{u}, \mathbf{x}, \boldsymbol{\theta}_L) \\ \dot{m} = -\frac{T(\mathbf{u}, \mathbf{x}, \boldsymbol{\theta}_T)}{I_{sp}(\mathbf{u}, \mathbf{x}, \boldsymbol{\theta}_{Isp})} \end{array} \right.$$

- Nonlinear in states and controls
- Nonlinear in parameters

# LEVERAGING THE DYNAMICS STRUCTURE

$$\left\{ \begin{array}{l} \dot{h} = V \sin \gamma \\ m \dot{V} + mg \sin \gamma = T(\mathbf{u}, \mathbf{x}, \boldsymbol{\theta}_T) \cos \alpha - D(\mathbf{u}, \mathbf{x}, \boldsymbol{\theta}_D) \\ mV \dot{\gamma} + mg \cos \gamma = T(\mathbf{u}, \mathbf{x}, \boldsymbol{\theta}_T) \sin \alpha + L(\mathbf{u}, \mathbf{x}, \boldsymbol{\theta}_L) \\ \dot{m} I_{sp}(\mathbf{u}, \mathbf{x}, \boldsymbol{\theta}_{Isp}) = -T(\mathbf{u}, \mathbf{x}, \boldsymbol{\theta}_T) \end{array} \right.$$

- Nonlinear in states and controls
- Nonlinear in parameters

# LEVERAGING THE DYNAMICS STRUCTURE

$$\left\{ \begin{array}{l} \dot{h} = V \sin \gamma \\ m\dot{V} + mg \sin \gamma = T(\mathbf{u}, \mathbf{x}, \boldsymbol{\theta}_T) \cos \alpha - D(\mathbf{u}, \mathbf{x}, \boldsymbol{\theta}_D) \\ mV\dot{\gamma} + mg \cos \gamma = T(\mathbf{u}, \mathbf{x}, \boldsymbol{\theta}_T) \sin \alpha + L(\mathbf{u}, \mathbf{x}, \boldsymbol{\theta}_L) \\ 0 = T(\mathbf{u}, \mathbf{x}, \boldsymbol{\theta}_T) + \dot{m}l_{sp}(\mathbf{u}, \mathbf{x}, \boldsymbol{\theta}_{lsp}) \end{array} \right.$$

- Nonlinear in states and controls
- Nonlinear in parameters

# LEVERAGING THE DYNAMICS STRUCTURE

$$\left\{ \begin{array}{l} \dot{h} = V \sin \gamma \\ m \dot{V} + mg \sin \gamma = (X_T \cdot \theta_T) \cos \alpha - X_D \cdot \theta_D \\ m V \dot{\gamma} + mg \cos \gamma = (X_T \cdot \theta_T) \sin \alpha + X_L \cdot \theta_L \\ 0 = X_T \cdot \theta_T + \dot{m}(X_{Isp} \cdot \theta_{Isp}) \end{array} \right.$$

- Nonlinear in states and controls
- Nonlinear in parameters

# LEVERAGING THE DYNAMICS STRUCTURE

$$\left\{ \begin{array}{l} \dot{h} = V \sin \gamma \\ m \dot{V} + mg \sin \gamma = (X_T \cdot \theta_T) \cos \alpha - X_D \cdot \theta_D + \varepsilon_1 \\ m V \dot{\gamma} + mg \cos \gamma = (X_T \cdot \theta_T) \sin \alpha + X_L \cdot \theta_L + \varepsilon_2 \\ 0 = X_T \cdot \theta_T + \dot{m}(X_{Isp} \cdot \theta_{Isp}) + \varepsilon_3 \end{array} \right.$$

- Nonlinear in states and controls
- Nonlinear in parameters

# LEVERAGING THE DYNAMICS STRUCTURE

$$\left\{ \begin{array}{l} \dot{h} = V \sin \gamma \\ m \dot{V} + mg \sin \gamma = (X_T \cdot \theta_T) \cos \alpha - X_D \cdot \theta_D + \varepsilon_1 \\ m V \dot{\gamma} + mg \cos \gamma = (X_T \cdot \theta_T) \sin \alpha + X_L \cdot \theta_L + \varepsilon_2 \\ 0 = X_T \cdot \theta_T + \dot{m}(X_{Isp} \cdot \theta_{Isp}) + \varepsilon_3 \end{array} \right.$$

$$\overbrace{Y(\mathbf{u}, \mathbf{x}, \dot{\mathbf{x}})}$$

$$\overbrace{G(\mathbf{u}, \mathbf{x}, \dot{\mathbf{x}}, \boldsymbol{\theta})}$$

- Nonlinear in states and controls
- Nonlinear in parameters

# LEVERAGING THE DYNAMICS STRUCTURE

$$\left\{ \begin{array}{l} \dot{h} = V \sin \gamma \\ m \dot{V} + mg \sin \gamma = (X_T \cdot \theta_T) \cos \alpha - X_D \cdot \theta_D + \varepsilon_1 \\ m V \dot{\gamma} + mg \cos \gamma = (X_T \cdot \theta_T) \sin \alpha + X_L \cdot \theta_L + \varepsilon_2 \\ 0 = X_T \cdot \theta_T + \dot{m}(X_{Isp} \cdot \theta_{Isp}) + \varepsilon_3 \end{array} \right.$$

$$\overbrace{Y(\mathbf{u}, \mathbf{x}, \dot{\mathbf{x}})}$$

$$\overbrace{G(\mathbf{u}, \mathbf{x}, \dot{\mathbf{x}}, \boldsymbol{\theta})}$$

- Nonlinear in states and controls
- ~~Nonlinear in parameters~~ → Linear in parameters

# LEVERAGING THE DYNAMICS STRUCTURE

$$\left\{ \begin{array}{l} \dot{h} = V \sin \gamma \\ m \dot{V} + mg \sin \gamma = (X_T \cdot \theta_T) \cos \alpha - X_D \cdot \theta_D + \varepsilon_1 \\ m V \dot{\gamma} + mg \cos \gamma = (X_T \cdot \theta_T) \sin \alpha + X_L \cdot \theta_L + \varepsilon_2 \\ 0 = X_T \cdot \theta_T + \dot{m}(X_{Isp} \cdot \theta_{Isp}) + \varepsilon_3 \end{array} \right.$$

$$\overbrace{Y(\mathbf{u}, \mathbf{x}, \dot{\mathbf{x}})}$$

$$\overbrace{G(\mathbf{u}, \mathbf{x}, \dot{\mathbf{x}}, \boldsymbol{\theta})}$$

- Nonlinear in states and controls
- ~~Nonlinear in parameters~~ → Linear in parameters
- Structured

# LEVERAGING THE DYNAMICS STRUCTURE

$$\left\{ \begin{array}{l} \dot{h} = V \sin \gamma \\ m \dot{V} + mg \sin \gamma = (X_T \cdot \theta_T) \cos \alpha - X_D \cdot \theta_D + \varepsilon_1 \\ m V \dot{\gamma} + mg \cos \gamma = (X_T \cdot \theta_T) \sin \alpha + X_L \cdot \theta_L + \varepsilon_2 \\ 0 = X_T \cdot \theta_T + \dot{m}(X_{Isp} \cdot \theta_{Isp}) + \varepsilon_3 \end{array} \right.$$

$$\overbrace{Y(\mathbf{u}, \mathbf{x}, \dot{\mathbf{x}})} \quad \overbrace{G(\mathbf{u}, \mathbf{x}, \dot{\mathbf{x}}, \boldsymbol{\theta})}$$

- Nonlinear in states and controls
- ~~Nonlinear in parameters~~ → Linear in parameters
- Structured

# LEVERAGING THE DYNAMICS STRUCTURE

$$\left\{ \begin{array}{l} \dot{h} = V \sin \gamma \\ m \dot{V} + mg \sin \gamma = (\mathbf{X}_T \cdot \boldsymbol{\theta}_T) \cos \alpha - X_D \cdot \theta_D + \varepsilon_1 \\ m V \dot{\gamma} + mg \cos \gamma = (\mathbf{X}_T \cdot \boldsymbol{\theta}_T) \sin \alpha + X_L \cdot \theta_L + \varepsilon_2 \\ 0 = X_T \cdot \boldsymbol{\theta}_T + \dot{m}(\mathbf{X}_{Isp} \cdot \boldsymbol{\theta}_{Isp}) + \varepsilon_3 \end{array} \right.$$

$$\overbrace{Y(\mathbf{u}, \mathbf{x}, \dot{\mathbf{x}})} \quad \overbrace{G(\mathbf{u}, \mathbf{x}, \dot{\mathbf{x}}, \boldsymbol{\theta})}$$

- Nonlinear in states and controls
- ~~Nonlinear in parameters~~ → Linear in parameters
- Structured

# LEVERAGING THE DYNAMICS STRUCTURE

$$\left\{ \begin{array}{l} \dot{h} = V \sin \gamma \\ m \dot{V} + mg \sin \gamma = (\textcolor{green}{X_T} \cos \alpha) \cdot \textcolor{red}{\theta_T} - X_D \cdot \theta_D + \varepsilon_1 \\ m V \dot{\gamma} + mg \cos \gamma = (\textcolor{green}{X_T} \sin \alpha) \cdot \textcolor{red}{\theta_T} + X_L \cdot \theta_L + \varepsilon_2 \\ 0 = X_T \cdot \textcolor{red}{\theta_T} + (\dot{m} \textcolor{green}{X_{Isp}}) \cdot \theta_{Isp} + \varepsilon_3 \end{array} \right.$$

$$\overbrace{Y(\boldsymbol{u}, \boldsymbol{x}, \dot{\boldsymbol{x}})} \quad \overbrace{G(\boldsymbol{u}, \boldsymbol{x}, \dot{\boldsymbol{x}}, \boldsymbol{\theta})}$$

- Nonlinear in states and controls
- ~~Nonlinear in parameters~~ → Linear in parameters
- Structured

# LEVERAGING THE DYNAMICS STRUCTURE

$$\left\{ \begin{array}{l} \dot{h} = V \sin \gamma \\ m\dot{V} + mg \sin \gamma = X_{T1} \cdot \theta_T - X_D \cdot \theta_D + \varepsilon_1 \\ mV\dot{\gamma} + mg \cos \gamma = X_{T2} \cdot \theta_T + X_L \cdot \theta_L + \varepsilon_2 \\ 0 = X_T \cdot \theta_T + X_{Ispm} \cdot \theta_{Isp} + \varepsilon_3 \end{array} \right.$$

$$\overbrace{Y(\mathbf{u}, \mathbf{x}, \dot{\mathbf{x}})} \quad \overbrace{G(\mathbf{u}, \mathbf{x}, \dot{\mathbf{x}}, \boldsymbol{\theta})}$$

- Nonlinear in states and controls
- ~~Nonlinear in parameters~~ → Linear in parameters
- Structured

# LEVERAGING THE DYNAMICS STRUCTURE

$$\left\{ \begin{array}{l} \dot{h} = V \sin \gamma \\ m\dot{V} + mg \sin \gamma = X_{T1} \cdot \theta_T - X_D \cdot \theta_D + \varepsilon_1 \\ mV\dot{\gamma} + mg \cos \gamma = X_{T2} \cdot \theta_T + X_L \cdot \theta_L + \varepsilon_2 \\ 0 = X_T \cdot \theta_T + X_{Ispm} \cdot \theta_{Isp} + \varepsilon_3 \end{array} \right.$$

$$\underbrace{Y(\mathbf{u}, \mathbf{x}, \dot{\mathbf{x}})} \quad \underbrace{G(\mathbf{u}, \mathbf{x}, \dot{\mathbf{x}}, \boldsymbol{\theta})}$$

- Nonlinear in states and controls
- ~~Nonlinear in parameters~~ → Linear in parameters
- Structured

# LEVERAGING THE DYNAMICS STRUCTURE

$$\left\{ \begin{array}{l} \dot{h} = V \sin \gamma \\ Y_1 = X_{T1} \cdot \theta_T - X_D \cdot \theta_D + \varepsilon_1 \\ Y_2 = X_{T2} \cdot \theta_T + X_L \cdot \theta_L + \varepsilon_2 \\ Y_3 = X_T \cdot \theta_T + X_{Ispm} \cdot \theta_{Isp} + \varepsilon_3 \end{array} \right.$$

- Nonlinear in states and controls
- ~~Nonlinear in parameters~~ → Linear in parameters
- Structured

# LEVERAGING THE DYNAMICS STRUCTURE

$$\left\{ \begin{array}{l} \dot{h} = V \sin \gamma \\ Y_1 = X_{T1} \cdot \theta_T - X_D \cdot \theta_D + \varepsilon_1 \\ Y_2 = X_{T2} \cdot \theta_T + X_L \cdot \theta_L + \varepsilon_2 \\ Y_3 = X_T \cdot \theta_T + X_{Ispm} \cdot \theta_{Isp} + \varepsilon_3 \end{array} \right.$$

- Nonlinear in states and controls
- ~~Nonlinear in parameters~~ → Linear in parameters
- Structured  $\rightsquigarrow$  **Multi-task Learning**

# MULTI-TASK REGRESSION

Aircraft:

$$\begin{cases} Y_1 = X_{T1} \cdot \theta_T - X_D \cdot \theta_D + \varepsilon_1 \\ Y_2 = X_{T2} \cdot \theta_T + X_L \cdot \theta_L + \varepsilon_2 \\ Y_3 = X_T \cdot \theta_T + X_{lspm} \cdot \theta_{lsp} + \varepsilon_3 \end{cases}$$

# MULTI-TASK REGRESSION

Aircraft:

$$\begin{cases} Y_1 = X_{T1} \cdot \theta_T - X_D \cdot \theta_D + \varepsilon_1 \\ Y_2 = X_{T2} \cdot \theta_T + X_L \cdot \theta_L + \varepsilon_2 \\ Y_3 = X_T \cdot \theta_T + X_{lspm} \cdot \theta_{lsp} + \varepsilon_3 \end{cases}$$

General:

$$\begin{cases} Y_1 = X_{c,1} \cdot \theta_c + X_1 \cdot \theta_1 + \varepsilon_1 \\ Y_2 = X_{c,2} \cdot \theta_c + X_2 \cdot \theta_2 + \varepsilon_2 \\ \vdots \\ Y_K = X_{c,K} \cdot \theta_c + X_K \cdot \theta_K + \varepsilon_K \end{cases}$$

# MULTI-TASK REGRESSION

Aircraft:

General:

$$\left\{ \begin{array}{l} Y_1 = X_{T1} \cdot \theta_T - X_D \cdot \theta_D + \varepsilon_1 \\ Y_2 = X_{T2} \cdot \theta_T + X_L \cdot \theta_L + \varepsilon_2 \\ Y_3 = X_T \cdot \theta_T + X_{lspm} \cdot \theta_{lsp} + \varepsilon_3 \end{array} \right. \quad \left\{ \begin{array}{l} Y_1 = X_{c,1} \cdot \theta_c + X_1 \cdot \theta_1 + \varepsilon_1 \\ Y_2 = X_{c,2} \cdot \theta_c + X_2 \cdot \theta_2 + \varepsilon_2 \\ \vdots \qquad \qquad \vdots \\ Y_K = X_{c,K} \cdot \theta_c + X_K \cdot \theta_K + \varepsilon_K \end{array} \right.$$

Coupling parameters , Task specific parameters

# MULTI-TASK REGRESSION

Aircraft:

General:

$$\begin{cases} Y_1 = X_{T1} \cdot \theta_T - X_D \cdot \theta_D + \varepsilon_1 \\ Y_2 = X_{T2} \cdot \theta_T + X_L \cdot \theta_L + \varepsilon_2 \\ Y_3 = X_T \cdot \theta_T + X_{lspm} \cdot \theta_{lsp} + \varepsilon_3 \end{cases} \quad \begin{cases} Y_1 = X_{c,1} \cdot \theta_c + X_1 \cdot \theta_1 + \varepsilon_1 \\ Y_2 = X_{c,2} \cdot \theta_c + X_2 \cdot \theta_2 + \varepsilon_2 \\ \vdots \qquad \qquad \vdots \\ Y_K = X_{c,K} \cdot \theta_c + X_K \cdot \theta_K + \varepsilon_K \end{cases}$$

**Coupling parameters**, **Task specific parameters**

Many other examples:

- *Giant squid neurons* [??],

$$\begin{cases} \dot{V} = \theta_1 \left( V - \frac{V^3}{3} + R \right), \\ \dot{R} = -\frac{1}{\theta_1} (V - \theta_2 + \theta_3 R), \end{cases}$$

# MULTI-TASK REGRESSION

Aircraft:

General:

$$\begin{cases} Y_1 = X_{T1} \cdot \theta_T - X_D \cdot \theta_D + \varepsilon_1 \\ Y_2 = X_{T2} \cdot \theta_T + X_L \cdot \theta_L + \varepsilon_2 \\ Y_3 = X_T \cdot \theta_T + X_{lspm} \cdot \theta_{lsp} + \varepsilon_3 \end{cases} \quad \begin{cases} Y_1 = X_{c,1} \cdot \theta_c + X_1 \cdot \theta_1 + \varepsilon_1 \\ Y_2 = X_{c,2} \cdot \theta_c + X_2 \cdot \theta_2 + \varepsilon_2 \\ \vdots \qquad \vdots \\ Y_K = X_{c,K} \cdot \theta_c + X_K \cdot \theta_K + \varepsilon_K \end{cases}$$

**Coupling parameters**, **Task specific parameters**

Many other examples:

- *Giant squid neurons* [??],
- *Susceptible-infectious-recovered models* [?],

$$\begin{cases} \dot{S} = \theta_1 - (\theta_2 I + \theta_3) S, \\ \dot{I} = \theta_2 I S - (\theta_4 + \theta_3) I, \\ \dot{R} = \theta_4 I - \theta_3 R. \end{cases}$$

# MULTI-TASK REGRESSION

Aircraft:

General:

$$\left\{ \begin{array}{l} Y_1 = X_{T1} \cdot \theta_T - X_D \cdot \theta_D + \varepsilon_1 \\ Y_2 = X_{T2} \cdot \theta_T + X_L \cdot \theta_L + \varepsilon_2 \\ Y_3 = X_T \cdot \theta_T + X_{lspm} \cdot \theta_{lsp} + \varepsilon_3 \end{array} \right. \quad \left\{ \begin{array}{l} Y_1 = X_{c,1} \cdot \theta_c + X_1 \cdot \theta_1 + \varepsilon_1 \\ Y_2 = X_{c,2} \cdot \theta_c + X_2 \cdot \theta_2 + \varepsilon_2 \\ \vdots \qquad \vdots \\ Y_K = X_{c,K} \cdot \theta_c + X_K \cdot \theta_K + \varepsilon_K \end{array} \right.$$

**Coupling parameters**, **Task specific parameters**

Many other examples:

- *Giant squid neurons* [??],
- *Susceptible-infectious-recovered models* [?],
- *Mechanical systems*,...

$$m \left( \frac{d\vec{V}}{dt} \right)_x = (\vec{F}(\theta_F))_x, \quad m \left( \frac{d\vec{V}}{dt} \right)_y = (\vec{F}(\theta_F))_y.$$

# MULTI-TASK REGRESSION

Aircraft:

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$$\begin{cases} Y_1 = X_{T1} \cdot \theta_T - X_D \cdot \theta_D + \varepsilon_1 \\ Y_2 = X_{T2} \cdot \theta_T + X_L \cdot \theta_L + \varepsilon_2 \\ Y_3 = X_T \cdot \theta_T + X_{lspm} \cdot \theta_{lsp} + \varepsilon_3 \end{cases} \quad \begin{cases} Y_1 = X_{c,1} \cdot \theta_c + X_1 \cdot \theta_1 + \varepsilon_1 \\ Y_2 = X_{c,2} \cdot \theta_c + X_2 \cdot \theta_2 + \varepsilon_2 \\ \vdots \qquad \qquad \vdots \\ Y_K = X_{c,K} \cdot \theta_c + X_K \cdot \theta_K + \varepsilon_K \end{cases}$$

Coupling parameters , Task specific parameters

Multi-task Linear Least-Squares:

$$\min_{\theta} \sum_{k=1}^K \sum_{i=1}^N (Y_{k,i} - X_{c,k,i} \cdot \theta_c - X_{k,i} \cdot \theta_k)^2$$

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Multi-task Linear Least-Squares:

Block-sparse Coupling Structure

$$\min_{\theta} \sum_{i=1}^N \left\| \begin{pmatrix} Y_{1,i} \\ \vdots \\ Y_{K,i} \end{pmatrix} - \begin{pmatrix} X_{c,1,i} & X_{1,i} & 0 & 0 & \dots & 0 \\ X_{c,2,i} & 0 & X_{2,i} & 0 & \dots & 0 \\ \vdots & 0 & 0 & \ddots & 0 & 0 \\ X_{c,K,i} & 0 & 0 & \dots & 0 & X_{K,i} \end{pmatrix} \begin{pmatrix} \theta_c \\ \theta_1 \\ \vdots \\ \theta_K \end{pmatrix} \right\|_2^2$$

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Multi-task Linear Least-Squares:

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**Coupling parameters , Task specific parameters**

Multi-task Linear Least-Squares:

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$$\min_{\theta} \sum_{i=1}^N \|Y_i - X_i \theta\|_2^2$$

with  $\theta = (\theta_c, \theta_1, \dots, \theta_K) \in \mathbb{R}^p$ ,  $p = d_c + \sum_{k=1}^K d_k$ ,  
 $Y_i \in \mathbb{R}^K$  and  $X_i \in \mathbb{R}^{K \times p}$ .

# FEATURE SELECTION

Our model:

$$T = N_1(\theta_{T,1} + \theta_{T,2}\rho + \theta_{T,3}M + \theta_{T,4}\rho^2 + \theta_{T,5}\rho M + \theta_{T,6}M^2 + \\ \theta_{T,7}\rho^3 + \theta_{T,8}\rho^2M + \theta_{T,9}\rho M^2 + \theta_{T,10}M^3 + \theta_{T,11}\rho^4 + \\ \theta_{T,12}\rho^3M + \theta_{T,13}\rho^2M^2 + \theta_{T,14}\rho M^3 + \theta_{T,15}M^4).$$

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⇒ High risk of overfitting

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Sparse models are:

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- More compliant with physical models,
- More interpretable,
- Lighter/Faster.

# BLOCK-SPARSE LASSO

Lasso [?]:  $\{(X_i, Y_i)\}_{i=1}^N \subset \mathbb{R}^{d+1}$  i.i.d sample,

$$\min_{\boldsymbol{\theta}} \sum_{i=1}^N (Y_i - X_i \cdot \boldsymbol{\theta})^2 + \lambda \|\boldsymbol{\theta}\|_1.$$

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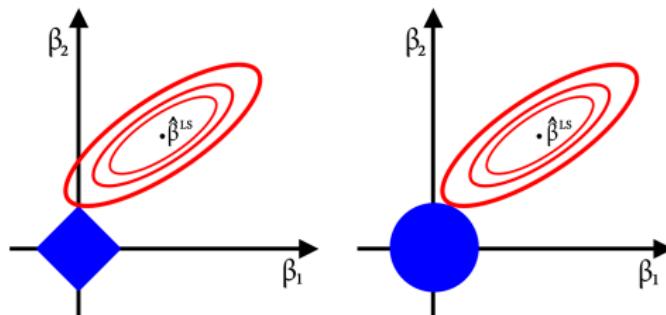


FIGURE:  $^1$ Sparsity induced by  $L^1$  norm in Lasso.

# BLOCK-SPARSE LASSO

$$\min_{\boldsymbol{\theta}} \sum_{k=1}^K \sum_{i=1}^N (Y_{k,i} - X_{c,k,i} \cdot \boldsymbol{\theta}_c - X_{k,i} \cdot \boldsymbol{\theta}_k)^2$$

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Block-sparse structure preserved

$$\min_{\boldsymbol{\theta}} \sum_{k=1}^K \sum_{i=1}^N (Y_{k,i} - X_{c,k,i} \cdot \boldsymbol{\theta}_c - X_{k,i} \cdot \boldsymbol{\theta}_k)^2 + \lambda_c \|\boldsymbol{\theta}_c\|_1 + \sum_{k=1}^K \lambda_k \|\boldsymbol{\theta}_k\|_1$$

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Block-sparse structure preserved  $\rightsquigarrow$  **Equivalent to Lasso problem**

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$$+ \lambda_c \left\| \begin{pmatrix} \beta_c \\ \beta_1 \\ \vdots \\ \beta_K \end{pmatrix} \right\|_1,$$

$$B_{c,k,i} := X_{c,k,i}, \quad \beta_c := \theta_c,$$

$$B_{k,i} := \frac{\lambda_c}{\lambda_k} X_{k,i}, \quad \beta_k := \frac{\lambda_k}{\lambda_c} \theta_k.$$

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In practice:  $\lambda_k = \lambda_c$ , for all  $k = 1, \dots, K$ .

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- Group Lasso [?] —  $\{(X_i, Y_i)\}_{i=1}^N \subset \mathbb{R}^{d+1}$  i.i.d sample,  $\boldsymbol{\theta} = (\boldsymbol{\theta}_g)_{g \in \mathcal{G}}$  partition:

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- Multi-task Lasso [?] —  $K$  tasks,  $X_i \in \mathbb{R}^d$ ,  $Y_i \in \mathbb{R}^K$ ,  $\boldsymbol{\theta} \in \mathbb{R}^{d \times K}$ ,  $\boldsymbol{\theta}^k = (\boldsymbol{\theta}_j^k)_{j=1}^d$  parameter of task  $k$ :

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$\Rightarrow$  Shared sparsity among tasks

# STRUCTURED FEATURE SELECTION STATE-OF-THE-ART

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Other methods

Similarity/Difference with Block-sparse Lasso

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<u>Sparse Group Lasso</u>	Sparsity induced within group,
<u>Multi-task Lasso</u>	Multi-task setting, but not same pattern for every task.

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# BOOTSTRAP IMPLEMENTATION

**High correlations among between polynomial features...**

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 $L^1$  penalty parameter  $\lambda_c$ ,

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$$\min_{\theta} \sum_{i=1}^N \|Y_i - X_i \theta\|_2^2 + \lambda_c \|\theta\|_1 \Rightarrow \hat{\theta}_T = \hat{\theta}_{Isp} = 0!$$

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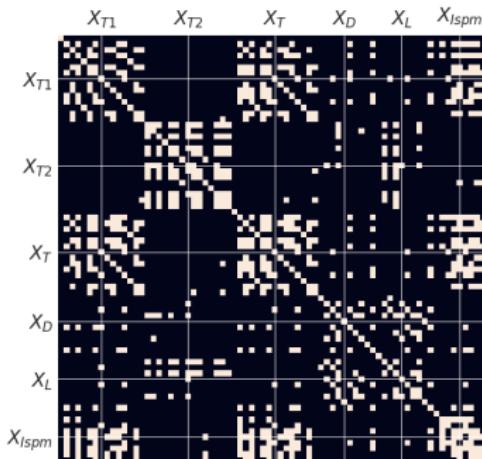
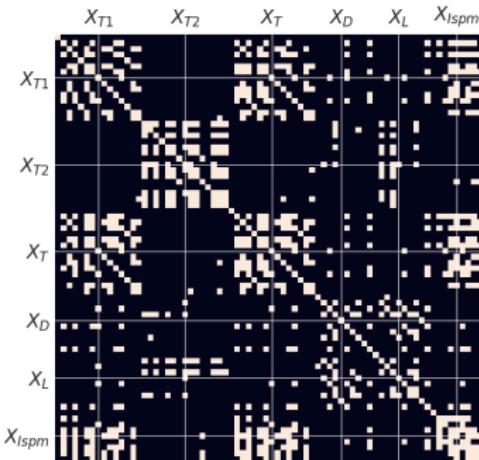


FIGURE: Features correlations  
higher than 0.9 in absolute value.

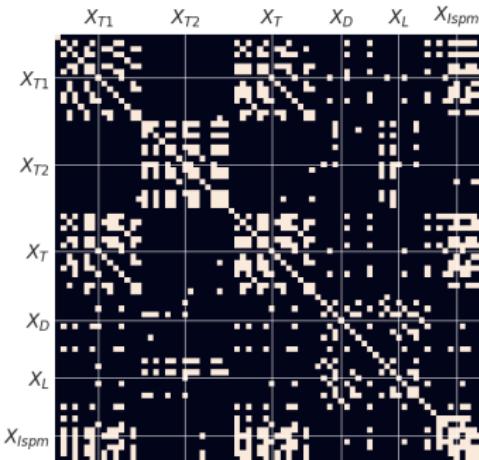
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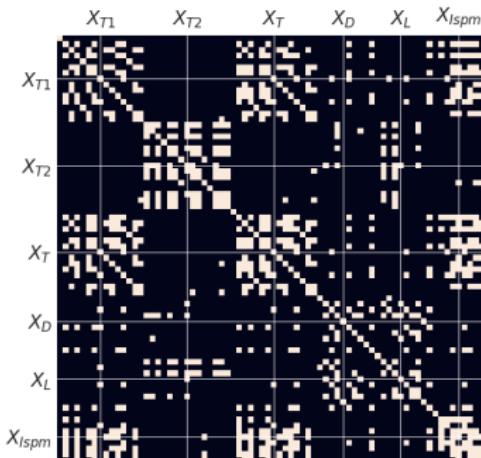


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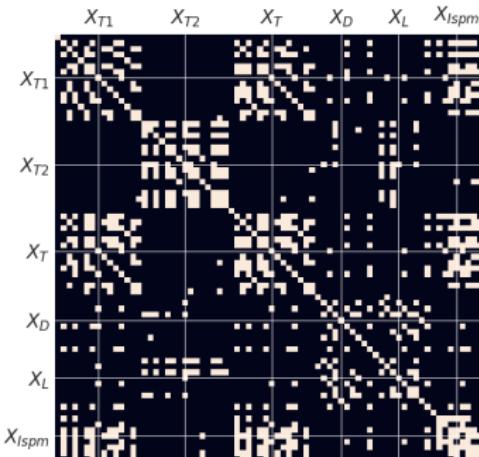
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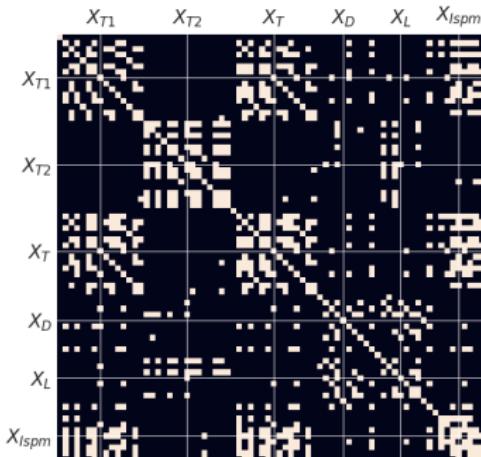


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$\Rightarrow$  Generalized Tikhonov

$$(\boldsymbol{\theta} - \tilde{\boldsymbol{\theta}})^\top Q (\boldsymbol{\theta} - \tilde{\boldsymbol{\theta}}) = \|\Gamma(\boldsymbol{\theta} - \tilde{\boldsymbol{\theta}})\|_2^2,$$

where  $\tilde{\boldsymbol{\theta}}$  is a prior and  $Q \in \mathbb{R}^{p \times p}$ .

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Equivalent to  $\|\Gamma(\theta - \tilde{\theta})\|_2^2$  with  $\Gamma_i = (\underbrace{0, \dots, 0}_{d_T+d_D+d_L}, X_{Isp}^\top)$  and  $\Gamma_i \tilde{\theta} = \tilde{I}_{sp,i}$ .

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$$\tilde{Y}_i = \begin{pmatrix} Y_{1,i} \\ Y_{2,i} \\ 0 \\ \lambda_t \tilde{I}_{sp,i} \end{pmatrix}, \quad \tilde{X}_i = \begin{pmatrix} (X_{T1,i})^\top & -(X_{D,i})^\top & 0 & 0 \\ (X_{T2,i})^\top & 0 & (X_{L,i})^\top & 0 \\ (X_{T,i})^\top & 0 & 0 & (X_{Ispm,i})^\top \\ 0 & 0 & 0 & \lambda_t (X_{Isp,i})^\top \end{pmatrix},$$

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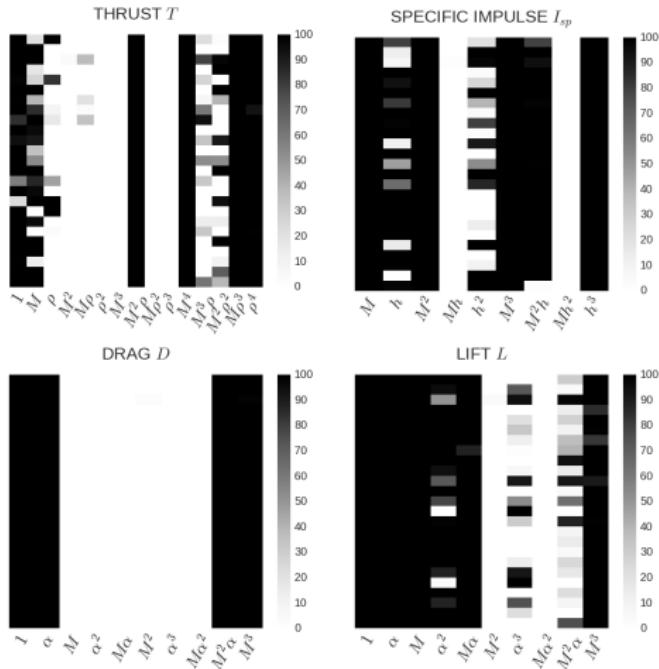
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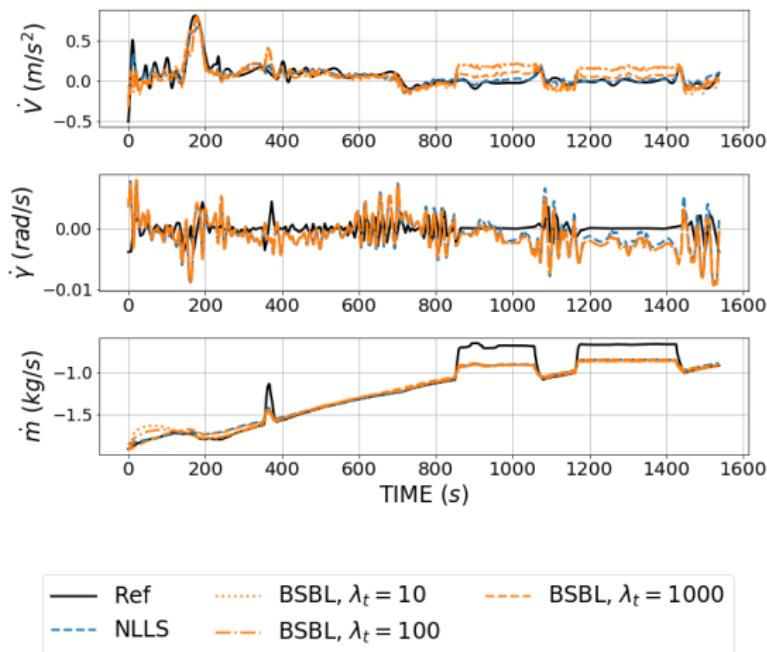
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# FEATURE SELECTION RESULTS



Feature selection results for the thrust, drag, lift and specific impulse models.

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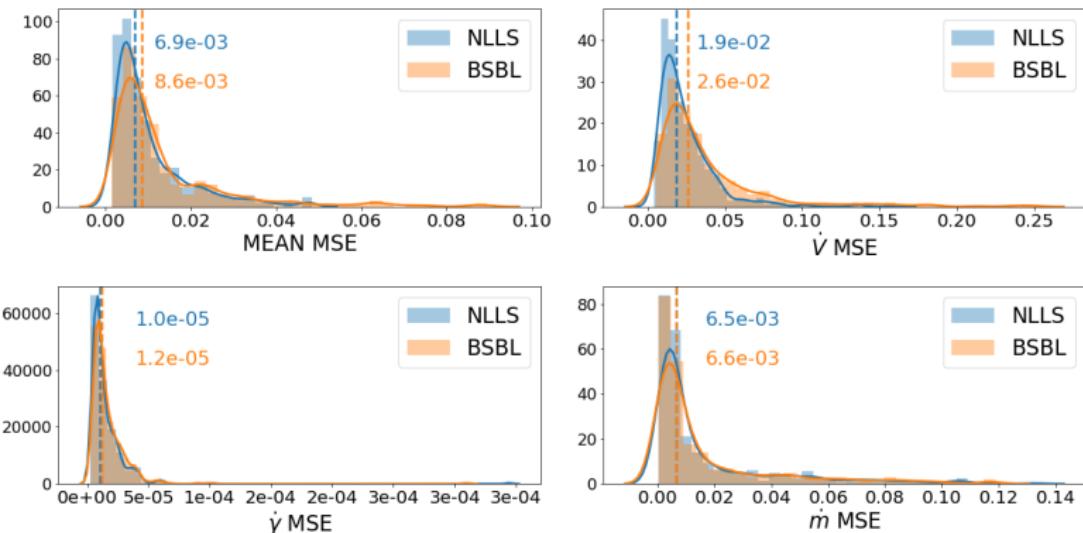
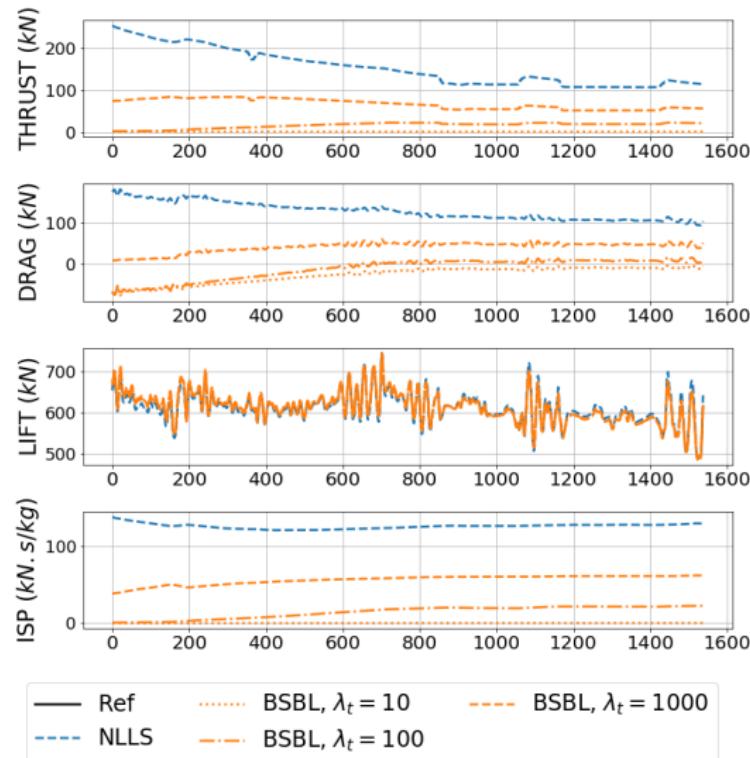


FIGURE: Leave-one-out off-sample errors distributions for nonlinear least-squares NLLS and block-sparse bolasso BSBL. Median errors are annotated and marked by dashed vertical lines.

# REALISM OF HIDDEN ELEMENTS



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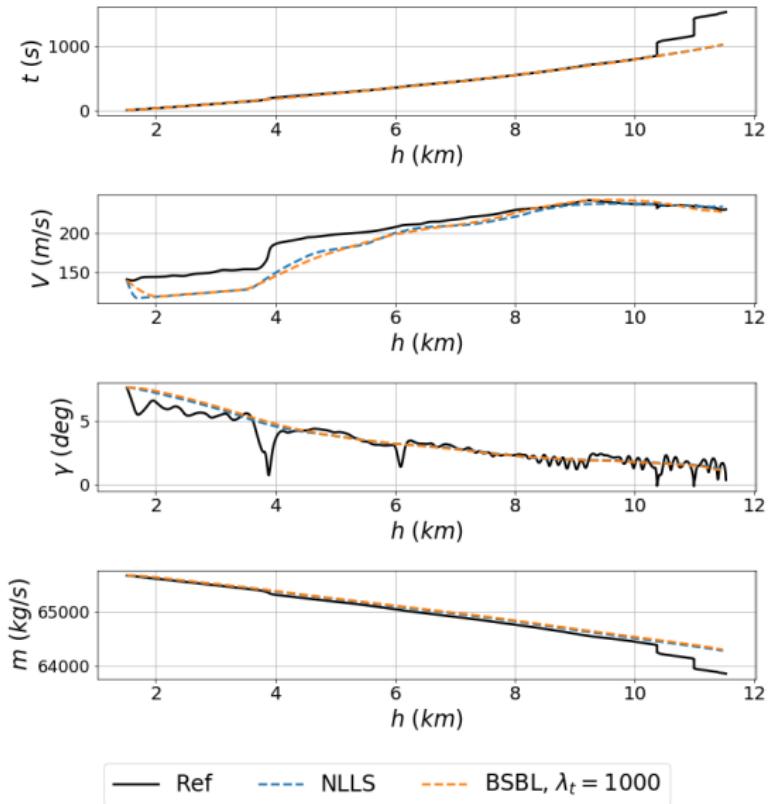
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$$\text{s.t. } \dot{\mathbf{x}}(t) = g(\mathbf{x}(t), \mathbf{u}(t), \hat{\theta}),$$

where  $\|\cdot\|_{\mathbf{u}}$ ,  $\|\cdot\|_{\mathbf{x}}$  denote scaling norms.

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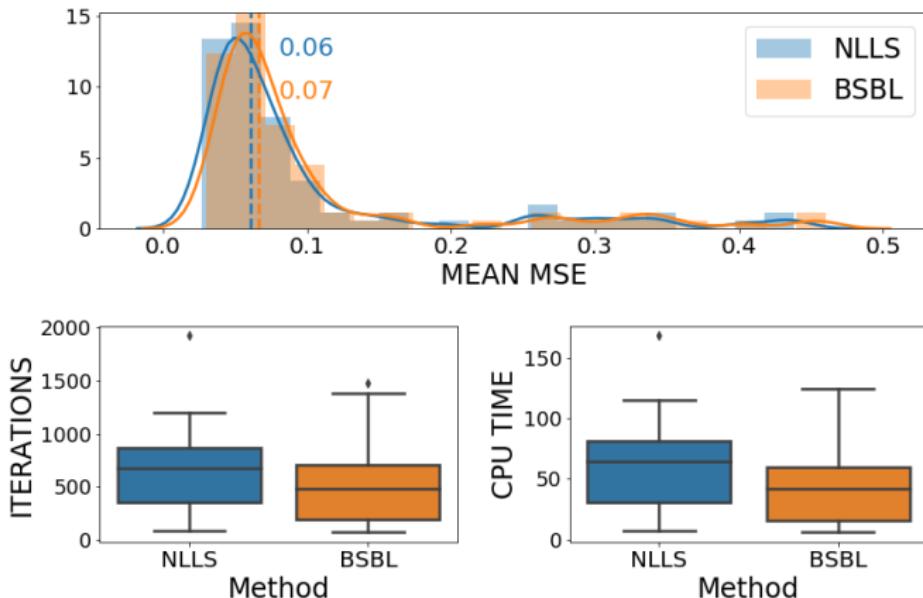


FIGURE: Distribution of the off-sample simulation error and boxplot of the optimization number of iterations and CPU time.

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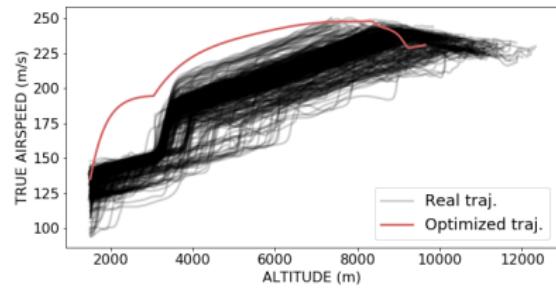
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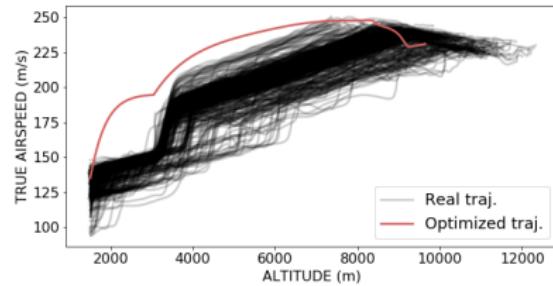
**Is  $\hat{\mathbf{z}}$  inside the validity region of the dynamics model  $\hat{g}$  ?**

**Does it look like a real trajectory ?**

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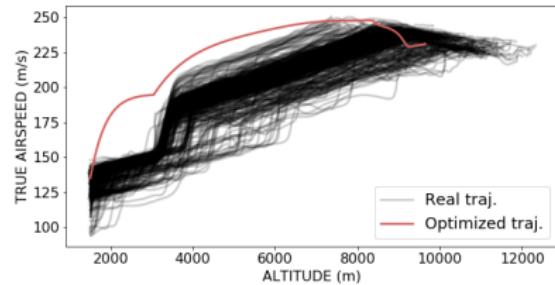


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Pilots acceptance

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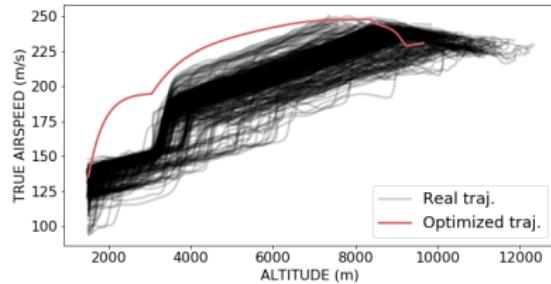


Pilots acceptance



Air Traffic Control<sup>2</sup>

# TRAJECTORY ACCEPTABILITY



Pilots acceptance



Air Traffic Control<sup>2</sup>

**How can we quantify the closeness from the optimized trajectory to the set of real flights?**

<sup>2</sup>NATS UK air traffic control

## LIKELIHOOD

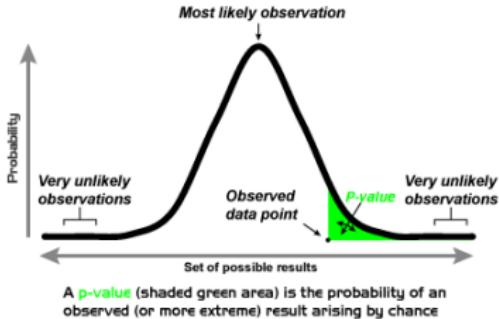
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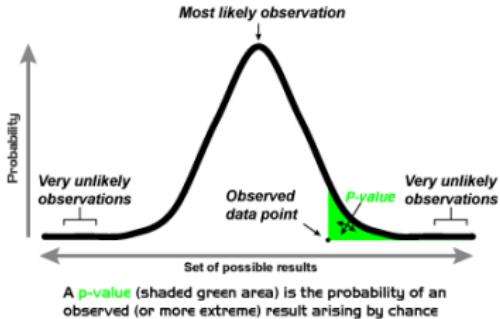


<sup>0</sup>Picture source: wikipedia, P-Value, author: Repapetilo CC.

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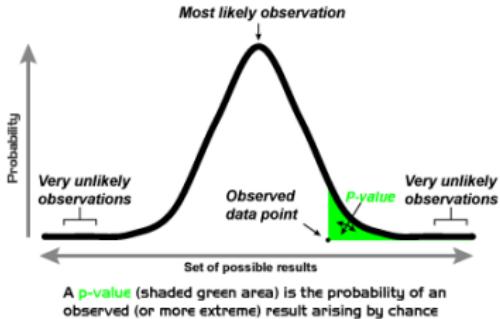
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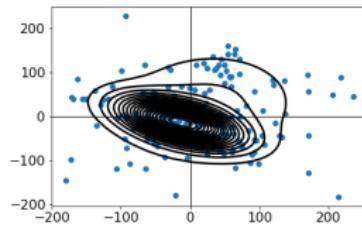
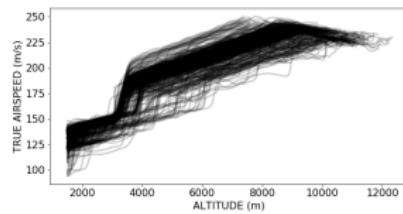
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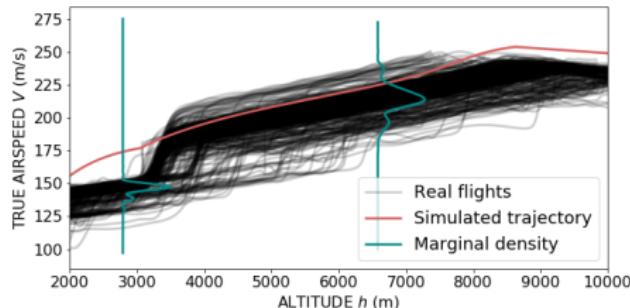


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- Or: we can aggregate the marginal densities



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**Marginal densities may have really different shapes**

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## MEAN MARGINAL LIKELIHOOD

$$\text{MML}(Z, \mathbf{y}) = \frac{1}{t_f} \int_0^{t_f} \psi[f_t, \mathbf{y}(t)] dt,$$

where  $\psi : L^1(E, \mathbb{R}_+) \times \mathbb{R} \rightarrow [0; 1]$  is a continuous scaling map.

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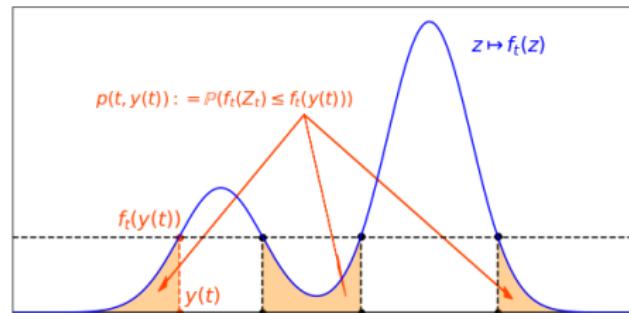
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$$\psi[f_t, \mathbf{y}(t)] := \mathbb{P}(f_t(Z_t) \leq f_t(\mathbf{y}(t))).$$



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In practice, the  $m$  trajectories are sampled at variable discrete times:

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Hence, we approximate the MML using a Riemann sum which aggregates consistent estimators  $\hat{f}_{\tilde{t}_j}^m$  of the marginal densities  $f_{\tilde{t}_j}$ :

$$\text{EMML}_m(\mathcal{T}^D, \mathcal{Y}) := \frac{1}{t_f} \sum_{j=1}^{\tilde{n}} \psi[\hat{f}_{\tilde{t}_j}^m, y_j] \Delta \tilde{t}_j.$$

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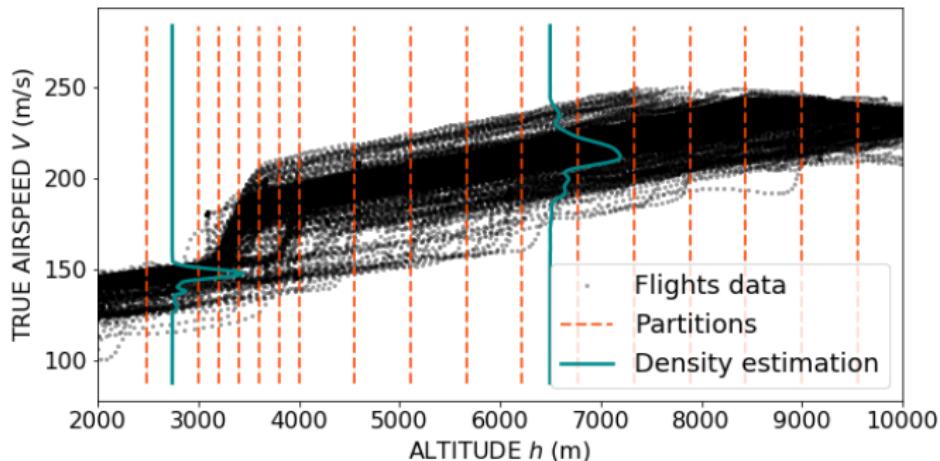
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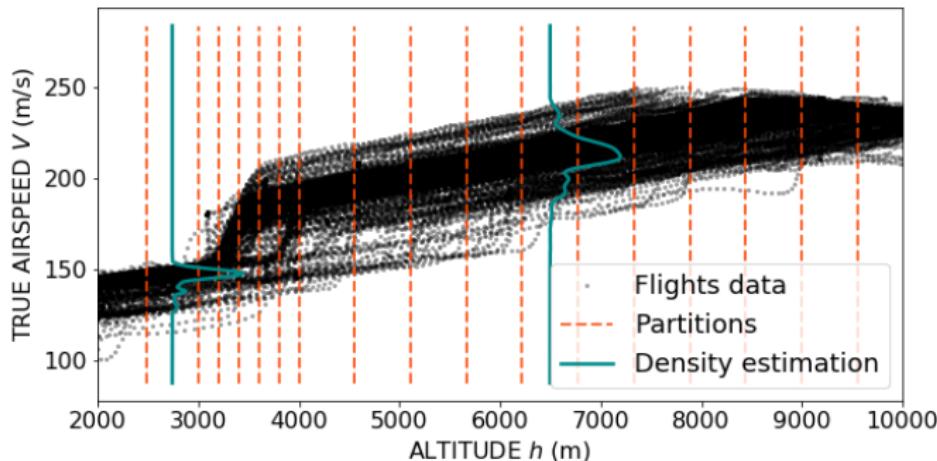
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⇒ Instead, we choose to use a fine partitioning of the time domain.

# PARTITION BASED MARGINAL DENSITY ESTIMATION



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Idea: to average in time the marginal densities over small bins by applying classical multivariate density estimation techniques to each subset.

# CONSISTENCY

We denote by:

- $\Theta : \mathcal{S} \rightarrow L^1(E, \mathbb{R}_+)$  multivariate density estimation statistic,
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- $\hat{f}_t^m := \Theta[\mathcal{T}_t^m]$  estimator trained using  $\mathcal{T}_t^m$ .

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ASSUMPTION 1 - POSITIVE TIME DENSITY

$\nu \in L^\infty(E, \mathbb{R}_+)$  density function of  $T$ , s.t.

$$\nu_+ := \text{ess} \sup_{t \in \mathbb{T}} \nu(t) < \infty, \quad \nu_- := \text{ess} \inf_{t \in \mathbb{T}} \nu(t) > 0.$$

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Function  $(t, z) \in \mathbb{T} \times E \mapsto f_t(z)$  is continuous and

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ASSUMPTION 3 - SHRINKING BINS

The homogeneous partition  $\{B_\ell^m\}_{\ell=1}^{q_m}$  of  $[0; t_f]$ , with binsize  $b_m$ , is s.t.

$$\lim_{m \rightarrow \infty} b_m = 0, \quad \lim_{m \rightarrow \infty} mb_m = \infty.$$

# CONSISTENCY

## ASSUMPTION 4 - I.I.D. CONSISTENCY

- $\mathcal{G}$  arbitrary family of probability density functions on  $E$ ,  $\rho \in \mathcal{G}$ ,
- $S_\rho^N$  i.i.d sample of size  $N$  drawn from  $\rho$  valued in  $\mathcal{S}$ .

The estimator obtained by applying  $\Theta$  to  $S_\rho^N$ , denoted by

$$\hat{\rho}^N := \Theta[S_\rho^N] \in L^1(E, \mathbb{R}_+),$$

is a (pointwise) consistent density estimator, uniformly in  $\rho$ :

For all  $z \in E, \varepsilon > 0, \alpha_1 > 0$ , there is  $N_{\varepsilon, \alpha_1} > 0$  such that, for any  $\rho \in \mathcal{G}$ ,

$$N \geq N_{\varepsilon, \alpha_1} \Rightarrow \mathbb{P} \left( \left| \hat{\rho}^N(z) - \rho(z) \right| < \varepsilon \right) > 1 - \alpha_1.$$

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## THEOREM 1 - [?]

Under assumptions 1 to 4, for any  $z \in E$  and  $t \in \mathbb{T}$ ,  $\hat{f}_{\ell^m(t)}^m(z)$  consistently approximates the marginal density  $f_t(z)$  as the number of curves  $m$  grows:

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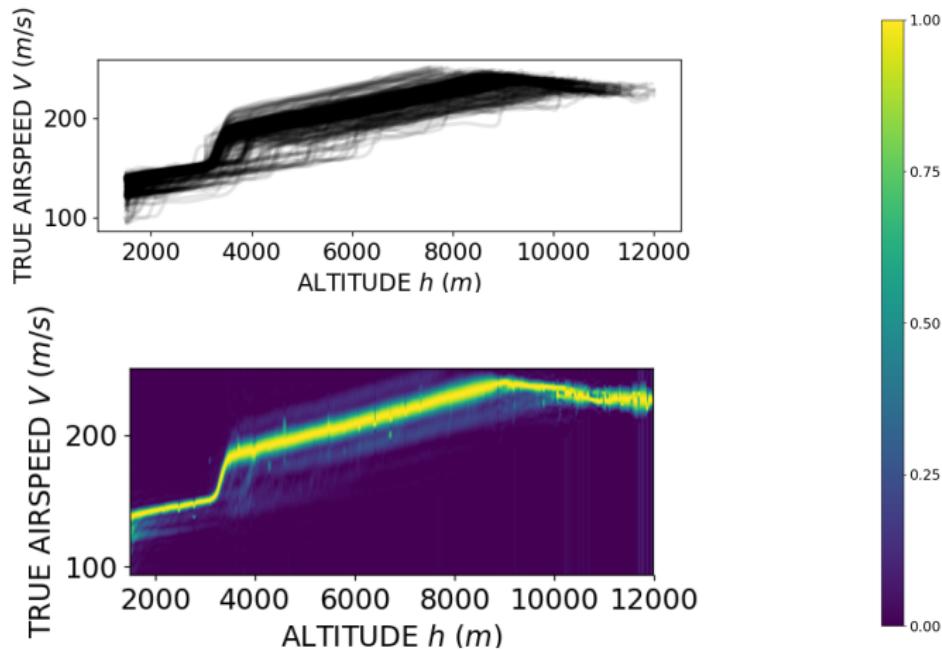
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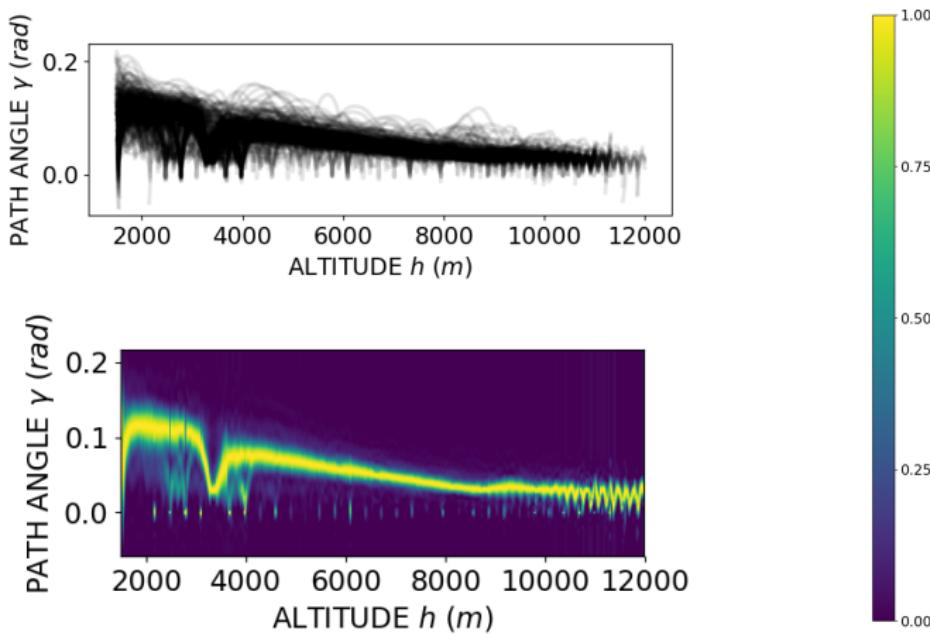
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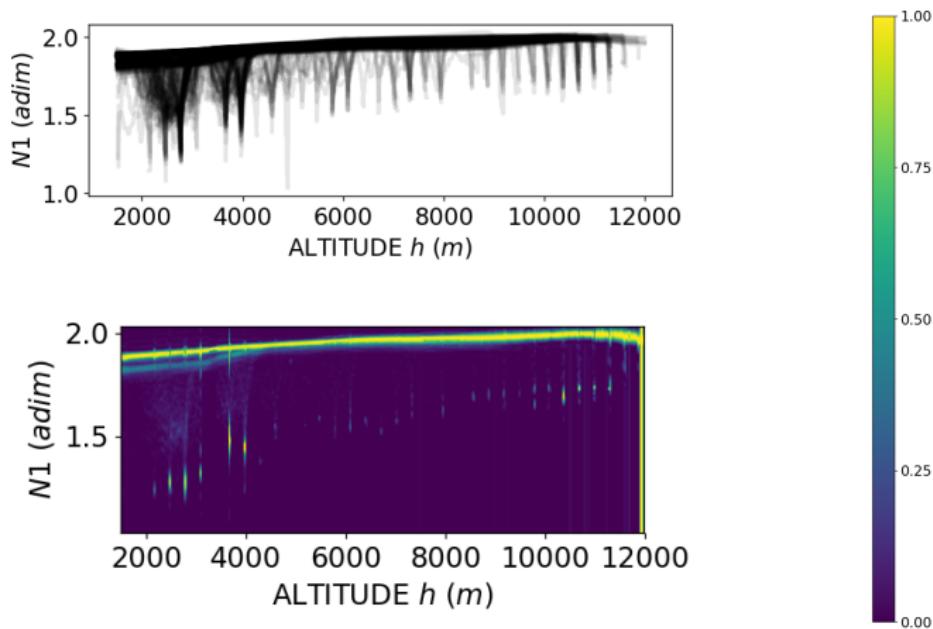
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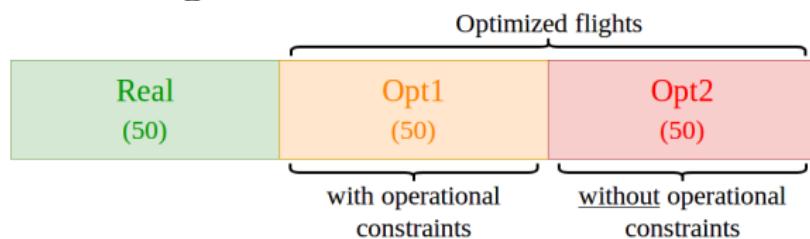
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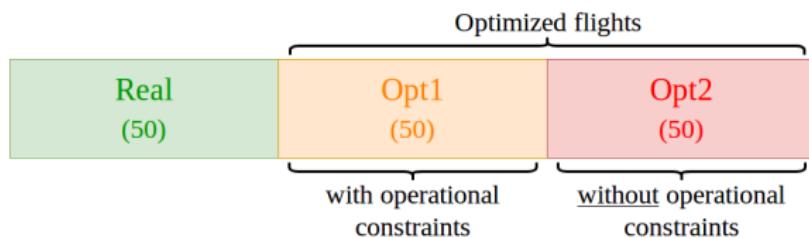
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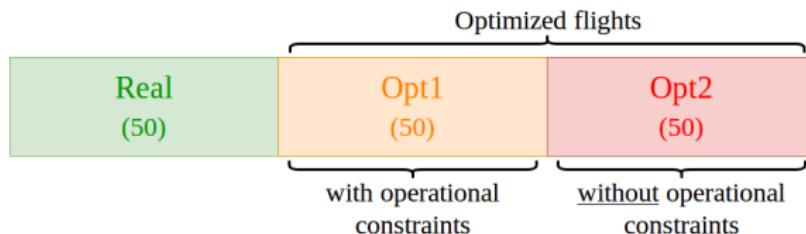
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VAR.	ESTIMATED LIKELIHOODS		
	REAL	OPT1	OPT2
MML	<b><math>0.63 \pm 0.07</math></b>	<b><math>0.43 \pm 0.08</math></b>	<b><math>0.13 \pm 0.02</math></b>
FPCA	$0.16 \pm 0.12$	$6.4\text{E-}03 \pm 3.8\text{E-}03$	$3.6\text{E-}03 \pm 5.4\text{E-}03$
LS-CDE	$0.77 \pm 0.05$	$0.68 \pm 0.04$	$0.49 \pm 0.06$

# MML PENALTY

The MML can be used not only to assess the optimization solutions, but also to penalize the optimization itself:

$$\begin{aligned} & \min_{(\mathbf{x}, \mathbf{u}) \in \mathbb{X} \times \mathbb{U}} \int_0^{t_f} C(\mathbf{u}(t), \mathbf{x}(t)) dt \\ \text{s.t. } & \left\{ \begin{array}{l} \dot{\mathbf{x}}(t) = g(\mathbf{u}(t), \mathbf{x}(t)) + \varepsilon(t), \quad \text{for a.e. } t \in [0, t_f], \\ \text{Other constraints...} \end{array} \right. \end{aligned} \tag{OCP}$$

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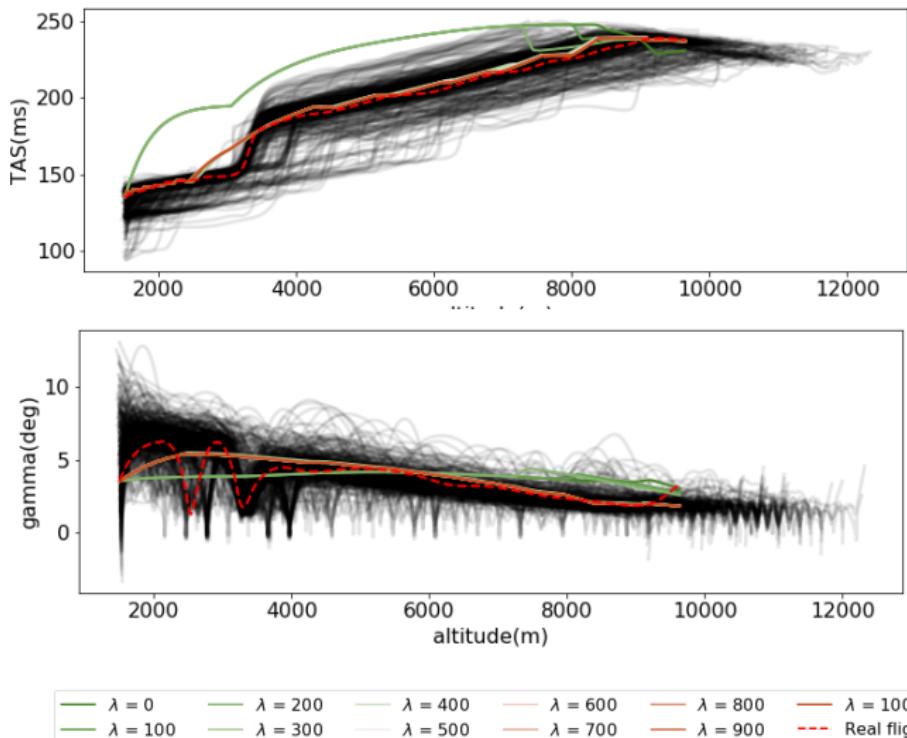
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- $\lambda$  sets trade-off between a fuel minimization and a likelihood maximization,

# PENALTY EFFECT



# CONSUMPTION X ACCEPTABILITY TRADE-OFF

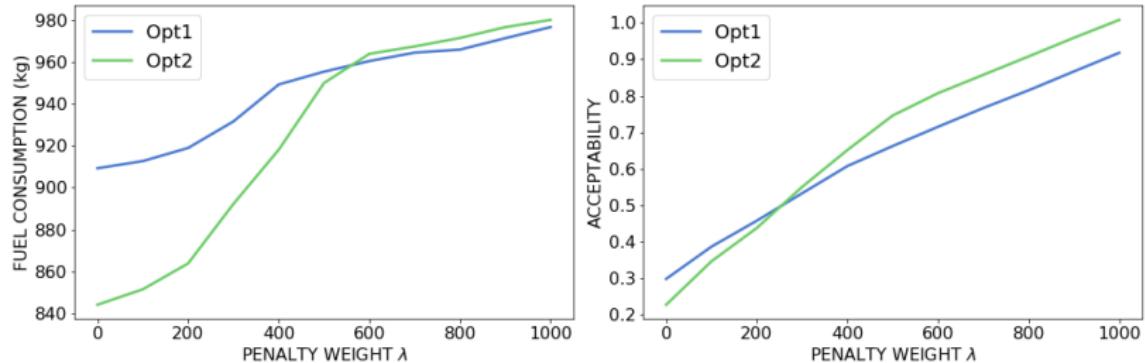


FIGURE: Average over 20 flights of the fuel consumption and MML score (called acceptability here) of optimized trajectories with varying MML-penalty weight  $\lambda$ .

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- 4 Particular Adaptive Kernel and Gaussian mixture implementation,
  - Showed that it can be used in optimal control problems to obtain solutions close to optimal, and still realistic.

**THANK YOU FOR YOUR ATTENTION**

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