Exploration de données pour l'optimisation de trajectoires aériennes

Cédric Rommel

Directeurs de thèse: Frédéric Bonnans, Pierre Martinon
Encadrant Safety Line: Baptiste Gregorutti

Soutenance de thèse, 26 octobre 2018

Context

Motivation

FIGURE: World air traffic - source: Flight Radar

Motivation

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- 20 000 airplanes — 80 000 flights per day,

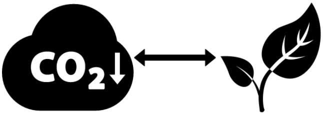
Motivation

Figure: World air traffic - source: Flight Radar

- 20 000 airplanes — 80 000 flights per day,
- Should double until 2033,

Motivation

- Most polluting means of transportation,



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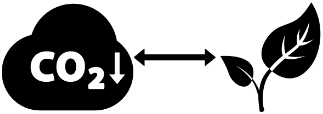
- Most polluting means of transportation,
- Responsible for 3% of CO_2 emissions,



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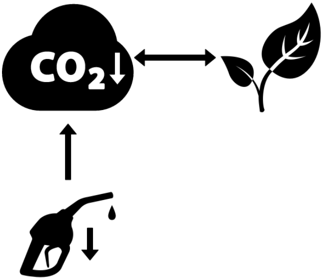

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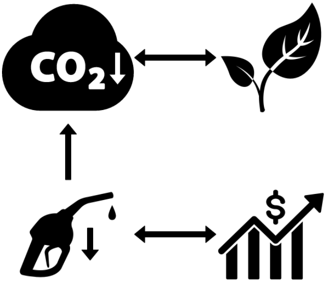
- Fuel $\simeq$ 30% of an airline operational cost,



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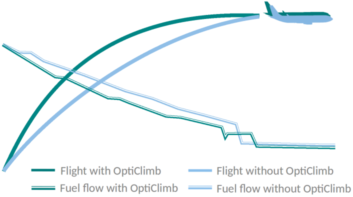

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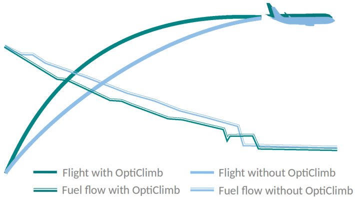

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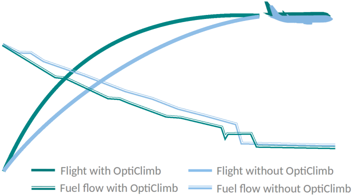

OPTI CLIMB
More savings, less CO_2

OptiClimb

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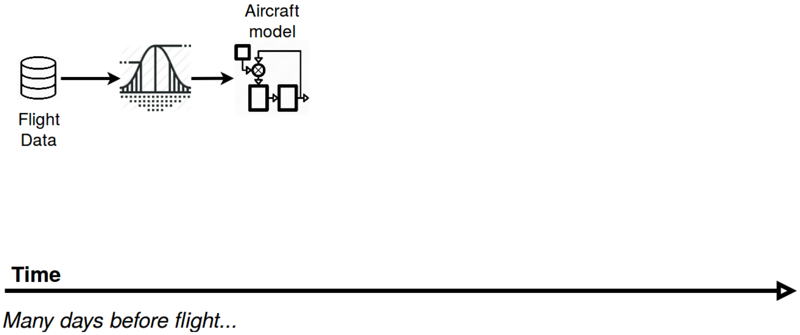

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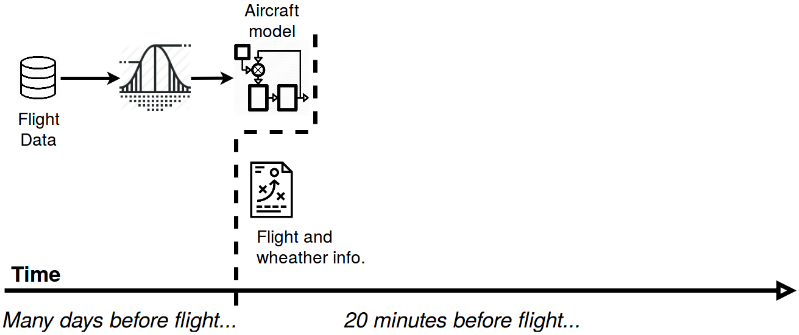
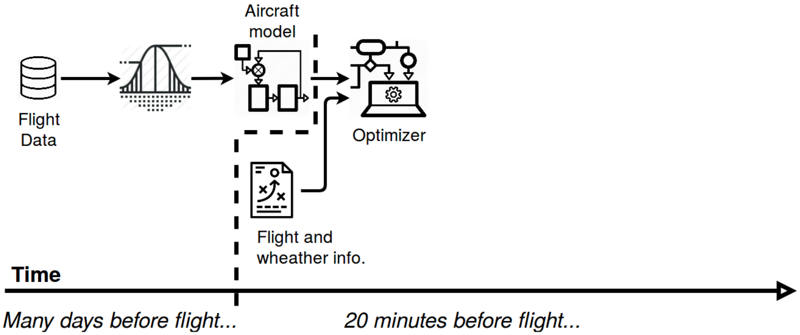


Aircraft
model
Flight
Data
Flight and
wheather info.
Time
Many days before flight...
20 minutes before flight...

OptiClimb



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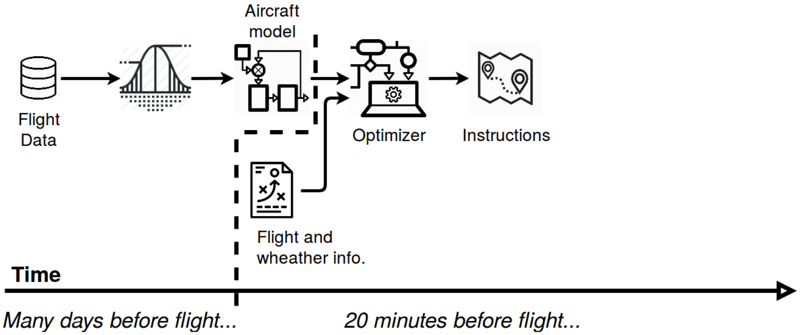

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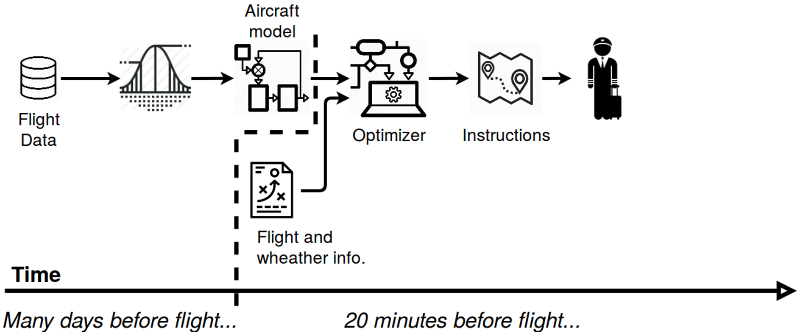
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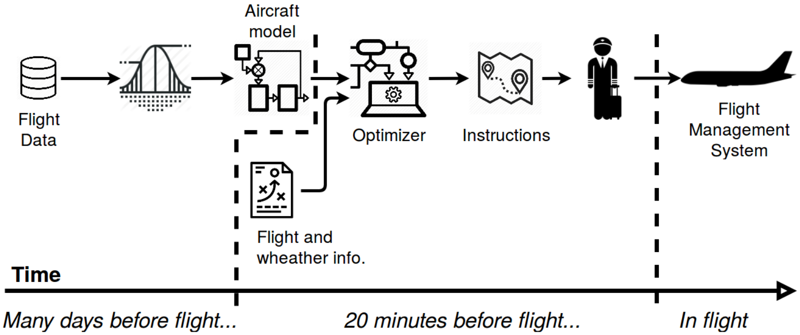

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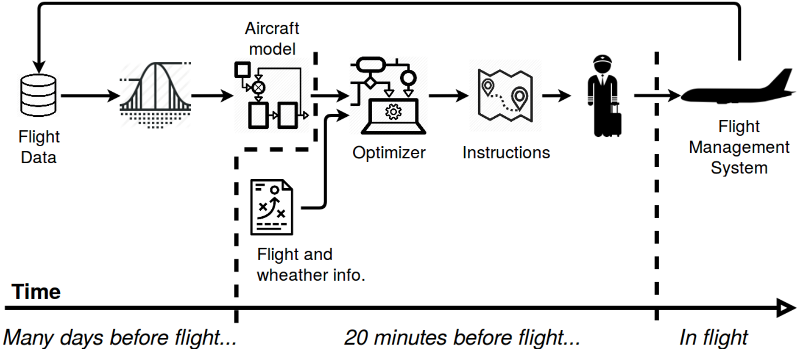

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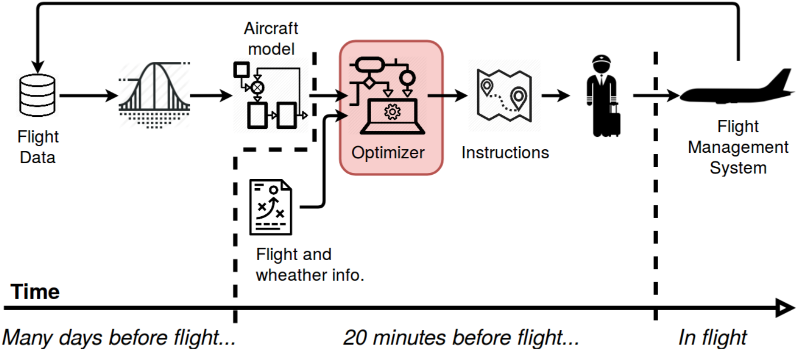

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Trajectory Optimization

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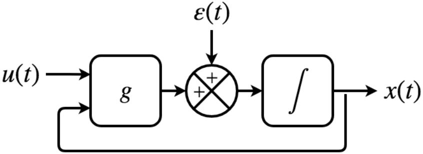
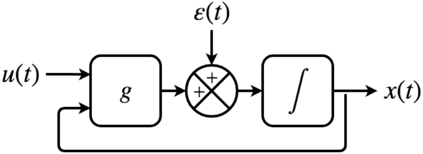


Trajectory Optimization

Dynamics:

$$\dot{x}(t) = g(\boldsymbol{u}(t), \boldsymbol{x}(t)) + \varepsilon(t)$$



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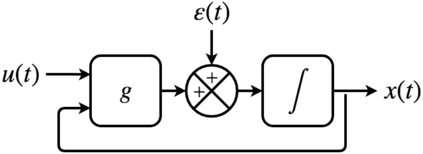

Optimization objective: $C(\boldsymbol{u}(t), \boldsymbol{x}(t))$

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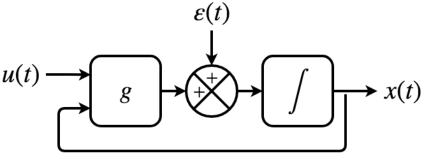

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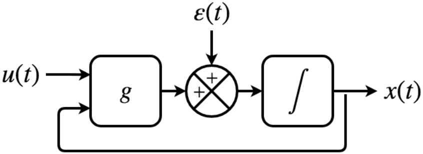

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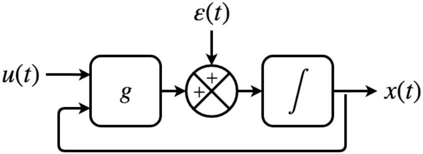

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$$\left\{ \right.$$

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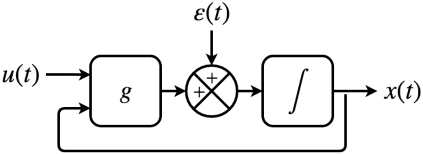

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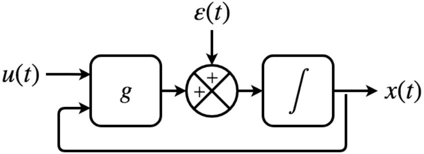

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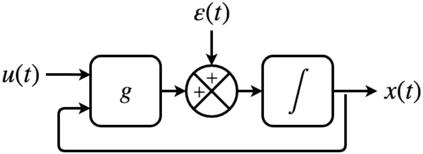

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Trajectory Optimization

Optimal Control Problem

$$\min_{(\boldsymbol{x},\boldsymbol{u})\in\mathbb{X}\times\mathbb{U}} \int_0^{t_f} C(\boldsymbol{u}(t),\boldsymbol{x}(t))dt,$$

$$\text{s.t.} \begin{cases} \dot{\boldsymbol{x}}(t) = g(\boldsymbol{u}(t),\boldsymbol{x}(t)) + \varepsilon(t), & \text{a.e. } t\in[0,t_f], \\ \Phi(\boldsymbol{x}(0),\boldsymbol{x}(t_f)) \in K_\Phi, & \\ \boldsymbol{u}(t)\in U_{ad}, \quad \boldsymbol{x}(t)\in X_{ad}, & \text{a.e. } t\in[0,t_f], \\ c(\boldsymbol{u}(t),\boldsymbol{x}(t)) \le 0, & \text{a.e. } t\in[0,t_f]. \end{cases} \tag{OCP}$$

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System Identification

Black box

Trajectory Optimization

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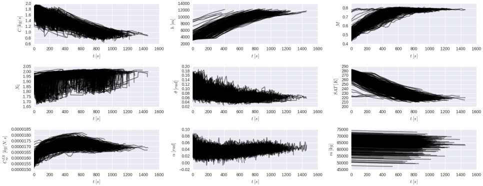

QAR data

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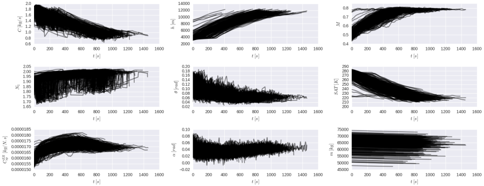

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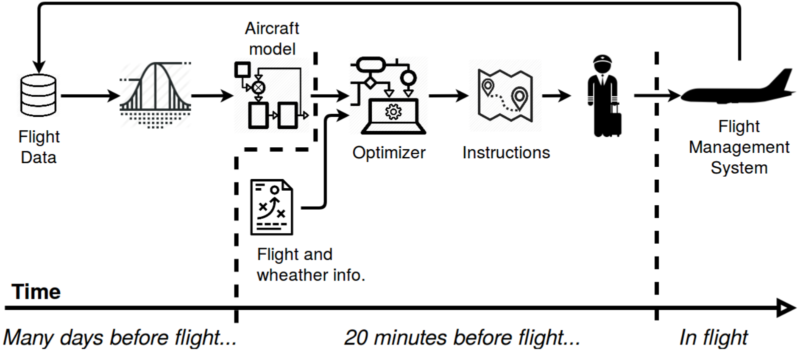

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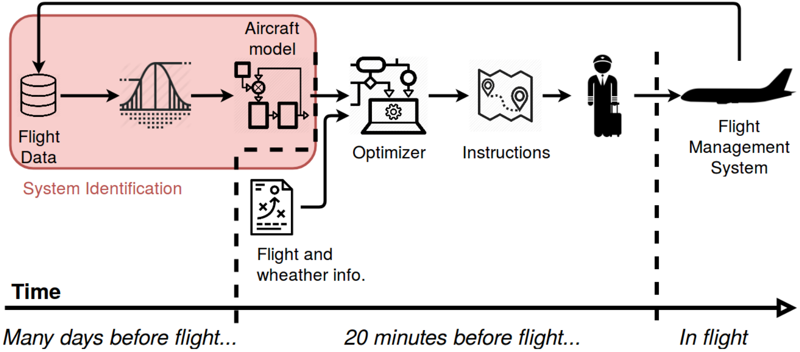
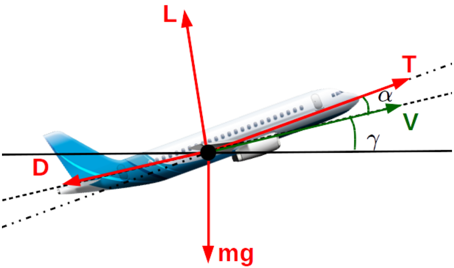


System Identification

Flight dynamics



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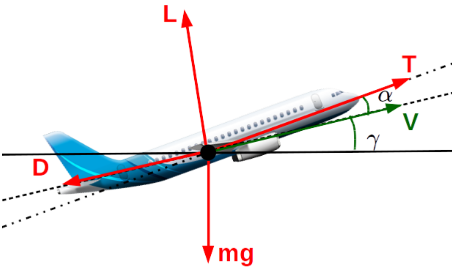

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\dot{h} = & V \sin\gamma \\
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\dot{\gamma} = & \dfrac{T\sin\alpha + L - mg\cos\gamma}{mV} \\
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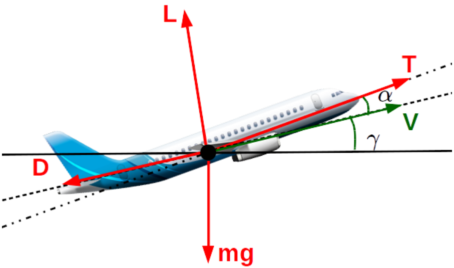
FLIGHT DYNAMICS



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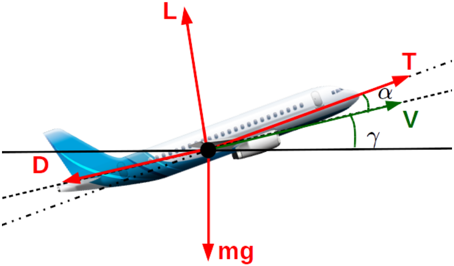

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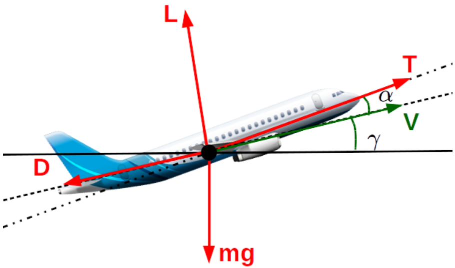

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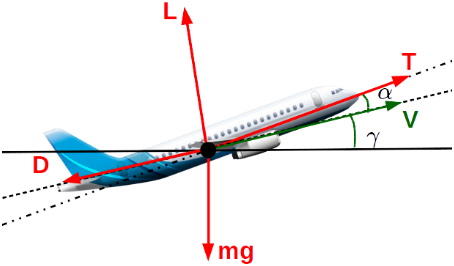

States: $\boldsymbol{x} = (h, V, \gamma, m)$

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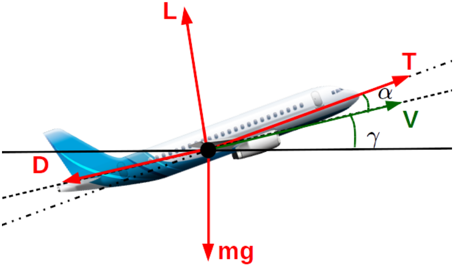

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Controls: $\boldsymbol{u} = (\alpha, N_1)$

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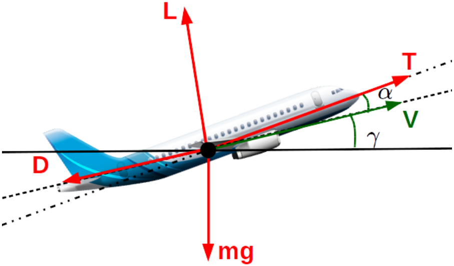

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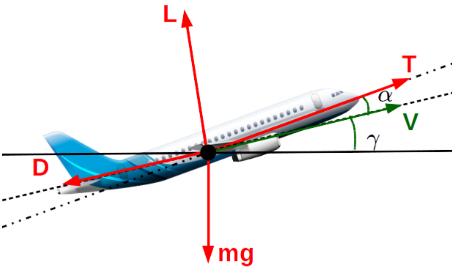

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Physical models of nested functions

$$\begin{cases} T & \text{function of } \; (N_1, M, \rho) \\ D & \text{function of } \; (q, M, \alpha) \\ L & \text{function of } \; (q, M, \alpha) \\ I_{sp} & \text{function of } \; (SAT, M, h) \end{cases}$$

Physical models of nested functions

$$\begin{cases} T & \text{function of } (N_1, M, \rho) = \varphi_T(\boldsymbol{x}, \boldsymbol{u}) \\ D & \text{function of } (q, M, \alpha) = \varphi_D(\boldsymbol{x}, \boldsymbol{u}) \\ L & \text{function of } (q, M, \alpha) = \varphi_L(\boldsymbol{x}, \boldsymbol{u}) \\ I_{sp} & \text{function of } (SAT, M, h) = \varphi_{Isp}(\boldsymbol{x}, \boldsymbol{u}) \end{cases}$$

Physical models of nested functions

$$
\begin{cases}
T = N_1 \times P_T(\rho, M), \\
D = q \times P_D(\alpha, M), \\
L = q \times P_L(\alpha, M), \\
I_{sp} = SAT \times P_{Isp}(h, M),
\end{cases}
$$

Physical models of nested functions

$$\begin{cases} T = X_T \cdot \boldsymbol{\theta}_T, \\ D = X_D \cdot \boldsymbol{\theta}_D, \\ L = X_L \cdot \boldsymbol{\theta}_L, \\ I_{sp} = X_{Isp} \cdot \boldsymbol{\theta}_{Isp}, \end{cases}$$

Physical models of nested functions

$$
\begin{cases}
T = X_T \cdot \boldsymbol{\theta}_T, \\
D = X_D \cdot \boldsymbol{\theta}_D, \\
L = X_L \cdot \boldsymbol{\theta}_L, \\
I_{sp} = X_{Isp} \cdot \boldsymbol{\theta}_{Isp},
\end{cases}
$$

$$
\begin{aligned}
X_T &= N_1(1, \rho, M, \rho^2, \rho M, M^2, \ldots) \\
X_D &= q(1, \alpha, M, \alpha^2, \alpha M, M^2, \ldots) \\
X_L &= q(1, \alpha, M, \alpha^2, \alpha M, M^2, \ldots) \\
X_{Isp} &= SAT(1, h, M, h^2, hM, M^2, \ldots)
\end{aligned}
$$

Physical models of nested functions

$$
\begin{cases}
T(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{\theta}_T) & = X_T \cdot \boldsymbol{\theta}_T, \\
D(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{\theta}_D) & = X_D \cdot \boldsymbol{\theta}_D, \\
L(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{\theta}_L) & = X_L \cdot \boldsymbol{\theta}_L, \\
I_{sp}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{\theta}_{Isp}) & = X_{Isp} \cdot \boldsymbol{\theta}_{Isp},
\end{cases}
$$

$$
\begin{aligned}
X_T &= N_1(1, \rho, M, \rho^2, \rho M, M^2, \ldots) \\
X_D &= q(1, \alpha, M, \alpha^2, \alpha M, M^2, \ldots) \\
X_L &= q(1, \alpha, M, \alpha^2, \alpha M, M^2, \ldots) \\
X_{Isp} &= SAT(1, h, M, h^2, hM, M^2, \ldots)
\end{aligned}
$$

State-of-the-art

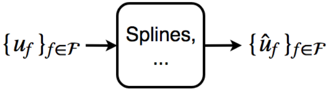
STATE-OF-THE-ART

- Output-Error Method

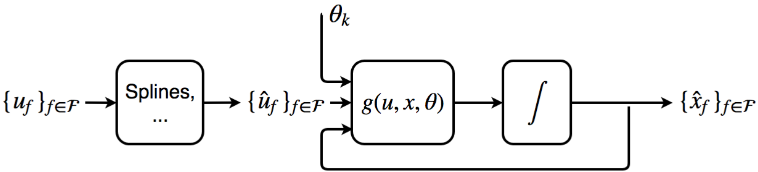
$\{u_f\}_{f \in \mathcal{F}}$

State-of-the-art

- Output-Error Method

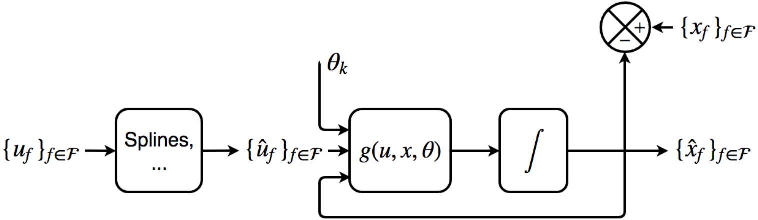


- Output-Error Method



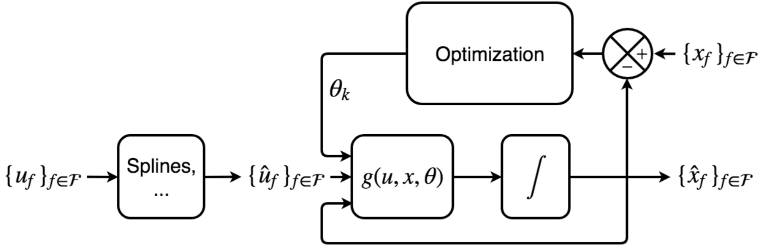
State-of-the-art

- Output-Error Method



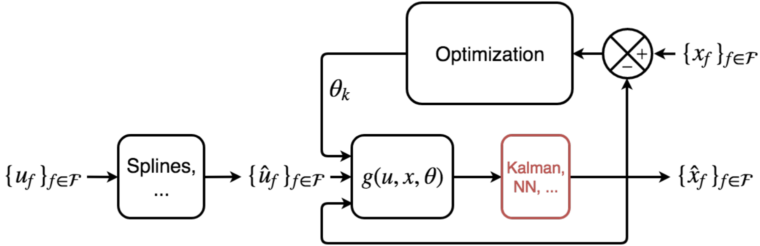
State-of-the-art

- Output-Error Method



State-of-the-art

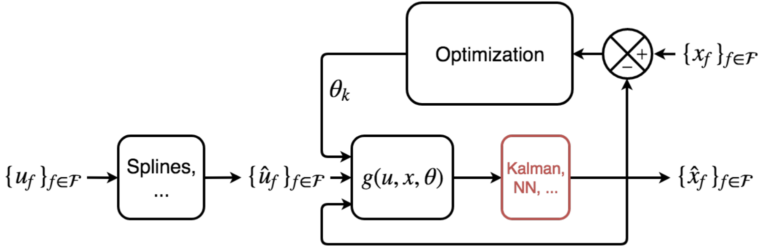
- Output-Error Method
- Filter-Error Method



State-of-the-art

- Output-Error Method
- Filter-Error Method

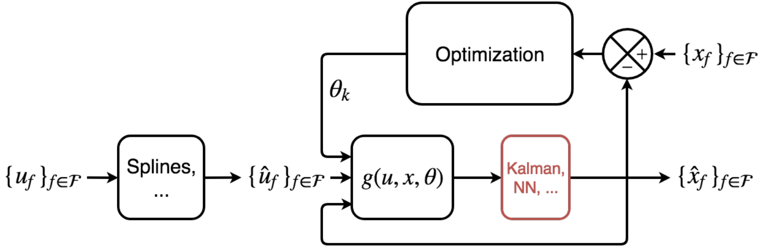
Not scalable to many trajectories



State-of-the-art

- Output-Error Method
- Filter-Error Method

} Not scalable to many trajectories



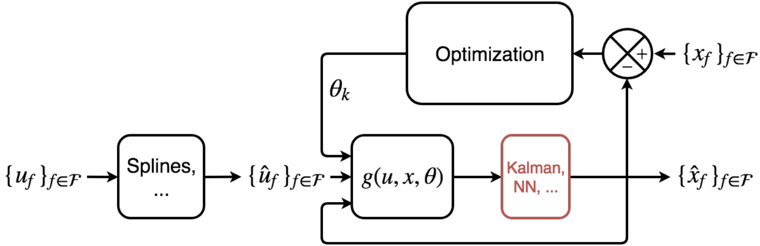
- Equation-Error Method

$$\dot{\boldsymbol{x}}(t) = g(\boldsymbol{u}(t), \boldsymbol{x}(t), \boldsymbol{\theta}) + \varepsilon(t), \quad t \in [0, t_f]$$

State-of-the-art

- Output-Error Method
- Filter-Error Method

Not scalable to many trajectories



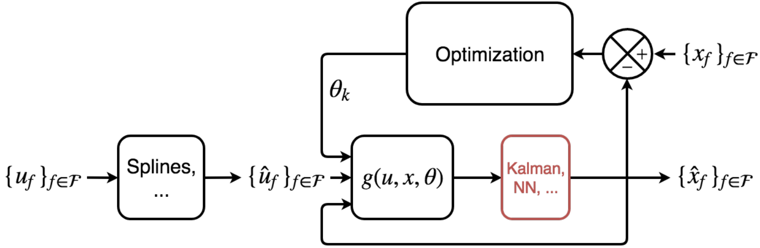
- Equation-Error Method

$$\dot{\boldsymbol{x}}_i = g\left(\boldsymbol{u}_i, \boldsymbol{x}_i, \boldsymbol{\theta}\right) + \varepsilon_i, \quad i = 1, \ldots, N$$

State-of-the-art

- Output-Error Method
- Filter-Error Method

} Not scalable to many trajectories



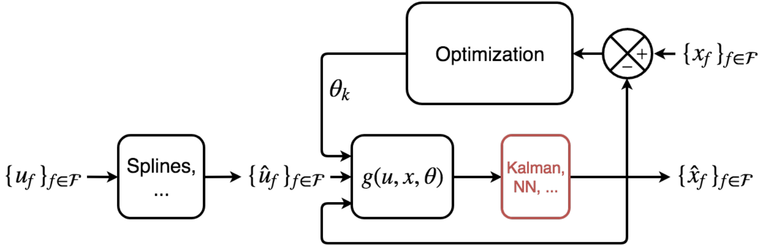
- Equation-Error Method

$$\min_{\boldsymbol{\theta}} \sum_{i=1}^{N} \ell\left(\dot{\boldsymbol{x}}_i, g\left(\boldsymbol{u}_i, \boldsymbol{x}_i, \boldsymbol{\theta}\right)\right)$$

STATE-OF-THE-ART

- Output-Error Method
- Filter-Error Method

Not scalable to many trajectories



- Equation-Error Method

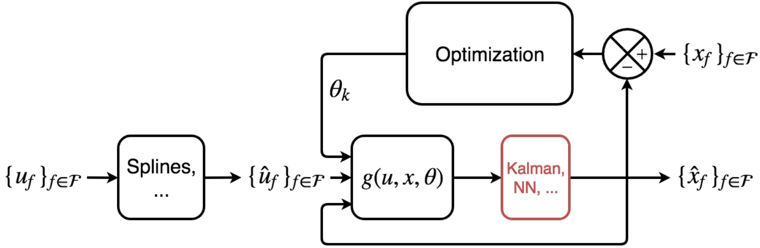
Ex: (Nonlinear) Least-Squares

$$\min_{\boldsymbol{\theta}} \sum_{i=1}^{N} \left\| \dot{\boldsymbol{x}}_i - g\left(\boldsymbol{u}_i, \boldsymbol{x}_i, \boldsymbol{\theta}\right) \right\|_2^2$$

State-of-the-art

- Output-Error Method
- Filter-Error Method

} Not scalable to many trajectories

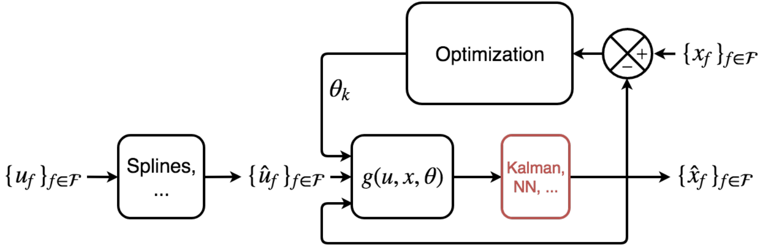


- Equation-Error Method *Ex: (Nonlinear) Least-Squares*

$$\min_{\boldsymbol{\theta}} \sum_{i=1}^{N} \left\| Y(\boldsymbol{u}_i, \boldsymbol{x}_i, \dot{\boldsymbol{x}}_i) - G(\boldsymbol{u}_i, \boldsymbol{x}_i, \dot{\boldsymbol{x}}_i, \boldsymbol{\theta}) \right\|_2^2$$

- Output-Error Method
- Filter-Error Method

} Not scalable to many trajectories



- **Equation-Error Method** *Ex: (Nonlinear) Least-Squares*

$$\min_{\boldsymbol{\theta}} \sum_{i=1}^{N} \left\| Y(\boldsymbol{u}_i, \boldsymbol{x}_i, \dot{\boldsymbol{x}}_i) - G(\boldsymbol{u}_i, \boldsymbol{x}_i, \dot{\boldsymbol{x}}_i, \boldsymbol{\theta}) \right\|_2^2$$

LEVERAGING THE DYNAMICS STRUCTURE

$$\begin{cases} \dot{h} = V \sin\gamma \\ \dot{V} = \dfrac{T(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{\theta}_T)\cos\alpha - D(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{\theta}_D) - mg\sin\gamma}{m} \\ \dot{\gamma} = \dfrac{T(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{\theta}_T)\sin\alpha + L(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{\theta}_L) - mg\cos\gamma}{mV} \\ \dot{m} = -\dfrac{T(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{\theta}_T)}{I_{sp}(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{\theta}_{Isp})} \end{cases}$$

Leveraging the dynamics structure

$$
\begin{cases}
\dot{h} = V \sin \gamma \\
\dot{V} = \dfrac{T(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{\theta}_T) \cos \alpha - D(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{\theta}_D) - mg \sin \gamma}{m} \\
\dot{\gamma} = \dfrac{T(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{\theta}_T) \sin \alpha + L(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{\theta}_L) - mg \cos \gamma}{mV} \\
\dot{m} = -\dfrac{T(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{\theta}_T)}{I_{sp}(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{\theta}_{Isp})}
\end{cases}
$$

- Nonlinear in states and controls

Leveraging the dynamics structure

$$
\begin{cases}
\dot{h} = V \sin\gamma \\
\dot{V} = \dfrac{T(\boldsymbol{u},\boldsymbol{x},\boldsymbol{\theta}_T)\cos\alpha - D(\boldsymbol{u},\boldsymbol{x},\boldsymbol{\theta}_D) - mg\sin\gamma}{m} \\
\dot{\gamma} = \dfrac{T(\boldsymbol{u},\boldsymbol{x},\boldsymbol{\theta}_T)\sin\alpha + L(\boldsymbol{u},\boldsymbol{x},\boldsymbol{\theta}_L) - mg\cos\gamma}{mV} \\
\dot{m} = -\dfrac{T(\boldsymbol{u},\boldsymbol{x},\boldsymbol{\theta}_T)}{I_{sp}(\boldsymbol{u},\boldsymbol{x},\boldsymbol{\theta}_{Isp})}
\end{cases}
$$

- Nonlinear in states and controls
- Nonlinear in parameters

Leveraging the dynamics structure

$$\begin{cases} \dot{h} &= V \sin\gamma \\ m\dot{V} &= T(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{\theta}_T)\cos\alpha - D(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{\theta}_D) - mg\sin\gamma \\ mV\dot{\gamma} &= T(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{\theta}_T)\sin\alpha + L(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{\theta}_L) - mg\cos\gamma \\ \dot{m} &= -\dfrac{T(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{\theta}_T)}{I_{sp}(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{\theta}_{Isp})} \end{cases}$$

- Nonlinear in states and controls
- Nonlinear in parameters

Leveraging the dynamics structure

$$\begin{cases} \dot{h} & = V \sin\gamma \\ m\dot{V} & = T(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{\theta}_T)\cos\alpha - D(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{\theta}_D) - mg\sin\gamma \\ mV\dot{\gamma} & = T(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{\theta}_T)\sin\alpha + L(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{\theta}_L) - mg\cos\gamma \\ \dot{m} & = -\dfrac{T(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{\theta}_T)}{I_{sp}(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{\theta}_{Isp})} \end{cases}$$

- Nonlinear in states and controls
- Nonlinear in parameters

Leveraging the dynamics structure

$$\begin{cases} \dot{h} = V \sin\gamma \\ m\dot{V} + mg\sin\gamma = T(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{\theta}_T)\cos\alpha - D(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{\theta}_D) \\ mV\dot{\gamma} + mg\cos\gamma = T(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{\theta}_T)\sin\alpha + L(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{\theta}_L) \\ \dot{m} = -\dfrac{T(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{\theta}_T)}{I_{sp}(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{\theta}_{Isp})} \end{cases}$$

- Nonlinear in states and controls
- Nonlinear in parameters

Leveraging the dynamics structure

$$
\begin{cases}
\dot{h} = V \sin\gamma \\
m\dot{V} + mg\sin\gamma = T(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{\theta}_T)\cos\alpha - D(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{\theta}_D) \\
mV\dot{\gamma} + mg\cos\gamma = T(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{\theta}_T)\sin\alpha + L(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{\theta}_L) \\
\dot{m}I_{sp}(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{\theta}_{Isp}) = -T(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{\theta}_T)
\end{cases}
$$

- Nonlinear in states and controls
- Nonlinear in parameters

Leveraging the dynamics structure

$$
\begin{cases}
\dot{h} = V \sin\gamma \\
m\dot{V} + mg\sin\gamma = T(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{\theta}_T)\cos\alpha - D(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{\theta}_D) \\
mV\dot{\gamma} + mg\cos\gamma = T(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{\theta}_T)\sin\alpha + L(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{\theta}_L) \\
0 = T(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{\theta}_T) + \dot{m} I_{sp}(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{\theta}_{Isp})
\end{cases}
$$

- Nonlinear in states and controls
- Nonlinear in parameters

Leveraging the dynamics structure

$$\begin{cases} \dot{h} = V \sin\gamma \\ m\dot{V} + mg\sin\gamma = (X_T \cdot \boldsymbol{\theta}_T)\cos\alpha - X_D \cdot \boldsymbol{\theta}_D \\ mV\dot{\gamma} + mg\cos\gamma = (X_T \cdot \boldsymbol{\theta}_T)\sin\alpha + X_L \cdot \boldsymbol{\theta}_L \\ 0 = X_T \cdot \boldsymbol{\theta}_T + \dot{m}(X_{Isp} \cdot \boldsymbol{\theta}_{Isp}) \end{cases}$$

- Nonlinear in states and controls
- Nonlinear in parameters

Leveraging the dynamics structure

$$
\begin{cases}
\dot{h} = V \sin\gamma \\
m\dot{V} + mg\sin\gamma = (X_T \cdot \boldsymbol{\theta}_T)\cos\alpha - X_D \cdot \boldsymbol{\theta}_D + \varepsilon_1 \\
mV\dot{\gamma} + mg\cos\gamma = (X_T \cdot \boldsymbol{\theta}_T)\sin\alpha + X_L \cdot \boldsymbol{\theta}_L + \varepsilon_2 \\
0 = X_T \cdot \boldsymbol{\theta}_T + \dot{m}(X_{Isp} \cdot \boldsymbol{\theta}_{Isp}) + \varepsilon_3
\end{cases}
$$

- Nonlinear in states and controls
- Nonlinear in parameters

Leveraging the dynamics structure

$$
\begin{cases}
\dot{h} = V \sin\gamma \\
\underbrace{m\dot{V} + mg\sin\gamma}_{} = \underbrace{(X_T \cdot \boldsymbol{\theta}_T)\cos\alpha - X_D \cdot \boldsymbol{\theta}_D}_{} + \varepsilon_1 \\
mV\dot{\gamma} + mg\cos\gamma = (X_T \cdot \boldsymbol{\theta}_T)\sin\alpha + X_L \cdot \boldsymbol{\theta}_L + \varepsilon_2 \\
0 = X_T \cdot \boldsymbol{\theta}_T + \dot{m}(X_{Isp} \cdot \boldsymbol{\theta}_{Isp}) + \varepsilon_3
\end{cases}
$$

$$\underbrace{\qquad\qquad\qquad}_{Y(\boldsymbol{u},\boldsymbol{x},\dot{\boldsymbol{x}})} \qquad \underbrace{\qquad\qquad\qquad\qquad}_{G(\boldsymbol{u},\boldsymbol{x},\dot{\boldsymbol{x}},\boldsymbol{\theta})}$$

- Nonlinear in states and controls
- Nonlinear in parameters

Leveraging the dynamics structure

$$
\begin{cases}
\dot{h} = V \sin\gamma \\
m\dot{V} + mg\sin\gamma = (X_T \cdot \boldsymbol{\theta}_T)\cos\alpha - X_D \cdot \boldsymbol{\theta}_D + \varepsilon_1 \\
mV\dot{\gamma} + mg\cos\gamma = (X_T \cdot \boldsymbol{\theta}_T)\sin\alpha + X_L \cdot \boldsymbol{\theta}_L + \varepsilon_2 \\
0 = X_T \cdot \boldsymbol{\theta}_T + \dot{m}(X_{Isp} \cdot \boldsymbol{\theta}_{Isp}) + \varepsilon_3
\end{cases}
$$

$$
\underbrace{\qquad\qquad\qquad}_{Y(\boldsymbol{u},\boldsymbol{x},\dot{\boldsymbol{x}})} \qquad \underbrace{\qquad\qquad\qquad\qquad\qquad}_{G(\boldsymbol{u},\boldsymbol{x},\dot{\boldsymbol{x}},\boldsymbol{\theta})}
$$

- Nonlinear in states and controls
- ~~Nonlinear in parameters~~ $\rightarrow$ Linear in parameters

Leveraging the dynamics structure

$$
\begin{cases}
\dot{h} = V \sin\gamma \\
m\dot{V} + mg\sin\gamma = (X_T \cdot \boldsymbol{\theta}_T)\cos\alpha - X_D \cdot \boldsymbol{\theta}_D + \varepsilon_1 \\
mV\dot{\gamma} + mg\cos\gamma = (X_T \cdot \boldsymbol{\theta}_T)\sin\alpha + X_L \cdot \boldsymbol{\theta}_L + \varepsilon_2 \\
0 = X_T \cdot \boldsymbol{\theta}_T + \dot{m}(X_{Isp} \cdot \boldsymbol{\theta}_{Isp}) + \varepsilon_3
\end{cases}
$$

$$
\underbrace{\qquad\qquad\qquad\qquad}_{Y(\boldsymbol{u},\boldsymbol{x},\dot{\boldsymbol{x}})} \qquad \underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{G(\boldsymbol{u},\boldsymbol{x},\dot{\boldsymbol{x}},\boldsymbol{\theta})}
$$

- Nonlinear in states and controls
- ~~Nonlinear in parameters~~ $\rightarrow$ Linear in parameters
- Structured

LEVERAGING THE DYNAMICS STRUCTURE

$$
\begin{cases}
\dot{h} = V \sin\gamma \\
m\dot{V} + mg\sin\gamma = (X_T \cdot \boldsymbol{\theta}_T)\cos\alpha - X_D \cdot \boldsymbol{\theta}_D + \varepsilon_1 \\
mV\dot{\gamma} + mg\cos\gamma = (X_T \cdot \boldsymbol{\theta}_T)\sin\alpha + X_L \cdot \boldsymbol{\theta}_L + \varepsilon_2 \\
0 = X_T \cdot \boldsymbol{\theta}_T + \dot{m}(X_{Isp} \cdot \boldsymbol{\theta}_{Isp}) + \varepsilon_3
\end{cases}
$$

$$
\underbrace{\qquad\qquad\qquad\qquad}_{Y(\boldsymbol{u},\boldsymbol{x},\dot{\boldsymbol{x}})} \qquad \underbrace{\qquad\qquad\qquad\qquad\qquad}_{G(\boldsymbol{u},\boldsymbol{x},\dot{\boldsymbol{x}},\boldsymbol{\theta})}
$$

- Nonlinear in states and controls
- ~~Nonlinear in parameters~~ $\rightarrow$ Linear in parameters
- Structured

Leveraging the dynamics structure

$$\begin{cases} \dot{h} = V \sin\gamma \\ m\dot{V} + mg\sin\gamma = (X_T \cdot \boldsymbol{\theta}_T)\cos\alpha - X_D \cdot \boldsymbol{\theta}_D + \varepsilon_1 \\ mV\dot{\gamma} + mg\cos\gamma = (X_T \cdot \boldsymbol{\theta}_T)\sin\alpha + X_L \cdot \boldsymbol{\theta}_L + \varepsilon_2 \\ 0 = X_T \cdot \boldsymbol{\theta}_T + \dot{m}(X_{Isp} \cdot \boldsymbol{\theta}_{Isp}) + \varepsilon_3 \end{cases}$$

$$\underbrace{\qquad\qquad\qquad}_{Y(\boldsymbol{u},\boldsymbol{x},\dot{\boldsymbol{x}})} \qquad \underbrace{\qquad\qquad\qquad\qquad}_{G(\boldsymbol{u},\boldsymbol{x},\dot{\boldsymbol{x}},\boldsymbol{\theta})}$$

- Nonlinear in states and controls
- ~~Nonlinear in parameters~~ $\rightarrow$ Linear in parameters
- Structured

Leveraging the dynamics structure

$$
\begin{cases}
\dot{h} = V \sin \gamma \\
m\dot{V} + mg \sin \gamma = (X_T \cos \alpha) \cdot \boldsymbol{\theta}_T - X_D \cdot \boldsymbol{\theta}_D + \varepsilon_1 \\
mV\dot{\gamma} + mg \cos \gamma = (X_T \sin \alpha) \cdot \boldsymbol{\theta}_T + X_L \cdot \boldsymbol{\theta}_L + \varepsilon_2 \\
0 = X_T \cdot \boldsymbol{\theta}_T + (\dot{m} X_{Isp}) \cdot \boldsymbol{\theta}_{Isp} + \varepsilon_3
\end{cases}
$$

$$
\underbrace{\qquad\qquad\qquad}_{Y(\boldsymbol{u},\boldsymbol{x},\dot{\boldsymbol{x}})} \qquad \underbrace{\qquad\qquad\qquad\qquad}_{G(\boldsymbol{u},\boldsymbol{x},\dot{\boldsymbol{x}},\boldsymbol{\theta})}
$$

- Nonlinear in states and controls
- ~~Nonlinear in parameters~~ $\rightarrow$ Linear in parameters
- Structured

LEVERAGING THE DYNAMICS STRUCTURE

$$
\begin{cases}
\dot{h} = V \sin\gamma \\
m\dot{V} + mg\sin\gamma = X_{T1} \cdot \boldsymbol{\theta}_T - X_D \cdot \boldsymbol{\theta}_D + \varepsilon_1 \\
mV\dot{\gamma} + mg\cos\gamma = X_{T2} \cdot \boldsymbol{\theta}_T + X_L \cdot \boldsymbol{\theta}_L + \varepsilon_2 \\
0 = X_T \cdot \boldsymbol{\theta}_T + X_{Ispm} \cdot \boldsymbol{\theta}_{Isp} + \varepsilon_3
\end{cases}
$$

$$\underbrace{\qquad\qquad\qquad}_{Y(\boldsymbol{u},\boldsymbol{x},\dot{\boldsymbol{x}})} \qquad \underbrace{\qquad\qquad\qquad\qquad}_{G(\boldsymbol{u},\boldsymbol{x},\dot{\boldsymbol{x}},\boldsymbol{\theta})}$$

- Nonlinear in states and controls
- ~~Nonlinear in parameters~~ → Linear in parameters
- Structured

Leveraging the dynamics structure

$$
\left\{
\begin{aligned}
\dot{h} &= V \sin\gamma \\
\underbrace{m\dot{V} + mg\sin\gamma}_{} &= \underbrace{X_{T1} \cdot \boldsymbol{\theta}_T - X_D \cdot \boldsymbol{\theta}_D}_{} + \varepsilon_1 \\
mV\dot{\gamma} + mg\cos\gamma &= X_{T2} \cdot \boldsymbol{\theta}_T + X_L \cdot \boldsymbol{\theta}_L + \varepsilon_2 \\
0 &= X_T \cdot \boldsymbol{\theta}_T + X_{Ispm} \cdot \boldsymbol{\theta}_{Isp} + \varepsilon_3
\end{aligned}
\right.
$$

$$\underbrace{\qquad\qquad\qquad}_{Y(\boldsymbol{u},\boldsymbol{x},\dot{\boldsymbol{x}})} \qquad \underbrace{\qquad\qquad\qquad\qquad\qquad}_{G(\boldsymbol{u},\boldsymbol{x},\dot{\boldsymbol{x}},\boldsymbol{\theta})}$$

- Nonlinear in states and controls
- ~~Nonlinear in parameters~~ $\rightarrow$ Linear in parameters
- Structured

Leveraging the dynamics structure

$$
\begin{cases}
\dot{h} = V \sin \gamma \\
Y_1 = X_{T1} \cdot \boldsymbol{\theta}_T - X_D \cdot \boldsymbol{\theta}_D + \varepsilon_1 \\
Y_2 = X_{T2} \cdot \boldsymbol{\theta}_T + X_L \cdot \boldsymbol{\theta}_L + \varepsilon_2 \\
Y_3 = X_T \cdot \boldsymbol{\theta}_T + X_{Ispm} \cdot \boldsymbol{\theta}_{Isp} + \varepsilon_3
\end{cases}
$$

- Nonlinear in states and controls
- ~~Nonlinear in parameters~~ $\rightarrow$ Linear in parameters
- Structured

Leveraging the dynamics structure

$$\begin{cases} \dot{h} = V \sin\gamma \\ Y_1 = X_{T1} \cdot \boldsymbol{\theta}_T - X_D \cdot \boldsymbol{\theta}_D + \varepsilon_1 \\ Y_2 = X_{T2} \cdot \boldsymbol{\theta}_T + X_L \cdot \boldsymbol{\theta}_L + \varepsilon_2 \\ Y_3 = X_T \cdot \boldsymbol{\theta}_T + X_{Ispm} \cdot \boldsymbol{\theta}_{Isp} + \varepsilon_3 \end{cases}$$

- Nonlinear in states and controls
- ~~Nonlinear in parameters~~ $\rightarrow$ Linear in parameters
- Structured $\rightsquigarrow$ $\boxed{\textbf{Multi-task Learning}}$

Multi-task Regression

Aircraft:

$$\begin{cases} Y_1 = X_{T1} \cdot \boldsymbol{\theta}_T - X_D \cdot \boldsymbol{\theta}_D + \varepsilon_1 \\ Y_2 = X_{T2} \cdot \boldsymbol{\theta}_T + X_L \cdot \boldsymbol{\theta}_L + \varepsilon_2 \\ Y_3 = X_T \cdot \boldsymbol{\theta}_T + X_{Ispm} \cdot \boldsymbol{\theta}_{Isp} + \varepsilon_3 \end{cases}$$

Multi-task Regression

Aircraft:

$$\begin{cases} Y_1 = X_{T1} \cdot \boldsymbol{\theta}_T - X_D \cdot \boldsymbol{\theta}_D + \varepsilon_1 \\ Y_2 = X_{T2} \cdot \boldsymbol{\theta}_T + X_L \cdot \boldsymbol{\theta}_L + \varepsilon_2 \\ Y_3 = X_T \cdot \boldsymbol{\theta}_T + X_{Ispm} \cdot \boldsymbol{\theta}_{Isp} + \varepsilon_3 \end{cases}$$

General:

$$\begin{cases} Y_1 = X_{c,1} \cdot \boldsymbol{\theta}_c + X_1 \cdot \boldsymbol{\theta}_1 + \varepsilon_1 \\ Y_2 = X_{c,2} \cdot \boldsymbol{\theta}_c + X_2 \cdot \boldsymbol{\theta}_2 + \varepsilon_2 \\ \vdots \qquad \vdots \\ Y_K = X_{c,K} \cdot \boldsymbol{\theta}_c + X_K \cdot \boldsymbol{\theta}_K + \varepsilon_K \end{cases}$$

Multi-task Regression

Aircraft:

$$\begin{cases} Y_1 = X_{T1} \cdot \boldsymbol{\theta}_T - X_D \cdot \boldsymbol{\theta}_D + \varepsilon_1 \\ Y_2 = X_{T2} \cdot \boldsymbol{\theta}_T + X_L \cdot \boldsymbol{\theta}_L + \varepsilon_2 \\ Y_3 = X_T \cdot \boldsymbol{\theta}_T + X_{Ispm} \cdot \boldsymbol{\theta}_{Isp} + \varepsilon_3 \end{cases}$$

General:

$$\begin{cases} Y_1 = X_{c,1} \cdot \boldsymbol{\theta}_c + X_1 \cdot \boldsymbol{\theta}_1 + \varepsilon_1 \\ Y_2 = X_{c,2} \cdot \boldsymbol{\theta}_c + X_2 \cdot \boldsymbol{\theta}_2 + \varepsilon_2 \\ \vdots \qquad\qquad \vdots \\ Y_K = X_{c,K} \cdot \boldsymbol{\theta}_c + X_K \cdot \boldsymbol{\theta}_K + \varepsilon_K \end{cases}$$

Coupling parameters , **Task specific parameters**

Multi-task Regression

Aircraft:

$$\begin{cases} Y_1 = X_{T1} \cdot \boldsymbol{\theta}_T - X_D \cdot \boldsymbol{\theta}_D + \varepsilon_1 \\ Y_2 = X_{T2} \cdot \boldsymbol{\theta}_T + X_L \cdot \boldsymbol{\theta}_L + \varepsilon_2 \\ Y_3 = X_T \cdot \boldsymbol{\theta}_T + X_{Ispm} \cdot \boldsymbol{\theta}_{Isp} + \varepsilon_3 \end{cases}$$

General:

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Coupling parameters , **Task specific parameters**

Many other examples:

- *Giant squid neurons* [FitzHugh, 1961, Nagumo et al., 1962],

$$\begin{cases} \dot{V} = \boldsymbol{\theta}_1 \left(V - \dfrac{V^3}{3} + R \right), \\ \dot{R} = -\dfrac{1}{\boldsymbol{\theta}_1} (V - \boldsymbol{\theta}_2 + \boldsymbol{\theta}_3 R), \end{cases}$$

Multi-task Regression

Aircraft:

$$\begin{cases} Y_1 = X_{T1} \cdot \boldsymbol{\theta}_T - X_D \cdot \boldsymbol{\theta}_D + \varepsilon_1 \\ Y_2 = X_{T2} \cdot \boldsymbol{\theta}_T + X_L \cdot \boldsymbol{\theta}_L + \varepsilon_2 \\ Y_3 = X_T \cdot \boldsymbol{\theta}_T + X_{Ispm} \cdot \boldsymbol{\theta}_{Isp} + \varepsilon_3 \end{cases}$$

General:

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Coupling parameters , **Task specific parameters**

Many other examples:

- *Giant squid neurons* [FitzHugh, 1961, Nagumo et al., 1962],
- *Susceptible-infectious-recovered models* [Anderson and May, 1992],

$$\begin{cases} \dot{S} = \boldsymbol{\theta}_1 - (\boldsymbol{\theta}_2 I + \boldsymbol{\theta}_3) S, \\ \dot{I} = \boldsymbol{\theta}_2 IS - (\boldsymbol{\theta}_4 + \boldsymbol{\theta}_3) I, \\ \dot{R} = \boldsymbol{\theta}_4 I - \boldsymbol{\theta}_3 R. \end{cases}$$

Multi-task Regression

Aircraft:

$$\begin{cases} Y_1 = X_{T1} \cdot \boldsymbol{\theta}_T - X_D \cdot \boldsymbol{\theta}_D + \varepsilon_1 \\ Y_2 = X_{T2} \cdot \boldsymbol{\theta}_T + X_L \cdot \boldsymbol{\theta}_L + \varepsilon_2 \\ Y_3 = X_T \cdot \boldsymbol{\theta}_T + X_{lspm} \cdot \boldsymbol{\theta}_{lsp} + \varepsilon_3 \end{cases}$$

General:

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Coupling parameters , Task specific parameters

Many other examples:

- *Giant squid neurons* [FitzHugh, 1961, Nagumo et al., 1962],
- *Susceptible-infectious-recovered models* [Anderson and May, 1992],
- *Mechanical systems,...*

$$m\left(\frac{d\vec{V}}{dt}\right)_x = (\vec{F}(\boldsymbol{\theta}_F))_x, \quad m\left(\frac{d\vec{V}}{dt}\right)_y = (\vec{F}(\boldsymbol{\theta}_F))_y.$$

Multi-task Regression

Aircraft:

$$\begin{cases} Y_1 = X_{T1} \cdot \theta_T - X_D \cdot \theta_D + \varepsilon_1 \\ Y_2 = X_{T2} \cdot \theta_T + X_L \cdot \theta_L + \varepsilon_2 \\ Y_3 = X_T \cdot \theta_T + X_{lspm} \cdot \theta_{lsp} + \varepsilon_3 \end{cases}$$

General:

$$\begin{cases} Y_1 = X_{c,1} \cdot \theta_c + X_1 \cdot \theta_1 + \varepsilon_1 \\ Y_2 = X_{c,2} \cdot \theta_c + X_2 \cdot \theta_2 + \varepsilon_2 \\ \vdots \qquad \vdots \\ Y_K = X_{c,K} \cdot \theta_c + X_K \cdot \theta_K + \varepsilon_K \end{cases}$$

Coupling parameters , Task specific parameters

Multi-task Linear Least-Squares:

$$\min_{\theta} \sum_{k=1}^{K} \sum_{i=1}^{N} \left(Y_{k,i} - X_{c,k,i} \cdot \theta_c - X_{k,i} \cdot \theta_k \right)^2$$

Multi-task Regression

Aircraft:

$$\begin{cases} Y_1 = X_{T1} \cdot \theta_T - X_D \cdot \theta_D + \varepsilon_1 \\ Y_2 = X_{T2} \cdot \theta_T + X_L \cdot \theta_L + \varepsilon_2 \\ Y_3 = X_T \cdot \theta_T + X_{lspm} \cdot \theta_{lsp} + \varepsilon_3 \end{cases}$$

General:

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Coupling parameters , Task specific parameters

Multi-task Linear Least-Squares: Block-sparse Coupling Structure

$$\min_{\theta} \sum_{i=1}^{N} \left\| \begin{pmatrix} Y_{1,i} \\ \vdots \\ Y_{K,i} \end{pmatrix} - \begin{pmatrix} X_{c,1,i} & X_{1,i} & 0 & 0 & \dots & 0 \\ X_{c,2,i} & 0 & X_{2,i} & 0 & \dots & 0 \\ \vdots & 0 & 0 & \ddots & 0 & 0 \\ X_{c,K,i} & 0 & 0 & \dots & 0 & X_{K,i} \end{pmatrix} \begin{pmatrix} \theta_c \\ \theta_1 \\ \vdots \\ \theta_K \end{pmatrix} \right\|_2^2$$

Multi-task Regression

Aircraft:

$$\begin{cases} Y_1 = X_{T1} \cdot \theta_T - X_D \cdot \theta_D + \varepsilon_1 \\ Y_2 = X_{T2} \cdot \theta_T + X_L \cdot \theta_L + \varepsilon_2 \\ Y_3 = X_T \cdot \theta_T + X_{Ispm} \cdot \theta_{Isp} + \varepsilon_3 \end{cases}$$

General:

$$\begin{cases} Y_1 = X_{c,1} \cdot \theta_c + X_1 \cdot \theta_1 + \varepsilon_1 \\ Y_2 = X_{c,2} \cdot \theta_c + X_2 \cdot \theta_2 + \varepsilon_2 \\ \vdots \qquad\qquad \vdots \\ Y_K = X_{c,K} \cdot \theta_c + X_K \cdot \theta_K + \varepsilon_K \end{cases}$$

Coupling parameters , **Task specific parameters**

Multi-task Linear Least-Squares: Block-sparse Coupling Structure

$$\min_{\theta} \sum_{i=1}^{N} \left\| \begin{pmatrix} Y_{1,i} \\ Y_{2,i} \\ Y_{3,i} \end{pmatrix} - \begin{pmatrix} X_{T1,i} & -X_{D,i} & 0 & 0 \\ X_{T2,i} & 0 & X_{L,i} & 0 \\ X_{T,i} & 0 & 0 & X_{Ispm,i} \end{pmatrix} \begin{pmatrix} \theta_T \\ \theta_D \\ \theta_L \\ \theta_{Isp} \end{pmatrix} \right\|_2^2$$

Multi-task Regression

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$$\begin{cases} Y_1 = X_{T1} \cdot \boldsymbol{\theta}_T - X_D \cdot \boldsymbol{\theta}_D + \varepsilon_1 \\ Y_2 = X_{T2} \cdot \boldsymbol{\theta}_T + X_L \cdot \boldsymbol{\theta}_L + \varepsilon_2 \\ Y_3 = X_T \cdot \boldsymbol{\theta}_T + X_{lspm} \cdot \boldsymbol{\theta}_{lsp} + \varepsilon_3 \end{cases}$$

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Coupling parameters , **Task specific parameters**

Multi-task Linear Least-Squares: Block-sparse Coupling Structure

$$\min_{\boldsymbol{\theta}} \sum_{i=1}^{N} \| Y_i - X_i \boldsymbol{\theta} \|_2^2$$

with $\boldsymbol{\theta} = (\boldsymbol{\theta}_c, \boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_K) \in \mathbb{R}^p$, $p = d_c + \sum_{k=1}^{K} d_k$,
$Y_i \in \mathbb{R}^K$ and $X_i \in \mathbb{R}^{K \times p}$.

Feature Selection

Our model:

$$
\begin{aligned}
T =& N_1(\boldsymbol{\theta}_{T,1} + \boldsymbol{\theta}_{T,2}\rho + \boldsymbol{\theta}_{T,3}M + \boldsymbol{\theta}_{T,4}\rho^2 + \boldsymbol{\theta}_{T,5}\rho M + \boldsymbol{\theta}_{T,6}M^2 + \\
& \boldsymbol{\theta}_{T,7}\rho^3 + \boldsymbol{\theta}_{T,8}\rho^2 M + \boldsymbol{\theta}_{T,9}\rho M^2 + \boldsymbol{\theta}_{T,10}M^3 + \boldsymbol{\theta}_{T,11}\rho^4 + \\
& \boldsymbol{\theta}_{T,12}\rho^3 M + \boldsymbol{\theta}_{T,13}\rho^2 M^2 + \boldsymbol{\theta}_{T,14}\rho M^3 + \boldsymbol{\theta}_{T,15}M^4).
\end{aligned}
$$

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Mattingly's model [Mattingly et al., 1992]:

$$T = N_1(\boldsymbol{\theta}_{T,1}\rho + \boldsymbol{\theta}_{T,2}\rho M^3).$$

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$\Rightarrow$ High risk of overfitting

Feature Selection

Our (sparse) model:

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$$T = N_1(\boldsymbol{\theta}_{T,1}\rho + \boldsymbol{\theta}_{T,2}\rho M^3).$$

Sparse models are:

- Less susceptible to overfitting,

Feature Selection

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Sparse models are:

- Less susceptible to overfitting,
- More compliant with physical models,

FEATURE SELECTION

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Mattingly's model [Mattingly et al., 1992]:

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Sparse models are:

- Less susceptible to overfitting,
- More compliant with physical models,
- More interpretable,

Feature Selection

Our (sparse) model:

$$T = N_1(\boldsymbol{\theta}_{T,1} + \theta_{T,2}\rho + \boldsymbol{\theta}_{T,3}M + \theta_{T,4}\rho^2 + \theta_{T,5}\rho M + \boldsymbol{\theta}_{T,6}M^2 + \theta_{T,7}\rho^3 + \theta_{T,8}\rho^2 M + \theta_{T,9}\rho M^2 + \theta_{T,10}M^3 + \boldsymbol{\theta}_{T,11}\rho^4 + \boldsymbol{\theta}_{T,12}\rho^3 M + \theta_{T,13}\rho^2 M^2 + \theta_{T,14}\rho M^3 + \theta_{T,15}M^4).$$

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Sparse models are:

- Less susceptible to overfitting,
- More compliant with physical models,
- More interpretable,
- Lighter/Faster.

Block-sparse Lasso

Lasso [Tibshirani, 1994]: $\{(X_i, Y_i)\}_{i=1}^{N} \subset \mathbb{R}^{d+1}$ i.i.d sample,

$$\min_{\boldsymbol{\theta}} \sum_{i=1}^{N} (Y_i - X_i \cdot \boldsymbol{\theta})^2 + \lambda \|\boldsymbol{\theta}\|_1.$$

Block-sparse Lasso

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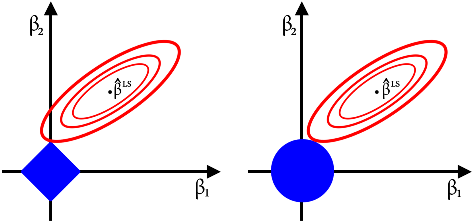

FIGURE: [1]Sparsity induced by L^1 norm in Lasso.

[1] Source : Wikipedia, Lasso(statistics)

Block-sparse Lasso

$$\min_{\boldsymbol{\theta}} \sum_{k=1}^{K} \sum_{i=1}^{N} \left(Y_{k,i} - X_{c,k,i} \cdot \boldsymbol{\theta}_c - X_{k,i} \cdot \boldsymbol{\theta}_k \right)^2$$

Block-sparse Lasso

$$\min_{\boldsymbol{\theta}} \sum_{k=1}^{K} \sum_{i=1}^{N} \left(Y_{k,i} - X_{c,k,i} \cdot \boldsymbol{\theta}_c - X_{k,i} \cdot \boldsymbol{\theta}_k \right)^2 + \lambda_c \|\boldsymbol{\theta}_c\|_1 + \sum_{k=1}^{K} \lambda_k \|\boldsymbol{\theta}_k\|_1$$

Block-sparse Lasso

Block-sparse structure preserved

$$\min_{\boldsymbol{\theta}} \sum_{k=1}^{K} \sum_{i=1}^{N} \left(Y_{k,i} - X_{c,k,i} \cdot \boldsymbol{\theta}_c - X_{k,i} \cdot \boldsymbol{\theta}_k \right)^2 + \lambda_c \|\boldsymbol{\theta}_c\|_1 + \sum_{k=1}^{K} \lambda_k \|\boldsymbol{\theta}_k\|_1$$

Block-sparse Lasso

Block-sparse structure preserved $\rightsquigarrow$ **Equivalent to Lasso problem**

$$\min_{\boldsymbol{\theta}} \sum_{k=1}^{K} \sum_{i=1}^{N} \left(Y_{k,i} - X_{c,k,i} \cdot \boldsymbol{\theta}_c - X_{k,i} \cdot \boldsymbol{\theta}_k \right)^2 + \lambda_c \|\boldsymbol{\theta}_c\|_1 + \sum_{k=1}^{K} \lambda_k \|\boldsymbol{\theta}_k\|_1$$

Block-sparse Lasso

Block-sparse structure preserved $\rightsquigarrow$ **Equivalent to Lasso problem**

$$\min_{\boldsymbol{\theta}} \sum_{i=1}^{N} \left\| \begin{pmatrix} Y_{1,i} \\ \vdots \\ Y_{K,i} \end{pmatrix} - \begin{pmatrix} B_{c,1,i} & B_{1,i} & 0 & 0 & \dots & 0 \\ B_{c,2,i} & 0 & B_{2,i} & 0 & \dots & 0 \\ \vdots & 0 & 0 & \ddots & 0 & 0 \\ B_{c,K,i} & 0 & 0 & \dots & 0 & B_{K,i} \end{pmatrix} \begin{pmatrix} \beta_c \\ \beta_1 \\ \vdots \\ \beta_K \end{pmatrix} \right\|_2^2$$

$$+ \lambda_c \left\| \begin{pmatrix} \beta_c \\ \beta_1 \\ \vdots \\ \beta_K \end{pmatrix} \right\|_1 ,$$

$$B_{c,k,i} := X_{c,k,i}, \qquad \beta_c := \boldsymbol{\theta}_c,$$

$$B_{k,i} := \frac{\lambda_c}{\lambda_k} X_{k,i}, \qquad \beta_k := \frac{\lambda_k}{\lambda_c} \boldsymbol{\theta}_k .$$

Block-sparse Lasso

Block-sparse structure preserved $\rightsquigarrow$ **Equivalent to Lasso problem**

$$\min_{\boldsymbol{\theta}} \sum_{i=1}^{N} \left\| \begin{pmatrix} Y_{1,i} \\ Y_{2,i} \\ Y_{3,i} \end{pmatrix} - \begin{pmatrix} X_{T1,i} & -\frac{\lambda_c}{\lambda_1} X_{D,i} & 0 & 0 \\ X_{T2,i} & 0 & \frac{\lambda_c}{\lambda_2} X_{L,i} & 0 \\ X_{T,i} & 0 & 0 & \frac{\lambda_c}{\lambda_3} X_{Ispm,i} \end{pmatrix} \begin{pmatrix} \boldsymbol{\theta}_T \\ \frac{\lambda_1}{\lambda_c} \boldsymbol{\theta}_D \\ \frac{\lambda_2}{\lambda_c} \boldsymbol{\theta}_L \\ \frac{\lambda_3}{\lambda_c} \boldsymbol{\theta}_{Isp} \end{pmatrix} \right\|_2^2 + \left\| \begin{pmatrix} \boldsymbol{\theta}_T \\ \frac{\lambda_1}{\lambda_c} \boldsymbol{\theta}_D \\ \frac{\lambda_2}{\lambda_c} \boldsymbol{\theta}_L \\ \frac{\lambda_3}{\lambda_c} \boldsymbol{\theta}_{Isp} \end{pmatrix} \right\|_1 .$$

Block-sparse Lasso

Block-sparse structure preserved $\rightsquigarrow$ **Equivalent to Lasso problem**

$$\min_{\boldsymbol{\theta}} \sum_{i=1}^{N} \left\| \begin{pmatrix} Y_{1,i} \\ Y_{2,i} \\ Y_{3,i} \end{pmatrix} - \begin{pmatrix} X_{T1,i} & -\frac{\lambda_c}{\lambda_1} X_{D,i} & 0 & 0 \\ X_{T2,i} & 0 & \frac{\lambda_c}{\lambda_2} X_{L,i} & 0 \\ X_{T,i} & 0 & 0 & \frac{\lambda_c}{\lambda_3} X_{Ispm,i} \end{pmatrix} \begin{pmatrix} \boldsymbol{\theta}_T \\ \frac{\lambda_1}{\lambda_c} \boldsymbol{\theta}_D \\ \frac{\lambda_2}{\lambda_c} \boldsymbol{\theta}_L \\ \frac{\lambda_3}{\lambda_c} \boldsymbol{\theta}_{Isp} \end{pmatrix} \right\|_2^2 + \left\| \begin{pmatrix} \boldsymbol{\theta}_T \\ \frac{\lambda_1}{\lambda_c} \boldsymbol{\theta}_D \\ \frac{\lambda_2}{\lambda_c} \boldsymbol{\theta}_L \\ \frac{\lambda_3}{\lambda_c} \boldsymbol{\theta}_{Isp} \end{pmatrix} \right\|_1 .$$

In practice: $\lambda_k = \lambda_c$, for all $k = 1, \ldots, K$.

Block-sparse Lasso

Block-sparse structure preserved $\rightsquigarrow$ **Equivalent to Lasso problem**

$$\min_{\boldsymbol{\theta}} \sum_{i=1}^{N} \left\| \begin{pmatrix} Y_{1,i} \\ Y_{2,i} \\ Y_{3,i} \end{pmatrix} - \begin{pmatrix} X_{T1,i} & - & X_{D,i} & 0 & 0 \\ X_{T2,i} & & 0 & X_{L,i} & 0 \\ X_{T,i} & & 0 & 0 & X_{lspm,i} \end{pmatrix} \begin{pmatrix} \boldsymbol{\theta}_T \\ \boldsymbol{\theta}_D \\ \boldsymbol{\theta}_L \\ \boldsymbol{\theta}_{lsp} \end{pmatrix} \right\|_2^2 + \left\| \begin{pmatrix} \boldsymbol{\theta}_T \\ \boldsymbol{\theta}_D \\ \boldsymbol{\theta}_L \\ \boldsymbol{\theta}_{lsp} \end{pmatrix} \right\|_1 .$$

In practice: $\lambda_k = \lambda_c$, for all $k = 1, \ldots, K$.

Block-sparse Lasso

Block-sparse structure preserved $\rightsquigarrow$ **Equivalent to Lasso problem**

$$\min_{\boldsymbol{\theta}} \sum_{i=1}^{N} \| Y_i - X_i \boldsymbol{\theta} \|_2^2 + \lambda_c \|\boldsymbol{\theta}\|_1$$

with $\boldsymbol{\theta} = (\boldsymbol{\theta}_c, \boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_K) \in \mathbb{R}^p$, $p = d_c + \sum_{k=1}^{K} d_k$,
$Y_i \in \mathbb{R}^K$ and $X_i \in \mathbb{R}^{K \times p}$.

Structured feature selection

state-of-the-art

- Group Lasso [Yuan and Lin, 2005] — $\{(X_i, Y_i)\}_{i=1}^{N} \subset \mathbb{R}^{d+1}$ i.i.d sample, $\boldsymbol{\theta} = (\boldsymbol{\theta}_g)_{g \in \mathcal{G}}$ partition:

$$\min_{\boldsymbol{\theta}} \sum_{i=1}^{N} (Y_i - X_i \cdot \boldsymbol{\theta})^2 + \sum_{g \in \mathcal{G}} \lambda_g \|\boldsymbol{\theta}_g\|_2,$$

STRUCTURED FEATURE SELECTION

STATE-OF-THE-ART

- Group Lasso [Yuan and Lin, 2005] — $\{(X_i, Y_i)\}_{i=1}^N \subset \mathbb{R}^{d+1}$ i.i.d sample, $\boldsymbol{\theta} = (\boldsymbol{\theta}_g)_{g \in \mathcal{G}}$ partition:

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$\Rightarrow$ Sparsity among groups

Structured feature selection

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  $\Rightarrow$ Sparsity among groups

- Sparse group Lasso [Friedman et al., 2010]:

$$\min_{\boldsymbol{\theta}} \sum_{i=1}^{N} (Y_i - X_i \cdot \boldsymbol{\theta})^2 + \sum_{g \in \mathcal{G}} \lambda_g \|\boldsymbol{\theta}_g\|_2 + \lambda \|\boldsymbol{\theta}\|_1,$$

Structured feature selection

State-of-the-art

- Group Lasso [Yuan and Lin, 2005] — $\{(X_i, Y_i)\}_{i=1}^{N} \subset \mathbb{R}^{d+1}$ i.i.d sample, $\boldsymbol{\theta} = (\boldsymbol{\theta}_g)_{g\in\mathcal{G}}$ partition:

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Structured feature selection
state-of-the-art

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Multi-task Lasso	Multi-task setting, but not same pattern for every task.

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Bolasso - Bach [2008]

Require: training data $\mathcal{T} = \{(X_i, Y_i)\}_{i=1}^N \subset \mathbb{R}^{K\times(K+1)} \times \mathbb{R}^K$,
number of bootstrap replicates b,
L^1 penalty parameter λ_c,

1: **for** $k = 1$ **to** b **do**
2: Generate bootstrap sample $\mathcal{T}_k$,
3: Compute Block sparse Lasso estimate $\hat{\boldsymbol{\theta}}^k$ from $\mathcal{T}_k$,
4: Compute support $J_k = \{j, \hat{\boldsymbol{\theta}}_j^k \neq 0\}$,
5: **end for**
6: Compute intersection $J = \bigcap_{k=1}^b J_k$,
7: Compute $\hat{\boldsymbol{\theta}}_J$ from selected features using Least-Squares.

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- Consistency under high correlations proved in Bach [2008],
- Efficient implementations exists: LARS [Efron et al., 2004].

PROBLEM WITH INTRA-GROUP CORRELATIONS

Problem with intra-group correlations

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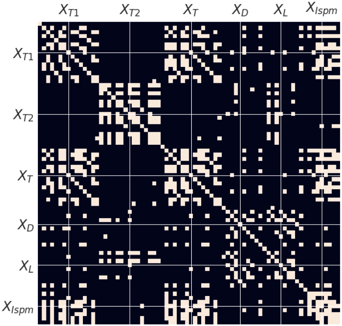

Figure: Features correlations higher than 0.9 in absolute value.

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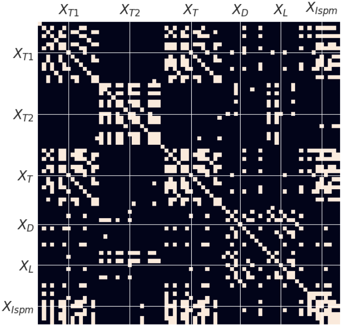

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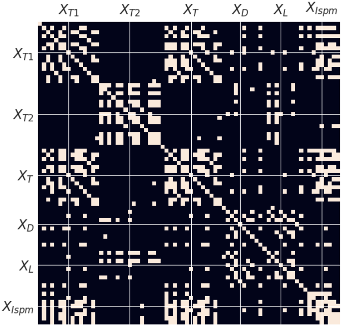

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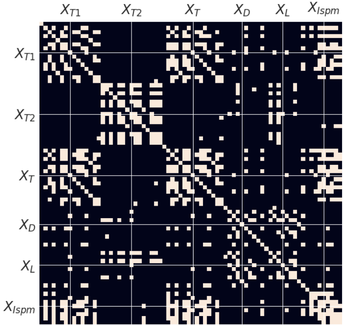

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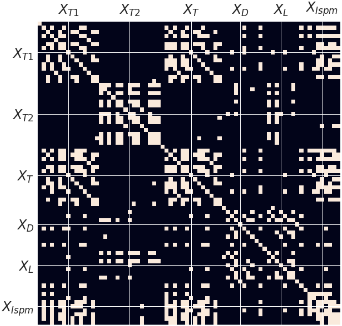

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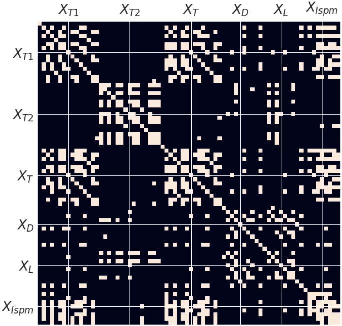

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$\Rightarrow$ Generalized Tikhonov

$$\boxed{(\boldsymbol{\theta} - \tilde{\boldsymbol{\theta}})^\top Q (\boldsymbol{\theta} - \tilde{\boldsymbol{\theta}}) = \|\Gamma(\boldsymbol{\theta} - \tilde{\boldsymbol{\theta}})\|_2^2,}$$

where $\tilde{\boldsymbol{\theta}}$ is a prior and $Q \in \mathbb{R}^{p \times p}$.

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Equivalent to $\|\Gamma(\boldsymbol{\theta} - \tilde{\boldsymbol{\theta}})\|_2^2$ with $\Gamma_i = (\underbrace{0, \ldots, 0,}_{d_T + d_D + d_L} X_{Isp}^\top)$ and $\Gamma_i \tilde{\boldsymbol{\theta}} = \tilde{I}_{sp,i}$.

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$$\tilde{Y}_i = \begin{pmatrix} Y_{1,i} \\ Y_{2,i} \\ 0 \\ \lambda_t \tilde{I}_{sp,i} \end{pmatrix}, \quad \tilde{X}_i = \begin{pmatrix} (X_{T1,i})^\top & -(X_{D,i})^\top & 0 & 0 \\ (X_{T2,i})^\top & 0 & (X_{L,i})^\top & 0 \\ (X_{T,i})^\top & 0 & 0 & (X_{Ispm,i})^\top \\ 0 & 0 & 0 & \lambda_t (X_{Isp,i})^\top \end{pmatrix},$$

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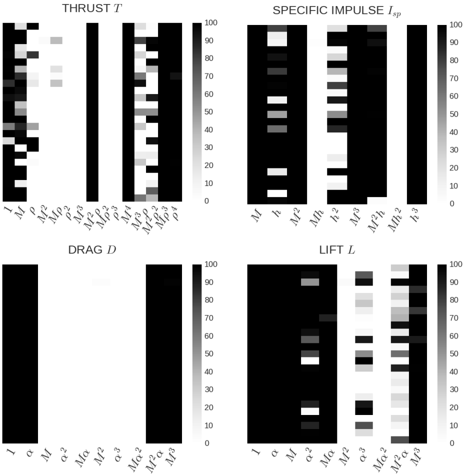
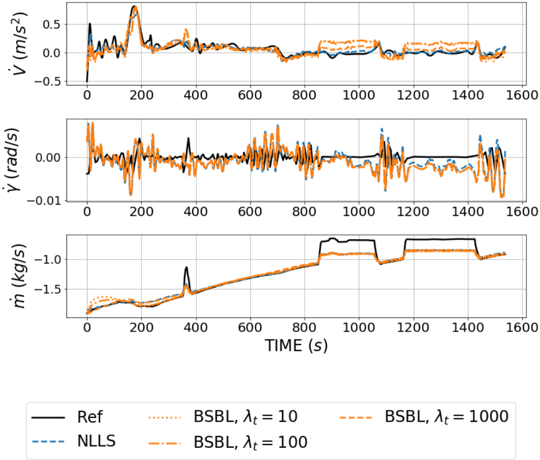


Feature selection results for the thrust, drag, lift and specific impulse models.

ACCURACY OF DYNAMICS PREDICTIONS



Accuracy of dynamics predictions

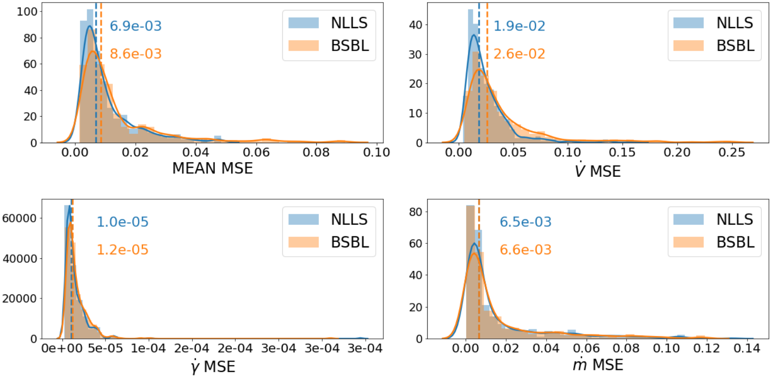
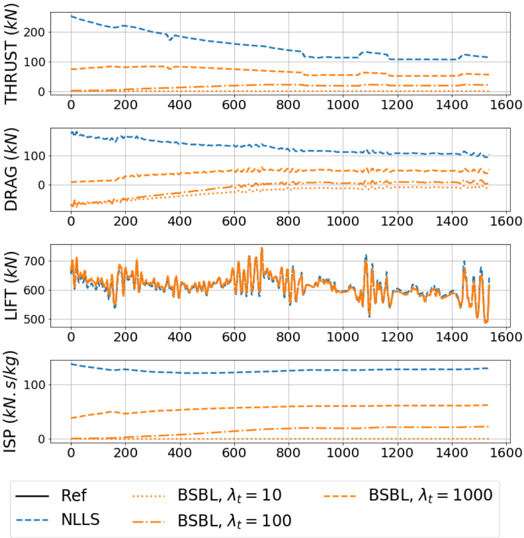


FIGURE: Leave-one-out off-sample errors distributions for nonlinear least-squares NLLS and block-sparse bolasso BSBL. Median errors are annotated and marked by dashed vertical lines.

Realism of hidden elements



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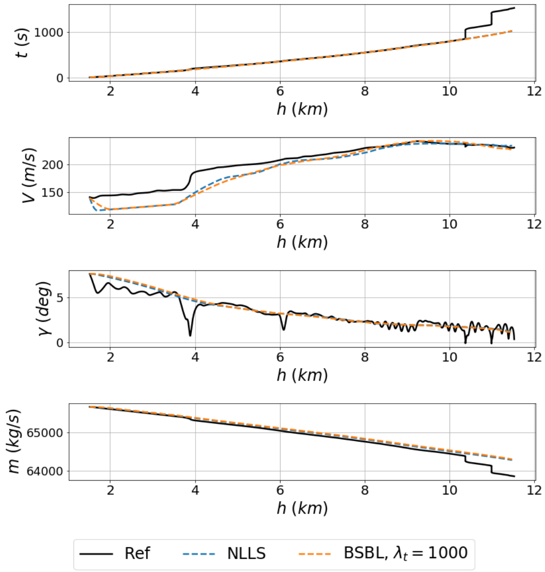
$$\min_{(\boldsymbol{x},\boldsymbol{u})} \int_{t_0}^{t_n} \left(\|\boldsymbol{u}(t) - \boldsymbol{u}_{test}(t)\|_{\boldsymbol{u}}^2 + \|\boldsymbol{x}(t) - \boldsymbol{x}_{test}(t)\|_{\boldsymbol{x}}^2 \right) dt$$

$$\text{s.t.} \quad \dot{\boldsymbol{x}}(t) = g(\boldsymbol{x}(t), \boldsymbol{u}(t), \hat{\boldsymbol{\theta}}),$$

where $\|\cdot\|_{\boldsymbol{u}}$, $\|\cdot\|_{\boldsymbol{x}}$ denote scaling norms.

Flight resimulation



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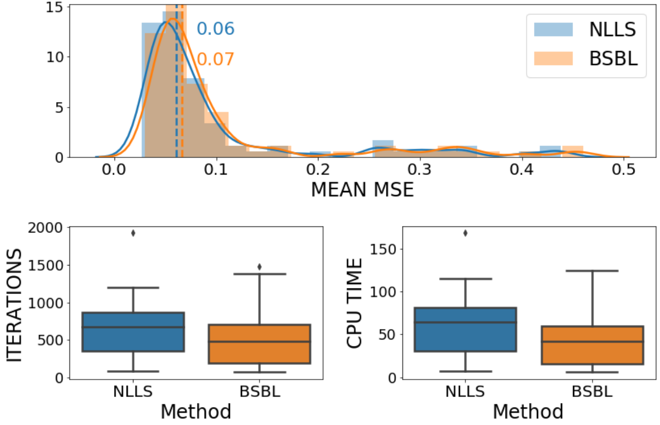

Figure: Distribution of the off-sample simulation error and boxplot of the optimization number of iterations and CPU time.

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Trajectory acceptability

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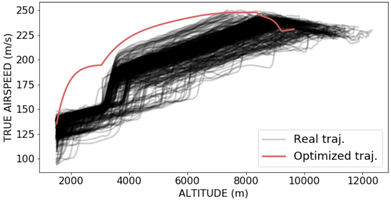

[2]NATS UK air traffic control

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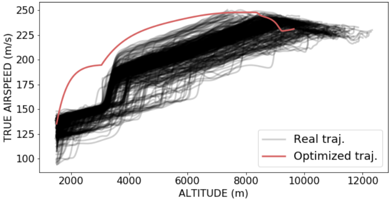

Pilots acceptance

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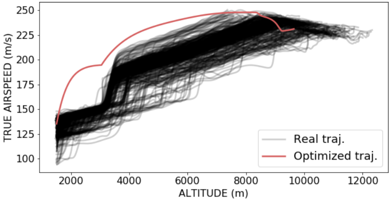

Pilots acceptance

Air Traffic Control[2]

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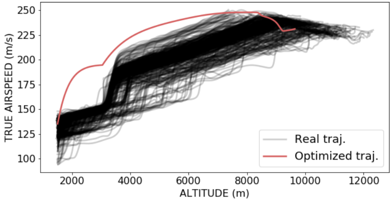

Pilots acceptance Air Traffic Control[2]

How can we quantify the closeness from the optimized trajectory to the set of real flights?

[2]NATS UK air traffic control

Likelihood

- $X \sim f_{\theta^*}$,
- x: observation of X,
- Likelihood function of θ, given x:

$$\mathcal{L}(\theta|x) = f_\theta(x).$$

[0]Picture source: wikipedia, P-Value, author: Repapetilto CC.

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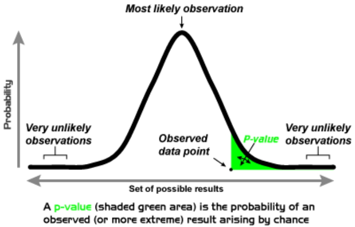

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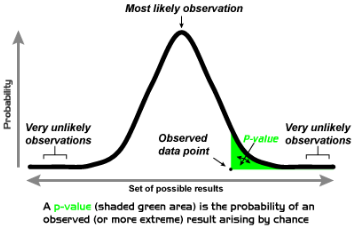

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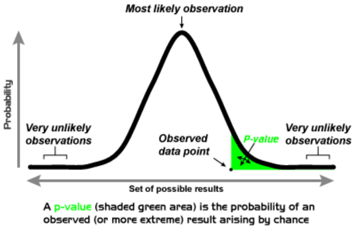

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How to apply this to functional data?

Assumption: We suppose that the real flights are observations of the same functional random variable $Z = (Z_t)$ valued in $\mathcal{C}(\mathbb{T}, E)$, with E compact subset of $\mathbb{R}^d$ and $\mathbb{T} = [0, t_f]$.

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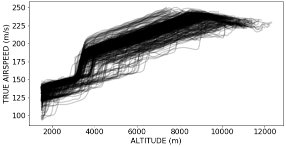

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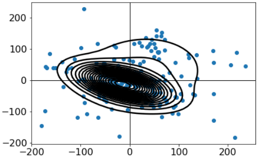

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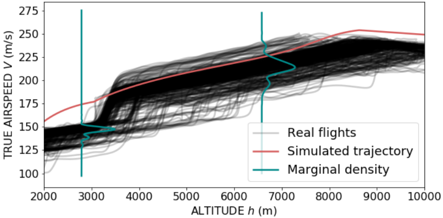

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Marginal densities may have really different shapes

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Mean marginal likelihood

$$\mathrm{MML}(Z, \boldsymbol{y}) = \frac{1}{t_f} \int_0^{t_f} \psi[f_t, \boldsymbol{y}(t)] dt,$$

where $\psi : L^1(E, \mathbb{R}_+) \times \mathbb{R} \to [0; 1]$ is a continuous scaling map.

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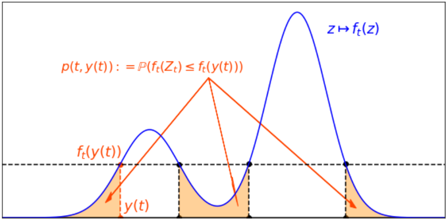
$$\psi[f_t, \boldsymbol{y}(t)] := \frac{\boldsymbol{y}(t)}{\max\limits_{z \in E} f_t(z)},$$

or the confidence level

$$\psi[f_t, \boldsymbol{y}(t)] := \mathbb{P}\left(f_t(Z_t) \leq f_t(\boldsymbol{y}(t))\right).$$



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In practice, the m trajectories are sampled at variable discrete times:

$$\mathcal{T}^D := \{(t_j^r, z_j^r)\}_{\substack{1\leq j\leq n\\ 1\leq r\leq m}} \subset \mathbb{T} \times E, \qquad z_j^r := \boldsymbol{z}^r(t_j^r),$$

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Hence, we approximate the MML using a Riemann sum which aggregates consistent estimators $\hat{f}^m_{\tilde{t}_j}$ of the marginal densities $f_{\tilde{t}_j}$:

$$\boxed{\mathrm{EMML}_m(\mathcal{T}^D, \mathcal{Y}) := \frac{1}{t_f} \sum_{j=1}^{\tilde{n}} \psi[\hat{f}^m_{\tilde{t}_j}, y_j] \Delta \tilde{t}_j.}$$

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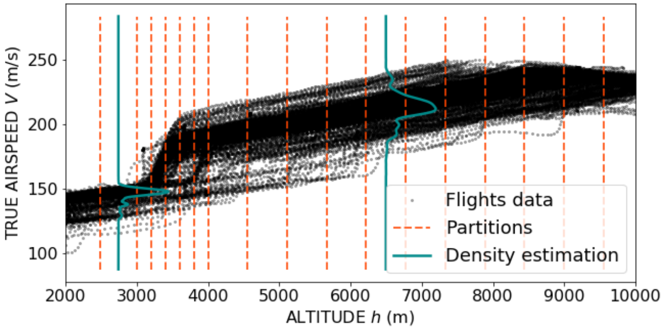
$\Rightarrow$ **Instead, we choose to use a fine partitioning of the time domain.**

Partition based marginal density estimation



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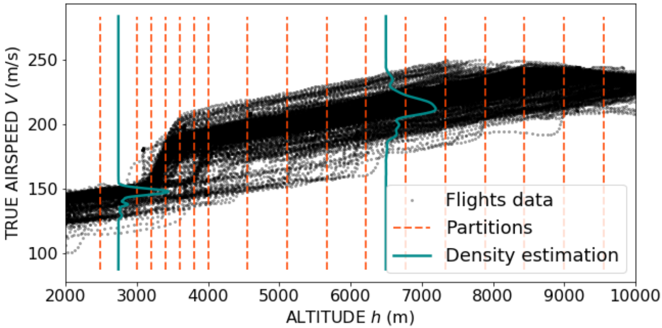

Idea: to average in time the marginal densities over small bins by applying classical multivariate density estimation techniques to each subset.

Consistency

We denote by:

- $\Theta : \mathcal{S} \to L^1(E, \mathbb{R}_+)$ multivariate density estimation statistic,
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- $\hat{f}_t^m := \Theta[\mathcal{T}_t^m]$ estimator trained using $\mathcal{T}_t^m$.

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Assumtion 1 - Positive time density
$\nu \in L^\infty(E, \mathbb{R}_+)$ density function of T, s.t.

$$\nu_+ := \operatorname*{ess\,sup}_{t \in \mathbb{T}} \nu(t) < \infty, \qquad \nu_- := \operatorname*{ess\,inf}_{t \in \mathbb{T}} \nu(t) > 0.$$

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Function $(t, z) \in \mathbb{T} \times E \mapsto f_t(z)$ is continuous and

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Assumption 3 - Shrinking bins
The homogeneous partition $\{B_\ell^m\}_{\ell=1}^{q_m}$ of $[0; t_f]$, with binsize b_m, is s.t.

$$\lim_{m\to\infty} b_m = 0, \qquad \lim_{m\to\infty} mb_m = \infty.$$

Consistency

Assumption 4 - i.i.d. consistency

- $\mathcal{G}$ arbitrary family of probability density functions on E, $\rho \in \mathcal{G}$,
- S_ρ^N i.i.d sample of size N drawn from ρ valued in $\mathcal{S}$.

The estimator obtained by applying Θ to S_ρ^N, denoted by

$$\hat{\rho}^N := \Theta[S_\rho^N] \in L^1(E, \mathbb{R}_+),$$

is a (pointwise) consistent density estimator, uniformly in ρ:

For all $z \in E, \varepsilon > 0, \alpha_1 > 0$, there is $N_{\varepsilon,\alpha_1} > 0$ such that, for any $\rho \in \mathcal{G}$,

$$N \geq N_{\varepsilon,\alpha_1} \Rightarrow \mathbb{P}\left(\left|\hat{\rho}^N(z) - \rho(z)\right| < \varepsilon\right) > 1 - \alpha_1.$$

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Theorem 1
Under assumptions 1 to 4, for any $z \in E$ and $t \in \mathbb{T}$, $\hat{f}^m_{\ell^m(t)}(z)$ consistently approximates the marginal density $f_t(z)$ as the number of curves m grows:

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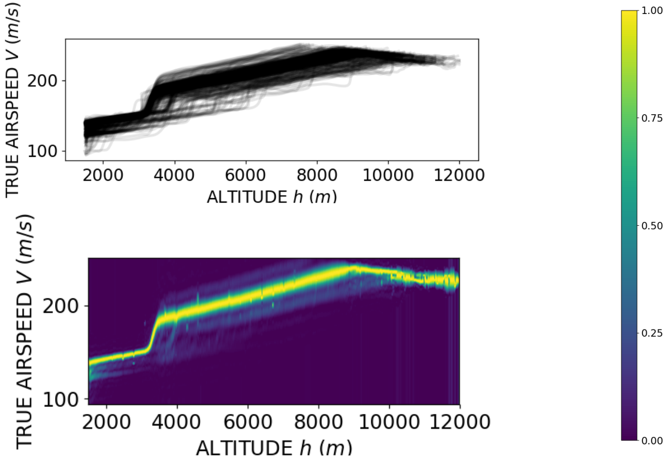
Marginal density estimation results



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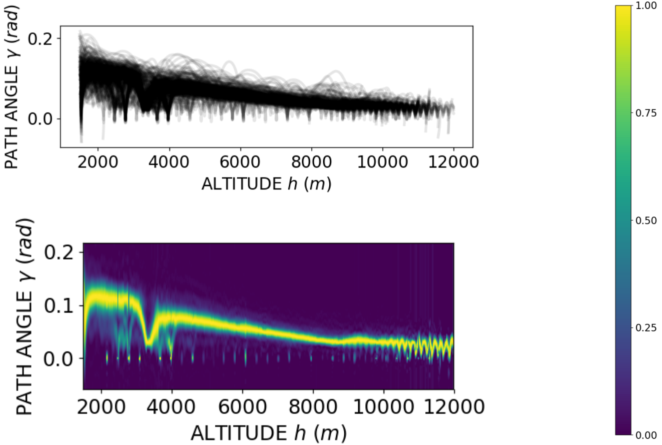

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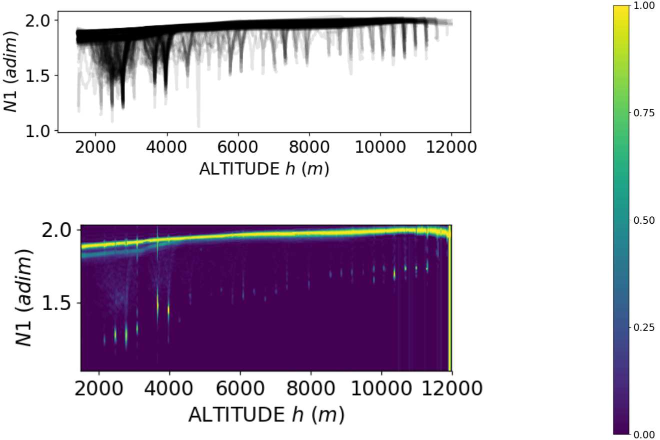

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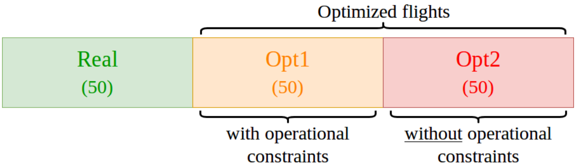

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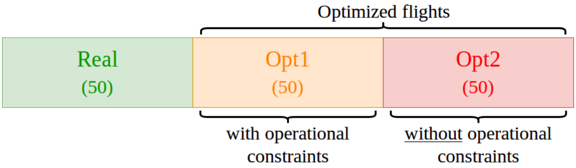

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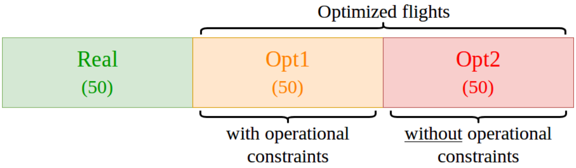

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Var.	Estimated Likelihoods		
	Real	Opt1	Opt2
MML	**0.63 $\pm$ 0.07**	**0.43 $\pm$ 0.08**	**0.13 $\pm$ 0.02**
FPCA	0.16 $\pm$ 0.12	6.4E-03 $\pm$ 3.8E-03	3.6E-03 $\pm$ 5.4E-03
LS-CDE	0.77 $\pm$ 0.05	0.68 $\pm$ 0.04	0.49 $\pm$ 0.06

MML PENALTY

The MML can be used not only to assess the optimization solutions, but also to penalize the optimization itself:

$$\min_{(\boldsymbol{x},\boldsymbol{u})\in\mathbb{X}\times\mathbb{U}} \int_0^{t_f} C(\boldsymbol{u}(t),\boldsymbol{x}(t))dt$$
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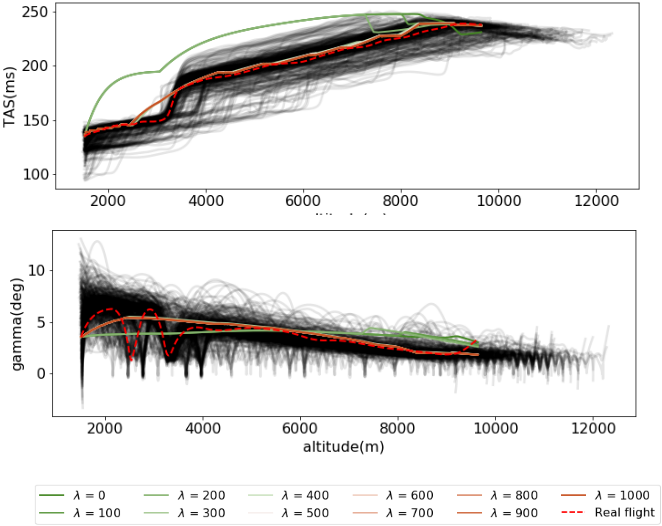
- λ sets trade-off between a fuel minimization and a likelihood maximization,

Penalty effect



Consumption x Acceptability trade-off

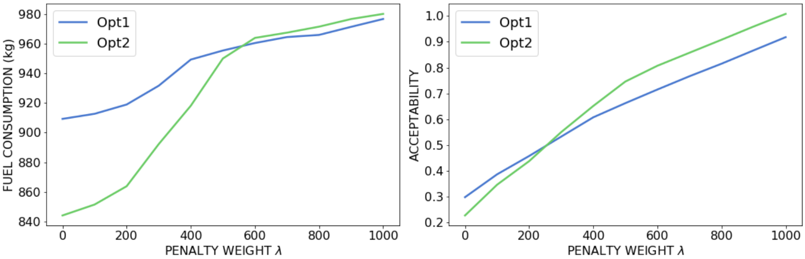


Figure: Average over 20 flights of the fuel consumption and MML score (called acceptability here) of optimized trajectories with varying MML-penalty weight λ.

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   - Showed that it can be used in optimal control problems to obtain solutions close to optimal, and still realistic.

THANK YOU FOR YOUR ATTENTION

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