MEEUUW - Mantle modelling Early Earth Utrecht University Work in progress

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1 Generic mass, momentum, energy conservation equations

We focus on the system of equations in a $d = 2$- or $d = 3$-dimensional domain Ω that describes the motion of a highly viscous fluid (i.e. near infinite Prandlt number) driven by differences in the gravitational force due to a density variations. In the following, we largely follow the exposition of this material in Schubert, Turcotte and Olson Schubert et al. 2001.

Specifically, we consider the following set of equations for velocity $\vec{\mathrm{v}}$, pressure p and temperature T:

$$-\vec{\nabla} p + \vec{\nabla} \cdot \left[2\eta \left(\dot{\boldsymbol{\varepsilon}}(\vec{\mathrm{v}}) - \frac{1}{3}(\vec{\nabla} \cdot \vec{\mathrm{v}})\mathbf{1} \right) \right] + \rho \vec{g} = \vec{0} \qquad \text{in } \Omega, \tag{1}$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{\mathrm{v}}) = 0 \qquad \text{in } \Omega, \tag{2}$$

$$\rho C_p \left(\frac{\partial T}{\partial t} + \vec{\mathrm{v}} \cdot \vec{\nabla} T \right) - \vec{\nabla} \cdot k \vec{\nabla} T = \rho H$$

$$+ 2\eta \left(\dot{\varepsilon}(\vec{\mathrm{v}}) - \frac{1}{3}(\vec{\nabla} \cdot \vec{\mathrm{v}})\mathbf{1} \right) : \left(\dot{\varepsilon}(\vec{\mathrm{v}}) - \frac{1}{3}(\vec{\nabla} \cdot \vec{\mathrm{v}})\mathbf{1} \right) \tag{3}$$

$$+ \alpha T \left(\frac{\partial p}{\partial t} + \vec{\mathrm{v}} \cdot \vec{\nabla} p \right) \qquad \text{in } \Omega, \tag{4}$$

where $\dot{\boldsymbol{\varepsilon}}(\vec{\mathrm{v}}) = \frac{1}{2}(\vec{\nabla}\vec{\mathrm{v}} + \vec{\nabla}\vec{\mathrm{v}}^T)$ is the symmetric gradient of the velocity (often called the *strain rate* tensor).

In this set of equations, (1) and (2) represent the compressible Stokes equations in which $\vec{\mathrm{v}} = \vec{\mathrm{v}}(\mathbf{x}, t)$ is the velocity field and $p = p(\mathbf{x}, t)$ the pressure field. Both fields depend on space $\mathbf{x}$ and time t. Fluid flow is driven by the gravity force that acts on the fluid and that is proportional to both the density of the fluid and the strength of the gravitational pull.

Coupled to this Stokes system is equation (3) for the temperature field $T = T(\mathbf{x}, t)$ that contains heat conduction terms as well as advection with the flow velocity $\vec{\mathrm{v}}$. The right hand side terms of this equation correspond to

- internal heat production for example due to radioactive decay;
- friction (shear) heating;
- adiabatic compression of material;

2 Equation of state

The equation of state gives the density as a function of the pressure and temperature: $\rho = \rho(p, T)$. The density is then either obtained from precomputed lookup tables or a simpler functional approach is often taken by means of a linearisation.

The widely used Boussinesq approximation (see below) linearizes these basic conservation laws near the reference hydrostatic state. If density changes caused by the pressure deviations $p' = p - p_0$ are neglected, we may linearize the state equation with respect to the temperature deviations $T - T_0$, where T_0 is a reference temperature, and write:

$$\rho = \rho_0(1 - \alpha(T - T_0))$$

This approximation thus means that the influence of hydrostatic pressure (as well as temperature T_0) on density is hidden in a spatial dependence of the reference density ρ_0.

The reference density ρ_0 is assumed to be a time-independent function. Considering only the largest term in the equation of continuity, that is, neglecting thermal expansion, we arrive at the simplified equation:

$$\vec{\nabla} \cdot (\rho_0 \vec{v}) = 0$$

3 Anelastic liquid approximation (ALA)

This comes from Matyska et al. (2007).

If we assume that there is a reference hydrostatic state characterized by $\vec{v} = 0$ in which the hydrostatic pressure p_0, hydrostatic density ρ_0, and hydrostatic gravity acceleration g_0 are related by $\vec{\nabla} p_0 = \rho_0 \vec{g}_0$, and moreover that pressure deviations $p' = p - p_0$ are negligible in the heat equation, the transfer of heat in a homogeneous material (i.e., entropy may be considered as a function of only p and T) is then described by the well-known equation:

$$\rho C_p \left(\frac{\partial T}{\partial t} + \vec{v} \cdot \vec{\nabla} T \right) = \vec{\nabla} \cdot k \vec{\nabla} T - \alpha T \vec{v} \cdot \rho \vec{g} + \boldsymbol{\tau} : \vec{\nabla} \vec{v} + Q$$

where cp is the isobaric specific heat, α is the thermal expansion coefficient and vr denotes the radial component of velocity. The left-hand side of equation 8 represents local changes of heat balance; the second (third) term on the right-hand side describes ad- vection of heat (adiabatic heating and/or cooling).

4 Boussinesq approximation (BA)

In the case of an incompressible flow, then $\partial \rho/\partial t = 0$ and $\vec{\nabla}\rho = 0$, i.e. $D\rho/Dt = 0$ and the mass conservation equation becomes:

$$\vec{\nabla}\cdot\vec{\mathrm{v}} = 0$$

A vector field that is divergence-free is also called solenoidal[1].

In this case the equations above can now be written

$$-\vec{\nabla}p + \vec{\nabla}\cdot[2\eta\,(\dot{\varepsilon}(\vec{\mathrm{v}}))] + \rho\vec{g} = \vec{0} \qquad \text{in } \Omega, \tag{5}$$

$$\vec{\nabla}\cdot\vec{\mathrm{v}} = 0 \qquad \text{in } \Omega, \tag{6}$$

$$\rho C_p\left(\frac{\partial T}{\partial t} + \vec{\mathrm{v}}\cdot\vec{\nabla}T\right) - \vec{\nabla}\cdot k\vec{\nabla}T = \rho H + 2\eta\dot{\varepsilon}(\vec{\mathrm{v}}):\dot{\varepsilon}(\vec{\mathrm{v}}) + \alpha T\left(\frac{\partial p}{\partial t} + \vec{\mathrm{v}}\cdot\vec{\nabla}p\right) \qquad \text{in } \Omega, \tag{7}$$

As nicely explained in Spiegel et al. (1960):

> In the study of problems of thermal convection it is a frequent practice to simplify the basic equations by introducing certain approximations which are attributed to Boussinesq (1903). The Boussinesq approximations can best be summarized by two statements:
>
> 1. The fluctuations in density which appear with the advent of motion result principally from thermal (as opposed to pressure) effects.
> 2. In the equations for the rate of change of momentum and mass, density variations may be neglected except when they are coupled to the gravitational acceleration in the buoyancy force."

Note that their paper examines the Boussinesq approximation for compressible fluids.

The Boussinesq approximation assumes that the density can be considered constant in all occurrences in the equations with the exception of the buoyancy term on the right hand side of (1). The primary result of this assumption is that the continuity equation (2) will now read $\vec{\nabla}\cdot\vec{\mathrm{v}} = 0$. This implies that the strain rate tensor is deviatoric. Under the Boussinesq approximation, the equations are much simplified:

$$-\vec{\nabla}p + \vec{\nabla}\cdot[2\eta\dot{\varepsilon}(\vec{\mathrm{v}})] + \rho\vec{g} = \vec{0} \qquad \text{in } \Omega, \tag{8}$$

$$\vec{\nabla}\cdot\vec{\mathrm{v}} = 0 \qquad \text{in } \Omega, \tag{9}$$

$$\rho_0 C_p\left(\frac{\partial T}{\partial t} + \vec{\mathrm{v}}\cdot\vec{\nabla}T\right) - \vec{\nabla}\cdot k\vec{\nabla}T = \rho H \qquad \text{in } \Omega \tag{10}$$

Note that all terms on the rhs of the temperature equations have disappeared, with the exception of the source term.

In Zelst et al. (2022) we read:

> The Boussinesq approximation (Oberbeck (1879); Boussinesq, 1903; Rayleigh, 1916) assumes that density variations are so small that they can be neglected everywhere except in the buoyancy term in the momentum equation, which is equivalent to using a constant reference density profile. This implies incompressibility [...]. In addition, adiabatic heating and shear heating are not considered in the energy equation. This approximation is valid as long as density variations are small and the modelled processes would cause no substantial shear or adiabatic heating. The Boussinesq approximation is often used in lithosphere-scale models. Due to its simplicity, the approximation of in- compressibility is sometimes also adopted for whole-mantle convection models, wherein it is only approximately valid, and it has been shown that compressibility can have a large effect on the pattern of convective flow Tackley 1996.

[1] https://en.wikipedia.org/wiki/Solenoidal_vector_field

5 Extended Boussinesq approximation (EBA)

In Zelst et al. (2022) we read:

> The extended Boussinesq approximation (Christensen et al. 1985; Oxburgh et al. 1978) is based on the same assumptions as the BA but does consider adiabatic and shear heating. Since it includes adiabatic heating, but not the associated volume and density changes, it can lead to artificial changes of energy in the model, i.e. material is being heated or cooled based on the assumption that it is compressed or it expands, but the mechanical work that causes compression or expansion is not done. Consequently, the extended Boussinesq approximation should only be used in models without substantial adiabatic temperature changes.
>
> For a comparison between some of these approximations using benchmark models, see e.g. Steinbach et al. (1989), Leng et al. (2008), King et al. (2010), Gassmöller et al. (2020). In addition, the choice of approximation may also be limited by the numerical methods being employed (for example, the accuracy of the solution for the variables that affect the density). Also note that, technically, these approximations are all internally inconsistent to varying degrees, since they do not fulfil the definitions of thermodynamic variables but use linearised versions instead, and they use different density formulations in the different equations. Nevertheless, many of them are generally accepted and widely used in geodynamic modelling studies, as they allow for simpler equations and more easily obtained solutions.

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