MEEUUW - Mantle modelling Early Earth Utrecht University Work in progress

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1 Generic mass, momentum, energy conservation equations

We focus on the system of equations in a $d = 2$- or $d = 3$-dimensional domain Ω that describes the motion of a highly viscous fluid driven by differences in the gravitational force due to a density variations. In the following, we largely follow the exposition of this material in Schubert, Turcotte and Olson Schubert et al. 2001.

Specifically, we consider the following set of equations for velocity $\vec{\mathrm{v}}$, pressure p and temperature T:

$$-\vec{\nabla} p + \vec{\nabla}\cdot\left[2\eta\left(\dot{\boldsymbol{\varepsilon}}(\vec{\mathrm{v}}) - \frac{1}{3}(\vec{\nabla}\cdot\vec{\mathrm{v}})\mathbf{1}\right)\right] + \rho\vec{g} = \vec{0} \qquad \text{in } \Omega, \tag{1}$$

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{\mathrm{v}}) = 0 \qquad \text{in } \Omega, \tag{2}$$

$$\begin{aligned} \rho C_p\left(\frac{\partial T}{\partial t} + \vec{\mathrm{v}}\cdot\vec{\nabla}T\right) - \vec{\nabla}\cdot k\vec{\nabla}T = \rho H & \\ + 2\eta\left(\dot{\varepsilon}(\vec{\mathrm{v}}) - \frac{1}{3}(\vec{\nabla}\cdot\vec{\mathrm{v}})\mathbf{1}\right) : \left(\dot{\varepsilon}(\vec{\mathrm{v}}) - \frac{1}{3}(\vec{\nabla}\cdot\vec{\mathrm{v}})\mathbf{1}\right) & \\ + \alpha T\left(\frac{\partial p}{\partial t} + \vec{\mathrm{v}}\cdot\vec{\nabla}p\right) & \qquad \text{in } \Omega, \end{aligned} \tag{3}$$

where $\dot{\boldsymbol{\varepsilon}}(\vec{\mathrm{v}}) = \frac{1}{2}(\vec{\nabla}\vec{\mathrm{v}} + \vec{\nabla}\vec{\mathrm{v}}^T)$ is the symmetric gradient of the velocity (often called the *strain rate* tensor).

In this set of equations, (4) and (5) represent the compressible Stokes equations in which $\vec{\mathrm{v}} = \vec{\mathrm{v}}(\mathbf{x}, t)$ is the velocity field and $p = p(\mathbf{x}, t)$ the pressure field. Both fields depend on space $\mathbf{x}$ and time t. Fluid flow is driven by the gravity force that acts on the fluid and that is proportional to both the density of the fluid and the strength of the gravitational pull.

Coupled to this Stokes system is equation (3) for the temperature field $T = T(\mathbf{x}, t)$ that contains heat conduction terms as well as advection with the flow velocity $\vec{\mathrm{v}}$. The right hand side terms of this equation correspond to

- internal heat production for example due to radioactive decay;
- friction (shear) heating;
- adiabatic compression of material;

2 Boussinesq approximation

In the case of an incompressible flow, then $\partial\rho/\partial t = 0$ and $\vec{\nabla}\rho = 0$, i.e. $D\rho/Dt = 0$ and the mass conservation equation becomes:

$$\vec{\nabla}\cdot\vec{\mathrm{v}} = 0$$

A vector field that is divergence-free is also called solenoidal[1].

In this case the equations above can now be written

$$-\vec{\nabla} p + \vec{\nabla}\cdot\left[2\eta\left(\dot{\boldsymbol{\varepsilon}}(\vec{\mathrm{v}})\right)\right] + \rho\vec{g} = \vec{0} \qquad \text{in } \Omega, \tag{4}$$

$$\vec{\nabla}\cdot\vec{\mathrm{v}} = 0 \qquad \text{in } \Omega, \tag{5}$$

$$\rho C_p\left(\frac{\partial T}{\partial t} + \vec{\mathrm{v}}\cdot\vec{\nabla}T\right) - \vec{\nabla}\cdot k\vec{\nabla}T = \rho H + 2\eta\dot{\varepsilon}(\vec{\mathrm{v}}) : \dot{\varepsilon}(\vec{\mathrm{v}}) + \alpha T\left(\frac{\partial p}{\partial t} + \vec{\mathrm{v}}\cdot\vec{\nabla}p\right) \qquad \text{in } \Omega,$$

[1] `https://en.wikipedia.org/wiki/Solenoidal_vector_field`

3 Extended Boussinesq approximation

References

Schubert, G. et al. (2001). *Mantle Convection in the Earth and Planets*. Cambridge University Press. ISBN: 0-521-70000-0. DOI: 10.1017/CBO9780511612879.