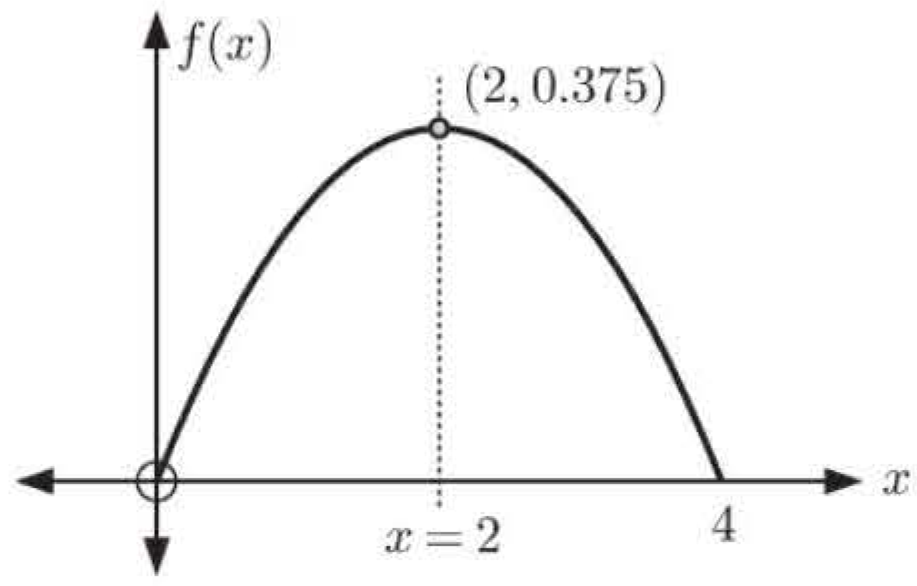
Chapter 26
CONTINUOUS RANDOM VARIABLES

EXERCISE 26A

1 **a**
$$\int_0^4 ax(x-4)\,dx = 1$$
$$\therefore\ a\int_0^4 (x^2-4x)\,dx = 1$$
$$\therefore\ a\left[\frac{x^3}{3}-\frac{4x^2}{2}\right]_0^4 = 1$$
$$\therefore\ a\left(\tfrac{64}{3}-32\right) = 1$$
$$\therefore\ a\left(\tfrac{-32}{3}\right) = 1$$
$$\therefore\ a = -\tfrac{3}{32}$$

b $f(x) = -\frac{3}{32}x(x-4),\quad 0 \leqslant x \leqslant 4$



c **i**
$$\begin{aligned}\mu &= \int_0^4 x\,f(x)\,dx\\ &= \int_0^4 -\tfrac{3}{32}x^2(x-4)\,dx\\ &= -\tfrac{3}{32}\int_0^4 (x^3-4x^2)\,dx\\ &= -\tfrac{3}{32}\left[\tfrac{1}{4}x^4-\tfrac{4}{3}x^3\right]_0^4\\ &= -\tfrac{3}{32}\left(\tfrac{1}{4}(4)^4-\tfrac{4}{3}(4)^3\right)\\ &= -\tfrac{3}{32}\left(4^3-\tfrac{4}{3}\times 4^3\right)\\ &= -\tfrac{3}{32}\left(-\tfrac{64}{3}\right)\\ &= 2\end{aligned}$$

ii mode $= 2$ {symmetry of graph}

iii If $\int_0^m -\frac{3}{32}x(x-4)\,dx = \frac{1}{2}$

then $\int_0^m (x^2-4x)\,dx = -\frac{16}{3}$
$$\therefore\ \left[\frac{x^3}{3}-\frac{4x^2}{2}\right]_0^m = -\tfrac{16}{3}$$
$$\therefore\ \frac{m^3}{3}-2m^2-0 = -\tfrac{16}{3}$$
$$\therefore\ m^3-6m^2 = -16$$
$$\therefore\ m^3-6m^2+16 = 0$$
$$\therefore\ (m-2)(m^2-4m-8) = 0$$
$$\therefore\ m = 2\ \text{ or }\ \frac{4\pm\sqrt{16+32}}{2}$$
$$\therefore\ m = 2\ \text{ or }\ 2\pm 2\sqrt{3}$$
$$\therefore\ m = 2\quad \{\text{as } 0 < m < 4\}$$
$\therefore$ the median is 2

iv
$$\begin{aligned}&\int_0^4 x^2\,f(x)\,dx\\ &= \int_0^4 -\tfrac{3}{32}x^3(x-4)\,dx\\ &= -\tfrac{3}{32}\int_0^4 (x^4-4x^3)\,dx\\ &= -\tfrac{3}{32}\left[\tfrac{1}{5}x^5-x^4\right]_0^4\\ &= -\tfrac{3}{32}\left(\tfrac{4}{5}(4)^4-4^4\right)\\ &= -\tfrac{3}{32}\left(-\tfrac{256}{5}\right) = \tfrac{24}{5}\end{aligned}$$
$$\therefore\ \text{Var}(X) = \tfrac{24}{5}-2^2 = \tfrac{4}{5} = 0.8$$

2 **a**
$$\int_0^b -0.2x(x-b)\,dx = 1$$
$$\therefore\ -0.2\int_0^b (x^2-bx)\,dx = 1$$
$$\therefore\ \left[\tfrac{1}{3}x^3-\tfrac{1}{2}bx^2\right]_0^b = -5$$
$$\therefore\ \tfrac{1}{3}b^3-\tfrac{1}{2}b^3-0 = -5$$
$$\therefore\ 2b^3-3b^3 = -30$$
$$\therefore\ -b^3 = -30$$
$$\therefore\ b^3 = 30$$
$$\therefore\ b = \sqrt[3]{30}$$

b **i**
$$\begin{aligned}\mu &= \int_0^{\sqrt[3]{30}} -0.2x^2(x-\sqrt[3]{30})\,dx\\ &\approx 1.5536 \quad \{\text{using technology}\}\\ &\approx 1.55\end{aligned}$$

ii
$$\begin{aligned}&\int_0^{\sqrt[3]{30}} x^2\,f(x)\,dx\\ &= \int_0^{\sqrt[3]{30}} -0.2x^3(x-\sqrt[3]{30})\,dx\\ &\approx 2.8965 \quad \{\text{using technology}\}\end{aligned}$$
$$\therefore\ \text{Var}(X) \approx 2.8965-\mu^2 \approx 0.483$$

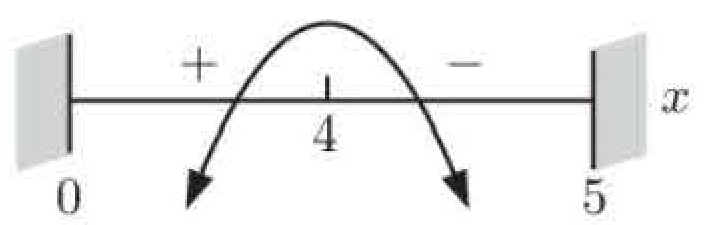
3 **a**
$$\int_0^3 ke^{-x}\,dx = 1$$
$$\therefore\ k\int_0^3 e^{-x}\,dx = 1$$
$$\therefore\ k\left[\frac{e^{-x}}{-1}\right]_0^3 = 1$$
$$\therefore\ k(-e^{-3}-(-1)) = 1$$
$$\therefore\ k(1-e^{-3}) = 1$$
$$\therefore\ k \approx 1.0524$$

b If m is the median then
$$\int_0^m ke^{-x}\,dx = \tfrac{1}{2}$$
$$\therefore\ \int_0^m e^{-x}\,dx = \frac{1}{2k}$$
$$\therefore\ \left[\frac{e^{-x}}{-1}\right]_0^m = \frac{1}{2k}$$
$$\therefore\ -e^{-m}-(-1) = \frac{1}{2k}$$
$$\therefore\ e^{-m} \approx 1 - \frac{1}{2(1.0524)}$$
$$\therefore\ e^{-m} \approx 0.524\,89$$
$$\therefore\ -m \approx \ln(0.524\,89)$$
$$\therefore\ m \approx 0.645$$

4 **a**
$$\int_0^5 kx^2(x-6)\,dx = 1$$
$$\therefore\ k\int_0^5 (x^3-6x^2)\,dx = 1$$
$$\therefore\ k\left[\tfrac{1}{4}x^4 - \tfrac{6}{3}x^3\right]_0^5 = 1$$
$$\therefore\ k\left(\tfrac{625}{4} - 250\right) = 1$$
$$\therefore\ k\left(\tfrac{-375}{4}\right) = 1$$
$$\therefore\ k = -\tfrac{4}{375}$$

b
$$f(x) = -\tfrac{4}{375}x^2(x-6)$$
$$= -\tfrac{4}{375}(x^3-6x^2)$$
$$\therefore\ f'(x) = -\tfrac{4}{375}(3x^2-12x)$$
$$\therefore\ f'(x) = 0 \quad \text{when} \quad 3x(x-4) = 0$$
$$\therefore\ x = 0 \text{ or } 4$$

$f'(x)$ has sign diagram:



There is a maximum when $x = 4$, so the mode is 4.

c If m is the median,

then $\int_0^m -\tfrac{4}{375}\,x^2(x-6)\,dx = \tfrac{1}{2}$
$$\therefore\ \int_0^m (x^3-6x^2)\,dx = -\tfrac{375}{8}$$
$$\therefore\ \left[\tfrac{1}{4}x^4 - \tfrac{6}{3}x^3\right]_0^m = -\tfrac{375}{8}$$
$$\therefore\ \tfrac{1}{4}m^4 - 2m^3 = -\tfrac{375}{8}$$
$$\therefore\ 2m^4 - 16m^3 + 375 = 0$$

Using technology, $m \approx 3.46$

d
$$\mu = \int_0^5 x\,f(x)\,dx$$
$$= \int_0^5 -\tfrac{4}{375}\,x^3(x-6)\,dx$$
$$= 3\tfrac{1}{3} \quad \{\text{using technology}\}$$

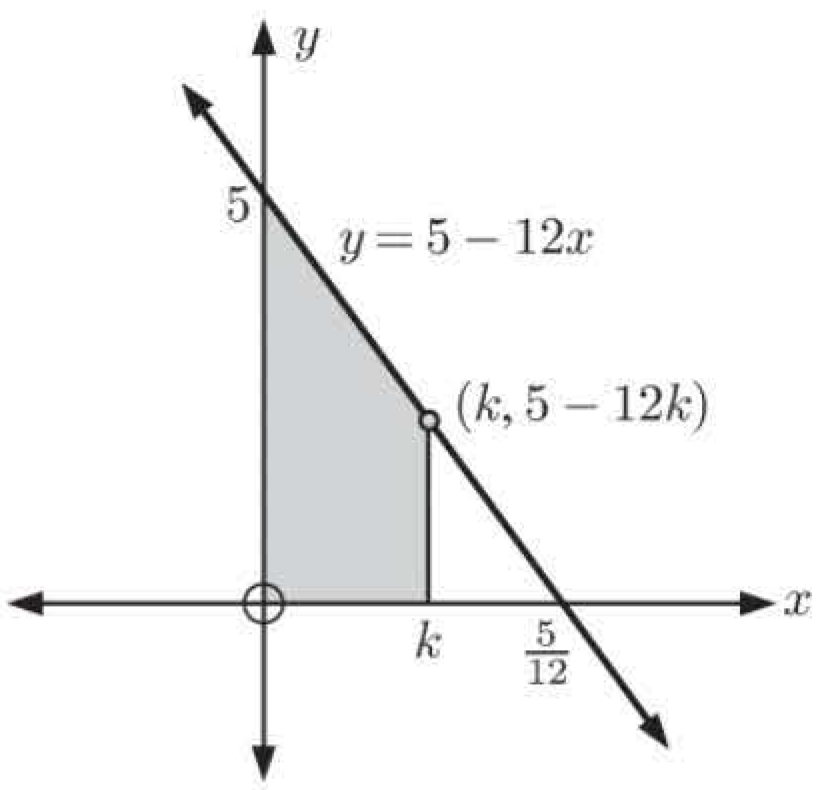
e
$$\mathrm{E}(X^2) = \int_0^5 x^2\,f(x)\,dx$$
$$= \int_0^5 -\tfrac{4}{375}\,x^4(x-6)\,dx$$
$$= 12\tfrac{2}{9} \quad \{\text{using technology}\}$$
$$\therefore\ \mathrm{Var}(X) = 12\tfrac{2}{9} - \left(3\tfrac{1}{3}\right)^2 = 1\tfrac{1}{9}$$

5 **a** Y is a continuous random variable if $5-12y \geqslant 0$ for all $0 \leqslant y \leqslant k$ and $\int_0^k (5-12y)\,dy = 1$.

Since $f(y) = 5-12y$ is a decreasing function, $f(k) = 5-12k$ is the smallest value of $f(y)$ on $0 \leqslant y \leqslant k$.
$$\therefore\ 5-12k \geqslant 0$$
$$\therefore\ 12k \leqslant 5$$
$$\therefore\ k \leqslant \tfrac{5}{12}$$

So, $k \leqslant \tfrac{5}{12}$ and $\int_0^k (5-12y)\,dy = 1$

b



$\int_0^k (5-12y)\,dy = \text{shaded area} = 1$

$\therefore\ k \times \left(\dfrac{5+(5-12k)}{2}\right) = 1$

$\therefore\ k(5-6k) = 1$

$\therefore\ 5k - 6k^2 = 1$

$\therefore\ 6k^2 - 5k + 1 = 0$

$\therefore\ (3k-1)(2k-1) = 0$

$\therefore\ k = \frac{1}{3}$ or $\frac{1}{2}$

But $k \leqslant \frac{5}{12}$, so $k = \frac{1}{3}$

c If $k = \frac{1}{2}$, the graph $f(y) = 5 - 12y$ falls below the horizontal axis.

d $\mu = \int_0^{\frac{1}{3}} y\,f(y)\,dy$

$= \int_0^{\frac{1}{3}} (5y - 12y^2)\,dy$

$= \left[\frac{5}{2}y^2 - 4y^3\right]_0^{\frac{1}{3}}$

$= \frac{5}{2}\left(\frac{1}{9}\right) - 4\left(\frac{1}{27}\right)$

$= \frac{7}{54}$

If $\int_0^m (5-12y)\,dy = \frac{1}{2}$

then $\left[5y - 6y^2\right]_0^m = \frac{1}{2}$

$\therefore\ 5m - 6m^2 = \frac{1}{2}$

$\therefore\ 12m^2 - 10m + 1 = 0$

$\therefore\ m = \frac{5 \pm \sqrt{13}}{12}$

But $m < \frac{5}{12}$, so $m = \frac{5-\sqrt{13}}{12} \approx 0.116$

$\therefore$ the median ≈ 0.116

6 **a** $\int_a^b k\,dx = 1$

$\therefore\ [kx]_a^b = 1$

$\therefore\ bk - ak = 1$

$\therefore\ k = \dfrac{1}{b-a}$

b $\mu = \int_a^b kx\,dx$

$= \dfrac{1}{b-a}\left[\frac{1}{2}x^2\right]_a^b$

$= \dfrac{1}{b-a}\left(\frac{1}{2}b^2 - \frac{1}{2}a^2\right)$

$= \frac{1}{2}\dfrac{(b-a)(b+a)}{(b-a)}$

$\therefore\ \text{mean} = \dfrac{a+b}{2}$

If $\int_a^m k\,dx = \frac{1}{2}$ then $[kx]_a^m = \frac{1}{2}$

$\therefore\ \dfrac{m}{b-a} - \dfrac{a}{b-a} = \frac{1}{2}$

$\therefore\ m - a = \dfrac{b-a}{2}$

$\therefore\ m = a + \dfrac{b-a}{2}$

$= \dfrac{a+b}{2}$

$\therefore\ \text{median} = \dfrac{a+b}{2}$

The mode is undefined as the function is constant for all $a \leqslant x \leqslant b$.

c $\int_a^b kx^2\,dx = \dfrac{1}{b-a}\left[\dfrac{x^3}{3}\right]_a^b$

$= \dfrac{1}{b-a}\left(\dfrac{b^3}{3} - \dfrac{a^3}{3}\right)$

$= \frac{1}{3}\,\dfrac{b^3-a^3}{b-a}$

$= \frac{1}{3}\,\dfrac{(b-a)(b^2+ab+a^2)}{(b-a)}$

$= \dfrac{a^2+ab+b^2}{3}$

$\therefore\ \text{Var}(X) = \dfrac{a^2+ab+b^2}{3} - \left(\dfrac{a+b}{2}\right)^2$

$= \dfrac{4(a^2+ab+b^2) - 3(a+b)^2}{12}$

$= \dfrac{4a^2+4ab+4b^2-3a^2-6ab-3b^2}{12}$

$= \dfrac{a^2-2ab+b^2}{12}$

$= \dfrac{(a-b)^2}{12}$

$\therefore\ \sigma_X = \sqrt{\dfrac{(a-b)^2}{12}} = \dfrac{b-a}{\sqrt{12}}$ {as $b > a$}

7 **a** If $\int_0^m 2e^{-2x}\,dx = \frac{1}{2}$

then $\left[-e^{-2x}\right]_0^m = \frac{1}{2}$

$\therefore\ -e^{-2m} - (-e^0) = \frac{1}{2}$

$\therefore\ \frac{1}{2} = e^{-2m}$

$\therefore\ -2m = \ln\frac{1}{2}$

$\therefore\ m = -\frac{1}{2}\ln\frac{1}{2} \approx 0.347$

b $f(x) = 2e^{-2x}$

$\therefore\ f'(x) = -4e^{-2x}$

$\therefore\ f'(x) < 0$ for all $x \geqslant 0$ $\{e^{-2x} > 0\}$

$\therefore\ f(x)$ is always decreasing for $x \geqslant 0$

$\therefore$ the mode $= 0$

8 **a** $\int_0^a 6\cos 3x\,dx = 1$

$\therefore\ [2\sin 3x]_0^a = 1$

$\therefore\ 2\sin 3a - 2\sin 0 = 1$

$\therefore\ \sin 3a = \frac{1}{2}$

$\therefore\ 3a = \frac{\pi}{6}$

$\therefore\ a = \frac{\pi}{18}$

b $\mu = \int_0^{\frac{\pi}{18}} 6x\cos 3x\,dx$

We integrate by parts with $u = 6x$ $v' = \cos 3x$

$u' = 6$ $v = \frac{1}{3}\sin 3x$

$\therefore\ \int 6x\cos 3x\,dx = 2x\sin 3x - \int 2\sin 3x\,dx$

$= 2x\sin 3x + \frac{2}{3}\cos 3x + c$

$\therefore\ \mu = \int_0^{\frac{\pi}{18}} 6x\cos 3x\,dx$

$= \left[2x\sin 3x + \frac{2}{3}\cos 3x\right]_0^{\frac{\pi}{18}}$

$= \left(\frac{\pi}{9}\sin\frac{\pi}{6} + \frac{2}{3}\cos\frac{\pi}{6}\right) - \left(0 + \frac{2}{3}\cos 0\right)$

$= \frac{\pi}{9}\left(\frac{1}{2}\right) + \frac{2}{3}\left(\frac{\sqrt{3}}{2}\right) - \frac{2}{3}$

$= \frac{\pi}{18} + \frac{\sqrt{3}-2}{3}$

≈ 0.0852

c If k is the 20th percentile of X,

then $\int_0^k 6\cos 3x\,dx = 0.2$

$\therefore\ [2\sin 3x]_0^k = 0.2$

$\therefore\ 2\sin 3k - 2\sin 0 = 0.2$

$\therefore\ \sin 3k = 0.1$

$\therefore\ 3k \approx 0.100$

$\therefore\ k \approx 0.0334$

So, the 20th percentile of $X \approx 0.0334$

d $\mathrm{E}(X^2) = \int_0^{\frac{\pi}{18}} 6x^2\cos 3x\,dx$

$\approx 0.009\,773$ {using technology}

$\therefore\ \mathrm{Var}(X) \approx 0.009\,773 - (0.0852)^2$

$\approx 0.002\,511$

$\therefore\ \sigma_X \approx \sqrt{0.002\,511}$

≈ 0.0501

9 $\mathrm{P}\left(X \leqslant \frac{2}{3}\right) = \frac{1}{243}$

$\therefore\ \int_0^{\frac{2}{3}} ax^4\,dx = \frac{1}{243}$

$\therefore\ \left[\frac{1}{5}ax^5\right]_0^{\frac{2}{3}} = \frac{1}{243}$

$\therefore\ \frac{1}{5}a \times \left(\frac{2}{3}\right)^5 = \frac{1}{243}$

$\therefore\ \frac{1}{5}a \times \frac{32}{243} = \frac{1}{243}$

$\therefore\ a = \frac{5}{32}$

So, $\int_0^k \frac{5}{32}x^4\,dx = 1$

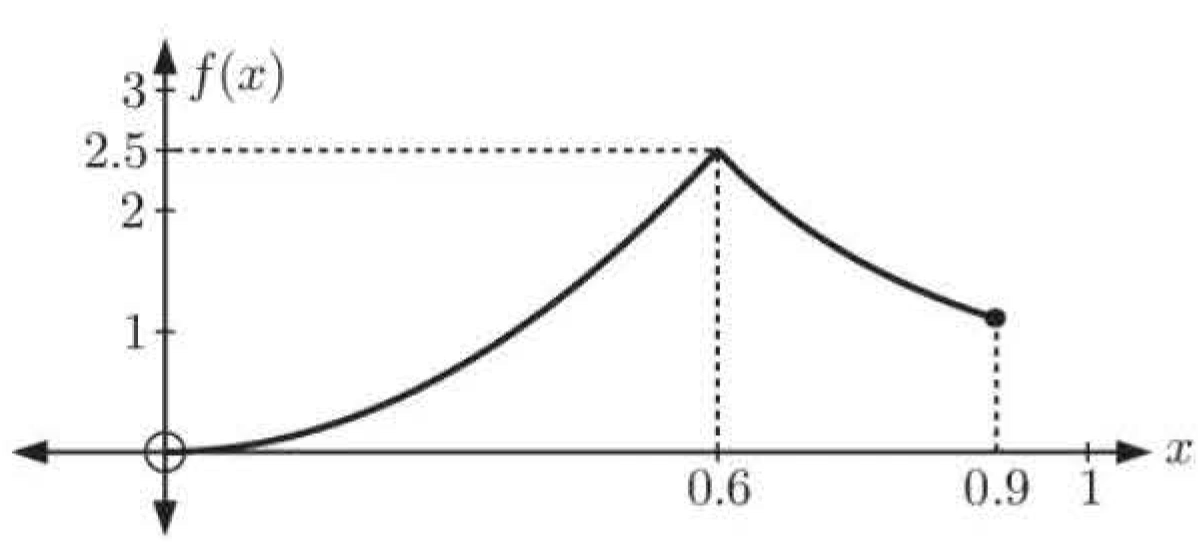
$\therefore\ \left[\frac{1}{32}x^5\right]_0^k = 1$

$\therefore\ \frac{1}{32}k^5 = 1$

$\therefore\ k^5 = 32$

$\therefore\ k = 2$

10 **a**



b From the graph in **a**, $f(x) \geqslant 0$ for all $x \in [0,\ 0.9]$.

Also, the area under the curve
$$\begin{aligned}
&= \int_0^{0.6} \tfrac{125}{18}x^2\,dx + \int_{0.6}^{0.9} \frac{9}{10x^2}\,dx \\
&= \left[\tfrac{125}{54}x^3\right]_0^{0.6} + \left[-\frac{9}{10x}\right]_{0.6}^{0.9} \\
&= \tfrac{125}{54}\left(\tfrac{3}{5}\right)^3 + \left(-\tfrac{9}{9}\right) - \left(-\tfrac{9}{6}\right) \\
&= \tfrac{1}{2} - 1 + \tfrac{3}{2} \\
&= 1 \quad \text{as required}
\end{aligned}$$

c
$$\begin{aligned}
\mu &= \textstyle\int_0^{0.9} x\,f(x)\,dx \\
&= \int_0^{0.6} \tfrac{125}{18}\,x^3\,dx + \int_{0.6}^{0.9} \frac{9}{10x}\,dx \\
&= \left[\tfrac{125}{72}x^4\right]_0^{0.6} + \left[\tfrac{9}{10}\ln x\right]_{0.6}^{0.9} \\
&= \tfrac{125}{72}(0.6)^4 + \tfrac{9}{10}\ln(0.9) - \tfrac{9}{10}\ln(0.6) \\
&\approx 0.590
\end{aligned}$$

From the calculations in **b**,

$$\textstyle\int_0^{0.6} f(x)\,dx = \int_{0.6}^{0.9} f(x)\,dx = \tfrac{1}{2}$$

$\therefore$ median $= 0.6$

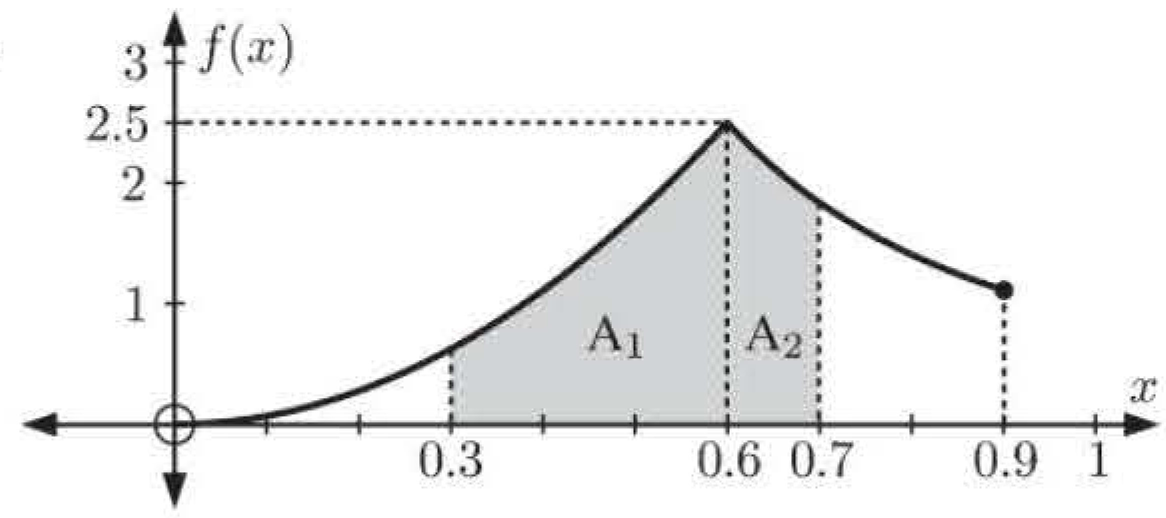
From the graph in **a**, the highest value of $f(x)$ occurs at $x = 0.6$

$\therefore$ mode $= 0.6$

d
$$\begin{aligned}
\mathrm{E}(X^2) &= \textstyle\int_0^{0.9} x^2\,f(x)\,dx \\
&= \textstyle\int_0^{0.6} \tfrac{125}{18}x^4\,dx + \int_{0.6}^{0.9} \tfrac{9}{10}\,dx \\
&= \left[\tfrac{25}{18}x^5\right]_0^{0.6} + \left[\tfrac{9}{10}x\right]_{0.6}^{0.9} \\
&= \tfrac{25}{18}\left(\tfrac{3}{5}\right)^5 + \tfrac{9}{10}\left(\tfrac{9}{10}\right) - \tfrac{9}{10}\left(\tfrac{3}{5}\right) \\
&= \tfrac{189}{500}
\end{aligned}$$

$$\begin{aligned}
\therefore\ \mathrm{Var}(X) &\approx \tfrac{189}{500} - 0.589\,92^2 \\
&\approx 0.0300 \\
\therefore\ \sigma_X &\approx \sqrt{0.029\,994} \\
&\approx 0.173
\end{aligned}$$

e

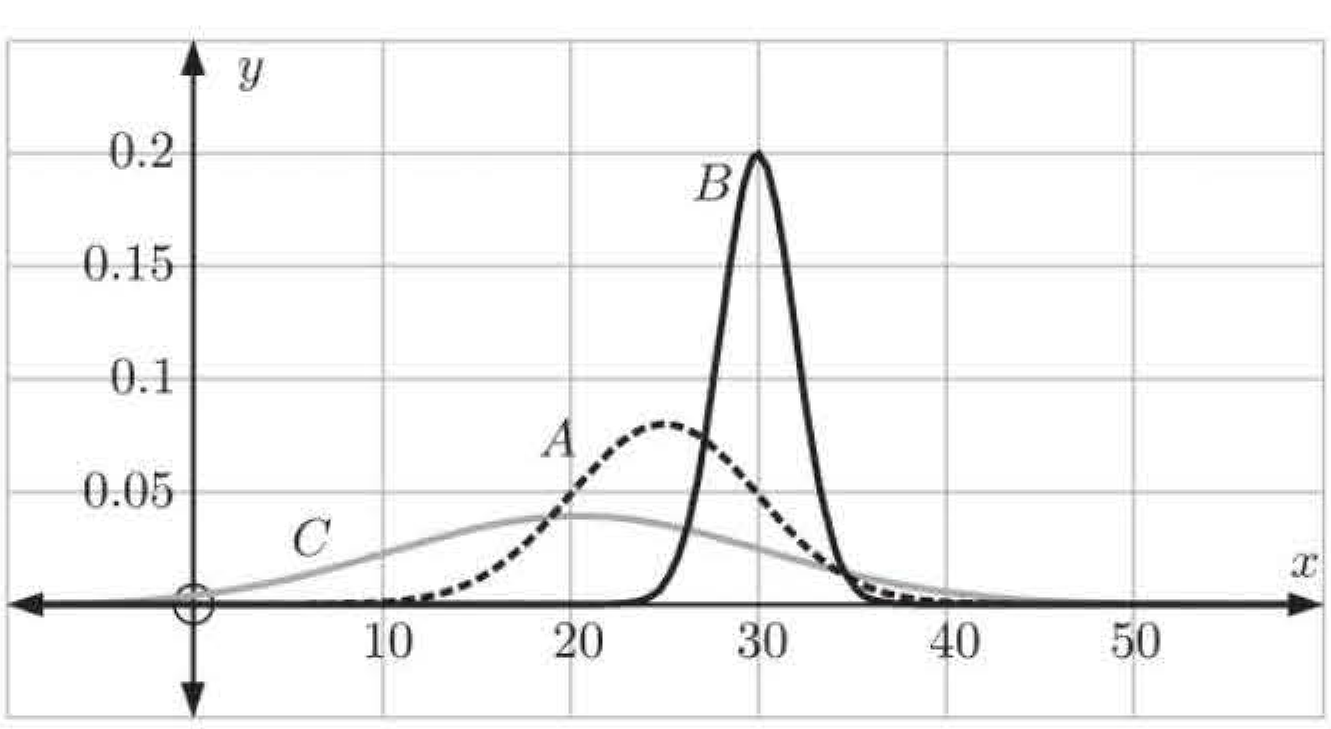


$$\begin{aligned}
&\mathrm{P}(0.3 < X < 0.7) \\
&= \text{shaded area} \\
&= \mathrm{A}_1 + \mathrm{A}_2 \\
&= \int_{0.3}^{0.6} \tfrac{125}{18}x^2\,dx + \int_{0.6}^{0.7} \frac{9}{10x^2}\,dx \\
&= \left[\tfrac{125}{54}x^3\right]_{0.3}^{0.6} + \left[-\frac{9}{10x}\right]_{0.6}^{0.7} \\
&= \tfrac{125}{54}(0.6^3 - 0.3^3) + \left(-\tfrac{9}{7}\right) - \left(-\tfrac{9}{6}\right) \\
&\approx 0.652
\end{aligned}$$

$\therefore$ it takes between 0.3 hours ($=$ 18 minutes) and 0.7 hours ($=$ 42 minutes) to perform the task 65.2% of the time.

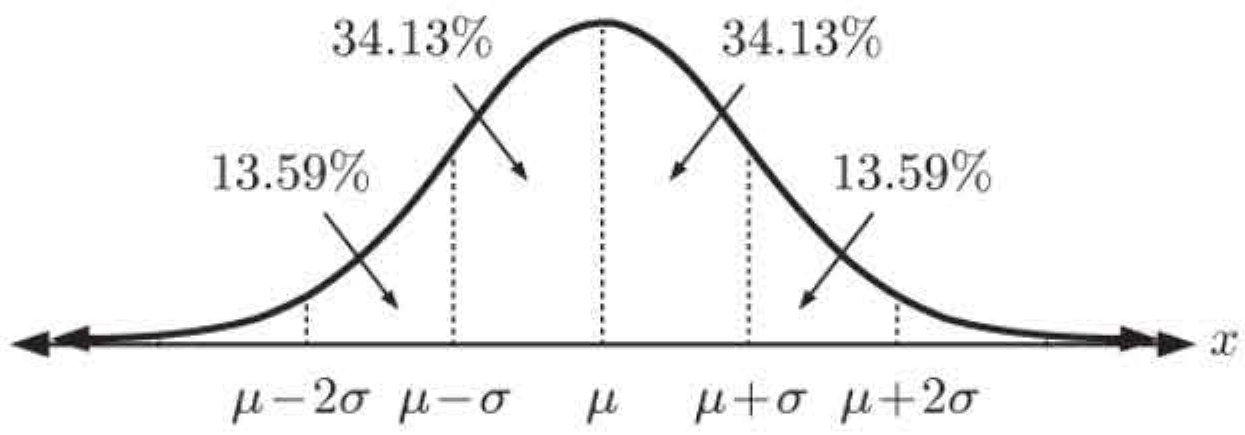
EXERCISE 26B

1



2 a, b The mean volume (or diameter) is likely to occur most often with variations around the mean occurring symmetrically as a result of random variations in the production process.

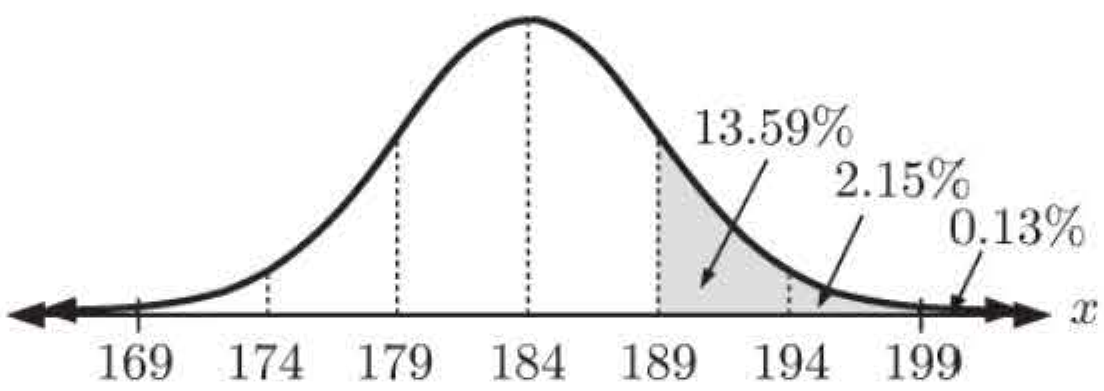
3



a P(value between $\mu-\sigma$ and $\mu+\sigma$)
$\approx 34.13\% + 34.13\%$
≈ 0.683

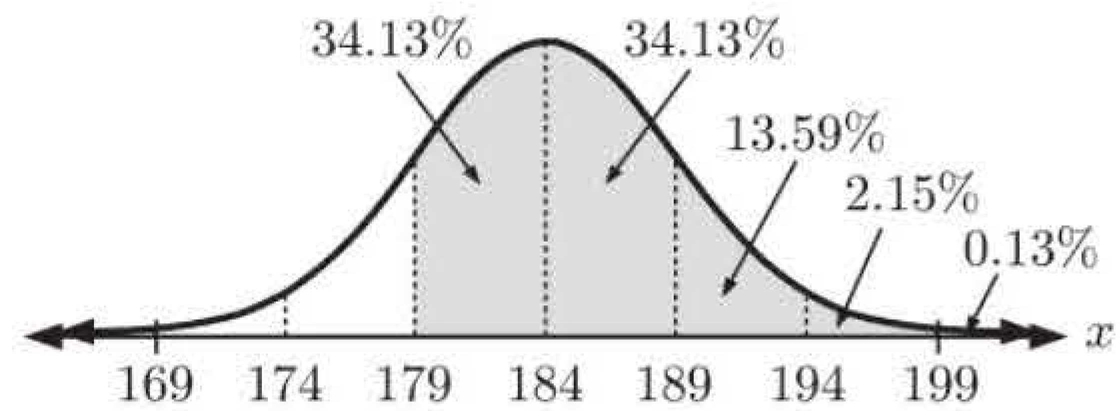
b P(value between μ and $\mu+2\sigma$)
$\approx 34.13\% + 13.59\%$
≈ 0.477

4 **a**



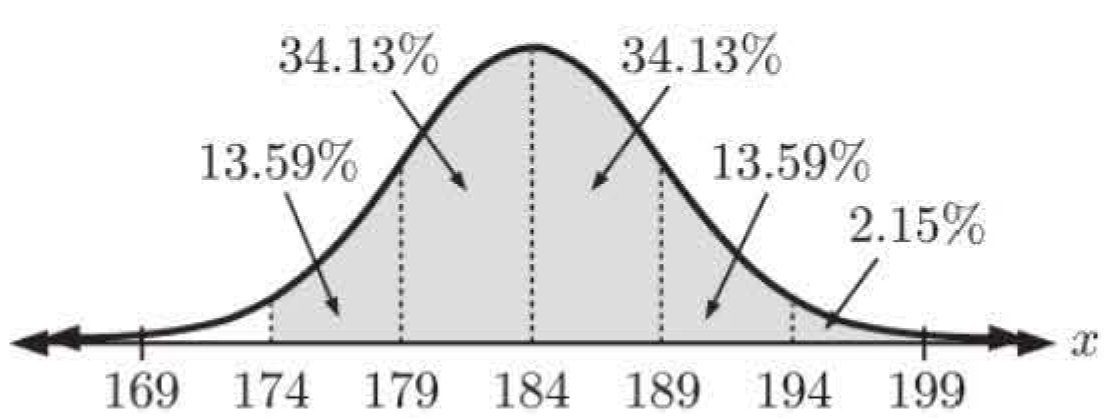
We need the percentage greater than 189 cm.
This is $13.59\% + 2.15\% + 0.13\%$
$\approx 15.9\%$

b



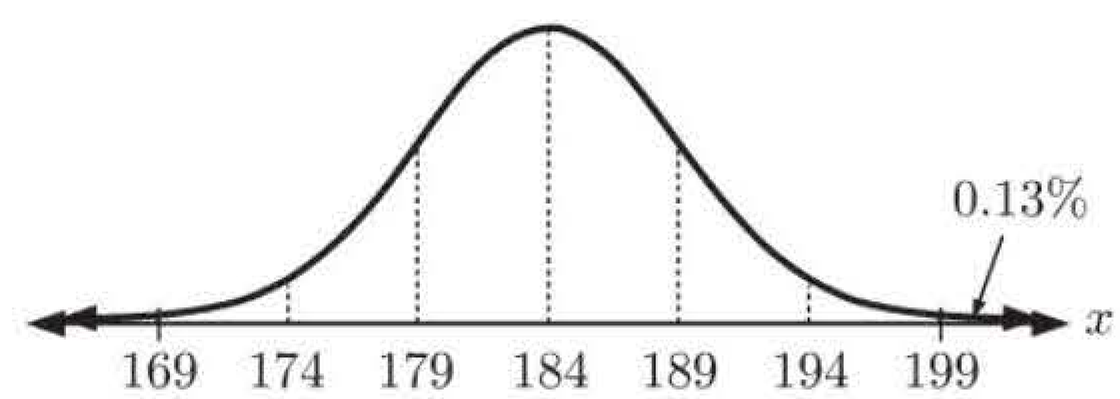
We need the percentage greater than 179 cm.
This is $34.13\% + 34.13\% + 13.59\%$
$+ 2.15\% + 0.13\%$
$\approx 84.1\%$

c



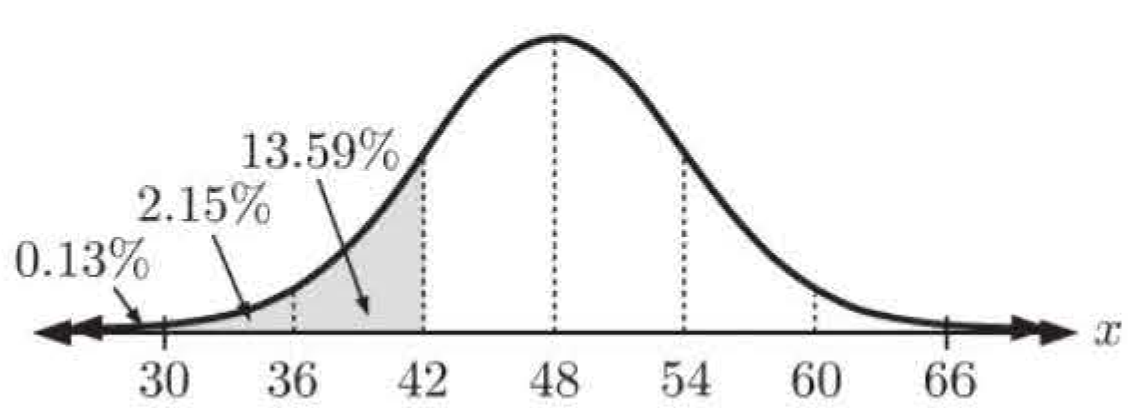
We need the percentage between 174 cm and 199 cm.
This is $13.59\% + 34.13\% + 34.13\%$
$+ 13.59\% + 2.15\%$
$\approx 97.6\%$

d



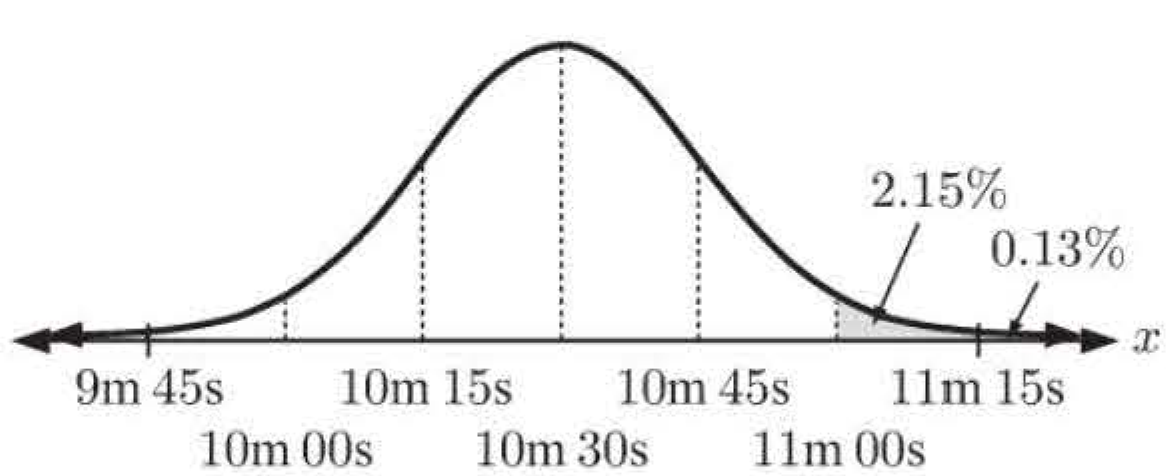
We need the percentage greater than 199 cm.
This is 0.13%.

5



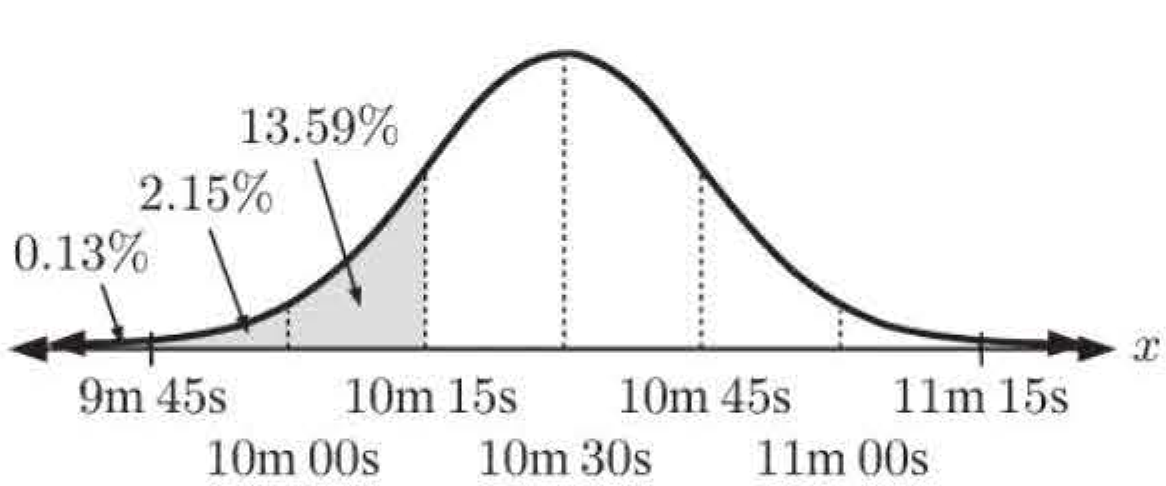
The chance of there being less than 42 mm of rain during August is
$0.13\% + 2.15\% + 13.59\% = 15.87\%$
and 15.87% of $20 = 3.174$
So, over a 20 year period, there would be less than 42 mm of rain during August three times.

6 **a**



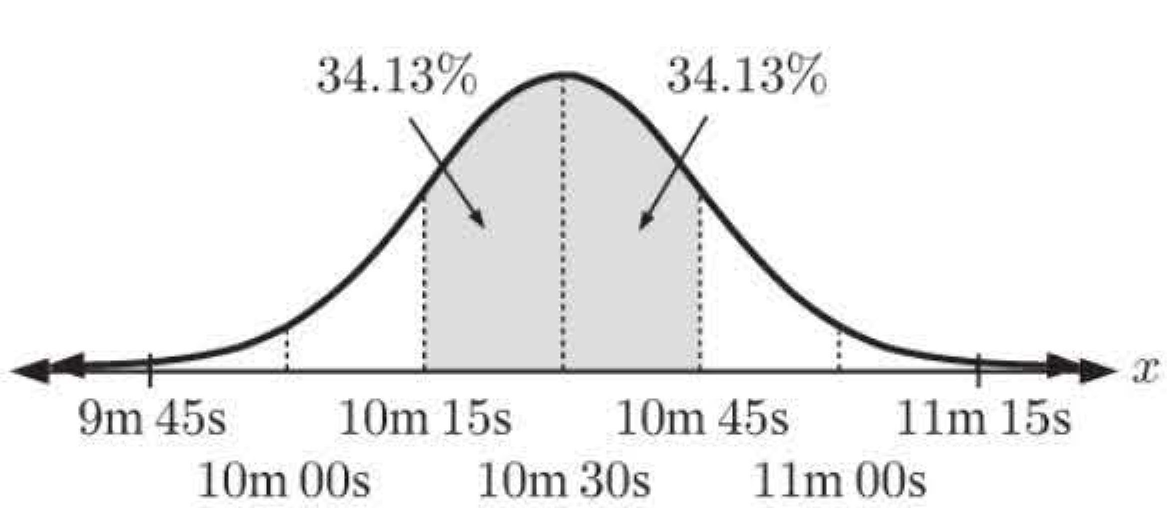
$2.15\% + 0.13\% = 2.28\%$ of competitors took over 11 minutes,
and 2.28% of $200 = 4.56$
So, 5 competitors took longer than 11 minutes.

b



$0.13\% + 2.15\% + 13.59\% = 15.87\%$ of competitors took less than 10 minutes 15 seconds,
and 15.87% of $200 = 31.74$
So, 32 competitors took less than 10 minutes 15 seconds.

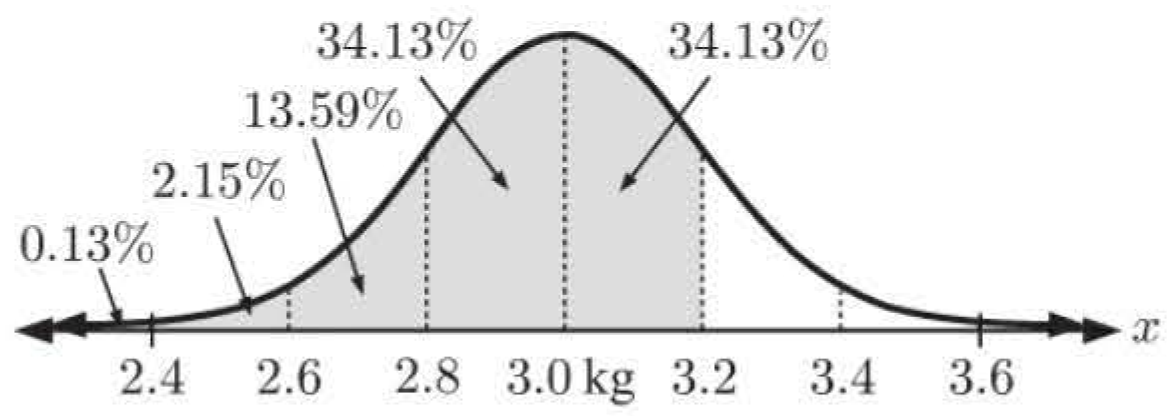
c



$34.13\% + 34.13\% = 68.26\%$ of competitors took between 10 minutes 15 seconds and 10 minutes 45 seconds, and 68.26% of $200 = 136.52$

So, 137 competitors took between 10 minutes 15 seconds and 10 minutes 45 seconds.

7 **a**

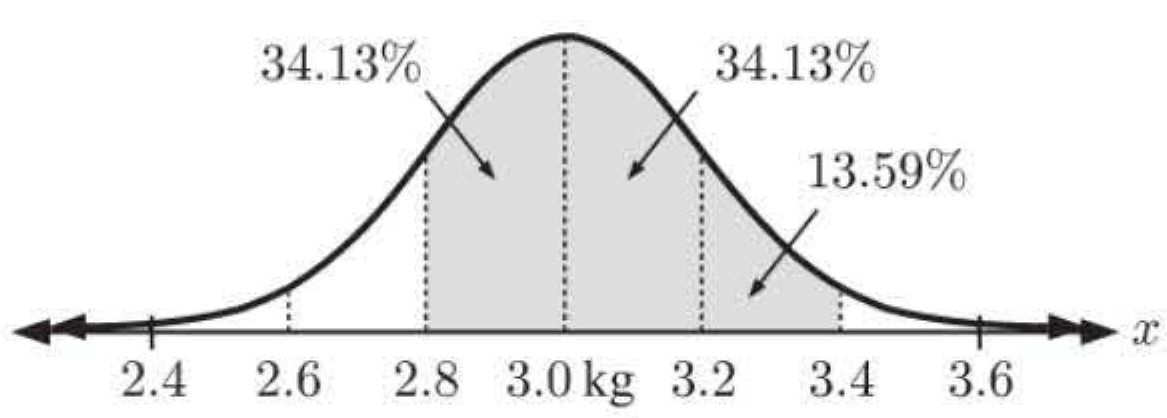


$0.13\% + 2.15\% + 13.59\% + 34.13\% + 34.13\% = 84.13\%$ of babies born weighed less than 3.2 kg, and

84.13% of $545 = 458.5085$

So, 459 babies born weighed less than 3.2 kg.

b

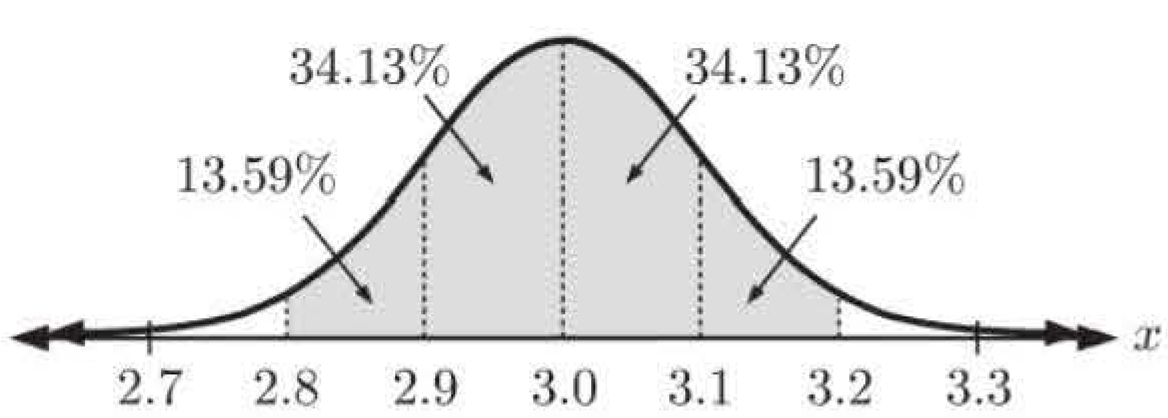


$34.13\% + 34.13\% + 13.59\% = 81.85\%$ of babies born weighed between 2.8 kg and 3.4 kg, and

81.85% of $545 = 446.0825$

So, 446 babies born weighed between 2.8 kg and 3.4 kg.

8



a P(value is within 2 standard deviations of the mean)

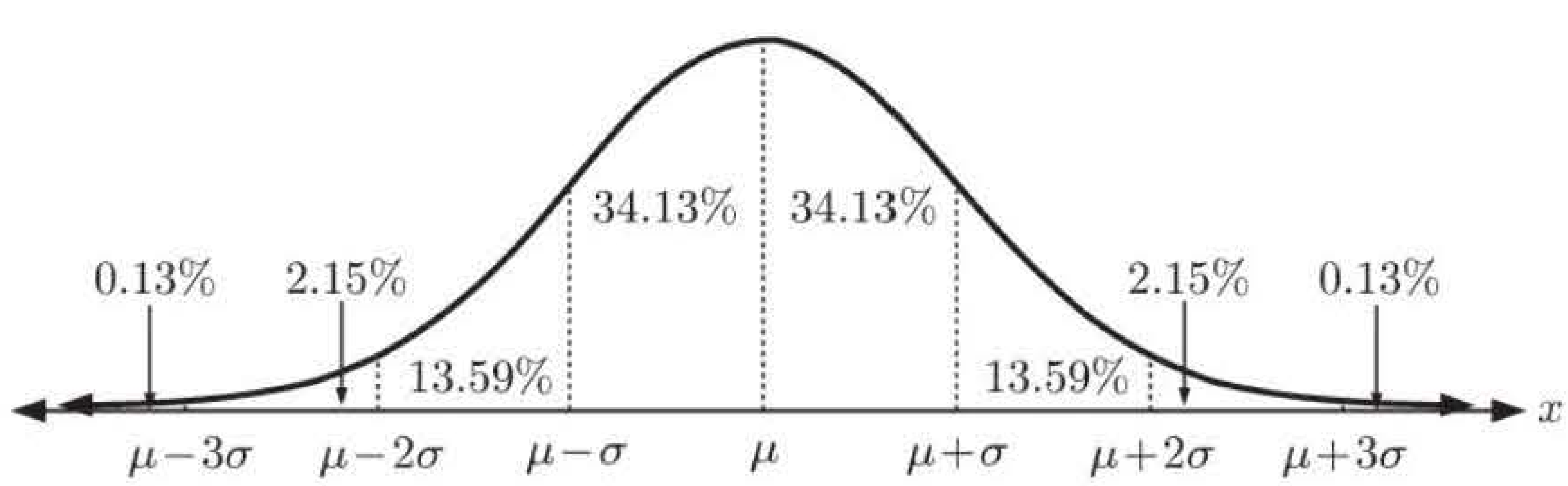
$= \text{P}(2.8 \leqslant X \leqslant 3.2)$

$\approx 13.59\% + 34.13\% + 34.13\% + 13.59\%$

≈ 0.954

b The value 1 standard deviation below the mean is $X = 3 - 0.1 = 2.9$

9 **a**



84% of the crop weigh more than 152 g $\therefore\ \mu - \sigma = 152$

16% of the crop weigh more than 200 g $\therefore\ \mu + \sigma = 200$ .... (1)

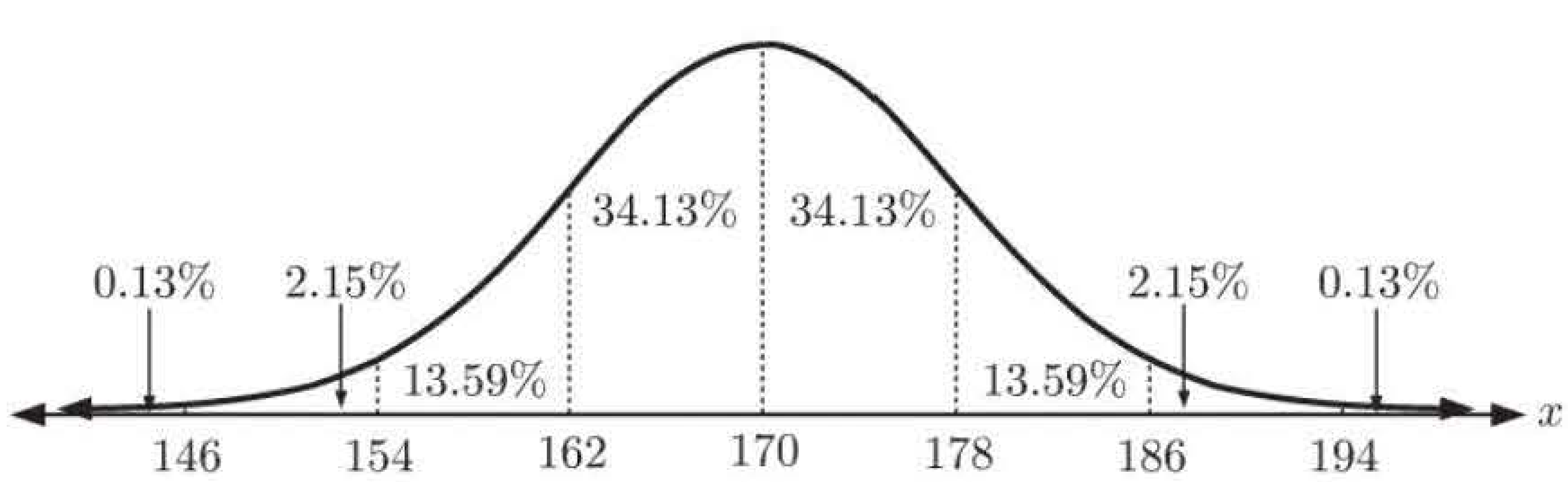
Adding: $2\mu = 352$, and so $\mu = 176$ g

Substituting $\mu = 176$ into (1) gives $\sigma = 200 - \mu = 24$ g.

b For $\mu = 176$ g and $\sigma = 24$ g, $152 \text{ g} = \mu - \sigma$, and $224 \text{ g} = \mu + 2\sigma$.

$\therefore$ between 152 g and 224 g, the percentage is $34.13\% + 34.13\% + 13.59\% \approx 81.9\%$

10



a **i** $P(162 < X < 170) \approx 34.1\%$

ii $P(170 < X < 186) \approx 34.13\% + 13.59\%$
$\approx 47.7\%$

b **i** $P(178 < X < 186)$
$\approx 13.59\%$
≈ 0.136

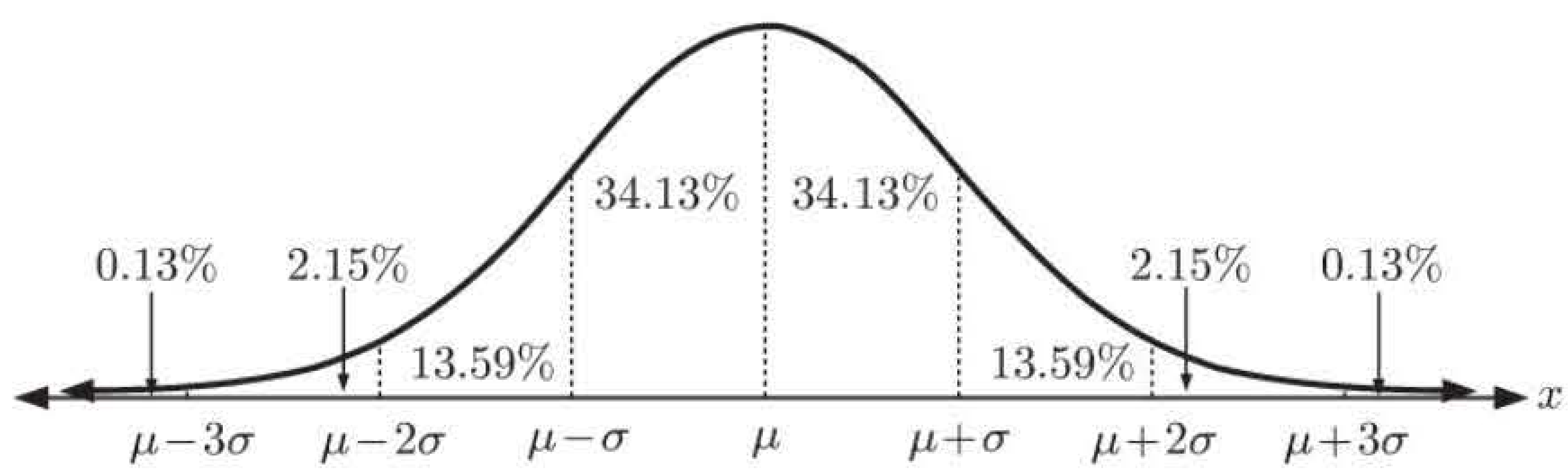
ii $P(X < 162)$
$\approx 1 - (0.5 + 0.3413)$
≈ 0.159

iii $P(X < 154)$
$\approx 0.0215 + 0.0013$
≈ 0.0228

iv $P(X > 162)$
$\approx 1 - 0.159$ {using **b ii**}
≈ 0.841

c 16% of students are taller than 178 cm $\{13.59\% + 2.15\% + 0.13\% \approx 16\%\}$
$\therefore\ k = 178$

11

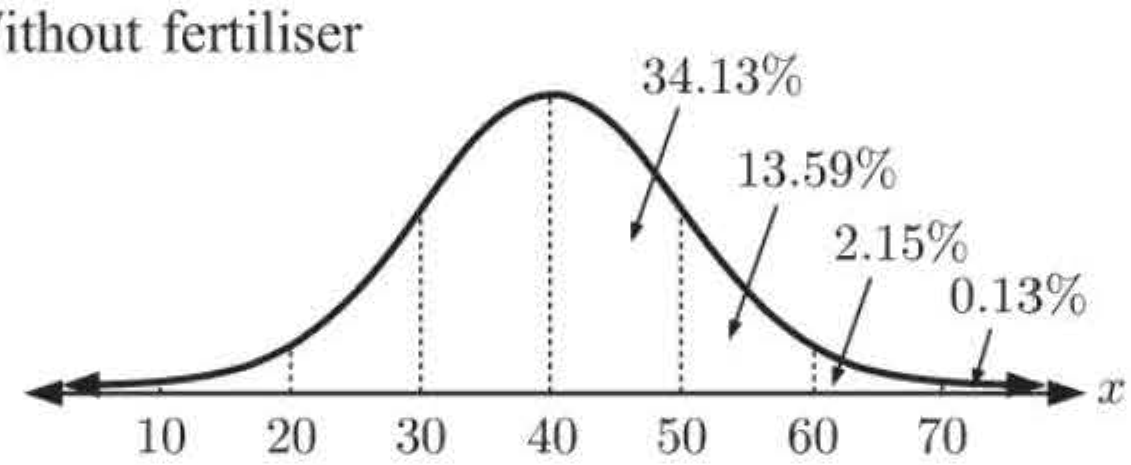


a 97.72% of 13 year old boys are taller than 131 cm $\therefore\ \mu - 2\sigma = 131$
2.28% of 13 year old boys are taller than 179 cm $\therefore\ \mu + 2\sigma = 179$ .... (1)
Adding: $2\mu = 310$, and so $\mu = 155$ cm

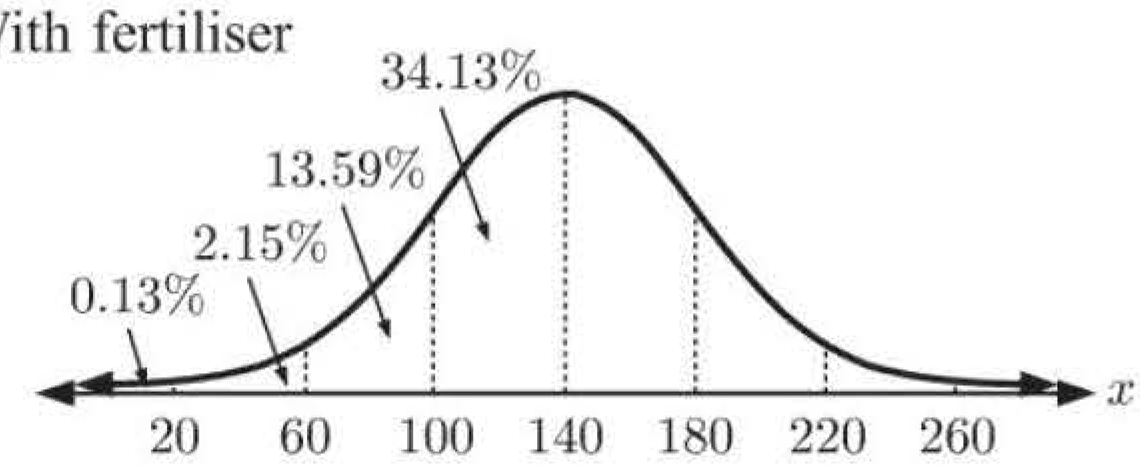
Substituting $\mu = 155$ cm into (1) gives $\sigma = \dfrac{179 - \mu}{2} = \dfrac{24}{2} = 12$ cm

b For $\mu = 155$ cm and $\sigma = 12$ cm, $143 \text{ cm} = \mu - \sigma$, and $191 \text{ cm} = \mu + 3\sigma$
$\therefore$ between 143 cm and 191 cm, the percentage is $34.13\% + 34.13\% + 13.59\% + 2.15\% \approx 84.0\%$
So the probability is 0.84.

12 Without fertiliser



With fertiliser



a P(without and < 50)
$\approx 50\% + 34.13\%$
$\approx 84.1\%$

b P(with and < 60)
$\approx 0.13\% + 2.15\%$
$\approx 2.28\%$

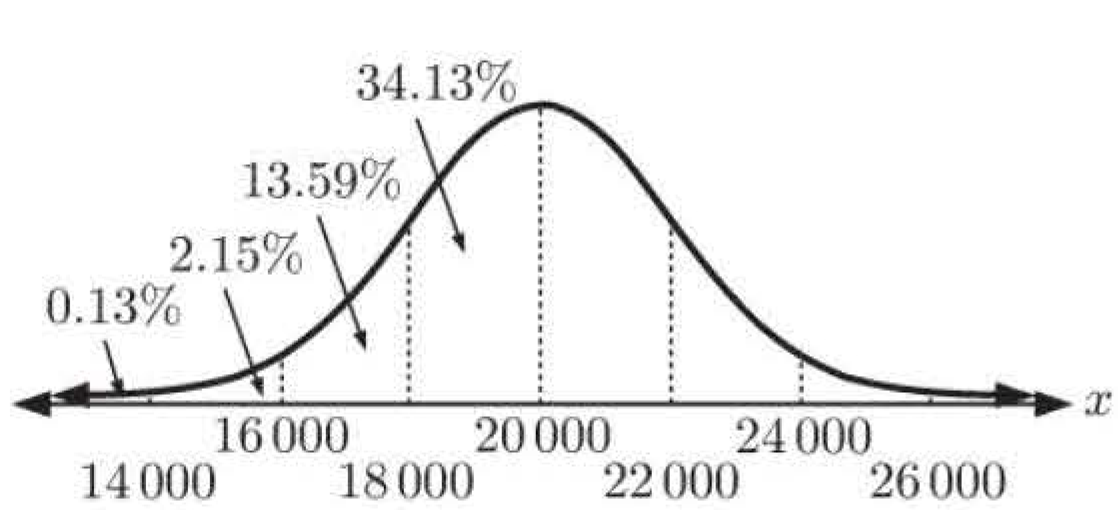
c **i** P(with and $20 \leqslant X \leqslant 60$)
$\approx 2.15\%$

ii P(without and $20 \leqslant X \leqslant 60$)
$\approx 2(34.13\% + 13.59\%)$
$\approx 95.4\%$

d **i** P(with and $X \geqslant 60$)
$\approx 13.59\% + 34.13\% + 50\%$
$\approx 97.7\%$

ii P(without and $X \geqslant 60$)
$\approx 2.15\% + 0.13\%$
$\approx 2.28\%$

13



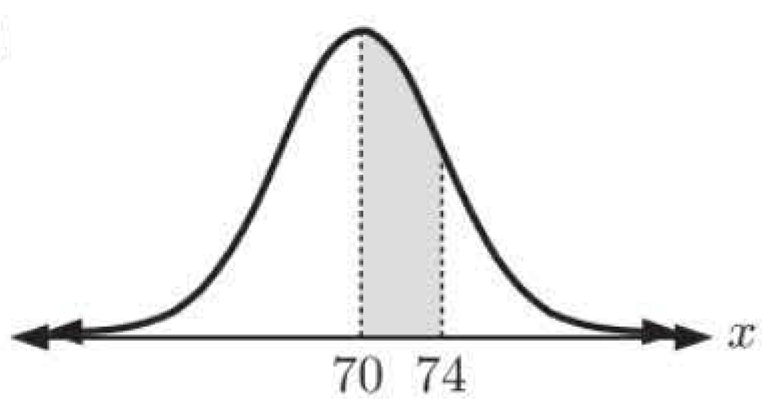
a $P(X < 18\,000)$
$\approx 1 - 0.5 - 0.3413$
≈ 0.1587
$\therefore$ we expect that less than 18 000 bottles are filled on $260 \times 0.1587 \approx 41$ days.

b $\mathrm{P}(X > 16\,000)$
$\approx 0.1359 + 0.3413 + 0.5$
≈ 0.9772
$\therefore$ we expect that over 16 000 bottles are filled on $260 \times 0.9772 \approx 254$ days.

c $\mathrm{P}(18\,000 \leqslant X \leqslant 24\,000)$
$\approx 0.3413 \times 2 + 0.1359$
≈ 0.8185
$\therefore$ we expect that between 18 000 and 24 000 bottles are filled on 260×0.8185
≈ 213 days.

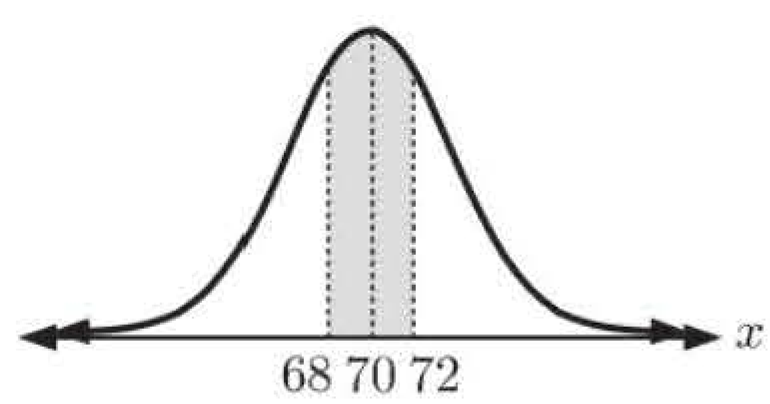
EXERCISE 26C

1 **a**



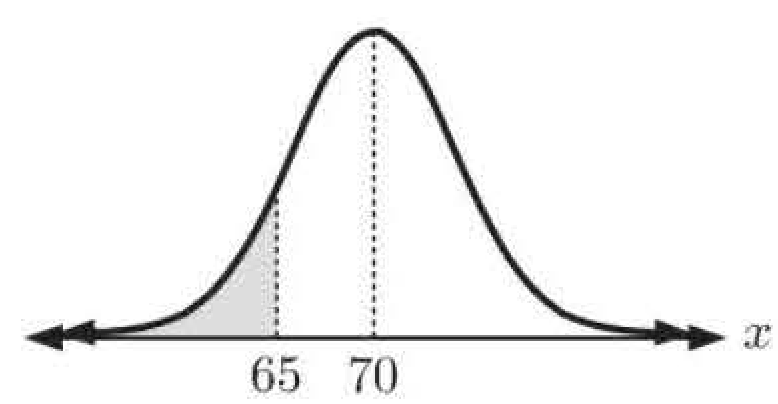
$\mathrm{P}(70 \leqslant X \leqslant 74) \approx 0.341$

b



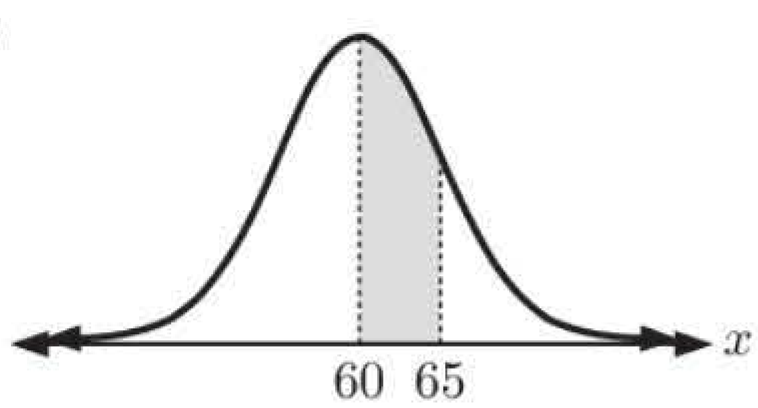
$\mathrm{P}(68 \leqslant X \leqslant 72) \approx 0.383$

c



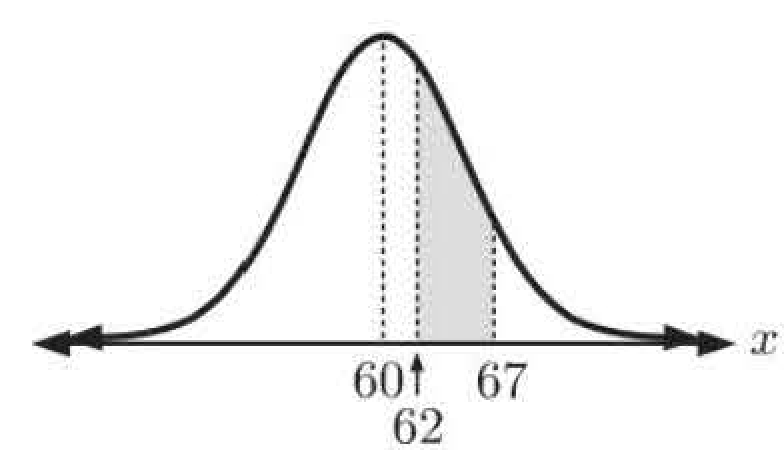
$\mathrm{P}(X \leqslant 65) \approx 0.106$

2 **a**



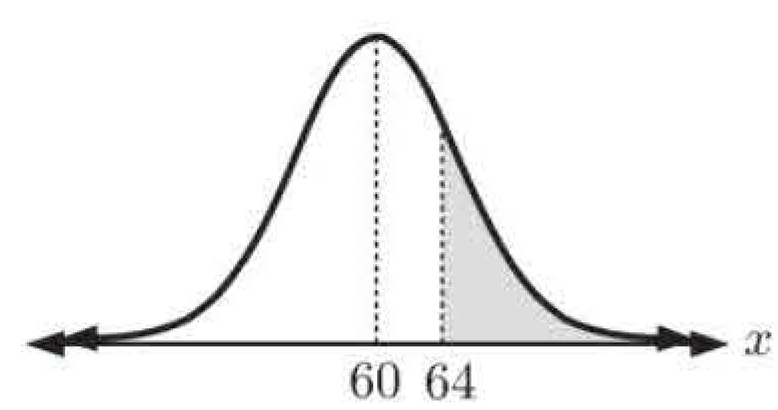
$\mathrm{P}(60 \leqslant X \leqslant 65) \approx 0.341$

b



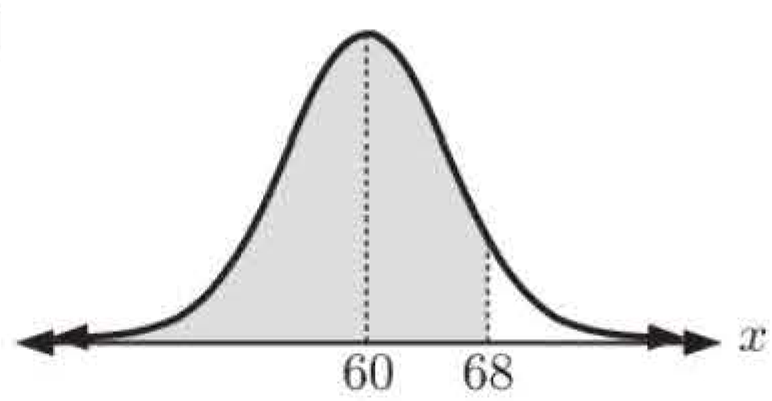
$\mathrm{P}(62 \leqslant X \leqslant 67) \approx 0.264$

c



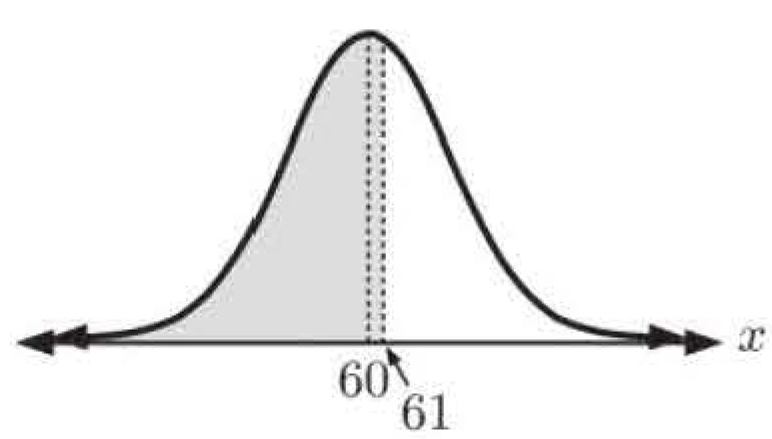
$\mathrm{P}(X \geqslant 64) \approx 0.212$

d



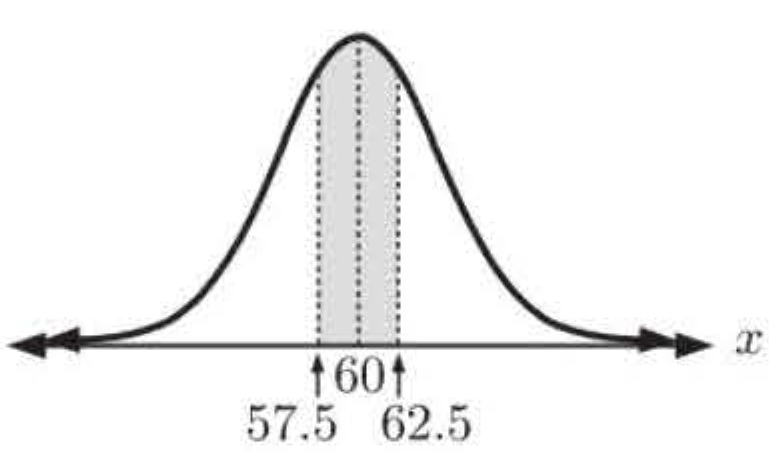
$\mathrm{P}(X \leqslant 68) \approx 0.945$

e



$\mathrm{P}(X \leqslant 61) \approx 0.579$

f



$\mathrm{P}(57.5 \leqslant X \leqslant 62.5) \approx 0.383$

3 If X is the length of a bolt in cm, then X is normally distributed with $\mu = 19.8$ and $\sigma = 0.3$.
$\therefore$ $\mathrm{P}(19.7 < X < 20) \approx 0.378$

4 If X is the money collected in dollars, then X is normally distributed with $\mu = 40$ and $\sigma = 6$.

a $\mathrm{P}(30.00 < X < 50.00) \approx 0.904$
$\approx 90.4\%$

b $\mathrm{P}(X \geqslant 50) \approx 0.0478$
$\approx 4.78\%$

5 If X is the length of an eel in cm, then X is normally distributed with $\mu = 41$ and $\sigma = \sqrt{11}$.

a $\mathrm{P}(X \geqslant 50) \approx 0.003\,33$

b $\mathrm{P}(40 \leqslant X \leqslant 50) \approx 0.615$
$\approx 61.5\%$

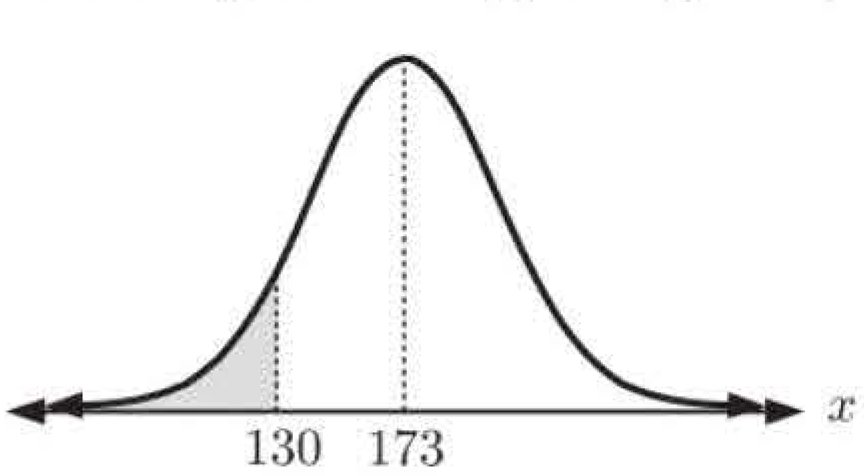
c $\mathrm{P}(X \geqslant 45) \approx 0.114$
So, we would expect $200 \times 0.114 \approx 23$ eels to be at least 45 cm long.

6 If X is the speed of a car in $\mathrm{km\,h^{-1}}$ then X is normally distributed with $\mu = 56.3$ and $\sigma = 7.4$.

a $\mathrm{P}(60 < X < 75) \approx 0.303$ **b** $\mathrm{P}(X \leqslant 70) \approx 0.968$ **c** $\mathrm{P}(X \geqslant 60) \approx 0.309$

7 If X is the weight of an apple in grams, then X is normally distributed with $\mu = 173$ and $\sigma = 34$.

a



$\mathrm{P}(X < 130) \approx 0.102\,988\,39$
≈ 0.103

So, 10.3% of the apples from this crop were too small to sell.

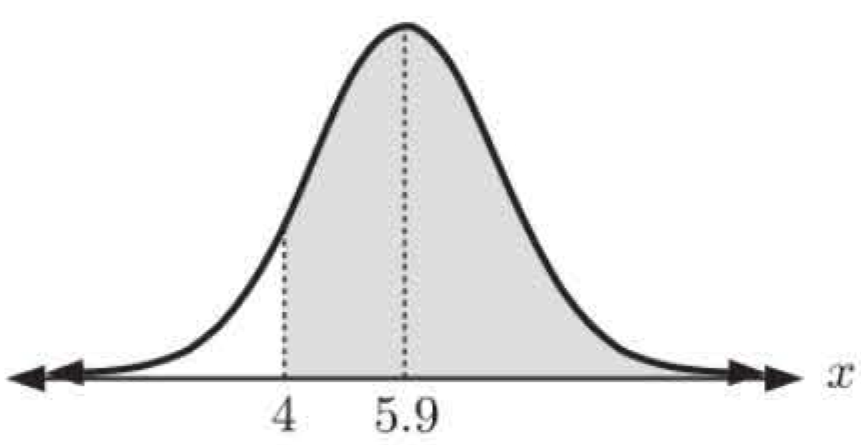
b The chance of one apple being too small to sell is 0.102 988 39.

$\therefore$ the distribution is B(100, 0.102 988 39)

$\therefore$ $P(X \leqslant 10) \approx 0.544$

So, the probability that up to 10 apples were too small to sell is 0.544.

8



If X is the drop in blood pressure (in units) then X is normally distributed with $\mu = 5.9$ and $\sigma = 1.9$

a $P(X \geqslant 4) \approx 0.841\,344\,74$

So, 84.1% of people show a drop of more than 4 units.

b The chance of one person showing a drop of more than 4 units is 0.841 344 74.

$\therefore$ the distribution is B(8, 0.841 344 74)

$$\therefore\ P(X > 5) = P(X \geqslant 6) \approx 0.880$$

So, the probability that more than 5 people show a drop of more than 4 units is 0.880.

EXERCISE 26D

1 **a** For English, $z\text{-score} = \dfrac{48-40}{4.4} \approx 1.82$

For Mandarin, $z\text{-score} = \dfrac{81-60}{9} \approx 2.33$

For Geography, $z\text{-score} = \dfrac{84-55}{18} \approx 1.61$

For Biology, $z\text{-score} = \dfrac{68-50}{20} = 0.9$

For Maths, $z\text{-score} = \dfrac{84-50}{15} \approx 2.27$

b Mandarin, Maths, English, Geography, Biology

2 **a** For Physics, $Z = \dfrac{73-78}{10.8} \approx -0.463$

For Chemistry, $Z = \dfrac{77-72}{11.6} \approx 0.431$

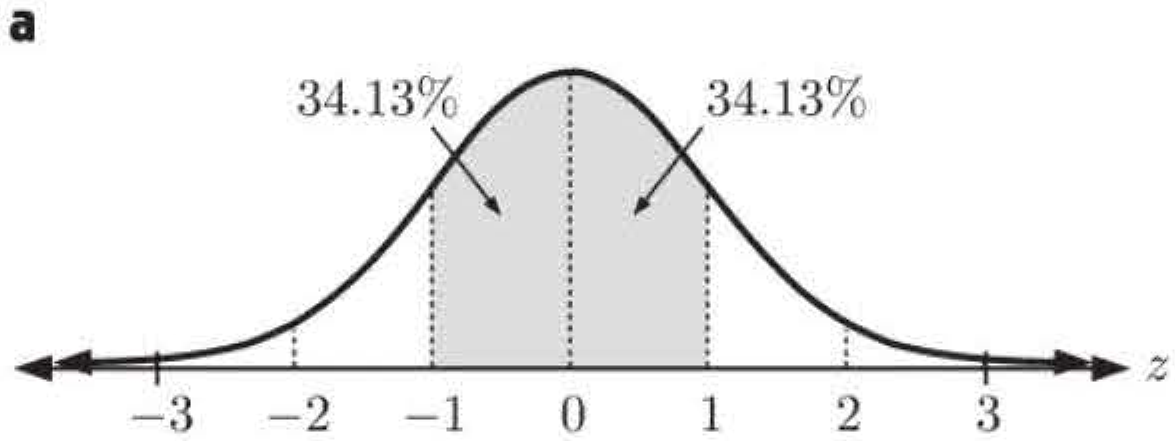
For Maths, $Z = \dfrac{76-74}{10.1} \approx 0.198$

For German, $Z = \dfrac{91-86}{9.6} \approx 0.521$

For Biology, $Z = \dfrac{58-62}{5.2} \approx -0.769$

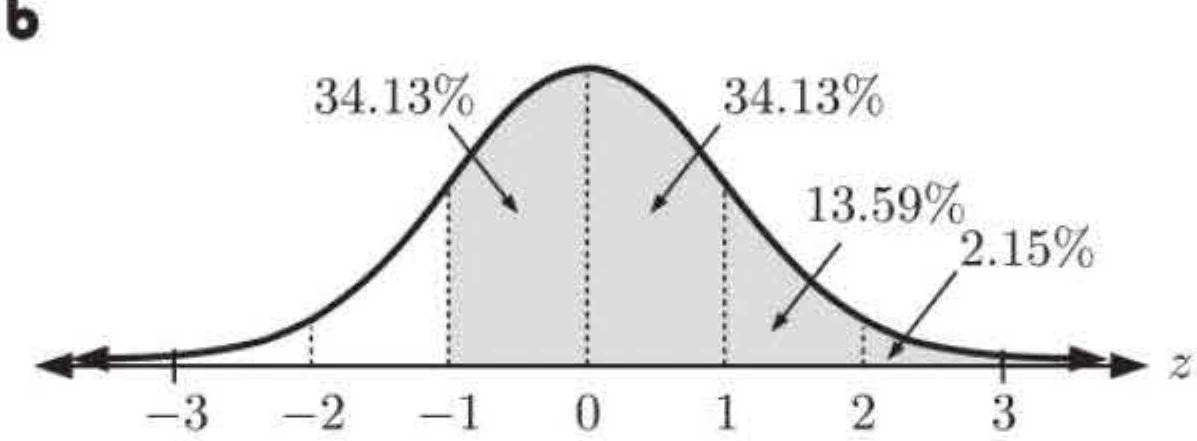
b German, Chemistry, Maths, Physics, Biology

3 **a**



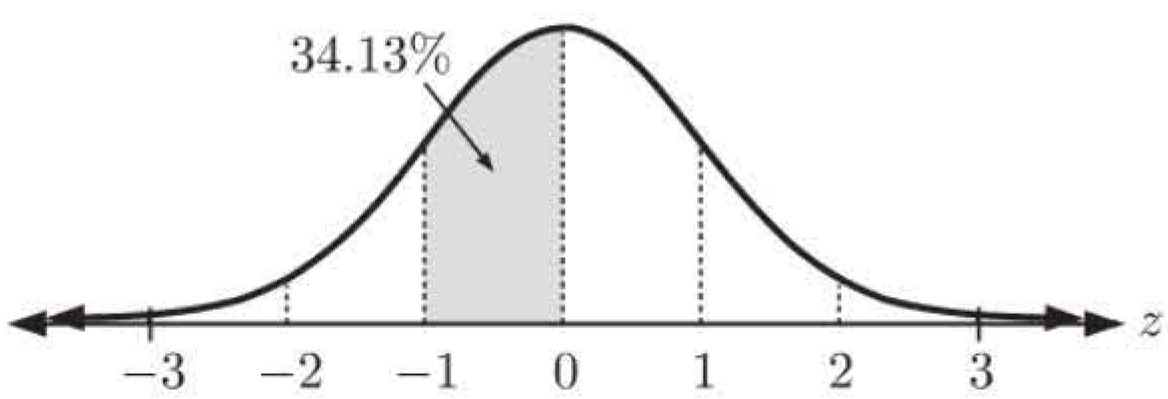
$$\therefore\ P(-1 < Z < 1) \approx 34.13\% + 34.13\% \approx 0.683$$

b



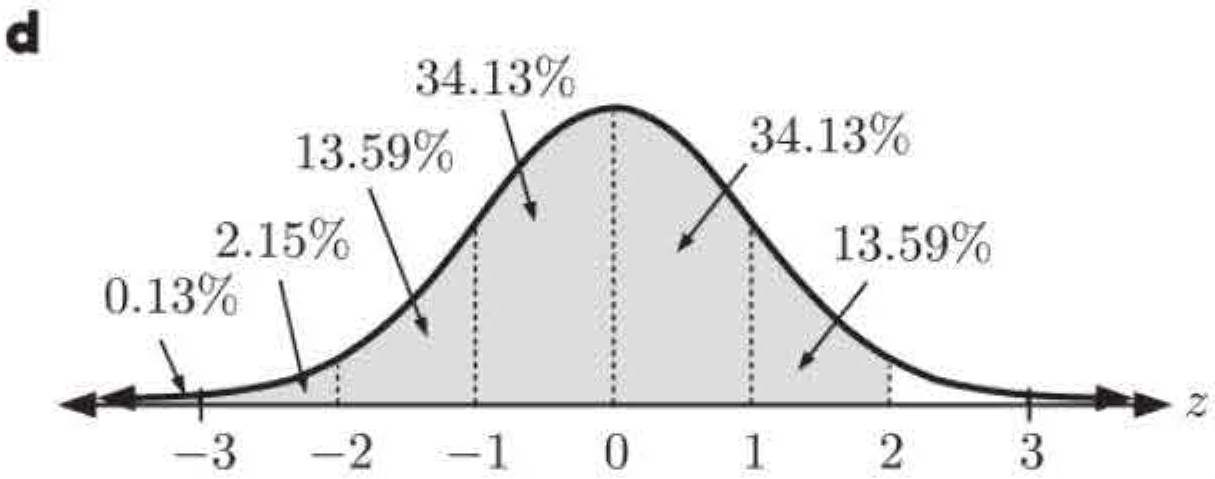
$$\therefore\ P(-1 \leqslant Z \leqslant 3) \approx 34.13\% + 34.13\% + 13.59\% + 2.15\% \approx 0.84$$

c



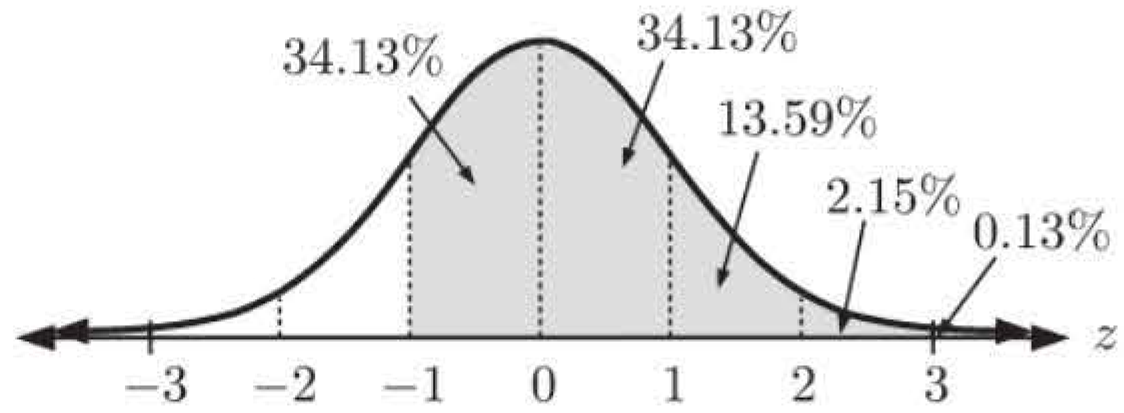
$\therefore\ \mathrm{P}(-1 < Z < 0) \approx 34.13\%$
≈ 0.341

d



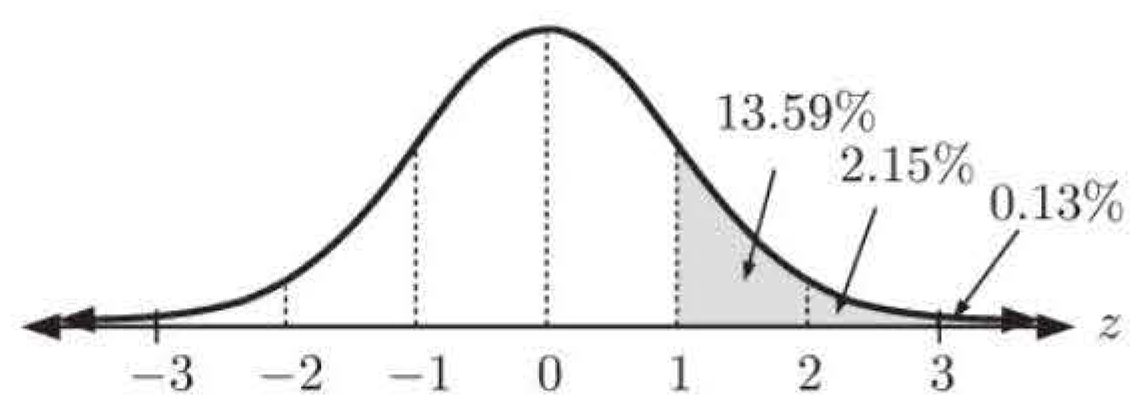
$\therefore\ \mathrm{P}(Z < 2)$
$\approx 0.13\% + 2.15\% + 13.59\% + 34.13\%$
$+ 34.13\% + 13.59\%$
$\approx 97.72\%$
≈ 0.977

e



$\therefore\ \mathrm{P}(-1 < Z)$
$= \mathrm{P}(Z > -1)$
$\approx 34.13\% + 34.13\% + 13.59\% + 2.15\%$
$+ 0.13\%$
≈ 0.841

f



$\therefore\ \mathrm{P}(Z \geqslant 1)$
$\approx 13.59\% + 2.15\% + 0.13\%$
≈ 0.159

4 **a**
$$\begin{aligned}\mathrm{E}\left(\frac{X-\mu}{\sigma}\right) &= \mathrm{E}\left(\frac{1}{\sigma}X - \frac{\mu}{\sigma}\right)\\ &= \frac{1}{\sigma}\,\mathrm{E}(X) - \frac{\mu}{\sigma}\\ &= \frac{1}{\sigma}\mu - \frac{\mu}{\sigma}\\ &= 0\end{aligned}$$

b
$$\begin{aligned}\mathrm{Var}\left(\frac{X-\mu}{\sigma}\right) &= \mathrm{Var}\left(\frac{1}{\sigma}X - \frac{\mu}{\sigma}\right)\\ &= \left(\frac{1}{\sigma}\right)^2 \mathrm{Var}(X)\\ &= \frac{1}{\sigma^2} \times \sigma^2\\ &= 1\end{aligned}$$

5 **a** If $\mathrm{P}(\mu - \sigma < X < \mu + 2\sigma) = \mathrm{P}(a < Z < b)$

then $\dfrac{(\mu-\sigma)-\mu}{\sigma} = a$ and $\dfrac{(\mu+2\sigma)-\mu}{\sigma} = b$

$\therefore\ a = \dfrac{-\sigma}{\sigma} = -1$ $\qquad \therefore\ b = \dfrac{2\sigma}{\sigma} = 2$

$\therefore\ a = -1,\quad b = 2$

b If $\mathrm{P}(\mu - 0.5\sigma < X < \mu) = \mathrm{P}(a < Z < b)$

then $\dfrac{(\mu-0.5\sigma)-\mu}{\sigma} = a$ and $\dfrac{\mu-\mu}{\sigma} = b$

$\therefore\ a = \dfrac{-0.5\sigma}{\sigma} = -0.5$ $\qquad \therefore\ b = 0$

$\therefore\ a = -0.5,\quad b = 0$

c If $P(0 \leqslant Z \leqslant 3) = P(\mu - a\sigma \leqslant X \leqslant \mu + b\sigma)$

then $\dfrac{(\mu - a\sigma) - \mu}{\sigma} = 0$ and $\dfrac{(\mu + b\sigma) - \mu}{\sigma} = 3$

$\therefore\ \mu - a\sigma - \mu = 0$ $\qquad \therefore\ \mu + b\sigma - \mu = 3\sigma$

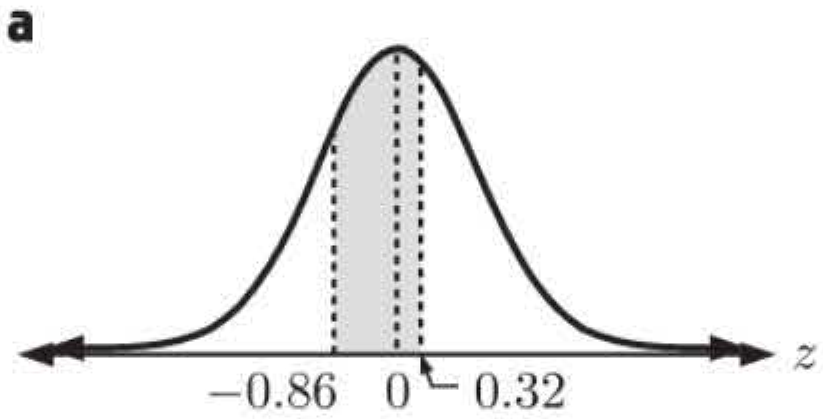
$\therefore\ -a\sigma = 0$ $\qquad \therefore\ b\sigma = 3\sigma$

$\therefore\ a = 0$ $\qquad \therefore\ b = 3$

$\therefore\ a = 0,\quad b = 3$

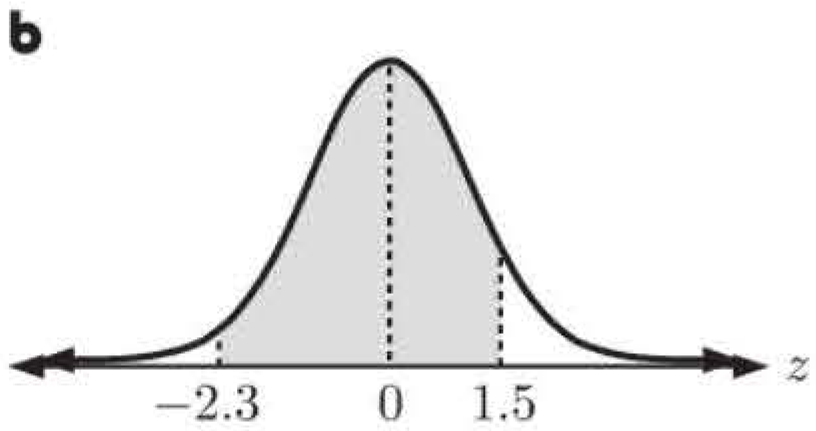
6 $Z \sim N(0,\ 1)$

a



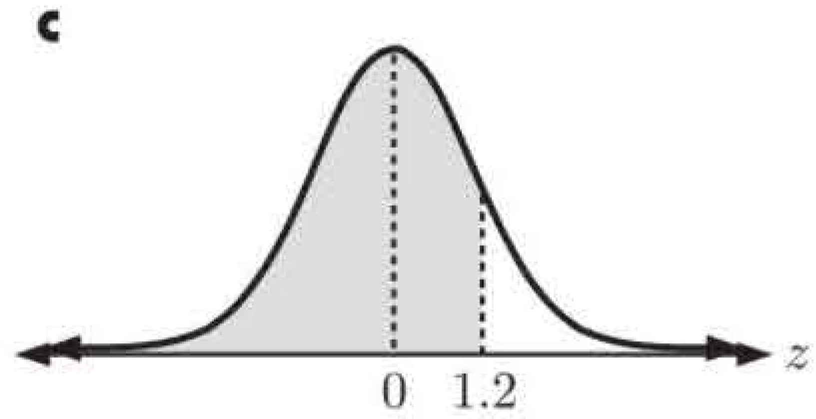
$\therefore\ P(-0.86 \leqslant Z \leqslant 0.32)$
≈ 0.431

b



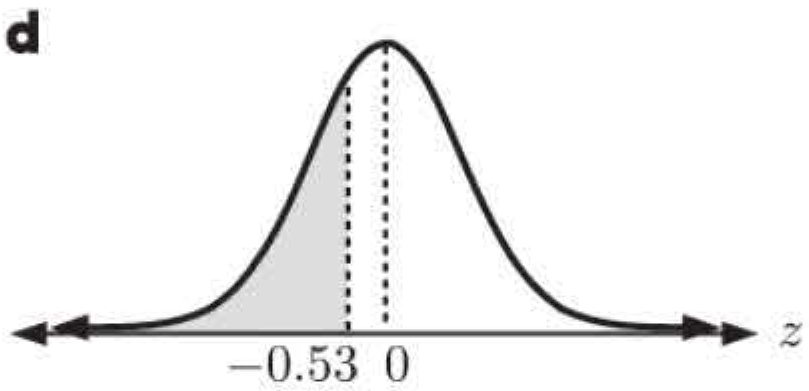
$\therefore\ P(-2.3 \leqslant Z \leqslant 1.5)$
≈ 0.922

c



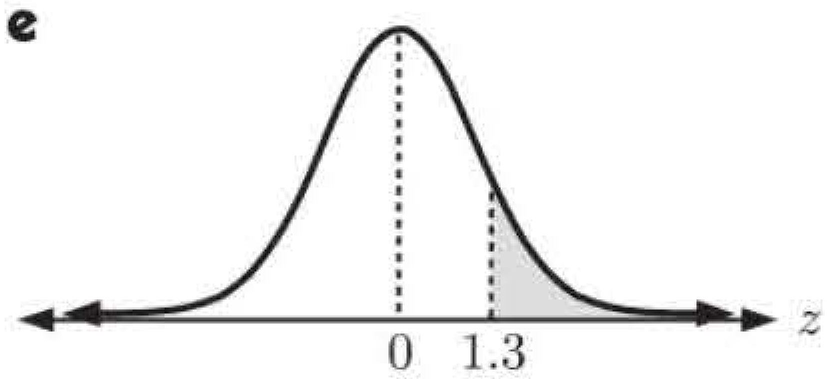
$\therefore\ P(Z \leqslant 1.2)$
≈ 0.885

d



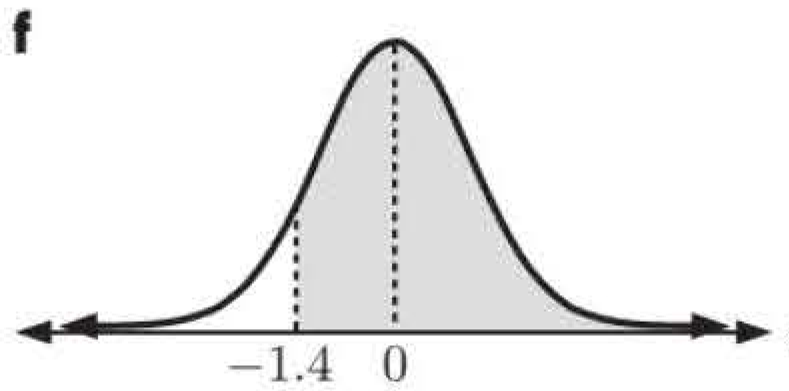
$\therefore\ P(Z \leqslant -0.53)$
≈ 0.298

e



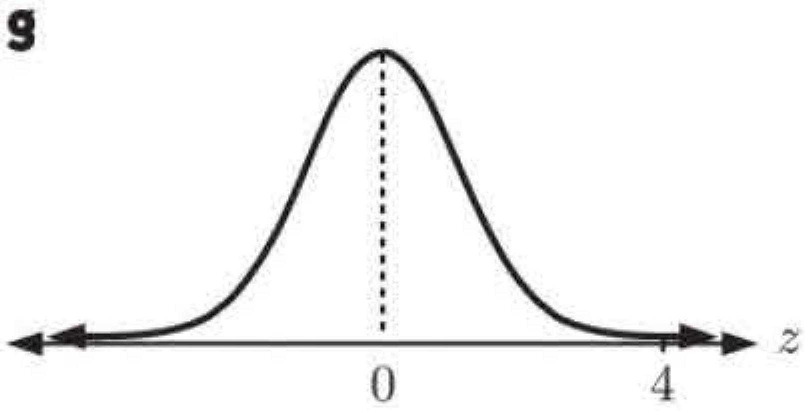
$\therefore\ P(Z \geqslant 1.3)$
≈ 0.0968

f



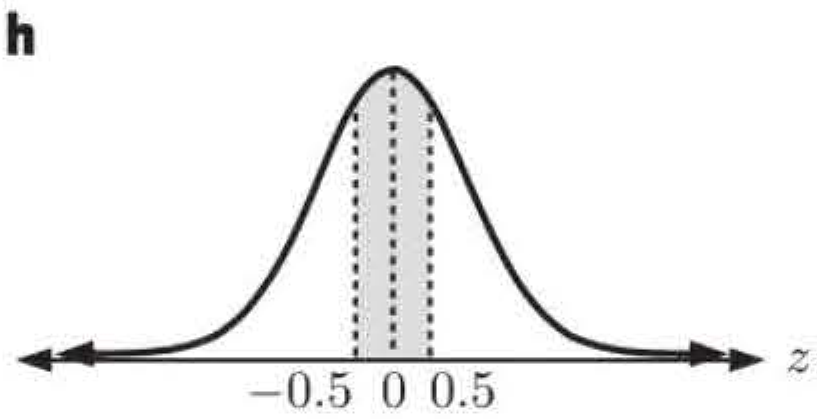
$\therefore\ P(Z \geqslant -1.4)$
≈ 0.919

g



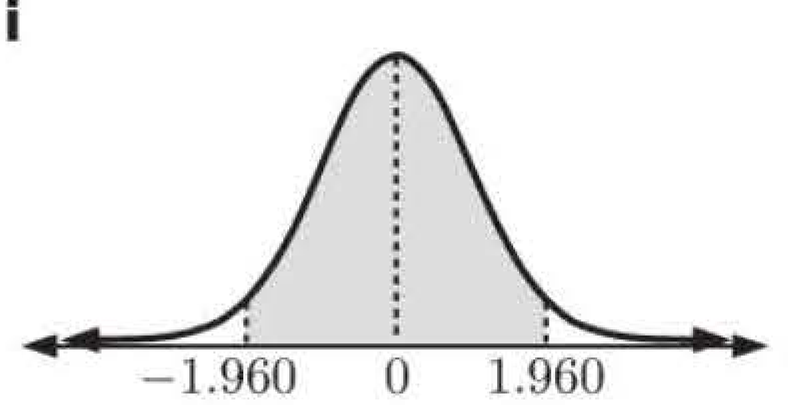
$\therefore\ P(Z > 4)$
$\approx 0.000\,031\,7$
(3.17×10^{-5})

h



$\therefore\ P(-0.5 < Z < 0.5)$
≈ 0.383

i



$\therefore\ P(-1.960 \leqslant Z \leqslant 1.960)$
≈ 0.950

7 **a** **i**

$$z_1 = \frac{50.6 - 58.3}{8.96}$$
$$\therefore\ z_1 \approx -0.859\,375$$
$$\approx -0.859$$
$$z_2 = \frac{68.9 - 58.3}{8.96}$$
$$\approx 1.183\,035\,714$$
$$\approx 1.18$$

ii $Z \sim N(0,\ 1)$

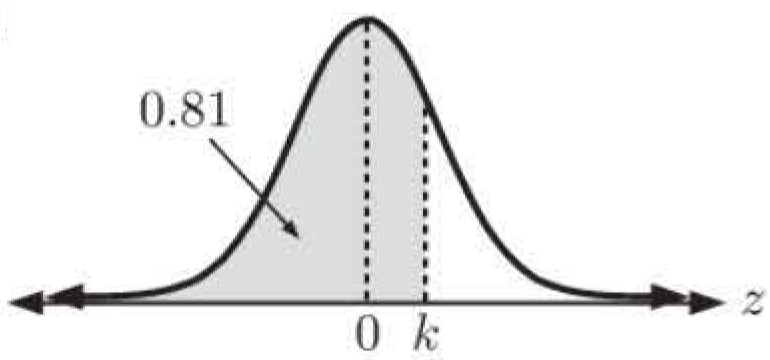
$P(-0.859\,375 \leqslant Z \leqslant 1.183\,035\,714)$
$\approx 0.686\,535\,67$
≈ 0.687

b $X \sim N(58.3,\ 8.96^2)$

$\therefore\ P(50.6 \leqslant X \leqslant 68.9) \approx 0.687$ ✓

EXERCISE 26E.1

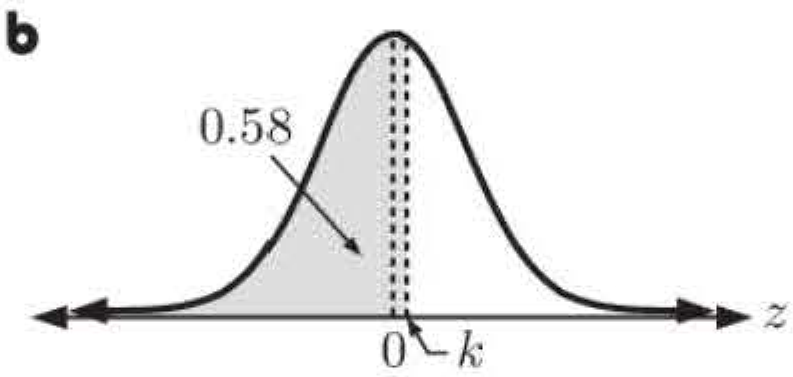
1 **a**



$\mathrm{P}(Z \leqslant k) = 0.81$

$\therefore \; k \approx 0.878$

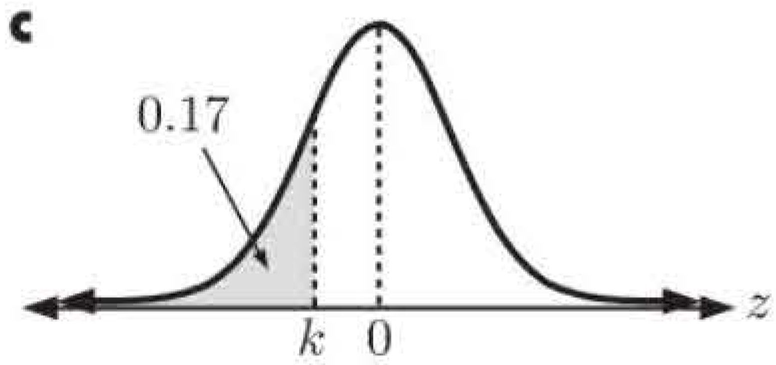
b



$\mathrm{P}(Z \leqslant k) = 0.58$

$\therefore \; k \approx 0.202$

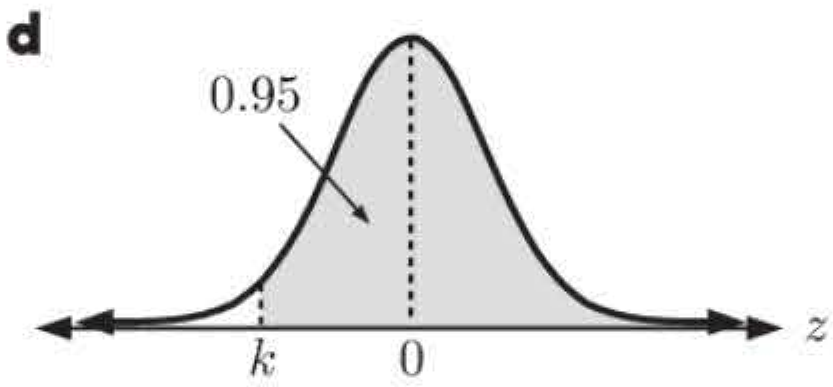
c



$\mathrm{P}(Z \leqslant k) = 0.17$

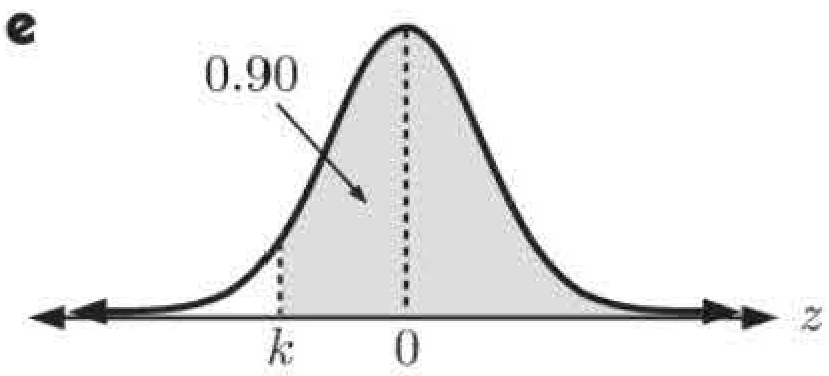
$\therefore \; k \approx -0.954$

d



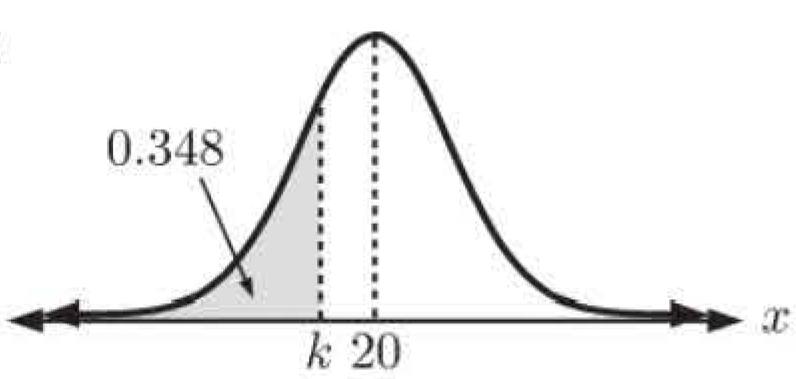
$$\begin{aligned} \mathrm{P}(Z \geqslant k) &= 0.95 \\ \therefore \; \mathrm{P}(Z \leqslant k) &= 1 - 0.95 \\ &= 0.05 \\ \therefore \; k &\approx -1.64 \end{aligned}$$

e



$$\begin{aligned} \mathrm{P}(Z \geqslant k) &= 0.90 \\ \therefore \; \mathrm{P}(Z \leqslant k) &= 1 - 0.90 \\ &= 0.1 \\ \therefore \; k &\approx -1.28 \end{aligned}$$

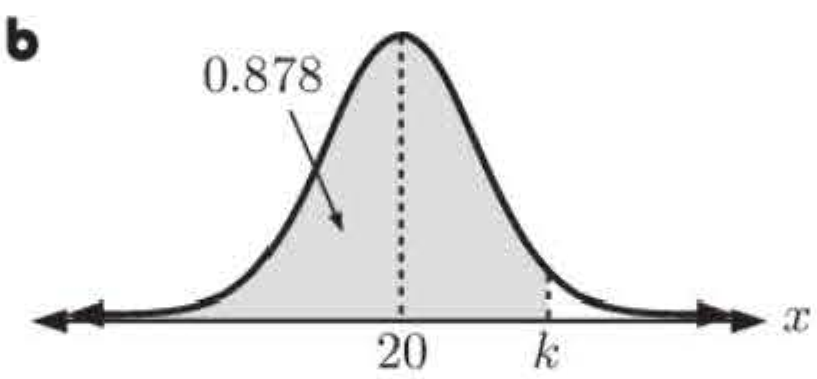
2 **a**



$\mathrm{P}(X \leqslant k) = 0.348$

$\therefore \; k \approx 18.8$

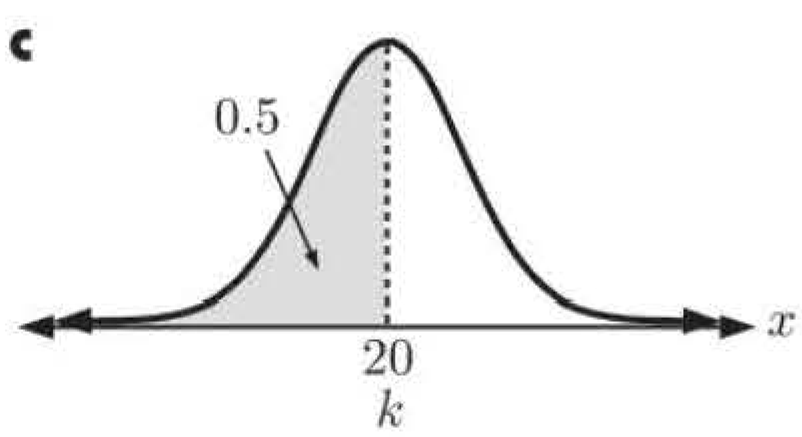
b



$\mathrm{P}(X \leqslant k) = 0.878$

$\therefore \; k \approx 23.5$

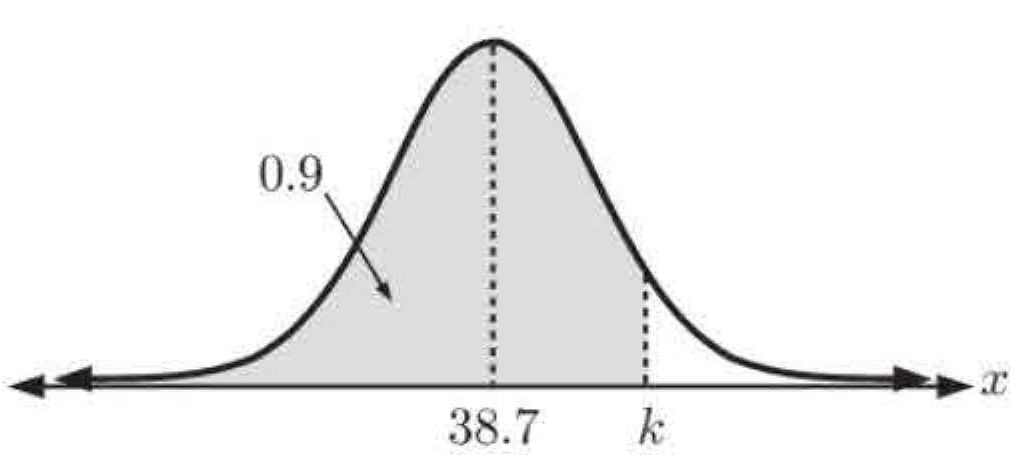
c



$\mathrm{P}(X \leqslant k) = 0.5$

$\therefore \; k = 20$

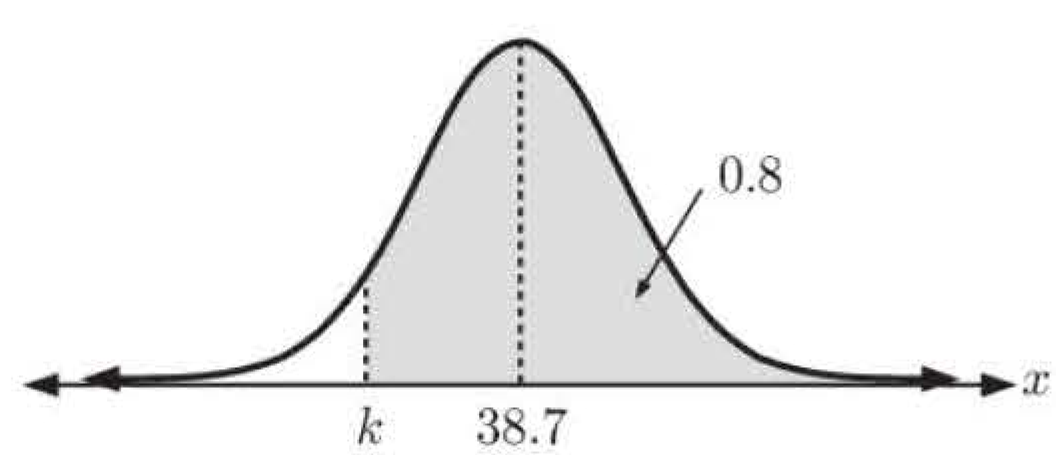
3 **a**



$\mathrm{P}(X \leqslant k) = 0.9$

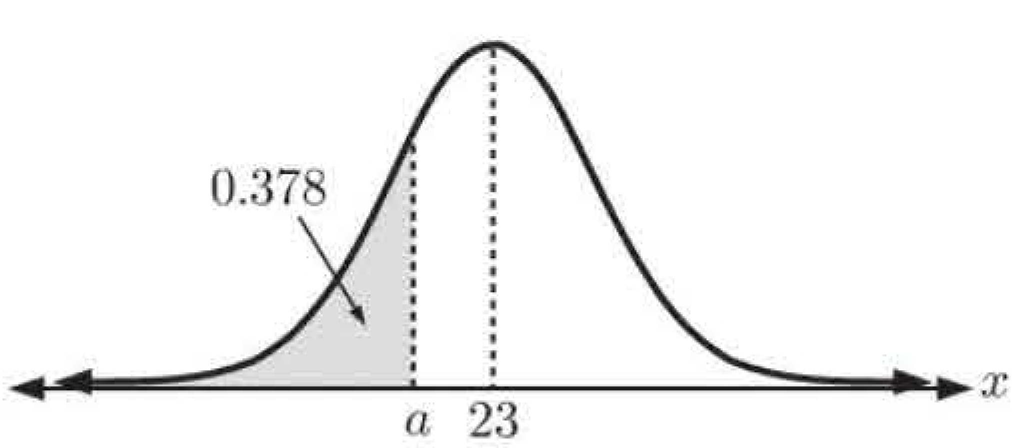
$\therefore \; k \approx 49.2$

b



$$\begin{aligned} \mathrm{P}(X \geqslant k) &= 0.8 \\ \therefore \; \mathrm{P}(X \leqslant k) &= 1 - 0.8 = 0.2 \\ \therefore \; k &\approx 31.8 \end{aligned}$$

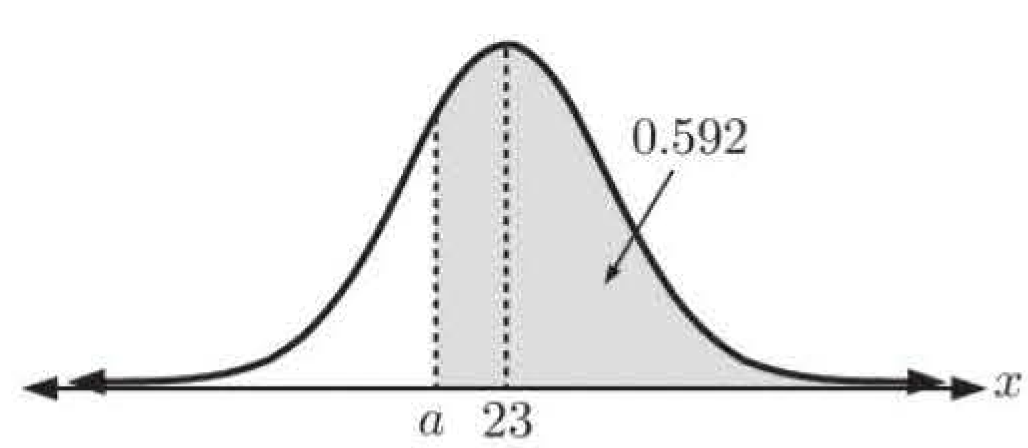
4 **a**



$\mathrm{P}(X < a) = 0.378$

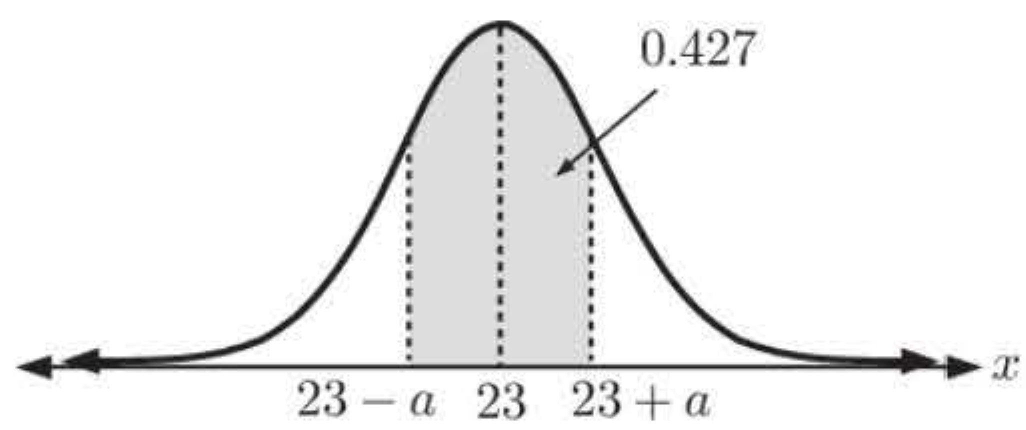
$\therefore \; a \approx 21.4$

b



$$\begin{aligned} \mathrm{P}(X \geqslant a) &= 0.592 \\ \therefore \; \mathrm{P}(X \leqslant a) &= 1 - 0.592 \\ &= 0.408 \\ \therefore \; a &\approx 21.8 \end{aligned}$$

c



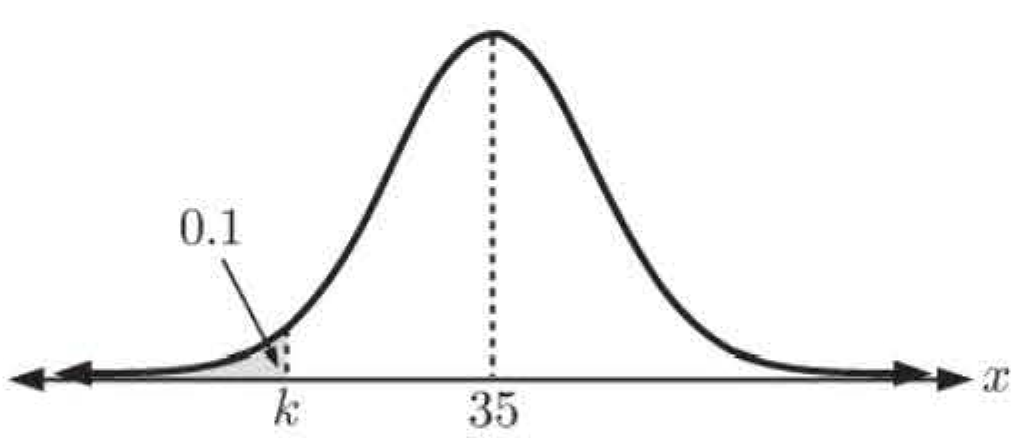
$$\begin{aligned} \text{P}(23 - a < X < 23 + a) &= 0.427 \\ \therefore\ 1 - 2 \times \text{P}(X \leqslant 23 - a) &= 0.427 \\ \therefore\ -2 \times \text{P}(X \leqslant 23 - a) &= -0.573 \\ \therefore\ \text{P}(X \leqslant 23 - a) &= 0.2865 \\ \therefore\ 23 - a &= 20.181\,806\,2 \\ \therefore\ a &\approx 23 - 20.181\,806\,2 \\ \therefore\ a &\approx 2.82 \end{aligned}$$

5 Let X be the result of the Physics test, so $X \sim \text{N}(46,\, 25^2)$.

We need to find k such that $\text{P}(X \geqslant k) = 0.07$

$$\begin{aligned} \therefore\ 1 - \text{P}(X < k) &= 0.07 \\ \therefore\ \text{P}(X < k) &= 0.93 \\ \therefore\ k &\approx 82.894 \\ \therefore\ k &\approx 82.9 \end{aligned}$$

6



$X \sim \text{N}(35,\, 8^2)$

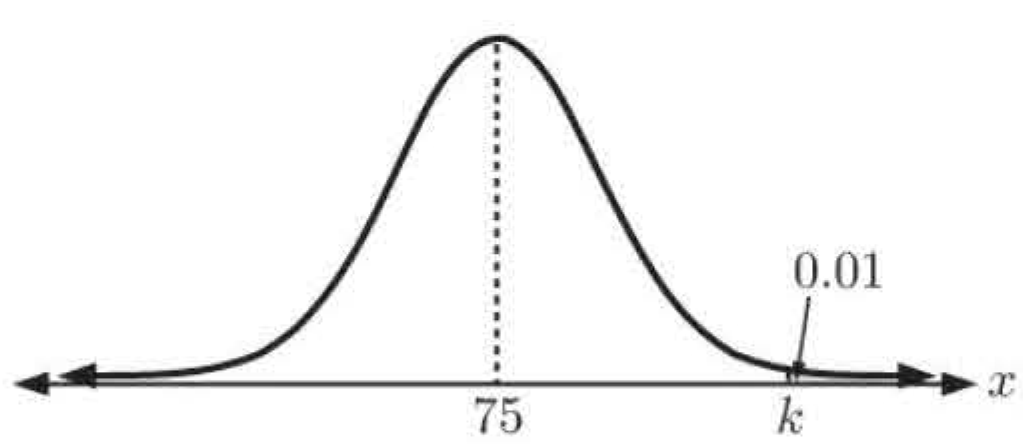
We need to find k such that

$\text{P}(X \leqslant k) = 0.1$

$\therefore\ k \approx 24.747\,587\,5$

So, the length of the smallest fish to be harvested is 24.7 cm.

7



$X \sim \text{N}(75,\, 0.1^2)$

We need to find k such that

$\text{P}(X \geqslant k) = 0.01$

$\therefore\ \text{P}(X \leqslant k) = 0.99$

$\therefore\ k \approx 75.232\,634\,8$

So, the length of the smallest screw to be rejected is 75.2 mm.

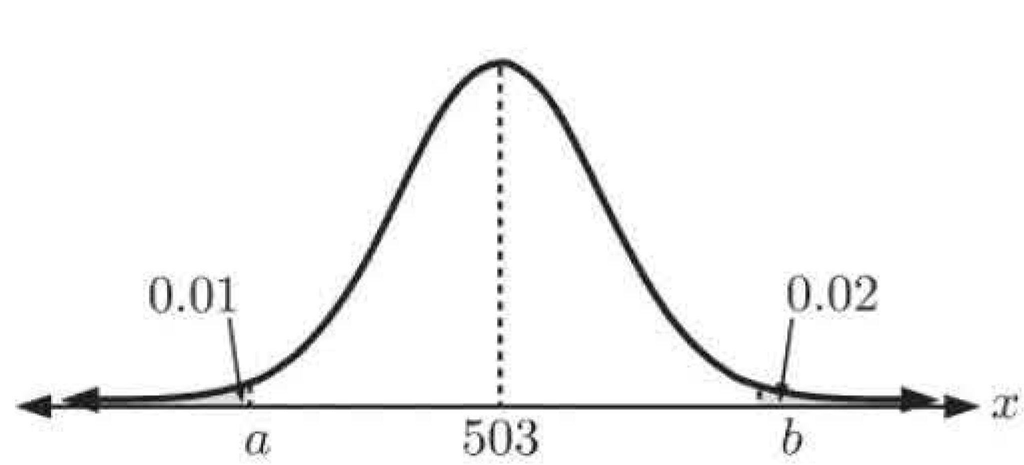
8 Z-score for algebra $= \dfrac{56 - 50.2}{15.8} \approx 0.3671$ $\qquad$ Z-score for geometry $= \dfrac{x - 58.7}{18.7}$

$\therefore$ we need to solve $\dfrac{x - 58.7}{18.7} = 0.3671$

$$\begin{aligned} \therefore\ x - 58.7 &\approx 6.86 \\ \therefore\ x &\approx 65.6 \end{aligned}$$

So, Pedro needs a result of 65.6%.

9



$X \sim \text{N}(503,\, 0.5^2)$

We need to find a such that

$\text{P}(X \leqslant a) = 0.01$

$\therefore\ a \approx 502$

We also need to find b such that $\text{P}(X \geqslant b) = 0.02$

$\therefore\ \text{P}(X \leqslant b) = 0.98$

$\therefore\ b \approx 504$

So, the range of volumes in the bottles that are kept is 502 mL to 504 mL.

EXERCISE 26E.2

1 Let the mean IQ of a student at school be μ.
If X is the IQ of a student at the school, then $X \sim \text{N}(\mu,\, 15^2)$.

$$\text{Now,} \quad \text{P}(X \geqslant 125) = 0.2$$
$$\therefore \quad \text{P}\left(\frac{X-\mu}{15} \geqslant \frac{125-\mu}{15}\right) = 0.2$$
$$\therefore \quad \text{P}\left(Z \geqslant \frac{125-\mu}{15}\right) = 0.2$$
$$\therefore \quad \text{P}\left(Z < \frac{125-\mu}{15}\right) = 0.8$$
$$\therefore \quad \frac{125-\mu}{15} \approx 0.8416$$
$$\therefore \quad \mu \approx 112.4$$

The mean IQ at the school is 112.4.

2 Let the standard deviation of the distances jumped be σ m.
If X is the distance jumped by the athlete, then $X \sim \text{N}(5.2,\, \sigma^2)$.

$$\text{Now,} \quad \text{P}(X < 5) = 0.15$$
$$\therefore \quad \text{P}\left(\frac{X-5.2}{\sigma} < \frac{5-5.2}{\sigma}\right) = 0.15$$
$$\therefore \quad \text{P}\left(Z < -\frac{0.2}{\sigma}\right) = 0.15$$
$$\therefore \quad -\frac{0.2}{\sigma} \approx -1.036$$
$$\therefore \quad \sigma \approx 0.193$$

So, the standard deviation of the distances jumped is 0.193 m.

3 Let the standard deviation of the weekly income be $\$\sigma$.
If X denotes the weekly income of the bakery, then $X \sim \text{N}(6100,\, \sigma^2)$.

$$\text{Now,} \quad \text{P}(X \geqslant 6000) = 0.85$$
$$\therefore \quad \text{P}\left(Z \geqslant \frac{6000-6100}{\sigma}\right) = 0.85$$
$$\therefore \quad \text{P}\left(Z < \frac{6000-6100}{\sigma}\right) = 0.15$$

Using invNorm for $\text{N}(0,\, 1^2)$,

$$\frac{-100}{\sigma} \approx -1.036\,433\,4$$
$$\therefore \quad \sigma \approx \frac{-100}{-1.036\,433\,4}$$
$$\therefore \quad \sigma \approx 96.5$$

So, the standard deviation is \$96.50.

4 Let the mean arrival time be μ minutes after midday.
If X denotes the arrival time of a bus, then $X \sim \text{N}(\mu,\, 5^2)$.

Now, $\text{P}(X \leqslant 235) = 0.1$
{3:55 pm $= 3 \times 60 + 55 = 235$ minutes after midday}

$$\therefore \quad \text{P}\left(Z \leqslant \frac{235-\mu}{5}\right) = 0.1$$

Using invNorm for $\text{N}(0,\, 1^2)$,

$$\frac{235-\mu}{5} \approx -1.281\,551\,6$$
$$\therefore \quad 235-\mu \approx -6.407\,758$$
$$\therefore \quad \mu \approx 235 + 6.407\,758$$
$$\therefore \quad \mu \approx 241.407\,758 \text{ minutes after midday}$$

and $241.407\,758$ minutes $= 4$ h 1 m 24 s

So, the mean arrival time of buses at the depot is 4:01:24 pm.

5 $X \sim \text{N}(\mu,\, \sigma^2)$ where we have to find μ and σ.
We start by finding z_1 and z_2 which correspond to $x_1 = 35$ and $x_2 = 8$.

$$\text{Now} \quad \text{P}(X \geqslant 35) = 0.32$$
$$\therefore \quad \text{P}(X < 35) = 0.68$$
$$\therefore \quad \text{P}\left(\frac{X-\mu}{\sigma} < \frac{35-\mu}{\sigma}\right) = 0.68$$
$$\therefore \quad \text{P}\left(Z < \frac{35-\mu}{\sigma}\right) = 0.68$$
$$\therefore \quad z_1 = \frac{35-\mu}{\sigma} \approx 0.4677$$
$$\therefore \quad 35-\mu \approx 0.4677\sigma \quad .... \ (1)$$

and

$$\text{P}(X \leqslant 8) = 0.26$$
$$\therefore \quad \text{P}\left(\frac{X-\mu}{\sigma} \leqslant \frac{8-\mu}{\sigma}\right) = 0.26$$
$$\therefore \quad \text{P}\left(Z \leqslant \frac{8-\mu}{\sigma}\right) = 0.26$$
$$\therefore \quad z_2 = \frac{8-\mu}{\sigma} \approx -0.6433$$
$$\therefore \quad 8-\mu \approx -0.6433\sigma \quad .... \ (2)$$

Solving (1) and (2) simultaneously we get $\mu \approx 23.6$ and $\sigma \approx 24.3$.

6 **a** $X \sim \mathrm{N}(\mu, \sigma^2)$ where we have to find μ and σ.

We start by finding z_1 and z_2 which correspond to $x_1 = 30$ and $x_2 = 80$.

Now $\mathrm{P}(X \leqslant 30) = 0.15$

$\therefore \mathrm{P}\left(\frac{X-\mu}{\sigma} \leqslant \frac{30-\mu}{\sigma}\right) = 0.15$

$\therefore \mathrm{P}\left(Z \leqslant \frac{30-\mu}{\sigma}\right) = 0.15$

$\therefore z_1 = \frac{30-\mu}{\sigma} \approx -1.0364$

$\therefore 30 - \mu \approx -1.0364\sigma$ .... (1)

and $\mathrm{P}(X \geqslant 80) = 0.1$

$\therefore \mathrm{P}(X < 80) = 0.9$

$\therefore \mathrm{P}\left(\frac{X-\mu}{\sigma} < \frac{80-\mu}{\sigma}\right) = 0.9$

$\therefore \mathrm{P}\left(Z < \frac{80-\mu}{\sigma}\right) = 0.9$

$\therefore z_2 = \frac{80-\mu}{\sigma} \approx 1.2816$

$\therefore 80 - \mu \approx 1.2816\sigma$ .... (2)

Solving (1) and (2) simultaneously, $\mu \approx 52.36 \approx 52.4$ and $\sigma \approx 21.57 \approx 21.6$.

b Let X be the result of the mathematics exam.

X is normally distributed with mean μ and standard deviation σ.

We know that $\mathrm{P}(X \geqslant 80) = 0.1$ and $\mathrm{P}(X \leqslant 30) = 0.15$.

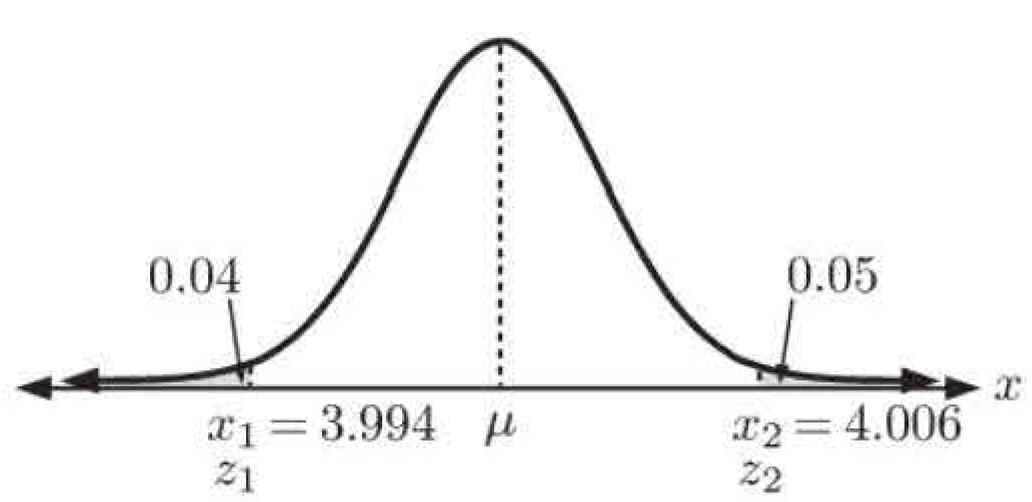
So, from **a**, $\mu \approx 52.4$ and $\sigma \approx 21.6$.

If part marks can be given, $\mathrm{P}(X > 50) \approx 0.544$

$\approx 54.4\%$

So, 54.4% of students scored more than 50.

7 **a**



$X \sim \mathrm{N}(\mu, \sigma^2)$ where we have to find μ and σ.

We find z_1 and z_2 which correspond to $x_1 = 3.994$ and $x_2 = 4.006$

Now $\mathrm{P}(X \leqslant x_1) = 0.04$

$\therefore \mathrm{P}\left(Z \leqslant \frac{3.994-\mu}{\sigma}\right) = 0.04$

$\therefore \frac{3.994-\mu}{\sigma} = -1.750\,686\,1$

$\therefore 3.994 - \mu = -1.750\,686\,1\sigma$ .... (1)

and $\mathrm{P}(X \geqslant x_2) = 0.05$

$\therefore \mathrm{P}\left(Z \leqslant \frac{4.006-\mu}{\sigma}\right) = 0.95$

$\therefore \frac{4.006-\mu}{\sigma} = 1.644\,853\,63$

$\therefore 4.006 - \mu = 1.644\,853\,63\sigma$ .... (2)

Solving simultaneously, $\mu \approx 4.000\,187\,009$ and $\sigma \approx 0.003\,534\,047\,88$

$\therefore \mu \approx 4.00$ cm and $\sigma \approx 0.003\,53$ cm

b From **a**, $\mu \approx 4.000$ and $\sigma \approx 0.003\,534$

$\therefore X \sim \mathrm{N}(4.000, 0.003\,534^2)$

$\therefore \mathrm{P}(3.997 \leqslant X \leqslant 4.003) \approx 0.604$

So, the probability that a randomly chosen piston has diameter between 3.997 cm and 4.003 cm is 0.604.

8 **a** $X \sim \mathrm{N}(\mu, \sigma^2)$ where we have to find μ and σ.

We start by finding z_1 and z_2 which correspond to $x_1 = 1.94$ and $x_2 = 2.06$.

Now $\mathrm{P}(X < 1.94) = 0.02$

$\therefore \mathrm{P}\left(\frac{X-\mu}{\sigma} < \frac{1.94-\mu}{\sigma}\right) = 0.02$

$\therefore \mathrm{P}\left(Z < \frac{1.94-\mu}{\sigma}\right) = 0.02$

$\therefore z_1 = \frac{1.94-\mu}{\sigma} \approx -2.054$

$\therefore 1.94 - \mu \approx -2.054\sigma$ .... (1)

and

$\mathrm{P}(X > 2.06) = 0.03$

$\therefore \mathrm{P}\left(\frac{X-\mu}{\sigma} > \frac{2.06-\mu}{\sigma}\right) = 0.03$

$\therefore \mathrm{P}\left(Z > \frac{2.06-\mu}{\sigma}\right) = 0.03$

$\therefore \mathrm{P}\left(Z \leqslant \frac{2.06-\mu}{\sigma}\right) = 0.97$

$\therefore z_2 = \frac{2.06-\mu}{\sigma} \approx 1.881$

$2.06 - \mu \approx 1.881\sigma$ .... (2)

Solving (1) and (2) simultaneously, we get $\mu \approx 2.00$ cm and $\sigma \approx 0.0305$ cm.

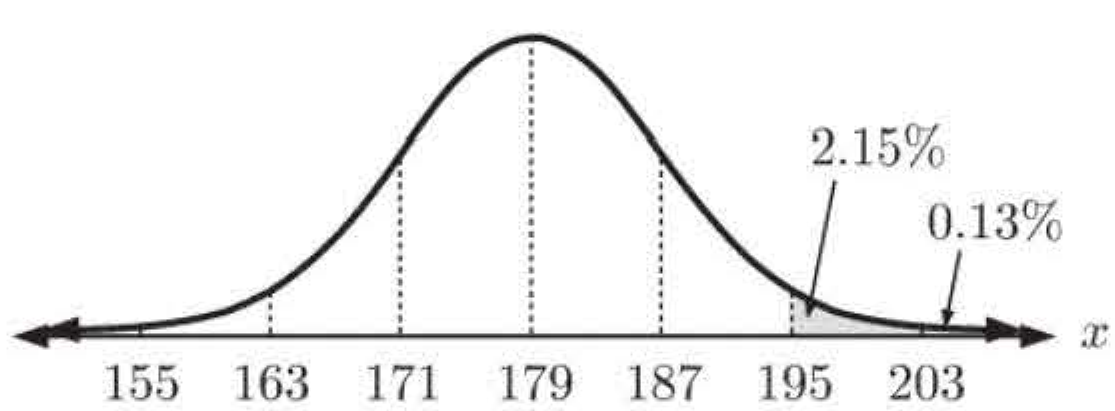
b Let Y be the number of tokens which will not operate the machine. This is a binomial situation with the probability $p = 0.02 + 0.03 = 0.05$ of failure to operate and $n = 20$. So, $Y \sim \mathrm{B}(20, 0.05)$.

$\therefore$ P(at most one will not operate) $= \mathrm{P}(Y \leqslant 1)$
≈ 0.736

REVIEW SET 26A

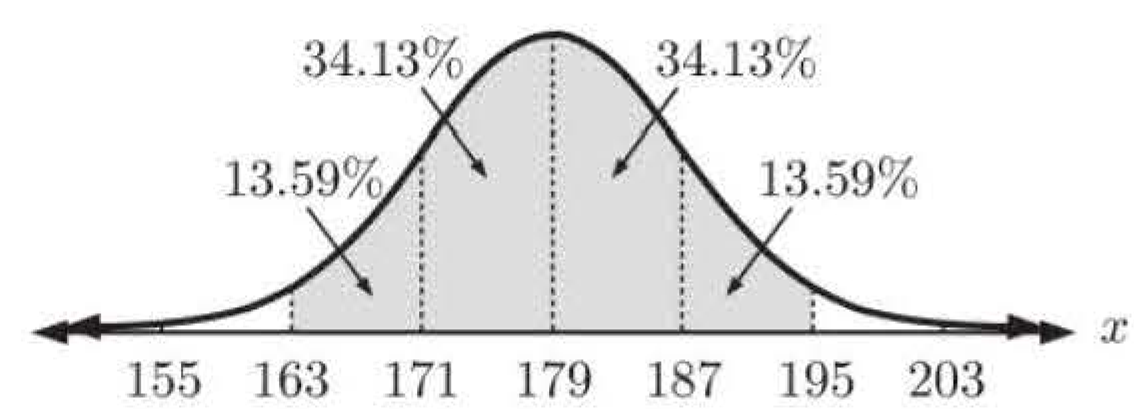
1 X is the height of a 17 year old boy.
X is normally distributed with $\mu = 179$ cm and $\sigma = 8$ cm.

a



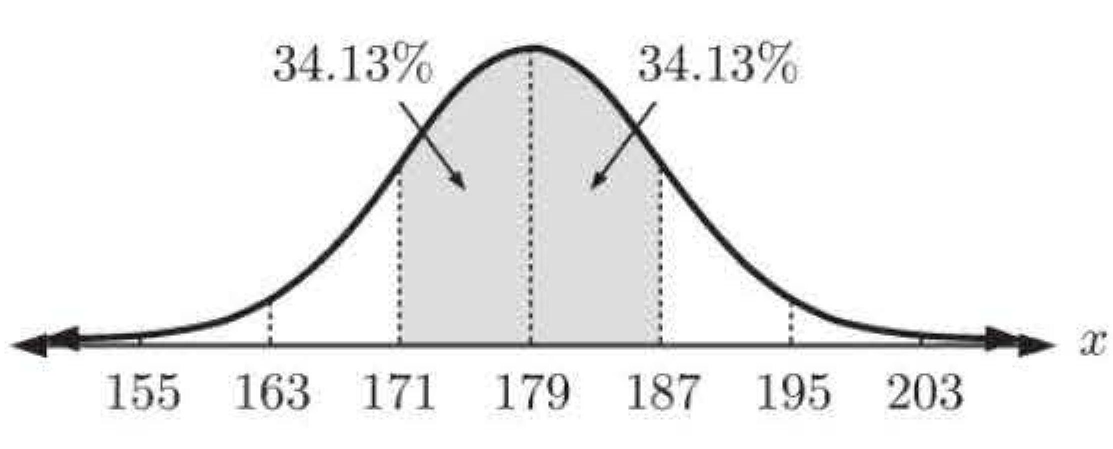
$\mathrm{P}(X \geqslant 195) \approx 2.15\% + 0.13\%$
$\approx 2.28\%$

b



$\mathrm{P}(163 \leqslant X \leqslant 195)$
$\approx 13.59\% + 34.13\% + 34.13\% + 13.59\%$
$\approx 95.44\%$
$\approx 95.4\%$

c



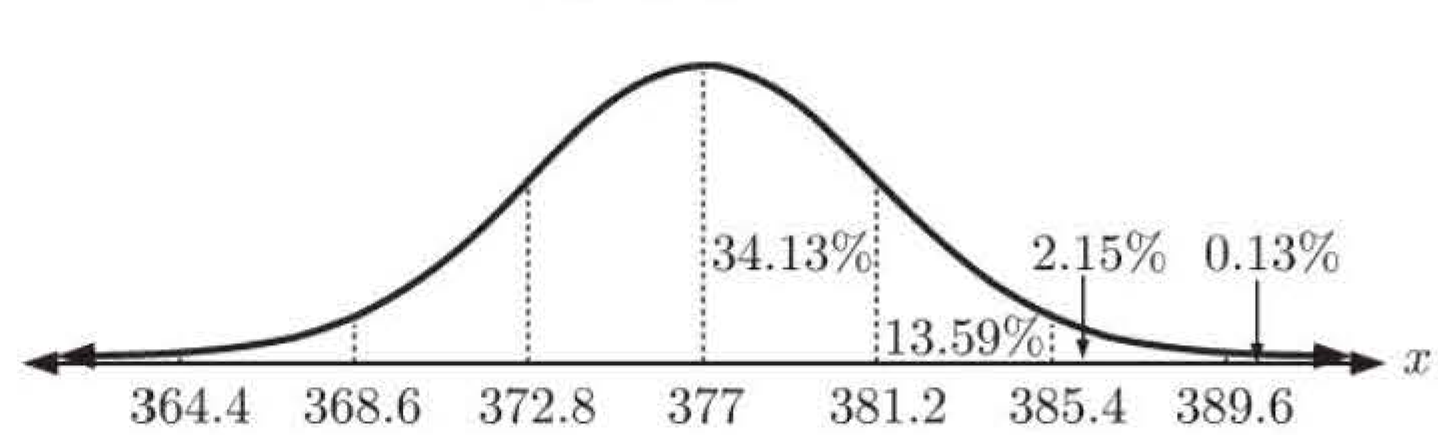
$\mathrm{P}(171 \leqslant X \leqslant 187) \approx 34.13\% + 34.13\%$
$\approx 68.26\%$
$\approx 68.3\%$

2 If X is the contents of the container in mL, then $X \sim \mathrm{N}(377, 4.2^2)$.

a **i** $\mathrm{P}(X < 368.6)$
$\approx 2.15\% + 0.13\%$
$\approx 2.28\%$

ii $\mathrm{P}(372.8 < X < 389.6)$
$\approx 2 \times 34.13\% + 13.59\% + 2.15\%$
$\approx 84.0\%$

b $\mathrm{P}(377 < X < 381.2)$
≈ 0.341



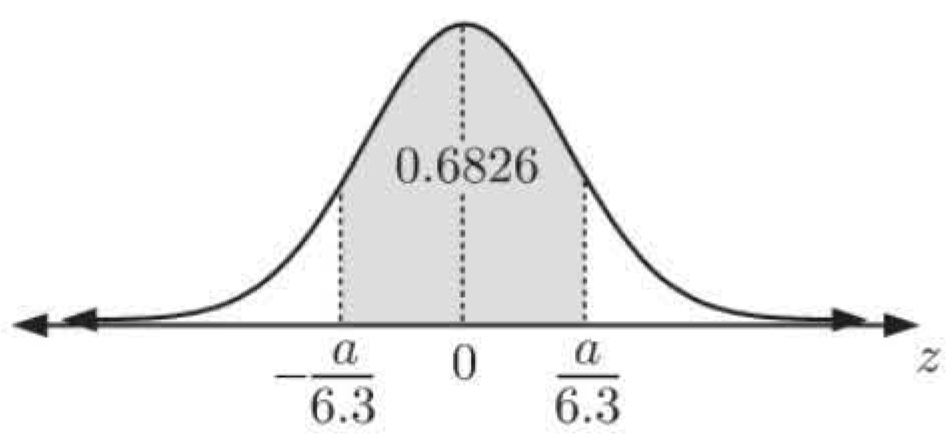
3 If X is the mass of a Coffin Bay Oyster, then $X \sim \mathrm{N}(38.6,\, 6.3^2)$.

a

$$\mathrm{P}(38.6 - a \leqslant X \leqslant 38.6 + a) = 0.6826$$

$$\therefore\ \mathrm{P}\left(\frac{38.6 - a - 38.6}{6.3} \leqslant \frac{X - 38.6}{6.3} \leqslant \frac{38.6 + a - 38.6}{6.3}\right) = 0.6826$$

$$\therefore\ \mathrm{P}\left(-\frac{a}{6.3} \leqslant Z \leqslant \frac{a}{6.3}\right) = 0.6826$$



$$\therefore\ \text{by symmetry,}\quad \mathrm{P}\left(Z \leqslant -\frac{a}{6.3}\right) = \frac{1 - 0.6826}{2}$$

$$\therefore\ \mathrm{P}\left(Z \leqslant -\frac{a}{6.3}\right) = 0.1587 \quad .... (*)$$

$$\therefore\ -\frac{a}{6.3} \approx -1.00$$

$$\therefore\ a \approx 6.30 \text{ g}$$

b

$$\mathrm{P}(X \geqslant b) = 0.8413$$

$$\therefore\ \mathrm{P}(X < b) = 0.1587$$

$$\therefore\ \mathrm{P}\left(\frac{X - 38.6}{6.3} < \frac{b - 38.6}{6.3}\right) = 0.1587$$

$$\therefore\ \mathrm{P}\left(Z < \frac{b - 38.6}{6.3}\right) = 0.1587$$

Comparing with $(*)$, $\frac{b - 38.6}{6.3} = -\frac{a}{6.3}$

$$\therefore\ b - 38.6 \approx -6.30$$

$$\therefore\ b \approx 32.3 \text{ g}$$

4 $f(x) = a(x+1)x(x-1)(x-2), \quad 0 < x < 1$
$= a(x^2 + x)(x^2 - 3x + 2)$
$= a(x^4 - 2x^3 - x^2 + 2x)$

a

$$\int_0^1 f(x)\, dx = 1$$

$$\therefore\ a\int_0^1 x^4 - 2x^3 - x^2 + 2x\, dx = 1$$

$$\therefore\ a\left[\frac{x^5}{5} - \frac{x^4}{2} - \frac{x^3}{3} + x^2\right]_0^1 = 1$$

$$\therefore\ a\left[\tfrac{1}{5} - \tfrac{1}{2} - \tfrac{1}{3} + 1\right] = 1$$

$$\therefore\ a\left(\tfrac{11}{30}\right) = 1$$

$$\therefore\ a = \tfrac{30}{11}$$

b The mode is the value of x when $f(x)$ is a maximum.

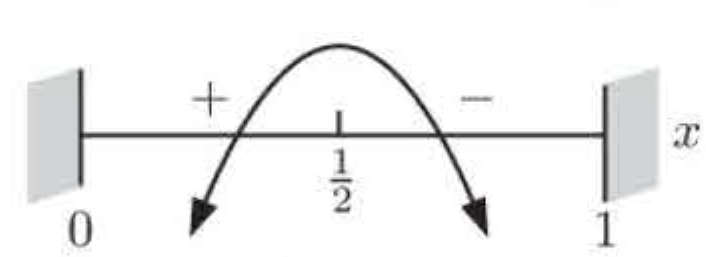
$$f(x) = a(x^4 - 2x^3 - x^2 + 2x)$$

$$\therefore\ f'(x) = a(4x^3 - 6x^2 - 2x + 2)$$

$$= 2a(2x^3 - 3x^2 - x + 1)$$

$$= 2a(2x - 1)(x^2 - x - 1)$$

$$\therefore\ f'(x) = 0 \quad \text{when} \quad x = \tfrac{1}{2} \quad \{0 < x < 1\}$$



$\therefore$ the mode is $\frac{1}{2}$.

c

$$f(\tfrac{1}{2} + x) = a(\tfrac{1}{2} + x + 1)(\tfrac{1}{2} + x)(\tfrac{1}{2} + x - 1)(\tfrac{1}{2} + x - 2)$$

$$= a(\tfrac{3}{2} + x)(\tfrac{1}{2} + x)(-\tfrac{1}{2} + x)(-\tfrac{3}{2} + x)$$

$$f(\tfrac{1}{2} - x) = a(\tfrac{1}{2} - x + 1)(\tfrac{1}{2} - x)(\tfrac{1}{2} - x - 1)(\tfrac{1}{2} - x - 2)$$

$$= a(\tfrac{3}{2} - x)(\tfrac{1}{2} - x)(-\tfrac{1}{2} - x)(-\tfrac{3}{2} - x)$$

$$= a(-1)(-\tfrac{3}{2} + x)(-1)(-\tfrac{1}{2} + x)(-1)(\tfrac{1}{2} + x)(-1)(\tfrac{3}{2} + x)$$

$$= a(-\tfrac{3}{2} + x)(-\tfrac{1}{2} + x)(\tfrac{1}{2} + x)(\tfrac{3}{2} + x)$$

$$\therefore\ f(\tfrac{1}{2} - x) = f(\tfrac{1}{2} + x)$$

d From **c**, $f(\frac{1}{2} - x) = f(\frac{1}{2} + x)$ for all $0 < x < 1$.

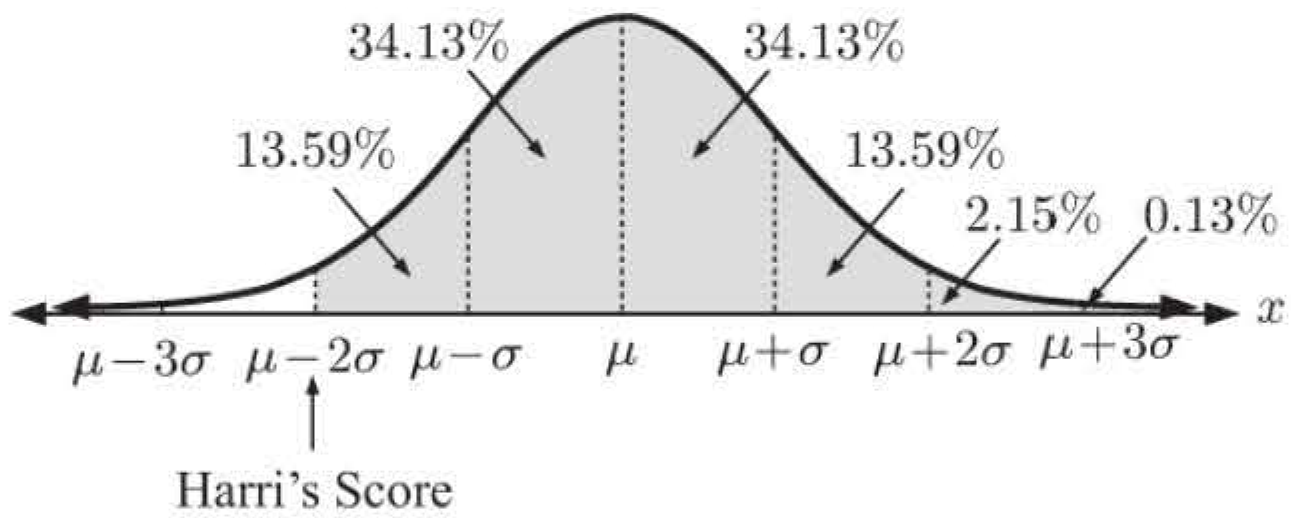
$\therefore$ $f(x)$ is symmetric about $x = \frac{1}{2}$.

$\therefore$ half the data values lie below $x = \frac{1}{2}$ and half lie above.

$\therefore$ median $= \frac{1}{2}$

5 **a** Harri's score is 2 standard deviations below the mean.

b



Proportion of students who scored better than Harri

$\approx 13.59\% + 34.13\% + 34.13\% + 13.59\% + 2.15\% + 0.13\%$

$\approx 97.72\%$

$\approx 97.7\%$

c $\mu = 151$ and $\mu - 2\sigma = 117$

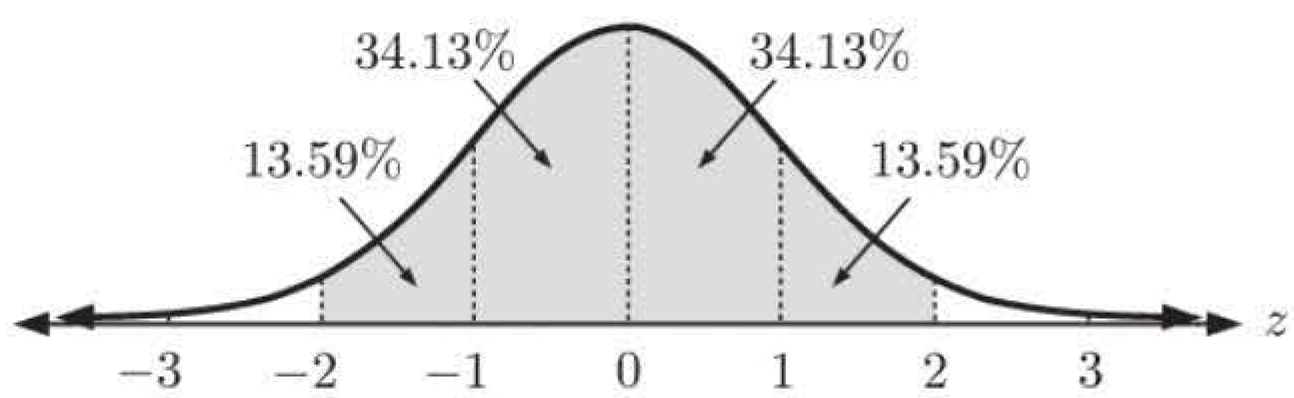
$\therefore\ 151 - 2\sigma = 117$

$\therefore\ -2\sigma = -34$

$\therefore\ \sigma = 17$

The standard deviation was 17.

6



The shaded part of the diagram has an area of approximately 0.95 .

$\therefore\ \text{P}(-2 \leqslant Z \leqslant 2) \approx 0.95$

$\therefore\ k \approx 2$

7 **a**

$$\int_0^2 ax(4-x^2)\,dx = 1$$

$$\therefore\ a\int_0^2 (4x - x^3)\,dx = 1$$

$$\therefore\ a\left[2x^2 - \frac{x^4}{4}\right]_0^2 = 1$$

$$\therefore\ a(8-4) = 1$$

$$\therefore\ a = \tfrac{1}{4}$$

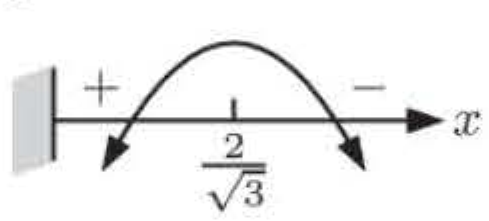
b The mode is the value of x when $f(x)$ is a maximum.

$$f(x) = \tfrac{1}{4}x(4-x^2) = x - \tfrac{1}{4}x^3$$

$$\therefore\ f'(x) = 1 - \tfrac{3}{4}x^2$$

$$\therefore\ f'(x) = 0 \text{ when } x^2 = \tfrac{4}{3}$$

$$\therefore\ x = \tfrac{2}{\sqrt{3}} \quad \{0 \leqslant x \leqslant 2\}$$



$\therefore$ the mode is $\frac{2}{\sqrt{3}}$

c If the median is m, then

$$\int_0^m (x - \tfrac{1}{4}x^3)\,dx = \tfrac{1}{2}$$

$$\therefore\ \left[\frac{x^2}{2} - \frac{x^4}{16}\right]_0^m = \tfrac{1}{2}$$

$$\therefore\ \frac{m^2}{2} - \frac{m^4}{16} = \tfrac{1}{2}$$

$$\therefore\ m^4 - 8m^2 + 8 = 0$$

$$\therefore\ m^2 = \frac{8 \pm \sqrt{64-32}}{2}$$

$$= \frac{8 \pm 4\sqrt{2}}{2}$$

$$= 4 \pm 2\sqrt{2}$$

$$\therefore\ m^2 = 4 - 2\sqrt{2} \quad \{\text{as } 0 < m < 2\}$$

$$\therefore\ m = \pm\sqrt{4 - 2\sqrt{2}}$$

$$\therefore\ m = \sqrt{4 - 2\sqrt{2}} \quad \{\text{as } 0 < m < 2\}$$

$\therefore$ the median is $\sqrt{4 - 2\sqrt{2}}$.

d

$$\mu = \int_0^2 (x^2 - \tfrac{1}{4}x^4)\,dx$$

$$= \left[\frac{x^3}{3} - \frac{x^5}{20}\right]_0^2$$

$$= \tfrac{8}{3} - \tfrac{32}{20}$$

$$= \tfrac{16}{15}$$

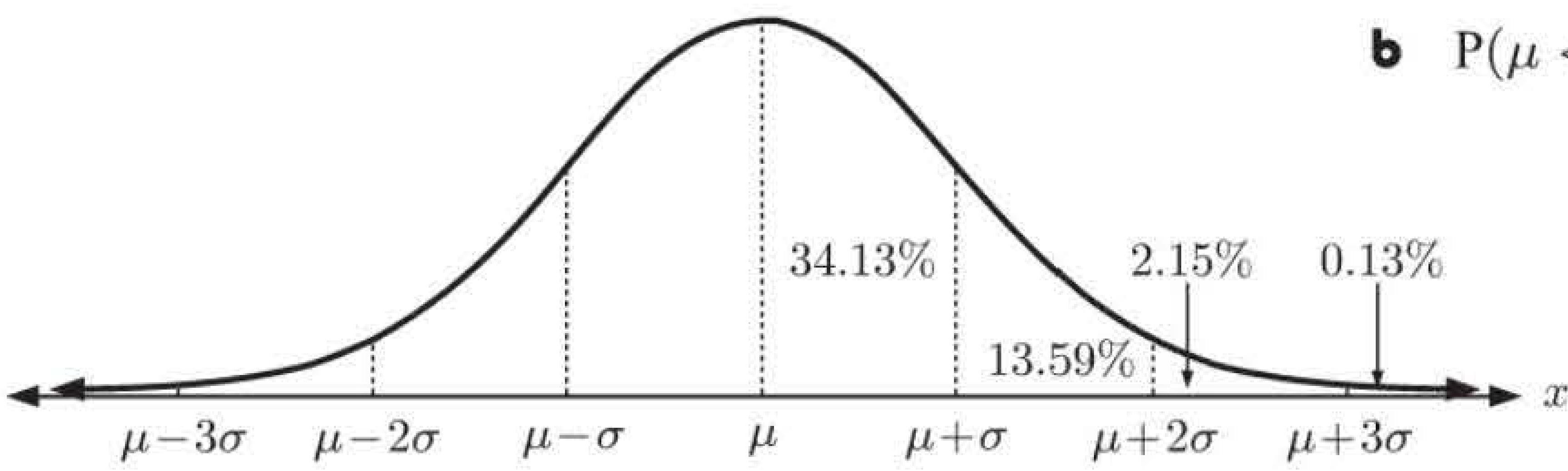
8 Jarrod's z-score is $\dfrac{41-35}{4} = 1.5$

$\therefore$ Paul needs x such that $\dfrac{x-25}{3} = 1.5$

$\therefore\ x = 25 + 4.5 = 29.5$

Paul needs to throw a tennis ball 29.5 m to perform as well as Jarrod.

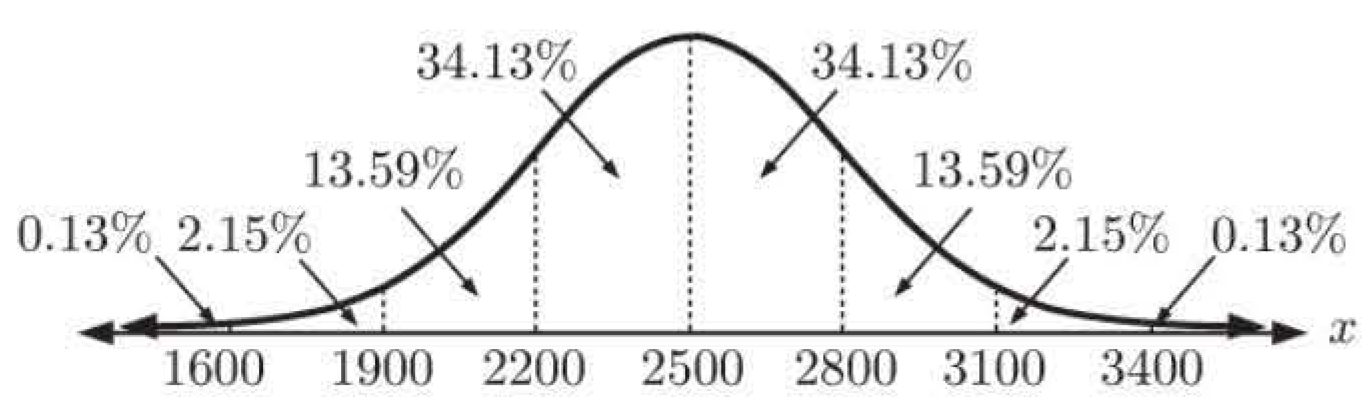
9



a $\text{P}(\mu+\sigma < X < \mu+2\sigma) \approx 13.59\%$
≈ 0.136

b $\text{P}(\mu < X < \mu+\sigma) \approx 34.13\%$
≈ 0.341

10 If X is the number of bottles sold per day, then $X \sim \text{N}(2500,\ 300^2)$.



a $\text{P}(X < 1900) \approx 0.13\% + 2.15\%$
$\approx 2.28\%$

b $\text{P}(X > 2200)$
$\approx 2 \times 34.13\% + 13.59\% + 2.15\% + 0.13\%$
$\approx 84.13\%$
$\approx 84.1\%$

c $\text{P}(2200 \leqslant X \leqslant 3100) \approx 34.13\% + 34.13\% + 13.59\%$
$\approx 81.85\%$
$\approx 81.9\%$

11 $f(x) = 2e^{-x},\quad 0 \leqslant x \leqslant k$

a
$$\begin{aligned}
&\int_0^k f(x)\,dx = 1\\
\therefore\ &\int_0^k 2e^{-x}\,dx = 1\\
\therefore\ &\left[-2e^{-x}\right]_0^k = 1\\
\therefore\ &-2e^{-k} + 2e^0 = 1\\
\therefore\ &2e^{-k} = 1\\
\therefore\ &-k = \ln\tfrac{1}{2}\\
\therefore\ &k = \ln 2
\end{aligned}$$

b
$$\begin{aligned}
&\text{P}(\ln\tfrac{4}{3} < X < \ln\tfrac{5}{3})\\
&= \int_{\ln\frac{4}{3}}^{\ln\frac{5}{3}} 2e^{-x}\,dx\\
&= \left[-2e^{-x}\right]_{\ln\frac{4}{3}}^{\ln\frac{5}{3}}\\
&= -2e^{-\ln\frac{5}{3}} + 2e^{-\ln\frac{4}{3}}\\
&= -2e^{\ln\frac{3}{5}} + 2e^{\ln\frac{3}{4}}\\
&= -2 \times \tfrac{3}{5} + 2 \times \tfrac{3}{4}\\
&= \tfrac{3}{10}\\
&= 0.3
\end{aligned}$$

c $\mu = \int_0^{\ln 2} x f(x)\, dx$

$= \int_0^{\ln 2} 2xe^{-x}\, dx$

We integrate by parts with

$u = 2x \qquad v' = e^{-x}$

$u' = 2 \qquad v = -e^{-x}$

$\therefore \int 2xe^{-x}\, dx$

$= -2xe^{-x} - \int -2e^{-x}\, dx$

$= -2xe^{-x} - 2e^{-x} + c$

$= -2e^{-x}(x+1) + c \quad .... \ (*)$

So, $\mu = \int_0^{\ln 2} 2xe^{-x}\, dx$

$= \left[-2e^{-x}(x+1)\right]_0^{\ln 2}$

$= -2e^{-\ln 2}(\ln 2 + 1) + 2e^0(1)$

$= -2(\frac{1}{2})(\ln 2 + 1) + 2$

$= -\ln 2 - 1 + 2$

$= 1 - \ln 2$

d Now consider $E(X^2) = \int_0^{\ln 2} 2x^2 e^{-x}\, dx$

We integrate by parts with

$u = 2x^2 \qquad v' = e^{-x}$

$u' = 4x \qquad v = -e^{-x}$

$\therefore \int 2x^2 e^{-x}\, dx$

$= -2x^2 e^{-x} - \int -4xe^{-x}\, dx$

$= -2x^2 e^{-x} + 2\int 2xe^{-x}\, dx$

$= -2x^2 e^{-x} + 2(-2e^{-x}(x+1)) + c$

{using $*$}

$= -2e^{-x}(x^2 + 2x + 2) + c$

$\therefore E(X^2)$

$= \int_0^{\ln 2} 2x^2 e^{-x}\, dx$

$= \left[-2e^{-x}(x^2 + 2x + 2)\right]_0^{\ln 2}$

$= -2e^{-\ln 2}((\ln 2)^2 + 2\ln 2 + 2) + 2e^0(2)$

$= -2(\frac{1}{2})((\ln 2)^2 + 2\ln 2 + 2) + 4$

$= 2 - 2\ln 2 - (\ln 2)^2$

$\therefore \mathrm{Var}(X)$

$= E(X^2) - (E(X))^2$

$= E(X^2) - \mu^2$

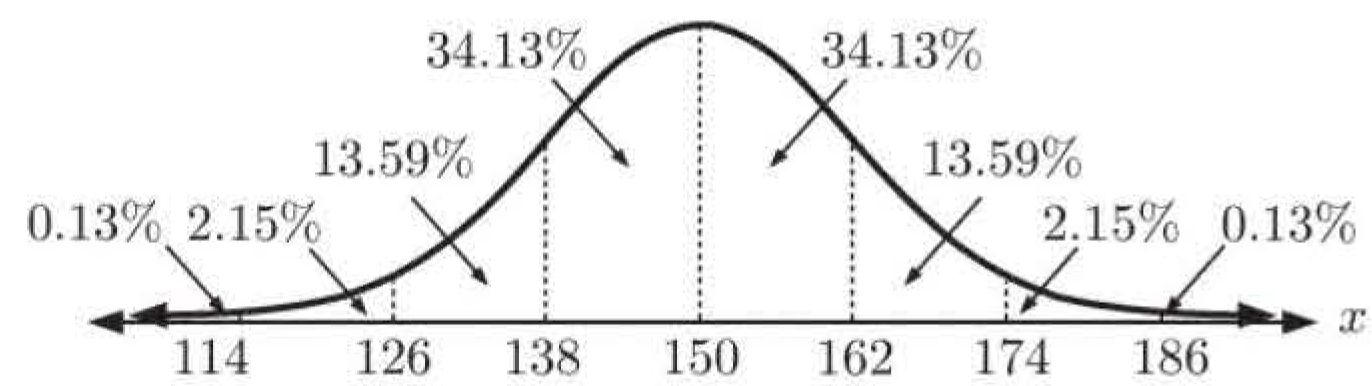
$= 2 - 2\ln 2 - (\ln 2)^2 - (1 - \ln 2)^2$

$= 2 - \cancel{2\ln 2} - (\ln 2)^2 - (1 - \cancel{2\ln 2} + (\ln 2)^2)$

$= 1 - 2(\ln 2)^2$

REVIEW SET 26B

1 $X \sim N(150, 12^2)$



a $P(138 \leqslant X \leqslant 162)$
$\approx 34.13\% + 34.13\%$
$\approx 68.26\%$
$\approx 68.3\%$

b $P(126 \leqslant X \leqslant 174)$
$\approx 13.59\% + 34.13\% + 34.13\% + 13.59\%$
$\approx 95.44\%$
$\approx 95.4\%$

c $P(126 \leqslant X \leqslant 162)$
$\approx 13.59\% + 34.13\% + 34.13\%$
$\approx 81.85\%$
$\approx 81.9\%$

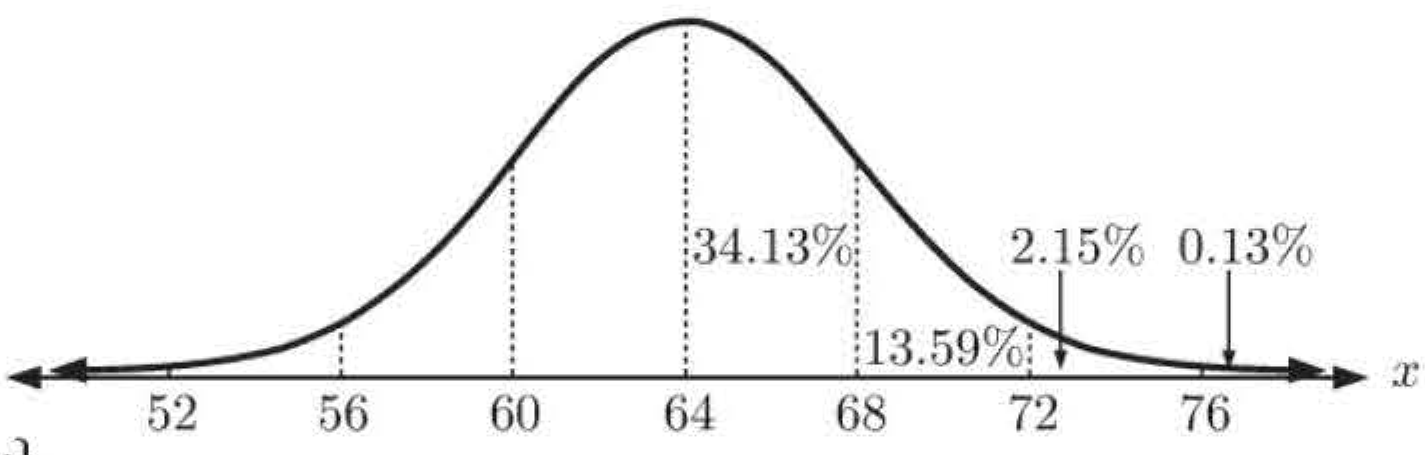
d $P(162 \leqslant X \leqslant 174)$
$\approx 13.59\%$
$\approx 13.6\%$

2 If random variable X is the arm length in cm then $X \sim N(64, 4^2)$.

a **i** $P(60 < X < 72)$
$\approx 2 \times 34.13\% + 13.59\%$
$\approx 81.9\%$

ii $P(X > 60)$
$\approx 50\% + 34.13\%$
$\approx 84.1\%$

b $P(56 < X < 64) \approx 0.3413 + 0.1359$
≈ 0.477



c $P(X > x) = 0.7$
$\therefore \; P(X \leqslant x) = 0.3$
$\therefore \; x \approx 61.9$ {using technology}

3 If X is the rod length in mm, then $X \sim N(\mu, 3^2)$.

Now $P(X < 25) = 0.02$

$\therefore \; P\left(\frac{X-\mu}{3} < \frac{25-\mu}{3}\right) = 0.02$

$\therefore \; P\left(Z < \frac{25-\mu}{3}\right) = 0.02$

$\therefore \; \frac{25-\mu}{3} \approx -2.0537$

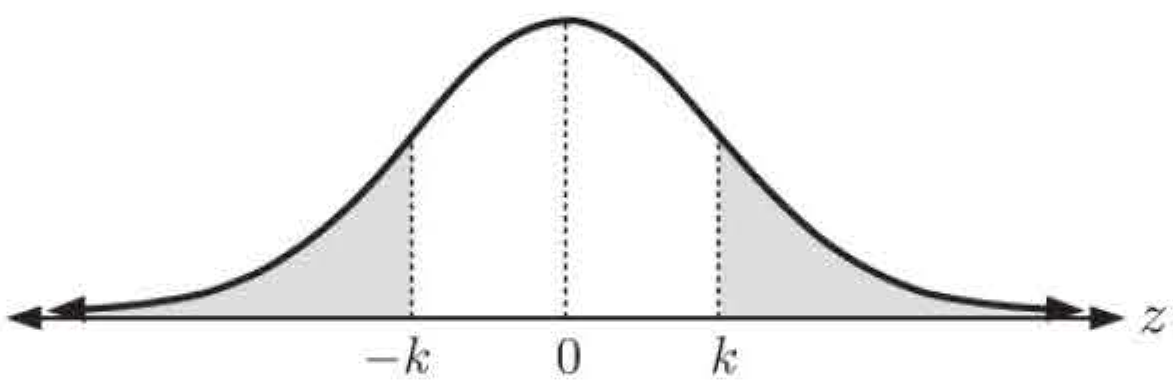
$\therefore \; 25 - \mu \approx -6.161$

$\therefore \; \mu \approx 31.2$

$\therefore$ the mean rod length is 31.2 mm.

4 $P(|Z| > k) = 0.376$

$\therefore \; P(Z > k \text{ or } Z < -k) = 0.376$



$\therefore \; P(Z < -k) = \frac{1}{2}(0.376) = 0.188$

$\therefore \; -k \approx -0.885$

$\therefore \; k \approx 0.885$

5 **a** $\int_0^2 ax(x-3)\,dx = 1$

$\therefore \; a\int_0^2 (x^2 - 3x)\,dx = 1$

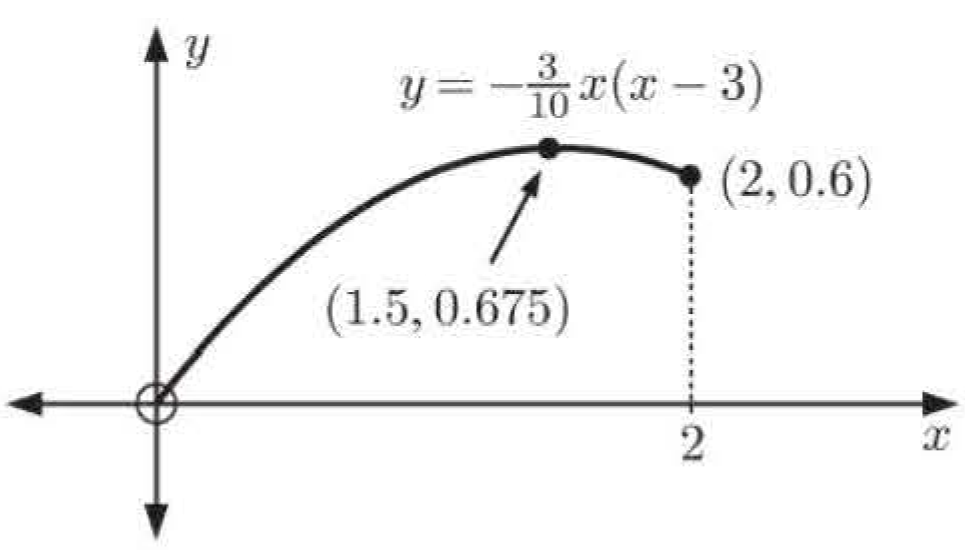
$\therefore \; a\left[\frac{1}{3}x^3 - \frac{3}{2}x^2\right]_0^2 = 1$

$\therefore \; a\left[\frac{8}{3} - 6\right] = 1$

$\therefore \; a\left(-\frac{10}{3}\right) = 1$

$\therefore \; a = -\frac{3}{10}$

b



c **i** $\mu = \int_0^2 x\,f(x)\,dx$

$= \int_0^2 -\frac{3}{10}x^2(x-3)\,dx$

$= -\frac{3}{10}\int_0^2 (x^3 - 3x^2)\,dx$

$= -\frac{3}{10}\left[\frac{1}{4}x^4 - x^3\right]_0^2\,dx$

$= -\frac{3}{10}\left(\frac{1}{4}(16) - 8\right)dx$

$= -\frac{3}{10}(-4)$

$= \frac{6}{5} = 1.2$

ii $f(x)$ has maximum value when $x = 1.5$
$\therefore$ the mode $= 1.5$

iii If the median is m, then

$\int_0^m f(x)\,dx = \frac{1}{2}$

$\int_0^m -\frac{3}{10}x(x-3)\,dx = \frac{1}{2}$

$\therefore \; \int_0^m (x^2 - 3x)\,dx = -\frac{5}{3}$

$\left[\frac{1}{3}x^3 - \frac{3}{2}x^2\right]_0^m = -\frac{5}{3}$

$\therefore \; \frac{1}{3}m^3 - \frac{3}{2}m^2 + \frac{5}{3} = 0$

$\therefore \; 2m^3 - 9m^2 + 10 = 0$

$\therefore \; m \approx -0.957, 1.24, 4.22$
{using technology}

But $0 \leqslant m \leqslant 2$, so $m \approx 1.24$

iv $E(X^2) = \int_0^2 x^2\,f(x)\,dx$

$= \int_0^2 -\frac{3}{10}x^3(x-3)\,dx$

$= 1.68$ {using technology}

$\therefore \; \text{Var}(X) = E(X^2) - \{E(X)\}^2$

$= 1.68 - (1.2)^2$

$= 0.24$

d $\quad \mathrm{P}(1 \leqslant x \leqslant 2)$

$= \int_1^2 -\frac{3}{10}x(x-3)\,dx$

$= 0.65 \qquad$ {using technology}

$= \frac{13}{20}$

6 **a** Since Area $A =$ Area B, 20 and 38 must be equal distances away from the mean μ, because of the symmetry of the normal distribution.

$\therefore$ μ is halfway between 20 and 38, so $\mu = \dfrac{20+38}{2} = 29$

Now $\mathrm{P}(X \leqslant 20) = 0.2$

$$\therefore\ \mathrm{P}\left(\frac{X-29}{\sigma} \leqslant \frac{20-29}{\sigma}\right) = 0.2$$

$$\therefore\ \mathrm{P}\left(Z \leqslant -\frac{9}{\sigma}\right) = 0.2$$

$$\therefore\ -\frac{9}{\sigma} \approx -0.8416$$

$$\therefore\ \sigma \approx 10.69$$

$$\therefore\ \mu = 29, \quad \sigma \approx 10.7$$

b Using the values obtained for μ and σ in **a** and technology:

i $\mathrm{P}(X \leqslant 35) \approx 0.713$

ii $\mathrm{P}(23 \leqslant X \leqslant 30) \approx 0.250$

7 $X \sim \mathrm{N}(503,\ 2^2)$

a $\quad \mathrm{P}(X < 500)$

$\approx 0.066\,807\,2$

≈ 0.0668

So, approximately 6.68% of the bags are underweight.

b This is a binomial distribution where X is the number of underweight bags,

$n = 20$ and $p = 0.066\,807\,2$

$\therefore\ \mathrm{P}(X \leqslant 2) \approx 0.854$

{using technology}

8 If X is the marks in the examination, then $X \sim \mathrm{N}(49,\ 15^2)$.

a $\mathrm{P}(X \geqslant 45) \approx 0.6051$

So, $2376 \times 0.6051 \approx 1438$ candidates passed the examination.

b Let k be the minimum mark required for a '7'.

$\therefore\ \mathrm{P}(X \geqslant k) = 0.07$

$\therefore\ \mathrm{P}(X < k) = 1 - 0.07 = 0.93$

$\therefore\ k \approx 71.1$

So the minimum mark required to obtain a '7' is 71.1 marks.

c Let L and U be the lower and upper quartiles of the distribution.

$\therefore\ \mathrm{P}(X \leqslant L) = 0.25 \quad$ and $\quad \mathrm{P}(X \leqslant U) = 0.75$

$\therefore\ L \approx 38.88 \qquad\qquad \therefore\ U \approx 59.12$

$\therefore$ the interquartile range $= U - L \approx 59.12 - 38.88 \approx 20.2$ marks

9 X is the life of a battery in weeks.

X is normally distributed with $\mu = 33.2$ and $\sigma = 2.8$.

a $\mathrm{P}(X \geqslant 35) \approx 0.260$

b We need to find k such that $\mathrm{P}(X \leqslant k) = 0.08$

$\therefore\ k \approx 29.3$

So, the manufacturer can expect the batteries to last 29.3 weeks before 8% of them fail.

10 **a** $P(X \leqslant 30) = 0.0832$ and $P(X \geqslant 90) = 0.101$

$$\therefore\ P\left(\frac{X-\mu}{\sigma} \leqslant \frac{30-\mu}{\sigma}\right) = 0.0832$$

$$\therefore\ P\left(Z \leqslant \frac{30-\mu}{\sigma}\right) = 0.0832$$

$$\therefore\ \frac{30-\mu}{\sigma} \approx -1.383\,864$$

$$\therefore\ 30-\mu \approx -1.383\,864\sigma \quad .... (1)$$

$$\therefore\ P(X < 90) = 0.899$$

$$\therefore\ P\left(\frac{X-\mu}{\sigma} < \frac{90-\mu}{\sigma}\right) = 0.899$$

$$\therefore\ P\left(Z < \frac{90-\mu}{\sigma}\right) = 0.899$$

$$\therefore\ \frac{90-\mu}{\sigma} \approx 1.275\,874$$

$$\therefore\ 90-\mu \approx 1.275\,874\sigma \quad .... (2)$$

Solving (1) and (2) simultaneously, we get $\mu \approx 61.218 \approx 61.2$ and $\sigma \approx 22.559 \approx 22.6$.

b $P(-7 \leqslant X-\mu \leqslant 7) \approx P(-7 \leqslant X - 61.218 \leqslant 7)$

$\approx P(54.218 \leqslant X \leqslant 68.218)$

≈ 0.244

11 **a** The relative difficulty of each test is not known. We would need the mean mark and standard deviation for each test.

b Kerry's English z-score $= \dfrac{26-22}{4} = \dfrac{4}{4} = 1$

Kerry's Chemistry z-score $= \dfrac{82-75}{7} = \dfrac{7}{7} = 1$

Since the z-scores are the same, Kerry's performance relative to the rest of the class is the same in both tests.

REVIEW SET 26C

1 **a** The middle 68% of the distribution lies between 16.2 and 21.4, and the middle 68% of data lies between one standard deviation of the mean.

$$\therefore\ \mu \approx \frac{16.2+21.4}{2}$$

$$\therefore\ \mu \approx \frac{37.6}{2}$$

$$\therefore\ \mu \approx 18.8$$

and $\sigma \approx 18.8 - 16.2$

$\therefore\ \sigma \approx 2.6$

b The middle 95% of the data lies between 2 standard deviations of the mean.

$\mu - 2\sigma \approx 18.8 - 2 \times 2.6 \approx 13.6$ and $\mu + 2\sigma \approx 18.8 + 2 \times 2.6 \approx 24.0$

$\therefore$ the middle 95% of the data lies between 13.6 and 24.0.

2 Using technology:

a $P(X \geqslant 22) \approx 0.364$

b $P(18 \leqslant X \leqslant 22) \approx 0.356$

c $P(X \leqslant k) = 0.3$

$\therefore\ k \approx 18.2$

3 $$P(-0.524 < X-\mu < 0.524) = P\left(\frac{-0.524}{2} < \frac{X-\mu}{2} < \frac{0.524}{2}\right)$$

$$= P(-0.262 < Z < 0.262)$$

$$\approx 0.207 \quad \{\text{using technology}\}$$

4 If X is the length of a rod, then $X \sim \mathrm{N}(\mu, 6^2)$.

$$\begin{aligned}
\text{Now}\quad & \mathrm{P}(X \geqslant 89.52) = 0.0563 \\
\therefore\quad & \mathrm{P}(X < 89.52) = 1 - 0.0563 \\
\therefore\quad & \mathrm{P}\left(\frac{X-\mu}{6} < \frac{89.52-\mu}{6}\right) = 0.9437 \\
\therefore\quad & \mathrm{P}\left(Z < \frac{89.52-\mu}{6}\right) = 0.9437 \\
\therefore\quad & \frac{89.52-\mu}{6} \approx 1.5866 \\
\therefore\quad & 89.52-\mu \approx 9.52 \\
\therefore\quad & \mu \approx 80.0
\end{aligned}$$

So, the mean is 80.0 cm.

Since the normal distribution is symmetrical and bell-shaped, the median and modal lengths are also 80.0 cm.

5 **a** T is the lifetime in years of a solar cell component.

$$\begin{aligned}
\therefore\quad \mathrm{P}(T \leqslant 1) &= \int_0^1 0.4e^{-0.4t}\, dt \\
&= \left[-e^{-0.4t}\right]_0^1 \\
&= -e^{-0.4} - (-e^0) \\
&= 1 - e^{-0.4} \\
&\approx 0.329\,68 \\
&\approx 0.330
\end{aligned}$$

b Let X be the number of components not working after one year.

Then $X \sim \mathrm{B}(5, 0.329\,68)$

$$\begin{aligned}
\therefore\quad & \mathrm{P(solar\ cell\ still\ operates)} \\
&= \mathrm{P}(X \leqslant 2) \quad \{\text{at least 3 work}\} \\
&\approx 0.796
\end{aligned}$$

6

$$\begin{aligned}
& \mathrm{P}(X < 90) \approx 0.975 \\
\therefore\quad & \mathrm{P}\left(\frac{X-50}{\sigma} < \frac{90-50}{\sigma}\right) \approx 0.975 \\
\therefore\quad & \mathrm{P}\left(Z < \frac{40}{\sigma}\right) \approx 0.975 \\
\therefore\quad & \frac{40}{\sigma} \approx 1.959\,96 \\
\therefore\quad & \sigma \approx 20.409
\end{aligned}$$

So, $X \sim \mathrm{N}(50, 20.409^2)$

Now, the shaded area $= \mathrm{P}(X \geqslant 80)$

≈ 0.0708 units^2

7 If X is the weight of an apple, then $X \sim \mathrm{N}(300, 50^2)$.

a
$$\begin{aligned}
& \mathrm{P}(250 \leqslant X \leqslant 350) \\
&\approx 0.682\,689\,49 \\
&\approx 68.3\%
\end{aligned}$$

b This is a binomial distribution where X is the number of apples that are fit for sale.

$n = 100$ and $p \approx 0.682\,689\,49$

$$\begin{aligned}
\mathrm{P}(X \geqslant 75) &= 1 - \mathrm{P}(X \leqslant 74) \\
&\approx 1 - 0.911\,645\,43 \\
&\approx 0.0884
\end{aligned}$$

8 **a** Consider the integral $\displaystyle\int_0^1 \frac{4}{1+x^2}\, dx$.

Let $x = \tan\theta$, $\dfrac{dx}{d\theta} = \sec^2\theta$

When $x = 0$, $\theta = 0$, and when $x = 1$, $\theta = \frac{\pi}{4}$

$$\therefore \int_0^1 \frac{4}{1+x^2}\,dx = \int_0^{\frac{\pi}{4}} \frac{4}{1+\tan^2\theta}\sec^2\theta\,d\theta$$
$$= \int_0^{\frac{\pi}{4}} 4\,d\theta \qquad \{1+\tan^2\theta = \sec^2\theta\}$$
$$= \Big[4\theta\Big]_0^{\frac{\pi}{4}} = \pi$$

$\therefore \displaystyle\int_0^1 \frac{4}{1+x^2}\,dx \neq 1$, and so $f(x)$ cannot be a probability density function.

b $k\,f(x) = \begin{cases} \dfrac{4k}{1+x^2} & \text{for } 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$

For a probability density function, $\displaystyle\int_0^1 \frac{4k}{1+x^2}\,dx = 1$

$$\therefore\ k\int_0^1 \frac{4}{1+x^2}\,dx = 1$$
$$\therefore\ k(\pi) = 1$$
$$\therefore\ k = \tfrac{1}{\pi}$$

c $\mu = \frac{1}{\pi}\displaystyle\int_0^1 \frac{4x}{1+x^2}\,dx$

$= \frac{2}{\pi}\displaystyle\int_0^1 \frac{2x}{1+x^2}\,dx$

$= \frac{2}{\pi}\left[\ln(1+x^2)\right]_0^1 \quad \{\text{as } 1+x^2>0\}$

$= \frac{2}{\pi}(\ln 2 - \ln 1)$

$= \frac{2}{\pi}\ln 2$

$$\mathrm{E}(X^2) = \tfrac{4}{\pi}\int_0^1 \frac{x^2}{1+x^2}\,dx$$
$$= \tfrac{4}{\pi}\int_0^1 \left(1-\frac{1}{1+x^2}\right)dx$$
$$= \tfrac{4}{\pi}\int_0^1 1\,dx - \tfrac{4}{\pi}\int_0^1 \frac{1}{1+x^2}\,dx$$
$$= \tfrac{4}{\pi}[\,x\,]_0^1 - \tfrac{4}{\pi}[\arctan x\,]_0^1$$
$$= \tfrac{4}{\pi} - \tfrac{4}{\pi}\times\tfrac{\pi}{4}$$
$$= \tfrac{4}{\pi} - 1$$
$$\therefore\ \mathrm{Var}(X) = \mathrm{E}(X^2) - \mu^2$$
$$= \tfrac{4}{\pi} - 1 - \left(\tfrac{2}{\pi}\ln 2\right)^2$$
$$= \tfrac{4}{\pi} - 1 - \left(\tfrac{2\ln 2}{\pi}\right)^2$$

9 If X is the volume of drink in mL, then $X \sim \mathrm{N}(376, \sigma^2)$.

Now $\mathrm{P}(X < 375) = 0.023$

$$\therefore\ \mathrm{P}\left(\frac{X-376}{\sigma} < \frac{375-376}{\sigma}\right) = 0.023$$
$$\therefore\ \mathrm{P}\left(Z < \frac{-1}{\sigma}\right) = 0.023$$
$$\therefore\ -\frac{1}{\sigma} \approx -1.995$$
$$\therefore\ \sigma \approx 0.501$$

$\therefore$ the standard deviation is 0.501 mL.

10 If X is the height of an 18 year old boy, then $X \sim \mathrm{N}(187, \sigma^2)$.

Now $\mathrm{P}(X > 193) = 0.15$

$$\therefore\ \mathrm{P}(X \leqslant 193) = 0.85$$
$$\therefore\ \mathrm{P}\left(\frac{X-187}{\sigma} \leqslant \frac{193-187}{\sigma}\right) = 0.85$$
$$\therefore\ \mathrm{P}\left(Z \leqslant \frac{6}{\sigma}\right) = 0.85$$
$$\therefore\ \frac{6}{\sigma} \approx 1.0364$$
$$\therefore\ \sigma \approx 5.789$$

So, $\mathrm{P}(X > 185) \approx 0.635$

$\therefore$ the probability that two 18 year old boys are taller than 185 cm $\approx 0.635^2$

≈ 0.403

11 **a**

$$\int_0^k f(x)\,dx = 1$$

$$\therefore \int_0^2 \frac{x}{5}\,dx + \int_2^k \frac{8}{5x^2}\,dx = 1$$

$$\therefore \left[\frac{x^2}{10}\right]_0^2 + \left[-\frac{8}{5x}\right]_2^k = 1$$

$$\therefore \tfrac{4}{10} + \left(-\frac{8}{5k}\right) - \left(-\tfrac{8}{10}\right) = 1$$

$$\therefore -\frac{8}{5k} = -\tfrac{2}{10}$$

$$\therefore 10k = 80$$

$$\therefore k = 8$$

b If m is the median of X, then $\int_0^m f(x)\,dx = \frac{1}{2}$

$\therefore$ since $\int_0^2 \frac{x}{5}\,dx < \frac{1}{2}$,

$$\int_0^2 \frac{x}{5}\,dx + \int_2^m \frac{8}{5x^2}\,dx = \tfrac{1}{2}$$

$$\therefore \tfrac{4}{10} + \left[-\frac{8}{5x}\right]_2^m = \tfrac{1}{2}$$

$$\therefore \tfrac{4}{10} + \left(-\frac{8}{5m}\right) - \left(-\tfrac{8}{10}\right) = \tfrac{1}{2}$$

$$\therefore -\frac{8}{5m} = -\tfrac{7}{10}$$

$$\therefore 35m = 80$$

$$\therefore m = \tfrac{16}{7}$$

$\therefore$ the median is $2\frac{2}{7}$

c

$$\begin{aligned}\mu &= \int_0^8 x\, f(x)\,dx \\ &= \int_0^2 \frac{x^2}{5}\,dx + \int_2^8 \frac{8}{5x}\,dx \\ &= \left[\frac{x^3}{15}\right]_0^2 + \left[\tfrac{8}{5}\ln|x|\right]_2^8 \\ &= \tfrac{8}{15} + \tfrac{8}{5}\ln 8 - \tfrac{8}{5}\ln 2 \\ &\approx 2.75\end{aligned}$$

$$\begin{aligned}\mathrm{E}(X^2) &= \int_0^8 x^2 f(x)\,dx \\ &= \int_0^2 \frac{x^3}{5}\,dx + \int_2^8 \tfrac{8}{5}\,dx \\ &= \left[\frac{x^4}{20}\right]_0^2 + \left[\tfrac{8}{5}x\right]_2^8 \\ &= \tfrac{16}{20} + \tfrac{64}{5} - \tfrac{16}{5} \\ &= \tfrac{52}{5}\end{aligned}$$

$$\begin{aligned}\therefore \mathrm{Var}(X) &= \mathrm{E}(X^2) - \mu^2 \\ &\approx \tfrac{52}{5} - 2.751^2 \\ &\approx 2.83\end{aligned}$$