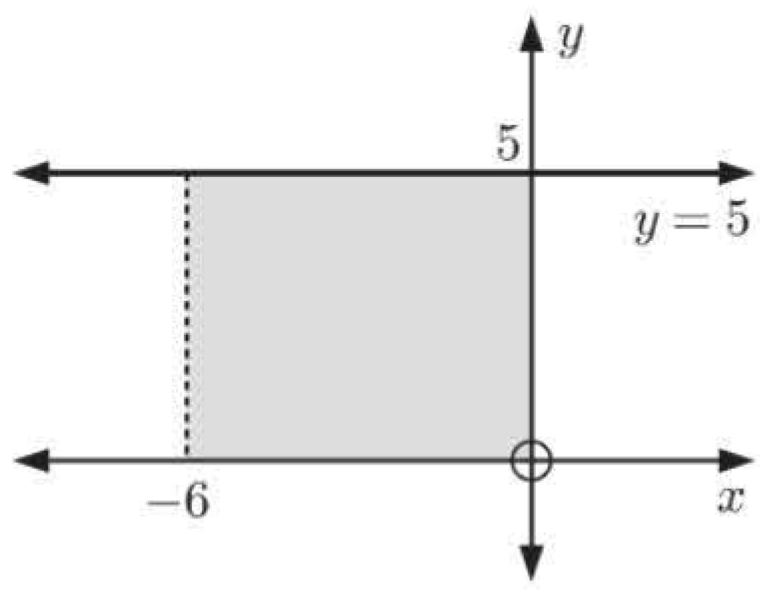
Chapter 22

APPLICATIONS OF INTEGRATION

EXERCISE 22A

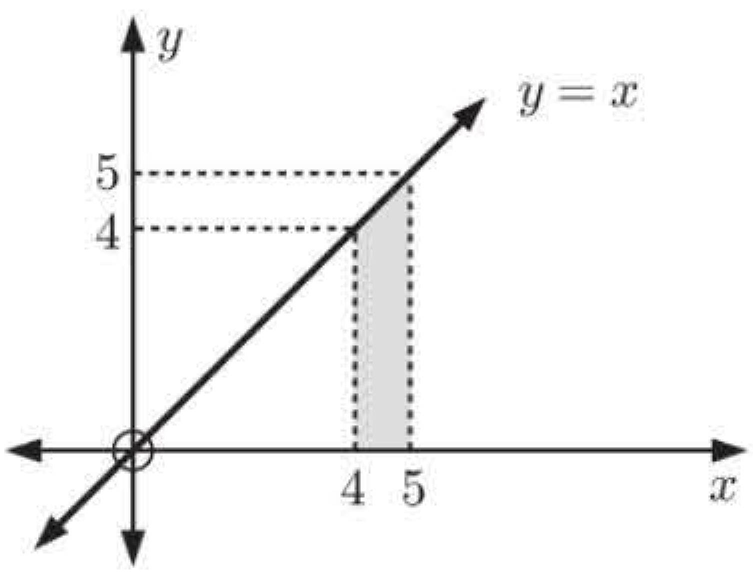
1 **a**



i $\text{Area} = 5 \times 6$
$= 30 \text{ units}^2$

ii $\text{Area} = \int_{-6}^{0} 5\,dx$
$= [5x]_{-6}^{0}$
$= 5(0) - 5(-6)$
$= 30 \text{ units}^2$

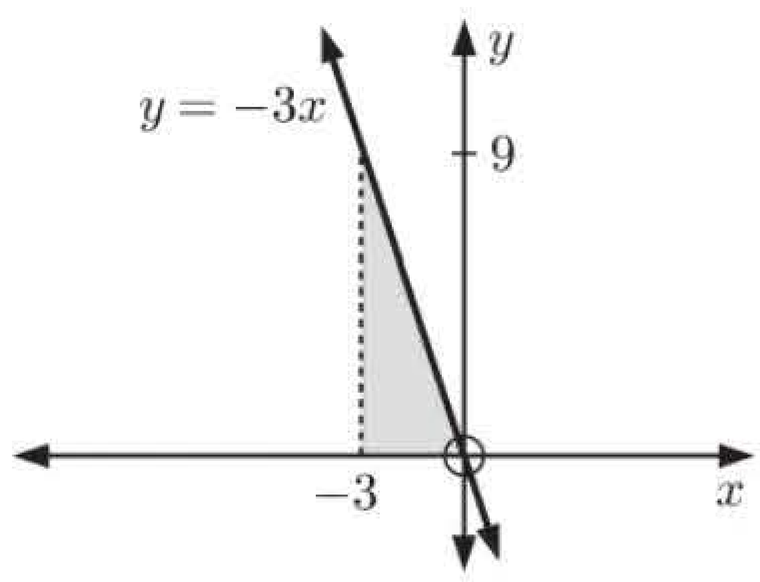
b



i $\text{Area} = \left(\dfrac{4+5}{2}\right) \times 1$
$= \frac{9}{2} \text{ units}^2$

ii $\text{Area} = \int_{4}^{5} x\,dx$
$= \left[\frac{1}{2}x^2\right]_4^5$
$= \frac{1}{2}(25) - \frac{1}{2}(16)$
$= \frac{9}{2} \text{ units}^2$

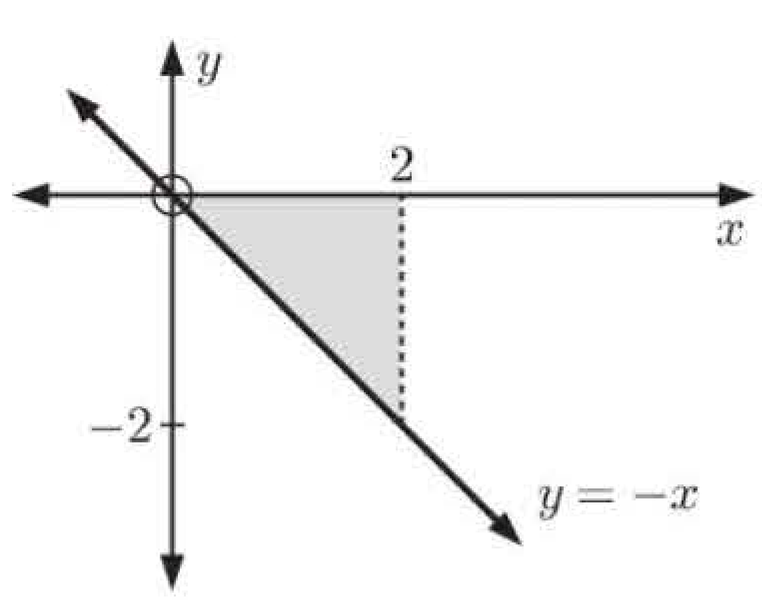
c



i $\text{Area} = \frac{1}{2} \times 3 \times 9$
$= \frac{27}{2} \text{ units}^2$

ii $\text{Area} = \int_{-3}^{0} (-3x)\,dx$
$= \left[-\frac{3}{2}x^2\right]_{-3}^{0}$
$= -\frac{3}{2}(0) - (-\frac{3}{2})(9)$
$= \frac{27}{2} \text{ units}^2$

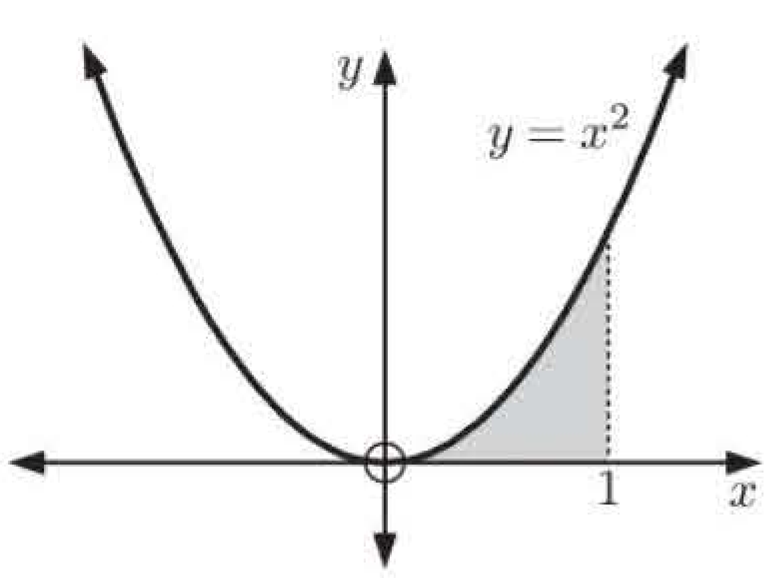
d



i $\text{Area} = \frac{1}{2} \times 2 \times 2$
$= 2 \text{ units}^2$

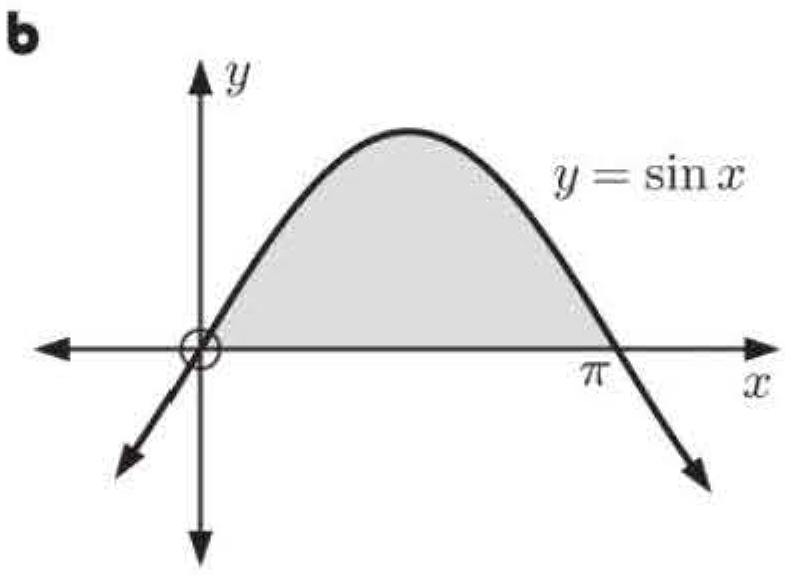
ii $\text{Area} = -\int_{0}^{2} -x\,dx$
$= -\left[-\frac{1}{2}x^2\right]_0^2$
$= -\left(-\frac{1}{2}(4) - (-\frac{1}{2})(0)\right)$
$= 2 \text{ units}^2$

2 **a**



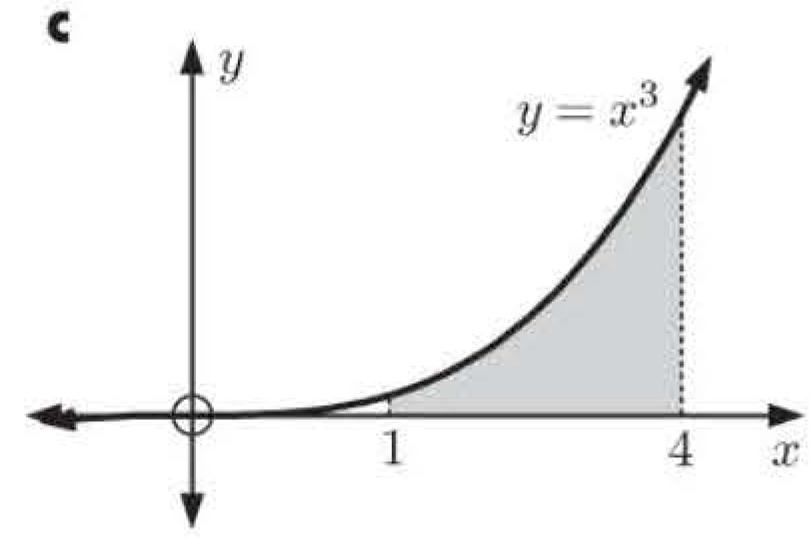
$\text{Area} = \int_0^1 x^2\,dx$
$= \left[\dfrac{x^3}{3}\right]_0^1$
$= \frac{1}{3} - 0$
$= \frac{1}{3} \text{ units}^2$

b



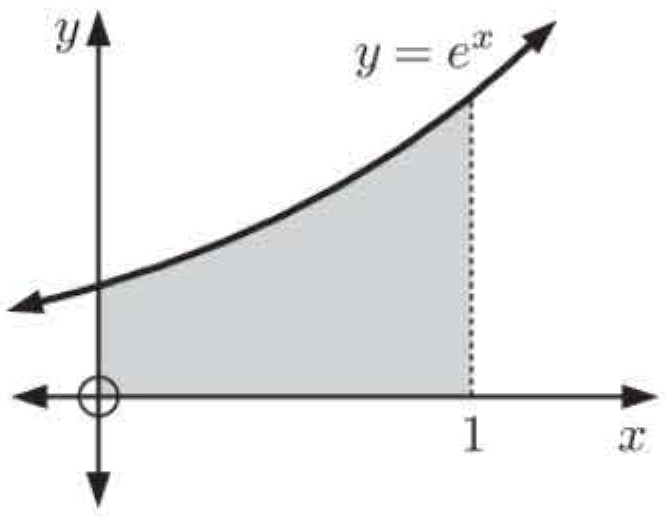
$\text{Area} = \int_0^{\pi} \sin x\,dx$
$= [-\cos x]_0^{\pi}$
$= -\cos\pi - (-\cos 0)$
$= 2 \text{ units}^2$

c



$\text{Area} = \int_1^4 x^3\,dx$
$= \left[\dfrac{x^4}{4}\right]_1^4$
$= \frac{256}{4} - \frac{1}{4}$
$= 63\frac{3}{4} \text{ units}^2$

d



Area $= \int_0^1 e^x \, dx$

$= [e^x]_0^1$

$= (e-1)$ units^2

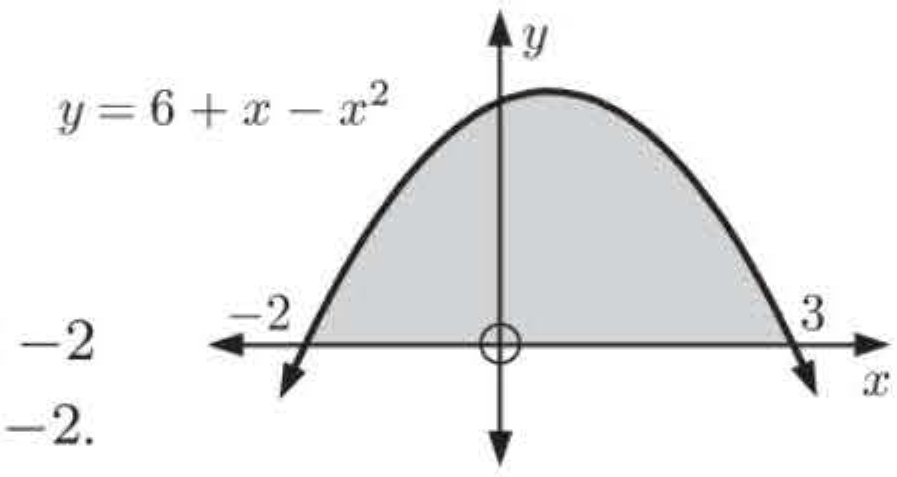
e The graph cuts the x-axis at $y = 0$.

$\therefore \quad 6 + x - x^2 = 0$

$\therefore \quad (3-x)(2+x) = 0$

$\therefore \quad x = 3$ or -2

The x-intercepts are 3 and -2.



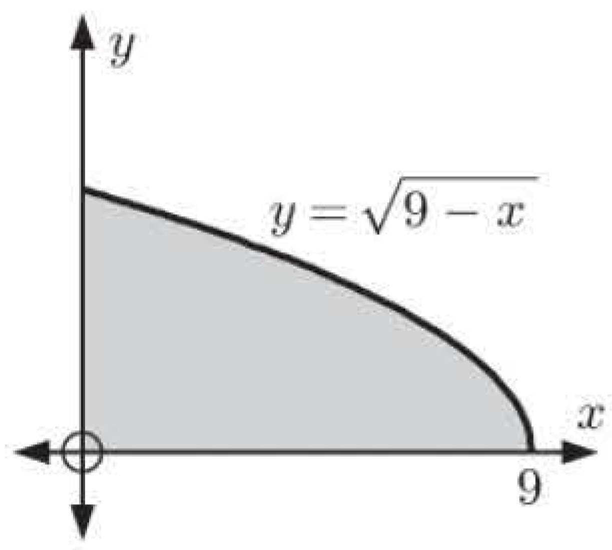
Area $= \int_{-2}^{3} (6 + x - x^2) \, dx$

$$= \left[6x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-2}^{3}$$

$= (18 + \frac{9}{2} - 9) - (-12 + 2 + \frac{8}{3})$

$= 20\frac{5}{6}$ units^2

f



Area $= \int_0^9 (9-x)^{\frac{1}{2}} \, dx$

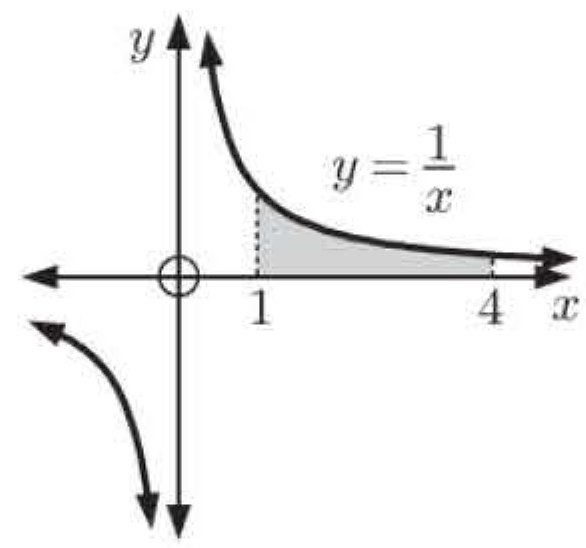
$$= \left[\left(\tfrac{1}{-1}\right) \frac{(9-x)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^9$$

$$= -\tfrac{2}{3}\left[(9-x)^{\frac{3}{2}}\right]_0^9$$

$= -\frac{2}{3}(0 - 27)$

$= 18$ units^2

g



$$\text{Area} = \int_1^4 \frac{1}{x} \, dx$$

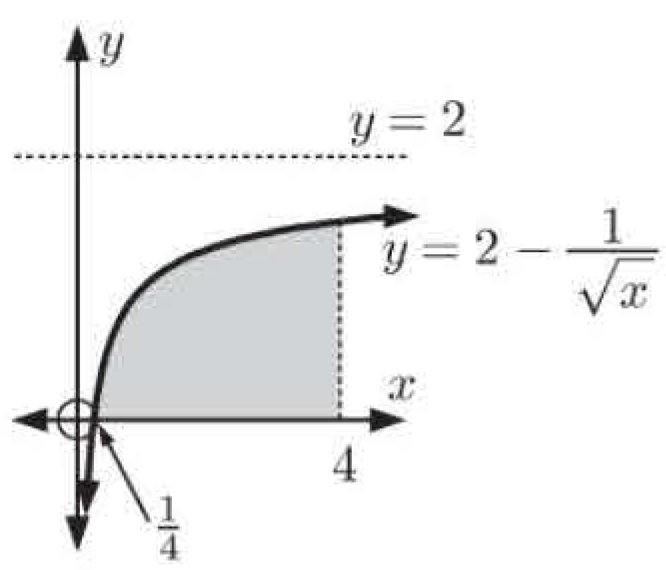
$= [\ln x]_1^4 \quad \{x > 0\}$

$= \ln 4 - \ln 1$

$= \ln 4 - 0$

$= \ln 4$ units^2

h



$$\text{Area} = \int_{\frac{1}{4}}^{4} \left(2 - \tfrac{1}{\sqrt{x}}\right) dx$$

$$= \int_{\frac{1}{4}}^{4} (2 - x^{-\frac{1}{2}}) \, dx$$

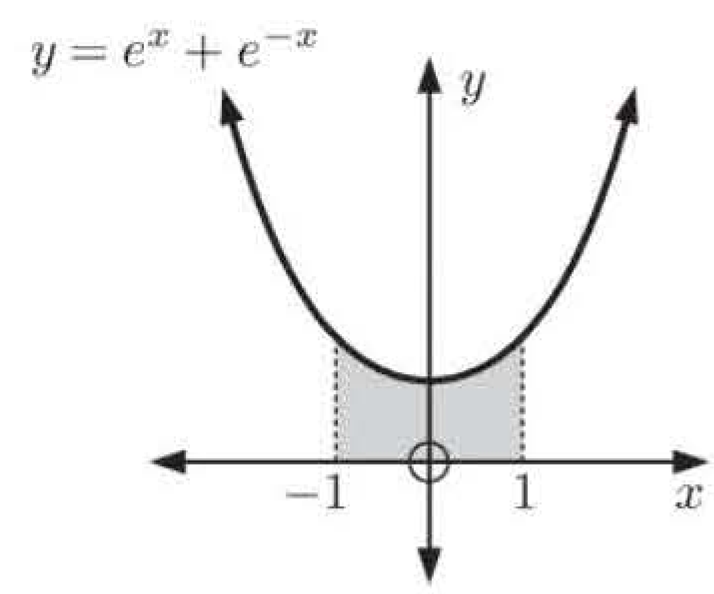
$$= \left[2x - \frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right]_{\frac{1}{4}}^{4}$$

$$= \left[2x - 2\sqrt{x}\right]_{\frac{1}{4}}^{4}$$

$= (8 - 4) - (\frac{1}{2} - 1)$

$= 4\frac{1}{2}$ units^2

i



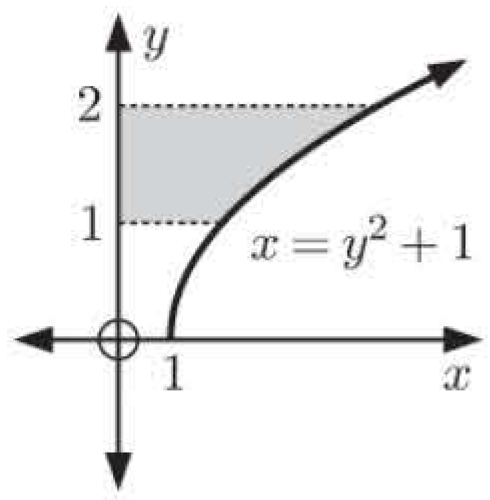
Area $= \int_{-1}^{1} (e^x + e^{-x}) \, dx$

$= \left[e^x - e^{-x}\right]_{-1}^{1}$

$= (e - e^{-1}) - (e^{-1} - e)$

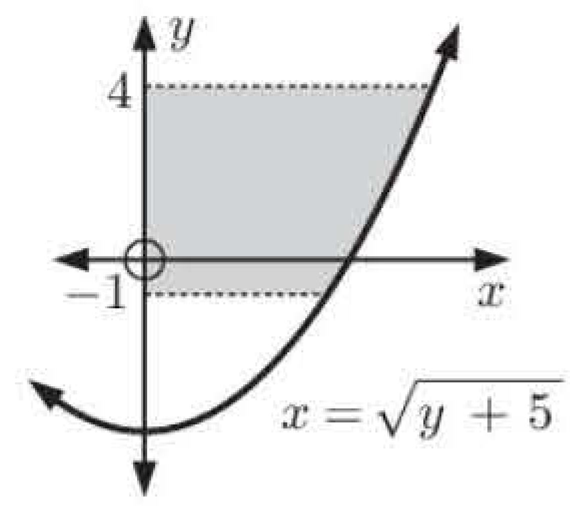
$$= \left(2e - \frac{2}{e}\right) \text{ units}^2$$

3 **a**



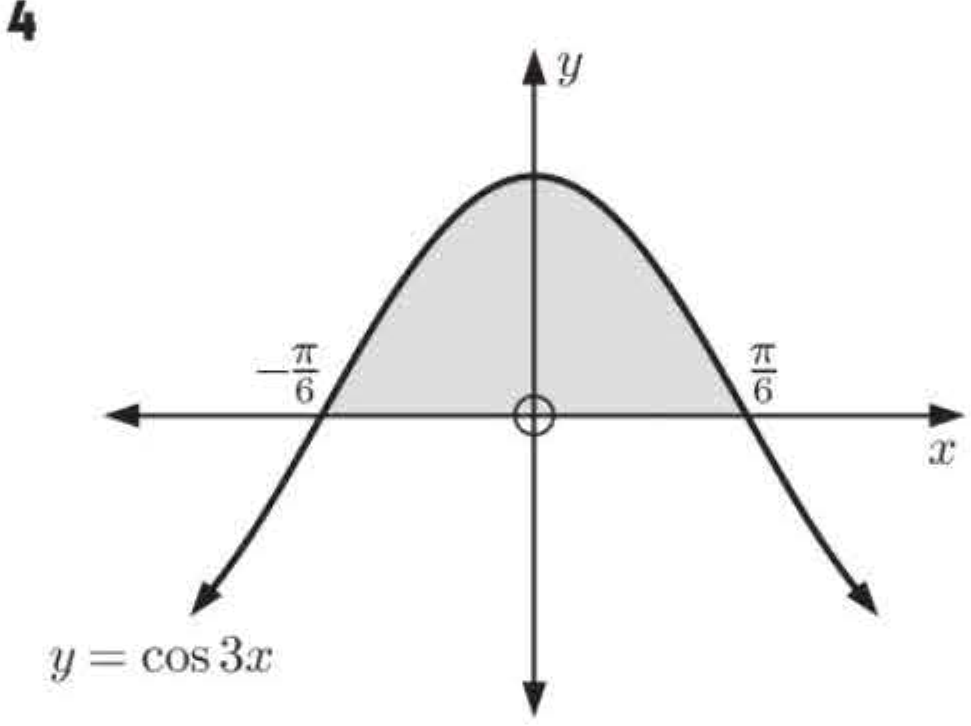
Area
$= \int_1^2 x\ dy$
$= \int_1^2 (y^2+1)\ dy$
$= \left[\dfrac{y^3}{3} + y\right]_1^2$
$= \left(\frac{8}{3}+2\right) - \left(\frac{1}{3}+1\right)$
$= 3\frac{1}{3}$ units^2

b



Area
$= \int_{-1}^{4} x\ dy$
$= \int_{-1}^{4} (y+5)^{\frac{1}{2}}\ dy$
$= \left[\dfrac{(y+5)^{\frac{3}{2}}}{\frac{3}{2}}\right]_{-1}^{4}$
$= \left[\frac{2}{3}(y+5)^{\frac{3}{2}}\right]_{-1}^{4}$
$= \frac{2}{3}(9)^{\frac{3}{2}} - \frac{2}{3}(4)^{\frac{3}{2}}$
$= \frac{2}{3}(27-8)$
$= \frac{2}{3} \times 19$
$= 12\frac{2}{3}$ units^2

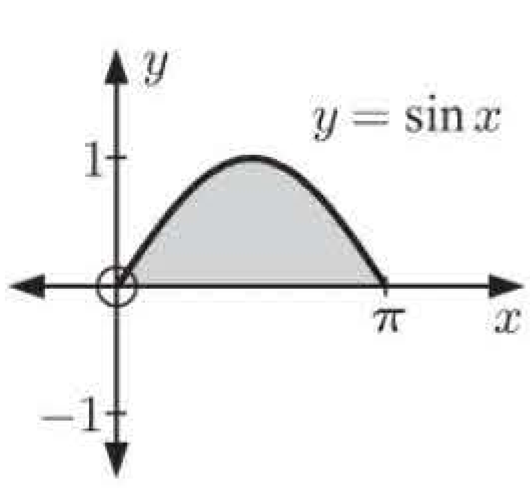
4



$y = \cos 3x$ has zeros at $\begin{cases} \frac{\pi}{6} \\ -\frac{\pi}{6} \end{cases} + \frac{2k\pi}{3}$, k an integer

$\therefore$ area $= \displaystyle\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos 3x\ dx$
$= \left[\frac{1}{3}\sin 3x\right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}}$
$= \frac{1}{3}\left(\sin(\frac{\pi}{2}) - \sin(-\frac{\pi}{2})\right)$
$= \frac{1}{3}(1-(-1))$
$= \frac{2}{3}$ units^2

5 **a**

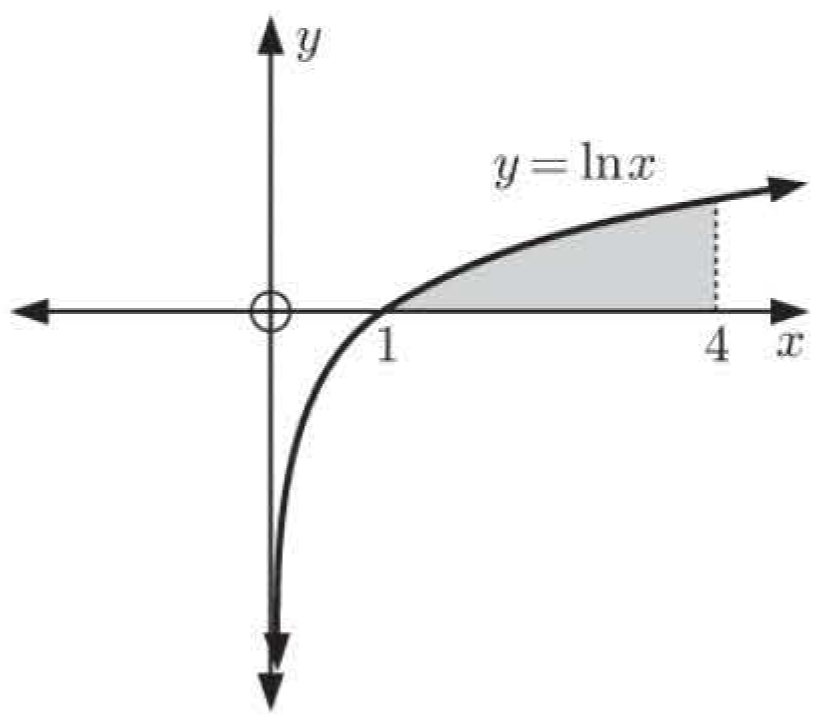


Area $= \int_0^{\pi} \sin x\ dx$
$= [-\cos x]_0^{\pi}$
$= [-\cos \pi + \cos 0]$
$= -(-1) + 1$
$= 2$ units^2

b Since $\sin^2 x \geqslant 0$ always, the function never drops below the x-axis.

$\therefore$ area $= \int_0^{\pi} \sin^2 x\ dx$
$= \int_0^{\pi} (\frac{1}{2} - \frac{1}{2}\cos(2x))\ dx$
$= \left[\dfrac{x}{2} - \frac{1}{4}\sin(2x)\right]_0^{\pi}$
$= \left[\frac{\pi}{2} - \frac{1}{4}\sin(2\pi)\right] - \left[0 - \frac{1}{4}\sin 0\right]$
$= \frac{\pi}{2}$ units^2

6



$$\begin{aligned}\text{Area} &= \int_1^4 \ln x \, dx \\ &= [x \ln x - x]_1^4 \\ &= 4\ln 4 - 4 - \ln 1 + 1 \\ &= (4 \ln 4 - 3) \text{ units}^2\end{aligned}$$

7 **a** $u = x \qquad v' = \sin x$

$\therefore\ u' = 1 \qquad v = -\cos x$

$\therefore\ \int x \sin x \, dx = -x\cos x - \int -\cos x \, dx$

$= -x \cos x + \sin x + c$

b $$\begin{aligned}\text{Area} &= \int_1^{\frac{\pi}{2}} x \sin x \, dx \\ &= \Big[-x\cos x + \sin x\Big]_1^{\frac{\pi}{2}} \\ &= -\tfrac{\pi}{2}\cos(\tfrac{\pi}{2}) + \sin(\tfrac{\pi}{2}) + \cos 1 - \sin 1 \\ &= (1 + \cos 1 - \sin 1) \text{ units}^2\end{aligned}$$

EXERCISE 22B

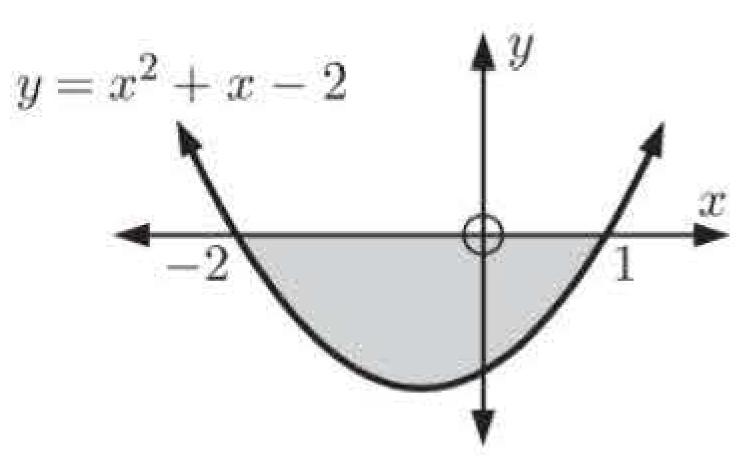
1 **a** The curve cuts the x-axis when $y = 0$.

$\therefore\ x^2 + x - 2 = 0$

$\therefore\ (x+2)(x-1) = 0$

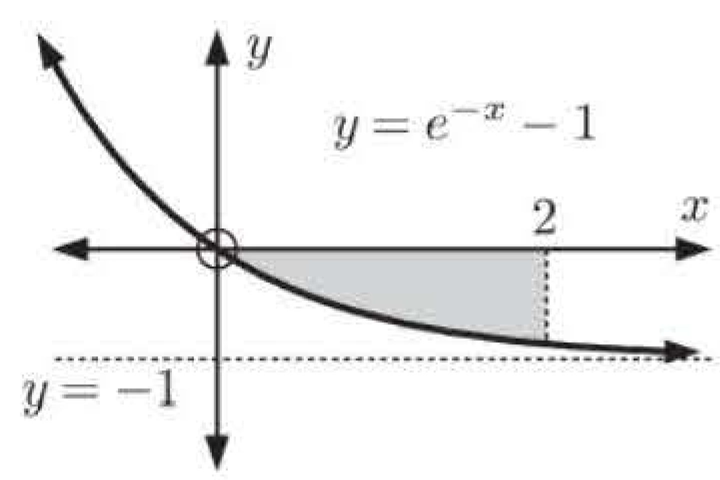
$\therefore\ x = -2$ or 1

$\therefore$ the x-intercepts are -2 and 1



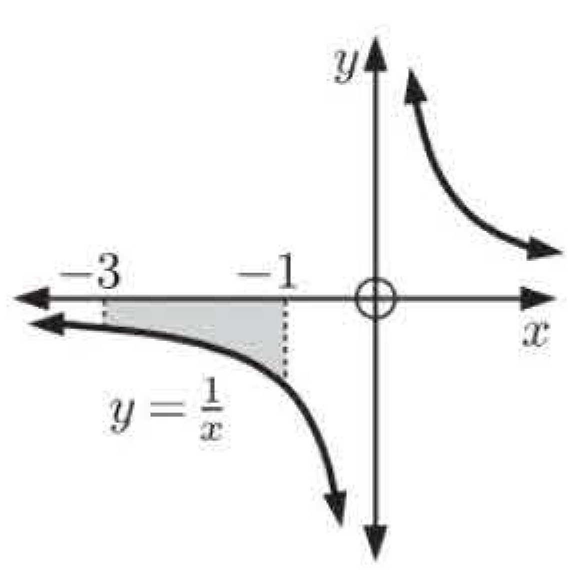
$$\begin{aligned}\text{Area} &= \int_{-2}^1 [0 - (x^2 + x - 2)] \, dx \\ &= \int_{-2}^1 (-x^2 - x + 2) \, dx \\ &= \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x\right]_{-2}^1 \\ &= (-\tfrac{1}{3} - \tfrac{1}{2} + 2) - (\tfrac{8}{3} - 2 - 4) \\ &= 4\tfrac{1}{2} \text{ units}^2\end{aligned}$$

b The curve cuts the x-axis at $(0, 0)$.



$$\begin{aligned}\text{Area} &= \int_0^2 [0 - (e^{-x} - 1)] \, dx \\ &= \int_0^2 (1 - e^{-x}) \, dx \\ &= \left[x + e^{-x}\right]_0^2 \\ &= \left(2 + e^{-2}\right) - \left(0 + e^0\right) \\ &= (1 + e^{-2}) \text{ units}^2\end{aligned}$$

c



Area

$$\begin{aligned} &= \int_{-3}^{-1} \left(0 - \frac{1}{x}\right) dx \\ &= -\int_{-3}^{-1} \left(\frac{1}{x}\right) dx \\ &= -\left[\ln|x|\right]_{-3}^{-1} \\ &= -(\ln 1 - \ln 3) \\ &= \ln 3 \text{ units}^2\end{aligned}$$

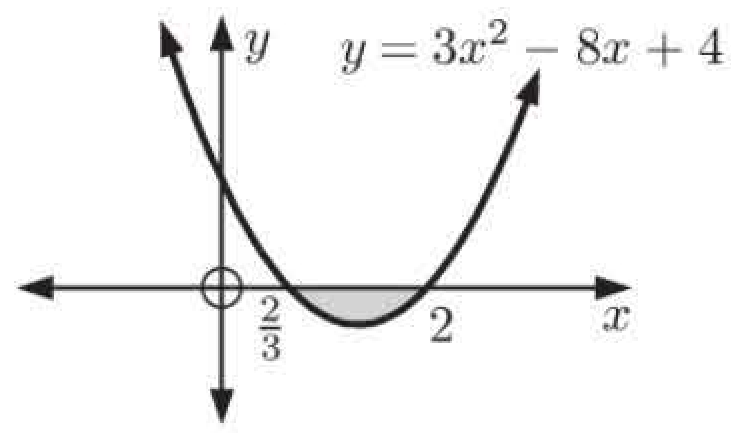
d The curve cuts the x-axis when $y = 0$.

$\therefore\ 3x^2 - 8x + 4 = 0$

$\therefore\ (3x-2)(x-2) = 0$

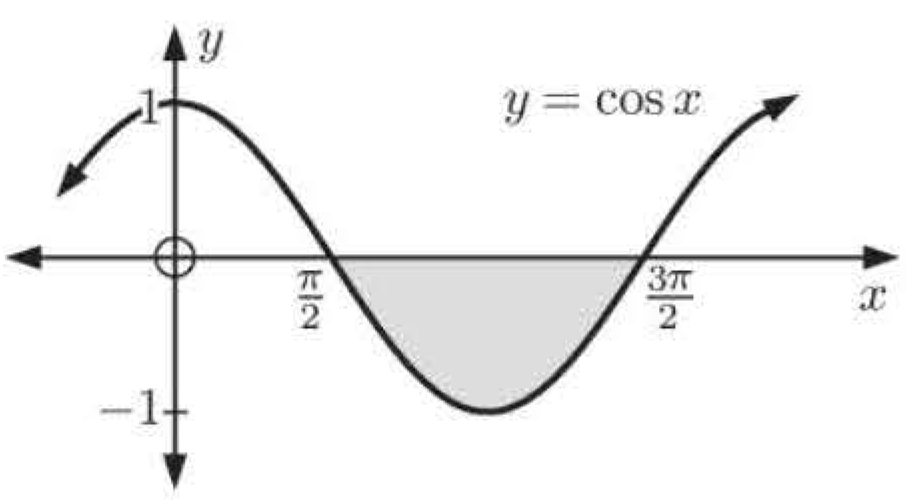
$\therefore\ x = 2$ or $\frac{2}{3}$

$\therefore$ the x-intercepts are 2 and $\frac{2}{3}$.



$$\text{Area} = \int_{\frac{2}{3}}^{2} [0 - (3x^2 - 8x + 4)]\, dx$$
$$= \int_{\frac{2}{3}}^{2} (-3x^2 + 8x - 4)\, dx$$
$$= \Big[-x^3 + 4x^2 - 4x\Big]_{\frac{2}{3}}^{2}$$
$$= (-8 + 16 - 8) - (-\tfrac{8}{27} + \tfrac{16}{9} - \tfrac{8}{3})$$
$$= 1\tfrac{5}{27} \text{ units}^2$$

e



$$\text{Area} = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} [0 - \cos x]\, dx$$
$$= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} -\cos x\, dx$$
$$= \Big[-\sin x\Big]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$
$$= -\sin(\tfrac{3\pi}{2}) - \big(-\sin(\tfrac{\pi}{2})\big)$$
$$= -(-1) - (-1)$$
$$= 2 \text{ units}^2$$

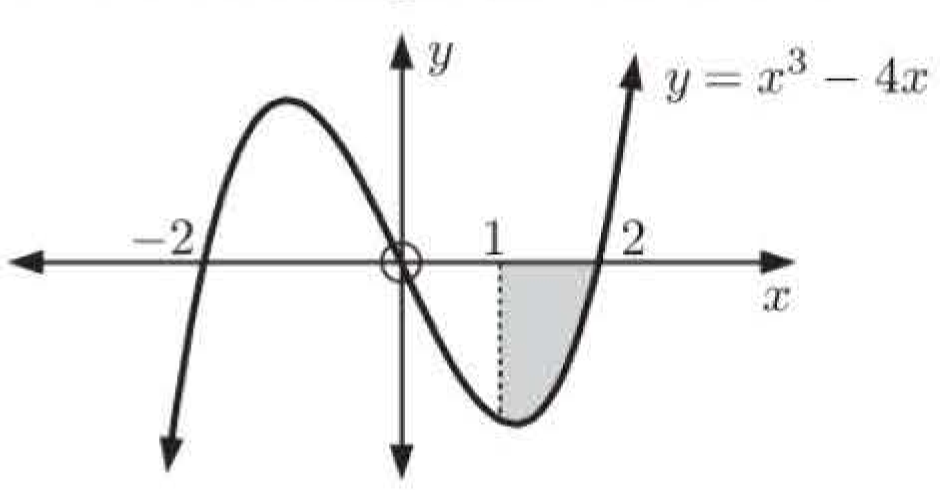
f The curve cuts the x-axis when $y = 0$.

$\therefore\ x^3 - 4x = 0$

$\therefore\ x(x^2 - 4) = 0$

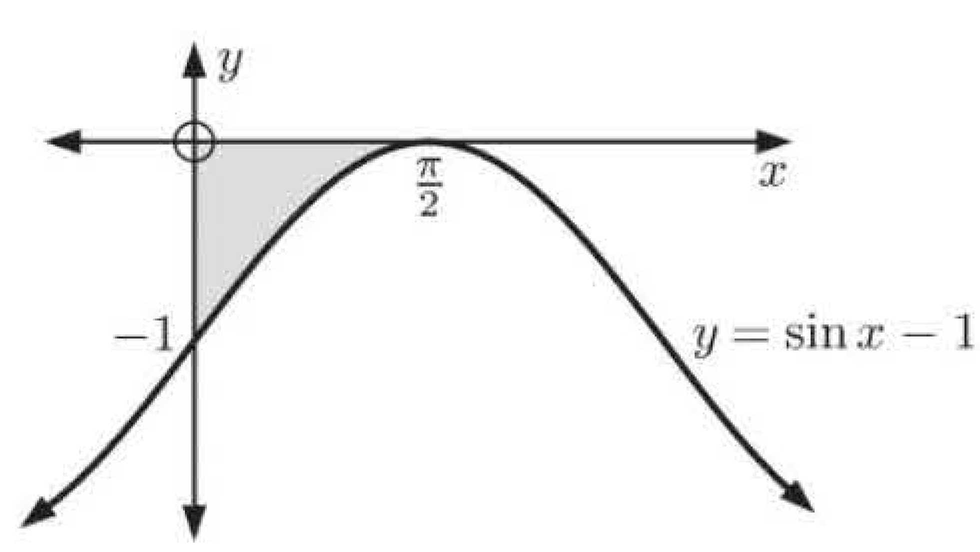
$\therefore\ x(x+2)(x-2) = 0$

$\therefore$ the x-intercepts are 0 and ± 2.



$$\text{Area} = \int_1^2 [0 - (x^3 - 4x)]\, dx$$
$$= \int_1^2 (-x^3 + 4x)\, dx$$
$$= \left[-\frac{x^4}{4} + 2x^2\right]_1^2$$
$$= (-4 + 8) - (-\tfrac{1}{4} + 2)$$
$$= 2\tfrac{1}{4} \text{ units}^2$$

g $y = \sin x - 1$ is the graph of $\sin x$ translated vertically -1 unit downwards.



$$\text{Area} = \int_0^{\frac{\pi}{2}} [0 - (\sin x - 1)]\, dx$$
$$= \int_0^{\frac{\pi}{2}} (1 - \sin x)\, dx$$
$$= \Big[x + \cos x\Big]_0^{\frac{\pi}{2}}$$
$$= (\tfrac{\pi}{2} + \cos\tfrac{\pi}{2}) - (0 + \cos 0)$$
$$= (\tfrac{\pi}{2} - 1) \text{ units}^2$$

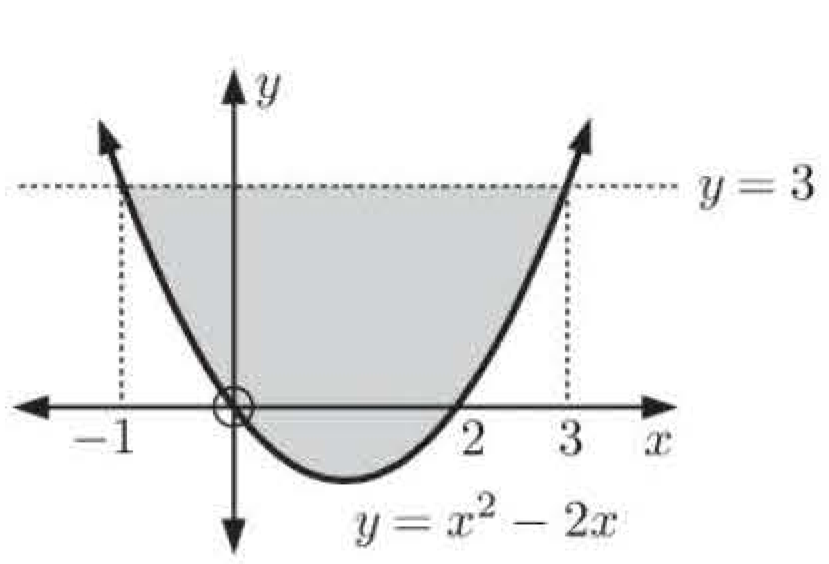
2 $y = x^2 - 2x$ meets $y = 3$

when $x^2 - 2x = 3$

$\therefore\ x^2 - 2x - 3 = 0$

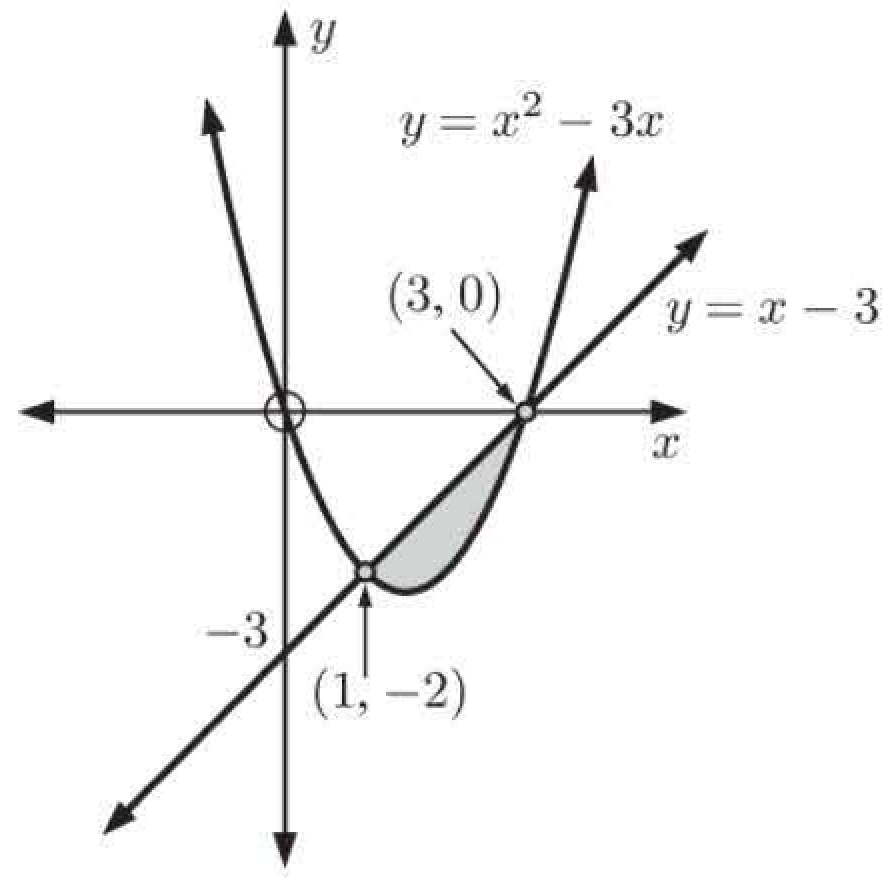
$\therefore\ (x-3)(x+1) = 0$

$\therefore\ x = 3$ or -1



$$A = \int_{-1}^{3} [3 - (x^2 - 2x)]\, dx$$
$$= \int_{-1}^{3} (3 + 2x - x^2)\, dx$$
$$= \left[3x + x^2 - \frac{x^3}{3}\right]_{-1}^{3}$$
$$= (9 + 9 - 9) - (-3 + 1 + \tfrac{1}{3})$$
$$= 10\tfrac{2}{3} \text{ units}^2$$

3 **a**



b The graphs meet where $x - 3 = x^2 - 3x$

$$\therefore \quad x^2 - 3x - x + 3 = 0$$
$$\therefore \quad x^2 - 4x + 3 = 0$$
$$\therefore \quad (x-1)(x-3) = 0$$
$$\therefore \quad x = 1 \text{ or } 3$$

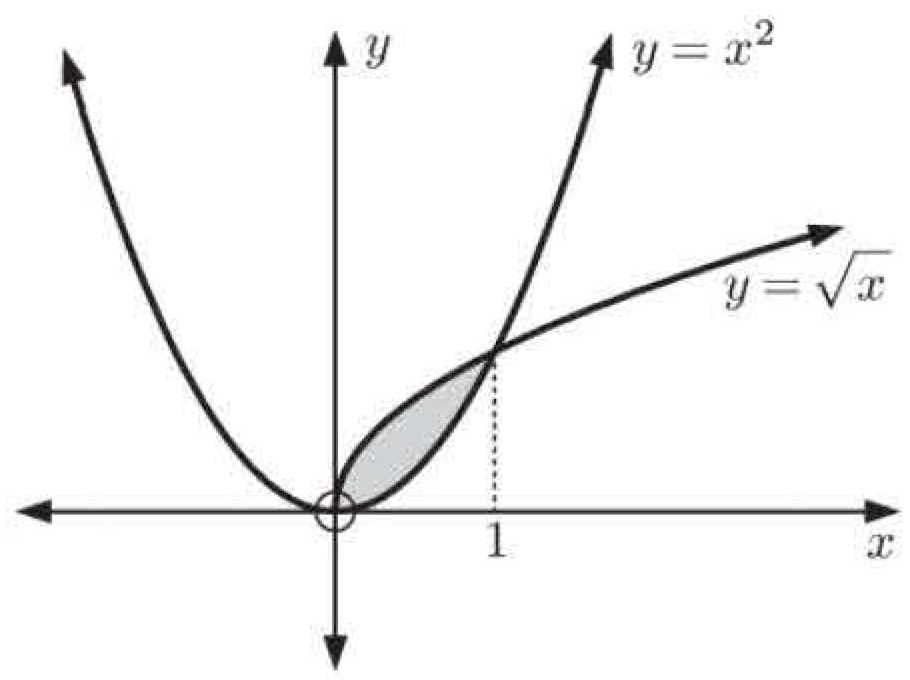
$\therefore$ the graphs meet at $(1, -2)$ and $(3, 0)$.

c
$$\begin{aligned}\text{Area} &= \int_1^3 [(x-3) - (x^2 - 3x)]\, dx \\ &= \int_1^3 (-3 + 4x - x^2)\, dx \\ &= \left[-3x + 2x^2 - \frac{x^3}{3}\right]_1^3 \\ &= (-9 + 18 - 9) - (-3 + 2 - \tfrac{1}{3}) \\ &= 1\tfrac{1}{3} \text{ units}^2\end{aligned}$$

4 $y = \sqrt{x}$ meets $y = x^2$ where $\sqrt{x} = x^2$

$$\therefore \quad x = x^4$$
$$\therefore \quad x^4 - x = 0$$
$$\therefore \quad x(x^3 - 1) = 0$$
$$\therefore \quad x(x-1)(x^2 + x + 1) = 0$$
$$\therefore \quad x = 0 \text{ or } 1$$

The factor $(x^2 + x + 1)$ has no real root since $\Delta = -3$ which is < 0.



$$\begin{aligned}\text{Area} &= \int_0^1 \left(\sqrt{x} - x^2\right) dx \\ &= \int_0^1 (x^{\frac{1}{2}} - x^2)\, dx \\ &= \left[\tfrac{2}{3}x^{\frac{3}{2}} - \frac{x^3}{3}\right]_0^1 \\ &= \tfrac{2}{3} - \tfrac{1}{3} \\ &= \tfrac{1}{3} \text{ unit}^2\end{aligned}$$

5 **a** $y = e^x - 1$ has no vertical asymptotes.

As $x \to \infty$, $e^x - 1 \to \infty$

As $x \to -\infty$, $e^x \to 0$

so $e^x - 1 \to -1^+$

$\therefore$ $y = -1$ is a horizontal asymptote.

$y = 0$ when $e^x - 1 = 0$

$$\therefore \quad e^x = 1$$
$$\therefore \quad x = 0$$

$\therefore$ x-intercept is $(0, 0)$.

This is also the y-intercept.

$y = 2 - 2e^{-x}$ has no vertical asymptotes.

As $x \to \infty$, $e^{-x} \to 0$

so $2 - 2e^{-x} \to 2^-$

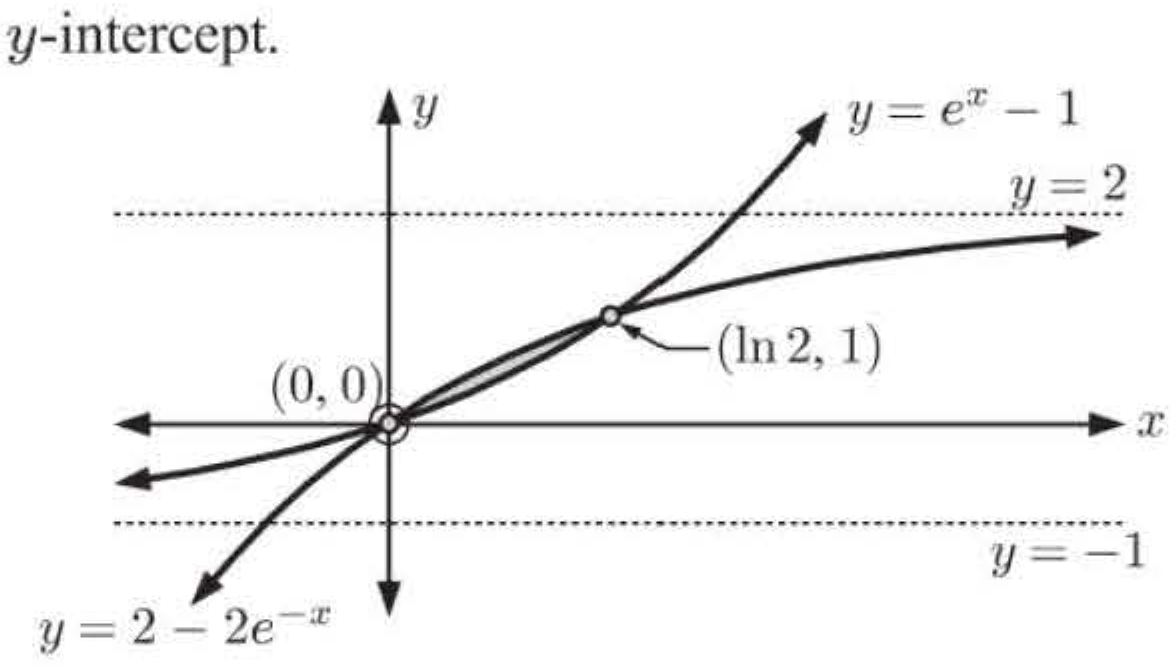
$\therefore$ $y = 2$ is a horizontal asymptote.

$y = 0$ when $2 - 2e^{-x} = 0$

$$\therefore \quad e^{-x} = 1$$
$$\therefore \quad x = 0$$

$\therefore$ x-intercept is $(0, 0)$.

This is also the y-intercept.



b $y = e^x - 1$ meets $y = 2 - 2e^{-x}$

where $e^x - 1 = 2 - 2e^{-x}$

$\therefore\ e^{2x} - e^x = 2e^x - 2 \quad \{\times\ e^x\}$

$\therefore\ e^{2x} - 3e^x + 2 = 0$

$\therefore\ (e^x - 1)(e^x - 2) = 0$

$\therefore\ e^x = 1$ or 2

$\therefore\ x = 0$ or $\ln 2$

$\therefore$ the graphs meet at $(0, 0)$ and $(\ln 2, 1)$.

c $\text{Area} = \int_0^{\ln 2} [(2 - 2e^{-x}) - (e^x - 1)]\, dx$

$= \int_0^{\ln 2} (3 - e^x - 2e^{-x})\, dx$

$= \left[3x - e^x + 2e^{-x}\right]_0^{\ln 2}$

$= (3\ln 2 - 2 + 1) - (0 - 1 + 2)$

$= 3\ln 2 - 2$

≈ 0.0794 units^2

6 $y = 2e^x$ meets $y = e^{2x}$ where

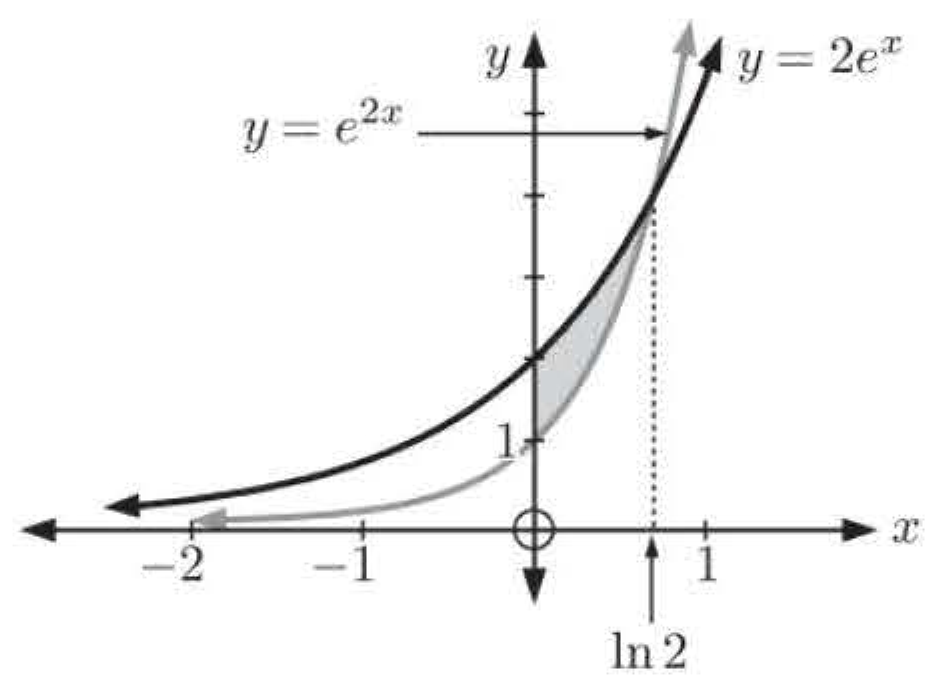
$2e^x = e^{2x}$

$\therefore\ e^{2x} - 2e^x = 0$

$\therefore\ e^x(e^x - 2) = 0$

$\therefore\ e^x = 2 \qquad \{e^x > 0 \text{ for all } x\}$

$\therefore\ x = \ln 2$



$\text{Area} = \int_0^{\ln 2} (2e^x - e^{2x})\, dx$

$= \left[2e^x - \frac{1}{2}e^{2x}\right]_0^{\ln 2}$

$= (4 - 2) - (2 - \frac{1}{2})$

$= \frac{1}{2}$ unit^2

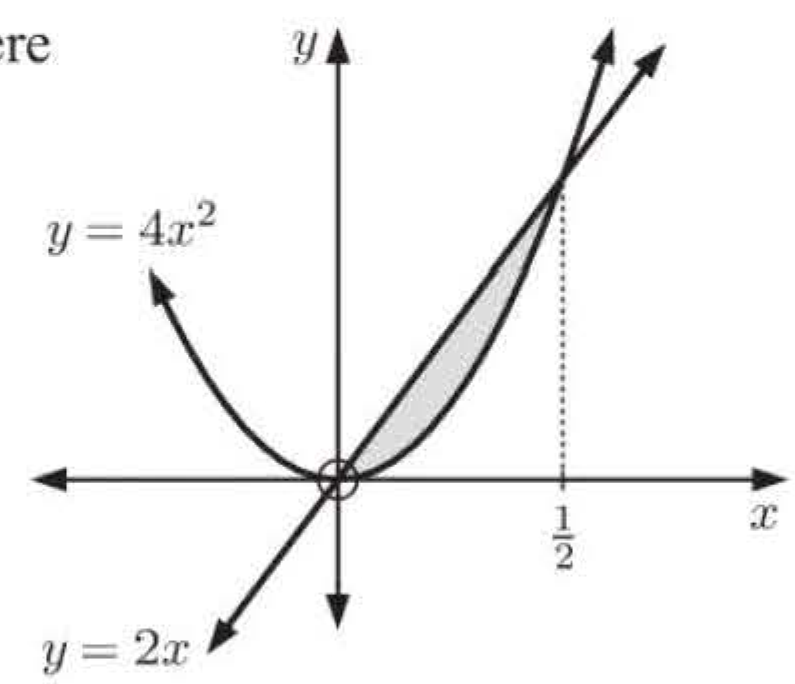
7 **a** $y = 2x$ meets $y = 4x^2$ where

$2x = 4x^2$

$\therefore\ 4x^2 - 2x = 0$

$\therefore\ 2x(2x - 1) = 0$

$\therefore\ x = 0$ or $\frac{1}{2}$



Area

$= \int_0^{\frac{1}{2}} (2x - 4x^2)\, dx$

$= \left[x^2 - \frac{4}{3}x^3\right]_0^{\frac{1}{2}}$

$= \left(\frac{1}{4} - \frac{4}{3}(\frac{1}{8})\right) - (0 - 0)$

$= \frac{1}{12}$ unit^2

b $y = 2x$ meets $y^2 = 4x$ where

$(2x)^2 = 4x$

$\therefore\ 4x^2 = 4x$

$\therefore\ 4x^2 - 4x = 0$

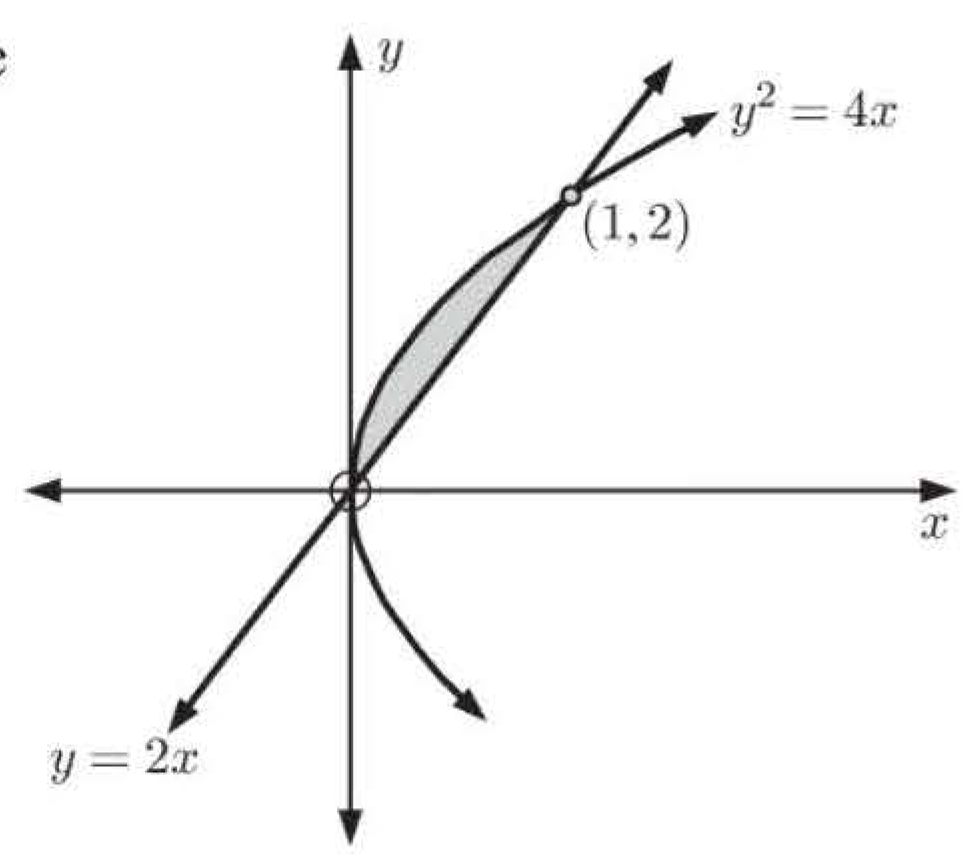
$\therefore\ 4x(x - 1) = 0$

$\therefore\ x = 0$ or 1

The upper part of $y^2 = 4x$

is $y = \sqrt{4x}$

or $y = 2\sqrt{x}$



Area

$= \int_0^1 (2\sqrt{x} - 2x)\, dx$

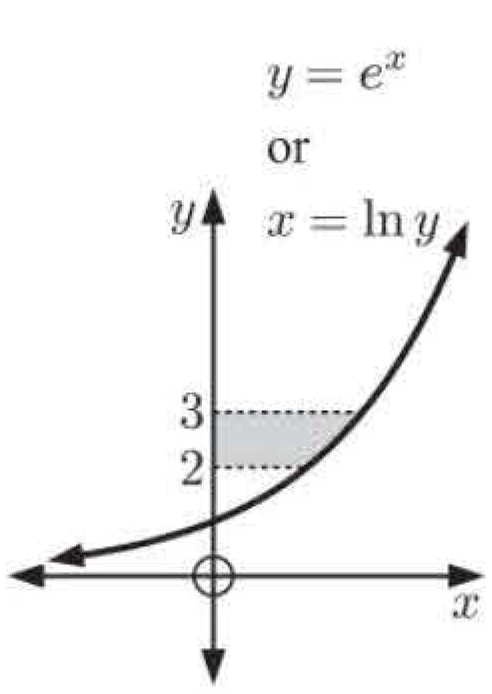
$= \int_0^1 (2x^{\frac{1}{2}} - 2x)\, dx$

$= \left[\frac{4}{3}x^{\frac{3}{2}} - x^2\right]_0^1$

$= \frac{4}{3} - 1$

$= \frac{1}{3}$ unit^2

8



$\text{Area} = \int_2^3 (x - 0)\ dy$

$= \int_2^3 \ln y\ dy$

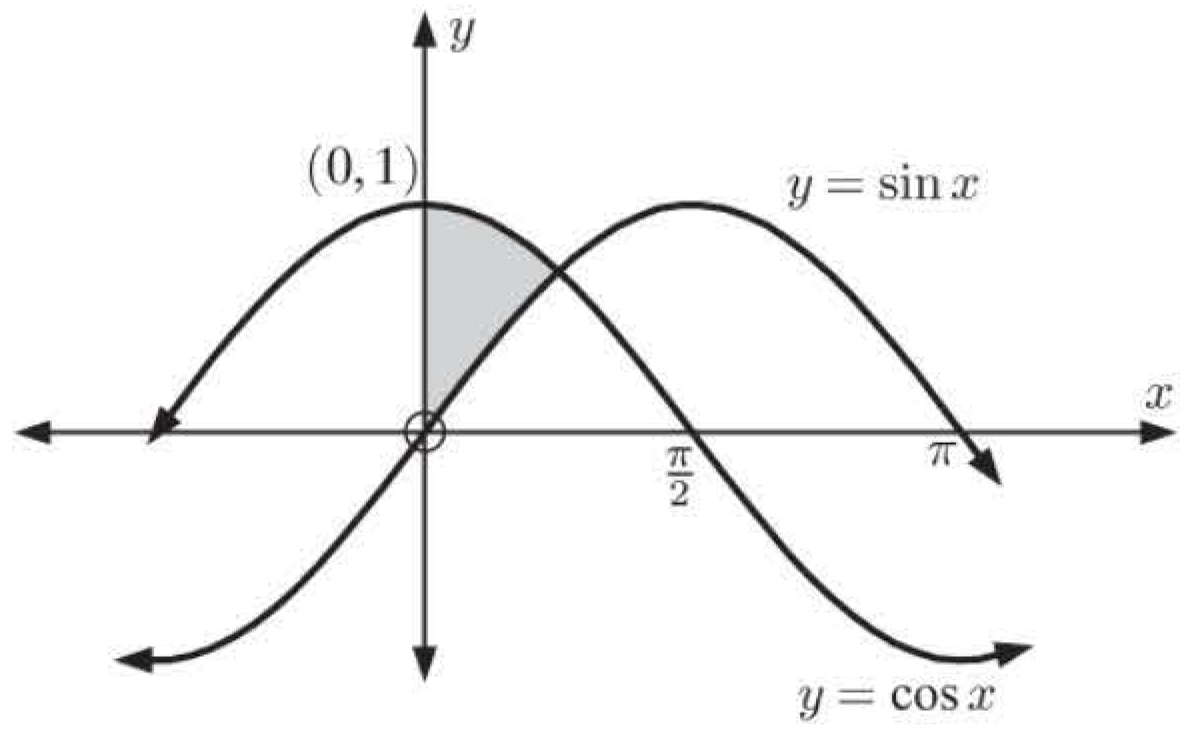
$= [y \ln y - y]_2^3$

$= (3\ln 3 - 3) - (2\ln 2 - 2)$

$= \ln 27 - \ln 4 - 1$

$= (\ln\left(\frac{27}{4}\right) - 1)$ units^2

9



The curves $y = \cos x$ and $y = \sin x$ meet when $x = \frac{\pi}{4}$.

$$\therefore \quad A = \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\, dx$$
$$= [\sin x + \cos x]_0^{\frac{\pi}{4}}$$
$$= (\sin \tfrac{\pi}{4} + \cos \tfrac{\pi}{4}) - (\sin 0 + \cos 0)$$
$$= \left(\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}\right) - (0 + 1)$$
$$= (\sqrt{2} - 1) \text{ units}^2$$

10 **a** Point A has y-coordinate 1 and lies on the graph of $y = \tan x$ on the interval $[0, \frac{\pi}{2}[$.

At this point, $\tan x = 1$ $\quad \therefore \; x = \frac{\pi}{4}$ $\quad \therefore$ A is at $(\frac{\pi}{4}, 1)$.

b Consider $\tan x = \dfrac{\sin x}{\cos x}$

Let $u = \cos x$, $\dfrac{du}{dx} = -\sin x$

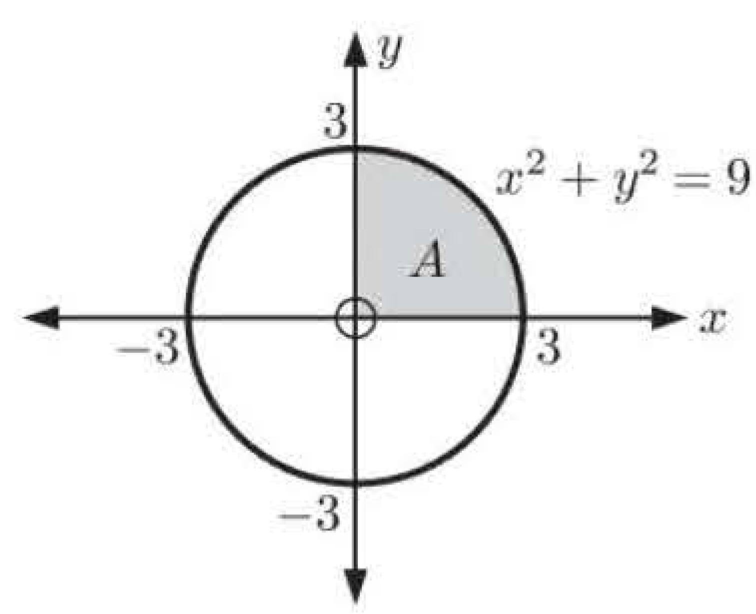
When $x = 0$, $u = \cos 0 = 1$

When $x = \frac{\pi}{4}$, $u = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$$\therefore \quad \text{area} = \int_0^{\frac{\pi}{4}} \tan x \, dx$$
$$= -\int_0^{\frac{\pi}{4}} \frac{-\sin x}{\cos x}\, dx$$
$$= -\int_0^{\frac{\pi}{4}} \frac{1}{u}\frac{du}{dx}\, dx$$
$$= -\int_1^{\frac{1}{\sqrt{2}}} \frac{1}{u}\, du$$
$$= \int_{\frac{1}{\sqrt{2}}}^{1} \frac{1}{u}\, du$$
$$= [\ln |u|]_{\frac{1}{\sqrt{2}}}^{1}$$
$$= \ln 1 - \ln \tfrac{1}{\sqrt{2}}$$
$$= \ln \sqrt{2} \text{ units}^2$$

11 **a** Now $x^2 + y^2 = 9 \quad \therefore \; y^2 = 9 - x^2$

$$\therefore \quad y = \pm\sqrt{9 - x^2}$$



In the upper half of the circle all y-values are $\geqslant 0$

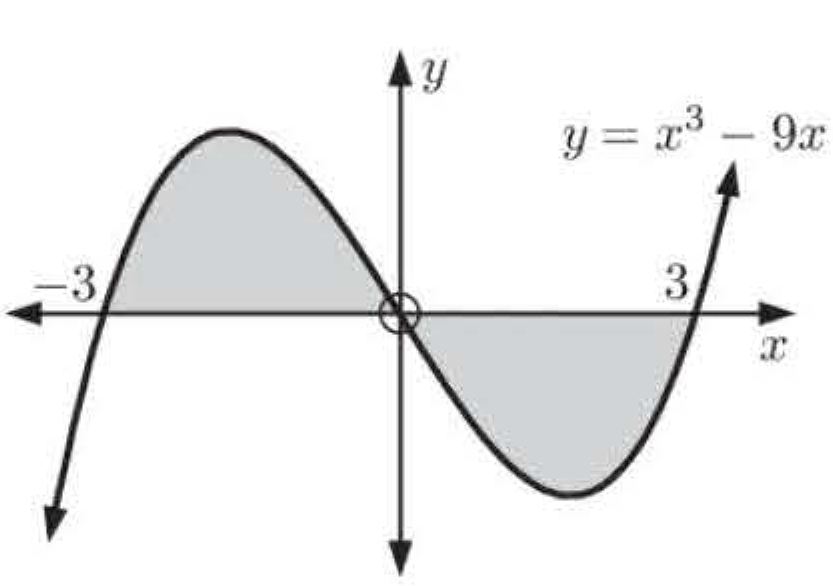
$\therefore \; y = +\sqrt{9 - x^2}$ is the required equation.

b The shaded area is A where $A = \int_0^3 \sqrt{9 - x^2}\, dx$

This is a quarter of the area of a circle with radius 3 units.

$\therefore \; A = \frac{1}{4}(\pi \times 3^2) = \frac{9\pi}{4} \approx 7.07 \text{ units}^2$

12 **a**
$$f(x) = x^3 - 9x$$
$$= x(x^2 - 9)$$
$$= x(x + 3)(x - 3)$$



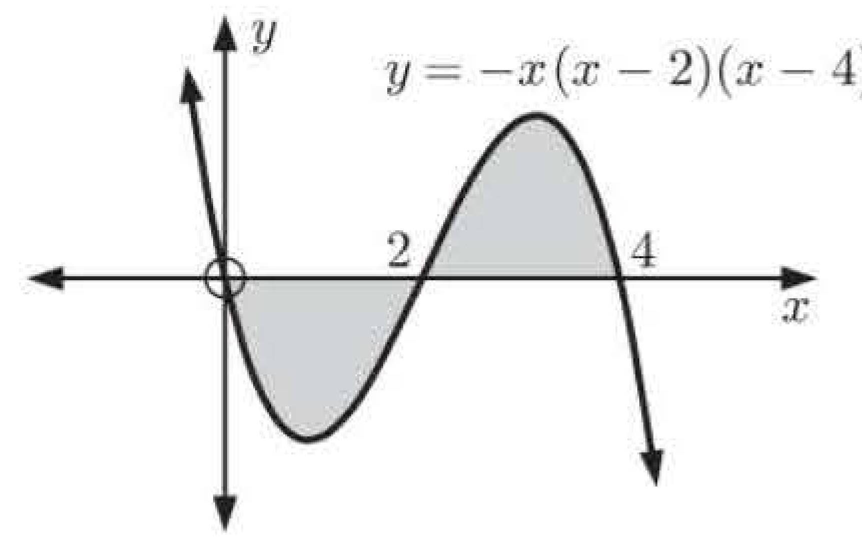
$\therefore \; y = f(x)$ cuts the x-axis at $0, \pm 3$

$$\text{Area} = \int_{-3}^{0} (x^3 - 9x)\, dx + \int_0^3 [0 - (x^3 - 9x)]\, dx$$
$$= \left[\frac{x^4}{4} - \frac{9x^2}{2}\right]_{-3}^{0} + \left[-\frac{x^4}{4} + \frac{9x^2}{2}\right]_0^3$$
$$= 0 - (\tfrac{81}{4} - \tfrac{81}{2}) + (-\tfrac{81}{4} + \tfrac{81}{2}) - 0$$
$$= 40\tfrac{1}{2} \text{ units}^2$$

b $f(x) = -x(x-2)(x-4)$

$= -x^3 + 6x^2 - 8x$

$\therefore \quad y = f(x)$ cuts the x-axis at 0, 2, and 4



Area $= \int_0^2 [0 - (-x^3 + 6x^2 - 8x)]\, dx$

$\quad + \int_2^4 (-x^3 + 6x^2 - 8x)\, dx$

$= \int_0^2 (x^3 - 6x^2 + 8x)\, dx + \int_2^4 (-x^3 + 6x^2 - 8x)\, dx$

$= \left[\dfrac{x^4}{4} - 2x^3 + 4x^2\right]_0^2 + \left[-\dfrac{x^4}{4} + 2x^3 - 4x^2\right]_2^4$

$= ([4 - 16 + 16] - 0) + ([-64 + 128 - 64] - [-4 + 16 - 16])$

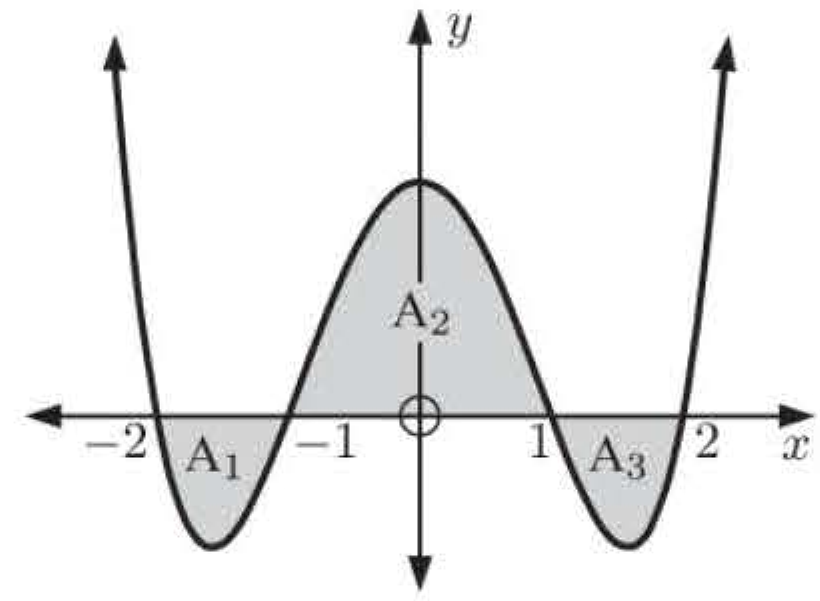
$= 8$ units^2

c $f(x) = x^4 - 5x^2 + 4$

$= (x^2 - 1)(x^2 - 4)$

$= (x+1)(x-1)(x+2)(x-2)$

$\therefore \quad y = f(x)$ cuts the x-axis at ± 1, ± 2



$A_1 = \int_{-2}^{-1} [0 - (x^4 - 5x^2 + 4)]\, dx$

$= \int_{-2}^{-1} (-x^4 + 5x^2 - 4)\, dx$

$= \left[-\dfrac{x^5}{5} + \dfrac{5x^3}{3} - 4x\right]_{-2}^{-1}$

$= (\frac{1}{5} - \frac{5}{3} + 4) - (\frac{32}{5} - \frac{40}{3} + 8)$

$= \frac{22}{15}$ units^2

$A_2 = \int_{-1}^{1} (x^4 - 5x^2 + 4)\, dx$

$= \left[\dfrac{x^5}{5} - \dfrac{5x^3}{3} + 4x\right]_{-1}^{1}$

$= (\frac{1}{5} - \frac{5}{3} + 4) - (-\frac{1}{5} + \frac{5}{3} - 4)$

$= \frac{76}{15}$ units^2

By symmetry, $A_3 = A_1$

$\therefore \quad$ area $= \frac{22}{15} + \frac{76}{15} + \frac{22}{15} = \frac{120}{15} = 8$ units^2

13 **a** $y = \sin(2x)$ is the curve C_1 and $y = \sin x$ is the curve C_2.

b The curves meet when $\sin(2x) = \sin x$

$\therefore \quad 2\sin x \cos x - \sin x = 0$

$\therefore \quad \sin x(2\cos x - 1) = 0$

$\therefore \quad \sin x = 0$ or $\cos x = \frac{1}{2}$

$\therefore \quad x = 0 + k\pi$ or $x = \begin{cases} \frac{\pi}{3} \\ \frac{5\pi}{3} \end{cases} + 2k\pi$, k an integer

$\therefore$ the x-coordinate of A $= \frac{\pi}{3}$

{smallest positive solution}

$\therefore \quad$ A is at $(\frac{\pi}{3}, \frac{\sqrt{3}}{2})$

c Area $= \int_0^{\frac{\pi}{3}} (\sin(2x) - \sin x)\, dx + \int_{\frac{\pi}{3}}^{\pi} (\sin x - \sin(2x))\, dx$

$= \left[-\frac{1}{2}\cos(2x) + \cos x\right]_0^{\frac{\pi}{3}} + \left[-\cos x + \frac{1}{2}\cos(2x)\right]_{\frac{\pi}{3}}^{\pi}$

$= \left(-\frac{1}{2}\cos\frac{2\pi}{3} + \cos\frac{\pi}{3}\right) - \left(-\frac{1}{2}\cos 0 + \cos 0\right) + \left(-\cos\pi + \frac{1}{2}\cos 2\pi\right)$

$\quad - \left(-\cos\frac{\pi}{3} + \frac{1}{2}\cos\frac{2\pi}{3}\right)$

$= (\frac{1}{4} + \frac{1}{2}) - (-\frac{1}{2} + 1) + (1 + \frac{1}{2}) - (-\frac{1}{2} - \frac{1}{4})$

$= 2\frac{1}{2}$ units^2

14 **a** **i** The graphs meet where $x^3 - 4x = 3x + 6$

$$\therefore \quad x^3 - 7x - 6 = 0$$
$$\therefore \quad (x+2)(x^2 - 2x - 3) = 0 \quad \{\text{diagram shows intersection at } -2\}$$
$$\therefore \quad (x+2)(x+1)(x-3) = 0$$
$$\therefore \quad x = -2, -1 \text{ or } 3$$

$$\therefore \quad \text{area} = \int_{-2}^{-1} \left([x^3 - 4x] - [3x+6]\right)\, dx + \int_{-1}^{3} \left([3x+6] - [x^3 - 4x]\right)\, dx$$
$$= \int_{-2}^{-1} \left(x^3 - 7x - 6\right)\, dx + \int_{-1}^{3} \left(-x^3 + 7x + 6\right)\, dx$$

ii Area $= \int_{-2}^{3} \left|x^3 - 7x - 6\right|\, dx$

b Using technology, area $= 32\frac{3}{4}$ units^2

15 **a** The graphs meet where

$$x^3 - 5x = 2x^2 - 6$$
$$\therefore \quad x^3 - 2x^2 - 5x + 6 = 0$$
$$\therefore \quad (x-1)(x^2 - x - 6) = 0$$
$$\therefore \quad (x-1)(x-3)(x+2) = 0$$
$$\therefore \quad x = -2,\ 1, \text{ or } 3$$

So, area $= \int_{-2}^{3} \left|x^3 - 2x^2 - 5x + 6\right|\, dx$

$= 21\frac{1}{12}$ units^2 {technology}

b The graphs meet where

$$-x^3 + 3x^2 + 6x - 8 = 5x - 5$$
$$\therefore \quad x^3 - 3x^2 - x + 3 = 0$$
$$\therefore \quad (x-1)(x^2 - 2x - 3) = 0$$
$$\therefore \quad (x-1)(x-3)(x+1) = 0$$
$$\therefore \quad x = -1,\ 1, \text{ or } 3$$

So, area $= \int_{-1}^{3} \left|x^3 - 3x^2 - x + 3\right|\, dx$

$= 8$ units^2 {technology}

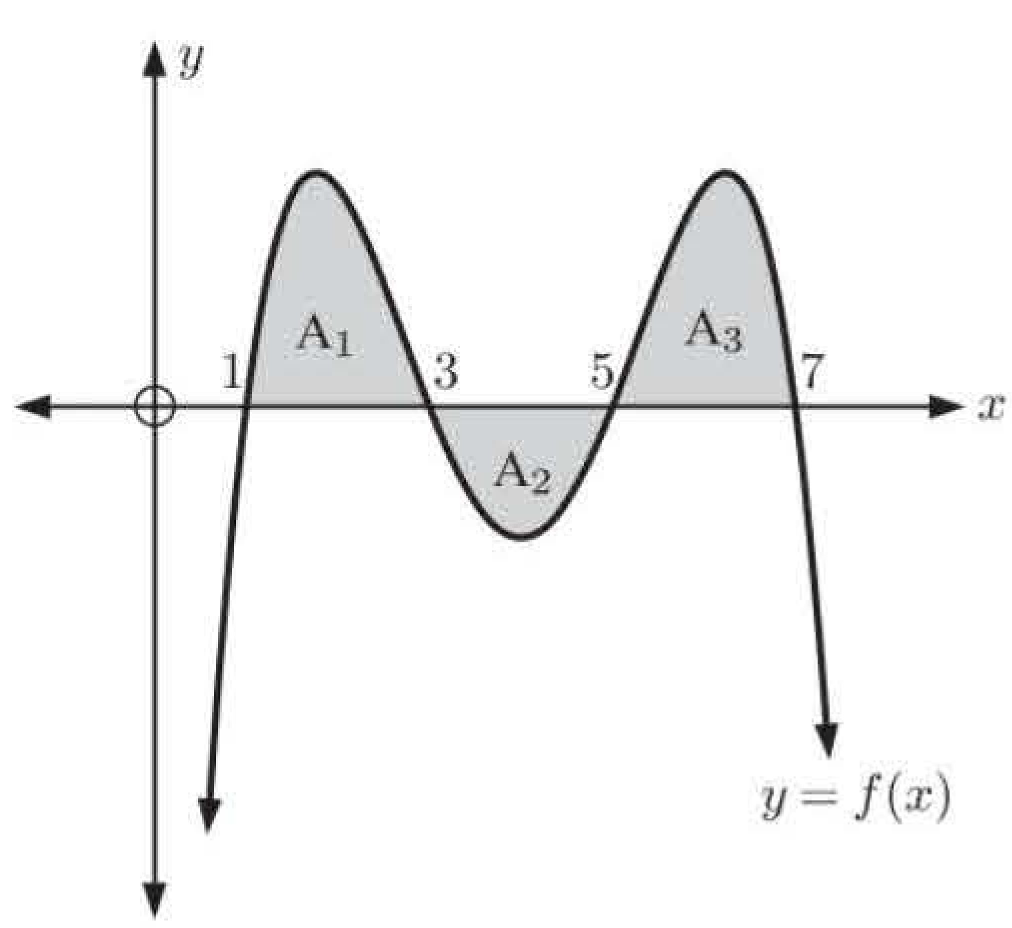
c The graphs meet where

$$2x^3 - 3x^2 + 18 = x^3 + 10x - 6$$
$$\therefore \quad x^3 - 3x^2 - 10x + 24 = 0$$
$$\therefore \quad (x-2)(x^2 - x - 12) = 0$$
$$\therefore \quad (x-2)(x-4)(x+3) = 0$$
$$\therefore \quad x = -3,\ 2, \text{ or } 4$$

So, area $= \int_{-3}^{4} \left|x^3 - 3x^2 - 10x + 24\right|\, dx$

$= 101\frac{3}{4}$ units^2 {technology}

16



a $\int_1^7 f(x)\, dx$ only gives us the correct area provided that $f(x)$ is positive on the interval $1 \leqslant x \leqslant 7$. But $f(x)$ is not positive for $3 \leqslant x \leqslant 5$, so $\int_1^7 f(x)\, dx = A_1 - A_2 + A_3$ which is *not* the shaded area.

b Shaded area

$$= \int_1^3 f(x)\, dx + \int_3^5 [0 - f(x)]\, dx + \int_5^7 f(x)\, dx$$
$$= \int_1^3 f(x)\, dx - \int_3^5 f(x)\, dx + \int_5^7 f(x)\, dx$$

17 **a** $y = \cos(2x)$ is the curve C_2 and $y = \cos^2 x$ is the curve C_1.

b Point A lies on $y = \cos(2x)$. When $x = 0$, $y = \cos 0 = 1$. $\therefore$ A is at $(0, 1)$.

Point B lies on $y = \cos(2x)$. When $x = \frac{\pi}{4}$, $y = \cos\frac{\pi}{2} = 0$. $\therefore$ B is at $(\frac{\pi}{4}, 0)$.

Point C lies on $y = \cos^2 x$. When $x = \frac{\pi}{2}$, $y = \cos^2\frac{\pi}{2} = 0$. $\therefore$ C is at $(\frac{\pi}{2}, 0)$.

Point D lies on $y = \cos(2x)$. When $x = \frac{3\pi}{4}$, $y = \cos\frac{3\pi}{2} = 0$. $\therefore$ D is at $(\frac{3\pi}{4}, 0)$.

Point E lies where the curves meet. Now $\cos(2\pi) = \cos^2\pi = 1$. $\therefore$ E is at $(\pi, 1)$.

c Area $= \int_0^{\pi} (\cos^2 x - \cos(2x))\,dx$

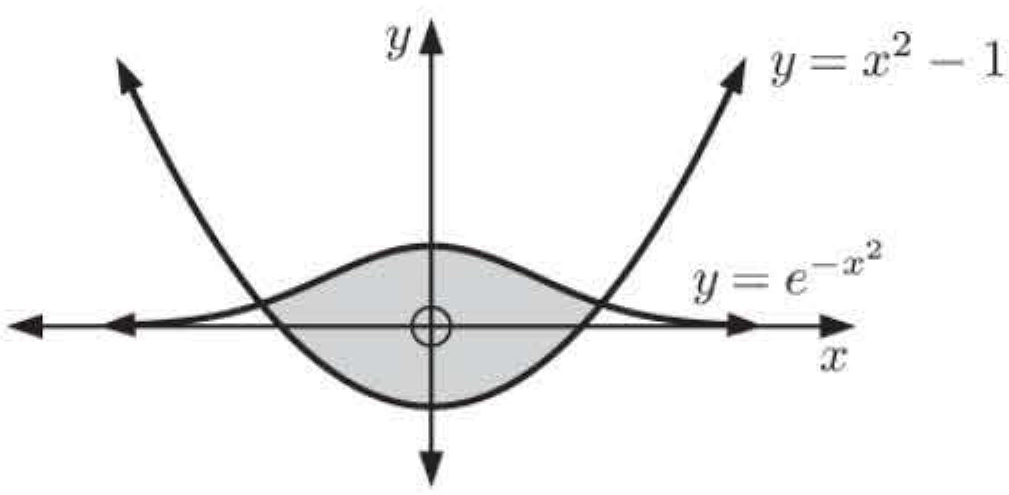
$= \int_0^{\pi} \left(\frac{1}{2} + \frac{1}{2}\cos(2x) - \cos(2x)\right)\,dx$

$= \int_0^{\pi} \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right)\,dx$

$= \left[\frac{x}{2} - \frac{1}{4}\sin(2x)\right]_0^{\pi} = (\frac{\pi}{2} - 0) - (0 - 0) = \frac{\pi}{2}$ units^2

18 **a** The graphs meet when $e^{-x^2} = x^2 - 1$

$\therefore\ x = \pm 1.1307$ {technology}

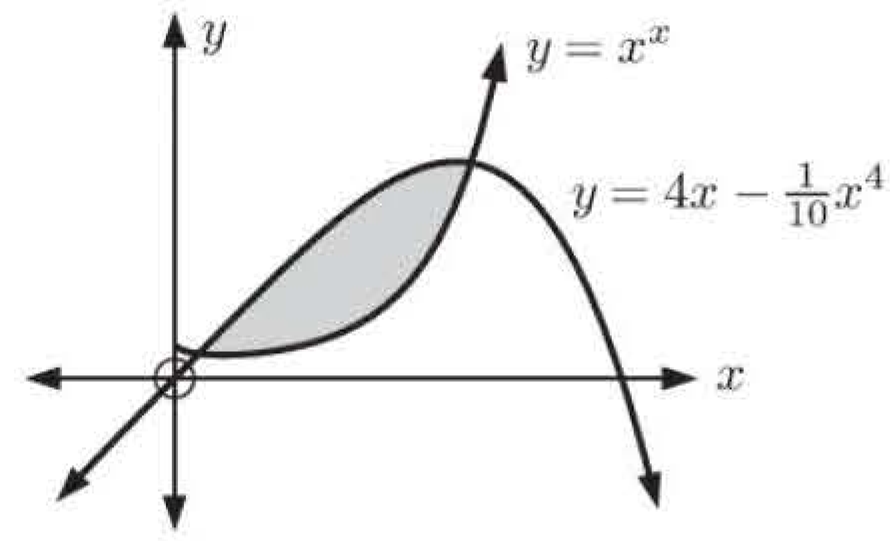


$\therefore$ area $= \int_{-1.1307}^{1.1307} [e^{-x^2} - (x^2 - 1)]\,dx$

≈ 2.88 units^2 {technology}

b The graphs meet when $x^x = 4x - \frac{1}{10}x^4$

$\therefore\ x \approx 0.1832$ or 2.2696 {technology}



$\therefore$ area $= \int_{0.1832}^{2.2696} (4x - \frac{1}{10}x^4 - x^x)\,dx$

≈ 4.97 units^2 {technology}

19 **a** Area $= \displaystyle\int_1^k \frac{1}{1+2x}\,dx = 0.2$ units^2

$\therefore\ \left[\frac{1}{2}\ln(1+2x)\right]_1^k = 0.2,\ 1 + 2x > 0$

$\therefore\ [\ln(1+2x)]_1^k = 0.4$

$\therefore\ \ln(1+2k) - \ln 3 = 0.4$

{since $k \geqslant 1,\ 1 + 2x > 0$ for all x in the shaded region}

$\therefore\ \ln\left(\dfrac{1+2k}{3}\right) = 0.4$

$\therefore\ \dfrac{1+2k}{3} = e^{0.4}$

$\therefore\ 1 + 2k = 3e^{0.4}$

$\therefore\ k = \dfrac{3e^{0.4} - 1}{2} \approx 1.7377$

b Area $= \int_0^b \sqrt{x}\,dx$

$\therefore\ \int_0^b x^{\frac{1}{2}}\,dx = 1$

$\therefore\ \left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^b = 1$

$\therefore\ \frac{2}{3}b\sqrt{b} - 0 = 1$

$\therefore\ b\sqrt{b} = \frac{3}{2}$

$\therefore\ b^{\frac{3}{2}} = 1.5$

$\therefore\ b = (1.5)^{\frac{2}{3}}$

≈ 1.3104

20 **a** $y = x^2$ meets $y = k$ where $x^2 = k$

$\therefore\ x = \pm\sqrt{k}$

Now, the area $= \int_0^{\sqrt{k}} (k - x^2)\,dx$

$\therefore\ \int_0^{\sqrt{k}} (k - x^2)\,dx = 2.4$

$\therefore\ \left[kx - \dfrac{x^3}{3}\right]_0^{\sqrt{k}} = 2.4$

$\therefore\ k\sqrt{k} - \dfrac{k\sqrt{k}}{3} - 0 = 2.4$

$\therefore\ \dfrac{2k\sqrt{k}}{3} = 2.4$

$\therefore\ k^{\frac{3}{2}} = 3.6$

$\therefore\ k = (3.6)^{\frac{2}{3}}$

≈ 2.3489

b By symmetry, the area bounded by $x = 0$ and $x = a$ is $\frac{1}{2}(6a)$ units^2.

$\therefore\ \int_0^a (x^2 + 2)\,dx = 3a$

$\therefore\ \left[\dfrac{x^3}{3} + 2x\right]_0^a = 3a$

$\therefore\ \dfrac{a^3}{3} + 2a - 0 = 3a$

$\therefore\ a^3 + 6a = 9a$

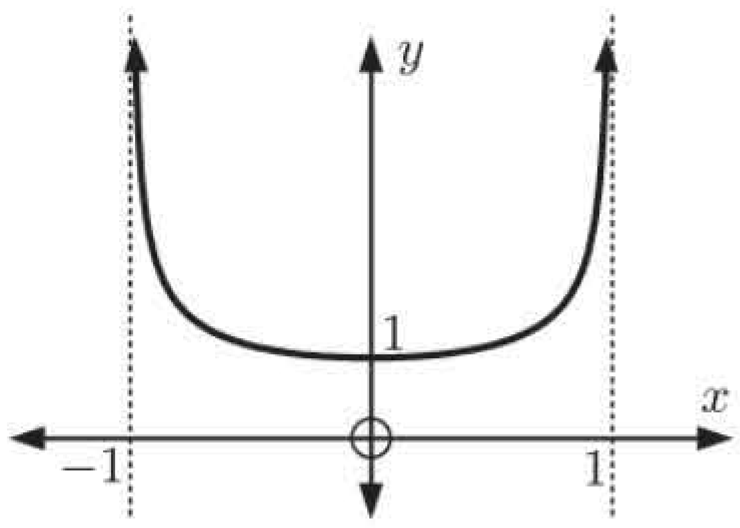
$\therefore\ a^3 - 3a = 0$

$\therefore\ a(a^2 - 3) = 0$

$\therefore\ a = 0$ or $\pm\sqrt{3}$

$\therefore\ a = \sqrt{3}$ {as $a > 0$}

21 **a**



b **i** If $f(x)=\dfrac{1}{\sqrt{1-x^2}}$ then $f(-x)=\dfrac{1}{\sqrt{1-(-x)^2}}$

$$=\frac{1}{\sqrt{1-x^2}}$$

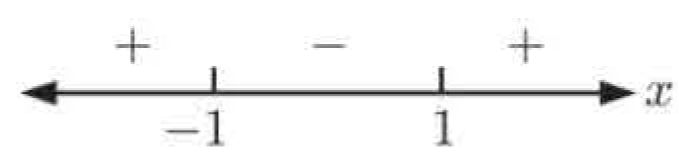
$$=f(x) \text{ for all } x$$

$\therefore$ $f(x)$ is an even function, and so it is symmetric about the y-axis.

ii $f(x)$ is defined when $1-x^2>0$

$\therefore$ $x^2-1<0$

$\therefore$ $(x+1)(x-1)<0$



$\therefore$ $x\in\,]-1,\,1\,[$

c Area $=\displaystyle\int_0^{\frac{1}{2}}\frac{1}{\sqrt{1-x^2}}\,dx$

$$=\Big[\arcsin(x)\Big]_0^{\frac{1}{2}}$$

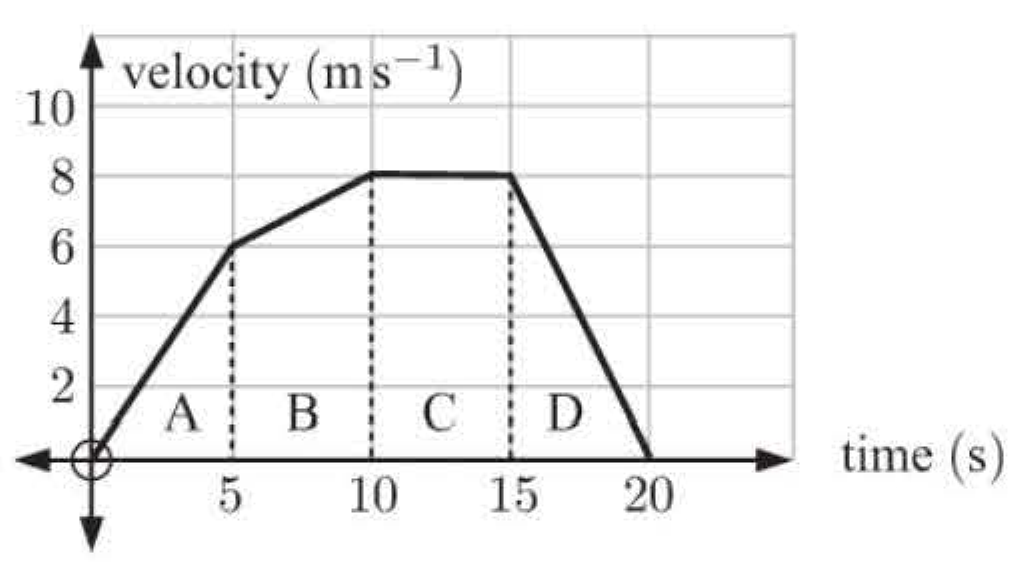
$$=\arcsin\left(\tfrac{1}{2}\right)-\arcsin(0)$$

$$=\tfrac{\pi}{6}-0$$

$$=\tfrac{\pi}{6}\text{ units}^2$$

EXERCISE 22C.1

1



Total distance travelled

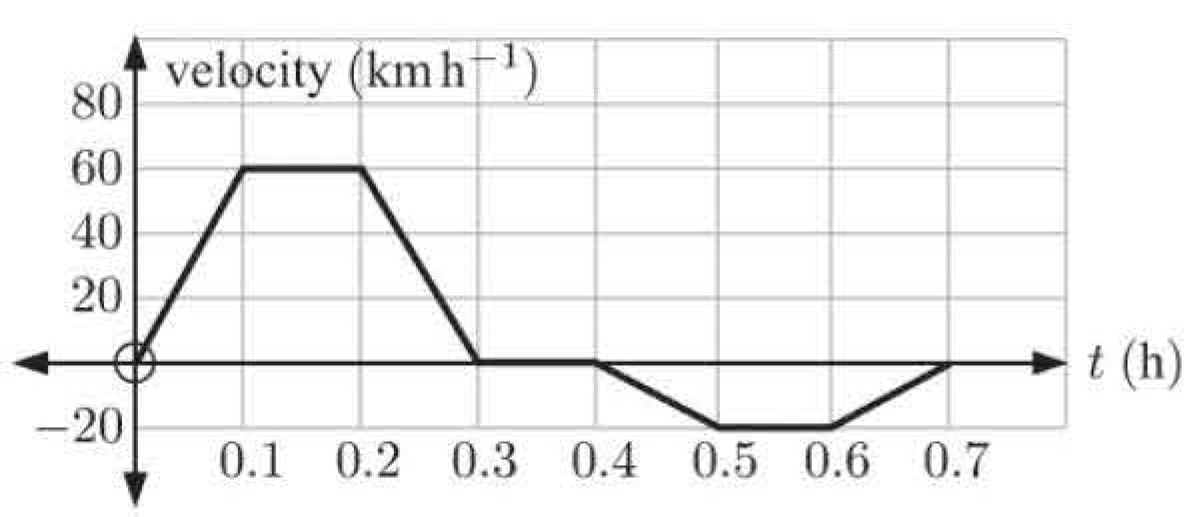
$=$ area A + area B + area C + area D

$=\frac{1}{2}(5\times6)+\left(\dfrac{6+8}{2}\right)5+5\times8+\frac{1}{2}(5\times8)$

$=15+35+40+20$

$=110$ m

2



a **i** The graph above the t-axis indicates that the velocity is positive and the car is travelling forwards.

ii The graph below the t-axis indicates that the velocity is negative and the car is travelling backwards (opposite direction).

b Total distance travelled $=$ area above the t-axis $+$ area below the t-axis

$$=\left(\tfrac{0.1}{2}+0.1+\tfrac{0.1}{2}\right)60+\left(\tfrac{0.1}{2}+0.1+\tfrac{0.1}{2}\right)20$$

$$=12+4$$

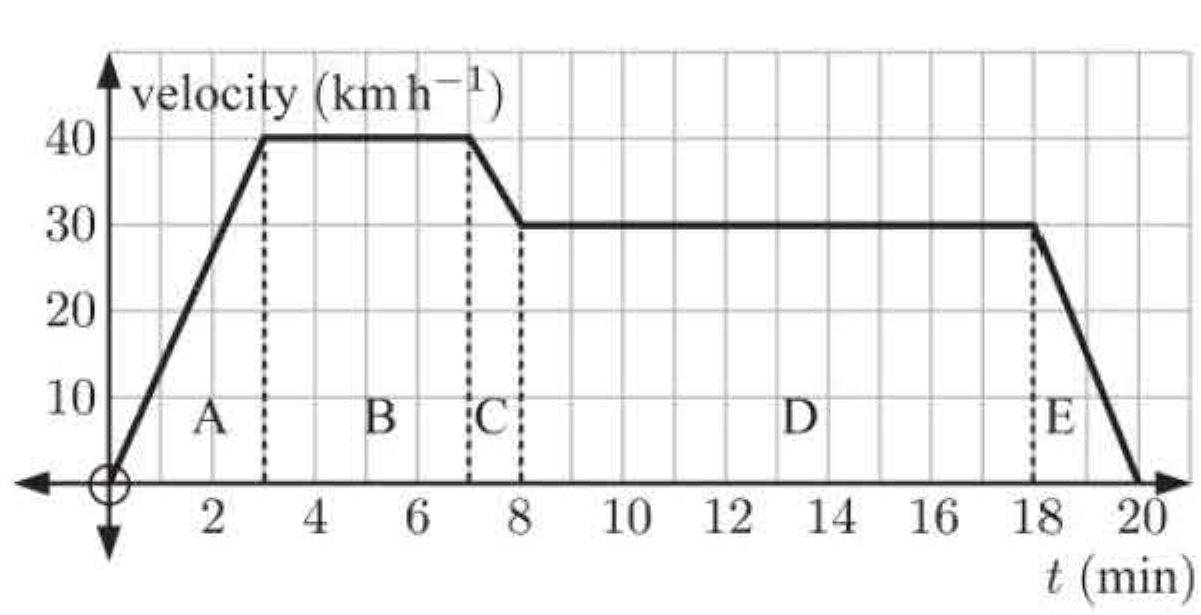
$$=16 \text{ km}$$

c Final displacement $=$ area above the t-axis $-$ area below the t-axis

$$=12-4$$

$=8$ km from the starting point (on the positive side)

3 **a**



b Total distance travelled

$= \text{area A} + \text{area B} + \text{area C} + \text{area D} + \text{area E}$

$= \frac{1}{60}\left[\frac{1}{2}(3\times 40) + (40\times 4) + \left(\frac{40+30}{2}\right)1 + (10\times 30) + \frac{1}{2}(2\times 30)\right]$

{the factor $\frac{1}{60}$ accounts for the fact that the times are in minutes while the speeds are in km h^{-1}}

$= \frac{1}{60}(60+160+35+300+30)$

$= 9.75$ km

EXERCISE 22C.2

1 **a**

$$\begin{aligned} s(t) &= \int (1-2t)\,dt \\ &= t - 2\left(\frac{t^2}{2}\right) + c \\ &= t - t^2 + c \end{aligned}$$

But $s(0) = 2$

$\therefore\ 0 - 0^2 + c = 2$

$\therefore\ c = 2$

$\therefore\ s(t) = t - t^2 + 2$ cm

b The particle changes direction when

$v(t) = 0$

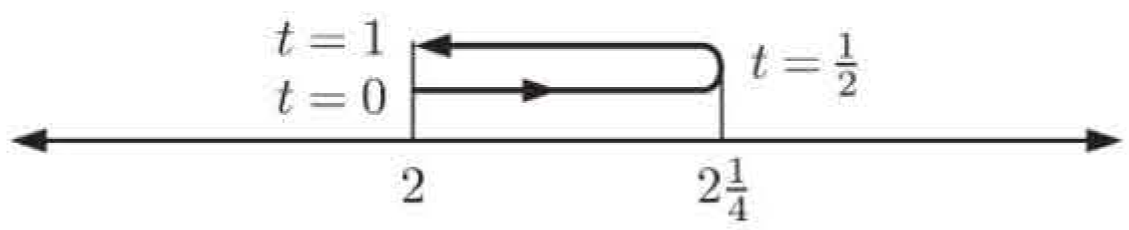
$\therefore\ 1 - 2t = 0$

$\therefore\ t = \frac{1}{2}$ s

Now $s(\frac{1}{2}) = \frac{1}{2} - (\frac{1}{2})^2 + 2 = 2\frac{1}{4}$ cm

and $s(1) = 1 - 1 + 2 = 2$ cm

$\therefore$ motion diagram is:



$\therefore$ total distance travelled $= \frac{1}{4} + \frac{1}{4}$

$= \frac{1}{2}$ cm

c Displacement $= s(1) - s(0)$

$= 2 - 2$

$= 0$ cm

2 **a**

$$\begin{aligned} s(t) &= \int (t^2 - t - 2)\,dt \\ &= \tfrac{1}{3}t^3 - \tfrac{1}{2}t^2 - 2t + c \end{aligned}$$

But $s(0) = 0$, $\therefore\ c = 0$

$\therefore\ s(t) = \frac{1}{3}t^3 - \frac{1}{2}t^2 - 2t$ cm

b P changes direction when $v(t) = 0$

$\therefore\ t^2 - t - 2 = 0$

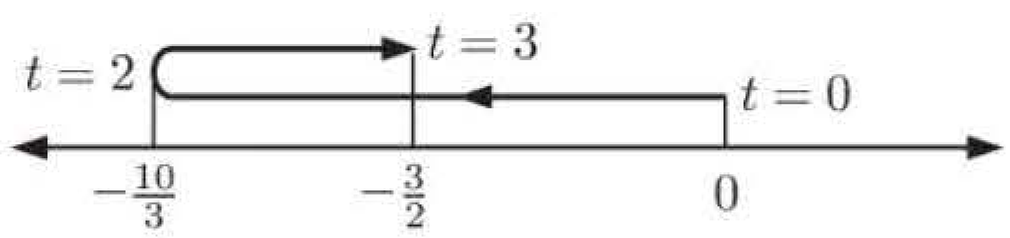
$\therefore\ (t-2)(t+1) = 0$

$\therefore$ P changes direction when $t = 2$ (since $t \geqslant 0$)

Now $s(2) = \dfrac{2^3}{3} - \dfrac{2^2}{2} - 2(2) = -\frac{10}{3}$

and $s(3) = \dfrac{3^3}{3} - \dfrac{3^2}{2} - 2(3) = -\frac{3}{2}$

$\therefore$ motion diagram is:



$\therefore$ total distance travelled $= \frac{10}{3} + (\frac{10}{3} - \frac{3}{2})$

$= 5\frac{1}{6}$ cm

c Displacement

$= s(3) - s(0)$

$= -\frac{3}{2} - 0$

$= -\frac{3}{2}$ cm ($1\frac{1}{2}$ cm left of its starting point)

3 **a**

$$\begin{aligned} s(t) &= \int (32 + 4t)\,dt \\ &= 32t + 4\left(\frac{t^2}{2}\right) + c \end{aligned}$$

$\therefore\ s(t) = 32t + 2t^2 + c$

But $s(0) = 16$

$\therefore\ 0 + 0 + c = 16$

$\therefore\ c = 16$

$\therefore\ s(t) = 32t + 2t^2 + 16$ m

b We check for any changes in direction.

These occur when $v(t) = 0$

$$\therefore\ 32 + 4t = 0$$
$$\therefore\ 4t = -32$$
$$\therefore\ t = -8$$

But $0 \leqslant t \leqslant t_1$, so the object does not change direction on the interval.

$$\therefore\ \text{displacement} = s(t_1) - s(0) = \int_0^{t_1} (32 + 4t)\, dt$$

c $a(t) = v'(t) = 4$

$\therefore$ the object is travelling with constant acceleration of $4\ \text{m s}^{-2}$.

4 $s(t) = \int (\cos(2t))\, dt = \frac{1}{2}\sin(2t) + c$

But $s(\frac{\pi}{4}) = 1$ $\therefore$ $\frac{1}{2}\sin(\frac{\pi}{2}) + c = 1$

$$\therefore\ c + \tfrac{1}{2} = 1$$
$$\therefore\ c = \tfrac{1}{2}$$
$$\therefore\ s(t) = \tfrac{1}{2}\sin(2t) + \tfrac{1}{2}$$

$$\therefore\ s(\tfrac{\pi}{3}) = \tfrac{1}{2}\sin(\tfrac{2\pi}{3}) + \tfrac{1}{2} = \tfrac{1}{2} \times \tfrac{\sqrt{3}}{2} + \tfrac{1}{2} = \tfrac{\sqrt{3}}{4} + \tfrac{1}{2} = \tfrac{\sqrt{3}+2}{4}\ \text{m}$$

5 $x'(t) = 16t - 4t^3$ units s^{-1}, $t \geqslant 0$

$= 4t(4 - t^2)$

$= 4t(2 + t)(2 - t)$ which has sign diagram:

$\therefore$ a direction reversal occurs at $t = 2$.

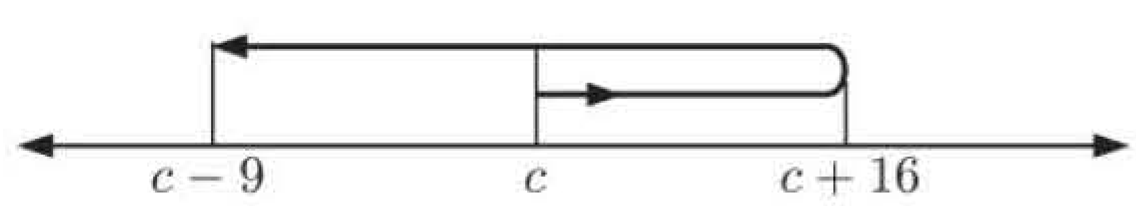
Now $x(t) = \int (16t - 4t^3)\, dt = 8t^2 - t^4 + c$

a $x(0) = c$

$x(2) = 32 - 16 + c = c + 16$

$x(3) = 72 - 81 + c = c - 9$

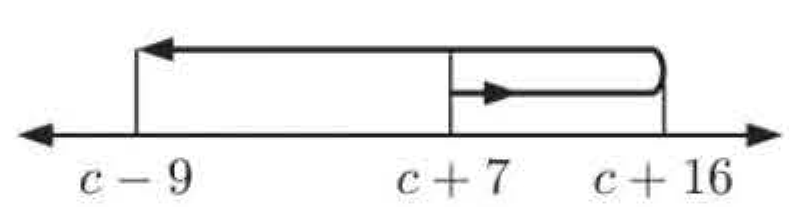
$\therefore$ motion diagram for $0 \leqslant t \leqslant 3$ is:



$\therefore$ total distance travelled $= (c + 16 - c) + (c + 16 - [c - 9])$
$= 41$ units

b $x(1) = 7 + c = c + 7$

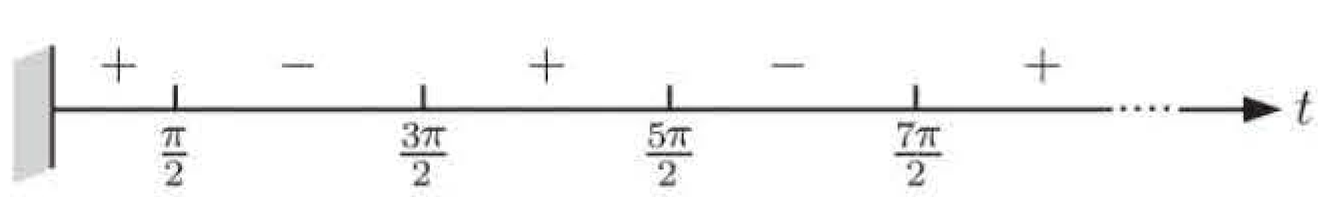
$\therefore$ motion diagram for $1 \leqslant t \leqslant 3$ is:



$\therefore$ total distance travelled $= (c + 16 - [c + 7]) + (c + 16 - [c - 9])$
$= 34$ units

6 **a** $v(t) = \cos t\ \text{m s}^{-1}$, $t \geqslant 0$

$\therefore$ $v(t)$ has sign diagram:



$\therefore$ a direction reversal occurs at $t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots$

$s(t) = \int \cos t\, dt = \sin t + c$

$\therefore$ $s(0) = c$

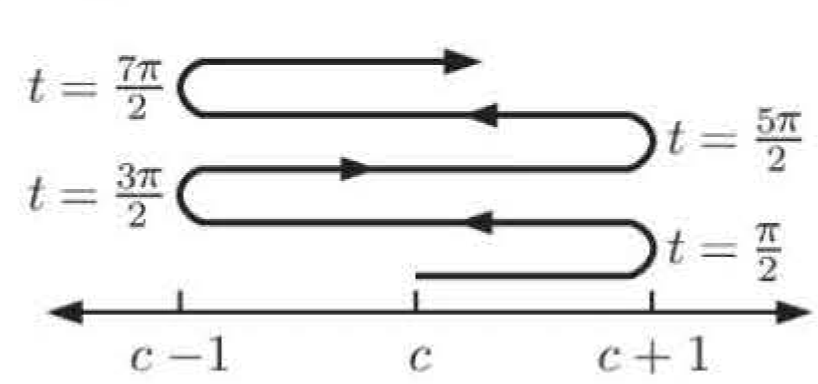
$s\left(\frac{\pi}{2}\right) = c + 1$

$s\left(\frac{3\pi}{2}\right) = c - 1$

$s\left(\frac{5\pi}{2}\right) = c + 1$

$s\left(\frac{7\pi}{2}\right) = c - 1$

The motion diagram is:



$\therefore$ the particle oscillates between the points $(c - 1)$ and $(c + 1)$.

b distance $= (c+1) - (c-1)$
$= 2$ m

7 $v(t) = 50 - 10e^{-0.5t}$ m s^{-1}, $t \geqslant 0$

a $v(0) = 50 - \dfrac{10}{e^0} = 50 - 10 = 40$ m s^{-1}

b $v(3) = 50 - \dfrac{10}{e^{1.5}} \approx 47.8$ m s^{-1}

c The velocity reaches 45 m s^{-1}

when $45 = 50 - 10e^{-0.5t}$

$\therefore\ 10e^{-\frac{t}{2}} = 5$

$\therefore\ e^{\frac{t}{2}} = 2$

$\therefore\ \dfrac{t}{2} = \ln 2$

$\therefore\ t = 2\ln 2 \approx 1.39$ seconds

d $v(t) = 50 - \dfrac{10}{e^{\frac{t}{2}}}$

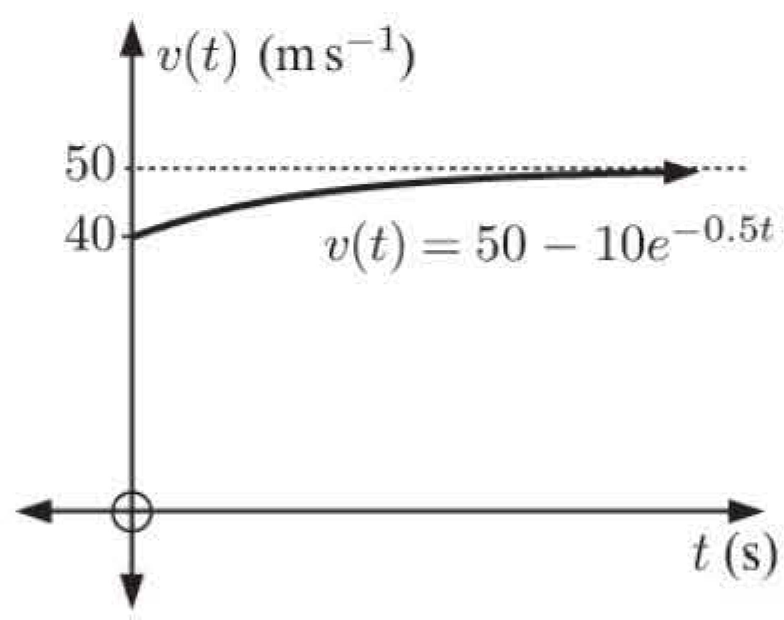
As $t \to \infty$, $\dfrac{10}{e^{\frac{t}{2}}} \to 0^+$

$\therefore\ v(t) \to 50^-$

e $a(t) = v'(t)$
$= -10e^{-0.5t}(-0.5)$
$= 5e^{-0.5t}$ m s^{-2}

$\therefore\ a(t) > 0$ for all t $\{e^x > 0$ for all $x\}$
$\therefore$ the acceleration is always positive

f



g total distance travelled
$= \int_0^3 (50 - 10e^{-0.5t})\,dt$
$= \left[50t + 20e^{-0.5t}\right]_0^3$
$= 150 + 20e^{-1.5} - 20$
≈ 134.5 m

8 **a**
$$v(t) = \int \frac{-1}{(t+1)^2}\,dt$$
$$= \int -(t+1)^{-2}\,dt$$
$$= (t+1)^{-1} + c$$
But $v(0) = 0$
$$\therefore\ \frac{1}{0+1} + c = 0$$
$\therefore\ c + 1 = 0$
$\therefore\ c = -1$
$$\therefore\ v(t) = \frac{1}{t+1} - 1 \ \text{ m s}^{-1}$$

b
$$s(t) = \int \left(\frac{1}{t+1} - 1\right) dt$$
$$= \ln|t+1| - t + c$$
But $s(0) = 0$
$\therefore\ \ln 1 - 0 + c = 0$
$\therefore\ c = 0$
$\therefore\ s(t) = \ln|t+1| - t$ m

c $s(2) = \ln 3 - 2$ m
≈ -0.901 m

$v(2) = \dfrac{1}{2+1} - 1$
$= -\frac{2}{3}$ m s^{-1}

$a(2) = \dfrac{-1}{(2+1)^2}$
$= -\frac{1}{9}$ m s^{-2}

The object is approximately 0.901 m to the left of the origin, travelling left at $\frac{2}{3}$ m s^{-1}, with acceleration $-\frac{1}{9}$ m s^{-2}.

9 **a**
$$v(t) = \int \left(\frac{t}{10} - 3\right) dt$$
$$= \tfrac{1}{10}\left(\frac{t^2}{2}\right) - 3t + c$$
$$= \tfrac{1}{20}t^2 - 3t + c \ \text{ m s}^{-1}$$

But $v(0) = 45$
$\therefore\ \frac{1}{20}(0)^2 - 3(0) + c = 45$
$\therefore\ c = 45$
$\therefore\ v(t) = \frac{1}{20}t^2 - 3t + 45$ m s^{-1}

b $\int_0^{60} (\frac{1}{20}t^2 - 3t + 45)\, dt = \left[\frac{1}{60}t^3 - \frac{3}{2}t^2 + 45t\right]_0^{60}$
$= \frac{1}{60}(60)^3 - \frac{3}{2}(60)^2 + 45(60)$
$= 900$

The train travels a total of 900 m in the first 60 seconds.

10 $a(t) = 4e^{-\frac{t}{20}}$ m s^{-2}

$\therefore\ v(t) = \int 4e^{-\frac{t}{20}}\, dt$
$= 4 \times \dfrac{1}{-\frac{1}{20}} e^{-\frac{t}{20}} + c$
$= -80e^{-\frac{t}{20}} + c$

Now $v(0) = 20$ m s^{-1}
$\therefore\ c = 100$
$\therefore\ v(t) = 100 - 80e^{-\frac{t}{20}}$ m s^{-1}

a As $t \to \infty$, $e^{-\frac{t}{20}} \to 0^+$ $\therefore\ v(t) \to 100^-$ m s^{-1}
$\therefore$ the object approaches a limiting velocity of 100 m s^{-1}.

b The total distance travelled $= \int_0^{10} (100 - 80e^{-\frac{t}{20}})\, dt$ $\{v(t) > 0$ for $0 \leqslant t \leqslant 10\}$
$= \left[100t + 1600e^{-\frac{t}{20}}\right]_0^{10}$
$= \left(1000 + 1600e^{-\frac{1}{2}}\right) - (0 + 1600)$
≈ 370.4 m

EXERCISE 22D

1 The marginal cost is $C'(x)$ and $C'(x) = 3.15 + 0.004x$ € per gadget
$\therefore\ C(x) = \int (3.15 + 0.004x)\, dx$
$= 3.15x + 0.002x^2 + c$
But $C(0) = 450$ so $c = 450$
$\therefore\ C(x) = 3.15x + 0.002x^2 + 450$ euros
$\therefore\ C(800) = 3.15(800) + 0.002(800)^2 + 450$
$=$ €4250

So, the total cost is €4250.

2 **a** The marginal profit is $P'(x)$ and $P'(x) = 15 - 0.03x$ dollars per plate
$\therefore\ P(x) = \int (15 - 0.03x)\, dx$
$= 15x - 0.015x^2 + c$
But $P(0) = -650$ so $c = -650$
$\therefore\ P(x) = 15x - 0.015x^2 - 650$ dollars

b The maximum profit occurs when $P'(x) = 0$, which is when $15 - 0.03x = 0$
$\therefore\ 0.03x = 15$
$\therefore\ x = \dfrac{15}{0.03}$
$\therefore\ x = 500$

Now $P''(x) = -0.03 < 0$ $\therefore$ the profit is at a maximum when $x = 500$ plates.
The maximum profit $= P(500) = 15(500) - 0.015(500)^2 - 650$
$= \$3100$

c In order for a profit to be made, $P(x)$ must be greater than 0
$\therefore\ 15x - 0.015x^2 - 650 > 0$
Using technology, the x-intercepts of $P(x)$ are $x_1 = 45.39$ and $x_2 = 954.6$
Since we cannot produce part plates, a profit is made for $46 \leqslant x \leqslant 954$.

3 $E'(t) = 350(80+0.15t)^{0.8} - 120(80+0.15t)$ calories per day

Total energy needs over the first week $= \int_0^7 E'(t)\,dt$

$$= \int_0^7 [350(80+0.15t)^{0.8} - 120(80+0.15t)]\,dt$$

$$= \left[\tfrac{1}{0.15} \times \frac{350(80+0.15t)^{1.8}}{1.8} - 9600t - 9t^2\right]_0^7$$

$$\approx 14\,400 \text{ calories}$$

4 $\dfrac{dT}{dx} = \dfrac{-20}{x^{0.63}} = -20x^{-0.63}$ $\quad\therefore\quad T = \int -20x^{-0.63}\,dx = \dfrac{-20x^{0.37}}{0.37} + c$

Now when $x = 3$, $T = 100$

$$\therefore \quad \frac{-20(3^{0.37})}{0.37} + c = 100$$

$$\therefore \quad c = 100 + \frac{20(3^{0.37})}{0.37} \approx 181.1639$$

$$\therefore \quad T \approx \frac{-20x^{0.37}}{0.37} + 181.1639$$

So, when $x = 6$, $T \approx -104.8925 + 181.1639 \approx 76.27$

$\therefore$ the outer surface temperature is about $76.3°\text{C}$.

5 **a** When $x = 0$, deflection $= 0$ $\therefore$ $y = 0$.

And when $x = 0$, the tangent is horizontal $\therefore$ $\dfrac{dy}{dx} = 0$.

b $\dfrac{d^2y}{dx^2} = -\tfrac{1}{10}(1-x)^2$

$$\therefore \quad \frac{dy}{dx} = \int -\tfrac{1}{10}(1-x)^2\,dx$$

$$= -\tfrac{1}{10}\left(\tfrac{1}{-1}\right) \times \frac{(1-x)^3}{3} + c$$

$$= \tfrac{1}{30}(1-x)^3 + c$$

From **a**, when $x = 0$, $\dfrac{dy}{dx} = 0$

$$\therefore \quad \tfrac{1}{30}(1-0)^3 + c = 0$$

$$\therefore \quad c = -\tfrac{1}{30}$$

$$\therefore \quad \frac{dy}{dx} = \tfrac{1}{30}(1-x)^3 - \tfrac{1}{30}$$

$$\therefore \quad y = \int [\tfrac{1}{30}(1-x)^3 - \tfrac{1}{30}]\,dx$$

$$= \tfrac{1}{30}\left(\tfrac{1}{-1}\right)\frac{(1-x)^4}{4} - \tfrac{1}{30}x + d$$

$$= -\frac{(1-x)^4}{120} - \frac{x}{30} + d$$

Also from **a**, when $x = 0$, $y = 0$

$$\therefore \quad -\tfrac{1}{120} - 0 + d = 0$$

$$\therefore \quad d = \tfrac{1}{120}$$

$$\therefore \quad y = \frac{1}{120} - \frac{(1-x)^4}{120} - \frac{x}{30}$$

c Maximum deflection occurs at the right hand end where $x \approx 1$

and at $x \approx 1$, $y \approx \tfrac{1}{120} - 0 - \tfrac{1}{30} \approx -0.025$ m

$\therefore$ the maximum deflection is about 2.5 cm.

6 **a** $\dfrac{d^2y}{dx^2} = 0.01\left(2x - \dfrac{x^2}{2}\right) = 0.02x - 0.005x^2$

$$\therefore \quad \frac{dy}{dx} = \int (0.02x - 0.005x^2)\,dx = 0.01x^2 - \tfrac{0.005}{3}x^3 + c$$

The sag, $y = \int (0.01x^2 - \tfrac{0.005}{3}x^3 + c)\,dx$

$$\therefore \quad y = \tfrac{0.01}{3}x^3 - \tfrac{0.005}{12}x^4 + cx + d$$

Now when $x = 0,\ y = 0$ $\quad \therefore \quad 0 - 0 + 0 + d = 0$

$\therefore \quad d = 0$

$\therefore \quad y = \frac{0.01}{3}x^3 - \frac{0.005}{12}x^4 + cx$

Also, when $x = 4,\ y = 0$ $\quad \therefore \quad \frac{0.01}{3}\left(4^3\right) - \frac{0.005}{12}\left(4^4\right) + 4c = 0$

$\therefore \quad 4c = \frac{0.005}{12}\left(4^4\right) - \frac{0.01}{3}\left(4^3\right)$

$\therefore \quad c = \frac{0.005}{12}\left(4^3\right) - \frac{0.01}{3}\left(4^2\right)$

$\therefore \quad c = -\frac{0.08}{3}$

$\therefore \quad y = \left(\frac{0.01}{3}x^3 - \frac{0.005}{12}x^4 - \frac{0.08}{3}x\right)$ m

b The maximum sag occurs when $\dfrac{dy}{dx} = 0$ $\quad \therefore \quad 0.01x^2 - \frac{0.005}{3}x^3 - \frac{0.08}{3} = 0$

$\therefore \quad 6x^2 - x^3 - 16 = 0$

Using technology, the three solutions are $x = -1.464,\ 2,$ and 5.464

But the maximum lies between 0 and 4, so it must occur when $x = 2$.

When $x = 2,\quad y = \frac{0.01}{3}\left(2^3\right) - \frac{0.005}{12}\left(2^4\right) - \frac{0.08}{3}(2)$

$\approx -0.033\,33$ m

≈ -3.333 cm $\quad \therefore$ the maximum sag is ≈ 3.33 cm

c At the point 1 m from P, $x = 3$ m, so $y = \frac{0.01}{3}\left(3^3\right) - \frac{0.005}{12}\left(3^4\right) - \frac{0.08}{3}(3)$

$= -0.023\,75$ m

$= -2.375$ cm $\quad \therefore$ the sag is 2.375 cm

d At the point 1 m from P, $x = 3$ m, so $\dfrac{dy}{dx} = 0.01\left(3^2\right) - \frac{0.005}{3}\left(3^3\right) - \frac{0.08}{3} \approx 0.0183$

$\therefore$ the angle θ that the plank makes with the horizontal is such that $\tan\theta \approx 0.0183$

$\therefore \quad \theta \approx \tan^{-1}(0.0183) \approx 1.05°$

7 The cost per unit volume, $\dfrac{dC}{dV} = \frac{1}{2}x^2 + 4$ dollars per m^3 (at depth x).

Since the volume of a well x m deep is $V = \pi r^2 x,\quad \dfrac{dV}{dx} = \pi r^2$

Now $\dfrac{dC}{dx} = \dfrac{dC}{dV}\dfrac{dV}{dx}$ $\quad$ {chain rule}

$\therefore \quad \dfrac{dC}{dx} = \left(\frac{1}{2}x^2 + 4\right)\pi r^2$

$\therefore \quad C = \displaystyle\int \frac{dC}{dx}\,dx$

$= \int [\pi r^2(\frac{1}{2}x^2 + 4)]\,dx$

$= \pi r^2\left(\dfrac{x^3}{6} + 4x\right) + c$

So, the cost of digging a well h metres deep

$= \pi r^2\left(\dfrac{h^3}{6} + 4h\right) + c$

Now if the initial cost $= C_0$ when $h = 0$,

$\pi r^2(\frac{0}{6} + 0) + c = C_0$

$\therefore \quad c = C_0$

$\therefore \quad C(h) = \pi r^2\left(\dfrac{h^3 + 24h}{6}\right) + C_0$

8 $\quad y = \sin x,\ 0 \leqslant x \leqslant \pi$

$\therefore \quad \dfrac{dy}{dx} = \cos x$

$\therefore \quad L = \int_0^\pi \sqrt{1 + \cos^2 x}\,dx \approx 3.820\,20$ units $\quad$ {technology}

9 **a** The yield Y per unit area A is proportional to $\dfrac{1}{\sqrt{x+4}}$.

$\therefore \quad \dfrac{dY}{dA} \propto \dfrac{1}{\sqrt{x+4}}$

$\therefore \quad \dfrac{dY}{dA} = \dfrac{k}{\sqrt{x+4}}$ for some constant k.

b The shaded area $A = \text{length} \times \text{width}$

$\therefore \quad A = (4-2p)x$

$\therefore \quad \dfrac{dA}{dx} = 4-2p$

Now $\dfrac{dY}{dx} = \dfrac{dY}{dA}\dfrac{dA}{dx}$ {chain rule}

$\therefore \quad \dfrac{dY}{dx} = \dfrac{k}{\sqrt{x+4}} \times (4-2p)$

$\therefore \quad \dfrac{dY}{dx} = \dfrac{k(4-2p)}{\sqrt{x+4}}$

c $\dfrac{dY}{dx}$ is the instantaneous rate of change of the yield with respect to the distance x from the canal.

$\therefore$ total yield $= Y = \displaystyle\int_0^p \frac{dY}{dx}\,dx = \int_0^p \frac{k(4-2p)}{\sqrt{x+4}}\,dx$

d Using **c**, $Y = k(4-2p)\int_0^p (x+4)^{-\frac{1}{2}}\,dx$

$= k(4-2p) \times \left[\dfrac{(x+4)^{\frac{1}{2}}}{\frac{1}{2}}\right]_0^p$

$= 2k(4-2p)\left[\sqrt{x+4}\right]_0^p$

$= 4k(2-p)\left[\sqrt{p+4}-\sqrt{4}\right]$

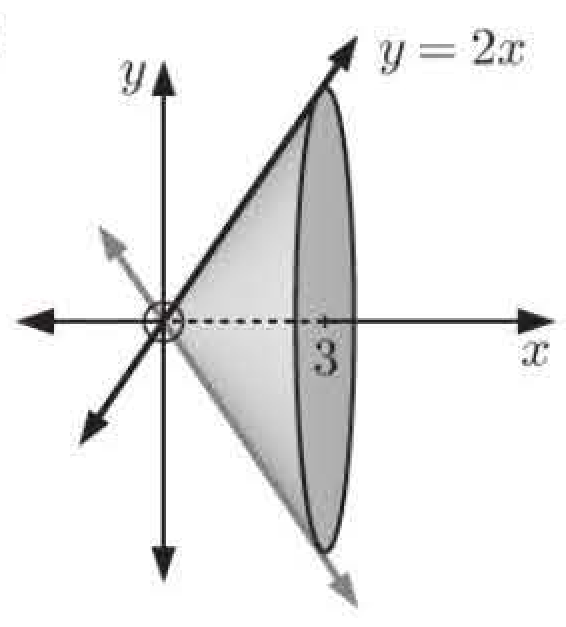
$\therefore \quad Y = 4k(2-p)\left(\sqrt{p+4}-2\right)$

e Using technology to graph Y and find its maximum, we find that the maximum occurs when $p \approx 0.9735$ km

$\therefore$ the orchard is $0.974 \text{ km} \times 2.05 \text{ km}$.

EXERCISE 22E.1

1 **a**



Volume $= \pi\int_0^3 (2x)^2\,dx$

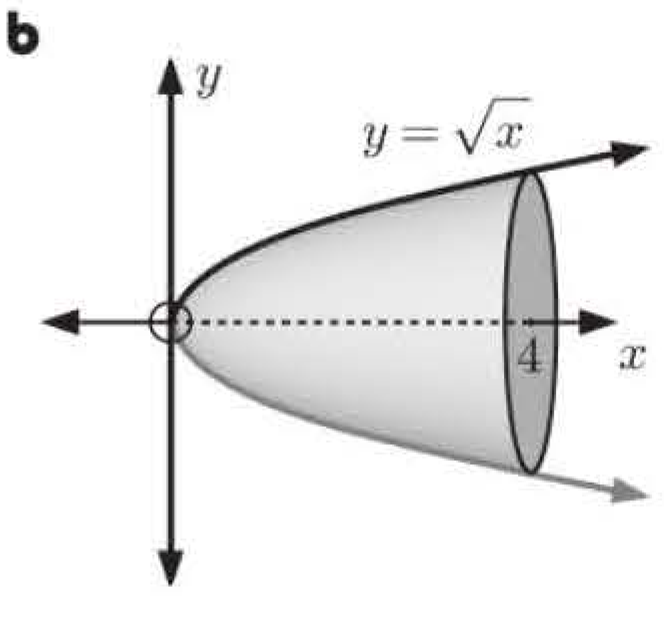
$= 4\pi\int_0^3 x^2\,dx$

$= 4\pi\left[\frac{1}{3}x^3\right]_0^3$

$= 4\pi(9-0)$

$= 36\pi$ units^3

b



Volume $= \pi\int_0^4 (\sqrt{x})^2\,dx$

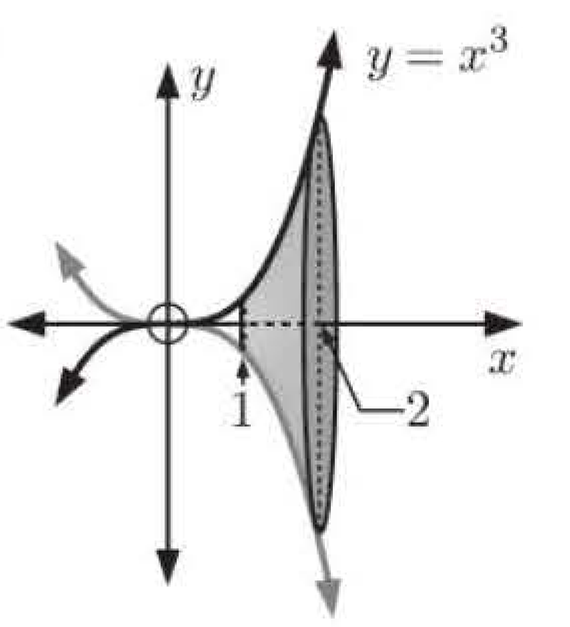
$= \pi\int_0^4 x\,dx$

$= \pi\left[\frac{1}{2}x^2\right]_0^4$

$= \pi(8-0)$

$= 8\pi$ units^3

c



Volume $= \pi\int_1^2 (x^3)^2\,dx$

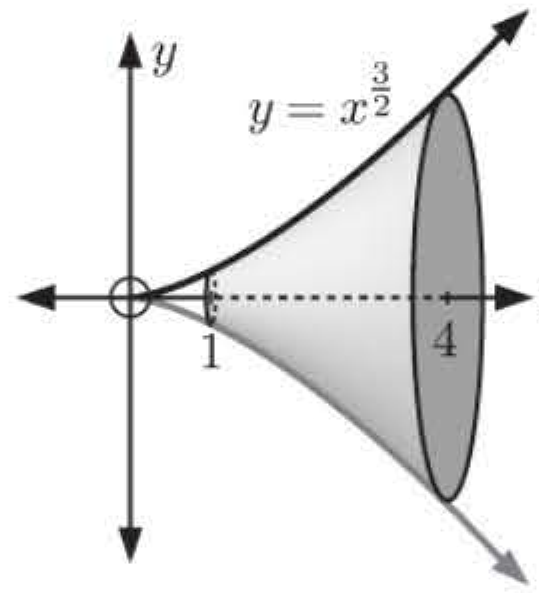
$= \pi\int_1^2 x^6\,dx$

$= \pi\left[\frac{1}{7}x^7\right]_1^2$

$= \pi\left(\frac{128}{7} - \frac{1}{7}\right)$

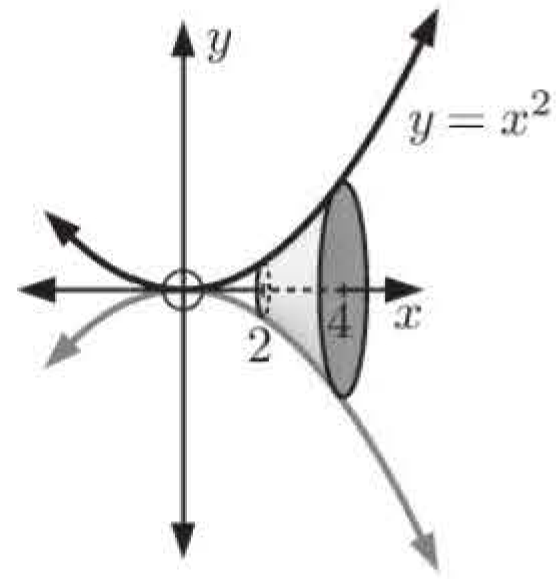
$= \frac{127\pi}{7}$ units^3

d



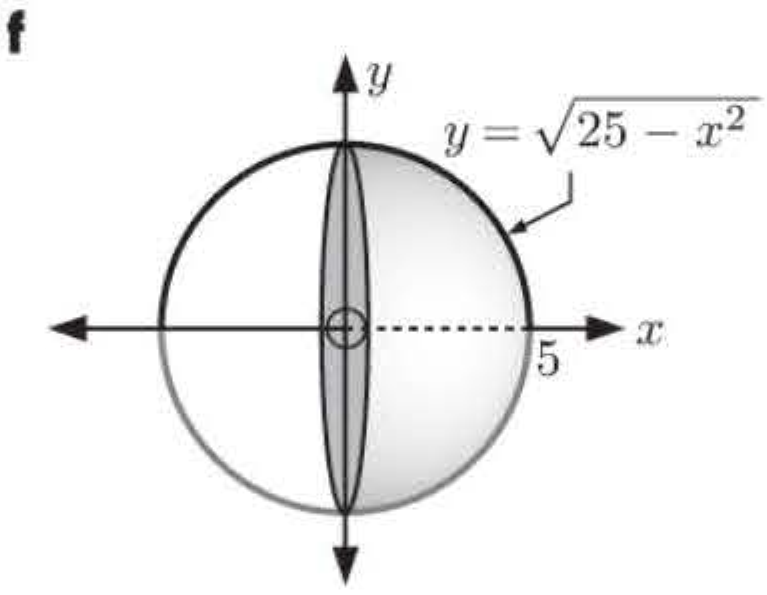
$$\begin{aligned}\text{Volume} &= \pi \int_1^4 (x^{\frac{3}{2}})^2 \, dx \\ &= \pi \int_1^4 x^3 \, dx \\ &= \pi \left[\tfrac{1}{4}x^4\right]_1^4 \\ &= \pi \left(\tfrac{256}{4} - \tfrac{1}{4}\right) \\ &= \tfrac{255\pi}{4} \text{ units}^3\end{aligned}$$

e



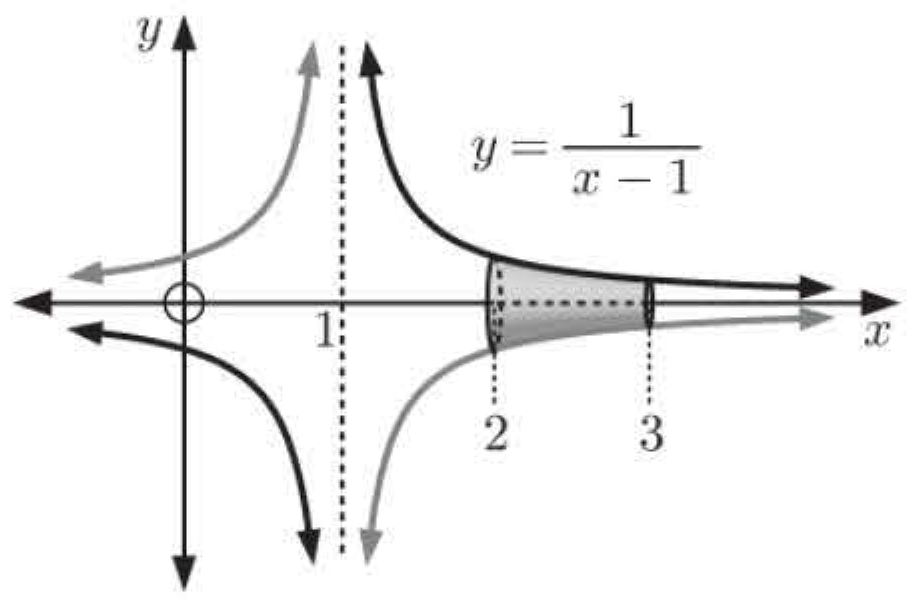
$$\begin{aligned}\text{Volume} &= \pi \int_2^4 (x^2)^2 \, dx \\ &= \pi \int_2^4 x^4 \, dx \\ &= \pi \left[\tfrac{1}{5}x^5\right]_2^4 \\ &= \pi \left(\tfrac{1024}{5} - \tfrac{32}{5}\right) \\ &= \tfrac{992\pi}{5} \text{ units}^3\end{aligned}$$

f



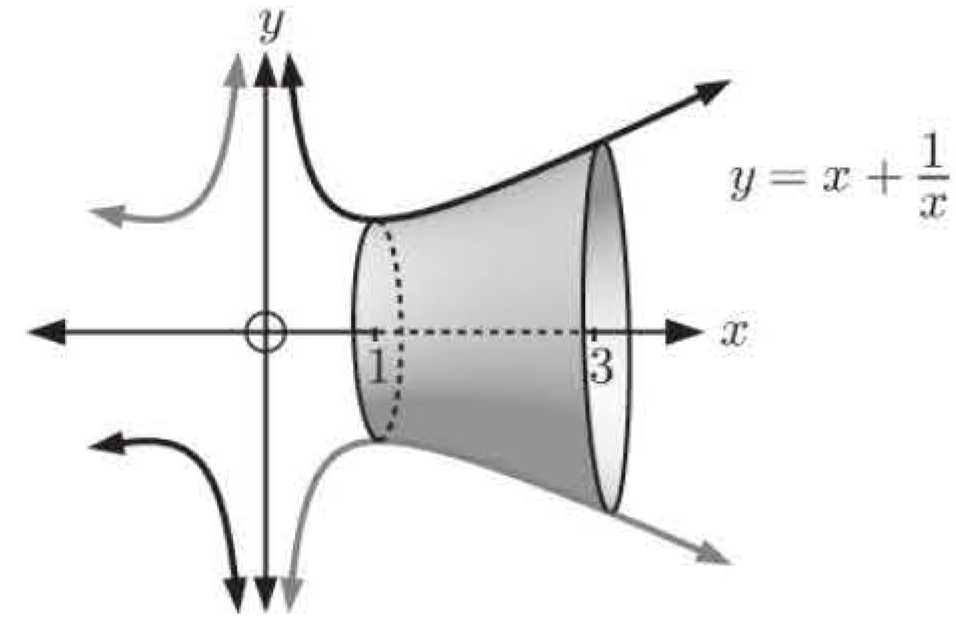
$$\begin{aligned}\text{Volume} &= \pi \int_0^5 (25 - x^2) \, dx \\ &= \pi \left[25x - \frac{x^3}{3}\right]_0^5 \\ &= \pi \left(125 - \tfrac{125}{3}\right) \\ &= \pi \left(\tfrac{2}{3}\right) 125 \\ &= \tfrac{250\pi}{3} \text{ units}^3\end{aligned}$$

g



$$\begin{aligned}\text{Volume} &= \pi \int_2^3 \left(\frac{1}{x-1}\right)^2 dx \\ &= \pi \int_2^3 (x-1)^{-2} \, dx \\ &= \pi \left[-\frac{1}{x-1}\right]_2^3 \\ &= \pi \left(-\tfrac{1}{2} + 1\right) \\ &= \tfrac{\pi}{2} \text{ units}^3\end{aligned}$$

h



$$\begin{aligned}\text{Volume} &= \pi \int_1^3 \left(x + \frac{1}{x}\right)^2 dx \\ &= \pi \int_1^3 \left(x^2 + 2 + x^{-2}\right) \, dx \\ &= \pi \left[\frac{x^3}{3} + 2x - \frac{1}{x}\right]_1^3 \\ &= \pi \left[9 + 6 - \tfrac{1}{3} - \left(\tfrac{1}{3} + 2 - 1\right)\right] \\ &= \tfrac{40\pi}{3} \text{ units}^3\end{aligned}$$

2 **a**
$$\begin{aligned}\text{Volume} &= \pi \int_1^3 \left(\frac{x^3}{x^2+1}\right)^2 dx \\ &\approx 5.926\pi \quad \{\text{using technology}\} \\ &\approx 18.6 \text{ units}^3\end{aligned}$$

b
$$\begin{aligned}\text{Volume} &= \pi \int_0^2 \left(e^{\sin x}\right)^2 \, dx \\ &\approx 9.613\pi \quad \{\text{using technology}\} \\ &\approx 30.2 \text{ units}^3\end{aligned}$$

3 **a**
$$\begin{aligned}V &= \pi \int_0^6 \left(\frac{x}{2} + 4\right)^2 dx \\ &= \pi \int_0^6 \left(\tfrac{1}{4}x^2 + 4x + 16\right) \, dx \\ &= \pi \left[\frac{x^3}{12} + \frac{4x^2}{2} + 16x\right]_0^6 \\ &= \pi(18 + 72 + 96) - 0 \\ &= 186\pi \text{ units}^3\end{aligned}$$

b
$$\begin{aligned}V &= \pi \int_1^2 (x^2 + 3)^2 \, dx \\ &= \pi \int_1^2 \left(x^4 + 6x^2 + 9\right) \, dx \\ &= \pi \left[\frac{x^5}{5} + \frac{6x^3}{3} + 9x\right]_1^2 \\ &= \pi \left[\left(\tfrac{32}{5} + 16 + 18\right) - \left(\tfrac{1}{5} + 2 + 9\right)\right] \\ &= \pi \left(\tfrac{146}{5}\right) \\ &= \tfrac{146\pi}{5} \text{ units}^3\end{aligned}$$

c $V = \pi \int_0^4 (e^x)^2 \, dx$

$= \pi \int_0^4 e^{2x} \, dx$

$= \pi \left[\frac{1}{2}e^{2x}\right]_0^4$

$= \pi \left(\frac{1}{2}e^8 - \frac{1}{2}\right)$

$= \frac{\pi}{2}(e^8 - 1)$ units^3

4 **a** If we take a vertical slice of the bowl, we get a circle.

b Volume of revolution

$= \pi \int_a^b y^2 \, dx$

$= \pi \int_0^4 \left(4\sqrt{x}\right)^2 \, dx$

$= \int_0^4 \pi \left(4\sqrt{x}\right)^2 \, dx$

c Capacity $= \int_0^4 \pi \times 16x \, dx$

$= \int_0^4 16\pi x \, dx$

$= \left[8\pi x^2\right]_0^4$

$= 8\pi \times 16$

$= 128\pi$ units^3

≈ 402 units^3

5 **a** Volume $= \pi \int_5^8 y^2 \, dx$

$= \pi \int_5^8 (64 - x^2) \, dx$

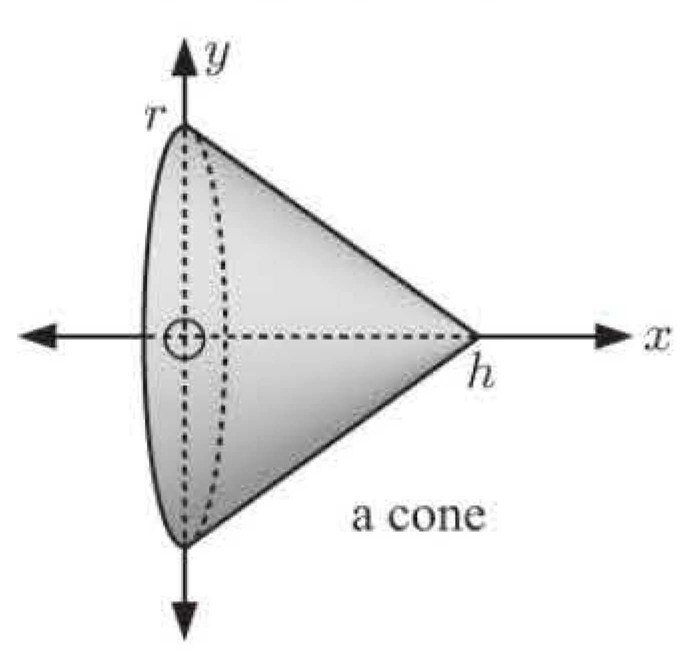
$= \pi \left[64x - \frac{x^3}{3}\right]_5^8$

$= \pi \left[\left(512 - \frac{512}{3}\right) - \left(320 - \frac{125}{3}\right)\right]$

$= 63\pi$ units^3

b $63\pi \text{ cm}^3 \approx 198 \text{ cm}^3$

6 **a** a cone of base radius r and height h



b [AB] has gradient $= \frac{r-0}{0-h} = -\frac{r}{h}$

$\therefore$ its equation is $y = -\left(\frac{r}{h}\right)x + r$

c $V = \pi \int_0^h \left(\frac{-r}{h}x + r\right)^2 dx$

$= \pi r^2 \int_0^h \left(-\frac{x}{h} + 1\right)^2 dx$

$= \pi r^2 \int_0^h \left(\frac{x^2}{h^2} - \frac{2x}{h} + 1\right) dx$

$= \pi r^2 \left[\frac{x^3}{3h^2} - \frac{2x^2}{2h} + x\right]_0^h$

$= \pi r^2 \left[\left(\frac{h}{3} - h + h\right) - 0\right]$

$= \frac{1}{3}\pi r^2 h$ units^3

7 **a** a sphere of radius r

b $V = \pi \int_{-r}^{r} y^2 \, dx = 2\pi \int_0^r (r^2 - x^2) \, dx$

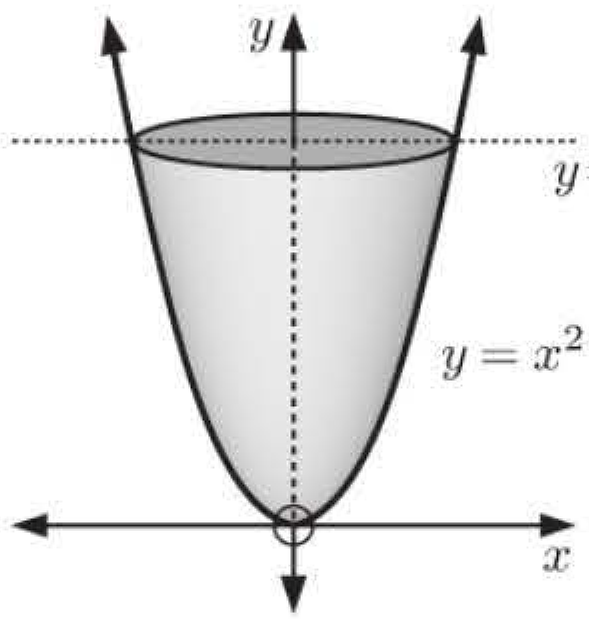
$= 2\pi \left[r^2 x - \frac{x^3}{3}\right]_0^r$

$= 2\pi(r^3 - \frac{r^3}{3} - 0)$

$= 2\pi \times \frac{2}{3}r^3$

$= \frac{4}{3}\pi r^3$ units^3

8 **a**



Volume $= \pi \int_0^4 x^2\, dy$

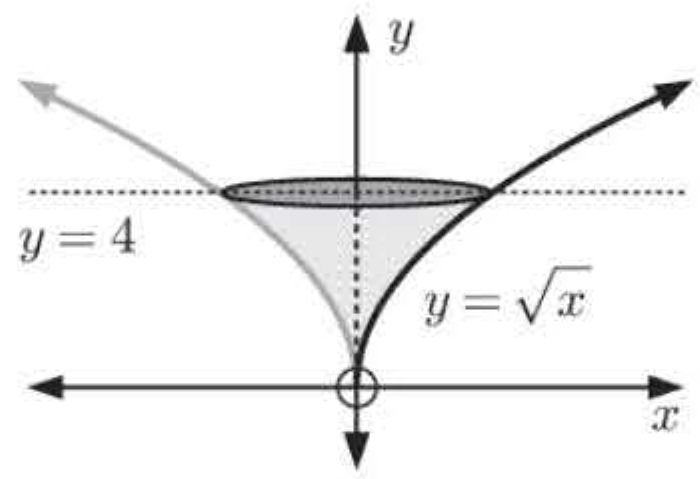
$= \pi \int_0^4 y\, dy$

$= \pi \left[\dfrac{y^2}{2}\right]_0^4$

$= \pi(8-0)$

$= 8\pi$ units^3

b



Volume $= \pi \int_1^4 x^2\, dy$

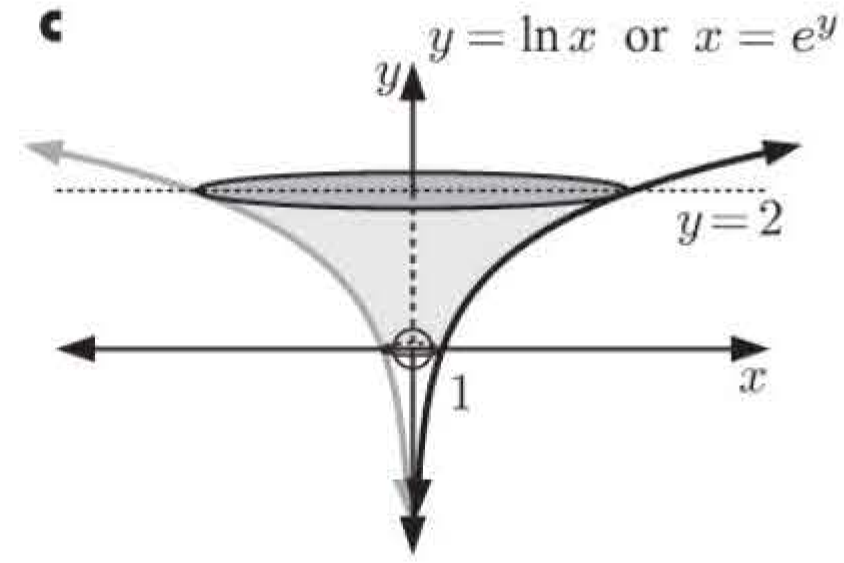
$= \pi \int_1^4 y^4\, dy$

$= \pi \left[\dfrac{y^5}{5}\right]_1^4$

$= \pi\left(\frac{4^5}{5} - \frac{1}{5}\right)$

$= \frac{1023}{5}\pi$ units^3

c



Volume $= \pi \int_0^2 x^2\, dy$

$= \pi \int_0^2 (e^y)^2\, dy$

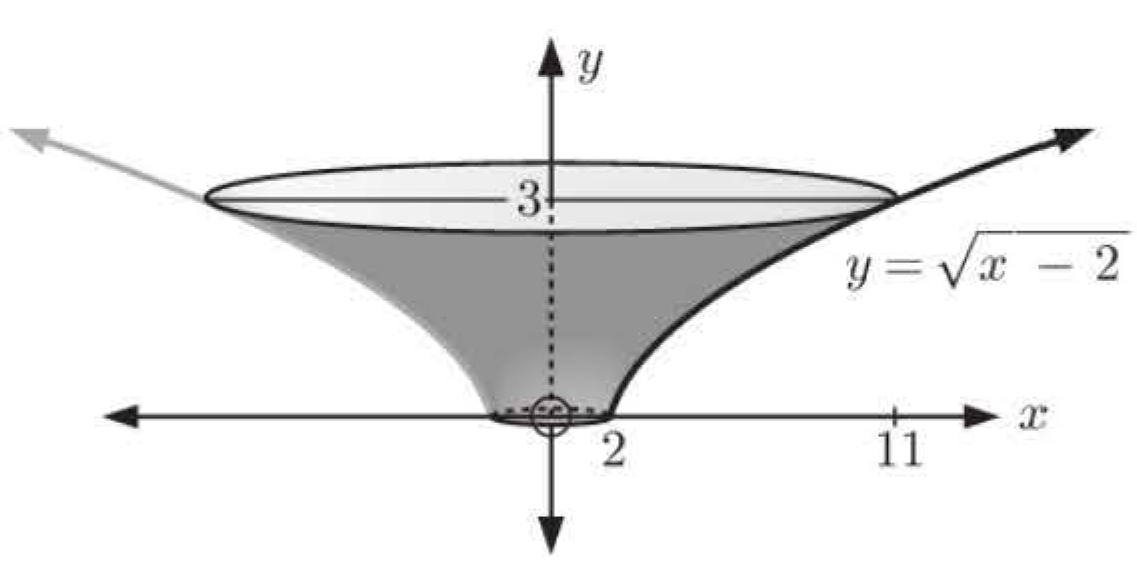
$= \pi \int_0^2 e^{2y}\, dy$

$= \pi \left[\frac{1}{2}e^{2y}\right]_0^2$

$= \pi\left(\frac{1}{2}e^4 - \frac{1}{2}\right)$

$= \frac{\pi}{2}(e^4 - 1)$ units^3

d



When $x = 2,\quad y = 0$

When $x = 11,\quad y = 3$

Now $y = \sqrt{x-2}$

$\therefore\ y^2 = x - 2$

$\therefore\ x = y^2 + 2$

$\therefore$ volume $= \pi \int_0^3 x^2\, dy$

$= \pi \int_0^3 (y^2+2)^2\, dy$

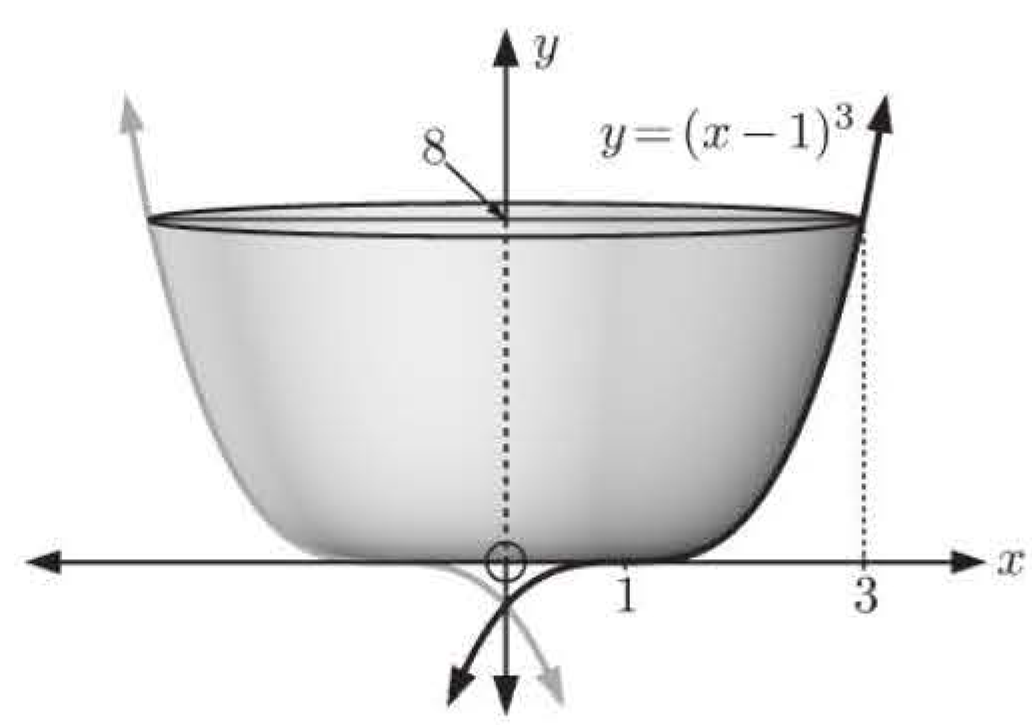
$= \pi \int_0^3 (y^4 + 4y^2 + 4)\, dy$

$= \pi \left[\dfrac{y^5}{5} + \dfrac{4}{3}y^3 + 4y\right]_0^3$

$= \pi\left(\frac{3^5}{5} + 36 + 12 - 0\right)$

$= \frac{483}{5}\pi$ units^3

e



When $x = 1,\quad y = 0$

When $x = 3,\quad y = 8$

Now $y = (x-1)^3$

$\therefore\ x - 1 = y^{\frac{1}{3}}$

$\therefore\ x = y^{\frac{1}{3}} + 1$

$\therefore$ volume $= \pi \int_0^8 x^2\, dy$

$= \pi \int_0^8 (y^{\frac{1}{3}} + 1)^2\, dy$

$= \pi \int_0^8 \left(y^{\frac{2}{3}} + 2y^{\frac{1}{3}} + 1\right) dy$

$= \left[\frac{3}{5}y^{\frac{5}{3}} + \frac{3}{2}y^{\frac{4}{3}} + y\right]_0^8$

$= \pi\left(\frac{3}{5} \times 32 + \frac{3}{2} \times 16 + 8 - 0\right)$

$= \frac{256}{5}\pi$ units^3

9 $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1,\ x \geqslant 0 \qquad \therefore\ x^2 = 9\left(1 - \dfrac{y^2}{16}\right)$

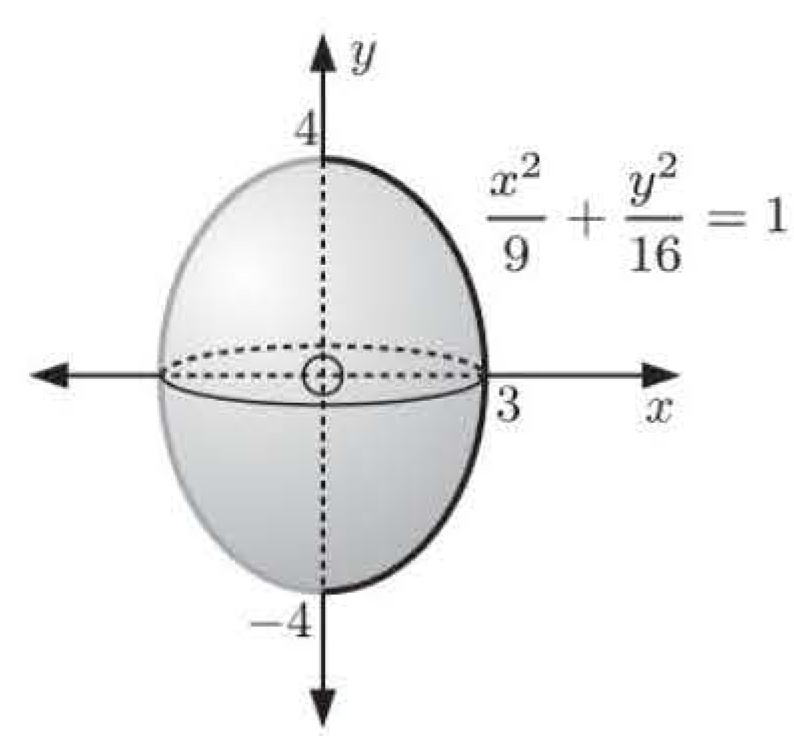
$\therefore$ volume $= \pi \int_{-4}^4 x^2\, dy$

$= \pi \int_{-4}^4 \left(9 - \frac{9}{16}y^2\right) dy$

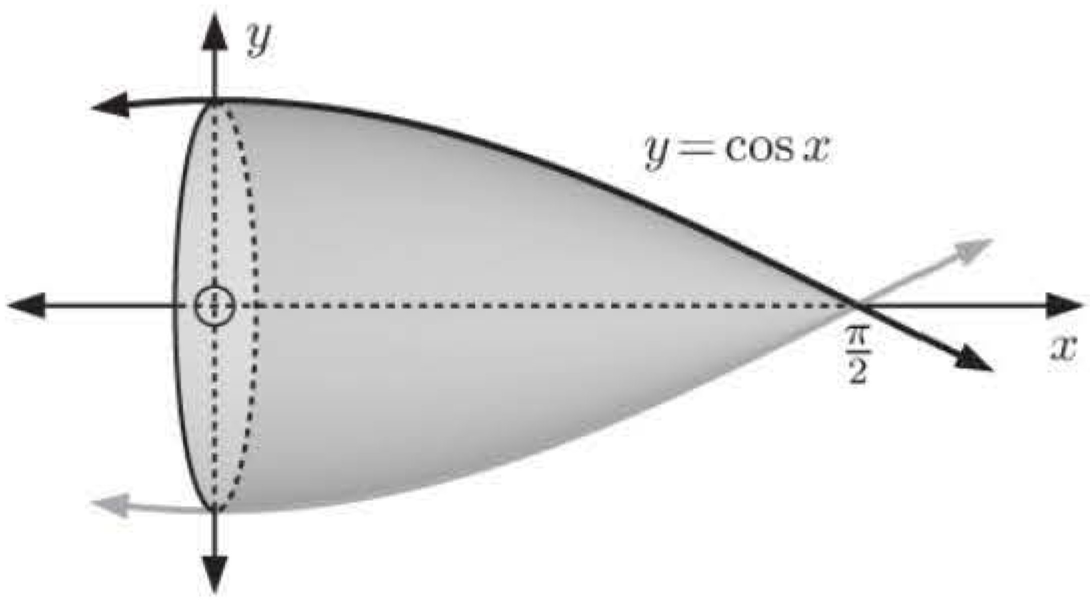
$= \pi \left[9y - \frac{3}{16}y^3\right]_{-4}^4$

$= \pi\,[(36 - 12) - (-36 + 12)]$

$= 48\pi$ units^3

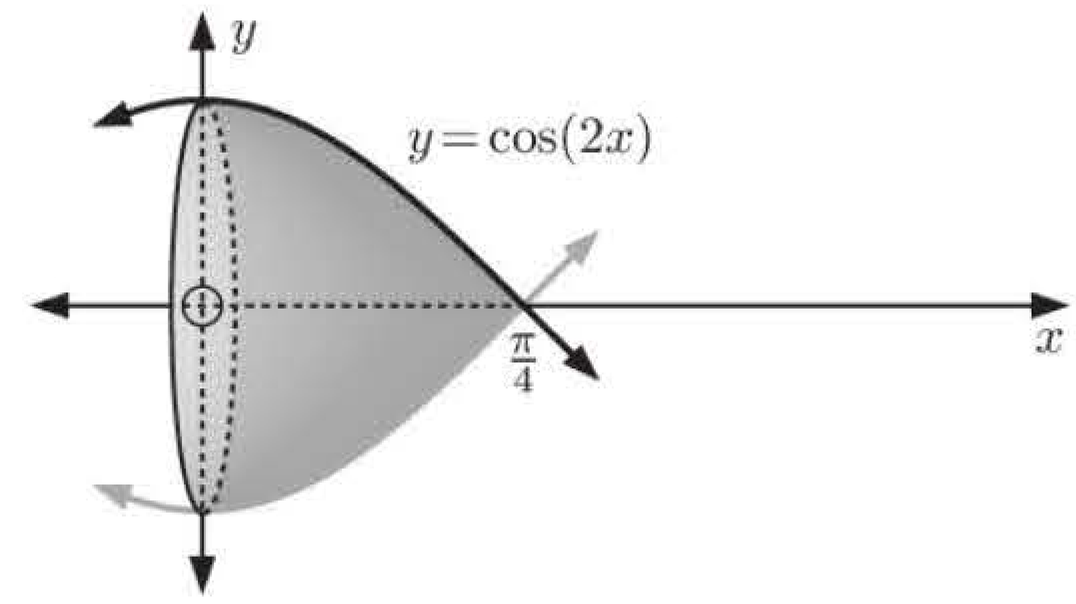


10 **a**



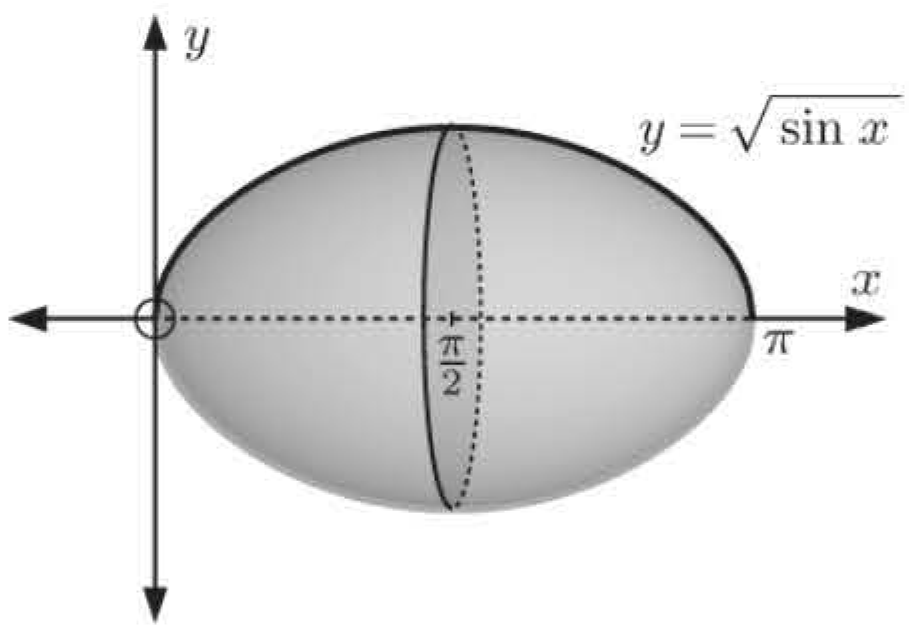
$$\begin{aligned}\text{Volume} &= \pi \int_0^{\frac{\pi}{2}} (\cos x)^2 \, dx \\ &= \pi \int_0^{\frac{\pi}{2}} \cos^2 x \, dx \\ &= \pi \int_0^{\frac{\pi}{2}} \left(\tfrac{1}{2} + \tfrac{1}{2}\cos(2x)\right) dx \\ &= \pi \left[\tfrac{1}{2}x + \tfrac{1}{2}\left(\tfrac{1}{2}\right)\sin(2x)\right]_0^{\frac{\pi}{2}} \\ &= \pi \left[\tfrac{\pi}{4} + \tfrac{1}{4}\sin\pi - 0\right] \\ &= \tfrac{\pi^2}{4} \text{ units}^3\end{aligned}$$

b



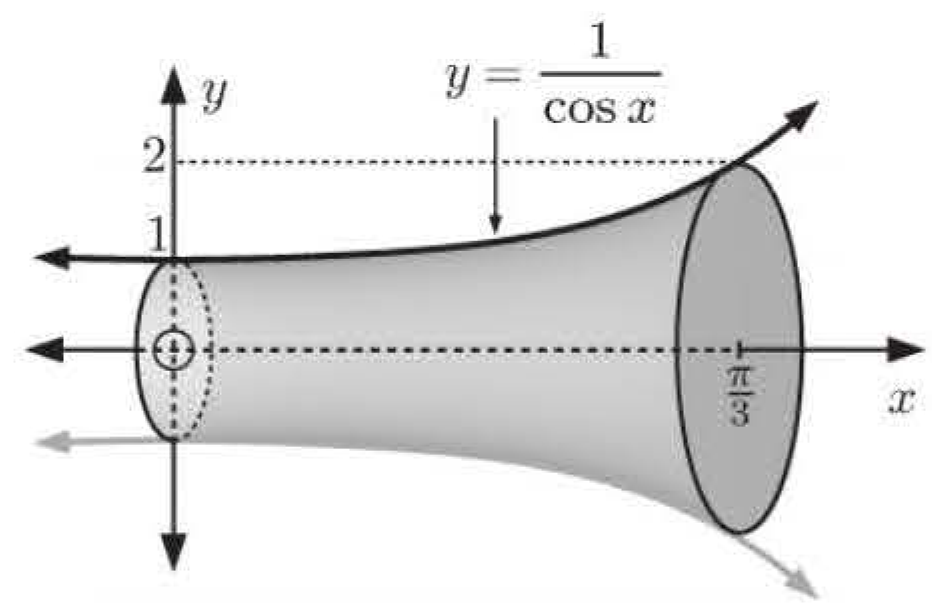
$$\begin{aligned}\text{Volume} &= \pi \int_0^{\frac{\pi}{4}} \cos^2(2x) \, dx \\ &= \pi \int_0^{\frac{\pi}{4}} \left(\tfrac{1}{2} + \tfrac{1}{2}\cos(4x)\right) dx \\ &= \pi \left[\tfrac{1}{2}x + \tfrac{1}{2}\left(\tfrac{1}{4}\right)\sin(4x)\right]_0^{\frac{\pi}{4}} \\ &= \pi \left[\tfrac{\pi}{8} + \tfrac{1}{8}\sin\pi - 0\right] \\ &= \tfrac{\pi^2}{8} \text{ units}^3\end{aligned}$$

c



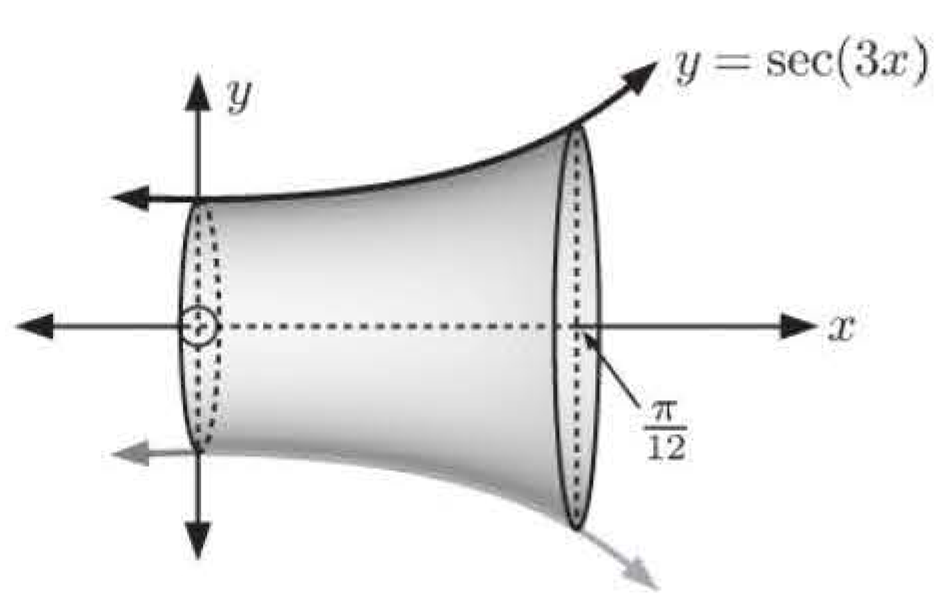
$$\begin{aligned}\text{Volume} &= \pi \int_0^{\pi} \sin x \, dx \\ &= \pi \left[-\cos x\right]_0^{\pi} \\ &= \pi \left[-\cos\pi - -\cos 0\right] \\ &= \pi(2) \\ &= 2\pi \text{ units}^3\end{aligned}$$

d



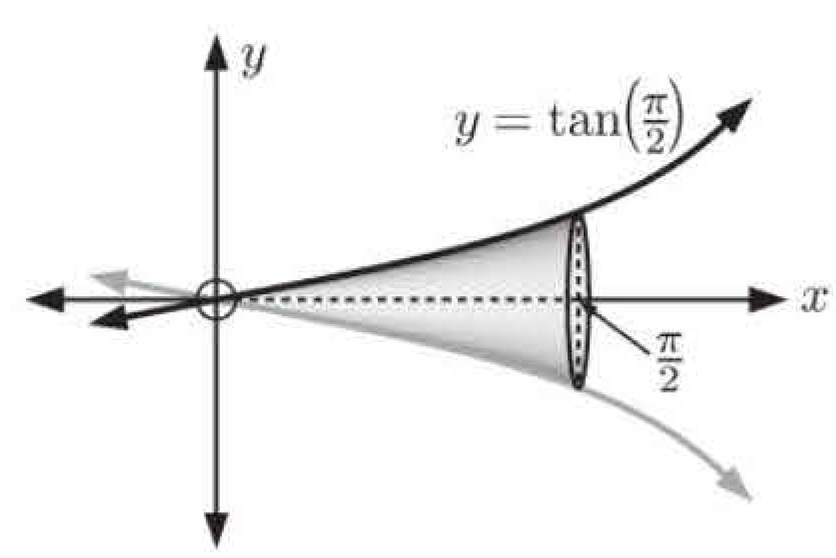
$$\begin{aligned}\text{Volume} &= \pi \int_0^{\frac{\pi}{3}} \frac{1}{\cos^2 x} \, dx \\ &= \pi \left[\tan x\right]_0^{\frac{\pi}{3}} \\ &= \pi \left(\tan\tfrac{\pi}{3} - \tan 0\right) \\ &= \pi(\sqrt{3} - 0) \\ &= \pi\sqrt{3} \text{ units}^3\end{aligned}$$

e



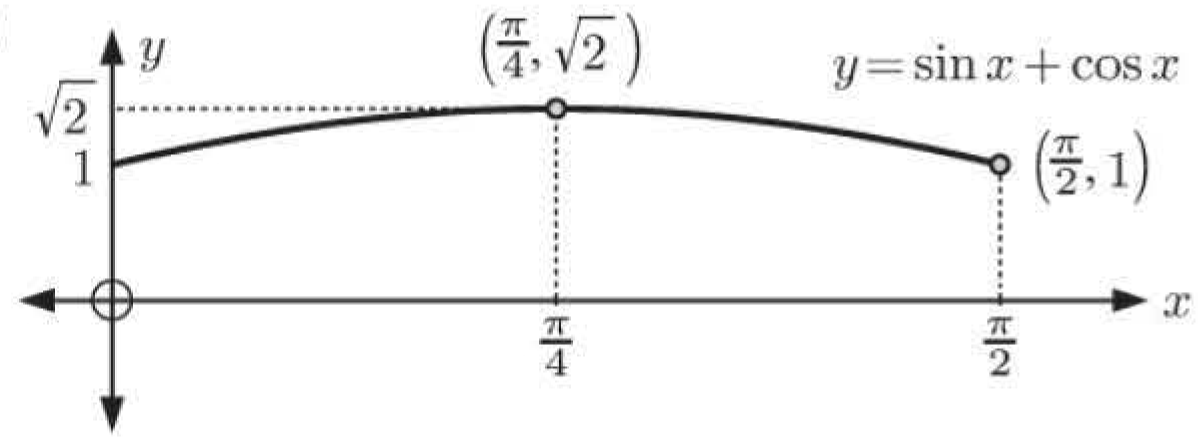
$$\begin{aligned}\text{Volume} &= \pi \int_0^{\pi\frac{\pi}{12}} \sec^2(3x) \, dx \\ &= \pi \left[\tfrac{1}{3}\tan(3x)\right]_0^{\frac{\pi}{12}} \\ &= \tfrac{\pi}{3}\left(\tan\left(\tfrac{\pi}{4}\right) - 0\right) \\ &= \tfrac{\pi}{3} \text{ units}^3\end{aligned}$$

f



$$\begin{aligned}\text{Volume} &= \pi \int_0^{\frac{\pi}{2}} \tan^2\left(\tfrac{x}{2}\right) dx \\ &= \pi \int_0^{\frac{\pi}{2}} \left(\sec^2\left(\tfrac{x}{2}\right) - 1\right) dx \\ &= \pi \left[2\tan\left(\tfrac{x}{2}\right) - x\right]_0^{\frac{\pi}{2}} \\ &= \pi \left(2\tan\tfrac{\pi}{4} - \tfrac{\pi}{2} - 0\right) \\ &= \pi\left(2 - \tfrac{\pi}{2}\right) \text{ units}^3\end{aligned}$$

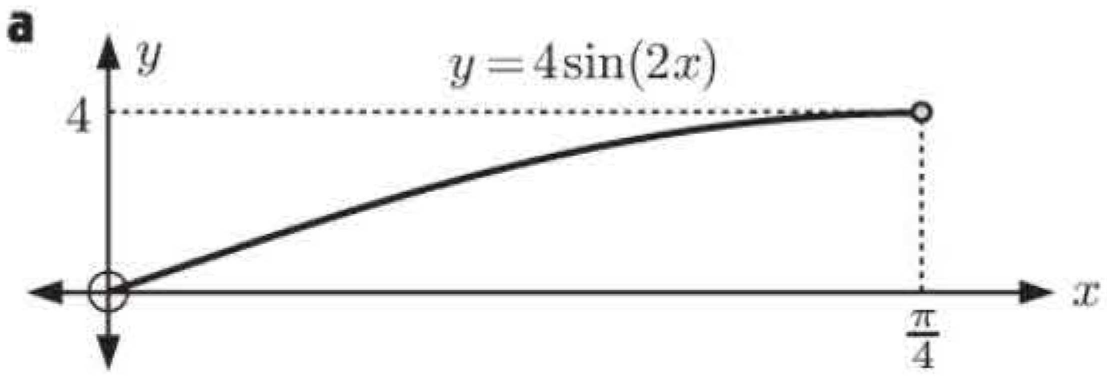
11 **a**



b Volume

$$= \pi \int_0^{\frac{\pi}{4}} (\sin x + \cos x)^2 \; dx$$
$$= \pi \int_0^{\frac{\pi}{4}} \left(\sin^2 x + 2\sin x \cos x + \cos^2 x\right) \, dx$$
$$= \pi \int_0^{\frac{\pi}{4}} (1 + \sin(2x)) \; dx$$
$$= \pi \left[x - \tfrac{1}{2}\cos(2x)\right]_0^{\frac{\pi}{4}}$$
$$= \pi \left[\left(\tfrac{\pi}{4} - \tfrac{1}{2}\cos\left(\tfrac{\pi}{2}\right)\right) - \left(0 - \tfrac{1}{2}\cos 0\right)\right]$$
$$= \pi \left(\tfrac{\pi}{4} + \tfrac{1}{2}\right) \text{ units}^3$$

12 **a**



b Volume

$$= \pi \int_0^{\frac{\pi}{4}} (4\sin(2x))^2 \; dx$$
$$= 16\pi \int_0^{\frac{\pi}{4}} \sin^2(2x) \, dx$$
$$= 16\pi \int_0^{\frac{\pi}{4}} \left(\tfrac{1}{2} - \tfrac{1}{2}\cos(4x)\right) \, dx$$
$$= 16\pi \left[\frac{x}{2} - \tfrac{1}{2}\left(\tfrac{1}{4}\right)\sin(4x)\right]_0^{\frac{\pi}{4}}$$
$$= 16\pi \left[\left(\tfrac{\pi}{8} - \tfrac{1}{8}\sin\pi\right) - \left(0 - \tfrac{1}{8}\sin 0\right)\right]$$
$$= 2\pi^2 \text{ units}^3$$

EXERCISE 22E.2

1 **a** The graphs meet where $4 - x^2 = 3$

$$\therefore \quad x^2 = 1$$
$$\therefore \quad x = \pm 1$$

$\therefore$ A is at $(-1, 3)$ and B is at $(1, 3)$.

b
$$V = \pi \int_{-1}^{1} \left((4 - x^2)^2 - 3^2\right) \, dx$$
$$= \pi \int_{-1}^{1} \left(16 - 8x^2 + x^4 - 9\right) \, dx$$
$$= \pi \int_{-1}^{1} \left(x^4 - 8x^2 + 7\right) \, dx$$
$$= \pi \left[\frac{x^5}{5} - \frac{8x^3}{3} + 7x\right]_{-1}^{1}$$
$$= \pi \left(\tfrac{1}{5} - \tfrac{8}{3} + 7 - \left(\tfrac{-1}{5} - \tfrac{-8}{3} - 7\right)\right)$$
$$= \tfrac{136\pi}{15} \text{ units}^3$$

2 **a** The graphs meet where $e^{\frac{x}{2}} = e$

$$\therefore \quad e^{\frac{x}{2}} = e^1$$
$$\therefore \quad \frac{x}{2} = 1$$
$$\therefore \quad x = 2$$

$\therefore$ A is at $(2, e)$.

b
$$V = \pi \int_0^2 \left(e^2 - \left(e^{\frac{x}{2}}\right)^2\right) \, dx$$
$$= \pi \int_0^2 \left(e^2 - e^x\right) \, dx$$
$$= \pi \left[e^2 x - e^x\right]_0^2$$
$$= \pi \left[(2e^2 - e^2) - (0 - 1)\right]$$
$$= \pi(e^2 + 1) \text{ units}^3$$

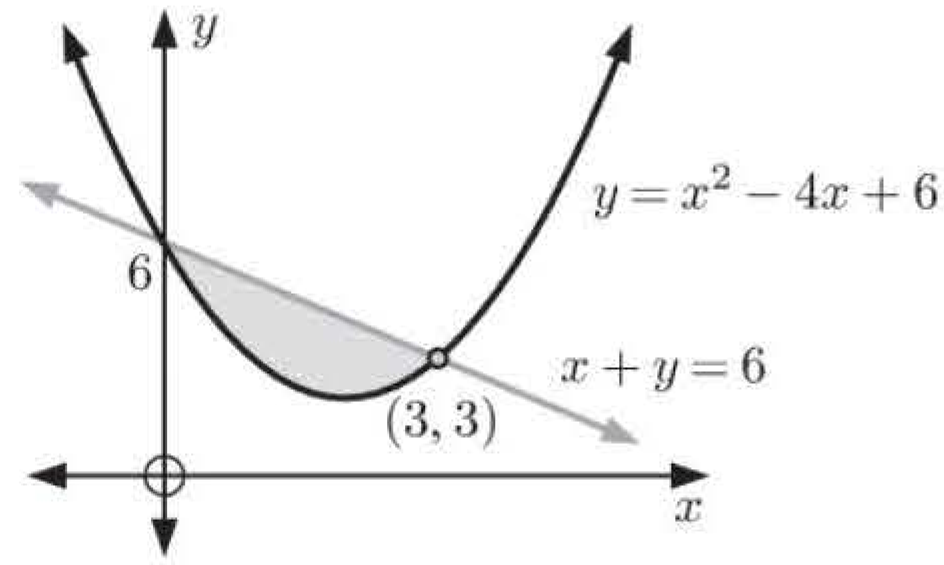
3 **a** The graphs meet where $x = \dfrac{1}{x}$

$$\therefore \quad x^2 = 1$$
$$\therefore \quad x = \pm 1$$
$$\therefore \quad x = 1 \quad \{\text{as } x > 0\}$$

$\therefore$ A is at $(1, 1)$.

b
$$V = \pi \int_1^2 \left(x^2 - \left(\frac{1}{x}\right)^2\right) \, dx$$
$$= \pi \int_1^2 \left(x^2 - x^{-2}\right) \, dx$$
$$= \pi \left[\frac{x^3}{3} - \frac{x^{-1}}{-1}\right]_1^2$$
$$= \pi \left[\left(\tfrac{8}{3} + \tfrac{1}{2}\right) - \left(\tfrac{1}{3} + 1\right)\right]$$
$$= \tfrac{11\pi}{6} \text{ units}^3$$

4 **a** The graphs meet where $x^2 - 4x + 6 = 6 - x$
$\therefore\ x^2 - 3x = 0$
$\therefore\ x(x-3) = 0$
$\therefore\ x = 0$ or 3



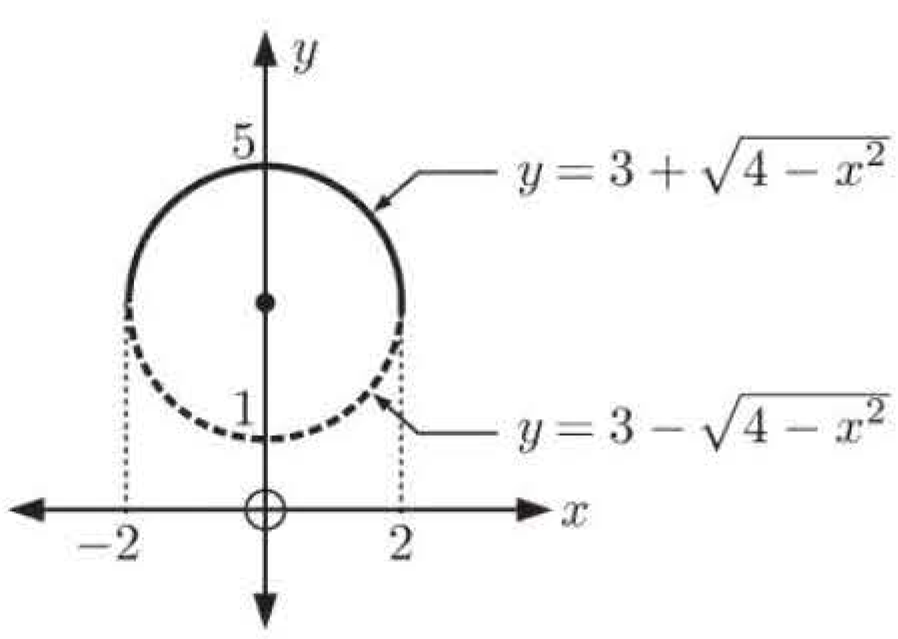
b $V = \pi \int_0^3 \left[(6-x)^2 - (x^2-4x+6)^2\right] dx$
$= \pi \int_0^3 \left[(36 - 12x + x^2) - (x^4 - 4x^3 + 6x^2 - 4x^3 + 16x^2 - 24x + 6x^2 - 24x + 36)\right] dx$
$= \pi \int_0^3 \left(-x^4 + 8x^3 - 27x^2 + 36x\right) dx$
$= \pi \left[-\dfrac{x^5}{5} + 2x^4 - 9x^3 + 18x^2\right]_0^3$
$= \pi \left(-\frac{3^5}{5} + 2(3^4) - 9(27) + 18(9) - 0\right)$
$= \frac{162}{5}\pi$ units^3

5 **a** The curves meet where $\sqrt{x-4} = 1$
$\therefore\ x - 4 = 1$
$\therefore\ x = 5$
$\therefore$ A is at $(5, 1)$.

b $V = \pi \int_5^8 \left(\left(\sqrt{x-4}\right)^2 - 1^2\right) dx$
$= \pi \int_5^8 (x - 4 - 1)\, dx$
$= \pi \int_5^8 (x-5)\, dx$
$= \pi \left[\dfrac{x^2}{2} - 5x\right]_5^8$
$= \pi \left[(32 - 40) - (\frac{25}{2} - 25)\right]$
$= \frac{9\pi}{2}$ units^3

6 **a** $x^2 + (y-3)^2 = 4$
$\therefore\ (y-3)^2 = 4 - x^2$
$\therefore\ y - 3 = \pm\sqrt{4-x^2}$
$\therefore\ y = 3 \pm \sqrt{4-x^2}$

b



c $V = \pi \int_{-2}^2 \left[\left(3 + \sqrt{4-x^2}\right)^2 - \left(3 - \sqrt{4-x^2}\right)^2\right] dx$
$= 2\pi \int_0^2 \left[\left(3 + \sqrt{4-x^2}\right)^2 - \left(3 - \sqrt{4-x^2}\right)^2\right] dx$
$= 2\pi \int_0^2 \left[\left(9 + 6\sqrt{4-x^2} + 4 - x^2\right) - \left(9 - 6\sqrt{4-x^2} + 4 - x^2\right)\right] dx$
$= 2\pi \int_0^2 12\sqrt{4-x^2}\, dx$
$= 24\pi \int_0^2 \sqrt{4-x^2}\, dx$

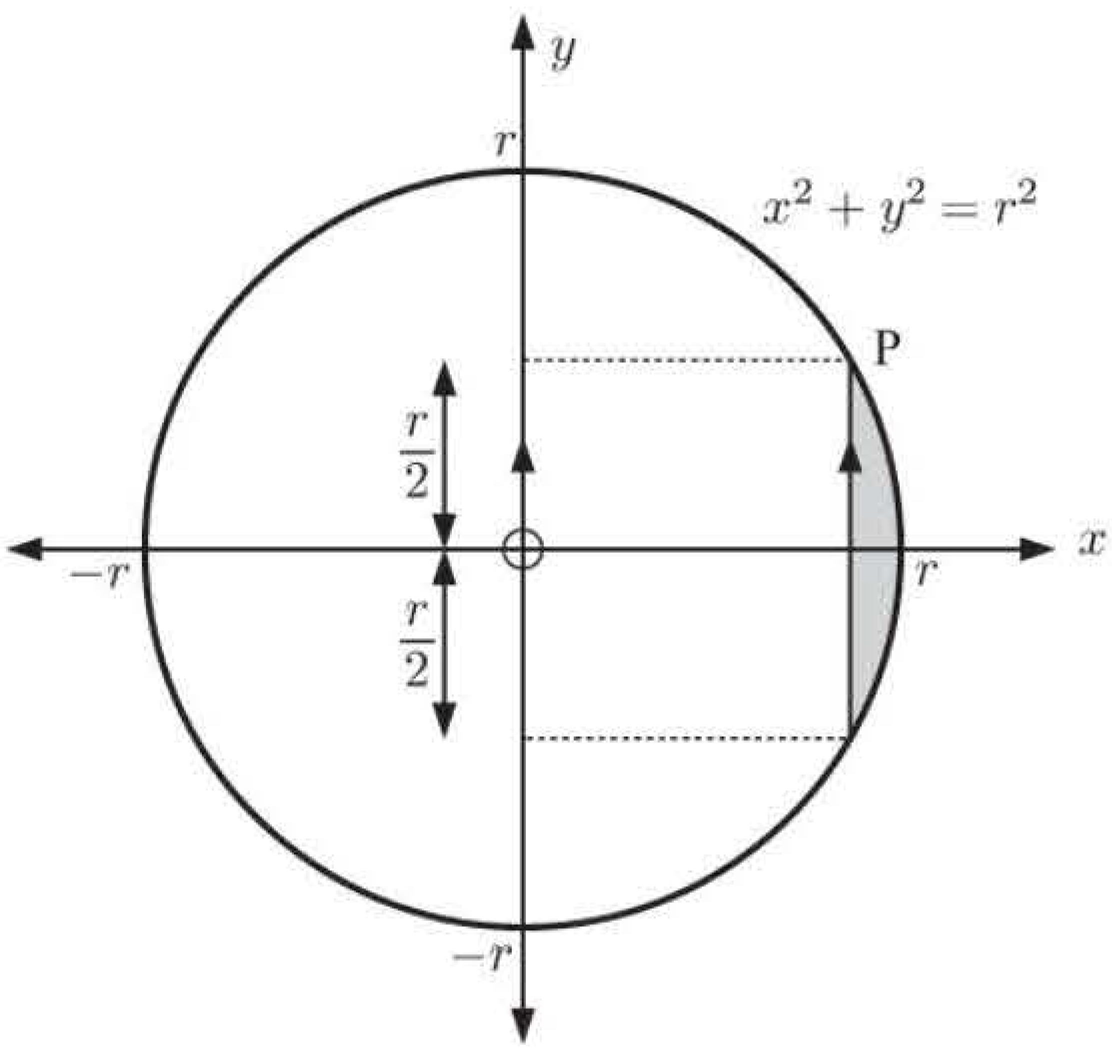
Let $x = 2\sin u$, $\dfrac{dx}{du} = 2\cos u$

when $x = 0$, $u = 0$

when $x = 2$, $u = \frac{\pi}{2}$

$$\begin{aligned}
\therefore\quad V &= 24\pi \int_0^2 \sqrt{4-(2\sin u)^2}\,dx \\
&= 24\pi \int_0^{\frac{\pi}{2}} \sqrt{4-4\sin^2 u}\,\frac{dx}{du}\,du \\
&= 48\pi \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2 u}\,(2\cos u)\,du \\
&= 48\pi \int_0^{\frac{\pi}{2}} 2\cos^2 u\,du \qquad \{\sqrt{1-\sin^2 u}=\cos u\} \\
&= 48\pi \int_0^{\frac{\pi}{2}} (1+\cos 2u)\,du \\
&= 48\pi \left[u+\tfrac{1}{2}\sin 2u\right]_0^{\frac{\pi}{2}} \\
&= 48\pi \left(\tfrac{\pi}{2}+\tfrac{1}{2}(0)-0\right) \\
&= 24\pi^2 \text{ units}^3 \quad (\approx 237 \text{ units}^3)
\end{aligned}$$

7 Since the chord is parallel to the y-axis, the y-coordinate of P is $\dfrac{r}{2}$.



When $y=\dfrac{r}{2}$, $x^2+\left(\dfrac{r}{2}\right)^2=r^2$

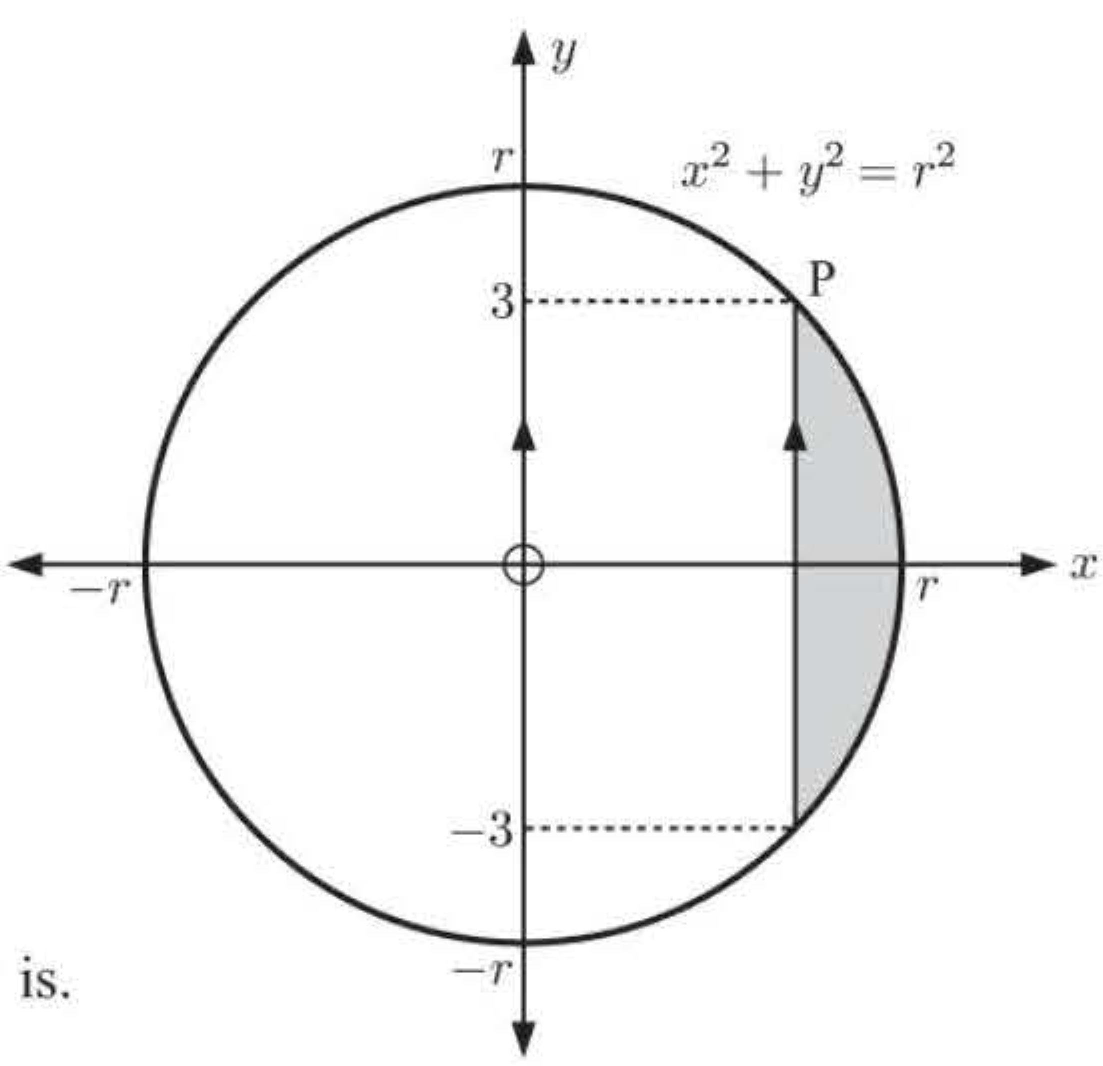
$\therefore\quad x^2=\frac{3}{4}r^2$

$\therefore\quad x=\pm\frac{\sqrt{3}}{2}r$

$\therefore$ the coordinates of P are $\left(\frac{\sqrt{3}}{2}r, \dfrac{r}{2}\right)$.

$$\begin{aligned}
\therefore\quad V &= 2\pi \int_0^{\frac{r}{2}} \left[(r^2-y^2)-\tfrac{3}{4}r^2\right] dy \\
&= 2\pi \left[r^2y-\frac{y^3}{3}-\tfrac{3}{4}r^2y\right]_0^{\frac{r}{2}} \\
&= 2\pi\left(\frac{r^3}{2}-\frac{r^3}{24}-\frac{3r^3}{8}-0\right) \\
&= 2\pi\left(\frac{r^3}{12}\right) \\
&= \frac{\pi r^3}{6} \text{ units}^3
\end{aligned}$$

8 When $y=3$, $x^2+3^2=r^2$



$\therefore\quad x^2=r^2-9$

$\therefore\quad x=\pm\sqrt{r^2-9}$

$\therefore$ the coordinates of P are $(\sqrt{r^2-9}, 3)$.

$$\begin{aligned}
\therefore\quad V &= 2\pi \int_0^3 \left[(r^2-y^2)-(r^2-9)\right] dy \\
&= 2\pi \int_0^3 (9-y^2)\,dy \\
&= 2\pi \left[9y-\frac{y^3}{3}\right]_0^3 \\
&= 2\pi(27-9-0) \\
&= 36\pi \text{ units}^3, \text{ no matter what the value of } r \text{ is.}
\end{aligned}$$

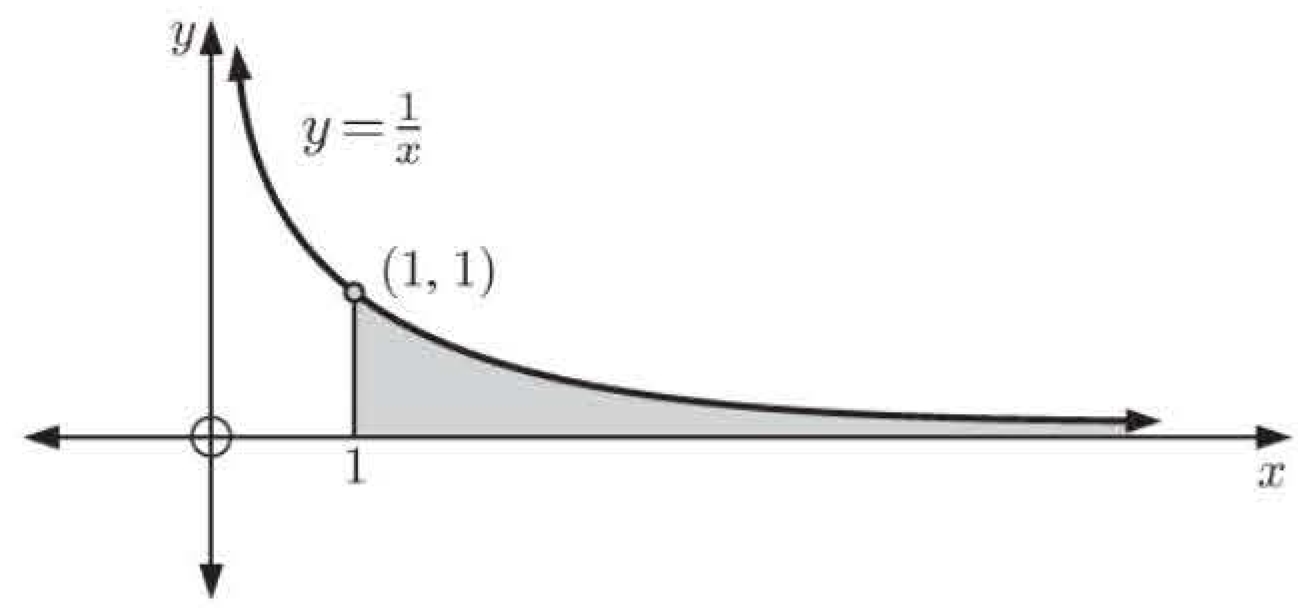
$\therefore$ the volume is independent of r.

9 The shaded area $= \int_1^{\infty} \frac{1}{x}\, dx$

$= \lim_{t\to\infty} \int_1^{t} \frac{1}{x}\, dx$

$= \lim_{t\to\infty} [\ln(x)]_1^t, \ x > 0$

$= \lim_{t\to\infty} \ln t,$ which is infinite



The volume of revolution $= \pi \int_1^{\infty} \left(\frac{1}{x}\right)^2 dx$

$= \pi \lim_{t\to\infty} \int_1^t x^{-2}\, dx$

$= \pi \lim_{t\to\infty} \left[-\frac{1}{x}\right]_1^t$

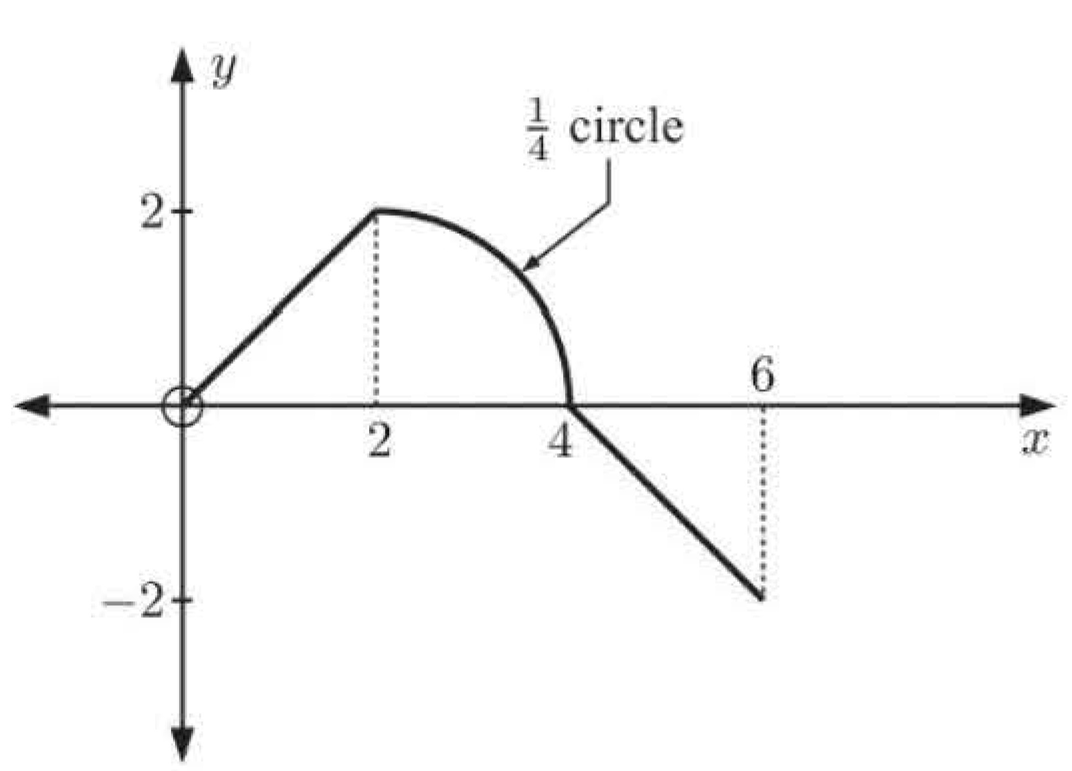
$= \pi \lim_{t\to\infty} \left(-\frac{1}{t}+1\right)$

$= \pi,$ which is finite

REVIEW SET 22A

1 shaded area $= \int_a^b [f(x)-g(x)]\, dx + \int_b^c [g(x)-f(x)]\, dx + \int_c^d [f(x)-g(x)]\, dx$

2



a $\int_0^4 f(x)\, dx =$ area of triangle + area of $\frac{1}{4}$ circle

$= \frac{1}{2}(2\times 2) + \frac{1}{4}\pi(2)^2$

$= (2+\pi)$ units^2

b $\int_4^6 f(x)\, dx = -$area of triangle below x-axis

$= -\frac{1}{2}(2\times 2)$

$= -2$ units^2

c $\int_0^6 f(x)\, dx = \int_0^4 f(x)\, dx + \int_4^6 f(x)\, dx$

$= (2+\pi)+(-2)$

$= \pi$ units^2

3 $\int_{-1}^3 f(x)\, dx$ gives us the correct area only if $f(x)$ is non-negative on the interval $-1 \leqslant x \leqslant 3$.

In this case $f(x)$ is negative for $1 < x < 3$, so $\int_{-1}^3 f(x)\, dx$ does not provide the correct answer.

(The shaded area which is below the x-axis is given by $\int_1^3 [0 - f(x)]\, dx = -\int_1^3 f(x)\, dx$.

$\therefore$ total area shaded $= \int_{-1}^1 f(x)\, dx - \int_1^3 f(x)\, dx$)

4 $y = k$ meets $y = x^2$ where $x^2 = k$ $\quad \therefore\ x = \pm\sqrt{k}$

By symmetry, $\int_0^{\sqrt{k}} (k - x^2)\, dx = \frac{1}{2} \times 5\frac{1}{3} = \frac{1}{2} \times \frac{16}{3}$

$\therefore \left[kx - \frac{x^3}{3}\right]_0^{\sqrt{k}} = \frac{8}{3}$

$\therefore k\sqrt{k} - \frac{k\sqrt{k}}{3} = \frac{8}{3}$

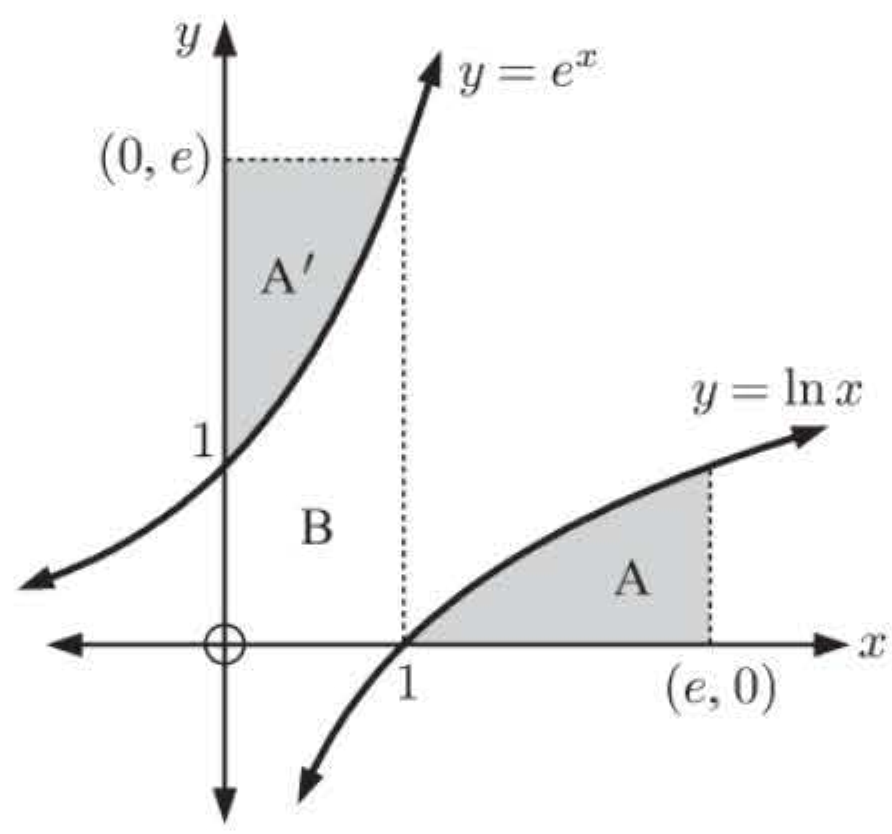
$\therefore \frac{2}{3}k\sqrt{k} = \frac{8}{3}$

$\therefore k\sqrt{k} = 4$

$\therefore k^{\frac{3}{2}} = 4$

$\therefore k = 4^{\frac{2}{3}} = \sqrt[3]{16}$

5



$y = e^x$ and $y = \ln x$ are inverse functions, so they are symmetrical about $y = x$

$\therefore$ area A = area A′

But area A′ + area B = area of rectangle

$\therefore$ area A + area B $= e \times 1 = e$

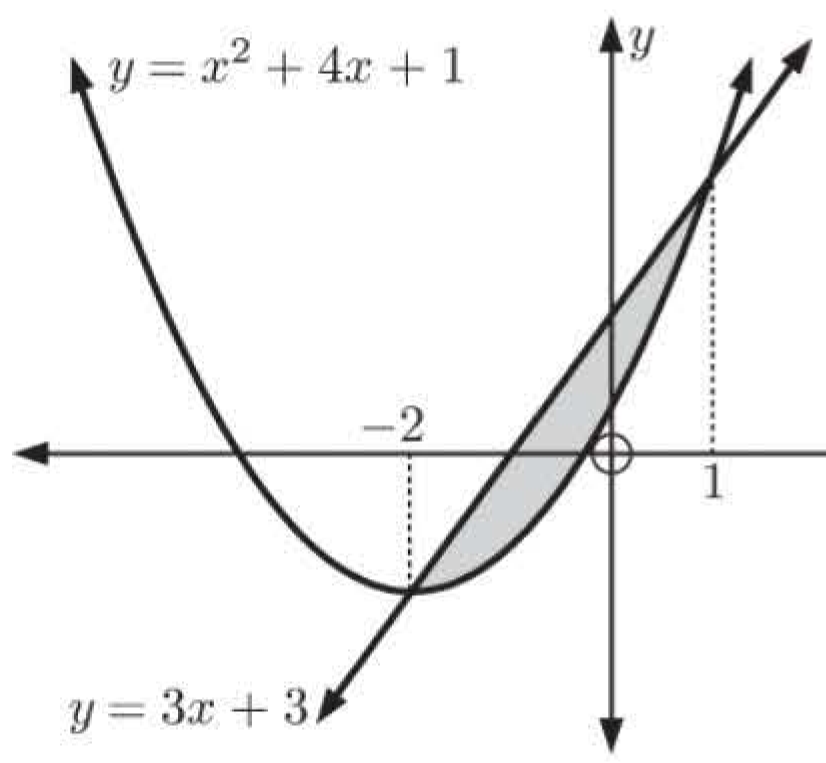
Since area A $= \int_1^e \ln x \, dx$

and area B $= \int_0^1 e^x \, dx$,

$$\int_0^1 e^x \, dx + \int_1^e \ln x \, dx = e$$

6 $y = x^2 + 4x + 1$ meets $y = 3x + 3$ where

$$x^2 + 4x + 1 = 3x + 3$$
$$\therefore \quad x^2 + x - 2 = 0$$
$$\therefore \quad (x+2)(x-1) = 0$$
$$\therefore \quad x = -2 \text{ or } 1$$



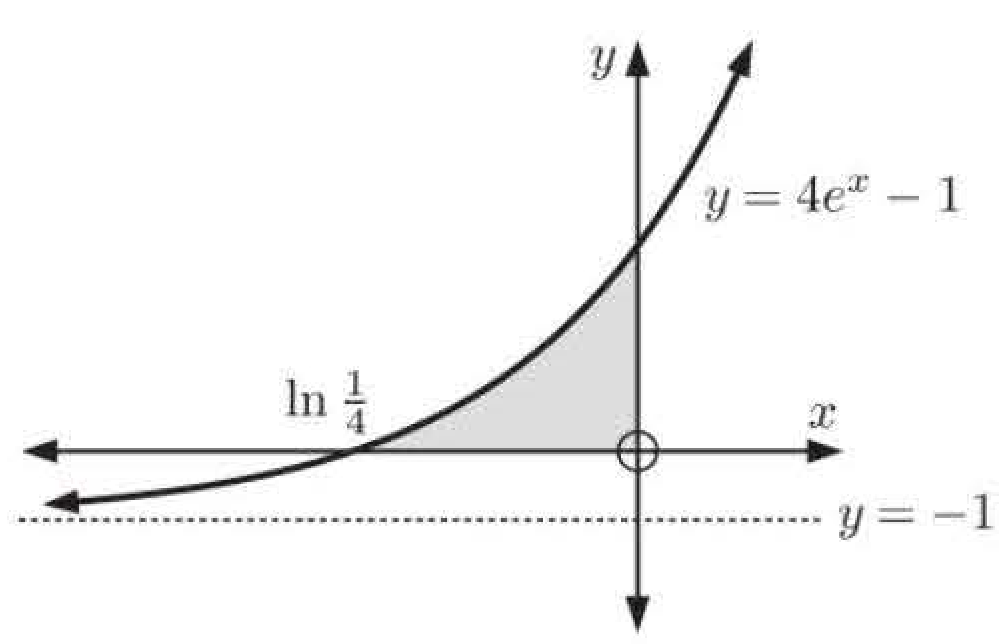
$$\begin{aligned}
\therefore \quad \text{area} &= \int_{-2}^{1} [(3x+3) - (x^2+4x+1)] \, dx \\
&= \int_{-2}^{1} (-x^2 - x + 2) \, dx \\
&= \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x \right]_{-2}^{1} \\
&= \left(-\tfrac{1}{3} - \tfrac{1}{2} + 2\right) - \left(\tfrac{8}{3} - 2 - 4\right) \\
&= -\tfrac{1}{3} - \tfrac{1}{2} + 2 - \tfrac{8}{3} + 2 + 4 \\
&= 4\tfrac{1}{2} \text{ units}^2
\end{aligned}$$

7 Consider $y = 4e^x - 1$.

The x-intercept occurs when $y = 0$

$$\therefore \quad 4e^x - 1 = 0$$
$$\therefore \quad e^x = \tfrac{1}{4}$$
$$\therefore \quad x = \ln \tfrac{1}{4} < 0$$

$y = 4e^x - 1$ is the graph of $y = e^x$ with a vertical stretch of factor 4 and a vertical translation of -1.

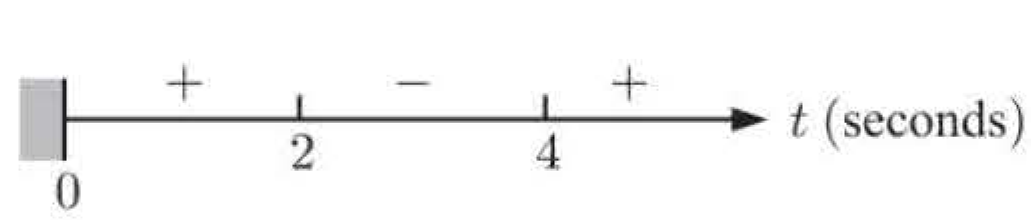


$$\begin{aligned}
\text{Area} &= \int_{\ln \frac{1}{4}}^{0} (4e^x - 1) \, dx \\
&= [4e^x - x]_{\ln \frac{1}{4}}^{0} \\
&= (4e^0 - 0) - \left(4e^{\ln \frac{1}{4}} - \ln \tfrac{1}{4}\right) \\
&= 4 - 0 - 4(\tfrac{1}{4}) + \ln \tfrac{1}{4} \\
&= 3 + \ln \tfrac{1}{4} \text{ units}^2 \\
&= (3 - \ln 4) \text{ units}^2
\end{aligned}$$

8 **a** $v(t) = t^2 - 6t + 8 \text{ m s}^{-1}, \quad t \geqslant 0$

$= (t-4)(t-2)$

which has sign diagram:



b Now $s(t) = \int (t^2 - 6t + 8)\,dt$

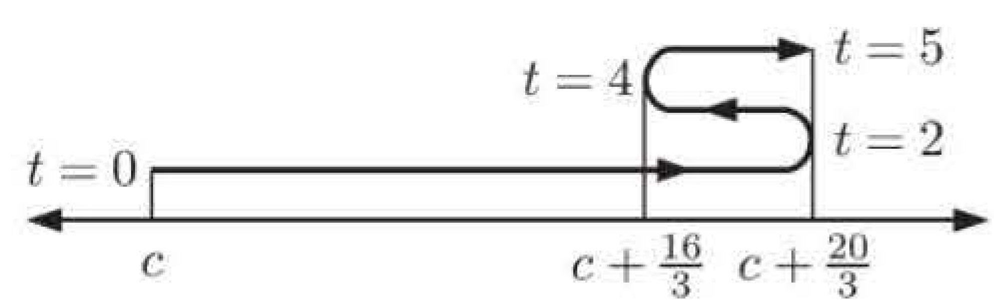
$$= \frac{t^3}{3} - 3t^2 + 8t + c$$

$\therefore \quad s(0) = c$

$s(2) = c + 6\frac{2}{3}$

$s(4) = c + 5\frac{1}{3}$ the motion diagram is:

$s(5) = c + 6\frac{2}{3}$



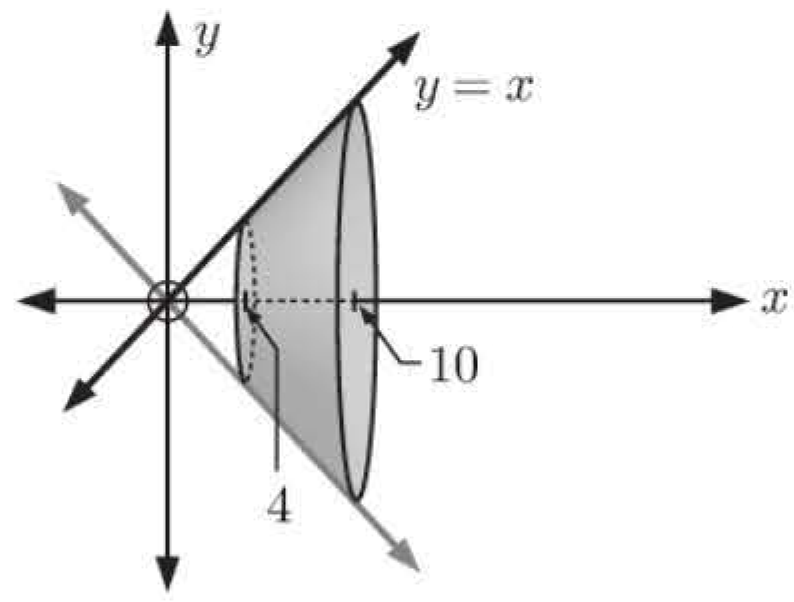
The particle moves in the positive direction initially, then at $t = 2$, $6\frac{2}{3}$ m from its starting point, it changes direction. It changes direction again at $t = 4$, $5\frac{1}{3}$ m from its starting point. When $t = 5$ it is $6\frac{2}{3}$ m from its starting point.

c After 5 seconds, the particle is $6\frac{2}{3}$ m to the right of its starting point.

d The total distance travelled $= (c + \frac{20}{3} - c) + [(c + \frac{20}{3}) - (c + \frac{16}{3})] + [(c + \frac{20}{3}) - (c + \frac{16}{3})]$

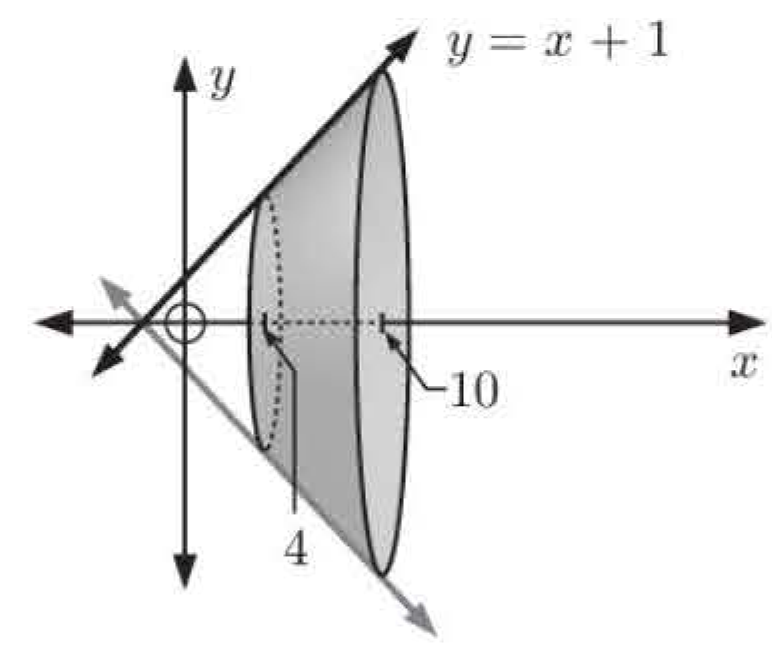
$= 9\frac{1}{3}$ m

9 **a**



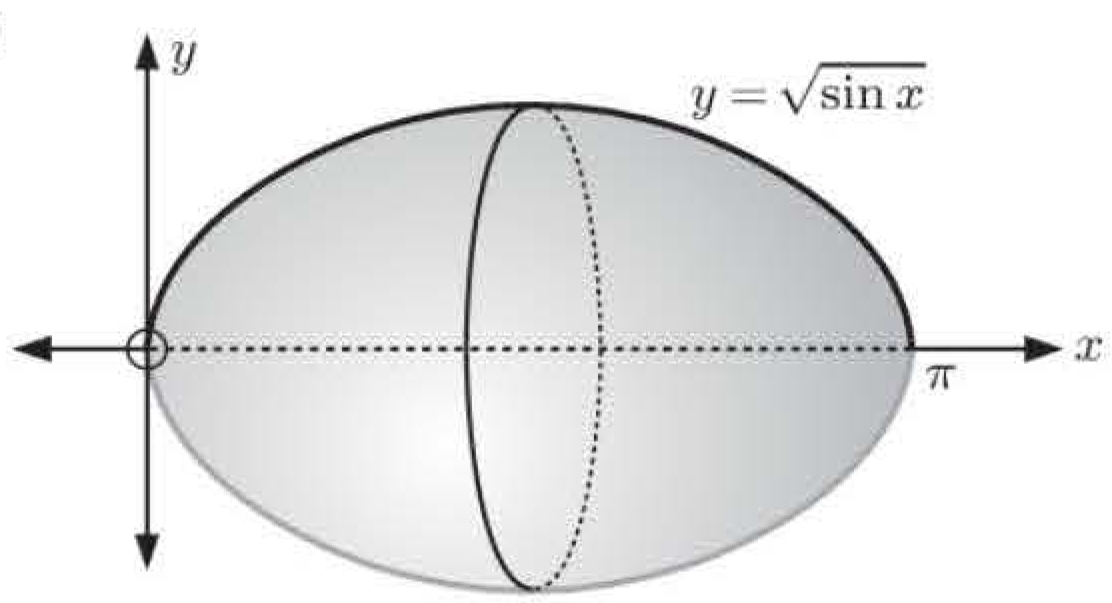
$$V = \pi \int_4^{10} x^2\,dx$$
$$= \pi \left[\frac{x^3}{3}\right]_4^{10}$$
$$= \pi \left(\frac{1000}{3} - \frac{64}{3}\right)$$
$$= \frac{936\pi}{3}$$
$$= 312\pi \text{ units}^3$$

b



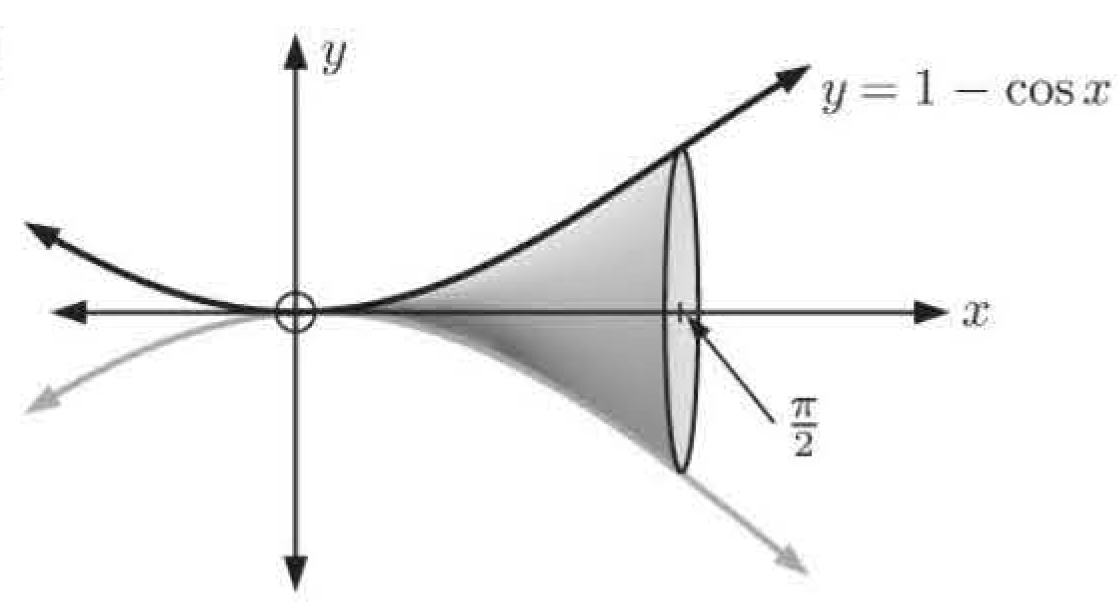
$$V = \pi \int_4^{10} (x+1)^2\,dx$$
$$= \pi \left[\frac{(x+1)^3}{3}\right]_4^{10}$$
$$= \pi \left(\frac{11^3}{3} - \frac{5^3}{3}\right)$$
$$= \frac{1206\pi}{3} = 402\pi \text{ units}^3$$

c



$$V = \pi \int_0^{\pi} \left(\sqrt{\sin x}\right)^2\,dx$$
$$= \pi \int_0^{\pi} \sin x\,dx$$
$$= \pi \left[-\cos x\right]_0^{\pi}$$
$$= \pi (1 - (-1))$$
$$= 2\pi \text{ units}^3$$

d



$$V = \pi \int_0^{\frac{\pi}{2}} (1 - \cos x)^2\,dx$$
$$= \pi \int_0^{\frac{\pi}{2}} (1 - 2\cos x + \cos^2 x)\,dx$$
$$= \pi \int_0^{\frac{\pi}{2}} (1 - 2\cos x + \tfrac{1}{2} + \tfrac{1}{2}\cos 2x)\,dx$$
$$= \pi \left[\tfrac{3}{2}x - 2\sin x + \tfrac{1}{4}\sin 2x\right]_0^{\frac{\pi}{2}}$$
$$= \pi \left[\left(\tfrac{3}{2}(\tfrac{\pi}{2}) - 2\sin(\tfrac{\pi}{2}) + \tfrac{1}{4}\sin\pi\right) - \left(\tfrac{3}{2}(0) - 2\sin 0 + \tfrac{1}{4}\sin 0\right)\right]$$
$$= \pi \left(\frac{3\pi}{4} - 2\right) \text{ units}^3$$
$$= \pi \left(\frac{3\pi - 8}{4}\right) \text{ units}^3$$

10 $\dfrac{d^2y}{dx^2} = k(L-x)^2$

$\therefore \quad \dfrac{dy}{dx} = \displaystyle\int k(L-x)^2\,dx = \dfrac{-k(L-x)^3}{3} + c$

But when $x = 0$ the tangent is horizontal and so $\dfrac{dy}{dx} = 0$.

$\therefore \quad \dfrac{-kL^3}{3} + c = 0$ and so $c = \dfrac{kL^3}{3}$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{-k(L-x)^3}{3} + \dfrac{kL^3}{3}$

$$\therefore \quad y = \int \left(\frac{-k(L-x)^3}{3} + \frac{kL^3}{3}\right) dx$$
$$= \frac{k(L-x)^4}{12} + \frac{kL^3}{3}x + d$$

But when $x = 0,\ y = 0$ $\quad \therefore \quad \dfrac{kL^4}{12} + d = 0$

$$\therefore \quad d = -\frac{kL^4}{12}$$
$$\therefore \quad y = \frac{k(L-x)^4}{12} + \frac{kL^3x}{3} - \frac{kL^4}{12}$$

The greatest deflection occurs when $x \approx L$

$\therefore \quad y \approx \dfrac{k(0)^4}{12} + \dfrac{kL^4}{3} - \dfrac{kL^4}{12} = \dfrac{kL^4}{4}$

$\therefore$ the greatest deflection is about $\dfrac{kL^4}{4}$ metres.

11 **a** The shaded area $= \int_0^2 ax(x-2)\,dx$

$= 4$ units^2

$\therefore \quad \int_0^2 (ax^2 - 2ax)\,dx = 4$

$\therefore \quad \left[\dfrac{ax^3}{3} - ax^2\right]_0^2 = 4$

$\therefore \quad \left(\dfrac{8a}{3} - 4a\right) - 0 = 4$

$\therefore \quad \dfrac{8a}{3} - \dfrac{12a}{3} = 4$

$\therefore \quad -\dfrac{4a}{3} = 4$

$\therefore \quad a = -3$

$\therefore \quad y = -3x(x-2)$

b Suppose A has coordinates $(k, -3k(k-2))$.

$\therefore$ gradient of [OA] $= \dfrac{-3k(k-2) - 0}{k - 0}$

$= -3(k-2)$

$\therefore$ equation of [OA] is $y = -3(k-2)x$

If [OA] divides the shaded region into equal areas,

$\int_0^k [-3x(x-2) - (-3(k-2)x)]\,dx = 2$

$\therefore \quad \int_0^k (-3x^2 + 6x + 3kx - 6x)\,dx = 2$

$\therefore \quad \int_0^k (-3x^2 + 3kx)\,dx = 2$

$\therefore \quad \left[-x^3 + \dfrac{3kx^2}{2}\right]_0^k = 2$

$\therefore \quad -k^3 + \dfrac{3k^3}{2} = 2$

$\therefore \quad \dfrac{k^3}{2} = 2$

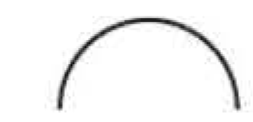
$\therefore \quad k^3 = 4$

$\therefore \quad k = \sqrt[3]{4}$

$\therefore$ the x-coordinate of A is $\sqrt[3]{4}$.

12 **a** A is the *upper half* of a circle centre $(2,\ 0)$ and radius 2.

$$\therefore \quad (x-2)^2+(y-0)^2=2^2$$
$$(x-2)^2+y^2=4$$
$$y^2=4-(x-2)^2$$
$$y^2=4-x^2+4x-4$$
$$y^2=4x-x^2$$
$$\therefore \quad y=\pm\sqrt{4x-x^2}$$

So, $y=\sqrt{4x-x^2}$ 

or $y=-\sqrt{4x-x^2}$ 

Since A is the upper half of the circle,

$$y_A=\sqrt{4x-x^2}$$

b Now B is the *lower half* of a circle centre $(5,\ 0)$ and radius 1.

$$\therefore \quad (x-5)^2+(y-0)^2=1^2$$
$$(x-5)^2+y^2=1$$
$$y^2=1-(x-5)^2$$
$$y^2=1-x^2+10x-25$$
$$\therefore \quad y=\pm\sqrt{10x-x^2-24}$$
$$\therefore \quad y_B=-\sqrt{10x-x^2-24}$$

c $\int_0^4 y_A\,dx$

$=\frac{1}{2}\pi r^2$ where $r=2$

$=\frac{1}{2}\pi(2)^2$

$=2\pi$

$\int_4^6 y_B\,dx$

$=-\frac{1}{2}\pi r^2$ where $r=1$

$=-\frac{1}{2}\pi(1)^2$

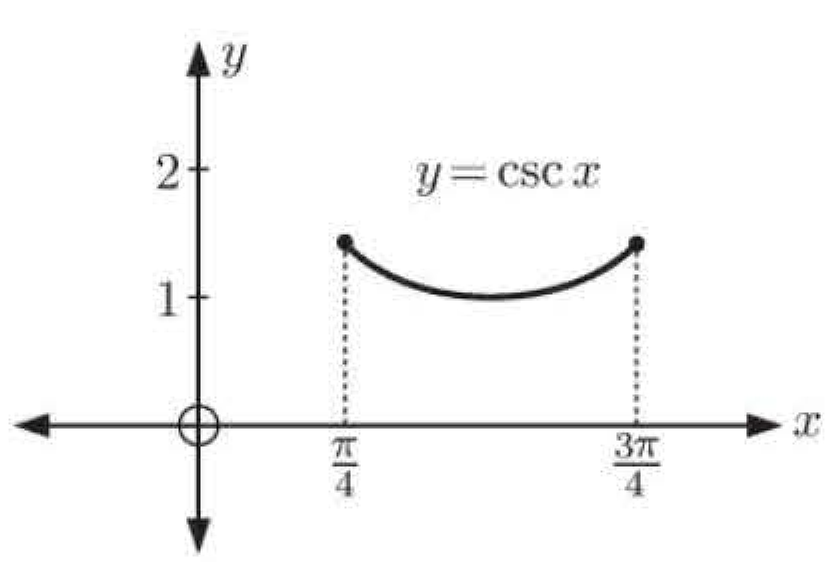
$=-\frac{\pi}{2}$

d $\int_0^6 f(x)\,dx$

$=\int_0^4 y_A\,dx+\int_4^6 y_B\,dx$

$=2\pi+(-\frac{\pi}{2})$

$=\frac{3\pi}{2}$

13 

$$V=\pi\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\csc^2 x\,dx$$
$$=\pi\left[-\cot x\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}$$
$$=\pi\left(-\cot\left(\tfrac{3\pi}{4}\right)--\cot\left(\tfrac{\pi}{4}\right)\right)$$
$$=\pi\left(-(-1)+1\right)$$
$$=2\pi \text{ units}^3$$

14 $\dfrac{dT}{dx}=\dfrac{k}{x}=kx^{-1}$

$\therefore \quad T=k\ln x+c \quad \{x>0\}$

When $x=r_1,\ T=T_0$

$\therefore \quad k\ln r_1+c=T_0$

$\therefore \quad c=T_0-k\ln r_1$

$\therefore \quad T=k\ln x+T_0-k\ln r_1$

$=T_0+k\ln\left(\dfrac{x}{r_1}\right)$

So, when $x=r_2$,

$$T=T_0+k\ln\left(\frac{r_2}{r_1}\right)$$

$\therefore$ the outer surface has temperature $T_0+k\ln\left(\dfrac{r_2}{r_1}\right)$.

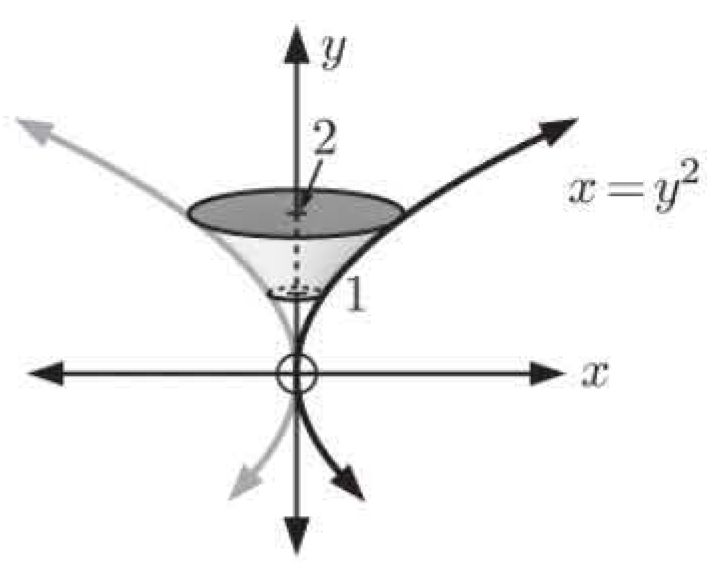
15 The gradient of the straight line is $\dfrac{0-8}{4-0} = \frac{-8}{4} = -2$

$\therefore$ the straight line has equation $y = -2x + 8$

$\therefore$ the volume of revolution

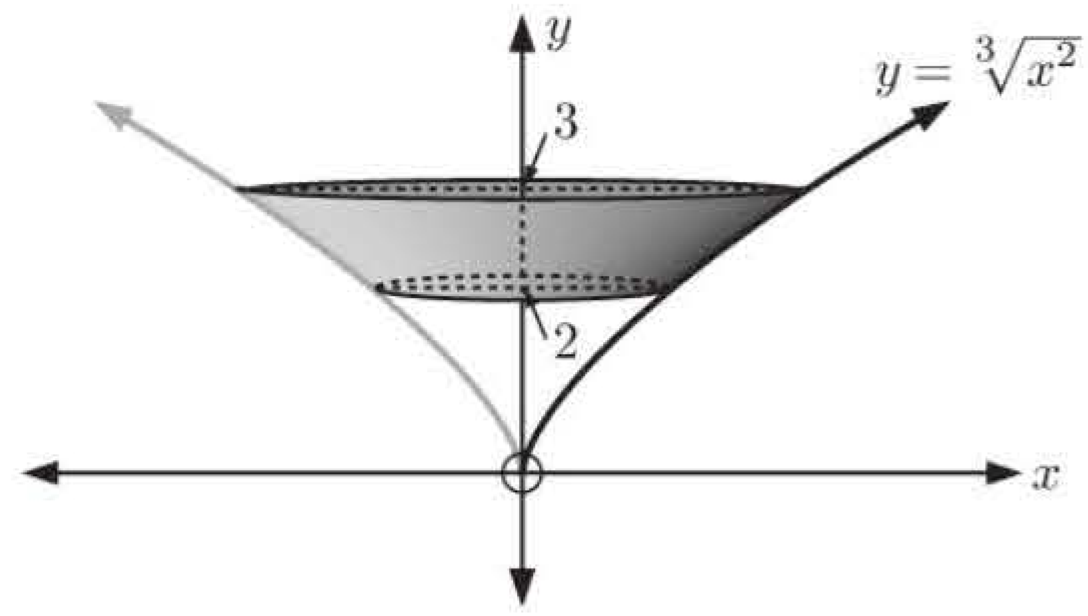
$$= \pi \int_0^2 \left(x^2\right)^2 \, dx + \pi \int_2^4 (-2x+8)^2 \, dx$$
$$= \pi \int_0^2 x^4 \, dx + \pi \int_2^4 (4x^2 - 32x + 64) \, dx$$
$$= \pi \left[\tfrac{1}{5}x^5\right]_0^2 + \pi \left[\tfrac{4}{3}x^3 - 16x^2 + 64x\right]_2^4$$
$$= \pi \left(\tfrac{1}{5}(2)^5 - 0\right) + \pi \left[\tfrac{4}{3}(4)^3 - 16(4)^2 + 64(4) - \left(\tfrac{4}{3}(2)^3 - 16(2)^2 + 64(2)\right)\right]$$
$$= \pi \times \tfrac{32}{5} + \pi \left(\tfrac{256}{3} - \tfrac{224}{3}\right)$$
$$= \tfrac{32\pi}{5} + \tfrac{32\pi}{3}$$
$$= \tfrac{256\pi}{15} \quad \text{as required}$$

16 **a**



$$V = \pi \int_1^2 x^2 \, dy$$
$$= \pi \int_1^2 y^4 \, dy$$
$$= \pi \left[\frac{y^5}{5}\right]_1^2$$
$$= \pi \left(\tfrac{32}{5} - \tfrac{1}{5}\right)$$
$$= \tfrac{31\pi}{5} \text{ units}^3$$

b



$$y = \sqrt[3]{x^2} \quad \therefore \ x^2 = y^3$$
$$V = \pi \int_2^3 x^2 \, dy$$
$$= \pi \int_2^3 y^3 \, dy$$
$$= \pi \left[\frac{y^4}{4}\right]_2^3$$
$$= \pi \left(\tfrac{81}{4} - \tfrac{16}{4}\right)$$
$$= \tfrac{65\pi}{4} \text{ units}^3$$

REVIEW SET 22B

1 **a** $a(t) = v'(t)$

$\therefore \ a(t) = 2 - 6t \ \text{m s}^{-2}$

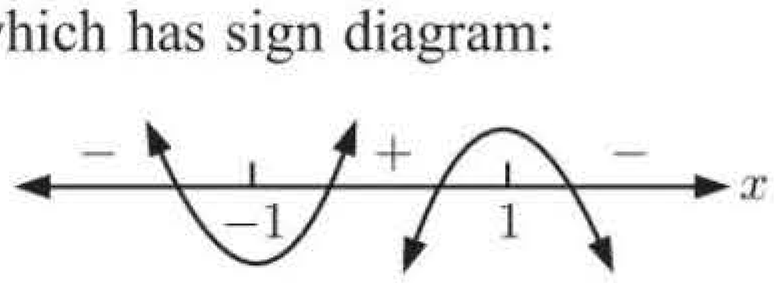
b $s(t) = \int (2t - 3t^2) \, dt$

$\therefore \ s(t) = t^2 - t^3 + c \ \text{m}$

c Change in displacement after two seconds $= s(2) - s(0)$
$$= 2^2 - 2^3 + c - (0^2 - 0^3 + c)$$
$$= 4 - 8 + c - c$$
$$= -4 \text{ m} \quad (4 \text{ m to the left})$$

2 **a** $f(x) = \dfrac{x}{1+x^2} \quad \therefore \quad f'(x) = \dfrac{1(1+x^2) - x(2x)}{(1+x^2)^2}$ {quotient rule}

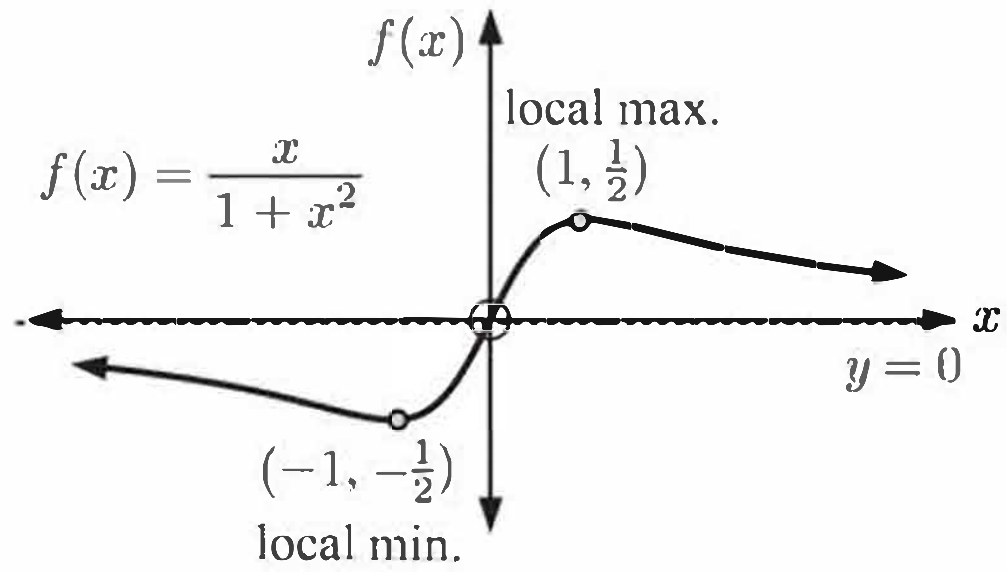
$$= \frac{1 + x^2 - 2x^2}{(1+x^2)^2}$$
$$= \frac{1 - x^2}{(1+x^2)^2}$$
which has sign diagram:
$$= \frac{(1+x)(1-x)}{(1+x^2)^2}$$



$\therefore$ there is a local minimum at $\left(-1, -\frac{1}{2}\right)$ and a local maximum at $\left(1, \frac{1}{2}\right)$.

b As $x \to \infty$, $f(x) \to 0^+$.
As $x \to -\infty$, $f(x) \to 0^-$.

c



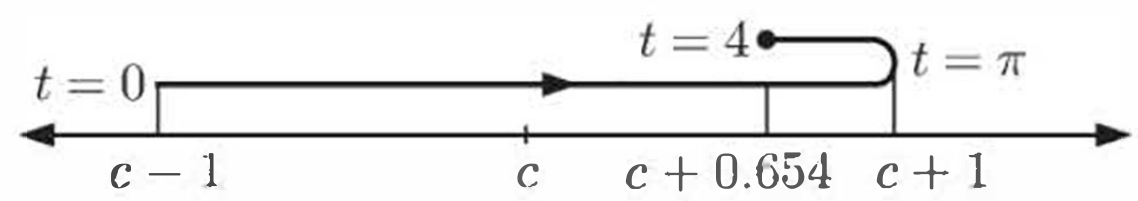
d $$\text{Area} = \int_{-2}^{0} \left[0 - \frac{x}{1+x^2}\right] dx$$
$$= \int_{-2}^{0} \frac{-x}{1+x^2}\, dx$$
$$\approx 0.805 \text{ units}^2$$

3 $v(t) = \sin t$ which has sign diagram:

Now $s(t) = \int \sin t\, dt$
$= -\cos t + c$ metres

$\therefore$ $s(0) = -1 + c$
$s(\pi) = 1 + c$
$s(4) = -\cos 4 + c \approx c + 0.654$

motion diagram:



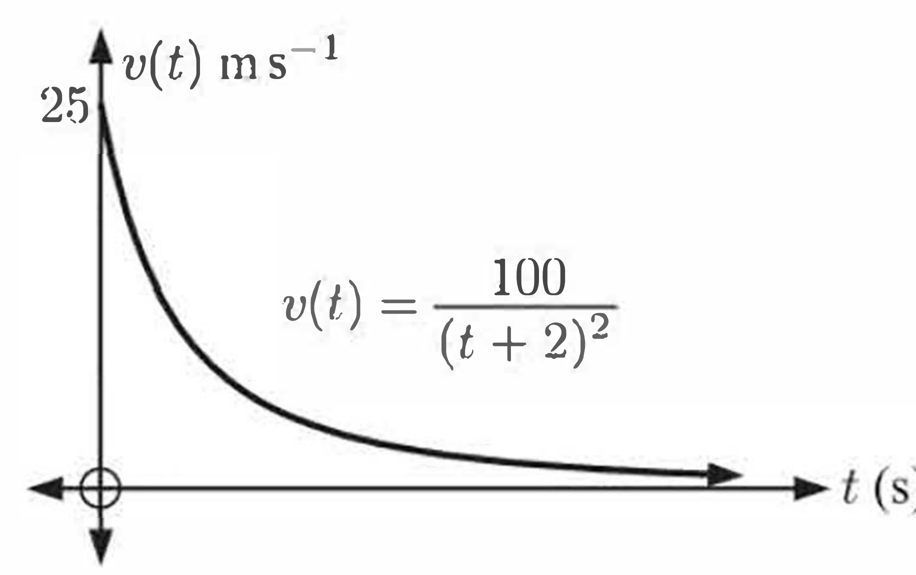
$\therefore$ total distance travelled $= [(c+1) - (c-1)] + [(c+1) - (c+0.654)]$
≈ 2.35 m

4 $v(t) = \dfrac{100}{(t+2)^2} = 100(t+2)^{-2}$ m s^{-1}

a At $t = 0$, $v(0) = \dfrac{100}{2^2} = 25$ m s^{-1}
At $t = 3$, $v(3) = \dfrac{100}{5^2} = 4$ m s^{-1}

b As $t \to \infty$, $v(t) \to 0^+$

c



d As $v(t)$ is always positive, the boat is always travelling forwards.
$$s(t) = \int v(t)\, dt$$
$$= \int 100(t+2)^{-2}\, dt$$
$$= -100(t+2)^{-1} + c$$
$$= \frac{-100}{t+2} + c$$
$\therefore$ $s(0) = c - 50$ m
$\therefore$ when the boat has travelled 30 m,
$s(t) = c - 20$ m
$\therefore$ $c - 20 = \dfrac{-100}{t+2} + c$
$\therefore$ $\dfrac{-100}{t+2} = -20$
$\therefore$ $t + 2 = 5$
$\therefore$ $t = 3$ seconds

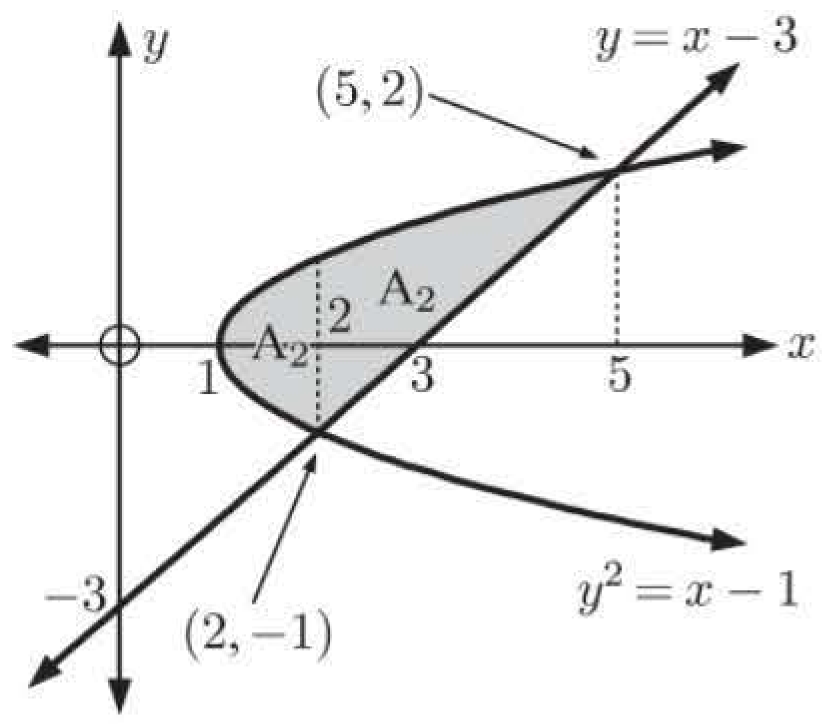
e $$a(t) = v'(t)$$
$$= -200(t+2)^{-3}$$
$$= \frac{-200}{(t+2)^3} \text{ m s}^{-2}, \quad t \geqslant 0$$

f $$\frac{dv}{dt} = \frac{-200}{(t+2)^3} = -\tfrac{1}{5}\,\frac{1000}{(t+2)^3}$$
$$= -\tfrac{1}{5}\left(\frac{100}{(t+2)^2}\right)^{\frac{3}{2}}$$
$$= -\tfrac{1}{5}\, v^{\frac{3}{2}}$$
$\therefore$ $\dfrac{dv}{dt} = -kv^{\frac{3}{2}}$ where $k = \frac{1}{5}$

5 **a** The graphs meet when $\cos 2x = e^{3x}$
Using technology, $x = 0$ and
$x \approx -0.7292$

b Shaded area $\approx \int_{-0.7292}^{0} (\cos 2x - e^{3x})\, dx$
≈ 0.2009 units^2 {using technology}

6 **a**



b $y^2 = x - 1$ meets $y = x - 3$ where

$$x - 1 = (x-3)^2$$
$$\therefore \quad x - 1 = x^2 - 6x + 9$$
$$\therefore \quad x^2 - 7x + 10 = 0$$
$$\therefore \quad (x-5)(x-2) = 0$$
$$\therefore \quad x = 2 \quad \text{or} \quad x = 5$$

$\therefore$ the graphs meet at $(5, 2)$ and $(2, -1)$.

c Area $= A_1 + A_2$

$$= 2\int_1^2 (x-1)^{\frac{1}{2}}\,dx + \int_2^5 [(x-1)^{\frac{1}{2}} - (x-3)]\,dx$$
$$= 2\left[\tfrac{2}{3}(x-1)^{\frac{3}{2}}\right]_1^2 + \left[\tfrac{2}{3}(x-1)^{\frac{3}{2}} - \frac{x^2}{2} + 3x\right]_2^5$$
$$= 2\left[\tfrac{2}{3} - 0\right] + \left[(\tfrac{2}{3}(8) - \tfrac{25}{2} + 15) - (\tfrac{2}{3} - 2 + 6)\right] = 4\tfrac{1}{2} \text{ units}^2$$

7

$$\int_0^m \sin x\,dx = \tfrac{1}{2}$$
$$\therefore \quad [-\cos x]_0^m = \tfrac{1}{2}$$
$$\therefore \quad -\cos m + \cos 0 = \tfrac{1}{2}$$
$$\therefore \quad \cos m = \tfrac{1}{2}$$
$$\therefore \quad m = \tfrac{\pi}{3} \quad \{0 < m < \tfrac{\pi}{2}\}$$

8 **a** The graphs meet where

$$x^2 = \sin x$$
$$\therefore \quad x = 0 \quad \text{or} \quad \approx 0.8767 \quad \{\text{using technology}\}$$
$$\therefore \quad a \approx 0.8767$$

b area $\approx \int_0^{0.8767} (\sin x - x^2)\,dx$

≈ 0.1357 units^2 {using technology}

9 **a** $y = \cos(2x)$ meets the x-axis where $2x = \frac{\pi}{2}$, or $x = \frac{\pi}{4}$.

$$\therefore \quad V = \pi\int_{\frac{\pi}{16}}^{\frac{\pi}{4}} \cos^2(2x)\,dx = \pi\int_{\frac{\pi}{16}}^{\frac{\pi}{4}} \left(\tfrac{1}{2} + \tfrac{1}{2}\cos(4x)\right)dx$$
$$= \pi\left[\tfrac{1}{2}x + \tfrac{1}{8}\sin(4x)\right]_{\frac{\pi}{16}}^{\frac{\pi}{4}}$$
$$= \pi\left[\left(\tfrac{\pi}{8} + \tfrac{1}{8}\sin\pi\right) - \left(\tfrac{\pi}{32} + \tfrac{1}{8}\sin\left(\tfrac{\pi}{4}\right)\right)\right]$$
$$= \pi\left(\tfrac{\pi}{8} - \tfrac{\pi}{32} - \tfrac{1}{8}\left(\tfrac{1}{\sqrt{2}}\right)\right)$$
$$= \pi\left(\tfrac{3\pi}{32} - \tfrac{1}{8\sqrt{2}}\right) \text{ units}^3$$

b $V = \pi\int_0^2 (e^{-x} + 4)^2\,dx$

$$= \pi\int_0^2 \left(e^{-2x} + 8e^{-x} + 16\right)dx$$
$$= \pi\left[\tfrac{1}{-2}e^{-2x} + \tfrac{8}{-1}e^{-x} + 16x\right]_0^2$$
$$= \pi\left[\left(-\tfrac{1}{2}e^{-4} - 8e^{-2} + 32\right) - \left(-\tfrac{1}{2} - 8\right)\right]$$
$$= \pi\left(\frac{81}{2} - \frac{1}{2e^4} - \frac{8}{e^2}\right) \text{ units}^3$$
$$\approx 124 \text{ units}^3$$

10 $y = 2x^3 - 9x$ meets $y = 3x^2 - 10$ when $2x^3 - 9x = 3x^2 - 10$

$\therefore\ 2x^3 - 3x^2 - 9x + 10 = 0$

$\therefore\ (x-1)(2x^2 - x - 10) = 0$

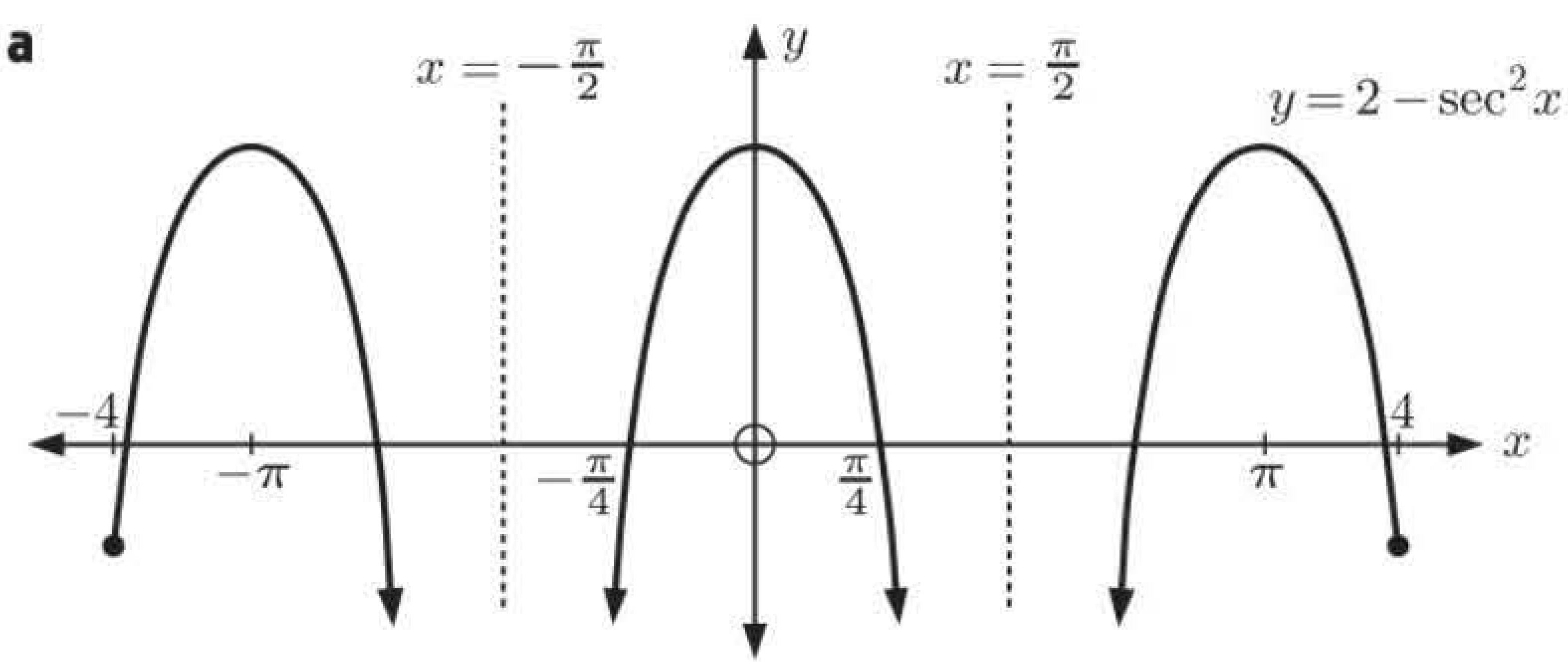
$\therefore\ (x-1)(2x-5)(x+2) = 0$

$\therefore\ x = -2,\ 1,$ or $\frac{5}{2}$

$\therefore$ total area $= \displaystyle\int_{-2}^{\frac{5}{2}} \left|2x^3 - 3x^2 - 9x + 10\right| dx$

≈ 31.2 units^2

11 **a**



b $f(x) = 2 - \sec^2 x$ is undefined when $\cos x = 0$.
On the domain $x \in [-4, 4]$ this is when $x = -\frac{\pi}{2}, \frac{\pi}{2}$.
$\therefore$ the vertical asymptotes are $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$

c When $y = 0$, $2 - \sec^2 x = 0$

$\therefore\ \cos^2 x = \frac{1}{2}$

$\therefore\ \cos x = \pm\frac{1}{\sqrt{2}}$

$\therefore\ x = -\frac{5\pi}{4}, -\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$ {for $x \in [-4, 4]$}

$\therefore$ the x-intercepts are $-\frac{5\pi}{4}, -\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$

When $x = 0$, $y = 2 - \sec^2(0) = 2 - 1^2 = 1$

$\therefore$ the y-intercept is 1.

d area

$= \displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (2 - \sec^2 x)\, dx$

$= [2x - \tan x]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$

$= \left(\frac{\pi}{2} - 1\right) - \left(-\frac{\pi}{2} - (-1)\right)$

$= (\pi - 2)$ units^2

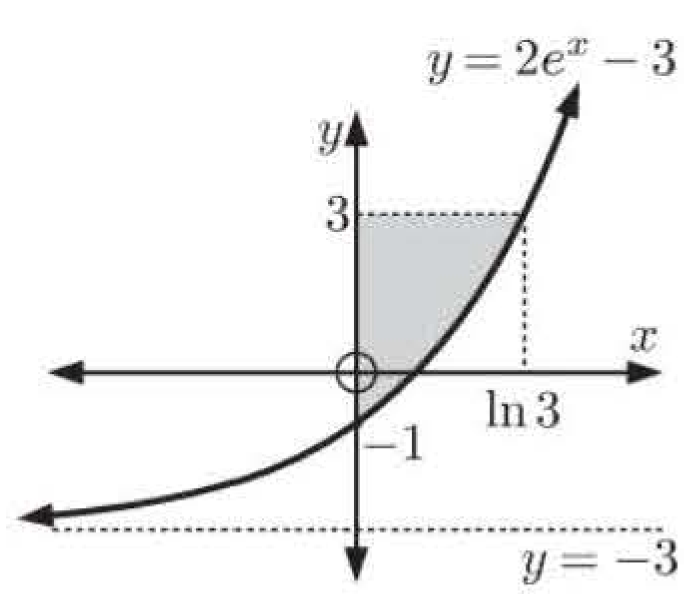
12 $x = \ln\left(\dfrac{y+3}{2}\right)$

$\therefore\ \dfrac{y+3}{2} = e^x$

$\therefore\ y = 2e^x - 3$

When $y = 3$, $x = \ln\left(\dfrac{3+3}{2}\right)$

$\therefore\ x = \ln 3$



Area $= \int_0^{\ln 3} (3 - [2e^x - 3])\, dx$

$= \int_0^{\ln 3} (6 - 2e^x)\, dx$

$= [6x - 2e^x]_0^{\ln 3}$

$= \left(6\ln 3 - 2e^{\ln 3}\right) - (0 - 2)$

$= (6\ln 3 - 2 \times 3 + 2)$

$= 6\ln 3 - 4$ units^2

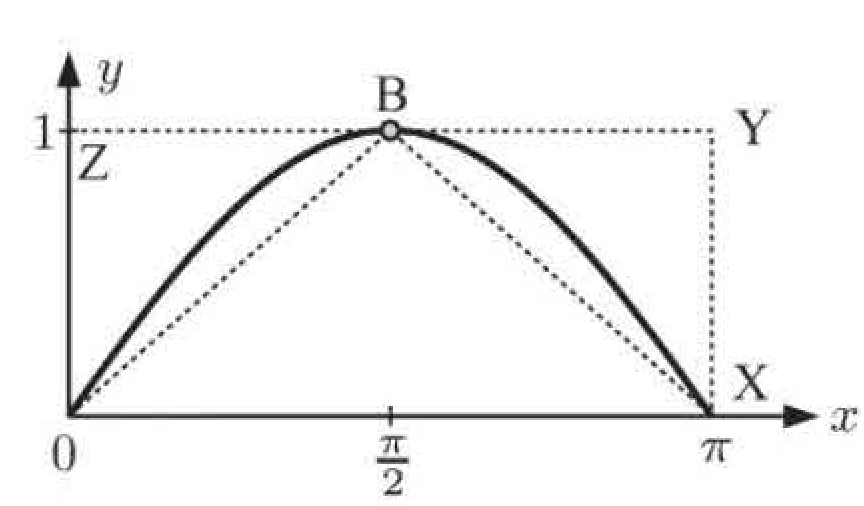
≈ 2.59 units^2

13 **a** From the graph,

area $\triangle$OBX $<$ area under the curve $<$ area OXYZ

$\therefore\ \frac{1}{2}\pi(1) < \int_0^{\pi} \sin x\, dx < \pi(1)$

$\therefore\ \frac{\pi}{2} < \int_0^{\pi} \sin x\, dx < \pi$



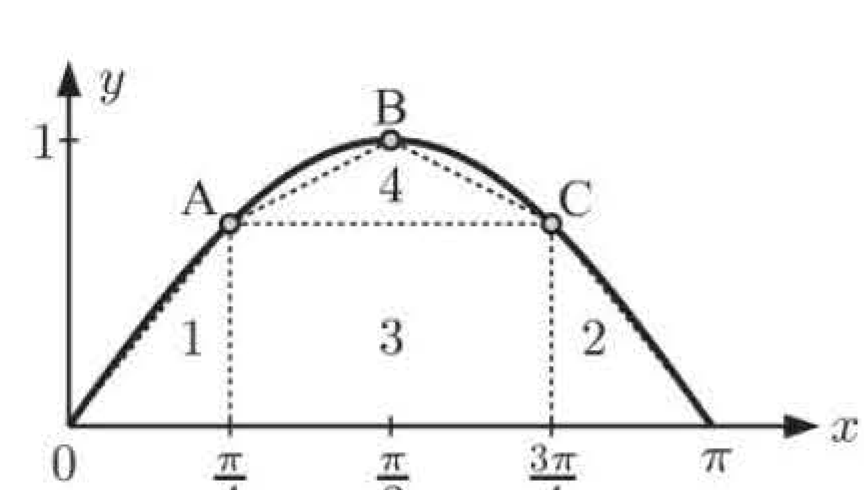
b If we partition the diagram as shown:

Area $1 = \frac{1}{2}\left(\frac{\pi}{4}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi\sqrt{2}}{16}$

Area $2 = \frac{\pi\sqrt{2}}{16}$

Area $3 = \frac{\pi}{2} \times \frac{\sqrt{2}}{2} = \frac{\pi\sqrt{2}}{4}$

Area $4 = \frac{1}{2}\left(\frac{\pi}{2}\right)\left(1 - \frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}\left(1 - \frac{\sqrt{2}}{2}\right)$

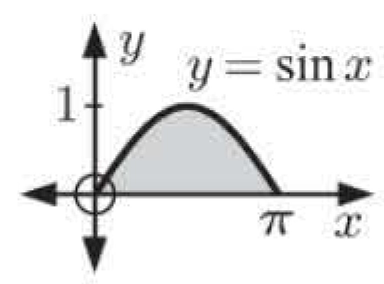


The area under the arch is greater than the total area of the 4 sections.

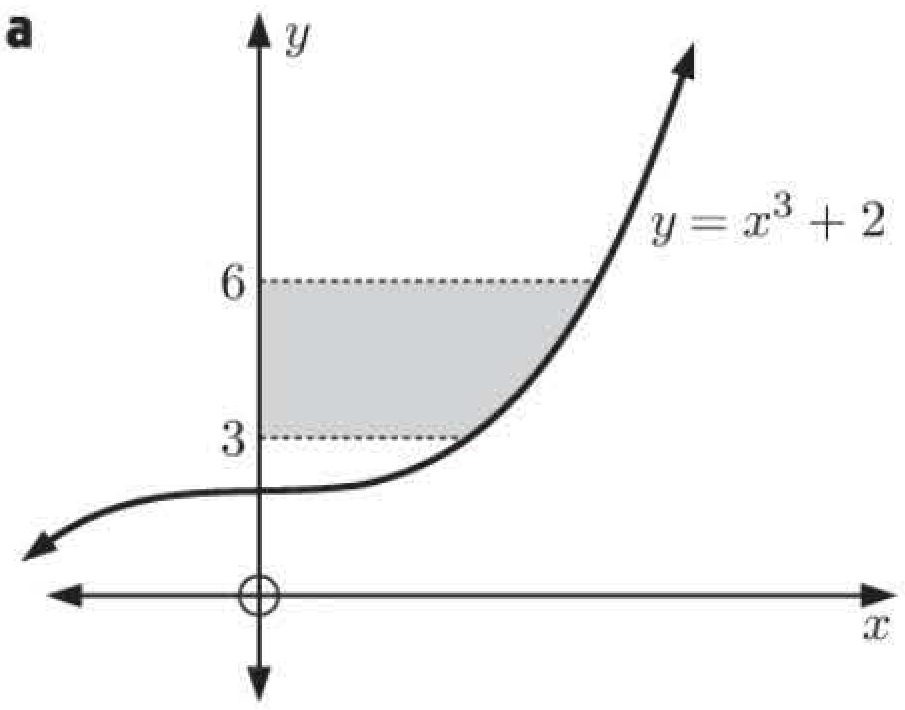
$$\begin{aligned}\text{Total area of the sections} &= \tfrac{\pi\sqrt{2}}{16} + \tfrac{\pi\sqrt{2}}{16} + \tfrac{\pi\sqrt{2}}{4} + \tfrac{\pi}{4}\left(1 - \tfrac{\sqrt{2}}{2}\right) \\ &= \frac{\pi\sqrt{2} + \pi\sqrt{2} + 4\pi\sqrt{2} + 4\pi - 2\pi\sqrt{2}}{16} \\ &= \frac{4\pi + 4\pi\sqrt{2}}{16} \\ &= \tfrac{\pi + \pi\sqrt{2}}{4} \\ &= \tfrac{\pi}{4}(1+\sqrt{2}) \text{ units}^2\end{aligned}$$

$\therefore$ the area under the arch is greater than $\frac{\pi}{4}(1+\sqrt{2})$ units^2.

c $$\begin{aligned}A &= \int_0^\pi \sin x\, dx \\ &= [-\cos x]_0^\pi \\ &= [-\cos\pi + \cos 0] \\ &= -(-1) + 1 \\ &= 2 \text{ units}^2\end{aligned}$$



14 **a**



b $$\begin{aligned}y &= x^3 + 2 \\ \therefore\ x^3 &= y - 2 \\ \therefore\ x &= (y-2)^{\frac{1}{3}} \\ \therefore\ x &= f(y) \\ &= \sqrt[3]{y-2}\end{aligned}$$

c $$\begin{aligned}\text{Area} &= \int_3^6 x\, dy \\ &= \int_3^6 (y-2)^{\frac{1}{3}}\, dy \\ &= \left[\frac{(y-2)^{\frac{4}{3}}}{\frac{4}{3}}\right]_3^6 \\ &= \tfrac{3}{4}\left(4^{\frac{4}{3}} - 1^{\frac{4}{3}}\right) \\ &= \tfrac{3}{4}\left(4\sqrt[3]{4} - 1\right) \text{ units}^2 \\ &\approx 4.01 \text{ units}^2\end{aligned}$$

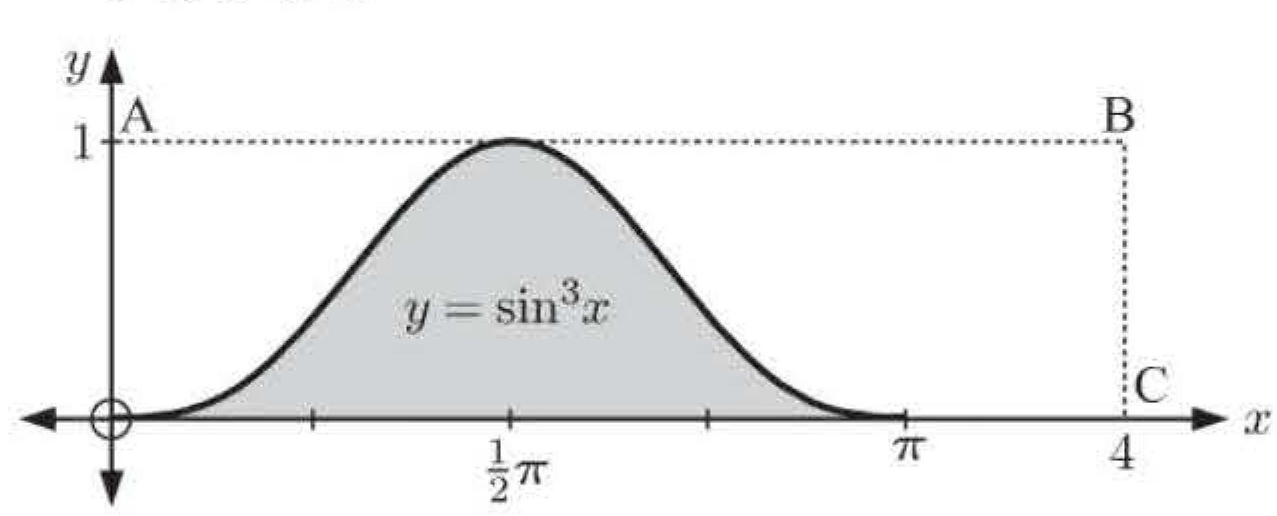
15 The curves meet when

$$\begin{aligned}x^3 + x^2 + 2x + 6 &= 7x^2 - x - 4 \\ \therefore\ x^3 - 6x^2 + 3x + 10 &= 0 \\ \therefore\ (x+1)(x^2 - 7x + 10) &= 0 \\ \therefore\ (x+1)(x-2)(x-5) &= 0 \\ \therefore\ x &= -1,\ 2, \text{ or } 5\end{aligned}$$

$\therefore$ area enclosed

$$\begin{aligned}&= \int_{-1}^5 \left|x^3 - 6x^2 + 3x + 10\right|\, dx \\ &= 40\tfrac{1}{2} \text{ units}^2\end{aligned}$$

16 Consider the graph of $y = \sin^3 x$, $0 \leqslant x \leqslant \pi$



Now $\int_0^\pi \sin^3 x\, dx =$ shaded area

But the shaded area $<$ area of rectangle ABCO

$\therefore\ \int_0^\pi \sin^3 x\, dx < 4$

REVIEW SET 22C

1 **a** $$\begin{aligned}a(t) &= 6t - 30 \text{ cm s}^{-2} \\ v(t) &= \int (6t - 30)\, dt \\ &= 3t^2 - 30t + c\end{aligned}$$

But $v(0) = 27$

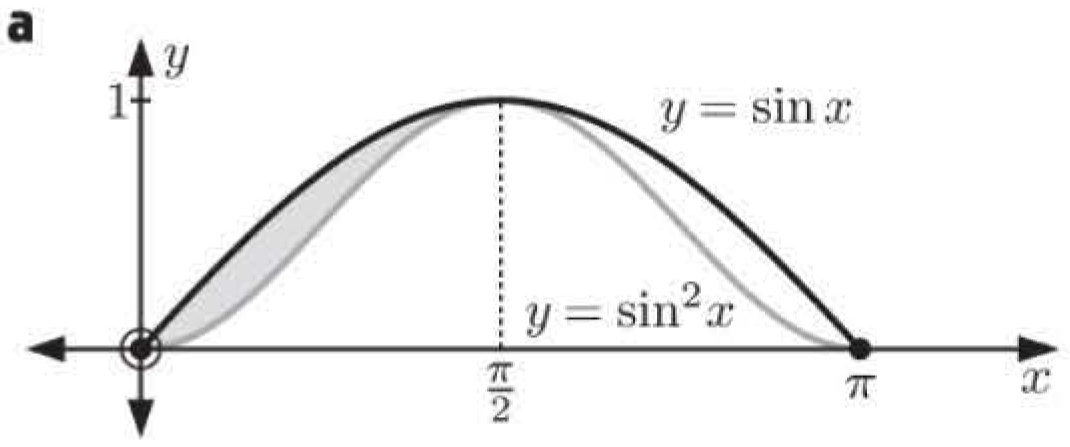
$$\begin{aligned}\therefore\ 0 - 0 + c &= 27 \\ \therefore\ c &= 27 \\ \therefore\ v(t) &= 3t^2 - 30t + 27 \text{ cm s}^{-1}\end{aligned}$$

b Displacement after 6 seconds

$$\begin{aligned}&= \int_0^6 (3t^2 - 30t + 27)\, dt \\ &= \left[t^3 - 15t^2 + 27t\right]_0^6 \\ &= 6^3 - 15(6)^2 + 27(6) - 0 \\ &= -162 \text{ cm}\end{aligned}$$

(162 cm to the left of the origin)

2 **a**



b $\text{Area} = \int_0^{\frac{\pi}{2}} (\sin x - \sin^2 x)\,dx$

$= \int_0^{\frac{\pi}{2}} (\sin x - (\frac{1}{2} - \frac{1}{2}\cos 2x))\,dx$

$= \int_0^{\frac{\pi}{2}} (\sin x + \frac{1}{2}\cos 2x - \frac{1}{2})\,dx$

$= \left[-\cos x + \frac{1}{4}\sin 2x - \frac{1}{2}x\right]_0^{\frac{\pi}{2}}$

$= \left(0 + \frac{1}{4}(0) - \frac{\pi}{4}\right) - (-1 + 0 - 0)$

$= \left(1 - \frac{\pi}{4}\right)$ units^2

3 The area between $x = 0$ and $x = a$ is 2 units^2.

$\therefore \int_0^a e^x\,dx = 2$

$\therefore [e^x]_0^a = 2$

$\therefore e^a - e^0 = 2$

$\therefore e^a = 3$

$\therefore a = \ln 3$

The area between $x = a = \ln 3$ and $x = b$ is 2 units^2.

$\therefore \int_{\ln 3}^b e^x\,dx = 2$

$\therefore [e^x]_{\ln 3}^b = 2$

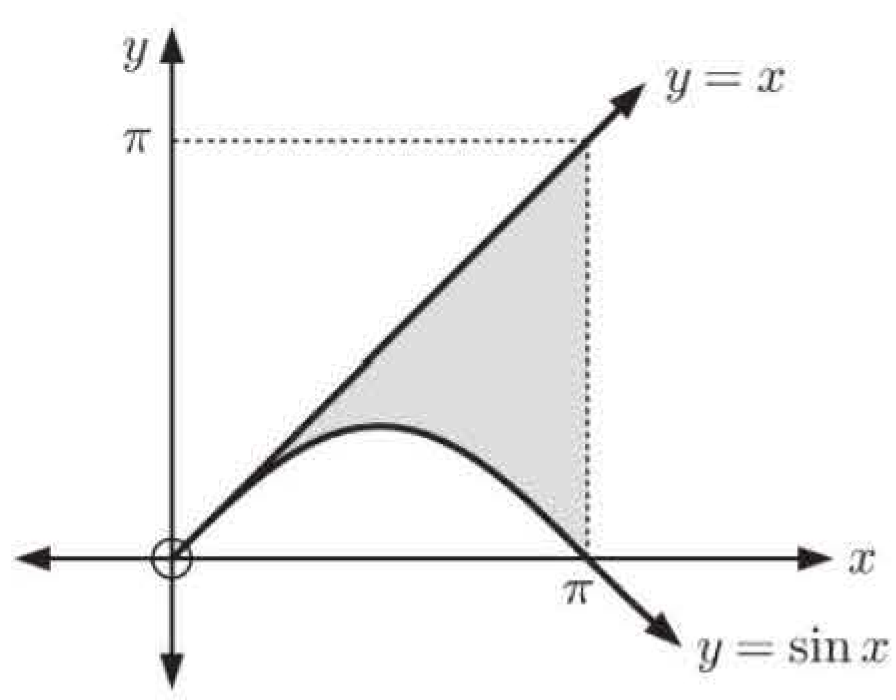
$\therefore e^b - e^{\ln 3} = 2$

$\therefore e^b - 3 = 2$

$\therefore e^b = 5$

$\therefore b = \ln 5$

4



Required area = area of △ − area under sine curve

$= \frac{1}{2}\pi \times \pi - \int_0^{\pi} \sin x\,dx$

$= \frac{\pi^2}{2} - [-\cos x]_0^{\pi}$

$= \frac{\pi^2}{2} - [-\cos \pi + \cos 0]$

$= \left(\frac{\pi^2}{2} - 2\right)$ units^2

5 The graphs meet when $\frac{2}{\pi}x = \sin x$

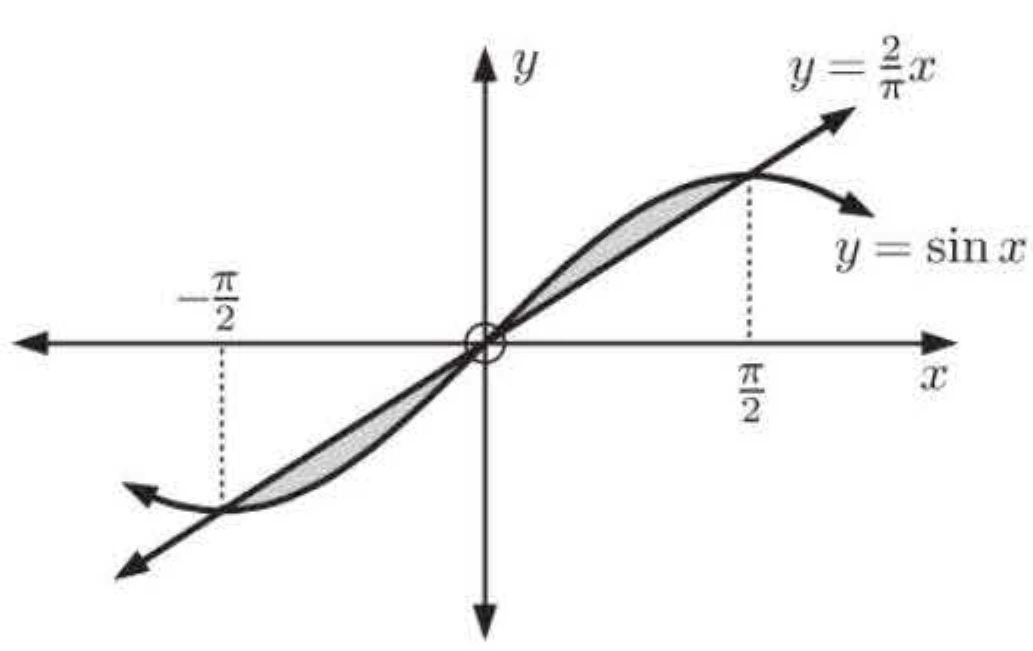
$\therefore x = -\frac{\pi}{2}, 0, \frac{\pi}{2}$ {using technology}

$\therefore \text{area} = \int_{-\frac{\pi}{2}}^{0} (\frac{2}{\pi}x - \sin x)\,dx + \int_0^{\frac{\pi}{2}} (\sin x - \frac{2}{\pi}x)\,dx$

$= \left[\frac{x^2}{\pi} + \cos x\right]_{-\frac{\pi}{2}}^{0} + \left[-\cos x - \frac{x^2}{\pi}\right]_0^{\frac{\pi}{2}}$

$= (0 + 1) - (\frac{\pi}{4} + 0) + (0 - \frac{\pi}{4}) - (-1 - 0)$

$= (2 - \frac{\pi}{2})$ units^2



6 The coordinates of B are $(2, 4 + k)$

$\therefore$ area rectangle OABC $= 2 \times (4 + k)$

$= 8 + 2k$

$\therefore$ since the two shaded regions are equal in area, each area is $4 + k$ units^2.

$\therefore \int_0^2 (x^2 + k)\,dx = 4 + k$

$\therefore \left[\frac{x^3}{3} + kx\right]_0^2 = 4 + k$

$\therefore \frac{8}{3} + 2k = 4 + k$

$\therefore k = 4 - \frac{8}{3}$

$\therefore k = 1\frac{1}{3}$

7 $y = \sqrt{4-x^2}$ is a semi-circle above the x-axis with centre O and radius 2.

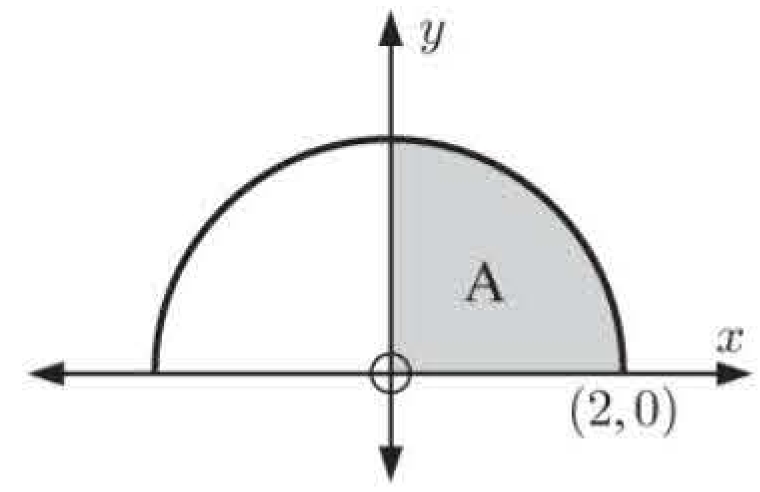
Now $\int_0^2 \sqrt{4-x^2}\,dx$

$= \text{shaded area}$

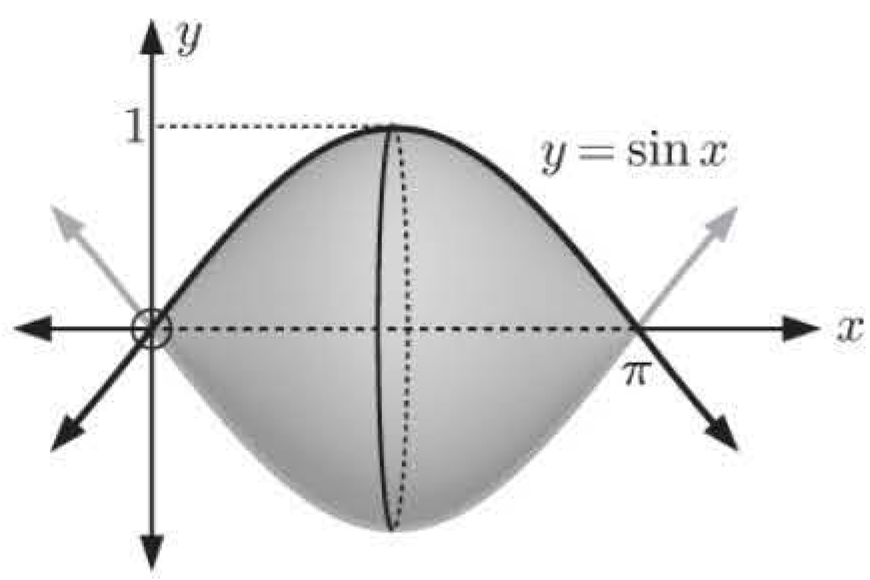
$= \frac{1}{4}$ of the area of a circle of radius 2 units

$= \frac{1}{4}\pi(2^2)$

$= \pi$ units^2

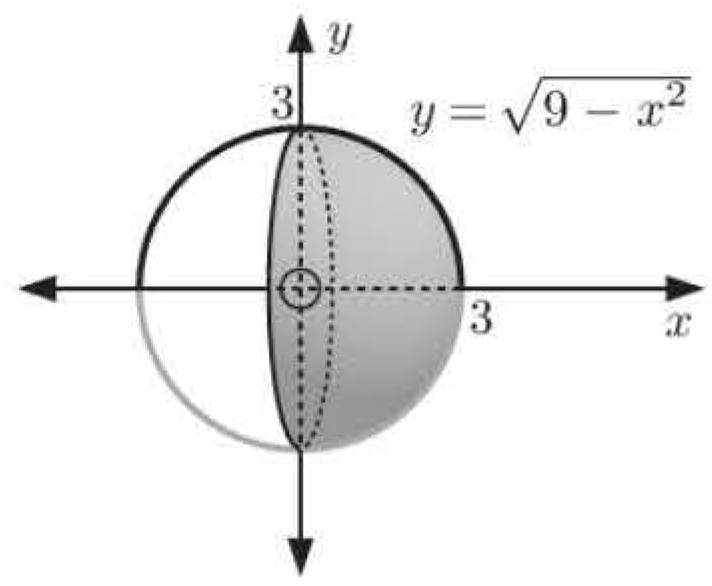


8 **a**



$$V = \pi \int_0^{\pi} \sin^2 x\,dx$$
$$= \pi \int_0^{\pi} \left(\tfrac{1}{2} - \tfrac{1}{2}\cos(2x)\right) dx$$
$$= \pi \left[\tfrac{1}{2}x - \tfrac{1}{2}\left(\tfrac{1}{2}\right)\sin(2x)\right]_0^{\pi}$$
$$= \pi \left[\tfrac{1}{2}\pi - \tfrac{1}{4}\sin 2\pi - 0\right]$$
$$= \tfrac{\pi^2}{2} \text{ units}^3$$

b



$$V = \pi \int_0^3 (9 - x^2)\,dx$$
$$= \pi \left[9x - \frac{x^3}{3}\right]_0^3$$
$$= \pi \left[27 - \tfrac{27}{3} - 0\right]$$
$$= 18\pi \text{ units}^3$$

9 $y = x^3$ meets $y = 7x^2 - 10x$

when $x^3 = 7x^2 - 10x$

$\therefore\ x^3 - 7x^2 + 10x = 0$

$\therefore\ x(x^2 - 7x + 10) = 0$

$\therefore\ x(x-2)(x-5) = 0$

$\therefore\ x = 0, 2, \text{ or } 5$

$\therefore$ total area $= \int_0^5 \left|x^3 - 7x^2 + 10x\right|\,dx$

$= 21\frac{1}{12}$ units^2

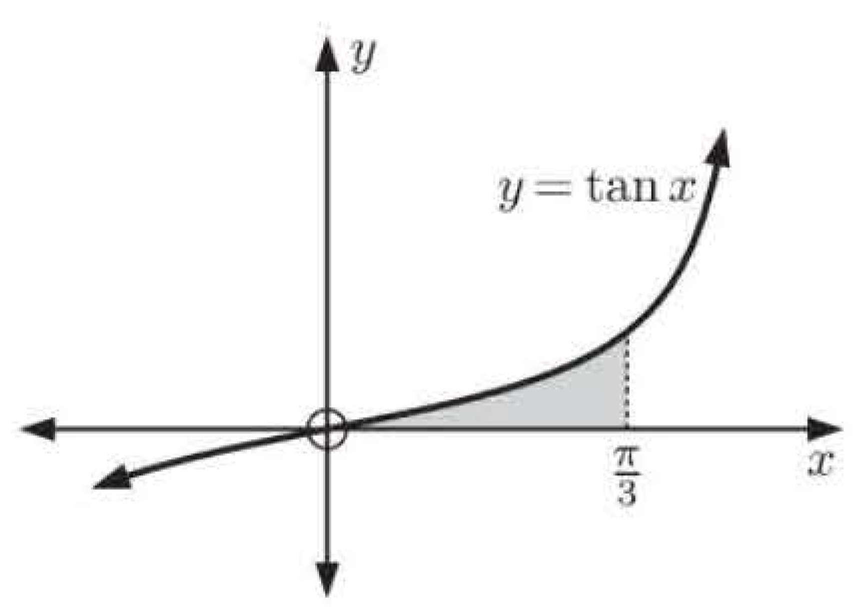
10 area $= \int_0^{\frac{\pi}{3}} \tan x\,dx$

$= [-\ln|\cos x|\,]_0^{\frac{\pi}{3}}$ {see **Exercise 21G.1** Q **6 c**}

$= -\ln\cos\frac{\pi}{3} + \ln\cos 0$

$= -\ln(\frac{1}{2}) + \ln 1$

$= \ln 2$ units^2



11 **a** $\dfrac{dI}{dt} = -\dfrac{100}{t^2}$, $t \geqslant 0.2$ seconds

$$\therefore\ I(t) = \int \frac{dI}{dt}\,dt = \int -100t^{-2}\,dt = 100t^{-1} + c$$

Now $I(2) = 150$ milliamps, so $\dfrac{100}{2} + c = 150$ and so $c = 100$

$$\therefore\ I(t) = \left(\frac{100}{t} + 100\right) \text{ milliamps}$$

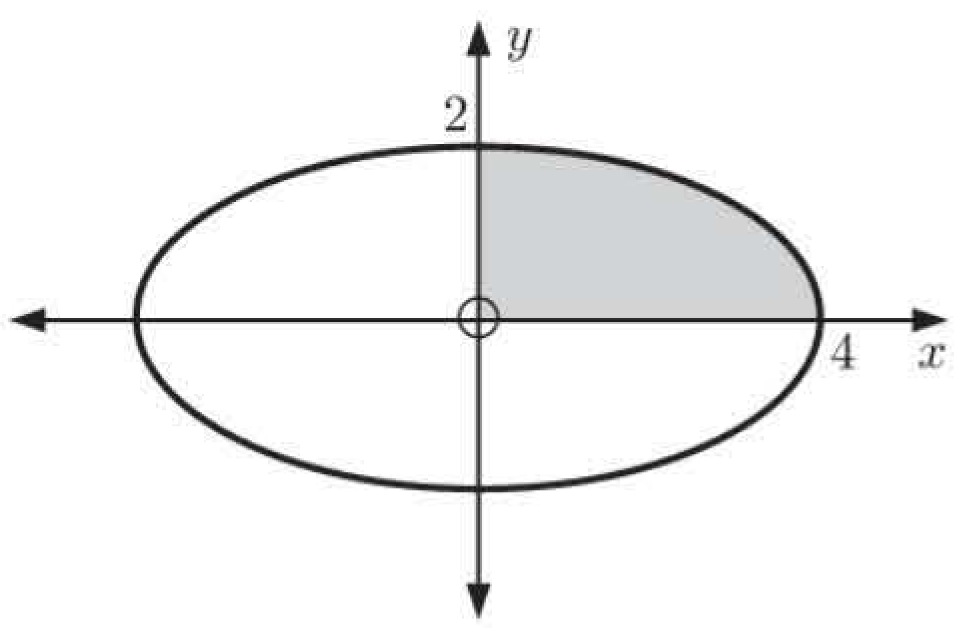
b **i** $I(20) = \frac{100}{20} + 100$

$= 105$ milliamps

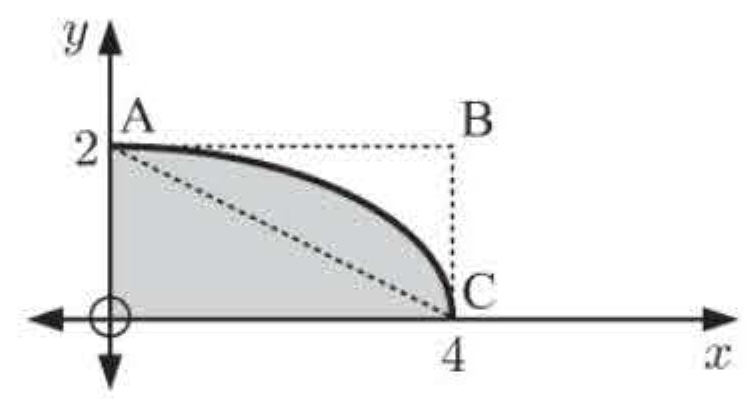
ii As $t \to \infty$,

$I(t) \to 100$ milliamps (above)

12 **a**



b



Now area $\triangle$AOC < shaded area < area ABCO

$\therefore \quad \frac{1}{2}(2 \times 4) < \int_0^4 \frac{1}{2}\sqrt{16-x^2}\, dx < 2 \times 4$

$\therefore \quad 4 < \int_0^4 \frac{1}{2}\sqrt{16-x^2}\, dx < 8$

$\therefore \quad 8 < \int_0^4 \sqrt{16-x^2}\, dx < 16$

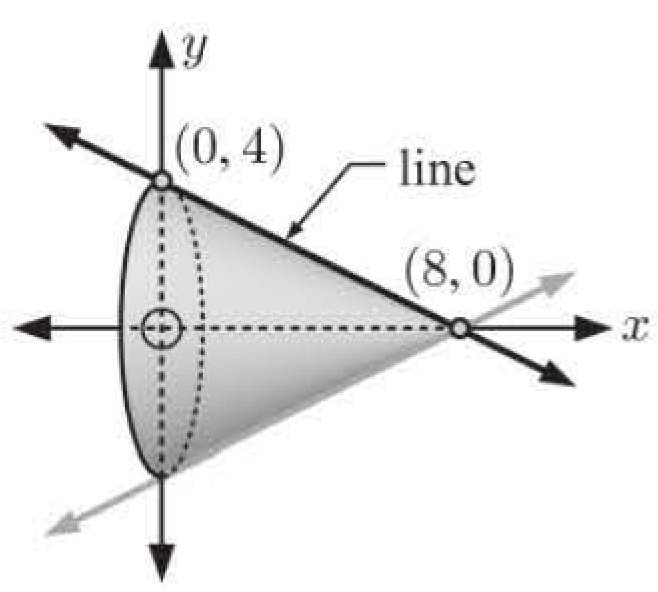
13 **a** $V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi \times 4^2 \times 8$

$= \frac{1}{3}\pi \times 128$

$= \frac{128\pi}{3}$ units^3

b



gradient $= \frac{0-4}{8-0} = -\frac{1}{2}$

$\therefore$ the line has equation $y = -\frac{1}{2}x + 4$

$\therefore \quad V = \pi \int_0^8 \left(-\frac{1}{2}x+4\right)^2 dx$

$$= \pi \int_0^8 \left(\frac{x^2}{4} - 4x + 16\right) dx$$

$$= \pi \left[\frac{x^3}{12} - \frac{4x^2}{2} + 16x\right]_0^8$$

$= \pi\left(\frac{128}{3} - 128 + 128 - 0\right)$

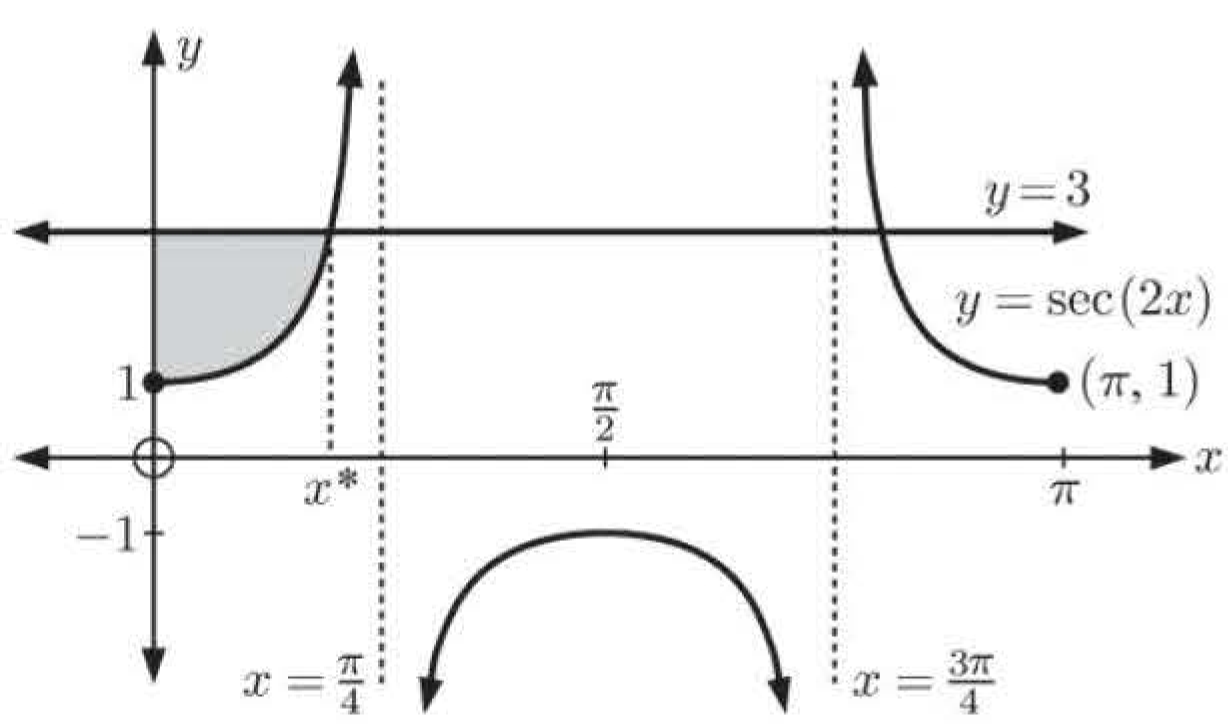
$= \frac{128\pi}{3}$ units^3 ✓

14 **a** $$\frac{d}{dx}\left[\ln(\tan x + \sec x)\right] = \frac{\sec^2 x + \sec x \tan x}{\tan x + \sec x}$$

$$= \sec x$$

$\therefore \quad \int \sec x\, dx = \ln|\tan x + \sec x| + c$

b **i**



ii $y = \sec(2x)$ and $y = 3$ meet when $\sec(2x) = 3$

$\therefore \quad \cos(2x) = \frac{1}{3}$

$\therefore \quad x^* \approx 0.615$ {see the graph}

$$\begin{aligned}\therefore\quad \text{shaded area} &\approx (\text{rectangle area}) - \int_0^{0.615} \sec(2x)\,dx\\ &\approx 0.615\times 3 - \left[\tfrac{1}{2}\ln|\tan(2x)+\sec(2x)|\right]_0^{0.615}\\ &\approx 1.846 - \left[\tfrac{1}{2}\ln|\tan(1.23)+\sec(1.23)| - \tfrac{1}{2}\ln|0+1|\right]\\ &\approx 0.965 \text{ units}^2\end{aligned}$$

15 $y=\sin x$ and $y=\cos x$ meet where $\sin x=\cos x$

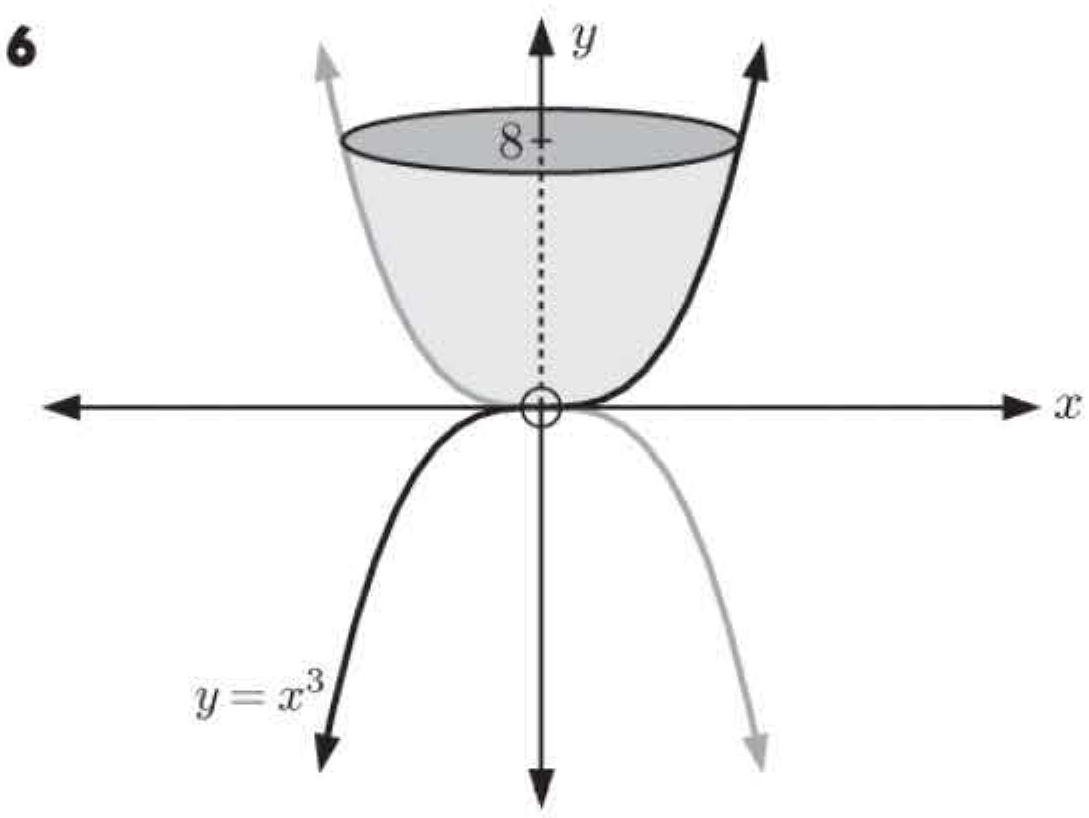
$$\therefore\quad \frac{\sin x}{\cos x}=1$$

$$\therefore\quad \tan x = 1$$

$$\therefore\quad x=\tfrac{\pi}{4}$$

$$\begin{aligned}\text{Hence}\quad V &= \pi\int_0^{\frac{\pi}{4}} (\cos^2 x - \sin^2 x)\,dx\\ &= \pi\int_0^{\frac{\pi}{4}} \cos(2x)\,dx\\ &= \pi\left[\tfrac{1}{2}\sin(2x)\right]_0^{\frac{\pi}{4}}\\ &= \pi\left(\tfrac{1}{2}\sin\left(\tfrac{\pi}{2}\right)-\tfrac{1}{2}\sin 0\right)\\ &= \pi\left(\tfrac{1}{2}(1)-0\right)\\ &= \tfrac{\pi}{2}\text{ units}^3\end{aligned}$$

16



Now $x^3=y$

$$\therefore\quad (x^3)^{\frac{2}{3}} = y^{\frac{2}{3}}$$

$$\therefore\quad x^2 = y^{\frac{2}{3}}$$

$$\begin{aligned}\text{Volume}\quad V &= \pi\int_0^8 x^2\,dy\\ &= \pi\int_0^8 y^{\frac{2}{3}}\,dy\\ &= \pi\left[\frac{y^{\frac{5}{3}}}{\frac{5}{3}}\right]_0^8\\ &= \tfrac{3\pi}{5}\left(8^{\frac{5}{3}}-0^{\frac{5}{3}}\right)\\ &= \tfrac{3\pi}{5}\times(2^3)^{\frac{5}{3}}\\ &= \tfrac{3\pi}{5}\times 2^5\\ &= \tfrac{96\pi}{5}\text{ units}^3\end{aligned}$$