Chapter 6
COMPLEX NUMBERS AND POLYNOMIALS

EXERCISE 6A

1 **a** $\sqrt{-9}$
$= \sqrt{9} \times \sqrt{-1}$
$= 3i$

b $\sqrt{-64}$
$= \sqrt{64} \times \sqrt{-1}$
$= 8i$

c $\sqrt{-\frac{1}{4}}$
$= \sqrt{\frac{1}{4}} \times \sqrt{-1}$
$= \frac{1}{2}i$

d $\sqrt{-5}$
$= \sqrt{5} \times \sqrt{-1}$
$= i\sqrt{5}$

e $\sqrt{-8}$
$= \sqrt{8} \times \sqrt{-1}$
$= i\sqrt{8}$

2 **a** $x^2 - 9$
$= (x+3)(x-3)$

b $x^2 + 9$
$= x^2 - 9i^2$
$= (x+3i)(x-3i)$

c $x^2 - 7$
$= (x+\sqrt{7})(x-\sqrt{7})$

d $x^2 + 7$
$= x^2 - (i\sqrt{7})^2$
$= (x+i\sqrt{7})(x-i\sqrt{7})$

e $4x^2 - 1$
$= (2x+1)(2x-1)$

f $4x^2 + 1$
$= 4x^2 - i^2$
$= (2x+i)(2x-i)$

g $2x^2 - 9$
$= (\sqrt{2}x+3)(\sqrt{2}x-3)$

h $2x^2 + 9$
$= 2x^2 - 9i^2$
$= (\sqrt{2}x+3i)(\sqrt{2}x-3i)$

i $x^3 - x$
$= x(x^2-1)$
$= x(x+1)(x-1)$

j $x^3 + x$
$= x(x^2+1)$
$= x(x^2-i^2)$
$= x(x+i)(x-i)$

k $x^4 - 1$
$= (x^2+1)(x^2-1)$
$= (x^2-i^2)(x^2-1)$
$= (x+i)(x-i)(x+1)(x-1)$

l $x^4 - 16$
$= (x^2+4)(x^2-4)$
$= (x^2-4i^2)(x^2-4)$
$= (x+2i)(x-2i)(x+2)(x-2)$

3 **a**
$$x^2 - 25 = 0$$
$$\therefore \ (x+5)(x-5) = 0$$
$$\therefore \ x = \pm 5$$

b
$$x^2 + 25 = 0$$
$$\therefore \ x^2 - 25i^2 = 0$$
$$\therefore \ (x+5i)(x-5i) = 0$$
$$\therefore \ x = \pm 5i$$

c
$$x^2 - 5 = 0$$
$$\therefore \ (x+\sqrt{5})(x-\sqrt{5}) = 0$$
$$\therefore \ x = \pm\sqrt{5}$$

d
$$x^2 + 5 = 0$$
$$\therefore \ x^2 - 5i^2 = 0$$
$$\therefore \ (x+i\sqrt{5})(x-i\sqrt{5}) = 0$$
$$\therefore \ x = \pm i\sqrt{5}$$

e
$$4x^2 - 9 = 0$$
$$\therefore \ (2x+3)(2x-3) = 0$$
$$\therefore \ x = \pm\tfrac{3}{2}$$

f
$$4x^2 + 9 = 0$$
$$\therefore \ 4x^2 - 9i^2 = 0$$
$$\therefore \ (2x+3i)(2x-3i) = 0$$
$$\therefore \ x = \pm\tfrac{3}{2}i$$

g
$$x^3 - 4x = 0$$
$$\therefore \ x(x^2-4) = 0$$
$$\therefore \ x(x+2)(x-2) = 0$$
$$\therefore \ x = 0 \text{ or } \pm 2$$

h
$$x^3 + 4x = 0$$
$$\therefore \ x(x^2+4) = 0$$
$$\therefore \ x(x^2-4i^2) = 0$$
$$\therefore \ x(x+2i)(x-2i) = 0$$
$$\therefore \ x = 0 \text{ or } \pm 2i$$

i

$$\begin{aligned} x^3 - 3x &= 0 \\ \therefore\ x(x^2-3) &= 0 \\ \therefore\ x(x+\sqrt{3})(x-\sqrt{3}) &= 0 \\ \therefore\ x &= 0 \text{ or } \pm\sqrt{3} \end{aligned}$$

j

$$\begin{aligned} x^3 + 3x &= 0 \\ \therefore\ x(x^2+3) &= 0 \\ \therefore\ x(x^2-3i^2) &= 0 \\ \therefore\ x(x+i\sqrt{3})(x-i\sqrt{3}) &= 0 \\ \therefore\ x &= 0 \text{ or } \pm i\sqrt{3} \end{aligned}$$

k

$$\begin{aligned} x^4 - 1 &= 0 \\ \therefore\ (x^2+1)(x^2-1) &= 0 \\ \therefore\ (x^2-i^2)(x^2-1) &= 0 \\ \therefore\ (x+i)(x-i)(x+1)(x-1) &= 0 \\ \therefore\ x &= \pm i \text{ or } \pm 1 \end{aligned}$$

l

$$\begin{aligned} x^4 &= 81 \\ \therefore\ x^4 - 81 &= 0 \\ \therefore\ (x^2+9)(x^2-9) &= 0 \\ \therefore\ (x^2-9i^2)(x^2-9) &= 0 \\ \therefore\ (x+3i)(x-3i)(x+3)(x-3) &= 0 \\ \therefore\ x &= \pm 3i \text{ or } \pm 3 \end{aligned}$$

4 **a** If $x^2 - 10x + 29 = 0$

$$\begin{aligned} \text{then}\quad x &= \frac{10 \pm \sqrt{100 - 4\times 1\times 29}}{2} \\ &= \frac{10 \pm \sqrt{-16}}{2} \\ &= 5 \pm \sqrt{-4} \\ &= 5 \pm 2i \end{aligned}$$

b If $x^2 + 6x + 25 = 0$

$$\begin{aligned} \text{then}\quad x &= \frac{-6 \pm \sqrt{36 - 4\times 1\times 25}}{2} \\ &= \frac{-6 \pm \sqrt{-64}}{2} \\ &= -3 \pm \sqrt{-16} \\ &= -3 \pm 4i \end{aligned}$$

c If $x^2 + 14x + 50 = 0$,

$$\begin{aligned} \text{then}\quad x &= \frac{-14 \pm \sqrt{14^2 - 4\times 1\times 50}}{2} \\ &= \frac{-14 \pm \sqrt{-4}}{2} \\ &= -7 \pm \sqrt{-1} \\ &= -7 \pm i \end{aligned}$$

d If $2x^2 + 5 = 6x$,

$$\begin{aligned} \text{then}\quad 2x^2 - 6x + 5 &= 0 \\ x &= \frac{6 \pm \sqrt{36 - 4\times 2\times 5}}{4} \\ &= \frac{6 \pm \sqrt{-4}}{4} \\ &= \frac{3 \pm \sqrt{-1}}{2} \\ &= \tfrac{3}{2} \pm \tfrac{1}{2}i \end{aligned}$$

e If $x^2 - 2\sqrt{3}x + 4 = 0$,

$$\begin{aligned} \text{then}\quad x &= \frac{2\sqrt{3} \pm \sqrt{12 - 4\times 1\times 4}}{2} \\ &= \frac{2\sqrt{3} \pm \sqrt{-4}}{2} \\ &= \sqrt{3} \pm \sqrt{-1} \\ &= \sqrt{3} \pm i \end{aligned}$$

f If $2x + \frac{1}{x} = 1$,

$$\begin{aligned} \text{then}\quad 2x^2 + 1 &= x \\ \therefore\ 2x^2 - x + 1 &= 0 \\ x &= \frac{1 \pm \sqrt{1 - 4\times 2\times 1}}{4} \\ &= \frac{1 \pm \sqrt{-7}}{4} \\ &= \tfrac{1}{4} \pm i\tfrac{\sqrt{7}}{4} \end{aligned}$$

5 **a**

$$\begin{aligned} x^4 + 2x^2 &= 3 \\ \therefore\ x^4 + 2x^2 - 3 &= 0 \\ \therefore\ (x^2+3)(x^2-1) &= 0 \\ \therefore\ (x^2-3i^2)(x^2-1) &= 0 \\ \therefore\ (x+i\sqrt{3})(x-i\sqrt{3})(x+1)(x-1) &= 0 \\ \therefore\ x &= \pm i\sqrt{3} \text{ or } \pm 1 \end{aligned}$$

b

$$\begin{aligned} x^4 &= x^2 + 6 \\ \therefore\ x^4 - x^2 - 6 &= 0 \\ \therefore\ (x^2-3)(x^2+2) &= 0 \\ \therefore\ (x^2-3)(x^2-2i^2) &= 0 \\ \therefore\ (x+\sqrt{3})(x-\sqrt{3})(x+i\sqrt{2})(x-i\sqrt{2}) &= 0 \\ \therefore\ x &= \pm\sqrt{3} \text{ or } \pm i\sqrt{2} \end{aligned}$$

c
$$\begin{aligned} x^4+5x^2&=36\\ \therefore\ x^4+5x^2-36&=0\\ \therefore\ (x^2+9)(x^2-4)&=0\\ \therefore\ (x^2-9i^2)(x^2-4)&=0\\ \therefore\ (x+3i)(x-3i)(x+2)(x-2)&=0\\ \therefore\ x&=\pm 3i \text{ or } \pm 2 \end{aligned}$$

d
$$\begin{aligned} x^4+9x^2+14&=0\\ \therefore\ (x^2+7)(x^2+2)&=0\\ \therefore\ (x^2-7i^2)(x^2-2i^2)&=0\\ \therefore\ (x+i\sqrt{7})(x-i\sqrt{7})(x+i\sqrt{2})(x-i\sqrt{2})&=0\\ \therefore\ x&=\pm i\sqrt{7} \text{ or } \pm i\sqrt{2} \end{aligned}$$

e
$$\begin{aligned} x^4+1&=2x^2\\ \therefore\ x^4-2x^2+1&=0\\ \therefore\ (x^2-1)^2&=0\\ \therefore\ (x+1)^2(x-1)^2&=0\\ \therefore\ x&=\pm 1 \end{aligned}$$

f
$$\begin{aligned} x^4+2x^2+1&=0\\ \therefore\ (x^2+1)^2&=0\\ \therefore\ (x^2-i^2)^2&=0\\ \therefore\ (x+i)^2(x-i)^2&=0\\ \therefore\ x&=\pm i \end{aligned}$$

EXERCISE 6B.1

1

z	$\mathcal{Re}(z)$	$\mathcal{Im}(z)$	z	$\mathcal{Re}(z)$	$\mathcal{Im}(z)$
$3+2i$	3	2	$-3+4i$	-3	4
$5-i$	5	-1	$-7-2i$	-7	-2
3	3	0	$-11i$	0	-11
0	0	0	$i\sqrt{3}$	0	$\sqrt{3}$

2 **a** $z+w$
$=(5-2i)+(2+i)$
$=7-i$

b $2z$
$=2(5-2i)$
$=10-4i$

c $iw=i(2+i)$
$=2i+i^2$
$=-1+2i$

d $z-w$
$=(5-2i)-(2+i)$
$=5-2i-2-i$
$=3-3i$

e $2z-3w$
$=2(5-2i)-3(2+i)$
$=10-4i-6-3i$
$=4-7i$

f zw
$=(5-2i)(2+i)$
$=10-4i+5i-2i^2$
$=12+i$

g $w^2=(2+i)^2$
$=4+4i+i^2$
$=3+4i$

h $z^2=(5-2i)^2$
$=25-20i+4i^2$
$=21-20i$

3 **a** $z+2w$
$=(1+i)+2(-2+3i)$
$=1+i-4+6i$
$=-3+7i$

b z^2
$=(1+i)^2$
$=1+2i+i^2$
$=2i$

c $z^3=z^2\times z$
$=2i(1+i)$
$=2i+2i^2$
$=-2+2i$

d $iz=i(1+i)$
$=i+i^2$
$=-1+i$

e $w^2=(-2+3i)^2$
$=4-12i+9i^2$
$=-5-12i$

f zw
$=(1+i)(-2+3i)$
$=-2+3i-2i+3i^2$
$=-5+i$

g $z^2w=(1+i)^2(-2+3i)$
$=2i(-2+3i)$
$=-4i+6i^2$
$=-6-4i$

h $izw=i(1+i)(-2+3i)$
$=i(-5+i)$
$=-5i+i^2$
$=-1-5i$

4 **a** $i^0 = 1$ $\quad i^4 = 1$ $\quad i^8 = 1$ $\quad i^{-1} = -i$

$i^1 = i$ $\quad i^5 = i$ $\quad i^9 = i$ $\quad i^{-2} = -1$

$i^2 = -1$ $\quad i^6 = -1$ $\quad i^{-3} = i$

$i^3 = -i$ $\quad i^7 = -i$ $\quad i^{-4} = 1$

$i^{-5} = -i$

b $i^{4n+3} = -i$ where n is any integer

5
$$\begin{aligned}(1+i)^4 &= [(1+i)^2]^2\\ &= (1+2i+i^2)^2\\ &= (2i)^2\\ &= -4\end{aligned}$$

$$\begin{aligned}(1+i)^{101} &= (1+i)^{100}\times(1+i)\\ &= \left[(1+i)^4\right]^{25}\times(1+i)\\ &= [-4]^{25}\,(1+i)\\ &= -2^{50}(1+i)\end{aligned}$$

6 **a**
$$\begin{aligned}\frac{z}{w} &= \frac{2-i}{1+3i}\times\frac{1-3i}{1-3i}\\ &= \frac{2-6i-i+3i^2}{1-9i^2}\\ &= \frac{-1-7i}{10}\\ &= -\tfrac{1}{10}-\tfrac{7}{10}i\end{aligned}$$

b
$$\begin{aligned}\frac{i}{z} &= \frac{i}{2-i}\times\frac{2+i}{2+i}\\ &= \frac{2i+i^2}{4-i^2}\\ &= \frac{-1+2i}{5}\\ &= -\tfrac{1}{5}+\tfrac{2}{5}i\end{aligned}$$

c
$$\begin{aligned}\frac{w}{iz} &= \frac{1+3i}{i(2-i)}\\ &= \frac{1+3i}{2i-i^2}\\ &= \frac{1+3i}{1+2i}\times\frac{1-2i}{1-2i}\\ &= \frac{1-2i+3i-6i^2}{1-4i^2}\\ &= \frac{7+i}{5}\\ &= \tfrac{7}{5}+\tfrac{1}{5}i\end{aligned}$$

d
$$\begin{aligned}z^{-2} &= \frac{1}{(2-i)^2}\\ &= \frac{1}{4-4i+i^2}\\ &= \frac{1}{3-4i}\times\frac{3+4i}{3+4i}\\ &= \frac{3+4i}{9-16i^2}\\ &= \frac{3+4i}{25}\\ &= \tfrac{3}{25}+\tfrac{4}{25}i\end{aligned}$$

7 **a**
$$\begin{aligned}&\frac{i}{1-2i}\\ &= \frac{i}{1-2i}\times\frac{1+2i}{1+2i}\\ &= \frac{i+2i^2}{1-4i^2}\\ &= \frac{-2+i}{5}\\ &= -\tfrac{2}{5}+\tfrac{1}{5}i\end{aligned}$$

b
$$\begin{aligned}&\frac{i(2-i)}{3-2i}\\ &= \frac{2i-i^2}{3-2i}\\ &= \frac{1+2i}{3-2i}\times\frac{3+2i}{3+2i}\\ &= \frac{3+2i+6i+4i^2}{9-4i^2}\\ &= \frac{-1+8i}{13}\\ &= -\tfrac{1}{13}+\tfrac{8}{13}i\end{aligned}$$

c
$$\begin{aligned}&\frac{1}{2-i}-\frac{2}{2+i}\\ &= \frac{1}{2-i}\left(\frac{2+i}{2+i}\right)-\frac{2}{2+i}\left(\frac{2-i}{2-i}\right)\\ &= \frac{2+i-2(2-i)}{(2-i)(2+i)}\\ &= \frac{2+i-4+2i}{4-i^2}\\ &= \frac{-2+3i}{5}\\ &= -\tfrac{2}{5}+\tfrac{3}{5}i\end{aligned}$$

8 **a**
$$\begin{aligned}4z-3w &= 4(2+i)-3(-1+2i)\\ &= 8+4i+3-6i\\ &= 11-2i\end{aligned}$$
$\therefore\ \mathcal{Im}\,(4z-3w) = -2$

b
$$\begin{aligned}zw &= (2+i)(-1+2i)\\ &= -2+4i-i+2i^2\\ &= -4+3i\end{aligned}$$
$\therefore\ \mathcal{Re}\,(zw) = -4$

c
$$\begin{aligned} iz^2 &= i(2+i)^2 \\ &= i(4+4i+i^2) \\ &= i(3+4i) \\ &= 3i+4i^2 \\ &= -4+3i \end{aligned}$$
$\therefore\ \mathcal{Im}\,(iz^2) = 3$

d
$$\begin{aligned} \frac{z}{w} &= \frac{2+i}{-1+2i} \times \frac{-1-2i}{-1-2i} \\ &= \frac{-2-4i-i-2i^2}{1-4i^2} \\ &= \frac{0-5i}{5} \\ &= -i \end{aligned}$$
$\therefore\ \mathcal{Re}\left(\frac{z}{w}\right) = 0$

EXERCISE 6B.2

1 **a** $2x+3iy = -x-6i$

Equating real and imaginary parts,

$2x = -x$ and $3y = -6$ $\{x, y$ are real$\}$

$\therefore\ 3x = 0$ and $y = -2$

$\therefore\ x = 0$ and $y = -2$

b $x^2 + ix = 4-2i$

Equating real and imaginary parts,

$x^2 = 4$ and $x = -2$ $\{x$ is real$\}$

$\therefore\ x = \pm 2$ and $x = -2$

$\therefore\ x = -2$

c $(x+iy)(2-i) = 8+i$

$\therefore\ x+iy = \frac{8+i}{2-i} \times \frac{2+i}{2+i}$

$\therefore\ x+iy = \frac{16+8i+2i+i^2}{4-i^2}$

$\therefore\ x+iy = \frac{15+10i}{5}$

$\therefore\ x+iy = 3+2i$

Equating real and imag. parts, for real x, y

$x = 3$ and $y = 2$

d $(3+2i)(x+iy) = -i$

$\therefore\ x+iy = \frac{-i}{3+2i} \times \frac{3-2i}{3-2i}$

$\therefore\ x+iy = \frac{-3i+2i^2}{9-4i^2}$

$\therefore\ x+iy = \frac{-2-3i}{13}$

Equating real and imag. parts, for real x, y

$x = -\frac{2}{13}$ and $y = -\frac{3}{13}$

2 **a** $2(x+iy) = x-iy$

$\therefore\ 2x+2iy = x-iy$

Equating real and imaginary parts, $2x = x$ and $2y = -y$

$\therefore\ x = 0$ and $3y = 0$

$\therefore\ x = 0$ and $y = 0$

b $(x+2i)(y-i) = -4-7i$

$\therefore\ xy - ix + 2iy - 2i^2 = -4-7i$

$\therefore\ (xy+2) + i(2y-x) = -4-7i$

Equating real and imaginary parts,

$xy+2 = -4$ and $2y - x = -7$

$\therefore\ xy = -6$ and $x = 2y+7$

$\therefore\ (2y+7)y = -6$

$\therefore\ 2y^2 + 7y = -6$

$\therefore\ 2y^2+7y+6 = 0$

$\therefore\ (2y+3)(y+2) = 0$

$\therefore\ y = -\frac{3}{2}$ or $y = -2$

When $y = -2$, $x = 2(-2)+7 = 3$

When $y = -\frac{3}{2}$, $x = 2(-\frac{3}{2})+7 = 4$

$\therefore\ x = 3$ and $y = -2$

or $x = 4$ and $y = -\frac{3}{2}$

c $(x+i)(3-iy) = 1+13i$

$\therefore\ 3x - ixy + 3i - i^2y = 1+13i$

$\therefore\ (3x+y) + i(3-xy) = 1+13i$

Equating real and imaginary parts,

$3x+y = 1$ and $3 - xy = 13$

$\therefore\ y = 1-3x$ and $xy = -10$

$\therefore\ x(1-3x) = -10$

$\therefore\ x - 3x^2 = -10$

$\therefore\ 0 = 3x^2 - x - 10$

$\therefore\ 0 = (3x+5)(x-2)$

$\therefore\ x = -\frac{5}{3}$ or $x = 2$

When $x = -\frac{5}{3}$, $y = 1 - 3(-\frac{5}{3}) = 6$

and when $x = 2$, $y = 1 - 3 \times 2 = -5$

$\therefore\ x = -\frac{5}{3}$ and $y = 6$

or $x = 2$ and $y = -5$

d $$(x+iy)(2+i) = 2x - i(y+1)$$
$$\therefore\ 2x + ix + 2iy + yi^2 = 2x + i(-y-1)$$
$$\therefore\ (2x-y) + i(x+2y) = 2x + i(-y-1)$$
Equating real and imaginary parts, $2x - y = 2x$ and $x + 2y = -y - 1$
$$\therefore\ -y = 2x - 2x$$
$\therefore\ y = 0$ and consequently
$$x + 0 = 0 - 1 = -1$$
$\therefore\ x = -1$ and $y = 0$

3 $$3z + 17i = iz + 11$$
$$\therefore\ z(3-i) = 11 - 17i$$
$$\therefore\ z = \frac{11-17i}{3-i} \times \frac{(3+i)}{(3+i)}$$
$$= \frac{33 + 11i - 51i - 17i^2}{9 - i^2}$$
$$= \frac{50 - 40i}{10}$$
$$= 5 - 4i$$

4 $$\frac{4}{1+i} = \frac{4}{(1+i)} \times \frac{(1-i)}{(1-i)}$$
$$= \frac{4-4i}{1-i^2}$$
$$= \frac{4-4i}{2} = 2 - 2i$$
$$\therefore\ \sqrt{z} = (2-2i) + 7 - 2i$$
$$= 9 - 4i$$
$$\therefore\ z = (9-4i)^2$$
$$= 81 - 72i + 16i^2$$
$$= 65 - 72i$$

5 $$3(m+ni) = n - 2mi - (1-2i)$$
$$\therefore\ 3m + 3ni = n - 2mi - 1 + 2i$$
$$\therefore\ 3m + 3ni = (n-1) + i(2-2m)$$
Equating real and imaginary parts,
$3m = n - 1$ and $3n = 2 - 2m$
$\therefore\ n = 3m + 1$ and $3n = 2 - 2m$
$$\therefore\ 3(3m+1) = 2 - 2m$$
$$\therefore\ 9m + 3 = 2 - 2m$$
$$\therefore\ 11m = -1$$
$$\therefore\ m = -\tfrac{1}{11}$$
and $n = 3(-\tfrac{1}{11}) + 1 = \tfrac{8}{11}$

6 $$z = \frac{3i}{\sqrt{2}-i} + 1$$
$$= \left(\frac{3i}{\sqrt{2}-i}\right)\left(\frac{\sqrt{2}+i}{\sqrt{2}+i}\right) + 1$$
$$= \frac{3i\sqrt{2} + 3i^2}{2 - i^2} + 1$$
$$= \frac{3i\sqrt{2} - 3}{3} + \frac{3}{3}$$
$$= \frac{3i\sqrt{2}}{3}$$
$$= i\sqrt{2}$$

7 $$(a+bi)^2 = -16 - 30i$$
$$a^2 + 2abi + b^2i^2 = -16 - 30i$$
$\therefore\ a^2 - b^2 = -16$ and $2ab = -30$,
$\therefore\ ab = -15$ and $\therefore\ b = -\dfrac{15}{a}$
So, $$a^2 - \left(-\frac{15}{a}\right)^2 = -16$$
$$\therefore\ a^2 - \frac{225}{a^2} = -16$$
$$\therefore\ a^4 + 16a^2 - 225 = 0$$
$$\therefore\ (a^2+25)(a^2-9) = 0$$
$$\therefore\ a = \pm 3 \text{ or } \pm 5i$$
But a is real and > 0
$\therefore\ a = 3$ and $b = -\tfrac{15}{3} = -5$

EXERCISE 6B.3

1 **a** roots α and β are $3 \pm i$ $\therefore$ $\alpha + \beta = 6$ and $\alpha\beta = 9 - i^2 = 10$

$\therefore$ quadratics have form $a(x^2 - 6x + 10) = 0,\ a \neq 0$

b roots α and β are $1 \pm 3i$ $\therefore$ $\alpha + \beta = 2$ and $\alpha\beta = 1 - 9i^2 = 10$

$\therefore$ quadratics have form $a(x^2 - 2x + 10) = 0,\ a \neq 0$

c roots α and β are $-2 \pm 5i$ $\therefore$ $\alpha + \beta = -4$ and $\alpha\beta = 4 - 25i^2 = 29$

$\therefore$ quadratics have form $a(x^2 + 4x + 29) = 0,\ a \neq 0$

d roots α and β are $\sqrt{2} \pm i$ $\therefore$ $\alpha + \beta = 2\sqrt{2}$ and $\alpha\beta = 2 - i^2 = 3$

$\therefore$ quadratics have form $a(x^2 - 2\sqrt{2}x + 3) = 0,\ a \neq 0$

e roots α and β are $2 \pm \sqrt{3}$ $\therefore$ $\alpha + \beta = 4$ and $\alpha\beta = 4 - 3 = 1$

$\therefore$ quadratics have form $a(x^2 - 4x + 1) = 0,\ a \neq 0$

f roots α and β are 0 and $-\frac{2}{3}$ $\therefore$ factors are $x,\ 3x + 2$

$\therefore$ quadratics have form $ax(3x + 2) = 0$

$\therefore$ $a(3x^2 + 2x) = 0,\ a \neq 0$

g roots α and β are $\pm i\sqrt{2}$ $\therefore$ $\alpha + \beta = 0$ and $\alpha\beta = -2i^2 = 2$

$\therefore$ quadratics have form $a(x^2 + 2) = 0,\ a \neq 0$

h roots α and β are $-6 \pm i$ $\therefore$ $\alpha + \beta = -12$ and $\alpha\beta = 36 - i^2 = 37$

$\therefore$ quadratics have form $a(x^2 + 12x + 37) = 0,\ a \neq 0$

2 **a** If $3 + i$ is a root then so is $3 - i$ (if a and b are real).

$\therefore$ $\alpha + \beta = 6$ and $\alpha\beta = 9 - i^2 = 10$

$\therefore$ $x^2 - 6x + 10 = 0$ and so $a = -6,\ b = 10$

b If $1 - \sqrt{2}$ is a root then so is $1 + \sqrt{2}$ if a, b are rational.

$\therefore$ $\alpha + \beta = 2$ and $\alpha\beta = 1 - 2 = -1$

$\therefore$ $x^2 - 2x - 1 = 0$

$\therefore$ $a = -2$ and $b = -1$

c If $a + ai$ is a root then so is $a - ai$ (if a and b are real and $a \neq 0$).

$\therefore$ $\alpha + \beta = 2a$

and $\alpha\beta = (a + ai)(a - ai)$
$= a^2 - (ai)^2$
$= 2a^2$

$\therefore$ $x^2 - 2ax + 2a^2 = x^2 + 4x + b$

$\therefore$ $-2a = 4$ and $b = 2a^2$

$\therefore$ $a = -2$ and $b = 8$

However, if $a = 0,\ a + ai = 0$, which is *not* complex, so the other root could be any real number.

But $\alpha\beta = 0$ $\therefore$ $b = 0$

$\therefore$ $a = 0,\ b = 0$ is also a solution.

EXERCISE 6B.4

1 *To prove:* $(z_1 - z_2)^* = {z_1}^* - {z_2}^*$

Let $z_1 = a + ib$ and $z_2 = c + id$.

$$\begin{aligned}\therefore\ (z_1 - z_2)^* &= [(a + ib) - (c + id)]^* \\ &= [(a - c) + i(b - d)]^* \\ &= (a - c) - i(b - d) \\ &= a - c - bi + di \\ &= (a - bi) - (c - di) \\ &= {z_1}^* - {z_2}^*\end{aligned}$$

2

$$\begin{aligned}&(w^* - z)^* - (w - 2z^*) \\ &= w^{**} - z^* - w + 2z^* \\ &= w - z^* - w + 2z^* \\ &= -z^* + 2z^* \\ &= z^*\end{aligned}$$

3 Let $z = a + bi$ $\therefore$ $z^* = a - bi$

If $z^* = -z$, then $a - bi = -a - bi$

$\therefore$ $a = -a$

$\therefore$ $2a = 0$

$\therefore$ $a = 0$ and b is any real number

$\therefore$ z is purely imaginary or $a = 0$, $b = 0$ $\therefore$ z is zero.

4 **a** Let $z_1 = a + bi$ $z_2 = c + di$

$$\therefore \quad \frac{z_1}{z_2} = \frac{a+bi}{c+di} \times \frac{c-di}{c-di}$$
$$= \frac{(a+bi)(c-di)}{(c+di)(c-di)}$$
$$= \frac{ac - adi + bci - bdi^2}{c^2 - i^2 d^2}$$
$$= \frac{(ac+bd) + i(-ad+bc)}{c^2+d^2}$$
$$= \left(\frac{ac+bd}{c^2+d^2}\right) + \left(\frac{bc-ad}{c^2+d^2}\right) i$$

b
$$\frac{{z_1}^*}{{z_2}^*} = \frac{a-bi}{c-di} \times \frac{c+di}{c+di}$$
$$= \frac{(a-bi)(c+di)}{(c-di)(c+di)}$$
$$= \frac{ac + adi - bci - bdi^2}{c^2 - i^2 d^2}$$
$$= \frac{(ac+bd) - i(bc-ad)}{c^2+d^2}$$
$$= \left(\frac{ac+bd}{c^2+d^2}\right) - \left(\frac{bc-ad}{c^2+d^2}\right) i$$
$$= \left(\frac{z_1}{z_2}\right)^* \quad \text{for all } z_2 \neq 0$$

5 **a**
$$\left(\frac{z_1}{z_2}\right)^* \times {z_2}^* = \left(\frac{z_1}{z_2} \times z_2\right)^* \quad \{\text{from } \textbf{Example 9}\}$$
$$= {z_1}^*$$
$$\therefore \quad \left(\frac{z_1}{z_2}\right)^* = \frac{{z_1}^*}{{z_2}^*} \quad \{\text{dividing both sides by } {z_2}^*\}$$

b $z = a + bi$

i If $z = z^*$,

then $a + bi = a - bi$

$\therefore$ $bi = -bi$

$\therefore$ $b = -b$

$\therefore$ $2b = 0$

$\therefore$ $b = 0$

And if $b = 0$, then $z = a + i(0) = a$

$\therefore$ z is real.

ii If $z^* = -z$,

then $a - bi = -(a + bi)$

$\therefore$ $a - bi = -a - bi$

$\therefore$ $a = -a$

$\therefore$ $2a = 0$

$\therefore$ $a = 0$

And if $a = 0$, then $z = 0 + bi = bi$

$\therefore$ z is purely imaginary or zero.

6 **a** Let $z = a + bi$ and $w = c + di$

$$\therefore \quad zw^* + z^*w = (a+bi)(c-di) + (a-bi)(c+di)$$
$$= ac - adi + bci + bd + ac + adi - bci + bd$$
$$= ac + bd + ac + bd$$
$$= 2ac + 2bd \quad \text{which is a real number}$$

b Let $z = a + bi$ and $w = c + di$

$$\therefore \quad zw^* - z^*w = (a+bi)(c-di) - (a-bi)(c+di)$$
$$= ac - adi + bci + bd - [ac + adi - bci + bd]$$
$$= 2bci - 2adi$$
$$= (2bc - 2ad)i \quad \text{which is purely imaginary or zero}$$

7 **a** If $z = a + bi$

then $z^2 = (a + bi)(a + bi)$

$= a^2 + 2abi + b^2 i^2$

$= (a^2 - b^2) + 2abi$

b $(z^*)^2 = (a - bi)^2$

$= a^2 - 2abi + b^2 i^2$

$= (a^2 - b^2) - 2abi$

and $(z^2)^* = (a^2 - b^2) - 2abi$ {from **a**}

$\therefore$ $(z^2)^* = (z^*)^2$ as required

c
$$\begin{aligned} z^3 &= (z^2)z \\ &= ((a^2-b^2)+2abi)(a+bi) \\ &= a(a^2-b^2)+b(a^2-b^2)i+2a^2bi+2ab^2i^2 \\ &= a^3-ab^2+a^2bi-b^3i+2a^2bi-2ab^2 \\ &= (a^3-3ab^2)+(3a^2b-b^3)i \\ \therefore\ \left(z^3\right)^* &= (a^3-3ab^2)-(3a^2b-b^3)i \\ (z^*)^3 &= (z^*)^2\,z^* \\ &= \left[(a^2-b^2)-2abi\right](a-bi) \\ &= a(a^2-b^2)-b(a^2-b^2)i-2a^2bi+2ab^2i^2 \\ &= a^3-ab^2-a^2bi+b^3i-2a^2bi-2ab^2 \\ &= (a^3-3ab^2)-(3a^2b-b^3)i \quad \text{which is } \left(z^3\right)^* \text{ as required} \end{aligned}$$

8 $w = \dfrac{z-1}{z^*+1}$ where $z = a+bi$

$$\begin{aligned} \therefore\ w &= \frac{(a-1)+bi}{(a+1)-bi} \times \frac{(a+1)+bi}{(a+1)+bi} \\ &= \frac{(a^2-1)+(a-1)bi+(a+1)bi+b^2i^2}{(a+1)^2-b^2i^2} \\ &= \frac{(a^2-b^2-1)+2abi}{(a+1)^2+b^2} \end{aligned}$$

a w is real if $2ab = 0$,
that is, if $a = 0$ or $b = 0$, $a \neq -1$
However, if $b = 0$ and $a = -1$,
w is undefined and hence is not real.
$\therefore$ $a = 0$ or $(b = 0,\ a \neq -1)$.

b w is purely imaginary if
$a^2-b^2-1 = 0$, and $2ab \neq 0$
that is, if $a^2-b^2 = 1$
and neither a nor b is zero, and $a \neq -1$.

9 **a**
$$\begin{aligned} (z_1z_2z_3)^* &= [z_1 \times (z_2 \times z_3)]^* \\ &= {z_1}^*\,(z_2 \times z_3)^* && \{\text{as } (zw)^* = z^*w^*\} \\ &= {z_1}^* \times {z_2}^* \times {z_3}^* && \{\text{as } (zw)^* = z^*w^* \text{ again}\} \end{aligned}$$

b
$$\begin{aligned} (z_1z_2z_3z_4)^* &= (z_1z_2z_3)^* \times {z_4}^* && \{\text{as } (zw)^* = z^*w^*\} \\ &= {z_1}^* \times {z_2}^* \times {z_3}^* \times {z_4}^* && \{\text{using } \mathbf{a}\} \end{aligned}$$

c $(z_1 \times z_2 \times z_3 \ldots\ldots z_n)^* = {z_1}^* \times {z_2}^* \times {z_3}^* \ldots\ldots {z_n}^*$

d
$$\begin{aligned} (z^n)^* &= (z \times z \times z \times \ldots\ldots \times z)^* \\ &= z^* \times z^* \times z^* \times \ldots\ldots \times z^* \quad \{\text{using } \mathbf{c}\} \\ &= (z^*)^n \end{aligned}$$

EXERCISE 6C.1

1 **a**
$$\begin{aligned} &3P(x) \\ &= 3(x^2+2x+3) \\ &= 3x^2+6x+9 \end{aligned}$$

b
$$\begin{aligned} &P(x)+Q(x) \\ &= (x^2+2x+3)+(4x^2+5x+6) \\ &= 5x^2+7x+9 \end{aligned}$$

c
$$\begin{aligned} &P(x)-2Q(x) \\ &= (x^2+2x+3)-2(4x^2+5x+6) \\ &= x^2+2x+3-8x^2-10x-12 \\ &= -7x^2-8x-9 \end{aligned}$$

d
$$\begin{aligned} &P(x)\,Q(x) \\ &= (x^2+2x+3)(4x^2+5x+6) \\ &= 4x^4+8x^3+12x^2+5x^3+10x^2 \\ &\quad +15x+6x^2+12x+18 \\ &= 4x^4+13x^3+28x^2+27x+18 \end{aligned}$$

2 **a**
$$\begin{aligned} &f(x)+g(x) \\ &= (x^2-x+2)+(x^3-3x+5) \\ &= x^3+x^2-4x+7 \end{aligned}$$

b
$$\begin{aligned} &g(x)-f(x) \\ &= (x^3-3x+5)-(x^2-x+2) \\ &= x^3-3x+5-x^2+x-2 \\ &= x^3-x^2-2x+3 \end{aligned}$$

c
$$\begin{aligned}&2f(x)+3g(x)\\&=2(x^2-x+2)+3(x^3-3x+5)\\&=2x^2-2x+4+3x^3-9x+15\\&=3x^3+2x^2-11x+19\end{aligned}$$

d
$$\begin{aligned}&g(x)+x\,f(x)\\&=(x^3-3x+5)+x(x^2-x+2)\\&=x^3-3x+5+x^3-x^2+2x\\&=2x^3-x^2-x+5\end{aligned}$$

e
$$\begin{aligned}&f(x)\,g(x)\\&=(x^2-x+2)(x^3-3x+5)\\&=x^5-x^4+2x^3-3x^3+3x^2\\&\quad-6x+5x^2-5x+10\\&=x^5-x^4-x^3+8x^2-11x+10\end{aligned}$$

f
$$\begin{aligned}&[f(x)]^2\\&=(x^2-x+2)(x^2-x+2)\\&=x^4-x^3+2x^2-x^3+x^2\\&\quad-2x+2x^2-2x+4\\&=x^4-2x^3+5x^2-4x+4\end{aligned}$$

3 **a**
$$\begin{aligned}&(x^2-2x+3)(2x+1)\\&=2x^3-4x^2+6x+x^2-2x+3\\&=2x^3-3x^2+4x+3\end{aligned}$$

b
$$\begin{aligned}&(x-1)^2(x^2+3x-2)\\&=(x^2-2x+1)(x^2+3x-2)\\&=x^4-2x^3+x^2+3x^3-6x^2\\&\quad+3x-2x^2+4x-2\\&=x^4+x^3-7x^2+7x-2\end{aligned}$$

c
$$\begin{aligned}&(x+2)^3\\&=(x+2)(x+2)^2\\&=(x+2)(x^2+4x+4)\\&=x^3+2x^2+4x^2+8x+4x+8\\&=x^3+6x^2+12x+8\end{aligned}$$

d
$$\begin{aligned}&(2x^2-x+3)^2\\&=(2x^2-x+3)(2x^2-x+3)\\&=4x^4-2x^3+6x^2-2x^3+x^2\\&\quad-3x+6x^2-3x+9\\&=4x^4-4x^3+13x^2-6x+9\end{aligned}$$

e
$$\begin{aligned}&(2x-1)^4\\&=(2x-1)^2(2x-1)^2\\&=(4x^2-4x+1)(4x^2-4x+1)\\&=16x^4-16x^3+4x^2-16x^3+16x^2\\&\quad-4x+4x^2-4x+1\\&=16x^4-32x^3+24x^2-8x+1\end{aligned}$$

f
$$\begin{aligned}&(3x-2)^2(2x+1)(x-4)\\&=(9x^2-12x+4)(2x^2-7x-4)\\&=18x^4-24x^3+8x^2-63x^3+84x^2\\&\quad-28x-36x^2+48x-16\\&=18x^4-87x^3+56x^2+20x-16\end{aligned}$$

4 **a**
$$\begin{aligned}&(2x^2-3x+5)(3x-1)\\&=6x^3-11x^2+18x-5\end{aligned}$$

as
$$\begin{array}{rrrr}&2&-3&5\\&\times&3&-1\\\hline&-2&3&-5\\6&-9&15&\\\hline 6&-11&18&-5\\\hline\end{array}$$

b
$$\begin{aligned}&(4x^2-x+2)(2x+5)\\&=8x^3+18x^2-x+10\end{aligned}$$

as
$$\begin{array}{rrrr}&4&-1&2\\&\times&2&5\\\hline&20&-5&10\\8&-2&4&\\\hline 8&18&-1&10\\\hline\end{array}$$

c
$$\begin{aligned}&(2x^2+3x+2)(5-x)\\&=-2x^3+7x^2+13x+10\end{aligned}$$

as
$$\begin{array}{rrrr}&2&3&2\\&\times&-1&5\\\hline&10&15&10\\-2&-3&-2&\\\hline -2&7&13&10\\\hline\end{array}$$

d
$$\begin{aligned}&(x-2)^2(2x+1)\\&=(x^2-4x+4)(2x+1)\\&=2x^3-7x^2+4x+4\end{aligned}$$

as
$$\begin{array}{rrrr}&1&-4&4\\&\times&2&1\\\hline&1&-4&4\\2&-8&8&\\\hline 2&-7&4&4\\\hline\end{array}$$

e $(x^2-3x+2)(2x^2+4x-1)$
$= 2x^4-2x^3-9x^2+11x-2$

as
$$\begin{array}{rrrrr} & & 1 & -3 & 2 \\ & \times & 2 & 4 & -1 \\ \hline & & -1 & 3 & -2 \\ & 4 & -12 & 8 & \\ 2 & -6 & 4 & & \\ \hline 2 & -2 & -9 & 11 & -2 \\ \hline \end{array}$$

f $(3x^2-x+2)(5x^2+2x-3)$
$= 15x^4+x^3-x^2+7x-6$

as
$$\begin{array}{rrrrr} & & 3 & -1 & 2 \\ & \times & 5 & 2 & -3 \\ \hline & & -9 & 3 & -6 \\ & 6 & -2 & 4 & \\ 15 & -5 & 10 & & \\ \hline 15 & 1 & -1 & 7 & -6 \\ \hline \end{array}$$

g $(x^2-x+3)^2$
$= x^4-2x^3+7x^2-6x+9$

as
$$\begin{array}{rrrrr} & & 1 & -1 & 3 \\ & \times & 1 & -1 & 3 \\ \hline & & 3 & -3 & 9 \\ & -1 & 1 & -3 & \\ 1 & -1 & 3 & & \\ \hline 1 & -2 & 7 & -6 & 9 \\ \hline \end{array}$$

h $(2x^2+x-4)^2$
$= 4x^4+4x^3-15x^2-8x+16$

as
$$\begin{array}{rrrrr} & & 2 & 1 & -4 \\ & \times & 2 & 1 & -4 \\ \hline & & -8 & -4 & 16 \\ & 2 & 1 & -4 & \\ 4 & 2 & -8 & & \\ \hline 4 & 4 & -15 & -8 & 16 \\ \hline \end{array}$$

i $(2x+5)^3$
$= (2x+5)^2(2x+5)$
$= (4x^2+20x+25)(2x+5)$
$= 8x^3+60x^2+150x+125$

as
$$\begin{array}{rrrr} & 4 & 20 & 25 \\ & \times & 2 & 5 \\ \hline & 20 & 100 & 125 \\ 8 & 40 & 50 & \\ \hline 8 & 60 & 150 & 125 \\ \hline \end{array}$$

j $(x^3+x^2-2)^2$
$= x^6+2x^5+x^4-4x^3-4x^2+4$

as
$$\begin{array}{rrrrrrr} & & & 1 & 1 & 0 & -2 \\ & & \times & 1 & 1 & 0 & -2 \\ \hline & & & -2 & -2 & 0 & 4 \\ & & 0 & 0 & 0 & 0 & \\ & 1 & 1 & 0 & -2 & & \\ 1 & 1 & 0 & -2 & & & \\ \hline 1 & 2 & 1 & -4 & -4 & 0 & 4 \\ \hline \end{array}$$

EXERCISE 6C.2

1 **a**
$$\begin{array}{r l} & \;x \\ x+2 & \overline{\big)\; x^2+2x-3} \\ & \underline{-(x^2+2x)} \\ & \;-3 \end{array}$$

$\therefore\ Q(x)=x,\quad R=-3$
$\therefore\ x^2+2x-3=x(x+2)-3$

b
$$\begin{array}{r l} & \;x-4 \\ x-1 & \overline{\big)\; x^2-5x+1} \\ & \underline{-(x^2-\ \ x)} \\ & -4x+1 \\ & \underline{-(-4x+4)} \\ & \;-3 \end{array}$$

$\therefore\ Q(x)=x-4$
$R=-3$

$\therefore\ x^2-5x+1=(x-4)(x-1)-3$

c
$$\begin{array}{r l} & \;2x^2+10x+16 \\ x-2 & \overline{\big)\; 2x^3+\ 6x^2-\ 4x+\ 3} \\ & \underline{-(2x^3-\ 4x^2)} \\ & \;10x^2-\ 4x \\ & \underline{-(10x^2-20x)} \\ & \;16x+\ 3 \\ & \underline{-(16x-32)} \\ & \;35 \end{array}$$

$\therefore\ Q(x)=2x^2+10x+16$
$R=35$

$\therefore\ 2x^3+6x^2-4x+3=(2x^2+10x+16)(x-2)+35$

2 **a**

$$
\begin{array}{rr}
 & x+1 \\ \cline{2-2}
x-4 & \big)\; x^2-3x+6 \\
 & -(x^2-4x) \\ \cline{2-2}
 & x+6 \\
 & -(x-4) \\ \cline{2-2}
 & 10
\end{array}
$$

$\therefore\ x^2-3x+6=(x+1)(x-4)+10$

b

$$
\begin{array}{rr}
 & x+1 \\ \cline{2-2}
x+3 & \big)\; x^2+4x-11 \\
 & -(x^2+3x) \\ \cline{2-2}
 & x-11 \\
 & -(x+\ 3) \\ \cline{2-2}
 & -14
\end{array}
$$

$\therefore\ x^2+4x-11=(x+1)(x+3)-14$

c

$$
\begin{array}{rr}
 & 2x-3 \\ \cline{2-2}
x-2 & \big)\; 2x^2-7x+2 \\
 & -(2x^2-4x) \\ \cline{2-2}
 & -3x+2 \\
 & -(-3x+6) \\ \cline{2-2}
 & -4
\end{array}
$$

$\therefore\ 2x^2-7x+2=(2x-3)(x-2)-4$

d

$$
\begin{array}{rr}
 & x^2+x-2 \\ \cline{2-2}
2x+1 & \big)\; 2x^3+3x^2-3x-2 \\
 & -(2x^3+\ x^2) \\ \cline{2-2}
 & 2x^2-3x \\
 & -(2x^2+\ x) \\ \cline{2-2}
 & -4x-2 \\
 & -(-4x-2) \\ \cline{2-2}
 & 0
\end{array}
$$

$\therefore\ 2x^3+3x^2-3x-2=(x^2+x-2)(2x+1)$

e

$$
\begin{array}{rr}
 & x^2+4x+4 \\ \cline{2-2}
3x-1 & \big)\; 3x^3+11x^2+8x+7 \\
 & -(3x^3-\quad x^2) \\ \cline{2-2}
 & 12x^2+8x \\
 & -(12x^2-4x) \\ \cline{2-2}
 & 12x+7 \\
 & -(12x-4) \\ \cline{2-2}
 & 11
\end{array}
$$

$\therefore\ 3x^3+11x^2+8x+7$
$=(x^2+4x+4)(3x-1)+11$

f

$$
\begin{array}{rr}
 & x^3-2x^2+\frac{5}{2}x-\frac{1}{4} \\ \cline{2-2}
2x+3 & \big)\; 2x^4-\ x^3-\ x^2+7x+4 \\
 & -(2x^4+3x^3) \\ \cline{2-2}
 & -4x^3-\ x^2 \\
 & -(-4x^3-6x^2) \\ \cline{2-2}
 & 5x^2+7x \\
 & -(5x^2+\frac{15}{2}x) \\ \cline{2-2}
 & -\frac{1}{2}x+4 \\
 & -(-\frac{1}{2}x-\frac{3}{4}) \\ \cline{2-2}
 & \frac{19}{4}
\end{array}
$$

$\therefore\ 2x^4-x^3-x^2+7x+4$
$=(x^3-2x^2+\frac{5}{2}x-\frac{1}{4})(2x+3)+\frac{19}{4}$

3 **a**

$$
\begin{array}{rr}
 & x+2 \\ \cline{2-2}
x-2 & \big)\; x^2+0x+5 \\
 & -(x^2-2x) \\ \cline{2-2}
 & 2x+5 \\
 & -(2x-4) \\ \cline{2-2}
 & 9
\end{array}
$$

$\therefore\ \dfrac{x^2+5}{x-2}=x+2+\dfrac{9}{x-2}$

b

$$
\begin{array}{rr}
 & 2x+1 \\ \cline{2-2}
x+1 & \big)\; 2x^2+3x+0 \\
 & -(2x^2+2x) \\ \cline{2-2}
 & x+0 \\
 & -(x+1) \\ \cline{2-2}
 & -1
\end{array}
$$

$\therefore\ \dfrac{2x^2+3x}{x+1}=2x+1-\dfrac{1}{x+1}$

c

$$
\begin{array}{rr}
 & 3x-4 \\ \cline{2-2}
x+2 & \big)\; 3x^2+2x-5 \\
 & -(3x^2+6x) \\ \cline{2-2}
 & -4x-5 \\
 & -(-4x-8) \\ \cline{2-2}
 & 3
\end{array}
$$

$\therefore\ \dfrac{3x^2+2x-5}{x+2}=3x-4+\dfrac{3}{x+2}$

d

$$
\begin{array}{rr}
 & x^2+3x-2 \\ \cline{2-2}
x-1 & \big)\; x^3+2x^2-5x+2 \\
 & -(x^3-\ x^2) \\ \cline{2-2}
 & 3x^2-5x \\
 & -(3x^2-3x) \\ \cline{2-2}
 & -2x+2 \\
 & -(-2x+2) \\ \cline{2-2}
 & 0
\end{array}
$$

$\therefore\ \dfrac{x^3+2x^2-5x+2}{x-1}=x^2+3x-2$

e

$$
\begin{array}{r|r}
 & 2x^2 - 8x + 31 \\ \cline{2-2}
x+4 & 2x^3 + 0x^2 - x + 0 \\
 & -(2x^3 + 8x^2) \\ \cline{2-2}
 & -8x^2 - x \\
 & -(-8x^2 - 32x) \\ \cline{2-2}
 & 31x + 0 \\
 & -(31x + 124) \\ \cline{2-2}
 & -124
\end{array}
$$

$$\therefore \quad \frac{2x^3 - x}{x+4} = 2x^2 - 8x + 31 - \frac{124}{x+4}$$

f

$$
\begin{array}{r|r}
 & x^2 + 3x + 6 \\ \cline{2-2}
x-2 & x^3 + x^2 + 0x - 5 \\
 & -(x^3 - 2x^2) \\ \cline{2-2}
 & 3x^2 + 0x \\
 & -(3x^2 - 6x) \\ \cline{2-2}
 & 6x - 5 \\
 & -(6x - 12) \\ \cline{2-2}
 & 7
\end{array}
$$

$$\therefore \quad \frac{x^3 + x^2 - 5}{x-2} = x^2 + 3x + 6 + \frac{7}{x-2}$$

EXERCISE 6C.3

1 **a**

$$
\begin{array}{r|r}
 & x+1 \\ \cline{2-2}
x^2+x+1 & x^3 + 2x^2 + x - 3 \\
 & -(x^3 + x^2 + x) \\ \cline{2-2}
 & x^2 + 0x - 3 \\
 & -(x^2 + x + 1) \\ \cline{2-2}
 & -x - 4
\end{array}
$$

$\therefore \quad Q(x) = x+1, \quad R(x) = -x-4$

b

$$
\begin{array}{r|r}
 & 3 \\ \cline{2-2}
x^2-1 & 3x^2 - x + 0 \\
 & -(3x^2 - 3) \\ \cline{2-2}
 & -x + 3
\end{array}
$$

$\therefore \quad Q(x) = 3, \quad R(x) = -x+3$

c

$$
\begin{array}{r|r}
 & 3x \\ \cline{2-2}
x^2+1 & 3x^3 + 0x^2 + x - 1 \\
 & -(3x^3 + 3x) \\ \cline{2-2}
 & -2x - 1
\end{array}
$$

$\therefore \quad Q(x) = 3x, \quad R(x) = -2x-1$

d $Q(x) = 0, \quad R(x) = x-4$

2 **a**

$$
\begin{array}{r|r}
 & 1 \\ \cline{2-2}
x^2+x+1 & x^2 - x + 1 \\
 & -(x^2 + x + 1) \\ \cline{2-2}
 & -2x
\end{array}
$$

$$\therefore \quad \frac{x^2 - x + 1}{x^2 + x + 1} = 1 - \frac{2x}{x^2 + x + 1}$$

$\therefore \quad x^2 - x + 1 = (x^2 + x + 1) - 2x$

b

$$
\begin{array}{r|r}
 & x \\ \cline{2-2}
x^2+2 & x^3 + 0x^2 + 0x + 0 \\
 & -(x^3 + 2x) \\ \cline{2-2}
 & -2x + 0
\end{array}
$$

$$\therefore \quad \frac{x^3}{x^2+2} = x - \frac{2x}{x^2+2}$$

$\therefore \quad x^3 = x(x^2+2) - 2x$

c

$$
\begin{array}{r|r}
 & x^2 + x + 3 \\ \cline{2-2}
x^2-x+1 & x^4 + 0x^3 + 3x^2 + x - 1 \\
 & -(x^4 - x^3 + x^2) \\ \cline{2-2}
 & x^3 + 2x^2 + x \\
 & -(x^3 - x^2 + x) \\ \cline{2-2}
 & 3x^2 + 0x - 1 \\
 & -(3x^2 - 3x + 3) \\ \cline{2-2}
 & 3x - 4
\end{array}
$$

$$\therefore \quad \frac{x^4 + 3x^2 + x - 1}{x^2 - x + 1} = x^2 + x + 3 + \frac{3x - 4}{x^2 - x + 1}$$

$\therefore \quad x^4 + 3x^2 + x - 1 = (x^2 + x + 3)(x^2 - x + 1) + 3x - 4$

d $\dfrac{2x^3-x+6}{(x-1)^2}=\dfrac{2x^3-x+6}{x^2-2x+1}$

$$\begin{array}{r|l} & \phantom{2x^3+0x^2-{}}2x+4 \\ x^2-2x+1 & 2x^3+0x^2-x+6 \\ & -(2x^3-4x^2+2x) \\ & \phantom{-(2x^3-{}}4x^2-3x+6 \\ & -(4x^2-8x+4) \\ & \phantom{-(2x^3-4x^2-{}}5x+2 \end{array}$$

$\therefore \dfrac{2x^3-x+6}{(x-1)^2}=2x+4+\dfrac{5x+2}{(x-1)^2}$

$\therefore 2x^3-x+6=(2x+4)(x-1)^2+5x+2$

e $\dfrac{x^4}{(x+1)^2}=\dfrac{x^4}{x^2+2x+1}$

$$\begin{array}{r|l} & \phantom{x^4+0x^3+{}}x^2-2x+3 \\ x^2+2x+1 & x^4+0x^3+0x^2+0x+0 \\ & -(x^4+2x^3+x^2) \\ & -2x^3-x^2+0x \\ & -(-2x^3-4x^2-2x) \\ & \phantom{x^4+0x^3+{}}3x^2+2x+0 \\ & -(3x^2+6x+3) \\ & -4x-3 \end{array}$$

$\therefore \dfrac{x^4}{(x+1)^2}=x^2-2x+3-\dfrac{4x+3}{(x+1)^2}$

$\therefore x^4=(x^2-2x+3)(x+1)^2-4x-3$

f $\dfrac{x^4-2x^3+x+5}{(x-1)(x+2)}=\dfrac{x^4-2x^3+x+5}{x^2+x-2}$

$$\begin{array}{r|l} & \phantom{x^4-2x^3+{}}x^2-3x+5 \\ x^2+x-2 & x^4-2x^3+0x^2+x+5 \\ & -(x^4+x^3-2x^2) \\ & -3x^3+2x^2+x \\ & -(-3x^3-3x^2+6x) \\ & \phantom{x^4-2x^3+{}}5x^2-5x+5 \\ & -(5x^2+5x-10) \\ & -10x+15 \end{array}$$

$\therefore \dfrac{x^4-2x^3+x+5}{(x-1)(x+2)}=x^2-3x+5+\dfrac{15-10x}{(x-1)(x+2)}$

$\therefore x^4-2x^3+x+5=(x^2-3x+5)(x-1)(x+2)-10x+15$

3 $\dfrac{P(x)}{x-2}=\dfrac{(x-2)(x^2+2x+3)+7}{x-2}$

$=x^2+2x+3+\dfrac{7}{x-2}$

$\therefore$ quotient is x^2+2x+3,
remainder is 7

4 $\dfrac{f(x)}{x^2+x-2}=\dfrac{(x-1)(x+2)(x^2-3x+5)+15-10x}{(x-1)(x+2)}$

$=x^2-3x+5+\dfrac{15-10x}{(x-1)(x+2)}$

$\therefore$ quotient is x^2-3x+5, remainder is $15-10x$

EXERCISE 6D.1

1 **a** $2x^2-5x-12$ has zeros

$x=\dfrac{5\pm\sqrt{25-4(2)(-12)}}{4}$

$=\dfrac{5\pm\sqrt{121}}{4}$

$=\dfrac{5\pm 11}{4}$

$=4,\ -\frac{6}{4}$

$\therefore$ zeros are $4,\ -\frac{3}{2}$

b $x^2+6x+10$ has zeros

$x=\dfrac{-6\pm\sqrt{36-4(1)(10)}}{2}$

$=\dfrac{-6\pm\sqrt{-4}}{2}$

$=-3\pm i$

$\therefore$ zeros are $-3\pm i$

c $z^2 - 6z + 6$ has zeros

$$z = \frac{6 \pm \sqrt{36 - 4(1)(6)}}{2} = \frac{6 \pm \sqrt{12}}{2} = 3 \pm \sqrt{3}$$

$\therefore$ zeros are $3 \pm \sqrt{3}$

d

$$x^3 - 4x = x(x^2 - 4) = x(x+2)(x-2)$$

$\therefore$ zeros are $0, \pm 2$

e

$$z^3 + 2z = z(z^2 + 2) = z(z^2 - 2i^2) = z(z + i\sqrt{2})(z - i\sqrt{2})$$

$\therefore$ zeros are $0, \pm i\sqrt{2}$

f

$$z^4 + 4z^2 - 5 = (z^2 + 5)(z^2 - 1) = (z^2 - 5i^2)(z^2 - 1) = (z + i\sqrt{5})(z - i\sqrt{5})(z+1)(z-1)$$

$\therefore$ zeros are $\pm i\sqrt{5}, \pm 1$

2 **a**

$$5x^2 = 3x + 2$$
$$\therefore\ 5x^2 - 3x - 2 = 0$$
$$\therefore\ (5x+2)(x-1) = 0$$

$\therefore$ roots are $1, -\frac{2}{5}$

b

$$(2x+1)(x^2+3) = 0$$
$$\therefore\ (2x+1)(x^2 - 3i^2) = 0$$
$$\therefore\ (2x+1)(x + i\sqrt{3})(x - i\sqrt{3}) = 0$$

$\therefore$ roots are $-\frac{1}{2}, \pm i\sqrt{3}$

c $-2z(z^2 - 2z + 2) = 0$

$$z = 0 \text{ or } \frac{2 \pm \sqrt{4 - 4(1)(2)}}{2}$$
$$= 0 \text{ or } \frac{2 \pm \sqrt{-4}}{2}$$
$$= 0 \text{ or } 1 \pm i$$

$\therefore$ roots are $0, 1 \pm i$

d

$$x^3 = 5x$$
$$\therefore\ x^3 - 5x = 0$$
$$x(x^2 - 5) = 0$$
$$x(x + \sqrt{5})(x - \sqrt{5}) = 0$$

$\therefore$ roots are $0, \pm\sqrt{5}$

e

$$z^3 + 5z = 0$$
$$z(z^2 + 5) = 0$$
$$z(z^2 - 5i^2) = 0$$
$$z(z + i\sqrt{5})(z - i\sqrt{5}) = 0$$

$\therefore$ roots are $0, \pm i\sqrt{5}$

f

$$z^4 = 3z^2 + 10$$
$$\therefore\ z^4 - 3z^2 - 10 = 0$$
$$(z^2 - 5)(z^2 + 2) = 0$$
$$(z^2 - 5)(z^2 - 2i^2) = 0$$
$$(z + \sqrt{5})(z - \sqrt{5})(z + i\sqrt{2})(z - i\sqrt{2}) = 0$$

$\therefore$ roots are $\pm\sqrt{5}, \pm i\sqrt{2}$

3 **a**

$$2x^2 - 7x - 15 = (2x+3)(x-5)$$

b $z^2 - 6z + 16$ is zero when

$$z = \frac{6 \pm \sqrt{36 - 4(1)(16)}}{2} = 3 \pm i\sqrt{7}$$

$$\therefore\ z^2 - 6z + 16 = (z - 3 + i\sqrt{7})(z - 3 - i\sqrt{7})$$

c

$$x^3 + 2x^2 - 4x = x(x^2 + 2x - 4)$$

$x^2 + 2x - 4$ is zero when

$$x = \frac{-2 \pm \sqrt{4 + 16}}{2} = -1 \pm \sqrt{5}$$

$$\therefore\ x^3 + 2x^2 - 4x = x(x + 1 + \sqrt{5})(x + 1 - \sqrt{5})$$

d

$$6z^3 - z^2 - 2z = z(6z^2 - z - 2) = z(2z+1)(3z-2)$$

e

$$z^4 - 6z^2 + 5 = (z^2 - 1)(z^2 - 5) = (z+1)(z-1)(z + \sqrt{5})(z - \sqrt{5})$$

f

$$z^4 - z^2 - 2 = (z^2 - 2)(z^2 + 1) = (z + \sqrt{2})(z - \sqrt{2})(z + i)(z - i)$$

4 $P(x) = a(x-\alpha)(x-\beta)(x-\gamma)$

$\therefore$ $P(\alpha) = a \times 0 \times (\alpha-\beta)(\alpha-\gamma) = 0$

and $P(\beta) = a(\beta-\alpha) \times 0 \times (\beta-\gamma) = 0$ $\quad\therefore$ α, β, and γ all satisfy $P(x) = 0$

and $P(\gamma) = a(\gamma-\alpha)(\gamma-\beta) \times 0 = 0$ $\quad\therefore$ α, β, and γ are zeros of $P(x)$

5 **a** The zeros ± 2 have sum $= 0$ and product $= -4$

$\therefore$ come from quadratic factor $z^2 - 4$

and zero 3 comes from $(z-3)$

$\therefore$ $P(z) = a(z^2-4)(z-3),\ a \neq 0$

b The zeros $\pm i$ have sum $= 0$ and product $= 1$

$\therefore$ come from quadratic factor $z^2 + 1$

and zero -2 comes from $(z+2)$

$\therefore$ $P(z) = a(z^2+1)(z+2),\ a \neq 0$

c The zeros $-1 \pm i$ have sum $= -2$ and product $= 2$

$\therefore$ come from quadratic factor $z^2 + 2z + 2$

and zero 3 comes from $(z-3)$

$\therefore$ $P(z) = a(z-3)(z^2+2z+2),\ a \neq 0$

d The zeros $-2 \pm \sqrt{2}$ have sum $= -4$ and product $= 2$

$\therefore$ come from quadratic factor $z^2 + 4z + 2$

and zero -1 comes from $(z+1)$

$\therefore$ $P(z) = a(z+1)(z^2+4z+2),\ a \neq 0$

6 **a** For zeros of ± 1, sum $= 0$ and product $= -1$ $\quad\therefore$ come from $z^2 - 1$

For zeros of $\pm\sqrt{2}$, sum $= 0$ and product $= -2$ $\quad\therefore$ come from $z^2 - 2$

$\therefore$ $P(z) = a(z^2-1)(z^2-2),\quad a \neq 0$

b For zeros of $\pm i\sqrt{3}$, sum $= 0$ and product $= 3$ $\quad\therefore$ come from $z^2 + 3$

zeros of $2, -1$ come from $(z-2)(z+1)$

$\therefore$ $P(z) = a(z-2)(z+1)(z^2+3),\ a \neq 0$

c For zeros of $\pm\sqrt{3}$, sum $= 0$ and product $= -3$ $\quad\therefore$ come from $z^2 - 3$

For zeros of $1 \pm i$, sum $= 2$ and product $= 2$ $\quad\therefore$ come from $z^2 - 2z + 2$

$\therefore$ $P(z) = a(z^2-3)(z^2-2z+2),\quad a \neq 0$

d For zeros of $2 \pm \sqrt{5}$, sum $= 4$ and product $= -1$ $\quad\therefore$ come from $z^2 - 4z - 1$

For zeros of $-2 \pm 3i$, sum $= -4$ and product $= 13$ $\quad\therefore$ come from $z^2 + 4z + 13$

$\therefore$ $P(z) = a(z^2-4z-1)(z^2+4z+13),\quad a \neq 0$

EXERCISE 6D.2

1 **a** $2x^2 + 4x + 5 = ax^2 + (2b-6)x + c$

Equating coefficients gives

$a = 2,\ 2b - 6 = 4,$ and $c = 5$

$\therefore$ $2b = 10$

$\therefore$ $b = 5$

$\therefore$ $a = 2,\ b = 5,\ c = 5$

b
$$\begin{aligned}2x^3 - x^2 + 6 &= (x-1)^2(2x+a) + bx + c\\ &= (x^2-2x+1)(2x+a) + bx + c\\ &= 2x^3 + (a-4)x^2 + (2-2a)x + a + bx + c\\ &= 2x^3 + (a-4)x^2 + (2-2a+b)x + (a+c)\end{aligned}$$

Equating coefficients gives $a - 4 = -1$ $\quad 2 - 2a + b = 0$ $\quad a + c = 6$

$\therefore$ $a = 3$ $\quad\therefore$ $b = 2a - 2$ $\quad\therefore$ $c = 6 - a$

$\therefore$ $b = 4$ $\quad\therefore$ $c = 3$

2 **a**
$$\begin{aligned}z^4 + 4 &= (z^2+az+2)(z^2+bz+2)\\ &= z^4 + (a+b)z^3 + (4+ab)z^2 + (2a+2b)z + 4\end{aligned}$$

Equating coefficients gives: $a + b = 0$ $\quad 4 + ab = 0$

$\therefore$ $a = -b$ $\quad\therefore$ $ab = -4$

By inspection $a = 2$ and $b = -2$

or $a = -2$ and $b = 2$

		1	a	2
	$\times$	1	b	2
		2	$2a$	4
	b	ab	$2b$	
1	a	2		
1	$a+b$	$4+ab$	$2a+2b$	4

b $2z^4 + 5z^3 + 4z^2 + 7z + 6$
$= (z^2 + az + 2)(2z^2 + bz + 3)$
$= 2z^4 + (2a + b)z^3 + (ab + 7)z^2 + (3a + 2b)z + 6$

		1	a	2
$\times$		2	b	3
		3	$3a$	6
	b	ab	$2b$	
2	$2a$	4		
2	$2a+b$	$ab+7$	$3a+2b$	6

Equating coefficients gives: $2a + b = 5$ .... (1)
$3a + 2b = 7$ .... (2)
$ab + 7 = 4$ .... (3)
$\therefore\ 4a + 2b = 10 \quad \{(1) \times 2\}$
and $3a + 2b = 7$

and solving these two equations gives $a = 3,\ b = -1$
which checks with (3) as $ab + 7 = -3 + 7 = 4$ ✓

3 Consider
$z^4 + 64$
$= (z^2 + az + 8)(z^2 + bz + 8)$
$= z^4 + (a + b)z^3 + (ab + 16)z^2 + (8a + 8b)z + 64$

		1	a	8
	$\times$	1	b	8
		8	$8a$	64
	b	ab	$8b$	
1	a	8		
1	$a+b$	$ab+16$	$8a+8b$	64

Equating coefficients gives:
$a + b = 0$ and $ab + 16 = 0$
$\therefore\ a = -b \qquad \therefore\ ab = -16$

$\therefore$ by inspection $a = 4$ and $b = -4$
or $a = -4$ and $b = 4$

$\therefore\ z^4 + 64$ can be factorised into $(z^2 + 4z + 8)(z^2 - 4z + 8)$

Now consider
$z^4 + 64$
$= (z^2 + az + 16)(z^2 + bz + 4)$
$= z^4 + (a + b)z^3 + (ab + 20)z^2 + (4a + 16b)z + 64$

		1	a	16
	$\times$	1	b	4
		4	$4a$	64
	b	ab	$16b$	
1	a	16		
1	$a+b$	$ab+20$	$4a+16b$	64

Equating coefficients gives:
$a + b = 0$ .... (1) and $ab + 20 = 0$
$4a + 16b = 0$ .... (2) $\quad \therefore\ ab = -20$ .... (3)

Solution to (1), (2) is $a = b = 0$
But this does not satisfy (3)
$\therefore$ no values of a and b exist which obey the original assumption
$\therefore$ cannot be factorised in this way.

4 Consider
$x^4 - 4x^2 + 8x - 4$
$= (x^2 + ax + 2)(x^2 + bx - 2)$
$= x^4 + (a + b)x^3 + (ab)x^2 + (2b - 2a)x - 4$

		1	a	2
	$\times$	1	b	-2
		-2	$-2a$	-4
	b	ab	$2b$	
1	a	2		
1	$a+b$	ab	$2b-2a$	-4

Equating coefficients gives:
$a + b = 0$ and $ab = -4$ and $-2a + 2b = 8$
$\therefore\ 2a + 2b = 0$ .... (1)
$-2a + 2b = 8$ .... (2)

Adding (1) and (2) gives $4b = 8 \quad \therefore\ b = 2$ and hence $a = -2$, which checks with $ab = -4$ ✓
$\therefore\ P(x) = (x^2 - 2x + 2)(x^2 + 2x - 2)$

Now if $x^4 + 8x = 4x^2 + 4$
then $x^4 - 4x^2 + 8x - 4 = 0$
$\therefore\ (x^2 - 2x + 2)(x^2 + 2x - 2) = 0$
$\therefore\ x^2 - 2x + 2 = 0$ or $x^2 + 2x - 2 = 0$

$$\therefore\ x = \frac{2 \pm \sqrt{4 - 8}}{2} = 1 \pm i \quad \text{or} \quad x = \frac{-2 \pm \sqrt{4 + 8}}{2} = -1 \pm \sqrt{3}$$

$$\therefore\ x = 1 \pm i,\ -1 \pm \sqrt{3}$$

5 **a** $P(z) = 2z^3 - z^2 + az - 3$
$= (2z-3)(z^2+bz+1)$ for some value b
$= 2z^3 + (2b-3)z^2 + (2-3b)z - 3$

	1	b	1
	$\times$	2	-3
	-3	$-3b$	-3
2	$2b$	2	
2	$2b-3$	$2-3b$	-3

Equating coefficients gives:
$2b - 3 = -1$ and $2 - 3b = a$
$2b = 2$ $\quad \therefore \quad a = 2-3$
$b = 1$ $\quad a = -1$

$\therefore \quad P(z) = (2z-3)\underbrace{(z^2+z+1)}$

this quadratic has zeros $z = \dfrac{-1 \pm \sqrt{1-4}}{2} = \dfrac{-1 \pm i\sqrt{3}}{2}$

$\therefore \quad a = -1$ and zeros are $\frac{3}{2}$, $-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

b $P(z) = 3z^3 - z^2 + (a+1)z + a$
$= (3z+2)(z^2+bz+c)$
$= 3z^3 + (2+3b)z^2 + (2b+3c)z + 2c$

	1	b	c
	$\times$	3	2
	2	$2b$	$2c$
3	$3b$	$3c$	
3	$2+3b$	$2b+3c$	$2c$

Equating coefficients gives:
$\therefore \quad 2 + 3b = -1$, $2b + 3c = a + 1$, and $2c = a$
Now as $2 + 3b = -1$
$\therefore \quad 3b = -3$
$\therefore \quad b = -1$

Substituting $b = -1$ and $a = 2c$ into $2b + 3c = a + 1$ gives $2(-1) + 3c = 2c + 1$
$\therefore \quad -2 + 3c = 2c + 1$
$c = 3$
and so $a = 6$

$\therefore \quad P(z) = (3z+2)\underbrace{(z^2-z+3)}$

this quadratic has zeros $\dfrac{1 \pm \sqrt{1-4(3)(1)}}{2} = \dfrac{1 \pm i\sqrt{11}}{2}$

So, $a = 6$ and the zeros are $-\frac{2}{3}$, $\frac{1}{2} \pm i\frac{\sqrt{11}}{2}$.

6 **a** $P(x) = 2x^4 + ax^3 + bx^2 - 12x - 8$
$= (2x+1)(x-2)(x^2+cx+4)$
$= (2x^2-3x-2)(x^2+cx+4)$
$= 2x^4 + (2c-3)x^3 + (6-3c)x^2$
$\quad + (-2c-12)x - 8$

		2	-3	-2
	$\times$	1	c	4
		8	-12	-8
	$2c$	$-3c$	$-2c$	
2	-3	-2		
2	$2c-3$	$6-3c$	$-2c-12$	-8

Equating coefficients: $2c - 3 = a$,
$6 - 3c = b$, and $-2c - 12 = -12$
The last equation has solution $c = 0$, and consequently, $a = -3$ and $b = 6$

$\therefore \quad P(x) = (2x+1)(x-2)(x^2+4) = (2x+1)(x-2)(x+2i)(x-2i)$
$\therefore$ zeros are $-\frac{1}{2}$, 2, and $\pm 2i$ and $a = -3$, $b = 6$.

b $P(x) = 2x^4 + ax^3 + bx^2 + ax + 3$
$= (x+3)(2x-1)(x^2+cx-1)$
$= (2x^2+5x-3)(x^2+cx-1)$
$= 2x^4 + (2c+5)x^3 + (5c-5)x^2$
$\quad + (-5-3c)x + 3$

		2	5	-3
	$\times$	1	c	-1
		-2	-5	3
	$2c$	$5c$	$-3c$	
2	5	-3		
2	$2c+5$	$5c-5$	$-5-3c$	3

Equating coefficients: $a = 2c + 5$,
$b = 5c - 5$, $a = -5 - 3c$

$\therefore \quad 2c+5=-5-3c \qquad \{\text{equating } as\}$

$\therefore \quad 5c=-10$

$\therefore \quad c=-2$ and so, $a=1 \quad b=-15$

$$\therefore \quad P(x)=(x+3)(2x-1)\underbrace{(x^2-2x-1)}_{\text{this quadratic has zeros } \frac{2\pm\sqrt{4+4}}{2}=1\pm\sqrt{2}}$$

$\therefore$ zeros are $-3, \frac{1}{2}, 1\pm\sqrt{2}$ and $a=1,\ b=-15$

7 **a** x^3+3x^2-9x+c
$=(x+a)^2(x+b)$
$=(x^2+2ax+a^2)(x+b)$
$=x^3+(b+2a)x^2+(a^2+2ab)x+a^2b$

	1	$2a$	a^2
	$\times$	1	b
	b	$2ab$	a^2b
1	$2a$	a^2	
1	$b+2a$	a^2+2ab	a^2b

Equating coefficients gives
$2a+b=3, \quad a^2+2ab=-9,$ and $c=a^2b$

Substituting $b=3-2a$ into the second equation gives:

$a^2+2a(3-2a)=-9$

$\therefore \quad a^2+6a-4a^2=-9$

$\therefore \quad -3a^2+6a=-9$

$\therefore \quad 3a^2-6a-9=0$

$\therefore \quad a^2-2a-3=0$

$\therefore \quad (a-3)(a+1)=0$

$$\therefore \quad \begin{cases} a=3 \\ b=-3 \\ c=-27 \end{cases} \quad \text{or} \quad \begin{cases} a=-1 \\ b=5 \\ c=5 \end{cases}$$

If $c=-27, \quad P(x)=(x+3)^2(x-3)$

If $c=5, \quad P(x)=(x-1)^2(x+5)$

b $3x^3+4x^2-x+m$
$=(x+a)^2(3x+b)$
$=(x^2+2ax+a^2)(3x+b)$
$=3x^3+(6a+b)x^2+(3a^2+2ab)x+a^2b$

	1	$2a$	a^2
	$\times$	3	b
	b	$2ab$	a^2b
3	$6a$	$3a^2$	
3	$6a+b$	$3a^2+2ab$	a^2b

Equating coefficients gives
$6a+b=4, \ 3a^2+2ab=-1,$ and $a^2b=m$

Substituting $b=4-6a$ into the second equation gives: $3a^2+2a(4-6a)=-1$

$\therefore \quad 3a^2+8a-12a^2=-1$

$\therefore \quad 9a^2-8a-1=0$

$\therefore \quad (9a+1)(a-1)=0$

$\therefore \quad a=-\frac{1}{9}$ or $a=1$

When $a=1, \ b=-2$ and $m=-2$. So, $P(x)=(x+1)^2(3x-2) \quad \therefore$ zeros are $-1, \frac{2}{3}$

When $a=-\frac{1}{9}, \ b=\frac{14}{3}$ and $m=\frac{14}{243}$. So, $P(x)=(x-\frac{1}{9})^2(3x+\frac{14}{3})$

$\therefore$ zeros are $\frac{1}{9}, -\frac{14}{9}$

EXERCISE 6E.1

1 **a** If $P(2)=7$, then $P(x)=Q(x)(x-2)+7$ and $P(x)$ divided by $(x-2)$ leaves a remainder of 7.

b If $P(x)=Q(x)(x+3)-8$, then $P(-3)=-8$ and $P(x)$ divided by $(x+3)$ leaves a remainder of -8.

c If $P(x)$ when divided by $(x-5)$ has a remainder of 11, then $P(5)=11$ and $P(x)=Q(x)(x-5)+11$.

2 **a** $P(x)=x^3+2x^2-7x+5$

$\therefore \quad R=P(1) \quad \{\text{Remainder theorem}\}$
$=1^3+2(1)^2-7+5$
$=1$

b $P(x)=x^4-2x^2+3x-1$

$\therefore \quad R=P(-2) \quad \{\text{Remainder theorem}\}$
$=(-2)^4-2(-2)^2+3(-2)-1$
$=16-8-6-1$
$=1$

3 **a** $P(x) = x^3 - 2x + a$

Now $P(2) = 7$ {Remainder theorem}

$\therefore\ 2^3 - 2(2) + a = 7$

$4 + a = 7$

$\therefore\ a = 3$

b $P(x) = 2x^3 + x^2 + ax - 5$

Now $P(-1) = -8$

$\therefore\ 2(-1)^3 + (-1)^2 + a(-1) - 5 = -8$

$-2 + 1 - a - 5 = -8$

$\therefore\ -a - 6 = -8$

$-a = -2$

$\therefore\ a = 2$

4 $P(x) = x^3 + 2x^2 + ax + b$

Now $P(1) = 4$ and $P(-2) = 16$ {Remainder theorem}

If $P(1) = 4$ then $1 + 2 + a + b = 4$ and so $a + b = 1$ .... (1)

If $P(-2) = 16$ then $(-2)^3 + 2(-2)^2 + a(-2) + b = 16$

$\therefore\ -8 + 8 - 2a + b = 16$

$\therefore\ -2a + b = 16$ .... (2)

Solving (1) and (2) $-a - b = -1$

$-2a + b = 16$

$\therefore\ -3a = 15$ {adding}

$\therefore\ a = -5$ and so $b = 6$

$\therefore\ a = -5$ and $b = 6$

5 $P(x) = 2x^n + ax^2 - 6$

By the Remainder theorem, $P(1) = -7\ \therefore\ 2(1)^n + a(1)^2 - 6 = -7$

$\therefore\ 2 + a - 6 = -7$

$\therefore\ a = -3$

So, $P(x) = 2x^n - 3x^2 - 6$

and since $P(-3) = 129$, $\therefore\ 2(-3)^n - 3(-3)^2 - 6 = 129$

$2(-3)^n - 27 - 6 = 129$

$2(-3)^n = 162$

$(-3)^n = 81$

$\therefore\ n = 4$

$\therefore\ a = -3$ and $n = 4$

6 $P(z) = Q(z)(z^2 - 3z + 2) + (4z - 7) = Q(z)(z-2)(z-1) + (4z - 7)$

a Remainder is $P(1)$ {Remainder theorem}

$\therefore\ R = Q(1) \times (1 - 2) \times 0 + (4 - 7)$

$= -3$

b Remainder is $P(2)$ {Remainder theorem}

$\therefore\ R = Q(2) \times 0 \times (2 - 1) + [4(2) - 7]$

$= 0 + 1$

$= 1$

7 Suppose $P(z)$ is divided by $(z-3)(z+1)$

$\therefore\ P(z) = Q(z) \times (z-3)(z+1) + (Az + B)$

↑ the remainder must be of this form

Now $P(-1) = -8$ $\therefore\ Q(-1) \times 0 + (-A + B) = -8$

$\therefore\ -A + B = -8$ .... (1)

and $P(3) = 4$ $\therefore\ Q(3) \times 0 + (3A + B) = 4$

$\therefore\ 3A + B = 4$ .... (2)

Solving (1) and (2), $-A + B = -8$

$-3A - B = -4$

$-4A = -12$

$\therefore\ A = 3$ and so $B = -5$

$\therefore\ R(z) = 3z - 5$

8 Suppose $P(x)$ is divided by $(x-a)(x-b)$ and has remainder $Ex+F$
hence $P(x) = Q(x) \times (x-a)(x-b) + Ex + F$
Now $P(a) = Ea + F$ .... (1) and $P(b) = Eb + F$ .... (2)
Subtracting (1) from (2), $P(b) - P(a) = Eb - Ea = E(b-a)$

$$\therefore\ E = \frac{P(b)-P(a)}{b-a}$$

$$\therefore\ \text{from (1)},\quad F = P(a) - Ea = P(a) - \left(\frac{P(b)-P(a)}{b-a}\right)a$$

$$\text{Now}\quad R(x) = Ex + F$$

$$\therefore\ R(x) = \left(\frac{P(b)-P(a)}{b-a}\right)x + P(a) - \left(\frac{P(b)-P(a)}{b-a}\right)a$$

$$\therefore\ R(x) = \left(\frac{P(b)-P(a)}{b-a}\right)(x-a) + P(a)$$

EXERCISE 6E.2

1 **a** $P(x) = 2x^3 + x^2 + kx - 4$
if $(x+2)$ is a factor then $P(-2) = 0$
$\therefore\ -2k - 16 = 0$
$\therefore\ k = -8$

$$\begin{array}{r|rrr|r} -2 & 2 & 1 & k & -4 \\ & 0 & -4 & 6 & -2k-12 \\ \hline & 2 & -3 & k+6 & -2k-16 \end{array}$$

$\therefore\ P(x) = 2x^3 + x^2 - 8x - 4$
$= (x+2)(2x^2 - 3x - 2)$ {as when $k=-8$, $k+6=-2$}
$\therefore\ P(x) = (x+2)(2x+1)(x-2)$ and $k = -8$

b $P(x) = x^4 - 3x^3 - kx^2 + 6x$
if $(x-3)$ is a factor then $P(3) = 0$
$\therefore\ 18 - 9k = 0$
$\therefore\ 9k = 18$
$\therefore\ k = 2$

$$\begin{array}{r|rrrr|r} 3 & 1 & -3 & -k & 6 & 0 \\ & 0 & 3 & 0 & -3k & 18-9k \\ \hline & 1 & 0 & -k & 6-3k & 18-9k \end{array}$$

$\therefore\ P(x) = x^4 - 3x^3 - 2x^2 + 6x$
$\therefore\ P(x) = (x-3)(x^3 - 2x)$ {as when $k=2$, $-k=-2$, and $6-3k=0$}
$= x(x-3)(x^2-2)$
$= x(x-3)(x+\sqrt{2})(x-\sqrt{2})$ and $k = 2$

2 $P(x) = 2x^3 + ax^2 + bx + 5$
if $(x-1)$ is a factor, $P(1) = 0$
$\therefore\ 2(1)^3 + a(1)^2 + b(1) + 5 = 0$
$2 + a + b + 5 = 0$
$\therefore\ a + b = -7$ .... (1)

if $(x+5)$ is a factor, $P(-5) = 0$
$2(-5)^3 + a(-5)^2 + b(-5) + 5 = 0$
$-250 + 25a - 5b + 5 = 0$
$25a - 5b = 245$
$\therefore\ 5a - b = 49$ .... (2)

Adding (1) and (2) gives: $6a = 42$
$\therefore\ a = 7$ and $b = -14$

3 **a** $P(z) = z^3 - z^2 + (k-5)z + (k^2-7)$
if 3 is a zero, $R = P(3) = 0$
$\therefore\ k^2 + 3k - 4 = 0$
$(k+4)(k-1) = 0$
$\therefore\ k = -4$ or $k = 1$

$$\begin{array}{r|rrr|r} 3 & 1 & -1 & k-5 & k^2-7 \\ & 0 & 3 & 6 & 3k+3 \\ \hline & 1 & 2 & k+1 & k^2+3k-4 \end{array}$$

if $k = 1$, $P(z) = (z-3)(z^2+2z+2)$

the quadratic has zeros:

$$\frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$$

$\therefore$ zeros are $3, -1 \pm i$

if $k = -4$, $P(z) = (z-3)(z^2+2z-3)$

$= (z-3)(z+3)(z-1)$

$\therefore$ zeros are $3, -3$, and 1

b $P(z) = z^3 + mz^2 + (3m-2)z - 10m - 4$

if $(z-2)$ is a factor, $P(2) = 0$

since $* = 0$, R_1 is always 0

$\therefore$ $(z-2)$ is always a factor

now for $(z-2)^2$ to be a factor

$7m + 10 = 0$ $\{R_2$ is also $0\}$

$\therefore$ $m = -\frac{10}{7}$

$$\begin{array}{c|cccc}
2 & 1 & m & 3m-2 & -10m-4 \\
 & 0 & 2 & 2m+4 & 10m+4 \\ \hline
2 & 1 & m+2 & 5m+2 & 0\ldots \quad (*) \\
 & 0 & 2 & 2m+8 & \\ \hline
 & 1 & m+4 & 7m+10 &
\end{array}$$

4 **a** **i** $P(x) = x^3 - a^3$

$P(a) = a^3 - a^3$

$= 0$

$\therefore$ $(x-a)$ is a linear factor of $P(x)$ for all a

ii

$$\begin{array}{c|cccc}
a & 1 & 0 & 0 & -a^3 \\
 & 0 & a & a^2 & a^3 \\ \hline
 & 1 & a & a^2 & 0
\end{array}$$

$\therefore$ $P(x) = (x-a)(x^2+ax+a^2)$

b **i** $P(x) = x^3 + a^3$

$P(-a) = -a^3 + a^3$

$= 0$

$\therefore$ $(x+a)$ is a factor of $P(x)$ for all a

ii

$$\begin{array}{c|cccc}
-a & 1 & 0 & 0 & a^3 \\
 & 0 & -a & a^2 & -a^3 \\ \hline
 & 1 & -a & a^2 & 0
\end{array}$$

$\therefore$ $P(x) = (x+a)(x^2-ax+a^2)$

5 **a** Consider $P(x) = x^n + 1$

if $(x+1)$ is a factor then

$P(-1) = 0$

$\therefore$ $(-1)^n + 1 = 0$

$\therefore$ $(-1)^n = -1$

which is only true if n is odd

$\therefore$ $(x+1)$ is a factor of $x^n + 1 \Leftrightarrow n$ is odd.

if n is odd, $(-1)^n = -1$

$\therefore$ $(-1)^n + 1 = 0$

then $P(-1) = 0$ if $P(x) = x^n + 1$

$\therefore$ $x = -1$ is a zero of $P(x)$

$\therefore$ $(x+1)$ is a factor of $P(x)$

b $P(x) = x^3 - 3ax - 9$ and if $(x-1-a)$ is a factor then $P(1+a) = 0$

$$\begin{array}{c|cccc}
1+a & 1 & 0 & -3a & -9 \\
 & 0 & 1+a & 1+2a+a^2 & a^3+1 \\ \hline
 & 1 & 1+a & a^2-a+1 & a^3-8
\end{array}$$

$\therefore$ $a^3 - 8 = 0$

$\therefore$ $a = 2$ {the only real solution}

EXERCISE 6E.3

1 Since it is a real polynomial, the zeros must be $-\frac{1}{2}$, $1-3i$, and $1+3i$.

For $1 \pm 3i$, $\alpha + \beta = 2$, and $\alpha\beta = 1 - 9i^2 = 10$

$\therefore$ factors are $(2x+1)$ and $(x^2 - 2x + 10)$

$\therefore$ $P(x) = a(2x+1)(x^2-2x+10)$, $a \neq 0$

2 $p(1) = p(2+i) = 0$

Hence zeros of $p(x)$ must be $1, 2 \pm i$ {as $p(x)$ is real}

For $2 \pm i$, $\alpha + \beta = 4$ and $\alpha\beta = 4 - i^2 = 5$

$\therefore$ factors must be $(x-1)$ and $(x^2 - 4x + 5)$

$\therefore$ $p(x) = k(x-1)(x^2 - 4x + 5)$

Since $p(0) = -20$ then $-20 = k(-1)(5)$

$\therefore$ $k = 4$

$\therefore$ $p(x) = 4(x-1)(x^2 - 4x + 5)$

$\therefore$ $p(x) = 4x^3 - 20x^2 + 36x - 20$

	1	-4	5
	$\times$	4	-4
	-4	16	-20
4	-16	20	
4	-20	36	-20

3 **a** $2 - 3i$ is a zero of $z^3 + pz + q$ and as the cubic has real coefficients, $2 + 3i$ is also a zero.

For $2 \pm 3i$, $\alpha + \beta = 4$ and $\alpha\beta = 4 - 9i^2 = 13$

$\therefore$ $z^2 - 4z + 13$ is a factor

$\therefore$ $z^3 + pz + q = (z^2 - 4z + 13)(z + a)$ for some a

$= z^3 + (a-4)z^2 + (13 - 4a)z + 13a$

	1	-4	13
	$\times$	1	a
	a	$-4a$	$13a$
1	-4	13	
1	$a-4$	$13-4a$	$13a$

Equating coefficients:

$a - 4 = 0$, $\quad 13 - 4a = p$, $\quad$ and $\quad 13a = q$

$\therefore$ $a = 4$, $\quad p = -3$, $\quad q = 52$

$\therefore$ the other zeros are -4 and $2 + 3i$

b *Check*: Since $P(2-3i) = 0$, $(2-3i)^3 + p(2-3i) + q = 0$

Expanding, $(-46 - 9i) + p(2 - 3i) + q = 0$

$\therefore$ $(-46 + 2p + q) + (-9 - 3p)i = 0$

Equating real and imaginary parts, $-46 + 2p + q = 0$ .... (1)

and $-9 - 3p = 0$ .... (2)

From (2), $p = -3$, so in (1), $-46 - 6 + q = 0$ $\therefore$ $p = -3$, $q = 52$ ✓

4 $3 + i$ is a root of $z^4 - 2z^3 + az^2 + bz + 10 = 0$ where the coefficients are real.

$\therefore$ $3 - i$ is also a root

For $3 \pm i$, $\alpha + \beta = 6$ and $\alpha\beta = 9 - i^2 = 10$

$\therefore$ $z^2 - 6z + 10$ is a factor

$\therefore$ $z^4 - 2z^3 + az^2 + bz + 10$

$= (z^2 - 6z + 10)(z^2 + sz + 1)$ for some s

		1	-6	10
	$\times$	1	s	1
		1	-6	10
	s	$-6s$	$10s$	
1	-6	10		
1	$s-6$	$11-6s$	$10s-6$	10

Equating coefficients:

$s - 6 = -2$, $\quad 11 - 6s = a$ $\quad$ and $\quad 10s - 6 = b$

$\therefore$ $s = 4$ $\quad a = 11 - 6(4) = -13$ $\quad b = 10(4) - 6 = 34$

$\therefore$ the other factor is $z^2 + 4z + 1$ which has zeros $\dfrac{-4 \pm \sqrt{16-4}}{2} = -2 \pm \sqrt{3}$

$\therefore$ $a = -13$, $b = 34$ and the other roots are $3 - i$, $-2 \pm \sqrt{3}$.

5 Let the purely imaginary zero be bi. Since $P(z)$ is real, another zero is $-bi$ (b is real).

$\therefore$ $z^2 + b^2$ is a factor of $P(z)$

$\therefore$ $z^3 + az^2 + 3z + 9 = (z^2 + b^2)(z + c)$

$= z^3 + cz^2 + b^2 z + b^2 c$

Equating coefficients, $b^2 = 3$, $b^2 c = 9$, and $a = c$

$\therefore$ $c = 3$, $a = 3$, and $b = \pm\sqrt{3}$

$\therefore$ $P(z) = (z+3)(z^2+3)$

$\therefore$ $P(z) = (z+3)(z + i\sqrt{3})(z - i\sqrt{3})$, $a = 3$

6 Let ai be the purely imaginary zero of $3x^3+kx^2+15x+10$

$\therefore$ as $P(x)$ is real, $-ai$ is also a zero

For $\pm ai$, $\alpha+\beta=0$ and $\alpha\beta=-a^2i^2=a^2$

$\therefore$ x^2+a^2 is a factor

$$\therefore\ 3x^3+kx^2+15x+10=(x^2+a^2)(3x+b)$$
$$=3x^3+bx^2+3a^2x+a^2b$$

	1	0	a^2
	$\times$	3	b
	b	0	a^2b
3	0	$3a^2$	
3	b	$3a^2$	a^2b

Equating coefficients, $k=b$

and $3a^2=15$ $\therefore$ $a^2=5$

and $a^2b=10$ $\therefore$ $b=2$ $\therefore$ $k=2$

$\therefore$ $P(x)=(x^2+5)(3x+2)$

$\therefore$ $P(x)=(3x+2)(x-i\sqrt{5})(x+i\sqrt{5})$, $k=2$

EXERCISE 6E.4

1 **a** $2x^2-3x+4=0$

$\therefore$ the sum of the roots $=-\dfrac{(-3)}{2}=\frac{3}{2}$

The polynomial equation has degree 2.

$\therefore$ the product of the roots $=\dfrac{(-1)^2 4}{2}=2$

b $3x^3-4x^2+8x-5=0$

$\therefore$ the sum of the roots $=-\dfrac{(-4)}{3}=\frac{4}{3}$

The polynomial equation has degree 3.

$\therefore$ the product of the roots $=\dfrac{(-1)^3(-5)}{3}=\frac{5}{3}$

c $x^4-x^3+2x^2+3x-4=0$

$\therefore$ the sum of the roots $=-\dfrac{(-1)}{1}=1$

The polynomial equation has degree 4.

$\therefore$ the product of the roots $=\dfrac{(-1)^4(-4)}{1}=-4$

d $2x^5-3x^4+x^2-8=0$

$\therefore$ the sum of the roots $=-\dfrac{(-3)}{2}=\frac{3}{2}$

The polynomial equation has degree 5.

$\therefore$ the product of the roots $=\dfrac{(-1)^5(-8)}{2}=4$

e $x^7-x^5+2x-9=0$

$\therefore$ $x^7+(0)x^6-x^5+2x-9=0$

$\therefore$ the sum of the roots $=-\frac{0}{1}=0$

The polynomial equation has degree 7.

$\therefore$ the product of the roots $=\dfrac{(-1)^7(-9)}{1}=9$

f $x^6-1=0$

$\therefore$ $x^6+(0)x^5-1=0$

$\therefore$ the sum of the roots $=-\dfrac{0}{1}=0$

The polynomial equation has degree 6.

$\therefore$ the product of the roots $=\dfrac{(-1)^6(-1)}{1}=-1$

2 **a** The cubic polynomial $P(x)$ has zeros $3\pm i\sqrt{2}$ and $\frac{2}{3}$.

$\therefore$ the sum of the zeros $=(3+i\sqrt{2})+(3-i\sqrt{2})+\frac{2}{3}$
$=6+\frac{2}{3}$
$=\frac{20}{3}$

The product of the zeros $=\frac{2}{3}\times(3+i\sqrt{2})(3-i\sqrt{2})$
$=\frac{2}{3}(9+2)$
$=\frac{22}{3}$

b Let $P(x)=6x^3+ax^2+bx+c$
{leading coefficient is 6}.

$\therefore$ the sum of the zeros is $-\dfrac{a}{6}=\dfrac{20}{3}$

$\therefore$ $-a=\frac{120}{3}$

$\therefore$ $a=-40$

So, the coefficient of x^2 is -40.

c The product of the zeros is $\dfrac{(-1)^3c}{6}=\dfrac{22}{3}$

$\therefore$ $-c=\frac{132}{3}$

$\therefore$ $c=-44$

So, the constant term is -44.

3 **a** A polynomial has zeros $-2,\ 3 \pm i,$ and $\sqrt{k} \pm 1$.

The constant term is the y-intercept. $\therefore$ the constant term is 18.

$\therefore$ the product of the zeros is

$$\begin{aligned} -2(3+i)(3-i)(\sqrt{k}+1)(\sqrt{k}-1) &= \frac{(-1)^5 18}{-1} = 18 \\ \therefore\ -2(9+1)(k-1) &= 18 \\ \therefore\ -20(k-1) &= 18 \\ \therefore\ k-1 &= \tfrac{-9}{10} \\ \therefore\ k &= \tfrac{1}{10} \end{aligned}$$

b Let a be the coefficient of x^4.

$\therefore$ the sum of the zeros is

$$-2+(3+i)+(3-i)+\left(\sqrt{\tfrac{1}{10}}+1\right)+\left(\sqrt{\tfrac{1}{10}}-1\right) = -\frac{a}{-1} = a$$

$$\begin{aligned} \therefore\ a &= -2+6+2\sqrt{\tfrac{1}{10}} \\ &= 4+\frac{2}{\sqrt{10}} \\ &= \frac{4\sqrt{10}+2}{\sqrt{10}}\left(\frac{\sqrt{10}}{\sqrt{10}}\right) \\ &= \frac{40+2\sqrt{10}}{10} \\ &= \frac{20+\sqrt{10}}{5} \end{aligned}$$

So, the coefficient of x^4 is $\dfrac{20+\sqrt{10}}{5}$.

4 The polynomial has zeros $\frac{1}{2},\ 1 \pm \sqrt{2},$ and $m \pm ni$.

$\therefore$ the sum of the zeros is

$$\begin{aligned} \tfrac{1}{2}+(1+\sqrt{2})+(1-\sqrt{2})+(m+ni)+(m-ni) &= -\tfrac{3}{2} \\ \therefore\ \tfrac{1}{2}+2+2m &= -\tfrac{3}{2} \\ \therefore\ \tfrac{5}{2}+2m &= -\tfrac{3}{2} \\ \therefore\ 2m &= -4 \\ \therefore\ m &= -2 \end{aligned}$$

The constant term is the y-intercept. $\therefore$ the constant term is 5.

$\therefore$ the product of the zeros is

$$\begin{aligned} \tfrac{1}{2}\times(1+\sqrt{2})(1-\sqrt{2})(-2+ni)(-2-ni) &= \frac{(-1)^5 5}{2} \\ \therefore\ \tfrac{1}{2}(1-2)(4+n^2) &= -\tfrac{5}{2} \\ \therefore\ -\tfrac{1}{2}(4+n^2) &= -\tfrac{5}{2} \\ \therefore\ 4+n^2 &= 5 \\ \therefore\ n^2 &= 1 \\ \therefore\ n &= 1 \qquad \{n>0\} \end{aligned}$$

So, $m=-2$ and $n=1$.

5 The quartic polynomial has zeros $a \pm i$ and $3 \pm a$.

$\therefore$ the product of the zeros is $(a+i)(a-i)(3+a)(3-a) = \dfrac{(-1)^4 25}{1}$

$$\begin{aligned}
\therefore\ (a^2+1)(9-a^2) &= 25 \\
\therefore\ -a^4+8a^2+9 &= 25 \\
\therefore\ a^4-8a^2+16 &= 0 \\
\therefore\ (a^2)^2-8(a^2)+16 &= 0 \\
\therefore\ (a^2-4)^2 &= 0 \\
\therefore\ a^2 &= 4 \\
\therefore\ a &= \pm 2
\end{aligned}$$

6 **a** $x^3 - px^2 + qx - r = 0$ has non-zero roots p, q, and r.

$\therefore$ the sum of the roots is

$$\begin{aligned}
p+q+r &= -\frac{-p}{1} \\
\therefore\ p+q+r &= p \\
\therefore\ q+r &= 0 \\
\therefore\ q &= -r
\end{aligned}$$

The product of the roots is

$$\begin{aligned}
pqr &= \frac{(-1)^3(-r)}{1} \\
\therefore\ pqr &= r \\
\therefore\ pq &= 1 \\
\therefore\ p &= \frac{1}{q} \\
\therefore\ p &= -\frac{1}{r} \qquad \{\text{as } q=-r\}
\end{aligned}$$

So, $q = -r$ and $p = -\dfrac{1}{r}$.

b By the Factor Theorem, if p, q, and r are zeros of the polynomial, then $(x-p)$, $(x-q)$, and $(x-r)$ are factors.

$$\begin{aligned}
\therefore\ x^3-px^2+qx-r &= (x-p)(x-q)(x-r) \\
&= (x^2-(p+q)x+pq)(x-r) \\
&= x^3-(p+q+r)x^2+(pq+pr+qr)x-pqr
\end{aligned}$$

Equating coefficients of x:

$$\begin{aligned}
pq+pr+qr &= q \\
\therefore\ \left(-\frac{1}{r}\right)(-r)+\left(-\frac{1}{r}\right)r+(-r)r &= -r \qquad \left\{\text{substituting } -\frac{1}{r} \text{ for } p, \text{ and } -r \text{ for } q\right\} \\
\therefore\ 1-1-r^2 &= -r \\
\therefore\ r^2-r &= 0 \\
\therefore\ r(r-1) &= 0 \\
\therefore\ r=0 \text{ or } r &= 1 \\
\therefore\ r &= 1 \qquad \{r \text{ is non-zero}\}
\end{aligned}$$

$\therefore\ p = -\dfrac{1}{r} = -1$ and $q = -r = -1$

So, $p=-1$, $q=-1$, and $r=1$.

EXERCISE 6F.1

1 **a** A single factor such as $(x-\alpha)$ indicates that the graph *cuts* the x-axis at α.

b A squared factor such as $(x-\alpha)^2$ indicates that the graph *touches* the x-axis at α.

c A cubed factor such as $(x-\alpha)^3$ indicates that the graph *cuts* the x-axis at α, and at α the graph changes shape.

2 **a** The x-intercepts are: -1, 2, and 3 $\quad\therefore\ P(x) = a(x+1)(x-2)(x-3),\ a \neq 0$

As the curve passes through (0, 12), $\quad 12 = a(1)(-2)(-3) \quad\therefore\ a = 2$

$\therefore\ P(x) = 2(x+1)(x-2)(x-3)$

b The x-intercepts are: -3, $-\frac{1}{2}$, and $\frac{1}{2}$ $\quad\therefore\ P(x) = a(x+3)(2x+1)(2x-1),\ \ a \neq 0$

As the curve passes through $(0, 6)$, $\quad 6 = a(3)(1)(-1) \quad \therefore\ a = -2$

$\therefore\ P(x) = -2(x+3)(2x+1)(2x-1)$

c The x-intercepts are: -4, -4, and 3 $\quad\therefore\ P(x) = a(x+4)^2(x-3),\ \ a \neq 0$

As the curve passes through $(0, -12)$, $\quad -12 = a(4)^2(-3) \quad \therefore\ a = \frac{1}{4}$

$\therefore\ P(x) = \frac{1}{4}(x+4)^2(x-3)$

d The x-intercepts are: -5, -2, and 5 $\quad\therefore\ P(x) = a(x+5)(x+2)(x-5),\ \ a \neq 0$

As the curve passes through $(0, -5)$, $\quad -5 = a(5)(2)(-5) \quad \therefore\ a = \frac{1}{10}$

$\therefore\ P(x) = \frac{1}{10}(x+5)(x+2)(x-5)$

e The x-intercepts are: -4, 3, and 3 $\quad\therefore\ P(x) = a(x+4)(x-3)^2,\ \ a \neq 0$

As the curve passes through $(0, 9)$, $\quad 9 = a(4)(-3)^2 \quad \therefore\ a = \frac{1}{4}$

$\therefore\ P(x) = \frac{1}{4}(x+4)(x-3)^2$

f The x-intercepts are: -3, -2, and $-\frac{1}{2}$ $\quad\therefore\ P(x) = a(x+3)(x+2)(2x+1),\ \ a \neq 0$

As the curve passes through $(0, -12)$, $\quad -12 = a(3)(2)(1) \quad \therefore\ a = -2$

$\therefore\ P(x) = -2(x+3)(x+2)(2x+1)$

3 **a** $P(x) = a(x-3)(x-1)(x+2)$

Since $P(x)$ passes through $(2, -4)$,

$-4 = a(-1)(1)(4)$

$\therefore\ -4 = -4a$

$\therefore\ a = 1$

$\therefore\ P(x) = (x-3)(x-1)(x+2)$

b $P(x) = ax(x+2)(2x-1)$

Since $P(x)$ passes through $(-3, -21)$,

$-21 = -3a(-1)(-7)$

$\therefore\ -21 = -21a$

$\therefore\ a = 1$

$\therefore\ P(x) = x(x+2)(2x-1)$

c $P(x) = a(x-1)^2(x+2)$

Since $P(x)$ passes through $(4, 54)$,

$54 = a(9)(6)$

$\therefore\ a = 1$

$\therefore\ P(x) = (x-1)^2(x+2)$

d $P(x) = a(3x+2)^2(x-4)$

Since $P(x)$ passes through $(-1, -5)$,

$-5 = a(1)(-5)$

$\therefore\ a = 1$

$\therefore\ P(x) = (3x+2)^2(x-4)$

4 **a** $y = 2(x-1)(x+2)(x+4)$

has x-intercepts $1, -2, -4$

has y-intercept $2(-1)(2)(4) = -16$

$\therefore$ matches graph **F**

b $y = -(x+1)(x-2)(x-4)$

has x-intercepts $-1, 2, 4$

has y-intercept $-(1)(-2)(-4) = -8$

$\therefore$ matches graph **C**

c $y = (x-1)(x-2)(x+4)$

has x-intercepts $1, 2, -4$

has y-intercept $(-1)(-2)(4) = 8$

$\therefore$ matches graph **A**

d $y = -2(x-1)(x+2)(x+4)$

has x-intercepts $1, -2, -4$

has y-intercept $-2(-1)(2)(4) = 16$

$\therefore$ matches graph **E**

e $y = -(x-1)(x+2)(x+4)$

has x-intercepts $1, -2, -4$

has y-intercept $-(-1)(2)(4) = 8$

$\therefore$ matches graph **D**

f $y = 2(x-1)(x-2)(x+4)$

has x-intercepts $1, 2, -4$

has y-intercept $2(-1)(-2)(4) = 16$

$\therefore$ matches graph **B**

5 **a** $\frac{1}{2}$ and -3 are zeros, and so $(2x-1)$ and $(x+3)$ are factors

$\therefore\ P(x) = (2x-1)(x+3)(ax+b)$

But $P(0) = 30$

$\therefore\ b(-1)(3) = 30$ and so $b = -10$

$\therefore\ P(x) = (2x-1)(x+3)(ax-10)$

Now $P(1) = (1)(4)(a-10) = -20$

$\therefore \; a - 10 = -5$ and so $a = 5$

$\therefore \; P(x) = (2x-1)(x+3)(5x-10)$

$\therefore \; P(x) = 5(x-2)(2x-1)(x+3)$

b 1 is a zero and so $(x-1)$ is a factor, touches at -2 indicates that $(x+2)^2$ is a factor

$\therefore \; P(x) = k(x-1)(x+2)^2$

But $P(0) = 8$ $\therefore$ $8 = k(-1)(2)^2$ and so $k = -2$

$\therefore \; P(x) = -2(x-1)(x+2)^2$

c cuts the x-axis at $(2, 0)$ and so $(x-2)$ is a factor

$\therefore \; P(x) = (x-2)(ax^2+bx+c)$

But $P(0) = -4$ $\therefore$ $-2c = -4$ and so $c = 2$

Also $P(1) = -1$ $\therefore$ $(-1)(a+b+2) = -1$

$\therefore \; a+b+2 = 1$ $\therefore$ $a+b = -1$ .... (1)

Also $P(-1) = -21$ $\therefore$ $(-3)(a-b+2) = -21$

$\therefore \; a-b+2 = 7$ $\therefore$ $a-b = 5$ .... (2)

Adding (1) and (2) gives $2a = 4$ $\therefore$ $a = 2$ and so $b = -3$

$\therefore \; P(x) = (x-2)(2x^2-3x+2)$

EXERCISE 6F.2

1 **a** $P(x) = a(x+1)^2(x-1)^2$

where $a \neq 0$, and passes through $(0, 2)$

$2 = a(1)(1)$

$\therefore \; a = 2$

$\therefore \; P(x) = 2(x+1)^2(x-1)^2$

b $P(x) = a(x+3)(x+1)^2(3x-2)$

where $a \neq 0$, and passes through $(0, -6)$

$-6 = a(3)(1)(-2)$

$\therefore \; a = 1$

$\therefore \; P(x) = (x+3)(x+1)^2(3x-2)$

c $P(x) = a(x+2)(x+1)(x-2)^2$

where $a \neq 0$, and passes through $(0, -16)$

$-16 = a(2)(1)(4)$

$\therefore \; a = -2$

$\therefore \; P(x) = -2(x+2)(x+1)(x-2)^2$

d $P(x) = a(x+3)(x+1)(2x-3)(x-3)$

where $a \neq 0$, and passes through $(0, -9)$

$-9 = a(3)(1)(-3)(-3)$

$\therefore \; a = -\frac{1}{3}$

$\therefore \; P(x) = -\frac{1}{3}(x+3)(x+1)(2x-3)(x-3)$

e $P(x) = a(x+1)(x-4)^3$

where $a \neq 0$, and passes through $(0, -16)$

$-16 = a(1)(-4)^3$

$\therefore \; a = \frac{1}{4}$

$\therefore \; P(x) = \frac{1}{4}(x+1)(x-4)^3$

f $P(x) = ax^2(x+2)(x-3)$

where $a \neq 0$, and passes through $(-3, 54)$

$54 = a(9)(-1)(-6)$

$\therefore \; 54 = 54a$

$\therefore \; a = 1$

$\therefore \; P(x) = x^2(x+2)(x-3)$

2 **a** $y = (x-1)^2(x+1)(x+3)$

has x-intercepts $-1, -3$, touches at 1

has y-intercept $(-1)^2(1)(3) = 3$ (> 0)

$\therefore$ matches graph **C**

b $y = -2(x-1)^2(x+1)(x+3)$

has x-intercepts $-1, -3$, touches at 1

has y-intercept $-2(-1)^2(1)(3) = -6$ (< 0)

$\therefore$ matches graph **F**

c $y = (x-1)(x+1)^2(x+3)$

has x-intercepts $1, -3$, touches at -1

has y-intercept $(-1)(1)^2(3) = -3$ (< 0)

$\therefore$ matches graph **A**

d $y = (x-1)(x+1)^2(x-3)$

has x-intercepts $1, 3$, touches at -1

has y-intercept $(-1)(1)^2(-3) = 3$ (> 0)

$\therefore$ matches graph **E**

e $y = -\frac{1}{3}(x-1)(x+1)(x+3)^2$

has x-intercepts $1, -1$, touches at -3

has y-intercept $-\frac{1}{3}(-1)(1)(3)^2 = 3$ (> 0)

$\therefore$ matches graph **B**

f $y = -(x-1)(x+1)(x-3)^2$

has x-intercepts $1, -1$, touches at 3

has y-intercept $-(-1)(1)(3)^2 = 9$ (> 0)

$\therefore$ matches graph **D**

3 **a** $P(x) = a(x+4)(2x-1)(x-2)^2$
where $a \neq 0$, and passes through $(1, 5)$
$5 = a \times 5 \times 1 \times 1$
$\therefore \quad a = 1$
$\therefore \quad P(x) = (x+4)(2x-1)(x-2)^2$

b $P(x) = a(3x-2)^2(x+3)^2$
where $a \neq 0$, and passes through $(-4, 49)$
$49 = a(-14)^2(1)$
$\therefore \quad a = \frac{1}{4}$
$\therefore \quad P(x) = \frac{1}{4}(3x-2)^2(x+3)^2$

c $P(x) = a(2x+1)(2x-1)(x+2)(x-2)$
where $a \neq 0$, and passes through $(1, -18)$
$-18 = a(3)(1)(3)(-1)$
$\therefore \quad a = 2$
$\therefore \quad P(x) = 2(2x+1)(2x-1)(x+2)(x-2)$

d $P(x) = (x-1)^2(ax^2+bx+c)$
where $a \neq 0$, and cuts y-axis at $(0, -1)$
$-1 = 1 \times (0+0+c)$
$\therefore \quad c = -1$
$\therefore \quad P(x) = (x-1)^2(ax^2+bx-1)$
But $P(-1) = -4$
$\therefore \quad -4 = 4(a-b-1)$
$\therefore \quad a - b = 0$ .... (1)
Also $P(2) = 15$
$\therefore \quad 15 = 1(4a+2b-1)$
$\therefore \quad 16 = 4a + 2b$
$\therefore \quad 2a + b = 8$ .... (2)
Adding (1) and (2) we get:
$\therefore \quad a = \frac{8}{3}$ and so $b = \frac{8}{3}$ also
$\therefore \quad P(x) = (x-1)^2(\frac{8}{3}x^2 + \frac{8}{3}x - 1)$

EXERCISE 6F.3

1 **a** $P(x) = x^3 - 3x^2 - 3x + 1$
From technology, -1 is a zero.
Check: $P(-1) = -1 - 3 + 3 + 1 = 0$ ✓
$\therefore \quad x + 1$ is a factor
$\therefore \quad x^3 - 3x^2 - 3x + 1 = (x+1)(x^2 - 4x + 1)$

$$\begin{array}{r|rrr|r} -1 & 1 & -3 & -3 & 1 \\ & 0 & -1 & 4 & -1 \\ \hline & 1 & -4 & 1 & 0 \end{array}$$

and the quadratic has zeros of $\dfrac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$
$\therefore \quad$ zeros are $-1,\ 2 \pm \sqrt{3}$

b $P(x) = x^3 - 3x^2 + 4x - 2$
From technology, 1 is a zero
Check: $P(1) = 1 - 3 + 4 - 2 = 0$ ✓
$\therefore \quad x - 1$ is a factor

$$\begin{array}{r|rrr|r} 1 & 1 & -3 & 4 & -2 \\ & 0 & 1 & -2 & 2 \\ \hline & 1 & -2 & 2 & 0 \end{array}$$

From the division process $x^2 - 2x + 2$ is a quadratic factor
and it has zeros of $\dfrac{2 \pm \sqrt{4 - 4 \times 2}}{2} = \dfrac{2 \pm \sqrt{-4}}{2} = 1 \pm i$
$\therefore \quad$ zeros are $1,\ 1 \pm i$

c $P(x) = 2x^3 - 3x^2 - 4x - 35$
From technology, $\frac{7}{2}$ is a zero.

$$\begin{array}{r|rrr|r} \frac{7}{2} & 2 & -3 & -4 & -35 \\ & 0 & 7 & 14 & 35 \\ \hline & 2 & 4 & 10 & 0 \end{array}$$

Check: $P(\frac{7}{2}) = \frac{343}{4} - \frac{147}{4} - 14 - 35$
$= \dfrac{343 - 147 - 56 - 140}{4}$
$= 0$ ✓

From the division process $2x^2 + 4x + 10$ is a quadratic factor
$\therefore \quad P(x) = (x - \frac{7}{2})(2x^2 + 4x + 10)$
$= (2x-7)(x^2 + 2x + 5)$
where the quadratic has zeros $\dfrac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$
$\therefore \quad$ zeros are $\frac{7}{2},\ -1 \pm 2i$

d $P(x) = 2x^3 - x^2 + 20x - 10$

From technology, $\frac{1}{2}$ is a zero.

Check: $P(\frac{1}{2}) = \frac{1}{4} - \frac{1}{4} + 10 - 10 = 0$ ✓

$$\begin{array}{r|rrrr} \frac{1}{2} & 2 & -1 & 20 & -10 \\ & 0 & 1 & 0 & 10 \\ \hline & 2 & 0 & 20 & 0 \end{array}$$

$\therefore\ P(x) = (x - \frac{1}{2})(2x^2 + 20)$
$= (2x - 1)(x^2 + 10)$

$\therefore$ zeros are $\frac{1}{2},\ \pm i\sqrt{10}$

e $P(x) = 4x^4 - 4x^3 - 25x^2 + x + 6$

From technology, -2 and 3 are zeros

Check: $P(-2) = 64 + 32 - 100 - 2 + 6 = 0$ ✓
$P(3) = 324 - 108 - 225 + 3 + 6 = 0$ ✓

$$\begin{array}{r|rrrrr} -2 & 4 & -4 & -25 & 1 & 6 \\ & 0 & -8 & 24 & 2 & -6 \\ \hline 3 & 4 & -12 & -1 & 3 & 0 \\ & 0 & 12 & 0 & -3 & \\ \hline & 4 & 0 & -1 & 0 & \end{array}$$

$\therefore\ P(x) = (x + 2)(x - 3)(4x^2 - 1)$

$\therefore$ zeros are $-2,\ 3,\ \pm\frac{1}{2}$

f $P(x) = x^4 - 6x^3 + 22x^2 - 48x + 40$

From technology, 2 seems to be a double zero.
{Graph touches the x-axis at 2}

Check: $P(2) = 16 - 48 + 88 - 96 + 40 = 0$ ✓

$$\begin{array}{r|rrrrr} 2 & 1 & -6 & 22 & -48 & 40 \\ & 0 & 2 & -8 & 28 & -40 \\ \hline 2 & 1 & -4 & 14 & -20 & 0 \\ & 0 & 2 & -4 & 20 & \\ \hline & 1 & -2 & 10 & 0 & \end{array}$$

$\therefore\ P(x) = (x - 2)^2(x^2 - 2x + 10)$

where the quadratic has zeros of $\dfrac{2 \pm \sqrt{4 - 40}}{2}$
$= 1 \pm 3i$

$\therefore$ zeros are 2 (repeated), $1 \pm 3i$

2 **a** $P(x) = x^3 + 2x^2 + 3x + 6$

From technology, -2 is a zero.

Check: $P(-2) = -8 + 8 - 6 + 6 = 0$ ✓

$$\begin{array}{r|rrrr} -2 & 1 & 2 & 3 & 6 \\ & 0 & -2 & 0 & -6 \\ \hline & 1 & 0 & 3 & 0 \end{array}$$

$\therefore\ P(x) = (x + 2)(x^2 + 3)$
$= (x + 2)(x + i\sqrt{3})(x - i\sqrt{3})$

$\therefore$ roots of $P(x) = 0$ are $x = -2$ and $x = \pm i\sqrt{3}$

b $P(x) = 2x^3 + 3x^2 - 3x - 2$

From technology, 1 is a zero.

Check: $P(1) = 2 + 3 - 3 - 2 = 0$ ✓

$$\begin{array}{r|rrrr} 1 & 2 & 3 & -3 & -2 \\ & 0 & 2 & 5 & 2 \\ \hline & 2 & 5 & 2 & 0 \end{array}$$

$\therefore\ P(x) = (x - 1)(2x^2 + 5x + 2)$
$= (x - 1)(2x + 1)(x + 2)$

$\therefore$ roots of $P(x) = 0$ are $1,\ -\frac{1}{2},\ -2$

c $P(x) = x^3 - 6x^2 + 12x - 8$

From technology, 2 is a zero.

Check: $P(2) = 8 - 24 + 24 - 8 = 0$ ✓

$$\begin{array}{r|rrrr} 2 & 1 & -6 & 12 & -8 \\ & 0 & 2 & -8 & 8 \\ \hline & 1 & -4 & 4 & 0 \end{array}$$

$\therefore\ P(x) = (x - 2)(x^2 - 4x + 4)$
$= (x - 2)(x - 2)(x - 2)$

$\therefore$ only root of $P(x) = 0$ is $x = 2$ (a treble root)

d $P(x) = 2x^3 - 5x^2 - 9x + 18$

From technology, 3 is a zero.

Check: $P(3) = 54 - 45 - 27 + 18 = 0$ ✓

$$\begin{array}{r|rrrr} 3 & 2 & -5 & -9 & 18 \\ & 0 & 6 & 3 & -18 \\ \hline & 2 & 1 & -6 & 0 \end{array}$$

$\therefore\ P(x) = (x - 3)(2x^2 + x - 6)$
$= (x - 3)(2x - 3)(x + 2)$

$\therefore$ roots of $P(x) = 0$ are $3,\ \frac{3}{2}$, and -2

e $P(x) = x^4 - x^3 - 9x^2 + 11x + 6$

From technology, 2 and -3 are zeros.

Check: $P(2) = 16 - 8 - 36 + 22 + 6 = 0$ ✓

$P(-3) = 81 + 27 - 81 - 33 + 6 = 0$ ✓

$\therefore\ P(x) = (x-2)(x+3)(x^2 - 2x - 1)$

where the quadratic has zeros of $\dfrac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}$

$\therefore$ roots of $P(x) = 0$ are $2, -3, 1 \pm \sqrt{2}$

$$\begin{array}{r|rrrr|r} 2 & 1 & -1 & -9 & 11 & 6 \\ & 0 & 2 & 2 & -14 & -6 \\ \hline -3 & 1 & 1 & -7 & -3 & 0 \\ & 0 & -3 & 6 & 3 & \\ \hline & 1 & -2 & -1 & 0 & \end{array}$$

f $P(x) = 2x^4 - 13x^3 + 27x^2 - 13x - 15$

From technology, $-\frac{1}{2}$ and 3 are zeros.

Check: $P(-\frac{1}{2}) = \frac{1}{8} + \frac{13}{8} + \frac{27}{4} + \frac{13}{2} - 15 = 0$ ✓

$P(3) = 162 - 351 + 243 - 39 - 15 = 0$ ✓

$\therefore\ P(x) = (x + \frac{1}{2})(x-3)(2x^2 - 8x + 10)$

$= (2x+1)(x-3)(x^2 - 4x + 5)$

where the quadratic has zeros of $\dfrac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$

$\therefore$ roots of $P(x) = 0$ are $-\frac{1}{2}, 3, 2 \pm i$

$$\begin{array}{r|rrrr|r} -\frac{1}{2} & 2 & -13 & 27 & -13 & -15 \\ & 0 & -1 & 7 & -17 & 15 \\ \hline 3 & 2 & -14 & 34 & -30 & 0 \\ & 0 & 6 & -24 & 30 & \\ \hline & 2 & -8 & 10 & 0 & \end{array}$$

3 **a** Consider $P(x) = x^3 - 3x^2 + 4x - 2$

From technology, 1 is a zero.

Check: $P(1) = 1 - 3 + 4 - 2 = 0$ ✓

Now $P(x) = (x-1)(x^2 - 2x + 2)$

where the quadratic has zeros of $\dfrac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$

$\therefore\ P(x) = (x-1)(x-1+i)(x-1-i)$

$$\begin{array}{r|rrr|r} 1 & 1 & -3 & 4 & -2 \\ & 0 & 1 & -2 & 2 \\ \hline & 1 & -2 & 2 & 0 \end{array}$$

b Consider $P(x) = x^3 + 3x^2 + 4x + 12$

From technology, -3 is a zero.

Check: $P(-3) = -27 + 27 - 12 + 12 = 0$ ✓

Now $P(x) = (x+3)(x^2+4)$

$\therefore\ P(x) = (x+3)(x-2i)(x+2i)$

$$\begin{array}{r|rrr|r} -3 & 1 & 3 & 4 & 12 \\ & 0 & -3 & 0 & -12 \\ \hline & 1 & 0 & 4 & 0 \end{array}$$

c Consider $P(x) = 2x^3 - 9x^2 + 6x - 1$

From technology, $\frac{1}{2}$ is a zero.

Check: $P(\frac{1}{2}) = \frac{1}{4} - \frac{9}{4} + 3 - 1 = 0$ ✓

Now $P(x) = (x - \frac{1}{2})(2x^2 - 8x + 2)$

$= (2x-1)(x^2 - 4x + 1)$

where the quadratic has zeros of $\dfrac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$

$\therefore\ P(x) = (2x-1)(x - 2 + \sqrt{3})(x - 2 - \sqrt{3})$

$$\begin{array}{r|rrr|r} \frac{1}{2} & 2 & -9 & 6 & -1 \\ & 0 & 1 & -4 & 1 \\ \hline & 2 & -8 & 2 & 0 \end{array}$$

d $P(x) = x^3 - 4x^2 + 9x - 10$

From technology, 2 is a zero.

Check: $P(2) = 8 - 16 + 18 - 10 = 0$ ✓

Now $P(x) = (x-2)(x^2 - 2x + 5)$

where the quadratic has zeros of $\dfrac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$

$\therefore\ P(x) = (x-2)(x-1+2i)(x-1-2i)$

$$\begin{array}{r|rrr|r} 2 & 1 & -4 & 9 & -10 \\ & 0 & 2 & -4 & 10 \\ \hline & 1 & -2 & 5 & 0 \end{array}$$

e $P(x) = 4x^3 - 8x^2 + x + 3$

From technology, 1 is a zero.

Check: $P(1) = 4 - 8 + 1 + 3 = 0$ ✓

$$\begin{array}{r|rrr|r} 1 & 4 & -8 & 1 & 3 \\ & 0 & 4 & -4 & -3 \\ \hline & 4 & -4 & -3 & 0 \end{array}$$

Now $P(x) = (x-1)(4x^2 - 4x - 3)$
$= (x-1)(2x-3)(2x+1)$

$\therefore \; P(x) = (x-1)(2x+1)(2x-3)$

f $P(x) = 3x^4 + 4x^3 + 5x^2 + 12x - 12$

From technology, -2 and $\frac{2}{3}$ are zeros.

Check: $P(-2) = 48 - 32 + 20 - 24 - 12 = 0$ ✓
$P(\frac{2}{3}) = \frac{16}{27} + \frac{32}{27} + \frac{20}{9} + 8 - 12 = 0$ ✓

$$\begin{array}{r|rrrr|r} -2 & 3 & 4 & 5 & 12 & -12 \\ & 0 & -6 & 4 & -18 & 12 \\ \hline \frac{2}{3} & 3 & -2 & 9 & -6 & 0 \\ & 0 & 2 & 0 & 6 & \\ \hline & 3 & 0 & 9 & 0 & \end{array}$$

Now $P(x) = (x+2)(x-\frac{2}{3})(3x^2+9)$
$= (x+2)(3x-2)(x^2+3)$

$\therefore \; P(x) = (x+2)(3x-2)(x+i\sqrt{3})(x-i\sqrt{3})$

g $P(x) = 2x^4 - 3x^3 + 5x^2 + 6x - 4$

From technology, -1 and $\frac{1}{2}$ are zeros.

Check: $P(-1) = 2 + 3 + 5 - 6 - 4 = 0$ ✓
$P(\frac{1}{2}) = \frac{1}{8} - \frac{3}{8} + \frac{5}{4} + 3 - 4 = 0$ ✓

$$\begin{array}{r|rrrr|r} -1 & 2 & -3 & 5 & 6 & -4 \\ & 0 & -2 & 5 & -10 & 4 \\ \hline \frac{1}{2} & 2 & -5 & 10 & -4 & 0 \\ & 0 & 1 & -2 & 4 & \\ \hline & 2 & -4 & 8 & 0 & \end{array}$$

Now $P(x) = (x+1)(x-\frac{1}{2})(2x^2 - 4x + 8)$
$= (x+1)(2x-1)(x^2 - 2x + 4)$

where the quadratic has zeros of $\dfrac{2 \pm \sqrt{4-16}}{2} = 1 \pm i\sqrt{3}$

$\therefore \; P(x) = (x+1)(2x-1)(x-1+i\sqrt{3})(x-1-i\sqrt{3})$

h $P(x) = 2x^3 + 5x^2 + 8x + 20$

From technology, $-\frac{5}{2}$ is a zero.

Check: $P(-\frac{5}{2}) = -\frac{125}{4} + \frac{125}{4} - 20 + 20 = 0$ ✓

$$\begin{array}{r|rrr|r} -\frac{5}{2} & 2 & 5 & 8 & 20 \\ & 0 & -5 & 0 & -20 \\ \hline & 2 & 0 & 8 & 0 \end{array}$$

Now $P(x) = (x+\frac{5}{2})(2x^2+8)$
$= (2x+5)(x^2+4)$

$\therefore \; P(x) = (2x+5)(x-2i)(x+2i)$

4 **a** Using technology, $x^3 + 2x^2 - 6x - 6$ has zeros of -0.860, 2.13, and -3.27

b Using technology, $x^3 + x^2 - 7x - 8$ has zeros of -2.52, -1.18, and 2.70

5 **a** $f(t) = kt(t-a)^2$

From the graph a is the t-value at the point where the graph touches the t-axis.

$\therefore \; a = 700$ milliseconds

This represents the time when the barrier has returned to its original position.

b when $t = 100$ ms $f(t) = 85$ mm

$\therefore \; 85 = k \times 100(100-700)^2$

$85 = 100 \times k \times 360\,000$

$k = \dfrac{85}{36\,000\,000}$

$\therefore \; f(t) = \dfrac{85t}{36\,000\,000}(t-700)^2$

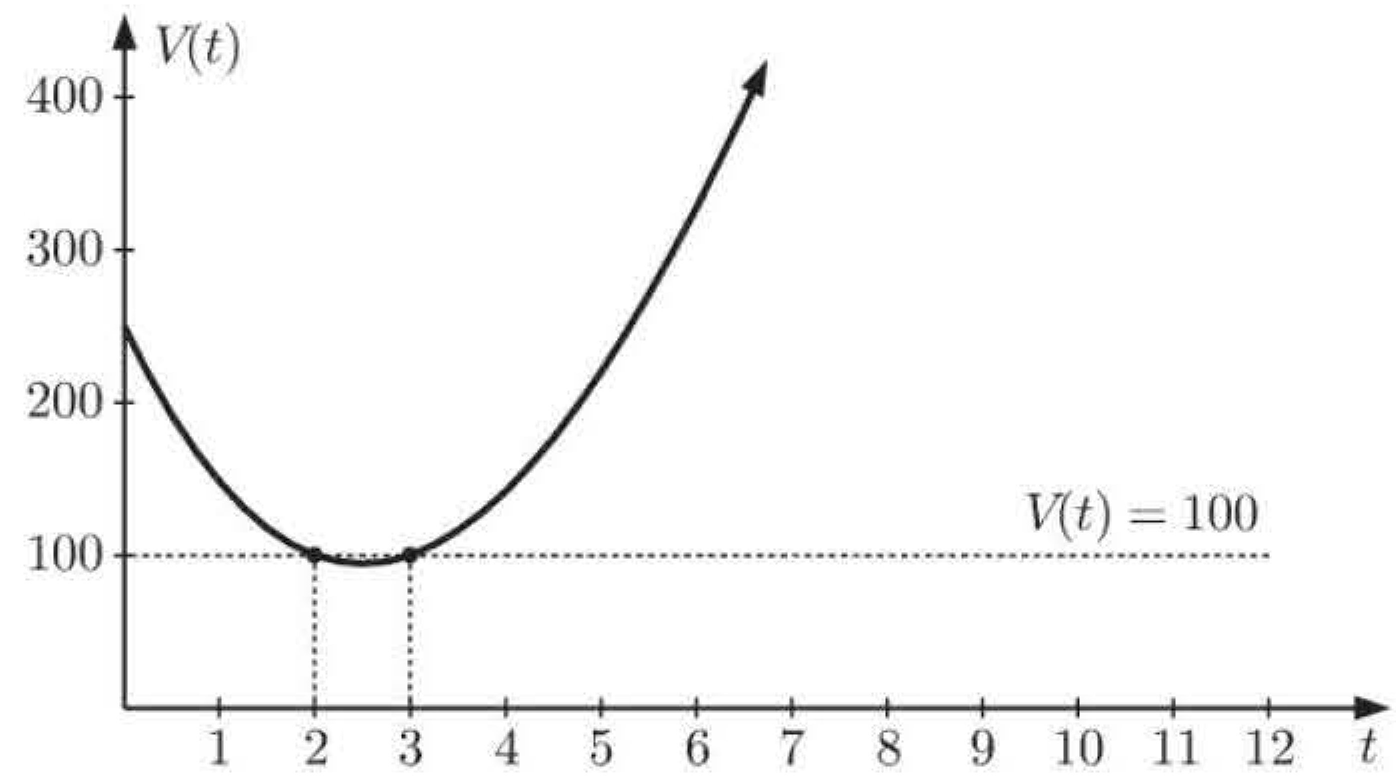
6 $V(t) = -t^3 + 30t^2 - 131t + 250$

We graph $V(t)$ against t and add the graph of $V(t) = 100$.

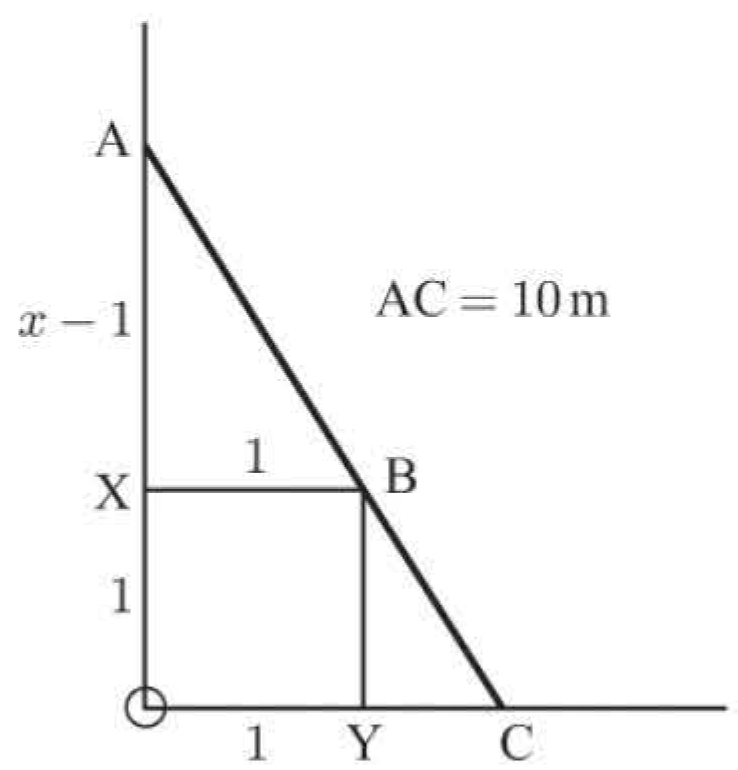
From the graph, the level drops below 100 ML when $t = 2$ and rises above 100 ML again when $t = 3$.

Now if $t = 0$ is Jan 1st, $0 \leqslant t < 1$ is January.

$\therefore$ as irrigation is prohibited for $2 < t < 3$, it is banned during March.



7



Let the height of the wall where the ladder touches be x m.

Using similar triangles AXB, AOC:

$$\frac{x-1}{1} = \frac{x}{\text{OC}}$$

$$\therefore \quad \text{OC} = \frac{x}{x-1}$$

but $x^2 + \text{OC}^2 = 10^2$

$$x^2 + \left(\frac{x}{x-1}\right)^2 = 100$$

Using technology to find the intersection of $y = x^2 + \left(\dfrac{x}{x-1}\right)^2$ and $y = 100$

$x \approx 1.112$ or 9.938

So, distance ≈ 9.938 m or 1.112 m

REVIEW SET 6A

1 **a** $a + bi = 4 = 4 + 0i, \quad \therefore \quad a = 4, \quad b = 0$

b $(1-2i)(a+bi) = -5 - 10i$

$$\therefore \quad a + bi = \frac{-5-10i}{1-2i} \times \frac{1+2i}{1+2i}$$

$$= \frac{-5-10i-10i-20i^2}{1-4i^2}$$

$$= \frac{15-20i}{5}$$

$$= 3 - 4i$$

$\therefore \quad a = 3 \quad b = -4$

c $(a+2i)(1+bi) = 17 - 19i$

$\therefore \quad a + 2i + abi + 2i^2 b = 17 - 19i$

$\therefore \quad (a - 2b) + i(ab + 2) = 17 - 19i$

Equating real and imaginary parts,

$a - 2b = 17$ and $ab + 2 = -19$

$\therefore \quad a = 2b + 17$ $\qquad ab = -21$

$\therefore \quad b(2b+17) = -21$

$\therefore \quad 2b^2 + 17b + 21 = 0$

$\therefore \quad (2b+3)(b+7) = 0$

$\therefore \quad b = -\frac{3}{2}$ or $b = -7$

When $b = -7$, $a = 3$ and when $b = -\frac{3}{2}$, $a = 14$

2 $z = 3 + i, \quad w = -2 - i$

a $2z - 3w$

$= 2(3+i) - 3(-2-i)$

$= 6 + 2i + 6 + 3i$

$= 12 + 5i$

b $\dfrac{z^*}{w}$

$$= \frac{3-i}{-2-i} \times \frac{-2+i}{-2+i}$$

$$= \frac{-6+2i+3i-i^2}{4-i^2}$$

$$= \frac{-5+5i}{5}$$

$$= -1 + i$$

c z^3

$= (3+i)^3$

$= 3^3 + 3(3^2)(i) + 3(3)(i^2) + i^3$

$= 27 + 27i + 9i^2 - i$

$= 27 - 9 + 26i$

$= 18 + 26i$

3
$$\begin{aligned} z &= \frac{3}{i+\sqrt{3}} + \sqrt{3} \\ &= \frac{3}{i+\sqrt{3}}\,\frac{(i-\sqrt{3})}{(i-\sqrt{3})} + \sqrt{3} \\ &= \frac{3i-3\sqrt{3}}{i^2-3} + \sqrt{3} \\ &= \frac{3i-3\sqrt{3}}{-4} - \frac{4\sqrt{3}}{-4} \\ &= \frac{3i-7\sqrt{3}}{-4} \end{aligned}$$
$\therefore\ \mathcal{R}e\,(z) = \frac{7\sqrt{3}}{4},\quad \mathcal{I}m\,(z) = -\frac{3}{4}$

4
$$\begin{aligned} 2z-1 &= iz-i \\ \therefore\ 2(a+bi)-1 &= i(a+bi)-i \\ \therefore\ 2a+2bi-1 &= ai+bi^2-i \\ \therefore\ (2a-1)+2bi &= -b+i(a-1) \end{aligned}$$
Equating real and imaginary parts,
$2a-1=-b \quad$ and $\quad 2b=a-1$
$\therefore\ b=1-2a \quad$ and $\quad 2b=a-1$
$\therefore\ 2(1-2a)=a-1$
$\therefore\ 2-4a=a-1$
$\therefore\ 3=5a$
$\therefore\ a=\frac{3}{5}$
and $b=1-2(\frac{3}{5})=-\frac{1}{5}$
$\therefore\ z=\frac{3}{5}-\frac{1}{5}i$

5 Let $z=a+bi,\quad w=c+di$
$$\begin{aligned} \therefore\ zw^*-z^*w &= (a+bi)(c-di)-(a-bi)(c+di) \\ &= ac-adi+bci-bdi^2-ac-adi+bci+bdi^2 \\ &= 2bci-2adi \\ &= 2i(bc-ad) \end{aligned}$$
which is purely imaginary if $bc-ad\neq 0$ and zero if $bc-ad=0$.

6
$$\begin{aligned} w &= \frac{z+1}{z^*+1} \\ &= \frac{a+1+bi}{a+1-bi}\times\frac{(a+1+bi)}{(a+1+bi)} \\ &= \frac{(a+1)^2+2(a+1)bi+b^2i^2}{(a+1)^2-b^2i^2} \\ &= \frac{(a+1)^2+2(a+1)bi-b^2}{(a+1)^2+b^2} \\ &= \frac{(a+1)^2-b^2}{(a+1)^2+b^2}+i\left(\frac{2(a+1)b}{(a+1)^2+b^2}\right) \end{aligned}$$
w is purely imaginary when
$(a+1)^2-b^2=0$ and $2(a+1)b\neq 0$
$\therefore\ b^2=(a+1)^2$ and $a\neq -1,\ b\neq 0$
$\therefore\ b=\pm(a+1),\quad a\neq -1,\ b\neq 0$

7 **a**
$$\begin{aligned} &(3x^3+2x-5)(4x-3) \\ &= 12x^4-9x^3+8x^2-6x-20x+15 \\ &= 12x^4-9x^3+8x^2-26x+15 \end{aligned}$$

b $(2x^2-x+3)^2=4x^4-4x^3+13x^2-6x+9$

$$\begin{array}{rrrrr} & & 2 & -1 & 3 \\ & \times & 2 & -1 & 3 \\ \hline & & 6 & -3 & 9 \\ & -2 & 1 & -3 & \\ 4 & -2 & 6 & & \\ \hline 4 & -4 & 13 & -6 & 9 \\ \hline \end{array}$$

8 **a**
$$\begin{array}{r} x^2-2x+4 \\ x+2\overline{\big)\,x^3+0x^2+0x+0} \\ \underline{-(x^3+2x^2)}\quad\quad\quad \\ -2x^2+0x\quad\ \\ \underline{-(-2x^2-4x)}\quad\ \\ 4x+0 \\ \underline{-(4x+8)} \\ -8 \end{array}$$
$\therefore\ \dfrac{x^3}{x+2}=x^2-2x+4-\dfrac{8}{x+2}$

b $(x+2)(x+3)=x^2+5x+6$
$$\begin{array}{r} x-5 \\ x^2+5x+6\overline{\big)\,x^3+0x^2+\ 0x+0} \\ \underline{-(x^3+5x^2+\ 6x)}\quad \\ -5x^2-\ 6x+0 \\ \underline{-(-5x^2-25x-30)} \\ 19x+30 \end{array}$$
$\therefore\ \dfrac{x^3}{(x+2)(x+3)}=x-5+\dfrac{19x+30}{(x+2)(x+3)}$

9 **a** $3x^4-4x^3+3x^2+8$

$\therefore$ the sum of the zeros is $-\dfrac{-4}{3}=\frac{4}{3}$

The polynomial has degree 4.

$\therefore$ the product of the zeros is $\dfrac{(-1)^4 8}{3}=\frac{8}{3}$

b $2x^6+2x^4-x^3+7x-10$

$=2x^6+(0)x^5+2x^4-x^3+7x-10$

$\therefore$ the sum of the zeros is $-\dfrac{0}{2}=0$

The polynomial has degree 6.

$\therefore$ the product of the zeros is $\dfrac{(-1)^6(-10)}{2}=-5$

10 The Remainder theorem:

"When a polynomial $P(x)$ is divided by $x-k$ until a constant remainder R is obtained then $R=P(k)$."

Proof: From the division process, $P(x)=(x-k)Q(x)+R$

Now, letting $x=k$, $P(k)=(k-k)\times Q(k)+R$

$$\therefore\ P(k)=0\times Q(k)+R$$
$$\therefore\ P(k)=R$$
$$\therefore\ R=P(k)$$

11 Let $P(z)=z^2+az+(3+a)$

if $-2+bi$ is a zero then $P(-2+bi)=0$

$$\therefore\ (-2+bi)^2+a(-2+bi)+3+a=0$$
$$4-4bi+b^2i^2-2a+abi+3+a=0$$
$$(4-b^2-2a+3+a)+i(-4b+ab)=0$$

$\therefore$ $4-b^2-2a+3+a=0$ and $-4b+ab=0$

$a=7-b^2$ $\therefore$ $b(a-4)=0$

$\therefore$ $b=0$ or $a=4$

If $b=0$ then $a=7-0=7$.

If $a=4$ then $b^2=3$ and so $b=\pm\sqrt{3}$.

12 As $(x-3)(x+2)=x^2-x-6$,

$P(x)=Q(x)(x^2-x-6)+(Ax+B)$, where $Q(x)$ is the quotient and $Ax+B$ is the remainder.

Now $P(x)$ has remainder 2 when divided by $x-3$ and so $P(3)=2$ {Remainder theorem}

$\therefore$ $Q(3)(9-3-6)+(3A+B)=2$

$\therefore$ $3A+B=2$ .... (1)

Also $P(x)$ has remainder -13 when divided by $x+2$ and so $P(-2)=-13$

$\therefore$ $Q(-2)(4+2-6)+(-2A+B)=-13$

$\therefore$ $-2A+B=-13$ .... (2)

Solving (1) and (2): $5A=15$

$\therefore$ $A=3$ and $B=-7$

So, the remainder is $3x-7$.

13 $2-i\sqrt{3}$ and $\sqrt{2}+1$

Since the quartic has real rational coefficients, $2-i\sqrt{3}$ and $2+i\sqrt{3}$ are zeros

$\sqrt{2}+1$ and $-\sqrt{2}+1$ are zeros

$\therefore$ the four zeros are: $2\pm i\sqrt{3}$ and $\pm\sqrt{2}+1$

For $2\pm i\sqrt{3}$, $\alpha+\beta=4$

$\alpha\beta=4-3i^2=7$

For $\pm\sqrt{2}+1$, $\alpha+\beta=2$

$\alpha\beta=-2+1=-1$

$\therefore$ $P(z)=(z^2-4z+7)(z^2-2z-1)$

$\therefore$ $P(z)=z^4-6z^3+14z^2-10z-7$

$$\begin{array}{rrrrr}
 & & 1 & -4 & 7 \\
 & \times & 1 & -2 & -1 \\
\hline
 & & -1 & 4 & -7 \\
 & -2 & 8 & -14 & \\
1 & -4 & 7 & & \\
\hline
1 & -6 & 14 & -10 & -7 \\
\hline
\end{array}$$

14 $f(x) = x^3 - 3x^2 - 9x + b$ .... (1)

$= (x-k)^2(x+a)$

$= (x^2 - 2kx + k^2)(x+a)$

$= x^3 + (a-2k)x^2 + (k^2 - 2ak)x + ak^2$ .... (2)

	1	$-2k$	k^2
	$\times$	1	a
	a	$-2ak$	ak^2
1	$-2k$	k^2	
1	$a-2k$	k^2-2ak	ak^2

Equating coefficients of (1) and (2) gives

$\therefore\ a - 2k = -3,\ \ k^2 - 2ak = -9,$ and $ak^2 = b$

Since $a = 2k - 3$, then as $k^2 - 2ak = -9$

$$k^2 - 2k(2k-3) = -9$$
$$k^2 - 4k^2 + 6k = -9$$
$$\therefore\ 3k^2 - 6k - 9 = 0$$
$$k^2 - 2k - 3 = 0$$
$$(k-3)(k+1) = 0$$
$$\therefore\ k = -1 \text{ or } k = 3$$

If $k = -1$, $a = -5$ and $b = ak^2 = -5$ and so $f(x) = (x+1)^2(x-5)$

and the roots of $f(x) = 0$ are $-1, 5$

If $k = 3$, $a = 3$ and $b = ak^2 = 3 \times 9 = 27$ and so $f(x) = (x-3)^2(x+3)$

and the roots of $f(x) = 0$ are $3, -3$

15 Since the coefficients are rational, $3 + i\sqrt{2}$ and $1 + \sqrt{2}$ also have to be zeros.

For zeros of $3 \pm i\sqrt{2}$, $\alpha + \beta = 6$ and $\alpha\beta = 9 - 2i^2 = 11$

For zeros of $1 \pm \sqrt{2}$, $\alpha + \beta = 2$ and $\alpha\beta = 1 - 2 = -1$

$\therefore\ P(x) = a(x^2 - 6x + 11)(x^2 - 2x - 1),\ a \neq 0$

$\therefore\ P(x) = a(x^4 - 8x^3 + 22x^2 - 16x - 11),\ a \neq 0$

		1	-6	11
	$\times$	1	-2	-1
		-1	6	-11
	-2	12	-22	
1	-6	11		
1	-8	22	-16	-11

16 $P(x) = x^n + 3x^2 + kx + 6$

$P(-1) = 12$ {Remainder theorem} $\therefore\ 12 = (-1)^n + 3 - k + 6$

$\therefore\ k = (-1)^n - 3$ .... (1)

$P(1) = 8$ {Remainder theorem} $\therefore\ 8 = 1^n + 3 + k + 6$

$\therefore\ 8 = 10 + k$

$\therefore\ k = -2$

$\therefore$ (1) becomes $-2 = (-1)^n - 3$

$\therefore\ (-1)^n = 1$ $\quad\therefore\ n$ is even

If $34 < n < 38$, then $n = 36$.

17 Let $x^3 - x + 1 = (x-\alpha)(x-\beta)(x-\gamma)$

$= (x^2 - [\alpha+\beta]x + \alpha\beta)(x-\gamma)$

$= x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta + \beta\gamma + \alpha\gamma) - \alpha\beta\gamma$

$\alpha + \beta + \gamma = 0$

$\alpha\beta + \beta\gamma + \alpha\gamma = -1$

$\therefore\ \alpha\beta\gamma = -1$

Now $\gamma = \dfrac{-1}{\alpha\beta}$ .... (1) and $\alpha\beta + \gamma(\alpha+\beta) = -1$ .... (2)

$$\therefore\ \alpha\beta + \gamma(-\gamma) = -1$$
$$\therefore\ \alpha\beta - \gamma^2 = -1$$
$$\therefore\ \alpha\beta - \frac{1}{(\alpha\beta)^2} = -1 \quad \text{\{using (1)\}}$$
$$\therefore\ (\alpha\beta)^3 - 1 = -(\alpha\beta)^2$$
$$\therefore\ (\alpha\beta)^3 + (\alpha\beta)^2 - 1 = 0$$

$\therefore\ \alpha\beta$ is a root of $x^3 + x^2 - 1 = 0$

REVIEW SET 6B

1 $\dfrac{5}{2-i}=\dfrac{5}{2-i}\dfrac{(2+i)}{(2+i)}$
$=\dfrac{10+5i}{4-i^2}$
$=\dfrac{10+5i}{5}=2+i$

$\therefore\ \sqrt[3]{z}=(2+i)-3-2i$
$=-1-i$

$\therefore\ z=(-1-i)^3$
$=-(i+1)^3$
$=-(i^3+3i^2+3i+1)$
$=-(-i-3+3i+1)$
$=-(2i-2)$
$=2-2i$
$\therefore\ x=2,\ y=-2$

2 Let $z=a+bi=\sqrt{5-12i},\quad a, b$ real
$\therefore\ (a+bi)^2=5-12i$
$\therefore\ a^2+2abi+b^2i^2=5-12i$
$\therefore\ a^2-b^2+2abi=5-12i$

Equating real and imaginary parts,
$a^2-b^2=5$ and $2ab=-12$
$\therefore\ a^2-b^2=5$ and $b=-\dfrac{6}{a}$

$\therefore\ a^2-\left(-\dfrac{6}{a}\right)^2=5$
$\therefore\ a^2-\dfrac{36}{a^2}=5$
$\therefore\ a^4-36=5a^2$
$\therefore\ a^4-5a^2-36=0$
$\therefore\ (a^2-9)(a^2+4)=0$
$\therefore\ a=\pm 3\ \ (a \text{ real})$
using $b=-\dfrac{6}{a}$, if $a=3,\ b=-2$
and if $a=-3,\ b=2$
$\therefore\ z=3-2i$ or $z=-3+2i$

3 If $z=a+bi$
then $z+z^*=(a+bi)+(a-bi)$
$=2a$ which is real.

Also, $zz^*=(a+bi)(a-bi)$
$=a^2+abi-abi-b^2i^2$
$=a^2+b^2$ which is real.

4 $z=4+i\qquad w=3-2i\qquad 2w^*-iz=2(3+2i)-i(4+i)$
$=6+4i-4i-i^2$
$=7$

5 If $\dfrac{2-3i}{2a+bi}=3+2i$ then $2a+bi=\dfrac{2-3i}{3+2i}\times\dfrac{3-2i}{3-2i}$
$\therefore\ 2a+bi=\dfrac{6-4i-9i+6i^2}{9-4i^2}=\dfrac{-13i}{13}$
$\therefore\ 2a+bi=0-i$
$\therefore\ a=0$ and $b=-1$

6 If $a+ai$ is a root of $x^2-6x+b=0$ then $(a+ai)^2-6(a+ai)+b=0$
$\therefore\ a^2+2a^2i-a^2-6a-6ai+b=0$
$\therefore\ (b-6a)+(2a^2-6a)i=0$
$\therefore\ b=6a$ and $2a^2-6a=0$
$\therefore\ b=6a$ and $2a(a-3)=0$
$\therefore\ a=0$ or 3, and when $a=0,\ b=0$
and when $a=3,\ b=18$

7 Let $P(x)=x^{47}-3x^{26}+5x^3+11$
$\therefore\ R=P(-1)$ {Remainder theorem}
$=(-1)^{47}-3(-1)^{26}+5(-1)^3+11$
$=-1-3-5+11$
$\therefore$ remainder $=2$

8 $P(z) = 2z^3 + z^2 + 10z + 5$

Using technology, $-\frac{1}{2}$ is a zero.

Check: $P(-\frac{1}{2}) = -\frac{1}{4} + \frac{1}{4} - 5 + 5 = 0$ ✓

$\therefore\ P(z) = (z + \frac{1}{2})(2z^2 + 10)$

$P(z) = (2z + 1)(z^2 + 5)$

$P(z) = (2z + 1)(z - i\sqrt{5})(z + i\sqrt{5})$

$$\begin{array}{r|rrr|r} -\frac{1}{2} & 2 & 1 & 10 & 5 \\ & 0 & -1 & 0 & -5 \\ \hline & 2 & 0 & 10 & 0 \end{array}$$

9 Touches the x-axis at $(-2, 0)$ and cuts it at $(1, 0)$.

$\therefore\ P(x) = (x + 2)^2(x - 1)(ax + b)$

But $P(0) = 12 \quad \therefore\ 4(-1)b = 12 \quad \therefore\ b = -3$

$\therefore\ P(x) = (x + 2)^2(x - 1)(ax - 3)$

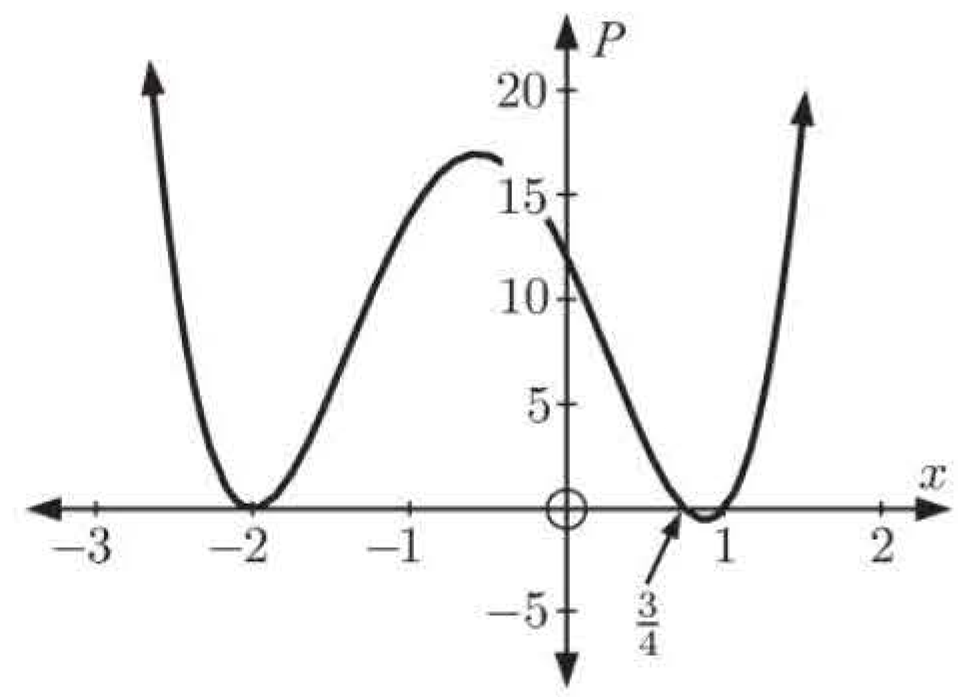
Also $P(2) = 80 \quad \therefore\ 80 = 16(1)(2a - 3)$

$\therefore\ 2a - 3 = 5$

$\therefore\ 2a = 8$

$\therefore\ a = 4$

$\therefore\ P(x) = (x + 2)^2(x - 1)(4x - 3)$



10 Let $P(z) = 2z^4 - 5z^3 + 13z^2 - 4z - 6$

From technology, 1 and $-\frac{1}{2}$ are zeros.

Check: $P(1) = 2 - 5 + 13 - 4 - 6 = 0$ ✓

$P(-\frac{1}{2}) = \frac{1}{8} + \frac{5}{8} + \frac{13}{4} + 2 - 6 = 0$ ✓

$\therefore\ P(z) = (z - 1)(z + \frac{1}{2})(2z^2 - 4z + 12)$

$= (z - 1)(2z + 1)(z^2 - 2z + 6)$

$$\begin{array}{r|rrrr|r} 1 & 2 & -5 & 13 & -4 & -6 \\ & 0 & 2 & -3 & 10 & 6 \\ \hline -\frac{1}{2} & 2 & -3 & 10 & 6 & 0 \\ & 0 & -1 & 2 & -6 & \\ \hline & 2 & -4 & 12 & 0 & \end{array}$$

where the quadratic has zeros of $\dfrac{2 \pm \sqrt{4 - 24}}{2} = 1 \pm i\sqrt{5}$. $\therefore$ zeros are $1, -\frac{1}{2}, 1 \pm i\sqrt{5}$.

11 Let $P(z) = z^4 + 2z^3 - 2z^2 + 8$

From technology, -2 seems to be a double zero.

Check: $P(-2) = 16 - 16 - 8 + 8 = 0$ ✓

$\therefore\ P(z) = (z + 2)^2(z^2 - 2z + 2)$

where the quadratic has zeros of $\dfrac{2 \pm \sqrt{4 - 8}}{2} = 1 \pm i$

$\therefore\ P(z) = (z + 2)^2(z - 1 + i)(z - 1 - i)$

$$\begin{array}{r|rrrr|r} -2 & 1 & 2 & -2 & 0 & 8 \\ & 0 & -2 & 0 & 4 & -8 \\ \hline -2 & 1 & 0 & -2 & 4 & 0 \\ & 0 & -2 & 4 & -4 & \\ \hline & 1 & -2 & 2 & 0 & \end{array}$$

12 Zeros are $2 + i$ and $-1 + 3i$.

Since we have a real polynomial, the other zeros are $2 - i$ and $-1 - 3i$.

For zeros of $2 \pm i$, $\alpha + \beta = 4$ and $\alpha\beta = 4 - i^2 = 5$

For zeros of $-1 \pm 3i$, $\alpha + \beta = -2$ and $\alpha\beta = 1 - 9i^2 = 10$

$\therefore\ P(z) = a(z^2 - 4z + 5)(z^2 + 2z + 10), \quad a \neq 0$

13 Since $3 - 2i$ is a zero, so is $3 + 2i$. These have $\alpha + \beta = 6$ and $\alpha\beta = 9 - 4i^2 = 13$.

$\therefore\ z^2 - 6z + 13$ is a factor

$\therefore\ P(z) = (z^2 - 6z + 13)(z^2 + Az + B)$

$= z^4 + kz^3 + 32z + 3k - 1$

Equating coefficients gives

$$\begin{cases} A - 6 = k \\ B - 6A + 13 = 0 \\ 13A - 6B = 32 \\ 3k - 1 = 13B \end{cases}$$

$$\begin{array}{ccccc} & & 1 & -6 & 13 \\ & \times & 1 & A & B \\ \hline & & B & -6B & 13B \\ & A & -6A & 13A & \\ 1 & -6 & 13 & & \\ \hline 1 & A-6 & B-6A+13 & 13A-6B & 13B \\ \hline \end{array}$$

$\therefore \quad -6A + B = -13$ .... (1)
and $\quad 13A - 6B = 32$ .... (2)
(1) $\times$ 6 gives $\quad -36A + 6B = -78$ .... (3)
Adding (2) and (3) gives: $\quad -23A = -46 \quad \therefore \quad A = 2$
$\therefore \quad k = A - 6 = 2 - 6 = -4$ and $B = 6A - 13 = 12 - 13 = -1$

$$\therefore \quad P(z) = (z^2 - 6z + 13)\underbrace{(z^2 + 2z - 1)}_{\text{this quadratic has zeros } \frac{-2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2}}$$

$\therefore \quad k = -4,$ zeros are $3 \pm 2i, \quad -1 \pm \sqrt{2}$

14 Consider $P(z) = z^4 + 2z^3 + 6z^2 + 8z + 8$.
If one zero is purely imaginary, for example, ai, then $-ai$ is also a zero (a is real).
$\pm ai$ have $\alpha + \beta = 0$ and $\alpha\beta = -a^2 i^2 = a^2$
$\therefore \quad z^2 + a^2$ is a factor, $\therefore \quad z^2 + A$ is a factor
$\therefore \quad P(z) = (z^2 + A)(z^2 + Bz + C)$
$\qquad\qquad = z^4 + Bz^3 + (A + C)z^2 + ABz + AC$

		1	B	C
	$\times$	1	0	A
		A	AB	AC
1	B	C		
1	B	$A+C$	AB	AC

Equating coefficients gives:
$B = 2, \quad A + C = 6, \quad AB = 8$ and $AC = 8$
$\therefore \quad B = 2 \qquad \therefore \quad A = 4 \qquad \therefore \quad C = 2$

$$\therefore \quad P(z) = (z^2 + 4)\underbrace{(z^2 + 2z + 2)}_{\text{this quadratic has zeros } \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i}$$

$\therefore \quad$ zeros are $\pm 2i, \quad -1 \pm i$

15 $\dfrac{P(x)}{x^2 - 3x + 2} = Q(x) + \dfrac{2x + 3}{x^2 - 3x + 2}$

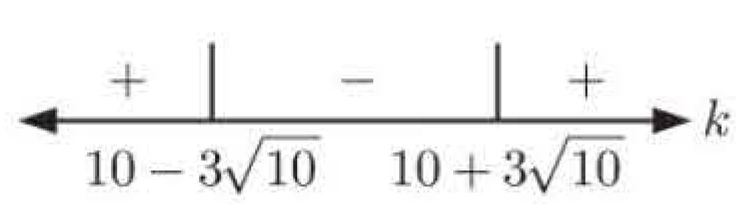
$\therefore \quad P(x) = Q(x)(x^2 - 3x + 2) + 2x + 3$
and $\quad P(2) = Q(2)(4 - 6 + 2) + 4 + 3$
$\qquad\qquad = Q(2) \times (0) + 7$
$\qquad\qquad = 7,$ so the remainder is 7.

16 They meet where $\quad x^2 + (2x + k)^2 + 8x - 4(2x + k) + 2 = 0$
$\therefore \quad x^2 + 4x^2 + 4kx + k^2 + 8x - 8x - 4k + 2 = 0$
$\therefore \quad 5x^2 + 4kx + (k^2 - 4k + 2) = 0$

Now $\quad \Delta = (4k)^2 - 4(5)(k^2 - 4k + 2)$
$\qquad = 16k^2 - 20k^2 + 80k - 40$
$\qquad = -4k^2 + 80k - 40$

and $\Delta < 0$ when $-4k^2 + 80k - 40 < 0$
$\therefore \quad k^2 - 20k + 10 > 0$
$\therefore \quad k^2 - 20k + 10^2 + 10 - 10^2 > 0$
$\therefore \quad (k - 10)^2 - 90 > 0$
$\therefore \quad (k - 10 + 3\sqrt{10})(k - 10 - 3\sqrt{10}) > 0$

$\therefore \quad k \in \,]-\infty, 10 - 3\sqrt{10}[\;$ or $\; k \in \,]10 + 3\sqrt{10}, \infty[$



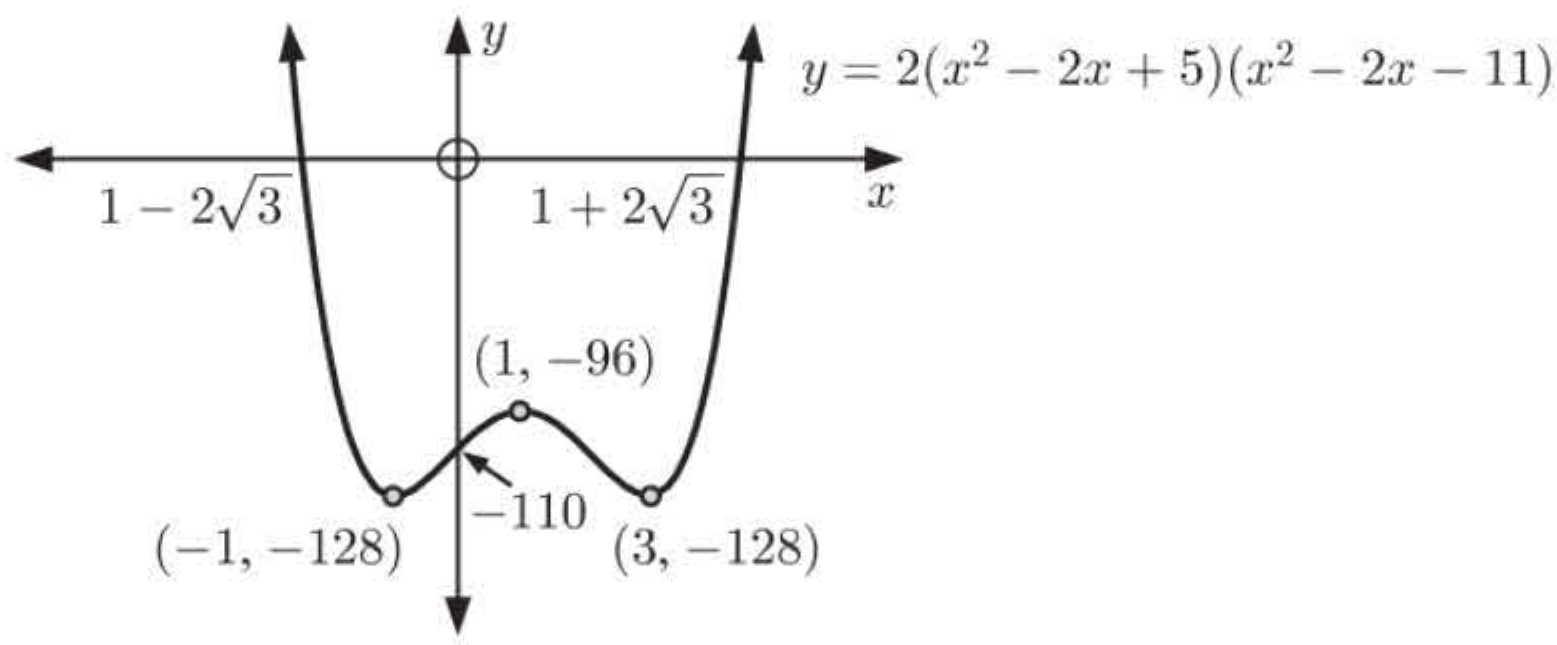
17 **a** $P(x) = 2x^4 - 8x^3 + ax^2 + bx - 110$ has zeros $m \pm 2i$ and $1 \pm n\sqrt{3}$.
$\therefore \quad$ the sum of the zeros is $\quad (m + 2i) + (m - 2i) + (1 + n\sqrt{3}) + (1 - n\sqrt{3}) = -\frac{-8}{2}$
$\therefore \quad 2m + 2 = 4$
$\therefore \quad 2m = 2$
$\therefore \quad m = 1$

$\therefore$ the product of the zeros is $(1+2i)(1-2i)(1+n\sqrt{3})(1-n\sqrt{3}) = \dfrac{(-1)^4(-110)}{2}$

$$\begin{aligned}\therefore\ (1+4)(1-3n^2) &= -55\\ \therefore\ 5(1-3n^2) &= -55\\ \therefore\ 1-3n^2 &= -11\\ \therefore\ 3n^2 &= 12\\ \therefore\ n^2 &= 4\\ \therefore\ n &= \pm 2\end{aligned}$$

So, $m=1$ and $n=\pm 2$.

b $P(x) = 2(x-1+2i)(x-1-2i)(x-1+2\sqrt{3})(x-1-2\sqrt{3})$
$= 2(x^2-2x+5)(x^2-2x-11)$



REVIEW SET 6C

1 $(3x+2yi)(1-i) = (3y+1)i - x$

$\therefore\ 3x - 3xi + 2yi - 2yi^2 = 3yi + i - x$

$\therefore\ (3x+2y) + i(2y-3x) = -x + i(3y+1)$

Equating real and imaginary parts, $3x+2y=-x$ and $2y-3x=3y+1$

$\therefore\ 4x=-2y$ and $-1-3x=y$

$\therefore\ 4x = -2(-1-3x)$

$\therefore\ 4x = 2+6x$

$\therefore\ -2x = 2$

$\therefore\ x=-1$ and $y=-1-3(-1)=2$

2 $z^2 + iz + 10 = 6z$

$\therefore\ z^2 + (i-6)z + 10 = 0$

$$\therefore\ z = \frac{6-i\pm\sqrt{(i-6)^2-4(10)}}{2} = \frac{6-i\pm\sqrt{i^2-12i+36-40}}{2} = \frac{6-i\pm\sqrt{-5-12i}}{2}$$

We must now find $\sqrt{-5-12i}$

Let $z = a+bi = \sqrt{-5-12i}$, a, b real

$\therefore\ (a+bi)^2 = -5-12i$

$\therefore\ a^2 + 2abi + b^2i^2 = -5-12i$

$\therefore\ a^2 - b^2 + 2abi = -5-12i$

Equating real and imaginary parts,

$a^2-b^2=-5$ and $2ab=-12$

$\therefore\ a^2-b^2=-5$ and $b=-\dfrac{6}{a}$

$\therefore\ a^2 - \left(-\dfrac{6}{a}\right)^2 = -5$

$\therefore\ a^2 - \dfrac{36}{a^2} = -5$

$\therefore\ a^4 + 5a^2 - 36 = 0$

$\therefore\ (a^2+9)(a^2-4) = 0$

$\therefore\ a = 2$ (a real, $a>0$)

and $b = \frac{-6}{2} = -3$

$\therefore\ \sqrt{-5-12i} = 2-3i$

so $z = \dfrac{6-i\pm(2-3i)}{2}$

$\therefore\ z = \dfrac{8-4i}{2}$ or $\dfrac{4+2i}{2}$

$\therefore\ z = 4-2i$ or $2+i$

3 Let $z = a + bi$ and $w = c + di$

$\therefore$ $zw^* + z^*w$

$= (a+bi)(c-di) + (a-bi)(c+di)$

$= ac - adi + bci - bdi^2 + ac + adi - bci - bdi^2$

$= ac + bd + ac + bd$

$= 2ac + 2bd$ which is a real number

4 **a** If $x + iy = 0$ then $x = 0$ and $y = 0$ {equating real and imaginary parts}

b
$$(3-2i)(x+i) = 17 + yi$$
$$3x + 3i - 2xi - 2i^2 = 17 + yi$$
$$(3x+2) + i(3-2x) = 17 + yi$$
$\therefore$ $3x + 2 = 17$ and $y = 3 - 2x$

$\therefore$ $3x = 15$

$\therefore$ $x = 5$

and so $\therefore$ $y = 3 - 10$

$\therefore$ $y = -7$

c
$$(x+iy)^2 = x - iy$$
$$x^2 + i^2y^2 + 2xyi = x - iy$$
$$x^2 - y^2 = x \quad \text{and} \quad 2xy = -y$$
Now if $2xy + y = 0$

then $y(2x+1) = 0$

$\therefore$ $y = 0$ or $x = -\frac{1}{2}$

If $y = 0$, then $x^2 = x$ and so $x = 0$ or 1

If $x = -\frac{1}{2}$, then $\frac{1}{4} - y^2 = -\frac{1}{2}$,

$\therefore$ $y^2 = \frac{3}{4}$

and so $y = \pm\frac{\sqrt{3}}{2}$

Possible solutions are:

x	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$
y	0	0	$\frac{\sqrt{3}}{2}$	$\frac{-\sqrt{3}}{2}$

5 Let $z = a + bi$ and $w = c + di$ where $b \neq 0$ and $d \neq 0$ .... (1)

Now $z + w = (a+c) + (b+d)i$ and $zw = (a+bi)(c+di)$

$= (ac - bd) + i(bc + ad)$

As $z + w$ is real, $b + d = 0$ $\therefore$ $b = -d$ .... (2)

As zw is real, $bc + ad = 0$ .... (3)

Substituting (2) into (3) $-dc + ad = 0$

$\therefore$ $d(a-c) = 0$

Since $d \neq 0$ {from (1)}

$\therefore$ $a = c$ and $b = -d$

$\therefore$ $z^* = a - bi = c + di = w$

6 **a** $2x^3 + 3x^2 - 4x + 6 = 0$

$\therefore$ the sum of the roots is $-\frac{3}{2}$

The polynomial equation has degree 3.

$\therefore$ the product of the roots is $\dfrac{(-1)^3 6}{2} = -3$

b $4x^4 = x^2 + 2x - 6$

$\therefore$ $4x^4 + (0)x^3 - x^2 - 2x + 6 = 0$

$\therefore$ the sum of the roots is $-\frac{0}{4} = 0$

The polynomial equation has degree 4.

$\therefore$ the product of the roots is $\dfrac{(-1)^4 6}{4} = \frac{3}{2}$

7
$$\sqrt{z} = \frac{2}{3-2i} + 2 + 5i$$
$$= \frac{2}{3-2i} \times \frac{3+2i}{3+2i} + 2 + 5i$$
$$= \frac{6+4i}{9-4i^2} + 2 + 5i$$
$$= \frac{6+4i}{13} + 2 + 5i$$
$$= \tfrac{32}{13} + \tfrac{69}{13}i$$

$\therefore$
$$z = \left(\tfrac{32}{13} + \tfrac{69}{13}i\right)^2$$
$$= \frac{32^2 - 69^2}{169} + \frac{2 \times 32 \times 69}{169}i$$
$$= -\tfrac{3737}{169} + \tfrac{4416}{169}i$$

8 Let $P(x) = 2x^{17} + 5x^{10} - 7x^3 + 6$

Now $R = P(2)$ {Remainder theorem}

$= 2^{18} + 5 \times 2^{10} - 7 \times 2^3 + 6$

$= 267\,214$

9 Let $P(z) = 2z^3 + az^2 + 62z + [a - 5]$

If $5 - i$ is a zero, then so is $5 + i$

{as $P(z)$ is a real polynomial}

and for these zeros $\alpha + \beta = 10$

$\alpha\beta = 25 - i^2 = 26$

$\therefore\ P(z) = (z^2 - 10z + 26)(2z + b)$

	1	-10	26
	$\times$	2	b
	b	$-10b$	$26b$
2	-20	52	
2	$b - 20$	$52 - 10b$	$26b$

Equating coefficients gives $a = b - 20$, $52 - 10b = 62$, and $a - 5 = 26b$.

From $52 - 10b = 62$,

$-10b = 10$

$\therefore\ b = -1$, and since $a = b - 20$ we find $a = -21$.

Check: $a - 5 = -26$ and $26b = -26$ ✓

$\therefore\ P(z) = 2z^3 - 21z^2 + 62z - 26$

$= (z^2 - 10z + 26)(2z - 1)$

$\therefore$ other two zeros are $\frac{1}{2}$, $5 + i$, and $a = -21$.

10 **a** Two zeros are $i\sqrt{2}$ and $\frac{1}{2}$, so another zero must be $-i\sqrt{2}$

Now $\pm i\sqrt{2}$ have $\alpha + \beta = 0$ and $\alpha\beta = -2i^2 = 2$

$\therefore\ x^2 + 2$ is a factor

$\therefore\ P(x) = a(2x - 1)(x^2 + 2),\ a \neq 0$

b Two zeros are $1 - i$ and $-3 - i$

$\therefore$ the other zeros must be $1 + i$ and $-3 + i$

$1 \pm i$ have $\alpha + \beta = 2$ and $-3 \pm i$ have $\alpha + \beta = -6$

and $\alpha\beta = 1 - i^2 = 2$ and $\alpha\beta = 9 - i^2 = 10$

$\therefore\ P(x) = a(x^2 - 2x + 2)(x^2 + 6x + 10),\ a \neq 0$

11 **a** $P(x) = 2x^3 + 7x^2 + kx - k$ .... (1)

$= (x + a)^2(2x + b)$ .... (2)

$= (x^2 + 2ax + a^2)(2x + b)$

$= 2x^3 + (b + 4a)x^2 + (2ab + 2a^2)x + a^2b$ .... (3)

	1	$2a$	a^2
	$\times$	2	b
	b	$2ab$	a^2b
2	$4a$	$2a^2$	
2	$b + 4a$	$2ab + 2a^2$	a^2b

Equating coefficients gives in (1) and (3):

$b + 4a = 7$, $2ab + 2a^2 = k$ and $a^2b = -k$

$\therefore\ 2ab + 2a^2 = -a^2b$ {equating ks}

$\therefore\ 2a(7 - 4a) + 2a^2 = -a^2(7 - 4a)$

$\therefore\ 14a - 8a^2 + 2a^2 + 7a^2 - 4a^3 = 0$

$\therefore\ 4a^3 - a^2 - 14a = 0$

$\therefore\ a(4a^2 - a - 14) = 0$

$\therefore\ a(4a + 7)(a - 2) = 0$

$\therefore\ a = 0,\ -\frac{7}{4}$, or 2

If $a = 0$, $b = 7$ and $k = 0$.

If $a = 2$, $b = -1$ and $k = 4$.

If $a = -\frac{7}{4}$, $b = 14$ and $k = -\frac{343}{8}$.

b Largest value of k is $k = 4$ and $P(x) = (x + 2)^2(2x - 1)$. {substituting into (2)}

12 $$2z^4 - 3z^3 + 2z^2 = 6z + 4$$
$\therefore\ 2z^4 - 3z^3 + 2z^2 - 6z - 4 = 0$

From technology, 2 and $-\frac{1}{2}$ are solutions.

Check: $P(2) = 32 - 24 + 8 - 12 - 4 = 0$ ✓
$P(-\frac{1}{2}) = \frac{1}{8} + \frac{3}{8} + \frac{1}{2} + 3 - 4 = 0$ ✓

$$\begin{array}{r|rrrr|r} 2 & 2 & -3 & 2 & -6 & -4 \\ & 0 & 4 & 2 & 8 & 4 \\ \hline -\frac{1}{2} & 2 & 1 & 4 & 2 & 0 \\ & 0 & -1 & 0 & -2 & \\ \hline & 2 & 0 & 4 & 0 & \end{array}$$

$\therefore\ P(z) = (z-2)(z+\frac{1}{2})(2z^2+4)$
$= (z-2)(2z+1)(z^2+2)$ $\quad\therefore$ roots are $2, -\frac{1}{2}, \pm i\sqrt{2}$

13 $P(z) = z^3 + az^2 + kz + ka$
$P(-a) = -a^3 + a^3 - ka + ka = 0$
$\therefore\ z + a$ is a factor

$$\begin{array}{r|rrr|r} -a & 1 & a & k & ka \\ & 0 & -a & 0 & -ka \\ \hline & 1 & 0 & k & 0 \end{array}$$

$\therefore\ P(z) = (z+a)(z^2+k)$

or $P(z) = z^3 + az^2 + kz + ka$
$= z^2(z+a) + k(z+a)$
$= (z+a)(z^2+k)$

a $P(z) = 0$ has one real root if $k > 0,\ a \in \mathbb{R}$

b and 3 real roots if $k \leqslant 0,\ a \in \mathbb{R}$

14 Let $P(x) = 6x^3 + ax^2 - 4ax + b$
$3x + 2$ and $x - 2$ are factors
$\therefore\ 6x^3 + ax^2 - 4ax + b = (3x+2)(x-2)(2x+c)$
$= (3x^2 - 4x - 4)(2x + c)$
$= 6x^3 - 8x^2 - 8x + 3cx^2 - 4cx - 4c$
$= 6x^3 + (3c-8)x^2 + (-8-4c)x - 4c$

Equating coefficients gives: $a = 3c - 8$, $-4a = -8 - 4c$, and $b = -4c$

Equating as gives: $3c - 8 = 2 + c$
$\therefore\ 2c = 10$
$\therefore\ c = 5$

Consequently, $a = 3(5) - 8 = 7$ and $b = -20$.

15 $y = x - k$ meets $(x-2)^2 + (y+3)^2 = 4$ where
$(x-2)^2 + (x-k+3)^2 = 4$
$\therefore\ x^2 - 4x + 4 + (x-k)^2 + 6(x-k) + 9 - 4 = 0$
$\therefore\ x^2 - 4x + 4 + x^2 - 2kx + k^2 + 6x - 6k + 5 = 0$
$\therefore\ 2x^2 + (-4 - 2k + 6)x + (4 + k^2 - 6k + 5) = 0$
$\therefore\ 2x^2 + (2-2k)x + (k^2 - 6k + 9) = 0$

which is a quadratic in x and in the tangent case $\Delta = 0$
$\therefore\ (2-2k)^2 - 4(2)(k^2 - 6k + 9) = 0$
$\therefore\ 4 - 8k + 4k^2 - 8k^2 + 48k - 72 = 0$
$\therefore\ -4k^2 + 40k - 68 = 0$
$\therefore\ k^2 - 10k + 17 = 0$
$\therefore\ k = 5 \pm 2\sqrt{2}$

16

$$
\begin{array}{r}
x^2+3x-9 \\
x^2+2 \overline{\left)\, x^4+3x^3-7x^2+11x-1 \right.} \\
\underline{-(x^4 \qquad\quad +2x^2) \qquad\qquad} \\
3x^3-9x^2+11x \\
\underline{-(3x^3 \qquad\quad +\ 6x) } \\
-9x^2+\ 5x-1 \\
\underline{-(-9x^2 \qquad\quad -18)} \\
5x+17
\end{array}
$$

$\therefore$ $Q(x) = x^2+3x-9$ $\quad R(x) = 5x+17$ and the new function would be divisible by x^2+2 if $x^4+3x^3-7x^2+(2+a)x+b = P(x)-R(x)$

and as $P(x)-R(x) = x^4+3x^3-7x^2+11x-1-(5x+17)$

$= x^4+3x^3-7x^2+6x-18$

then $2+a=6$ and $b=-18$ {equating coefficients}

$\therefore$ $a=4$ and $b=-18$

17 **a** The polynomial $P(x)$ of degree 5 has zeros $m \pm 2i$, $1 \pm mi$, and 2.
The constant term is the y-intercept. $\therefore$ the constant term is -56.

$\therefore$ the product of the zeros is $2(m+2i)(m-2i)(1+mi)(1-mi) = \dfrac{(-1)^5(-56)}{1}$

$$\therefore\ 2(m^2+4)(1+m^2) = 56$$
$$\therefore\ m^4+5m^2+4 = 28$$
$$\therefore\ (m^2)^2+5(m^2)-24 = 0$$
$$\therefore\ (m^2+8)(m^2-3) = 0$$
$$\therefore\ m^2=-8 \text{ or } m^2=3$$
$$\therefore\ m^2 = 3 \qquad \{m^2 \geqslant 0\}$$
$$\therefore\ m = \pm\sqrt{3}$$

b Let the coefficient of x^4 be a.
If $m=\sqrt{3}$, then the sum of the zeros is

$$2+(\sqrt{3}+2i)+(\sqrt{3}-2i)+(1+i\sqrt{3})+(1-i\sqrt{3}) = -\frac{a}{1}$$
$$\therefore\ 2+2\sqrt{3}+2 = -a$$
$$\therefore\ a = -4-2\sqrt{3}$$

If $m=-\sqrt{3}$, then the sum of the zeros is

$$2+(-\sqrt{3}+2i)+(-\sqrt{3}-2i)+(1-i\sqrt{3})+(1+i\sqrt{3}) = -a$$
$$\therefore\ 2-2\sqrt{3}+2 = -a$$
$$\therefore\ a = -4+2\sqrt{3}$$

So, the coefficient of x^4 is $-4-2\sqrt{3}$ if $m=\sqrt{3}$,
and $-4+2\sqrt{3}$ if $m=-\sqrt{3}$.