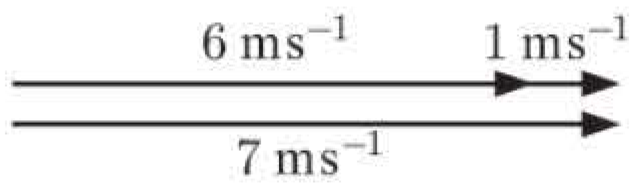
Chapter 15

VECTOR APPLICATIONS

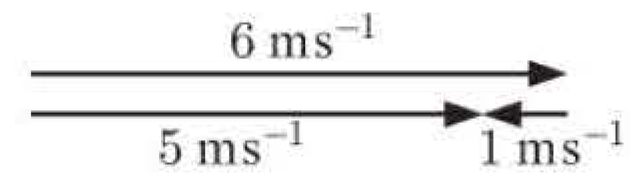
EXERCISE 15A

1 **a**



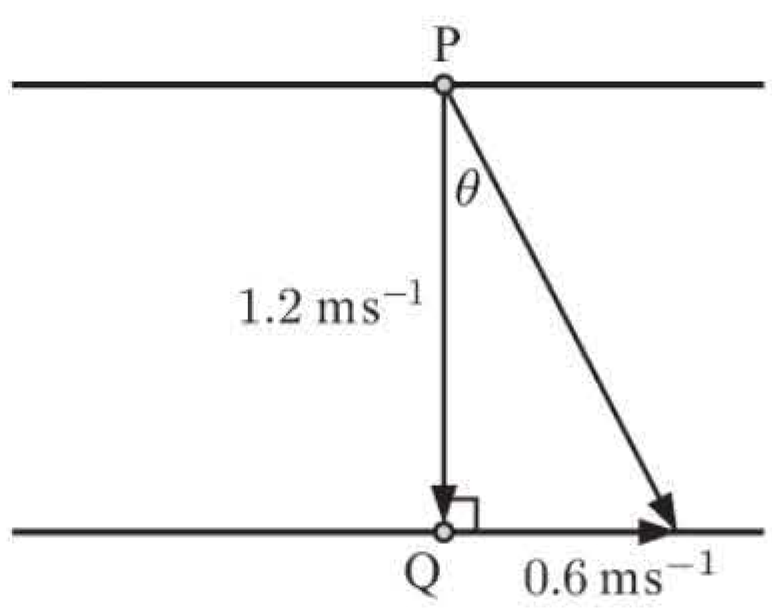
If the athlete is assisted by a wind of 1 m s^{-1} his speed will be 7 m s^{-1}.

b



If the athlete runs into a head wind of 1 m s^{-1} his speed will be 5 m s^{-1}.

2 **a**



$$\begin{aligned}(\text{actual speed})^2 &= (\text{swimming speed})^2 + (\text{current})^2\\ &= 1.2^2 + 0.6^2\\ &= 1.8\end{aligned}$$

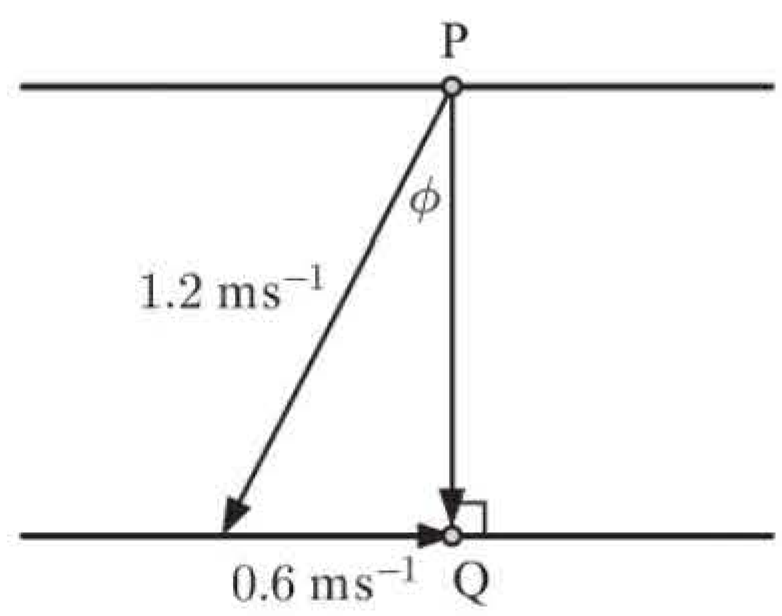
$\therefore$ actual speed $= \sqrt{1.8} \approx 1.34 \text{ m s}^{-1}$

$\tan\theta = \dfrac{0.6}{1.2}$

$\therefore \quad \theta \approx 26.6°$

$\therefore$ Mary's actual velocity is approximately 1.34 m s^{-1} in the direction $26.6°$ to the left of her intended line.

b **i**



Mary needs to aim to the right of Q so the current will correct her direction.

$\sin\phi = \dfrac{0.6}{1.2}$

$\therefore \quad \phi = 30°$

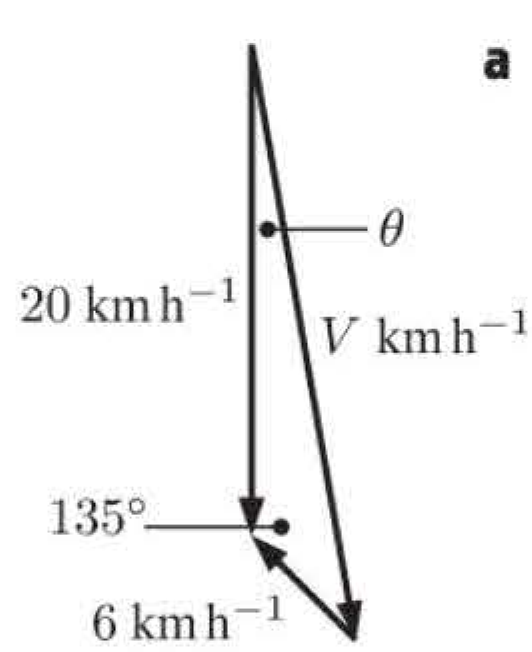
$\therefore$ Mary should aim to swim $30°$ to the right of Q.

ii $(\text{swimming speed})^2 = (\text{actual speed})^2 + (\text{current})^2$

$$\begin{aligned}\therefore\ (\text{actual speed})^2 &= 1.2^2 - 0.6^2\\ &= 1.08\end{aligned}$$

$\therefore$ Mary's actual speed $= \sqrt{1.08} \approx 1.04 \text{ m s}^{-1}$

3



a Using the cosine rule,

$V^2 = 20^2 + 6^2 - 2 \times 20 \times 6 \times \cos 135°$

$\therefore \quad V \approx 24.6$

$\therefore$ the equivalent speed in still water is 24.6 km h^{-1}.

b Using the sine rule,

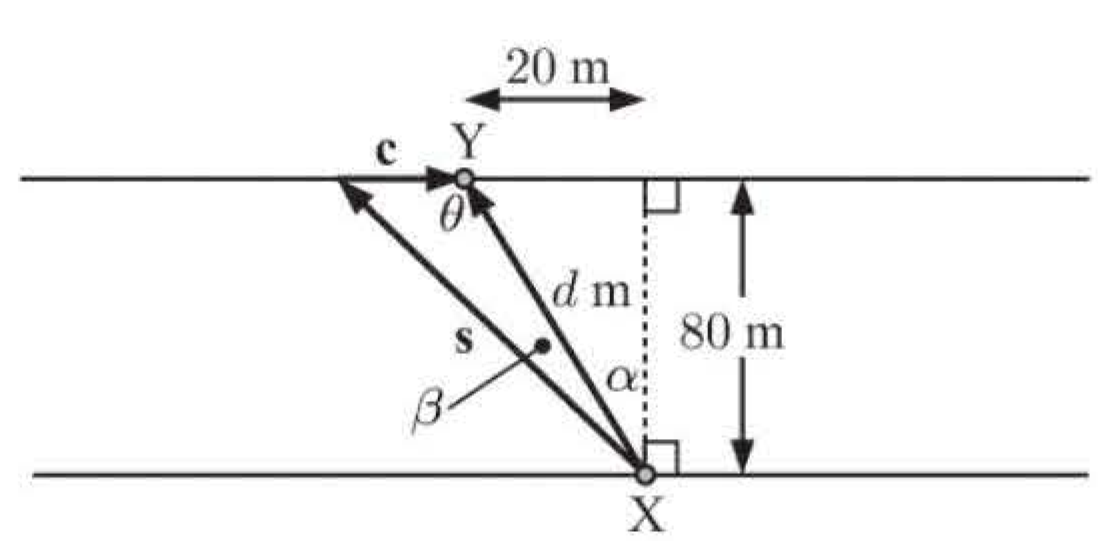
$\dfrac{\sin\theta}{6} \approx \dfrac{\sin 135°}{24.61}$

$\therefore \quad \theta \approx \sin^{-1}\left(\dfrac{6 \times \sin 135°}{24.61}\right)$

$\therefore \quad \theta \approx 9.93°$

$\therefore$ the boat should head $9.93°$ east of south.

4



a $d^2 = 80^2 + 20^2$ {Pythagoras}

$\therefore \quad d = \sqrt{80^2 + 20^2}$ $\{d > 0\}$

$\therefore \quad d \approx 82.5$

$\therefore$ the distance from X to Y is about 82.5 m.

b $\alpha = \tan^{-1}(\frac{20}{80}) \approx 14.04°$

$\therefore\ \theta \approx 90° + 14.04°$ {exterior angle of $\triangle$}

$\therefore\ \theta \approx 104.04°$

In t seconds, Stephanie can swim $1.8t$ metres, and the current will move $0.3t$ metres.

$\therefore\ |\mathbf{s}| = 1.8t$ and $|\mathbf{c}| = 0.3t$

Using the sine rule,

$$\frac{\sin\beta}{0.3t} = \frac{\sin\theta}{1.8t}$$

$$\beta \approx \sin^{-1}\left(\frac{0.3 \times \sin 104.04°}{1.8}\right)$$

$\therefore\ \beta \approx 9.31°$

$\therefore\ \alpha + \beta \approx 23.3°$

$\therefore$ Stephanie should head $23.3°$ to the left of the perpendicular across the river.

c $\tan(\alpha+\beta) = \dfrac{20 + 0.3t}{80}$

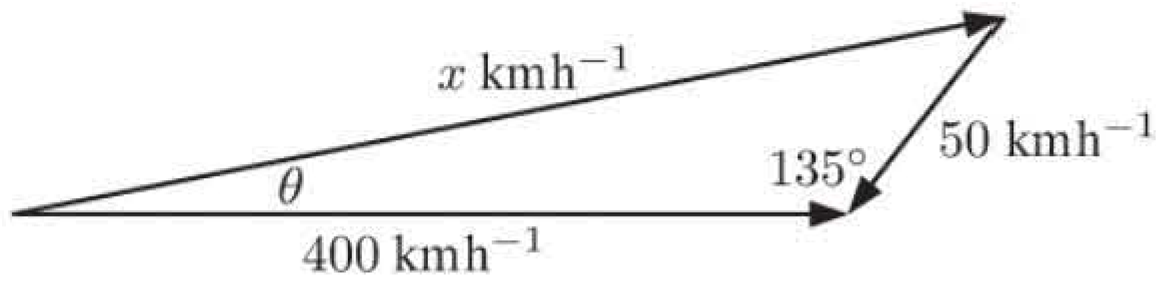
$\therefore\ 20 + 0.3t \approx 80\tan(23.34°)$

$\therefore\ t \approx \dfrac{80\tan(23.34°) - 20}{0.3}$

$\therefore\ t \approx 48.4$

$\therefore$ Stephanie will take 48.4 seconds to cross the river.

5



a Using the cosine rule,

$x^2 = 50^2 + 400^2 - 2 \times 50 \times 400\cos 135°$

$\therefore\ x \approx 436.79$

The aeroplane should fly so that its speed in still air would be 437 km h^{-1}.

The wind slows the aeroplane down to 400 km h^{-1}.

b Using the sine rule,

$$\frac{\sin\theta}{50} \approx \frac{\sin 135°}{436.79}$$

$\therefore\ \theta \approx 4.64°$

The aeroplane should head $4.64°$ north of east.

EXERCISE 15B

1 **a** Given A(2, 1, 1), B(4, 3, 0), and C(1, 3, −2), $\overrightarrow{AB} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$.

$$\therefore\ \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & -1 \\ -1 & 2 & -3 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 2 & -3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & -1 \\ -1 & -3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 2 \\ -1 & 2 \end{vmatrix}\mathbf{k}$$

$= -4\mathbf{i} + 7\mathbf{j} + 6\mathbf{k}$

$\therefore$ area $= \frac{1}{2}|-4\mathbf{i} + 7\mathbf{j} + 6\mathbf{k}|$ {area $= \frac{1}{2}\,|\overrightarrow{AB} \times \overrightarrow{AC}|$}

$= \frac{1}{2}\sqrt{(-4)^2 + 7^2 + 6^2}$

$= \frac{1}{2}\sqrt{101}$ units^2

b Given A(0, 0, 0), B(−1, 2, 3), and C(1, 2, 6), $\overrightarrow{AB} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$.

$$\therefore\ \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 3 \\ 1 & 2 & 6 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 2 & 6 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & 3 \\ 1 & 6 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & 2 \\ 1 & 2 \end{vmatrix}\mathbf{k}$$

$= 6\mathbf{i} + 9\mathbf{j} - 4\mathbf{k}$

$\therefore$ area $= \frac{1}{2}|6\mathbf{i} + 9\mathbf{j} - 4\mathbf{k}|$ {area $= \frac{1}{2}\,|\overrightarrow{AB} \times \overrightarrow{AC}|$}

$= \frac{1}{2}\sqrt{6^2 + 9^2 + (-4)^2}$

$= \frac{1}{2}\sqrt{133}$ units^2

c Given A(1, 3, 2), B(2, −1, 0), and C(1, 10, 6), $\overrightarrow{AB} = \begin{pmatrix} 1 \\ -4 \\ -2 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} 0 \\ 7 \\ 4 \end{pmatrix}$.

$$\therefore \quad \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & -2 \\ 0 & 7 & 4 \end{vmatrix} = \begin{vmatrix} -4 & -2 \\ 7 & 4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -2 \\ 0 & 4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -4 \\ 0 & 7 \end{vmatrix}\mathbf{k}$$

$$= -2\mathbf{i} - 4\mathbf{j} + 7\mathbf{k}$$

$$\therefore \quad \text{area} = \tfrac{1}{2}\,|-2\mathbf{i} - 4\mathbf{j} + 7\mathbf{k}| = \tfrac{1}{2}\sqrt{(-2)^2 + (-4)^2 + 7^2} = \tfrac{1}{2}\sqrt{69} \text{ units}^2$$

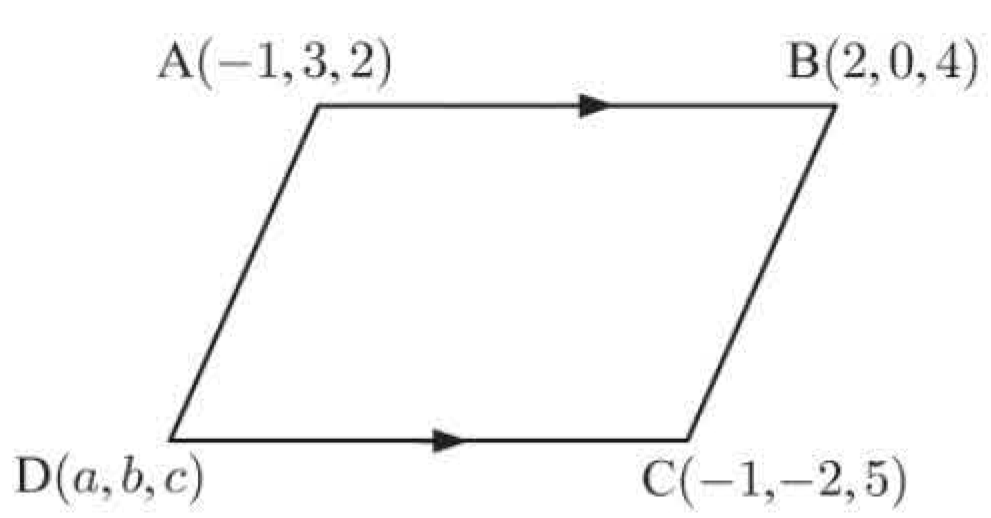
2 Given A(−1, 2, 2), B(2, −1, 4), and C(0, 1, 0), $\overrightarrow{AB} = \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$.

$$\therefore \quad \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -3 & 2 \\ 1 & -1 & -2 \end{vmatrix} = \begin{vmatrix} -3 & 2 \\ -1 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 2 \\ 1 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & -3 \\ 1 & -1 \end{vmatrix}\mathbf{k}$$

$$= 8\mathbf{i} + 8\mathbf{j}$$

$$\therefore \quad \text{area of parallelogram} = |\,8\mathbf{i} + 8\mathbf{j}\,| = \sqrt{8^2 + 8^2} = 8\sqrt{2} \text{ units}^2$$

3 **a**



Suppose D is at (a, b, c).

Since $\overrightarrow{AB} = \overrightarrow{DC}$,

$$\begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} -1-a \\ -2-b \\ 5-c \end{pmatrix}$$

$$\therefore \quad -1 - a = 3, \quad -2 - b = -3, \quad \text{and} \quad 5 - c = 2$$

$$\therefore \quad a = -4, \quad b = 1, \quad \text{and} \quad c = 3$$

$\therefore$ D is at (−4, 1, 3).

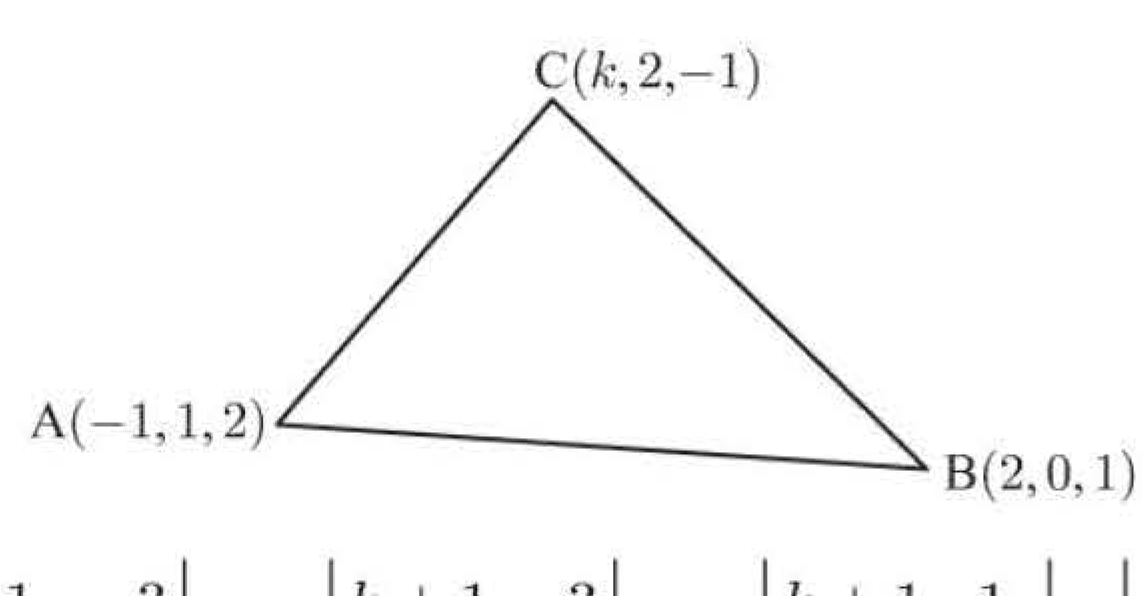
b $\overrightarrow{BC} = \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}$ and $\overrightarrow{BA} = \begin{pmatrix} -3 \\ 3 \\ -2 \end{pmatrix}$

$$\therefore \quad \overrightarrow{BC} \times \overrightarrow{BA} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & -2 & 1 \\ -3 & 3 & -2 \end{vmatrix} = \begin{vmatrix} -2 & 1 \\ 3 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -3 & 1 \\ -3 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -3 & -2 \\ -3 & 3 \end{vmatrix}\mathbf{k}$$

$$= \mathbf{i} - 9\mathbf{j} - 15\mathbf{k}$$

$$\therefore \quad \text{area} = |\mathbf{i} - 9\mathbf{j} - 15\mathbf{k}| = \sqrt{1^2 + (-9)^2 + (-15)^2} = \sqrt{307} \text{ units}^2$$

4 Now $\overrightarrow{AB} = \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} k+1 \\ 1 \\ -3 \end{pmatrix}$



Area of $\triangle$ABC $= \frac{1}{2}\,|\,\overrightarrow{AC} \times \overrightarrow{AB}\,|$

$$\therefore \quad \sqrt{88} = \tfrac{1}{2}\left|\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ k+1 & 1 & -3 \\ 3 & -1 & -1 \end{vmatrix}\right| = \tfrac{1}{2}\left|\begin{vmatrix} 1 & -3 \\ -1 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} k+1 & -3 \\ 3 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} k+1 & 1 \\ 3 & -1 \end{vmatrix}\mathbf{k}\right|$$

$$\therefore \quad \sqrt{352} = |(-1-3)\mathbf{i} - (-(k+1) - -9)\mathbf{j} + (-(k+1) - 3)\mathbf{k}|$$

$$\therefore \quad \sqrt{352} = |-4\mathbf{i} + (k-8)\mathbf{j} + (-k-4)\mathbf{k}|$$

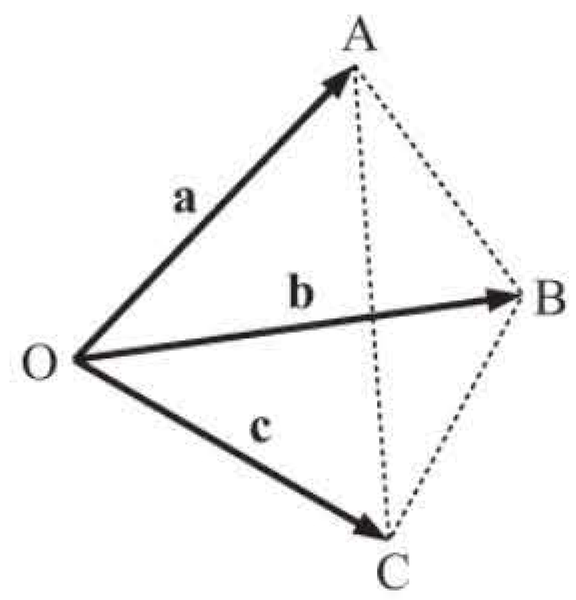
$$\therefore \quad \sqrt{352} = \sqrt{16 + (k-8)^2 + (-k-4)^2}$$

$$\therefore \quad 352 = 16 + k^2 - 16k + 64 + k^2 + 8k + 16$$

$$\therefore \quad 2k^2 - 8k - 256 = 0$$

$$\therefore \quad k^2 - 4k - 128 = 0$$

$$\therefore \quad k = \frac{4 \pm \sqrt{16 + 4(1)(128)}}{2} = 2 \pm \sqrt{132} = 2 \pm 2\sqrt{33}$$

5 

Total surface area S of the tetrahedron is the sum of the areas of the 4 triangular faces.

Now $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\mathbf{a} + \mathbf{b} = \mathbf{b} - \mathbf{a}$

and $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC} = -\mathbf{a} + \mathbf{c} = \mathbf{c} - \mathbf{a}$

$$\therefore\ S = \tfrac{1}{2}\,|\mathbf{a}\times\mathbf{b}| + \tfrac{1}{2}\,|\mathbf{a}\times\mathbf{c}| + \tfrac{1}{2}\,|\mathbf{b}\times\mathbf{c}| + \tfrac{1}{2}\,|(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})|$$
$$= \tfrac{1}{2}\{|\mathbf{a}\times\mathbf{b}| + |\mathbf{a}\times\mathbf{c}| + |\mathbf{b}\times\mathbf{c}| + |(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})|\}$$

EXERCISE 15C

1 **a** **i** $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 4 \end{pmatrix},\quad \lambda \in \mathbb{R}$

ii $x = 3 + \lambda,\quad y = -4 + 4\lambda,\quad \lambda \in \mathbb{R}$

iii $\lambda = x - 3 = \dfrac{y+4}{4}$

$\therefore\ 4x - 12 = y + 4$

$\therefore\ 4x - y = 16$

b **i** If the line has direction vector $\mathbf{b}$ perpendicular to $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$, then

$\mathbf{b} \bullet \begin{pmatrix} 5 \\ 2 \end{pmatrix} = 0$

$\therefore\ \mathbf{b} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$ is a reasonable choice

$\therefore\ \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 5 \end{pmatrix},\quad \lambda \in \mathbb{R}$

ii $x = 5 - 2\lambda,\quad y = 2 + 5\lambda,\quad \lambda \in \mathbb{R}$

iii $\lambda = \dfrac{x-5}{-2} = \dfrac{y-2}{5}$

$\therefore\ 5x - 25 = -2y + 4$

$\therefore\ 5x + 2y = 29$

c **i** $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -6 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 7 \end{pmatrix},\quad \lambda \in \mathbb{R}$

ii $x = -6 + 3\lambda,\quad y = 7\lambda,\quad \lambda \in \mathbb{R}$

iii $\lambda = \dfrac{x+6}{3} = \dfrac{y}{7}$

$\therefore\ 7x + 42 = 3y$

$\therefore\ 7x - 3y = -42$

d **i** Take $(-1,\ 11)$ as our fixed point,

so $\mathbf{a} = \begin{pmatrix} -1 \\ 11 \end{pmatrix}$.

The direction vector $\mathbf{b} = \begin{pmatrix} -3-(-1) \\ 12-11 \end{pmatrix}$

$= \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$\therefore\ \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 11 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \end{pmatrix},\quad \lambda \in \mathbb{R}$

ii $x = -1 - 2\lambda,\quad y = 11 + \lambda,\quad \lambda \in \mathbb{R}$

iii $\lambda = \dfrac{x+1}{-2} = y - 11$

$\therefore\ x + 1 = -2y + 22$

$\therefore\ x + 2y = 21$

2 **a** $x = -1 + 2\lambda,\quad y = 4 - \lambda,\quad \lambda \in \mathbb{R}$

b When $\lambda = 0,\ x = -1 + 2(0) = -1$ and $y = 4 - 0 = 4$ $\quad\therefore$ the point is $(-1,\ 4)$.

When $\lambda = 1,\ x = -1 + 2(1) = 1$ and $y = 4 - 1 = 3$ $\quad\therefore$ the point is $(1,\ 3)$.

When $\lambda = 3,\ x = -1 + 2(3) = 5$ and $y = 4 - 3 = 1$ $\quad\therefore$ the point is $(5,\ 1)$.

When $\lambda = -1,\ x = -1 + 2(-1) = -3$ and $y = 4 - (-1) = 5$ $\quad\therefore$ the point is $(-3,\ 5)$.

When $\lambda = -4,\ x = -1 + 2(-4) = -9$ and $y = 4 - (-4) = 8$ $\quad\therefore$ the point is $(-9,\ 8)$.

3 **a** If $\lambda + 2 = 3$ and $1 - 3\lambda = -2$,

then $\lambda = 1$ and $-3\lambda = -3$

$\therefore\ \lambda = 1$

Since $\lambda = 1$ in each case,

$(3,\ -2)$ lies on the line.

b If $(k,\ 4)$ lies on $x = 1 - 2\lambda,\quad y = 1 + \lambda,$ then

$k = 1 - 2\lambda$ and $4 = 1 + \lambda$

$\therefore\ \lambda = 3$

and $k = 1 - 6$

$\therefore\ k = -5$

4 **a** When $\lambda = 1$, $\mathbf{r} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} + 1 \times \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1-1 \\ 5+3 \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \end{pmatrix}$

$\therefore$ the point is $(0, 8)$.

b $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ is a non-zero scalar multiple of $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$. It is parallel and in the opposite direction, so it could also be used to describe the direction of the line.

c The line passes through point $(0, 8)$ and has direction vector $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$.

$\therefore$ $\mathbf{r} = \begin{pmatrix} 0 \\ 8 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -3 \end{pmatrix}$, $\mu \in \mathbb{R}$ is an alternative vector equation for the line.

5 **a** **i** $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -7 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, $\lambda \in \mathbb{R}$

ii $x = 1 + 2\lambda$, $y = 3 + \lambda$, $z = -7 + 3\lambda$, $\lambda \in \mathbb{R}$ **iii** $\lambda = \dfrac{x-1}{2} = y - 3 = \dfrac{z+7}{3}$

b **i** $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$, $\lambda \in \mathbb{R}$

ii $x = \lambda$, $y = 1 + \lambda$, $z = 2 - 2\lambda$, $\lambda \in \mathbb{R}$ **iii** $\lambda = x = y - 1 = \dfrac{-z+2}{2}$

c **i** Since the line is parallel to the X-axis, it has direction vector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\therefore$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\lambda \in \mathbb{R}$

ii $x = -2 + \lambda$, $y = 2$, $z = 1$, $\lambda \in \mathbb{R}$ **iii** $y = 2$, $z = 1$

d **i** $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$, $\lambda \in \mathbb{R}$

ii $x = 2\lambda$, $y = 2 - \lambda$, $z = -1 + 3\lambda$, $\lambda \in \mathbb{R}$ **iii** $\lambda = \dfrac{x}{2} = -y + 2 = \dfrac{z+1}{3}$

e **i** Since the line is perpendicular to the XOY plane, it has direction vector $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\therefore$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $\lambda \in \mathbb{R}$

ii $x = 3$, $y = 2$, $z = -1 + \lambda$, $\lambda \in \mathbb{R}$ **iii** $x = 3$, $y = 2$

6 **a** $\overrightarrow{AB} = \begin{pmatrix} -1-1 \\ 3-2 \\ 2-1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ $\therefore$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$, $\lambda \in \mathbb{R}$

b $\overrightarrow{CD} = \begin{pmatrix} 3-0 \\ 1-1 \\ -1-3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix}$ $\therefore$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix}$, $\lambda \in \mathbb{R}$

c $\overrightarrow{EF} = \begin{pmatrix} 1-1 \\ -1-2 \\ 5-5 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix}$ $\therefore$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix}$, $\lambda \in \mathbb{R}$

d $\overrightarrow{GH} = \begin{pmatrix} 5-0 \\ -1-1 \\ 3--1 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 4 \end{pmatrix}$ $\therefore$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ -2 \\ 4 \end{pmatrix}$, $\lambda \in \mathbb{R}$

7 **a** $\begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix}$ **b** $\begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix}$

c $\lambda = \dfrac{x-2}{3} = \dfrac{y+1}{2} = z - 1$

$\therefore\ x - 2 = 3\lambda$ and $y + 1 = 2\lambda$ and $z - 1 = \lambda$

$\therefore\ x = 2 + 3\lambda$ $\quad y = -1 + 2\lambda$ $\quad z = 1 + \lambda$

$\therefore$ direction vector is $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

d $\mu = \dfrac{1-x}{2} = \dfrac{y}{4} = \dfrac{z-3}{3}$

$\therefore\ 2\mu = 1 - x$ and $y = 4\mu$ and $z - 3 = 3\mu$

$\therefore\ x = 1 - 2\mu$ $\quad y = 4\mu$ $\quad z = 3 + 3\mu$

$\therefore$ direction vector is $\begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix}$

8 Given $x = 1 - \lambda,\ \ y = 3 + \lambda,\ \ z = 3 - 2\lambda$:

a The line meets the XOY plane when $z = 0$ $\quad \therefore\ 3 - 2\lambda = 0$

$\therefore\ \lambda = \frac{3}{2}$

Then $x = 1 - \frac{3}{2} = -\frac{1}{2}$ and $y = 3 + \frac{3}{2} = \frac{9}{2}$, so the point is $\left(-\frac{1}{2}, \frac{9}{2}, 0\right)$.

b The line meets the YOZ plane when $x = 0$ $\quad \therefore\ 1 - \lambda = 0$

$\therefore\ \lambda = 1$

Then $y = 3 + 1 = 4$ and $z = 3 - 2 = 1$, so the point is $(0, 4, 1)$.

c The line meets the XOZ plane when $y = 0$ $\quad \therefore\ 3 + \lambda = 0$

$\therefore\ \lambda = -3$

Then $x = 1 - (-3) = 4$ and $z = 3 - 2(-3) = 9$, so the point is $(4, 0, 9)$.

9 **a** When $\lambda = 0,\ x = x_0,\ y = y_0,$ and $z = z_0$

$\therefore\ (x_0, y_0, z_0)$

b $\begin{pmatrix} l \\ m \\ n \end{pmatrix}$

c $\lambda = \dfrac{x - x_0}{l} = \dfrac{y - y_0}{m} = \dfrac{z - z_0}{n}, \quad l, m, n \neq 0$

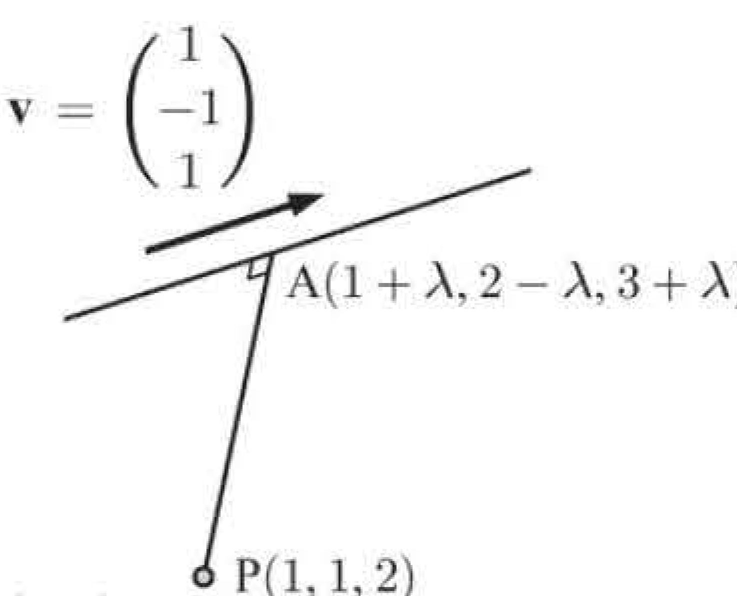
10 Given a line with equations $x = 2 - \lambda,\ \ y = 3 + 2\lambda,$ and $z = 1 + \lambda,$
the distance to the point $(1, 0, -2)$ is $\sqrt{(2 - \lambda - 1)^2 + (3 + 2\lambda - 0)^2 + (1 + \lambda + 2)^2}.$

But this distance $= 5\sqrt{3}$ units

$$\therefore\ \sqrt{(1-\lambda)^2 + (3+2\lambda)^2 + (\lambda+3)^2} = 5\sqrt{3}$$
$$\therefore\ (1-\lambda)^2 + (3+2\lambda)^2 + (\lambda+3)^2 = 75$$
$$\therefore\ 1 - 2\lambda + \lambda^2 + 9 + 12\lambda + 4\lambda^2 + \lambda^2 + 6\lambda + 9 = 75$$
$$\therefore\ 6\lambda^2 + 16\lambda - 56 = 0$$
$$\therefore\ 3\lambda^2 + 8\lambda - 28 = 0$$
$$\therefore\ (3\lambda + 14)(\lambda - 2) = 0$$
$$\therefore\ \lambda = -\tfrac{14}{3} \text{ or } \lambda = 2$$

When $\lambda = 2$, the point is $(0, 7, 3)$, and when $\lambda = -\frac{14}{3}$, the point is $\left(\frac{20}{3}, -\frac{19}{3}, -\frac{11}{3}\right)$.

11 **a** Let $A(1+\lambda, 2-\lambda, 3+\lambda)$ be a point on the line such that $\overrightarrow{PA}$ is perpendicular to the line.



Then $\overrightarrow{PA} = \begin{pmatrix} 1+\lambda-1 \\ 2-\lambda-1 \\ 3+\lambda-2 \end{pmatrix} = \begin{pmatrix} \lambda \\ 1-\lambda \\ 1+\lambda \end{pmatrix}$

and $\text{PA} = \sqrt{\lambda^2+(1-\lambda)^2+(1+\lambda)^2}$

$= \sqrt{\lambda^2+(1-2\lambda+\lambda^2)+(1+2\lambda+\lambda^2)}$

$= \sqrt{3\lambda^2+2}$ units

[PA] is perpendicular to the line when $\text{PA}^2 = 3\lambda^2+2$ is minimised,

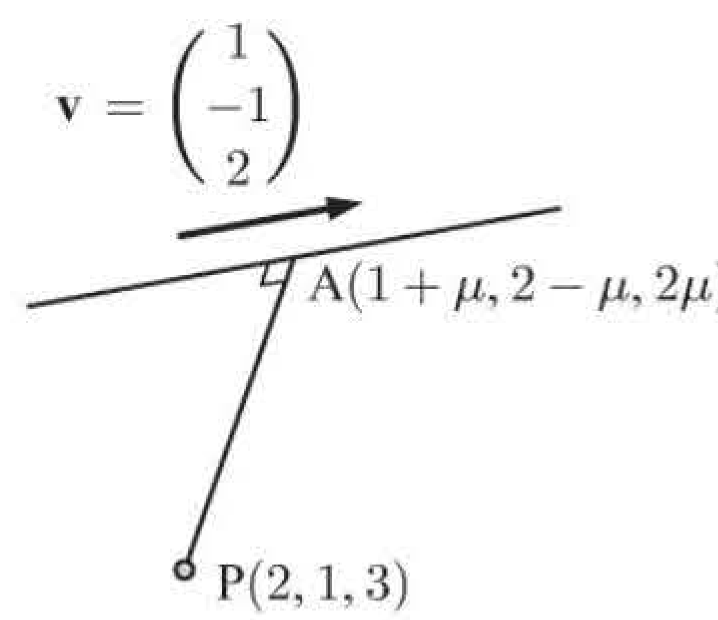
which occurs when $\lambda = -\dfrac{b}{2a} = -\dfrac{0}{6} = 0$

$\therefore$ A is at $(1+0, 2-0, 3+0)$

$\therefore$ the foot of the perpendicular is $(1, 2, 3)$.

b Let A be a point on the line such that $\overrightarrow{PA}$ is perpendicular to the line.

$\therefore$ A is at $(1+\mu, 2-\mu, 2\mu)$ for some μ.



Now $\overrightarrow{PA} = \begin{pmatrix} 1+\mu-2 \\ 2-\mu-1 \\ 2\mu-3 \end{pmatrix} = \begin{pmatrix} \mu-1 \\ 1-\mu \\ 2\mu-3 \end{pmatrix}$

and $\text{PA} = \sqrt{(\mu-1)^2+(1-\mu)^2+(2\mu-3)^2}$

$= \sqrt{\mu^2-2\mu+1+1-2\mu+\mu^2+4\mu^2-12\mu+9}$

$= \sqrt{6\mu^2-16\mu+11}$ units

[PA] is perpendicular to the line when $\text{PA}^2 = 6\mu^2-16\mu+11$ is minimised,

which occurs when $\mu = -\dfrac{b}{2a} = -\dfrac{-16}{12} = \frac{4}{3}$

$\therefore$ A is at $\left(1+\frac{4}{3}, 2-\frac{4}{3}, 2(\frac{4}{3})\right)$

$\therefore$ the foot of the perpendicular is $\left(\frac{7}{3}, \frac{2}{3}, \frac{8}{3}\right)$.

EXERCISE 15D

1 L_1 has direction vector $\begin{pmatrix} 12 \\ 5 \end{pmatrix}$ and L_2 has direction vector $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$.

If θ is the angle between them, $\cos\theta = \dfrac{\left|\begin{pmatrix} 12 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -4 \end{pmatrix}\right|}{\sqrt{144+25}\sqrt{9+16}}$

$= \dfrac{|36+(-20)|}{13 \times 5}$

$= \frac{16}{65}$

$\therefore \theta = \cos^{-1}\left(\frac{16}{65}\right)$

$\therefore \theta \approx 75.7^\circ$

2 Line 1 has direction vector $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$ and line 2 has direction vector $\begin{pmatrix} 4 \\ 10 \end{pmatrix}$

and $\begin{pmatrix} 5 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 4 \\ 10 \end{pmatrix} = 20+(-20) = 0$

$\therefore$ the lines are perpendicular.

3 L_1 has direction vector $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ and L_2 has direction vector $\begin{pmatrix} 5 \\ 4 \end{pmatrix}$.

If θ is the angle between them,
$$\cos\theta = \frac{\left|\begin{pmatrix} 4 \\ -3 \end{pmatrix} \bullet \begin{pmatrix} 5 \\ 4 \end{pmatrix}\right|}{\sqrt{16+9}\sqrt{25+16}} = \frac{|20+(-12)|}{\sqrt{25\times 41}} = \frac{8}{\sqrt{25\times 41}}$$
$$\therefore\ \theta = \cos^{-1}\left(\frac{8}{\sqrt{25\times 41}}\right)$$
$$\therefore\ \theta \approx 75.5°$$

$\therefore$ the required angle measures $75.5°$.

4 **a** L_1: $s = \dfrac{x-8}{3} = \dfrac{9-y}{16} = \dfrac{z-10}{7}$

$3s = x - 8$ and $16s = 9 - y$ and $7s = z - 10$

$\therefore\ x = 8 + 3s$ $\quad y = 9 - 16s$ $\quad z = 10 + 7s$

$\therefore$ line 1 has direction vector $\begin{pmatrix} 3 \\ -16 \\ 7 \end{pmatrix}$ and line 2 has direction vector $\begin{pmatrix} 3 \\ 8 \\ -5 \end{pmatrix}$.

If θ is the angle between them,

$$\cos\theta = \frac{\left|\begin{pmatrix} 3 \\ -16 \\ 7 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 8 \\ -5 \end{pmatrix}\right|}{\sqrt{9+256+49}\sqrt{9+64+25}} = \frac{|9-128-35|}{\sqrt{314}\sqrt{98}} = \frac{154}{\sqrt{314\times 98}}$$

$\therefore\ \theta \approx 28.6°$

b Since $L_1 \perp L_3$,
$$\begin{pmatrix} 3 \\ -16 \\ 7 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ -3 \\ x \end{pmatrix} = 0$$
$$\therefore\ 48 + 7x = 0$$
$$\therefore\ x = -\tfrac{48}{7}$$

5 **a** $x - y = 3$ has gradient $+\frac{1}{1}$ and so has direction vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$3x + 2y = 11$ has gradient $-\frac{3}{2}$ and so has direction vector $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

$$\therefore\ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \sqrt{1+1}\sqrt{4+9}\cos\theta$$
$$\therefore\ 2 - 3 = \sqrt{2}\sqrt{13}\cos\theta$$
$$\therefore\ \tfrac{-1}{\sqrt{26}} = \cos\theta$$
$$\therefore\ \theta \approx 101.3°$$

$\therefore$ the angle is $180° - 101.3° \approx 78.7°$

b $y = x + 2$ has gradient $1 = \frac{1}{1}$ and so has direction vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$y = 1 - 3x$ has gradient $-3 = \frac{-3}{1}$ and so has direction vector $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$.

$$\therefore\ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \sqrt{1+1}\sqrt{1+9}\cos\theta$$
$$\therefore\ 1 - 3 = \sqrt{2}\sqrt{10}\cos\theta$$
$$\therefore\ \tfrac{-2}{\sqrt{20}} = \cos\theta$$
$$\therefore\ \theta \approx 116.6°$$

$\therefore$ the angle is $180° - 116.6° \approx 63.4°$

c $y+x=7$ has gradient $-1=\frac{-1}{1}$ and so has direction vector $\begin{pmatrix}1\\-1\end{pmatrix}$.

$x-3y+2=0$ has gradient $\frac{1}{3}$ and so has direction vector $\begin{pmatrix}3\\1\end{pmatrix}$.

$$\therefore \begin{pmatrix}1\\-1\end{pmatrix}\bullet\begin{pmatrix}3\\1\end{pmatrix}=\sqrt{1+1}\sqrt{9+1}\cos\theta$$
$$\therefore 3-1=\sqrt{2}\sqrt{10}\cos\theta$$
$$\therefore \frac{2}{\sqrt{20}}=\cos\theta$$
$$\therefore \theta\approx 63.4^\circ$$

$\therefore$ the angle is 63.4°.

d $y=2-x$ has gradient $-1=\frac{-1}{1}$ and so has direction vector $\begin{pmatrix}1\\-1\end{pmatrix}$.

$x-2y=7$ has gradient $\frac{1}{2}$ and so has direction vector $\begin{pmatrix}2\\1\end{pmatrix}$.

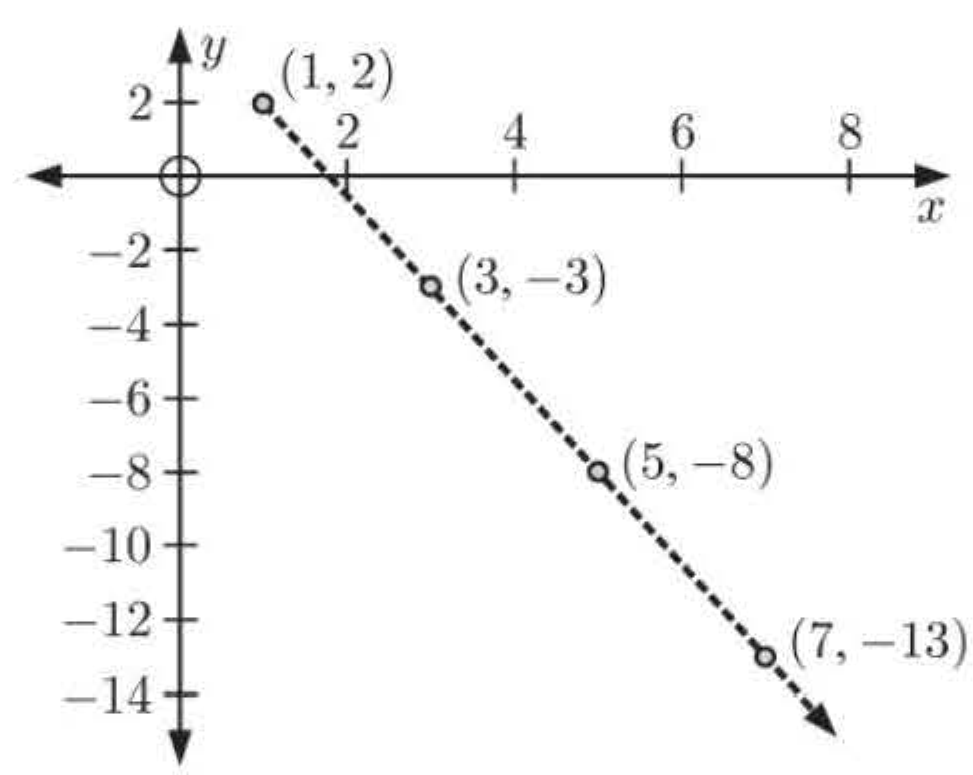
$$\therefore \begin{pmatrix}1\\-1\end{pmatrix}\bullet\begin{pmatrix}2\\1\end{pmatrix}=\sqrt{1+1}\sqrt{4+1}\cos\theta$$
$$\therefore 2-1=\sqrt{2}\sqrt{5}\cos\theta$$
$$\therefore \frac{1}{\sqrt{10}}=\cos\theta$$
$$\therefore \theta\approx 71.6^\circ$$

$\therefore$ the angle is 71.6°.

EXERCISE 15E

1 **a** $x(0)=1$ and $y(0)=2$,
$\therefore$ the initial position is $(1, 2)$

b



c The velocity vector is $\begin{pmatrix}2\\-5\end{pmatrix}$.

d In 1 second, the x-step is 2 and y-step is -5, which is a distance of $\sqrt{2^2+(-5)^2}=\sqrt{29}$ cm
$\therefore$ the speed is $\sqrt{29}$ cm s^{-1}.

2 **a** $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2\\3\end{pmatrix}+t\begin{pmatrix}4\\-5\end{pmatrix},\quad t\geqslant 0$

b 90 minutes $=1.5$ hours
When $t=1.5$, $x=2+4(1.5)=8$
and $y=3-5(1.5)=-4.5$
$\therefore$ the boat is at $(8, -4.5)$ after 90 minutes.

c When the boat reaches the point $(5, -0.75)$,
$2+4t=5$ and $3-5t=-0.75$
$\therefore 4t=3$ $\quad -5t=-3.75$
$\therefore t=0.75$ $\quad t=0.75$
It will take 0.75 hours $=45$ minutes for the boat to reach point $(5, -0.75)$.

3 **a**
$$\mathbf{r}=\mathbf{a}+t\mathbf{b}$$
$$\therefore \begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-3\\-2\end{pmatrix}+t\begin{pmatrix}2\\4\end{pmatrix},\quad t\geqslant 0$$
$$\therefore \begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-3+2t\\-2+4t\end{pmatrix}$$

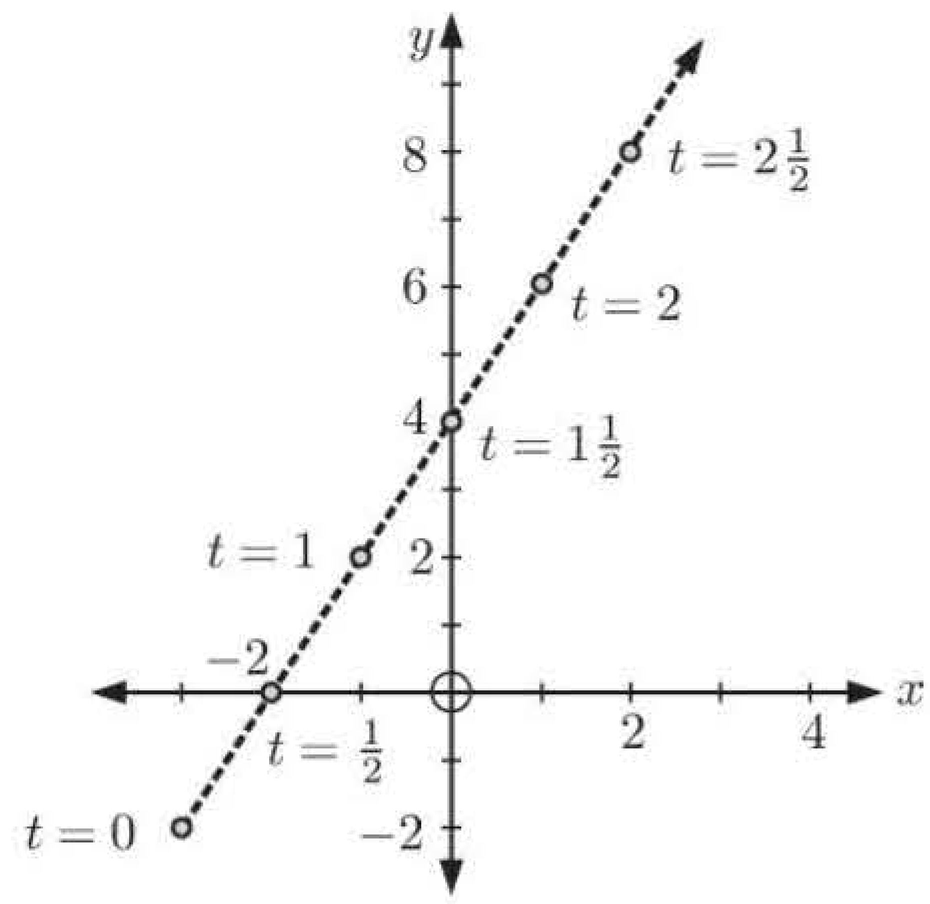
b At $t=2.5$, $-3+2t=-3+5=2$
and $-2+4t=-2+10=8$

So, the position vector is $\begin{pmatrix}2\\8\end{pmatrix}$.

c **i** When the car is due north, $x=0$
$\therefore -3+2t=0$
$\therefore t=1.5$ seconds

ii When the car is due west, $y=0$
$\therefore -2+4t=0$
$\therefore t=0.5$ seconds

d



4 **a** **i** When $t=0$, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$

$\therefore$ the object is at $(-4, 3)$.

ii The velocity vector is $\begin{pmatrix} 12 \\ 5 \end{pmatrix}$.

iii The speed is $\sqrt{12^2+5^2} = 13 \text{ m s}^{-1}$

b **i** When $t=0$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$

$\therefore$ the object is at $(3, 0, 4)$.

ii The velocity vector is $\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$.

iii The speed is $\sqrt{2^2+(-1)^2+(-2)^2} = 3 \text{ m s}^{-1}$

5 **a** $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ has length $\sqrt{4^2+(-3)^2} = 5$

$\therefore$ $30\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ has length 150

$\therefore$ the velocity vector is $\begin{pmatrix} 120 \\ -90 \end{pmatrix}$.

b $2\mathbf{i}+\mathbf{j} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ has length $\sqrt{2^2+1^2} = \sqrt{5}$

$\therefore$ $10\sqrt{5}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ has length 50

$\therefore$ the velocity vector is $\begin{pmatrix} 20\sqrt{5} \\ 10\sqrt{5} \end{pmatrix}$.

6 $-2\mathbf{i}+5\mathbf{j}-14\mathbf{k}$ has length $\sqrt{(-2)^2+5^2+(-14)^2} = \sqrt{4+25+196} = \sqrt{225} = 15$

$\therefore$ $6\begin{pmatrix} -2 \\ 5 \\ -14 \end{pmatrix}$ has length 90, so the velocity vector is $\begin{pmatrix} -12 \\ 30 \\ -84 \end{pmatrix}$.

7 Yacht A: $\begin{pmatrix} x_A \\ y_A \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ Yacht B: $\begin{pmatrix} x_B \\ y_B \end{pmatrix} = \begin{pmatrix} 1 \\ -8 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $t \geqslant 0$

a When $t=0$, $\begin{pmatrix} x_A \\ y_A \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$ $\therefore$ A is at $(4, 5)$

and $\begin{pmatrix} x_B \\ y_B \end{pmatrix} = \begin{pmatrix} 1 \\ -8 \end{pmatrix}$ $\therefore$ B is at $(1, -8)$.

b For A, the velocity vector is $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$, and for B it is $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

c Speed of A $= \sqrt{1^2+(-2)^2} = \sqrt{5} \text{ km h}^{-1}$. Speed of B $= \sqrt{2^2+1^2} = \sqrt{5} \text{ km h}^{-1}$.

d A has direction vector $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and B has direction vector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Since $\begin{pmatrix} 1 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2-2 = 0$, the paths of the yachts are at right angles to each other.

8 **a** P's torpedo has position $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} -5 \\ 4 \end{pmatrix} + t\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ and at $t=0$, the time is 1:34 pm

$\therefore$ $x_1(t) = -5+3t$, $y_1(t) = 4-t$.

b Speed $= \sqrt{3^2+(-1)^2} = \sqrt{10} \text{ km min}^{-1}$

c Q fires its torpedo after a minutes.

$\therefore$ at time t, its torpedo has travelled for $(t-a)$ minutes.

$\therefore$ $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 15 \\ 7 \end{pmatrix} + (t-a)\begin{pmatrix} -4 \\ -3 \end{pmatrix}$, $t \geqslant a$

$\therefore$ $x_2(t) = 15-4(t-a)$ and $y_2(t) = 7-3(t-a)$

$\therefore$ position is $Q(15-4(t-a), 7-3(t-a))$.

d They meet when $x_1(t) = x_2(t)$ and $y_1(t) = y_2(t)$

$\therefore\ -5 + 3t = 15 - 4(t - a)$ and $4 - t = 7 - 3(t - a)$

$\therefore\ 7t - 4a = 20$ .... (1) and $2t - 3a = 3$ .... (2)

Solving simultaneously,

$$\begin{array}{rl} 21t - 12a = 60 & \{3 \times (1)\} \\ -8t + 12a = -12 & \{-4 \times (2)\} \\ \hline \text{adding} \quad 13t \qquad = 48 & \end{array}$$

$\therefore\ t = \frac{48}{13}$ and $7\left(\frac{48}{13}\right) - 4a = 20$

$\therefore\ t \approx 3.6923$ $\quad\therefore\ 5.8462 = 4a$

$\therefore\ t \approx 3$ min 41.54 sec $\quad\therefore\ a \approx 1.4615 \approx 1$ min 27.7 sec

So, as $a \approx 1.4615$, Q fired at 1:35:28 pm, and the explosion occurred at 1:37:42 pm.

9 **a** $\overrightarrow{\mathrm{AB}} = \begin{pmatrix} 3-6 \\ 10-9 \\ 2.5-3 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ -0.5 \end{pmatrix}$

b $|\overrightarrow{\mathrm{AB}}| = \sqrt{(-3)^2 + 1^2 + (-0.5)^2}$

$= \sqrt{10.25}$ km

c $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix} + t\begin{pmatrix} -3 \\ 1 \\ -0.5 \end{pmatrix}, \quad t \in \mathbb{R}$

The helicopter travels $\sqrt{10.25}$ km in 10 minutes.

$\therefore$ the helicopter's speed is

$6 \times \sqrt{10.25} \approx 19.2\ \mathrm{km\,h^{-1}}$.

d If $z = 0,\ 3 + (-0.5)t = 0$

$\therefore\ t = 6$

$t = 1$ represents 10 minutes, so $t = 6$ represents 60 minutes.

$\therefore$ the helicopter lands on the helipad after 1 hour.

EXERCISE 15F

1 **a** Let N be the point on the line closest to P.
N has coordinates $(2 + t, 3 + 2t)$ for some $t \in \mathbb{R}$.

$\overrightarrow{\mathrm{PN}}$ is $\begin{pmatrix} 2+t-3 \\ 3+2t-2 \end{pmatrix} = \begin{pmatrix} t-1 \\ 2t+1 \end{pmatrix}$.

Now $\overrightarrow{\mathrm{PN}} \bullet \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 0$, as $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is the direction vector of the line.

$\therefore\ \begin{pmatrix} t-1 \\ 2t+1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 0$

$\therefore\ (t-1) + 2(2t+1) = 0$

$\therefore\ t - 1 + 4t + 2 = 0$

$\therefore\ 5t = -1$

$\therefore\ t = -\frac{1}{5}$

Thus $\overrightarrow{\mathrm{PN}} = \begin{pmatrix} -\frac{1}{5} - 1 \\ -\frac{2}{5} + 1 \end{pmatrix} = \begin{pmatrix} -\frac{6}{5} \\ \frac{3}{5} \end{pmatrix}$

$= \frac{3}{5}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

and $|\overrightarrow{\mathrm{PN}}| = \frac{3}{5}\left|\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right| = \frac{3}{5}\sqrt{(-2)^2 + 1^2}$

$= \frac{3}{5}\sqrt{5}$

So the shortest distance from P to the line is $\frac{3}{5}\sqrt{5}$ units.

b Let N be the point on the line closest to Q.
N has coordinates $(t, 1-t)$ for some $t \in \mathbb{R}$.

$\overrightarrow{\mathrm{QN}}$ is $\begin{pmatrix} t-(-1) \\ 1-t-1 \end{pmatrix} = \begin{pmatrix} t+1 \\ -t \end{pmatrix}$.

Now $\overrightarrow{\mathrm{QN}} \bullet \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$, as $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is the direction vector of the line.

$\therefore\ \begin{pmatrix} t+1 \\ -t \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$

$\therefore\ (t+1) + (-1)(-t) = 0$

$\therefore\ t + 1 + t = 0$

$\therefore\ 2t = -1$

$\therefore\ t = -\frac{1}{2}$

Thus $\overrightarrow{\mathrm{QN}} = \begin{pmatrix} -\frac{1}{2} + 1 \\ -\left(-\frac{1}{2}\right) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

and $|\overrightarrow{\mathrm{QN}}| = \frac{1}{2}\left|\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right| = \frac{1}{2}\sqrt{1^2 + 1^2}$

$= \frac{1}{2}\sqrt{2}$

So the shortest distance from Q to the line is $\frac{1}{2}\sqrt{2}$ units.

c Let N be the point on the line closest to R.
N has coordinates $(2+s, 3-s)$ for some $s \in \mathbb{R}$.

$\overrightarrow{\mathrm{RN}}$ is $\begin{pmatrix} 2+s-(-3) \\ 3-s-(-1) \end{pmatrix} = \begin{pmatrix} s+5 \\ 4-s \end{pmatrix}$.

Now $\overrightarrow{\mathrm{RN}} \bullet \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$, as $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is the direction vector of the line.

$$\therefore \begin{pmatrix} s+5 \\ 4-s \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$$

$$\therefore (s+5)+(-1)(4-s)=0$$
$$\therefore s+5-4+s=0$$
$$\therefore 2s=-1$$
$$\therefore s=-\tfrac{1}{2}$$

Thus $\overrightarrow{\mathrm{RN}} = \begin{pmatrix} -\frac{1}{2}+5 \\ 4-(-\frac{1}{2}) \end{pmatrix} = \begin{pmatrix} \frac{9}{2} \\ \frac{9}{2} \end{pmatrix}$
$= \frac{9}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

and $|\overrightarrow{\mathrm{RN}}| = \frac{9}{2}\left|\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right| = \frac{9}{2}\sqrt{1^2+1^2}$
$= \frac{9}{2}\sqrt{2}$

So the shortest distance from R to the line is $\frac{9}{2}\sqrt{2}$ units.

d Let N be the point on the line closest to S.
N has coordinates $(2+3t, 5-7t)$ for some $t \in \mathbb{R}$.

$\overrightarrow{\mathrm{SN}}$ is $\begin{pmatrix} 2+3t-5 \\ 5-7t-(-2) \end{pmatrix} = \begin{pmatrix} 3t-3 \\ 7-7t \end{pmatrix}$.

Now $\overrightarrow{\mathrm{SN}} \bullet \begin{pmatrix} 3 \\ -7 \end{pmatrix} = 0$, as $\begin{pmatrix} 3 \\ -7 \end{pmatrix}$ is the direction vector of the line.

$$\therefore \begin{pmatrix} 3t-3 \\ 7-7t \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -7 \end{pmatrix} = 0$$

$$\therefore 3(3t-3)-7(7-7t)=0$$
$$\therefore 9t-9-49+49t=0$$
$$\therefore 58t=-58$$
$$\therefore t=1$$

Thus $\overrightarrow{\mathrm{SN}} = \begin{pmatrix} 3(1)-3 \\ 7-7(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\therefore |\overrightarrow{\mathrm{SN}}| = 0$

So S actually lies on the line, and the shortest distance is 0 units.

2 **a** $6\mathbf{i}-6\mathbf{j}$

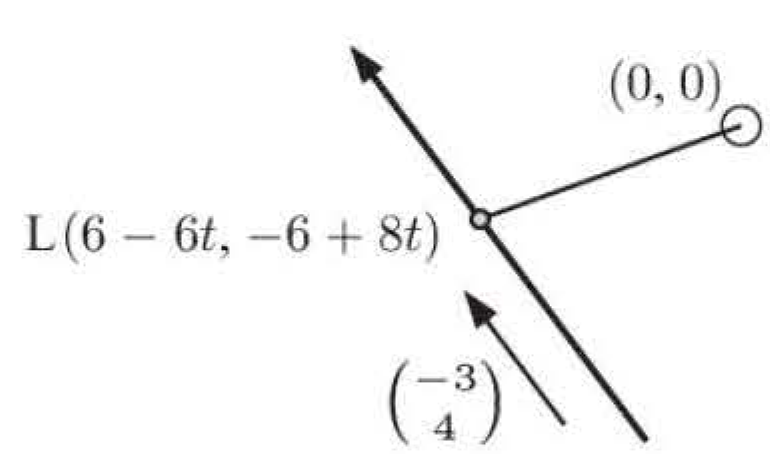
b The length of $\begin{pmatrix} -3 \\ 4 \end{pmatrix} = \sqrt{(-3)^2+4^2} = \sqrt{25} = 5$

As the speed is 10 $\mathrm{km\,h^{-1}}$, the liner has velocity vector $2\begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} -6 \\ 8 \end{pmatrix}$.

$\therefore$ the liner has position $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \end{pmatrix} + t\begin{pmatrix} -6 \\ 8 \end{pmatrix} = \begin{pmatrix} 6-6t \\ -6+8t \end{pmatrix}$, $t \geqslant 0$, t in hours.

c The liner is due east when $y=0$
$$\therefore -6+8t=0$$
$$\therefore \text{at } t=\tfrac{3}{4} \text{ hours}$$

d The liner L is nearest the fishing boat O when $\overrightarrow{\mathrm{OL}} \perp \binom{-3}{4}$



$$\therefore \overrightarrow{\mathrm{OL}} \bullet \begin{pmatrix} -3 \\ 4 \end{pmatrix} = 0$$

$$\therefore \begin{pmatrix} 6-6t \\ -6+8t \end{pmatrix} \bullet \begin{pmatrix} -3 \\ 4 \end{pmatrix} = 0$$

$$\therefore (-18+18t)+(-24+32t)=0$$
$$\therefore 50t=42$$
$$\therefore t=0.84 \text{ hours} = 50.4 \text{ minutes}$$

and when $t=0.84$, $\overrightarrow{\mathrm{OL}} = \begin{pmatrix} 6-6(0.84) \\ -6+8(0.84) \end{pmatrix} = \begin{pmatrix} 0.96 \\ 0.72 \end{pmatrix}$

$\therefore$ the liner is closest to the fishing boat after 0.84 hours or 50.4 minutes, when it is at (0.96, 0.72).

3 **a** $|\mathbf{b}| = \sqrt{(-3)^2 + (-1)^2} = \sqrt{10}$

As the speed is $40\sqrt{10}$ km h^{-1}, the velocity vector is $40\begin{pmatrix}-3\\-1\end{pmatrix} = \begin{pmatrix}-120\\-40\end{pmatrix}$.

b $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}200\\100\end{pmatrix} + t\begin{pmatrix}-120\\-40\end{pmatrix}$, $t \geqslant 0$ $\{t = 0$ at 12:00 noon$\}$

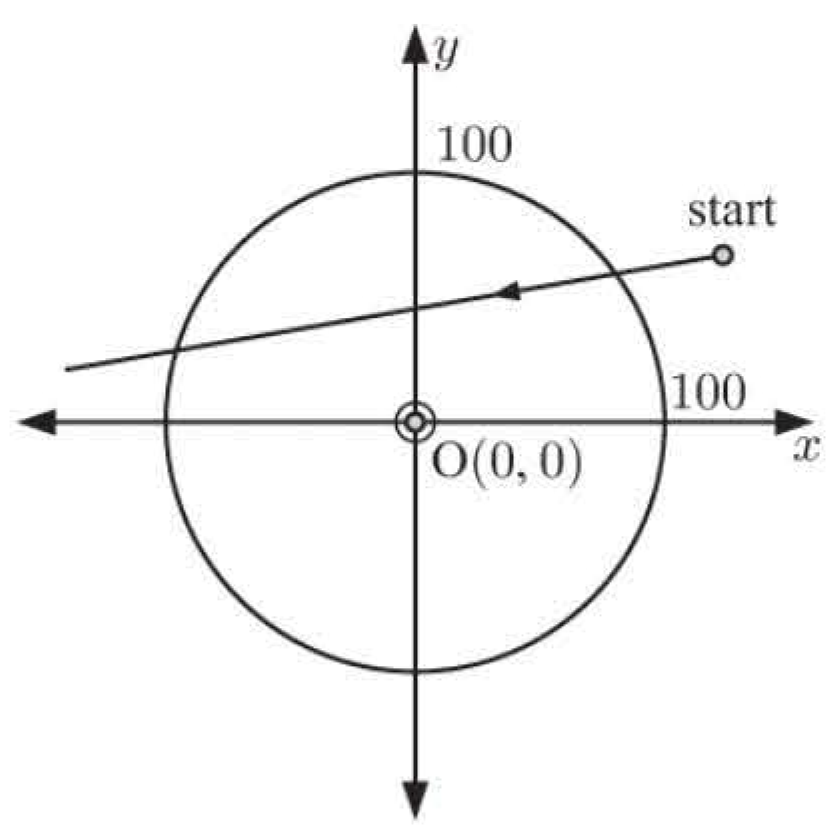
c At 1:00 pm, $t = 1$ and $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}200-120\\100-40\end{pmatrix} = \begin{pmatrix}80\\60\end{pmatrix}$

$\therefore$ the aircraft is at (80, 60).

d The distance from O(0, 0) to $P_1(80, 60)$ is $\left|\begin{pmatrix}80\\60\end{pmatrix}\right| = \sqrt{80^2+60^2} = 100$ km,

which is when it becomes visible to radar. {within 100 km of O(0, 0)}

e



A general point on the path is P$(200 - 120t, 100 - 40t)$.

Now $\overrightarrow{OP} = \begin{pmatrix}200-120t\\100-40t\end{pmatrix}$,

and for the closest point $\overrightarrow{OP} \bullet \begin{pmatrix}-3\\-1\end{pmatrix} = 0$

$$\therefore\ -3(200-120t) - 1(100-40t) = 0$$
$$\therefore\ -700 + 400t = 0$$
$$\therefore\ t = \tfrac{7}{4} = 1\tfrac{3}{4} \text{ hours}$$

$\therefore$ the time when the aircraft is closest is 1:45 pm, and

at this time $\overrightarrow{OP} = \begin{pmatrix}200-120(\frac{7}{4})\\100-40(\frac{7}{4})\end{pmatrix} = \begin{pmatrix}-10\\30\end{pmatrix}$

$$\therefore\ d_{\min} = \sqrt{(-10)^2+30^2} \approx 31.6 \text{ km}$$

f It disappears from radar when $|\overrightarrow{OP}| = 100$ and $t > 1\frac{3}{4}$

$$\therefore\ \sqrt{(200-120t)^2 + (100-40t)^2} = 100$$
$$\therefore\ 40\,000 - 48\,000t + 14\,400t^2 + 10\,000 - 8000t + 1600t^2 = 10\,000$$
$$\therefore\ 16\,000t^2 - 56\,000t + 40\,000 = 0$$
$$\therefore\ 16t^2 - 56t + 40 = 0 \qquad \{\div 1000\}$$
$$\therefore\ 2t^2 - 7t + 5 = 0 \qquad \{\div 8\}$$
$$\therefore\ (2t-5)(t-1) = 0$$
$$\therefore\ t = \tfrac{5}{2} \qquad \{\text{as } t > 1\tfrac{3}{4}\}$$

So, the aircraft disappears from the radar screen $2\frac{1}{2}$ hours after noon, or at 2:30 pm.

4 **a** At A, $y = 0$
$\therefore\ 2x = 36$
$\therefore\ x = 18$

At B, $x = 0$
$\therefore\ 3y = 36$
$\therefore\ y = 12$

So A is (18, 0) and B is (0, 12).

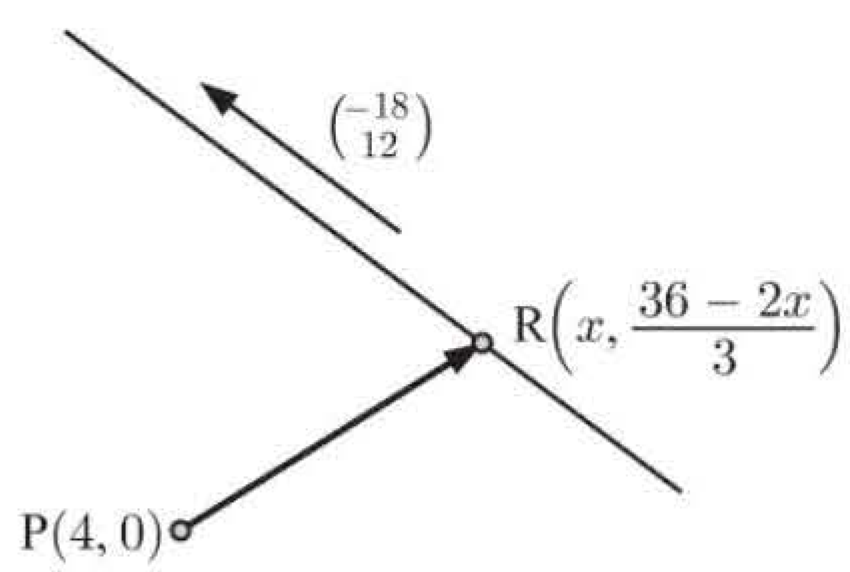
b $2x + 3y = 36$
$\therefore\ 3y = 36 - 2x$
$\therefore\ y = \dfrac{36-2x}{3}$

$\therefore$ any point R on the railway track can be written $R\left(x, \dfrac{36-2x}{3}\right)$.

c $\overrightarrow{PR} = \begin{pmatrix}x-4\\\frac{36-2x}{3}-0\end{pmatrix} = \begin{pmatrix}x-4\\\frac{36-2x}{3}\end{pmatrix}$,

$\overrightarrow{AB} = \begin{pmatrix}0-18\\12-0\end{pmatrix} = \begin{pmatrix}-18\\12\end{pmatrix}$

d The point closest to the railway track is R such that $\overrightarrow{PR} \perp \overrightarrow{AB}$.



$$\therefore \quad \overrightarrow{PR} \bullet \overrightarrow{AB} = 0$$

$$\therefore \quad \begin{pmatrix} x-4 \\ \frac{36-2x}{3} \end{pmatrix} \bullet \begin{pmatrix} -18 \\ 12 \end{pmatrix} = 0$$

$$\therefore \quad -18(x-4) + 4(36-2x) = 0$$

$$\therefore \quad -18x + 72 + 144 - 8x = 0$$

$$\therefore \quad 26x = 216$$

$$\therefore \quad x = \tfrac{108}{13}$$

Now when $x = \frac{108}{13}$, $\dfrac{36-2x}{3} = 12 - \frac{2}{3}x = 12 - \frac{2}{3}\left(\frac{108}{13}\right) = \frac{84}{13}$. So R is $\left(\frac{108}{13}, \frac{84}{13}\right)$.

$$|\overrightarrow{PR}| = \sqrt{\left(\tfrac{108}{13} - 4\right)^2 + \left(\tfrac{84}{13} - 0\right)^2} = \sqrt{\tfrac{784}{13}} \approx 7.77 \text{ km}$$

The closest point on the track to the camp is $\left(\frac{108}{13}, \frac{84}{13}\right)$, a distance of 7.77 km.

5 For A, $x_A(t) = 3 - t$, $y_A(t) = 2t - 4$ For B, $x_B(t) = 4 - 3t$, $y_B(t) = 3 - 2t$

a When $t = 0$, $x_A(0) = 3$, $y_A(0) = -4$ and $x_B(0) = 4$, $y_B(0) = 3$

$\therefore$ A is at $(3, -4)$. $\therefore$ B is at $(4, 3)$.

b The velocity vector of A is $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and the velocity vector of B is $\begin{pmatrix} -3 \\ -2 \end{pmatrix}$.

c If the angle is θ, $\begin{pmatrix} -1 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} -3 \\ -2 \end{pmatrix} = \sqrt{1+4}\sqrt{9+4}\cos\theta$

$$\therefore \quad 3 - 4 = \sqrt{5}\sqrt{13}\cos\theta$$

$$\therefore \quad \tfrac{-1}{\sqrt{65}} = \cos\theta \quad \text{and so} \quad \theta \approx 97.1^\circ$$

$\therefore$ the acute angle between the paths is $\approx 82.9^\circ$.

d If D is the distance between them, then

$$\begin{aligned} D &= \sqrt{[(4-3t)-(3-t)]^2 + [(3-2t)-(2t-4)]^2} \\ &= \sqrt{[1-2t]^2 + [7-4t]^2} \\ &= \sqrt{1 - 4t + 4t^2 + 49 - 56t + 16t^2} \\ &= \sqrt{20t^2 - 60t + 50} \end{aligned}$$

Under the square root we have a quadratic in t, so D is a minimum when

$$t = -\frac{b}{2a} = \tfrac{60}{40} = 1\tfrac{1}{2}$$

$\therefore$ $t = 1.5$ hours

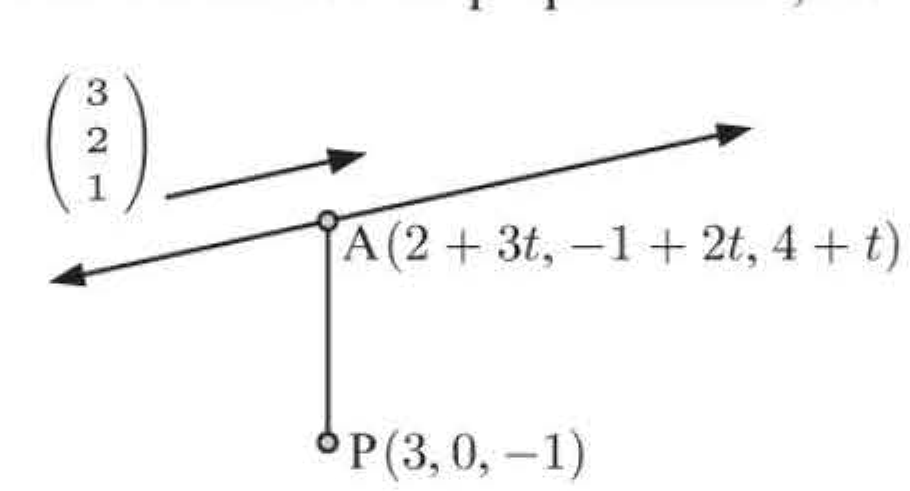
$\therefore$ the boats are closest after $1\frac{1}{2}$ hours.

6 a The direction vector of the line is $\mathbf{b} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$.

Let the point $(3, 0, -1)$ be P, and $A(2+3t, -1+2t, 4+t)$ be any point on the line.

$$\therefore \quad \overrightarrow{PA} = \begin{pmatrix} 2+3t-3 \\ -1+2t-0 \\ 4+t-(-1) \end{pmatrix} = \begin{pmatrix} -1+3t \\ -1+2t \\ 5+t \end{pmatrix}$$

Now $\overrightarrow{PA}$ and $\mathbf{b}$ are perpendicular, so $\overrightarrow{PA} \bullet \mathbf{b} = 0$.



$$\therefore \quad \begin{pmatrix} -1+3t \\ -1+2t \\ 5+t \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = 0$$

$$\therefore \quad -3 + 9t - 2 + 4t + 5 + t = 0$$

$$\therefore \quad 14t = 0$$

$$\therefore \quad t = 0$$

Substituting $t = 0$ into the parametric equations, we obtain the foot of the perpendicular $(2, -1, 4)$.

b When $t = 0$, $\overrightarrow{PA} = \begin{pmatrix} -1 \\ -1 \\ 5 \end{pmatrix}$, so $PA = \sqrt{1+1+25}$
$= \sqrt{27}$ units

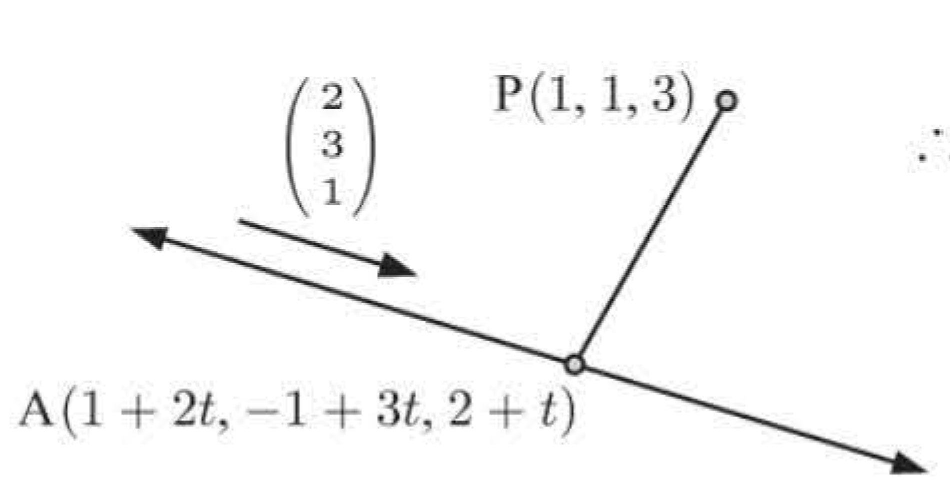
$\therefore$ the shortest distance from the point to the line is $\sqrt{27}$ units.

7 **a** The line has direction vector $\mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$.

Let the point $(1, 1, 3)$ be P and $A(1+2t, -1+3t, 2+t)$ be any point on the line.

$\therefore \overrightarrow{PA} = \begin{pmatrix} 1+2t-1 \\ -1+3t-1 \\ 2+t-3 \end{pmatrix} = \begin{pmatrix} 2t \\ -2+3t \\ -1+t \end{pmatrix}$.

Now $\overrightarrow{PA}$ and $\mathbf{b}$ are perpendicular, so $\overrightarrow{PA} \bullet \mathbf{b} = 0$



$\therefore \begin{pmatrix} 2t \\ -2+3t \\ -1+t \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = 0$

$\therefore 4t - 6 + 9t - 1 + t = 0$

$\therefore 14t = 7$

$\therefore t = \frac{1}{2}$

Substituting $t = \frac{1}{2}$ into the parametric equations, we obtain the foot of the perpendicular $(2, \frac{1}{2}, \frac{5}{2})$.

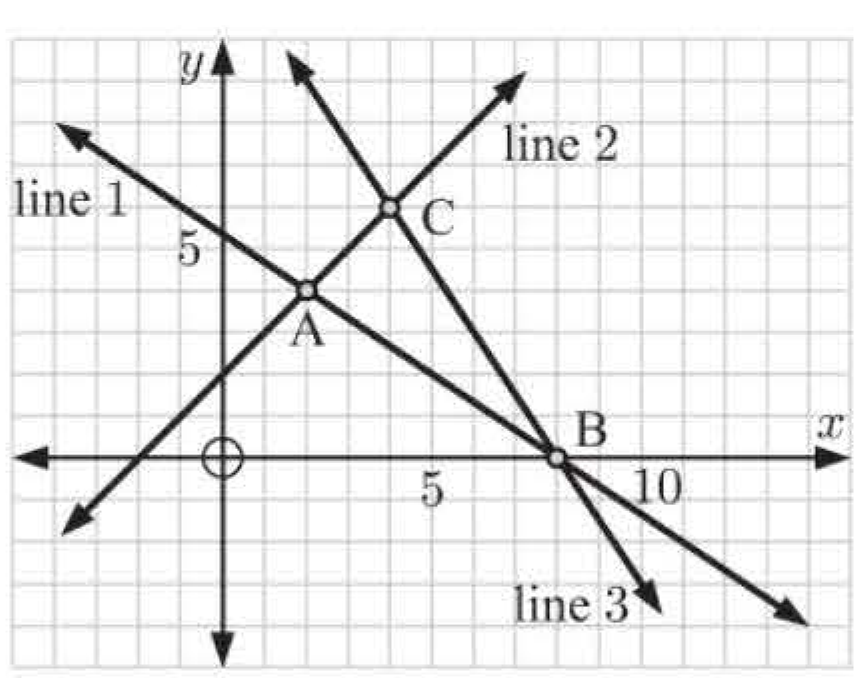
b When $t = \frac{1}{2}$, $\overrightarrow{PA} = \begin{pmatrix} 1 \\ -2+\frac{3}{2} \\ -1+\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$

$\therefore PA = \sqrt{1 + \frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{3}{2}}$ units

$\therefore$ the shortest distance from the point to the line is $\sqrt{\frac{3}{2}}$ units.

EXERCISE 15G

1 **a**



b A is $(2, 4)$, B is $(8, 0)$, C is $(4, 6)$

c $BC = \sqrt{(4-8)^2 + (6-0)^2} = \sqrt{16+36}$
$= \sqrt{52}$ units

$AB = \sqrt{(8-2)^2 + (0-4)^2} = \sqrt{36+16}$
$= \sqrt{52}$ units

$\therefore$ BC = AB and so $\triangle$ABC is isosceles.

d Line 1 and Line 2 meet at A.

$\therefore \begin{pmatrix} -1 \\ 6 \end{pmatrix} + r\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + s\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\therefore \begin{pmatrix} 3r - s \\ -2r - s \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$

$\therefore 3r - s = 1$

and $2r + s = 4$

Adding, $5r = 5$ $\therefore r = 1$

$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \end{pmatrix} + \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ ✓

Line 2 and Line 3 meet at C.

$\therefore \begin{pmatrix} 0 \\ 2 \end{pmatrix} + s\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ -3 \end{pmatrix} + t\begin{pmatrix} -2 \\ 3 \end{pmatrix}$

$\therefore \begin{pmatrix} s + 2t \\ s - 3t \end{pmatrix} = \begin{pmatrix} 10 \\ -5 \end{pmatrix}$

$\therefore s + 2t = 10$

$-s + 3t = 5$

Adding, $5t = 15$ $\therefore t = 3$

$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10 \\ -3 \end{pmatrix} + 3\begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$ ✓

Line 1 and Line 3 meet at B.

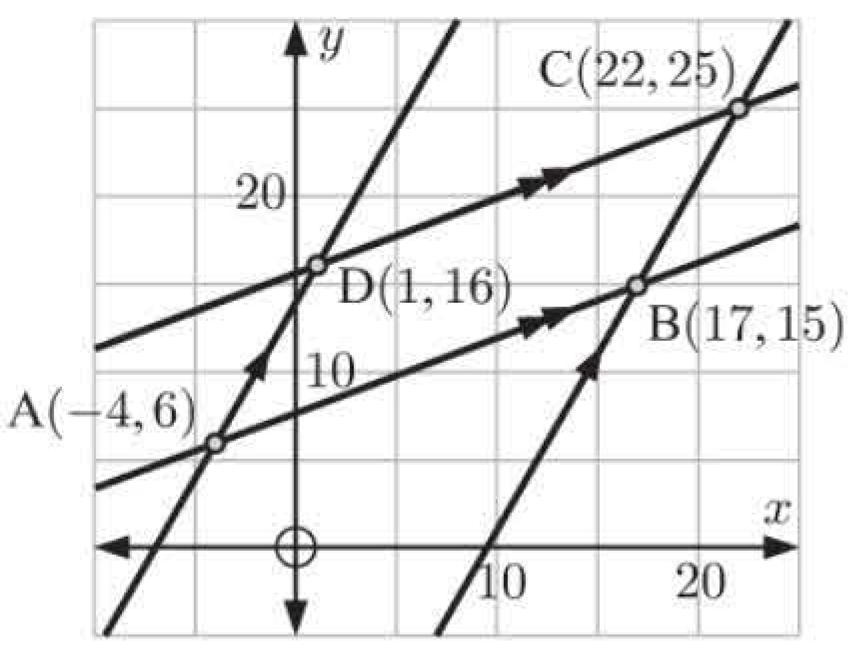
$$\therefore \begin{pmatrix}-1\\6\end{pmatrix}+r\begin{pmatrix}3\\-2\end{pmatrix}=\begin{pmatrix}10\\-3\end{pmatrix}+t\begin{pmatrix}-2\\3\end{pmatrix}$$

$$\therefore \begin{pmatrix}3r+2t\\-2r-3t\end{pmatrix}=\begin{pmatrix}11\\-9\end{pmatrix}$$

$$\begin{aligned}\therefore\ 3r+2t&=11 \quad .... (1)\\ -2r-3t&=-9 \quad .... (2)\\ \therefore\ 9r+6t&=33 \quad \{3\times(1)\}\\ -4r-6t&=-18 \quad \{2\times(2)\}\\ \text{Adding,}\quad 5r&=15\\ \therefore\ r&=3\end{aligned}$$

So, $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-1\\6\end{pmatrix}+3\begin{pmatrix}3\\-2\end{pmatrix}=\begin{pmatrix}8\\0\end{pmatrix}$ ✓

2 **a**



b A(−4, 6), B(17, 15), C(22, 25), D(1, 16)

c Lines (AB) and (AD) meet at A.

$$\therefore \begin{pmatrix}-4\\6\end{pmatrix}+r\begin{pmatrix}7\\3\end{pmatrix}=\begin{pmatrix}-4\\6\end{pmatrix}+s\begin{pmatrix}1\\2\end{pmatrix}$$

$$\therefore \begin{pmatrix}7r-s\\3r-2s\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{aligned}\therefore\ 7r-s&=0 \quad .... (1)\\ \text{and}\quad 3r-2s&=0\\ -14r+2s&=0 \quad \{-2\times(1)\}\\ \text{Adding,}\quad -11r&=0\\ \therefore\ r&=0\end{aligned}$$

$$\therefore \begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-4\\6\end{pmatrix} \checkmark$$

Lines (AB) and (CB) meet at B.

$$\therefore \begin{pmatrix}-4\\6\end{pmatrix}+r\begin{pmatrix}7\\3\end{pmatrix}=\begin{pmatrix}22\\25\end{pmatrix}+u\begin{pmatrix}-1\\-2\end{pmatrix}$$

$$\therefore \begin{pmatrix}7r+u\\3r+2u\end{pmatrix}=\begin{pmatrix}26\\19\end{pmatrix}$$

$$\begin{aligned}\therefore\ 7r+u&=26 \quad .... (1)\\ \text{and}\quad 3r+2u&=19\\ -14r-2u&=-52 \quad \{-2\times(1)\}\\ \text{Adding,}\quad -11r&=-33\\ \therefore\ r&=3\end{aligned}$$

$$\therefore \begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-4\\6\end{pmatrix}+3\begin{pmatrix}7\\3\end{pmatrix}=\begin{pmatrix}17\\15\end{pmatrix} \checkmark$$

Lines (CD) and (CB) meet at C.

$$\therefore \begin{pmatrix}22\\25\end{pmatrix}+t\begin{pmatrix}-7\\-3\end{pmatrix}=\begin{pmatrix}22\\25\end{pmatrix}+u\begin{pmatrix}-1\\-2\end{pmatrix}$$

$$\therefore \begin{pmatrix}-7t+u\\-3t+2u\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{aligned}\therefore\ -7t+u&=0 \quad .... (1)\\ \text{and}\quad -3t+2u&=0\\ 14t-2u&=0 \quad \{-2\times(1)\}\\ \text{Adding,}\quad 11t&=0\\ \therefore\ t&=0\end{aligned}$$

$$\therefore \begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}22\\25\end{pmatrix} \checkmark$$

Lines (AD) and (CD) meet at D.

$$\therefore \begin{pmatrix}-4\\6\end{pmatrix}+s\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}22\\25\end{pmatrix}+t\begin{pmatrix}-7\\-3\end{pmatrix}$$

$$\therefore \begin{pmatrix}s+7t\\2s+3t\end{pmatrix}=\begin{pmatrix}26\\19\end{pmatrix}$$

$$\begin{aligned}\therefore\ s+7t&=26 \quad .... (1)\\ \text{and}\quad 2s+3t&=19\\ -2s-14t&=-52 \quad \{-2\times(1)\}\\ \text{Adding,}\quad -11t&=-33\\ \therefore\ t&=3\end{aligned}$$

$$\therefore \begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}22\\25\end{pmatrix}+3\begin{pmatrix}-7\\-3\end{pmatrix}=\begin{pmatrix}1\\16\end{pmatrix} \checkmark$$

3 **a** Lines (AB) and (AC) meet at A.

$$\therefore \begin{pmatrix} 0 \\ 2 \end{pmatrix} + r\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 2r-t \\ r+t \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

$$\therefore 2r - t = 0$$
$$r + t = 3$$
Adding, $3r = 3$

$$\therefore r = 1$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$\therefore$ A is (2, 3)

Lines (AB) and (BC) meet at B.

$$\therefore \begin{pmatrix} 0 \\ 2 \end{pmatrix} + r\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \end{pmatrix} + s\begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 2r+s \\ r+2s \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$$

$$\therefore -4r - 2s = -16$$
$$r + 2s = 4$$
Adding, $-3r = -12$

$$\therefore r = 4$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + 4\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \end{pmatrix}$$

$\therefore$ B is (8, 6)

Lines (BC) and (AC) meet at C.

$$\therefore \begin{pmatrix} 8 \\ 6 \end{pmatrix} + s\begin{pmatrix} -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} -s-t \\ -2s+t \end{pmatrix} = \begin{pmatrix} -8 \\ -1 \end{pmatrix}$$

$$\therefore -s - t = -8$$
$$-2s + t = -1$$
Adding, $-3s = -9$

$$\therefore s = 3$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \end{pmatrix} + 3\begin{pmatrix} -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

$\therefore$ C is (5, 0)

b A(2, 3), B(8, 6), C(5, 0)

$$\text{AB} = \sqrt{(8-2)^2 + (6-3)^2} = \sqrt{36+9} = \sqrt{45}$$

$$\text{BC} = \sqrt{(5-8)^2 + (0-6)^2} = \sqrt{9+36} = \sqrt{45}$$

$$\text{AC} = \sqrt{(5-2)^2 + (0-3)^2} = \sqrt{9+9} = \sqrt{18}$$

The two equal sides are [AB] and [BC] and they have length $\sqrt{45}$ units. [AC] has length $\sqrt{18}$ units.

4 **a** Lines (QP) and (PR) meet at P.

$$\therefore \begin{pmatrix} 3 \\ -1 \end{pmatrix} + r\begin{pmatrix} 14 \\ 10 \end{pmatrix} = \begin{pmatrix} 0 \\ 18 \end{pmatrix} + t\begin{pmatrix} 5 \\ -7 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 14r-5t \\ 10r+7t \end{pmatrix} = \begin{pmatrix} -3 \\ 19 \end{pmatrix}$$

$$\therefore 14r - 5t = -3 \quad .... (1)$$
$$10r + 7t = 19 \quad .... (2)$$

$$\therefore 98r - 35t = -21 \quad \{7 \times (1)\}$$
$$50r + 35t = 95 \quad \{5 \times (2)\}$$
Adding, $148r = 74$

$$\therefore r = \tfrac{1}{2}$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + \tfrac{1}{2}\begin{pmatrix} 14 \\ 10 \end{pmatrix} = \begin{pmatrix} 10 \\ 4 \end{pmatrix}$$

$\therefore$ P is (10, 4)

Lines (QR) and (PR) meet at R.

$$\therefore \begin{pmatrix} 3 \\ -1 \end{pmatrix} + s\begin{pmatrix} 17 \\ -9 \end{pmatrix} = \begin{pmatrix} 0 \\ 18 \end{pmatrix} + t\begin{pmatrix} 5 \\ -7 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 17s-5t \\ -9s+7t \end{pmatrix} = \begin{pmatrix} -3 \\ 19 \end{pmatrix}$$

$$\therefore 17s - 5t = -3 \quad .... (1)$$
$$-9s + 7t = 19 \quad .... (2)$$

$$\therefore 119s - 35t = -21 \quad \{7 \times (1)\}$$
$$-45s + 35t = 95 \quad \{5 \times (2)\}$$
Adding, $74s = 74$

$$\therefore s = 1$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + \begin{pmatrix} 17 \\ -9 \end{pmatrix} = \begin{pmatrix} 20 \\ -10 \end{pmatrix}$$

$\therefore$ R is (20, −10)

Lines (QP) and (PR) meet at Q.

$$\begin{pmatrix} 3 \\ -1 \end{pmatrix} + r\begin{pmatrix} 14 \\ 10 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + s\begin{pmatrix} 17 \\ -9 \end{pmatrix}$$

$$\therefore \quad r\begin{pmatrix} 14 \\ 10 \end{pmatrix} = s\begin{pmatrix} 17 \\ -9 \end{pmatrix}$$

$$\therefore \quad r = s = 0$$

$$\text{So,} \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$\therefore$ Q is $(3, -1)$

b $\overrightarrow{PQ} = \begin{pmatrix} 3-10 \\ -1-4 \end{pmatrix} = \begin{pmatrix} -7 \\ -5 \end{pmatrix}$

$\overrightarrow{PR} = \begin{pmatrix} 20-10 \\ -10-4 \end{pmatrix} = \begin{pmatrix} 10 \\ -14 \end{pmatrix}$

and $\overrightarrow{PQ} \bullet \overrightarrow{PR} = -70 + 70 = 0$

c $[PQ] \perp [PR] \quad \therefore \quad \widehat{QPR} = 90°$

d $\text{Area} = \frac{1}{2} \mid \overrightarrow{PQ} \mid \mid \overrightarrow{PR} \mid$

$= \frac{1}{2}\sqrt{49+25}\sqrt{100+196}$

$= 74 \text{ units}^2$

5 **a** Lines (AB) and (AD) meet at A.

$$\therefore \quad \begin{pmatrix} 2 \\ 5 \end{pmatrix} + r\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + u\begin{pmatrix} -3 \\ 12 \end{pmatrix}$$

$$\therefore \quad \begin{pmatrix} 4r+3u \\ r-12u \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

$\therefore \quad 4r + 3u = 1$

$r - 12u = -4 \quad .... (1)$

$\therefore \quad 4r + 3u = 1$

$-4r + 48u = 16 \quad \{-4 \times (1)\}$

Adding, $\quad 51u = 17$

$\therefore \quad u = \frac{1}{3}$

$$\therefore \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \tfrac{1}{3}\begin{pmatrix} -3 \\ 12 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$\therefore$ A is $(2, 5)$

Lines (AB) and (BC) meet at B.

$$\therefore \quad \begin{pmatrix} 2 \\ 5 \end{pmatrix} + r\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 18 \\ 9 \end{pmatrix} + s\begin{pmatrix} -8 \\ 32 \end{pmatrix}$$

$$\therefore \quad \begin{pmatrix} 4r+8s \\ r-32s \end{pmatrix} = \begin{pmatrix} 16 \\ 4 \end{pmatrix}$$

$\therefore \quad 4r + 8s = 16 \quad .... (1)$

$r - 32s = 4 \quad .... (2)$

$\therefore \quad r + 2s = 4 \quad \{(1) \div 4\}$

$-r + 32s = -4 \quad \{-1 \times (2)\}$

Adding, $\quad 34s = 0$

$\therefore \quad s = 0$

$$\therefore \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 18 \\ 9 \end{pmatrix}$$

$\therefore$ B is $(18, 9)$

Lines (BC) and (CD) meet at C.

$$\therefore \quad \begin{pmatrix} 18 \\ 9 \end{pmatrix} + s\begin{pmatrix} -8 \\ 32 \end{pmatrix} = \begin{pmatrix} 14 \\ 25 \end{pmatrix} + t\begin{pmatrix} -8 \\ -2 \end{pmatrix}$$

$$\therefore \quad \begin{pmatrix} -8s+8t \\ 32s+2t \end{pmatrix} = \begin{pmatrix} -4 \\ 16 \end{pmatrix}$$

$\therefore \quad -8s + 8t = -4 \quad .... (1)$

$32s + 2t = 16$

$\therefore \quad 2s - 2t = 1 \quad \{(1) \div -4\}$

$32s + 2t = 16$

Adding, $\quad 34s \quad = 17$

$\therefore \quad s = \frac{1}{2}$

$$\therefore \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 18 \\ 9 \end{pmatrix} + \tfrac{1}{2}\begin{pmatrix} -8 \\ 32 \end{pmatrix} = \begin{pmatrix} 14 \\ 25 \end{pmatrix}$$

$\therefore$ C is $(14, 25)$

Lines (CD) and (AD) meet at D.

$$\therefore \quad \begin{pmatrix} 14 \\ 25 \end{pmatrix} + t\begin{pmatrix} -8 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + u\begin{pmatrix} -3 \\ 12 \end{pmatrix}$$

$$\therefore \quad \begin{pmatrix} -8t+3u \\ -2t-12u \end{pmatrix} = \begin{pmatrix} -11 \\ -24 \end{pmatrix}$$

$\therefore \quad -8t + 3u = -11 \quad .... (1)$

$-2t - 12u = -24 \quad .... (2)$

$\therefore \quad 16t - 6u = 22 \quad \{(-2) \times (1)\}$

$t + 6u = 12 \quad \{(2) \div -2\}$

Adding, $\quad 17t \quad = 34$

$\therefore \quad t = 2$

$$\therefore \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 14 \\ 25 \end{pmatrix} + 2\begin{pmatrix} -8 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ 21 \end{pmatrix}$$

$\therefore$ D is $(-2, 21)$

b $\overrightarrow{AC} = \begin{pmatrix} 14-2 \\ 25-5 \end{pmatrix} = \begin{pmatrix} 12 \\ 20 \end{pmatrix}$

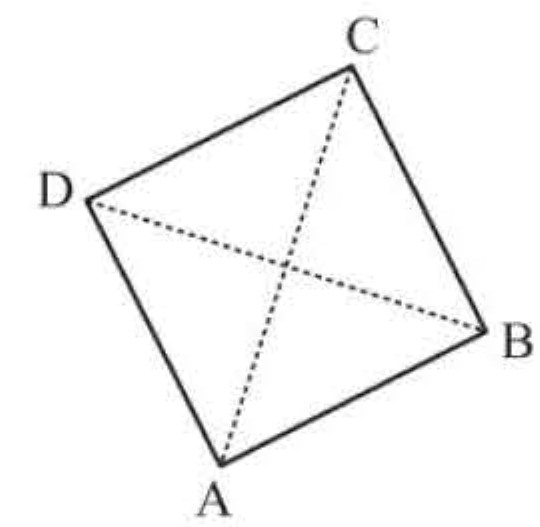
$\overrightarrow{DB} = \begin{pmatrix} 18--2 \\ 9-21 \end{pmatrix} = \begin{pmatrix} 20 \\ -12 \end{pmatrix}$

i $|\overrightarrow{AC}| = \sqrt{12^2+20^2} = \sqrt{544}$ units

ii $|\overrightarrow{DB}| = \sqrt{20^2+(-12)^2} = \sqrt{544}$ units

iii $\overrightarrow{AC} \bullet \overrightarrow{DB} = 240 - 240 = 0$

c The diagonals are perpendicular and equal in length, and as their midpoints are the same (at (8, 15)), ABCD is a square.



EXERCISE 15H.1

1 **a** In augmented matrix form, the system is:

$$\left[\begin{array}{cc|c} 1 & -2 & 8 \\ 4 & 1 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & 8 \\ 0 & 9 & -27 \end{array}\right] \quad R_2 \to R_2 - 4R_1 \qquad \begin{array}{ccc} 4 & 1 & 5 \\ -4 & 8 & -32 \\ \hline 0 & 9 & -27 \end{array}$$

From R_2, $9y = -27$
$\therefore\ y = -3$

Now $x - 2y = 8$
$\therefore\ x - 2(-3) = 8$
$\therefore\ x = 2$

So, the solution is $x = 2,\ \ y = -3$.

b In augmented matrix form, the system is:

$$\left[\begin{array}{cc|c} 4 & 5 & 21 \\ 5 & -3 & -20 \end{array}\right] \sim \left[\begin{array}{cc|c} 4 & 5 & 21 \\ 0 & -37 & -185 \end{array}\right] \quad R_2 \to 4R_2 - 5R_1 \qquad \begin{array}{ccc} 20 & -12 & -80 \\ -20 & -25 & -105 \\ \hline 0 & -37 & -185 \end{array}$$

From R_2, $-37y = -185$
$\therefore\ y = 5$

Now $4x + 5y = 21$
$\therefore\ 4x + 25 = 21$
$\therefore\ 4x = -4$
$\therefore\ x = -1$

So, the solution is $x = -1,\ \ y = 5$.

c In augmented matrix form, the system is:

$$\left[\begin{array}{cc|c} 3 & 1 & -10 \\ 2 & 5 & -24 \end{array}\right] \sim \left[\begin{array}{cc|c} 3 & 1 & -10 \\ 0 & 13 & -52 \end{array}\right] \quad R_2 \to 3R_2 - 2R_1 \qquad \begin{array}{ccc} 6 & 15 & -72 \\ -6 & -2 & 20 \\ \hline 0 & 13 & -52 \end{array}$$

From R_2, $13y = -52$
$\therefore\ y = -4$

Now $3x + y = -10$
$\therefore\ 3x + (-4) = -10$
$\therefore\ 3x = -6$
$\therefore\ x = -2$

So, the solution is $x = -2,\ \ y = -4$.

2 **a** One equation is not a multiple of the other and their gradients are not the same, so the lines are intersecting.

b $x + y = 7$ can be written as $3x + 3y = 21$ and the other line is $3x + 3y = 1$,
$\therefore$ the lines are parallel.

c The lines intersect at $(2\frac{1}{2}, 2)$.

d $x - 2y = 4$ can be written as $2x - 4y = 8$, so the lines are coincident.

e The lines are intersecting.

f $3x - 4y = 5$ can be written as $-3x + 4y = -5$ and the other line is $-3x + 4y = 2$,
$\therefore$ the lines are parallel.

3 **a** $x + 2y = 3$ can be written as $2x + 4y = 6$, $\therefore$ the equations represent coincident lines.
So, there are an infinite number of solutions (all the points on the line).

b As the second equation is an exact multiple of the first, it will give the same solutions as the first so it can be ignored.

c **i** If $x = t$, $t + 2y = 3$

$\therefore\ 2y = 3 - t$

$\therefore\ y = \dfrac{3-t}{2}$ $\therefore$ the solutions are $x = t$, $y = \dfrac{3-t}{2}$, $t \in \mathbb{R}$.

ii If $y = s$, $x + 2s = 3$

$\therefore\ x = 3 - 2s$ $\therefore$ the solutions are $x = 3 - 2s$, $y = s$, $s \in \mathbb{R}$.

4 **a** In augmented matrix form, the system is:

$$\left[\begin{array}{cc|c} 2 & 3 & 5 \\ 2 & 3 & 11 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & 3 & 5 \\ 0 & 0 & 6 \end{array}\right] \quad R_2 \to R_2 - R_1$$

$$\begin{array}{rrr} 2 & 3 & 11 \\ -2 & -3 & -5 \\ \hline 0 & 0 & 6 \end{array}$$

b R_2 shows $0x + 0y = 6$ $\therefore$ there are no solutions

c the lines are parallel

5 **a** In augmented matrix form, the system is:

$$\left[\begin{array}{cc|c} 2 & 3 & 5 \\ 4 & 6 & 10 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & 3 & 5 \\ 0 & 0 & 0 \end{array}\right] \quad R_2 \to R_2 - 2R_1$$

$$\begin{array}{rrr} 4 & 6 & 10 \\ -4 & -6 & -10 \\ \hline 0 & 0 & 0 \end{array}$$

b R_2 shows $0x + 0y = 0$, which is true for all x and y.

All solutions come from $2x + 3y = 5$. Let $x = t$, $y = \dfrac{5-2t}{3}$ for all values of t

$\therefore$ there are infinitely many solutions of the form $x = t$, $y = \dfrac{5-2t}{3}$, $t \in \mathbb{R}$.

The lines are coincident.

6 **a** In augmented matrix form, the system is:

$$\left[\begin{array}{cc|c} 3 & -1 & 2 \\ 6 & -2 & 4 \end{array}\right] \sim \left[\begin{array}{cc|c} 3 & -1 & 2 \\ 0 & 0 & 0 \end{array}\right] \quad R_2 \to R_2 - 2R_1$$

$$\begin{array}{rrr} 6 & -2 & 4 \\ -6 & 2 & -4 \\ \hline 0 & 0 & 0 \end{array}$$

R_2 shows $0x + 0y = 0$, which is true for all x and y.

So, there are infinitely many solutions {the lines are coincident}.

Substitute $x = t$ in the first equation $3x - y = 2$

$\therefore\ 3t - y = 2$

$y = 3t - 2$

So, the solutions have form $x = t$, $y = 3t - 2$, $t \in \mathbb{R}$.

b $3x - y = 2$ .... (1)

$6x - 2y = k$ .... (2)

If $k = 4$ then $6x - 2y = 4$, which is an exact multiple $(\times 2)$ of equation (1), $\therefore$ the lines are coincident and there are an infinite number of solutions of the form $x = t$, $y = 3t - 2$, $t \in \mathbb{R}$.

If $k \neq 4$ then the equations represent parallel lines $\therefore$ there are no solutions.

7 **a** In augmented matrix form, the system is:

$$\left[\begin{array}{cc|c} 3 & -1 & 8 \\ 6 & -2 & k \end{array}\right] \sim \left[\begin{array}{cc|c} 3 & -1 & 8 \\ 0 & 0 & k-16 \end{array}\right] \quad R_2 \to R_2 - 2R_1$$

$$\begin{array}{rrr} 6 & -2 & k \\ -6 & 2 & -16 \\ \hline 0 & 0 & k-16 \end{array}$$

b **i** If $k = 16$ there are infinitely many solutions.

ii Substitute $x = t$ in $3x - y = 8$, then $3t - y = 8$ $\therefore$ $y = 3t - 8$.

The solutions are $x = t$, $y = 3t - 8$, $t \in \mathbb{R}$.

c **i** The system has no solutions when $k - 16 \neq 0$, $\therefore$ $k \neq 16$.

ii The lines are parallel but not coincident.

8 **a** In augmented matrix form, the system is:

$$\left[\begin{array}{cc|c} 4 & 8 & 1 \\ 2 & -a & 11 \end{array}\right] \sim \left[\begin{array}{cc|c} 4 & 8 & 1 \\ 0 & -2a-8 & 21 \end{array}\right] \quad R_2 \to 2R_2 - R_1$$

$$\begin{array}{ccc} 4 & -2a & 22 \\ -4 & -8 & -1 \\ \hline 0 & -2a-8 & 21 \end{array}$$

$$\sim \left[\begin{array}{cc|c} 4 & 8 & 1 \\ 0 & 2a+8 & -21 \end{array}\right] \quad R_2 \to -R_2$$

b A unique solution exists provided $2a + 8 \neq 0 \quad \therefore \quad a \neq -4$.

c From R_2, $(2a+8)y = -21$

$$\therefore \quad y = \frac{-21}{2a+8}$$

and $4x + 8y = 1$

$$\therefore \quad 4x + 8\left(\frac{-21}{2a+8}\right) = 1$$

$$\therefore \quad 4x(2a+8) - 168 = 2a + 8$$

$$\therefore \quad 2x(2a+8) - 84 = a + 4$$

$$\therefore \quad 2x(2a+8) = a + 88$$

$$\therefore \quad x = \frac{a+88}{4a+16}$$

The solution is $x = \dfrac{a+88}{4a+16}$, $y = \dfrac{-21}{2a+8}$, $a \neq -4$.

d When $a = -4$ there are no solutions as the lines are parallel.

9 **a** In augmented matrix form, the system is:

$$\left[\begin{array}{cc|c} m & 2 & 6 \\ 2 & m & 6 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} m & 2 & 6 \\ 0 & m^2-4 & 6m-12 \end{array}\right] \quad R_2 \to mR_2 - 2R_1$$

$$\begin{array}{ccc} 2m & m^2 & 6m \\ -2m & -4 & -12 \\ \hline 0 & m^2-4 & 6m-12 \end{array}$$

A unique solution exists provided $m^2 - 4 \neq 0$.

So, there is a unique solution for all m except $m = \pm 2$.

b In R_2, $(m^2 - 4)y = 6m - 12$

$$\therefore \quad y = \frac{6(m-2)}{(m-2)(m+2)}$$

$$\therefore \quad y = \frac{6}{m+2} \quad \text{provided} \quad m \neq \pm 2$$

Substituting in $mx + 2y = 6$

gives $mx + 2\left(\dfrac{6}{m+2}\right) = 6$

$$\therefore \quad m(m+2)x + 12 = 6(m+2)$$

$$\therefore \quad m(m+2)x = 6m + 12 - 12$$

$$\therefore \quad m(m+2)x = 6m$$

$$\therefore \quad x = \frac{6}{m+2}$$

So, the unique solution is $x = \dfrac{6}{m+2}$, $y = \dfrac{6}{m+2}$ when $m \neq \pm 2$.

c When $m = 2$, the equations are $2x + 2y = 6$ and $2x + 2y = 6$, $\therefore$ the lines are coincident. So, there are an infinite number of solutions of the form $x = t$, $y = \dfrac{6-2t}{2} = 3 - t$ for all $t \in \mathbb{R}$.

When $m = -2$, the equations are $-2x + 2y = 6$ and $2x - 2y = 6$
or $-2x + 2y = -6$

$\therefore$ the lines are parallel and there are no solutions.

EXERCISE 15H.2

1 **a** Line 1 has direction vector $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ and line 2 has direction vector $\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$.

As one vector is not a scalar multiple of the other, the lines are not parallel.

Now $\quad 1 + 2t = -2 + 3s \qquad 2 - t = 3 - s \qquad 3 + t = 1 + 2s$

$\therefore \quad 2t - 3s = -3 \;\; .... \;(1) \qquad \therefore \quad -t + s = 1 \;\; .... \;(2) \qquad \therefore \quad t - 2s = -2 \;\; .... \;(3)$

Solving (2) and (3) simultaneously:
$$\begin{aligned} -t+s&=1\\ t-2s&=-2\\ \hline -s&=-1 \quad \therefore\ s=1 \text{ and } t=0\end{aligned}$$

and in (1), $\text{LHS} = 2t-3s = 2(0)-3(1) = -3$ ✓

$\therefore$ $s=1$, $t=0$ satisfies all three equations

$\therefore$ the two lines meet at $(1, 2, 3)$ $\{$using $t=0$ or $s=1\}$

The acute angle between the lines has $\cos\theta = \dfrac{|6+1+2|}{\sqrt{4+1+1}\,\sqrt{9+1+4}} = \dfrac{9}{\sqrt{84}}$

and so $\theta \approx 10.9°$

b Line 1 has direction vector $\begin{pmatrix}2\\-12\\12\end{pmatrix}$ and line 2 has direction vector $\begin{pmatrix}4\\3\\-1\end{pmatrix}$.

As one vector is not a scalar multiple of the other, the lines are not parallel.

Now $-1+2\lambda = 4\mu-3$ $\quad$ $2-12\lambda = 3\mu+2$ $\quad$ $4+12\lambda = -\mu-1$

$\therefore$ $2\lambda-4\mu=-2$ $\quad$ $-12\lambda-3\mu=0$ $\quad$ $12\lambda+\mu=-5$ .... (3)

$\therefore$ $\lambda-2\mu=-1$ .... (1) $\quad$ $\mu=-4\lambda$ .... (2)

Solving (1) and (2) simultaneously: $\lambda-2(-4\lambda)=-1$

$\therefore$ $9\lambda=-1$

$\therefore$ $\lambda=-\frac{1}{9}$ and so $\mu=\frac{4}{9}$

In (3), $12\lambda+\mu = 12\left(-\frac{1}{9}\right)+\frac{4}{9} = -\frac{12}{9}+\frac{4}{9} = -\frac{8}{9}$, which is not -5.

Since the system is inconsistent, the lines do not intersect, so the lines are skew.

The acute angle between the lines has $\cos\theta = \dfrac{|8-36-12|}{\sqrt{292}\,\sqrt{26}} = \dfrac{40}{\sqrt{7592}}$ and so $\theta \approx 62.7°$.

c Line 1 has direction vector $\begin{pmatrix}6\\8\\2\end{pmatrix}$ and line 2 has direction vector $\begin{pmatrix}3\\4\\1\end{pmatrix}$.

As $\begin{pmatrix}6\\8\\2\end{pmatrix} = 2\begin{pmatrix}3\\4\\1\end{pmatrix}$ the two lines are parallel. Hence, $\theta = 0°$.

To see if the lines are coincident, try to find a shared point.

The point on line 1 where $t=1$ is $(6, 11, 1)$.

The unique point on line 2 with z-coordinate 1 is the point where $1+s=1$ $\therefore$ $s=0$.

This point is $(2, 0, 1)$. Since $(6, 11, 1) \neq (2, 0, 1)$ the lines are not coincident.

d In line 1 let $x = 2-y = z+2 = t$, so $x=t$, $y=2-t$, and $z=t-2$, $t\in\mathbb{R}$.

Line 1 has direction vector $\begin{pmatrix}1\\-1\\1\end{pmatrix}$ and line 2 has direction vector $\begin{pmatrix}3\\-2\\2\end{pmatrix}$.

As one vector is not a scalar multiple of the other, the lines are not parallel.

Now $t=1+3s$ .... (1) $\quad$ $2-t=-2-2s$ $\quad$ $-2+t=2s+\frac{1}{2}$

$-t+2s=-4$ .... (2) $\quad$ $t-2s=2\frac{1}{2}$ .... (3)

Solving (1) and (2) simultaneously: $-(1+3s)+2s=-4$

$\therefore$ $-1-3s+2s=-4$

$\therefore$ $-s=-3$

$\therefore$ $s=3$ and so $t=1+3(3)=10$

Substituting in (3), $t-2s = 10-2(3) = 4 \neq 2\frac{1}{2}$

Since the system is inconsistent, the lines do not meet. $\therefore$ they are skew.

The acute angle between the lines has $\cos\theta = \dfrac{|3+2+2|}{\sqrt{1+1+1}\sqrt{9+4+4}} = \dfrac{7}{\sqrt{3}\sqrt{17}}$

$\therefore$ $\theta \approx 11.4°$

e Line 1 has direction vector $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ and line 2 has direction vector $\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$.

As one vector is not a scalar multiple of the other, the lines are not parallel.

$$\begin{array}{lll} 1+\lambda = 2+3\mu & 2-\lambda = 3-2\mu & 3+2\lambda = \mu - 5 \\ \lambda - 3\mu = 1 \;\; .... \;(1) & -\lambda + 2\mu = 1 \;\; .... \;(2) & 2\lambda - \mu = -8 \;\; .... \;(3) \end{array}$$

Solving (1) and (2) simultaneously:
$$\begin{aligned} \lambda - 3\mu &= 1 \\ -\lambda + 2\mu &= 1 \\ \hline \text{Adding,}\quad -\mu &= 2 \end{aligned}$$

$\therefore\ \mu = -2$ and $\lambda - 3(-2) = 1$ $\therefore\ \lambda = -5$

Checking in (3), $2\lambda - \mu = 2(-5) - (-2) = -10 + 2 = -8$ ✓

Since $\mu = -2$, $\lambda = -5$ satisfies all three equations, the lines meet.

They meet at $x = 1 + (-5)$, $y = 2 - (-5)$, $z = 3 + 2(-5)$, or at $(-4, 7, -7)$.

The acute angle between the lines has $\cos\theta = \dfrac{|3+2+2|}{\sqrt{1+1+4}\sqrt{9+4+1}} = \dfrac{7}{\sqrt{84}}$

and so $\theta \approx 40.2°$

f Line 1 has direction vector $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$ and line 2 has direction vector $\begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix}$.

Now $\begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix} = -2\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$, so the lines are parallel and hence $\theta = 0°$.

All points on line 1 have z-coordinate 5 and all points on line 2 have z-coordinate 3.

$\therefore$ the lines are not coincident.

g Line 1 has direction vector $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and line 2 has direction vector $\begin{pmatrix} -4 \\ 2 \\ -6 \end{pmatrix}$.

As $\begin{pmatrix} -4 \\ 2 \\ -6 \end{pmatrix} = -2\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$, the two lines are parallel. Hence $\theta = 0°$.

The point on line 1 where $\lambda = 1$ is $(3, -1, 4)$.

The unique point on line 2 with x-coordinate 3 is the point where $3 - 4\mu = 3$ $\therefore\ \mu = 0$.

This point is $(3, -1, 4)$.

Lines 1 and 2 are parallel and share the point $(3, -1, 4)$. $\therefore$ they are coincident and $\theta = 0°$.

2 Line 1 is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$ with direction vector $\mathbf{a} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$.

Line 2 is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \mu\begin{pmatrix} -2 \\ 4 \\ 2 \end{pmatrix}$ with direction vector $\mathbf{b} = \begin{pmatrix} -2 \\ 4 \\ 2 \end{pmatrix}$.

Line 3 is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ with direction vector $\mathbf{c} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$.

Line 1 and line 2:

Since $\begin{pmatrix} -2 \\ 4 \\ 2 \end{pmatrix} = -2\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$, line 1 and line 2 are parallel.

When $\lambda = 0$, the point on line 1 is $(3, -1, 2)$.

For line 2, $y = 4\mu$, so the unique point on line 2 with y-coordinate -1 is the point where $\mu = -\frac{1}{4}$.

This point is $(\frac{3}{2}, -1, -\frac{3}{2})$.

Since $(\frac{3}{2}, -1, -\frac{3}{2}) \neq (3, -1, 2)$, line 1 and line 2 are not coincident.

Line 1 and line 3:

Equating x, y, and z values in lines 1 and 3 gives

$$\underbrace{\begin{array}{ll} 3+\lambda = t & -1-2\lambda = 1+2t \\ \therefore\ t = 3+\lambda & \therefore\ 2t = -2-2\lambda \\ & \therefore\ t = -1-\lambda \end{array}} \qquad \begin{array}{l} 2-\lambda = 1+t \\ \therefore\ \lambda + t = 1 \quad .... (1) \\ \\ \end{array}$$

Solving these we get $3+\lambda = -1-\lambda$

$\therefore\ 2\lambda = -4$

$\therefore\ \lambda = -2$ and so $t = 3-2$ $\quad \therefore\ t = 1$

Checking in (1): $\lambda + t = -2+1 = -1 \neq 1$

So, there is no simultaneous solution to all 3 equations.

$\therefore$ the lines do not intersect.

$\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} \neq k \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ for any $k \in \mathbb{R}$.

$\therefore$ lines 1 and 3 are not parallel.

Since they do not intersect and are not parallel, they are skew.

If θ is the acute angle between $\mathbf{a}$ and $\mathbf{c}$, then $\cos\theta = \dfrac{|\mathbf{a} \bullet \mathbf{c}|}{|\mathbf{a}|\,|\mathbf{c}|} = \dfrac{|1-4-1|}{\sqrt{1+4+1}\sqrt{1+4+1}} = \dfrac{4}{\sqrt{6}\sqrt{6}} = \frac{2}{3}$

$\therefore\ \theta \approx 48.2^\circ$

So, line 1 and line 3 are skew with an angle of about 48.2° between them.

Line 2 and line 3:

Equating x, y, and z values in lines 2 and 3 gives

$$\underbrace{\begin{array}{ll} 1-2\mu = t & 4\mu = 1+2t \\ \therefore\ t = 1-2\mu & \therefore\ 2t = -1+4\mu \\ \therefore\ 2t = 2-4\mu & \end{array}} \qquad \begin{array}{l} -1+2\mu = 1+t \\ \therefore\ 2\mu - t = 2 \quad .... (2) \\ \\ \end{array}$$

Solving these we get $2-4\mu = -1+4\mu$

$\therefore\ 8\mu = 3$

$\therefore\ \mu = \frac{3}{8}$ and so $t = 1-2(\frac{3}{8})$ $\quad \therefore\ t = \frac{1}{4}$

Checking in (2): $2\mu - t = 2(\frac{3}{8}) - \frac{1}{4} = \frac{6}{8} - \frac{2}{8} = \frac{4}{8} = \frac{1}{2} \neq 2$

So, there is no simultaneous solution to all 3 equations.

$\therefore$ the lines do not intersect.

$\begin{pmatrix} -2 \\ 4 \\ 2 \end{pmatrix} \neq k \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ for any $k \in \mathbb{R}$.

$\therefore$ lines 2 and 3 are not parallel.

Since they do not intersect and are not parallel, they are skew.

If ϕ is the acute angle between $\mathbf{b}$ and $\mathbf{c}$ then

$$\cos\phi = \frac{|\mathbf{b} \bullet \mathbf{c}|}{|\mathbf{b}|\,|\mathbf{c}|} = \frac{|-2+8+2|}{\sqrt{4+16+4}\sqrt{1+4+1}} = \frac{8}{\sqrt{144}} = \frac{2}{3}$$

$\therefore\ \phi \approx 48.2^\circ$

So, line 2 and line 3 are skew with an angle of about 48.2° between them.

EXERCISE 15I

1 **a** $\mathbf{n} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$, and the point $(-1, 2, 4)$ lies on the plane.

$\therefore$ the equation is $2x - y + 3z = 2(-1) - 2 + 3(4)$ which is $2x - y + 3z = 8$.

b $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$ is a vector normal to the plane, and $(2, 3, 1)$ lies on the plane.

$\therefore$ the equation is $\quad 3x + 4y + z = 3(2) + 4(3) + 1$

$\therefore \quad 3x + 4y + z = 19$

c The line $\quad x = 1 + t, \quad y = 2 - t, \quad z = 3 + 2t \quad$ has direction vector $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$.

Also, letting $t = 0$, the point $(1, 2, 3)$ lies on the plane and we call this point B.

$\therefore \quad \overrightarrow{AB} = \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix}$ and so a vector normal to the plane is $\overrightarrow{AB} \times \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

$$\therefore \quad \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 0 & 2 \\ 1 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 2 \\ -1 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -2 & 2 \\ 1 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -2 & 0 \\ 1 & -1 \end{vmatrix} \mathbf{k}$$

$$= 2\mathbf{i} + 6\mathbf{j} + 2\mathbf{k} \quad \text{or} \quad 2(\mathbf{i} + 3\mathbf{j} + \mathbf{k})$$

$\therefore$ since A$(3, 2, 1)$ lies on the plane, it has equation $\quad x + 3y + z = 3 + 3(2) + 1$

or $\quad x + 3y + z = 10$

2 **a** $2x + 3y - z = 8$ has $\mathbf{n} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

b $3x - y + 0z = 11$ has $\mathbf{n} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$

c $0x + 0y + z = 2$ has $\mathbf{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

d $1x + 0y + 0z = 0$ has $\mathbf{n} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

3 **a** The y-axis is perpendicular to the XOZ plane $\quad \therefore$ a normal vector is $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\therefore$ since the origin lies on the plane, it has equation $y = 0$.

b Since the plane is perpendicular to the Z-axis, it has normal vector $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\therefore$ since $(2, -1, 4)$ lies on the plane, it has equation $z = 4$.

4 **a** **i** $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix}$, so $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix}$

ii If $\mathbf{n}$ is the normal vector, then

$$\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -4 \\ -1 & 0 & -2 \end{vmatrix} = \begin{vmatrix} 1 & -4 \\ 0 & -2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -4 \\ -1 & -2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} \mathbf{k} = -2\mathbf{i} + 6\mathbf{j} + \mathbf{k}$$

$\therefore$ since A$(0, 2, 6)$ lies on the plane, it has equation $\quad -2x + 6y + z = -2(0) + 6(2) + 6$

$\therefore \quad -2x + 6y + z = 18$

b **i** $\overrightarrow{AB} = \begin{pmatrix} -3 \\ 3 \\ -2 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} -3 \\ -1 \\ -1 \end{pmatrix}$, so $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 3 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ -1 \\ -1 \end{pmatrix}$

ii If $\mathbf{n}$ is the normal vector, then

$$\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 3 & -2 \\ -3 & -1 & -1 \end{vmatrix} = \begin{vmatrix} 3 & -2 \\ -1 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -3 & -2 \\ -3 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -3 & 3 \\ -3 & -1 \end{vmatrix} \mathbf{k}$$

$$= -5\mathbf{i} + 3\mathbf{j} + 12\mathbf{k}$$

$\therefore$ since C$(0, 0, 1)$ lies on the plane, it has equation $\quad -5x + 3y + 12z = 12$.

c **i** $\overrightarrow{AB} = \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} 2 \\ -3 \\ -3 \end{pmatrix}$, so $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -3 \\ -3 \end{pmatrix}$

ii If $\mathbf{n}$ is the normal vector, then

$$\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -1 & -1 \\ 2 & -3 & -3 \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ -3 & -3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -2 & -1 \\ 2 & -3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -2 & -1 \\ 2 & -3 \end{vmatrix} \mathbf{k}$$

$$= -8\mathbf{j} + 8\mathbf{k} \text{ or } -8(\mathbf{j} - \mathbf{k})$$

$\therefore$ since A(2, 0, 3) lies on the plane, it has equation $y - z = -3$.

5 **a** The normal to $x - 3y + 4z = 8$ is $\begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}$, and this is the direction vector of the line.

$\therefore$ since the line passes through $(1, -2, 0)$, it has equation

$$x = 1 + \lambda, \quad y = -2 - 3\lambda, \quad z = 4\lambda, \quad \lambda \in \mathbb{R}.$$

b The normal to $x - y - 2z = 11$ is $\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$.

$\therefore$ since the line passes through $(3, 4, -1)$, it has equation

$$x = 3 + \lambda, \quad y = 4 - \lambda, \quad z = -1 - 2\lambda, \quad \lambda \in \mathbb{R}.$$

6 The line has direction vector $\overrightarrow{AB} = \begin{pmatrix} -1 \\ 3 \\ -3 \end{pmatrix}$.

$\therefore$ since the line passes through $A(2, -1, 3)$, it has parametric equations

$$x = 2 - t, \quad y = -1 + 3t, \quad z = 3 - 3t, \quad t \in \mathbb{R}.$$

This line meets $x + 2y - z = 5$ when $(2 - t) + 2(-1 + 3t) - (3 - 3t) = 5$

$$\therefore\ 2 - t - 2 + 6t - 3 + 3t = 5$$

$\therefore$ $8t = 8$ $\therefore$ $t = 1$, and so they meet at $(1, 2, 0)$.

7 **a** The direction vector of the line is $\overrightarrow{PQ} = \begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix}$.

$\therefore$ since it passes through $P(1, -2, 4)$, it has parametric equations

$$x = 1 + t, \quad y = -2 + 2t, \quad z = 4 - 5t, \quad t \in \mathbb{R}.$$

b **i** The line meets the YOZ plane when $x = 0$, or when $t = -1$.
This corresponds to the point $(0, -4, 9)$.

ii The line meets $y + z = 2$ when $-2 + 2t + 4 - 5t = 2$ $\therefore$ $-3t = 0$ $\therefore$ $t = 0$
This corresponds to the point $(1, -2, 4)$.

iii The line meets $\dfrac{x-3}{2} = \dfrac{y+2}{3} = \dfrac{z-30}{-1}$

when $\dfrac{1+t-3}{2} = \dfrac{-2+2t+2}{3} = \dfrac{4-5t-30}{-1}$

$$\therefore\ \frac{t-2}{2} = \frac{2t}{3} = 5t + 26$$

$$\therefore\ 3t - 6 = 4t = 30t + 156$$

$$\therefore\ 3t - 6 = 4t \text{ and } 4t = 30t + 156$$

$$\therefore\ t = -6 \text{ and } -26t = 156$$

$$\therefore\ t = -6 \text{ is a common solution}$$

$\therefore$ the lines meet at the point corresponding to $t = -6$, which is $(-5, -14, 34)$.

8 **a** The plane $2x + y - 2z = -11$ has $\mathbf{n} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$.

$\therefore$ the parametric equations of (AN) are $x = 1 + 2t,\ y = 0 + t,\ z = 2 - 2t,\ t \in \mathbb{R}$.

This line meets the plane when
$$\begin{aligned} 2(1+2t) + t - 2(2-2t) &= -11 \\ \therefore\ 2 + 4t + t - 4 + 4t &= -11 \\ \therefore\ 9t &= -9 \\ \therefore\ t &= -1 \end{aligned}$$

Thus N is $(-1, -1, 4)$ and $\therefore$ $\overrightarrow{\mathrm{AN}} = \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}$ and $\mathrm{AN} = \sqrt{(-2)^2 + (-1)^2 + 2^2}$
$= \sqrt{9} = 3$ units

b The plane $x - y + 3z = -10$ has $\mathbf{n} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$

$\therefore$ the parametric equations of (AN) are $x = 2 + t,\ y = -1 - t,\ z = 3 + 3t,\ t \in \mathbb{R}$.

This line meets the plane when
$$\begin{aligned} (2+t) - (-1-t) + 3(3+3t) &= -10 \\ \therefore\ 2 + t + 1 + t + 9 + 9t &= -10 \\ \therefore\ 11t &= -22 \\ \therefore\ t &= -2 \end{aligned}$$

$\therefore$ N is $(0, 1, -3)$

$\therefore$ $\overrightarrow{\mathrm{AN}} = \begin{pmatrix} -2 \\ 2 \\ -6 \end{pmatrix}$ and $\mathrm{AN} = \sqrt{(-2)^2 + 2^2 + (-6)^2} = \sqrt{44} = 2\sqrt{11}$ units.

c The plane $4x - y - 2z = 8$ has $\mathbf{n} = \begin{pmatrix} 4 \\ -1 \\ -2 \end{pmatrix}$.

$\therefore$ the parametric equations of (AN) are
$$x = 1 + 4t, \quad y = -4 - t, \quad z = -3 - 2t, \quad t \in \mathbb{R}.$$

This line meets the plane when
$$\begin{aligned} 4(1+4t) - (-4-t) - 2(-3-2t) &= 8 \\ \therefore\ 4 + 16t + 4 + t + 6 + 4t &= 8 \\ \therefore\ 21t &= -6 \\ \therefore\ t &= -\tfrac{2}{7} \end{aligned}$$

$\therefore$ N is $\left(-\frac{1}{7}, -\frac{26}{7}, -\frac{17}{7}\right)$, $\therefore$ $\overrightarrow{\mathrm{AN}} = \begin{pmatrix} -\frac{8}{7} \\ \frac{2}{7} \\ \frac{4}{7} \end{pmatrix}$ and so $\mathrm{AN} = \sqrt{\left(-\frac{8}{7}\right)^2 + \left(\frac{2}{7}\right)^2 + \left(\frac{4}{7}\right)^2}$
$= \sqrt{\frac{84}{49}} = 2\sqrt{\frac{3}{7}}$ units

9 The mirror image lies on the normal line to the plane through the object point.

Now $x + 2y + z = 1$ has $\mathbf{n} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

$\therefore$ the normal at A has parametric equations $x = 3 + t,\ y = 1 + 2t,\ z = 2 + t,\ t \in \mathbb{R}$.

This line meets the plane when
$$\begin{aligned} (3+t) + 2(1+2t) + 2 + t &= 1 \\ \therefore\ 3 + t + 2 + 4t + 2 + t &= 1 \\ \therefore\ 6t &= -6 \\ \therefore\ t &= -1 \end{aligned}$$

$\therefore$ N is $(2, -1, 1)$

If A$'$ is the mirror image of A, then $\overrightarrow{\mathrm{AN}} = \overrightarrow{\mathrm{NA'}}$

$\therefore$ letting A$'$ have coordinates (a, b, c), $\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} a-2 \\ b+1 \\ c-1 \end{pmatrix}$

$\therefore$ $a - 2 = -1,\ b + 1 = -2,\ c - 1 = -1$ and so A$'$ is at $(1, -3, 0)$.

10 The plane $x+4y-z=-2$ has normal $\mathbf{n}=\begin{pmatrix}1\\4\\-1\end{pmatrix}$ which passes through $(3, 4, -1)$.

$\therefore$ the normal has parametric equations $x=3+t,\ y=4+4t,\ z=-1-t,\ t\in\mathbb{R}$ and will meet any of the coordinate axes if any two of the values of x, y, and z are zero at the same time.

$\therefore$ since $x=0$ when $t=-3$ and $y=z=0$ when $t=-1$, the normal meets the X-axis when $t=-1$, at the point $(2, 0, 0)$.

11 $\overrightarrow{\mathrm{AB}}=\begin{pmatrix}-1\\-3\\-1\end{pmatrix}$

a The normal $\mathbf{n}$ is perpendicular to both the X-axis and $\overrightarrow{\mathrm{AB}}$.

Since the X-axis has direction vector $\begin{pmatrix}1\\0\\0\end{pmatrix}$,

$$\mathbf{n}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&0&0\\-1&-3&-1\end{vmatrix}$$

$$=\begin{vmatrix}0&0\\-3&-1\end{vmatrix}\mathbf{i}-\begin{vmatrix}1&0\\-1&-1\end{vmatrix}\mathbf{j}+\begin{vmatrix}1&0\\-1&-3\end{vmatrix}\mathbf{k}$$

$$=\mathbf{j}-3\mathbf{k}$$

Since $\mathrm{A}(1, 2, 3)$ is in the plane, the plane has equation $y-3z=1(2)-3(3)$

or $y-3z=-7$

b The normal $\mathbf{n}$ is perpendicular to both the Y-axis and $\overrightarrow{\mathrm{AB}}$.

Since the Y-axis has direction vector $\begin{pmatrix}0\\1\\0\end{pmatrix}$,

$$\mathbf{n}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&1&0\\-1&-3&-1\end{vmatrix}$$

$$=\begin{vmatrix}1&0\\-3&-1\end{vmatrix}\mathbf{i}-\begin{vmatrix}0&0\\-1&-1\end{vmatrix}\mathbf{j}+\begin{vmatrix}0&1\\-1&-3\end{vmatrix}\mathbf{k}$$

$$=-\mathbf{i}+\mathbf{k}$$

Since $\mathrm{A}(1, 2, 3)$ is in the plane, the plane has equation $-x+z=-1(1)+1(3)=2$

$\therefore\ x-z=-2$

c The normal $\mathbf{n}$ is perpendicular to both the Z-axis and $\overrightarrow{\mathrm{AB}}$.

Since the Z-axis has direction vector $\begin{pmatrix}0\\0\\1\end{pmatrix}$,

$$\mathbf{n}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&0&1\\-1&-3&-1\end{vmatrix}$$

$$=\begin{vmatrix}0&1\\-3&-1\end{vmatrix}\mathbf{i}-\begin{vmatrix}0&1\\-1&-1\end{vmatrix}\mathbf{j}+\begin{vmatrix}0&0\\-1&-3\end{vmatrix}\mathbf{k}$$

$$=3\mathbf{i}-\mathbf{j}$$

Since $\mathrm{A}(1, 2, 3)$ is in the plane, the plane has equation $3x-y=3(1)-1(2)$

or $3x-y=1$

12 Now $x-1=\dfrac{y-2}{2}=z+3$ has direction vector $\begin{pmatrix}1\\2\\1\end{pmatrix}$

and $x+1=y-3=2z+5$ has direction vector $\begin{pmatrix}1\\1\\\frac{1}{2}\end{pmatrix}$ $\left\{\text{since } 2z+5=\dfrac{z+\frac{5}{2}}{\frac{1}{2}}\right\}$

$\therefore$ a vector perpendicular to both lines is:

$$\begin{pmatrix}1\\2\\1\end{pmatrix}\times\begin{pmatrix}1\\1\\\frac{1}{2}\end{pmatrix}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&2&1\\1&1&\frac{1}{2}\end{vmatrix}=\begin{vmatrix}2&1\\1&\frac{1}{2}\end{vmatrix}\mathbf{i}-\begin{vmatrix}1&1\\1&\frac{1}{2}\end{vmatrix}\mathbf{j}+\begin{vmatrix}1&2\\1&1\end{vmatrix}\mathbf{k}$$

$$=0\mathbf{i}+\tfrac{1}{2}\mathbf{j}-\mathbf{k}$$

$\therefore$ $\begin{pmatrix}0\\1\\-2\end{pmatrix}$ is perpendicular to both lines.

A plane with normal $\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$ has equation $y - 2z = c$ for some c.

Now for line 1, $\dfrac{y-2}{2} = z + 3$ $\quad$ and for line 2, $\quad y - 3 = 2z + 5$

$$\therefore \; y - 2 = 2z + 6 \qquad\qquad \therefore \; y - 2z = 8 \text{ also.}$$

$$\therefore \; y - 2z = 8$$

$\therefore$ $y - 2z = 8$ is a plane containing both lines, so the lines are coplanar.

13 **a** Since $A(1, 2, k)$ lies on $x + 2y - 2z = 8$, $1 + 2(2) - 2k = 8$

$$\therefore \; 1 + 4 - 2k = 8$$
$$\therefore \; -2k = 3$$
$$\therefore \; k = -\tfrac{3}{2}$$

b Since $x + 2y - 2z = 8$, the plane has normal vector $\mathbf{n} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$.

$\therefore$ the normal from A has parametric equations

$$x = 1 + t, \quad y = 2 + 2t, \quad z = -\tfrac{3}{2} - 2t, \quad t \in \mathbb{R}.$$

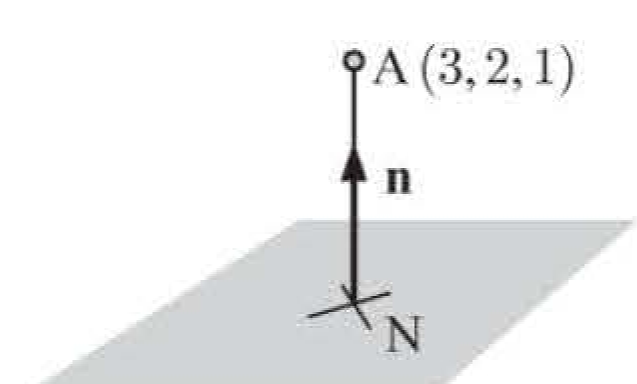
$\therefore$ points of the normal that are 6 units from A have

$$\sqrt{(1+t-1)^2 + (2+2t-2)^2 + (-\tfrac{3}{2} - 2t + \tfrac{3}{2})^2} = 6$$
$$\therefore \; \sqrt{t^2 + 4t^2 + 4t^2} = 6$$
$$\therefore \; 9t^2 = 36$$
$$\therefore \; t^2 = 4$$
$$\therefore \; t = \pm 2$$

$\therefore$ B is $(3, 6, -\frac{11}{2})$ or $(-1, -2, \frac{5}{2})$

14 **a** The normal from $A(3, 2, 1)$ to the plane has direction vector

$$\mathbf{n} = (2\mathbf{i} + \mathbf{j} + \mathbf{k}) \times (4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ 4 & 2 & -2 \end{vmatrix}$$



$$= \begin{vmatrix} 1 & 1 \\ 2 & -2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 1 \\ 4 & -2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} \mathbf{k}$$
$$= -4\mathbf{i} + 8\mathbf{j}$$

$\therefore$ if N is the foot of the normal from A, (AN) has equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + t \begin{pmatrix} -4 \\ 8 \\ 0 \end{pmatrix}$

So, N has coordinates of the form $(3 - 4t, 2 + 8t, 1)$

But N lies on the plane $\therefore$ $\begin{pmatrix} 3-4t \\ 2+8t \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix}$

$$\therefore \; \begin{cases} 3 - 4t = 3 + 2\lambda + 4\mu \\ 2 + 8t = 1 + \lambda + 2\mu \\ 1 = 2 + \lambda - 2\mu \end{cases} \text{ and so } \begin{cases} 2\lambda + 4\mu + 4t = 0 \\ \lambda + 2\mu - 8t = 1 \\ \lambda - 2\mu \qquad = -1 \end{cases}$$

Solving simultaneously using technology gives $\lambda = -0.4, \quad \mu = 0.3, \quad t = -0.1$

$\therefore$ N is $(3 - 4(-0.1), 2 + 8(-0.1), 1)$ or $(3.4, 1.2, 1)$

$\therefore$ $\overrightarrow{AN} = \begin{pmatrix} 3.4 - 3 \\ 1.2 - 2 \\ 1 - 1 \end{pmatrix} = \begin{pmatrix} 0.4 \\ -0.8 \\ 0 \end{pmatrix}$ and $|\overrightarrow{AN}| = \sqrt{(0.4)^2 + (0.8)^2 + 0^2} = \frac{2}{\sqrt{5}}$ units

b The normal from $A(1,\,0,\,-2)$ to the plane has direction vector

$$\mathbf{n} = (3\mathbf{i}-\mathbf{j}+2\mathbf{k})\times(-\mathbf{i}+\mathbf{j}-\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 2 \\ -1 & 1 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 2 \\ -1 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & -1 \\ -1 & 1 \end{vmatrix}\mathbf{k}$$

$$= -\mathbf{i}+\mathbf{j}+2\mathbf{k}$$

$\therefore$ (AN) has equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$

So, N has coordinates of the form $(1-t,\,t,\,-2+2t)$

But N lies on the plane $\therefore$ $\begin{pmatrix} 1-t \\ t \\ -2+2t \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \mu\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$

$$\therefore \quad \begin{cases} 1+3\lambda-\mu = 1-t \\ -1-\lambda+\mu = t \\ 1+2\lambda-\mu = -2+2t \end{cases} \quad \text{and so} \quad \begin{cases} 3\lambda-\mu+t = 0 \\ -\lambda+\mu-t = 1 \\ 2\lambda-\mu-2t = -3 \end{cases}$$

Solving simultaneously using technology gives $\lambda = \frac{1}{2}$, $\mu = 2\frac{1}{3}$, $t = \frac{5}{6}$

$\therefore$ N is $\left(1-\frac{5}{6},\,\frac{5}{6},\,-2+\frac{5}{3}\right)$ or $\left(\frac{1}{6},\,\frac{5}{6},\,-\frac{1}{3}\right)$

$$\therefore \quad \overrightarrow{AN} = \begin{pmatrix} \frac{1}{6}-1 \\ \frac{5}{6}-0 \\ -\frac{1}{3}--2 \end{pmatrix} = \begin{pmatrix} -\frac{5}{6} \\ \frac{5}{6} \\ \frac{5}{3} \end{pmatrix} \quad \text{and} \quad |\overrightarrow{AN}| = \sqrt{\tfrac{25}{36}+\tfrac{25}{36}+\tfrac{25}{9}} = \tfrac{5\sqrt{6}}{6}$$

$$= \tfrac{5}{\sqrt{6}} \text{ units}$$

15 **b** and **c** are non-parallel vectors in the plane. The normal vector to the plane is $\mathbf{n} = \mathbf{b}\times\mathbf{c}$.

We want to show that $\mathbf{n} \perp (\lambda\mathbf{b}+\mu\mathbf{c})$ for all $\lambda, \mu \in \mathbb{R}$ except when $\lambda = \mu = 0$.

So, we want to show that $\mathbf{n} \bullet (\lambda\mathbf{b}+\mu\mathbf{c}) = 0$ $\{\mathbf{v}\bullet\mathbf{w} = 0 \Leftrightarrow$ **v** and **w** are perpendicular$\}$

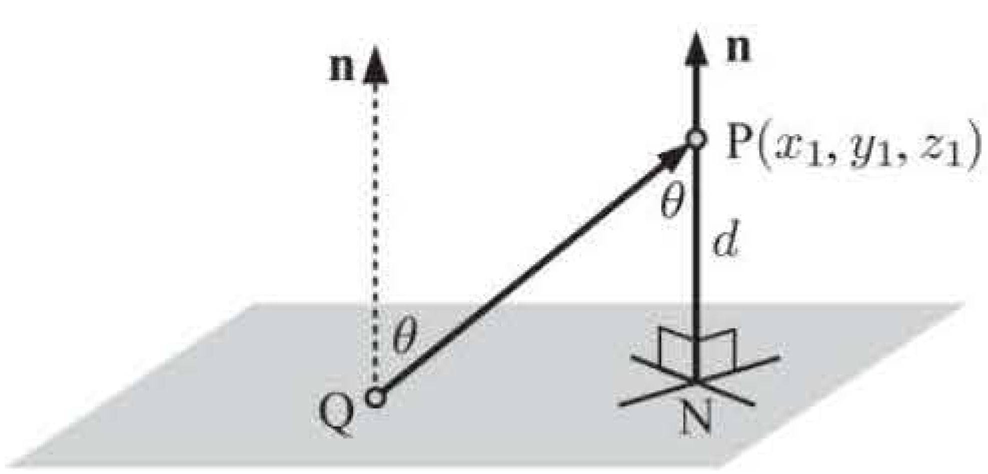
$$\begin{aligned} \mathbf{n}\bullet(\lambda\mathbf{b}+\mu\mathbf{c}) &= (\mathbf{b}\times\mathbf{c})\bullet(\lambda\mathbf{b}+\mu\mathbf{c}) && \{\mathbf{n} = \mathbf{b}\times\mathbf{c}\} \\ &= (\mathbf{b}\times\mathbf{c})\bullet\lambda\mathbf{b} + (\mathbf{b}\times\mathbf{c})\bullet\mu\mathbf{c} && \{\mathbf{v}\bullet(\mathbf{w}+\mathbf{x}) = \mathbf{v}\bullet\mathbf{w}+\mathbf{v}\bullet\mathbf{x}\} \\ &= 0+0 && \{(\mathbf{b}\times\mathbf{c}) \perp \mathbf{b} \quad \therefore \ (\mathbf{b}\times\mathbf{c}) \perp \text{ to any non-zero scalar multiple of } \mathbf{b}\text{. Similarly for } \mathbf{c}.\} \\ &= 0 \quad \text{for all } \lambda, \mu \in \mathbb{R} \text{ except when } \lambda = \mu = 0 \end{aligned}$$

16 **a** If N is the point on the plane such that (NP) is a normal to it, then $\triangle$NPQ is right angled at N.

Draw a line parallel to **n** through Q.

Now θ is the angle between vectors **n** and $\overrightarrow{QP}$.

$\therefore$ $\cos\theta = \dfrac{|\overrightarrow{QP}\bullet\mathbf{n}|}{|\overrightarrow{QP}|\,|\mathbf{n}|}$ .... (1)



But $\widehat{QPN} = \theta$ {alternate angles}

$\therefore$ $\cos\theta = \dfrac{d}{|\overrightarrow{QP}|}$ .... (2)

Equating (1) and (2), $\dfrac{d}{|\overrightarrow{QP}|} = \dfrac{|\overrightarrow{QP}\bullet\mathbf{n}|}{|\overrightarrow{QP}|\,|\mathbf{n}|}$

$$\therefore \quad d = \frac{|\overrightarrow{QP}\bullet\mathbf{n}|}{|\mathbf{n}|}$$

b Since Q is any point on the plane, it has coordinates (x, y, z) such that $Ax+By+Cz+D=0$.

The normal vector to the plane is $\mathbf{n}=\begin{pmatrix}A\\B\\C\end{pmatrix}$.

$$\therefore \text{ using } \mathbf{a}, \quad d=\frac{|\overrightarrow{\mathrm{QP}}\bullet\mathbf{n}|}{|\mathbf{n}|}=\frac{\left|\begin{pmatrix}x_1-x\\y_1-y\\z_1-z\end{pmatrix}\bullet\begin{pmatrix}A\\B\\C\end{pmatrix}\right|}{\sqrt{A^2+B^2+C^2}}$$

$$=\frac{|Ax_1-Ax+By_1-By+Cz_1-Cz|}{\sqrt{A^2+B^2+C^2}}$$

$$=\frac{|Ax_1+By_1+Cz_1-(Ax+By+Cz)|}{\sqrt{A^2+B^2+C^2}}$$

$$=\frac{|Ax_1+By_1+Cz_1+D|}{\sqrt{A^2+B^2+C^2}}$$

c **8 a** *check*: Given A(1, 0, 2) and the plane $2x+y-2z+11=0$,

$$d=\frac{|2x_1+y_1-2z_1+11|}{\sqrt{2^2+1^2+(-2)^2}}=\frac{|2(1)+1(0)-2(2)+11|}{\sqrt{9}}=\tfrac{9}{3}=3 \text{ units}$$

8 b *check*: Given A(2, −1, 3) and the plane $x-y+3z=-10$,

$$d=\frac{|x_1-y_1+3z_1+10|}{\sqrt{1^2+(-1)^2+3^2}}$$

$$=\frac{|2-(-1)+3(3)+10|}{\sqrt{11}}$$

$$=\tfrac{22}{\sqrt{11}}=2\sqrt{11} \text{ units}$$

8 c *check*: Given A(1, −4, −3) and the plane $4x-y-2z=8$,

$$d=\frac{|4x_1-y_1-2z_1-8|}{\sqrt{4^2+(-1)^2+(-2)^2}}$$

$$=\frac{|4-(-4)-2(-3)-8|}{\sqrt{21}}$$

$$=\tfrac{6}{\sqrt{21}} \text{ units or } 2\sqrt{\tfrac{3}{7}} \text{ units}$$

d Using the formula derived in **b**,

i $d=\dfrac{|x_1+2y_1-z_1-10|}{\sqrt{1^2+2^2+(-1)^2}}=\dfrac{|0+2(0)-0-10|}{\sqrt{6}}=\dfrac{10}{\sqrt{6}}$ units

ii $d=\dfrac{|x_1+y_1-z_1-2|}{\sqrt{1^2+1^2+(-1)^2}}=\dfrac{|1+(-3)-2-2|}{\sqrt{3}}=\dfrac{|-6|}{\sqrt{3}}=\dfrac{6}{\sqrt{3}}$ units or $2\sqrt{3}$ units

17 **a** First choose a point on the first plane $x+y+2z=4$, for example, (0, 0, 2).
Using the formula obtained in **16 b** to calculate the distance from this point to the second plane,

$$d=\frac{|2x_1+2y_1+4z_1+11|}{\sqrt{2^2+2^2+4^2}}=\frac{|2(0)+2(0)+4(2)+11|}{\sqrt{24}}=\frac{19}{\sqrt{24}} \text{ units.}$$

b Choose a point on the plane $ax+by+cz+d_1=0$, for example, $\left(0, 0, -\dfrac{d_1}{c}\right)$.

Using the formula obtained in **16 b** to calculate the distance from this point to the second plane,

$$d=\frac{|ax_1+by_1+cz_1+d_2|}{\sqrt{a^2+b^2+c^2}}=\frac{\left|a(0)+b(0)+c\left(-\dfrac{d_1}{c}\right)+d_2\right|}{\sqrt{a^2+b^2+c^2}}=\frac{|d_2-d_1|}{\sqrt{a^2+b^2+c^2}} \text{ units}$$

18 The line $x=2+t,\ y=-1+2t,\ z=-3t$ has direction vector $\begin{pmatrix}1\\2\\-3\end{pmatrix}$,

and $\begin{pmatrix}11\\-4\\1\end{pmatrix}$ is a vector normal to the plane $11x-4y+z=0$.

But $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \bullet \begin{pmatrix} 11 \\ -4 \\ 1 \end{pmatrix} = 11 - 8 - 3 = 0$

$\therefore$ these vectors are perpendicular and so the line is parallel to the plane.

Choose any point on the line, say $t = 0$, which corresponds to the point $(2, -1, 0)$.

Then the distance $d = \dfrac{|11x_1 - 4y_1 + z_1|}{\sqrt{11^2 + (-4)^2 + 1^2}} = \dfrac{|11(2) - 4(-1) + 0|}{\sqrt{138}} = \dfrac{26}{\sqrt{138}}$ units

19 Since the planes are parallel to $2x - y + 2z = 5$, they have equation $2x - y + 2z = a$ for some a.

Choose any point on $2x - y + 2z = 5$, for example, $(0, -5, 0)$.

Then the distance from this point to the plane $2x - y + 2z = a$ is

$$d = \frac{|2x_1 - y_1 + 2z_1 - a|}{\sqrt{2^2 + (-1)^2 + 2^2}}$$

$$\therefore\ 2 = \frac{|2(0) - (-5) + 2(0) - a|}{3}$$

$$\therefore\ 6 = |5 - a|$$

$$\therefore\ 5 - a = \pm 6$$

$$\therefore\ a = 5 \pm 6$$

$$\therefore\ a = -1 \text{ or } a = 11$$

$\therefore$ the planes are $2x - y + 2z = -1$ and $2x - y + 2z = 11$

EXERCISE 15J

1 **a** $\mathbf{n} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ and $\mathbf{d} = \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}$

$\therefore\ \sin\phi = \dfrac{|\mathbf{n} \bullet \mathbf{d}|}{|\mathbf{n}||\mathbf{d}|} = \dfrac{|4 - 3 + 1|}{\sqrt{3}\sqrt{26}} = \dfrac{2}{\sqrt{78}}$

and so $\phi \approx 13.1^\circ$

b $\mathbf{n} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ and $\mathbf{d} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$

$\therefore\ \sin\phi = \dfrac{|\mathbf{n} \bullet \mathbf{d}|}{|\mathbf{n}||\mathbf{d}|} = \dfrac{|2 - 3 + 1|}{\sqrt{6}\sqrt{11}} = 0$

and so $\phi \approx 0^\circ$

So, the line and plane are parallel.

c $\mathbf{n} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$ So, if $x - 4 = 3 - y = 2(z + 1) = t$

or equivalently $x = 4 + t,\ y = 3 - t,\ z = -1 + \frac{1}{2}t$ then $\mathbf{d} = \begin{pmatrix} 1 \\ -1 \\ \frac{1}{2} \end{pmatrix}$.

$\therefore\ \sin\phi = \dfrac{|\mathbf{n} \bullet \mathbf{d}|}{|\mathbf{n}||\mathbf{d}|} = \dfrac{\left|3 + (-4) + (-\frac{1}{2})\right|}{\sqrt{26}\sqrt{\frac{9}{4}}} = \dfrac{\left|-\frac{3}{2}\right|}{\frac{3}{2}\sqrt{26}} = \dfrac{1}{\sqrt{26}}$ and so $\phi \approx 11.3^\circ$

d The plane has normal vector

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -4 & -1 \\ 1 & 1 & -2 \end{vmatrix} = \begin{vmatrix} -4 & -1 \\ 1 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & -1 \\ 1 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & -4 \\ 1 & 1 \end{vmatrix}\mathbf{k} = 9\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}$$

and the line has direction vector $\mathbf{d} = \mathbf{i} - \mathbf{j} + \mathbf{k}$

$$\therefore\ \sin\phi = \frac{|\mathbf{n} \bullet \mathbf{d}|}{|\mathbf{n}||\mathbf{d}|} = \frac{|9 - 5 + 7|}{\sqrt{81 + 25 + 49}\sqrt{1 + 1 + 1}} = \frac{11}{\sqrt{155}\sqrt{3}}$$

$$\therefore\ \phi \approx 30.7^\circ$$

2 **a** $\mathbf{n}_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ and $\mathbf{n}_2 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$

$\therefore\ \cos\theta = \dfrac{|\mathbf{n}_1 \bullet \mathbf{n}_2|}{|\mathbf{n}_1||\mathbf{n}_2|} = \dfrac{|2 - 3 + 2|}{\sqrt{6}\sqrt{14}} = \frac{1}{\sqrt{84}}$

$\therefore\ \theta \approx 83.7^\circ$

b $\mathbf{n}_1 = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{n}_2 = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$

$\therefore\ \cos\theta = \dfrac{|\mathbf{n}_1 \bullet \mathbf{n}_2|}{|\mathbf{n}_1||\mathbf{n}_2|} = \dfrac{|3 - 1 - 3|}{\sqrt{11}\sqrt{11}} = \frac{1}{11}$

$\therefore\ \theta \approx 84.8^\circ$

c $\mathbf{n}_1 = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$ and $\mathbf{n}_2 = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$ $\quad \therefore \cos\theta = \dfrac{|\mathbf{n}_1 \bullet \mathbf{n}_2|}{|\mathbf{n}_1||\mathbf{n}_2|} = \dfrac{|6-4-1|}{\sqrt{11}\sqrt{21}} = \dfrac{1}{\sqrt{231}}$

$\therefore \theta \approx 86.2^\circ$

d $$\mathbf{n}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & -1 \\ 2 & -4 & 3 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ -4 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & -1 \\ 2 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & 1 \\ 2 & -4 \end{vmatrix}\mathbf{k}$$
$$= -\mathbf{i} + \mathbf{j} + 2\mathbf{k}$$

$$\mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ 1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -2 & -1 \\ 1 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -2 & -1 \\ 1 & 1 \end{vmatrix}\mathbf{k}$$
$$= \mathbf{j} - \mathbf{k}$$

$$\therefore \cos\theta = \frac{|\mathbf{n}_1 \bullet \mathbf{n}_2|}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{|-1(0)+1(1)+2(-1)|}{\sqrt{1+1+4}\sqrt{0+1+1}} = \frac{|1-2|}{\sqrt{6}\sqrt{2}} = \frac{1}{\sqrt{12}}$$

$\therefore \theta \approx 73.2^\circ$

e $\mathbf{n}_1 = \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}$ and $\mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 0 \\ 2 & 1 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 0 \\ 2 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix}\mathbf{k}$

$$= -\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$$

$$\therefore \cos\theta = \frac{|\mathbf{n}_1 \bullet \mathbf{n}_2|}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{|(3)(-1)+(-4)(-3)+(1)(5)|}{\sqrt{9+16+1}\sqrt{1+9+25}} = \frac{14}{\sqrt{26\times 35}}$$

$\therefore \theta \approx 62.3^\circ$

EXERCISE 15K

1 **a** In augmented matrix form, the system is:

$$\left[\begin{array}{ccc|c} 1 & -2 & 5 & 1 \\ 2 & -4 & 8 & 2 \\ -3 & 6 & 7 & -3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 5 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 22 & 0 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + 3R_1 \end{array}$$

$$\begin{array}{rrrr} 2 & -4 & 8 & 2 \\ -2 & 4 & -10 & -2 \\ \hline 0 & 0 & -2 & 0 \end{array} \qquad \begin{array}{rrrr} -3 & 6 & 7 & -3 \\ 3 & -6 & 15 & 3 \\ \hline 0 & 0 & 22 & 0 \end{array}$$

Rows 2 and 3 show $-2z = 0$ and $22z = 0$, so $z = 0$.

Row 1 becomes $x - 2y + 5(0) = 1$

let $y = t$, then $x - 2t = 1$

$\therefore x = 1 + 2t$

$\therefore$ there are infinitely many solutions of the form $x = 1 + 2t$, $y = t$, $z = 0$, $t \in \mathbb{R}$.

b In augmented matrix form, the system is:

$$\left[\begin{array}{ccc|c} 1 & 4 & 11 & 7 \\ 1 & 6 & 17 & 9 \\ 1 & 4 & 8 & 4 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 4 & 11 & 7 \\ 0 & 2 & 6 & 2 \\ 0 & 0 & -3 & -3 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\begin{array}{rrrr} 1 & 6 & 17 & 9 \\ -1 & -4 & -11 & -7 \\ \hline 0 & 2 & 6 & 2 \end{array} \qquad \begin{array}{rrrr} 1 & 4 & 8 & 4 \\ -1 & -4 & -11 & -7 \\ \hline 0 & 0 & -3 & -3 \end{array}$$

The last row gives $-3z = -3 \quad \therefore z = 1$

$\therefore$ in row 2, $2y + 6z = 2$

$\therefore 2y + 6 = 2$

$\therefore y = -2$

and in row 1, $x + 4y + 11z = 7$

$\therefore x + 4(-2) + 11(1) = 7$

$\therefore x + 3 = 7$

$\therefore x = 4$

Thus we have a unique solution $x = 4$, $y = -2$, $z = 1$.

c In augmented matrix form, the system is:

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 17 \\ 2 & -2 & -5 & 4 \\ 3 & 2 & 2 & 10 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 2 & -1 & 3 & 17 \\ 0 & -1 & -8 & -13 \\ 0 & 7 & -5 & -31 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to 2R_3 - 3R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 2 & -1 & 3 & 17 \\ 0 & -1 & -8 & -13 \\ 0 & 0 & -61 & -122 \end{array}\right] \quad R_3 \to R_3 + 7R_2$$

$$\begin{array}{rrrr} 2 & -2 & -5 & 4 \\ -2 & 1 & -3 & -17 \\ \hline 0 & -1 & -8 & -13 \end{array} \qquad \begin{array}{rrrr} 6 & 4 & 4 & 20 \\ -6 & 3 & -9 & -51 \\ \hline 0 & 7 & -5 & -31 \end{array} \qquad \begin{array}{rrrr} 0 & 7 & -5 & -31 \\ 0 & -7 & -56 & -91 \\ \hline 0 & 0 & -61 & -122 \end{array}$$

The last row gives $-61z = -122 \quad \therefore \quad z = 2$

$\therefore$ in row 2, $-y - 8z = -13$
$\therefore \quad -y - 16 = -13$
$\therefore \quad y = -3$

and in row 1, $2x - y + 3z = 17$
$\therefore \quad 2x + 3 + 6 = 17$
$\therefore \quad 2x = 8$
$\therefore \quad x = 4$

Thus we have a unique solution $x = 4, \quad y = -3, \quad z = 2$.

d In augmented matrix form, the system is:

$$\left[\begin{array}{ccc|c} 2 & 3 & 4 & 1 \\ 5 & 6 & 7 & 2 \\ 8 & 9 & 10 & 4 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 2 & 3 & 4 & 1 \\ 0 & -3 & -6 & -1 \\ 0 & -3 & -6 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to 2R_2 - 5R_1 \\ R_3 \to R_3 - 4R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 2 & 3 & 4 & 1 \\ 0 & -3 & -6 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right] \quad R_3 \to R_3 - R_2$$

$$\begin{array}{rrrr} 10 & 12 & 14 & 4 \\ -10 & -15 & -20 & -5 \\ \hline 0 & -3 & -6 & -1 \end{array} \qquad \begin{array}{rrrr} 8 & 9 & 10 & 4 \\ -8 & -12 & -16 & -4 \\ \hline 0 & -3 & -6 & 0 \end{array}$$

In row 3, $0z = 1$

$\therefore$ the system is inconsistent and has no real solutions.

2 **a** Either (1) no solutions or (2) an infinite number of solutions.

b **i** They are parallel if $a_1 = ka_2$, $b_1 = kb_2$, $c_1 = kc_2$ and $d_1 \neq kd_2$ for some k.

ii They are coincident if $a_1 = ka_2$, $b_1 = kb_2$, $c_1 = kc_2$ and $d_1 = kd_2$ for some k.

c **i** $\left[\begin{array}{ccc|c} 1 & -3 & 2 & 8 \\ 3 & -9 & 2 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -3 & 2 & 8 \\ 0 & 0 & -4 & -20 \end{array}\right] \quad R_2 \to R_2 - 3R_1$

$\therefore \quad -4z = -20$ and $x - 3y + 2z = 8$ or $x = 3y - 2z + 8$

$\therefore \quad z = 5$ and if we let $y = t$, then $x = 3t - 2(5) + 8 = -2 + 3t$

$\therefore$ the planes meet in the line $x = -2 + 3t, \; y = t, \; z = 5, \; t \in \mathbb{R}$

ii $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 1 & -1 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & -3 & 1 & 1 \end{array}\right] \quad R_2 \to 2R_2 - R_1$

$\therefore \quad -3y + z = 1$ and $2x + y + z = 5$

$\therefore$ if we let $y = t$, then $z = 1 + 3y = 1 + 3t$ and
$$\begin{aligned} 2x &= 5 - y - z \\ &= 5 - t - (1 + 3t) \\ &= 4 - 4t \\ \therefore \quad x &= 2 - 2t \end{aligned}$$

$\therefore$ the planes meet in the line $x = 2 - 2t, \; y = t, \; z = 1 + 3t, \; t \in \mathbb{R}$

iii $\left[\begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 3 & 6 & -9 & 18 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_2 \to R_2 - 3R_1$

$\therefore$ there are infinitely many solutions, as the planes are coincident.

Let $y = s$ and $z = t$ in $x + 2y - 3z = 6$, $s, t \in \mathbb{R}$
$\therefore\ x = 3t - 2s + 6$
$\therefore\ x = 3t - 2s + 6,\ y = s,\ z = t,\quad s, t \in \mathbb{R}$ is the general solution of the plane.

3 **a** $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 6 \\ 2 & 4 & k & 12 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 6 \\ 0 & 0 & k+2 & 0 \end{array}\right]$ $R_2 \to R_2 - 2R_1$

If $k = -2$, the two planes are coincident.
$\therefore$ infinitely many solutions.
If $k \neq -2$, the two planes meet in a line.
$\therefore$ infinitely many solutions.

b $\left[\begin{array}{ccc|c} 1 & -1 & 3 & 8 \\ 2 & -2 & 6 & k \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & -1 & 3 & 8 \\ 0 & 0 & 0 & k-16 \end{array}\right]$ $R_2 \to R_2 - 2R_1$

If $k = 16$, the planes are coincident.
$\therefore$ infinitely many solutions.
If $k \neq 16$, the planes are parallel but not coincident.
$\therefore$ no solutions exist.

4 (1) $P_1 = P_2 = P_3$: infinitely many solutions where x, y, and z are in terms of two parameters, s and t say. The solution is a plane.

(2) $P_1 = P_2$ are coincident and cut by P_3: infinitely many solutions where x, y, and z are in terms of one parameter, t say. The solution is a line.

(3) $P_1 = P_2$ with P_3 parallel but not coincident: no solutions exist.

(4) P_1 and P_2 are parallel but not coincident, and P_3 cuts both planes: no solutions exist.

(5) P_1, P_2, and P_3 are all parallel but not coincident: no solutions exist.

(6) P_1, P_2, and P_3 meet in a unique point (a, b, c), so that $x = a$, $y = b$, $z = c$.

(7) P_1, P_2, and P_3 meet in a common line: infinitely many solutions where x, y, and z are in terms of one parameter, t say.

(8) P_1, P_2, and P_3 are such that the line of intersection between any two is parallel to the third plane: no solutions exist.

5 **a** The system has augmented matrix

$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -5 \\ 1 & -1 & 2 & 11 \\ 4 & 1 & -5 & -18 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & -5 \\ 0 & -2 & 3 & 16 \\ 0 & -3 & -1 & 2 \end{array}\right]$ $\begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 4R_1 \end{array}$

$\sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & -5 \\ 0 & -2 & 3 & 16 \\ 0 & 0 & -11 & -44 \end{array}\right]$ $R_3 \to 2R_3 - 3R_2$

$\therefore$ the planes meet at the unique point $(1, -2, 4)$

Now $-11z = -44 \quad \therefore\ z = 4$
and $-2y + 3z = 16$
$\therefore\ -2y + 12 = 16$
$\therefore\ -2y = 4$
$\therefore\ y = -2$
and $x + y - z = -5$
$\therefore\ x = -5 - (-2) + 4$
$\therefore\ x = 1$

b The system has augmented matrix

$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 2 & 1 & -1 & 8 \\ 5 & -2 & 5 & 11 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 3 & -5 & 6 \\ 0 & 3 & -5 & 6 \end{array}\right]$ $\begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 5R_1 \end{array}$

$\sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 3 & -5 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $R_3 \to R_3 - R_2$

$\therefore$ the three planes meet in a common line

$x = \dfrac{9-t}{3},\quad y = \dfrac{5t+6}{3},\quad z = t,\quad t \in \mathbb{R}$

Let $z = t$
As $3y - 5z = 6$
$3y = 5t + 6$
$\therefore\ y = \dfrac{5t+6}{3}$
But $x - y + 2z = 1$
$\therefore\ x = 1 + \dfrac{5t+6}{3} - 2t$
$\therefore\ x = \dfrac{3 + 5t + 6 - 6t}{3}$
$\therefore\ x = \dfrac{9-t}{3}$

c The system has augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 8 \\ 2 & -1 & -1 & 5 \\ 3 & -4 & -1 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 8 \\ 0 & -5 & 1 & -11 \\ 0 & -10 & 2 & -22 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 8 \\ 0 & -5 & 1 & -11 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ R_3 \to R_3 - 2R_2 \end{array}$$

Let $y = t$

Now $-5y + z = -11$

$\therefore\ z = -11 + 5t$

Also $x + 2y - z = 8$

$\therefore\ x = 8 - 2y + z$

$\therefore\ x = 8 - 2t - 11 + 5t$

$\therefore\ x = -3 + 3t$

$\therefore$ the three planes meet in a common line $x = 3t - 3,\quad y = t,\quad z = 5t - 11,\quad t \in \mathbb{R}$

d The system has augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 8 \\ 2 & -2 & 2 & 11 \\ 1 & 3 & -1 & -2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 8 \\ 0 & 0 & 0 & -5 \\ 0 & 4 & -2 & -10 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

The first two planes are parallel and are cut by the third plane.

$\therefore$ the equations are inconsistent and there are no solutions.

e The system has augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 1 & -1 & 1 & 4 \\ 3 & 3 & -6 & 3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & -2 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

There are two coincident planes cut by a third plane.

$\therefore$ infinitely many solutions in a line:

$$x = \frac{t+5}{2},\quad y = \frac{3t-3}{2},\quad z = t,\quad t \in \mathbb{R}$$

Let $z = t$

Now $-2y + 3z = 3$

$\therefore\ 2y = 3z - 3$

$\therefore\ y = \dfrac{3t-3}{2}$

and as $x + y - 2z = 1$

$\therefore\ x = 1 - y + 2z$

$\therefore\ x = 1 - \dfrac{3t-3}{2} + 2t$

$\therefore\ x = \dfrac{2 - 3t + 3 + 4t}{2}$

$\therefore\ x = \dfrac{t+5}{2}$

f The system has augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 5 \\ 1 & 1 & 1 & 1 \\ 5 & -1 & 2 & 17 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & -1 & 5 \\ 0 & 2 & 2 & -4 \\ 0 & 4 & 7 & -8 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 5R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & -1 & 5 \\ 0 & 2 & 2 & -4 \\ 0 & 0 & 3 & 0 \end{array}\right] \begin{array}{l} \\ \\ R_3 \to R_3 - 2R_2 \end{array}$$

Now $3z = 0$

$\therefore\ z = 0$

As $2y + 2z = -4$

$\therefore\ 2y = -4$

$\therefore\ y = -2$

and as $x - y - z = 5$

$\therefore\ x = 5 + (-2) + 0$

$\therefore\ x = 3$

$\therefore$ the planes meet at the unique point $(3, -2, 0)$.

6 The system has augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 1 \\ 2 & -3 & -1 & 3 \\ 3 & -5 & -5 & k \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 3 & -1 \\ 0 & -1 & -7 & 1 \\ 0 & -2 & -14 & k-3 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 3 & -1 \\ 0 & -1 & -7 & 1 \\ 0 & 0 & 0 & k-5 \end{array}\right] \begin{array}{l} \\ \\ R_3 \to R_3 - 2R_2 \end{array}$$

(1) If $k = 5$, the planes meet in a line {as we have a row of zeros}

$$\begin{aligned} &\text{Let } z = t \\ \text{Now } &-y - 7z = 1 \\ &\therefore\ y = -1 - 7t \\ \text{and } &x - y + 3z = 1 \\ &\therefore\ x = 1 + y - 3z \\ &\therefore\ x = 1 - 1 - 7t - 3t \\ &\therefore\ x = -10t \end{aligned}$$

$\therefore\ x = -10t,\ y = -1 - 7t,\ z = t,\ t \in \mathbb{R}$

(2) If $k \neq 5$ there are no solutions.

Since no two planes are parallel, the line of intersection of any two planes is parallel to the third plane.

7 **a** In augmented matrix form, the system is:

$$\left[\begin{array}{ccc|c} 1 & 3 & 3 & a-1 \\ 2 & -1 & 1 & 7 \\ 3 & -5 & a & 16 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 3 & a-1 \\ 0 & -7 & -5 & 9-2a \\ 0 & -14 & a-9 & 19-3a \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 3 & a-1 \\ 0 & -7 & -5 & 9-2a \\ 0 & 0 & a+1 & a+1 \end{array}\right] \begin{array}{l} \\ \\ R_3 \to R_3 - 2R_2 \end{array}$$

$$\begin{array}{rrrr} 2 & -1 & 1 & 7 \\ -2 & -6 & -6 & -2a+2 \\ \hline 0 & -7 & -5 & 9-2a \end{array}$$

$$\begin{array}{rrrr} 3 & -5 & a & 16 \\ -3 & -9 & -9 & -3a+3 \\ \hline 0 & -14 & a-9 & 19-3a \end{array}$$

$$\begin{array}{rrrr} 0 & -14 & a-9 & 19-3a \\ 0 & 14 & 10 & 4a-18 \\ \hline 0 & 0 & a+1 & a+1 \end{array}$$

b If $a = -1$, row 3 is a row of zeros, so there are infinitely many solutions, as we have 2 equations in 3 unknowns.

Let $z = t$ in row 2, then $-7y - 5t = 9 - 2(-1)$

$$\therefore\ -7y - 5t = 11$$

$$\therefore\ y = \frac{-5t - 11}{7}$$

and substituting in row 1 gives $x + 3\left(\dfrac{-5t-11}{7}\right) + 3t = (-1) - 1$

$$\therefore\ x - \frac{15t}{7} - \frac{33}{7} + 3t = -2 \text{ and so, } x = \frac{19}{7} - \frac{6t}{7}$$

$\therefore$ there are infinitely many solutions of form $x = \dfrac{19-6t}{7},\ y = \dfrac{-5t-11}{7},\ z = t,\ t \in \mathbb{R}$.

In this case we have three planes which meet in a line.

c If $a \neq -1$, then $(a+1)z = a+1 \quad \therefore\ z = 1$

From row 2, $-7y - 5(1) = 9 - 2a$

$$\therefore\ -7y = -2a + 14$$

$$y = \frac{2a}{7} - 2$$

and substituting in row 1 gives $x + 3\left(\dfrac{2a}{7} - 2\right) + 3(1) = a - 1$

$$\therefore\ x + \frac{6a}{7} - 6 + 3 = a - 1$$

$$\therefore\ x = \frac{a}{7} + 2$$

$\therefore$ the unique solution is $x = \frac{1}{7}a + 2,\ y = \frac{2}{7}a - 2,\ z = 1$.

In this case we have three planes which meet at a point.

8 In augmented matrix form, the system is:

$$\left[\begin{array}{ccc|c} 1 & 2 & m & -1 \\ m & -2 & 1 & 1 \\ 2 & 1 & -1 & 3 \end{array}\right]$$

$$\begin{array}{cccc} m & -2 & 1 & 1 \\ -m & -2m & -m^2 & m \\ \hline 0 & -2(m+1) & 1-m^2 & 1+m \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & m & -1 \\ 0 & -2(m+1) & 1-m^2 & 1+m \\ 0 & -3 & -1-2m & 5 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 - mR_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\begin{array}{cccc} 2 & 1 & -1 & 3 \\ -2 & -4 & -2m & 2 \\ \hline 0 & -3 & -1-2m & 5 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & m & -1 \\ 0 & -2(m+1) & 1-m^2 & 1+m \\ 0 & 0 & (m+1)(m+5) & -7(m+1) \end{array}\right] \quad R_3 \to -2(m+1)R_3 + 3R_2$$

$$\begin{array}{cccc} 0 & 6(m+1) & 2(m+1)(1+2m) & -10(m+1) \\ 0 & -6(m+1) & 3(1-m^2) & 3(1+m) \\ \hline 0 & 0 & 2(2m^2+3m+1)+3-3m^2 & -7(m+1) \\ & & = m^2+6m+5 & \\ & & = (m+1)(m+5) & \end{array}$$

a If $m=-5$, row 3 becomes $0x+0y+0z=28$

$\therefore$ the system is inconsistent and there are no solutions.

In this case we have three planes with no common point of intersection. No two planes are coincident or parallel. So, the line of intersection of any two planes is parallel to the third.

b If $m=-1$, row 3 is a row of zeros, so we have 2 equations in 3 unknowns.

$\therefore$ there are infinitely many solutions.

When $m=-1$ the equations become

$$\begin{cases} x+2y-z=-1 \\ -x-2y+z=1 \\ 2x+y-z=3 \end{cases}$$

equations 1 and 2 are equivalent.
In this case two planes are coincident and the third meets in a line.

c **i** If $m \neq -5$ or -1, there is a unique solution. The planes meet at a point.

ii If $m \neq -1$ and $m \neq -5$, then row 3 becomes

$(m+1)(m+5)z = -7(m+1)$

$$\therefore\ z = \frac{-7}{m+5}$$ and substituting in row 2 gives

$$-2(m+1)y + (1-m^2)\left(\frac{-7}{m+5}\right) = 1+m$$

$$\therefore\ -2(m+1)(m+5)y - 7(1-m^2) = (1+m)(m+5)$$
$$\therefore\ -2(m+1)(m+5)y = m^2+6m+5+7-7m^2$$
$$\therefore\ -2(m+1)(m+5)y = -6m^2+6m+12$$
$$\therefore\ -2(m+1)(m+5)y = -6(m^2-m-2)$$
$$\therefore\ (m+1)(m+5)y = 3(m+1)(m-2)$$
$$\therefore\ y = \frac{3(m-2)}{m+5}$$

and substituting in row 1 gives $$x + \frac{6(m-2)}{m+5} + \frac{-7m}{m+5} = -1$$

$$\therefore\ x(m+5) + 6(m-2) - 7m = -(m+5)$$
$$\therefore\ x(m+5) + 6m - 12 - 7m = -m-5$$
$$\therefore\ x(m+5) = 7$$
$$\therefore\ x = \frac{7}{m+5}$$

$\therefore$ the system has a unique solution for all m except $m=-5$ and $m=-1$, and the solution is $x = \dfrac{7}{m+5}$, $y = \dfrac{3(m-2)}{m+5}$, $z = \dfrac{-7}{m+5}$.

9 P_1 meets P_2 where

$$\begin{pmatrix}2\\-1\\0\end{pmatrix}+\lambda\begin{pmatrix}3\\0\\1\end{pmatrix}+\mu\begin{pmatrix}1\\1\\-1\end{pmatrix}=\begin{pmatrix}3\\-1\\3\end{pmatrix}+r\begin{pmatrix}2\\0\\-1\end{pmatrix}+s\begin{pmatrix}1\\1\\0\end{pmatrix}$$

$$\therefore \begin{cases}2+3\lambda+\mu=3+2r+s\\ -1+\mu=-1+s\\ \lambda-\mu=3-r\end{cases} \quad \text{which gives} \quad \begin{cases}3\lambda+\mu=2r+s+1\\ \mu=s\\ \lambda-\mu=3-r\end{cases}$$

If $\mu=a$ say, then $s=a$, $3\lambda+a=2r+a+1$, and $\lambda-a=3-r$

$$\therefore r=3-\lambda+a$$

$$\therefore 3\lambda+a=6-2\lambda+2a+a+1$$
$$\therefore 5\lambda=2a+7$$
$$\therefore \lambda=\frac{2a+7}{5} \quad \text{and} \quad r=3+a-\frac{2a+7}{5}$$
$$\therefore r=\frac{3a+8}{5}$$

$\therefore$ if $\mu=a$, $\lambda=\dfrac{2a+7}{5}$, $r=\dfrac{3a+8}{5}$, $s=a$ .... (1)

P_2 meets P_3 where

$$\begin{pmatrix}3\\-1\\3\end{pmatrix}+r\begin{pmatrix}2\\0\\-1\end{pmatrix}+s\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}2\\-1\\2\end{pmatrix}+t\begin{pmatrix}1\\-1\\0\end{pmatrix}-u\begin{pmatrix}0\\-1\\2\end{pmatrix}$$

$$\therefore \begin{cases}3+2r+s=2+t\\ -1+s=-1-t+u\\ 3-r=2-2u\end{cases} \quad \text{which gives} \quad \begin{cases}2r+s+1=t\\ s=u-t\\ 2u-r=-1\end{cases}$$

So, if $u=b$ say, then $r=2b+1$ and $4b+2+b-t+1=t$

$$\therefore 5b+3=2t \quad \text{and so} \quad t=\frac{5b+3}{2}$$
$$\text{and} \quad s=u-t=b-\frac{5b+3}{2}=\frac{-3b-3}{2}$$

So, if $u=b$, $r=2b+1$, $t=\dfrac{5b+3}{2}$, $s=\dfrac{-3b-3}{2}$ .... (2)

From (1) and (2), $\dfrac{3a+8}{5}=2b+1$ and $a=\dfrac{-3b-3}{2}$

$$\therefore 3a+8=10b+5 \quad \text{and} \quad 2a=-3b-3$$

$$\therefore \begin{cases}3a-10b=-3\\ 2a+3b=-3\end{cases} \quad \text{which has solutions} \quad a=-\tfrac{39}{29},\ b=-\tfrac{3}{29}$$

In (2), $u=-\frac{3}{29}$, $t=\dfrac{5\left(-\frac{3}{29}\right)+3}{2}=1\frac{7}{29}$ or $\frac{36}{29}$

$$\therefore \mathbf{r}_3=\begin{pmatrix}2\\-1\\2\end{pmatrix}+\tfrac{36}{29}\begin{pmatrix}1\\-1\\0\end{pmatrix}+\tfrac{3}{29}\begin{pmatrix}0\\-1\\2\end{pmatrix}=\begin{pmatrix}3\frac{7}{29}\\-2\frac{10}{29}\\2\frac{6}{29}\end{pmatrix}=\begin{pmatrix}\frac{94}{29}\\-\frac{68}{29}\\\frac{64}{29}\end{pmatrix}$$

$\therefore$ all 3 planes meet at $\left(\frac{94}{29}, -\frac{68}{29}, \frac{64}{29}\right)$.

REVIEW SET 15A

1 **a** The vector equation is

$$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-6\\3\end{pmatrix}+t\begin{pmatrix}4\\-3\end{pmatrix},\quad t\in\mathbb{R}$$

b The parametric equations are
$x=-6+4t$, $y=3-3t$, $t\in\mathbb{R}$

c $\dfrac{x+6}{4} = \dfrac{y-3}{-3} = t$

$\therefore\ -3x - 18 = 4y - 12$

So, the Cartesian equation is $3x + 4y = -6$.

2 $(-3,\ m)$ lies on the line, so $\begin{pmatrix} -3 \\ m \end{pmatrix} = \begin{pmatrix} 18 \\ -2 \end{pmatrix} + \begin{pmatrix} -7t \\ 4t \end{pmatrix}$

$\therefore\ -3 = 18 - 7t$ and $m = -2 + 4t$

$\therefore\ 7t = 21$

$\therefore\ t = 3$ and so $m = -2 + 4(3) = 10$

3 **a** When $t = 1$, $\mathbf{r} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} + 1\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 3+2 \\ -3+5 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$

$\therefore$ the point is $(5,\ 2)$.

b $\begin{pmatrix} 4 \\ 10 \end{pmatrix}$ is a non-zero scalar multiple of $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$, so it could also be used to describe the direction of the line.

c The line passes through point $(5,\ 2)$ and has direction vector $\begin{pmatrix} 4 \\ 10 \end{pmatrix}$.

$\therefore\ \mathbf{r} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} + s\begin{pmatrix} 4 \\ 10 \end{pmatrix}$, $s \in \mathbb{R}$ is an alternative vector equation for the line.

4 P(2, 0, 1), Q(3, 4, −2), R(−1, 3, 2)

a $\overrightarrow{\mathrm{PQ}} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$

Since $\overrightarrow{\mathrm{PQ}} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$ and P is at $(2,\ 0,\ 1)$, the line has parametric equations

$x = 2 + t,\quad y = 0 + 4t,\quad z = 1 - 3t$

$\therefore\ x = 2 + t,\quad y = 4t,\quad z = 1 - 3t,\quad t \in \mathbb{R}$

b $\overrightarrow{\mathrm{QR}} = \begin{pmatrix} -4 \\ -1 \\ 4 \end{pmatrix}$

$|\,\overrightarrow{\mathrm{PQ}}\,| = \sqrt{1^2 + 4^2 + (-3)^2} = \sqrt{26}$

$|\,\overrightarrow{\mathrm{QR}}\,| = \sqrt{(-4)^2 + (-1)^2 + 4^2} = \sqrt{33}$

$$\begin{aligned}\overrightarrow{\mathrm{PQ}} \bullet \overrightarrow{\mathrm{QR}} &= \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} \bullet \begin{pmatrix} -4 \\ -1 \\ 4 \end{pmatrix} \\ &= 1 \times (-4) + 4 \times (-1) + (-3) \times 4 \\ &= -20\end{aligned}$$

If $\theta = \mathrm{P}\widehat{\mathrm{Q}}\mathrm{R}$, then $\cos\theta = \dfrac{|\,\overrightarrow{\mathrm{PQ}} \bullet \overrightarrow{\mathrm{QR}}\,|}{|\,\overrightarrow{\mathrm{PQ}}\,|\,|\,\overrightarrow{\mathrm{QR}}\,|} = \dfrac{|-20|}{\sqrt{26}\sqrt{33}}$

$= \dfrac{20}{\sqrt{26}\sqrt{33}}$

5 **a** Lines (AB) and (AC) meet at A.

$$\therefore \begin{pmatrix}4\\-1\end{pmatrix} + t\begin{pmatrix}1\\3\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix} + u\begin{pmatrix}3\\1\end{pmatrix}$$

$$\therefore \begin{pmatrix}4+t\\-1+3t\end{pmatrix} = \begin{pmatrix}-1+3u\\u\end{pmatrix}$$

$$\therefore\ t - 3u = -5 \quad .... (1)$$
$$3t - u = 1$$
$$\therefore\ -3t + 9u = 15 \quad \{-3 \times (1)\}$$
$$3t - u = 1$$
$$\text{Adding,}\quad 8u = 16$$
$$\therefore\ u = 2$$

So, $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix} + 2\begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}5\\2\end{pmatrix}$

$\therefore$ A is $(5, 2)$.

Lines (AB) and (BC) meet at B.

$$\therefore \begin{pmatrix}4\\-1\end{pmatrix} + t\begin{pmatrix}1\\3\end{pmatrix} = \begin{pmatrix}7\\4\end{pmatrix} + s\begin{pmatrix}1\\-1\end{pmatrix}$$

$$\therefore \begin{pmatrix}4+t\\-1+3t\end{pmatrix} = \begin{pmatrix}7+s\\4-s\end{pmatrix}$$

$$\therefore\ t - s = 3$$
$$3t + s = 5$$
$$\text{Adding,}\quad 4t = 8$$
$$\therefore\ t = 2$$

So, $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}4\\-1\end{pmatrix} + 2\begin{pmatrix}1\\3\end{pmatrix} = \begin{pmatrix}6\\5\end{pmatrix}$

$\therefore$ B is $(6, 5)$.

Lines (BC) and (AC) meet at C.

$$\therefore \begin{pmatrix}7\\4\end{pmatrix} + s\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix} + u\begin{pmatrix}3\\1\end{pmatrix}$$

$$\therefore \begin{pmatrix}7+s\\4-s\end{pmatrix} = \begin{pmatrix}-1+3u\\u\end{pmatrix}$$

$$\therefore\ s - 3u = -8$$
$$-s - u = -4$$
$$\text{Adding,}\quad -4u = -12$$
$$\therefore\ u = 3$$

So, $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix} + 3\begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}8\\3\end{pmatrix}$ $\quad\therefore$ C is $(8, 3)$.

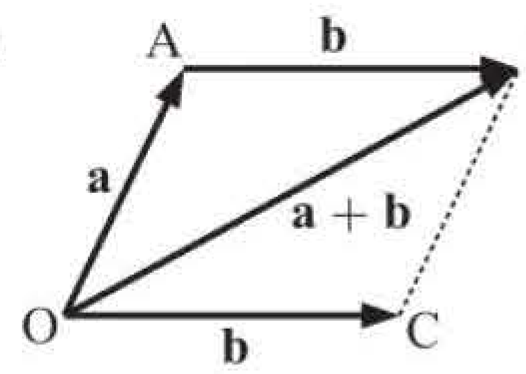
b $\overrightarrow{AB} = \begin{pmatrix}1\\3\end{pmatrix}$, so $|\overrightarrow{AB}| = \sqrt{1+9} = \sqrt{10}$ units

$\overrightarrow{BC} = \begin{pmatrix}2\\-2\end{pmatrix}$, so $|\overrightarrow{BC}| = \sqrt{4+4} = \sqrt{8}$ units

$\overrightarrow{AC} = \begin{pmatrix}3\\1\end{pmatrix}$, so $|\overrightarrow{AC}| = \sqrt{9+1} = \sqrt{10}$ units

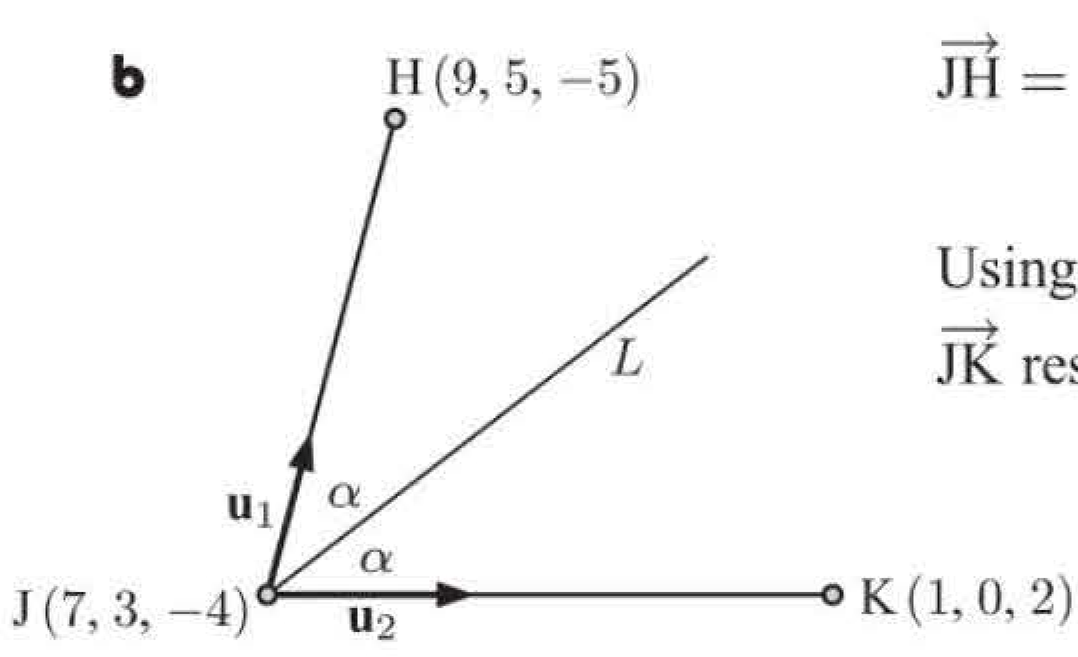
c Triangle ABC is isosceles.

6 **a**



As **a** and **b** are unit vectors, OABC is a rhombus.
But the angles of a rhombus are bisected by its diagonals, so $\mathbf{a} + \mathbf{b}$ bisects the angle between vector **a** and vector **b**.

b



$$\overrightarrow{JH} = \begin{pmatrix}9-7\\5-3\\-5--4\end{pmatrix} = \begin{pmatrix}2\\2\\-1\end{pmatrix}, \quad \overrightarrow{JK} = \begin{pmatrix}1-7\\0-3\\2--4\end{pmatrix} = \begin{pmatrix}-6\\-3\\6\end{pmatrix}$$

Using **a**, we write unit vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ in the direction of $\overrightarrow{JH}$ and $\overrightarrow{JK}$ respectively.

$$\therefore \quad \mathbf{u}_1 = \frac{1}{\sqrt{4+4+1}}\begin{pmatrix}2\\2\\-1\end{pmatrix} = \begin{pmatrix}\frac{2}{3}\\\frac{2}{3}\\-\frac{1}{3}\end{pmatrix} \quad \text{and} \quad \mathbf{u}_2 = \frac{1}{\sqrt{36+9+36}}\begin{pmatrix}-6\\-3\\6\end{pmatrix} = \begin{pmatrix}-\frac{2}{3}\\-\frac{1}{3}\\\frac{2}{3}\end{pmatrix}$$

and $\mathbf{u}_1 + \mathbf{u}_2 = \begin{pmatrix}0\\\frac{1}{3}\\\frac{1}{3}\end{pmatrix}$, which bisects $\widehat{\text{HJK}}$, by **a**.

$\therefore$ the equation of the line L is $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}7\\3\\-4\end{pmatrix} + \lambda\begin{pmatrix}0\\\frac{1}{3}\\\frac{1}{3}\end{pmatrix}$, $\lambda \in \mathbb{R}$

c $\overrightarrow{\text{HK}} = \begin{pmatrix}1-9\\0-5\\2--5\end{pmatrix} = \begin{pmatrix}-8\\-5\\7\end{pmatrix}$ so (HK) has equation $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\0\\2\end{pmatrix} + s\begin{pmatrix}-8\\-5\\7\end{pmatrix}$, $s \in \mathbb{R}$.

This line meets L where

$7 = 1 - 8s, \qquad 3 + \dfrac{t}{3} = -5s, \qquad$ and $\qquad -4 + \dfrac{t}{3} = 2 + 7s \quad .... (*)$

$\therefore \quad 8s = -6$

$\therefore \quad s = -\frac{3}{4}$ and so $3 + \dfrac{t}{3} = \frac{15}{4}$

$\therefore \quad \dfrac{t}{3} = \frac{3}{4}$

$\therefore \quad t = \frac{9}{4}$

In $(*)$, $\quad \text{LHS} = -4 + \dfrac{t}{3} = -4 + \frac{3}{4} = -\frac{13}{4} \qquad \text{RHS} = 2 + 7s = 2 + 7\left(-\frac{3}{4}\right) = \frac{8}{4} - \frac{21}{4} = -\frac{13}{4}$ ✓

$\therefore \quad s = -\frac{3}{4}, \quad t = \frac{9}{4}$ satisfy all 3 equations.

So, $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\0\\2\end{pmatrix} - \frac{3}{4}\begin{pmatrix}-8\\-5\\7\end{pmatrix} = \begin{pmatrix}1+6\\0+\frac{15}{4}\\2-\frac{21}{4}\end{pmatrix} = \begin{pmatrix}7\\3\frac{3}{4}\\-3\frac{1}{4}\end{pmatrix}$

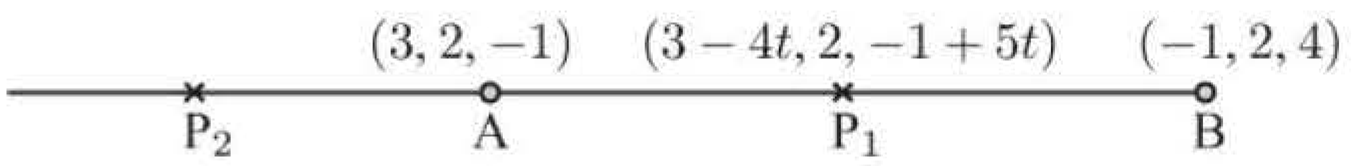
$\therefore \quad L$ meets (HK) at $(7, 3\frac{3}{4}, -3\frac{1}{4})$.

7 **a** $\overrightarrow{\text{AB}} = \begin{pmatrix}-4\\0\\5\end{pmatrix} \quad \therefore$ the line is $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}3\\2\\-1\end{pmatrix} + t\begin{pmatrix}-4\\0\\5\end{pmatrix}$, $t \in \mathbb{R}$

b The equation of the plane is

$-4x + 0y + 5z = -4(-1) + 5(4)$

$\therefore \quad -4x + 5z = 24$



c The distance from a point on the line to A is $d = \sqrt{(-4t)^2 + 0^2 + (5t)^2} = \sqrt{41t^2}$

$\therefore$ since $d = 2\sqrt{41}$ units, $\sqrt{41t^2} = 2\sqrt{41}$

$\therefore \quad t^2 = 4$

$\therefore \quad t = \pm 2 \qquad \therefore$ the points are $(-5, 2, 9)$ and $(11, 2, -11)$.

8 Given $\text{C}(-3, 2, -1)$ and $\text{D}(0, 1, -4)$, $\overrightarrow{\text{CD}} = \begin{pmatrix}3\\-1\\-3\end{pmatrix}$

$\therefore$ the line passing through C and D has parametric equations

$x = -3 + 3t, \quad y = 2 - t, \quad z = -1 - 3t$

The line meets $2x - y + z = 3$ when $2(-3+3t) - (2-t) + (-1-3t) = 3$

$$\therefore\ -6 + 6t - 2 + t - 1 - 3t = 3$$
$$\therefore\ 4t = 12$$
$$\therefore\ t = 3$$

$\therefore$ they meet at $(6, -1, -10)$

9 $|\mathbf{a}| = 3$, $|\mathbf{b}| = \sqrt{7}$ and $\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$

a
$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin\theta$$
$$\sqrt{1+4+9} = 3 \times \sqrt{7} \times \sin\theta$$
$$\sin\theta = \tfrac{\sqrt{14}}{3\sqrt{7}} = \tfrac{\sqrt{2}}{3}$$

But $\cos^2\theta = 1 - \sin^2\theta$

$\therefore\ \cos\theta = \pm\frac{\sqrt{7}}{3}$

Hence
$$\begin{aligned} \mathbf{a} \bullet \mathbf{b} &= |\mathbf{a}|\,|\mathbf{b}| \cos\theta \\ &= 3 \times \sqrt{7} \times (\pm\tfrac{\sqrt{7}}{3}) \\ &= \pm 7 \end{aligned}$$

b Area
$$\begin{aligned} \triangle\text{OAB} &= \tfrac{1}{2}|\mathbf{a} \times \mathbf{b}| \\ &= \tfrac{1}{2}\sqrt{14}\ \text{units}^2 \end{aligned}$$

10 **a** The distance of $X(-1, 1, 3)$ from $x - 2y - 2z = 8$

is $d = \dfrac{|x_1 - 2y_1 - 2z_1 - 8|}{\sqrt{1^2 + (-2)^2 + (-2)^2}} = \dfrac{|-1-2-6-8|}{3} = \dfrac{|-17|}{3} = \frac{17}{3}$ units

b Since $2 - x = y - 3 = -\frac{1}{2}z$, $\dfrac{x-2}{-1} = \dfrac{y-3}{1} = \dfrac{z}{-2}$

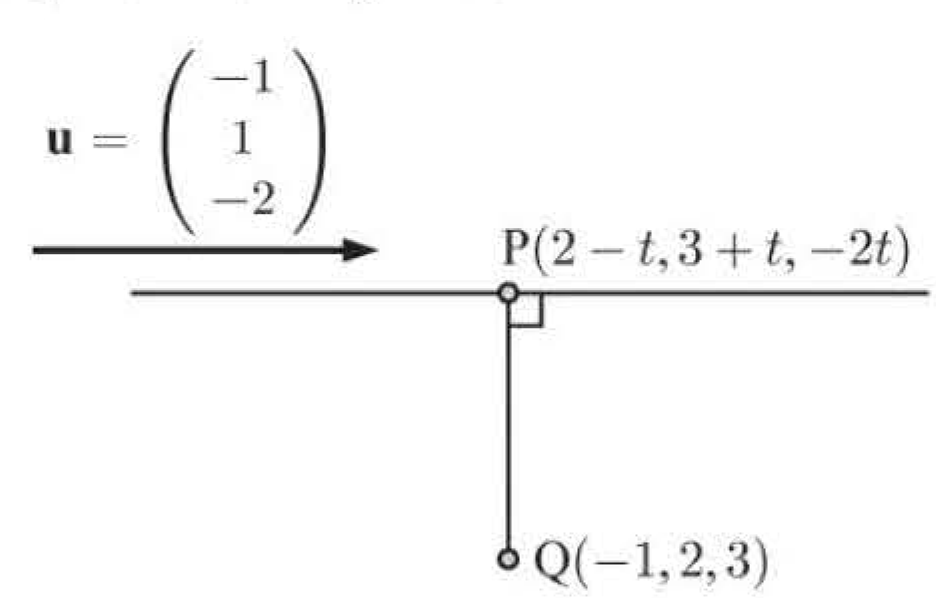
$\therefore$ the line has direction vector $\mathbf{u} = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$, and passes through $(2, 3, 0)$

$\therefore$ if P is a point on the line with coordinates $(2-t, 3+t, -2t)$, then

$$\overrightarrow{\text{QP}} = \begin{pmatrix} 2-t--1 \\ 3+t-2 \\ -2t-3 \end{pmatrix} = \begin{pmatrix} 3-t \\ 1+t \\ -2t-3 \end{pmatrix}$$

If P is chosen such that $\overrightarrow{\text{QP}}$ is perpendicular to the line, then $\mathbf{u} \bullet \overrightarrow{\text{QP}} = 0$

$$\therefore\ \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 3-t \\ 1+t \\ -2t-3 \end{pmatrix} = 0$$



$$\therefore\ -(3-t) + 1(1+t) - 2(-2t-3) = 0$$
$$\therefore\ -3 + t + 1 + t + 4t + 6 = 0$$
$$\therefore\ 6t = -4$$
$$\therefore\ t = -\tfrac{2}{3}$$

$\therefore$ P is at $(2 + \frac{2}{3}, 3 - \frac{2}{3}, 2(\frac{2}{3}))$, so the foot of the perpendicular is at $(\frac{8}{3}, \frac{7}{3}, \frac{4}{3})$.

11 **a** $\overrightarrow{\text{LM}} = \begin{pmatrix} -2 \\ 2 \\ -2 \end{pmatrix} = -2\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$\therefore$ since L lies on the line, it has parametric equations

$$x = 1 + t, \quad y = 0 - t, \quad z = 1 + t, \quad t \in \mathbb{R}$$

The line meets $x - 2y - 3z = 14$ if

$$(1+t) - 2(-t) - 3(1+t) = 14$$
$$\therefore\ 1 + t + 2t - 3 - 3t = 14$$
$$\therefore\ -2 = 14 \quad \text{which is absurd}$$

$\therefore$ the line and plane do not meet, but rather are parallel.

b The distance $d = \dfrac{|x_1 - 2y_1 - 3z_1 - 14|}{\sqrt{1+4+9}} = \dfrac{|1 - 2(0) - 3(1) - 14|}{\sqrt{14}} = \frac{16}{\sqrt{14}}$ units

12 **a** L_1 meets $2x + y - z = 2$

where $2(3t-4) + (t+2) - (2t-1) = 2$

$\therefore\ 6t - 8 + t + 2 - 2t + 1 = 2$

$\therefore\ 5t = 7$

$\therefore\ t = \frac{7}{5}$

$\therefore$ the lines meet at

$\left(3(\frac{7}{5}) - 4,\ \frac{7}{5} + 2,\ 2(\frac{7}{5}) - 1\right)$

which is $(\frac{1}{5}, \frac{17}{5}, \frac{9}{5})$

b L_1 meets L_2

where $3t - 4 = \dfrac{t+2-5}{2} = \dfrac{-(2t-1)-1}{2}$

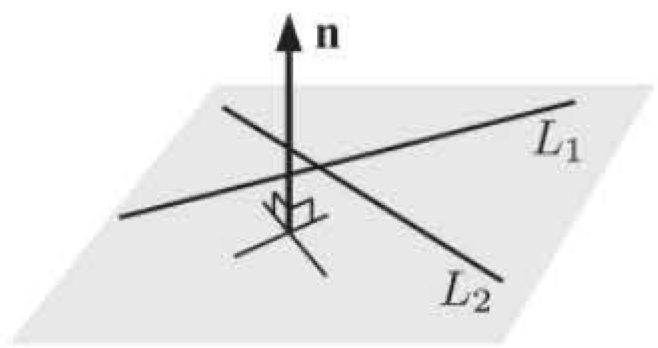
$\therefore\ 6t - 8 = t - 3 = -2t$

$\therefore\ 5t = 5$ and $3t = 3$

$\therefore\ t = 1$

So, L_1 and L_2 meet at $(-1, 3, 1)$.

c



$$\mathbf{n} = \begin{pmatrix}3\\1\\2\end{pmatrix} \times \begin{pmatrix}1\\2\\-2\end{pmatrix} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3 & 1 & 2\\ 1 & 2 & -2\end{vmatrix}$$

$$= \begin{vmatrix}1 & 2\\ 2 & -2\end{vmatrix}\mathbf{i} - \begin{vmatrix}3 & 2\\ 1 & -2\end{vmatrix}\mathbf{j} + \begin{vmatrix}3 & 1\\ 1 & 2\end{vmatrix}\mathbf{k} = -6\mathbf{i} + 8\mathbf{j} + 5\mathbf{k}$$

$\therefore$ equation is $-6x + 8y + 5z = -6(-1) + 8(3) + 5(1)$

$\therefore\ -6x + 8y + 5z = 35$

$\therefore\ 6x - 8y - 5z = -35$

13 $x - 1 = \dfrac{y+2}{2} = \dfrac{z-3}{4}$ has direction vector $\begin{pmatrix}1\\2\\4\end{pmatrix}$ and $6x + 7y - 5z = 8$ has $\mathbf{n} = \begin{pmatrix}6\\7\\-5\end{pmatrix}$.

Now $\begin{pmatrix}1\\2\\4\end{pmatrix} \bullet \begin{pmatrix}6\\7\\-5\end{pmatrix} = 6 + 14 - 20 = 0$

$\therefore$ since these two vectors are perpendicular, the line is parallel to the plane.

Choose any point on the line, for example, $(1, -2, 3)$.

Then the distance from the line to the plane is $d = \dfrac{|6x_1 + 7y_1 - 5z_1 - 8|}{\sqrt{6^2 + 7^2 + (-5)^2}}$

$= \dfrac{|6(1) + 7(-2) - 5(3) - 8|}{\sqrt{110}}$

$= \frac{31}{\sqrt{110}}$ units

14 If A is $(3, -1, -2)$ and $B(5, 3, -4)$ then $\overrightarrow{AB} = \begin{pmatrix}5-3\\3--1\\-4--2\end{pmatrix} = \begin{pmatrix}2\\4\\-2\end{pmatrix} = 2\begin{pmatrix}1\\2\\-1\end{pmatrix}$

$\therefore$ the line has equation $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}3\\-1\\-2\end{pmatrix} + t\begin{pmatrix}1\\2\\-1\end{pmatrix},\ t \in \mathbb{R}$

and it meets $x^2 + y^2 + z^2 = 26$ where

$(3+t)^2 + (-1+2t)^2 + (-2-t)^2 = 26$

$\therefore\ 9 + 6t + t^2 + 1 - 4t + 4t^2 + 4 + 4t + t^2 - 26 = 0$

$\therefore\ 6t^2 + 6t - 12 = 0$

$\therefore\ t^2 + t - 2 = 0$

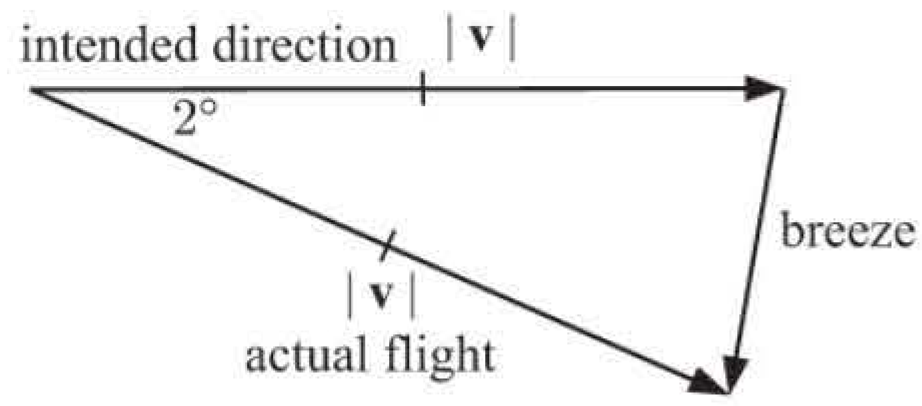
$\therefore\ (t+2)(t-1) = 0$

$\therefore\ t = -2$ or 1

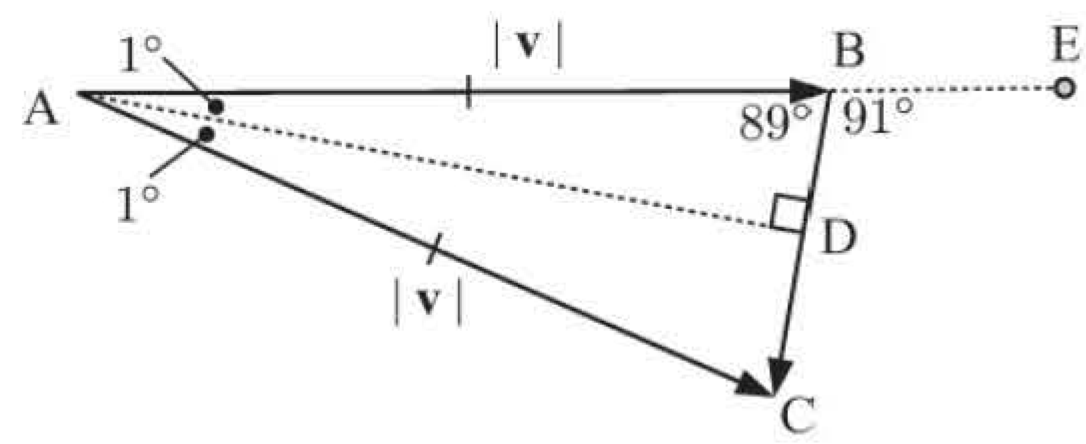
$\therefore\ \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}3\\-1\\-2\end{pmatrix} - 2\begin{pmatrix}1\\2\\-1\end{pmatrix}$ or $\begin{pmatrix}3\\-1\\-2\end{pmatrix} + \begin{pmatrix}1\\2\\-1\end{pmatrix}$

$\therefore$ the line meets the sphere at $(1, -5, 0)$ and $(4, 1, -3)$.

15 **a**



b **i**



Triangle ABC is isosceles.

$\therefore$ line (AD) that meets the base at the midpoint, D, bisects the angle at A and is perpendicular to the base.

The triangle ABD is right angled at D and angle DAB $= 1°$.

$\therefore$ angle ABD $= 180 - 90 - 1 = 89°$.

If line (AB) is extended to E, then angle DBE $= 180 - 89 = 91°$.

Line (AB) is the arrow's intended path and line (BC) is the breeze, so the breeze is $91°$ to the intended direction of the arrow.

ii The speed of the breeze is the length of (BC) $= 2\times |\overrightarrow{BD}|$.

$$\sin 1° = \frac{|\overrightarrow{BD}|}{|\mathbf{v}|}$$

$$\therefore |\overrightarrow{BD}| = |\mathbf{v}| \sin 1°$$

$$\therefore |\overrightarrow{BC}| = 2|\mathbf{v}| \sin 1°$$

So the speed of the breeze is $2|\mathbf{v}|\sin 1°$.

16 **a** X is $\left(\frac{4+10}{2}, \frac{4+2}{2}, \frac{-2+0}{2}\right)$ or $(7, 3, -1)$

If D is (a, b, c), $\overrightarrow{AD} = \begin{pmatrix} a-1 \\ b-3 \\ c--4 \end{pmatrix} = \begin{pmatrix} a-1 \\ b-3 \\ c+4 \end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix} 6 \\ -2 \\ 2 \end{pmatrix}$

Since $\overrightarrow{AD} = \overrightarrow{BC}$, $a - 1 = 6$, $b - 3 = -2$, $c + 4 = 2$

$\therefore a = 7$, $b = 1$, $c = -2$ $\therefore$ D is $(7, 1, -2)$

b $$\overrightarrow{OY} = \overrightarrow{OA} + \overrightarrow{AY} = \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix} + \tfrac{2}{3}\overrightarrow{AX} = \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix} + \tfrac{2}{3}\begin{pmatrix} 7-1 \\ 3-3 \\ -1--4 \end{pmatrix}$$

$\therefore \overrightarrow{OY} = \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} 6 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix}$ and so Y is $(5, 3, -2)$

c $\overrightarrow{BY} = \begin{pmatrix} 5-4 \\ 3-4 \\ -2--2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\overrightarrow{BD} = \begin{pmatrix} 7-4 \\ 1-4 \\ -2--2 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix}$

$\therefore \overrightarrow{BD} = 3\overrightarrow{BY}$ $\therefore \overrightarrow{BD} \parallel \overrightarrow{BY}$ and so B, D, and Y are collinear

17 The system has augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 2 & 1 & -1 & -1 \\ 7 & 2 & k & -k \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 3 & -3 & -11 \\ 0 & 9 & k-7 & -k-35 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 7R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 3 & -3 & -11 \\ 0 & 0 & k+2 & -k-2 \end{array}\right] \begin{array}{l} \\ \\ R_3 \to R_3 - 3R_2 \end{array}$$

Thus $(k+2)z = -(k+2)$

If $k \neq -2$ then $z=-1$, and as $3y - 3z = -11$,

then $3y = -14$

$\therefore\ y = -\frac{14}{3}$

and $x - y + z = 5$,

so $x = 5 - \frac{14}{3} + 1 = \frac{4}{3}$

$\therefore$ we have three planes that meet at the unique point $(\frac{4}{3}, -\frac{14}{3}, -1)$.

If $k=-2$, then the 3 planes meet in a common line and hence there are an infinite number of solutions.

In this case, let $z=t,\ t \in \mathbb{R}$.

Now $3y - 3z = -11$,

$\therefore\ 3y = -11 + 3t$

$\therefore\ y = -\frac{11}{3} + t$

and as $x - y + z = 5$

$\therefore\ x = 5 + y - z$

$\therefore\ x = 5 - \frac{11}{3} + t - t$

$\therefore\ x = \frac{4}{3}$

$\therefore\ x = \frac{4}{3},\quad y = -\frac{11}{3} + t,\quad z = t,\quad t \in \mathbb{R}$

REVIEW SET 15B

1 The vector equation is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 4 \end{pmatrix},\quad \lambda \in \mathbb{R}$

2 **a** **i** The yacht is initially at $(-6, 10)$, so its initial position vector is $\begin{pmatrix} -6 \\ 10 \end{pmatrix}$ or $-6\mathbf{i} + 10\mathbf{j}$.

ii $-\mathbf{i} - 3\mathbf{j}$ has length $\sqrt{(-1)^2 + (-3)^2} = \sqrt{10}$

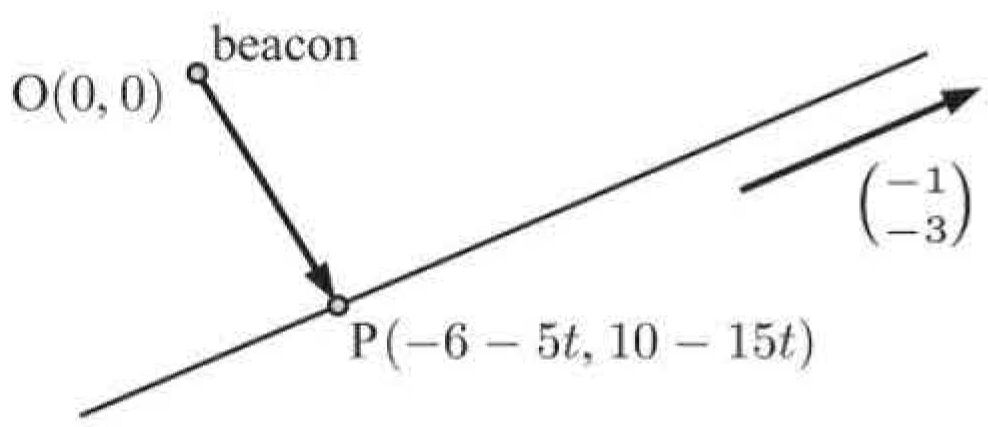
$\therefore\ 5(-\mathbf{i} - 3\mathbf{j})$ has length $5\sqrt{10}$

$\therefore$ the velocity vector is $-5\mathbf{i} - 15\mathbf{j}$

iii $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -6 \\ 10 \end{pmatrix} + t\begin{pmatrix} -5 \\ -15 \end{pmatrix}$ $\therefore$ the position vector is $-6\mathbf{i} + 10\mathbf{j} + t(-5\mathbf{i} - 15\mathbf{j})$

$= (-6-5t)\mathbf{i} + (10-15t)\mathbf{j},\quad t \geqslant 0$

b Let P be the point on the yacht's path when it is closest to the beacon.



Then $\overrightarrow{OP} = \begin{pmatrix} -6-5t \\ 10-15t \end{pmatrix}$ and $\overrightarrow{OP} \bullet \begin{pmatrix} -1 \\ -3 \end{pmatrix} = 0$

$\therefore\ -1(-6-5t) - 3(10-15t) = 0$

$\therefore\ 6 + 5t - 30 + 45t = 0$

$\therefore\ 50t = 24$

$\therefore\ t = 0.48$ h

(or 28.8 min)

c When $t = 0.48$, $\overrightarrow{OP} = \begin{pmatrix} -6-5(0.48) \\ 10-15(0.48) \end{pmatrix} = \begin{pmatrix} -8.4 \\ 2.8 \end{pmatrix}$

and $|\overrightarrow{OP}| = \sqrt{(-8.4)^2 + (2.8)^2} \approx 8.85$ km

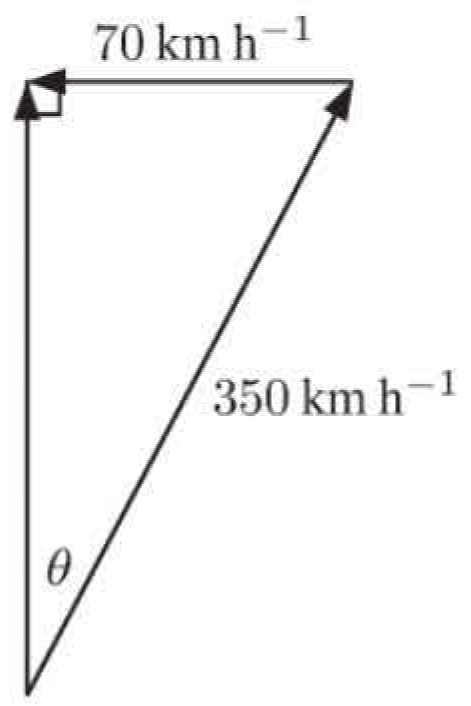
As the closest distance is 8.85 km and the radius is 8 km, the yacht will miss the reef.

3 **a** **i** $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} + t\begin{pmatrix} 4 \\ -1 \end{pmatrix},\quad t \in \mathbb{R}$ **ii** $x = 2 + 4t,\quad y = -3 - t,\quad t \in \mathbb{R}$

b **i** The line has direction vector $\begin{pmatrix} 5-(-1) \\ -2-6 \\ 0-3 \end{pmatrix} = \begin{pmatrix} 6 \\ -8 \\ -3 \end{pmatrix}$

$\therefore\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \\ 3 \end{pmatrix} + t\begin{pmatrix} 6 \\ -8 \\ -3 \end{pmatrix},\quad t \in \mathbb{R}$

ii $x = -1 + 6t,\quad y = 6 - 8t,\quad z = 3 - 3t,\quad t \in \mathbb{R}$

4 

a $\sin\theta = \dfrac{70}{350}$

$\therefore\ \theta \approx 11.5°$

So the pilot should face the plane $11.5°$ east of north.

b $x^2 + 70^2 = 350^2$

$\therefore\ x^2 = 350^2 - 70^2$

$\therefore\ x \approx 343\ \text{km h}^{-1}$

So the speed of the plane will be $343\ \text{km h}^{-1}$.

5 L_1 has direction vector $\mathbf{b}_1 = \begin{pmatrix} 5-0 \\ -2-3 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \end{pmatrix}$

L_2 has direction vector $\mathbf{b}_2 = \begin{pmatrix} -6-(-2) \\ 7-4 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$

If the angle between the lines is θ,

$$\cos\theta = \frac{|\,\mathbf{b}_1 \bullet \mathbf{b}_2\,|}{|\,\mathbf{b}_1\,|\,|\,\mathbf{b}_2\,|} = \frac{|-20-15|}{\sqrt{25+25}\sqrt{16+9}} = \frac{35}{\sqrt{50}\times 5}$$

$$\therefore\ \theta \approx 8.13°$$

$\therefore$ the angle between L_1 and L_2 is about $8.13°$.

6 **a** $\begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} + t\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ where $t \geqslant 0$. When $t = 0$, the time is 2:17 pm.

$\therefore\ x_1(t) = 2 + t,\quad y_1(t) = 4 - 3t,\quad t \geqslant 0$

b After time t has passed, submarine Y18's torpedo has been moving for time $(t-2)$.

$\therefore\ x_2(t) = 11 - (t-2),\qquad y_2(t) = 3 + a(t-2)$

$\therefore\ x_2(t) = 13 - t\qquad y_2(t) = (3-2a) + at,\quad t \geqslant 2$

c They meet where $2 + t = 13 - t$ and $4 - 3t = (3-2a) + at$

$\therefore\ 2t = 11$

$\therefore\ t = \frac{11}{2}$ $\quad\therefore$ the time would be 2:17 pm plus $5\frac{1}{2}$ min, or 2:22:30 pm

d When $t = \frac{11}{2}$,

$4 - 3\left(\frac{11}{2}\right) = (3-2a) + a\left(\frac{11}{2}\right)$

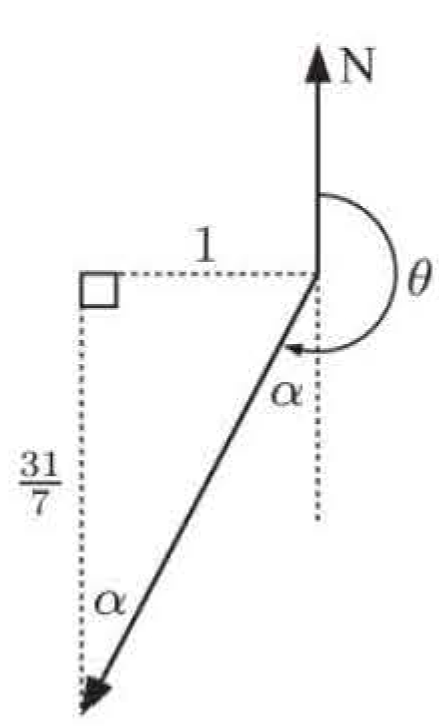
$\therefore\ -\frac{25}{2} = 3 + \dfrac{7a}{2}$

$\therefore\ -25 = 6 + 7a$

$\therefore\ 7a = -31$

$\therefore\ a = -\frac{31}{7}$

Y18's torpedo has velocity vector $\begin{pmatrix} -1 \\ -\frac{31}{7} \end{pmatrix}$



with speed $= \sqrt{(-1)^2 + \left(-\frac{31}{7}\right)^2}$

≈ 4.54 units per minute

$\tan\alpha = \dfrac{1}{\frac{31}{7}} = \frac{7}{31}$

$\therefore\ \alpha = \tan^{-1}\left(\frac{7}{31}\right) \approx 12.7°$

So, the torpedo has speed 4.54 units per minute and direction $12.7°$ west of south.

7 **a** $\mathbf{n} = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$ and $\mathbf{l} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

$\therefore\ \sin\phi = \dfrac{|2 + (-2) + 2|}{\sqrt{9}\sqrt{6}}$

$= \dfrac{2}{3\sqrt{6}}$

$\therefore\ \phi \approx 15.8°$

b The planes have normals

$\mathbf{n}_1 = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$ and $\mathbf{n}_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

$\therefore\ \cos\theta = \dfrac{|2 + (-1) + (-4)|}{\sqrt{9}\sqrt{6}}$

$= \dfrac{3}{3\sqrt{6}}$ or $\frac{1}{\sqrt{6}}$

$\therefore\ \theta \approx 65.9°$

8 **a** $\dfrac{x-8}{3}=\dfrac{y+9}{-16}=\dfrac{z-10}{7}$ has direction vector $\begin{pmatrix}3\\-16\\7\end{pmatrix}$

$x=15+3t,\quad y=29+8t,\quad z=5-5t$ has direction vector $\begin{pmatrix}3\\8\\-5\end{pmatrix}$

$\therefore$ since the direction vectors are not scalar multiples of each other, the lines are not parallel.

If they intersect then $\dfrac{15+3t-8}{3}=\dfrac{29+8t+9}{-16}=\dfrac{5-5t-10}{7}$

$\therefore\ t+\frac{7}{3}=-\frac{1}{2}t-\frac{38}{16}=-\frac{5}{7}t-\frac{5}{7}$

Now $t+\frac{7}{3}=-\frac{1}{2}t-\frac{38}{16}$ requires $\frac{3}{2}t=-\frac{19}{8}-\frac{7}{3}=-\frac{113}{24}$ $\therefore\ t=-\frac{113}{36}$

and $t+\frac{7}{3}=-\frac{5}{7}t-\frac{5}{7}$ requires $\frac{12}{7}t=-\frac{5}{7}-\frac{7}{3}=-\frac{64}{21}$ $\therefore\ t=-\frac{16}{9}$

Hence the lines do not intersect, and since they are not parallel, they are skew.

b If θ is the acute angle between the two lines, and $\mathbf{v}_1$ and $\mathbf{v}_2$ are their direction vectors,

then $\cos\theta=\dfrac{|\mathbf{v}_1\bullet\mathbf{v}_2|}{|\mathbf{v}_1|\,|\mathbf{v}_2|}=\dfrac{\left|\begin{pmatrix}3\\-16\\7\end{pmatrix}\bullet\begin{pmatrix}3\\8\\-5\end{pmatrix}\right|}{\sqrt{3^2+(-16)^2+7^2}\sqrt{3^2+8^2+(-5)^2}}$

$\therefore\ \cos\theta=\dfrac{|9-128-35|}{\sqrt{314}\sqrt{98}}=\dfrac{154}{\sqrt{30\,772}}$

and so $\theta\approx 28.6°$

c $\mathbf{v}_1\times\mathbf{v}_2=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\3&-16&7\\3&8&-5\end{vmatrix}=\begin{vmatrix}-16&7\\8&-5\end{vmatrix}\mathbf{i}-\begin{vmatrix}3&7\\3&-5\end{vmatrix}\mathbf{j}+\begin{vmatrix}3&-16\\3&8\end{vmatrix}\mathbf{k}$

$=24\mathbf{i}+36\mathbf{j}+72\mathbf{k}$ or $12(2\mathbf{i}+3\mathbf{j}+6\mathbf{k})$

$\therefore\ \begin{pmatrix}2\\3\\6\end{pmatrix}$ is the normal of a plane containing L_1 and parallel to L_2.

$\therefore$ the plane with normal $\begin{pmatrix}2\\3\\6\end{pmatrix}$ and containing L_2 also contains L_3.

$\therefore$ the plane has equation $2x+3y+6z=2(15)+3(29)+6(5)$

$\therefore\ 2x+3y+6z=147$

d The shortest distance between L_1 and L_2 is the shortest distance between the plane containing L_2 and L_3 and a point on L_1.

From **Exercise 15I** question **16 b**,

$$d=\frac{|Ax_1+By_1+Cz_1+D|}{\sqrt{A^2+B^2+C^2}}=\frac{|2(8)+3(-9)+6(10)-147|}{\sqrt{2^2+3^2+6^2}}$$

$$=\tfrac{98}{\sqrt{49}}=14\text{ units}$$

9 **a** Given A$(-1, 2, 3)$, B$(1, 0, -1)$, and C$(0, -1, 5)$,

$\overrightarrow{AB}=\begin{pmatrix}2\\-2\\-4\end{pmatrix}$ and $\overrightarrow{AC}=\begin{pmatrix}1\\-3\\2\end{pmatrix}$

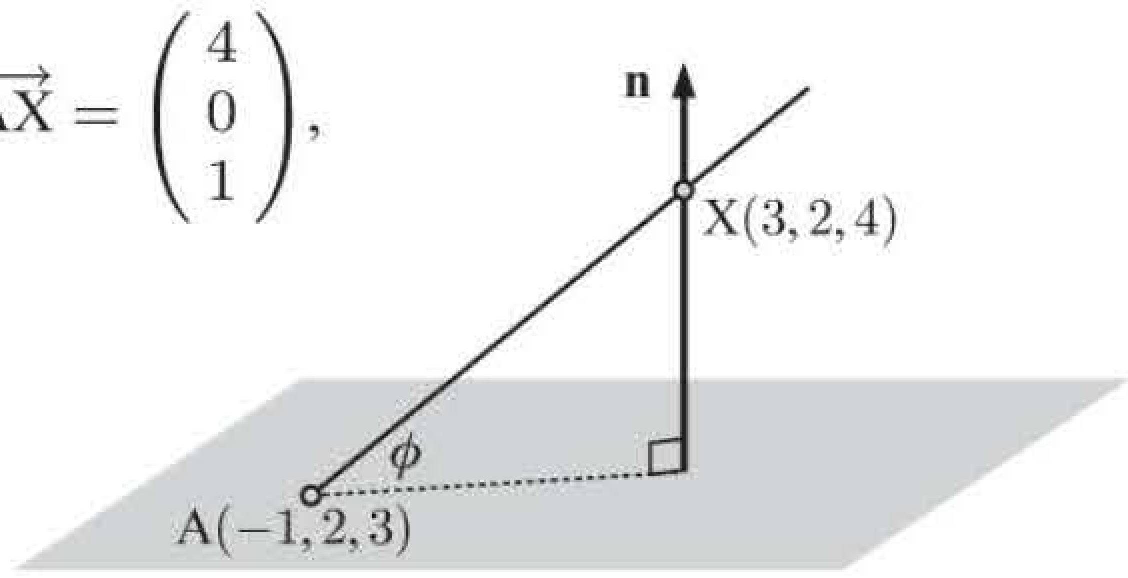
$\therefore$ a normal to the plane is $\mathbf{n}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&-2&-4\\1&-3&2\end{vmatrix}=\begin{vmatrix}-2&-4\\-3&2\end{vmatrix}\mathbf{i}-\begin{vmatrix}2&-4\\1&2\end{vmatrix}\mathbf{j}+\begin{vmatrix}2&-2\\1&-3\end{vmatrix}\mathbf{k}$

$=-16\mathbf{i}-8\mathbf{j}-4\mathbf{k}$ or $-4(4\mathbf{i}+2\mathbf{j}+\mathbf{k})$

$\therefore$ the plane has equation $4x+2y+z=4(1)+2(0)+1(-1)$

$\therefore\ 4x+2y+z=3$

b Given the plane has normal $\mathbf{n} = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$ and $\overrightarrow{AX} = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$,



$$\sin\phi = \frac{|\,\mathbf{n} \bullet \overrightarrow{AX}\,|}{|\,\mathbf{n}\,|\,|\overrightarrow{AX}|} = \frac{|4\times 4 + 2\times 0 + 1\times 1|}{\sqrt{21}\sqrt{17}} = \frac{17}{\sqrt{21}\sqrt{17}} \quad \text{and so} \quad \phi \approx 64.1^\circ$$

10 **a** All vectors normal to $x - y + z = 6$ have the form $t\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} t \\ -t \\ t \end{pmatrix}$, $t \in \mathbb{R}$

$\therefore$ if the vector has length 3 units, $\sqrt{t^2+t^2+t^2} = 3$

$$\therefore\ 3t^2 = 9$$
$$\therefore\ t^2 = 3$$
$$\therefore\ t = \pm\sqrt{3}$$

$\therefore$ the vectors are $\begin{pmatrix} \sqrt{3} \\ -\sqrt{3} \\ \sqrt{3} \end{pmatrix}$ and $\begin{pmatrix} -\sqrt{3} \\ \sqrt{3} \\ -\sqrt{3} \end{pmatrix}$.

b Any vector parallel to $\mathbf{i} + r\mathbf{j} + 3\mathbf{k}$ has the form $t\begin{pmatrix} 1 \\ r \\ 3 \end{pmatrix} = \begin{pmatrix} t \\ rt \\ 3t \end{pmatrix}$, $t \in \mathbb{R}$.

This is perpendicular to $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ if $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} t \\ rt \\ 3t \end{pmatrix} = 0$

$$\therefore\ 2t - rt + 6t = 0$$
$$\therefore\ 8t - rt = 0$$
$$\therefore\ t(8-r) = 0$$
$$\therefore\ t = 0 \text{ or } r = 8$$

But if $t = 0$, the vector has zero length. $\therefore$ $r = 8$ and so a vector is $\begin{pmatrix} 1 \\ 8 \\ 3 \end{pmatrix}$.

$\therefore$ the unit vectors are $\mathbf{u} = \frac{1}{\sqrt{74}}\mathbf{i} + \frac{8}{\sqrt{74}}\mathbf{j} + \frac{3}{\sqrt{74}}\mathbf{k}$ or $-\frac{1}{\sqrt{74}}\mathbf{i} - \frac{8}{\sqrt{74}}\mathbf{j} - \frac{3}{\sqrt{74}}\mathbf{k}$

c The distance from the plane to A is $d = \dfrac{|2x_1 - y_1 + 2z_1 - k|}{\sqrt{9}}$

$$\therefore\ \frac{|2(-1) - (2) + 2(3) - k|}{3} = 3$$
$$\therefore\ |2-k| = 9$$
$$\therefore\ 2-k = 9 \text{ or } k-2 = 9$$
$$\therefore\ k = -7 \text{ or } 11$$

11 $4x - 5y = 11$ has gradient $\frac{4}{5}$ $\therefore$ it has direction vector $\begin{pmatrix} 5 \\ 4 \end{pmatrix}$.

$2x + 3y = 7$ has gradient $-\frac{2}{3}$ $\therefore$ it has direction vector $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$.

If the angle is θ, $\begin{pmatrix} 5 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \sqrt{5^2+4^2}\sqrt{3^2+(-2)^2}\cos\theta$

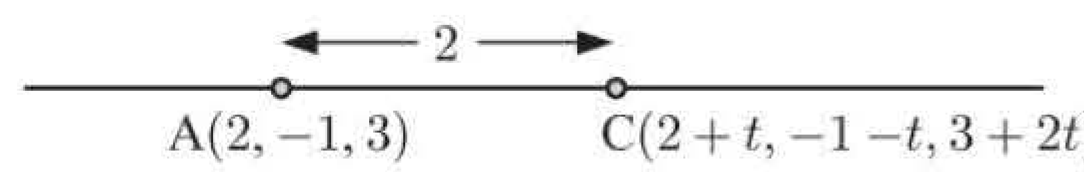
$$\therefore\ 15 - 8 = \sqrt{41}\sqrt{13}\cos\theta$$
$$\therefore\ \frac{7}{\sqrt{41\times 13}} = \cos\theta$$
$$\therefore\ \theta \approx 72.35^\circ \qquad \therefore \text{ the angle is } 72.35^\circ \text{ (or } 107.65^\circ)$$

12 **a** $\overrightarrow{AB} = \begin{pmatrix} -2 \\ 2 \\ -4 \end{pmatrix} = -2\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

$\therefore$ since A lies on the line, it has equations $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \quad \lambda \in \mathbb{R}$

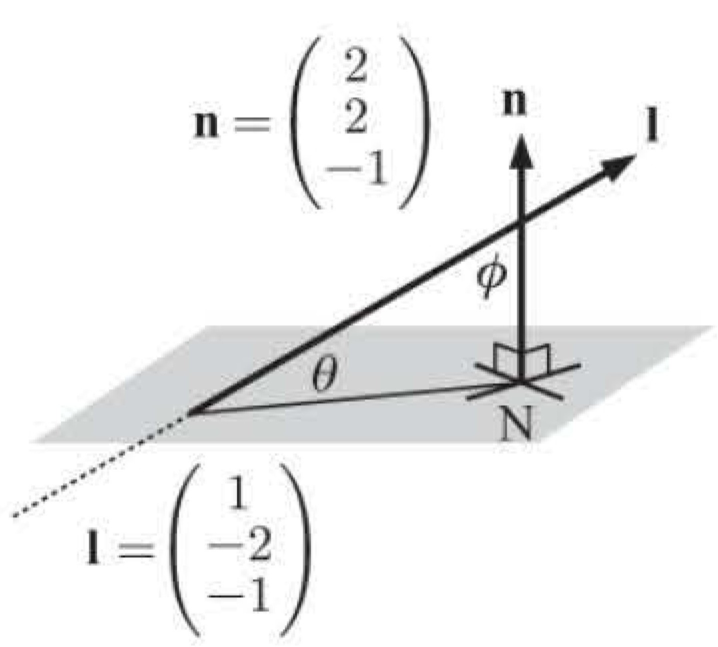
b If C lies on (AB) and is 2 units from A, then C corresponds to λ such that

$$\begin{aligned}
\sqrt{(\lambda)^2 + (-\lambda)^2 + (2\lambda)^2} &= 2 \\
\therefore \quad \sqrt{6\lambda^2} &= 2 \\
\therefore \quad 6\lambda^2 &= 4 \\
\therefore \quad \lambda^2 &= \tfrac{4}{6} \\
\therefore \quad \lambda &= \pm\tfrac{2}{\sqrt{6}}
\end{aligned}$$



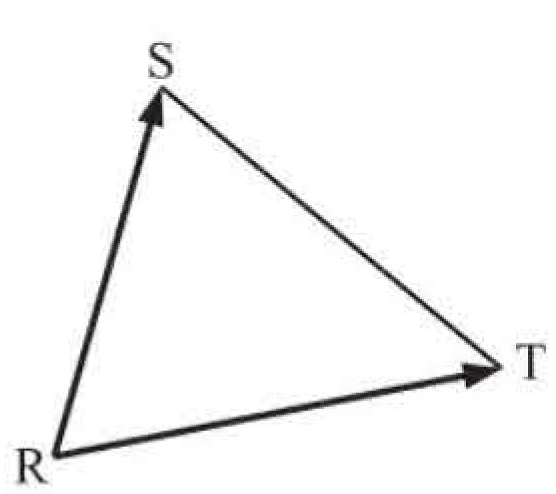
$\therefore$ C is $\left(2 + \frac{2}{\sqrt{6}}, -1 - \frac{2}{\sqrt{6}}, 3 + \frac{4}{\sqrt{6}}\right)$ or $\left(2 - \frac{2}{\sqrt{6}}, -1 + \frac{2}{\sqrt{6}}, 3 - \frac{4}{\sqrt{6}}\right)$

13



$$\begin{aligned}
\mathbf{n} \bullet \mathbf{l} &= |\mathbf{n}|\,|\mathbf{l}| \cos\phi \\
\therefore \quad \cos\phi &= \frac{\mathbf{n} \bullet \mathbf{l}}{|\mathbf{n}|\,|\mathbf{l}|} \\
\therefore \quad \sin\theta &= \frac{|\mathbf{n} \bullet \mathbf{l}|}{|\mathbf{n}|\,|\mathbf{l}|} \quad \text{as} \quad \cos\phi = \cos\left(\tfrac{\pi}{2} - \theta\right) = \sin\theta \\
&= \frac{|2 - 4 + 1|}{\sqrt{4+4+1}\sqrt{1+4+1}} = \tfrac{1}{\sqrt{54}} \\
\therefore \quad \theta &\approx 7.82^\circ
\end{aligned}$$

14



$$\begin{aligned}
\overrightarrow{RS} &= \overrightarrow{RO} + \overrightarrow{OS} = \overrightarrow{OS} - \overrightarrow{OR} \\
\therefore \quad \overrightarrow{RS} &= 2\mathbf{i} + \mathbf{j} + 2\mathbf{k} - 2\mathbf{i} + 2\mathbf{j} + \mathbf{k} \\
&= 3\mathbf{j} + 3\mathbf{k}
\end{aligned}$$

Likewise $\overrightarrow{RT} = \overrightarrow{OT} - \overrightarrow{OR} = \mathbf{i} + 2\mathbf{j} - \mathbf{k} - 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$
$= -\mathbf{i} + 4\mathbf{j}$

$$\begin{aligned}
\text{Now area} &= \tfrac{1}{2}\,|\overrightarrow{RS} \times \overrightarrow{RT}| \\
&= \tfrac{1}{2}\left\| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 3 & 3 \\ -1 & 4 & 0 \end{matrix} \right\| \\
&= \tfrac{1}{2}\left| \begin{vmatrix} 3 & 3 \\ 4 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0 & 3 \\ -1 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0 & 3 \\ -1 & 4 \end{vmatrix}\mathbf{k} \right| \\
&= \tfrac{1}{2}\,|-12\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}| \\
&= \tfrac{1}{2}\sqrt{144 + 9 + 9} \\
&= \tfrac{1}{2}\sqrt{162} \\
&= \tfrac{1}{2}\,9\sqrt{2} \\
&= \tfrac{9\sqrt{2}}{2} \text{ units}^2
\end{aligned}$$

15 **a** Line 1 has direction vector $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and line 2 has direction vector $\begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$.

As one vector is not a scalar multiple of the other, the lines are not parallel.

Now, $2 + t = -8 + 4s$ .... (1) $\quad -1 + 2t = s$ .... (2) $\quad 3 - t = 7 - 2s$ .... (3)

Substituting (2) into (1), $2+t=-8+4(-1+2t)$

$$\therefore\ 2+t=-8-4+8t$$
$$\therefore\ 7t=14$$
$$\therefore\ t=2$$
$$\therefore\ s=-1+2(2)=3$$

In (3), $\text{LHS}=3-2=1$ $\quad\text{RHS}=7-2(3)=1\ \checkmark$

$\therefore\ s=3,\ t=2$ satisfies all three equations.

$\therefore$ the lines meet at $(4,\ 3,\ 1)$ {substituting $t=2$ into line 1}

The angle θ between the lines has $\cos\theta=\dfrac{\left|\begin{pmatrix}1\\2\\-1\end{pmatrix}\bullet\begin{pmatrix}4\\1\\-2\end{pmatrix}\right|}{\sqrt{1+4+1}\sqrt{16+1+4}}=\dfrac{|4+2+2|}{\sqrt{6}\sqrt{21}}=\dfrac{8}{3\sqrt{14}}$

$$\therefore\ \theta\approx 44.5^\circ$$

b Line 1 has direction vector $\begin{pmatrix}1\\-2\\3\end{pmatrix}$ and line 2 has direction vector $\begin{pmatrix}-1\\3\\1\end{pmatrix}$.

As one vector is not a scalar multiple of the other, the lines are not parallel.

Now, $3+t=2-s$ $\qquad 5-2t=1+3s$ $\qquad -1+3t=4+s$

$\therefore\ t+s=-1$ .... (1) $\qquad 2t+3s=4$ .... (2) $\qquad 3t-s=5$ .... (3)

Solving (1) and (3) simultaneously:

$$\begin{aligned}t+s&=-1\\3t-s&=5\\\hline\text{Adding, }\ 4t\quad&=4\end{aligned}$$

$$\therefore\ t=1 \qquad \therefore\ s=-2$$

In (2), $\text{LHS}=2(1)+3(-2)=-4$ ✗

$\therefore$ the system of equations is inconsistent and so the lines are skew.

The angle θ between them has $\cos\theta=\dfrac{\left|\begin{pmatrix}1\\-2\\3\end{pmatrix}\bullet\begin{pmatrix}-1\\3\\1\end{pmatrix}\right|}{\sqrt{1+4+9}\sqrt{1+9+1}}=\dfrac{|-1-6+3|}{\sqrt{14}\sqrt{11}}=\dfrac{4}{\sqrt{154}}$

$$\therefore\ \theta\approx 71.2^\circ$$

16 **a** $\mathbf{p}\times\mathbf{q}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&-1&2\\2&3&-1\end{vmatrix}=\begin{vmatrix}-1&2\\3&-1\end{vmatrix}\mathbf{i}-\begin{vmatrix}1&2\\2&-1\end{vmatrix}\mathbf{j}+\begin{vmatrix}1&-1\\2&3\end{vmatrix}\mathbf{k}$

$$=-5\mathbf{i}+5\mathbf{j}+5\mathbf{k}$$
$$=5\begin{pmatrix}-1\\1\\1\end{pmatrix}$$

b l has direction vector $\begin{pmatrix}2\\1\\m\end{pmatrix}$

$\therefore$ $\mathbf{p}\times\mathbf{q}$ is perpendicular to l if $\begin{pmatrix}-1\\1\\1\end{pmatrix}\bullet\begin{pmatrix}2\\1\\m\end{pmatrix}=0$

$$\therefore\ -2+1+m=0$$
$$\therefore\ m=1$$

c P has normal vector $\mathbf{n}=\begin{pmatrix}-1\\1\\1\end{pmatrix}$, and $(1,\ -2,\ 3)$ lies on the plane {letting $\lambda=0$}

$\therefore$ P has equation $-x+y+z=-1+(-2)+3$

$$\therefore\ x-y-z=0$$

d A lies on the plane if $4 - t - 2 = 0$

$$\therefore\ t = 2$$

e $\overrightarrow{AB} = \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix}$ and P has normal vector $\mathbf{n} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

$\therefore$ if θ is the angle between (AB) and P then

$$\sin\theta = \frac{\left|\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix}\right|}{\sqrt{(-1)^2+1^2+1^2}\,\sqrt{2^2+(-5)^2+3^2}} = \frac{|-2-5+3|}{\sqrt{3}\sqrt{38}} = \frac{4}{\sqrt{114}}$$

17 **a** We substitute L_1 into the LHS of the plane's equation

$$2(-2t+2) + (t) + (3t+1) = -4t + 4 + t + 3t + 1 = 5 \quad \checkmark$$

$\therefore$ the plane contains the line.

b If $x + ky + z = 3$ contains L_1 then

$$\begin{aligned} (-2t+2) + k(t) + 3t + 1 &= 3 \\ \therefore\ t[-2+k+3] + 2 + 1 &= 3 \\ \therefore\ t[k+1] &= 0 \\ \therefore\ k &= -1 \text{ as } t \in \mathbb{R} \end{aligned}$$

c From **a** and **b**, both $2x + y + z = 5$ and $x - y + z = 3$ contain L_1.
So, substituting L_1 into plane 3 gives

$$\begin{aligned} -2(-2t+2) + pt + 2(3t+1) &= q \quad \text{for all } t \in \mathbb{R} \\ \therefore\ 4t - 4 + pt + 6t + 2 &= q \quad \text{for all } t \in \mathbb{R} \\ \therefore\ (10+p)t - 2 &= q \quad \text{for all } t \end{aligned}$$

This equation has infinitely many solutions for t
when $10 + p = 0$ and $-2 = q$ {equating coefficients}
$\therefore\ p = -10$ and $q = -2$

18 **a** In augmented matrix form, the system is:

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & -5 \\ 3 & 1 & (2-k) & 10 \\ -2 & 6 & k & 5 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & 2 & -5 \\ 0 & 10 & -4-k & 25 \\ 0 & 0 & k+4 & -5 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 + 2R_1 \end{array}$$

$$\begin{array}{cccc} 3 & 1 & 2-k & 10 \\ -3 & 9 & -6 & 15 \\ \hline 0 & 10 & -4-k & 25 \end{array}$$

$$\begin{array}{cccc} -2 & 6 & k & 5 \\ 2 & -6 & 4 & -10 \\ \hline 0 & 0 & k+4 & -5 \end{array}$$

b If $k = -4$, row 3 becomes $0x + 0y + 0z = -5$
$\therefore$ the system is inconsistent and there are no solutions.
When $k = -4$ the system becomes

$$\begin{cases} x - 3y + 2z = -5 \\ 3x + y + 6z = 10 \\ -2x + 6y - 4z = 5 \end{cases}$$

Planes 1 and 3 are parallel.
In this case two planes are parallel and they are intersected by the third plane.

c **i** There is a unique solution when $k \neq -4$.

ii If $k \neq -4$, then row 3 becomes $(k+4)z = -5$

$$\therefore\ z = \frac{-5}{k+4}$$

and substituting in row 2 gives $10y + -(4+k)\left(\dfrac{-5}{k+4}\right) = 25$

$$\begin{aligned} \therefore\ 10y + 5 &= 25 \\ \therefore\ 10y &= 20 \\ \therefore\ y &= 2 \end{aligned}$$

and substituting in row 1 gives $x - 3(2) + 2\left(\frac{-5}{k+4}\right) = -5$

$$\therefore \quad x = 1 + \frac{10}{k+4}$$

$\therefore$ the unique solution is $x = 1 + \frac{10}{k+4}$, $y = 2$, $z = \frac{-5}{k+4}$.

In this case we have three planes which meet at a point.

iii When $k = 1$, $x = 1 + \frac{10}{1+4} = 3$, $y = 2$, and $z = \frac{-5}{1+4} = -1$

So, when $k = 1$ the unique solution is $(3,\ 2,\ -1)$.

REVIEW SET 15C

1 The direction vector is $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ which has length $\sqrt{3^2 + (-1)^2} = \sqrt{10}$ units

$\therefore$ $2\sqrt{10}\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ has length 20. So, the velocity vector is $\begin{pmatrix} 6\sqrt{10} \\ -2\sqrt{10} \end{pmatrix}$ or $2\sqrt{10}(3\mathbf{i} - \mathbf{j})$.

2 **a** $x(0) = -4$ and $y(0) = 3$, so the initial position is $(-4,\ 3)$.

b $x(4) = -4 + 8(4) = 28$ and $y(4) = 3 + 6(4) = 27$, so at $t = 4$ the position is $(28,\ 27)$.

c The velocity vector is $\begin{pmatrix} 8 \\ 6 \end{pmatrix}$.

d The speed is $\sqrt{8^2 + 6^2} = 10\ \text{m s}^{-1}$.

3 **a** (KL) has direction vector $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$ and (MN) has direction vector $\begin{pmatrix} -5 \\ 2 \end{pmatrix}$.

Now $\begin{pmatrix} 5 \\ -2 \end{pmatrix} = -\begin{pmatrix} -5 \\ 2 \end{pmatrix}$, so (KL) $\parallel$ (MN).

b $\overrightarrow{\text{KL}} = a\begin{pmatrix} 5 \\ -2 \end{pmatrix}$, $\overrightarrow{\text{NK}} = b\begin{pmatrix} 4 \\ 10 \end{pmatrix}$, $\overrightarrow{\text{MN}} = c\begin{pmatrix} -5 \\ 2 \end{pmatrix}$ {for some constants a, b, c}

$\therefore$ $\overrightarrow{\text{KL}} \bullet \overrightarrow{\text{NK}} = ab(20 - 20) = 0$ and $\overrightarrow{\text{NK}} \bullet \overrightarrow{\text{MN}} = bc(-20 + 20) = 0$

$\therefore$ (NK) is perpendicular to both (KL) and (MN).

c (KL) and (NK) meet at K.

$$\therefore \begin{pmatrix} 2 \\ 19 \end{pmatrix} + p\begin{pmatrix} 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix} + r\begin{pmatrix} 4 \\ 10 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 5p - 4r \\ -2p - 10r \end{pmatrix} = \begin{pmatrix} 1 \\ -12 \end{pmatrix}$$

$$\begin{aligned} \therefore\ 5p - 4r &= 1 \quad \text{.... (1)} \\ 2p + 10r &= 12 \quad \text{.... (2)} \\ \therefore\ 25p - 20r &= 5 \quad \{5 \times (1)\} \\ 4p + 20r &= 24 \quad \{2 \times (2)\} \\ \text{Adding,}\quad 29p \qquad &= 29 \\ \therefore\ p &= 1 \end{aligned}$$

and $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 19 \end{pmatrix} + \begin{pmatrix} 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 7 \\ 17 \end{pmatrix}$

$\therefore$ K is $(7,\ 17)$.

(KL) and (ML) meet at L.

$$\therefore \begin{pmatrix} 2 \\ 19 \end{pmatrix} + p\begin{pmatrix} 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 33 \\ -5 \end{pmatrix} + q\begin{pmatrix} -11 \\ 16 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 5p + 11q \\ -2p - 16q \end{pmatrix} = \begin{pmatrix} 31 \\ -24 \end{pmatrix}$$

$$\begin{aligned} \therefore\ 5p + 11q &= 31 \quad \text{.... (1)} \\ -2p - 16q &= -24 \quad \text{.... (2)} \\ \therefore\ 10p + 22q &= 62 \quad \{2 \times (1)\} \\ -10p - 80q &= -120 \quad \{5 \times (2)\} \\ \text{Adding,}\quad -58q &= -58 \\ \therefore\ q &= 1 \end{aligned}$$

and $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 33 \\ -5 \end{pmatrix} + \begin{pmatrix} -11 \\ 16 \end{pmatrix} = \begin{pmatrix} 22 \\ 11 \end{pmatrix}$

$\therefore$ L is $(22,\ 11)$.

(ML) and (MN) meet at M.

$$\therefore \begin{pmatrix}33\\-5\end{pmatrix} + q\begin{pmatrix}-11\\16\end{pmatrix} = \begin{pmatrix}43\\-9\end{pmatrix} + s\begin{pmatrix}-5\\2\end{pmatrix}$$

$$\therefore \begin{pmatrix}-11q+5s\\16q-2s\end{pmatrix} = \begin{pmatrix}10\\-4\end{pmatrix}$$

$$\begin{aligned} \therefore\ -11q+5s &= 10 \quad .... (1)\\ 16q-2s &= -4 \quad .... (2)\\ \therefore\ -22q+10s &= 20 \quad \{2\times(1)\}\\ 80q-10s &= -20 \quad \{5\times(2)\}\\ \text{Adding, } 58q \quad &= 0 \end{aligned}$$

$$\therefore\ q=0 \quad \text{and so} \quad \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}33\\-5\end{pmatrix}$$

$\therefore$ M is $(33, -5)$.

(NK) and (MN) meet at N.

$$\therefore \begin{pmatrix}3\\7\end{pmatrix} + r\begin{pmatrix}4\\10\end{pmatrix} = \begin{pmatrix}43\\-9\end{pmatrix} + s\begin{pmatrix}-5\\2\end{pmatrix}$$

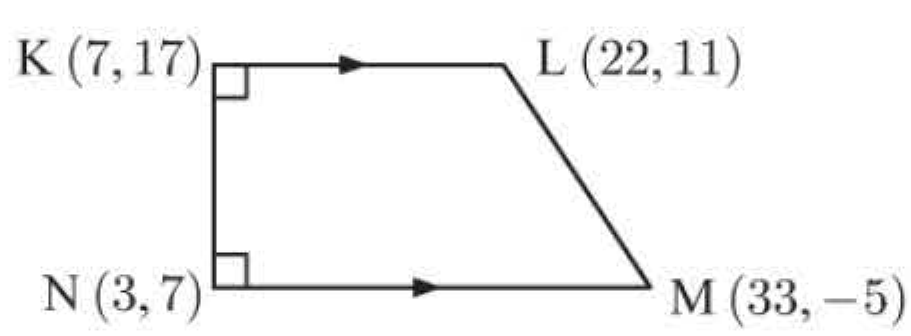
$$\therefore \begin{pmatrix}4r+5s\\10r-2s\end{pmatrix} = \begin{pmatrix}40\\-16\end{pmatrix}$$

$$\begin{aligned} \therefore\ 4r+5s &= 40 \quad .... (1)\\ 10r-2s &= -16 \quad .... (2)\\ \therefore\ 8r+10s &= 80 \quad \{2\times(1)\}\\ 50r-10s &= -80 \quad \{5\times(2)\}\\ \text{Adding, } 58r \quad &= 0 \end{aligned}$$

$$\therefore\ r=0 \quad \text{and so} \quad \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}3\\7\end{pmatrix}$$

$\therefore$ N is $(3, 7)$.

d



$$\begin{aligned} \text{KL} &= \sqrt{(22-7)^2+(11-17)^2}\\ &= \sqrt{225+36}\\ &= \sqrt{261} \text{ units} \end{aligned}$$

$$\begin{aligned} \text{KN} &= \sqrt{(7-3)^2+(17-7)^2}\\ &= \sqrt{16+100}\\ &= \sqrt{116} \text{ units} \end{aligned}$$

$$\begin{aligned} \text{NM} &= \sqrt{(33-3)^2+(-5-7)^2}\\ &= \sqrt{900+144}\\ &= \sqrt{1044} \text{ units} \end{aligned}$$

$$\begin{aligned} \therefore\ \text{area} &= \left(\frac{\sqrt{261}+\sqrt{1044}}{2}\right)\times\sqrt{116}\\ &= 261 \text{ units}^2 \end{aligned}$$

4 L_1 has direction vector $\mathbf{b}_1 = \begin{pmatrix}-4\\3\end{pmatrix}$, L_2 has direction vector $\mathbf{b}_2 = \begin{pmatrix}5\\-12\end{pmatrix}$

If θ is the angle between them,

$$\cos\theta = \frac{|\,\mathbf{b}_1 \bullet \mathbf{b}_2\,|}{|\,\mathbf{b}_1\,|\,|\,\mathbf{b}_2\,|} = \frac{|-20-36|}{\sqrt{16+9}\sqrt{25+144}} = \frac{56}{5\times 13}$$

$$\therefore\ \theta \approx 30.5°$$

$\therefore$ the angle between L_1 and L_2 is about $30.5°$.

5 **a**

$$\overrightarrow{\text{AB}} = \begin{pmatrix}0-3\\2--1\\-2-1\end{pmatrix} = \begin{pmatrix}-3\\3\\-3\end{pmatrix}$$

$$\therefore\ |\,\overrightarrow{\text{AB}}\,| = \sqrt{9+9+9} = \sqrt{27} \text{ units}$$

b

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\2\\-2\end{pmatrix} + t\begin{pmatrix}-3\\3\\-3\end{pmatrix}, \quad t\in\mathbb{R}$$

$$\therefore \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\2\\-2\end{pmatrix} + \lambda\begin{pmatrix}-1\\1\\-1\end{pmatrix}, \quad \text{where } \lambda = 3t$$

$$\therefore\ \mathbf{r} = 2\mathbf{j} - 2\mathbf{k} + \lambda(-\mathbf{i}+\mathbf{j}-\mathbf{k}), \quad \text{where } \lambda\in\mathbb{R}$$

A lies on the line $\mathbf{r}$ when $\lambda = -3$ and B lies on $\mathbf{r}$ when $\lambda = 0$.

$\therefore$ the line between A and B is the same line as $\mathbf{r}$, so it can be described by $\mathbf{r}$.

c The line with equation $t(\mathbf{i}+\mathbf{j}+\mathbf{k})$ has direction vector $\begin{pmatrix}1\\1\\1\end{pmatrix}$.

$$\therefore \quad \overrightarrow{AB} \bullet \begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}-3\\3\\-3\end{pmatrix} \bullet \begin{pmatrix}1\\1\\1\end{pmatrix}$$
$$= -3+3-3$$
$$= -3$$

$$\left|\begin{pmatrix}1\\1\\1\end{pmatrix}\right| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$$

$$\therefore \quad \cos\theta = \frac{|-3|}{3\sqrt{3}\times\sqrt{3}} = \frac{3}{9} = \frac{1}{3}$$
$$\therefore \quad \theta \approx 70.5^\circ$$

$\therefore$ the angle between the two lines is about 70.5°.

6 **a** Road A has direction vector $\begin{pmatrix}15--9\\-16-2\end{pmatrix} = \begin{pmatrix}24\\-18\end{pmatrix} = 6\begin{pmatrix}4\\-3\end{pmatrix}$.

So, Road A has equation $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-9\\2\end{pmatrix} + t\begin{pmatrix}4\\-3\end{pmatrix}$, $t\in\mathbb{R}$.

Road B has direction vector $\begin{pmatrix}21-6\\18--18\end{pmatrix} = \begin{pmatrix}15\\36\end{pmatrix} = 3\begin{pmatrix}5\\12\end{pmatrix}$

So, Road B has equation $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}6\\-18\end{pmatrix} + s\begin{pmatrix}5\\12\end{pmatrix}$, $s\in\mathbb{R}$.

b Let $A(-9+4t,\ 2-3t)$ be any point on Road A.

$$\therefore \quad \overrightarrow{HA} = \begin{pmatrix}-9+4t-4\\2-3t-11\end{pmatrix} = \begin{pmatrix}-13+4t\\-9-3t\end{pmatrix}$$

The closest point to $H(4, 11)$ on Road A is such that $\overrightarrow{HA} \perp \begin{pmatrix}4\\-3\end{pmatrix}$.

$$\therefore \quad \begin{pmatrix}-13+4t\\-9-3t\end{pmatrix} \bullet \begin{pmatrix}4\\-3\end{pmatrix} = 0$$
$$\therefore \quad -52+16t+27+9t = 0$$
$$\therefore \quad 25t = 25$$
$$\therefore \quad t = 1$$

So A is $(-9+4,\ 2-3)$ or $(-5, -1)$.

$$\therefore \quad \overrightarrow{HA} = \begin{pmatrix}-13+4\\-9-3\end{pmatrix} = \begin{pmatrix}-9\\-12\end{pmatrix}$$

$$\therefore \quad |\overrightarrow{HA}| = \sqrt{81+144} = \sqrt{225} = 15 \text{ km}$$

Now, let $B(6+5s,\ -18+12s)$ be any point on Road B.

$$\therefore \quad \overrightarrow{HB} = \begin{pmatrix}6+5s-4\\-18+12s-11\end{pmatrix} = \begin{pmatrix}2+5s\\-29+12s\end{pmatrix}$$

The closest point to $H(4, 11)$ on Road B is such that $\overrightarrow{HB} \perp \begin{pmatrix}5\\12\end{pmatrix}$.

$$\therefore \quad \begin{pmatrix}2+5s\\-29+12s\end{pmatrix} \bullet \begin{pmatrix}5\\12\end{pmatrix} = 0$$
$$\therefore \quad 10+25s-348+144s = 0$$
$$\therefore \quad 169s = 338$$
$$\therefore \quad s = 2$$

So B is $(6+5(2), -18+12(2))$ or $(16, 6)$.

$$\therefore\ \overrightarrow{HB} = \begin{pmatrix} 2+5(2) \\ -29+12(2) \end{pmatrix} = \begin{pmatrix} 12 \\ -5 \end{pmatrix}$$

$$\therefore\ |\overrightarrow{HB}| = \sqrt{144+25} = \sqrt{169} = 13 \text{ km}$$

The hiker should head toward Road B, a distance of 13 km.

7 **a** $\overrightarrow{AB} = \begin{pmatrix} 2-4 \\ 1-2 \\ 5--1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 6 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} 9-4 \\ 4-2 \\ 1--1 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 2 \end{pmatrix}$

$$\therefore\ \overrightarrow{AB} \bullet \overrightarrow{AC} = (-2)(5) + (-1)(2) + (6)(2) = -10 - 2 + 12 = 0$$

$$\therefore\ \overrightarrow{AB} \perp \overrightarrow{AC}$$

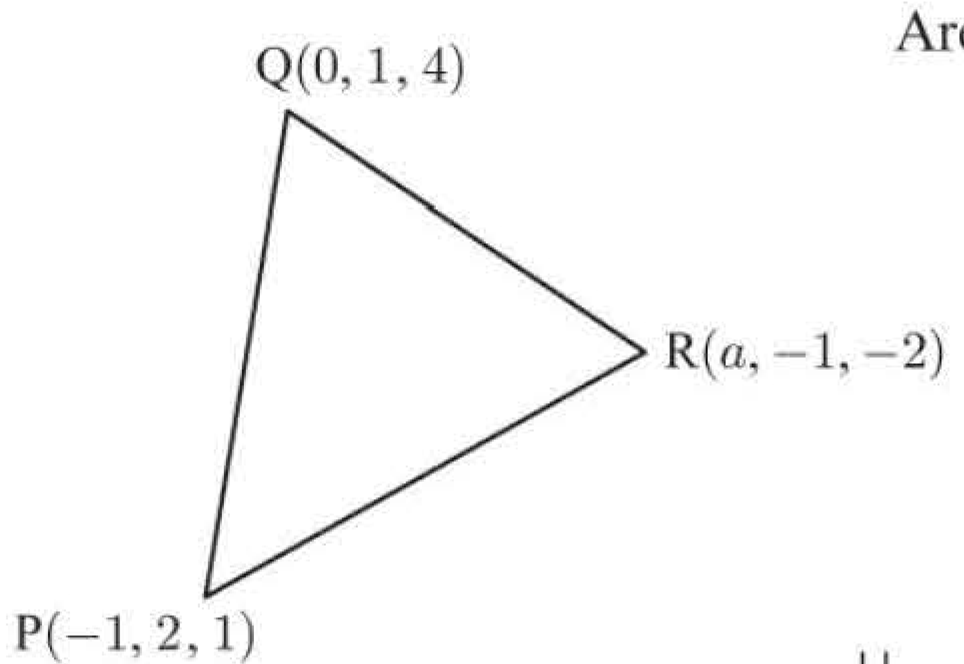
b **i** The equation is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix} + t\begin{pmatrix} -2 \\ -1 \\ 6 \end{pmatrix}$, $t \in \mathbb{R}$

or $x = 4-2t$, $y = 2-t$, $z = -1+6t$, $t \in \mathbb{R}$

ii The equation is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix} + s\begin{pmatrix} 5 \\ 2 \\ 2 \end{pmatrix}$, $s \in \mathbb{R}$

or $x = 4+5s$, $y = 2+2s$, $z = -1+2s$, $s \in \mathbb{R}$

8



$$\text{Area of } \triangle = \tfrac{1}{2}\,|\overrightarrow{PQ} \times \overrightarrow{PR}| = \tfrac{1}{2}\left|\begin{pmatrix} 0--1 \\ 1-2 \\ 4-1 \end{pmatrix} \times \begin{pmatrix} a--1 \\ -1-2 \\ -2-1 \end{pmatrix}\right|$$

$$= \tfrac{1}{2}\left|\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \times \begin{pmatrix} a+1 \\ -3 \\ -3 \end{pmatrix}\right|$$

$$\therefore\ \tfrac{1}{2}\left\|\begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 3 \\ a+1 & -3 & -3 \end{matrix}\right\| = \sqrt{118}$$

$$\therefore\ \left\|\begin{matrix} -1 & 3 \\ -3 & -3 \end{matrix}\right|\mathbf{i} - \left|\begin{matrix} 1 & 3 \\ a+1 & -3 \end{matrix}\right|\mathbf{j} + \left|\begin{matrix} 1 & -1 \\ a+1 & -3 \end{matrix}\right|\mathbf{k}\right| = 2\sqrt{118}$$

$$|12\mathbf{i} - (-3-3a-3)\mathbf{j} + (-3+a+1)\mathbf{k}| = 2\sqrt{118}$$

$$\therefore\ \sqrt{144 + (3a+6)^2 + (a-2)^2} = 2\sqrt{118}$$

$$\therefore\ 144 + 9a^2 + 36a + 36 + a^2 - 4a + 4 = 472$$

$$\therefore\ 10a^2 + 32a - 288 = 0$$

$$\therefore\ 5a^2 + 16a - 144 = 0$$

$$\therefore\ (5a+36)(a-4) = 0$$

$$\therefore\ a = -\tfrac{36}{5} \text{ or } 4$$

9 Given: A$(-1, 2, 3)$, B$(2, 0, -1)$, and C$(-3, 2, -4)$

a $\overrightarrow{AB} = \begin{pmatrix} 3 \\ -2 \\ -4 \end{pmatrix}$ $\overrightarrow{AC} = \begin{pmatrix} -2 \\ 0 \\ -7 \end{pmatrix}$ $\therefore$ a normal vector to the plane is

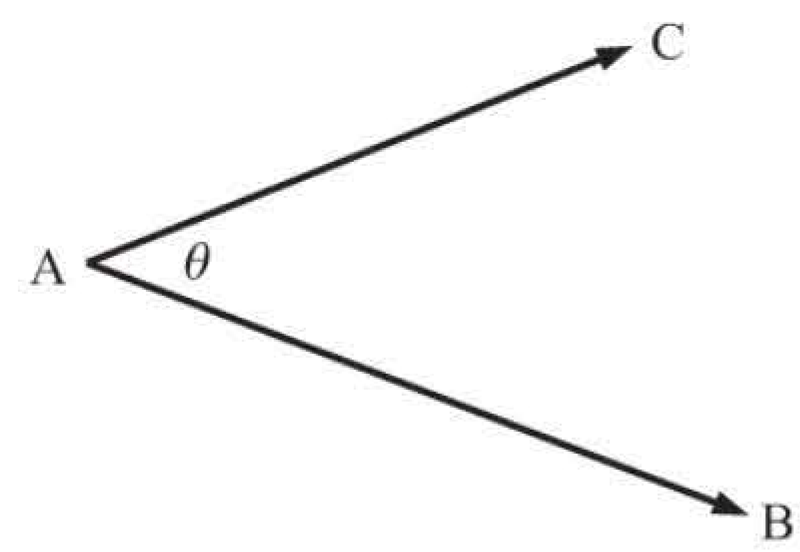
$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 0 & -7 \\ 3 & -2 & -4 \end{vmatrix} = \begin{vmatrix} 0 & -7 \\ -2 & -4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -2 & -7 \\ 3 & -4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -2 & 0 \\ 3 & -2 \end{vmatrix}\mathbf{k}$$

$$= -14\mathbf{i} - 29\mathbf{j} + 4\mathbf{k}$$

$\therefore$ since B lies on the plane, it has equation $14x + 29y - 4z = 14(2) + 29(0) - 4(-1)$

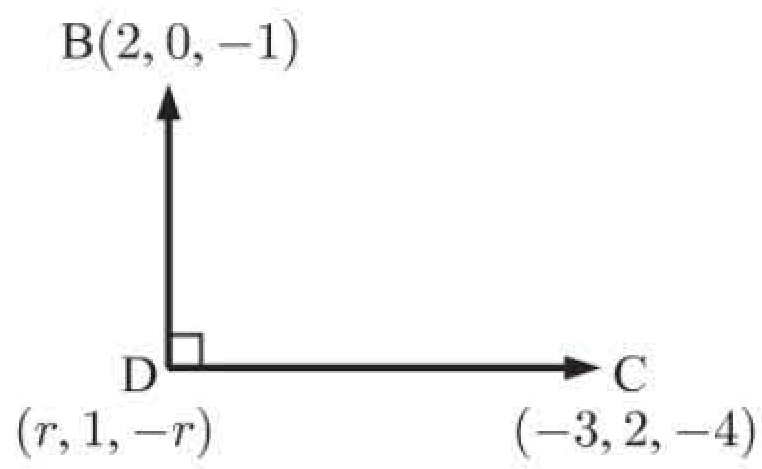
$$\therefore\ 14x + 29y - 4z = 32$$

b



$$\cos\theta = \frac{|\overrightarrow{AB} \bullet \overrightarrow{AC}|}{|\overrightarrow{AB}||\overrightarrow{AC}|}$$
$$= \frac{|3\times -2 + -2\times 0 + -4 \times -7|}{\sqrt{9+4+16}\sqrt{4+0+49}}$$
$$= \frac{22}{\sqrt{29\times 53}} \quad \text{and so} \quad \theta \approx 55.9^\circ$$

c



If D is at $(r,\ 1,\ -r)$ then $\overrightarrow{DB} = \begin{pmatrix} 2-r \\ -1 \\ -1+r \end{pmatrix}$

and $\overrightarrow{DC} = \begin{pmatrix} -3-r \\ 1 \\ -4+r \end{pmatrix}$

Now $B\widehat{D}C$ is a right angle, so $\overrightarrow{DB} \bullet \overrightarrow{DC} = 0$

$$\therefore\ (2-r)(-3-r) + (-1) + (-1+r)(-4+r) = 0$$
$$\therefore\ -6 - 2r + 3r + r^2 - 1 + 4 - r - 4r + r^2 = 0$$
$$\therefore\ 2r^2 - 4r - 3 = 0$$
$$\therefore\ r = \frac{4 \pm \sqrt{16+24}}{4}$$
$$\therefore\ r = \frac{2\pm\sqrt{10}}{2}$$

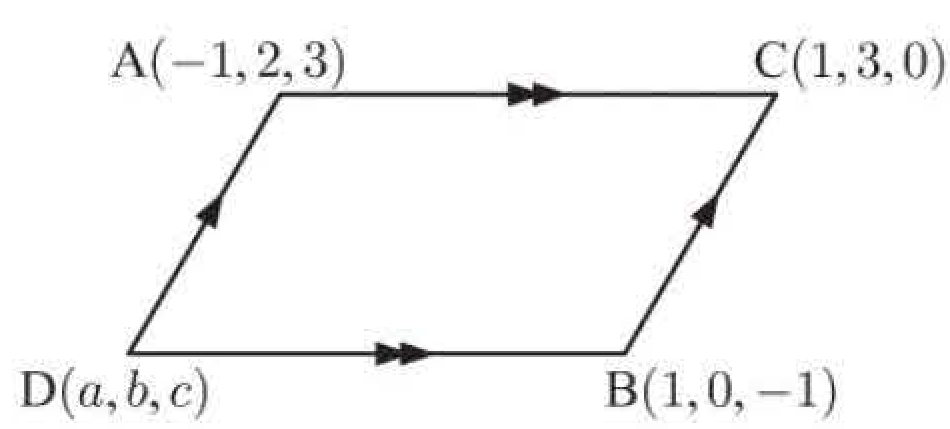
10 **a** Given $A(-1,\ 2,\ 3)$, $B(1,\ 0,\ -1)$, and $C(1,\ 3,\ 0)$,

$$\overrightarrow{AB} = \begin{pmatrix} 2 \\ -2 \\ -4 \end{pmatrix} = 2\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \quad \text{and} \quad \overrightarrow{AC} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$$

$\therefore$ a normal to the plane containing A, B, and C is

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -2 \\ 2 & 1 & -3 \end{vmatrix} = \begin{vmatrix} -1 & -2 \\ 1 & -3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -2 \\ 2 & -3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}\mathbf{k} = 5\mathbf{i} - \mathbf{j} + 3\mathbf{k} \quad \text{or} \quad \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}$$

b



Suppose D has coordinates $(a,\ b,\ c)$

$\therefore$ since $\overrightarrow{AD} = \overrightarrow{CB}$, $\begin{pmatrix} a+1 \\ b-2 \\ c-3 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ -1 \end{pmatrix}$

$\therefore\ a = -1,\quad b = -1 \quad \text{and} \quad c = 2$

$\therefore$ D is at $(-1,\ -1,\ 2)$

c From **a**, $\overrightarrow{AC} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$ and from **b**, $\overrightarrow{AD} = \begin{pmatrix} 0 \\ -3 \\ -1 \end{pmatrix}$

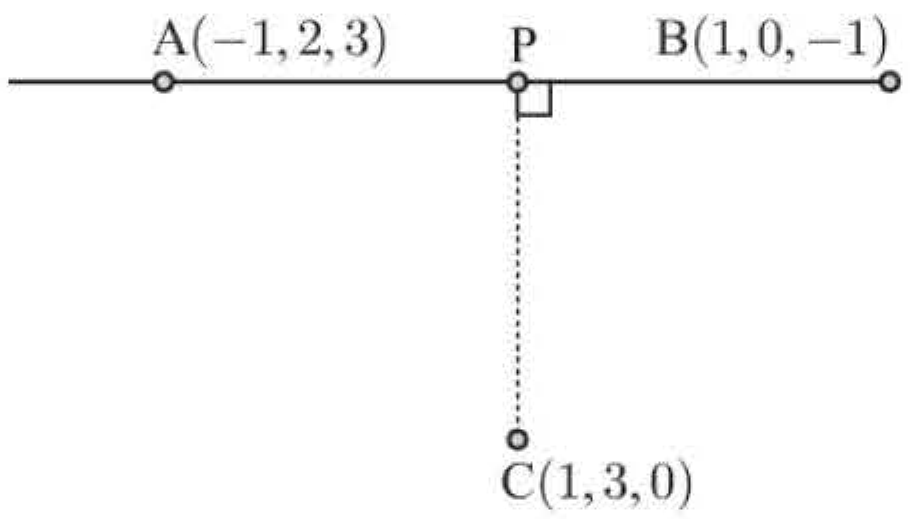
$$\overrightarrow{AC} \times \overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -3 \\ 0 & -3 & -1 \end{vmatrix} = \begin{vmatrix} 1 & -3 \\ -3 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & -3 \\ 0 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 1 \\ 0 & -3 \end{vmatrix}\mathbf{k}$$
$$= -10\mathbf{i} + 2\mathbf{j} - 6\mathbf{k}$$

$$\therefore\ \text{area of parallelogram} = |-10\mathbf{i} + 2\mathbf{j} - 6\mathbf{k}|$$
$$= \sqrt{100+4+36} = \sqrt{140} \approx 11.8 \text{ units}^2$$

d From **a**, $\overrightarrow{AB}$ has direction vector $\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$

$\therefore$ the line through A and B has parametric equations
$x = 1 + t,\quad y = 0 - t,\quad z = -1 - 2t,\quad t \in \mathbb{R}.$

If $P(1+t,\ -t,\ -1-2t)$ is the foot of the perpendicular,



then $\overrightarrow{CP} = \begin{pmatrix} t \\ -t-3 \\ -1-2t \end{pmatrix}$ and $\overrightarrow{CP} \bullet \overrightarrow{AB} = 0$

$$\therefore \begin{pmatrix} t \\ -t-3 \\ -1-2t \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = 0$$

$$\therefore t + t + 3 + 2 + 4t = 0$$

$$\therefore 6t = -5$$

$$\therefore t = -\tfrac{5}{6}$$

$\therefore$ P is $(1-\frac{5}{6},\ \frac{5}{6},\ -1+\frac{10}{6})$ or $(\frac{1}{6},\ \frac{5}{6},\ \frac{2}{3})$.

11 $P(2, 0, 1)$, $Q(3, 4, -2)$, $R(-1, 3, 2)$

a $\overrightarrow{PQ} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$

$|\overrightarrow{PQ}| = \sqrt{1+16+9} = \sqrt{26}$ units

and $\overrightarrow{QR} = \begin{pmatrix} -4 \\ -1 \\ 4 \end{pmatrix}$

b Since $\overrightarrow{PQ} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$ and P is at $(2, 0, 1)$,

the line has equation

$x = 2 + \lambda,\quad y = 0 + 4\lambda,\quad z = 1 - 3\lambda$

$\therefore\ x = 2 + \lambda,\quad y = 4\lambda,\quad z = 1 - 3\lambda,\quad \lambda \in \mathbb{R}$

c The vector equation of the plane is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} -4 \\ -1 \\ 4 \end{pmatrix}$, $\lambda, \mu \in \mathbb{R}$.

12 **a** Given $A(-1, 3, 2)$ and the plane $2x - y + 2z = 8$,

the distance from A to the plane is $d = \dfrac{|2x_1 - y_1 + 2z_1 - 8|}{\sqrt{2^2 + (-1)^2 + 2^2}} = \dfrac{|2(-1) - 3 + 2(2) - 8|}{3}$

$$= \frac{|-9|}{3} = 3 \text{ units}$$

b The point on the plane nearest A is the foot of the normal to the plane that passes through A.

Since the normal has direction vector $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ and passes through $(-1, 3, 2)$,

it has equation $x = -1 + 2t,\ y = 3 - t,\ z = 2 + 2t,\ t \in \mathbb{R}$

and meets the plane when $2(-1+2t) - (3-t) + 2(2+2t) = 8$

$$\therefore -2 + 4t - 3 + t + 4 + 4t = 8$$

$$\therefore 9t = 9$$

$\therefore t = 1 \quad \therefore$ the point is $(1, 2, 4)$.

c Suppose X is the foot of the perpendicular from A to the line, so X has coordinates $(7-2t,\ -6+t,\ 1+5t)$ for some $t \in \mathbb{R}$. Then the shortest distance from A to the line is AX.

Now $\overrightarrow{AX} = \begin{pmatrix} 8-2t \\ t-9 \\ -1+5t \end{pmatrix}$ and since the line has direction vector $\mathbf{u} = \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix}$,

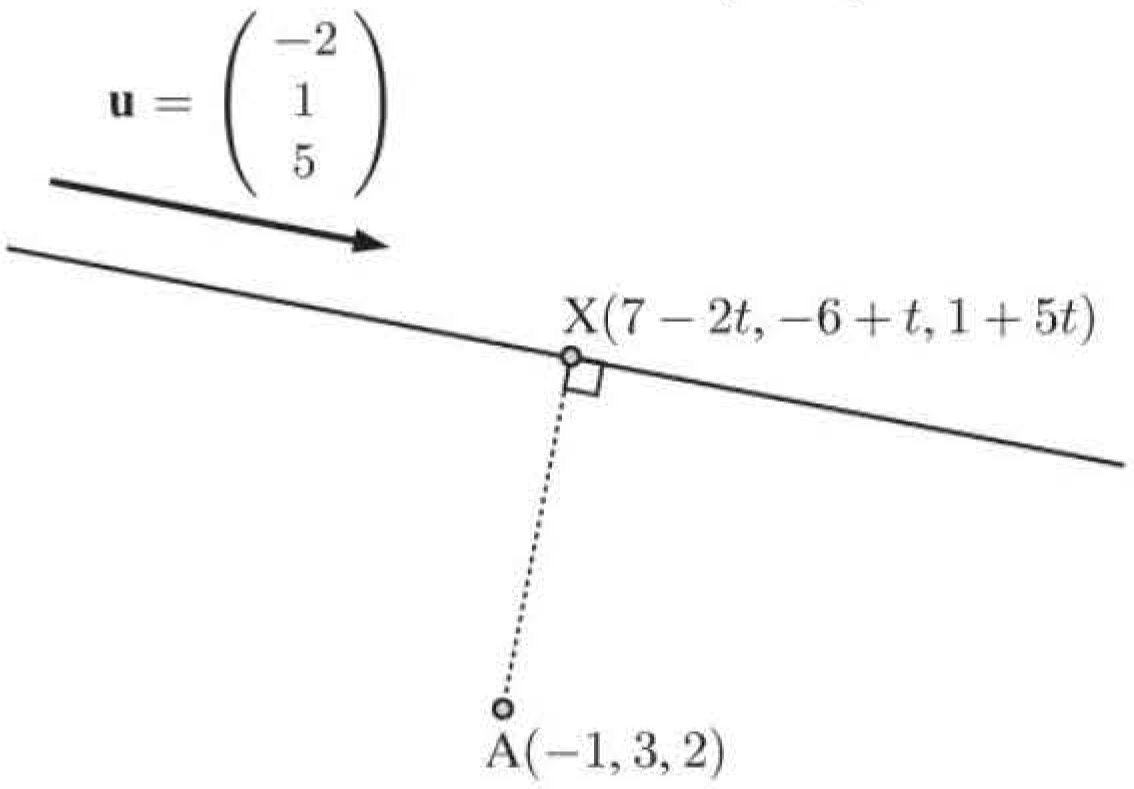
$$\mathbf{u} \bullet \overrightarrow{AX} = 0$$

$$\begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 8-2t \\ t-9 \\ -1+5t \end{pmatrix} = 0$$

$$-16 + 4t + t - 9 - 5 + 25t = 0$$

$$\therefore 30t = 30$$

$$\therefore t = 1$$



$$\therefore |AX| = \sqrt{6^2 + (-8)^2 + 4^2}$$

$$= \sqrt{36 + 64 + 16} = \sqrt{116} \text{ units}$$

13 Given A$(-1, 0, 2)$, B$(0, -1, 1)$, and C$(1, 2, -1)$

a $\overrightarrow{AB} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}$

$\therefore \quad \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -1 \\ 2 & 2 & -3 \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ 2 & -3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -1 \\ 2 & -3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix}\mathbf{k} = \begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix}$

$\therefore$ since A lies on the plane it has equation $5x + y + 4z = 5(-1) + 0 + 4(2)$

$\therefore \quad 5x + y + 4z = 3$

b Since the normal has direction $\begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix}$ and passes through $(0, 0, 0)$, it has equation

$x = 0 + 5t, \quad y = 0 + t, \quad z = 0 + 4t$

$\therefore \quad x = 5t, \quad y = t, \quad z = 4t, \quad t \in \mathbb{R}$

c The line meets the plane when $5(5t) + t + 4(4t) = 3$

$\therefore \quad 25t + t + 16t = 3$

$\therefore \quad 42t = 3$

$\therefore \quad t = \frac{1}{14}$

So, the line meets the plane at $\left(\frac{5}{14}, \frac{1}{14}, \frac{2}{7}\right)$.

14 **a** $\dfrac{x-3}{2} = \dfrac{y-4}{1} = \dfrac{z+1}{-2}$ has direction vector $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

while $x = -1 + 3t, \quad y = 2 + 2t, \quad z = 3 - t$ has direction vector $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$

$\therefore$ the lines are not parallel.

If the lines intersect, then $\dfrac{-1 + 3t - 3}{2} = \dfrac{2 + 2t - 4}{1} = \dfrac{3 - t + 1}{-2}$

$\therefore \quad \frac{3}{2}t - 2 = 2t - 2 = \frac{t}{2} - 2$

Now $t = 0$ satisfies this relation, so the lines intersect at $(-1, 2, 3)$.

b If θ is the acute angle between the lines, then

$\cos\theta = \dfrac{|\mathbf{v}_1 \bullet \mathbf{v}_2|}{|\mathbf{v}_1|\,|\mathbf{v}_2|} = \dfrac{|2 \times 3 + 1 \times 2 + -2 \times -1|}{\sqrt{9}\sqrt{14}} = \dfrac{|6 + 2 + 2|}{3\sqrt{14}} = \dfrac{10}{3\sqrt{14}}$

$\therefore \quad \theta \approx 27.0°$

15 **a** The lines meet when

$\dfrac{(15 + 3\lambda) - 8}{3} = \dfrac{(29 + 8\lambda) + 9}{-16} = \dfrac{(5 - 5\lambda) - 10}{7}$ for some $\lambda \in \mathbb{R}$

$\therefore \quad \dfrac{3\lambda + 7}{3} = \dfrac{8\lambda + 38}{-16} = \dfrac{-5\lambda - 5}{7}$

$\therefore \quad -48\lambda - 112 = 24\lambda + 114$ and $56\lambda + 266 = 80\lambda + 80$

$\therefore \quad -72\lambda = 226$ and $186 = 24\lambda$

$\therefore \quad \lambda = -\frac{113}{36}$ and $\lambda = \frac{31}{4}$

$\therefore$ no value of λ satisfies both equations.

$\therefore$ the lines do not meet.

Their direction vectors are $\begin{pmatrix} 3 \\ -16 \\ 7 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 8 \\ -5 \end{pmatrix}$ $\therefore$ they are not parallel.

$\therefore$ line 1 and line 2 are skew.

b Line 3 is parallel to line 1 and so has direction vector $\begin{pmatrix} 3 \\ -16 \\ 7 \end{pmatrix}$.

$\therefore$ the plane containing lines 2 and 3 has normal vector

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 8 & -5 \\ 3 & -16 & 7 \end{vmatrix} = \begin{vmatrix} 8 & -5 \\ -16 & 7 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & -5 \\ 3 & 7 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & 8 \\ 3 & -16 \end{vmatrix}\mathbf{k}$$
$$= -24\mathbf{i} - 36\mathbf{j} - 72\mathbf{k}$$
$$= -12(2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k})$$

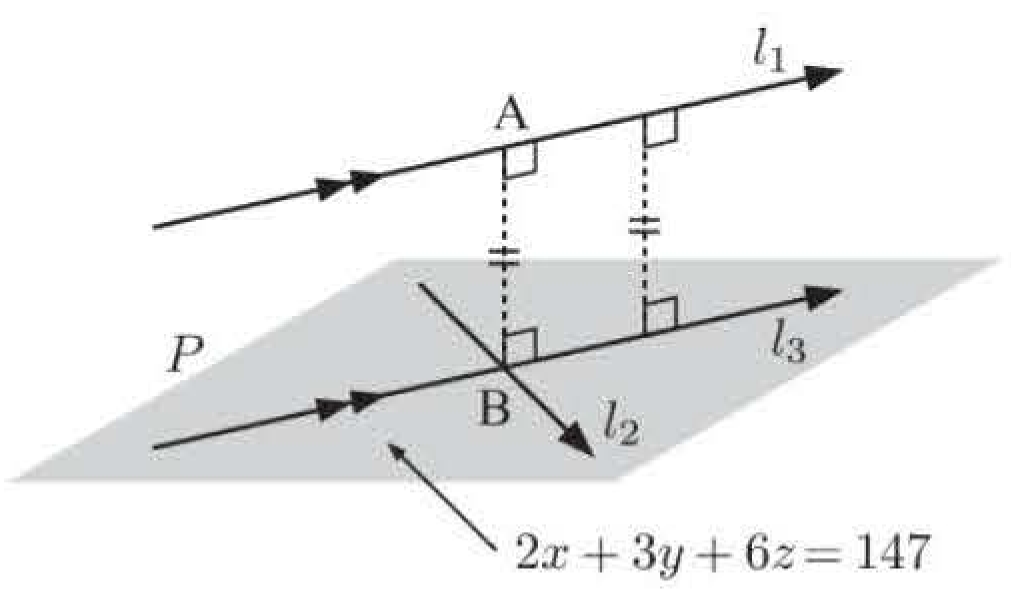
$\therefore$ the equation of the plane is $2x + 3y + 6z = 2(15) + 3(29) + 6(5)$

$\therefore$ $2x + 3y + 6z = 147$

c Since line 1 is parallel to line 3, line 1 is parallel to the plane containing lines 2 and 3.

$\therefore$ to find the shortest distance between lines 1 and 2, we choose a point on line 1 then find the shortest distance between this point and the plane.

$(8, -9, 10)$ is a point on line 1.



$$\therefore \text{ distance } d = \frac{|2(8) + 3(-9) + 6(10) - 147|}{\sqrt{2^2 + 3^2 + 6^2}}$$
$$= \frac{|-98|}{\sqrt{49}}$$
$$= \tfrac{98}{7}$$
$$= 14 \text{ units}$$

d Let the common perpendicular meet lines 1 and 2 at A and B respectively.

A is $(8 + 3\mu, -9 - 16\mu, 10 + 7\mu)$ for some $\mu \in \mathbb{R}$

B is $(15 + 3\lambda, 29 + 8\lambda, 5 - 5\lambda)$ for some $\lambda \in \mathbb{R}$

The common perpendicular has direction vector $\begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$

$\therefore$ we need to find values for μ and λ so that

$$\overrightarrow{AB} = k\begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} \quad \text{for some scalar } k$$

$$\therefore \begin{pmatrix} 15 + 3\lambda - (8 + 3\mu) \\ 29 + 8\lambda - (-9 - 16\mu) \\ 5 - 5\lambda - (10 + 7\mu) \end{pmatrix} = k\begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 7 + 3\lambda - 3\mu \\ 38 + 8\lambda + 16\mu \\ -5 - 5\lambda - 7\mu \end{pmatrix} = \begin{pmatrix} 2k \\ 3k \\ 6k \end{pmatrix}$$

$$\therefore \begin{cases} 3\lambda - 3\mu - 2k = -7 \\ 8\lambda + 16\mu - 3k = -38 \\ -5\lambda - 7\mu - 6k = 5 \end{cases}$$

Solving simultaneously using technology gives $\lambda = -2$, $\mu = -1$, $k = 2$.

$\therefore$ A is $(8 + 3(-1), -9 - 16(-1), 10 + 7(-1))$

and B is $(15 + 3(-2), 29 + 8(-2), 5 - 5(-2))$

$\therefore$ the common perpendicular meets lines 1 and 2 at $(5, 7, 3)$ and $(9, 13, 15)$.

16 **a** The lines meet where

$$\begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix} + s\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} + t\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

$$\therefore \quad \begin{pmatrix} -s \\ s \\ 2s \end{pmatrix} = \begin{pmatrix} -t \\ -t+2 \\ t+1 \end{pmatrix}$$

$$\therefore \quad \begin{cases} -s = -t \;\Rightarrow\; s = t & \text{.... (1)} \\ s = -t+2 & \text{.... (2)} \\ 2s = t+1 & \text{.... (3)} \end{cases}$$

Substituting (1) into (2) gives

$t = -t + 2$

$\therefore \ 2t = 2$

$\therefore \ t = 1$ and $s = 1$

Checking with (3): $2(1) = 1 + 1$ ✓

$$\therefore \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix} + 1\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

$\therefore$ A is $(2, -1, 0)$

b $B(0, -3, 2)$ lies on L_2 if

$$\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} + t\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix}$$

for some $t \in \mathbb{R}$

$$\therefore \quad \begin{cases} 3 - t = 0 \\ -t = -3 \\ -1 + t = 2 \end{cases}$$

$t = 3$ satisfies all these equations

$\therefore$ $B(0, -3, 2)$ lies on L_2

c $\overrightarrow{BC} = \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}$

$\therefore$ the equation of the line (BC) is

$$\mathbf{r} = \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix} + u\begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}, \quad u \in \mathbb{R}$$

d $\overrightarrow{AB} = \begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix}, \quad \overrightarrow{AC} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$

$\therefore$ the plane containing A, B, and C has normal vector

$$\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -2 & 2 \\ 1 & -1 & -2 \end{vmatrix} = \begin{vmatrix} -2 & 2 \\ -1 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -2 & 2 \\ 1 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -2 & -2 \\ 1 & -1 \end{vmatrix}\mathbf{k}$$

$$= 6\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$$

$\therefore$ the equation is $6x - 2y + 4z = 6(2) - 2(-1) + 4(0) = 14$ {using A}

or $3x - y + 2z = 7$

e Area of triangle ABC

$= \frac{1}{2}\,|\overrightarrow{AB} \times \overrightarrow{AC}|$

$= \frac{1}{2}\,|6\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}|$

$= \frac{1}{2}\sqrt{6^2 + (-2)^2 + 4^2}$

$= \frac{1}{2}\sqrt{56}$

$= \sqrt{14}$ units^2

f The normal to the plane has direction vector $\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$.

$\therefore$ the normal to the plane passing through $C(3, -2, -2)$ is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}, \quad \lambda \in \mathbb{R}$$

$D(9, -4, 2)$ lies on this line if

$$\begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 9 \\ -4 \\ 2 \end{pmatrix} \quad \text{for some } \lambda \in \mathbb{R}$$

$$\therefore \quad \begin{cases} 3 + 3\lambda = 9 \\ -2 - \lambda = -4 \\ -2 + 2\lambda = 2 \end{cases}$$

$\lambda = 2$ satisfies all three equations

$\therefore$ D lies on this line.

17 **a** $\left[\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 2 & 1 & 9 & 20 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & -5 & 5 & 10 \end{array}\right] \quad R_2 \to R_2 - 2R_1$

$\sim \left[\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & -1 & 1 & 2 \end{array}\right] \quad R_2 \to \frac{1}{5}R_2$

$\therefore\ -y + z = 2$ and $x + 3y + 2z = 5$

$\therefore$ if we let $y = s$, then $z = 2 + s$ and $x = 5 - 3y - 2z$

$= 5 - 3s - 2(2+s)$

$= 5 - 3s - 4 - 2s$

$\therefore\ x = 1 - 5s$

$\therefore$ planes A and B intersect in the line L_1: $x = 1 - 5s,\ y = s,\ z = 2 + s,\ s \in \mathbb{R}$

b $\left[\begin{array}{ccc|c} 1 & -1 & 6 & 8 \\ 2 & 1 & 9 & 20 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 6 & 8 \\ 0 & 3 & -3 & 4 \end{array}\right] \quad R_2 \to R_2 - 2R_1$

$\therefore\ 3y - 3z = 4$ and $x - y + 6z = 8$

$\therefore$ if we let $y = t$, then $z = -\frac{4}{3} + y = -\frac{4}{3} + t$ and $x = 8 + y - 6z$

$= 8 + t - 6(-\frac{4}{3} + t)$

$= 8 + t + 8 - 6t$

$\therefore\ x = 16 - 5t$

$\therefore$ planes B and C intersect in the line L_2: $x = 16 - 5t,\ y = t,\ z = -\frac{4}{3} + t,\ t \in \mathbb{R}$

c $\left[\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 1 & -1 & 6 & 8 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & -4 & 4 & 3 \end{array}\right] \quad R_2 \to R_2 - R_1$

$\therefore\ -4y + 4z = 3$ and $x + 3y + 2z = 5$

$\therefore$ if we let $y = u$, then $z = \frac{3}{4} + y = \frac{3}{4} + u$ and $x = 5 - 3y - 2z$

$= 5 - 3u - 2(\frac{3}{4} + u)$

$= 5 - 3u - \frac{3}{2} - 2u$

$\therefore\ x = \frac{7}{2} - 5u$

$\therefore$ planes A and C intersect in the line L_3: $x = \frac{7}{2} - 5u,\ y = u,\ z = \frac{3}{4} + u,\ u \in \mathbb{R}$

d L_1 is $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + s\begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix}$ with direction vector $\mathbf{a} = \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix}$.

L_2 is $\begin{pmatrix} 16 \\ 0 \\ -\frac{4}{3} \end{pmatrix} + t\begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix}$ with direction vector $\mathbf{b} = \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix}$.

L_3 is $\begin{pmatrix} \frac{7}{2} \\ 0 \\ \frac{3}{4} \end{pmatrix} + u\begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix}$ with direction vector $\mathbf{c} = \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix}$.

Since $\mathbf{a} = \mathbf{b} = \mathbf{c}$, L_1, L_2, and L_3 are a parallel.

When $s = 0$, the point on L_1 is $(1, 0, 2)$.

For L_2, $y = t$, so the unique point on L_2 with y-coordinate 0 is the point where $t = 0$.
This point is $(16, 0, -\frac{4}{3})$.

For L_3, $y = u$, so the unique point on L_3 with y-coordinate 0 is the point where $u = 0$.
This point is $(\frac{7}{2}, 0, \frac{3}{4})$.

$(1, 0, 2) \neq (16, 0, -\frac{4}{3})$, $(16, 0, -\frac{4}{3}) \neq (\frac{7}{2}, 0, \frac{3}{4})$, and $(1, 0, 2) \neq (\frac{7}{2}, 0, \frac{3}{4})$.

$\therefore$ none of the lines are coincident.

e The three planes have no common point of intersection. The line of intersection of any two planes is parallel to the third plane.