Chapter 20

APPLICATIONS OF DIFFERENTIAL CALCULUS

EXERCISE 20A.1

1 **a** $s(t) = t^2 + 3t - 2, \quad t \geqslant 0$

$$\begin{aligned}\text{Average velocity} &= \frac{s(t_2) - s(t_1)}{t_2 - t_1} \\ &= \frac{s(3) - s(1)}{3 - 1} \\ &= \frac{16 - 2}{2} \\ &= 7 \text{ m s}^{-1}\end{aligned}$$

b
$$\begin{aligned}\text{Average velocity} &= \frac{s(t_2) - s(t_1)}{t_2 - t_1} \\ &= \frac{s(1+h) - s(1)}{(1+h) - 1} \\ &= \frac{(1+h)^2 + 3(1+h) - 2 - 2}{h} \\ &= \frac{2h + h^2 + 3h}{h} \\ &= (h+5) \text{ m s}^{-1}, \quad h \neq 0\end{aligned}$$

c
$$\begin{aligned}&\lim_{h \to 0} \frac{s(1+h) - s(1)}{h} \\ &= \lim_{h \to 0} 5 + h \\ &= 5 \text{ m s}^{-1}\end{aligned}$$

This is the instantaneous velocity at $t = 1$ second, or $s'(1)$.

d
$$\begin{aligned}&\text{Average velocity} \\ &= \frac{s(t_2) - s(t_1)}{t_2 - t_1} \\ &= \frac{s(t+h) - s(t)}{(t+h) - t} \\ &= \frac{\left[(t+h)^2 + 3(t+h) - 2\right] - \left[t^2 + 3t - 2\right]}{h} \\ &= \frac{2ht + h^2 + 3h}{h} \\ &= (2t + h + 3) \text{ m s}^{-1}, \quad h \neq 0\end{aligned}$$

$$\begin{aligned}\text{Now } \lim_{h \to 0} \frac{s(t+h) - s(t)}{h} &= \lim_{h \to 0} (2t + h + 3) \\ &= (2t + 3) \text{ m s}^{-1}\end{aligned}$$

This is the instantaneous velocity at t seconds.

2 **a** $s(t) = 5 - 2t^2$ cm

$$\begin{aligned}\text{Average velocity} &= \frac{s(t_2) - s(t_1)}{t_2 - t_1} \\ &= \frac{s(5) - s(2)}{5 - 2} \\ &= \frac{(-45) - (-3)}{3} \\ &= -14 \text{ cm s}^{-1}\end{aligned}$$

b
$$\begin{aligned}\text{Average velocity} &= \frac{s(t_2) - s(t_1)}{t_2 - t_1} \\ &= \frac{s(2+h) - s(2)}{(2+h) - 2} \\ &= \frac{5 - 2(2+h)^2 + 3}{h} \\ &= \frac{-8h - 2h^2}{h} \\ &= (-8 - 2h) \text{ cm s}^{-1}, h \neq 0\end{aligned}$$

c
$$\begin{aligned}\lim_{h \to 0} \frac{s(2+h) - s(2)}{h} &= \lim_{h \to 0} (-8 - 2h) \\ &= -8 \text{ cm s}^{-1}\end{aligned}$$

This is the instantaneous velocity when $t = 2$ seconds, or $s'(2)$.

d
$$\begin{aligned}&\lim_{h \to 0} \frac{s(t+h) - s(t)}{h} \\ &= \lim_{h \to 0} \frac{\left[5 - 2(t+h)^2\right] - \left[5 - 2t^2\right]}{h} \\ &= \lim_{h \to 0} \frac{-4th - 2h^2}{h} \\ &= \lim_{h \to 0} (-4t - 2h) \\ &= -4t \text{ cm s}^{-1}\end{aligned}$$

This is the instantaneous velocity at t seconds.

3 $v(t) = 2\sqrt{t} + 3 \text{ cm s}^{-1}, \; t \geqslant 0$

a Average acceleration

$$= \frac{v(t_2) - v(t_1)}{t_2 - t_1} = \frac{v(4) - v(1)}{4 - 1} = \frac{7 - 5}{3} = \tfrac{2}{3} \text{ cm s}^{-2}$$

b Average acceleration

$$= \frac{v(t_2) - v(t_1)}{t_2 - t_1} = \frac{v(1+h) - v(1)}{(1+h) - 1} = \frac{[2\sqrt{1+h} + 3] - [2\sqrt{1} + 3]}{h} = \frac{2\sqrt{1+h} - 2}{h} \text{ cm s}^{-2}$$

c

$$\lim_{h \to 0} \frac{v(1+h) - v(1)}{(1+h) - 1}$$
$$= \lim_{h \to 0} \frac{2\sqrt{1+h} - 2}{h}$$
$$= \lim_{h \to 0} \frac{2\left(\sqrt{1+h} - 1\right)}{h} \times \frac{\sqrt{1+h} + 1}{\sqrt{1+h} + 1}$$
$$= \lim_{h \to 0} \frac{2(1 + h - 1)}{h\left(\sqrt{1+h} + 1\right)}$$
$$= \lim_{h \to 0} \frac{2h}{h\left(\sqrt{1+h} + 1\right)}$$
$$= \lim_{h \to 0} \frac{2}{\sqrt{1+h} + 1}$$
$$= \tfrac{2}{2}$$
$$= 1 \text{ cm s}^{-2}$$

This is the instantaneous acceleration when $t = 1$ second.

d

$$\lim_{h \to 0} \frac{v(t+h) - v(t)}{h}$$
$$= \lim_{h \to 0} \frac{2\sqrt{t+h} - 2\sqrt{t}}{h}$$
$$= \lim_{h \to 0} \frac{2(\sqrt{t+h} - \sqrt{t})}{h} \times \frac{\sqrt{t+h} + \sqrt{t}}{\sqrt{t+h} + t}$$
$$= \lim_{h \to 0} \frac{2h}{h(\sqrt{t+h} + \sqrt{t})}$$
$$= \frac{2}{2\sqrt{t}}$$
$$= \frac{1}{\sqrt{t}} \text{ cm s}^{-2}$$

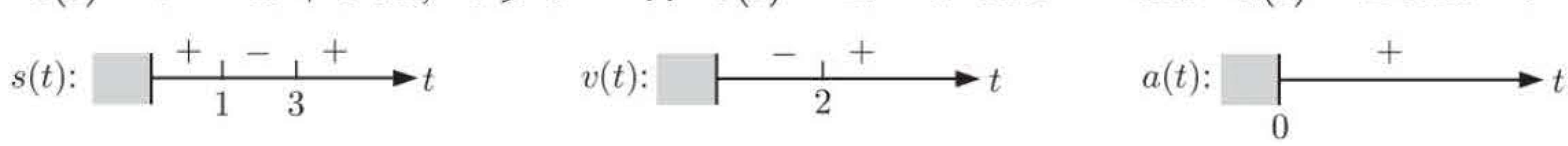
This is the instantaneous acceleration at t seconds.

4 **a** This is the instantaneous velocity at $t = 4$ seconds.

b This is the instantaneous acceleration at $t = 4$ seconds.

EXERCISE 20A.2

1 **a** $s(t) = t^2 - 4t + 3$ cm, $\; t \geqslant 0 \quad \therefore \; v(t) = 2t - 4 \text{ cm s}^{-1}$ and $a(t) = 2 \text{ cm s}^{-2}$.



b When $t = 0$, $s(0) = 3$ cm
$v(0) = -4 \text{ cm s}^{-1}$
$a(0) = 2 \text{ cm s}^{-2}$

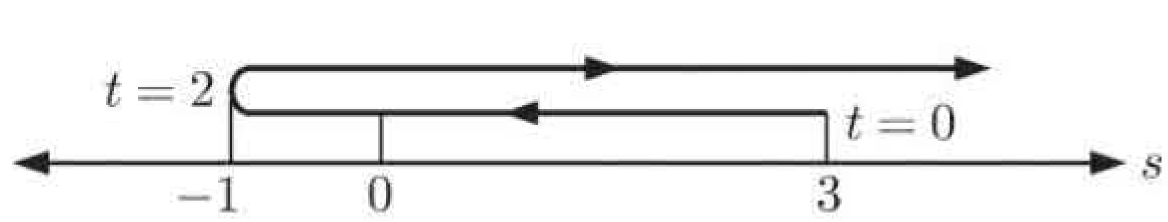
$\therefore$ the object is 3 cm right of O and is moving to the left with a velocity of 4 cm s^{-1} and slowing down, its acceleration being 2 cm s^{-2} to the right.

c When $t = 2$, $s(2) = -1$ cm
$v(2) = 0 \text{ cm s}^{-1}$
$a(2) = 2 \text{ cm s}^{-2}$

$\therefore$ the object is 1 cm left of O, instantaneously stationary and accelerating to the right at 2 cm s^{-2}.

d The object reverses direction when $v(t) = 0$, which occurs at $t = 2$ seconds.
At $t = 2$, the particle is 1 cm left of O.

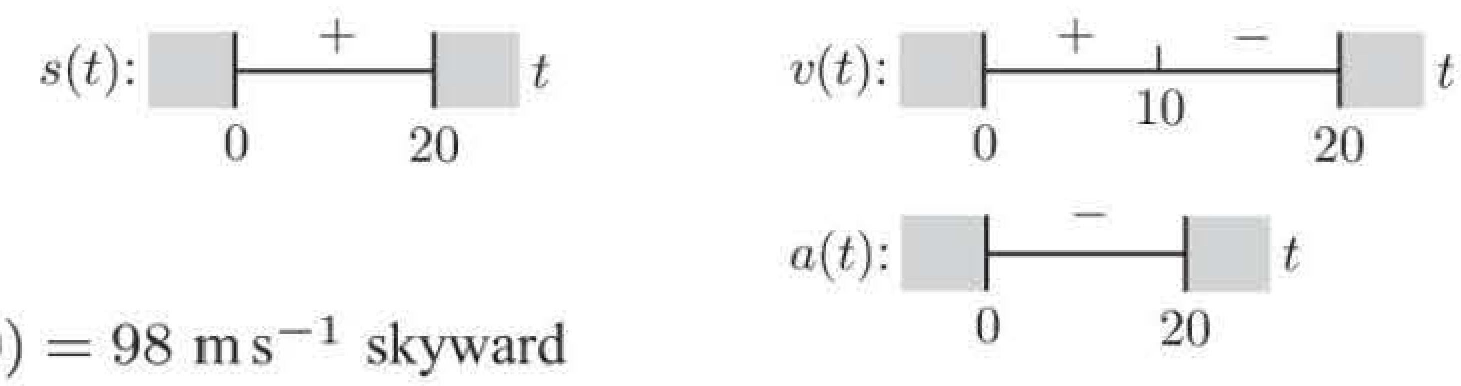
e



f Speed decreases when $v(t)$ and $a(t)$ have opposite signs, which is when $0 \leqslant t \leqslant 2$.

2 $s(t) = 98t - 4.9t^2$ m, $t \geqslant 0$

a $v(t) = 98 - 9.8t$ m s^{-1}
$a(t) = -9.8$ m s^{-2}



b When $t = 0$, $s(0) = 0$ m, $v(0) = 98$ m s^{-1} skyward

c When $t = 5$, $s(5) = 367.5$ m
$v(5) = 49$ m s^{-1}
$a(5) = -9.8$ m s^{-2}

The stone is 367.5 m above the ground and moving skyward at 49 m s^{-1}. Its speed is decreasing.

When $t = 12$, $s(12) = 470.4$ m
$v(12) = -19.6$ m s^{-1}
$a(12) = -9.8$ m s^{-2}

The stone is 470.4 m above the ground and moving groundward at 19.6 m s^{-1}. Its speed is increasing.

d The maximum height is reached when $v(t) = 0$ m s^{-1}

$\therefore$ $98 - 9.8t = 0$
$\therefore$ $9.8t = 98$
$\therefore$ $t = 10$ seconds

$\therefore$ the maximum height is
$s(10) = 98(10) - 4.9\,(100)$
$= 980 - 490$
$= 490$ m

e The stone is at ground level when $s(t) = 0$ which is when $98t - 4.9t^2 = 0$
$\therefore$ $4.9t(20 - t) = 0$
$\therefore$ $t = 0$ or 20 seconds
$\therefore$ it hits the ground after 20 seconds.

3 $s(t) = 1.2 + 28.1t - 4.9t^2$ metres

a When released, $t = 0$ and $s(0) = 1.2$ m $\therefore$ it is released 1.2 m above the ground.

b $s'(t) = 28.1 - 9.8t$ m s^{-1} is the instantaneous velocity of the ball at the time t seconds after release.

c When $s'(t) = 0$, $28.1 - 9.8t = 0$ $\therefore$ $t = \dfrac{28.1}{9.8} \approx 2.87$ seconds

So, after 2.87 seconds the ball has reached its maximum height, and is instantaneously at rest.

d $s(2.867) = 1.2 + 28.1 \times 2.867 - 4.9 \times 2.867^2 \approx 41.5$ m

So, the maximum height reached is about 41.5 m.

e **i** $s'(0) = 28.1$ m s^{-1} **ii** $s'(2) = 28.1 - 19.6$
$= 8.5$ m s^{-1} **iii** $s'(5) = 28.1 - 49$
$= -20.9$ m s^{-1}
$\therefore$ speed $= 20.9$ m s^{-1}

If $s'(t) \geqslant 0$, the ball is travelling upwards. If $s'(t) \leqslant 0$, the ball is travelling downwards.

f $s(t) = 0$ when $1.2 + 28.1t - 4.9t^2 = 0$
$\therefore$ $4.9t^2 - 28.1t - 1.2 = 0$

$$\therefore\ t = \frac{28.1 \pm \sqrt{28.1^2 - 4(4.9)(-1.2)}}{9.8} \approx -0.0424 \text{ or } 5.777$$

But $t > 0$, so the ball hits the ground after 5.78 seconds.

g $s''(t) = -9.8$ m s^{-2} and is the rate of change in $s'(t)$

$\therefore$ the instantaneous acceleration is constant at -9.8 m s^{-2} for the entire motion.

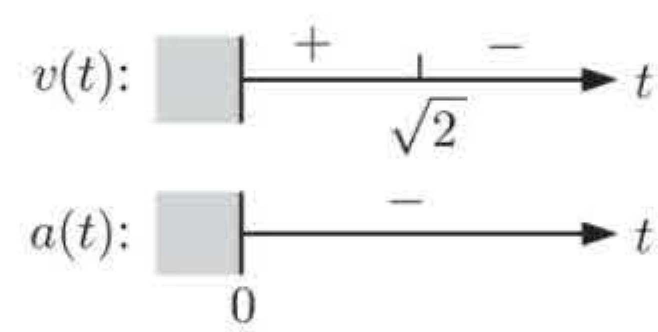
4 **a** $s(t) = bt - 4.9t^2$
$s'(t) = b - 9.8t$
$\therefore$ $s'(0) = b$
$\therefore$ the initial velocity is b m s^{-1} upwards.

b Since $s(14.2) = 0$,
$b(14.2) - 4.9(14.2)^2 = 0$
$\therefore$ $14.2\,[b - 4.9 \times 14.2] = 0$
$\therefore$ $b = 4.9 \times 14.2$
$\therefore$ $b = 69.58$
$\therefore$ the initial velocity is 69.6 m s^{-1}.

5 **a** $s(t) = 12t - 2t^3 - 1$ cm, $t \geqslant 0$

$\therefore\ v(t) = 12 - 6t^2$ cm s^{-1}

and $a(t) = -12t$ cm s^{-2}



b When $t = 0$, $s(0) = -1$ cm
$v(0) = 12$ cm s^{-1}
$a(0) = 0$ cm s^{-2}

The particle was 1 cm left of O and was moving right at a constant speed of 12 cm s^{-1}.

c The particle reverses direction when $v(t) = 0$ which is when $12 - 6t^2 = 0$

$\therefore\ t^2 = 2$

$\therefore\ t = \sqrt{2}\quad \{t > 0\}$

When $t = \sqrt{2}$, $s(\sqrt{2}) = 12\sqrt{2} - 2(2\sqrt{2}) - 1$

$= 8\sqrt{2} - 1$

$\therefore$ the particle is $(8\sqrt{2} - 1)$ cm to the right of O.

d **i** From the sign diagrams in **a**, the speed increases for $t \geqslant \sqrt{2}$ seconds.

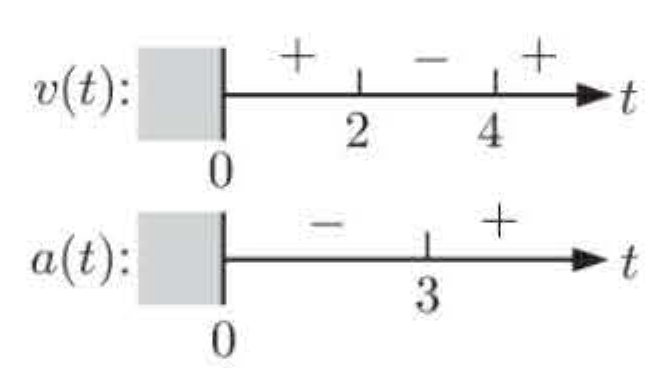
ii The velocity of the particle never increases $\{a(t) \leqslant 0\}$.

6 **a** $x(t) = t^3 - 9t^2 + 24t$ m, $t \geqslant 0$

$v(t) = 3t^2 - 18t + 24$ and $a(t) = 6t - 18$

$= 3(t^2 - 6t + 8)$ $\quad= 6(t - 3)$ m s^{-2}

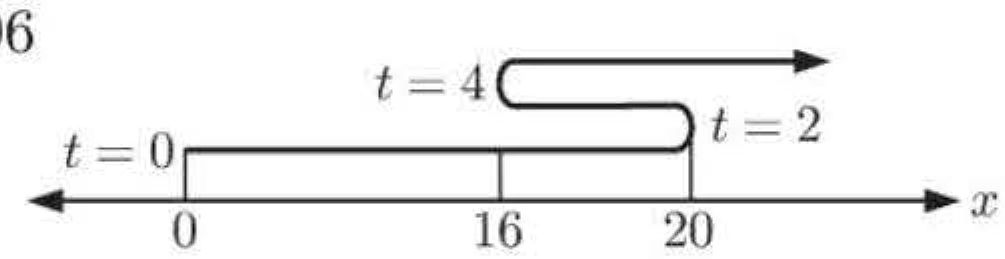
$= 3(t - 4)(t - 2)$ m s^{-1}



b The particle reverses direction when $v(t) = 0$, which occurs at $t = 2$ and $t = 4$ seconds.

$x(2) = 8 - 36 + 48$ m and $x(4) = 64 - 144 + 96$

$= 20$ m $\quad= 16$ m



c **i** The speed decreases when $v(t)$ and $a(t)$ have the opposite sign, which is when $0 \leqslant t \leqslant 2$ and $3 \leqslant t \leqslant 4$.

ii The velocity decreases when $a(t) \leqslant 0$, which is when $0 \leqslant t \leqslant 3$.

d When $t = 5$, $x(5) = 5^3 - 9(5)^2 + 24(5)$

$= 125 - 225 + 120$

$= 20$ m

$\therefore$ distance travelled $= 20 + 4 + 4$ m

$= 28$ m

7 **a** $s(t) = 100t + 200e^{-\frac{t}{5}}$ cm, $t \geqslant 0$

$v(t) = 100 - 40e^{-\frac{t}{5}}$ cm s^{-1}

$a(t) = 8e^{-\frac{t}{5}}$ cm s^{-2}

b When $t = 0$, $s(0) = 200$ cm (right of the origin)

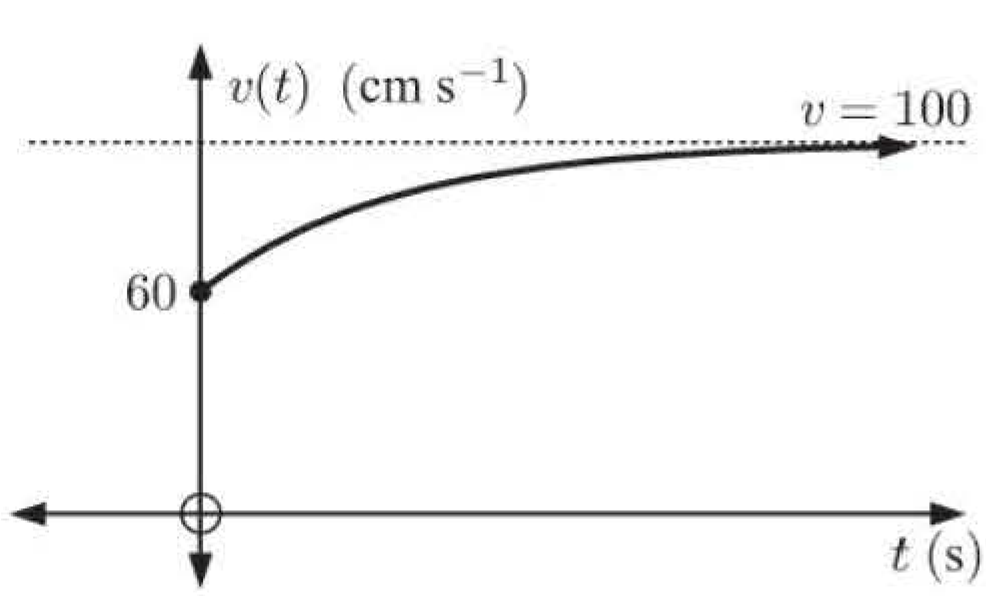
$v(0) = 60$ cm s^{-1}

$a(0) = 8$ cm s^{-2}

c As $t \to \infty$, $e^{-\frac{t}{5}} \to 0$,

$\therefore\ v(t) \to 100$ cm s^{-1} (below)

d



e When $v(t) = 80$ cm s^{-1},

$100 - 40e^{-\frac{t}{5}} = 80$

$\therefore\ -40e^{-\frac{t}{5}} = -20$

$\therefore\ e^{-\frac{t}{5}} = 0.5$

$\therefore\ -\dfrac{t}{5} = \ln 0.5$

$\therefore\ t = -5\ln 0.5 \approx 3.47$ s

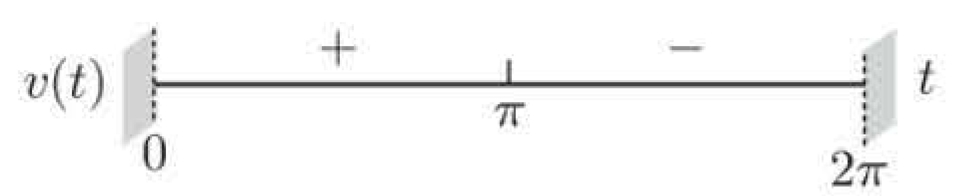
8 $x(t) = 1 - 2\cos t$ cm
$\therefore\ v(t) = x'(t) = 2\sin t$
$\therefore\ a(t) = v'(t) = 2\cos t$

a When $t = 0$,
$x(0) = 1 - 2\cos 0$
$= -1$ cm
$v(0) = 2\sin 0$
$= 0\ \text{cm s}^{-1}$
$a(0) = 2\cos 0$
$= 2\ \text{cm s}^{-2}$

b When $t = \frac{\pi}{4}$,
$x(\frac{\pi}{4}) = 1 - \frac{2}{\sqrt{2}}$
$= 1 - \sqrt{2}$ cm
$v(\frac{\pi}{4}) = \frac{2}{\sqrt{2}} = \sqrt{2}\ \text{cm s}^{-1}$
$a(\frac{\pi}{4}) = \frac{2}{\sqrt{2}} = \sqrt{2}\ \text{cm s}^{-2}$

The particle is $(\sqrt{2} - 1)$ cm left of the origin, moving right at $\sqrt{2}\ \text{cm s}^{-1}$ with increasing speed.

c We need to look for the points where the velocity equals zero.
If $v(t) = 2\sin t = 0$
then $\sin t = 0$
$\therefore\ t = \pi \quad (0 < t < 2\pi)$



The particle reverses direction when $t = \pi$.
At $t = \pi$, $x(\pi) = 3$ cm.

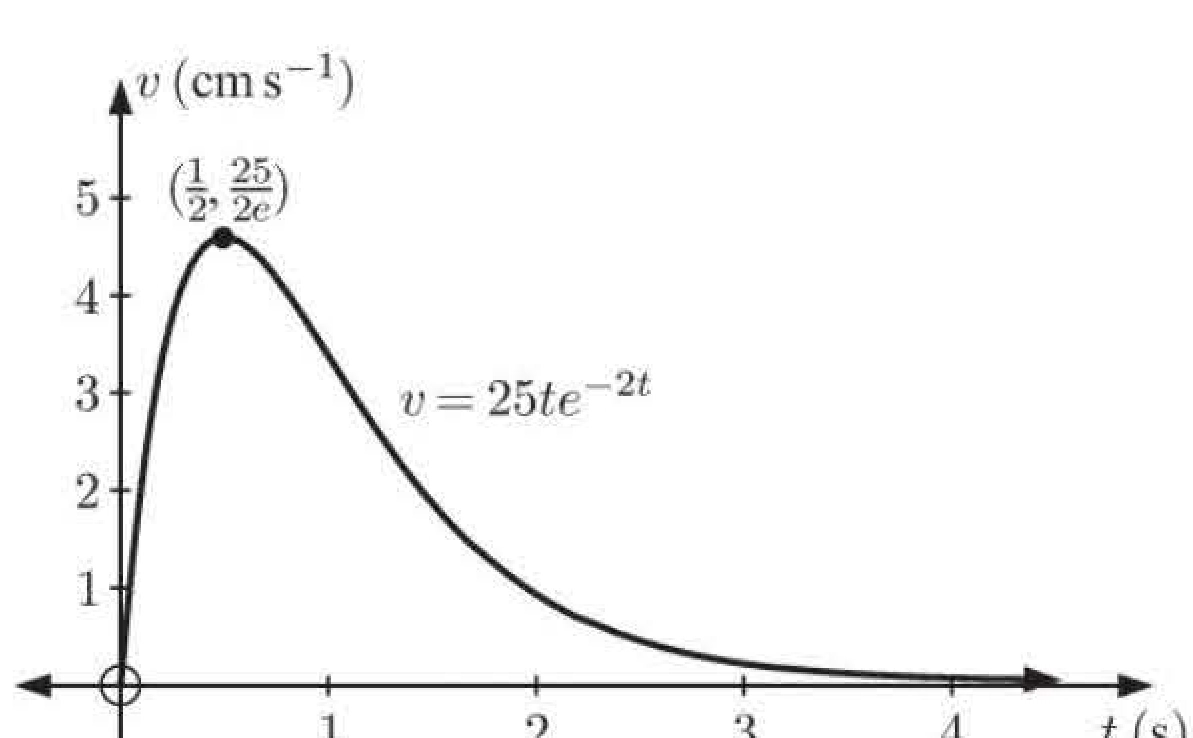
d The particle's speed is increasing when $v(t) = 2\sin t$ and $a(t) = 2\cos t$ have the same sign.
If $a(t) = 2\cos t = 0$
then $\cos t = 0$
$t = \frac{\pi}{2}, \frac{3\pi}{2} \quad (0 \leqslant t \leqslant 2\pi)$

$\therefore$ the particle's speed is increasing when $0 \leqslant t \leqslant \frac{\pi}{2}$ and $\pi \leqslant t \leqslant \frac{3\pi}{2}$.

9 **a** Let the equation be $s(t) = at^2 + bt + c$
$\therefore\ v(t) = 2at + b$
and $a(t) = 2a = g$ {gravitational acceleration}
$\therefore\ a = \frac{1}{2}g$ and $v(t) = gt + b$
But when $t = 0$, $v(0) = g \times 0 + b = b$
$\therefore$ the initial velocity is b
$\therefore\ v(t) = v(0) + gt$ as required

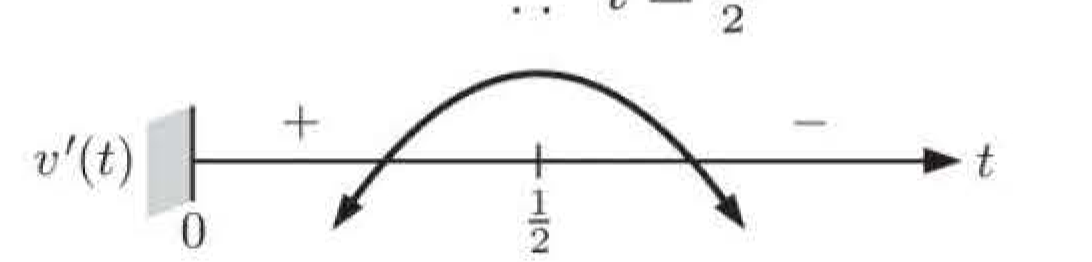
b Now when $t = 0$, $s(0) = 0$
$\therefore\ a \times 0^2 + b \times 0 + c = 0$
$\therefore\ c = 0$
and so $s(t) = \left(\frac{1}{2}g\right) t^2 + v(0)t$
$\therefore\ s(t) = v(0) \times t + \frac{1}{2}gt^2$ as required

10 **a**



b $a(t) = v'(t)$
$= 25 \times e^{-2t} + 25t(-2)e^{-2t}$
$= 25(1 - 2t)e^{-2t}\ \text{cm s}^{-2}$

c $v'(t) = 0$ when $1 - 2t = 0$
$\therefore\ t = \frac{1}{2}$



$\therefore$ the velocity is increasing for $0 \leqslant t \leqslant \frac{1}{2}$

EXERCISE 20B

1 $P(t) = 2t^2 - 12t + 118$ thousand dollars, $t \geqslant 0$

a $P(0) = \$118\,000$ is the current annual profit

b $\dfrac{dP}{dt} = 4t - 12$ thousand dollars per year

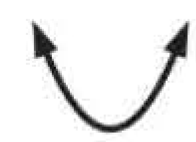
c $\dfrac{dP}{dt}$ is the rate of change in profit with time.

d **i** The profit decreases when $\dfrac{dP}{dt} \leqslant 0$, which occurs when $4t - 12 \leqslant 0$

$$\therefore\ 4t \leqslant 12$$
$$\therefore\ t \leqslant 3$$

But $t \geqslant 0$, so $0 \leqslant t \leqslant 3$ years.

ii The profit increases on the previous year when $\dfrac{dP}{dt} \geqslant 0$, which is for $t > 3$ years.

e The profit function is a quadratic with $a > 0$ $\therefore$ the shape is 

So, a minimum profit occurs when $\dfrac{dP}{dt} = 0$, which is when $t = 3$ years

and $P(3) = 18 - 36 + 118 = 100$ thousand dollars or $\$100\,000$.

f When $t = 4$, $\dfrac{dP}{dt} = 4$ thousand dollars per year.

So, the profit is increasing at \$4000 per year after 4 years.

When $t = 10$, $\dfrac{dP}{dt} = 28$ thousand dollars per year.

So, the profit is increasing at \$28 000 per year after 10 years.

When $t = 25$, $\dfrac{dP}{dt} = 88$ thousand dollars per year.

So, the profit is increasing at \$88 000 per year after 25 years.

2 $V = 200(50 - t)^2$ m^3

a average rate on $0 \leqslant t \leqslant 5$

$$= \frac{V(5) - V(0)}{5 - 0}$$
$$= \frac{200(45)^2 - 200(50)^2}{5}$$
$$= -19\,000 \text{ m}^3 \text{ per minute}$$

$\therefore$ leaving at $19\,000$ m^3 per minute

b $V'(t) = 400(50 - t)^1 \times (-1)$

$\therefore$ $V'(5) = 400 \times 45 \times -1$

$= -18\,000$ m^3 per minute

$\therefore$ leaving at $18\,000$ m^3 per minute

3 $Q = 100 - 10\sqrt{t}$, $t \geqslant 0$

a **i** At $t = 0$, $Q = 100$ units

ii At $t = 25$, $Q = 50$ units

iii At $t = 100$, $Q = 0$ units

b $\dfrac{dQ}{dt} = -5t^{-\frac{1}{2}} = -\dfrac{5}{\sqrt{t}}$

i At $t = 25$, $\dfrac{dQ}{dt} = -1$ unit per year

$\therefore$ decreasing at 1 unit per year

ii At $t = 50$, $\dfrac{dQ}{dt} = -\frac{5}{\sqrt{50}}$

$= -\frac{1}{\sqrt{2}}$ units per year

$\therefore$ decreasing at $\frac{1}{\sqrt{2}}$ units per year

c $\dfrac{dQ}{dt} = -\dfrac{5}{\sqrt{t}}$ $\therefore$ the skin *loses* the chemical at the rate $R = \dfrac{5}{\sqrt{t}} = 5t^{-\frac{1}{2}}$ units per year.

Now $\dfrac{dR}{dt} = -\frac{5}{2}t^{-\frac{3}{2}} = -\dfrac{5}{2t\sqrt{t}}$

Since $2t\sqrt{t} > 0$ for all $t > 0$, $\dfrac{dR}{dt} < 0$ for all $t > 0$.

$\therefore$ the rate at which the skin loses the chemical is decreasing for all $t > 0$.

4 $H = 20 - \dfrac{97.5}{t+5}$ m, $t \geqslant 0$

a At planting, $t = 0$ $\quad \therefore \quad H(0) = 20 - \dfrac{97.5}{0+5} = 0.5$ m

b $H(4) = 20 - \dfrac{97.5}{4+5} \approx 9.17$ m

$H(8) = 20 - \dfrac{97.5}{8+5} = 12.5$ m

$H(12) = 20 - \dfrac{97.5}{12+5} \approx 14.3$ m

c Now $\dfrac{dH}{dt} = 97.5(t+5)^{-2} = \dfrac{97.5}{(t+5)^2}$

When $t = 0$, $\dfrac{dH}{dt} = \frac{97.5}{25} = 3.9$ m year^{-1}

When $t = 5$, $\dfrac{dH}{dt} = \frac{97.5}{100} = 0.975$ m year^{-1}

When $t = 10$, $\dfrac{dH}{dt} = \frac{97.5}{225} \approx 0.433$ m year^{-1}

d Now $\dfrac{dH}{dt} = \dfrac{97.5}{(t+5)^2}$

Since $(t+5)^2 > 0$ for all $t \geqslant 0$, $\dfrac{dH}{dt} > 0$ for all $t \geqslant 0$

$\therefore$ the height of the tree is always increasing, which means that the tree is always growing.

5 **a** $C(x) = 0.0003x^3 + 0.02x^2 + 4x + 2250$

$\therefore \quad C'(x) = 0.0009x^2 + 0.04x + 4$ dollars per pair

b $C'(220) = 0.0009(220)^2 + 0.04(220) + 4 = \$56.36$ per pair

This estimates the cost of making the 221st pair of jeans if 220 pairs are currently being made.

c $C(221) - C(220) \approx \$7348.98 - \$7292.40 \approx \$56.58$

This is the actual cost to make the extra pair of jeans (221 instead of 220).

d $C''(x) = 0.0018x + 0.04$

$C''(x) = 0$ when $0.0018x + 0.04 = 0$

$$\therefore \quad x = -\frac{0.04}{0.0018} \approx -22.2$$

This is the point when the rate of change is a minimum. However, it is out of the bounds of our model, as we cannot make a negative quantity of jeans.

6 **a** $C(v) = \frac{1}{5}v^2 + 200\,000v^{-1}$ euros

i At $v = 50$ km h^{-1}, $\quad C = \frac{1}{5}(50)^2 + \dfrac{200\,000}{50} = €4500$

ii At $v = 100$ km h^{-1}, $\quad C = \frac{1}{5}(100)^2 + \dfrac{200\,000}{100} = €4000$

b $\dfrac{dC}{dv} = \frac{2}{5}v - 200\,000v^{-2} = \frac{2}{5}v - \dfrac{200\,000}{v^2}$

i At $v = 30$ km h^{-1},

$\dfrac{dC}{dv} = \frac{2}{5}(30) - \dfrac{200\,000}{30^2}$

$\approx -€210.22$ per km h^{-1}

So, a decrease of €210.22 per km h^{-1}.

ii At $v = 90$ km h^{-1},

$\dfrac{dC}{dv} = \frac{2}{5}(90) - \dfrac{200\,000}{90^2}$

$\approx €11.31$ per km h^{-1}

So, an increase of €11.31 per km h^{-1}.

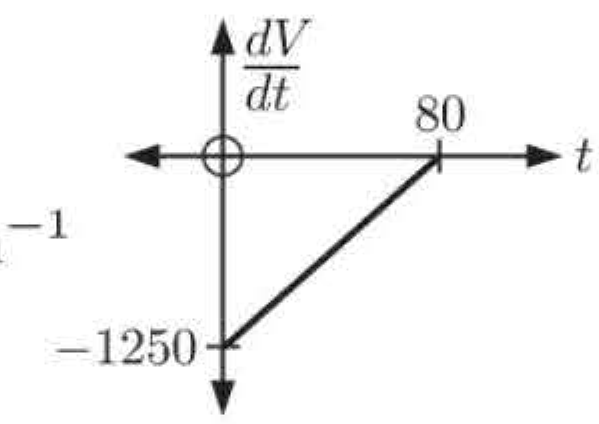
c The cost is a minimum when $\dfrac{dC}{dv} = 0$, which occurs when $\frac{2}{5}v - \dfrac{200\,000}{v^2} = 0$

$\therefore \quad \frac{2}{5}v = \dfrac{200\,000}{v^2}$

$\therefore \quad v^3 = 500\,000$

$\therefore \quad v \approx 79.4$ km h^{-1}

7 **a** $V = 50\,000\left(1 - \frac{t}{80}\right)^2, \quad 0 \leqslant t \leqslant 80$



$\therefore \frac{dV}{dt} = 2 \times 50\,000\left(1 - \frac{t}{80}\right)^1 \times \left(-\frac{1}{80}\right) = -1250\left(1 - \frac{t}{80}\right)$ L min^{-1}

b The outflow was fastest when $t = 0$, when the tap was first opened.

c $\frac{dV}{dt} = -1250 + \frac{1250}{80}t \qquad \therefore \frac{d^2V}{dt^2} = \frac{1250}{80} = \frac{125}{8}$ L min^{-2}

Since $\frac{d^2V}{dt^2}$ is constant and positive, $\frac{dV}{dt}$ is constantly increasing

$\therefore$ the outflow is decreasing at a constant rate.

8 $y = \frac{1}{10}x(x-2)(x-3) = \frac{1}{10}(x^3 - 5x^2 + 6x)$

a When $y = 0$, $x = 0$, 2, or 3

$\therefore$ the lake is between 2 and 3 km from the shoreline.

b $\frac{dy}{dx} = \frac{1}{10}(3x^2 - 10x + 6)$

$= \frac{3}{10}x^2 - x + \frac{3}{5}$

When $x = \frac{1}{2}$, $\frac{dy}{dx} = \frac{7}{40} = 0.175 \qquad \therefore$ land is sloping upwards.

When $x = 1\frac{1}{2}$, $\frac{dy}{dx} = -\frac{9}{40} = -0.225 \qquad \therefore$ land is sloping downwards.

This means the top of the hill is between $x = \frac{1}{2}$ and $x = 1\frac{1}{2}$.

c The deepest point of the lake occurs when the slope of the land is 0, which is when $\frac{dy}{dx} = 0$

$\therefore \frac{1}{10}(3x^2 - 10x + 6) = 0$

$\therefore 3x^2 - 10x + 6 = 0$

$$\therefore x = \frac{10 \pm \sqrt{100 - 72}}{6} = \frac{5 \pm \sqrt{7}}{3}$$

but it must be the value between 2 and 3 km, so $x = \frac{5 + \sqrt{7}}{3} \approx 2.55$ km from the sea

The depth at this point is $y(2.549) \approx \frac{1}{10}(2.549)(0.549)(-0.451)$

≈ -0.06311 km

≈ 63.1 m

9 **a** $\frac{dP}{dt} = aP\left(1 - \frac{P}{b}\right) - \left(\frac{c}{100}\right)P$ and when $\frac{dP}{dt} = 0$, the rate of change of population is zero, so the population is not changing and is stable.

b If $a = 0.06$, $b = 24\,000$, $c = 5$ then

$\frac{dP}{dt} = 0.06P\left(1 - \frac{P}{24\,000}\right) - \frac{5}{100}P$

$= 0.06P - 0.05P - \frac{0.06P^2}{24\,000}$

$= P\left(0.01 - \frac{P}{400\,000}\right)$

Now for a stable population, $\frac{dP}{dt} = 0$

$\therefore P = 0$ or $\frac{P}{400\,000} = 0.01$

$\therefore P = 0$ or 4000

$\therefore$ the stable population is 4000 fish.

c If the harvest rate is 4%, then $\frac{dP}{dt} = 0.06P\left(1 - \frac{P}{24\,000}\right) - \frac{4}{100}P = P\left(0.02 - \frac{0.06P}{24\,000}\right)$

For a stable population, $\frac{dP}{dt} = 0$ and so $0 = P\left(0.02 - \frac{0.06P}{24\,000}\right)$

$\therefore\ P = 0$ or $\dfrac{0.06P}{24\,000} = 0.02$

$\therefore\ P = 0$ or $\dfrac{0.02 \times 24\,000}{0.06}$

$\therefore\ P = 0$ or 8000

$\therefore$ the stable population is 8000 fish.

10 **a** $W = 20e^{-kt}$ so when $t = 50$ hours, $W = 10$ g

$\therefore\ 20e^{-50k} = 10$

$\therefore\ e^{-50k} = \frac{1}{2}$

$\therefore\ -50k = \ln\frac{1}{2} = -\ln 2 \qquad \therefore\ k = \frac{1}{50}\ln 2 \approx 0.0139$

b **i** When $t = 0$,
$W = 20e^0$
$= 20$ g

ii When $t = 24$,
$W = 20e^{-24k}$
$= 20e^{-24\frac{\ln 2}{50}}$
≈ 14.3 g

iii When $t = 1$ week
$= 7 \times 24$ hours
$= 168$ hours
$W = 20e^{-168\frac{\ln 2}{50}}$
≈ 1.95 g

c When $W = 1$ g, $\quad 20e^{-\frac{\ln 2}{50} \times t} = 1$

$\therefore\ e^{-\frac{\ln 2}{50} \times t} = 0.05$

$\therefore\ -\frac{\ln 2}{50} \times t = \ln 0.05$

$\therefore\ t = \dfrac{-50\ln 0.05}{\ln 2} = 216.096\,404\,7$

≈ 216 hours or 9 days and 6 minutes

d $\dfrac{dW}{dt}$
$= 20e^{-kt}(-k)$
$= \left(-20\dfrac{\ln 2}{50}\right) \times e^{-\frac{\ln 2}{50}t}$

i When $t = 100$ hours,
$\dfrac{dW}{dt} = \left(\dfrac{-20\ln 2}{50}\right)e^{-2\ln 2}$
≈ -0.0693 g h^{-1}

ii When $t = 1000$ hours,
$\dfrac{dW}{dt} = \left(\dfrac{-20\ln 2}{50}\right)e^{-20\ln 2}$
$\approx -2.64 \times 10^{-7}$ g h^{-1}

e $\dfrac{dW}{dt} = -k(20e^{-kt}) = -kW \qquad \therefore\ \dfrac{dW}{dt} \propto W$

11 $T = 5 + 95e^{-kt}$ °C

a $T = 20$°C when $t = 15$

$\therefore\ 20 = 5 + 95e^{-15k}$

$\therefore\ 15 = 95e^{-15k}$

$\therefore\ e^{15k} = \frac{95}{15}$

$\therefore\ 15k = \ln\left(\frac{19}{3}\right)$

$\therefore\ k = \frac{1}{15}\ln\left(\frac{19}{3}\right) \approx 0.123$

b When $t = 0$,
$T = 5 + 95e^0$
$= 5 + 95$
$= 100$°C

c $\dfrac{dT}{dt} = 0 + 95e^{-kt}(-k)$
$= -(95e^{-kt})k$
$= c(T - 5)$ where $c = -k$
$\therefore\ c \approx -0.123$

d $\dfrac{dT}{dt} = -95e^{-kt} \times k \approx -11.6902e^{-0.1231t}$

i When $t = 0$, $\dfrac{dT}{dt} \approx -11.69$, so the temperature is decreasing at 11.7°C min^{-1}.

ii When $t = 10$, $\dfrac{dT}{dt} \approx -11.6902e^{-1.231} \approx -3.415$,
so the temperature is decreasing at 3.42°C min^{-1}.

iii When $t = 20$, $\dfrac{dT}{dt} \approx -11.6902e^{-2.461} \approx -0.998$,
so the temperature is decreasing at 0.998°C min^{-1}.

12 $H(t) = 20\ln(3t+2) + 30$ cm, $t \geqslant 0$

a The shrubs were planted when $t = 0$. $H(0) = 20\ln(2) + 30 \approx 43.9$ cm

b When $H = 1$ m $= 100$ cm,

$$\begin{aligned} 20\ln(3t+2) + 30 &= 100 \\ \therefore\ 20\ln(3t+2) &= 70 \\ \therefore\ \ln(3t+2) &= 3.5 \\ \therefore\ 3t+2 &= e^{3.5} \\ \therefore\ 3t &= e^{3.5} - 2 \\ \therefore\ t &= \frac{e^{3.5}-2}{3} \text{ years} \\ \therefore\ t &\approx 10.4 \text{ years} \end{aligned}$$

c $\dfrac{dH}{dt} = 20 \times \dfrac{3}{(3t+2)} = \dfrac{60}{3t+2}$ cm year^{-1}

i When $t = 3$, $\dfrac{dH}{dt} = \frac{60}{11} \approx 5.4545$

$\therefore$ it is growing at 5.45 cm year^{-1}

ii When $t = 10$, $\dfrac{dH}{dt} = \frac{60}{32} = 1.875$

$\therefore$ it is growing at 1.88 cm year^{-1}

13 **a** $A = s(1 - e^{-kt})$, $t \geqslant 0$

When $t = 0$, $A = s(1 - e^0)$
$= s(1-1) = 0$

b **i** When $t = 3$, $A = 5$, and $s = 10$,

$$\begin{aligned} 5 &= 10(1 - e^{-3k}) \\ \therefore\ 0.5 &= 1 - e^{-3k} \\ \therefore\ e^{-3k} &= 0.5 \\ \therefore\ e^{3k} &= 2 \\ \therefore\ 3k &= \ln 2 \\ \therefore\ k &= \frac{\ln 2}{3} \approx 0.231 \end{aligned}$$

ii $\dfrac{dA}{dt} = ske^{-kt}$

$\therefore$ when $t = 5$ and $s = 10$,

$$\begin{aligned} \frac{dA}{dt} &= 10\left(\tfrac{1}{3}\ln 2\right)\left(e^{-\frac{5}{3}\ln 2}\right) \\ &\approx 0.728 \text{ litres per hour} \end{aligned}$$

c

$$\begin{aligned} A &= s\left(1 - e^{-kt}\right) \\ \therefore\ A &= s - se^{-kt} \\ \therefore\ A - s &= -se^{-kt} \end{aligned}$$

Now,
$$\begin{aligned} \frac{dA}{dt} &= ske^{-kt} \\ &= k\left(se^{-kt}\right) \\ &= -k\left(-se^{-kt}\right) \\ &= -k(A-s) \\ \therefore\ \frac{dA}{dt} &\propto (A-s) \end{aligned}$$

14 **a** When $t = 0$,

$$B(0) = \frac{C}{1+0.5e^0} = \frac{C}{1.5} = \frac{2C}{3} \text{ bees}$$

b When $t = 1$,
$$\begin{aligned} B(1) &= \frac{C}{1+0.5e^{-1.73}} \\ &\approx \frac{C}{1.089} \approx 0.919C \end{aligned}$$

$$\begin{aligned} \therefore\ \%\text{ increase} &= \left(\frac{0.919C - \frac{2}{3}C}{\frac{2}{3}C}\right) \times 100\% \\ &\approx 37.8\% \text{ increase} \end{aligned}$$

c As $t \to \infty$, $e^{-1.73t} \to 0$, and so

$B(t)$ approaches $\dfrac{C}{1+0} = C$

$\therefore$ the population is limited to C bees.

d
$$\begin{aligned} B(2) = \frac{C}{1+0.5e^{-1.73\times 2}} &= 4500 \\ \therefore\ \frac{C}{1.016} &= 4500 \\ \therefore\ C &\approx 4570.7 \end{aligned}$$

$\therefore$ from **a**, initial population $= \dfrac{2C}{3}$
≈ 3047 bees

e $B(t) = C(1 + 0.5e^{-1.73t})^{-1}$

$\therefore\ B'(t) = -C(1 + 0.5e^{-1.73t})^{-2} \times (0.5(-1.73)e^{-1.73t})$

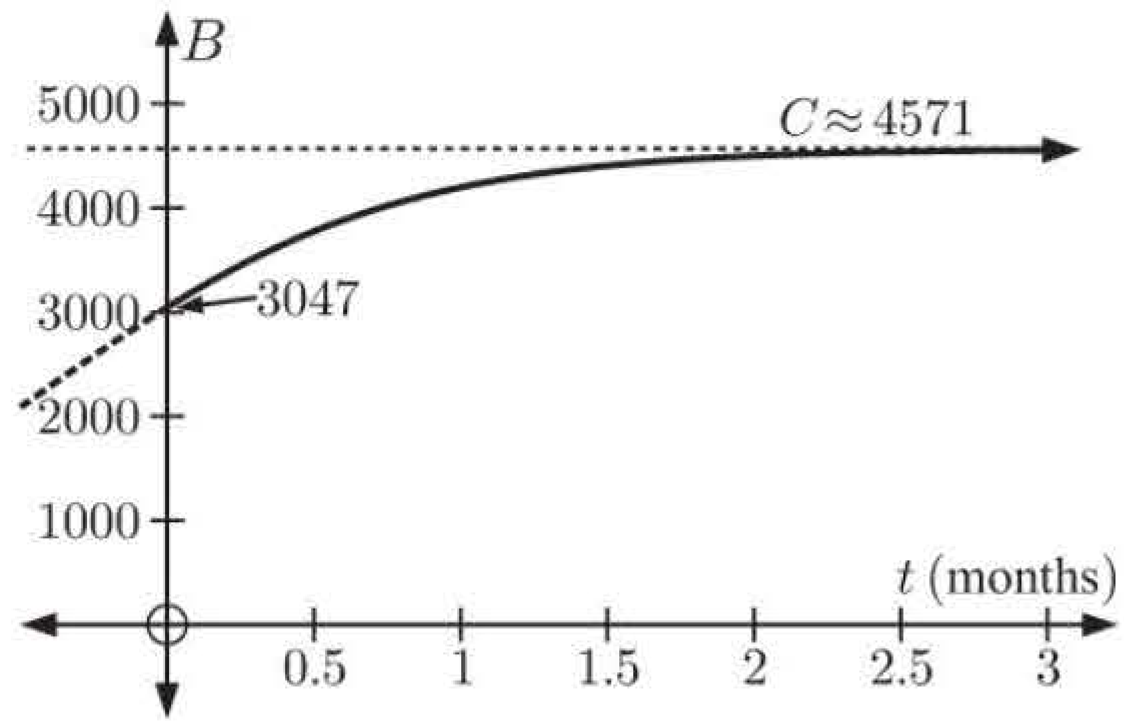
$= \dfrac{0.865Ce^{-1.73t}}{(1+0.5e^{-1.73t})^2}$

$= \dfrac{0.865C}{e^{1.73t}(1+0.5e^{-1.73})^2}$

Since $C > 0$, $B'(t) > 0$ for all t

$\therefore$ $B(t)$ is increasing over time.

f



15 Triangle PQR has area $A = \frac{1}{2} \times 6 \times 7 \times \sin\theta$

$\therefore\ A = 21\sin\theta\ \text{cm}^2$

$\therefore\ \dfrac{dA}{d\theta} = 21\cos\theta\ \text{cm}^2$ per radian

When $\theta = 45° = \frac{\pi}{4}$, $\dfrac{dA}{d\theta} = 21\cos(\frac{\pi}{4})\ \text{cm}^2$ per radian

$= 21 \times \frac{1}{\sqrt{2}}\ \text{cm}^2$ per radian

$\approx \frac{21}{\sqrt{2}}\ \text{cm}^2$ per radian

16 $d = 9.3 + 6.8\cos(0.507t)$ m

$\therefore\ \dfrac{dd}{dt} = -6.8\sin(0.507t) \times 0.507$

$= -3.4476\sin(0.507t)$

a When $t = 8$, $\dfrac{dd}{dt} = -3.4476\sin(0.507 \times 8)$

≈ 2.73

$\therefore$ the rate of change in the depth of water is 2.73 m per hour.

b When $t = 8$, $\dfrac{dd}{dt} \approx 2.73 > 0$

$\therefore$ the tide is rising.

17 **a** $V(t) = 340\sin(100\pi t)$

$\therefore\ V'(t) = 340\cos(100\pi t) \times 100\pi$

$= 34\,000\pi\cos(100\pi t)$

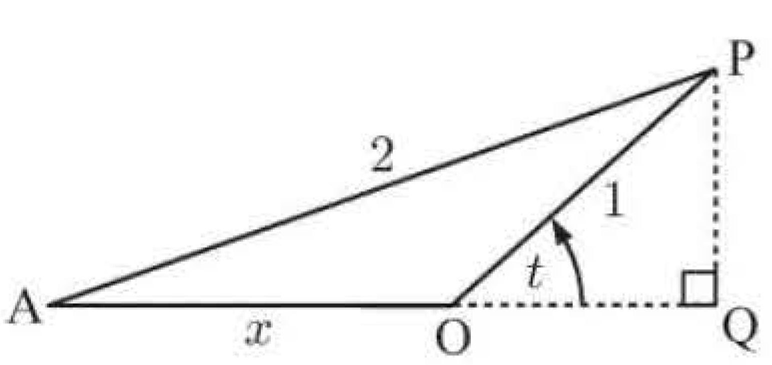
When $t = 0.01$,

$V'(0.01) = 34\,000\pi \times \cos\pi$

$= -34\,000\pi$ units per second

b When $V(t)$ is a maximum, $V'(t)$ must be 0 units per second.

18 **a** The distance from $A(-x, 0)$ to $P(\cos t, \sin t)$ is fixed at 2 m.



$\cos t = \dfrac{OQ}{1} = OQ$

$\therefore\ (\cos t + x)^2 + \sin^2 t = 2^2$ {Pythagoras in triangle APQ}

$\therefore\ (\cos t + x)^2 = 4 - \sin^2 t$

$\therefore\ x + \cos t = \pm\sqrt{4 - \sin^2 t}$

$\therefore$ since $x > 0$, $x = \sqrt{4 - \sin^2 t} - \cos t$

b Now $\dfrac{dx}{dt} = \frac{1}{2}(4 - \sin^2 t)^{-\frac{1}{2}}(-2\sin t\cos t) + \sin t$

$= \dfrac{-\sin t\cos t}{\sqrt{4 - \sin^2 t}} + \sin t$

i When $t = 0$,
$\sin t = 0$ and $\cos t = 1$
$$\therefore \quad \frac{dx}{dt} = 0 + 0 = 0 \text{ m s}^{-1}$$

ii When $t = \frac{\pi}{2}$,
$\sin t = 1$ and $\cos t = 0$
$$\therefore \quad \frac{dx}{dt} = 0 + \sin(\tfrac{\pi}{2}) = 1 \text{ m s}^{-1}$$

iii When $t = \frac{2\pi}{3}$,
$\sin t = \frac{\sqrt{3}}{2}$ and $\cos t = -\frac{1}{2}$
$$\therefore \quad \frac{dx}{dt} = \frac{-\frac{\sqrt{3}}{2}(-\frac{1}{2})}{\sqrt{4 - \frac{3}{4}}} + \frac{\sqrt{3}}{2} \approx 1.11 \text{ m s}^{-1}$$

EXERCISE 20C

1 $C(x) = 720 + 4x + 0.02x^2, \quad p(x) = 15 - 0.002x$

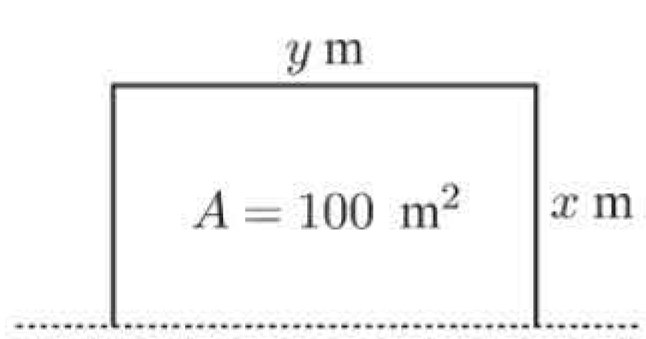
Revenue $R(x) = xp(x) = 15x - 0.002x^2$

Profit
$$\begin{aligned} P(x) &= \text{revenue} - \text{cost} \\ &= (15x - 0.002x^2) - (720 + 4x + 0.02x^2) \\ &= -0.022x^2 + 11x - 720 \end{aligned}$$
$$\therefore \quad P'(x) = -0.044x + 11$$

Now $P'(x) = 0$ when $x = \dfrac{11}{0.044} = 250$

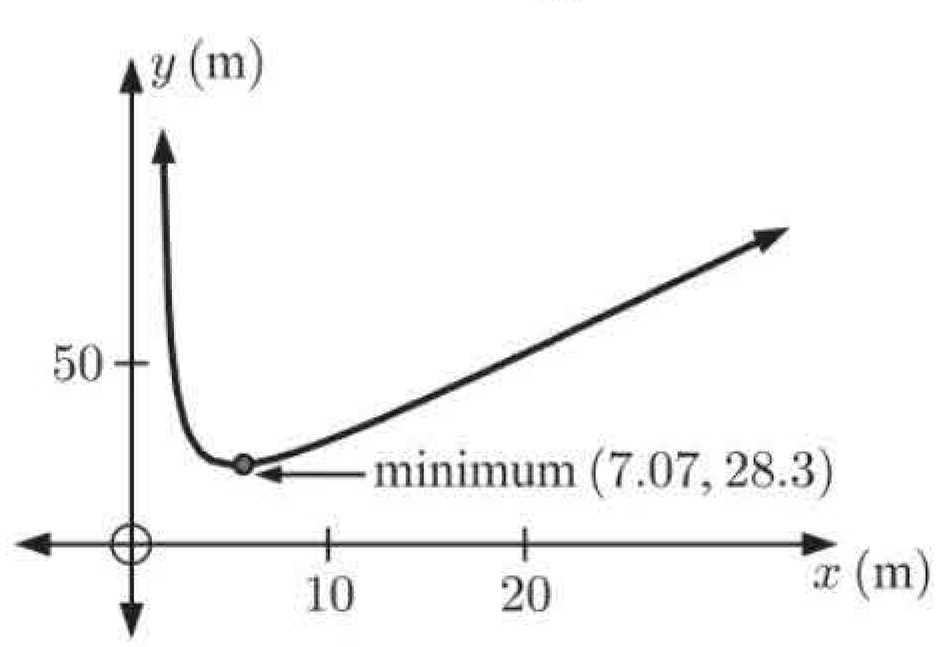
$\therefore$ as $P''(x) = -0.044 < 0$, P is maximised when 250 items are produced.

2 **a**



Now $xy = 100$
$$\therefore \quad y = \frac{100}{x}$$
$$\therefore \quad L = 2x + y$$
$$\therefore \quad L = 2x + \frac{100}{x}$$

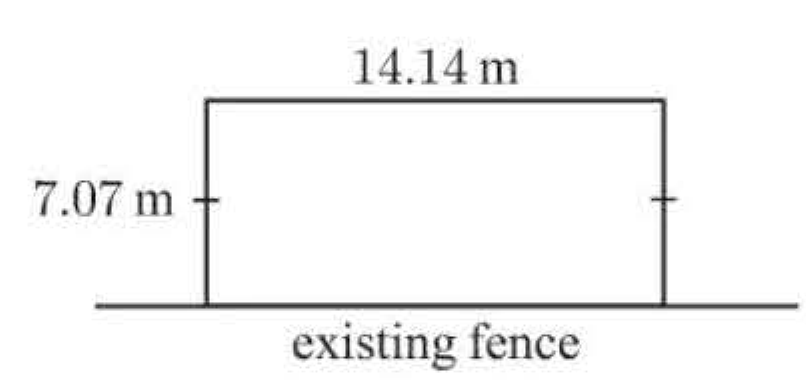
b



c $\dfrac{dL}{dx} = 2 - 100x^{-2} = 2 - \dfrac{100}{x^2}$

which is 0 when $\dfrac{100}{x^2} = 2$
$$\therefore \quad x^2 = 50$$
$$\therefore \quad x = \sqrt{50} \quad \{x > 0\}$$
$$\frac{d^2L}{dt^2} = 200x^{-3} = \frac{200}{x^3} > 0 \text{ for } x > 0$$
$$\begin{aligned} \therefore \quad L_{\min} &= 2\sqrt{50} + \frac{100}{\sqrt{50}} \\ &= 2\sqrt{50} + 2\sqrt{50} \\ &= 4\sqrt{50} \\ &= 20\sqrt{2} \text{ m when } x = 5\sqrt{2} \text{ m} \end{aligned}$$
$\therefore$ min $L \approx 28.3$ m when $x \approx 7.07$ m

d



3 Suppose x fittings are produced daily.
$$\begin{aligned} \therefore \quad C(x) &= 1000 + 2x + \frac{5000}{x} \\ &= 1000 + 2x + 5000x^{-1} \text{ euros} \end{aligned}$$
$$\therefore \quad C'(x) = 2 - \frac{5000}{x^2}$$

Now $C'(x) = 0$ when $x^2 = 2500$
$$\therefore \quad x = 50 \quad \{\text{as } x > 0\}$$

Also, $C''(x) = 10\,000x^{-3} = \dfrac{10\,000}{x^3}$

which is > 0 when $x > 0$.

$\therefore$ the cost is minimised when 50 fittings are produced.

4 $C(x) = \frac{1}{4}x^2 + 8x + 20$

$p(x) = 23 - \frac{1}{2}x$

Revenue $R(x) = xp(x) = 23x - \frac{1}{2}x^2$

Profit
$$\begin{aligned} P(x) &= \text{revenue} - \text{cost} \\ &= (23x - \tfrac{1}{2}x^2) - (\tfrac{1}{4}x^2 + 8x + 20) \\ &= -\tfrac{3}{4}x^2 + 15x - 20 \end{aligned}$$
$$\therefore \quad P'(x) = -\tfrac{3}{2}x + 15$$

Now $P'(x) = 0$ when $x = \dfrac{15}{\frac{3}{2}} = 10$

$\therefore$ as $P''(x) = -\frac{3}{2} < 0$, P is maximised when 10 blankets per day are produced.

5 Cost per hour $= \dfrac{v^2}{10} + 22$

Now, cost per km $= \dfrac{\text{cost per hour}}{\text{km per hour}}$

$$\therefore\ C(v) = \frac{\frac{v^2}{10} + 22}{v} = 0.1v + 22v^{-1}$$

$$\therefore\ C'(v) = 0.1 - 22v^{-2}$$

Now $C'(v) = 0$ when $0.1 = \dfrac{22}{v^2}$

$$\therefore\ v^2 = 220$$

$$\therefore\ v \approx 14.8\ \text{km h}^{-1}$$

6 **a** $A(t) = t\ln t + 1,\quad 0 < t \leqslant 5$

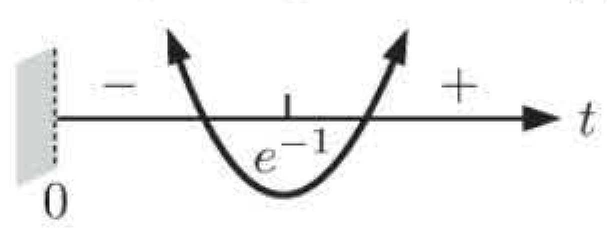
$$\therefore\ A'(t) = \ln t + t \times \frac{1}{t} + 0 \quad \{\text{product rule}\}$$

$$= \ln t + 1$$

$$\therefore\ A'(t) = 0 \text{ when } \ln t = -1$$

$$\therefore\ t = e^{-1}$$

and the sign diagram of $A'(t)$ is:



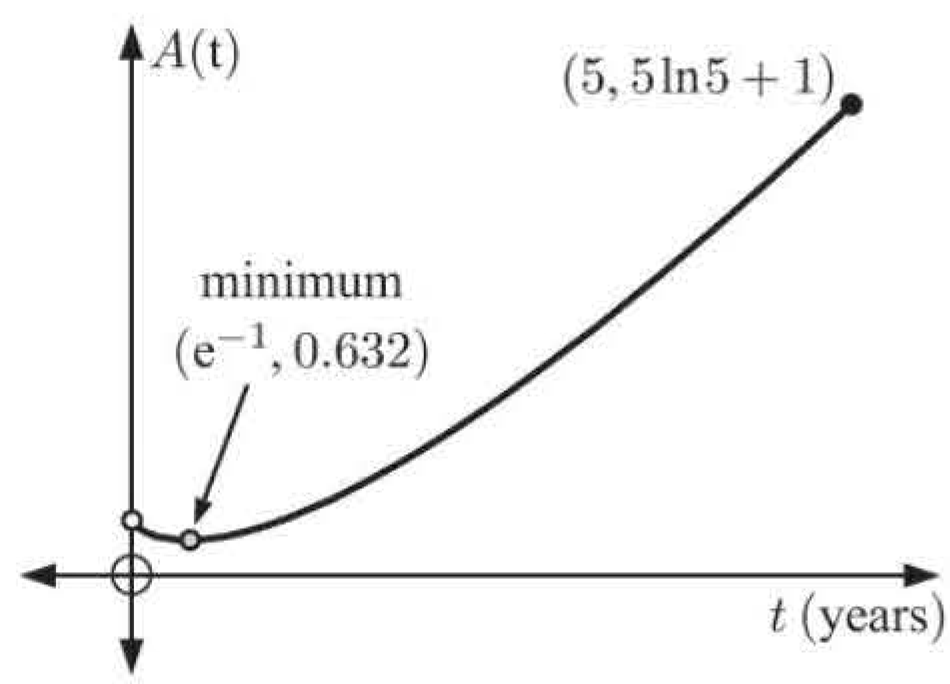
$$\therefore\ A(t) \text{ is a minimum when } t = \frac{1}{e}$$

$$\approx 0.3679 \text{ years}$$

$$\approx 4.41 \text{ months}$$

$\therefore$ the child's memorising ability is a minimum at 4.41 months old.

b



7 $C(x) = 0.0007x^3 - 0.1796x^2 + 14.663x + 160$ for $50 \leqslant x \leqslant 150$

$C'(x) = 0.0021x^2 - 0.3592x + 14.663$

$C'(x) = 0$ when $0.0021x^2 - 0.3592x + 14.663 = 0$

Using technology, $x \approx 103.74$ or $x \approx 67.30$

x	50	67.30	103.74	150
$C(x)$	531.65	546.73	529.80	680.95

$\therefore$ the maximum hourly cost is \$680.95 when 150 hinges are made per hour. The minimum hourly cost is \$529.80 when 104 hinges are made per hour.

8 $C(x) = 4\ln x + \left(\dfrac{30 - x}{10}\right)^2,\ x \geqslant 10$

$$\therefore\ C'(x) = \frac{4}{x} + 2\left(\frac{30-x}{10}\right)\left(-\tfrac{1}{10}\right)$$

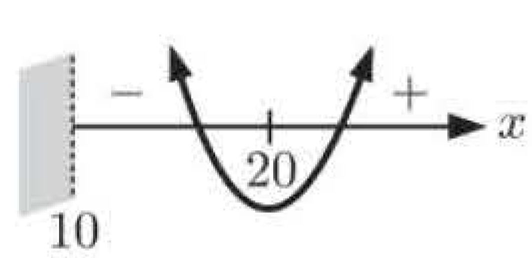
$$= \frac{4}{x} - \frac{30-x}{50}$$

$$= \frac{200 - x(30-x)}{50x}$$

$$= \frac{200 - 30x + x^2}{50x}$$

$$= \frac{(x-10)(x-20)}{50x}$$

$C'(x)$ has sign diagram:



$\therefore$ the minimum cost occurs when $x = 20$ or when 20 kettles per day are produced.

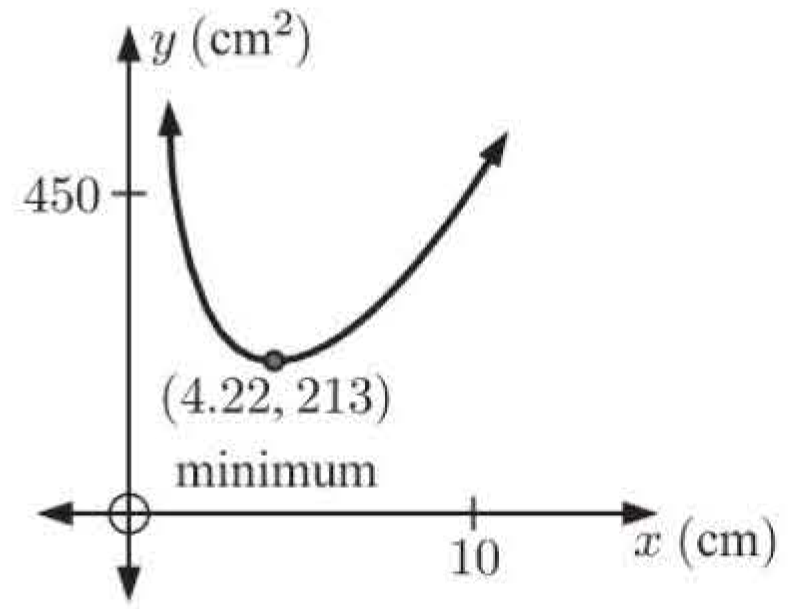
9 **a** Inner length of box $= 2x$ cm

b $\text{Volume} = 200 \text{ cm}^3$
$$\therefore\ x \times 2x \times h = 200$$
$$2x^2 h = 200$$
$$\therefore\ x^2 h = 100$$

c From **b**, $h = \dfrac{100}{x^2}$

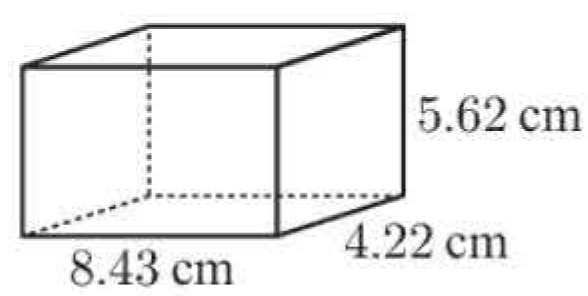
Area of inner surface is
$$\begin{aligned} A(x) &= 2(2x \times x) + 2(2x \times h) \\ &\quad + 2(x \times h) \\ &= 4x^2 + 4xh + 2xh \\ &= 4x^2 + 6xh \\ &= 4x^2 + \frac{600}{x} \text{ cm}^2 \end{aligned}$$

d



e
$$\begin{aligned} A(x) &= 4x^2 + 600x^{-1} \\ \therefore\ A'(x) &= 8x - 600x^{-2} \\ &= 8x - \frac{600}{x^2} \end{aligned}$$
$$\therefore\ A'(x) = 0 \quad \text{when} \quad 8x = \frac{600}{x^2}$$
$$\therefore\ 8x^3 = 600$$
$$\therefore\ x^3 = 75$$
$$\therefore\ x \approx 4.217 \text{ cm}$$
$$\begin{aligned} A''(x) &= 8 + 1200x^{-3} \\ &= 8 + \frac{1200}{x^3} \end{aligned}$$
$$\therefore\ A''(x) > 0 \qquad \{\text{as } x > 0\}$$
$\therefore$ area is minimised when $x \approx 4.22$ cm
$$\begin{aligned} \therefore\ A_{\min} &\approx 4(4.217)^2 + \frac{600}{(4.217)} \\ &\approx 213 \text{ cm}^2 \end{aligned}$$

f Height $h \approx \dfrac{100}{(4.217)^2}$
$$\approx 5.62 \text{ cm}$$



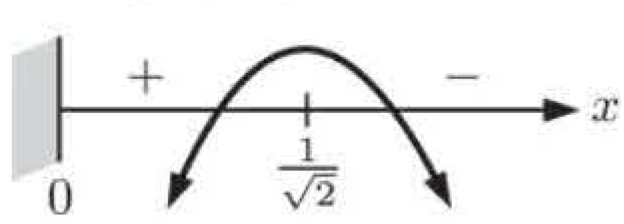
10 Let coordinates of D be $(x, 0)$ where $x > 0$.

$\therefore$ the coordinates of C are (x, e^{-x^2}).

$\therefore$ area ABCD $= 2xe^{-x^2}$

$$\begin{aligned} \therefore\ \frac{dA}{dx} &= 2e^{-x^2} + 2xe^{-x^2}(-2x) \quad \{\text{product rule}\} \\ &= 2e^{-x^2}\left(1 - 2x^2\right) \\ &= 2e^{-x^2}(1 + \sqrt{2}x)(1 - \sqrt{2}x) \end{aligned}$$

$\dfrac{dA}{dx}$ has sign diagram:

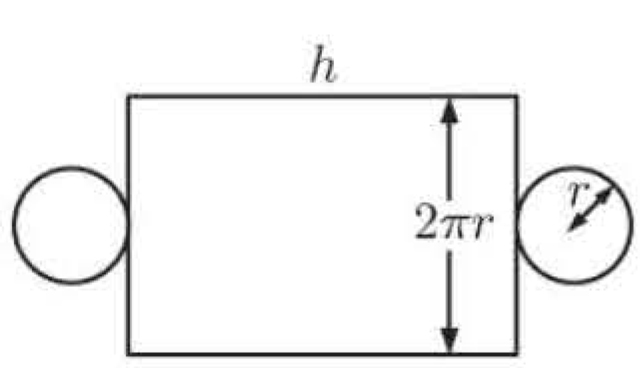


$\therefore$ the area is a maximum when $x = \frac{1}{\sqrt{2}}$

and so C is $\left(\frac{1}{\sqrt{2}}, e^{-\frac{1}{2}}\right)$.

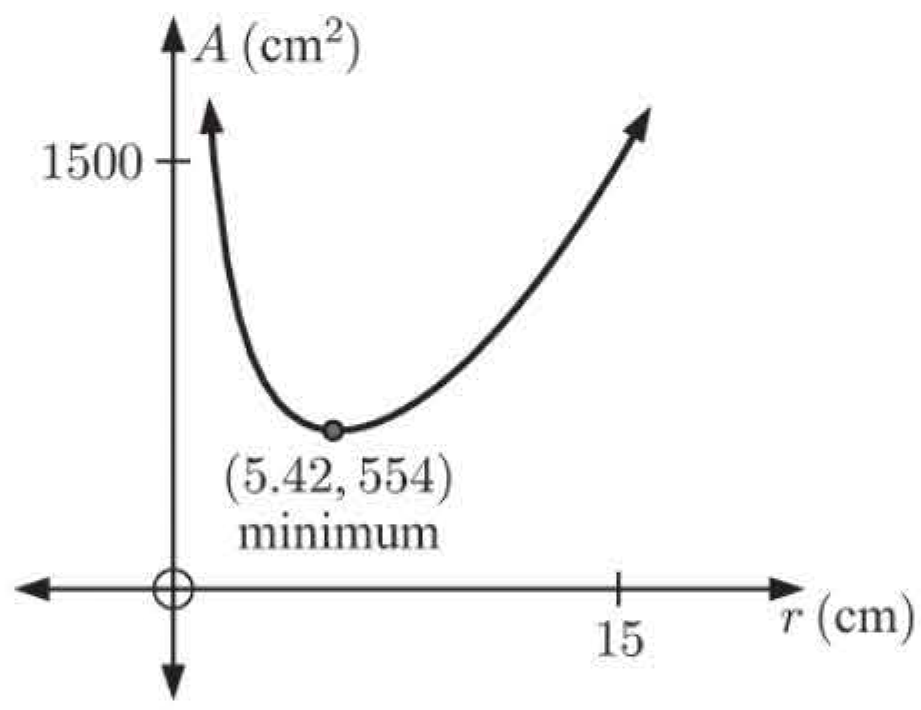
11 **a** Volume of can $= \pi r^2 h$
$$\therefore\ 1000 = \pi r^2 h \quad (\text{in cm})$$
$$\therefore\ h = \frac{1000}{\pi r^2} \text{ cm}$$

b Opening the can up we get:



$$\begin{aligned} \therefore\ A(r) &= \pi r^2 + \pi r^2 + 2\pi r h \\ &= 2\pi r^2 + 2\pi r h \\ &= 2\pi r^2 + \frac{2000}{r} \text{ cm}^2 \end{aligned}$$

c



d $A(r) = 2\pi r^2 + 2000r^{-1}$

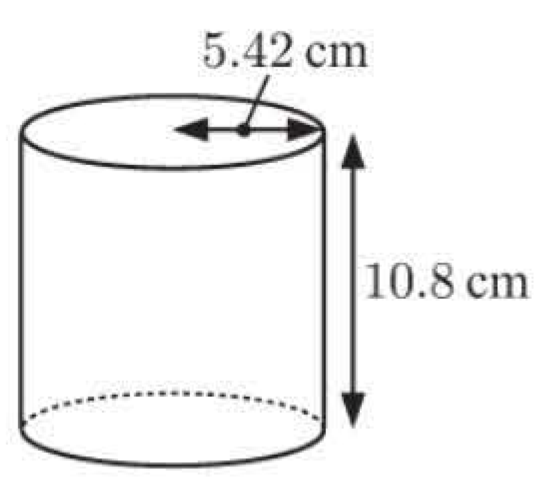
$$A'(r) = 4\pi r - 2000r^{-2} = 4\pi r - \frac{2000}{r^2}$$

So, $A'(r) = 0$ when $4\pi r = \dfrac{2000}{r^2}$

$$r^3 = \frac{2000}{4\pi}$$

$$r = \sqrt[3]{\frac{500}{\pi}} \approx 5.42 \text{ cm}$$

e



$$A''(r) = 4\pi + 4000r^{-3} = 4\pi + \frac{4000}{r^3}$$

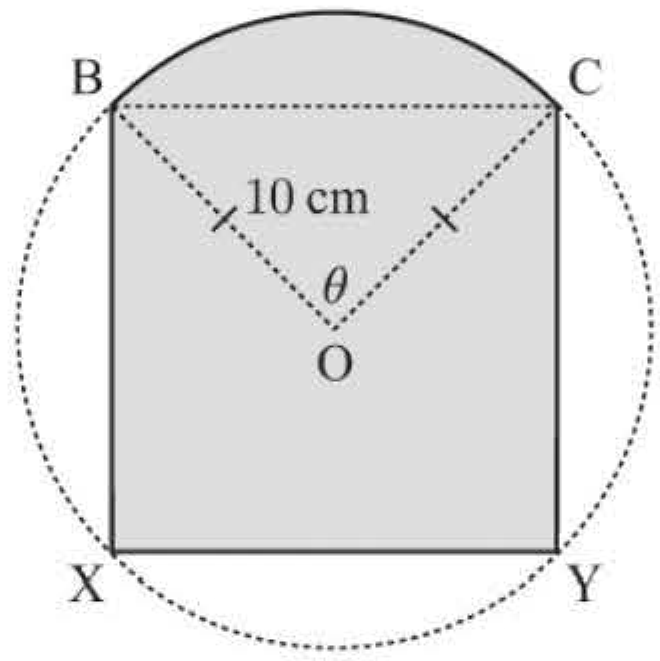
and as $r > 0$, $A''(r) > 0$

$\therefore$ area is a minimum when $r \approx 5.42$ cm

and $h = \dfrac{1000}{\pi r^2} \approx 10.8$ cm

$$A_{\min} = 2\pi r^2 + 2\pi rh \approx 554 \text{ cm}^2$$

12



a Using the cosine rule in $\triangle$BCO,

$$\text{BC}^2 = 10^2 + 10^2 - 2 \times 10 \times 10\cos\theta$$

$\therefore$ BC $= \sqrt{200 - 200\cos\theta}$

$\therefore$ XY $= \sqrt{200 - 200\cos\theta}$ also

Now $\text{BY}^2 = \text{BX}^2 + \text{XY}^2$ {Pythagoras}

$\therefore$ $400 = \text{BX}^2 + (200 - 200\cos\theta)$

$\therefore$ $\text{BX}^2 = 200 + 200\cos\theta$

$\therefore$ BX $= \sqrt{200 + 200\cos\theta}$

The shaded area is equal to the area of the sector plus $\frac{3}{4}$ of the area of BCYX.

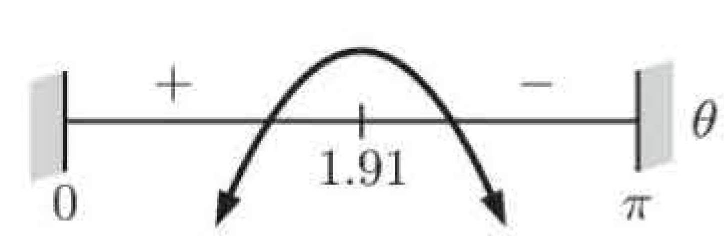
$$\begin{aligned}
\therefore\ A &= \tfrac{1}{2}(10)^2\theta + \tfrac{3}{4}(\text{BX} \times \text{BC}) \\
&= 50\theta + \tfrac{3}{4}\sqrt{200 + 200\cos\theta}\sqrt{200 - 200\cos\theta} \\
&= 50\theta + \tfrac{3}{4} \times 200\sqrt{1 + \cos\theta}\sqrt{1 - \cos\theta} \\
&= 50\theta + 150\sqrt{1 - \cos^2\theta} \\
&= 50\theta + 150\sin\theta \\
&= 50(\theta + 3\sin\theta) \quad \text{as required}
\end{aligned}$$

b $\dfrac{dA}{d\theta} = 50 + 150\cos\theta = 50(1 + 3\cos\theta)$

which is zero when $\cos\theta = -\frac{1}{3}$

$\therefore$ $\theta \approx 1.91$

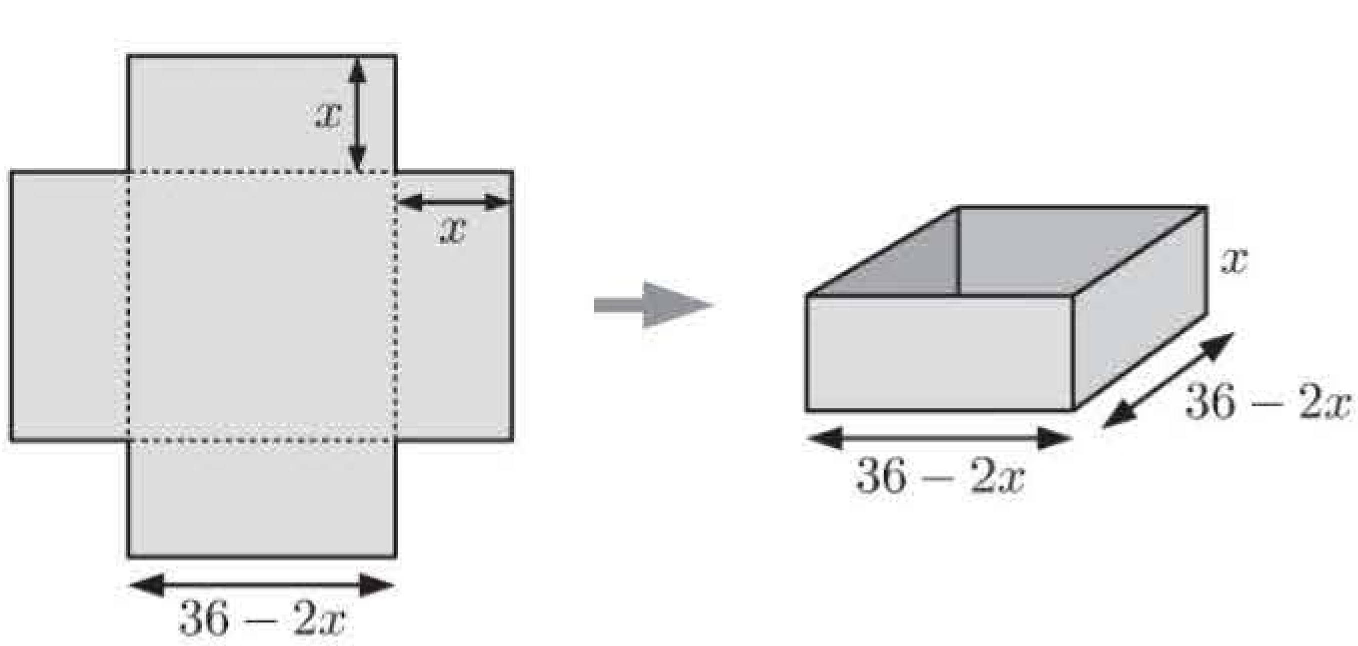
The sign diagram of $\dfrac{dA}{d\theta}$ is:



Since $0 < \theta < \pi$, A is maximised when $\theta \approx 1.91$.

When $\theta \approx 1.91$, $A \approx 237$ $\therefore$ the maximum area is 237 cm^2.

13 **a**



The volume of the container is

V = length × width × depth

$= x(36 - 2x)(36 - 2x)$

$\therefore$ $V = x(36 - 2x)^2$ cm^3

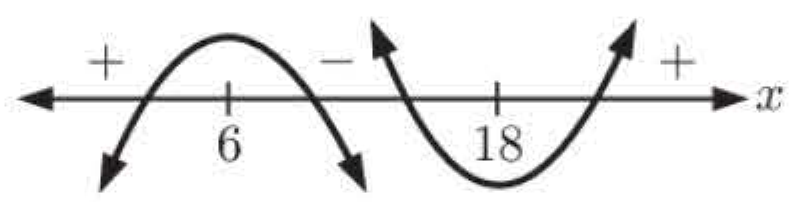
b The container will have the greatest capacity when $V(x)$ is maximised.

Using the product rule,

$$\begin{aligned} V'(x) &= (36-2x)^2 - 4x(36-2x) \\ &= (36-2x)[(36-2x)-4x] \\ &= (36-2x)(36-6x) \end{aligned}$$

$\therefore\ V'(x) = 0$ when $x = 6$ or $x = 18$

Sign diagram of $V'(x)$ is:



$\therefore$ the volume is maximised when $x = 6$ cm $\{0 \leqslant x < 18\}$

So, 6 cm × 6 cm squares should be cut out to maximise the capacity.

c The metal sheet is a cm by a cm and squares which are x cm by x cm are cut out.

$$\begin{aligned} V &= \text{length} \times \text{width} \times \text{depth} \\ &= (a-2x)\times(a-2x)\times x \end{aligned}$$

$\therefore\ V = x(a-2x)^2$

Now $$\begin{aligned} V'(x) &= (a-2x)^2 - 4x(a-2x) \\ &= (a-2x)[(a-2x)-4x] \\ &= (a-2x)(a-6x) \end{aligned}$$

$\therefore\ V'(x) = 0$ when $x = \dfrac{a}{2}$ or $x = \dfrac{a}{6}$

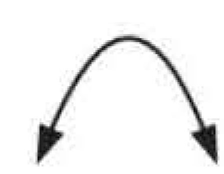
But $a - 2x > 0 \quad \therefore\ x < \dfrac{a}{2}$

So $x = \dfrac{a}{6}$ is the only value in $0 < x < \dfrac{a}{2}$ with $V'(x) = 0$.

Second derivative test:

$$\begin{aligned} V''(x) &= -2(a-6x) + (a-2x)(-6) \\ &= -2a + 12x - 6a + 12x \\ &= 24x - 8a \end{aligned}$$

$\therefore\ V''\left(\dfrac{a}{6}\right) = 4a - 8a = -4$ which is < 0

$\therefore$ the shape is  and we have a local maximum.

$\therefore$ the volume is maximised when $x = \dfrac{a}{6}$.

14 **a**

$$\begin{aligned} P &= 2\pi r + 2l \\ \therefore\ 400 &= 2\pi(x) + 2l \\ \therefore\ 200 &= \pi x + l \\ \therefore\ l &= 200 - \pi x \end{aligned}$$

Now clearly $x \geqslant 0$ and $l \geqslant 0$

$\therefore\ \pi x \leqslant 200$

$\therefore\ x \leqslant \frac{200}{\pi}$

So, $0 \leqslant x \leqslant \frac{200}{\pi}$

or $0 \leqslant x \leqslant 63.7$

b Area of shaded rectangle $A = 2xl\ \text{m}^2$

$$\begin{aligned} \therefore\ A &= 2x(200 - \pi x)\ \text{m}^2 \\ &= 400x - 2\pi x^2\ \text{m}^2 \end{aligned}$$

Now $\dfrac{dA}{dx} = 400 - 4\pi x$

which is 0 when $4\pi x = 400$

$\therefore\ x = \frac{100}{\pi} \approx 31.83$ m

and so $l = 200 - \pi\left(\frac{100}{\pi}\right)$

$= 100$ m

So, the maximum area of the rectangle is

$2 \times \frac{100}{\pi} \times 100 \approx 6366\ \text{m}^2$.

15

$$\begin{aligned} P(t) &= \frac{50\,000}{1+1000e^{-0.5t}}, \quad 0 \leqslant t \leqslant 25 \\ &= 50\,000(1+1000e^{-0.5t})^{-1} \\ \therefore\ P'(t) &= -50\,000(1+1000e^{-0.5t})^{-2}\left(-500e^{-0.5t}\right) \\ &= 2.5 \times 10^7 e^{-0.5t}(1+1000e^{-0.5t})^{-2} \end{aligned}$$

The wasp population is growing the fastest when $\dfrac{dP}{dt}$ is a maximum.

Using technology, the graph of $P'(t)$ can be drawn and the maximum obtained.

The maximum occurs when $t \approx 13.8$ weeks.

16 $E(t) = 750te^{-1.5t}$

$\therefore\ E'(t) = 750e^{-1.5t} + 750t \times -1.5e^{-1.5t}$

$= 750e^{-1.5t} - 1125te^{-1.5t}$

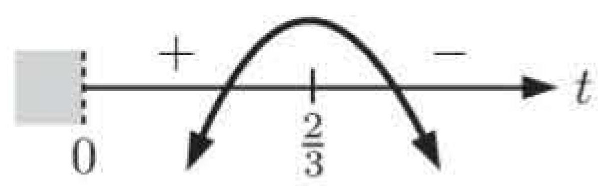
$= e^{-1.5t}(750 - 1125t)$

Now $E'(t) = 0$ when $750 - 1125t = 0$ {since $e^{-1.5t} > 0$ for all $t \in \mathbb{R}$}

$\therefore\ 1125t = 750$

$\therefore\ t = \frac{2}{3}$ hours (or 40 minutes)

Sign diagram for $E'(t)$:



So, the drug is most effective 40 minutes after the injection.

17 **a** $\text{AB} = x$ m

$\therefore\ \text{BC} = (24 - x)$ m $\qquad \therefore\ D(x) = \sqrt{x^2 + (24 - x)^2}$ {Pythagoras}

b $[D(x)]^2 = x^2 + (24 - x)^2$

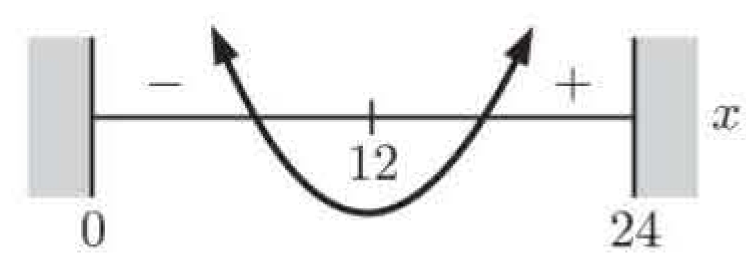
$= x^2 + 576 - 48x + x^2$

$= 2x^2 - 48x + 576$

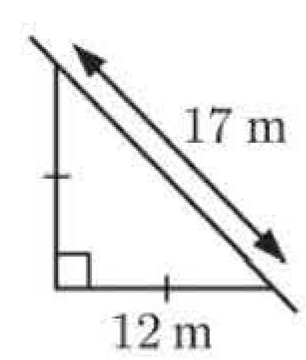
$\therefore\ \dfrac{d[D(x)]^2}{dx} = 4x - 48$

$\therefore\ \dfrac{d[D(x)]^2}{dx} = 0$ when $x = 12$

Sign diagram for $\dfrac{d[D(x)]^2}{dx}$:



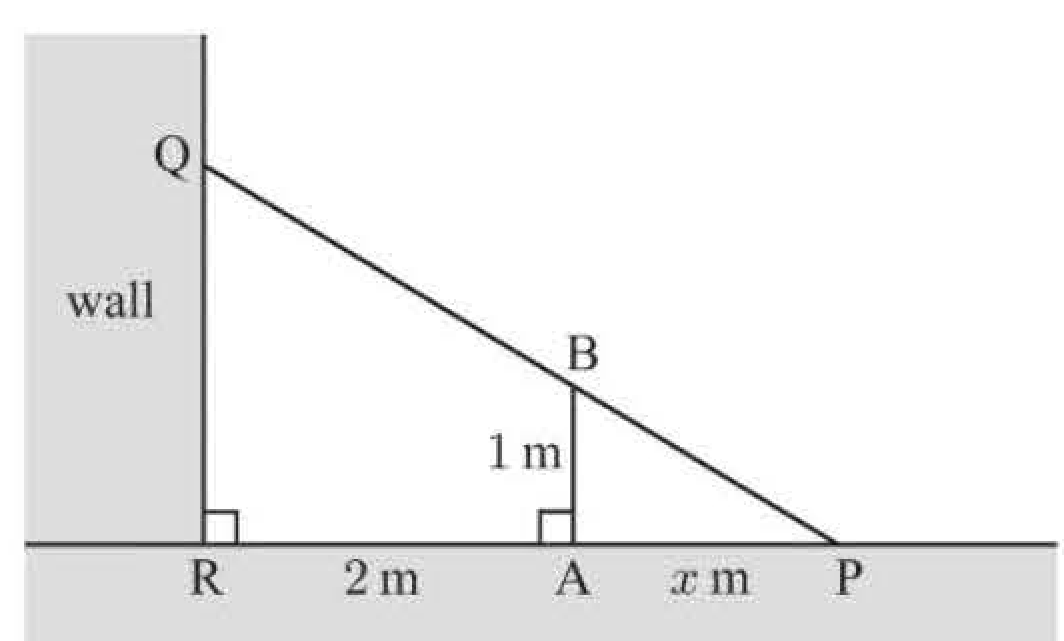
c When $\text{AB} = \text{BC} = 12$ m, $D(x)$ is a minimum, and the minimum $D(x) = 12\sqrt{2}$ m ≈ 17.0 m.



18 **a** $\triangle$s PAB and PRQ are similar.

$\therefore\ \dfrac{\text{PA}}{\text{PR}} = \dfrac{\text{PB}}{\text{PQ}} = \dfrac{\text{AB}}{\text{RQ}}$

$\therefore\ \dfrac{x}{x+2} = \dfrac{1}{\text{QR}}$ and $\therefore\ \text{QR} = \dfrac{x+2}{x}$



b Now $[L(x)]^2 = \text{RP}^2 + \text{QR}^2$ {Pythagoras}

$= (x+2)^2 + \left(\dfrac{x+2}{x}\right)^2$

$= (x+2)^2 \times 1 + (x+2)^2 \times \dfrac{1}{x^2}$

$\therefore\ [L(x)]^2 = (x+2)^2\left(1 + \dfrac{1}{x^2}\right)$ as required

c $[L(x)]^2 = (x+2)^2(1 + x^{-2})$

$\therefore\ \dfrac{d[L(x)]^2}{dx} = 2(x+2)(1 + x^{-2}) + (x+2)^2(-2x^{-3})$ {product rule}

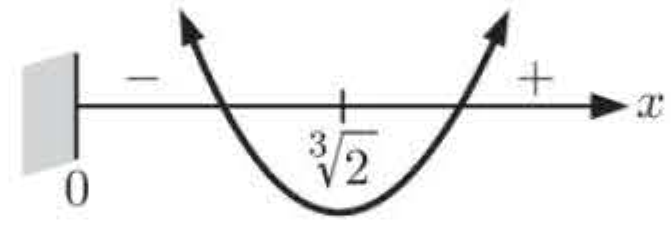
$= 2(x+2)(1 + x^{-2} - (x+2)x^{-3})$

$= 2(x+2)(1 + x^{-2} - x^{-2} - 2x^{-3})$

$= 2(x+2)\left(1 - \dfrac{2}{x^3}\right)$

$= \dfrac{2(x+2)(x^3 - 2)}{x^3}$

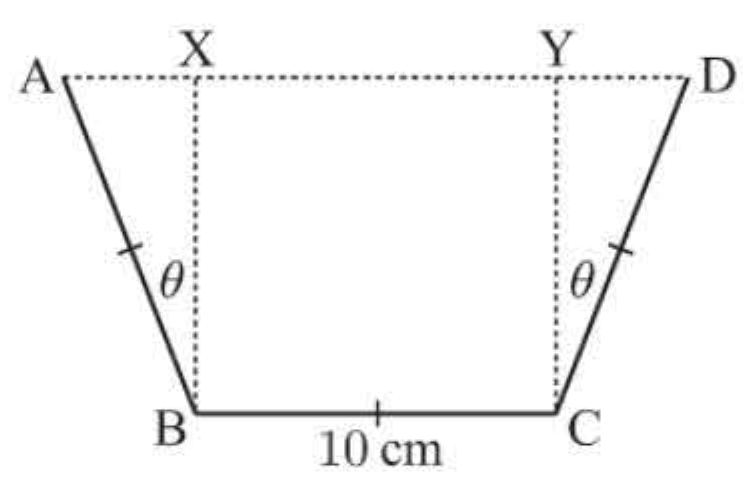
$\therefore\ \dfrac{d[L(x)]^2}{dx} = 0$ when $x = \sqrt[3]{2} \approx 1.2599$ {as $x > 0$ and $L(x) > 0$}

d Sign diagram of $\dfrac{d\,[L(x)]^2}{dx}$ is: 

$\therefore$ the ladder is shortest when $x = \sqrt[3]{2}$ m and

its length at this time is $L = \sqrt{(x+2)^2\left(1+\dfrac{1}{x^2}\right)} = \sqrt{(\sqrt[3]{2}+2)^2\left(1+\dfrac{1}{2^{\frac{2}{3}}}\right)} \approx 4.16$ m

19 **a**



The triangles have height $10\cos\theta$ and width $10\sin\theta$.

$\therefore$ area A

$= $ area of $\triangle$s + area of rectangle

$= 2 \times \frac{1}{2} \times 10\cos\theta \times 10\sin\theta + 10 \times 10\cos\theta$

$= 100\sin\theta\cos\theta + 100\cos\theta$

$= 100\cos\theta(1+\sin\theta)$

b $\dfrac{dA}{d\theta} = 100(-\sin\theta(1+\sin\theta) + \cos\theta \times \cos\theta)$

$= 100(-\sin\theta - \sin^2\theta + \cos^2\theta)$

$= 100(-\sin\theta - \sin^2\theta + 1 - \sin^2\theta)$

$= -100(2\sin^2\theta + \sin\theta - 1)$

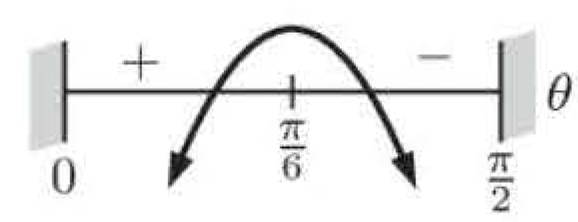
$= -100(2\sin\theta - 1)(\sin\theta + 1)$

$\therefore \dfrac{dA}{d\theta} = 0$ when $2\sin\theta - 1 = 0$ or $\sin\theta + 1 = 0$

$\therefore \sin\theta = \frac{1}{2}$ or $\sin\theta = -1$

c The maximum carrying capacity occurs when A is maximised.

Using **b**, $\dfrac{dA}{d\theta} = 0$ when $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$, or $\frac{3\pi}{2}$.

But $0 \leqslant \theta \leqslant \frac{\pi}{2}$, so the sign diagram for $\dfrac{dA}{d\theta}$ is: 

So, the maximum area occurs when $\theta = \frac{\pi}{6} = 30°$

When $\theta = 30°$, $A = 100\cos 30°(1+\sin 30°)$

$= 100 \times \frac{\sqrt{3}}{2} \times \frac{3}{2}$

$= 75\sqrt{3}$

≈ 130 cm^2

$\therefore$ the maximum cross-sectional area is 130 cm^2.

20 **a** Arc AC $= \dfrac{\theta}{360} \times (2\pi r_{\text{sector}})$

$= \dfrac{\theta}{360}(2 \times \pi \times 10)$

$= \dfrac{\theta\pi}{18}$

b Now arc AC forms the base of the cone.

$\therefore 2\pi r = \dfrac{\theta\pi}{18}$ {from **a**}

$\therefore r = \dfrac{\theta}{36}$

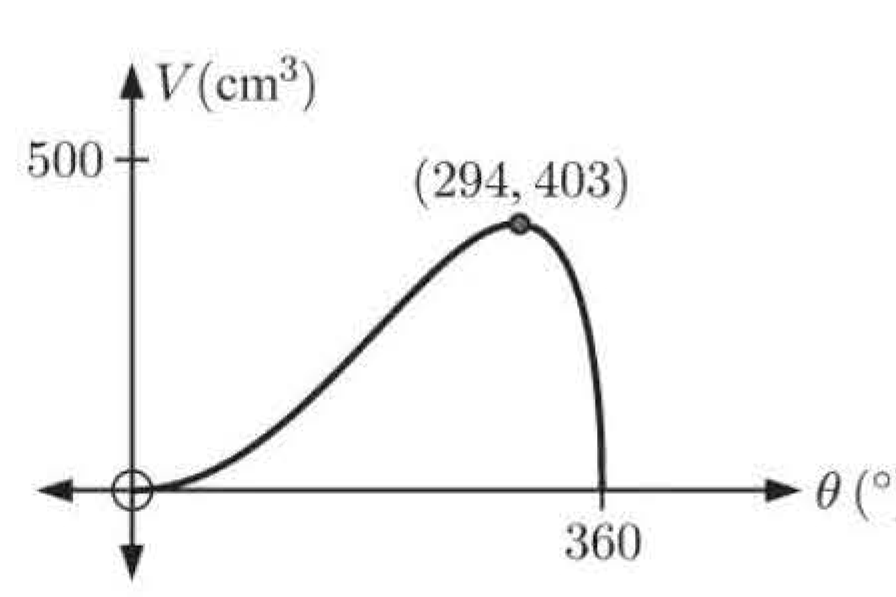
c Height of cone $= \sqrt{10^2 - r^2}$ {Pythagoras}

$\therefore h = \sqrt{100 - \left(\dfrac{\theta}{36}\right)^2}$

d $V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi\left(\dfrac{\theta}{36}\right)^2\sqrt{100 - \left(\dfrac{\theta}{36}\right)^2}$

e



f $V(\theta) = \frac{1}{3}\pi\left(\frac{\theta}{36}\right)^2\sqrt{100 - \left(\frac{\theta}{36}\right)^2}$

$= \dfrac{\pi\theta^2}{3 \times 36^2}\sqrt{\dfrac{100 \times 36^2 - \theta^2}{36^2}}$

$= \dfrac{\pi\theta^2}{3888} \times \dfrac{1}{36}\sqrt{129\,600 - \theta^2}$

$= \dfrac{\pi\theta^2}{139\,968}\sqrt{129\,600 - \theta^2}$

Now $V'(\theta) = \dfrac{2\pi\theta}{139\,968}\left(129\,600 - \theta^2\right)^{\frac{1}{2}} + \dfrac{\pi\theta^2}{139\,968}\left(\frac{1}{2}\right)\left(129\,600 - \theta^2\right)^{-\frac{1}{2}}(-2\theta)$ {product rule}

$= \dfrac{\pi\theta}{139\,968}\left(\dfrac{2\sqrt{129\,600 - \theta^2}}{1} - \dfrac{\theta^2}{\sqrt{129\,600 - \theta^2}}\right)$

$= \dfrac{\pi\theta}{139\,968}\left(\dfrac{2(129\,600 - \theta^2) - \theta^2}{\sqrt{129\,600 - \theta^2}}\right)$

and $V'(\theta) = 0$ when $\theta = 0$ or $2(129\,600 - \theta^2) = \theta^2$

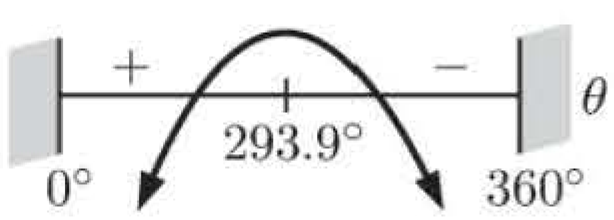
$259\,200 - 2\theta^2 = \theta^2$

$\therefore\ 3\theta^2 = 259\,200$

$\therefore\ \theta = \sqrt{86\,400}$ {as $\theta > 0$}

$\therefore\ \theta \approx 293.9°$

Sign diagram of $V'(\theta)$ is:

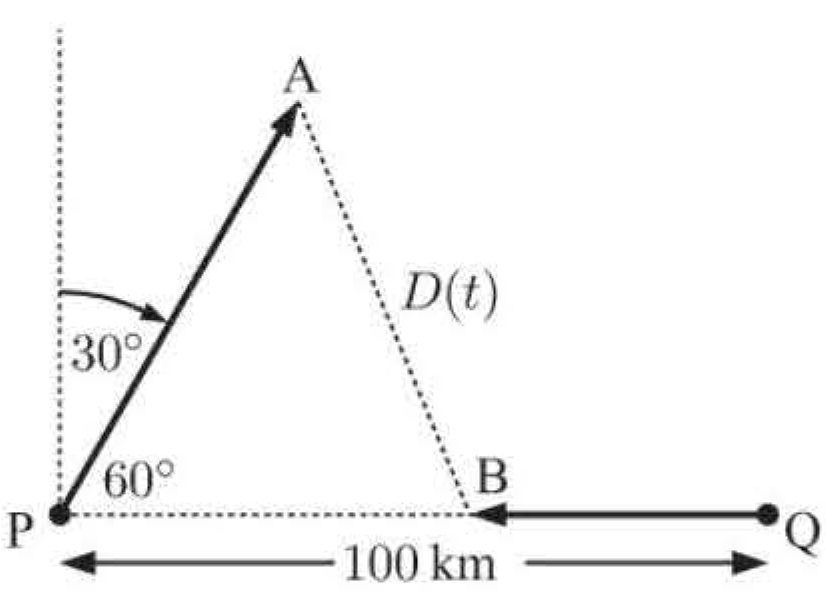


$\therefore$ maximum V occurs when $\theta \approx 294°$.

21 **a** Consider each boat's position t hours after 1:00 pm.

$PA = 12t$ and $QB = 8t$

$\therefore\ PB = 100 - 8t$



Using the cosine rule in $\triangle PAB$,

$D(t)^2 = AP^2 + BP^2 - 2AP \times BP\cos 60°$

$= (12t)^2 + (100 - 8t)^2 - 2(12t)(100 - 8t)\frac{1}{2}$

$= 144t^2 + (100 - 8t)^2 - 12t(100 - 8t)$

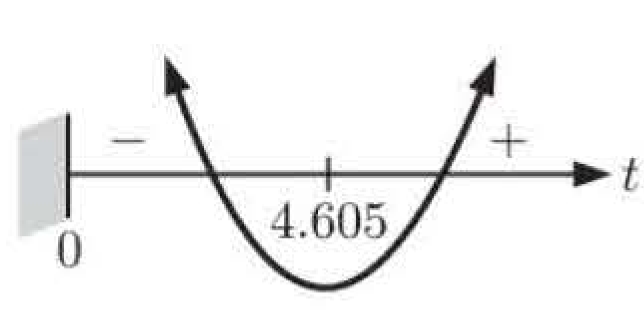
$= 144t^2 + 10\,000 - 1600t + 64t^2 - 1200t + 96t^2$

$= 304t^2 - 2800t + 10\,000$

$\therefore\ D(t) = \sqrt{304t^2 - 2800t + 10\,000}$

b Now $\dfrac{d[D(t)]^2}{dt} = 608t - 2800$

$\therefore\ \dfrac{d[D(t)]^2}{dt} = 0$ when $t = \dfrac{2800}{608} \approx 4.605\,26$



$\therefore\ D(t)$ is a minimum when $t \approx 4.605\,26$ hours after 1:00 pm

and $[D(t)]^2_{\text{min}} \approx 304\,(4.6053)^2 - 2800\,(4.6053) + 10\,000$

$\therefore\ [D(t)]^2_{\text{min}} \approx 3553\ \text{km}^2$

c The ships are closest when $t = 4.605\,26$ hours which occurs when the time is 4 hours 36 minutes after 1:00 pm.
So, the ships are closest at approximately 5:36 pm.

22 **a** X must lie either between A and C or else at one of the two points.
If $x = 0$, then he rows straight to the shore and runs to C.
If $x = 6$, then he rows straight to C. $\therefore\ 0 \leqslant x \leqslant 6$

b Now $\text{XC} = 6 - x$

$\therefore$ the time to row from B to $\text{X} = \dfrac{\text{BX}}{8} = \dfrac{\sqrt{5^2 + x^2}}{8}$

and the time to run from X to $\text{C} = \dfrac{\text{XC}}{17} = \dfrac{6 - x}{17}$

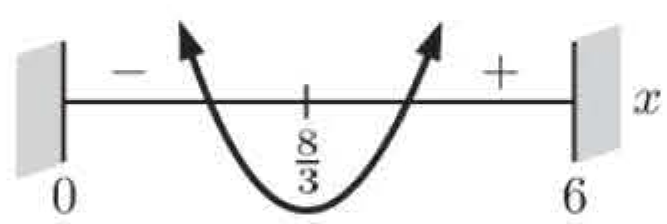
$$\therefore \text{ the total time } T(x) = \frac{\sqrt{25 + x^2}}{8} + \frac{6 - x}{17} \text{ hours}, \quad 0 \leqslant x \leqslant 6$$
$$= \tfrac{1}{8}(25 + x^2)^{\frac{1}{2}} + \tfrac{6}{17} - \frac{x}{17}$$

c $$\frac{dT}{dx} = \tfrac{1}{16}(25 + x^2)^{-\frac{1}{2}}(2x) - \tfrac{1}{17}$$
$$= \frac{x}{8\sqrt{25 + x^2}} - \tfrac{1}{17}$$

So, $\dfrac{dT}{dx} = 0$ when $\dfrac{x}{8\sqrt{25 + x^2}} = \tfrac{1}{17}$

$$17x = 8\sqrt{25 + x^2}$$
$$\therefore \ 289x^2 = 64(25 + x^2)$$
$$\therefore \ 225x^2 = 1600$$
$$\therefore \ x^2 = \tfrac{1600}{225}$$
$$\therefore \ x = \tfrac{40}{15}$$
$$= \tfrac{8}{3} \approx 2.67 \text{ km}$$

Sign diagram of $\dfrac{dT}{dx}$:



The time taken is minimised if Peter aims for X such that $x \approx 2.67$ km.

23 Let $\text{MX} = x$ km, so $\text{XN} = 5 - x$ km

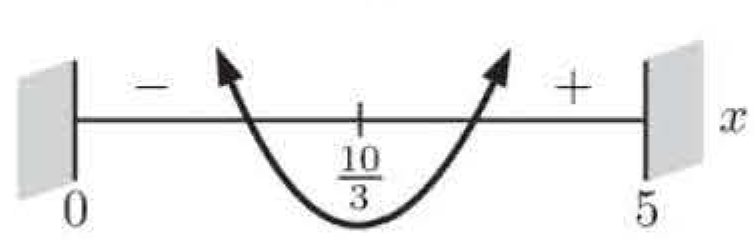
$\therefore$ $\text{AX} = \sqrt{4 + x^2}$ km and $\text{XB} = \sqrt{1 + (5 - x)^2}$ km {Pythagoras}

Now $P = \text{AX} + \text{XB}$
$$= (4 + x^2)^{\frac{1}{2}} + (26 - 10x + x^2)^{\frac{1}{2}}$$
$$\therefore \ \frac{dP}{dx} = \tfrac{1}{2}(4 + x^2)^{-\frac{1}{2}}(2x) + \tfrac{1}{2}(26 - 10x + x^2)^{-\frac{1}{2}}(2x - 10)$$
$$= \frac{x}{\sqrt{4 + x^2}} + \frac{x - 5}{\sqrt{x^2 - 10x + 26}}$$

Now $\dfrac{dP}{dx} = 0$ when $\dfrac{x}{\sqrt{4 + x^2}} = \dfrac{5 - x}{\sqrt{x^2 - 10x + 26}}$

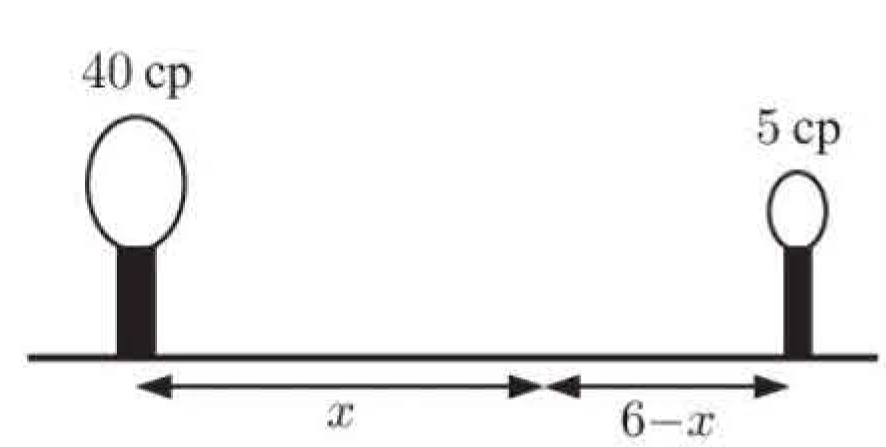
$$\therefore \ \frac{x^2}{4 + x^2} = \frac{(5 - x)^2}{x^2 - 10x + 26} \qquad \text{\{squaring both sides\}}$$
$$\therefore \ x^2(x^2 - 10x + 26) = (4 + x^2)(25 - 10x + x^2)$$
$$\therefore \ x^4 - 10x^3 + 26x^2 = 100 - 40x + 4x^2 + 25x^2 - 10x^3 + x^4$$
$$\therefore \ 3x^2 - 40x + 100 = 0$$
$$\therefore \ (3x - 10)(x - 10) = 0$$
$$\therefore \ x = \tfrac{10}{3} \approx 3.33 \qquad \text{\{as } x \text{ cannot be 10\}}$$

Sign diagram of $\dfrac{dP}{dx}$ is:



$\therefore$ the minimum length pipeline occurs when $x \approx 3.33$ km.

24 Now $I \propto \dfrac{s}{d^2}$ where s is the power of the source and d is the distance from it



$\therefore$ $I = \dfrac{ks}{d^2}$ where k is a constant

So, the intensity due to the 40 cp bulb $= \dfrac{40k}{x^2}$

and the intensity due to the 5 cp bulb $= \dfrac{5k}{(6 - x)^2}$

The total intensity $I = \dfrac{40k}{x^2} + \dfrac{5k}{(6-x)^2}$
$= k[40x^{-2} + 5(6-x)^{-2}]$

$\therefore \dfrac{dI}{dx} = k[-80x^{-3} - 10(6-x)^{-3}(-1)]$
$= k\left[\dfrac{-80}{x^3} + \dfrac{10}{(6-x)^3}\right]$

$\therefore \dfrac{dI}{dx} = 0$ when $\dfrac{80}{x^3} = \dfrac{10}{(6-x)^3}$

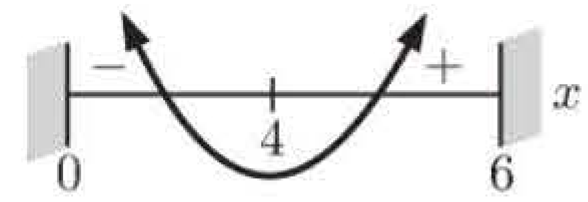
$\therefore 8(6-x)^3 = x^3$

$\therefore 2(6-x) = x$ {finding cube roots}

$\therefore 12 - 2x = x$

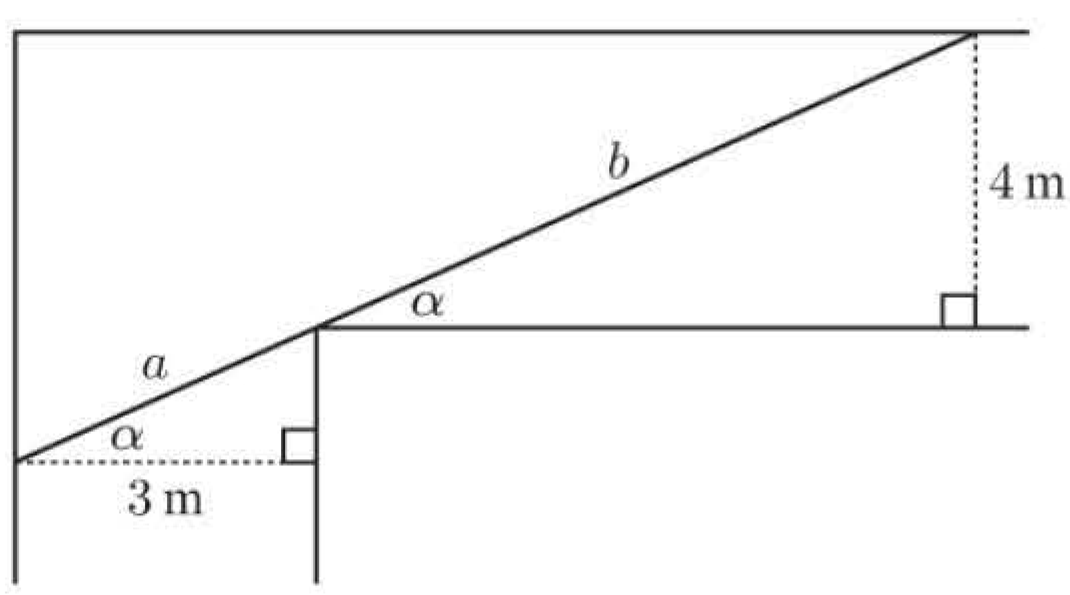
$\therefore x = 4$

Sign diagram of $\dfrac{dI}{dx}$ is:



$\therefore$ the darkest point occurs 4 m from the 40 cp lamp.

25



$\cos\alpha = \dfrac{3}{a}$ and $\sin\alpha = \dfrac{4}{b}$

$\therefore a = \dfrac{3}{\cos\alpha}$ and $b = \dfrac{4}{\sin\alpha}$

Now $L = a + b$

$\therefore L = \dfrac{3}{\cos\alpha} + \dfrac{4}{\sin\alpha}$
$= 3(\cos\alpha)^{-1} + 4(\sin\alpha)^{-1}$

$\therefore \dfrac{dL}{d\alpha} = -3(\cos\alpha)^{-2} \times (-\sin\alpha) - 4(\sin\alpha)^{-2} \times \cos\alpha$
$= \dfrac{3\sin\alpha}{\cos^2\alpha} - \dfrac{4\cos\alpha}{\sin^2\alpha}$
$= \dfrac{3\sin^3\alpha - 4\cos^3\alpha}{\cos^2\alpha\sin^2\alpha}$

$\therefore \dfrac{dL}{d\alpha} = 0$

when $3\sin^3\alpha - 4\cos^3\alpha = 0$

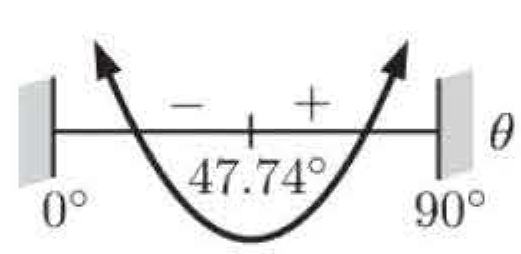
$\therefore 3\sin^3\alpha = 4\cos^3\alpha$

$\therefore \tan^3\alpha = \frac{4}{3}$

$\therefore \tan\alpha = \sqrt[3]{\frac{4}{3}}$

$\therefore \alpha \approx 47.74°$

Sign diagram of $\dfrac{dL}{d\alpha}$ is:



$\therefore$ L is minimised when $\alpha \approx 47.74°$ and $L = \dfrac{3}{\cos\alpha} + \dfrac{4}{\sin\alpha} \approx 9.87$ m

So, the maximum possible length of the X-ray screen is 9.87 m.

26 **a** $\tan\theta = \dfrac{2}{\text{AX}}$ and $\sin\theta = \dfrac{2}{\text{BX}}$

$\therefore \text{AX} = \dfrac{2}{\tan\theta} = \dfrac{2\cos\theta}{\sin\theta}$ and $\text{BX} = \dfrac{2}{\sin\theta}$

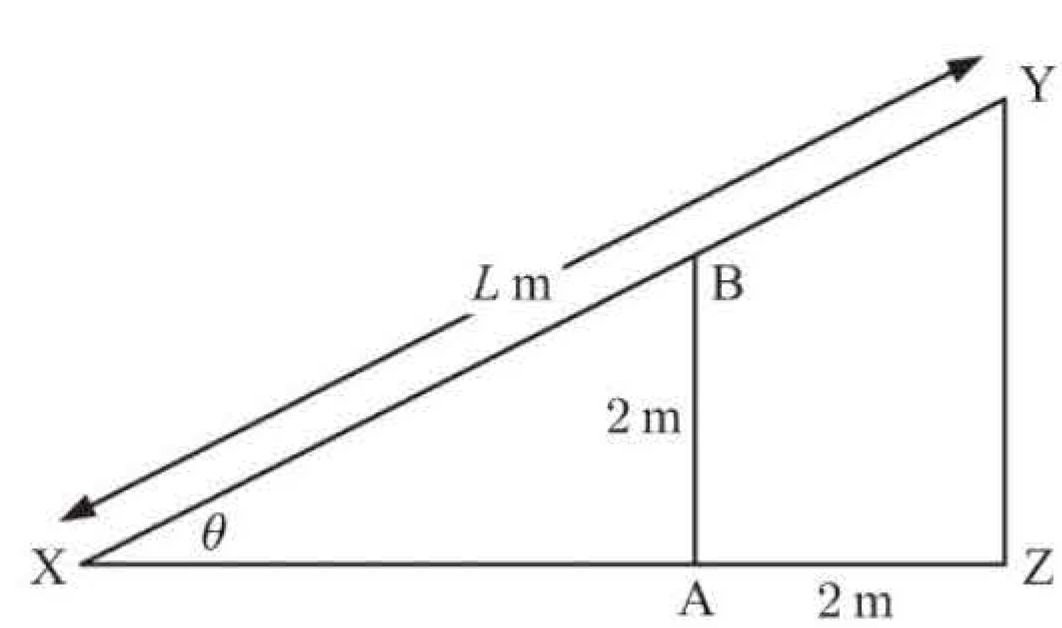
From the similar $\triangle$s, $\dfrac{L}{\text{BX}} = \dfrac{\text{AX}+2}{\text{AX}} = 1 + \dfrac{2}{\text{AX}}$

$\therefore L = \text{BX} + \dfrac{2\text{BX}}{\text{AX}}$

$= \dfrac{2}{\sin\theta} + \dfrac{2\left(\dfrac{2}{\sin\theta}\right)}{\left(\dfrac{2\cos\theta}{\sin\theta}\right)}$

$= \dfrac{2}{\sin\theta} + 2\left(\dfrac{2}{\sin\theta}\right)\left(\dfrac{\sin\theta}{2\cos\theta}\right)$

$= 2\sec\theta + 2\csc\theta$, as required

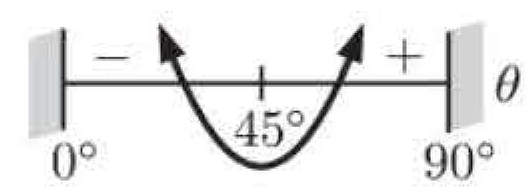


b Now $L = 2\sec\theta + 2\csc\theta$

$$\therefore \quad \frac{dL}{d\theta} = 2\sec\theta\tan\theta - 2\csc\theta\cot\theta$$
$$= \frac{2\sin\theta}{\cos^2\theta} - \frac{2\cos\theta}{\sin^2\theta}$$
$$= \frac{2\sin^3\theta - 2\cos^3\theta}{\sin^2\theta\cos^2\theta}$$

c Now $\dfrac{dL}{d\theta} = 0$ when $2\sin^3\theta - 2\cos^3\theta = 0$

$$\therefore \quad 2\sin^3\theta = 2\cos^3\theta$$
$$\therefore \quad \tan^3\theta = 1$$
$$\therefore \quad \tan\theta = 1$$
$$\therefore \text{ since } 0 < \theta < 90^\circ, \quad \theta = 45^\circ$$

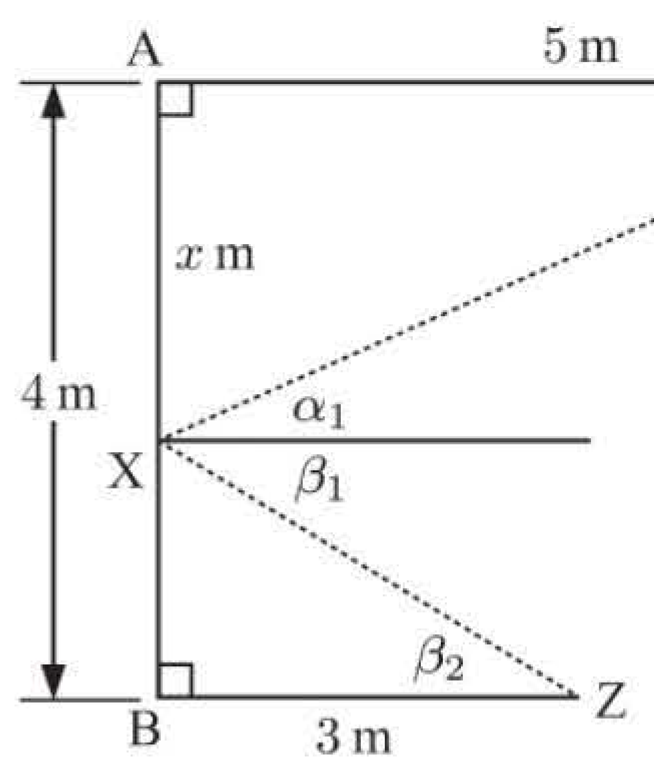
Sign diagram of $\dfrac{dL}{d\theta}$ is: 

$\therefore$ the ladder is shortest when $\theta = 45^\circ$

$\therefore$ $\sec\theta = \sqrt{2}$ and $\csc\theta = \sqrt{2}$

$\therefore$ $L_{\min} = 2\sqrt{2} + 2\sqrt{2} = 4\sqrt{2}$ m

27



Now $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$ {alternate angles}

$\therefore$ $\theta = \alpha + \beta$

Let AX $= x$ m

$\therefore$ XB $= (4 - x)$ m

$\therefore$ $\tan\alpha = \dfrac{x}{5}$ and $\tan\beta = \dfrac{4-x}{3}$

Now $\tan\theta = \tan(\alpha+\beta)$

$$= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$
$$= \frac{\frac{x}{5} + \frac{4-x}{3}}{1 - \frac{x}{5}\left(\frac{4-x}{3}\right)} \times \frac{15}{15}$$
$$= \frac{3x + 20 - 5x}{15 - x(4-x)} = \frac{20 - 2x}{x^2 - 4x + 15}$$

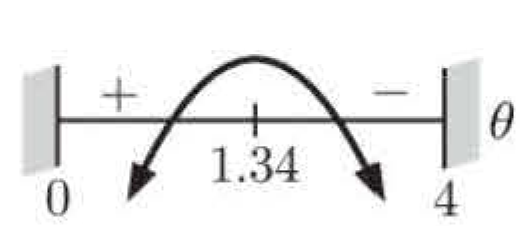
Differentiating both sides with respect to x

$$\sec^2\theta\,\frac{d\theta}{dx} = \frac{-2(x^2-4x+15) - (20-2x)(2x-4)}{(x^2-4x+15)^2} \quad \{\text{chain and quotient rules}\}$$
$$= \frac{-2x^2 + 8x - 30 - 40x + 80 + 4x^2 - 8x}{(x^2-4x+15)^2}$$
$$= \frac{2x^2 - 40x + 50}{(x^2-4x+15)^2}$$
$$\therefore \quad \frac{d\theta}{dx} = 2\cos^2\theta\,\frac{x^2 - 20x + 25}{(x^2-4x+15)^2}$$

Now by inspection, $\theta < 90^\circ$, so $\dfrac{d\theta}{dx} = 0$ when $x^2 - 20x + 25 = 0$

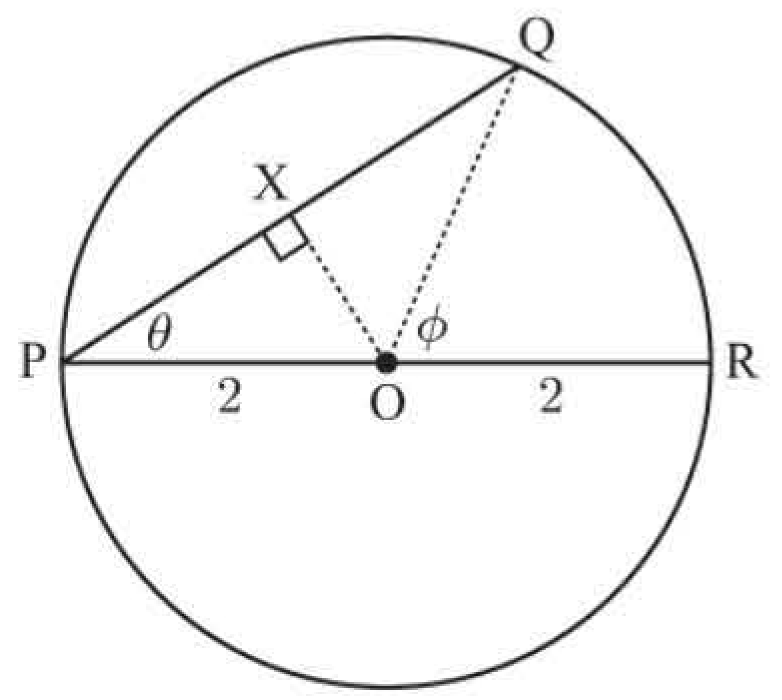
$\therefore$ $x \approx 1.3397$ or $x \approx 18.660$ (where 18.660 is not physically possible)

$\therefore$ $x \approx 1.34$ m from A

Sign diagram for $\dfrac{d\theta}{dx}$ is: 

$\therefore$ θ is a maximum when $x \approx 1.34$ m from A.

28



$\dfrac{PX}{2} = \cos\theta \quad \therefore \; PQ = 2PX = 4\cos\theta$

$\therefore$ the time taken to row from P to Q is $\dfrac{4\cos\theta}{3}$ hours

Now $\phi = 2\theta$ {angle at the centre}

But, arc length $QR_{arc} = 2\phi$

$\therefore \; QR_{arc} = 4\theta$

and the time taken to walk from Q to R is $\dfrac{4\theta}{5}$

$\therefore$ the total time from P to R, $T = \frac{4}{3}\cos\theta + \dfrac{4\theta}{5}$

$\therefore \; \dfrac{dT}{d\theta} = -\frac{4}{3}\sin\theta + \frac{4}{5}$

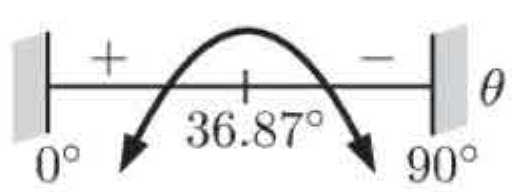
$\therefore \; \dfrac{dT}{d\theta} = 0$ when $-\frac{4}{3}\sin\theta = -\frac{4}{5}$

$\therefore \; \sin\theta = \frac{3}{5}$

$\therefore \; \theta \approx 0.6435$ radians

$\therefore \; \theta \approx 36.87°$

and the sign diagram of $\dfrac{dT}{d\theta}$ is:



So the maximum time occurs when $\theta \approx 36.9°$

and the maximum time is $\frac{4}{3}\cos 0.6435 + \frac{4}{5} \times 0.6435 \approx 1.581$ hours

≈ 1 hour 34 min 53 s

29 **a** $\tan\alpha = \dfrac{2}{x}$ and $\tan(\alpha+\theta) = \dfrac{3}{x}$

b Now $\theta = (\alpha+\theta) - \alpha = \arctan\left(\frac{3}{x}\right) - \arctan\left(\frac{2}{x}\right)$

c
$$\frac{d\theta}{dx} = \left(-\frac{3}{x^2}\right) \times \frac{1}{1+\left(\frac{3}{x}\right)^2} - \left(-\frac{2}{x^2}\right) \times \frac{1}{1+\left(\frac{2}{x}\right)^2}$$
$$= -\frac{3}{x^2+9} + \frac{2}{x^2+4}$$
$$= \frac{2}{x^2+4} - \frac{3}{x^2+9} \quad \text{as required}$$

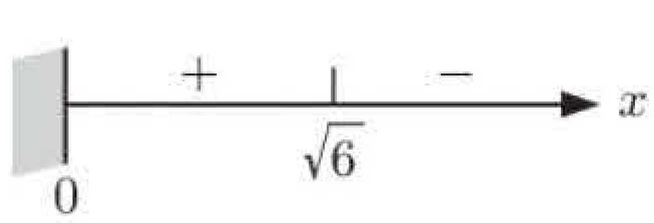
So, $\dfrac{d\theta}{dx} = 0$ when $2(x^2+9) - 3(x^2+4) = 0$

$\therefore \; 2x^2 + 18 - 3x^2 - 12 = 0$

$\therefore \; x^2 = 6$

$\therefore \; x = \sqrt{6} \quad \{x > 0\}$

and $\dfrac{d\theta}{dx}$ has sign diagram:



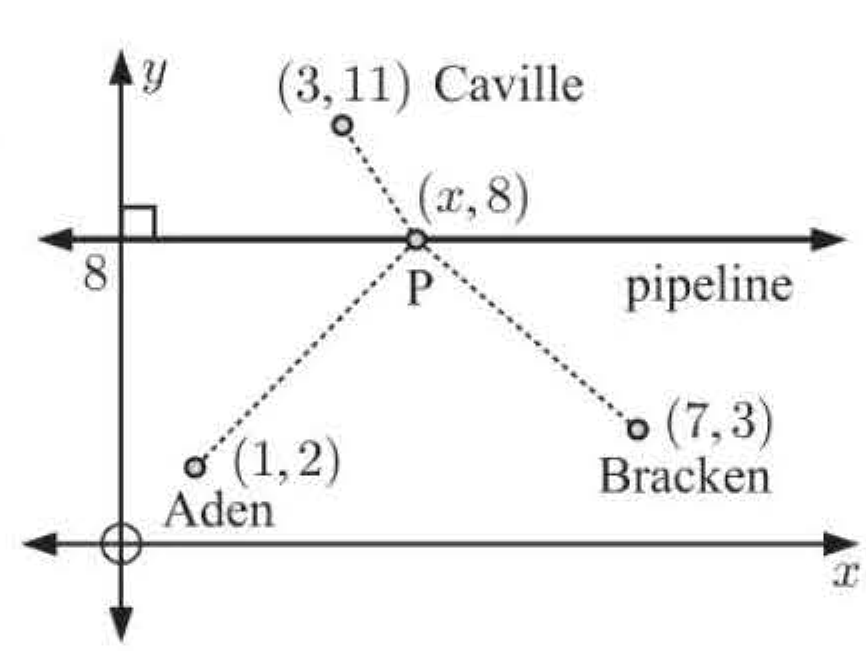
d The maximum viewing angle occurs when $x = \sqrt{6}$, which is when Sonia is $\sqrt{6}$ m from the wall.

30 Suppose P has coordinates $(x, 8)$

$\therefore \; CP = \sqrt{(x-3)^2 + (8-11)^2}$
$= \sqrt{x^2 - 6x + 9 + 9}$
$= \sqrt{x^2 - 6x + 18}$

$AP = \sqrt{(x-1)^2 + (8-2)^2}$
$= \sqrt{x^2 - 2x + 1 + 36}$
$= \sqrt{x^2 - 2x + 37}$

$BP = \sqrt{(x-7)^2 + (8-3)^2}$
$= \sqrt{x^2 - 14x + 49 + 25}$
$= \sqrt{x^2 - 14x + 74}$



$\therefore$ the length of pipeline $L = \sqrt{x^2 - 6x + 18} + \sqrt{x^2 - 2x + 37} + \sqrt{x^2 - 14x + 74}$

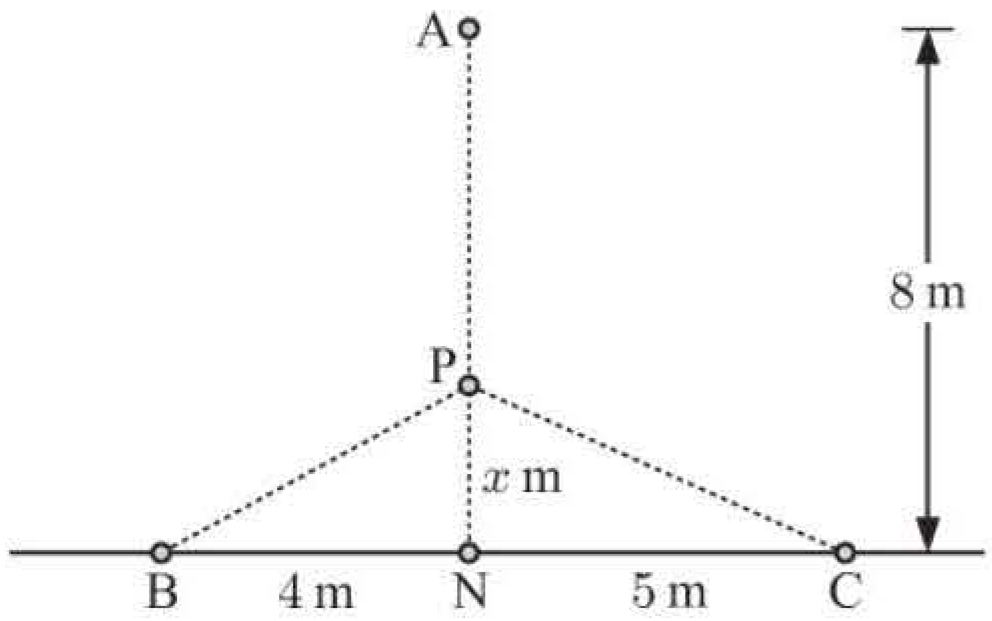
We use technology to graph L and to find its minimum. This occurs when $x \approx 3.543\,66$, and $L \approx 15.6$

$\therefore$ P is at $(3.54, 8)$ and the shortest total length of pipe required is 15.6 km.

31 Suppose $\text{PN} = x$ m

$\therefore$ length of cable $= \text{PA} + \text{PB} + \text{PC}$

$$\therefore \quad L = 8 - x + \sqrt{x^2+16} + \sqrt{x^2+25}$$



Using technology we graph and find the minimum value of the function.

$\therefore$ the minimum length occurs when $x \approx 2.577\,98$

$\therefore$ P should be ≈ 2.58 m from N.

32

$$r^2 + h^2 = s^2 \quad \{\text{Pythagoras}\}$$

$$\therefore \quad h^2 = s^2 - r^2$$

$$\therefore \quad h = \sqrt{s^2 - r^2}$$

But $V = \frac{1}{3}\pi r^2 h$

$$\therefore \quad V = \tfrac{1}{3}\pi r^2 \sqrt{s^2 - r^2}$$

$$\therefore \quad V^2 = \frac{\pi^2}{9} r^4 (s^2 - r^2)$$

$$= \frac{\pi^2}{9}\left(r^4 s^2 - r^6\right)$$

$$\therefore \quad \frac{d(V^2)}{dr} = \frac{\pi^2}{9}\left(4r^3 s^2 - 6r^5\right)$$

$$= \frac{\pi^2}{9} 2r^3 (2s^2 - 3r^2)$$

$\frac{d(V^2)}{dr} = 0$ when $2s^2 - 3r^2 = 0 \quad \{\text{as } r > 0\}$

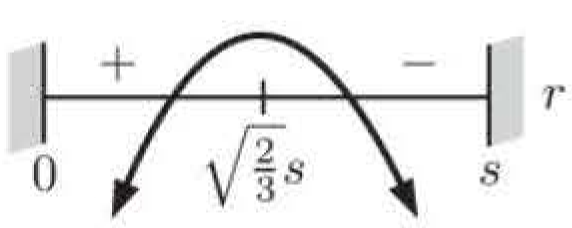
$$\therefore \quad 2s^2 = 3r^2$$

$$\therefore \quad \frac{s^2}{r^2} = \tfrac{3}{2}$$

$$\therefore \quad \frac{s}{r} = \sqrt{\tfrac{3}{2}}$$

$$\therefore \quad s : r = \sqrt{\tfrac{3}{2}} : 1$$

Sign diagram of $\frac{d(V^2)}{dr}$ is:



$\therefore$ V is a maximum when $s : r = \sqrt{\frac{3}{2}} : 1 = \sqrt{3} : \sqrt{2}$

33 **a**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\therefore \quad x^2 b^2 + y^2 a^2 = a^2 b^2 \quad \{\times \text{ by } a^2 b^2\}$$

$$\therefore \quad a^2 y^2 = a^2 b^2 - x^2 b^2$$

$$\therefore \quad y^2 = \frac{a^2 b^2 - x^2 b^2}{a^2}$$

$$\therefore \quad y = \pm\sqrt{\frac{a^2 b^2 - x^2 b^2}{a^2}}$$

Since A lies in Q_1, $y > 0$

$$\therefore \quad y = \sqrt{\left(\frac{b^2}{a^2}\right)(a^2 - x^2)}$$

$$\therefore \quad y = \frac{b}{a}\sqrt{a^2 - x^2}$$

b The seating area is $A = 2x \times 2y$

$$= 4xy$$

$$= 4x\left(\frac{b}{a}\sqrt{a^2 - x^2}\right)$$

$$\therefore \quad A(x) = \frac{4bx}{a}\sqrt{a^2 - x^2} \quad \text{as required}$$

$$\therefore \quad A^2 = \frac{16b^2 x^2}{a^2}(a^2 - x^2)$$

$$= \frac{16b^2}{a^2}(a^2 x^2 - x^4)$$

c

$$\frac{d(A^2)}{dx} = \frac{16b^2}{a^2}(2a^2 x - 4x^3)$$

which is 0 when

$$2a^2 x - 4x^3 = 0$$

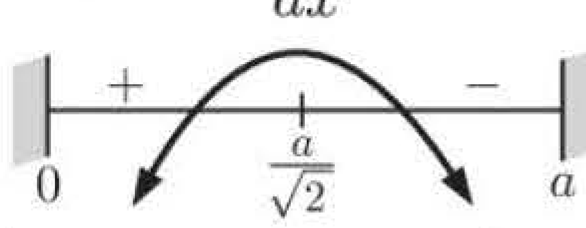
$$\therefore \quad 2x(a^2 - 2x^2) = 0$$

$$\therefore \quad 2x^2 = a^2 \quad \{\text{as } x > 0\}$$

$$\therefore \quad x = \pm\frac{a}{\sqrt{2}}$$

$$\therefore \quad x = \frac{a}{\sqrt{2}} \quad \{\text{as } x \text{ is in } Q_1\}$$

Sign diagram of $\frac{d(A^2)}{dx}$ is:



$\therefore$ maximum area occurs when $x = \frac{a}{\sqrt{2}}$

$$\text{Max. area} = \frac{4b}{a} \times \frac{a}{\sqrt{2}}\sqrt{a^2 - \left(\frac{a}{\sqrt{2}}\right)^2}$$

$$= \frac{4b}{\sqrt{2}} \times \sqrt{\frac{a^2}{2}}$$

$$= \frac{4b}{\sqrt{2}} \times \frac{a}{\sqrt{2}} = 2ab$$

d $\%$ occupied $= \dfrac{2ab}{\pi ab} \times 100\% = 63.7\%$

EXERCISE 20D

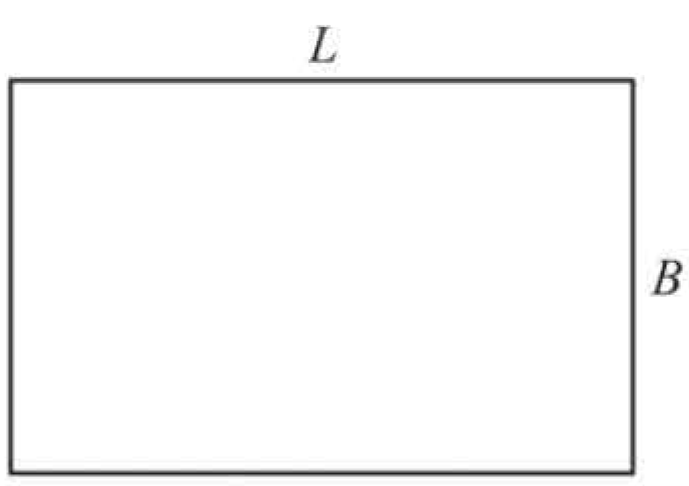
1 $ab^3 = 40 \quad \therefore \dfrac{da}{dt} b^3 + a(3b^2)\dfrac{db}{dt} = 0$

Particular case: When $a = 5$, $b = 2$, and $\dfrac{db}{dt} = +1$, so $8\dfrac{da}{dt} + 5(12)(1) = 0$

$$\therefore \frac{da}{dt} = -\tfrac{60}{8} = -7.5$$

$\therefore$ a decreases at 7.5 units per second

2



$A = LB = 100 \text{ cm}^2 \quad \therefore \dfrac{dL}{dt}B + L\dfrac{dB}{dt} = 0$

Particular case:

When a square, $L = B = 10$ cm and $\dfrac{dL}{dt} = -1$

$$\therefore 10\frac{dB}{dt} + 10(-1) = 0$$

$$\therefore \frac{dB}{dt} = 1 \text{ cm min}^{-1}$$

$\therefore$ the breadth is increasing at 1 cm min^{-1}.

3 $A = \pi r^2 \quad \therefore \dfrac{dA}{dt} = 2\pi r\dfrac{dr}{dt} = 2\pi r \quad \left\{\text{since } \dfrac{dr}{dt} = 1 \text{ m s}^{-1}\right\}$

Particular cases:

a When $t = 2$ and $r = 2$, $\dfrac{dA}{dt} = 2\pi(2) = 4\pi$ m^2 per second

b When $t = 4$ and $r = 4$, $\dfrac{dA}{dt} = 2\pi(4) = 8\pi$ m^2 per second

4 $V = \frac{4}{3}\pi r^3 \quad \therefore \dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt} = 6\pi \text{ m}^3 \text{ min}^{-1}$

$$\therefore \frac{dr}{dt} = \frac{6\pi}{4\pi r^2} = \frac{3}{2r^2} \text{ m min}^{-1}$$

Now $A = 4\pi r^2$

$$\therefore \frac{dA}{dt} = 8\pi r\frac{dr}{dt} = 8\pi r \times \frac{3}{2r^2}$$

Particular case: When $r = 2$, $\dfrac{dA}{dt} = \dfrac{8\pi \times 2 \times 3}{2 \times 4} \text{ m}^2 \text{ min}^{-1} = 6\pi \text{ m}^2 \text{ min}^{-1}$

$\therefore$ the surface area is increasing at 6π m^2 per minute.

5 $pV^{\frac{3}{2}} = 400 \quad \therefore \dfrac{dp}{dt}V^{\frac{3}{2}} + \frac{3}{2}pV^{\frac{1}{2}}\dfrac{dV}{dt} = 0$

Particular case: When $p = 50 \text{ Nm}^{-2}$, $V^{\frac{3}{2}} = 8$ and so $V = 4$

$$\therefore 3(8) + \tfrac{3}{2}(50)(2)\frac{dV}{dt} = 0 \qquad \left\{\text{as } \frac{dp}{dt} = +3 \text{ Nm}^{-2}\right\}$$

$$\therefore \frac{dV}{dt} = -\tfrac{24}{150} \text{ m}^3 \text{ min}^{-1}$$

$\therefore$ the volume is decreasing at 0.16 m^3 per minute.

6 $V = \frac{1}{3}\pi r^2 h$ and $r = 3h$

$\therefore\ V = \frac{1}{3}\pi(3h)^2 h = 3\pi h^3$ .... $(*)$

Particular case: After 1 min, the volume $V = 3\pi(20)^3\ \text{cm}^3 = 24\,000\pi\ \text{cm}^3$

$$\therefore\ \frac{dV}{dt} = 24\,000\pi\ \text{cm}^3\ \text{min}^{-1}$$

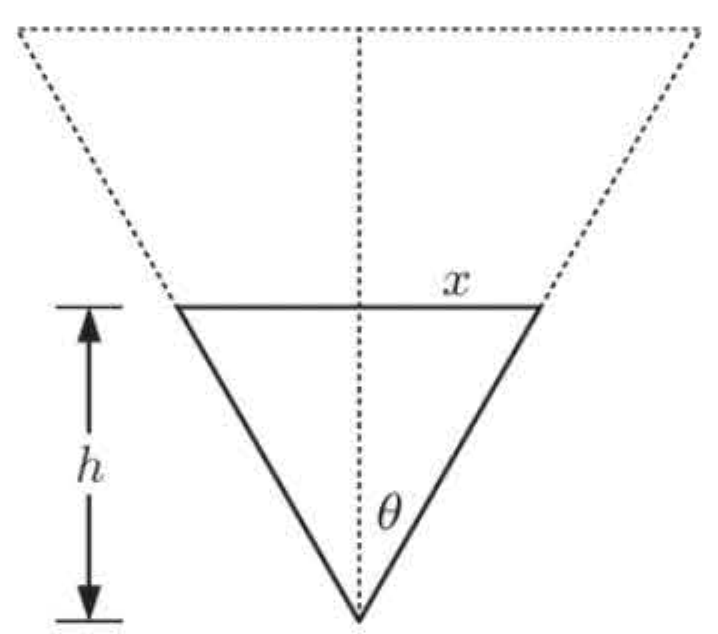
$$\text{But}\ \frac{dV}{dt} = 9\pi h^2 \frac{dh}{dt} \quad \{\text{from } (*)\}$$

$$\therefore\ \text{when}\ h = 20,\quad 24\,000\pi = 9\pi \times (20^2) \times \frac{dh}{dt}$$

$$\therefore\ \frac{dh}{dt} = \frac{24\,000\pi}{400 \times 9\pi} = \frac{20}{3}\ \text{cm min}^{-1}$$

$\therefore$ the height is rising at $\frac{20}{3}$ cm per minute.

7



$$\theta = 30^\circ \qquad \therefore\ \frac{x}{h} = \tan 30^\circ$$

$$\therefore\ x = h\tan 30^\circ = \frac{h}{\sqrt{3}}$$

$$\therefore\ V = \frac{h}{\sqrt{3}} \times h \times 600 = 200\sqrt{3}h^2\ \text{cm}^3$$

$$\therefore\ \frac{dV}{dt} = 400\sqrt{3}h\,\frac{dh}{dt}$$

Particular case: When $h = 20$, $-100\,000 = 400\sqrt{3}(20)\,\frac{dh}{dt}$ $\left\{\frac{dV}{dt} = -0.1\ \text{m}^3 = -100\,000\ \text{cm}^3\right\}$

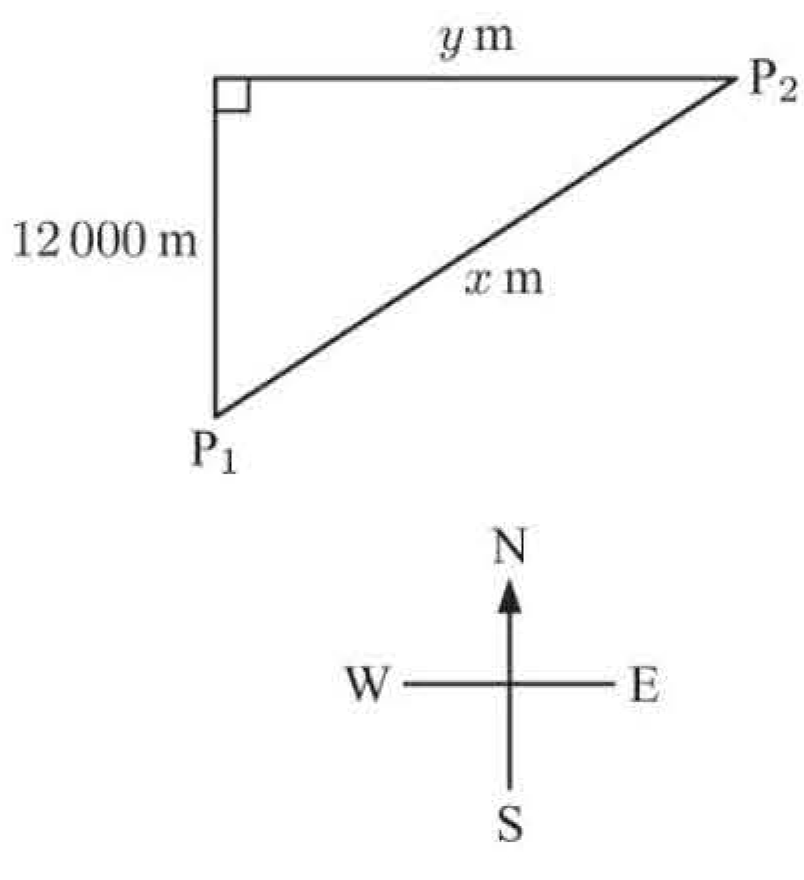
$$\therefore\ \frac{dh}{dt} = \frac{-100\,000}{400\sqrt{3} \times 20} = -\frac{25}{6}\sqrt{3}\ \text{cm min}^{-1}$$

$\therefore$ the water level is falling at $\frac{25\sqrt{3}}{6} \approx 7.22$ cm per minute

8 Let P_1 in the diagram be the faster jet and P_2 be the slower jet. Let y m be the distance that P_2 is ahead of P_1, and x m be the distance between them.

Now $x^2 = y^2 + (12\,000)^2$ {Pythagoras}

$$\therefore\ 2x\,\frac{dx}{dt} = 2y\,\frac{dy}{dt}$$



Particular case:

As P_1 is behind P_2, it is catching up at a rate of $50\ \text{m s}^{-1}$.

$$\therefore\ \frac{dy}{dt} = -50\ \text{m s}^{-1}$$

When $y = 5000$, $x = 13\,000$

$$\therefore\ 26\,000 \times \frac{dx}{dt} = 10\,000 \times (-50)$$

$$\frac{dx}{dt} = \frac{10}{26} \times (-50) = -\frac{250}{13}\ \text{m s}^{-1}$$

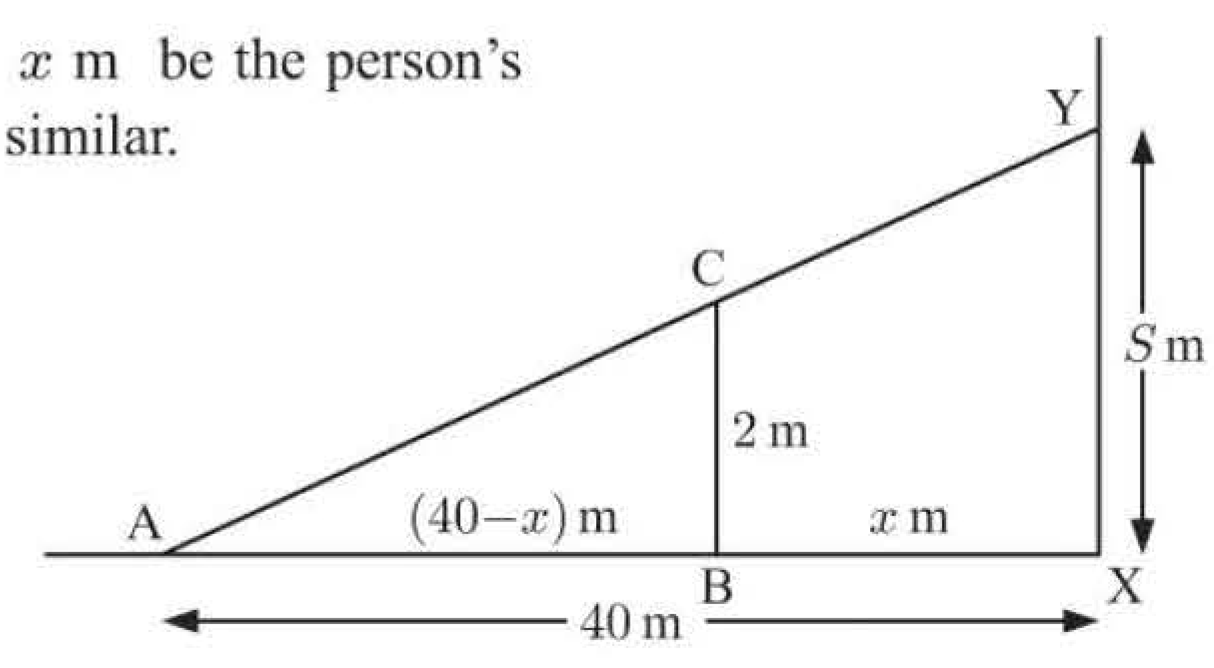
$\therefore$ their separation is decreasing at $\frac{250}{13} \approx 19.2\ \text{m s}^{-1}$.

9 Let S m be the height of the person's shadow and x m be the person's distance from the building. $\triangle$ABC and $\triangle$AXY are similar.

$$\therefore\ \frac{\text{AB}}{\text{AX}} = \frac{\text{BC}}{\text{XY}}$$

$$\therefore\ \frac{40 - x}{40} = \frac{2}{S}$$

$$\therefore\ S = \frac{80}{40 - x} = 80(40 - x)^{-1}$$



$\therefore \quad \dfrac{dS}{dt} = -80(40-x)^{-2}(-1)\dfrac{dx}{dt} = \dfrac{80}{(40-x)^2}\dfrac{dx}{dt}$

But $\dfrac{dx}{dt} = -1 \text{ m s}^{-1}$, so $\dfrac{dS}{dt} = -\dfrac{80}{(40-x)^2}$

Particular cases:

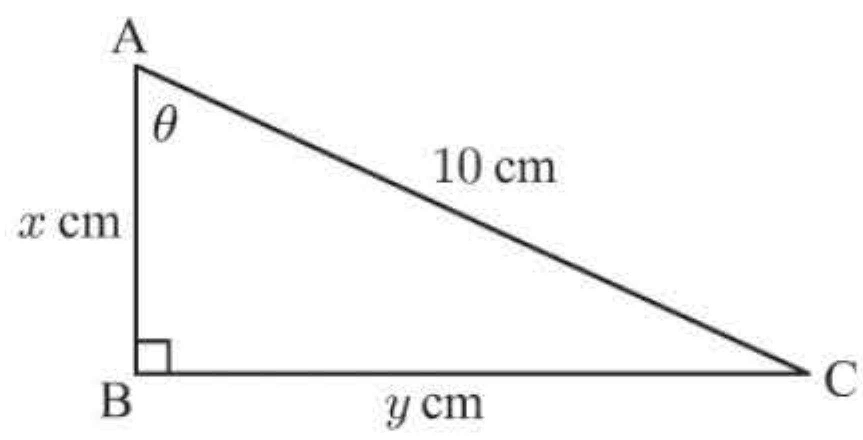
a When $x = 20$ m, $\dfrac{dS}{dt} = -\dfrac{80}{(40-20)^2} = -\frac{80}{400} = -0.2$

$\therefore$ the person's shadow is shortening at 0.2 m s^{-1}

b When $x = 10$ m, $\dfrac{dS}{dt} = -\dfrac{80}{(40-10)^2} = -\frac{80}{900} = -\frac{8}{90}$

$\therefore$ the person's shadow is shortening at $\frac{8}{90} \text{ m s}^{-1}$

10



$\cos\theta = \dfrac{x}{10}$

$\therefore \quad -\sin\theta\dfrac{d\theta}{dt} = \frac{1}{10}\dfrac{dx}{dt}$ {differentiating with respect to t}

If the length AB increases at 0.1 cm s^{-1}, $\dfrac{dx}{dt} = 0.1 \text{ cm s}^{-1}$

Particular case: When ABC is isosceles, $\theta = 45°$

$\therefore \quad \sin\theta = \frac{1}{\sqrt{2}}$

$\therefore \quad -\frac{1}{\sqrt{2}}\dfrac{d\theta}{dt} = \frac{1}{10} \times 0.1$

$\therefore \quad \dfrac{d\theta}{dt} = -\frac{\sqrt{2}}{100}$ radians s^{-1}

$\therefore$ $\widehat{CAB}$ is decreasing at $\frac{\sqrt{2}}{100}$ radians per second.

11

$\tan E = \dfrac{5000}{x} = 5000x^{-1}$

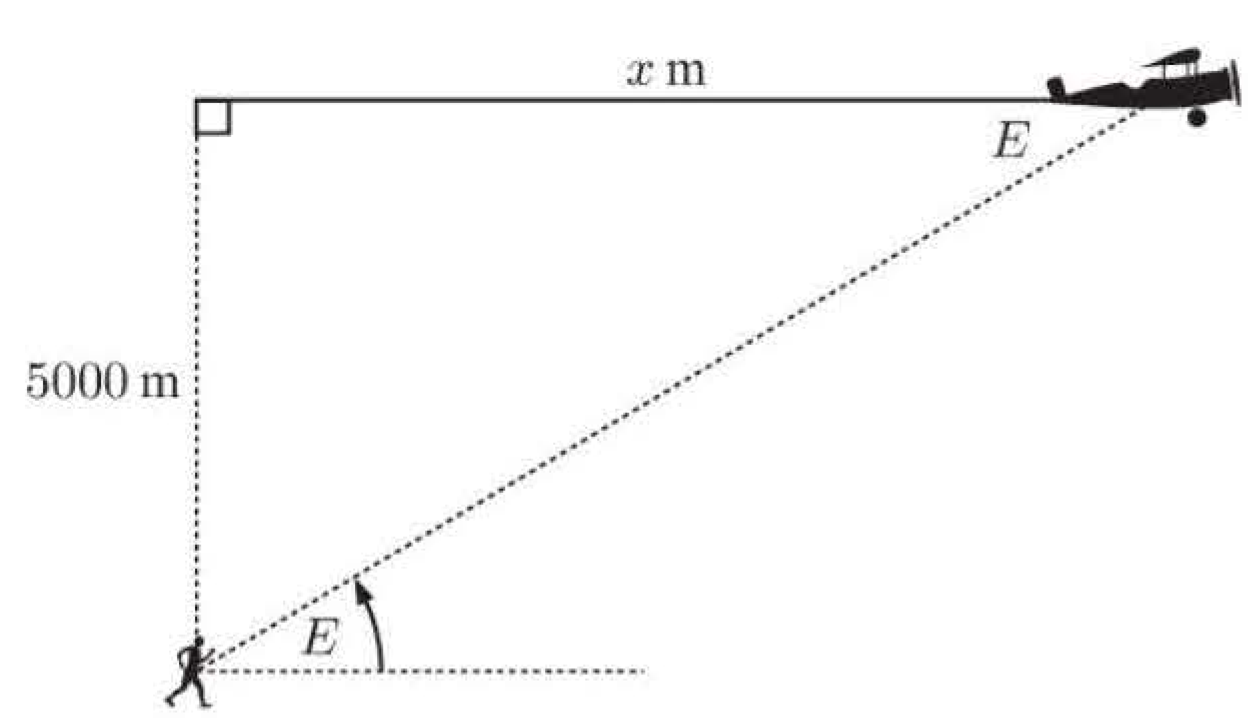
Differentiating with respect to t,

$\sec^2 E\,\dfrac{dE}{dt} = -5000x^{-2}\dfrac{dx}{dt}$

Now $\dfrac{dx}{dt} = 200 \text{ m s}^{-1}$,

$\therefore \quad \dfrac{1}{\cos^2 E}\,\dfrac{dE}{dt} = -5000 \times \dfrac{1}{x^2} \times 200$

$\therefore \quad \dfrac{dE}{dt} = -1\,000\,000 \times \dfrac{\cos^2 E}{x^2}$



Particular cases:

a When $E = 60°$, $\cos E = \frac{1}{2}$

and $\tan E = \sqrt{3} = \dfrac{5000}{x}$

$\therefore \quad x = \frac{5000}{\sqrt{3}}$

$\therefore \quad \dfrac{dE}{dt} = -1\,000\,000 \times \dfrac{\left(\frac{1}{2}\right)^2}{\left(\frac{5000}{\sqrt{3}}\right)^2}$

$= -0.03$

$\therefore$ the angle of elevation is decreasing at 0.03 radians per second

b When $E = 30°$, $\cos E = \frac{\sqrt{3}}{2}$

and $\tan E = \frac{1}{\sqrt{3}} = \dfrac{5000}{x}$

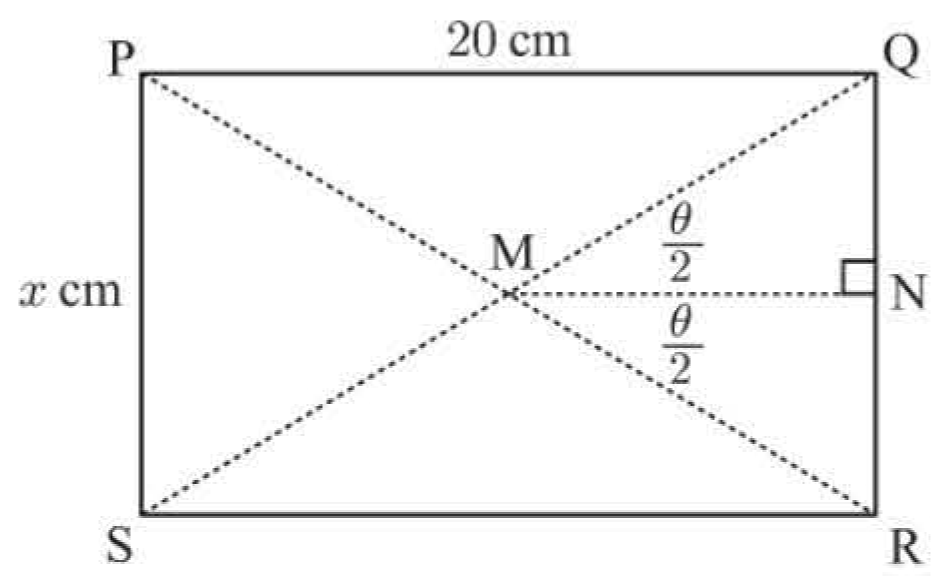
$\therefore \quad x = 5000\sqrt{3}$

$\therefore \quad \dfrac{dE}{dt} = -1\,000\,000 \times \dfrac{\left(\frac{\sqrt{3}}{2}\right)^2}{\left(5000\sqrt{3}\right)^2}$

$= -0.01$

$\therefore$ the angle of elevation is decreasing at 0.01 radians per second

12



Let N be the midpoint of [QR] in isosceles triangle QMR.

$\therefore$ MN $= 10$ cm

Let QR $= x$ cm and let $\widehat{QMR} = \theta$.

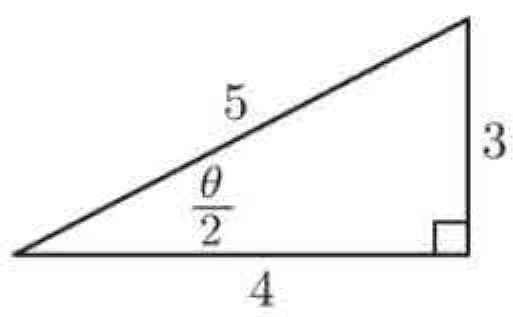
In triangle MNQ, $\tan\left(\frac{\theta}{2}\right) = \frac{QN}{MN} = \frac{\frac{x}{2}}{10} = \frac{x}{20}$

$$\therefore \frac{1}{2}\sec^2\left(\frac{\theta}{2}\right)\frac{d\theta}{dt} = \frac{1}{20}\frac{dx}{dt}$$

$$\therefore \frac{d\theta}{dt} = \frac{1}{10}\cos^2\left(\frac{\theta}{2}\right)\frac{dx}{dt}$$

where $\frac{dx}{dt} = 2\ \text{cm s}^{-1}$

Particular case: When $x = 15$ cm, $\tan\left(\frac{\theta}{2}\right) = \frac{15}{20} = \frac{3}{4}$

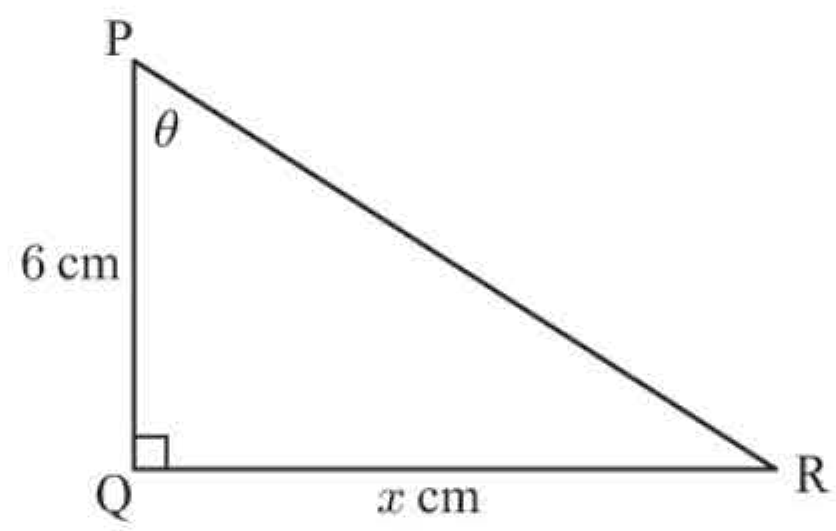


$$\therefore \cos\left(\frac{\theta}{2}\right) = \frac{4}{5}$$

$$\therefore \frac{d\theta}{dt} = \frac{1}{10}\left(\frac{4}{5}\right)^2 2 = 0.128$$

$\therefore$ θ is increasing at 0.128 radians per second

13



Let QR $= x$ cm and the angle at P be θ.

Then $\tan\theta = \frac{x}{6}$

$$\therefore \sec^2\theta\,\frac{d\theta}{dt} = \frac{1}{6}\frac{dx}{dt}$$

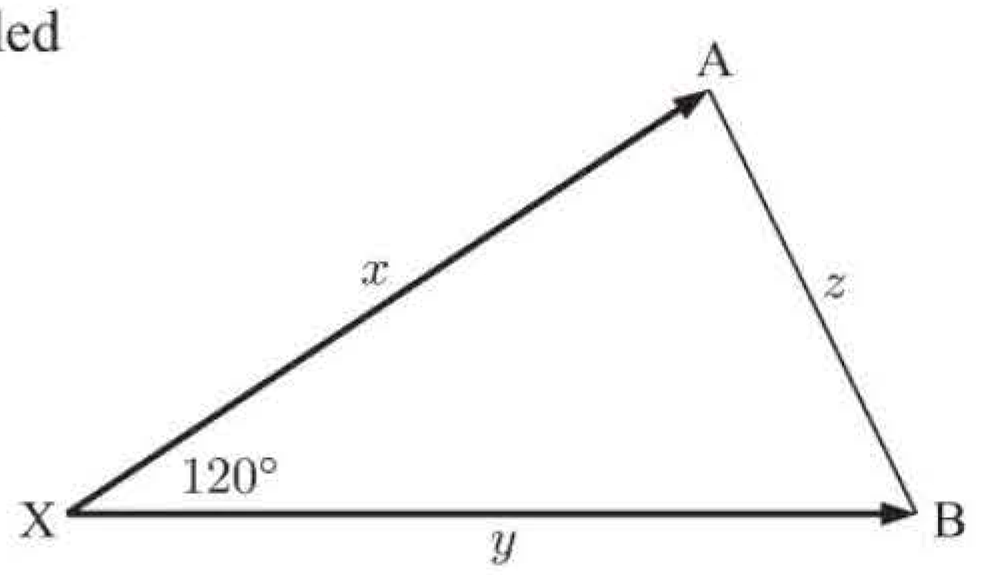
$$\therefore \frac{d\theta}{dt} = \frac{\cos^2\theta}{6}\frac{dx}{dt} \quad \text{where} \quad \frac{dx}{dt} = 2\ \text{cm min}^{-1}$$

Particular case: When $x = 8$ cm, PR $= 10$ cm

Now $\cos\theta = \frac{6}{10}$, so $\frac{d\theta}{dt} = \left(\frac{6}{10}\right)^2 \times \frac{1}{6} \times 2 = 0.12$

$\therefore$ the angle at P is increasing at a rate of 0.12 radians per minute.

14 Let x and y be the distances the cyclists A and B have travelled respectively at time t, and let z be the distance between them.



So, $z^2 = x^2 + y^2 - 2xy\cos 120°$ {cosine rule}

$\therefore$ $z^2 = x^2 + y^2 + xy$ .... (1)

$$\therefore 2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} + \frac{dx}{dt}y + x\frac{dy}{dt} \quad .... (2)$$

Particular case:

After 2 minutes, $t = 120$ s, $x = 1440$ m, $y = 1920$ m, $\frac{dx}{dt} = 12\ \text{m s}^{-1}$, and $\frac{dy}{dt} = 16\ \text{m s}^{-1}$.

Using (1), $z^2 = 1440^2 + 1920^2 + 1440 \times 1920 = 8\,524\,800$

$$\therefore z = \sqrt{8\,524\,800} = 480\sqrt{37}$$

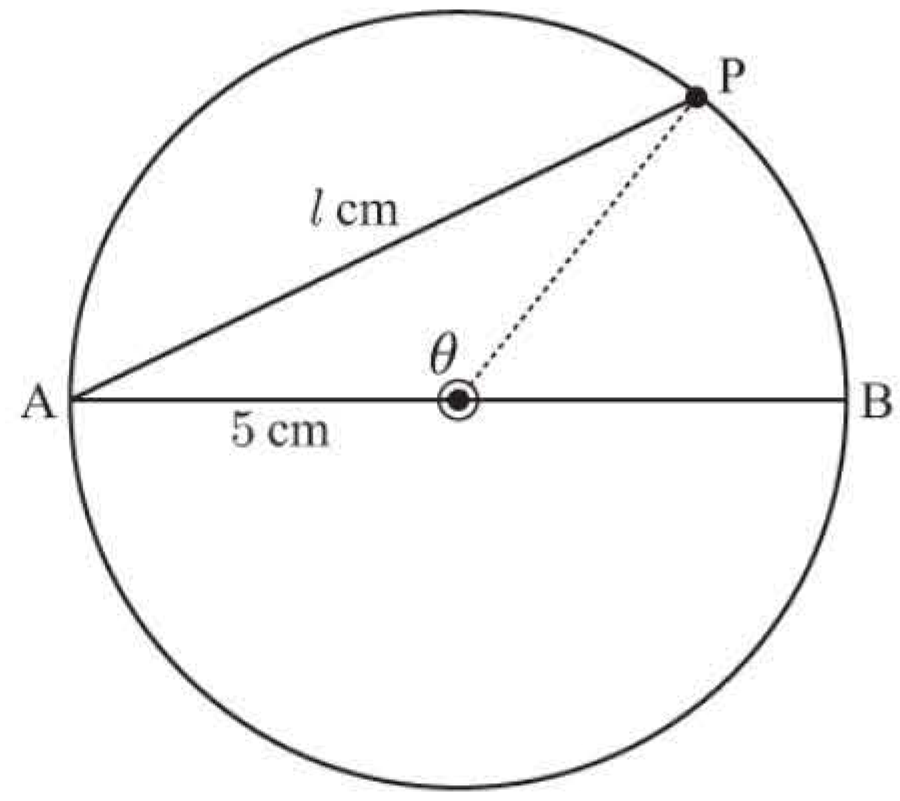
$$\therefore 2\left(480\sqrt{37}\right)\frac{dz}{dt} = 2880(12) + 3840(16) + (12)1920 + 1440(16) \quad \{\text{using (2)}\}$$

$$\therefore 960\sqrt{37}\,\frac{dz}{dt} = 142\,080$$

$$\therefore \frac{dz}{dt} = \frac{148}{\sqrt{37}} = 4\sqrt{37} \approx 24.33$$

$\therefore$ the distance between the cyclists is increasing at $24.3\ \text{m s}^{-1}$.

15



Let $\text{AP} = l$ cm and let $\widehat{\text{AOP}} = \theta$

$\therefore\ l^2 = 5^2 + 5^2 - 2 \times 5 \times 5\cos\theta$ {cosine rule}

$\therefore\ l^2 = 50 - 50\cos\theta$

$\therefore\ 2l\dfrac{dl}{dt} = 50\sin\theta\dfrac{d\theta}{dt}$

$\therefore\ \dfrac{dl}{dt} = \dfrac{25\sin\theta}{l}\dfrac{d\theta}{dt}$

Now the point moves at one revolution every 10 seconds.

$\therefore\ \dfrac{d\theta}{dt} = \frac{2\pi}{10} = \frac{\pi}{5}$ radians per second

Particular cases:

a If $\text{AP} = l = 5$ cm, $\dfrac{dl}{dt} > 0$,

then $\theta = \frac{\pi}{3}$ {$\triangle$APO is equilateral}

$\therefore\ \dfrac{dl}{dt} = \dfrac{25\sin(\frac{\pi}{3})}{5} \times \frac{\pi}{5}$

$= \frac{\sqrt{3}}{2}\pi\ \text{cm s}^{-1}$

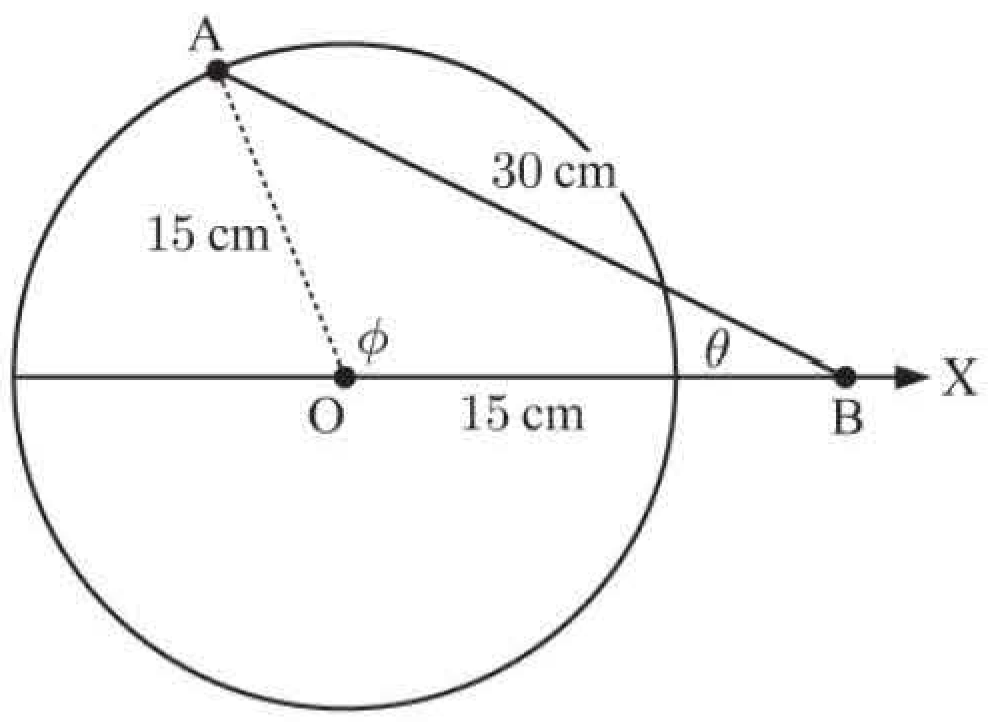
b If P is at B, then $l = 10$ cm

and $\theta = \pi$

$\therefore\ \dfrac{dl}{dt} = \dfrac{25\sin\pi}{10} \times \frac{\pi}{5}$

$= 0\ \text{cm s}^{-1}$

16



Let $\widehat{\text{AOB}} = \phi$ and $\widehat{\text{ABO}} = \theta$

Now $\dfrac{d\phi}{dt} = -100$ revolutions per second {negative for clockwise rotation}

$\therefore\ \dfrac{d\phi}{dt} = -200\pi$ radians per second

Also, $\dfrac{30}{\sin\phi} = \dfrac{15}{\sin\theta}$ {sine rule}

$\therefore\ \sin\phi = 2\sin\theta$

$\therefore\ \cos\phi\dfrac{d\phi}{dt} = 2\cos\theta\dfrac{d\theta}{dt}$

$\therefore\ \dfrac{d\theta}{dt} = \frac{1}{2}\dfrac{\cos\phi}{\cos\theta}\dfrac{d\phi}{dt}$

where $\dfrac{d\phi}{dt} = -200\pi$ radians s^{-1}

Particular cases:

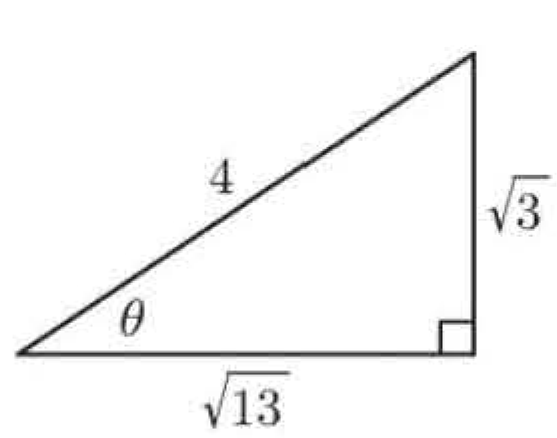
a When $\widehat{\text{AOX}} = 120°$, $\phi = \frac{2\pi}{3}$

$\therefore\ \cos\phi = -\frac{1}{2}$ and $\sin\phi = \frac{\sqrt{3}}{2}$

$\therefore\ \sin\theta = \frac{1}{2}\sin\phi$

$= \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{\sqrt{3}}{4}$



$\therefore\ \cos\theta = \frac{\sqrt{13}}{4}$

$\therefore\ \dfrac{d\theta}{dt} = \frac{1}{2} \times \dfrac{-\frac{1}{2}}{\frac{\sqrt{13}}{4}} \times (-200\pi) = \frac{200\pi}{\sqrt{13}}$

$\therefore$ $\widehat{\text{ABO}}$ is increasing at $\frac{200\pi}{\sqrt{13}}$ radians per second.

b When $\widehat{\text{AOX}} = 180°$, $\phi = \pi$,

$\therefore\ \cos\phi = -1$ and $\sin\phi = 0$

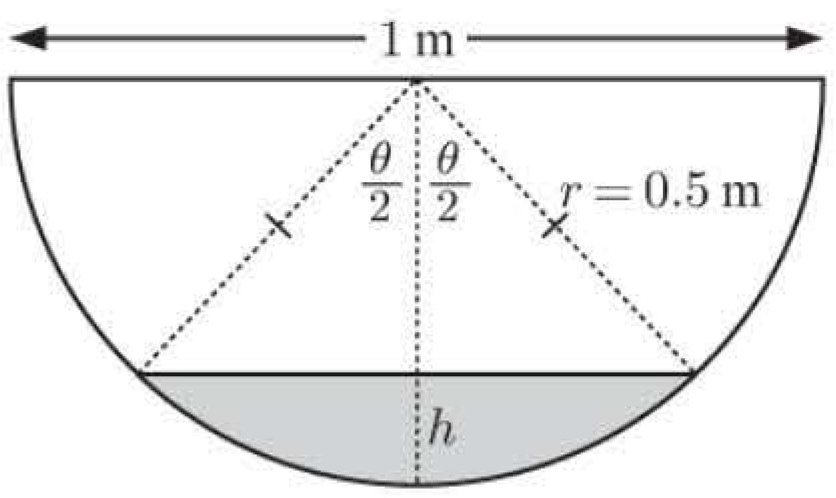
$\therefore\ \sin\theta = 0$ and $\cos\theta = 1$

$\therefore\ \dfrac{d\theta}{dt} = \frac{1}{2} \times \frac{-1}{1} \times (-200\pi)$

$= 100\pi$

$\therefore$ $\widehat{\text{ABO}}$ is increasing at 100π radians per second.

17



Denote the radius of the semi-circle $r = \frac{1}{2}$ m.

Let h be the depth and V be the volume of water in the trough at time t.

a The cross-sectional area of water in the trough

$$\begin{aligned} &= \text{area of sector} - \text{area of triangle} \\ &= \tfrac{1}{2}r^2\theta - \tfrac{1}{2}r^2 \sin\theta \\ &= \tfrac{1}{2}r^2(\theta - \sin\theta) \\ &= \tfrac{1}{8}(\theta - \sin\theta) \end{aligned}$$

$\therefore$ the volume of water,

$$\begin{aligned} V &= \text{area of water} \times \text{length of trough} \\ &= \tfrac{1}{8}(\theta - \sin\theta) \times 8 \\ &= \theta - \sin\theta \quad \text{as required} \end{aligned}$$

b Now $\dfrac{dV}{dt} = \dfrac{d\theta}{dt} - \cos\theta\,\dfrac{d\theta}{dt}$

$\therefore \dfrac{dV}{dt} = \dfrac{d\theta}{dt}(1 - \cos\theta)$

But $\dfrac{dV}{dt} = 0.1\ \text{m}^3\,\text{min}^{-1}$

$\therefore \dfrac{d\theta}{dt} = \dfrac{0.1}{1 - \cos\theta}$ .... (1)

Also, $\cos\left(\frac{\theta}{2}\right) = \dfrac{\frac{1}{2} - h}{\frac{1}{2}} = 1 - 2h$

Differentiating with respect to t,

$-\sin\left(\frac{\theta}{2}\right) \times \frac{1}{2}\,\dfrac{d\theta}{dt} = -2\,\dfrac{dh}{dt}$

$\therefore \dfrac{dh}{dt} = \frac{1}{4}\sin\left(\frac{\theta}{2}\right)\dfrac{d\theta}{dt}$ .... (2)

Particular case:

When $h = 0.25$ m, $\cos\left(\frac{\theta}{2}\right) = \dfrac{r - h}{r} = \dfrac{\frac{1}{2} - \frac{1}{4}}{\frac{1}{2}} = \dfrac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$

$\therefore \sin\left(\frac{\theta}{2}\right) = \frac{\sqrt{3}}{2}$ and $\cos\theta = 2\cos^2\left(\frac{\theta}{2}\right) - 1 = 2\left(\frac{1}{2}\right)^2 - 1 = -\frac{1}{2}$

Using (1), $\dfrac{d\theta}{dt} = \dfrac{0.1}{1 - \left(-\frac{1}{2}\right)} = \frac{1}{15}$ $\therefore$ θ is increasing at $\frac{1}{15}$ radians per minute.

Using (2), $\dfrac{dh}{dt} = \frac{1}{4}\sin\left(\frac{\theta}{2}\right)\dfrac{d\theta}{dt}$

$= \frac{1}{4} \times \frac{\sqrt{3}}{2} \times \frac{1}{15}$

$= \frac{\sqrt{3}}{120}$ $\therefore$ h is increasing at $\frac{\sqrt{3}}{120}$ metres per minute.

REVIEW SET 20A

1 **a** $x(t) = 3 + \sin(2t)$ cm, $t \geqslant 0$ s $\therefore x(0) = 3$ cm

$v(t) = x'(t) = 0 + 2\cos(2t)\ \text{cm}\,\text{s}^{-1}$ $v(0) = 2\ \text{cm}\,\text{s}^{-1}$

$a(t) = v'(t) = -4\sin(2t)\ \text{cm}\,\text{s}^{-2}$ $a(0) = 0\ \text{cm}\,\text{s}^{-2}$

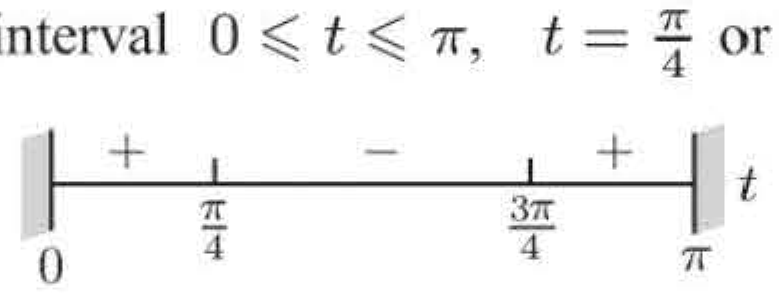
$\therefore$ initially the particle is 3 cm right of O, moving right at a speed of $2\ \text{cm}\,\text{s}^{-1}$.

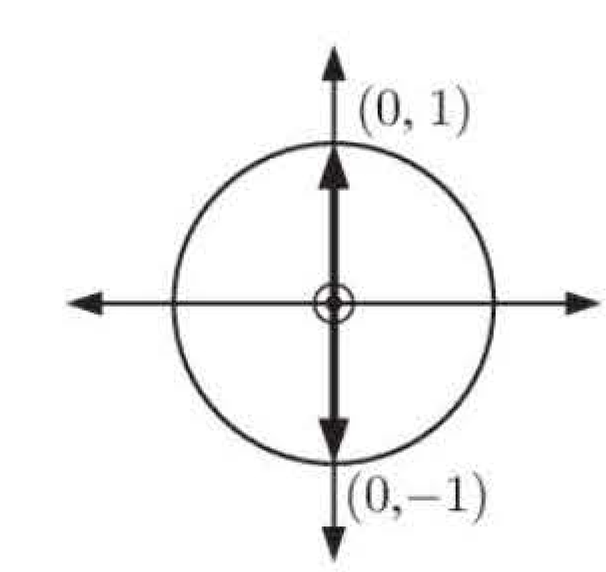
b $x'(t) = 0$ when $2\cos(2t) = 0$

$\therefore \cos(2t) = 0$

$\therefore 2t = \frac{\pi}{2} + k\pi$

For the interval $0 \leqslant t \leqslant \pi$, $t = \frac{\pi}{4}$ or $\frac{3\pi}{4}$



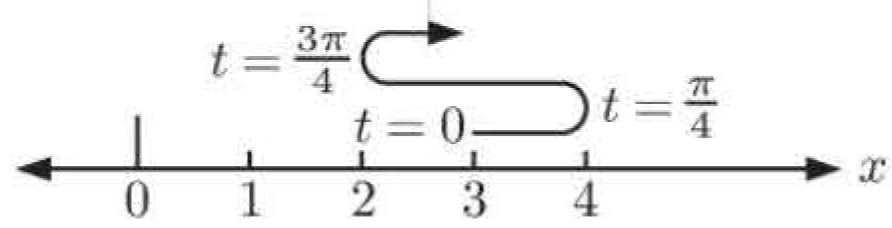


$\therefore$ the particle reverses direction at $t = \frac{\pi}{4}$ s, $\frac{3\pi}{4}$ s.

c $x(0) = 3, \quad x\left(\frac{\pi}{4}\right) = 3 + \sin\left(\frac{\pi}{2}\right) = 4,$

$x\left(\frac{3\pi}{4}\right) = 3 + \sin\left(\frac{3\pi}{2}\right) = 3 - 1 = 2,$

$x(\pi) = 3 + \sin(2\pi) = 3$

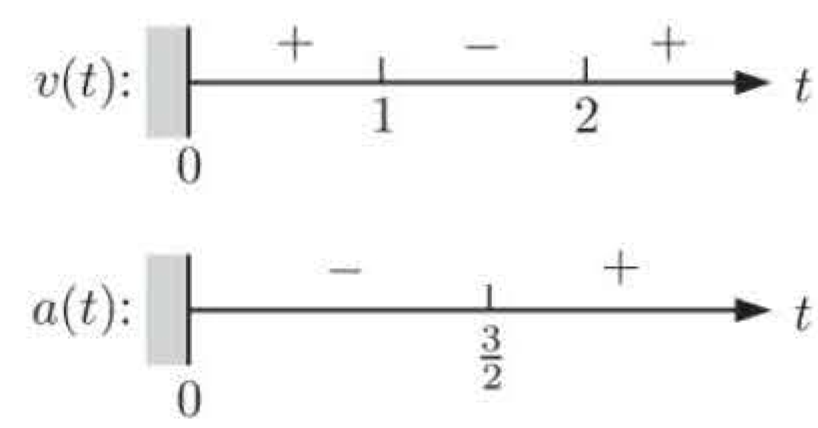


$\therefore$ the total distance travelled $= 1 + 2 + 1 = 4$ cm.

2 **a** $s(t) = 2t^3 - 9t^2 + 12t - 5$ cm, $\quad t \geqslant 0$

$$\begin{aligned} v(t) &= 6t^2 - 18t + 12 \\ &= 6(t^2 - 3t + 2) \\ &= 6(t-2)(t-1) \text{ cm s}^{-1} \end{aligned}$$

and $\begin{aligned} a(t) &= 12t - 18 \\ &= 6(2t - 3) \text{ cm s}^{-2} \end{aligned}$



b When $t = 0$, $\quad s(0) = -5$ cm
$v(0) = 12$ cm s^{-1}
$a(0) = -18$ cm s^{-2}

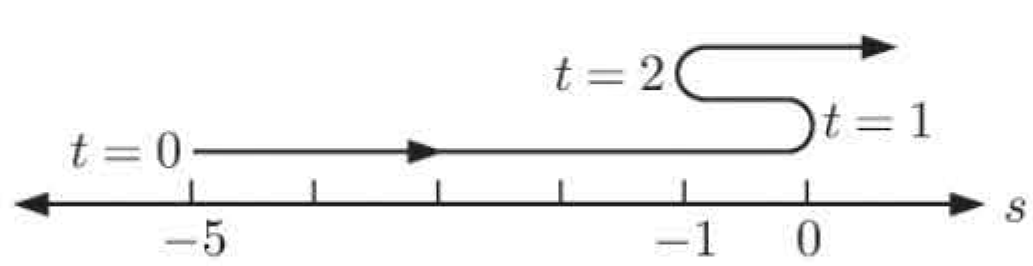
Initially, the particle is 5 cm to the left of O, moving at 12 cm s^{-1} towards the origin and decreasing in speed.

c When $t = 2$, $\quad s(2) = -1$ cm
$v(2) = 0$ cm s^{-1}
$a(2) = 6$ cm s^{-2}

When $t = 2$, the particle is 1 cm to the left of O, instantaneously at rest and increasing in speed towards O.

d The particle changes direction when $t = 1$ and $t = 2$, at $s(1) = 0$ cm, $\quad s(2) = -1$ cm.

e



f The speed is increasing when $1 \leqslant t \leqslant \frac{3}{2}$ and $t \geqslant 2$
$\{v(t)$ and $a(t)$ have the same sign$\}$

3 **a** Now if OD $= x$, the coordinates of C are $(x, k - x^2)$.

$\therefore$ the area of ABCD $= 2x \times (k - x^2)$

$\therefore \quad A = 2kx - 2x^3, \quad x > 0$

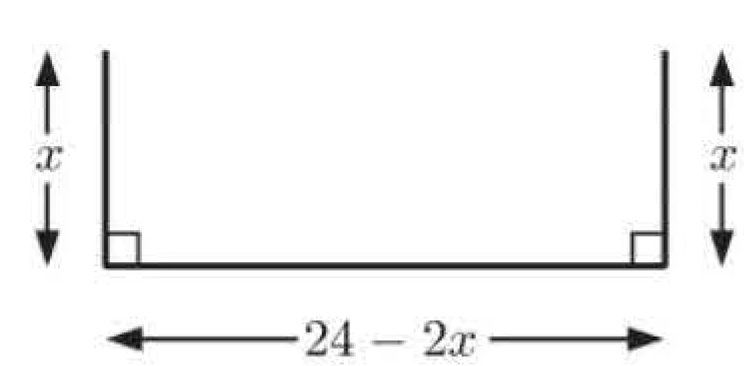
b Now $\dfrac{dA}{dx} = 2k - 6x^2$

But $\dfrac{dA}{dx} = 0$ when AD $= 2\sqrt{3}$, and this occurs when $x = \sqrt{3}$

$$\begin{aligned} \therefore \quad 2k - 6(\sqrt{3})^2 &= 0 \\ \therefore \quad 2k - 18 &= 0 \\ \therefore \quad 2k &= 18 \\ \therefore \quad k &= 9 \end{aligned}$$

Check: $\begin{aligned} \dfrac{dA}{dx} &= 18 - 6x^2 \\ &= 6(3 - x^2) \\ &= 6(\sqrt{3} + x)(\sqrt{3} - x) \end{aligned}$

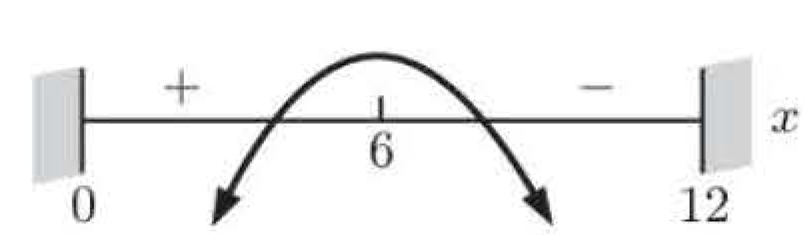
4 Suppose the sheet is bent x cm from each end. To maximise the water carried we need to maximise the area of cross-section.



$$\begin{aligned} A &= x(24 - 2x), \quad 0 \leqslant x \leqslant 12 \\ &= 24x - 2x^2 \end{aligned}$$

$\therefore \quad \dfrac{dA}{dx} = 24 - 4x$

So, $\dfrac{dA}{dx} = 0$ when $x = 6$, and $\dfrac{dA}{dx}$ has sign diagram:



The maximum water is held when $x = 6$ cm

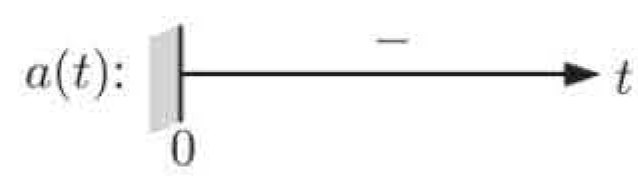
$\therefore$ the bends must be made 6 cm from each end.

5 **a** $s(t) = 2t - \dfrac{4}{t+1} = 2t - 4(t+1)^{-1}$

$$\begin{aligned} \therefore \quad v(t) &= 2 + 4(t+1)^{-2} \\ &= 2 + \frac{4}{(t+1)^2} \text{ cm s}^{-1} \end{aligned}$$

$\therefore\ a(t) = -8(t+1)^{-3}$

$= -\dfrac{8}{(t+1)^3}\ \text{cm s}^{-2}$



b $s(1) = 2(1) - \dfrac{4}{(1+1)} = 0$ cm

$v(1) = 2 + \dfrac{4}{(1+1)^2} = 3\ \text{cm s}^{-1}$

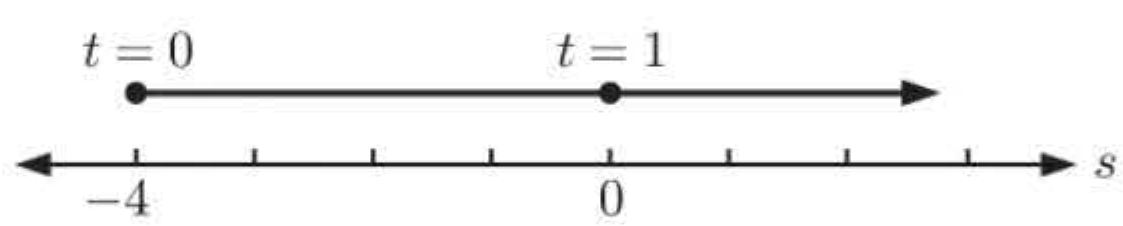
$a(1) = -\dfrac{8}{(1+1)^3} = -1\ \text{cm s}^{-2}$

$\therefore$ the particle is at the origin and is moving to the right with velocity $3\ \text{cm s}^{-1}$ and slowing down, its acceleration being $1\ \text{cm s}^{-2}$ to the left.

c $v(t) = 2 + \dfrac{4}{(t+1)^2} = \dfrac{2(t+1)^2+4}{(t+1)^2}$

$\therefore\ v(t) \neq 0$ for any real t, so the particle never changes direction.

d



e **i** The velocity is never increasing {acceleration is negative for all $t > 0$}.

ii The speed is never increasing, as $v(t)$ and $a(t)$ have different signs for all $t > 0$.

6 When the box is manufactured its base is $(2k - 2x)$ by $(k - 2x)$ and its height is x cm.

$\therefore\ V = x(2k-2x)(k-2x)$

$\therefore\ V = x(2k^2 - 4kx - 2xk + 4x^2)$

$= 2k^2x - 6kx^2 + 4x^3$

$\therefore\ \dfrac{dV}{dx} = 2k^2 - 12kx + 12x^2$

$= 2(6x^2 - 6kx + k^2)$

So, $\dfrac{dV}{dx} = 0$ when $x = \dfrac{6k \pm \sqrt{36k^2 - 4(6)k^2}}{12}$

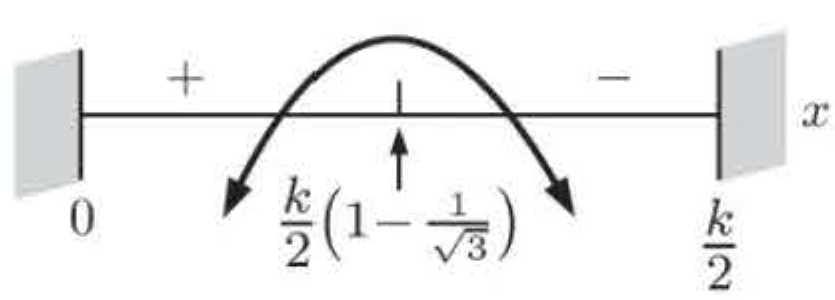
$= \dfrac{6k \pm k\sqrt{12}}{12}$

$= \dfrac{k}{2} \pm \dfrac{k}{\sqrt{12}}$

$= \dfrac{k}{2} - \dfrac{k}{2\sqrt{3}}$ $\{\text{as } x \leqslant \dfrac{k}{2}\}$

$= \dfrac{k}{2}\left(1 - \tfrac{1}{\sqrt{3}}\right)$

The sign diagram of $\dfrac{dV}{dx}$ is:



$\therefore$ the maximum capacity occurs when $x = \dfrac{k}{2}\left(1 - \tfrac{1}{\sqrt{3}}\right)$.

7 **a** $s(t) = 30 + \cos(\pi t)$ cm, $t \geqslant 0$

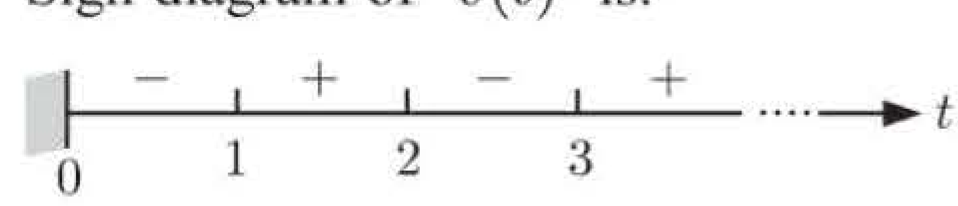
$\therefore\ v(t) = s'(t) = -\pi\sin(\pi t)$

So, $v(0) = 0\ \text{cm s}^{-1}$, $v(\tfrac{1}{2}) = -\pi\ \text{cm s}^{-1}$,

$v(1) = 0\ \text{cm s}^{-1}$, $v(\tfrac{3}{2}) = \pi\ \text{cm s}^{-1}$,

$v(2) = 0\ \text{cm s}^{-1}$

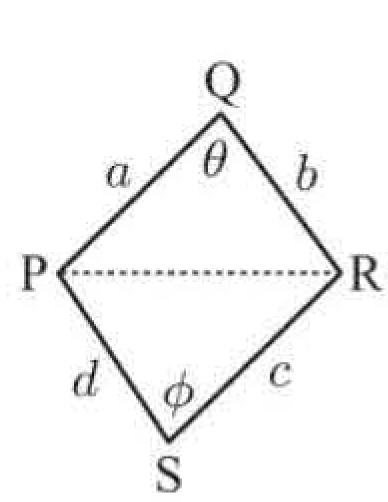
Sign diagram of $v(t)$ is:



b The cork is falling when $v(t) \leqslant 0$, which is for $0 \leqslant t \leqslant 1$, $2 \leqslant t \leqslant 3$, $4 \leqslant t \leqslant 5$, ...

$\therefore$ the cork is falling for $2n \leqslant t \leqslant 2n+1$, $n \in \{0, 1, 2, 3, ...\}$

8 **a**



Using the cosine rule: in $\triangle$PQR, $\text{PR}^2 = a^2 + b^2 - 2ab\cos\theta$

in $\triangle$PSR, $\text{PR}^2 = c^2 + d^2 - 2cd\cos\phi$

$\therefore\ a^2 + b^2 - 2ab\cos\theta = c^2 + d^2 - 2cd\cos\phi$

b Now a, b, c, and d are constants, so differentiating with respect to ϕ,

$$\therefore \quad 2ab\sin\theta\,\frac{d\theta}{d\phi} = 2cd\sin\phi$$

$$\therefore \quad \frac{d\theta}{d\phi} = \frac{2cd\sin\phi}{2ab\sin\theta} = \frac{cd\sin\phi}{ab\sin\theta} \quad \text{as required}$$

c Area of quadrilateral, $A =$ area of $\triangle$PQR $+$ area of $\triangle$PSR

$$= \tfrac{1}{2}ab\sin\theta + \tfrac{1}{2}cd\sin\phi$$

$$\therefore \quad \frac{dA}{d\phi} = \tfrac{1}{2}ab\cos\theta\,\frac{d\theta}{d\phi} + \tfrac{1}{2}cd\cos\phi$$

$$= \tfrac{1}{2}ab\cos\theta\left(\frac{cd\sin\phi}{ab\sin\theta}\right) + \tfrac{1}{2}cd\cos\phi \quad \{\text{using } \mathbf{a}\}$$

$$= \tfrac{1}{2}cd\left[\frac{\cos\theta\sin\phi}{\sin\theta} + \cos\phi\right]$$

$$= \frac{cd}{2\sin\theta}(\sin\phi\cos\theta + \cos\phi\sin\theta)$$

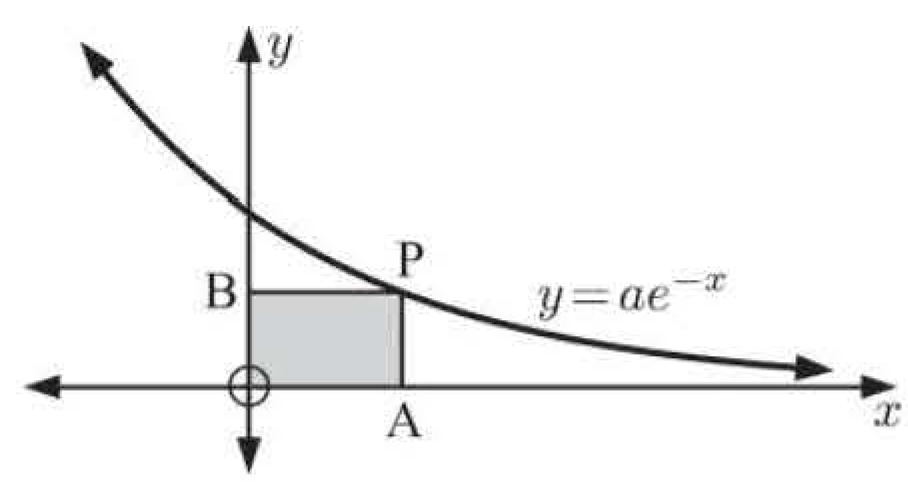
$$= \frac{cd}{2\sin\theta}\sin(\phi+\theta)$$

$$\therefore \quad \frac{dA}{d\phi} = 0 \text{ when } \sin(\phi+\theta) = 0, \quad \text{which is when} \quad \phi+\theta = \pi$$

$\therefore$ the area of PQRS is a maximum when the opposite angles are supplementary, which occurs when PQRS is a cyclic quadrilateral.

9 Let the coordinates of A be $(x, 0)$

$\therefore$ the coordinates of P are (x, ae^{-x})



OAPB has perimeter $P = 2(x + ae^{-x})$

$$= 2x + 2ae^{-x}$$

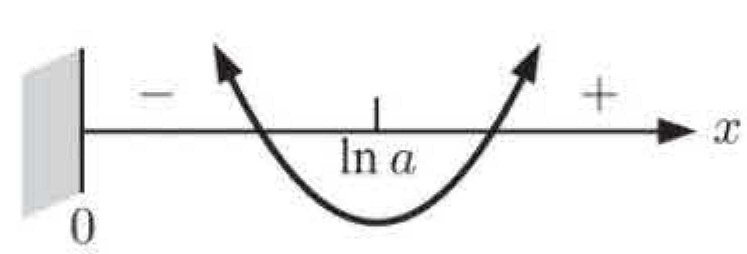
$$\therefore \quad \frac{dP}{dx} = 2 - 2ae^{-x}$$

$$= 2\left(1 - \frac{a}{e^x}\right)$$

Now $\dfrac{dP}{dx} = 0$ when $1 = \dfrac{a}{e^x}$

$$\therefore \quad e^x = a$$

$$\therefore \quad x = \ln a$$

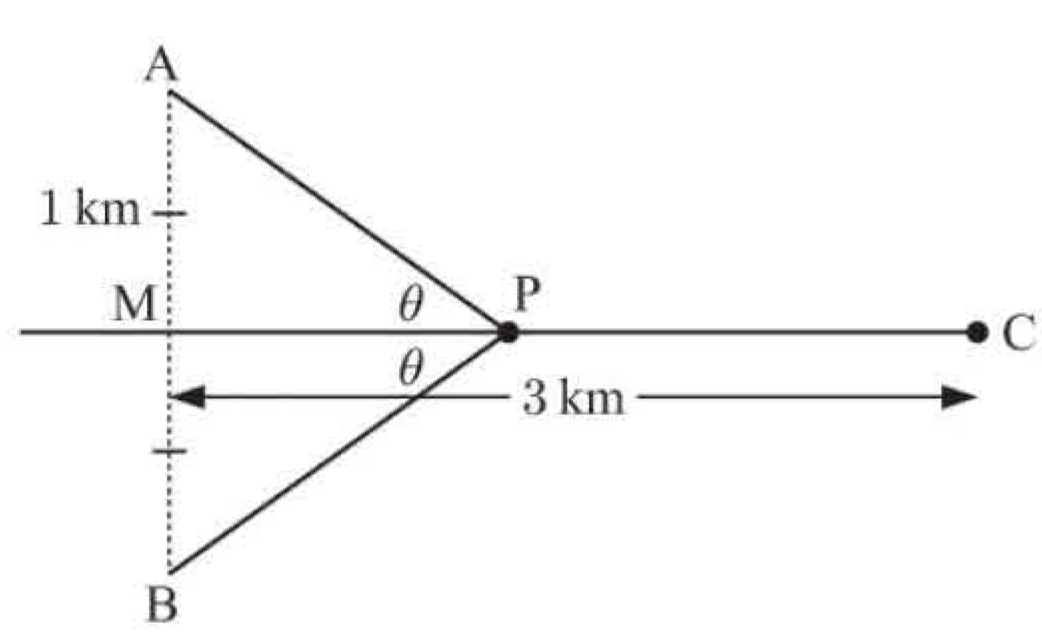


$\therefore$ there is a local minimum when $x = \ln a$

When $x = \ln a$, $y = ae^{-(\ln a)} = \dfrac{a}{e^{\ln a}} = 1$

$\therefore$ rectangle OAPB has minimum perimeter when P is at $(\ln a, 1)$.

10



a Length of cable required $=$ (PA $+$ PB $+$ PC) km

i If P is at M, then PA $=$ PB $= 1$ km and PC $= 3$ km
$\therefore$ 5 km of cable is required

ii If P is at C, then PA $=$ PB $= \sqrt{1^2+3^2} = \sqrt{10}$ km and PC $= 0$ km
$\therefore$ $2\sqrt{10}$ km of cable is required

b Now $\sin\theta = \dfrac{1}{\text{AP}} = \dfrac{1}{\text{BP}}$ and $\tan\theta = \dfrac{1}{\text{MP}}$

$$\therefore \quad \text{AP} = \text{BP} = \frac{1}{\sin\theta} \quad \text{and} \quad \text{MP} = \frac{1}{\tan\theta} = \cot\theta$$

$$\therefore \quad \text{AP} + \text{BP} + \text{CP} = \frac{2}{\sin\theta} + (\text{CM} - \text{MP}) \qquad \therefore \quad L(\theta) = 2\csc\theta + 3 - \cot\theta \quad \text{as required}$$

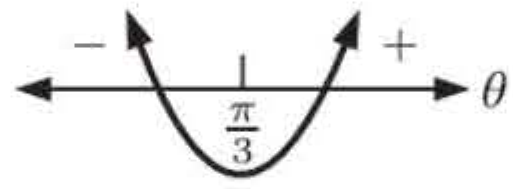
c Since $L(\theta) = 2\csc\theta + 3 - \cot\theta$, $\quad \dfrac{dL}{d\theta} = 2(-\csc\theta\cot\theta) - (-\csc^2\theta)$

$$= \frac{-2\cos\theta}{\sin^2\theta} + \frac{1}{\sin^2\theta}$$

$$= \frac{1 - 2\cos\theta}{\sin^2\theta} \quad \text{as required}$$

$\therefore \dfrac{dL}{d\theta} = 0$ if $\cos\theta = \frac{1}{2}$

$\therefore \theta = \frac{\pi}{3}$ and sign diagram of $\dfrac{dL}{d\theta}$ is:



$\therefore$ the minimum length of cable is required when $\theta = \frac{\pi}{3}$

When $\theta = \frac{\pi}{3}$, $\quad \sin\theta = \frac{\sqrt{3}}{2}$ and $\tan\theta = \sqrt{3}$,

$\therefore \csc\theta = \frac{2}{\sqrt{3}}$ and $\cot\theta = \frac{1}{\sqrt{3}}$

so $L_{\min} = \frac{4}{\sqrt{3}} + 3 - \frac{1}{\sqrt{3}} = (3 + \sqrt{3})$ km as required

REVIEW SET 20B

1 $H(t) = 60 + 40\ln(2t+1)$ cm, $t \geqslant 0$

a When first planted, $t = 0$ $\quad \therefore H(0) = 60 + 40\ln(1) = 60 + 40(0) = 60$ cm.

b **i** When $H(t) = 150$ cm,

$\therefore 60 + 40\ln(2t+1) = 150$

$\therefore 40\ln(2t+1) = 90$

$\therefore \ln(2t+1) = \frac{90}{40} = 2.25$

$\therefore 2t + 1 = e^{2.25}$

$\therefore 2t = e^{2.25} - 1$

$\therefore t = \frac{1}{2}(e^{2.25} - 1)$

$\therefore t \approx 4.24$ years

ii When $H(t) = 300$ cm,

$\therefore 60 + 40\ln(2t+1) = 300$

$\therefore 40\ln(2t+1) = 240$

$\therefore \ln(2t+1) = 6$

$\therefore 2t + 1 = e^6$

$\therefore 2t = e^6 - 1$

$\therefore t = \frac{1}{2}(e^6 - 1)$

$\therefore t \approx 201$ years

c $H'(t) = 40\left(\dfrac{2}{2t+1}\right) = \dfrac{80}{2t+1}$ cm per year

i When $t = 2$, $\quad H'(2) = \frac{80}{5} = 16$ cm per year

ii When $t = 20$, $\quad H'(20) = \frac{80}{41} \approx 1.95$ cm per year

2 $s(t) = 80e^{-\frac{t}{10}} - 40t$ metres, $t \geqslant 0$

a $v(t) = s'(t) = -8e^{-\frac{t}{10}} - 40$ m s^{-1}

$a(t) = v'(t) = \frac{4}{5}e^{-\frac{t}{10}}$ m s^{-2}

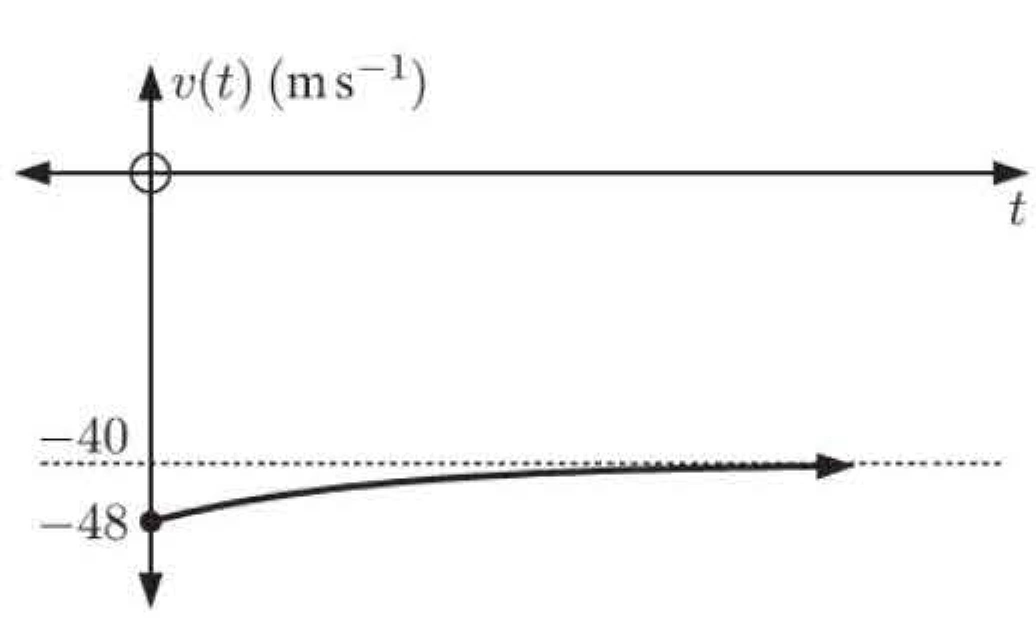
b When $t = 0$, $\quad s(0) = 80$ m

$v(0) = -48$ m s^{-1}

$a(0) = 0.8$ m s^{-2}

c As $t \to \infty$, $e^{-\frac{t}{10}} \to 0$ $\quad \therefore v(t) \to -40$ m s^{-1} (below)

d



e When $v(t) = -44$ m s^{-1}

$\therefore -8e^{-\frac{t}{10}} - 40 = -44$

$\therefore -8e^{-\frac{t}{10}} = -4$

$\therefore e^{-\frac{t}{10}} = 0.5$

$\therefore -\frac{t}{10} = \ln 0.5$

$\therefore t = -10\ln(2^{-1})$

$\therefore t = 10\ln 2$ seconds

3 $C(v) = \dfrac{v^2}{30} + \dfrac{9000}{v}$ dollars per hour

a **i** For $t = 2$ hours at $v = 45\ \text{km h}^{-1}$,

$$\text{cost} = \left(\frac{45^2}{30} + \frac{9000}{45}\right) \times 2 \text{ dollars}$$
$$= \$535.00$$

ii For $t = 5$ hours at $v = 64\ \text{km h}^{-1}$,

$$\text{cost} = \left(\frac{64^2}{30} + \frac{9000}{64}\right) \times 5 \text{ dollars}$$
$$\approx \$1385.79$$

b $C'(v) = \dfrac{2v}{30} - 9000v^{-2} = \dfrac{v}{15} - \dfrac{9000}{v^2}$

i For $v = 50\ \text{km h}^{-1}$

$$\therefore\ C'(50) = \frac{50}{15} - \frac{9000}{50^2}$$
$$\approx -\$0.267 \text{ per km h}^{-1}$$

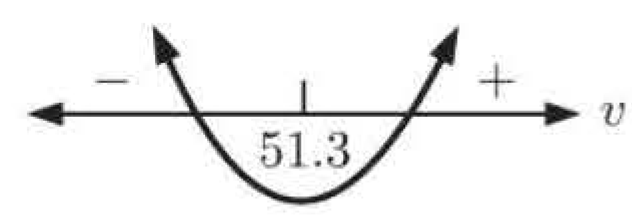
ii For $v = 66\ \text{km h}^{-1}$

$$\therefore\ C'(66) = \frac{66}{15} - \frac{9000}{66^2}$$
$$= \$2.33 \text{ per km h}^{-1}$$

c Now $C'(v) = \dfrac{v}{15} - \dfrac{9000}{v^2} = \dfrac{v^3 - 135\,000}{15v^2}$

$\therefore\ C'(v) = 0$ when $v^3 = 135\,000$

$\therefore\ v \approx 51.3$

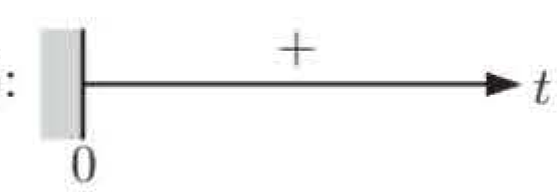
$C'(v)$: 

$\therefore$ the minimum cost occurs when $v \approx 51.3\ \text{km h}^{-1}$

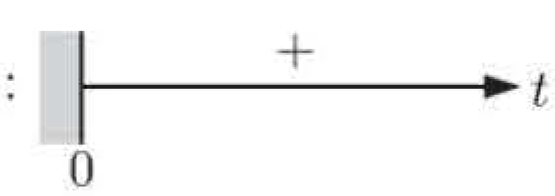
4 **a** $x(t) = 3t - \sqrt{t+1} = 3t - (t+1)^{\frac{1}{2}}$ cm, $t \geqslant 0$

$\therefore\ v(t) = 3 - \frac{1}{2}(t+1)^{-\frac{1}{2}} = 3 - \dfrac{1}{2\sqrt{t+1}}$

since $t \geqslant 0$, $\sqrt{t+1}$ exists and is > 0

$v(t)$: 

and $a(t) = \frac{1}{4}(t+1)^{-\frac{3}{2}} = \dfrac{1}{4(t+1)^{\frac{3}{2}}}$ which is always positive

$a(t)$: 

b $x(0) = 3(0) - \sqrt{0+1} = -1$ cm

$v(0) = 3 - \dfrac{1}{2\sqrt{0+1}} = 2.5\ \text{cm s}^{-1}$

$a(0) = \dfrac{1}{4(0+1)^{\frac{3}{2}}} = 0.25\ \text{cm s}^{-2}$

The particle is 1 cm to the left of the origin, is travelling to the right at $2.5\ \text{cm s}^{-1}$, and accelerating at $0.25\ \text{cm s}^{-2}$.

c $x(8) = 3(8) - \sqrt{8+1} = 21$ cm

$v(8) = 3 - \dfrac{1}{2\sqrt{8+1}} \approx 2.83\ \text{cm s}^{-1}$

$a(8) = \dfrac{1}{4(8+1)^{\frac{3}{2}}} \approx 0.009\,26\ \text{cm s}^{-2}$

The particle is 21 cm to the right of the origin, is travelling to the right at $2.83\ \text{cm s}^{-1}$, and accelerating at $0.009\,26\ \text{cm s}^{-2}$.

d Since $v(t)$ is > 0 for all $t \geqslant 0$, the particle never changes direction.

e $v(t)$ and $a(t)$ have the same sign for all $t \geqslant 0$, so the speed of the particle is always increasing.

$\therefore$ the speed of the particle is never decreasing.

5 **a** At time $t = 0$, $V = 20\,000e^{-0.4\times 0}$

$= 20\,000$ dollars

$\therefore$ the purchase price of the car was \$20 000.

b $V' = -0.4(20\,000)e^{-0.4t}$

$= -8000e^{-0.4t}$

At time $t = 10$, $V' = -8000e^{-0.4\times 10}$

≈ -146.53 dollars year^{-1}

$\therefore$ after 10 years, the car is decreasing in value at \$146.53 per year.

6 $P(x) = I(x) - C(x)$

$$= \left[200\ln\left(1+\frac{x}{100}\right)+1000\right] - \left[(x-100)^2+200\right]$$

$$= 200\ln(1+0.01x) - (x-100)^2 + 800$$

$$\therefore \quad \frac{dP}{dx} = 200\left(\frac{0.01}{1+0.01x}\right) - 2(x-100)^1$$

$$= \frac{2}{1+0.01x} - \frac{2(x-100)}{1}$$

$$= \frac{2-2(x-100)(1+0.01x)}{1+0.01x}$$

$$= \frac{2-2(x+0.01x^2-100-x)}{1+0.01x}$$

$$= \frac{2-0.02x^2+200}{1+0.01x}$$

$$= \frac{202-0.02x^2}{1+0.01x}$$

$\therefore \quad \dfrac{dP}{dx} = 0$ when $0.02x^2 = 202$

$\therefore \quad x^2 = 10\,100$

$\therefore \quad x = \sqrt{10\,100} \quad \{x > 0\}$

$\therefore \quad x \approx 100.49$

and the sign diagram of $\dfrac{dP}{dx}$ is:

$\therefore$ the maximum profit occurs when $x \approx 100.49$

Now $P(100) \approx \$938.63$ and $P(101) \approx \$938.63$

$\therefore$ the maximum daily profit is $938.63 when 100 or 101 shirts are made.

7 **a** $P = 200$ m

But $P = 2x + 2y + \pi x$

$\therefore \quad 200 = 2x + 2y + \pi x$

$\therefore \quad 2y = 200 - 2x - \pi x$

$\therefore \quad y = 100 - x - \frac{\pi}{2}x$

b Area of lawn $= 2x \times y + \frac{1}{2}\pi x^2$

$= 2x(100 - x - \frac{\pi}{2}x) + \frac{1}{2}\pi x^2$

$= 200x - 2x^2 - \pi x^2 + \frac{1}{2}\pi x^2$

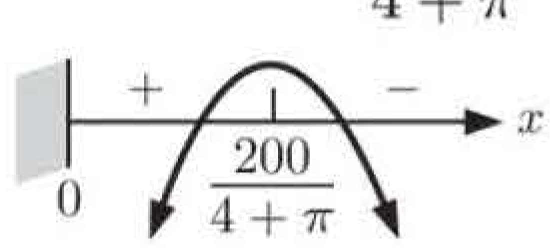
$\therefore \quad A = 200x - 2x^2 - \frac{1}{2}\pi x^2$

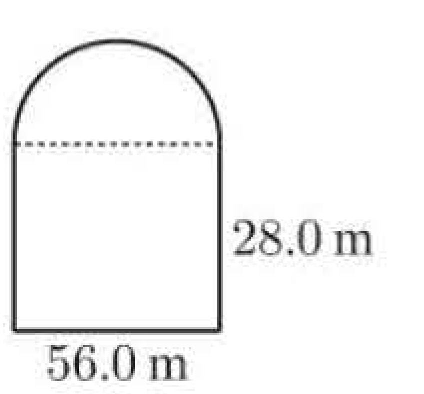
c $A = 200x - 2x^2 - \frac{1}{2}\pi x^2 = 200x - (2+\frac{\pi}{2})x^2$ m^2

$$\therefore \quad \frac{dA}{dx} = 200 - 2\left(2+\tfrac{\pi}{2}\right)x = 200 - (4+\pi)x$$

$\therefore \quad \dfrac{dA}{dx} = 0$ when $(4+\pi)x = 200 \qquad \therefore \quad x = \dfrac{200}{4+\pi}$

and the sign diagram for $\dfrac{dA}{dx}$ is:





$\therefore$ the maximum area occurs when $x = \dfrac{200}{4+\pi} \approx 28.0$ m

and $y = 100 - x - \frac{\pi}{2}x \approx 28.0$ m

8 **a** $\triangle$s LQX and XPM are similar.

$$\therefore \quad \frac{\text{LQ}}{\text{XP}} = \frac{\text{LX}}{\text{XM}} = \frac{\text{QX}}{\text{PM}}$$

$$\therefore \quad \frac{\text{LQ}}{1} = \frac{8}{\text{PM}}$$

$$\therefore \quad \text{LQ} = \frac{8}{\text{PM}}$$

$$\therefore \quad \text{LQ} = \frac{8}{x} \text{ km}$$

b $L^2 = (\text{LQ} + \text{QB})^2 + (\text{BP} + \text{PM})^2$

$$= \left(\frac{8}{x}+1\right)^2 + (8+x)^2$$

$$= \left(\frac{8}{x}+1\right)^2 + \left[x\left(\frac{8}{x}+1\right)\right]^2$$

$$= \left(\frac{8}{x}+1\right)^2 + x^2\left(\frac{8}{x}+1\right)^2$$

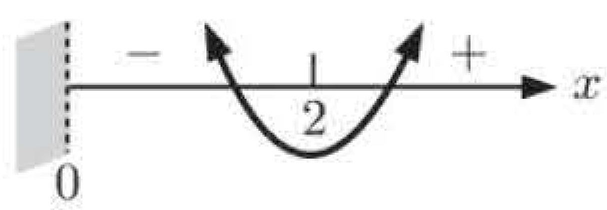
$$= \left(\frac{8}{x}+1\right)^2(1+x^2)$$

$$= (x^2+1)\left(\frac{8}{x}+1\right)^2$$

$$\therefore \quad L(x) = \sqrt{x^2+1}\left(\frac{8}{x}+1\right)$$

c $$\frac{d[L(x)]^2}{dx} = 2x\left(1+\frac{8}{x}\right)^2 + (x^2+1)2\left(1+\frac{8}{x}\right)\left(-\frac{8}{x^2}\right) \quad \{\text{product rule}\}$$
$$= 2\left(1+\frac{8}{x}\right)\left[x\left(1+\frac{8}{x}\right) - (x^2+1)\left(\frac{8}{x^2}\right)\right]$$
$$= 2\left(1+\frac{8}{x}\right)\left[x+8-8-\frac{8}{x^2}\right]$$
$$= 2\left(\frac{x+8}{x}\right)\left(\frac{x^3-8}{x^2}\right)$$

d $\dfrac{d[L(x)]^2}{dx} = 0$ when $x=-8$ or $x^3=8$, but $x>0$ so $\dfrac{d[L(x)]^2}{dx} = 0$ when $x = 2$

The sign diagram for $\dfrac{d[L(x)]^2}{dx}$ is: 

$\therefore$ minimum $L(x)$ occurs when $x=2$ and the shortest length is $\sqrt{2^2+1}\left(1+\frac{8}{2}\right) = 5\sqrt{5}$
≈ 11.2 km

9 **a** $(3\cos\theta, 2\sin\theta)$ lies on the curve

$\therefore$ $x = 3\cos\theta$ and $y = 2\sin\theta$

$\therefore$ $x^2 = 9\cos^2\theta$ and $y^2 = 4\sin^2\theta$

$\therefore$ $\dfrac{x^2}{9}+\dfrac{y^2}{4} = \cos^2\theta + \sin^2\theta = 1$

$\therefore$ the curve has equation $\dfrac{x^2}{9}+\dfrac{y^2}{4} = 1$

b $$\frac{x^2}{9}+\frac{y^2}{4} = 1$$
$$\therefore \frac{2x}{9}+\frac{2y}{4}\frac{dy}{dx} = 0$$
$$\therefore \frac{dy}{dx} = \left(\frac{2}{y}\right)\left(-\frac{2x}{9}\right)$$
$$= -\frac{4x}{9y} = -\frac{4\times 3\cos\theta}{9\times 2\sin\theta}$$
$$= -\frac{2\cos\theta}{3\sin\theta}$$

c The tangent has gradient $-\dfrac{2\cos\theta}{3\sin\theta}$ and passes through $(3\cos\theta, 2\sin\theta)$.

$\therefore$ the tangent has equation $\dfrac{y-2\sin\theta}{x-3\cos\theta} = -\dfrac{2\cos\theta}{3\sin\theta}$

$$\therefore 3y\sin\theta - 6\sin^2\theta = -2x\cos\theta + 6\cos^2\theta$$
$$\therefore 2x\cos\theta + 3y\sin\theta = 6 \qquad \{\sin^2\theta+\cos^2\theta = 1\}$$

The tangent meets the x-axis when $y=0$

$$\therefore 2x\cos\theta = 6$$
$$\therefore x = \frac{3}{\cos\theta}$$
$$\therefore \text{A is at } \left(\frac{3}{\cos\theta}, 0\right)$$

The tangent meets the y-axis when $x=0$

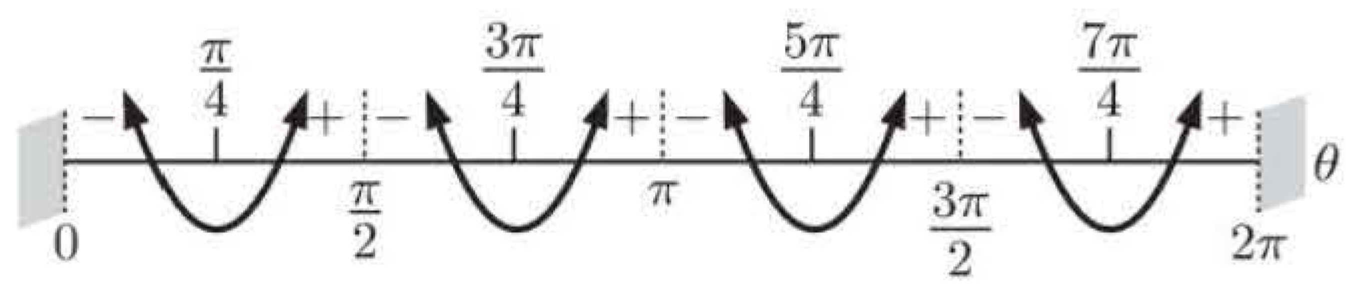
$$\therefore 3y\sin\theta = 6$$
$$\therefore y = \frac{2}{\sin\theta}$$
$$\therefore \text{B is at } \left(0, \frac{2}{\sin\theta}\right)$$

$\therefore$ triangle OAB has area $A = \left|\frac{1}{2}\left(\dfrac{3}{\cos\theta}\right)\left(\dfrac{2}{\sin\theta}\right)\right| = \left|\dfrac{6}{\sin 2\theta}\right|$ .... $(*)$

$$\therefore A^2 = 36(\sin 2\theta)^{-2}$$
$$\therefore \frac{d(A^2)}{d\theta} = -72(\sin 2\theta)^{-3}\times 2\cos 2\theta = -\frac{144\cos 2\theta}{\sin^3 2\theta}$$

Since $0 \leqslant \theta \leqslant 2\pi$, $0 \leqslant 2\theta \leqslant 4\pi$

$\therefore$ $\dfrac{d(A^2)}{d\theta} = 0$ when $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, and $\frac{7\pi}{4}$,
and is undefined when $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and 2π.

Sign diagram for $\dfrac{d(A^2)}{d\theta}$ is: 

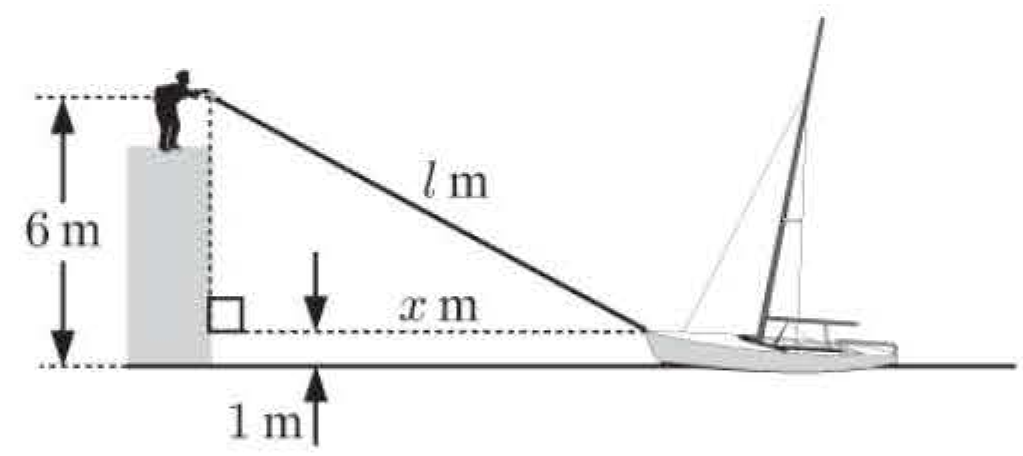
$\therefore$ there are local minima at $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

For all of these values of θ, $\sin 2\theta = -1$ or 1

$$\therefore \quad A = 6 \quad \{\text{using } (*)\}$$

$\therefore$ the smallest area of triangle OAB is 6 units^2, and this occurs when $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, or $\frac{7\pi}{4}$.

10 Let l m be the length of rope and x m be the distance of the boat from the jetty.



Then $x^2 + 5^2 = l^2$

$$\therefore \quad 2x\frac{dx}{dt} = 2l\frac{dl}{dt}$$

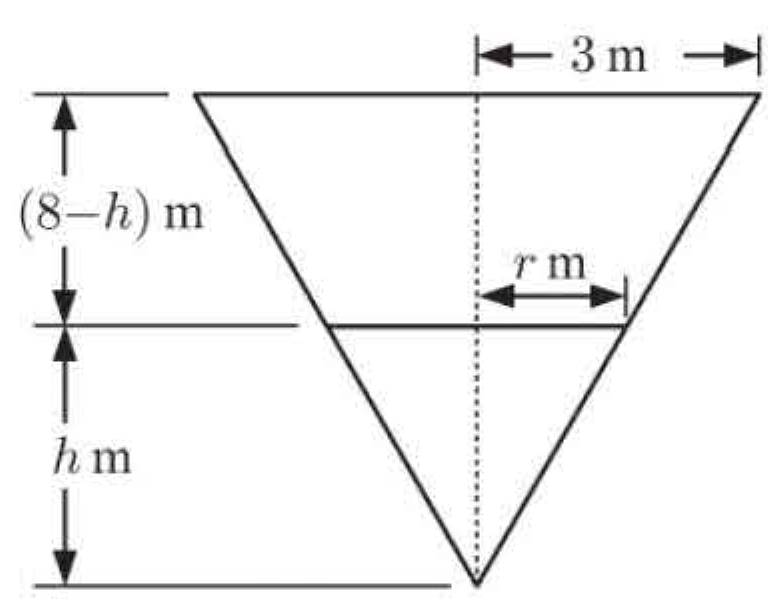
Particular case: When $x = 15$ m, $l = \sqrt{15^2 + 5^2} = \sqrt{250}$ and $\dfrac{dl}{dt} = -20 \text{ m min}^{-1}$

$$\therefore \quad 2(15)\frac{dx}{dt} = 2\sqrt{250}(-20)$$

$$\therefore \quad \frac{dx}{dt} = -\frac{40\sqrt{250}}{30} \approx -21.1$$

$\therefore$ the boat is approaching the jetty at 21.1 metres per minute.

11



a Volume $V = \frac{1}{3}\pi r^2 h$

Using similar triangles $\dfrac{h}{r} = \frac{8}{3}$

$$\therefore \quad h = \frac{8r}{3}$$

$$\therefore \quad V(r) = \tfrac{1}{3}\pi r^2\left(\frac{8r}{3}\right) = \tfrac{8\pi}{9}r^3 \text{ m}^3$$

b *Particular case:* When $h = 5$, $r = \dfrac{3h}{8} = \frac{15}{8}$ and $\dfrac{dV}{dt} = -0.2 = -\frac{1}{5} \text{ m}^3 \text{ min}^{-1}$

Now $\dfrac{dV}{dt} = \frac{8\pi}{3}r^2\dfrac{dr}{dt}$ $\quad\therefore\quad -\frac{1}{5} = \frac{8\pi}{3}\left(\frac{15}{8}\right)^2\dfrac{dr}{dt}$

$$\therefore \quad -\tfrac{1}{5} = \tfrac{225}{24}\pi\frac{dr}{dt}$$

$$\therefore \quad \frac{dr}{dt} = -\tfrac{8}{375\pi}$$

$$\therefore \quad \frac{dr}{dt} = -0.006\,79$$

$\therefore$ the radius is decreasing at 0.006 79 m per minute.

REVIEW SET 20C

1 **a** Volume = length × width × depth

$$\therefore \quad x^2y = 1$$

$$\therefore \quad y = \frac{1}{x^2}, \quad x > 0$$

b area $= x^2 + 4xy$

$$\therefore \quad \text{cost} = (x^2 + 4xy) \times 2$$

$$\therefore \quad C = 2x^2 + 8xy$$

$$= 2x^2 + \frac{8}{x} \text{ dollars} \qquad \{\text{using } \textbf{a}\}$$

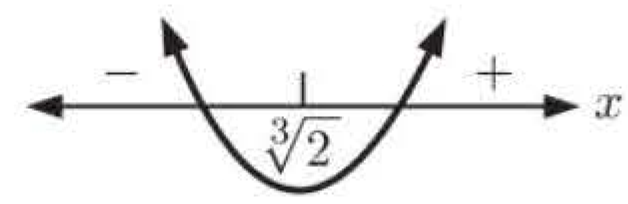
c $\dfrac{dC}{dx} = 4x - 8x^{-2}$

$= 4x - \dfrac{8}{x^2}$

$= \dfrac{4(x^3-2)}{x^2}$

So, $\dfrac{dC}{dx} = 0$ when $x = \sqrt[3]{2}$ m

$\dfrac{dC}{dx}$ has sign diagram:



The minimum cost is when $x = \sqrt[3]{2} \approx 1.26$ m

$\therefore\ y = \dfrac{1}{x^2} \approx 0.630$

and the box is 1.26 m by 1.26 m by 0.630 m.

2 **a** $s(t) = 15t - \dfrac{60}{(t+1)^2}$ cm, $t \geqslant 0$

$= 15t - 60(t+1)^{-2}$ cm

$\therefore\ v(t) = 15 + 120(t+1)^{-3}$ cm s^{-1}

$\therefore\ a(t) = -360(t+1)^{-4}$ cm s^{-2}

b When $t = 3$, $s(t) = 41.25$ cm

$v(t) \approx 16.88$ cm s^{-1}

$a(t) \approx -1.41$ cm s^{-2}

The particle is 41.25 cm right of O, travelling right at 16.88 cm s^{-1}, and is slowing down (decelerating) at 1.41 cm s^{-2}.

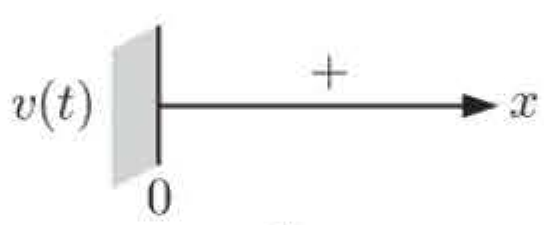
c $v(t) = 15 + \dfrac{120}{(t+1)^3}$ cm s^{-1}

$v(t) = 0$ when $15 + \dfrac{120}{(t+1)^3} = 0$

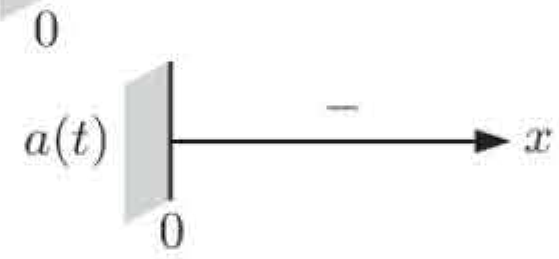
$\therefore\ 15(t+1)^3 + 120 = 0$

$\therefore\ (t+1)^3 = -8$

$\therefore\ t = -3$



$a(t) = -360(t+1)^{-4} = \dfrac{-360}{(t+1)^4}$ cm s^{-2}



where $(t+1)^4$ is always positive. $\therefore\ a(t) < 0$ for all $t > 0$

Since $v(t) > 0$ and $a(t) < 0$ for all $t > 0$, $v(t)$ is always decreasing.

$\therefore$ the particle's speed is never increasing.

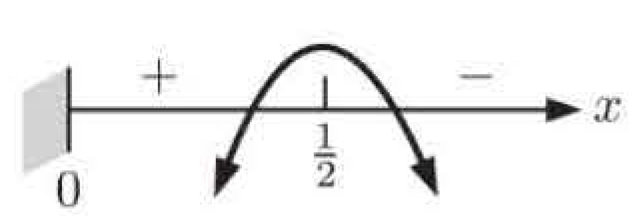
3 Let the coordinates of B be $(x, 0)$, so the coordinates of A are (x, e^{-2x}).

$\therefore$ the area OBAC is $A = xe^{-2x}$

$\therefore\ \dfrac{dA}{dx} = (1)e^{-2x} + x(-2e^{-2x})$ {product rule}

$= e^{-2x}(1-2x)$

$= \dfrac{1-2x}{e^{2x}}$ and has sign diagram:



So, the maximum area occurs when $x = \frac{1}{2}$ and $y = e^{-2(\frac{1}{2})} = e^{-1} = \frac{1}{e}$

$\therefore$ the coordinates of A are $\left(\frac{1}{2}, \frac{1}{e}\right)$.

4 **a** The tree was $H(0) = 6\left(1 - \frac{2}{3}\right) = 2$ metres tall when first planted.

b $t = 3$: $H(3) = 6\left(1 - \frac{2}{3+3}\right) = 4$ metres

$t = 6$: $H(6) = 6\left(1 - \frac{2}{6+3}\right) = 4\frac{2}{3}$ metres

$t = 9$: $H(9) = 6\left(1 - \frac{2}{9+3}\right) = 5$ metres

c $H(t) = 6\left(1 - \dfrac{2}{t+3}\right)$

$= 6 - 12(t+3)^{-1}$

$\therefore\ H'(t) = 12(t+3)^{-2}$

$= \dfrac{12}{(t+3)^2}$

So, $t = 0$: $H'(0) = \dfrac{12}{3^2} = \frac{4}{3}$ m year^{-1}

$t = 3$: $H'(3) = \dfrac{12}{6^2} = \frac{1}{3}$ m year^{-1}

$t = 6$: $H'(6) = \dfrac{12}{9^2} = \frac{4}{27}$ m year^{-1}

$t = 9$: $H'(9) = \dfrac{12}{12^2} = \frac{1}{12}$ m year^{-1}

d $$H'(t) = \frac{12}{(t+3)^2},$$

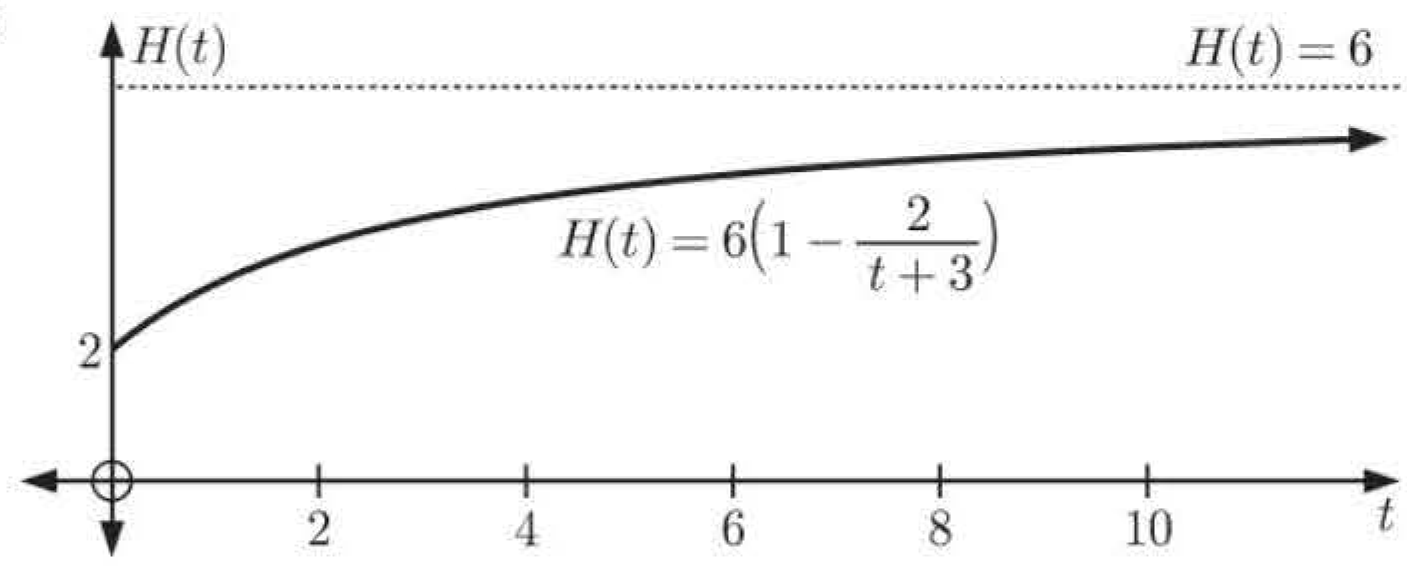
and $(t+3)^2 > 0$ for all $t \geqslant 0$

$\therefore \ \dfrac{12}{(t+3)^2} > 0$

$\therefore \ H'(t) > 0$ for all $t \geqslant 0$

This means that the height of the tree is always increasing.

e



5 **a** $s(t) = 25t - 10\ln t$ cm, $t \geqslant 1$

$\therefore \ v(t) = 25 - \dfrac{10}{t}$ cm min^{-1}

$\therefore \ a(t) = 10t^{-2} = \dfrac{10}{t^2}$ cm min^{-2}

b When $t = e$,

$s(e) = 25e - 10\ln e = (25e - 10)$ cm

≈ 58.0 cm

$v(e) = \left(25 - \dfrac{10}{e}\right)$ cm min^{-1}

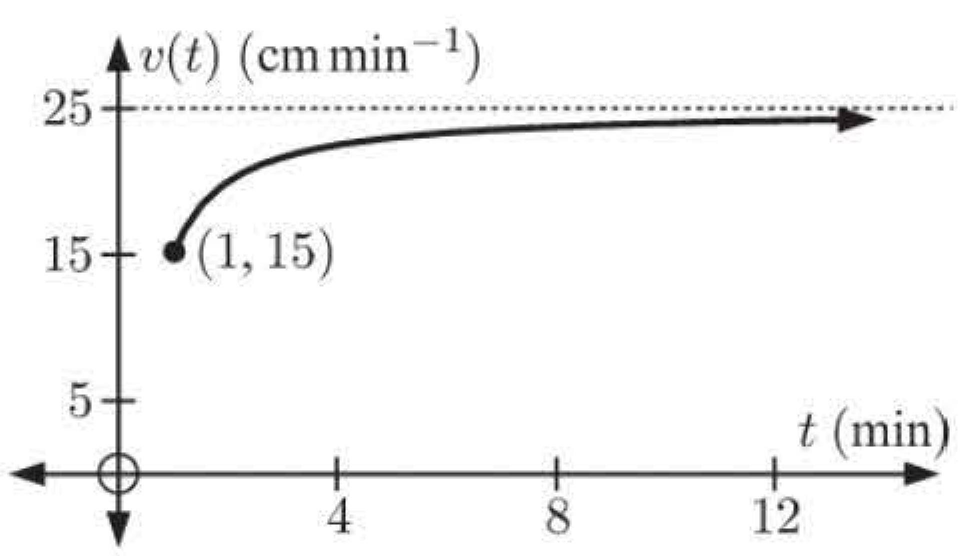
≈ 21.3 cm min^{-1}

$a(e) = \dfrac{10}{e^2}$ cm min^{-2} ≈ 1.35 cm min^{-2}

c As $t \to \infty$, $\dfrac{10}{t} \to 0$

$\therefore \ v(t) \to 25$ cm min^{-1} (below)

d



e When $v(t) = 20$ cm min^{-1},

$25 - \dfrac{10}{t} = 20$

$\therefore \ \dfrac{10}{t} = 5$

$\therefore \ t = 2$ minutes

6 **a** AC $= 2x$ m

Now ABC is an isosceles triangle.

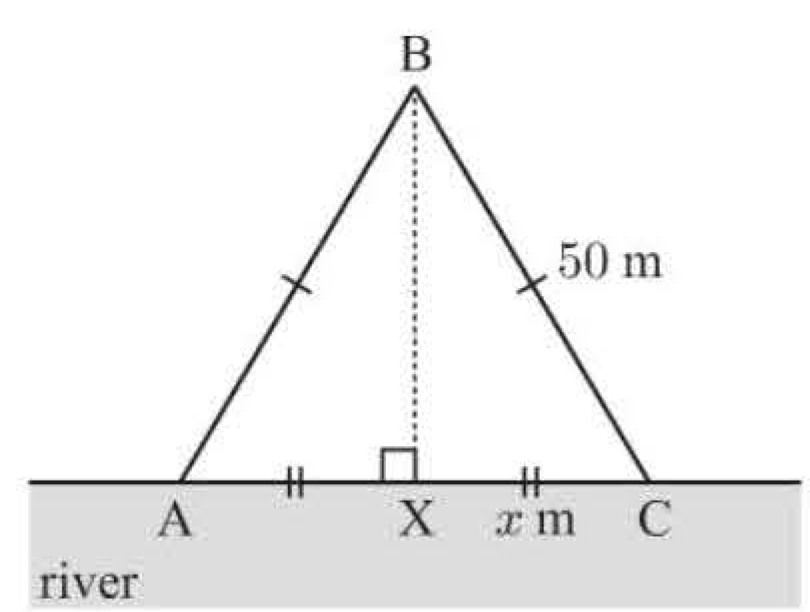
$\therefore$ XC $= x$

But BC^2 = BX^2 + XC^2 {Pythagoras}

$\therefore \ 2500 = \text{BX}^2 + x^2$

$\therefore \ \text{BX} = \sqrt{2500 - x^2}$

$\therefore \ A(x) = \frac{1}{2}(2x)\sqrt{2500 - x^2} = x\sqrt{2500 - x^2}$

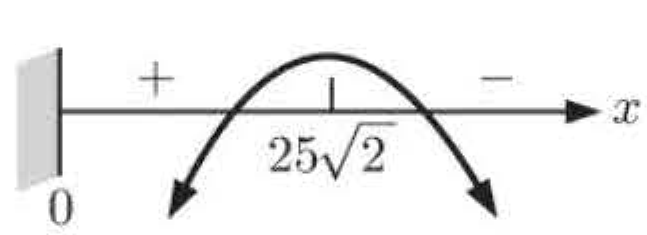


b Now $[A(x)]^2 = x^2(2500 - x^2)$

$\therefore \ A^2 = 2500x^2 - x^4$

$$\therefore \ \frac{d[(A(x)]^2}{dx} = 5000x - 4x^3$$
$$= 4x(1250 - x^2)$$
$$= 4x(\sqrt{1250} + x)(\sqrt{1250} - x)$$

c Sign diagram for $\dfrac{d\,[A(x)]^2}{dx}$ is:



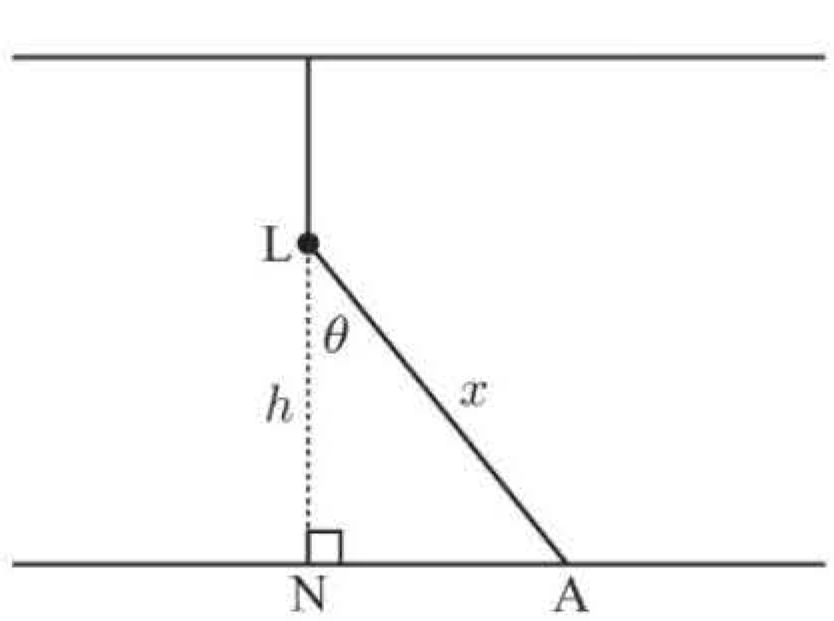
$\therefore$ the maximum area occurs when $x = 25\sqrt{2}$ m ≈ 35.4 m

The corresponding maximum area $= \sqrt{1250} \times \sqrt{1250} = 1250$ m^2.

7 **a** $\sin\theta = \dfrac{\text{NA}}{x} = \dfrac{1}{x}$

$\therefore \ \dfrac{1}{x^2} = \sin^2\theta$

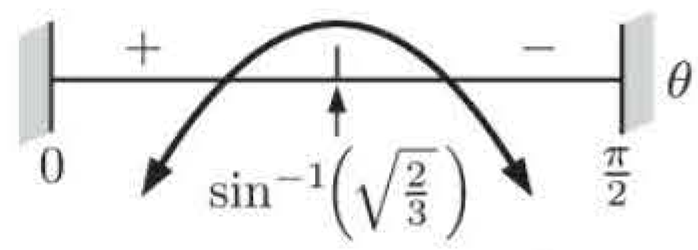
$\therefore$ at A, $I = \dfrac{\sqrt{8}\cos\theta}{x^2} = \sqrt{8}\cos\theta\sin^2\theta$



b $\dfrac{dI}{d\theta} = \sqrt{8}(-\sin\theta)\sin^2\theta + \sqrt{8}\cos\theta(2\sin\theta\cos\theta)$
$= \sqrt{8}\sin\theta[2\cos^2\theta - \sin^2\theta]$
$= \sqrt{8}\sin\theta[2(1-\sin^2\theta) - \sin^2\theta]$
$= \sqrt{8}\sin\theta[2 - 3\sin^2\theta]$

$\dfrac{dI}{d\theta} = 0$ when $\sin\theta = \sqrt{\frac{2}{3}},\ 0 < \theta < \frac{\pi}{2}$

and the sign diagram of $\dfrac{dI}{d\theta}$ is:

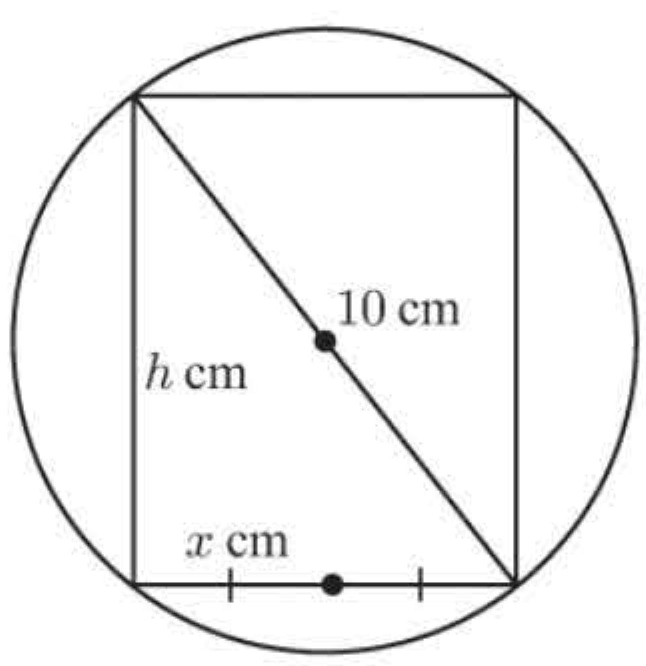


$\therefore$ the maximum illumination at A is obtained when $\sin\theta = \sqrt{\frac{2}{3}}$

$\therefore$ $x = \dfrac{1}{\sin\theta} = \sqrt{\frac{3}{2}}$ and $h = \sqrt{x^2 - \text{NA}^2} = \sqrt{\frac{3}{2} - 1} = \frac{1}{\sqrt{2}}$

$\therefore$ the bulb is $\frac{1}{\sqrt{2}}$ m above the floor.

8 **a**



Let the height of the cylinder be h cm.

$\therefore$ $(2x)^2 + h^2 = 10^2$ {Pythagoras}

$\therefore$ $h = \sqrt{100 - 4x^2}$

$\therefore$ $V(x) = $ area of base $\times$ height
$= \pi x^2 \times \sqrt{100 - 4x^2}$

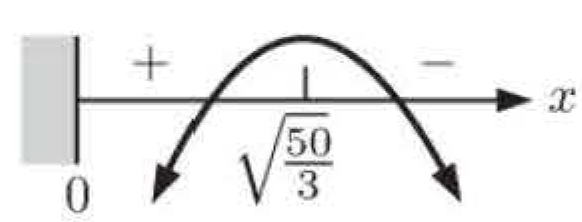
So, $V(x) = \pi x^2\sqrt{100 - 4x^2}$ cm^3

b Now $V^2 = \pi^2 x^4(100 - 4x^2)$
$= \pi^2(100x^4 - 4x^6)$

$\therefore$ $\dfrac{d(V^2)}{dx} = \pi^2(400x^3 - 24x^5)$
$= 8\pi^2x^3(50 - 3x^2)$
$= 8\pi^2x^3(\sqrt{50} + \sqrt{3}x)(\sqrt{50} - \sqrt{3}x)$

$\therefore$ $\dfrac{d(V^2)}{dx} = 0$ when $x = \sqrt{\frac{50}{3}}$ {as $x > 0$}

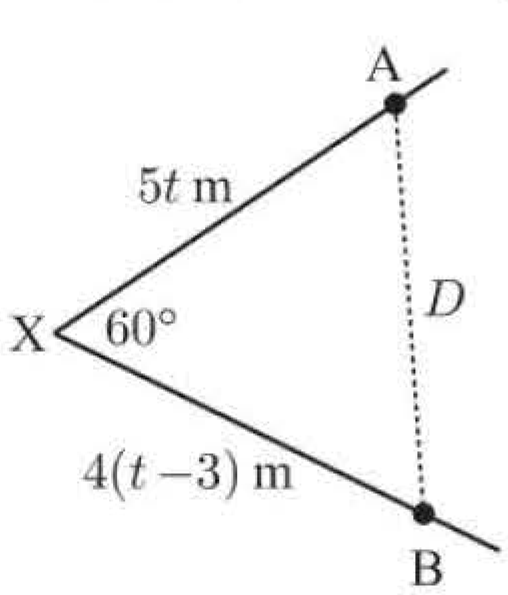
and $\dfrac{d(V^2)}{dx}$ has sign diagram:



$\therefore$ maximum V occurs when $x = \sqrt{\frac{50}{3}} \approx 4.08$

$\therefore$ radius ≈ 4.08 cm, height $= \sqrt{100 - 4\left(\frac{50}{3}\right)} \approx 5.77$ cm

9 Let t be the number of seconds after A passes through X. In this time, A travels $5t$ m.
B passes through X when $t = 3$.
$\therefore$ for $t > 3$, B is $4(t-3)$ m from X.



Using the cosine rule,

$$D^2 = 25t^2 + 16(t-3)^2 - 2 \times 5t \times 4(t-3) \times \cos 60°$$
$$= 25t^2 + 16(t-3)^2 - 20t(t-3)$$

$$\therefore\ 2D\frac{dD}{dt} = 50t + 32(t-3) - 20(t-3) - 20t$$

When $5t = 20$, $t = 4$ and $D^2 = 25 \times 16 + 16 - 20 \times 4 = 336$

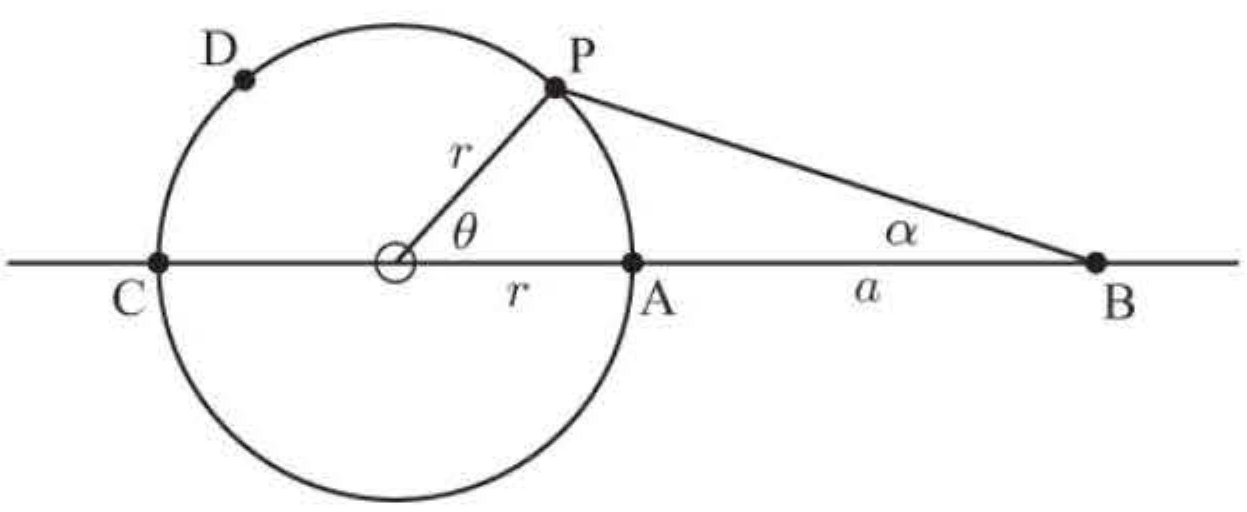
$$\therefore\ 2\sqrt{336}\,\frac{dD}{dt} = 200 + 32 - 20 - 80 = 132$$

$$\therefore\ \frac{dD}{dt} = \frac{66}{\sqrt{336}} \approx 3.60\ \text{m s}^{-1}$$

10 **a** Using the cosine rule for $\triangle$BPO,

$$\text{BP}^2 = r^2 + (a+r)^2 - 2r(a+r)\cos\theta$$

$$\therefore \ \text{BP} = \sqrt{r^2 + (a+r)^2 - 2r(a+r)\cos\theta}$$



$\therefore$ time taken to travel from B to P

$$= \frac{\text{distance}}{\text{speed}}$$

$$= \frac{\sqrt{r^2 + (a+r)^2 - 2r(a+r)\cos\theta}}{v}$$

Now arc AP $= r\theta$

$$\therefore \ \text{arc PC} = (\text{perimeter of semi-circle}) - \text{arc AP}$$
$$= \tfrac{1}{2} \times 2\pi r - r\theta$$
$$= r(\pi - \theta)$$

$$\therefore \ \text{the time taken to travel from P to C} = \frac{\text{distance}}{\text{speed}} = \frac{r(\pi-\theta)}{w}$$

The total time for the journey $T = \dfrac{\sqrt{r^2 + (a+r)^2 - 2r(a+r)\cos\theta}}{v} + \dfrac{r(\pi-\theta)}{w}$

b

$$T = \frac{w\left[r^2 + (a+r)^2 - 2r(a+r)\cos\theta\right]^{\frac{1}{2}} + rv(\pi-\theta)}{vw}$$

$$\therefore \ \frac{dT}{d\theta} = \frac{\frac{1}{2}w\left[r^2 + (a+r)^2 - 2r(a+r)\cos\theta\right]^{-\frac{1}{2}}(2r(a+r)\sin\theta) - rv}{vw}$$

$$= \frac{2r(a+r)\sin\theta}{2v\sqrt{r^2 + (a+r)^2 - 2r(a+r)\cos\theta}} - \frac{rv}{vw}$$

$$= \frac{r(a+r)\sin\theta}{v \times \text{BP}} - \frac{rv}{vw}$$

Now $\dfrac{\text{BP}}{\sin\theta} = \dfrac{r}{\sin\alpha}$ {sine rule}

$$\therefore \ \text{BP} = \frac{r\sin\theta}{\sin\alpha}$$

$$\therefore \ \frac{dT}{d\theta} = \frac{r(a+r)\sin\theta}{v \times \dfrac{r\sin\theta}{\sin\alpha}} - \frac{rv}{vw}$$

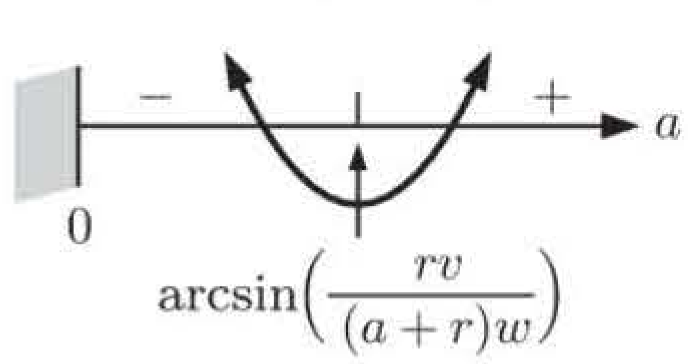
$$= \frac{a+r}{v}\sin\alpha - \frac{rv}{vw}$$

$$= \frac{a+r}{v}\left(\sin\alpha - \frac{rv}{(a+r)w}\right)$$

c Now $a + r \neq 0$ so $\dfrac{dT}{d\theta} = 0$ when $\sin\alpha - \dfrac{rv}{(a+r)w} = 0$

$$\therefore \ \sin\alpha = \frac{rv}{(a+r)w}$$

Sign diagram for $\dfrac{dT}{d\theta}$ is:



$\therefore$ T is minimised when $\sin\alpha = \dfrac{rv}{(a+r)w}$