Chapter 9

MATHEMATICAL INDUCTION

EXERCISE 9A

1 The nth term of the sequence $3, 7, 11, 15, 19, ...$ is $4n-1$ for $n \in \mathbb{Z}^+$.

2 **a**
$3^1 = 3 \qquad 1+2(1) = 3$
$3^2 = 9 \qquad 1+2(2) = 5$
$3^3 = 27 \qquad 1+2(3) = 7$
$3^4 = 81 \qquad 1+2(4) = 9$

Our proposition is:
$3^n > 1 + 2n$ for $n = 2, 3, 4, 5, ...$
or for all $n \in \mathbb{Z}^+, \quad n \geqslant 2$

b
$11^1 - 1 = 10$
$11^2 - 1 = 121 - 1 = 120$
$11^3 - 1 = 1331 - 1 = 1330$
$11^4 - 1 = 14\,641 - 1 = 14\,640$

Our proposition is:
$11^n - 1$ is divisible by 10 for all $n \in \mathbb{Z}^+$

c
$7^1 + 2 = 7 + 2 = 9 = 3 \times 3$
$7^2 + 2 = 49 + 2 = 51 = 3 \times 17$
$7^3 + 2 = 343 + 2 = 345 = 3 \times 115$
$7^4 + 2 = 2401 + 2 = 2403 = 3 \times 801$

Our proposition is:
$7^n + 2$ is divisible by 3 for all $n \in \mathbb{Z}^+$

d
$(1 - \frac{1}{2}) = \frac{1}{2}$
$(1 - \frac{1}{2})(1 - \frac{1}{3}) = \frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$
$(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{4}) = \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} = \frac{1}{4}$
$(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{4})(1 - \frac{1}{5}) = \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5} = \frac{1}{5}$

Our proposition is: $(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{4}) .... \left(1 - \dfrac{1}{n+1}\right) = \dfrac{1}{n+1}$ for all $n \in \mathbb{Z}^+$

3 **a**
$2 = 2 = 1 \times 2$
$2 + 4 = 6 = 2 \times 3$
$2 + 4 + 6 = 12 = 3 \times 4$
$2 + 4 + 6 + 8 = 20 = 4 \times 5$
$2 + 4 + 6 + 8 + 10 = 30 = 5 \times 6$

Our proposition is:
$2 + 4 + 6 + 8 + 10 + ... + \underbrace{2n}_{n\text{th term}} = n(n+1)$ for all $n \in \mathbb{Z}^+$

$\therefore \displaystyle\sum_{i=1}^{n} 2i = n(n+1)$ for all $n \in \mathbb{Z}^+$

b
$1! = 1$
$1! + 2 \times 2! = 1 + 2(2) = 5$
$1! + 2 \times 2! + 3 \times 3! = 1 + 4 + 18 = 23$
$1! + 2 \times 2! + 3 \times 3! + 4 \times 4! = 1 + 4 + 18 + 96 = 119$

where each number result is 1 less than a factorial number

$1 = 2! - 1$
$5 = 3! - 1$
$23 = 4! - 1$
$119 = 5! - 1$

Our proposition is:
$1! + 2 \times 2! + 3 \times 3! + 4 \times 4! + ... + n \times n! = (n+1)! - 1$
for all $n \in \mathbb{Z}^+$

$\therefore \displaystyle\sum_{i=1}^{n} i \times i! = (n+1)! - 1$ for all $n \in \mathbb{Z}^+$

c
$\dfrac{1}{2!} = \dfrac{1}{2} = \dfrac{2! - 1}{2!}$

$\dfrac{1}{2!} + \dfrac{2}{3!} = \dfrac{1}{2} + \dfrac{2}{6} = \dfrac{5}{6} = \dfrac{3! - 1}{3!}$

$\dfrac{1}{2!} + \dfrac{2}{3!} + \dfrac{3}{4!} = \dfrac{1}{2} + \dfrac{2}{6} + \dfrac{3}{24} = \dfrac{23}{24} = \dfrac{4! - 1}{4!}$

$\dfrac{1}{2!} + \dfrac{2}{3!} + \dfrac{3}{4!} + \dfrac{4}{5!} = \dfrac{1}{2} + \dfrac{2}{6} + \dfrac{3}{24} + \dfrac{4}{120} = \dfrac{119}{120} = \dfrac{5! - 1}{5!}$

Our proposition is: $\dfrac{1}{2!}+\dfrac{2}{3!}+\dfrac{3}{4!}+\dfrac{4}{5!}+....+\dfrac{n}{(n+1)!}=\dfrac{(n+1)!-1}{(n+1)!}$ for all $n\in\mathbb{Z}^+$

$$\therefore \sum_{i=1}^{n}\frac{i}{(i+1)!}=\frac{(n+1)!-1}{(n+1)!} \quad \text{for all } n\in\mathbb{Z}^+$$

d $\dfrac{1}{2\times5}=\dfrac{1}{10}$

$\dfrac{1}{2\times5}+\dfrac{1}{5\times8}=\dfrac{1}{10}+\dfrac{1}{40}=\dfrac{5}{40}=\dfrac{1}{8}=\dfrac{2}{16}$

$\dfrac{1}{2\times5}+\dfrac{1}{5\times8}+\dfrac{1}{8\times11}=\dfrac{1}{10}+\dfrac{1}{40}+\dfrac{1}{88}=\dfrac{3}{22}$

$\dfrac{1}{2\times5}+\dfrac{1}{5\times8}+\dfrac{1}{8\times11}+\dfrac{1}{11\times14}=\dfrac{1}{7}=\dfrac{4}{28}$

10, 16, 22, 28 is arithmetic with $u_1=10,\ d=6$

$$\begin{aligned}u_n&=u_1+(n-1)d\\&=10+6(n-1)\\&=6n+4\end{aligned}$$

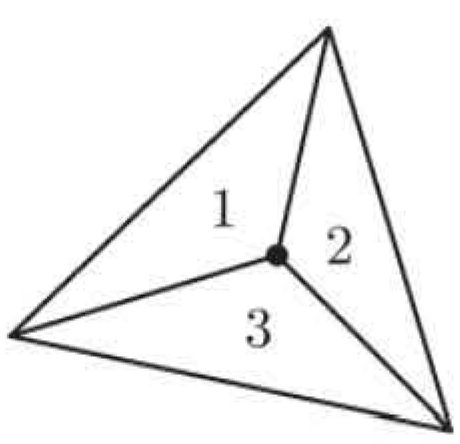
Our proposition is:

$\dfrac{1}{2\times5}+\dfrac{1}{5\times8}+\dfrac{1}{8\times11}+\dfrac{1}{11\times14}+....+\dfrac{1}{(3n-1)(3n+2)}=\dfrac{n}{6n+4}$ for all $n\in\mathbb{Z}^+$

{2, 5, 8, 11 are arithmetic with $u_1=2,\ \ d=3 \ \ \therefore\ \ u_n=2+(n-1)3=3n-1$}

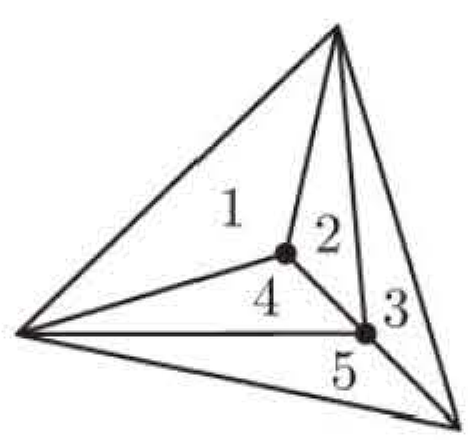
$$\therefore \sum_{i=1}^{n}\frac{1}{(3i-1)(3i+2)}=\frac{n}{6n+4} \quad \text{for all } n\in\mathbb{Z}^+$$

4 For $n=1$



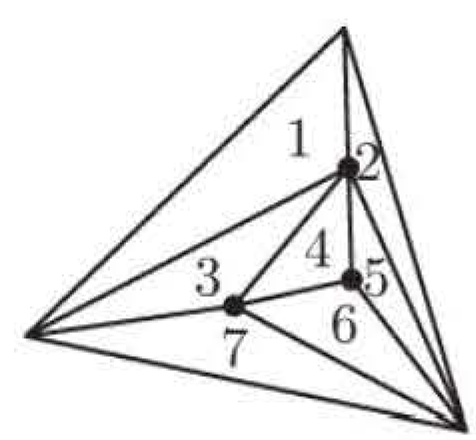
$T_1=3=2\times1+1$

For $n=2$



$T_2=5=2\times2+1$

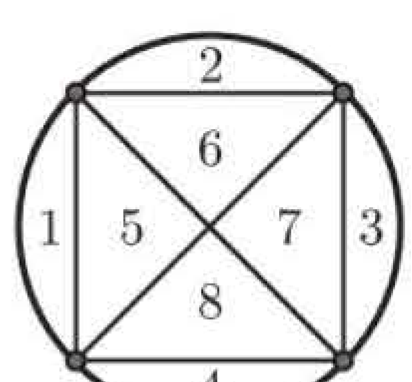
For $n=3$



$T_3=7=2\times3+1$

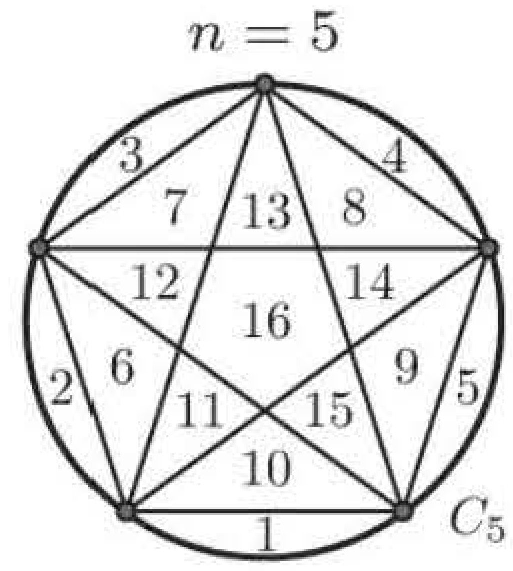
Our proposition is: The maximum number of triangles for n points within the original triangle is given by $T_n=2n+1$ for all $n\in\mathbb{Z}^+$.

5 **a** $n=4$



$C_4=8$

$n=5$

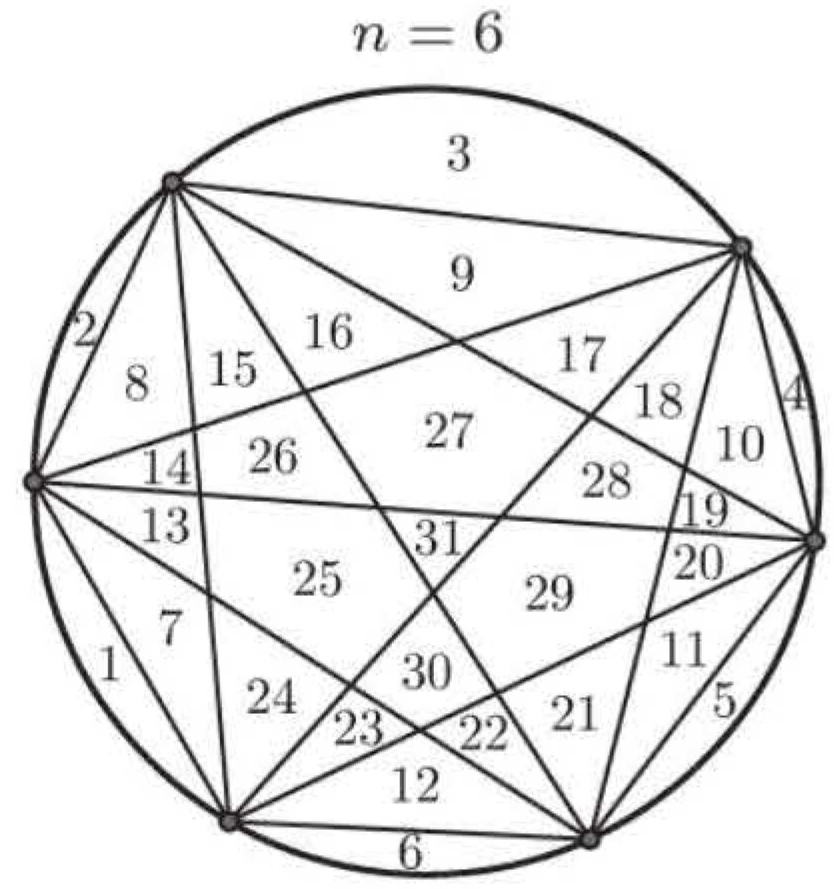


$C_5=16$

b When $n=1,\ \ C_1=1=2^0=2^{1-1}$
$n=2,\ \ C_2=2=2^1=2^{2-1}$
$n=3,\ \ C_3=4=2^2=2^{3-1}$
$n=4,\ \ C_4=8=2^3=2^{4-1}$
$n=5,\ \ C_5=16=2^4=2^{5-1}$

So, from the cases $n=1,\ 2,\ 3,\ 4,\ 5,$ our conjecture is:
The number of regions for n points placed around a circle is given by $C_n=2^{n-1}$ for all $n\in\mathbb{Z}^+$.

c $n=6$



By the conjecture we expect $2^{6-1}=2^5=32$ regions, but there are only 31.
So, we no longer believe the conjecture.

EXERCISE 9B.1

1 **a** If $n = 0$, $3^n + 1 = 3^0 + 1 = 2$ which is divisible by 2.

$$\begin{aligned} 3^n + 1 &= (1+2)^n + 1 \\ &= 1^n + \binom{n}{1}2 + \binom{n}{2}2^2 + \binom{n}{3}2^3 + ... + \binom{n}{n-1}2^{n-1} + \binom{n}{n}2^n + 1 \\ &= 2 + \binom{n}{1}2 + \binom{n}{2}2^2 + \binom{n}{3}2^3 + ... + \binom{n}{n-1}2^{n-1} + \binom{n}{n}2^n \\ &= 2\left(1 + \binom{n}{1} + \binom{n}{2}2 + \binom{n}{3}2^2 + ... + \binom{n}{n-1}2^{n-2} + \binom{n}{n}2^{n-1}\right) \end{aligned}$$

where the contents of the brackets is an integer.

$\therefore$ $3^n + 1$ is divisible by 2.

b P_n is: $3^n + 1$ is divisible by 2 for all integers $n \geqslant 0$.

Proof: (By the principle of mathematical induction)

(1) If $n = 0$, $3^0 + 1 = 2 = 1 \times 2$ $\therefore$ P_0 is true.

(2) If P_k is true, then $3^k + 1 = 2A$ where A is an integer, and $A \geqslant 1$.

Now
$$\begin{aligned} 3^{k+1} + 1 &= 3^1 3^k + 1 \\ &= 3(2A - 1) + 1 \quad \{\text{using } P_k\} \\ &= 6A - 3 + 1 \\ &= 6A - 2 \\ &= 2(3A - 1) \quad \text{where } 3A - 1 \text{ is an integer as } A \in \mathbb{Z} \end{aligned}$$

Thus $3^{k+1} + 1$ is divisible by 2 if $3^k + 1$ is divisible by 2.

Since P_0 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all integers $n \geqslant 0$ {Principle of mathematical induction}

2 **a** If $n = 0$, $6^n - 1 = 6^0 - 1 = 0$ which is divisible by 5.

$$\begin{aligned} 6^n - 1 &= (5+1)^n - 1 \\ &= 5^n + \binom{n}{1}5^{n-1} + \binom{n}{2}5^{n-2} + \binom{n}{3}5^{n-3} + ... + \binom{n}{n-1}5 + 1^n - 1 \\ &= 5^n + \binom{n}{1}5^{n-1} + \binom{n}{2}5^{n-2} + \binom{n}{3}5^{n-3} + ... + \binom{n}{n-1}5 \\ &= 5\left(5^{n-1} + \binom{n}{1}5^{n-2} + \binom{n}{2}5^{n-3} + \binom{n}{3}5^{n-4} + ... + \binom{n}{n-1}\right) \end{aligned}$$

where the contents of the brackets is an integer.

$\therefore$ $6^n - 1$ is divisible by 5.

b P_n is: $6^n - 1$ is divisible by 5 for all integers $n \geqslant 0$

Proof: (By the principle of mathematical induction)

(1) If $n = 0$, $6^0 - 1 = 0$ which is divisible by 5 $\therefore$ P_0 is true

(2) If P_k is true, then $6^k - 1 = 5A$ where $A \in \mathbb{N}$

Now $6^{k+1} - 1$
$$\begin{aligned} &= 6^1 6^k - 1 \\ &= 6(5A + 1) - 1 \quad \{\text{using } P_k\} \\ &= 30A + 6 - 1 \\ &= 30A + 5 \\ &= 5(6A + 1) \quad \text{where } 6A + 1 \text{ is an integer as } A \in \mathbb{N} \end{aligned}$$

Thus, $6^{k+1} - 1$ is divisible by 5 if $6^k - 1$ is divisible by 5.

Since P_0 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all integers $n \geqslant 0$ {Principle of mathematical induction}

3 **a** P_n is: $n^3 + 2n$ is divisible by 3 for all positive integers n

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $1^3 + 2(1) = 3$ which is divisible by 3

(2) If P_k is true, then $k^3 + 2k = 3A$ where $A \in \mathbb{Z}$

Now $(k+1)^3 + 2(k+1)$

$= k^3 + 3k^2 + 3k + 1 + 2k + 2$

$= (k^3 + 2k) + 3k^2 + 3k + 3$

$= 3A + 3k^2 + 3k + 3$ {using P_k}

$= 3(A + k^2 + k + 1)$ where $A + k^2 + k + 1$ is an integer as A and k are integers

Thus $(k+1)^3 + 2(k+1)$ is divisible by 3 if $k^3 + 2k$ is divisible by 3.

Since P_1 is true, and P_{k+1} is true whenever P_k is true, then P_n is true for all positive integers n {Principle of mathematical induction}

b P_n is: $n(n^2 + 5)$ is divisible by 6 for all integers $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $1(1^2 + 5) = 1 \times 6 = 6$ which is divisible by 6 $\therefore$ P_1 is true

(2) If P_k is true, then $k(k^2 + 5) = 6A$ where A is an integer

Now $(k+1)[(k+1)^2 + 5] = (k+1)(k^2 + 2k + 1 + 5)$

$= (k+1)(k^2 + 2k + 6)$

$= k^3 + 2k^2 + 6k + k^2 + 2k + 6$

$= k^3 + 5k + [3k^2 + 3k + 6]$

$= k(k^2 + 5) + 3k(k+1) + 6$

$= 6A + 6 + 3k(k+1)$

We notice that $k(k+1)$ is the product of consecutive integers, one of which must be even $\therefore$ $k(k+1) = 2B$ where $B \in \mathbb{Z}$

$\therefore$ $(k+1)[(k+1)^2 + 5] = 6A + 6 + 3(2B)$

$= 6(A + 1 + B)$ where $A + 1 + B \in \mathbb{Z}$

Thus $(k+1)[(k+1)^2 + 5]$ is divisible by 6 if $k(k^2 + 5)$ is divisible by 6.

Since P_1 is true, and P_{k+1} is true whenever P_k is true, then P_n is true for all integers $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

c P_n is: $7^n - 4^n - 3^n$ is divisible by 12 for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $7^1 - 4^1 - 3^1 = 0$ which is divisible by 12 $\therefore$ P_1 is true

(2) If P_k is true, then $7^k - 4^k - 3^k = 12A$ where $A \in \mathbb{Z}$

Now $7^{k+1} - 4^{k+1} - 3^{k+1}$

$= 7(7^k) - 4(4^k) - 3(3^k)$

$= 7[12A + 4^k + 3^k] - 4(4^k) - 3(3^k)$ {using P_k}

$= 84A + 7(4^k) + 7(3^k) - 4(4^k) - 3(3^k)$

$= 84A + 3(4^k) + 4(3^k)$

$= 84A + 3 \times 4 \times 4^{k-1} + 4 \times 3 \times 3^{k-1}$

$= 12(7A + 4^{k-1} + 3^{k-1})$ where $k \geqslant 2$, $k \in \mathbb{Z}^+$

$= 12 \times$ an integer {as 4^{k-1} and 3^{k-1} are integers}

Thus $7^{k+1} - 4^{k+1} - 3^{k+1}$ is divisible by 12 if $7^k - 4^k - 3^k$ is divisible by 12.

Since P_1 is true, and P_{k+1} is true whenever P_k is true, then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

EXERCISE 9B.2

1 **a** P_n is: $1+2+3+4+ .... +n = \dfrac{n(n+1)}{2}$ for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n=1$, LHS $=1$ and RHS $= \dfrac{1(2)}{2} = 1$, $\therefore$ P_1 is true

(2) If P_k is true then $1+2+3+4+ .... +k = \dfrac{k(k+1)}{2}$

Thus $1+2+3+4+ .... +k+(k+1)$

$$= \frac{k(k+1)}{2} + k + 1 \qquad \{\text{using } P_k\}$$

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \qquad \{\text{to equalise denominators}\}$$

$$= \frac{(k+1)(k+2)}{2} \qquad \{\text{common factor of } \frac{(k+1)}{2}\}$$

$$= \frac{(k+1)([k+1]+1)}{2}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

b P_n is: $1 \times 2 + 2 \times 3 + 3 \times 4 + ... + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$ for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n=1$, LHS $= 1 \times 2 = 2$ and RHS $= \dfrac{1(2)(3)}{3} = 2$, $\therefore$ P_1 is true

(2) If P_k is true then

$$1 \times 2 + 2 \times 3 + 3 \times 4 + ... + k(k+1) = \frac{k(k+1)(k+2)}{3}$$

$\therefore$ $1 \times 2 + 2 \times 3 + 3 \times 4 + ... + k(k+1) + (k+1)(k+2)$

$$= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \qquad \{\text{using } P_k\}$$

$$= \frac{k(k+1)(k+2)}{3} + \frac{3(k+1)(k+2)}{3} \qquad \{\text{to equalise denominators}\}$$

$$= \frac{(k+1)(k+2)(k+3)}{3} \qquad \{\text{common factor of } (k+1)(k+2)\}$$

$$= \frac{[k+1]([k+1]+1)([k+1]+2)}{3}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

c P_n is: $3 \times 5 + 6 \times 6 + 9 \times 7 + 12 \times 8 + ... + 3n(n+4) = \dfrac{n(n+1)(2n+13)}{2}$
for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n=1$, LHS $= 3 \times 5 = 15$, RHS $= \dfrac{1 \times 2 \times (2+13)}{2} = 15$, $\therefore$ P_1 is true

(2) If P_k is true, then

$$3 \times 5 + 6 \times 6 + 9 \times 7 + ... + 3k(k+4) = \frac{k(k+1)(2k+13)}{2}$$

Now $3 \times 5 + 6 \times 6 + 9 \times 7 + ... + 3k(k+4) + 3(k+1)(k+5)$

$$= \frac{k(k+1)(2k+13)}{2} + 3(k+1)(k+5) \quad \{\text{using } P_k\}$$

$$= \frac{k(k+1)(2k+13)}{2} + \frac{6(k+1)(k+5)}{2} \quad \{\text{to equalise denominators}\}$$

$$= \frac{(k+1)[k(2k+13)+6(k+5)]}{2} \quad \{\text{common factor}\}$$

$$= \frac{(k+1)[2k^2+19k+30]}{2}$$

$$= \frac{(k+1)(k+2)(2k+15)}{2}$$

$$= \frac{(k+1)([k+1]+1)(2[k+1]+13)}{2}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

d P_n is: $1^3 + 2^3 + 3^3 + 4^3 + ... + n^3 = \dfrac{n^2(n+1)^2}{4}$ for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= 1^3 = 1$, RHS $= \dfrac{1^2(2)^2}{4} = 1$ $\therefore$ P_1 is true

(2) If P_k is true, then $1^3 + 2^3 + 3^3 + ... + k^3 = \dfrac{k^2(k+1)^2}{4}$

Now $1^3 + 2^3 + 3^3 + ... + k^3 + (k+1)^3$

$$= \frac{k^2(k+1)^2}{4} + (k+1)^3 \quad \{\text{using } P_k\}$$

$$= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4} \quad \{\text{equalising denominators}\}$$

$$= \frac{(k+1)^2[k^2+4(k+1)]}{4} \quad \{\text{common factor}\}$$

$$= \frac{(k+1)^2(k^2+4k+4)}{4}$$

$$= \frac{(k+1)^2(k+2)^2}{4}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

2 **a** The sum of the first n odd numbers $1 + 3 + 5 + 7 + ... + 2n - 1$ is the sum of the first n terms of an arithmetic series.

$u_1 = 1$, $d = 2$. $\therefore$ $u_n = u_1 + (n-1)d = 1 + 2(n-1) = 2n - 1$.

Thus $S_n = \dfrac{n}{2}(2u_1 + (n-1)d)$

$$= \frac{n}{2}(2 \times 1 + 2(n-1))$$

$$= \frac{n}{2}(2 + 2n - 2)$$

$$= \frac{n}{2}(2n)$$

$$= n^2$$

So, the sum of the first n odd numbers is n^2.

b P_n is: $1+3+5+7+....+(2n-1)=n^2$ for all $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n=1$, LHS $=1$, RHS $=1^2=1$ $\therefore$ P_1 is true.

(2) If P_k is true, then $1+3+5+7+....+(2k-1)=k^2$

Now $1+3+5+7+....+(2k-1)+[2(k+1)-1]$

$$\begin{aligned}&= k^2+[2(k+1)-1] \quad \{\text{using } P_k\}\\&= k^2+2k+2-1\\&= k^2+2k+1\\&= (k+1)^2\end{aligned}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

3 P_n is: $1\times 2^0+2\times 2+3\times 2^2+4\times 2^3+....+n\times 2^{n-1}=(n-1)\times 2^n+1$ for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n=1$, LHS $=1$, RHS $=(0)2^0+1=1$, $\therefore$ P_1 is true

(2) If P_k is true, then $1\times 2^0+2\times 2+3\times 2^2+4\times 2^3+....+k\times 2^{k-1}=(k-1)2^k+1$

Now $1\times 2^0+2\times 2+3\times 2^2+4\times 2^3+....+k\,2^{k-1}+(k+1)2^k$

$$\begin{aligned}&= (k-1)2^k+1+(k+1)2^k \quad \{\text{using } P_k\}\\&= 2^k(k-1+k+1)+1\\&= 2^k(2k)+1\\&= k\,2^{k+1}+1\end{aligned}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

4 **a** P_n is: $\dfrac{1}{1\times 2}+\dfrac{1}{2\times 3}+\dfrac{1}{3\times 4}+....+\dfrac{1}{n(n+1)}=\dfrac{n}{n+1}$ for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n=1$, LHS $=\dfrac{1}{1\times 2}=\dfrac{1}{2}$, RHS $=\dfrac{1}{1+1}=\dfrac{1}{2}$ $\therefore$ P_1 is true

(2) If P_k is true, then $\dfrac{1}{1\times 2}+\dfrac{1}{2\times 3}+\dfrac{1}{3\times 4}+....+\dfrac{1}{k(k+1)}=\dfrac{k}{k+1}$

Now $\dfrac{1}{1\times 2}+\dfrac{1}{2\times 3}+\dfrac{1}{3\times 4}+....+\dfrac{1}{k(k+1)}+\dfrac{1}{(k+1)(k+2)}$

$$\begin{aligned}&= \frac{k}{k+1}+\frac{1}{(k+1)(k+2)} && \{\text{using } P_k\}\\&= \frac{k}{k+1}\left(\frac{k+2}{k+2}\right)+\frac{1}{(k+1)(k+2)} && \{\text{equalising denominators}\}\\&= \frac{k(k+2)+1}{(k+1)(k+2)}\\&= \frac{k^2+2k+1}{(k+1)(k+2)}\\&= \frac{(k+1)^2}{(k+1)(k+2)}\\&= \frac{k+1}{k+2}=\frac{k+1}{(k+1)+1}\end{aligned}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

b $\dfrac{1}{10\times 11}+\dfrac{1}{11\times 12}+\dfrac{1}{12\times 13}+\ldots.+\dfrac{1}{20\times 21}=S_{20}-S_9=\frac{20}{21}-\frac{9}{10}=\frac{11}{210}$

c P_n is: $\dfrac{1}{1\times 2\times 3}+\dfrac{1}{2\times 3\times 4}+\dfrac{1}{3\times 4\times 5}+\ldots.+\dfrac{1}{n(n+1)(n+2)}=\dfrac{n(n+3)}{4(n+1)(n+2)}$
for all $n\in\mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n=1$, LHS $=\dfrac{1}{1\times 2\times 3}=\dfrac{1}{6}$, RHS $=\dfrac{1(4)}{4(2)(3)}=\dfrac{1}{6}$ $\therefore$ P_1 is true.

(2) If P_k is true, then

$$\frac{1}{1\times 2\times 3}+\frac{1}{2\times 3\times 4}+\frac{1}{3\times 4\times 5}+\ldots.+\frac{1}{k(k+1)(k+2)}=\frac{k(k+3)}{4(k+1)(k+2)}$$

Now

$$\frac{1}{1\times 2\times 3}+\frac{1}{2\times 3\times 4}+\frac{1}{3\times 4\times 5}+\ldots.+\frac{1}{k(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)}$$

$$=\frac{k(k+3)}{4(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)} \qquad \{\text{using } P_k\}$$

$$=\frac{k(k+3)}{4(k+1)(k+2)}\left(\frac{k+3}{k+3}\right)+\frac{4}{4(k+1)(k+2)(k+3)} \qquad \{\text{equalising denominators}\}$$

$$=\frac{k(k+3)^2+4}{4(k+1)(k+2)(k+3)}$$

$$=\frac{k(k^2+6k+9)+4}{4(k+1)(k+2)(k+3)}$$

$$=\frac{k^3+6k^2+9k+4}{4(k+1)(k+2)(k+3)} \qquad \begin{array}{r|rrr|r} -1 & 1 & 6 & 9 & 4 \\ & 0 & -1 & -5 & -4 \\ \hline & 1 & 5 & 4 & 0 \end{array}$$

$$=\frac{\cancel{(k+1)}(k^2+5k+4)}{4\cancel{(k+1)}(k+2)(k+3)}$$

$$=\frac{(k+1)(k+4)}{4(k+2)(k+3)}$$

$$=\frac{(k+1)([k+1]+3)}{4([k+1]+1)([k+1]+2)}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n\in\mathbb{Z}^+$ {Principle of mathematical induction}

5 **a** P_n is: $1\times 1!+2\times 2!+3\times 3!+\ldots.+n\times n!=(n+1)!-1$ for all $n\in\mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n=1$, LHS $=1\times 1!$ $\quad$ RHS $=2!-1$
$\qquad\qquad\qquad\; =1\times 1 \qquad\quad =2-1$
$\qquad\qquad\qquad\; =1 \qquad\qquad\;\; =1$ $\therefore$ P_1 is true

(2) If P_k is true, then $1\times 1!+2\times 2!+3\times 3!+\ldots.+k\times k!=(k+1)!-1$

Now $1\times 1!+2\times 2!+3\times 3!+\ldots.+k\times k!+(k+1)(k+1)!$

$=(k+1)!-1+(k+1)(k+1)!$ {using P_k}

$=(k+1)!(1+k+1)-1$

$=(k+1)!(k+2)-1$

$=(k+2)!-1$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n\in\mathbb{Z}^+$ {Principle of mathematical induction}

b P_n is: $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+....+\frac{n}{(n+1)!}=\frac{(n+1)!-1}{(n+1)!}$ for all $n\in\mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n=1$, LHS $=\frac{1}{2!}=\frac{1}{2}$, RHS $=\frac{2!-1}{2!}=\frac{2-1}{2}=\frac{1}{2}$ $\therefore$ P_1 is true

(2) If P_k is true, then $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+....+\frac{k}{(k+1)!}=\frac{(k+1)!-1}{(k+1)!}$

Now $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+....+\frac{k}{(k+1)!}+\frac{k+1}{(k+2)!}$

$=\frac{(k+1)!-1}{(k+1)!}+\frac{k+1}{(k+2)!}$ {using P_k}

$=\left(\frac{k+2}{k+2}\right)\left[\frac{(k+1)!-1}{(k+1)!}\right]+\frac{k+1}{(k+2)!}$ {equalising denominators}

$=\frac{(k+2)!-(k+2)+k+1}{(k+2)!}$

$=\frac{(k+2)!-k-2+k+1}{(k+2)!}$

$=\frac{(k+2)!-1}{(k+2)!}$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n\in\mathbb{Z}^+$ {Principle of mathematical induction}

c $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+....+\frac{9}{10!}=\frac{10!-1}{10!}=\frac{3\,628\,799}{3\,628\,800}$

6 P_n is: $1\times n+2\times(n-1)+3\times(n-2)+....+(n-1)\times 2+n\times 1=\frac{n(n+1)(n+2)}{6}$
for all $n\in\mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n=1$, LHS $=1\times 1=1$, RHS $=\frac{1(2)(3)}{6}=1$ $\therefore$ P_1 is true

(2) If P_k is true, then $1\times k+2(k-1)+3(k-2)+....+(k-1)2+k\times 1=\frac{k(k+1)(k+2)}{6}$

Now $1(k+1)+2(k)+3(k-1)+....+k\,2+(k+1)1$

$=1(k)+2(k-1)+3(k-2)+....+k\,1+1+2+3+....+k+k+1$

{using the hint}

$=\frac{k(k+1)(k+2)}{6}+\frac{(k+1)(k+2)}{2}$ {using P_k and the sum of an arithmetic series}

$=\frac{k(k+1)(k+2)}{6}+\frac{3(k+1)(k+2)}{6}$ {equalising denominators}

$=\frac{(k+1)(k+2)(k+3)}{6}$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n\in\mathbb{Z}^+$ {Principle of mathematical induction}

7 **a** P_n is: if $u_1=5$ and $u_{n+1}=u_n+8n+5$ for all $n\in\mathbb{Z}^+$, then $u_n=4n^2+n$

Proof: (By the principle of mathematical induction)

(1) If $n=1$, $u_1=4(1)^2+1=5$ which is true and so P_1 is true.

(2) If P_k is true, then $u_k = 4k^2 + k$ and
$$\begin{aligned} u_{k+1} &= u_k + 8k + 5 \\ &= 4k^2 + k + 8k + 5 \quad \{\text{using } P_k\} \\ &= 4(k^2 + 2k + 1) + k + 1 \\ &= 4(k+1)^2 + (k+1) \end{aligned}$$

$\therefore$ P_{k+1} is also true.

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

b P_n is: if $u_1 = 1$ and $u_{n+1} = 2 + 3u_n$ for all $n \in \mathbb{Z}^+$, then $u_n = 2(3^{n-1}) - 1$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $u_1 = 2(3^{1-1}) - 1 = 1$ which is true and so P_1 is true.

(2) If P_k is true, then $u_k = 2(3^{k-1}) - 1$ and
$$\begin{aligned} u_{k+1} &= 2 + 3u_k \\ &= 2 + 3(2[3^{k-1}] - 1) \quad \{\text{using } P_k\} \\ &= 2 + 2 \times 3^k - 3 \\ &= 2(3^k) - 1 \end{aligned}$$

$\therefore$ P_{k+1} is also true.

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

c P_n is: if $u_1 = 2$ and $u_{n+1} = \dfrac{u_n}{2(n+1)}$ for all $n \in \mathbb{Z}^+$, then $u_n = \dfrac{2^{2-n}}{n!}$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $u_1 = \dfrac{2^{2-1}}{1!} = 2^1 = 2$ which is true and so P_1 is true.

(2) If P_k is true, then $u_k = \dfrac{2^{2-k}}{k!}$ and
$$\begin{aligned} u_{k+1} = \frac{u_k}{2(k+1)} &= \frac{2^{2-k}}{k!\,2(k+1)} \quad \{\text{using } P_k\} \\ &= \frac{2^{2-k-1}}{(k+1)k!} \\ &= \frac{2^{2-(k+1)}}{(k+1)!} \end{aligned}$$

$\therefore$ P_{k+1} is also true.

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

d P_n is: if $u_1 = 1$ and $u_{n+1} = u_n + (-1)^n(n+1)^2$ for all $n \in \mathbb{Z}^+$,
then $u_n = \dfrac{(-1)^{n-1}n(n+1)}{2}$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $u_1 = \dfrac{(-1)^0 \times 1 \times 2}{2} = 1$ which is true and so P_1 is true.

(2) If P_k is true, then $u_k = \dfrac{(-1)^{k-1}k(k+1)}{2}$

and
$$\begin{aligned} u_{k+1} &= u_k + (-1)^k(k+1)^2 \\ &= \frac{(-1)^{k-1}k(k+1)}{2} + (-1)^k(k+1)^2 \quad \{\text{using } P_k\} \\ &= \frac{(-1)^{k-1}k(k+1) + 2(-1)^k(k+1)^2}{2} \\ &= \frac{2(-1)^k(k+1)^2 - (-1)^k k(k+1)}{2} \end{aligned}$$

$$\therefore \quad u_{k+1} = \frac{(-1)^k(k+1)[2(k+1)-k]}{2}$$

$$= \frac{(-1)^k(k+1)(k+2)}{2} \qquad \therefore \quad P_{k+1} \text{ is also true.}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

8 **a** $u_1 = 1 = 1^2$
$u_2 = u_1 + 2(1) + 1 = 1 + 2 + 1 = 4 = 2^2$
$u_3 = u_2 + 2(2) + 1 = 4 + 4 + 1 = 9 = 3^2$
$u_4 = u_3 + 2(3) + 1 = 9 + 6 + 1 = 16 = 4^2$
We conjecture that $u_n = n^2$ for all $n \in \mathbb{Z}^+$

b P_n is: if $u_1 = 1$ and $u_{n+1} = u_n + (2n+1)$ for all $n \in \mathbb{Z}^+$, then $u_n = n^2$
Proof: (By the principle of mathematical induction)

(1) If $n = 1$, then $u_1 = 1 = 1^2$, so P_1 is true.

(2) If P_k is true, then $u_k = k^2$ and
$$u_{k+1} = u_k + (2k+1)$$
$$= k^2 + 2k + 1 \quad \{\text{using } P_k\}$$
$$= (k+1)^2 \quad \therefore \quad P_{k+1} \text{ is also true.}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

9 **a** $u_1 = \frac{1}{3}$

$$u_2 = u_1 + \frac{1}{(2(1)+1)(2(1)+3)} = \frac{1}{3} + \frac{1}{3 \times 5} = \frac{5+1}{15} = \frac{6}{15} = \frac{2}{5}$$

$$u_3 = u_2 + \frac{1}{(2(2)+1)(2(2)+3)} = \frac{2}{5} + \frac{1}{5 \times 7} = \frac{14+1}{35} = \frac{15}{35} = \frac{3}{7}$$

$$u_4 = u_3 + \frac{1}{(2(3)+1)(2(3)+3)} = \frac{3}{7} + \frac{1}{7 \times 9} = \frac{27+1}{63} = \frac{28}{63} = \frac{4}{9}$$

We conjecture that $u_n = \dfrac{n}{2n+1}$ for all $n \in \mathbb{Z}^+$

b P_n is: if $u_1 = \frac{1}{3}$ and $u_{n+1} = u_n + \dfrac{1}{(2n+1)(2n+3)}$ for all $n \in \mathbb{Z}^+$, then $u_n = \dfrac{n}{2n+1}$
Proof: (By the principle of mathematical induction)

(1) If $n = 1$, then $u_1 = \dfrac{1}{3} = \dfrac{1}{2(1)+1}$, so P_1 is true.

(2) If P_k is true, then $u_k = \dfrac{k}{2k+1}$

and
$$u_{k+1} = u_k + \frac{1}{(2k+1)(2k+3)}$$
$$= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} \quad \{\text{using } P_k\}$$
$$= \frac{k(2k+3)+1}{(2k+1)(2k+3)}$$
$$= \frac{2k^2+3k+1}{(2k+1)(2k+3)}$$
$$= \frac{\cancel{(2k+1)}(k+1)}{\cancel{(2k+1)}(2k+3)}$$
$$= \frac{k+1}{2(k+1)+1} \qquad \therefore \quad P_{k+1} \text{ is also true.}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

10 **a** $\left(2+\sqrt{3}\right)^1 = 2+\sqrt{3}$ $\quad\therefore\ A_1 = 2,\quad B_1 = 1$

$\left(2+\sqrt{3}\right)^2 = 4+4\sqrt{3}+3$

$= 7+4\sqrt{3}$ $\quad\therefore\ A_2 = 7,\quad B_2 = 4$

$\left(2+\sqrt{3}\right)^3 = (7+4\sqrt{3})(2+\sqrt{3})$

$= 14+7\sqrt{3}+8\sqrt{3}+4(3)$

$= 26+15\sqrt{3}$ $\quad\therefore\ A_3 = 26,\quad B_3 = 15$

$\left(2+\sqrt{3}\right)^4 = \left(7+4\sqrt{3}\right)^2$

$= 49+56\sqrt{3}+16(3)$

$= 97+56\sqrt{3}$ $\quad\therefore\ A_4 = 97,\quad B_4 = 56$

b $\left(2+\sqrt{3}\right)^n = A_n + B_n\sqrt{3}$

$$\begin{aligned}\therefore\quad \left(2+\sqrt{3}\right)^{n+1} &= \left(2+\sqrt{3}\right)^n\left(2+\sqrt{3}\right)\\ &= \left(A_n+B_n\sqrt{3}\right)\left(2+\sqrt{3}\right)\\ &= 2A_n + A_n\sqrt{3} + 2B_n\sqrt{3} + B_n(3)\\ &= 2A_n + 3B_n + (A_n+2B_n)\sqrt{3}\end{aligned}$$

$\therefore\ A_{n+1} = 2A_n + 3B_n,\quad B_{n+1} = A_n + 2B_n$

c $A_1^{\,2} - 3B_1^{\,2} = 2^2 - 3(1)^2 = 4-3 = 1$

$A_2^{\,2} - 3B_2^{\,2} = 7^2 - 3(4)^2 = 49 - 3\times 16 = 1$

$A_3^{\,2} - 3B_3^{\,2} = 26^2 - 3(15)^2 = 676 - 3\times 225 = 1$

$A_4^{\,2} - 3B_4^{\,2} = 97^2 - 3(56)^2 = 9409 - 3\times 3136 = 1$

$\therefore$ we conjecture $(A_n)^2 - 3(B_n)^2 = 1$ for all $n\in\mathbb{Z}^+$

d P_n is: if $\left(2+\sqrt{3}\right)^n = A_n + B_n\sqrt{3}$ for all $n\in\mathbb{Z}^+$, then $A_n^{\,2} - 3B_n^{\,2} = 1$

Proof: (By the principle of mathematical induction)

(1) If $n=1$, $A_1 = 2$, $B_1 = 1$, and $A_1^{\,2} - 3B_1^{\,2} = 2^2 - 3(1)^2 = 1$, so P_1 is true.

(2) If P_k is true, then $A_k^{\,2} - 3B_k^{\,2} = 1$, and

$$\begin{aligned}A_{k+1}^2 - 3B_{k+1}^2 &= (2A_k+3B_k)^2 - 3(A_k+2B_k)^2 \quad \{\text{using } \mathbf{b}\}\\ &= 4A_k^{\,2} + 12A_kB_k + 9B_k^{\,2} - 3(A_k^{\,2} + 4A_kB_k + 4B_k^{\,2})\\ &= 4A_k^{\,2} + \cancel{12A_kB_k} + 9B_k^{\,2} - 3A_k^{\,2} - \cancel{12A_kB_k} - 12B_k^{\,2}\\ &= A_k^{\,2} - 3B_k^{\,2}\\ &= 1 \qquad \{\text{using } P_k\}\end{aligned}$$

$\therefore\ P_{k+1}$ is true.

Since P_1 is true, and P_{k+1} is true whenever P_k is true, then P_n is true for all $n\in\mathbb{Z}^+$ {Principle of mathematical induction}

11 P_n is: $\dfrac{2^n - (-1)^n}{3}$ is an odd number for all $n\in\mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n=1$, $\dfrac{2^1-(-1)^1}{3} = \dfrac{3}{3} = 1$ which is odd $\therefore\ P_1$ is true

(2) If P_k is true, then $\dfrac{2^k - (-1)^k}{3} = 2A+1$ where $A\in\mathbb{Z}$

Now $$\begin{aligned}\frac{2^{k+1}-(-1)^{k+1}}{3} &= \frac{2(2^k) - (-1)^{k+1}}{3}\\ &= \frac{2[6A+3+(-1)^k] - (-1)^{k+1}}{3} \quad \{\text{using } P_k\}\end{aligned}$$

$$\therefore \quad \frac{2^{k+1}-(-1)^{k+1}}{3} = \frac{12A+6+2(-1)^k-(-1)(-1)^k}{3}$$
$$= \frac{12A+6+2(-1)^k+(-1)^k}{3}$$
$$= \frac{12A+6+3(-1)^k}{3}$$
$$= 4A+2+(-1)^k$$

Now $4A+2$ is always even and $(-1)^k$ is either $+1$ or -1

$\therefore$ $4A+2+(-1)^k$ is odd

$\therefore$ $\dfrac{2^{k+1}-(-1)^{k+1}}{3}$ is odd

Since P_1 is true, and P_{k+1} is true whenever P_k is true, then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

12 **a** P_n is: if $u_1 = 11$, $u_2 = 37$, and $u_{n+2} = 5u_{n+1} - 6u_n$ for all $n \in \mathbb{Z}^+$, then $u_n = 5(3^n) - 2^{n+1}$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $5(3^1) - 2^{1+1} = 15 - 2^2 = 11 = u_1$, so P_1 is true.
If $n = 2$, $5(3^2) - 2^{2+1} = 5 \times 9 - 2^3 = 37 = u_2$, so P_2 is true.

(2) If P_k and P_{k+1} are true, then $u_k = 5(3^k) - 2^{k+1}$ and $u_{k+1} = 5(3^{k+1}) - 2^{k+2}$ and

$$\begin{aligned} u_{k+2} &= 5u_{k+1} - 6u_k \\ &= 5[5(3^{k+1}) - 2^{k+2}] - 6[5(3^k) - 2^{k+1}] \quad \{\text{using } P_k \text{ and } P_{k+1}\} \\ &= 25(3^{k+1}) - 5(2^{k+2}) - 30(3^k) + 6(2^{k+1}) \\ &= 25(3^{k+1}) - 5(2^{k+2}) - 10(3^{k+1}) + 3(2^{k+2}) \\ &= 15(3^{k+1}) - 2(2^{k+2}) \\ &= 5(3^{k+2}) - 2^{k+3} \qquad \therefore \ P_{k+2} \text{ is true.} \end{aligned}$$

Since P_1 and P_2 are true, and P_{k+2} is true whenever P_k and P_{k+1} are true, then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

b

$$u_{n+2} = au_{n+1} + bu_n$$

$$\therefore \ \left(3+\sqrt{5}\right)^{n+2} + \left(3-\sqrt{5}\right)^{n+2} = a\left[\left(3+\sqrt{5}\right)^{n+1} + \left(3-\sqrt{5}\right)^{n+1}\right] + b\left[\left(3+\sqrt{5}\right)^{n} + \left(3-\sqrt{5}\right)^{n}\right]$$

$$\therefore \ \left(3+\sqrt{5}\right)^2\left(3+\sqrt{5}\right)^{n} + \left(3-\sqrt{5}\right)^2\left(3-\sqrt{5}\right)^{n} = a\left[\left(3+\sqrt{5}\right)\left(3+\sqrt{5}\right)^{n} + \left(3-\sqrt{5}\right)\left(3-\sqrt{5}\right)^{n}\right] + b\left[\left(3+\sqrt{5}\right)^{n} + \left(3-\sqrt{5}\right)^{n}\right]$$

$$\therefore \ \left(14+6\sqrt{5}\right)\left(3+\sqrt{5}\right)^{n} + \left(14-6\sqrt{5}\right)\left(3-\sqrt{5}\right)^{n} = \left(3a+a\sqrt{5}\right)\left(3+\sqrt{5}\right)^{n} + \left(3a-a\sqrt{5}\right)\left(3-\sqrt{5}\right)^{n} + b\left(3+\sqrt{5}\right)^{n} + b\left(3-\sqrt{5}\right)^{n}$$

$$\therefore \ \left(14+6\sqrt{5}\right)\left(3+\sqrt{5}\right)^{n} + \left(14-6\sqrt{5}\right)\left(3-\sqrt{5}\right)^{n} = \left(3a+b+a\sqrt{5}\right)\left(3+\sqrt{5}\right)^{n} + \left(3a+b-a\sqrt{5}\right)\left(3-\sqrt{5}\right)^{n}$$

Equating coefficients of $\left(3+\sqrt{5}\right)^n$, $14+6\sqrt{5} = 3a+b+a\sqrt{5}$

Equating rational and irrational parts,

$3a + b = 14$ and $a = 6$

$\therefore$ $a = 6$ and $b = -4$ {this checks with the coefficients of $\left(3-\sqrt{5}\right)^n$}

$\therefore$ $u_{n+2} = 6u_{n+1} - 4u_n$

P_n is: if $u_n = \left(3+\sqrt{5}\right)^n + \left(3-\sqrt{5}\right)^n$ where $n \in \mathbb{Z}^+$, then u_n is a multiple of 2^n

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $u_1 = (3+\sqrt{5}) + (3-\sqrt{5})$
$= 6$ which is a multiple of $2^1 = 2$, so P_1 is true.

If $n = 2$, $u_2 = (3+\sqrt{5})^2 + (3-\sqrt{5})^2$
$= 9 + \cancel{6\sqrt{5}} + 5 + 9 - \cancel{6\sqrt{5}} + 5$
$= 28$ which is a multiple of $2^2 = 4$, so P_2 is true.

(2) If P_k and P_{k+1} are true, then $u_k = A \times 2^k$, where $A \in \mathbb{Z}$
and $u_{k+1} = B \times 2^{k+1}$, where $B \in \mathbb{Z}$

Now $u_{k+2} = 6u_{k+1} - 4u_k$ {using the above result}

$$\begin{aligned} &= 6(B \times 2^{k+1}) - 4(A \times 2^k) \\ &= 3B \times 2^{k+2} - A \times 2^{k+2} \\ &= (3B - A) \times 2^{k+2} \end{aligned}$$

which is a multiple of 2^{k+2} since $3B - A \in \mathbb{Z}$ {as $A, B \in \mathbb{Z}$}
$\therefore$ P_{k+2} is true.

Since P_1 and P_2 are true, and P_{k+2} is true whenever P_k and P_{k+1} are true, then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

EXERCISE 9B.3

1 **a** P_n is: $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)....\left(1-\frac{1}{n+1}\right) = \frac{1}{n+1}$ for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= \left(1-\frac{1}{2}\right) = \frac{1}{2}$, RHS $= \dfrac{1}{1+1} = \dfrac{1}{2}$ $\therefore$ P_1 is true

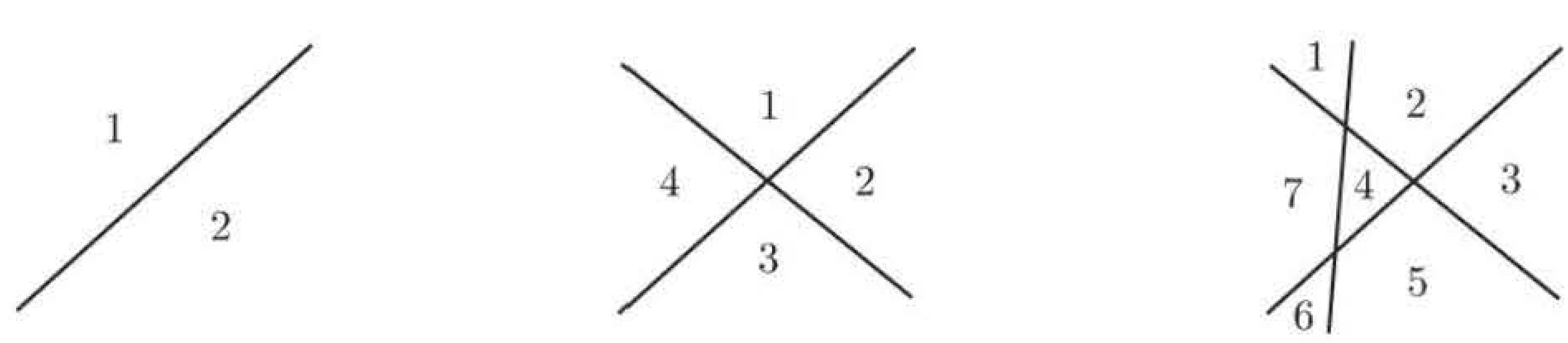
(2) If P_k is true, then $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)....\left(1-\frac{1}{k+1}\right) = \dfrac{1}{k+1}$

$\therefore$ $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)....\left(1-\frac{1}{k+1}\right)\left(1-\frac{1}{k+2}\right)$

$$\begin{aligned} &= \frac{1}{k+1}\left(1-\frac{1}{k+2}\right) \quad \{\text{using } P_k\} \\ &= \frac{1}{k+1}\left(\frac{k+2-1}{k+2}\right) \\ &= \frac{1}{k+1}\left(\frac{k+1}{k+2}\right) \\ &= \frac{1}{k+2} \end{aligned}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true, then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

b



1 line, $R_1 = 2$ 2 lines, $R_2 = 4$ 3 lines, $R_3 = 7$, and so on.

P_n is: For n lines as described, $R_n = \dfrac{n(n+1)}{2} + 1$ for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $R_1 = \dfrac{1(2)}{2} + 1 = 1 + 1 = 2 \quad \therefore \quad P_1$ is true

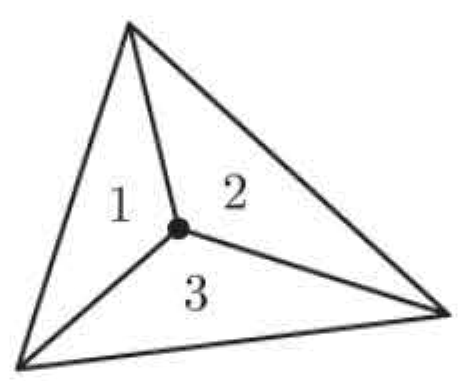
(2) If P_k is true, then $R_k = \dfrac{k(k+1)}{2} + 1$

The addition of another line creates another $k+1$ regions

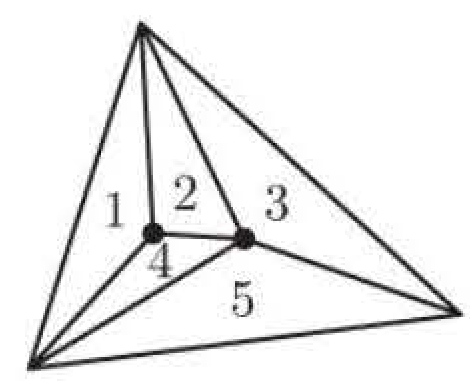
$$\begin{aligned}
\therefore \quad R_{k+1} &= \frac{k(k+1)}{2} + 1 + k + 1 \qquad \{\text{using } P_k\} \\
&= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} + 1 \\
&= \frac{k^2 + k + 2k + 2}{2} + 1 \\
&= \frac{k^2 + 3k + 2}{2} + 1 \\
&= \frac{(k+1)(k+2)}{2} + 1
\end{aligned}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

c



$n = 1, \; T_1 = 3$



$n = 2, \; T_2 = 5$

P_n is: For n points inside the triangle (as described) there are $T_n = 2n + 1$ triangular partitions for all $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

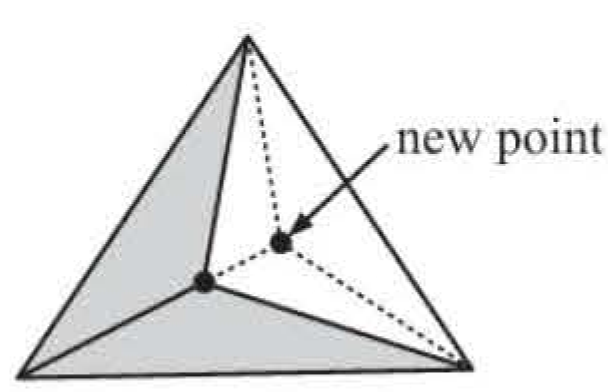
(1) If $n = 1$, $T_1 = 2(1) + 1 = 3 \quad \therefore \quad P_1$ is true

(2) If P_k is true, then $T_k = 2k + 1$

Adding an extra point within the triangle gives the $(k+1)$th case.

This point could be either

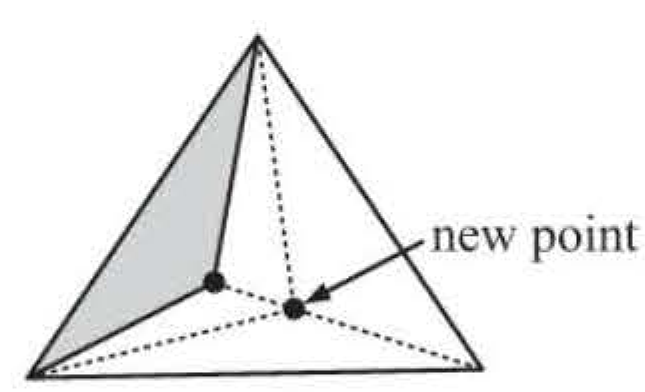
- in an existing triangle



So, 1 triangle becomes 3, a net increase of 2.

or

- on an existing line between 2 triangles



So, 2 triangles become 4, a net increase of 2.

In each case 2 triangles are added

$$\begin{aligned}
\therefore \quad T_{k+1} &= T_k + 2 \\
&= 2k + 1 + 2 \quad \{\text{using } P_k\} \\
&= 2(k+1) + 1
\end{aligned}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

d P_n is: $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)....\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$ for all $n \geqslant 2,\ n \in \mathbb{Z}$

Proof: (By the principle of mathematical induction)

(1) If $n=2$, $\quad \text{LHS} = 1-\frac{1}{2^2} = 1-\frac{1}{4}=\frac{3}{4}$ $\quad \text{RHS} = \frac{2+1}{2(2)} = \frac{3}{4}$ $\quad \therefore\ P_2$ is true

(2) If P_k is true, then $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)....\left(1-\frac{1}{k^2}\right)=\frac{k+1}{2k}$

Now $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)....\left(1-\frac{1}{k^2}\right)\left(1-\frac{1}{(k+1)^2}\right)$

$$\begin{aligned}
&= \frac{k+1}{2k}\left(1-\frac{1}{(k+1)^2}\right) \quad \{\text{using } P_k\}\\
&= \frac{k+1}{2k}\left(\frac{(k+1)^2-1}{(k+1)^2}\right)\\
&= \frac{k+1}{2k}\left(\frac{k^2+2k+1-1}{(k+1)^2}\right)\\
&= \frac{k^2+2k}{2k(k+1)}\\
&= \frac{k(k+2)}{2k(k+1)}\\
&= \frac{k+2}{2(k+1)}
\end{aligned}$$

Since P_2 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \geqslant 2,\ n \in \mathbb{Z}$ {Principle of mathematical induction}

2 **a** P_n is: $3^n \geqslant 1+2n$ for $n \in \mathbb{N}$

Proof: (By the principle of mathematical induction)

(1) If $n=0$, we have $3^0 \geqslant 1+2(0)$
$\therefore\ 1 \geqslant 1$ which is true $\therefore\ P_0$ is true

(2) If P_k is true, then $3^k \geqslant 1+2k$

Now $$\begin{aligned}3^{k+1} = 3^k \times 3 &\geqslant (1+2k)\times 3 \quad \{\text{using } P_k\}\\
&\geqslant 3+6k\\
&\geqslant 3+2k \quad \{k \geqslant 0\}\\
&\geqslant 1+2(k+1)\end{aligned}$$

$\therefore\ P_{k+1}$ is true.

Since P_0 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{N}$ {Principle of mathematical induction}

b P_n is: $n! \geqslant 2^n$ for $n \in \mathbb{Z},\ n \geqslant 4$

Proof: (By the principle of mathematical induction)

(1) If $n=4$, we have $4! \geqslant 2^4$
$\therefore\ 24 \geqslant 16$ which is true $\therefore\ P_4$ is true

(2) If P_k is true then $k! \geqslant 2^k$

Now $$\begin{aligned}(k+1)! = (k+1)\times k! &\geqslant (k+1)\times 2^k \quad \{\text{using } P_k\}\\
&\geqslant 2 \times 2^k \quad \{k \geqslant 4\}\\
&\geqslant 2^{k+1}\end{aligned}$$

$\therefore\ P_{k+1}$ is true.

Since P_4 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z},\ n \geqslant 4$ {Principle of mathematical induction}

c P_n is: $8^n \geqslant n^3$ for $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, we have $8^1 \geqslant 1^3$

$\therefore\ 8 \geqslant 1$ which is true, so P_1 is true

(2) If P_k is true then $8^k \geqslant k^3$

$$\begin{aligned} \text{Now } 8^{k+1} &= 8 \times 8^k \geqslant 8 \times k^3 && \{\text{using } P_k\} \\ &\geqslant (2k)^3 \\ &\geqslant (k+1)^3 && \{k \geqslant 1, \text{ so } 2k \geqslant k+1\} \end{aligned}$$

$\therefore\ P_{k+1}$ is true.

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

d P_n is: $(1-h)^n \leqslant \dfrac{1}{1+nh}$ for $0 \leqslant h \leqslant 1$ for $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, we have $(1-h) \leqslant \dfrac{1}{1+h}$

$$\therefore\ (1-h)(1+h) \leqslant 1 \qquad \{1+h \geqslant 0\}$$

$$\therefore\ 1 - h^2 \leqslant 1$$

$\therefore\ h^2 \geqslant 0$ which is true for $0 \leqslant h \leqslant 1$, so P_1 is true

(2) If P_k is true then $(1-h)^k \leqslant \dfrac{1}{1+kh}$ for $0 \leqslant h \leqslant 1$

Now $(1-h)^{k+1} = (1-h)(1-h)^k$

$$\therefore\ (1-h)^{k+1} \leqslant (1-h)\left(\frac{1}{1+kh}\right) \qquad \{\text{using } P_k\}$$

$$\therefore\ (1-h)^{k+1} \leqslant \left(\frac{1-h}{1+kh}\right) \times \left(\frac{1+kh+h}{1+kh+h}\right)$$

$$\therefore\ (1-h)^{k+1} \leqslant \frac{1+kh+h-h-kh^2-h^2}{(1+kh)(1+kh+h)}$$

$$\therefore\ (1-h)^{k+1} \leqslant \frac{(1+kh)-(kh^2+h^2)}{(1+kh)(1+(k+1)h)}$$

$$\therefore\ (1-h)^{k+1} \leqslant \frac{1 - \frac{kh^2+h^2}{1+kh}}{1+(k+1)h}$$

$$\therefore\ (1-h)^{k+1} \leqslant \frac{1}{1+(k+1)h} \qquad \left\{k, h \geqslant 0, \text{ so } \frac{kh^2+h^2}{1+kh} \geqslant 0\right\}$$

$\therefore\ P_{k+1}$ is true.

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

3 **a** P_n is: $(z_1 + z_2 + ... + z_n)^* = z_1^* + z_2^* + ... + z_n^*$ for all $n \in \mathbb{Z}^+$ and complex $z_1, z_2, ..., z_n$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $(z_1)^* = z_1^*$ $\therefore$ P_1 is true

(2) If P_k is true, then $(z_1 + z_2 + ... + z_k)^* = z_1^* + z_2^* + ... + z_k^*$

$$\begin{aligned} \text{Now } & (z_1 + z_2 + ... + z_k + z_{k+1})^* \\ &= ((z_1 + z_2 + ... + z_k) + z_{k+1})^* \\ &= (z_1 + z_2 + ... + z_k)^* + z_{k+1}^* && \{\text{using } (z_1+z_2)^* = z_1^* + z_2^*\} \\ &= z_1^* + z_2^* + ... + z_k^* + z_{k+1}^* && \{\text{using } P_k\} \end{aligned}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

b P_n is: $(z_1 z_2 .... z_n)^* = {z_1}^* {z_2}^* .... {z_n}^*$ for all $n \in \mathbb{Z}^+$ and complex $z_1, z_2, ..., z_n$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $(z_1)^* = {z_1}^*$ $\therefore$ P_1 is true

(2) If P_k is true, then $(z_1 z_2 .... z_k)^* = {z_1}^* {z_2}^* .... {z_k}^*$

Now $(z_1 z_2 .... z_k z_{k+1})^* = ((z_1 z_2 .... z_k) z_{k+1})^*$

$= (z_1 z_2 .... z_k)^* z_{k+1}^*$ {using $(z_1 z_2)^* = {z_1}^* {z_2}^*$}

$= {z_1}^* {z_2}^* .... {z_k}^* z_{k+1}^*$ {using P_k}

Since P_1 is true, and P_{k+1} is true whenever P_k is true, then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

c P_n is: $(z^n)^* = (z^*)^n$ for all $n \in \mathbb{Z}^+$ and complex z

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $(z^1)^* = z^* = (z^*)^1$ $\therefore$ P_1 is true

(2) If P_k is true, then $(z^k)^* = (z^*)^k$

Now $(z^{k+1})^* = (zz^k)^*$

$= z^*(z^k)^*$ {using $(z_1 z_2)^* = {z_1}^* {z_2}^*$}

$= z^*(z^*)^k$ {using P_k}

$= (z^*)^{k+1}$

Since P_1 is true, and P_{k+1} is true whenever P_k is true, then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

REVIEW SET 9A

1 P_n is: $1 + 3 + 5 + 7 + ... + (2n - 1) = n^2$ for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= 1$ and RHS $= 1^2 = 1$ $\therefore$ P_1 is true

(2) If P_k is true, then $1 + 3 + 5 + 7 + ... + (2k - 1) = k^2$

$\therefore$ $1 + 3 + 5 + 7 + ... + (2k - 1) + (2[k + 1] - 1)$

$= k^2 + 2k + 2 - 1$ {using P_k}

$= k^2 + 2k + 1$

$= (k + 1)^2$

Since P_1 is true, and P_{k+1} is true whenever P_k is true, then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

2 P_n is: $7^n + 2$ is divisible by 3 for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $7^1 + 2 = 9$ which is divisible by 3 $\therefore$ P_1 is true

(2) If P_k is true, then $7^k + 2 = 3A$ where $A \in \mathbb{Z}$

$\therefore$ $7^{k+1} + 2 = 7 \times 7^k + 2$

$= 7(3A - 2) + 2$ {using P_k}

$= 21A - 14 + 2$

$= 21A - 12$

$= 3(7A - 4)$ where $7A - 4$ is an integer as A is an integer

$\therefore$ $7^{k+1} + 2$ is divisible by 3

Since P_1 is true, and P_{k+1} is true whenever P_k is true, then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

3 P_n is: $1\times 2\times 3+2\times 3\times 4+3\times 4\times 5+ .... +n(n+1)(n+2)=\dfrac{n(n+1)(n+2)(n+3)}{4}$ for all $n\in\mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n=1$, LHS $=1\times 2\times 3=6$, RHS $=\dfrac{1\times 2\times 3\times 4}{4}=6$ $\therefore$ P_1 is true

(2) If P_k is true, then

$$1\times 2\times 3+2\times 3\times 4+ .... +k(k+1)(k+2)=\frac{k(k+1)(k+2)(k+3)}{4}$$

$$\begin{aligned}\therefore\quad & 1\times 2\times 3+2\times 3\times 4+ .... +k(k+1)(k+2)+(k+1)(k+2)(k+3)\\ &=\frac{k(k+1)(k+2)(k+3)}{4}+(k+1)(k+2)(k+3) \quad \{\text{using } P_k\}\\ &=\frac{k(k+1)(k+2)(k+3)}{4}+\frac{4(k+1)(k+2)(k+3)}{4} \quad \{\text{equalising denominators}\}\\ &=\frac{(k+1)(k+2)(k+3)(k+4)}{4}\end{aligned}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n\in\mathbb{Z}^+$ {Principle of mathematical induction}

4 P_n is: $1+r+r^2+r^3+ .... +r^{n-1}=\dfrac{1-r^n}{1-r}$ for all $n\in\mathbb{Z}^+$, $r\neq 1$

Proof: (By the principle of mathematical induction)

(1) If $n=1$, LHS $=1$ and RHS $=\dfrac{1-r}{1-r}=1$ as $r\neq 1$ $\therefore$ P_1 is true

(2) If P_k is true, then $1+r+r^2+r^3+ .... +r^{k-1}=\dfrac{1-r^k}{1-r}$

$$\begin{aligned}\text{Now}\quad 1+r+r^2+r^3+ .... +r^{k-1}+r^k&=\frac{1-r^k}{1-r}+r^k \quad \{\text{using } P_k\}\\ &=\frac{1-r^k}{1-r}+r^k\left(\frac{1-r}{1-r}\right) \quad \{\text{equalising denominators}\}\\ &=\frac{1-r^k+r^k-r^{k+1}}{1-r}\\ &=\frac{1-r^{k+1}}{1-r}\end{aligned}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n\in\mathbb{Z}^+$ {Principle of mathematical induction}

5 P_n is: $5^{2n}-1$ is divisible by 24 for all $n\in\mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n=1$, $5^2-1=25-1=24$ is divisible by 24 $\therefore$ P_1 is true

(2) If P_k is true, then $5^{2k}-1=24A$ where $A\in\mathbb{Z}$

$$\begin{aligned}\text{Now}\quad 5^{2(k+1)}-1&=5^{2k}5^2-1\\ &=25[24A+1]-1 \quad \{\text{using } P_k\}\\ &=25\times 24A+25-1\\ &=25\times 24A+24\\ &=24(25A+1) \quad \text{where } 25A+1 \text{ is an integer}\end{aligned}$$

$\therefore$ $5^{2(k+1)}-1$ is divisible by 24

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n\in\mathbb{Z}^+$ {Principle of mathematical induction}

6 P_n is: $5^n \geqslant 1+4n$ for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n=1$, we have $5^1 \geqslant 1+4(1)$

$\therefore\ 5 \geqslant 5$ which is true, so P_1 is true

(2) If P_k is true, then $5^k \geqslant 1+4k$

$$\begin{aligned}\text{Now}\quad 5^{k+1} = 5\times 5^k &\geqslant 5\times(1+4k) \quad \{\text{using } P_k\}\\ &\geqslant 5+20k\\ &\geqslant 5+4k \quad \{k \geqslant 0\}\\ &\geqslant 1+4(k+1)\end{aligned}$$

$\therefore\ P_{k+1}$ is true.

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

7 P_n is: if $u_1 = 1$ and $u_{n+1} = 3u_n + 2^n$ for all $n \in \mathbb{Z}^+$, then $u_n = 3^n - 2^n$

Proof: (By the principle of mathematical induction)

(1) If $n=1$, $u_1 = 1 = 3^1 - 2^1$, so P_1 is true

(2) If P_k is true, then $u_k = 3^k - 2^k$ and

$$\begin{aligned}u_{k+1} &= 3u_k + 2^k\\ &= 3(3^k-2^k)+2^k \quad \{\text{using } P_k\}\\ &= 3^{k+1} - 3\times 2^k + 2^k\\ &= 3^{k+1} - 2\times 2^k\\ &= 3^{k+1} - 2^{k+1}\end{aligned}$$

$\therefore\ P_{k+1}$ is true.

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

REVIEW SET 9B

1 P_n is: $1^2+3^2+5^2+7^2+ .... +(2n-1)^2 = \dfrac{n(2n+1)(2n-1)}{3}$ for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n=1$, LHS $= 1^2 = 1$, RHS $= \dfrac{1\times 3\times 1}{3} = 1$ $\therefore$ P_1 is true

(2) If P_k is true, then $1^2+3^2+5^2+ .... +(2k-1)^2 = \dfrac{k(2k+1)(2k-1)}{3}$

$\therefore\ 1^2+3^2+5^2+ .... +(2k-1)^2+(2k+1)^2$

$$\begin{aligned}&= \frac{k(2k+1)(2k-1)}{3} + (2k+1)^2 \quad \{\text{using } P_k\}\\ &= \frac{k(2k+1)(2k-1)}{3} + \frac{3(2k+1)^2}{3} \quad \{\text{equalising denominators}\}\\ &= \frac{(2k+1)[k(2k-1)+3(2k+1)]}{3}\\ &= \frac{(2k+1)[2k^2-k+6k+3]}{3}\\ &= \frac{(2k+1)(2k^2+5k+3)}{3}\\ &= \frac{(2k+1)(k+1)(2k+3)}{3}\\ &= \frac{(k+1)(2[k+1]+1)(2[k+1]-1)}{3}\end{aligned}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

2 P_n is: $3^{2n+2} - 8n - 9$ is divisible by 64 for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $3^4 - 8 - 9 = 81 - 17 = 64$ which is divisible by 64 $\therefore$ P_1 is true

(2) If P_k is true, then $3^{2k+2} - 8k - 9 = 64A$ where $A \in \mathbb{Z}$

Now $3^{2(k+1)+2} - 8(k+1) - 9$

$= 3^{2k+2} \times 3^2 - 8k - 8 - 9$

$= 9[64A + 8k + 9] - 8k - 17$ {using P_k}

$= 9 \times 64A + 72k + 81 - 8k - 17$

$= 9 \times 64A + 64k + 64$

$= 64(9A + k + 1)$ where $9A + k + 1$ is $\in \mathbb{Z}$ as A, $k \in \mathbb{Z}$

$\therefore$ $3^{2(k+1)+2} - 8(k+1) - 9$ is divisible by 64

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

3 P_n is: $3 + 5 \times 2 + 7 \times 2^2 + 9 \times 2^3 + ... + (2n+1)2^{n-1} = 1 + (2n-1) \times 2^n$ for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= 3$ and RHS $= 1 + 1 \times 2^1 = 1 + 2 = 3$ $\therefore$ P_1 is true

(2) If P_k is true, then $3 + 5 \times 2 + 7 \times 2^2 + 9 \times 2^3 + ... + (2k+1)2^{k-1} = 1 + (2k-1) \times 2^k$

$\therefore$ $3 + 5 \times 2 + 7 \times 2^2 + 9 \times 2^3 + ... + (2k+1)2^{k-1} + (2k+3)2^k$

$= 1 + (2k-1)2^k + (2k+3)2^k$ {using P_k}

$= 1 + 2^k(2k - 1 + 2k + 3)$

$= 1 + 2^k(4k+2)$

$= 1 + 2^k(2)(2k+1)$

$= 1 + (2k+1)2^{k+1}$

$= 1 + (2[k+1] - 1)2^{[k+1]}$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

4 P_n is: $5^n + 3$ is divisible by 4 for all $n \in \mathbb{Z}$, $n \geqslant 0$

Proof: (By the principle of mathematical induction)

(1) If $n = 0$, $5^0 + 3 = 4$ which is divisible by 4 $\therefore$ P_0 is true

(2) If P_k is true, then $5^k + 3 = 4A$ where $A \in \mathbb{Z}$

Now $5^{k+1} + 3 = 5 \times 5^k + 3$

$= 5[4A - 3] + 3$ {using P_k}

$= 20A - 15 + 3$

$= 20A - 12$

$= 4(5A - 3)$ where $5A - 3 \in \mathbb{Z}$, as $A \in \mathbb{Z}$

So, $5^{k+1} + 3$ is divisible by 4

Since P_0 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}$, $n \geqslant 0$ {Principle of mathematical induction}

5 P_n is: $1 \times 2^2 + 2 \times 3^2 + 3 \times 4^2 + ... + n(n+1)^2 = \dfrac{n(n+1)(n+2)(3n+5)}{12}$ for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= 1 \times 2^2 = 4$ and RHS $= \dfrac{1 \times 2 \times 3 \times 8}{12} = \dfrac{48}{12} = 4$ $\therefore$ P_1 is true

(2) If P_k is true, then $1 \times 2^2 + 2 \times 3^2 + 3 \times 4^2 + ... + k(k+1)^2 = \dfrac{k(k+1)(k+2)(3k+5)}{12}$

$$\begin{aligned}
\therefore\quad & 1\times 2^2 + 2\times 3^2 + 3\times 4^2 + \dots + k(k+1)^2 + (k+1)(k+2)^2 \\
&= \frac{k(k+1)(k+2)(3k+5)}{12} + (k+1)(k+2)^2 \qquad \{\text{using } P_k\} \\
&= \frac{k(k+1)(k+2)(3k+5)}{12} + \frac{12(k+1)(k+2)^2}{12} \qquad \{\text{equalising denominators}\} \\
&= \frac{(k+1)(k+2)[k(3k+5)+12(k+2)]}{12} \\
&= \frac{(k+1)(k+2)[3k^2+5k+12k+24]}{12} \\
&= \frac{(k+1)(k+2)(3k^2+17k+24)}{12} \\
&= \frac{(k+1)(k+2)(k+3)(3k+8)}{12} \\
&= \frac{(k+1)([k+1]+1)([k+1]+2)(3[k+1]+5)}{12}
\end{aligned}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

6 P_n is: $5^n + 3^n \geqslant 2^{2n+1}$ for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, we have $5^1 + 3^1 \geqslant 2^3$

$\therefore\ 8 \geqslant 8$ which is true, so P_1 is true

(2) If P_k is true, then $5^k + 3^k \geqslant 2^{2k+1}$, so $5^k \geqslant 2^{2k+1} - 3^k$

$$\begin{aligned}
\text{Now}\quad 5^{k+1} + 3^{k+1} &= 5\times 5^k + 3\times 3^k \\
&\geqslant 5(2^{2k+1} - 3^k) + 3\times 3^k \qquad \{\text{using } P_k\} \\
\therefore\quad 5^{k+1} + 3^{k+1} &\geqslant 5\times 2^{2k+1} - 5\times 3^k + 3\times 3^k \\
\therefore\quad 5^{k+1} + 3^{k+1} &\geqslant 5\times 2^{2k+1} - 2\times 3^k \\
&\geqslant 5\times 2^{2k+1} - 2\times 4^k \qquad \{4^k \geqslant 3^k \text{ as } k \geqslant 0\} \\
\therefore\quad 5^{k+1} + 3^{k+1} &\geqslant 5\times 2^{2k+1} - 2^{2k+1} \\
\therefore\quad 5^{k+1} + 3^{k+1} &\geqslant 4\times 2^{2k+1} \\
\therefore\quad 5^{k+1} + 3^{k+1} &\geqslant 2^{2k+3} \\
\therefore\quad 5^{k+1} + 3^{k+1} &\geqslant 2^{2(k+1)+1} \qquad \therefore\ P_{k+1} \text{ is true.}
\end{aligned}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

7 P_n is: if $u_1 = 9$ and $u_{n+1} = 2u_n + 3(5^n)$ then $u_n = 2^{n+1} + 5^n$ for all $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $u_1 = 2^2 + 5^1 = 9$ ✓ so P_1 is true

(2) If P_k is true, then $u_k = 2^{k+1} + 5^k$ and
$$\begin{aligned}
u_{k+1} &= 2u_k + 3(5^k) \\
&= 2(2^{k+1} + 5^k) + 3(5^k) \qquad \{\text{using } P_k\} \\
&= 2^{k+2} + 2(5^k) + 3(5^k) \\
&= 2^{k+2} + 5(5^k) \\
&= 2^{k+2} + 5^{k+1} \qquad \therefore\ P_{k+1} \text{ is true.}
\end{aligned}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

REVIEW SET 9C

1 P_n is: $1\times 3 + 2\times 4 + 3\times 5 + 4\times 6 + ... + n(n+2) = \dfrac{n(n+1)(2n+7)}{6}$ for all $n\in\mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n=1$, LHS $= 1\times 3 = 3$ and RHS $= \dfrac{1\times 2\times 9}{6} = \dfrac{18}{6} = 3$ $\therefore$ P_1 is true.

(2) If P_k is true, then $1\times 3 + 2\times 4 + 3\times 5 + ... + k(k+2) = \dfrac{k(k+1)(2k+7)}{6}$

$$\begin{aligned}
\therefore\quad & 1\times 3 + 2\times 4 + 3\times 5 + ... + k(k+2) + (k+1)(k+3)\\
&= \frac{k(k+1)(2k+7)}{6} + (k+1)(k+3) && \{\text{using } P_k\}\\
&= \frac{k(k+1)(2k+7)}{6} + \frac{6(k+1)(k+3)}{6} && \{\text{equalising denominators}\}\\
&= \frac{(k+1)[k(2k+7)+6(k+3)]}{6}\\
&= \frac{(k+1)[2k^2+13k+18]}{6}\\
&= \frac{(k+1)(k+2)(2k+9)}{6}\\
&= \frac{(k+1)([k+1]+1)(2[k+1]+7)}{6} && \therefore\ P_{k+1} \text{ is true.}
\end{aligned}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n\in\mathbb{Z}^+$ {Principle of mathematical induction}

2 P_n is: $7^n - 1$ is divisible by 6 for all $n\in\mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n=1$, $7^1 - 1 = 6$ which is divisible by 6 $\therefore$ P_1 is true.

(2) If P_k is true, then $7^k - 1 = 6A$ where $A\in\mathbb{Z}$

$$\begin{aligned}
\text{Now}\quad 7^{k+1} - 1 &= 7\times 7^k - 1\\
&= 7[6A+1] - 1 \quad \{\text{using } P_k\}\\
&= 42A + 7 - 1\\
&= 42A + 6\\
&= 6(7A+1) \quad \text{where } 7A+1\in\mathbb{Z}
\end{aligned}$$

Thus $7^{k+1} - 1$ is divisible by 6.

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n\in\mathbb{Z}^+$ {Principle of mathematical induction}

3 P_n is: $1^3 + 3^3 + 5^3 + ... + (2n-1)^3 = n^2(2n^2-1)$ for all $n\in\mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n=1$, LHS $= 1^3 = 1$ and RHS $= 1^2(2-1) = 1\times 1 = 1$ $\therefore$ P_1 is true.

(2) If P_k is true, then $1^3 + 3^3 + 5^3 + ... + (2k-1)^3 = k^2(2k^2-1)$

$$\begin{aligned}
\therefore\quad & 1^3 + 3^3 + 5^3 + ... + (2k-1)^3 + (2k+1)^3\\
&= k^2(2k^2-1) + (2k+1)^3 \quad \{\text{using } P_k\}\\
&= 2k^4 - k^2 + (2k)^3 + 3(2k)^2 1 + 3(2k)1^2 + 1^3\\
&= 2k^4 - k^2 + 8k^3 + 12k^2 + 6k + 1\\
&= 2k^4 + 8k^3 + 11k^2 + 6k + 1\\
&= (k+1)^2(2k^2+4k+1)\\
&= (k+1)^2(2[k^2+2k+1]-1)\\
&= (k+1)^2(2[k+1]^2-1)
\end{aligned}$$

$$\begin{array}{r|rrrr|r}
-1 & 2 & 8 & 11 & 6 & 1\\
 & 0 & -2 & -6 & -5 & -1\\
\hline
-1 & 2 & 6 & 5 & 1 & 0\\
 & 0 & -2 & -4 & -1 & \\
\hline
 & 2 & 4 & 1 & 0 &
\end{array}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n\in\mathbb{Z}^+$ {Principle of mathematical induction}

4 P_n is: $3^n - 1 - 2n$ is divisible by 4 for all $n \in \mathbb{N}$.

Proof: (By the principle of mathematical induction)

(1) If $n = 0$, $3^0 - 1 - 2(0) = 1 - 1 - 0 = 0$ which is divisible by 4 $\therefore$ P_0 is true

(2) If P_k is true, then $3^k - 1 - 2k = 4A$ where $A \in \mathbb{Z}$

$$\begin{aligned}\text{Now } 3^{k+1} - 1 - 2(k+1) &= 3 \times 3^k - 1 - 2k - 2 \\ &= 3[4A + 1 + 2k] - 2k - 3 \quad \{\text{using } P_k\} \\ &= 12A + 3 + 6k - 2k - 3 \\ &= 12A + 4k \\ &= 4(3A + k) \text{ where } 3A + k \in \mathbb{Z}\end{aligned}$$

$\therefore$ $3^{k+1} - 1 - 2(k+1)$ is divisible by 4

Since P_0 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{N}$ {Principle of mathematical induction}

5 P_n is: $\dfrac{1}{1 \times 3} + \dfrac{1}{3 \times 5} + \dfrac{1}{5 \times 7} + ... + \dfrac{1}{(2n-1)(2n+1)} = \dfrac{n}{2n+1}$ for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= \dfrac{1}{1 \times 3} = \dfrac{1}{3}$, RHS $= \dfrac{1}{2+1} = \dfrac{1}{3}$ $\therefore$ P_1 is true.

(2) If P_k is true, then $\dfrac{1}{1 \times 3} + \dfrac{1}{3 \times 5} + \dfrac{1}{5 \times 7} + ... + \dfrac{1}{(2k-1)(2k+1)} = \dfrac{k}{2k+1}$

$$\begin{aligned}\therefore \quad & \frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \frac{1}{5 \times 7} + ... + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2k+1)(2k+3)} \\ &= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} \qquad \{\text{using } P_k\} \\ &= \frac{k}{2k+1}\left(\frac{2k+3}{2k+3}\right) + \frac{1}{(2k+1)(2k+3)} \qquad \{\text{equalising denominators}\} \\ &= \frac{2k^2 + 3k + 1}{(2k+1)(2k+3)} \\ &= \frac{(k+1)(2k+1)}{(2k+1)(2k+3)} \\ &= \frac{(k+1)}{2(k+1)+1}\end{aligned}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

6 P_n is: if $u_1 = 5$ and $u_{n+1} = 2u_n - 3(-1)^n$ for $n \in \mathbb{Z}^+$, then $u_n = 3(2^n) + (-1)^n$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $3(2^1) + (-1)^1 = 6 - 1 = 5 = u_1$, so P_1 is true.

(2) If P_k is true, then $u_k = 3(2^k) + (-1)^k$,

$$\begin{aligned}\text{and } u_{k+1} &= 2u_k - 3(-1)^k \\ &= 2[3(2^k) + (-1)^k] - 3(-1)^k \quad \{\text{using } P_k\} \\ &= 6(2^k) + 2(-1)^k - 3(-1)^k \\ &= 3(2^{k+1}) - (-1)^k \\ &= 3(2^{k+1}) + (-1)^{k+1}\end{aligned}$$

$\therefore$ P_{k+1} is true.

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

7 $\sqrt[n]{n!} \leqslant \dfrac{n+1}{2} \quad \Leftrightarrow \quad n! \leqslant \left(\dfrac{n+1}{2}\right)^n$

$\therefore$ P_n is: $n! \leqslant \left(\dfrac{n+1}{2}\right)^n$ for $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, we have $1! \leqslant \left(\dfrac{1+1}{2}\right)^1$

$\therefore$ $1 \leqslant 1$ which is true, so P_1 is true

(2) If P_k is true, then $k! \leqslant \left(\dfrac{k+1}{2}\right)^k$

Now
$$\begin{aligned}(k+1)! &= (k+1)k! \\ &\leqslant (k+1)\left(\frac{k+1}{2}\right)^k \\ &\leqslant \frac{(k+1)^{k+1}}{2^k} \quad .... \text{ (1)}\end{aligned}$$

Also,
$$\begin{aligned}(k+1)^k &= k^k + \tbinom{k}{1} k^{k-1} + ... \\ &= k^k + k^k + ... \\ &= 2k^k + ...\end{aligned}$$

$\therefore$ $(k+1)^k \geqslant 2k^k$ for $k \geqslant 1$

$\therefore$ $\left(\dfrac{k+1}{k}\right)^k \geqslant 2$

$\therefore$ $\frac{1}{2}\left(\dfrac{k+1}{k}\right)^k \geqslant 1$

$\therefore$ $\frac{1}{2}\left(\dfrac{k+2}{k+1}\right)^{k+1} \geqslant 1$ .... (2)

{replacing k with $k+1$}

Using (1) and (2),
$$\begin{aligned}(k+1)! &\leqslant \frac{(k+1)^{k+1}}{2^k} \times \tfrac{1}{2}\left(\frac{k+2}{k+1}\right)^{k+1} \\ &\leqslant \frac{(k+1)^{k+1}}{2^{k+1}} \times \frac{(k+2)^{k+1}}{(k+1)^{k+1}} \\ &\leqslant \frac{(k+2)^{k+1}}{2^{k+1}} \\ &\leqslant \left(\frac{[k+1]+1}{2}\right)^{k+1}\end{aligned}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true, then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}