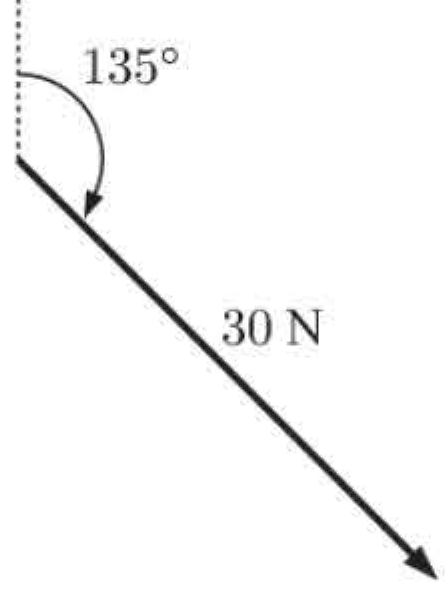
Chapter 14

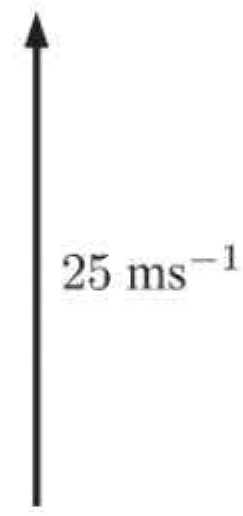
VECTORS

EXERCISE 14A.1

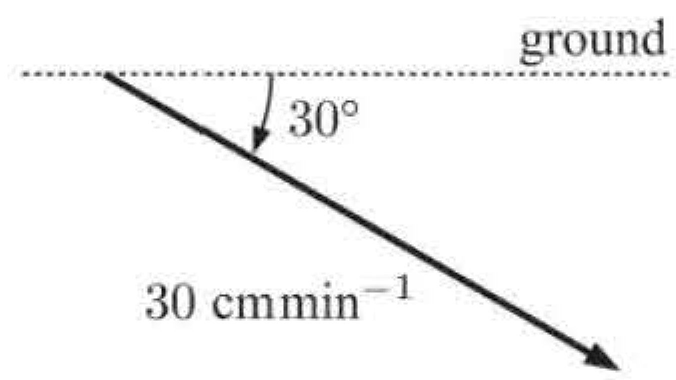
1 **a**



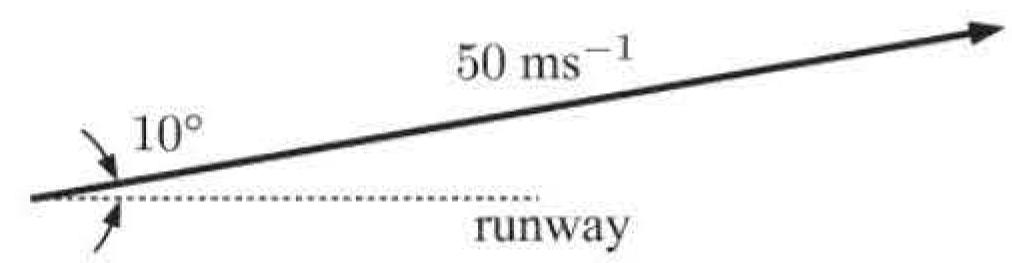
b



c

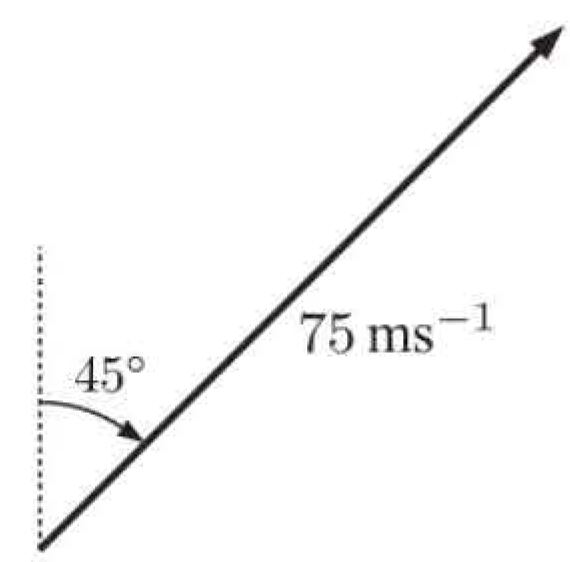


d

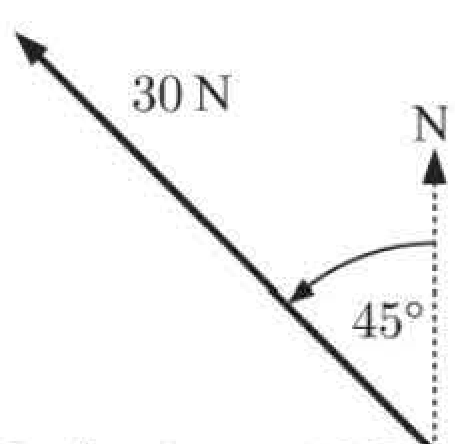


2 **a**

b

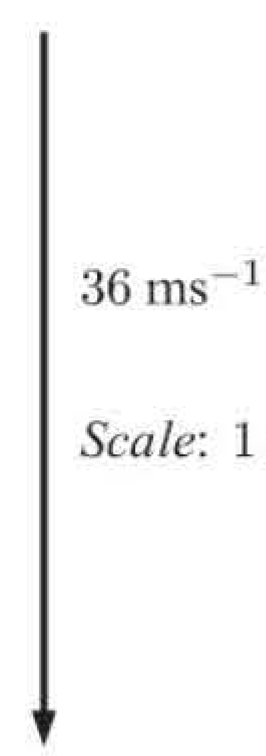


3 **a**



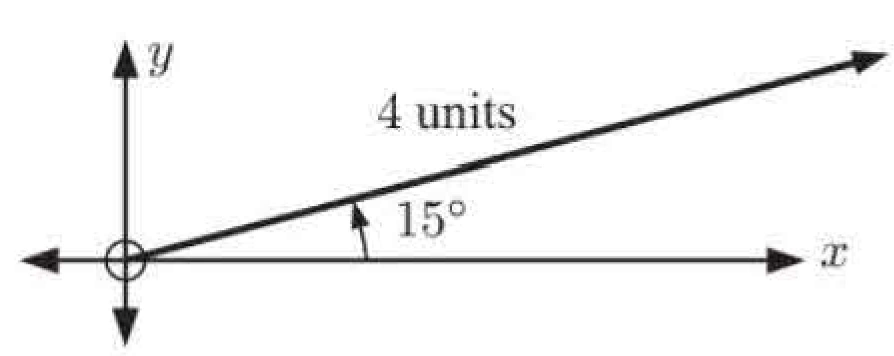
Scale: 1 cm ≡ 10 N

b



Scale: 1 cm ≡ 10 ms^{-1}

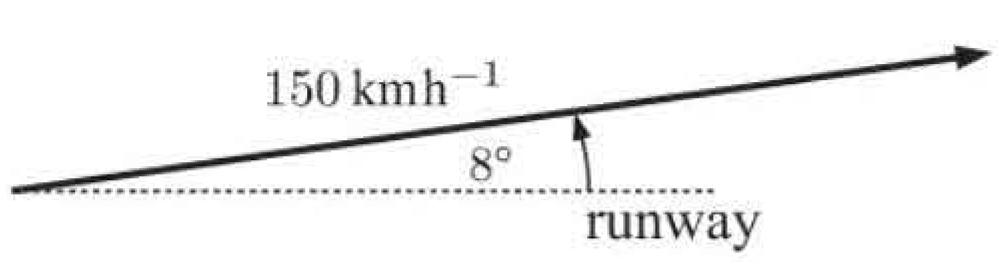
c



Scale: 1 cm ≡ 1 unit

d

Scale: 1 cm ≡ 30 kmh^{-1}



EXERCISE 14A.2

1 **a** If they are equal in magnitude, they have the same length. These are **p**, **q**, **s**, and **t**.

b Those parallel are **p**, **q**, **r**, and **t**.

c Those in the same direction are: **p** and **r**, **q** and **t**.

d To be equal they must have the same direction and be equal in length $\therefore$ $\mathbf{q} = \mathbf{t}$.

e **p** and **q** are negatives (equal length, but opposite direction). Likewise, **p** and **t** are negatives. We write $\mathbf{p} = -\mathbf{q}$ and $\mathbf{p} = -\mathbf{t}$.

2 **a** True, as they have the same length and direction.

b True, as they are sides of an equilateral triangle.

c False, as they do not have the same direction.

d False, as they have opposite directions.

e True, as they have the same length and direction.

f False, as they do not have the same direction.

3 **a** **i** $\overrightarrow{BC}$ is the vector which originates at B and terminates at C.

ii $\overrightarrow{ED} = \overrightarrow{AB}$, as they have the same length and direction.

b **i** $\overrightarrow{FE}$ and $\overrightarrow{BC}$ are negatives of $\overrightarrow{EF}$, as they both have the same length but opposite direction.

ii All sides of the hexagon are equal in length

$\therefore$ the vectors with the same length as $\overrightarrow{ED}$ are

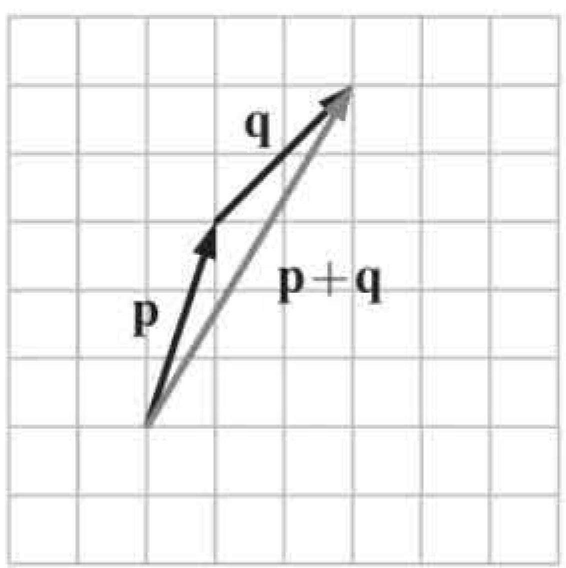
$\overrightarrow{DE}$, $\overrightarrow{EF}$, $\overrightarrow{FE}$, $\overrightarrow{FA}$, $\overrightarrow{AF}$, $\overrightarrow{AB}$, $\overrightarrow{BA}$, $\overrightarrow{BC}$, $\overrightarrow{CB}$, $\overrightarrow{CD}$, and $\overrightarrow{DC}$.

c The vector $\overrightarrow{FC}$ is parallel to $\overrightarrow{AB}$ and twice its length.

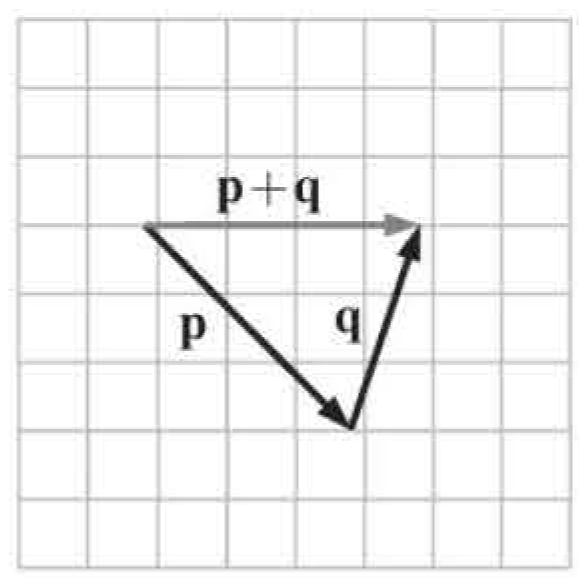
$\overrightarrow{CF}$ is also parallel to $\overrightarrow{AB}$ and twice its length (but in the opposite direction).

EXERCISE 14B.1

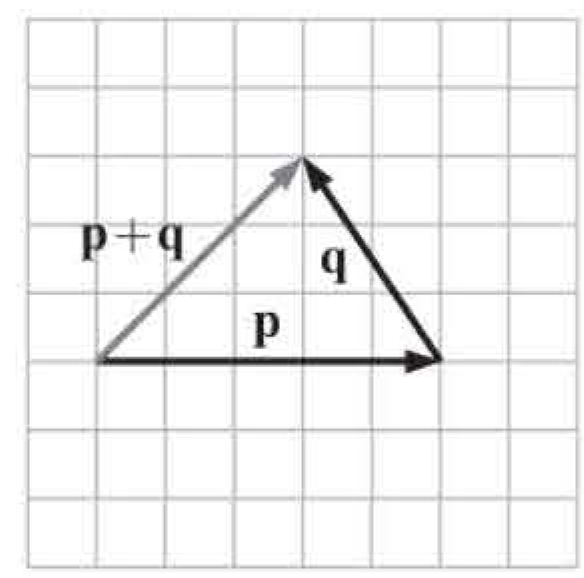
1 **a**



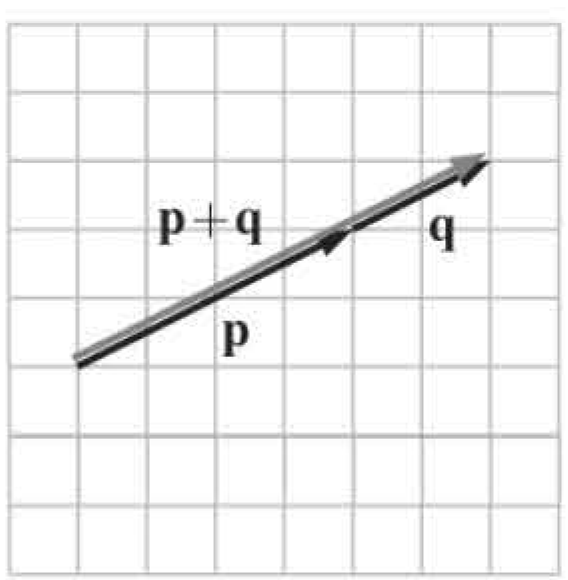
b



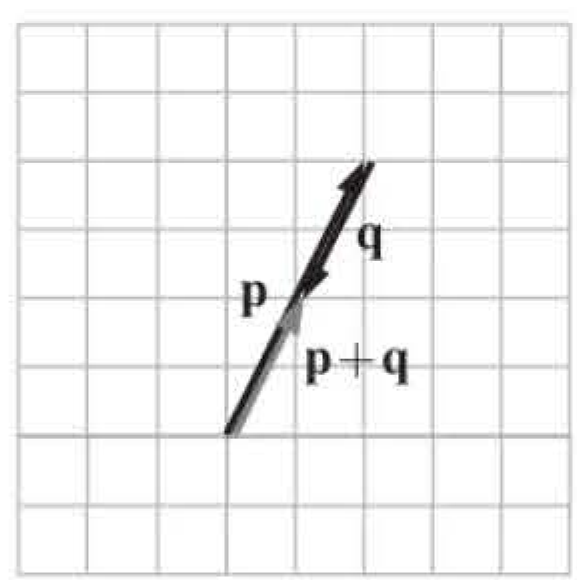
c



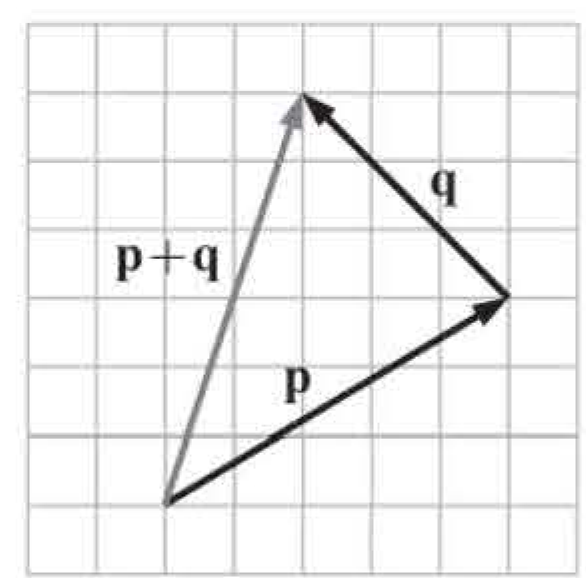
d



e



f



2 **a** $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

b $\overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{BD}$

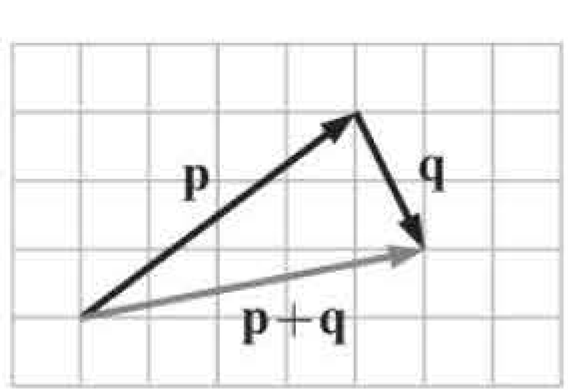
c $\overrightarrow{AB} + \overrightarrow{BA} = \overrightarrow{AA}$
$= \mathbf{0}$

d $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$
$= \overrightarrow{AC} + \overrightarrow{CD}$
$= \overrightarrow{AD}$

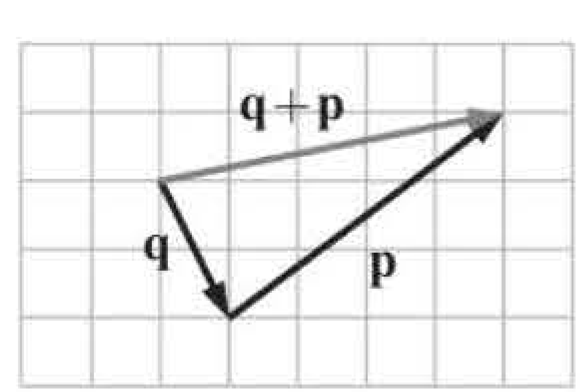
e $\overrightarrow{AC} + \overrightarrow{CB} + \overrightarrow{BD}$
$= \overrightarrow{AB} + \overrightarrow{BD}$
$= \overrightarrow{AD}$

f $\overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AB}$
$= \overrightarrow{BA} + \overrightarrow{AB}$
$= \overrightarrow{BB}$
$= \mathbf{0}$

3 **a** **i**



ii



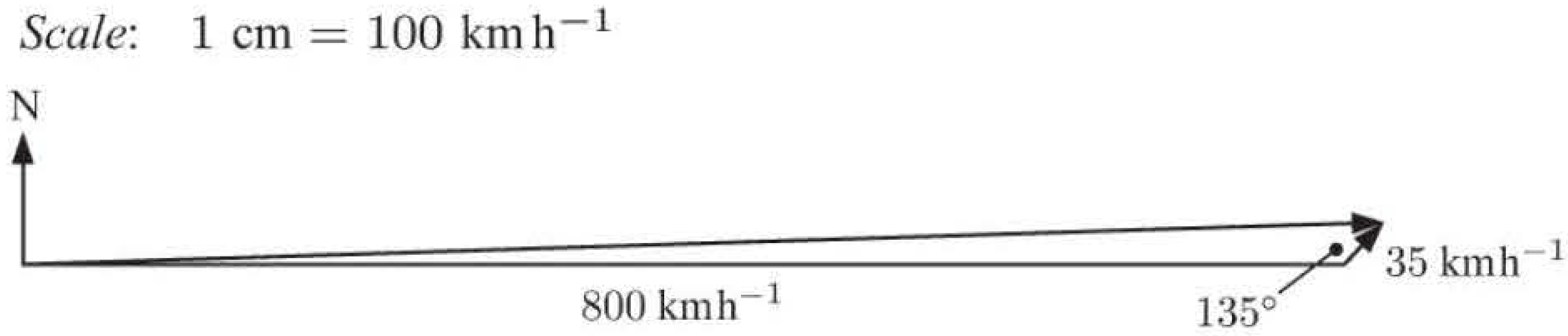
b yes

4 $\overrightarrow{PS} = \overrightarrow{PR} + \overrightarrow{RS}$
$= (\mathbf{a} + \mathbf{b}) + \mathbf{c}$

But $\overrightarrow{PS} = \overrightarrow{PQ} + \overrightarrow{QS}$
$= \mathbf{a} + (\mathbf{b} + \mathbf{c})$

$\therefore$ $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$ {as both are equal to $\overrightarrow{PS}$, associative law}

5 **a** *Scale*: 1 cm = 100 km h^{-1}



b We need to perform vector addition to find the effect of the wind on the aeroplane.

c Measuring the length of the resulting vector, we get 82.5 mm, or 8.25 cm.

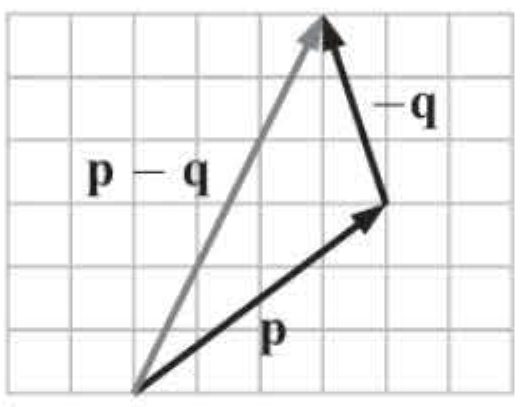
$\therefore$ the resulting speed of the plane is $8.25 \times 100 = 825\ \text{km h}^{-1}$.

Using a protractor to measure the angle between 'true north' and the resulting vector, we get $88°$.

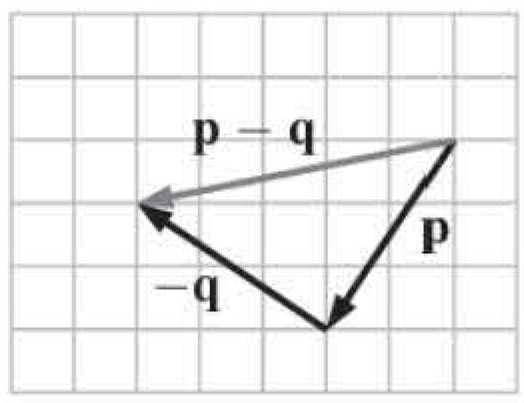
$\therefore$ the direction of the aeroplane is $88°$ east of north.

EXERCISE 14B.2

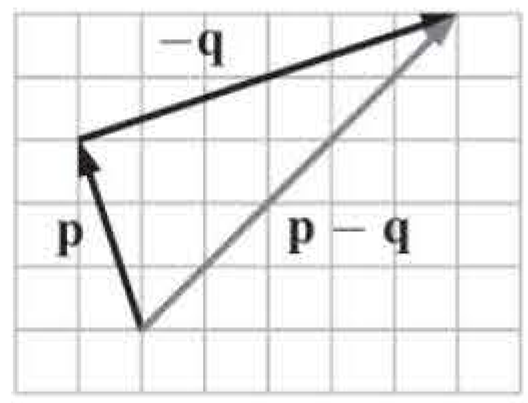
1 **a**



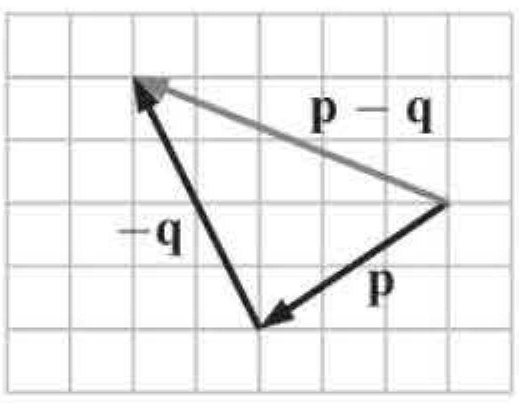
b



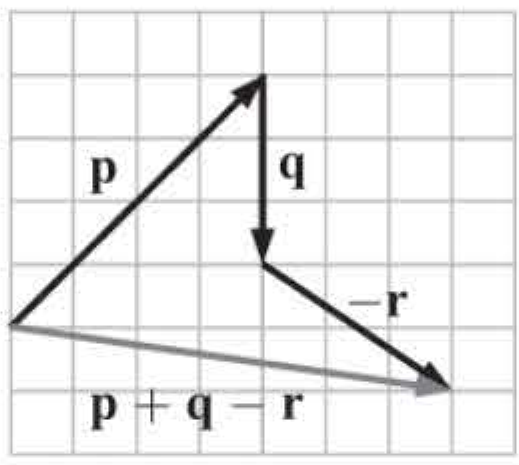
c



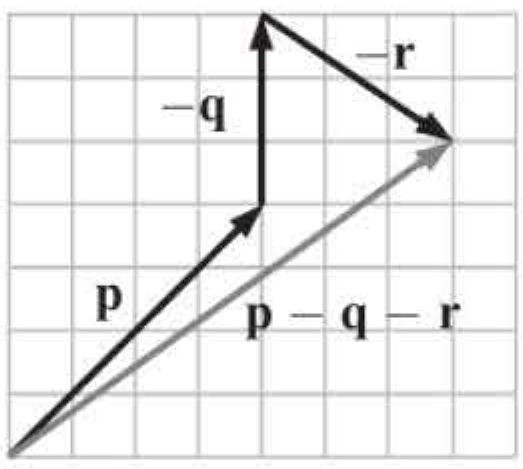
d



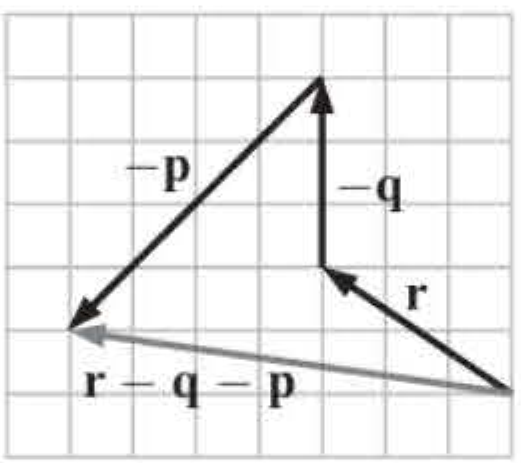
2 **a**



b



c



3 **a** $\overrightarrow{AC} + \overrightarrow{CB} = \overrightarrow{AB}$

b $\overrightarrow{AD} - \overrightarrow{BD} = \overrightarrow{AD} + \overrightarrow{DB}$
$= \overrightarrow{AB}$

c $\overrightarrow{AC} + \overrightarrow{CA} = \overrightarrow{AA}$
$= \mathbf{0}$

d $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$
$= \overrightarrow{AC} + \overrightarrow{CD}$
$= \overrightarrow{AD}$

e $\overrightarrow{BA} - \overrightarrow{CA} + \overrightarrow{CB}$
$= \overrightarrow{BA} + \overrightarrow{AC} + \overrightarrow{CB}$
$= \overrightarrow{BC} + \overrightarrow{CB}$
$= \overrightarrow{BB}$
$= \mathbf{0}$

f $\overrightarrow{AB} - \overrightarrow{CB} - \overrightarrow{DC}$
$= \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$
$= \overrightarrow{AC} + \overrightarrow{CD}$
$= \overrightarrow{AD}$

EXERCISE 14B.3

1 **a** $\mathbf{t} = \mathbf{r} + \mathbf{s}$ **b** $\mathbf{r} = -\mathbf{s} - \mathbf{t}$ **c** $\mathbf{r} = -\mathbf{p} - \mathbf{q} - \mathbf{s}$

d $\mathbf{r} = \mathbf{q} - \mathbf{p} + \mathbf{s}$ **e** $\mathbf{p} = \mathbf{t} + \mathbf{s} + \mathbf{r} - \mathbf{q}$ **f** $\mathbf{p} = -\mathbf{u} + \mathbf{t} + \mathbf{s} - \mathbf{r} - \mathbf{q}$

2 **a** **i** $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$
$= \mathbf{r} + \mathbf{s}$

ii $\overrightarrow{CA} = \overrightarrow{CB} + \overrightarrow{BA}$
$= -\overrightarrow{BC} - \overrightarrow{AB}$
$= -\mathbf{t} - \mathbf{s}$

iii $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC}$
$= \mathbf{r} + \mathbf{s} + \mathbf{t}$

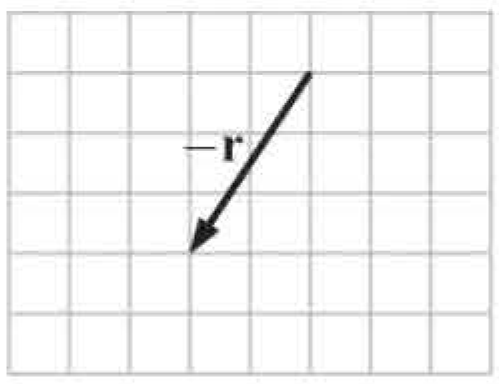
b **i** $\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD}$
$= \mathbf{p} + \mathbf{q}$

ii $\overrightarrow{BC} = \overrightarrow{BD} + \overrightarrow{DC}$
$= \mathbf{q} + \mathbf{r}$

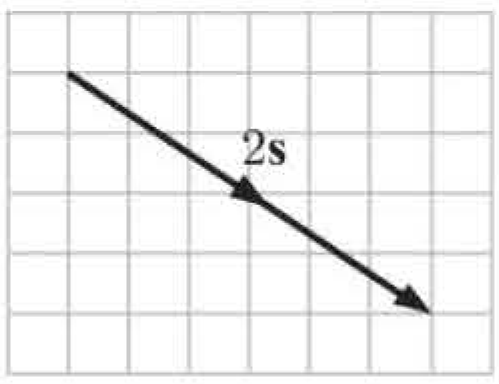
iii $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BD} + \overrightarrow{DC}$
$= \mathbf{p} + \mathbf{q} + \mathbf{r}$

EXERCISE 14B.4

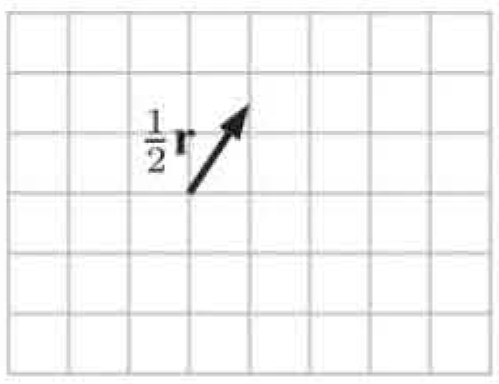
1 **a**



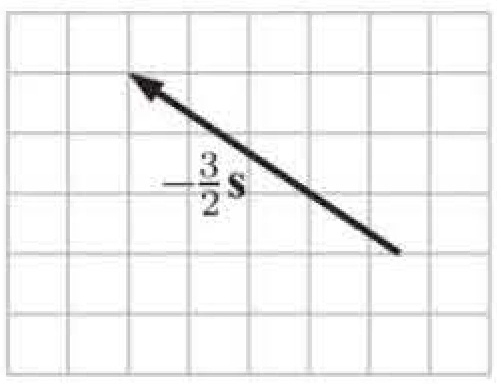
b



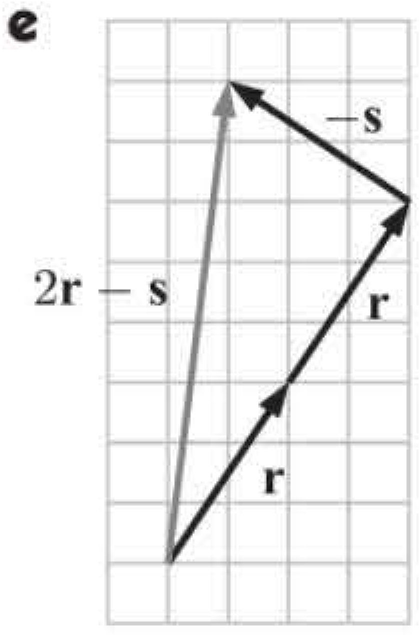
c



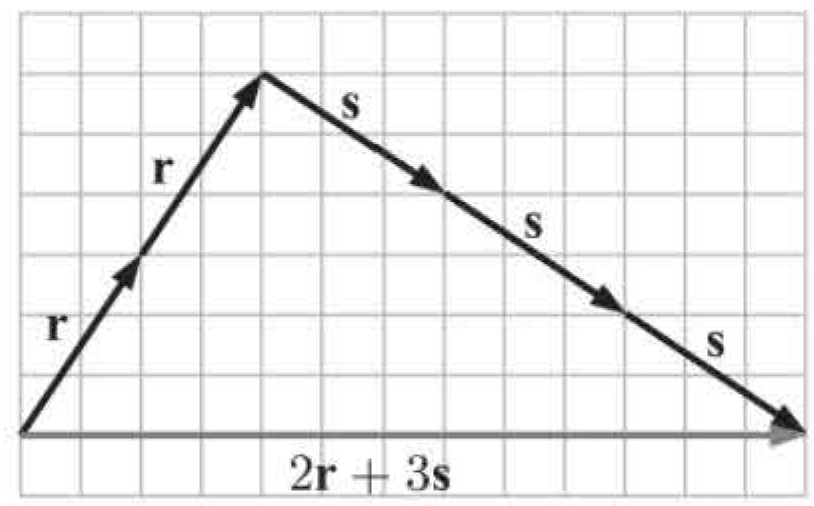
d



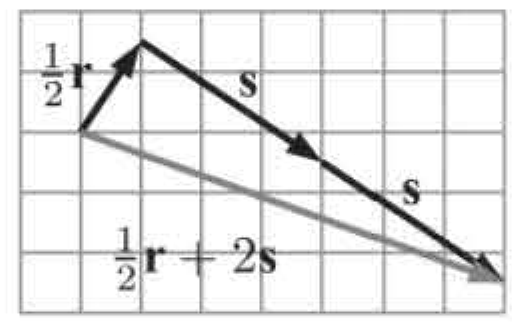
e



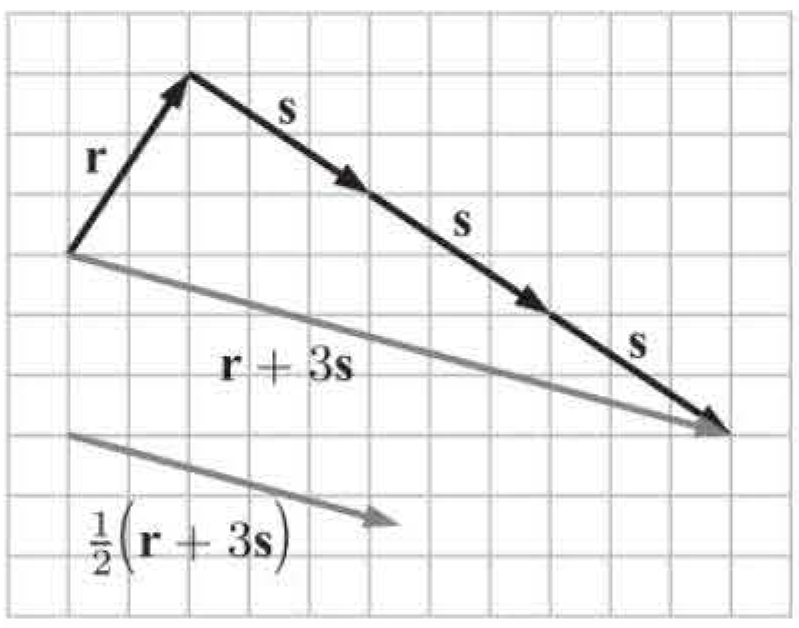
f



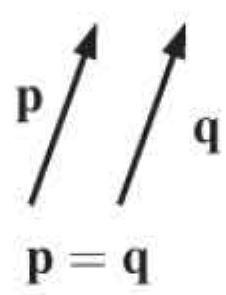
g



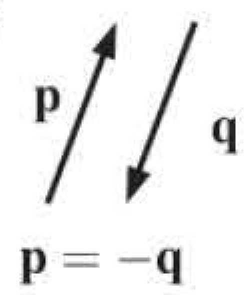
h



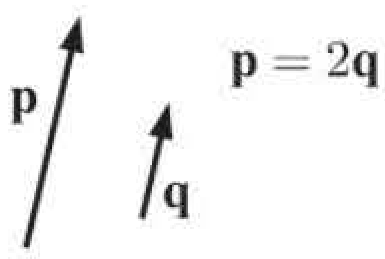
2 **a**



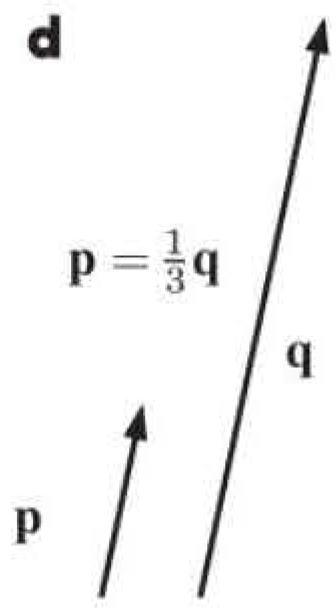
b



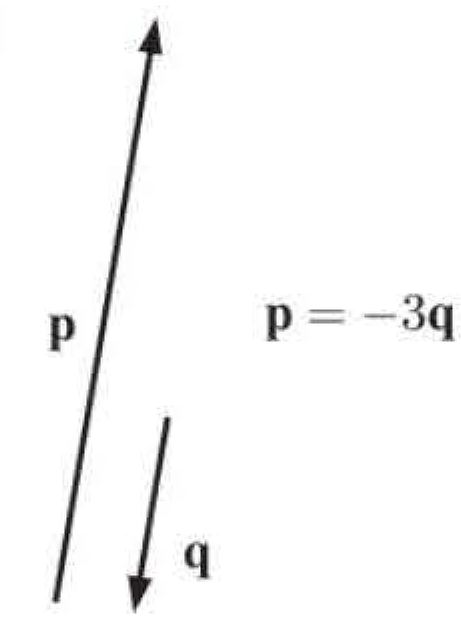
c



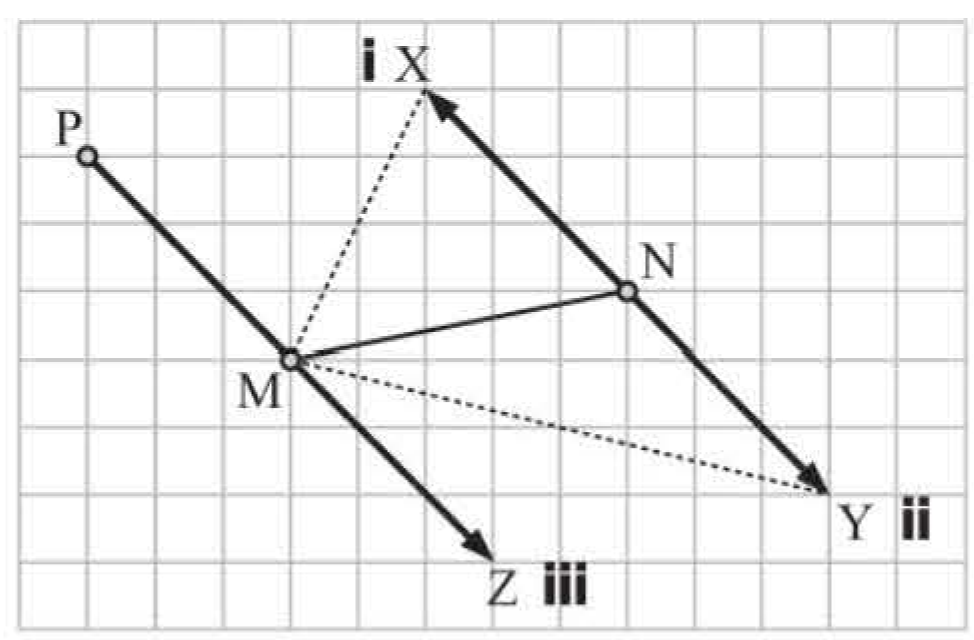
d



e

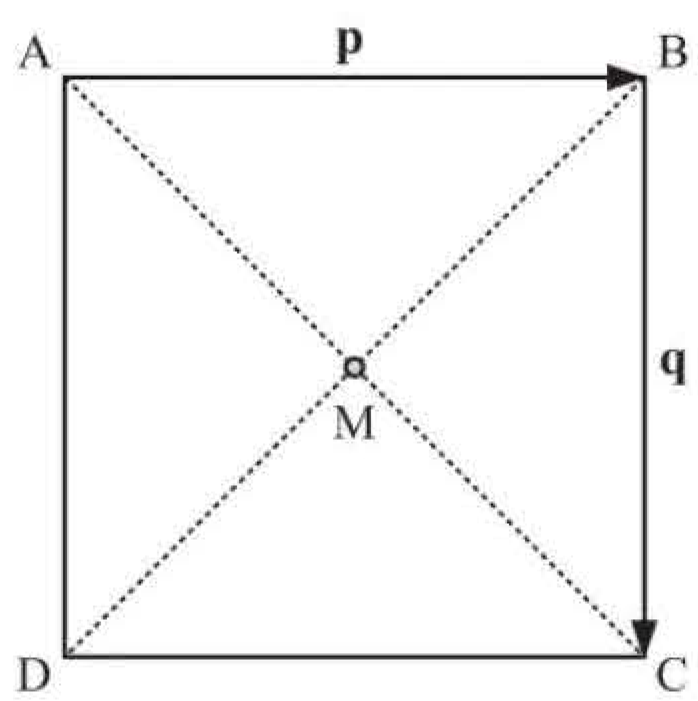


3 **a**



b a parallelogram

4



a $\overrightarrow{CD} = -\overrightarrow{AB}$
$= -\mathbf{p}$

b $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$
$= \mathbf{p} + \mathbf{q}$

c $\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AC}$
$= \frac{1}{2}(\mathbf{p} + \mathbf{q})$
{using part **b**}

d $\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD}$
$= \mathbf{q} + (-\mathbf{p})$ {using part **a**}
$= \mathbf{q} - \mathbf{p}$
and $\overrightarrow{BM} = \frac{1}{2}\overrightarrow{BD}$
$= \frac{1}{2}(\mathbf{q} - \mathbf{p})$

5 **a** $\overrightarrow{PX} = \overrightarrow{QR}$
$= \mathbf{b}$

b $\overrightarrow{PS} = 2\overrightarrow{PX}$
$= 2\mathbf{b}$
{using part **a**}

c $\overrightarrow{QX} = \overrightarrow{QR} + \overrightarrow{RX}$
$= \mathbf{b} + (-\mathbf{a})$
$= \mathbf{b} - \mathbf{a}$

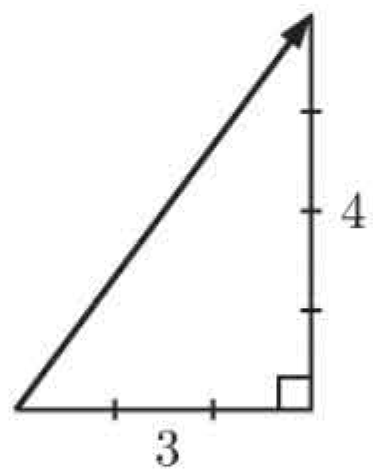
d $\overrightarrow{RS} = \overrightarrow{QX}$
$= \mathbf{b} - \mathbf{a}$
{using part **c**}

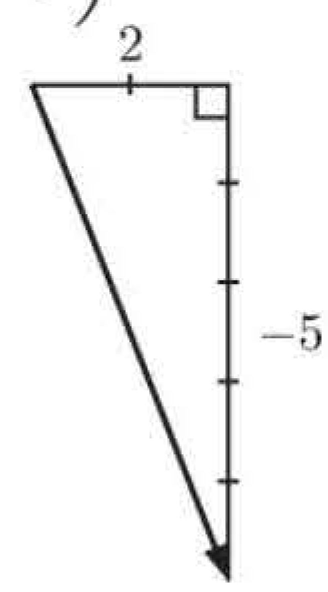
EXERCISE 14C

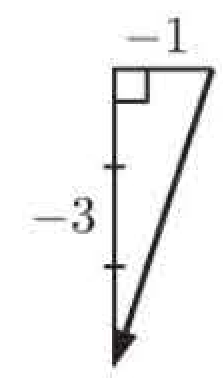
1 **a** $\begin{pmatrix} 7 \\ 3 \end{pmatrix}$, $7\mathbf{i} + 3\mathbf{j}$ **b** $\begin{pmatrix} -6 \\ 0 \end{pmatrix}$, $-6\mathbf{i}$ **c** $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$, $2\mathbf{i} - 5\mathbf{j}$

d $\begin{pmatrix} 0 \\ 6 \end{pmatrix}$, $6\mathbf{j}$ **e** $\begin{pmatrix} -6 \\ 3 \end{pmatrix}$, $-6\mathbf{i} + 3\mathbf{j}$ **f** $\begin{pmatrix} -5 \\ -5 \end{pmatrix}$, $-5\mathbf{i} - 5\mathbf{j}$

2 **a** $\begin{pmatrix} 3 \\ 4 \end{pmatrix} = 3\mathbf{i} + 4\mathbf{j}$ **b** $\begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2\mathbf{i}$ **c** $\begin{pmatrix} 2 \\ -5 \end{pmatrix} = 2\mathbf{i} - 5\mathbf{j}$ **d** $\begin{pmatrix} -1 \\ -3 \end{pmatrix} = -\mathbf{i} - 3\mathbf{j}$



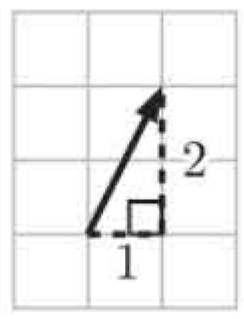


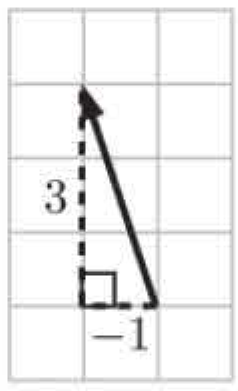


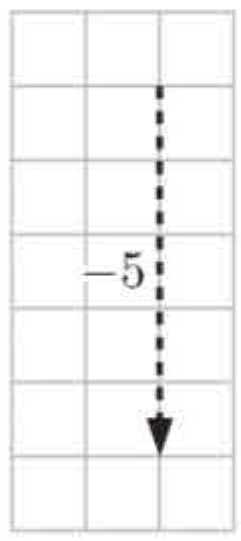
3 **a** $\overrightarrow{BA} = \begin{pmatrix} -4 \\ -1 \end{pmatrix} = -4\mathbf{i} - \mathbf{j}$ **b** $\overrightarrow{BC} = \begin{pmatrix} -1 \\ -5 \end{pmatrix} = -\mathbf{i} - 5\mathbf{j}$ **c** $\overrightarrow{DC} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2\mathbf{i}$

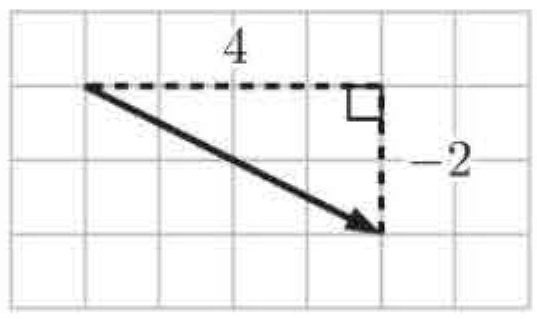
d $\overrightarrow{AC} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} = 3\mathbf{i} - 4\mathbf{j}$ **e** $\overrightarrow{CA} = \begin{pmatrix} -3 \\ 4 \end{pmatrix} = -3\mathbf{i} + 4\mathbf{j}$ **f** $\overrightarrow{DB} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} = 3\mathbf{i} + 5\mathbf{j}$

4 **a** $\mathbf{i} + 2\mathbf{j} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ **b** $-\mathbf{i} + 3\mathbf{j} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ **c** $-5\mathbf{j} = \begin{pmatrix} 0 \\ -5 \end{pmatrix}$ **d** $4\mathbf{i} - 2\mathbf{j} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$









5 The zero vector in component form is $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

EXERCISE 14D

1 **a** $\left|\begin{pmatrix} 3 \\ 4 \end{pmatrix}\right| = \sqrt{3^2 + 4^2}$
$= \sqrt{9+16}$
$= \sqrt{25} = 5$ units

b $\left|\begin{pmatrix} -4 \\ 3 \end{pmatrix}\right| = \sqrt{(-4)^2 + 3^2}$
$= \sqrt{16+9}$
$= \sqrt{25} = 5$ units

c $\left|\begin{pmatrix} 2 \\ 0 \end{pmatrix}\right| = \sqrt{2^2 + 0^2}$
$= \sqrt{4}$
$= 2$ units

d $\left|\begin{pmatrix} -2 \\ 2 \end{pmatrix}\right| = \sqrt{(-2)^2 + 2^2}$
$= \sqrt{4+4}$
$= \sqrt{8}$ units

e $\left|\begin{pmatrix} 0 \\ -3 \end{pmatrix}\right| = \sqrt{0^2 + (-3)^2}$
$= \sqrt{9}$
$= 3$ units

2 **a** As $\mathbf{i} + \mathbf{j} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$,
$|\mathbf{i} + \mathbf{j}| = \sqrt{1^2 + 1^2}$
$= \sqrt{2}$ units

b As $5\mathbf{i} - 12\mathbf{j} = \begin{pmatrix} 5 \\ -12 \end{pmatrix}$,
$|5\mathbf{i} - 12\mathbf{j}| = \sqrt{5^2 + (-12)^2}$
$= \sqrt{25+144}$
$= \sqrt{169} = 13$ units

c As $-\mathbf{i} + 4\mathbf{j} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$,
$|-\mathbf{i} + 4\mathbf{j}| = \sqrt{(-1)^2 + 4^2}$
$= \sqrt{1+16}$
$= \sqrt{17}$ units

d As $3\mathbf{i} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$,
$|3\mathbf{i}| = \sqrt{3^2 + 0^2}$
$= \sqrt{9}$
$= 3$ units

e As $k\mathbf{j} = \begin{pmatrix} 0 \\ k \end{pmatrix}$,
$|k\mathbf{j}| = \sqrt{0^2 + k^2}$
$= \sqrt{k^2}$
$= |k|$ units

3 **a** $\sqrt{0^2+(-1)^2}=1$

$\therefore \begin{pmatrix}0\\-1\end{pmatrix}$ is a unit vector.

b $\sqrt{(-\frac{1}{\sqrt{2}})^2+(\frac{1}{\sqrt{2}})^2}=\sqrt{\frac{1}{2}+\frac{1}{2}}=1$

$\therefore \begin{pmatrix}-\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$ is a unit vector.

c $\sqrt{(\frac{2}{3})^2+(\frac{1}{3})^2}=\sqrt{\frac{4}{9}+\frac{1}{9}}=\frac{\sqrt{5}}{3}$

$\therefore \begin{pmatrix}\frac{2}{3}\\ \frac{1}{3}\end{pmatrix}$ is not a unit vector.

d $\sqrt{(-\frac{3}{5})^2+(-\frac{4}{5})^2}=\sqrt{\frac{9}{25}+\frac{16}{25}}=1$

$\therefore \begin{pmatrix}-\frac{3}{5}\\ -\frac{4}{5}\end{pmatrix}$ is a unit vector.

e $\sqrt{(\frac{2}{7})^2+(-\frac{5}{7})^2}=\sqrt{\frac{4}{49}+\frac{25}{49}}=\frac{\sqrt{29}}{7}$

$\therefore \begin{pmatrix}\frac{2}{7}\\ -\frac{5}{7}\end{pmatrix}$ is not a unit vector.

4 **a**
$$\begin{aligned}\text{length}&=1\\ \therefore \sqrt{0^2+k^2}&=1\\ \therefore k^2&=1\\ \therefore k&=\pm1\end{aligned}$$

b
$$\begin{aligned}\text{length}&=1\\ \therefore \sqrt{k^2+0}&=1\\ \therefore k^2&=1\\ \therefore k&=\pm1\end{aligned}$$

c
$$\begin{aligned}\text{length}&=1\\ \therefore \sqrt{k^2+1}&=1\\ \therefore k^2+1&=1\\ \therefore k^2&=0\\ \therefore k&=0\end{aligned}$$

d
$$\begin{aligned}\text{length}&=1\\ \therefore \sqrt{k^2+k^2}&=1\\ \therefore 2k^2&=1\\ \therefore k^2&=\tfrac{1}{2}\\ \therefore k&=\pm\sqrt{\tfrac{1}{2}}=\pm\tfrac{1}{\sqrt{2}}\end{aligned}$$

e
$$\begin{aligned}\text{length}&=1\\ \therefore \sqrt{\left(\tfrac{1}{2}\right)^2+k^2}&=1\\ \therefore \tfrac{1}{4}+k^2&=1\\ \therefore k^2&=\tfrac{3}{4}\\ \therefore k&=\pm\sqrt{\tfrac{3}{4}}=\pm\tfrac{\sqrt{3}}{2}\end{aligned}$$

5 If $|\mathbf{v}|=\sqrt{73}$ units then $\sqrt{8^2+p^2}=\sqrt{73}$

$\therefore 64+p^2=73$

$\therefore p^2=9 \qquad \therefore p=\pm3$

EXERCISE 14E

1 **a** $\mathbf{a}+\mathbf{b}=\begin{pmatrix}-3\\2\end{pmatrix}+\begin{pmatrix}1\\4\end{pmatrix}=\begin{pmatrix}-2\\6\end{pmatrix}$

b $\mathbf{b}+\mathbf{a}=\begin{pmatrix}1\\4\end{pmatrix}+\begin{pmatrix}-3\\2\end{pmatrix}=\begin{pmatrix}-2\\6\end{pmatrix}$

c $\mathbf{b}+\mathbf{c}=\begin{pmatrix}1\\4\end{pmatrix}+\begin{pmatrix}-2\\-5\end{pmatrix}=\begin{pmatrix}-1\\-1\end{pmatrix}$

d $\mathbf{c}+\mathbf{b}=\begin{pmatrix}-2\\-5\end{pmatrix}+\begin{pmatrix}1\\4\end{pmatrix}=\begin{pmatrix}-1\\-1\end{pmatrix}$

e $\mathbf{a}+\mathbf{c}=\begin{pmatrix}-3\\2\end{pmatrix}+\begin{pmatrix}-2\\-5\end{pmatrix}=\begin{pmatrix}-5\\-3\end{pmatrix}$

f $\mathbf{c}+\mathbf{a}=\begin{pmatrix}-2\\-5\end{pmatrix}+\begin{pmatrix}-3\\2\end{pmatrix}=\begin{pmatrix}-5\\-3\end{pmatrix}$

g $\mathbf{a}+\mathbf{a}=\begin{pmatrix}-3\\2\end{pmatrix}+\begin{pmatrix}-3\\2\end{pmatrix}=\begin{pmatrix}-6\\4\end{pmatrix}$

h $\mathbf{b}+\mathbf{a}+\mathbf{c}=\begin{pmatrix}1\\4\end{pmatrix}+\begin{pmatrix}-3\\2\end{pmatrix}+\begin{pmatrix}-2\\-5\end{pmatrix}$

$=\begin{pmatrix}-2\\6\end{pmatrix}+\begin{pmatrix}-2\\-5\end{pmatrix}=\begin{pmatrix}-4\\1\end{pmatrix}$

2 **a** $\mathbf{p}-\mathbf{q}=\begin{pmatrix}-4\\2\end{pmatrix}-\begin{pmatrix}-1\\-5\end{pmatrix}=\begin{pmatrix}-3\\7\end{pmatrix}$

b $\mathbf{q}-\mathbf{r}=\begin{pmatrix}-1\\-5\end{pmatrix}-\begin{pmatrix}3\\-2\end{pmatrix}=\begin{pmatrix}-4\\-3\end{pmatrix}$

c $\mathbf{p}+\mathbf{q}-\mathbf{r}$

$=\begin{pmatrix}-4\\2\end{pmatrix}+\begin{pmatrix}-1\\-5\end{pmatrix}-\begin{pmatrix}3\\-2\end{pmatrix}=\begin{pmatrix}-8\\-1\end{pmatrix}$

d $\mathbf{p}-\mathbf{q}-\mathbf{r}$

$=\begin{pmatrix}-4\\2\end{pmatrix}-\begin{pmatrix}-1\\-5\end{pmatrix}-\begin{pmatrix}3\\-2\end{pmatrix}=\begin{pmatrix}-6\\9\end{pmatrix}$

e $\mathbf{q}-\mathbf{r}-\mathbf{p}=\begin{pmatrix}-1\\-5\end{pmatrix}-\begin{pmatrix}3\\-2\end{pmatrix}-\begin{pmatrix}-4\\2\end{pmatrix}$
$=\begin{pmatrix}0\\-5\end{pmatrix}$

f $\mathbf{r}+\mathbf{q}-\mathbf{p}=\begin{pmatrix}3\\-2\end{pmatrix}+\begin{pmatrix}-1\\-5\end{pmatrix}-\begin{pmatrix}-4\\2\end{pmatrix}$
$=\begin{pmatrix}6\\-9\end{pmatrix}$

3 **a** $\mathbf{a}+\mathbf{0}=\begin{pmatrix}a_1\\a_2\end{pmatrix}+\begin{pmatrix}0\\0\end{pmatrix}$
$=\begin{pmatrix}a_1+0\\a_2+0\end{pmatrix}=\begin{pmatrix}a_1\\a_2\end{pmatrix}=\mathbf{a}$

b $\mathbf{a}-\mathbf{a}=\begin{pmatrix}a_1\\a_2\end{pmatrix}-\begin{pmatrix}a_1\\a_2\end{pmatrix}$
$=\begin{pmatrix}a_1-a_1\\a_2-a_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}=\mathbf{0}$

4 **a** $-3\mathbf{p}$
$=-3\begin{pmatrix}1\\5\end{pmatrix}$
$=\begin{pmatrix}-3\\-15\end{pmatrix}$

b $\frac{1}{2}\mathbf{q}$
$=\frac{1}{2}\begin{pmatrix}-2\\4\end{pmatrix}$
$=\begin{pmatrix}-1\\2\end{pmatrix}$

c $2\mathbf{p}+\mathbf{q}$
$=2\begin{pmatrix}1\\5\end{pmatrix}+\begin{pmatrix}-2\\4\end{pmatrix}$
$=\begin{pmatrix}2\\10\end{pmatrix}+\begin{pmatrix}-2\\4\end{pmatrix}$
$=\begin{pmatrix}0\\14\end{pmatrix}$

d $\mathbf{p}-2\mathbf{q}$
$=\begin{pmatrix}1\\5\end{pmatrix}-2\begin{pmatrix}-2\\4\end{pmatrix}$
$=\begin{pmatrix}1\\5\end{pmatrix}-\begin{pmatrix}-4\\8\end{pmatrix}$
$=\begin{pmatrix}5\\-3\end{pmatrix}$

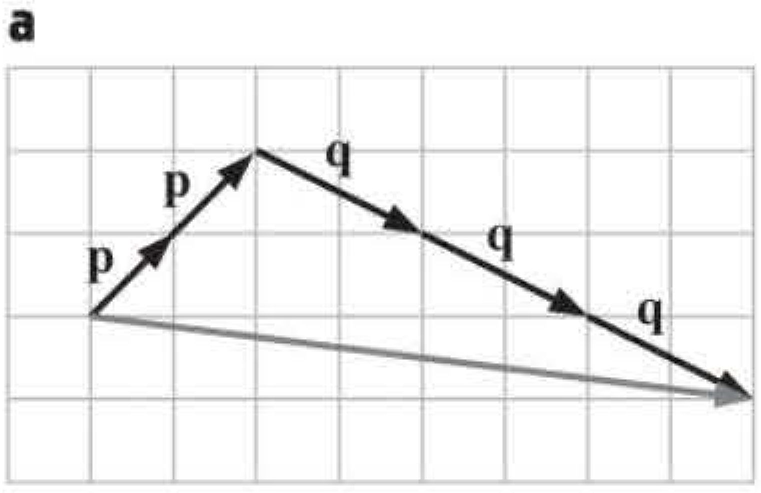
e $\mathbf{p}-\frac{1}{2}\mathbf{r}=\begin{pmatrix}1\\5\end{pmatrix}-\frac{1}{2}\begin{pmatrix}-3\\-1\end{pmatrix}$
$=\begin{pmatrix}1\\5\end{pmatrix}-\begin{pmatrix}-\frac{3}{2}\\-\frac{1}{2}\end{pmatrix}$
$=\begin{pmatrix}\frac{5}{2}\\\frac{11}{2}\end{pmatrix}$

f $2\mathbf{p}+3\mathbf{r}=2\begin{pmatrix}1\\5\end{pmatrix}+3\begin{pmatrix}-3\\-1\end{pmatrix}$
$=\begin{pmatrix}2\\10\end{pmatrix}+\begin{pmatrix}-9\\-3\end{pmatrix}$
$=\begin{pmatrix}-7\\7\end{pmatrix}$

g $2\mathbf{q}-3\mathbf{r}=2\begin{pmatrix}-2\\4\end{pmatrix}-3\begin{pmatrix}-3\\-1\end{pmatrix}$
$=\begin{pmatrix}-4\\8\end{pmatrix}-\begin{pmatrix}-9\\-3\end{pmatrix}$
$=\begin{pmatrix}5\\11\end{pmatrix}$

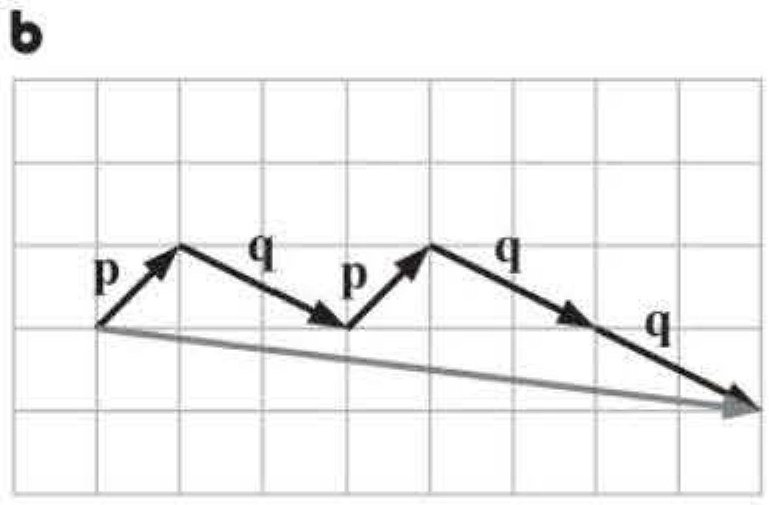
h $2\mathbf{p}-\mathbf{q}+\frac{1}{3}\mathbf{r}$
$=2\begin{pmatrix}1\\5\end{pmatrix}-\begin{pmatrix}-2\\4\end{pmatrix}+\frac{1}{3}\begin{pmatrix}-3\\-1\end{pmatrix}$
$=\begin{pmatrix}2\\10\end{pmatrix}-\begin{pmatrix}-2\\4\end{pmatrix}+\begin{pmatrix}-1\\-\frac{1}{3}\end{pmatrix}$
$=\begin{pmatrix}3\\\frac{17}{3}\end{pmatrix}$

5 **a**



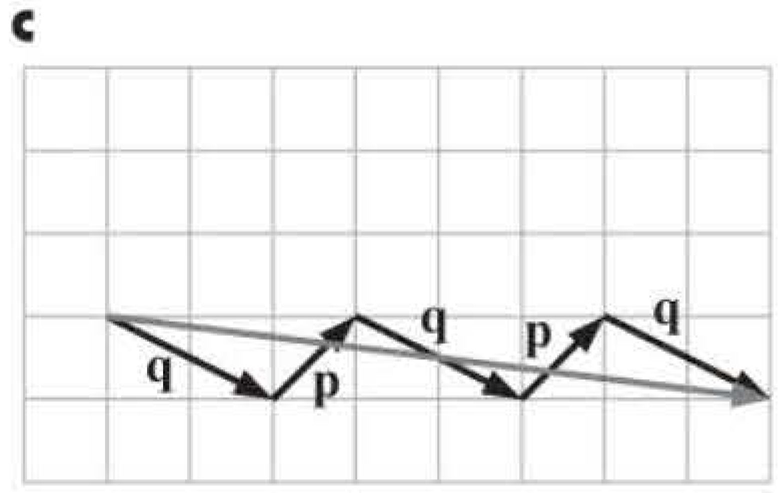
$=\begin{pmatrix}8\\-1\end{pmatrix}$

b



$=\begin{pmatrix}8\\-1\end{pmatrix}$

c



$=\begin{pmatrix}8\\-1\end{pmatrix}$

The vector expressions are equal, as each consists of 2 **p**s and 3 **q**s. Each expression is equal to $2\mathbf{p}+3\mathbf{q}$.

6 **a** $|\mathbf{r}|=\sqrt{2^2+3^2}$
$=\sqrt{13}$ units

b $|\mathbf{s}|=\sqrt{(-1)^2+4^2}$
$=\sqrt{17}$ units

c $\mathbf{r}+\mathbf{s}$
$= \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} -1 \\ 4 \end{pmatrix}$
$= \begin{pmatrix} 1 \\ 7 \end{pmatrix}$
$\therefore \ |\mathbf{r}+\mathbf{s}|$
$= \sqrt{1^2+7^2}$
$= \sqrt{50}$ units
$= 5\sqrt{2}$ units

d $\mathbf{r}-\mathbf{s}$
$= \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} -1 \\ 4 \end{pmatrix}$
$= \begin{pmatrix} 3 \\ -1 \end{pmatrix}$
$\therefore \ |\mathbf{r}-\mathbf{s}|$
$= \sqrt{3^2+(-1)^2}$
$= \sqrt{10}$ units

e $\mathbf{s}-2\mathbf{r}$
$= \begin{pmatrix} -1 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ 6 \end{pmatrix}$
$= \begin{pmatrix} -5 \\ -2 \end{pmatrix}$
$\therefore \ |\mathbf{s}-2\mathbf{r}|$
$= \sqrt{(-5)^2+(-2)^2}$
$= \sqrt{29}$ units

7 **a** $|\mathbf{p}| = \sqrt{1^2+3^2}$
$= \sqrt{10}$ units

b $2\mathbf{p} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$
$\therefore \ |2\mathbf{p}| = \sqrt{2^2+6^2}$
$= \sqrt{4+36}$
$= \sqrt{40}$
$= 2\sqrt{10}$ units

c $-2\mathbf{p} = \begin{pmatrix} -2 \\ -6 \end{pmatrix}$
$\therefore \ |-2\mathbf{p}| = \sqrt{(-2)^2+(-6)^2}$
$= \sqrt{4+36}$
$= \sqrt{40}$
$= 2\sqrt{10}$ units

d $3\mathbf{p} = \begin{pmatrix} 3 \\ 9 \end{pmatrix}$
$\therefore \ |3\mathbf{p}| = \sqrt{3^2+9^2}$
$= \sqrt{9+81}$
$= \sqrt{90}$
$= 3\sqrt{10}$ units

e $-3\mathbf{p} = \begin{pmatrix} -3 \\ -9 \end{pmatrix}$
$\therefore \ |-3\mathbf{p}| = \sqrt{(-3)^2+(-9)^2}$
$= \sqrt{9+81}$
$= \sqrt{90}$
$= 3\sqrt{10}$ units

f $|\mathbf{q}| = \sqrt{(-2)^2+4^2}$
$= \sqrt{4+16}$
$= \sqrt{20}$
$= 2\sqrt{5}$ units

g $4\mathbf{q} = \begin{pmatrix} -8 \\ 16 \end{pmatrix}$
$\therefore \ |4\mathbf{q}| = \sqrt{(-8)^2+16^2}$
$= \sqrt{64+256}$
$= \sqrt{320}$
$= 8\sqrt{5}$ units

h $-4\mathbf{q} = \begin{pmatrix} 8 \\ -16 \end{pmatrix}$
$\therefore \ |-4\mathbf{q}| = \sqrt{8^2+(-16)^2}$
$= \sqrt{64+256}$
$= \sqrt{320}$
$= 8\sqrt{5}$ units

i $\frac{1}{2}\mathbf{q} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$
$\therefore \ \left|\frac{1}{2}\mathbf{q}\right| = \sqrt{(-1)^2+2^2}$
$= \sqrt{5}$ units

j $-\frac{1}{2}\mathbf{q} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$
$\therefore \ \left|-\frac{1}{2}\mathbf{q}\right| = \sqrt{1^2+(-2)^2}$
$= \sqrt{5}$ units

8 $k\mathbf{x} = \mathbf{a}$
$\therefore \ k\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$
$\therefore \ kx_1 = a_1$ and $kx_2 = a_2$
$\therefore \ x_1 = \frac{1}{k}a_1$ and $x_2 = \frac{1}{k}a_2$
$\therefore \ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{k}a_1 \\ \frac{1}{k}a_2 \end{pmatrix} = \frac{1}{k}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$
and so $\mathbf{x} = \frac{1}{k}\mathbf{a}$

9 $k\mathbf{v} = \begin{pmatrix} kv_1 \\ kv_2 \end{pmatrix}$
$\therefore \ |k\mathbf{v}| = \sqrt{(kv_1)^2+(kv_2)^2}$
$= \sqrt{k^2{v_1}^2+k^2{v_2}^2}$
$= \sqrt{k^2({v_1}^2+{v_2}^2)}$
$= \sqrt{k^2}\sqrt{{v_1}^2+{v_2}^2}$
$= |k|\sqrt{{v_1}^2+{v_2}^2}$
$= |k|\,|\mathbf{v}|$

EXERCISE 14F

1 **a** $\overrightarrow{AB} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix} = \begin{pmatrix} 4-2 \\ 7-3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

b $\overrightarrow{AB} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix} = \begin{pmatrix} 1-3 \\ 4--1 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$

c $\overrightarrow{AB} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix} = \begin{pmatrix} 1--2 \\ 4-7 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$

d $\overrightarrow{AB} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix} = \begin{pmatrix} 3-2 \\ 0-5 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$

e $\overrightarrow{AB} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix} = \begin{pmatrix} 6-0 \\ -1-4 \end{pmatrix} = \begin{pmatrix} 6 \\ -5 \end{pmatrix}$

f $\overrightarrow{AB} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix} = \begin{pmatrix} 0--1 \\ 0--3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

2 **a** Let B have coordinates (b_1, b_2).

$\therefore \ \overrightarrow{AB} = \begin{pmatrix} b_1 - 1 \\ b_2 - 4 \end{pmatrix}$

$\therefore \ \begin{pmatrix} b_1 - 1 \\ b_2 - 4 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

$\therefore \ b_1 - 1 = 3$ and $b_2 - 4 = -2$

$\therefore \ b_1 = 4$ and $b_2 = 2$

$\therefore$ B has coordinates $(4, 2)$.

b Let C have coordinates (c_1, c_2).

$\therefore \ \overrightarrow{CA} = \begin{pmatrix} 1 - c_1 \\ 4 - c_2 \end{pmatrix}$

$\therefore \ \begin{pmatrix} 1 - c_1 \\ 4 - c_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$\therefore \ 1 - c_1 = -1$ and $4 - c_2 = 2$

$\therefore \ c_1 = 2$ and $c_2 = 2$

$\therefore$ C has coordinates $(2, 2)$.

3 **a** $\overrightarrow{PC} = \begin{pmatrix} 1-(-1) \\ 2-1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

b Let Q have coordinates (q_1, q_2).

$\therefore \ \overrightarrow{CQ} = \begin{pmatrix} q_1 - 1 \\ q_2 - 2 \end{pmatrix}$

But $\overrightarrow{CQ} = \overrightarrow{PC}$

$\therefore \ \begin{pmatrix} q_1 - 1 \\ q_2 - 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\therefore \ q_1 - 1 = 2$ and $q_2 - 2 = 1$

$\therefore \ q_1 = 3$ and $q_2 = 3$

$\therefore$ Q has coordinates $(3, 3)$.

4 **a** $\overrightarrow{AB} = \begin{pmatrix} 6-1 \\ 5-4 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$

b $\overrightarrow{CD} = -\overrightarrow{AB} = -\begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \end{pmatrix}$

c Let D have coordinates (d_1, d_2).

$\therefore \ \overrightarrow{CD} = \begin{pmatrix} d_1 - 4 \\ d_2 - (-1) \end{pmatrix} = \begin{pmatrix} d_1 - 4 \\ d_2 + 1 \end{pmatrix}$

But $\overrightarrow{CD} = \begin{pmatrix} -5 \\ -1 \end{pmatrix}$

$\therefore \ \begin{pmatrix} d_1 - 4 \\ d_2 + 1 \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \end{pmatrix}$

$\therefore \ d_1 - 4 = -5$ and $d_2 + 1 = -1$

$\therefore \ d_1 = -1$ and $d_2 = -2$

$\therefore$ D has coordinates $(-1, -2)$.

5 **a** $\overrightarrow{AB} = \begin{pmatrix} 3-(-1) \\ k-3 \end{pmatrix} = \begin{pmatrix} 4 \\ k-3 \end{pmatrix}$

Since A and B are 5 units apart,

$|\overrightarrow{AB}| = 5$ units and $|\overrightarrow{AB}| = \sqrt{4^2 + (k-3)^2}$

$= \sqrt{16 + (k-3)^2}$

b $|\overrightarrow{AB}| = 5$

$\therefore \sqrt{16+(k-3)^2} = 5$

$\therefore 16 + k^2 - 6k + 9 = 25$

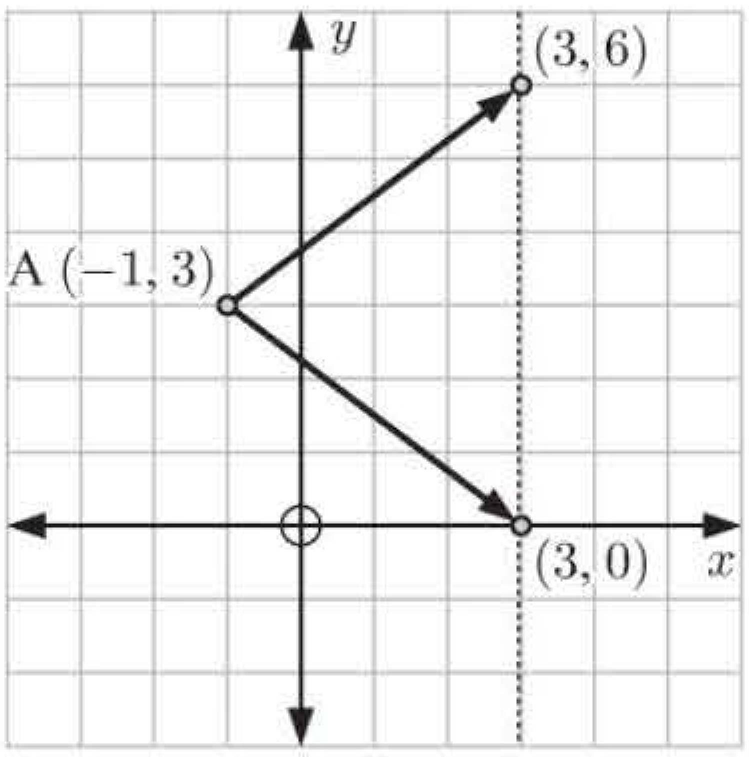
$\therefore k^2 - 6k = 0$

$\therefore k(k-6) = 0$

$\therefore k = 0$ or $k - 6 = 0$

$\therefore k = 0$ or 6

c



6 **a** $\overrightarrow{AB} = \begin{pmatrix} 3-1 \\ 5-2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$\overrightarrow{AC} = \begin{pmatrix} 4-1 \\ -1-2 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$

b $\overrightarrow{BC}$
$= \overrightarrow{BA} + \overrightarrow{AC}$
$= -\overrightarrow{AB} + \overrightarrow{AC}$

c $\overrightarrow{BC}$
$= -\overrightarrow{AB} + \overrightarrow{AC}$
$= -\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ -3 \end{pmatrix}$
$= \begin{pmatrix} -2 \\ -3 \end{pmatrix} + \begin{pmatrix} 3 \\ -3 \end{pmatrix}$
$= \begin{pmatrix} -2+3 \\ -3-3 \end{pmatrix}$
$= \begin{pmatrix} 1 \\ -6 \end{pmatrix}$

d $\overrightarrow{BC} = \begin{pmatrix} 4-3 \\ -1-5 \end{pmatrix} = \begin{pmatrix} 1 \\ -6 \end{pmatrix}$ ✓

7 **a** $\overrightarrow{AC}$
$= \overrightarrow{AB} + \overrightarrow{BC}$
$= -\overrightarrow{BA} + \overrightarrow{BC}$
$= -\begin{pmatrix} 2 \\ -3 \end{pmatrix} + \begin{pmatrix} -3 \\ 1 \end{pmatrix}$
$= \begin{pmatrix} -5 \\ 4 \end{pmatrix}$

b $\overrightarrow{CB}$
$= \overrightarrow{CA} + \overrightarrow{AB}$
$= \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \end{pmatrix}$
$= \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

c $\overrightarrow{SP}$
$= \overrightarrow{SR} + \overrightarrow{RQ} + \overrightarrow{QP}$
$= -\overrightarrow{RS} + \overrightarrow{RQ} - \overrightarrow{PQ}$
$= -\begin{pmatrix} -3 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 4 \end{pmatrix}$
$= \begin{pmatrix} 6 \\ -5 \end{pmatrix}$

8 **a** M is $\left(\frac{3+-1}{2}, \frac{6+2}{2}\right)$
$\therefore$ M is $(1, 4)$

b $\overrightarrow{CA} = \begin{pmatrix} 3--4 \\ 6-1 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$

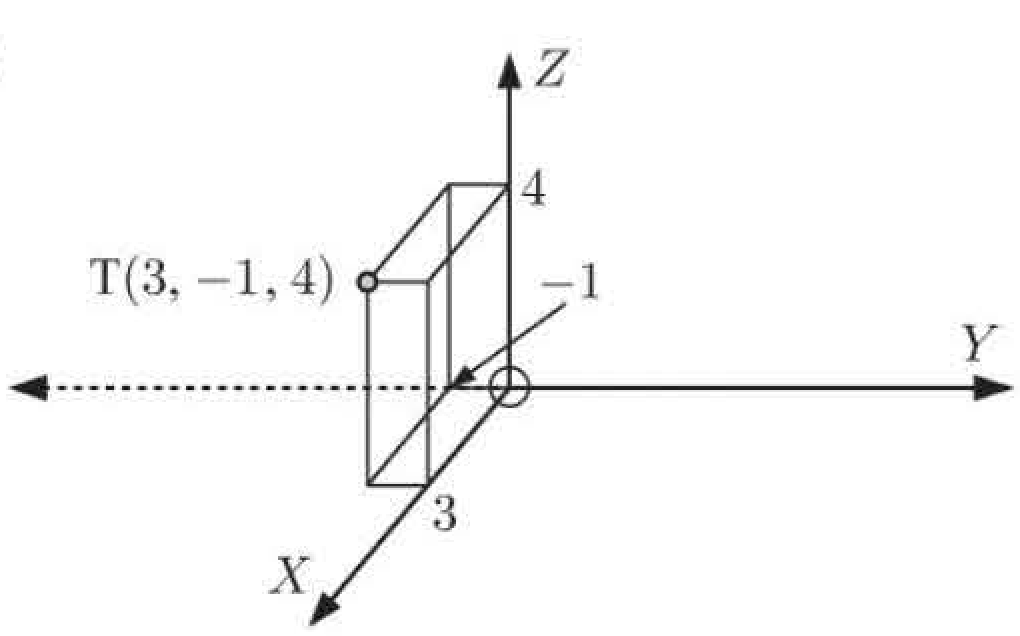
$\overrightarrow{CM} = \begin{pmatrix} 1--4 \\ 4-1 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$

$\overrightarrow{CB} = \begin{pmatrix} -1--4 \\ 2-1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

c $\frac{1}{2}\overrightarrow{CA} + \frac{1}{2}\overrightarrow{CB}$
$= \frac{1}{2}\begin{pmatrix} 7 \\ 5 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 3 \\ 1 \end{pmatrix}$
$= \begin{pmatrix} 5 \\ 3 \end{pmatrix}$ which is $\overrightarrow{CM}$

EXERCISE 14G

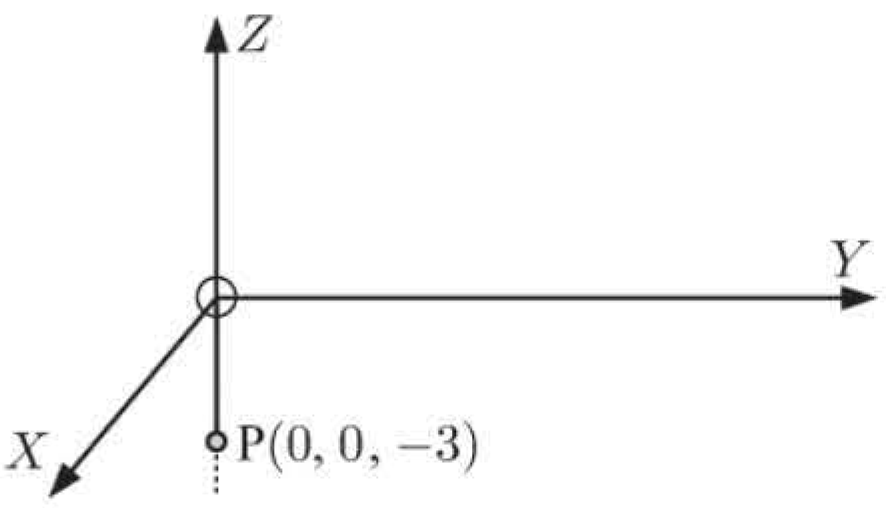
1 **a**



b $\overrightarrow{OT} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$

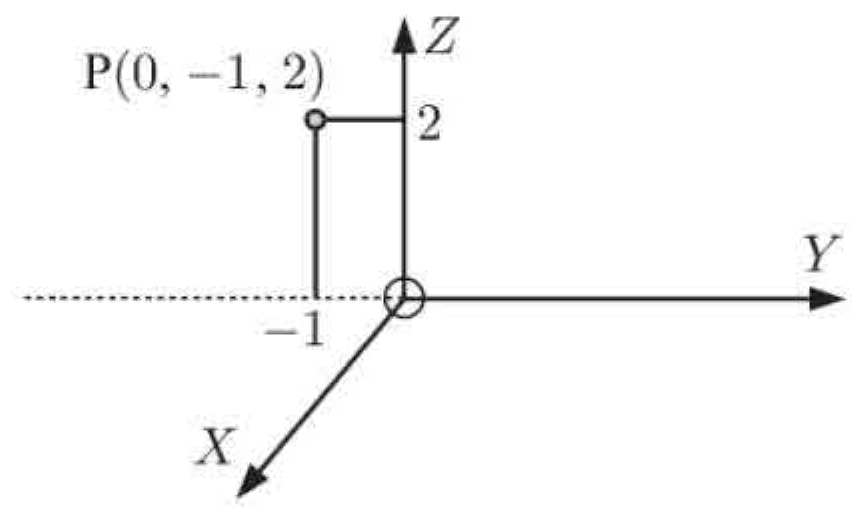
c $OT = \sqrt{(3-0)^2 + (-1-0)^2 + (4-0)^2}$
$= \sqrt{9+1+16}$
$= \sqrt{26}$ units

2 **a**



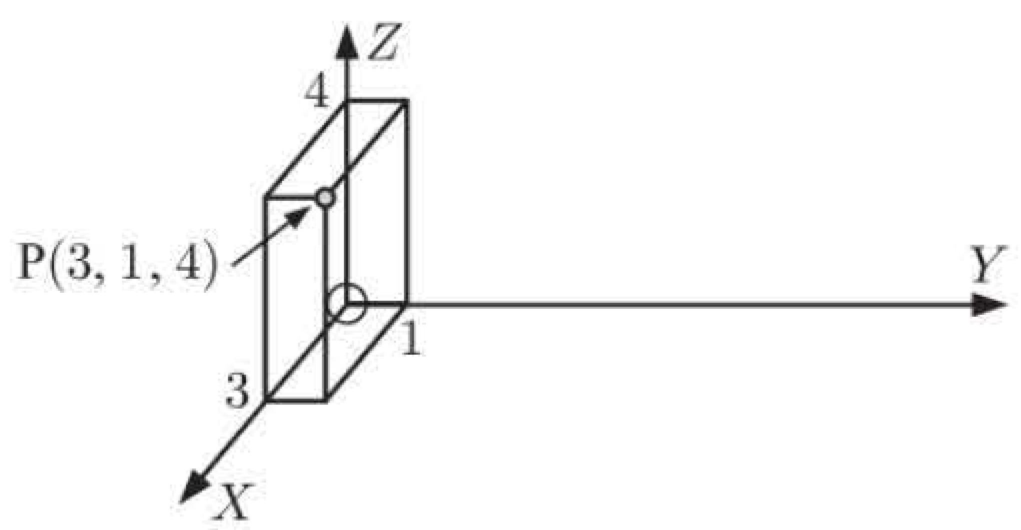
$\text{OP} = \sqrt{0^2 + 0^2 + (-3)^2} = 3$ units

b



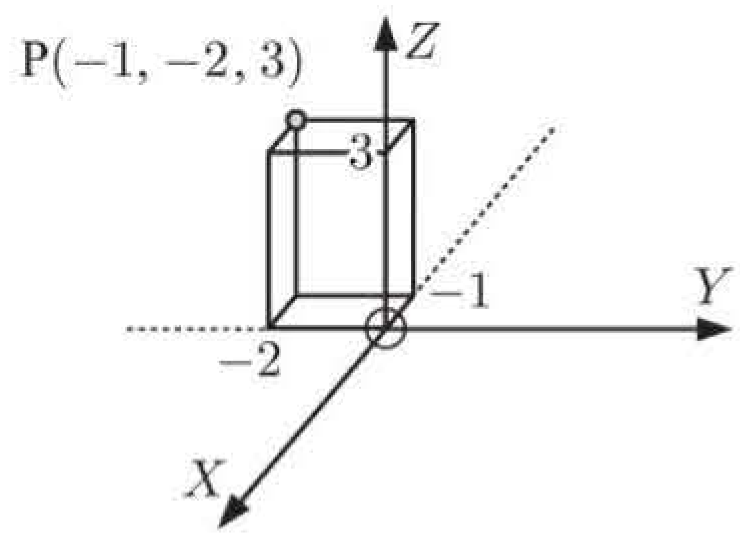
$\text{OP} = \sqrt{0^2 + (-1)^2 + 2^2} = \sqrt{5}$ units

c



$\text{OP} = \sqrt{3^2 + 1^2 + 4^2} = \sqrt{26}$ units

d



$\text{OP} = \sqrt{(-1)^2 + (-2)^2 + 3^2} = \sqrt{14}$ units

3 **a** $\overrightarrow{\text{AB}} = \begin{pmatrix} 1-(-3) \\ 0-1 \\ -1-2 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix}$, $\quad \overrightarrow{\text{BA}} = \begin{pmatrix} -3-1 \\ 1-0 \\ 2-(-1) \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix}$

b $|\overrightarrow{\text{AB}}| = \sqrt{4^2 + (-1)^2 + (-3)^2} = \sqrt{26}$ units, $\quad |\overrightarrow{\text{BA}}| = \sqrt{(-4)^2 + 1^2 + 3^2} = \sqrt{26}$ units

4 $\overrightarrow{\text{OA}} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$, $\quad \overrightarrow{\text{OB}} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$, $\quad \overrightarrow{\text{AB}} = \begin{pmatrix} -1-3 \\ 1-1 \\ 2-0 \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \\ 2 \end{pmatrix}$

5 **a** The position vector of M relative to N

$$= \overrightarrow{\text{NM}} = \begin{pmatrix} 4-(-1) \\ -2-2 \\ -1-0 \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \\ -1 \end{pmatrix}$$

b The position vector of N relative to M

$$= \overrightarrow{\text{MN}} = \begin{pmatrix} -1-4 \\ 2-(-2) \\ 0-(-1) \end{pmatrix} = \begin{pmatrix} -5 \\ 4 \\ 1 \end{pmatrix}$$

c $\text{MN} = \sqrt{(-5)^2 + 4^2 + 1^2} = \sqrt{25+16+1} = \sqrt{42}$ units

6 **a** The position vector of A relative to O

$$= \overrightarrow{\text{OA}} = \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}$$

$$\therefore\ \text{OA} = \sqrt{(-1)^2 + 2^2 + 5^2} = \sqrt{1+4+25} = \sqrt{30} \text{ units}$$

b The position vector of B relative to A

$$= \overrightarrow{\text{AB}} = \begin{pmatrix} 2-(-1) \\ 0-2 \\ 3-5 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix}$$

$$\therefore\ \text{AB} = \sqrt{3^2 + (-2)^2 + (-2)^2} = \sqrt{9+4+4} = \sqrt{17} \text{ units}$$

c The position vector of C relative to A

$$= \overrightarrow{\text{AC}} = \begin{pmatrix} -3-(-1) \\ 1-2 \\ 0-5 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ -5 \end{pmatrix}$$

$$\therefore\ \text{AC} = \sqrt{(-2)^2 + (-1)^2 + (-5)^2} = \sqrt{4+1+25} = \sqrt{30} \text{ units}$$

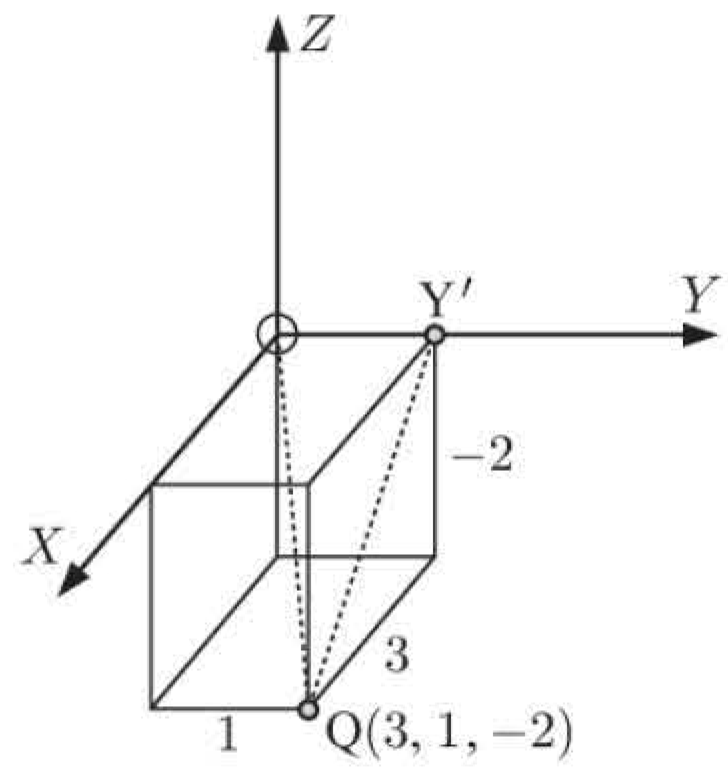
d The position vector of B relative to C

$$= \overrightarrow{\text{CB}} = \begin{pmatrix} 2-(-3) \\ 0-1 \\ 3-0 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}$$

$$\therefore\ \text{CB} = \sqrt{5^2 + (-1)^2 + 3^2} = \sqrt{25+1+9} = \sqrt{35} \text{ units}$$

e Triangle ABC has $\text{AC} = \sqrt{30}$ units, $\text{CB} = \sqrt{35}$ units, and $\text{AB} = \sqrt{17}$ units.
All the side lengths are different, and $\left(\sqrt{17}\right)^2 + \left(\sqrt{30}\right)^2 \neq \left(\sqrt{35}\right)^2$.
$\therefore$ triangle ABC is scalene, and not right angled.

7



a The distance from Q to the Y-axis is the distance from Q to $\text{Y}'(0,\ 1,\ 0)$.

$$\begin{aligned}\therefore\ \text{QY}' &= \sqrt{(3-0)^2+(1-1)^2+(-2-0)^2}\\ &= \sqrt{9+4}\\ &= \sqrt{13}\text{ units}\end{aligned}$$

b The distance from Q to the origin is

$$\begin{aligned}\text{QO} &= \sqrt{(3-0)^2+(1-0)^2+(-2-0)^2}\\ &= \sqrt{9+1+4}\\ &= \sqrt{14}\text{ units}\end{aligned}$$

c The distance from Q to the ZOY plane is the distance from Q to $(0,\ 1,\ -2)$, which is 3 units.

8 $\text{P}(0,\ 4,\ 4)$, $\text{Q}(2,\ 6,\ 5)$, $\text{R}(1,\ 4,\ 3)$

$$\begin{aligned}\text{PQ} &= \sqrt{(2-0)^2+(6-4)^2+(5-4)^2}\\ &= \sqrt{4+4+1}\\ &= 3\end{aligned}$$

$$\begin{aligned}\text{PR} &= \sqrt{(1-0)^2+(4-4)^2+(3-4)^2}\\ &= \sqrt{1+0+1}\\ &= \sqrt{2}\end{aligned}$$

$$\begin{aligned}\text{QR} &= \sqrt{(1-2)^2+(4-6)^2+(3-5)^2}\\ &= \sqrt{1+4+4}\\ &= 3\end{aligned}$$

$\therefore$ $\text{PQ} = \text{QR}$ and so $\triangle\text{PQR}$ is isosceles.

9 **a** $\text{A}(0,\ 0,\ 3)$, $\text{B}(2,\ 8,\ 1)$, $\text{C}(-9,\ 6,\ 18)$

$$\begin{aligned}\text{AB} &= \sqrt{(2-0)^2+(8-0)^2+(1-3)^2}\\ &= \sqrt{4+64+4}\\ &= \sqrt{72}\end{aligned}$$

$$\begin{aligned}\text{AC} &= \sqrt{(-9-0)^2+(6-0)^2+(18-3)^2}\\ &= \sqrt{81+36+225}\\ &= \sqrt{342}\end{aligned}$$

$$\begin{aligned}\text{BC} &= \sqrt{(-9-2)^2+(6-8)^2+(18-1)^2}\\ &= \sqrt{121+4+289}\\ &= \sqrt{414}\end{aligned}$$

Since $\text{BC}^2 = \text{AB}^2 + \text{AC}^2$, $\triangle\text{ABC}$ is right angled.

b $\text{A}(1,\ 0,\ -3)$, $\text{B}(2,\ 2,\ 0)$, $\text{C}(4,\ 6,\ 6)$

$$\begin{aligned}\text{AB} &= \sqrt{(2-1)^2+(2-0)^2+(0--3)^2}\\ &= \sqrt{1+4+9}\\ &= \sqrt{14}\end{aligned}$$

$$\begin{aligned}\text{AC} &= \sqrt{(4-1)^2+(6-0)^2+(6--3)^2}\\ &= \sqrt{9+36+81}\\ &= \sqrt{126} = 3\sqrt{14}\end{aligned}$$

$$\begin{aligned}\text{BC} &= \sqrt{(4-2)^2+(6-2)^2+(6-0)^2}\\ &= \sqrt{4+16+36}\\ &= \sqrt{56} = 2\sqrt{14}\end{aligned}$$

Since $\text{AB} + \text{BC} = \text{AC}$, the points A, B, and C lie on a straight line, so they do not form a triangle.

10 **a** $\overrightarrow{\text{AB}} = \begin{pmatrix} 6-5 \\ 12-6 \\ 9-(-2) \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ 11 \end{pmatrix}$

and so $$\begin{aligned}|\,\overrightarrow{\text{AB}}\,| &= \sqrt{1^2+6^2+11^2}\\ &= \sqrt{1+36+121}\\ &= \sqrt{158}\text{ units}\end{aligned}$$

$\overrightarrow{\text{BC}} = \begin{pmatrix} 2-6 \\ 4-12 \\ 2-9 \end{pmatrix} = \begin{pmatrix} -4 \\ -8 \\ -7 \end{pmatrix}$

and so $$\begin{aligned}|\,\overrightarrow{\text{BC}}\,| &= \sqrt{(-4)^2+(-8)^2+(-7)^2}\\ &= \sqrt{16+64+49}\\ &= \sqrt{129}\text{ units}\end{aligned}$$

$\overrightarrow{\text{AC}} = \begin{pmatrix} 2-5 \\ 4-6 \\ 2-(-2) \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \\ 4 \end{pmatrix}$

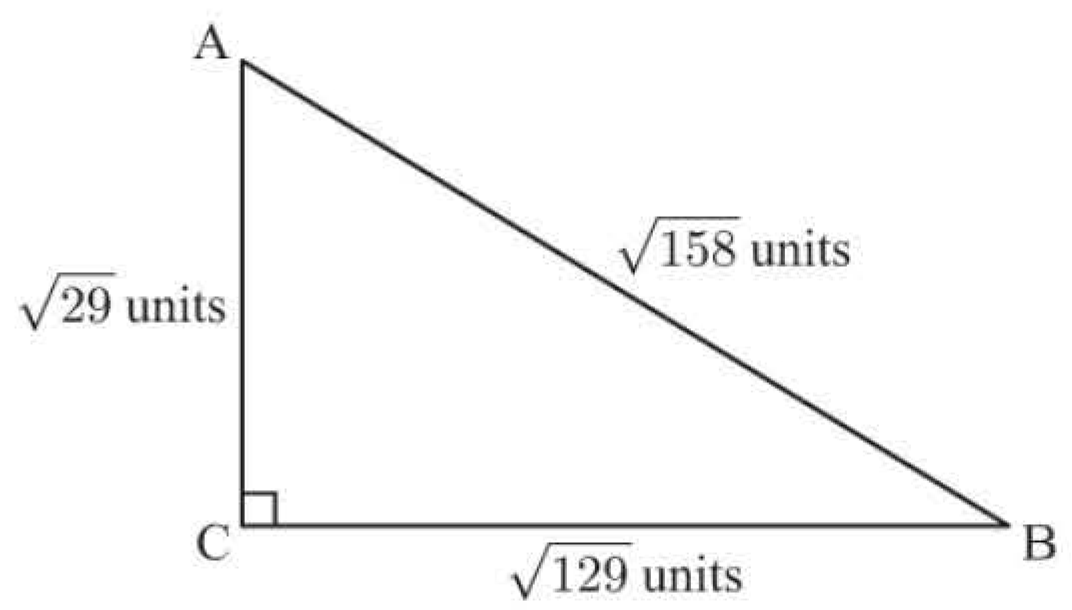
and so $$\begin{aligned}|\,\overrightarrow{\text{AC}}\,| &= \sqrt{(-3)^2+(-2)^2+4^2}\\ &= \sqrt{9+4+16}\\ &= \sqrt{29}\text{ units}\end{aligned}$$

Now, $$\begin{aligned}\left(\sqrt{29}\right)^2 + \left(\sqrt{129}\right)^2 &= 29+129\\ &= 158\\ &= \left(\sqrt{158}\right)^2\end{aligned}$$

So, $\text{AC}^2 + \text{BC}^2 = \text{AB}^2$

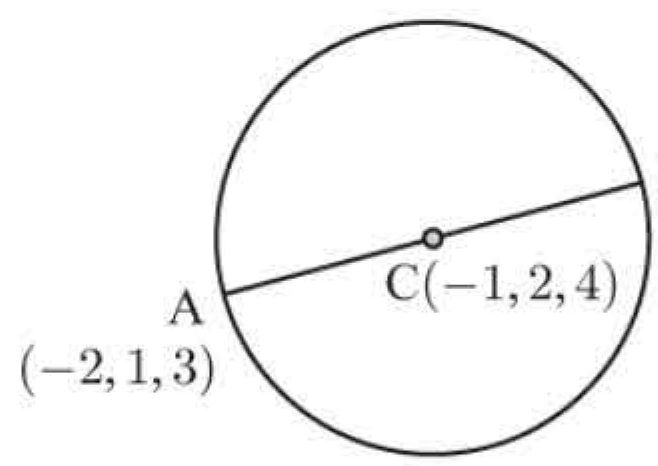
$\therefore$ triangle ABC is right angled with the right angle at C.

b



$$\begin{aligned}\text{Area} &= \tfrac{1}{2} \times \text{base} \times \text{height} \\ &= \tfrac{1}{2} \times \sqrt{129} \times \sqrt{29} \\ &\approx 30.6 \text{ units}^2\end{aligned}$$

11



If B is (a, b, c) then $\dfrac{a-2}{2} = -1, \quad \dfrac{b+1}{2} = 2, \quad \dfrac{c+3}{2} = 4$

$\therefore \quad a = 0, \quad b = 3, \quad c = 5$

$\therefore \quad$ B is $(0, 3, 5)$

$$\begin{aligned} r = \text{AC} &= \sqrt{(-1--2)^2 + (2-1)^2 + (4-3)^2} \\ &= \sqrt{1+1+1} \\ &= \sqrt{3} \text{ units}\end{aligned}$$

12 **a** $(0, y, 0)$ for any y

b The distance between $(0, y, 0)$ and $\text{B}(-1, -1, 2)$ is $\sqrt{(-1)^2 + (-1-y)^2 + 2^2}$.

$\therefore \quad \sqrt{1 + (y+1)^2 + 4} = \sqrt{14}$

$\therefore \quad (y+1)^2 = 9$

$\therefore \quad y + 1 = \pm 3$

$\therefore \quad y = -1 \pm 3$

$\therefore \quad y = -4$ or 2 $\qquad \therefore$ the two points are $(0, -4, 0)$ and $(0, 2, 0)$.

13 **a** $\begin{pmatrix} a-4 \\ b-3 \\ c+2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix}$

$\therefore \quad \begin{cases} a - 4 = 1 \\ b - 3 = 3 \\ c + 2 = -4 \end{cases}$

$\therefore \quad a = 5, \quad b = 6, \quad c = -6$

b $\begin{pmatrix} a-5 \\ b-2 \\ c+3 \end{pmatrix} = \begin{pmatrix} 3-a \\ 2-b \\ 5-c \end{pmatrix}$

$\therefore \quad \begin{cases} a - 5 = 3 - a \\ b - 2 = 2 - b \\ c + 3 = 5 - c \end{cases}$

$\therefore \quad 2a = 8, \quad 2b = 4, \quad 2c = 2$

$\therefore \quad a = 4, \quad b = 2, \quad c = 1$

14 **a** length $= 1$

$\therefore \quad \sqrt{\tfrac{1}{4} + k^2 + \tfrac{1}{16}} = 1$

$\therefore \quad \sqrt{k^2 + \tfrac{5}{16}} = 1$

$\therefore \quad k^2 = \tfrac{11}{16}$

$\therefore \quad k = \pm\tfrac{\sqrt{11}}{4}$

b length $= 1$

$\therefore \quad \sqrt{k^2 + \tfrac{4}{9} + \tfrac{1}{9}} = 1$

$\therefore \quad \sqrt{k^2 + \tfrac{5}{9}} = 1$

$\therefore \quad k^2 = \tfrac{4}{9}$

$\therefore \quad k = \pm\tfrac{2}{3}$

15 $\text{A}(-1, 3, 4), \quad \text{B}(2, 5, -1), \quad \text{C}(-1, 2, -2), \quad \text{D}(r, s, t)$

a If $\overrightarrow{\text{AC}} = \overrightarrow{\text{BD}}$ then $\begin{pmatrix} -1-(-1) \\ 2-3 \\ -2-4 \end{pmatrix} = \begin{pmatrix} r-2 \\ s-5 \\ t-(-1) \end{pmatrix}$

$\therefore \quad r - 2 = 0, \quad s - 5 = -1,$ and $t + 1 = -6 \qquad \therefore \quad r = 2, \quad s = 4,$ and $t = -7$

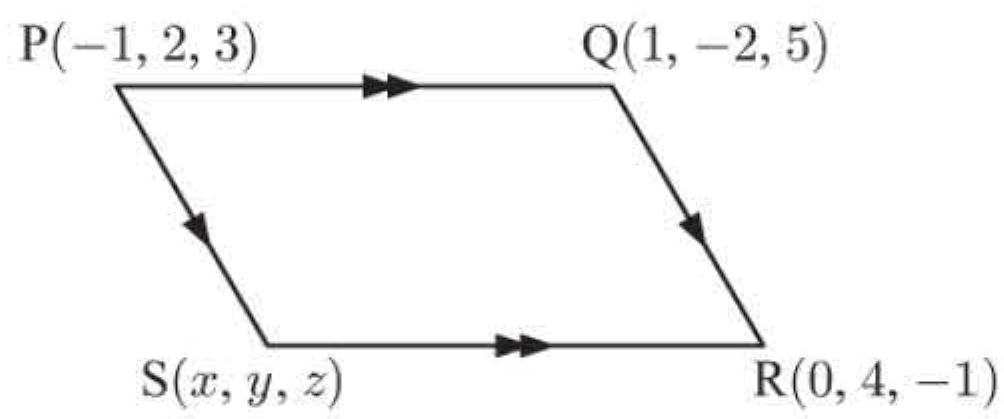
b If $\overrightarrow{\text{AB}} = \overrightarrow{\text{DC}}$ then $\begin{pmatrix} 2-(-1) \\ 5-3 \\ -1-4 \end{pmatrix} = \begin{pmatrix} -1-r \\ 2-s \\ -2-t \end{pmatrix}$

$\therefore \quad -1 - r = 3, \quad 2 - s = 2,$ and $-2 - t = -5 \qquad \therefore \quad r = -4, \quad s = 0,$ and $t = 3$

16 **a** $\overrightarrow{AB} = \begin{pmatrix} 3-1 \\ -3-2 \\ 2-3 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ -1 \end{pmatrix}$ and $\overrightarrow{DC} = \begin{pmatrix} 7-5 \\ -4-1 \\ 5-6 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ -1 \end{pmatrix}$.

b ABCD is a parallelogram since its opposite sides are parallel and equal in length.

17 **a** Suppose S is at (x, y, z). $\overrightarrow{PQ} = \overrightarrow{SR}$ {opposite sides are parallel and equal in length}



$$\therefore \begin{pmatrix} 1-(-1) \\ -2-2 \\ 5-3 \end{pmatrix} = \begin{pmatrix} 0-x \\ 4-y \\ -1-z \end{pmatrix}$$

$$\therefore \begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} -x \\ 4-y \\ -1-z \end{pmatrix}$$

$\therefore\ -x = 2 \qquad 4-y=-4 \qquad -1-z=2$

$\therefore\ x=-2 \qquad y=8 \qquad z=-3$

$\therefore$ S is at $(-2, 8, -3)$.

b The midpoint of [PR] is $\left(\dfrac{-1+0}{2}, \dfrac{2+4}{2}, \dfrac{3+(-1)}{2}\right)$ which is $(-\frac{1}{2}, 3, 1)$.

The midpoint of [QS] is $\left(\dfrac{1+(-2)}{2}, \dfrac{-2+8}{2}, \dfrac{5+(-3)}{2}\right)$ which is $(-\frac{1}{2}, 3, 1)$.

So, [PR] and [QS] have the same midpoint. ✓

EXERCISE 14H

1 **a** $2\mathbf{x} = \mathbf{q}$

$\therefore\ \frac{1}{2}(2\mathbf{x}) = \frac{1}{2}\mathbf{q}$

$\therefore\ \mathbf{x} = \frac{1}{2}\mathbf{q}$

b $\frac{1}{2}\mathbf{x} = \mathbf{n}$

$\therefore\ 2(\frac{1}{2}\mathbf{x}) = 2\mathbf{n}$

$\therefore\ \mathbf{x} = 2\mathbf{n}$

c $-3\mathbf{x} = \mathbf{p}$

$\therefore\ 3\mathbf{x} = -\mathbf{p}$

$\therefore\ \frac{1}{3}(3\mathbf{x}) = -\frac{1}{3}\mathbf{p}$

$\therefore\ \mathbf{x} = -\frac{1}{3}\mathbf{p}$

d $\mathbf{q} + 2\mathbf{x} = \mathbf{r}$

$\therefore\ 2\mathbf{x} = \mathbf{r} - \mathbf{q}$

$\therefore\ \mathbf{x} = \frac{1}{2}(\mathbf{r} - \mathbf{q})$

e $4\mathbf{s} - 5\mathbf{x} = \mathbf{t}$

$\therefore\ -5\mathbf{x} = \mathbf{t} - 4\mathbf{s}$

$\therefore\ 5\mathbf{x} = 4\mathbf{s} - \mathbf{t}$

$\therefore\ \mathbf{x} = \frac{1}{5}(4\mathbf{s} - \mathbf{t})$

f $4\mathbf{m} - \frac{1}{3}\mathbf{x} = \mathbf{n}$

$\therefore\ 4\mathbf{m} - \mathbf{n} = \frac{1}{3}\mathbf{x}$

$\therefore\ \mathbf{x} = 3(4\mathbf{m} - \mathbf{n})$

2 **a** $2\mathbf{a} + \mathbf{x} = \mathbf{b}$

$\therefore\ \mathbf{x} = \mathbf{b} - 2\mathbf{a}$

$$= \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} - 2\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} - \begin{pmatrix} -2 \\ 4 \\ 6 \end{pmatrix}$$

$$\therefore\ \mathbf{x} = \begin{pmatrix} 4 \\ -6 \\ -5 \end{pmatrix}$$

b $3\mathbf{x} - \mathbf{a} = 2\mathbf{b}$

$\therefore\ 3\mathbf{x} = \mathbf{a} + 2\mathbf{b}$

$\therefore\ \mathbf{x} = \frac{1}{3}(\mathbf{a} + 2\mathbf{b})$

$$= \tfrac{1}{3}\left[\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} + 2\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}\right]$$

$$= \tfrac{1}{3}\left[\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ -4 \\ 2 \end{pmatrix}\right]$$

$$\therefore\ \mathbf{x} = \tfrac{1}{3}\begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{2}{3} \\ \frac{5}{3} \end{pmatrix}$$

c $2\mathbf{b} - 2\mathbf{x} = -\mathbf{a}$

$\therefore\ \mathbf{a} + 2\mathbf{b} = 2\mathbf{x}$

$\therefore\ \mathbf{x} = \frac{1}{2}(\mathbf{a} + 2\mathbf{b}) = \frac{1}{2}\begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$ {using **b**}

$$= \begin{pmatrix} \frac{3}{2} \\ -1 \\ \frac{5}{2} \end{pmatrix}$$

3 $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\overrightarrow{OA} + \overrightarrow{OB} = -\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}$

$\therefore \; |\overrightarrow{AB}| = \sqrt{3^2 + 4^2 + (-2)^2} = \sqrt{9+16+4} = \sqrt{29}$ units

4 **a** $\overrightarrow{AB} = \begin{pmatrix} 3-(-1) \\ -2-3 \\ 1-2 \end{pmatrix} = \begin{pmatrix} 4 \\ -5 \\ -1 \end{pmatrix}$
$= 4\mathbf{i} - 5\mathbf{j} - \mathbf{k}$

b $|\overrightarrow{AB}| = \sqrt{4^2 + (-5)^2 + (-1)^2}$
$= \sqrt{16+25+1}$
$= \sqrt{42}$ units

5 **a** $|\mathbf{a}| = \sqrt{1^2 + 0^2 + 3^2}$
$= \sqrt{1+9}$
$= \sqrt{10}$ units

b $|\mathbf{b}| = \sqrt{(-2)^2 + 1^2 + 1^2}$
$= \sqrt{4+1+1}$
$= \sqrt{6}$ units

c $2|\mathbf{a}| = 2\sqrt{10}$ units {using part **a**}

d $2\mathbf{a} = 2\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix}$

$\therefore \; |2\mathbf{a}| = \sqrt{2^2 + 0^2 + 6^2}$
$= \sqrt{4+36}$
$= \sqrt{40}$
$= \sqrt{4}\sqrt{10}$
$= 2\sqrt{10}$ units

e $-3|\mathbf{b}| = -3\sqrt{6}$ units {using part **b**}

f $-3\mathbf{b} = -3\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ -3 \end{pmatrix}$

$\therefore \; |-3\mathbf{b}| = \sqrt{6^2 + (-3)^2 + (-3)^2}$
$= \sqrt{36+9+9}$
$= \sqrt{54}$
$= \sqrt{9}\sqrt{6}$
$= 3\sqrt{6}$ units

g $\mathbf{a}+\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$
$= \begin{pmatrix} 1-2 \\ 0+1 \\ 3+1 \end{pmatrix}$
$= \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}$

$\therefore \; |\mathbf{a}+\mathbf{b}| = \sqrt{(-1)^2 + 1^2 + 4^2}$
$= \sqrt{1+1+16}$
$= \sqrt{18}$
$= \sqrt{9}\sqrt{2}$ units $= 3\sqrt{2}$ units

h $\mathbf{a}-\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$
$= \begin{pmatrix} 1-(-2) \\ 0-1 \\ 3-1 \end{pmatrix}$
$= \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$

$\therefore \; |\mathbf{a}-\mathbf{b}| = \sqrt{3^2 + (-1)^2 + 2^2}$
$= \sqrt{9+1+4}$
$= \sqrt{14}$ units

6 $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$
$= (\mathbf{i} - \mathbf{j} + \mathbf{k}) + (-2\mathbf{i} + \mathbf{j} - 3\mathbf{k})$
$= -\mathbf{i} - 2\mathbf{k}$

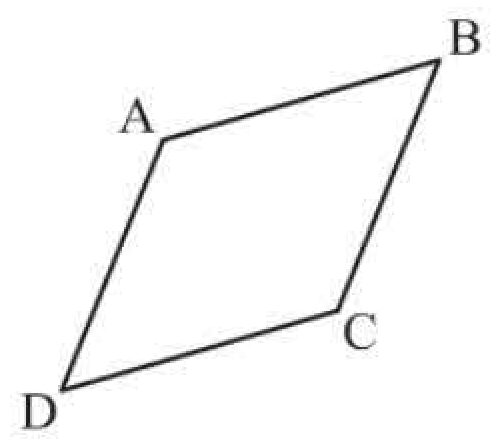
7 A(2, 1, −2), B(0, 3, −4), C(1, −2, 1), D(−2, −3, 2)

$\overrightarrow{AC} = \begin{pmatrix} 1-2 \\ -2-1 \\ 1-(-2) \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 3 \end{pmatrix}$

$\overrightarrow{BD} = \begin{pmatrix} -2-0 \\ -3-3 \\ 2-(-4) \end{pmatrix} = \begin{pmatrix} -2 \\ -6 \\ 6 \end{pmatrix} = 2\begin{pmatrix} -1 \\ -3 \\ 3 \end{pmatrix} = 2\overrightarrow{AC}$

8 $\overrightarrow{AB} = \begin{pmatrix} 2 - -1 \\ 3-5 \\ -3-2 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ -5 \end{pmatrix}$ $\therefore$ C is $(2+3, 3-2, -3-5)$, or $(5, 1, -8)$,
D is $(5+3, 1-2, -8-5)$, or $(8, -1, -13)$,
E is $(8+3, -1-2, -13-5)$, or $(11, -3, -18)$.

9



a $\overrightarrow{AB} = \begin{pmatrix} 4-3 \\ 2 - -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$\overrightarrow{DC} = \begin{pmatrix} -1 - -2 \\ 4-1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

Now $\overrightarrow{AB} = \overrightarrow{DC}$
$\therefore$ sides [AB] and [DC] are equal in length and parallel.
This is sufficient to deduce that ABCD is a parallelogram.

b $\overrightarrow{AB} = \begin{pmatrix} -1-5 \\ 2-0 \\ 4-3 \end{pmatrix} = \begin{pmatrix} -6 \\ 2 \\ 1 \end{pmatrix}$

$\overrightarrow{DC} = \begin{pmatrix} 4-10 \\ -3 - -5 \\ 6-5 \end{pmatrix} = \begin{pmatrix} -6 \\ 2 \\ 1 \end{pmatrix}$

So $\overrightarrow{AB} = \overrightarrow{DC}$
$\therefore$ sides [AB] and [DC] are equal in length and parallel.
This is sufficient to deduce that ABCD is a parallelogram.

c $\overrightarrow{AB} = \begin{pmatrix} 1-2 \\ 4 - -3 \\ -1-2 \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \\ -3 \end{pmatrix}$

$\overrightarrow{DC} = \begin{pmatrix} -2 - -1 \\ 6 - -1 \\ -2-2 \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \\ -4 \end{pmatrix}$

So, $\overrightarrow{AB} \neq \overrightarrow{DC}$
$\therefore$ ABCD cannot be a parallelogram.

10 **a** Let D be (a, b).

Now $\overrightarrow{CD} = \overrightarrow{BA}$

$\therefore \begin{pmatrix} a-8 \\ b - -2 \end{pmatrix} = \begin{pmatrix} 3-2 \\ 0 - -1 \end{pmatrix}$

$\therefore \begin{pmatrix} a-8 \\ b+2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\therefore$ $a = 9$, $b = -1$
So, D is $(9, -1)$.

b Let R be (a, b, c).

Now $\overrightarrow{SR} = \overrightarrow{PQ}$

$\therefore \begin{pmatrix} a-4 \\ b-0 \\ c-7 \end{pmatrix} = \begin{pmatrix} -2 - -1 \\ 5-4 \\ 2-3 \end{pmatrix}$

$\therefore \begin{pmatrix} a-4 \\ b \\ c-7 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$

$\therefore$ $a = 3$, $b = 1$, $c = 6$
So, R is $(3, 1, 6)$.

c Let X be (a, b, c).

Now $\overrightarrow{WX} = \overrightarrow{ZY}$

$\therefore \begin{pmatrix} a - -1 \\ b-5 \\ c-8 \end{pmatrix} = \begin{pmatrix} 3-0 \\ -2-4 \\ -2-6 \end{pmatrix}$

$\therefore \begin{pmatrix} a+1 \\ b-5 \\ c-8 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \\ -8 \end{pmatrix}$

$\therefore$ $a = 2$, $b = -1$, $c = 0$
So, X is $(2, -1, 0)$.

11 **a** $\overrightarrow{BD} = \frac{1}{2}\overrightarrow{OA}$
$= \frac{1}{2}\mathbf{a}$

b $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$
$= -\mathbf{a} + \mathbf{b}$
$= \mathbf{b} - \mathbf{a}$

c $\overrightarrow{BA} = -\overrightarrow{AB}$
$= -(\mathbf{b} - \mathbf{a})$ {using **b**}
$= -\mathbf{b} + \mathbf{a}$ or $\mathbf{a} - \mathbf{b}$

d $\overrightarrow{OD} = \overrightarrow{OB} + \overrightarrow{BD}$
$= \mathbf{b} + \frac{1}{2}\mathbf{a}$ {using **a**}

e $\overrightarrow{AD}$
$= \overrightarrow{AO} + \overrightarrow{OD}$
$= -\mathbf{a} + (\mathbf{b} + \frac{1}{2}\mathbf{a})$ {using **d**}
$= -\frac{1}{2}\mathbf{a} + \mathbf{b}$ or $\mathbf{b} - \frac{1}{2}\mathbf{a}$

f $\overrightarrow{DA} = -\overrightarrow{AD}$
$= \frac{1}{2}\mathbf{a} - \mathbf{b}$ {using **e**}

12 **a** $\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ -1 \end{pmatrix}$

b $\overrightarrow{CB} = \overrightarrow{CA} + \overrightarrow{AB} = -\overrightarrow{AC} + \overrightarrow{AB} = -\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \\ -2 \end{pmatrix}$

c $\overrightarrow{CD} = \overrightarrow{CB} + \overrightarrow{BD} = \begin{pmatrix} -3 \\ 4 \\ -2 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}$ {using **b**} $= \begin{pmatrix} -3 \\ 6 \\ -5 \end{pmatrix}$

13 **a** $\mathbf{a}+\mathbf{b}=\begin{pmatrix}2\\-1\\1\end{pmatrix}+\begin{pmatrix}1\\2\\-3\end{pmatrix}=\begin{pmatrix}3\\1\\-2\end{pmatrix}$

b $\mathbf{a}-\mathbf{b}=\begin{pmatrix}2\\-1\\1\end{pmatrix}-\begin{pmatrix}1\\2\\-3\end{pmatrix}=\begin{pmatrix}1\\-3\\4\end{pmatrix}$

c $\mathbf{b}+2\mathbf{c}=\begin{pmatrix}1\\2\\-3\end{pmatrix}+2\begin{pmatrix}0\\1\\-3\end{pmatrix}=\begin{pmatrix}1\\2\\-3\end{pmatrix}+\begin{pmatrix}0\\2\\-6\end{pmatrix}=\begin{pmatrix}1\\4\\-9\end{pmatrix}$

d $\mathbf{c}-\frac{1}{2}\mathbf{a}=\begin{pmatrix}0\\1\\-3\end{pmatrix}-\frac{1}{2}\begin{pmatrix}2\\-1\\1\end{pmatrix}=\begin{pmatrix}0\\1\\-3\end{pmatrix}-\begin{pmatrix}1\\-\frac{1}{2}\\\frac{1}{2}\end{pmatrix}=\begin{pmatrix}-1\\\frac{3}{2}\\-\frac{7}{2}\end{pmatrix}$

e $\mathbf{a}-\mathbf{b}-\mathbf{c}=\begin{pmatrix}2\\-1\\1\end{pmatrix}-\begin{pmatrix}1\\2\\-3\end{pmatrix}-\begin{pmatrix}0\\1\\-3\end{pmatrix}=\begin{pmatrix}1\\-4\\7\end{pmatrix}$

f $2\mathbf{b}-\mathbf{c}+\mathbf{a}=2\begin{pmatrix}1\\2\\-3\end{pmatrix}-\begin{pmatrix}0\\1\\-3\end{pmatrix}+\begin{pmatrix}2\\-1\\1\end{pmatrix}=\begin{pmatrix}2\\4\\-6\end{pmatrix}-\begin{pmatrix}0\\1\\-3\end{pmatrix}+\begin{pmatrix}2\\-1\\1\end{pmatrix}=\begin{pmatrix}4\\2\\-2\end{pmatrix}$

14 **a** $|\mathbf{a}|=\sqrt{(-1)^2+1^2+3^2}=\sqrt{11}$ units

b $|\mathbf{b}|=\sqrt{1^2+(-3)^2+2^2}=\sqrt{14}$ units

c $|\mathbf{b}+\mathbf{c}|=\left|\begin{pmatrix}1\\-3\\2\end{pmatrix}+\begin{pmatrix}-2\\2\\4\end{pmatrix}\right|=\left|\begin{pmatrix}-1\\-1\\6\end{pmatrix}\right|$

$=\sqrt{(-1)^2+(-1)^2+6^2}$
$=\sqrt{1+1+36}$
$=\sqrt{38}$ units

d $|\mathbf{a}-\mathbf{c}|=\left|\begin{pmatrix}-1\\1\\3\end{pmatrix}-\begin{pmatrix}-2\\2\\4\end{pmatrix}\right|=\left|\begin{pmatrix}1\\-1\\-1\end{pmatrix}\right|$

$=\sqrt{1^2+(-1)^2+(-1)^2}$
$=\sqrt{3}$ units

e $|\mathbf{a}|\,\mathbf{b}=\sqrt{11}\begin{pmatrix}1\\-3\\2\end{pmatrix}$ {using **a**}

$=\begin{pmatrix}\sqrt{11}\\-3\sqrt{11}\\2\sqrt{11}\end{pmatrix}$

f $\dfrac{1}{|\mathbf{a}|}\mathbf{a}=\dfrac{1}{\sqrt{11}}\begin{pmatrix}-1\\1\\3\end{pmatrix}$ {using **a**}

$=\begin{pmatrix}-\frac{1}{\sqrt{11}}\\\frac{1}{\sqrt{11}}\\\frac{3}{\sqrt{11}}\end{pmatrix}$

15 **a** $2\begin{pmatrix}1\\0\\3a\end{pmatrix}=\begin{pmatrix}b\\c-1\\2\end{pmatrix}$

$\therefore\ \begin{pmatrix}2\\0\\6a\end{pmatrix}=\begin{pmatrix}b\\c-1\\2\end{pmatrix}$

$\therefore\ 2=b,\quad 0=c-1,\quad$ and $\quad 6a=2$

$\therefore\ a=\frac{1}{3},\quad b=2,\quad$ and $\quad c=1$

b $a\begin{pmatrix}1\\1\\0\end{pmatrix}+b\begin{pmatrix}2\\0\\-1\end{pmatrix}+c\begin{pmatrix}0\\1\\1\end{pmatrix}=\begin{pmatrix}-1\\3\\3\end{pmatrix}$

$\therefore\ \begin{pmatrix}a\\a\\0\end{pmatrix}+\begin{pmatrix}2b\\0\\-b\end{pmatrix}+\begin{pmatrix}0\\c\\c\end{pmatrix}=\begin{pmatrix}-1\\3\\3\end{pmatrix}$

So, $a+2b=-1$ .... (1)
$a+c=3$
$\therefore\ c=3-a$ .... (2)
$-b+c=3$
$\therefore\ c=b+3$ .... (3)

Substituting (2) into (3), we get
$3-a=b+3$
$\therefore\ -a=b$

Substituting into (1), we get
$a+2(-a)=-1$
$\therefore\ -a=-1$
$\therefore\ a=1$
$\therefore\ b=-1$
and $c=-1+3=2$ {using (3)}
$\therefore\ a=1,\quad b=-1,\quad$ and $\quad c=2$

c $a\begin{pmatrix}2\\-3\\1\end{pmatrix}+b\begin{pmatrix}1\\7\\2\end{pmatrix}=\begin{pmatrix}7\\-19\\2\end{pmatrix}$

$\therefore \begin{pmatrix}2a\\-3a\\a\end{pmatrix}+\begin{pmatrix}b\\7b\\2b\end{pmatrix}=\begin{pmatrix}7\\-19\\2\end{pmatrix}$

$\therefore \begin{pmatrix}2a+b\\-3a+7b\\a+2b\end{pmatrix}=\begin{pmatrix}7\\-19\\2\end{pmatrix}$

So, $2a+b=7$

$\therefore\ b=7-2a$ .... (1)

$-3a+7b=-19$ .... (2)

$a+2b=2$ .... (3)

Substituting (1) into (3), we get

$a+2(7-2a)=2$

$\therefore\ a+14-4a=2$

$\therefore\ -3a=-12$

$\therefore\ a=4$

and so $b=7-2(4)=-1$

$\therefore\ a=4,\ \ b=-1$

Check: $-3(4)+7(-1)=-12-7=-19$ ✓

EXERCISE 14I

1 Since $\mathbf{a}$ and $\mathbf{b}$ are parallel, then $\mathbf{b}=k\mathbf{a}$. $\therefore \begin{pmatrix}-6\\r\\s\end{pmatrix}=k\begin{pmatrix}2\\-1\\3\end{pmatrix}=\begin{pmatrix}2k\\-k\\3k\end{pmatrix}$

$\therefore\ 2k=-6,\ r=-k,\ s=3k \qquad \therefore\ k=-3,\ r=3,\ s=-9$

2 If $\begin{pmatrix}3\\-1\\2\end{pmatrix}$ and $\begin{pmatrix}a\\2\\b\end{pmatrix}$ are parallel, then $\begin{pmatrix}a\\2\\b\end{pmatrix}=k\begin{pmatrix}3\\-1\\2\end{pmatrix}$.

$\therefore\ a=3k,\ 2=-k,\ b=2k \qquad \therefore\ k=-2,\ a=-6,$ and $b=-4$

3 **a** $\overrightarrow{AB}=3\overrightarrow{CD}$ means that $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$ and 3 times its length.

b $\overrightarrow{RS}=-\frac{1}{2}\overrightarrow{KL}$ means that $\overrightarrow{RS}$ is parallel to $\overrightarrow{KL}$, half its length, and in the opposite direction.

c

$\overrightarrow{AB}=2\overrightarrow{BC}$ means that A, B, and C are collinear and the length of $\overrightarrow{AB}$ is twice the length of $\overrightarrow{BC}$.

4 $\overrightarrow{OP}=\begin{pmatrix}3\\2\\-1\end{pmatrix},\ \overrightarrow{OQ}=\begin{pmatrix}1\\4\\-3\end{pmatrix},\ \overrightarrow{OR}=\begin{pmatrix}2\\-1\\2\end{pmatrix},\ \overrightarrow{OS}=\begin{pmatrix}-1\\-2\\3\end{pmatrix}$

a $\overrightarrow{PR}=\overrightarrow{PO}+\overrightarrow{OR}=-\begin{pmatrix}3\\2\\-1\end{pmatrix}+\begin{pmatrix}2\\-1\\2\end{pmatrix}=\begin{pmatrix}-1\\-3\\3\end{pmatrix}$

$\overrightarrow{QS}=\overrightarrow{QO}+\overrightarrow{OS}=-\begin{pmatrix}1\\4\\-3\end{pmatrix}+\begin{pmatrix}-1\\-2\\3\end{pmatrix}=\begin{pmatrix}-2\\-6\\6\end{pmatrix}=2\begin{pmatrix}-1\\-3\\3\end{pmatrix}=2\overrightarrow{PR}$ and so [QS] $\parallel$ [PR].

b Since $\overrightarrow{QS}=2\overrightarrow{PR}$, $|\overrightarrow{QS}|=2\,|\overrightarrow{PR}|$, and so [QS] is twice as long as [PR].

5 **a** The vector in the same direction as $\mathbf{a}$ and twice its length is $2\mathbf{a}$.

$$2\mathbf{a} = 2\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$$

b The vector in the opposite direction to $\mathbf{a}$ and half its length is $-\frac{1}{2}\mathbf{a}$.

$$-\tfrac{1}{2}\mathbf{a} = -\tfrac{1}{2}\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -\frac{2}{2} \\ -\frac{4}{2} \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

6 **a** $\mathbf{i} + 2\mathbf{j}$ has length $\sqrt{1^2 + 2^2} = \sqrt{5}$ units $\quad\therefore$ unit vector $= \frac{1}{\sqrt{5}}(\mathbf{i} + 2\mathbf{j})$

b $2\mathbf{i} - 3\mathbf{k}$ has length $\sqrt{2^2 + 0^2 + (-3)^2} = \sqrt{4+9} = \sqrt{13}$ units

$\therefore$ unit vector is $\frac{1}{\sqrt{13}}(2\mathbf{i} - 3\mathbf{k})$

c $2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ has length $\sqrt{2^2 + (-2)^2 + 1^2} = \sqrt{9} = 3$ units

$\therefore$ unit vector is $\frac{1}{3}(2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$

7 **a** $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ has length $\sqrt{2^2 + (-1)^2} = \sqrt{5}$ units

$\therefore$ the unit vector in the same direction is $\frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$\therefore$ the vector of length 3 units in the same direction is $\frac{3}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{6}{\sqrt{5}} \\ -\frac{3}{\sqrt{5}} \end{pmatrix}$

b $\begin{pmatrix} -1 \\ -4 \end{pmatrix}$ has length $\sqrt{(-1)^2 + (-4)^2} = \sqrt{17}$ units

$\therefore$ the unit vector in the opposite direction is $-\frac{1}{\sqrt{17}}\begin{pmatrix} -1 \\ -4 \end{pmatrix} = \frac{1}{\sqrt{17}}\begin{pmatrix} 1 \\ 4 \end{pmatrix}$

$\therefore$ the vector of length 2 units in the opposite direction is $\frac{2}{\sqrt{17}}\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{17}} \\ \frac{8}{\sqrt{17}} \end{pmatrix}$

8 **a** $\overrightarrow{AB}$ is a vector in the same direction as $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ with length 4 units.

Now, $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ has length $\sqrt{1^2 + (-1)^2} = \sqrt{2}$ units

$\therefore$ the unit vector in the same direction is $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\therefore$ the vector of length 4 units in the same direction is $\frac{4}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 2\sqrt{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$= \begin{pmatrix} 2\sqrt{2} \\ -2\sqrt{2} \end{pmatrix}$$

$\therefore\ \overrightarrow{AB} = \begin{pmatrix} 2\sqrt{2} \\ -2\sqrt{2} \end{pmatrix}$

b $\overrightarrow{OA} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\overrightarrow{AB} = \begin{pmatrix} 2\sqrt{2} \\ -2\sqrt{2} \end{pmatrix}$

Now $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$

$$\therefore\ \overrightarrow{OB} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 2\sqrt{2} \\ -2\sqrt{2} \end{pmatrix} = \begin{pmatrix} 3 + 2\sqrt{2} \\ 2 - 2\sqrt{2} \end{pmatrix}$$

c If $\overrightarrow{OB} = \begin{pmatrix} 3 + 2\sqrt{2} \\ 2 - 2\sqrt{2} \end{pmatrix}$, then the coordinates of B are $(3 + 2\sqrt{2},\ 2 - 2\sqrt{2})$.

9 **a** $|\mathbf{a}| = \sqrt{2^2+(-1)^2+(-2)^2}$
$= \sqrt{4+1+4}$
$= 3$ units

$\therefore$ the vectors of length 1 unit parallel to $\mathbf{a}$ are $\pm\frac{1}{3}\mathbf{a}$.

$\therefore$ the vectors are $\begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ -\frac{2}{3} \end{pmatrix}$ and $\begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$.

b $|\mathbf{b}| = \sqrt{(-2)^2+(-1)^2+2^2}$
$= \sqrt{4+1+4}$
$= 3$ units

$\therefore$ the vectors of length 2 units parallel to $\mathbf{b}$ are $\pm\frac{2}{3}\mathbf{b}$.

$\therefore$ the vectors are $\begin{pmatrix} -\frac{4}{3} \\ -\frac{2}{3} \\ \frac{4}{3} \end{pmatrix}$ and $\begin{pmatrix} \frac{4}{3} \\ \frac{2}{3} \\ -\frac{4}{3} \end{pmatrix}$.

10 **a** $\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$ has length $\sqrt{(-1)^2+4^2+1^2} = \sqrt{18} = 3\sqrt{2}$ units

$\therefore$ the unit vector in the same direction is $\frac{1}{3\sqrt{2}}\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$

$\therefore$ the vector of length 6 units in the same direction is $\frac{6}{3\sqrt{2}}\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} = \sqrt{2}\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sqrt{2} \\ 4\sqrt{2} \\ \sqrt{2} \end{pmatrix}$

b $\begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix}$ has length $\sqrt{(-1)^2+(-2)^2+(-2)^2} = \sqrt{9} = 3$ units

$\therefore$ the unit vector in the opposite direction is $-\frac{1}{3}\begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

$\therefore$ the vector of length 5 units in the opposite direction is $\frac{5}{3}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{5}{3} \\ \frac{10}{3} \\ \frac{10}{3} \end{pmatrix}$

11 **a** $\overrightarrow{AB} = \begin{pmatrix} 4-(-2) \\ 3-1 \\ 0-4 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ -4 \end{pmatrix} = 2\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$

$\overrightarrow{BC} = \begin{pmatrix} 19-4 \\ 8-3 \\ -10-0 \end{pmatrix} = \begin{pmatrix} 15 \\ 5 \\ -10 \end{pmatrix} = 5\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$

So, $\overrightarrow{AB} = \frac{2}{5}\overrightarrow{BC}$

$\therefore$ A, B, and C are collinear.

b $\overrightarrow{PQ} = \begin{pmatrix} 5-2 \\ -5-1 \\ -2-1 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \\ -3 \end{pmatrix} = 3\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$

$\overrightarrow{QR} = \begin{pmatrix} -1-5 \\ 7-(-5) \\ 4-(-2) \end{pmatrix} = \begin{pmatrix} -6 \\ 12 \\ 6 \end{pmatrix} = -6\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$

So, $\overrightarrow{PQ} = \frac{3}{-6}\overrightarrow{QR} = -\frac{1}{2}\overrightarrow{QR}$

$\therefore$ P, Q, and R are collinear.

c $\overrightarrow{AB} = \begin{pmatrix} 11-2 \\ -9-(-3) \\ 7-4 \end{pmatrix} = \begin{pmatrix} 9 \\ -6 \\ 3 \end{pmatrix}$

$\overrightarrow{BC} = \begin{pmatrix} -13-11 \\ a-(-9) \\ b-7 \end{pmatrix} = \begin{pmatrix} -24 \\ a+9 \\ b-7 \end{pmatrix}$

A, B, and C are collinear.

$\therefore$ $\overrightarrow{AB} = k\overrightarrow{BC}$
$9 = k \times -24$
$\therefore$ $k = -\frac{3}{8}$
$\therefore$ $\overrightarrow{AB} = -\frac{3}{8}\overrightarrow{BC}$
$\therefore$ $-\frac{8}{3}\overrightarrow{AB} = \overrightarrow{BC}$

So, $-\frac{8}{3} \times -6 = a+9$
$16 = a+9$
$a = 7$

and $-\frac{8}{3} \times 3 = b-7$
$-8 = b-7$
$b = -1$

d $\overrightarrow{KL} = \begin{pmatrix} 4-1 \\ -3-(-1) \\ 7-0 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 7 \end{pmatrix}$

$\overrightarrow{LM} = \begin{pmatrix} a-4 \\ 2-(-3) \\ b-7 \end{pmatrix} = \begin{pmatrix} a-4 \\ 5 \\ b-7 \end{pmatrix}$

K, L, and M are collinear.

$\therefore\ \overrightarrow{KL} = k\overrightarrow{LM}$

$-2 = k5$

$k = -\frac{2}{5}$

$\therefore\ \overrightarrow{KL} = -\frac{2}{5}\overrightarrow{LM}$

$\therefore\ -\frac{5}{2}\overrightarrow{KL} = \overrightarrow{LM}$

So, $-\frac{5}{2} \times 3 = a - 4$

$a = -\frac{15}{2} + \frac{8}{2}$

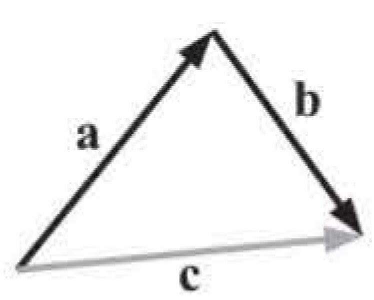
$= -\frac{7}{2}$

and $-\frac{5}{2} \times 7 = b - 7$

$b = -\frac{35}{2} + \frac{14}{2}$

$= -\frac{21}{2}$

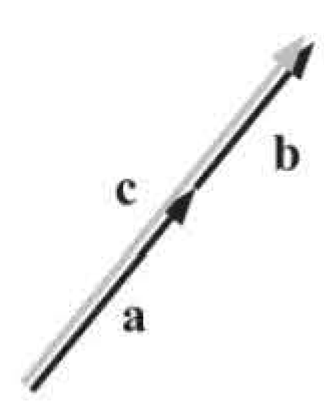
12 **Case:** **a** and **b** are not parallel.



Since **a**, **b**, and **c** form a triangle,
length of **a** + length of **b** > length of **c**
But $\mathbf{c} = \mathbf{a} + \mathbf{b}$, so length of **c** = length of $(\mathbf{a}+\mathbf{b})$
$\therefore$ length of **a** + length of **b** > length of $(\mathbf{a}+\mathbf{b})$

$\therefore\ |\mathbf{a}| + |\mathbf{b}| > |\mathbf{a}+\mathbf{b}|$

Case: **a** and **b** are parallel.



length of **a** + length of **b** = length of $(\mathbf{a}+\mathbf{b})$

$\therefore\ |\mathbf{a}| + |\mathbf{b}| = |\mathbf{a}+\mathbf{b}|$

These are the only possible cases, so $\therefore$ for any **a**, **b**, $|\mathbf{a}| + |\mathbf{b}| \geqslant |\mathbf{a}+\mathbf{b}|$

EXERCISE 14J

1 **a** $\mathbf{q} \bullet \mathbf{p}$
$= \begin{pmatrix} -1 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 2 \end{pmatrix}$
$= -1(3) + 5(2)$
$= -3 + 10$
$= 7$

b $\mathbf{q} \bullet \mathbf{r}$
$= \begin{pmatrix} -1 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 4 \end{pmatrix}$
$= -1(-2) + 5(4)$
$= 2 + 20$
$= 22$

c $\mathbf{q} \bullet (\mathbf{p} + \mathbf{r})$
$= \begin{pmatrix} -1 \\ 5 \end{pmatrix} \bullet \left[\begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} -2 \\ 4 \end{pmatrix}\right]$
$= \begin{pmatrix} -1 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 6 \end{pmatrix}$
$= -1(1) + 5(6)$
$= -1 + 30$
$= 29$

d $3\mathbf{r} \bullet \mathbf{q}$
$= 3\begin{pmatrix} -2 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 5 \end{pmatrix}$
$= \begin{pmatrix} -6 \\ 12 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 5 \end{pmatrix}$
$= -6(-1) + 12(5)$
$= 6 + 60$
$= 66$

e $2\mathbf{p} \bullet 2\mathbf{p}$
$= 2\begin{pmatrix} 3 \\ 2 \end{pmatrix} \bullet 2\begin{pmatrix} 3 \\ 2 \end{pmatrix}$
$= \begin{pmatrix} 6 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 6 \\ 4 \end{pmatrix}$
$= 6(6) + 4(4)$
$= 36 + 16$
$= 52$

f $\mathbf{i} \bullet \mathbf{p}$
$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 2 \end{pmatrix}$
$= 1(3) + 0(2)$
$= 3 + 0$
$= 3$

g $\mathbf{q} \bullet \mathbf{j} = \begin{pmatrix} -1 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
$= -1(0) + 5(1)$
$= 0 + 5 = 5$

h $\mathbf{i} \bullet \mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
$= 1(1) + 0(0)$
$= 1 + 0 = 1$

2 **a** $\mathbf{a} \bullet \mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$
$= 2(-1) + 1(1) + 3(1)$
$= -2 + 1 + 3$
$= 2$

b $\mathbf{b} \bullet \mathbf{a} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$
$= (-1)(2) + 1(1) + 1(3)$
$= -2 + 1 + 3$
$= 2$

c $|\mathbf{a}|^2 = \left(\sqrt{2^2 + 1^2 + 3^2}\right)^2$
$= 14$

d $\mathbf{a} \bullet \mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$
$= 2(2) + 1(1) + 3(3)$
$= 14$

e $\mathbf{a} \bullet (\mathbf{b} + \mathbf{c})$
$= \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \bullet \left[\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right]$
$= \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$
$= 2(-1) + 1(0) + 3(2) = 4$

f $\mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c}$
$= 2 + \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ {using **a**}
$= 2 + 2(0) + 1(-1) + 3(1)$
$= 4$

3 **a** $\mathbf{p} \bullet \mathbf{q}$
$= \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$
$= 3(-2) + (-1)(1) + 2(3)$
$= -6 - 1 + 6$
$= -1$

b If the angle between **p** and **q** is θ, then

$$\cos\theta = \frac{\mathbf{p} \bullet \mathbf{q}}{|\mathbf{p}||\mathbf{q}|} = \frac{-1}{\sqrt{3^2 + (-1)^2 + 2^2}\sqrt{(-2)^2 + 1^2 + 3^2}}$$
$$= \frac{-1}{\sqrt{14}\sqrt{14}}$$
$$\therefore \quad \theta = \cos^{-1}(-\tfrac{1}{14}) \approx 94.1^\circ$$

4 **a** If the angle between **m** and **n** is θ, then

$$\cos\theta = \frac{\mathbf{m} \bullet \mathbf{n}}{|\mathbf{m}||\mathbf{n}|} = \frac{2(-1) + (-1)(3) + (-1)(2)}{\sqrt{2^2 + (-1)^2 + (-1)^2}\sqrt{(-1)^2 + 3^2 + 2^2}}$$
$$= \frac{-7}{\sqrt{6}\sqrt{14}} = -\frac{7}{\sqrt{84}}$$
$$\therefore \quad \theta = \cos^{-1}\left(-\tfrac{7}{\sqrt{84}}\right) \approx 140^\circ$$

b $\mathbf{m} = 2\mathbf{j} - \mathbf{k} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$ and $\mathbf{n} = \mathbf{i} + 2\mathbf{k} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

If the angle between **m** and **n** is θ, then

$$\cos\theta = \frac{\mathbf{m} \bullet \mathbf{n}}{|\mathbf{m}||\mathbf{n}|} = \frac{(0)(1) + (2)(0) + (-1)(2)}{\sqrt{0^2 + 2^2 + (-1)^2}\sqrt{1^2 + 0^2 + 2^2}}$$
$$= \frac{-2}{\sqrt{5}\sqrt{5}} = -\frac{2}{5}$$
$$\therefore \quad \theta = \cos^{-1}\left(-\tfrac{2}{5}\right) \approx 114^\circ$$

5 **a** $(\mathbf{i}+\mathbf{j}-\mathbf{k}) \bullet (2\mathbf{j}+\mathbf{k})$
$$= \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$
$$= 1(0) + 1(2) - 1(1)$$
$$= 1$$

b $\mathbf{i} \bullet \mathbf{i}$
$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$
$$= 1(1) + 0(0) + 0(0)$$
$$= 1$$

c $\mathbf{i} \bullet \mathbf{j}$
$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$
$$= 1(0) + 0(1) + 0(0)$$
$$= 0$$

6 **a** $\mathbf{p} \bullet \mathbf{q} = |\mathbf{p}||\mathbf{q}|\cos\theta$
$$= 2 \times 5 \times \cos 60°$$
$$= 5$$

b $\mathbf{p} \bullet \mathbf{q} = |\mathbf{p}||\mathbf{q}|\cos\theta$
$$= 6 \times 3 \times \cos 120°$$
$$= -9$$

7 **a** **i** If $\mathbf{v}$ and $\mathbf{w}$ are parallel, then they are either in the same direction (so the angle between them is $0°$) or in opposite directions (so the angle between them is $180°$).

If the angle between them is $0°$, then $\mathbf{v} \bullet \mathbf{w} = |\mathbf{v}||\mathbf{w}|\cos 0°$
$$= 3 \times 4 \times 1$$
$$= 12$$

If the angle between them is $180°$, then $\mathbf{v} \bullet \mathbf{w} = |\mathbf{v}||\mathbf{w}|\cos 180°$
$$= 3 \times 4 \times -1$$
$$= -12$$

ii $\mathbf{v} \bullet \mathbf{w} = |\mathbf{v}||\mathbf{w}|\cos 60°$
$$= 3 \times 4 \times \tfrac{1}{2}$$
$$= 6$$

b **i** $\mathbf{a}$ and $\mathbf{b}$ are not perpendicular as their dot product is not equal to 0.

ii $\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$

If $\mathbf{a}$ and $\mathbf{b}$ are parallel, then $\theta = 0$ or 180
$$\therefore \cos\theta = \pm 1$$
$$\therefore -12 = |\mathbf{a}| \times 1 \times \pm 1$$
$$\therefore |\mathbf{a}| = \pm 12 \quad \text{but} \quad |\mathbf{a}| > 0$$
$$\therefore |\mathbf{a}| = 12 \text{ units}$$

{**Note:** This means that $\cos\theta$ must be -1, so the angle between $\mathbf{a}$ and $\mathbf{b}$ is $180°$
$\therefore$ $\mathbf{a}$ and $\mathbf{b}$ are in opposite directions.}

c **i** $\mathbf{c} \bullet \mathbf{d} = |\mathbf{c}||\mathbf{d}|\cos\theta$
$$\therefore 5 = \left|\sqrt{5}\right|\left|\sqrt{5}\right|\cos\theta$$
$$\therefore 5 = 5\cos\theta$$
$$\therefore \cos\theta = 1$$
$$\therefore \theta = 0°$$
So, $\mathbf{c} = \mathbf{d}$

ii $\mathbf{c} \bullet \mathbf{d} = |\mathbf{c}||\mathbf{d}|\cos\theta$
$$\therefore -5 = \left|\sqrt{5}\right|\left|\sqrt{5}\right|\cos\theta$$
$$\therefore -5 = 5\cos\theta$$
$$\therefore \cos\theta = -1$$
$$\therefore \theta = 180°$$
So, $\mathbf{c} = -\mathbf{d}$

8 **a** P has coordinates $(\cos\theta, \sin\theta)$.

b $\overrightarrow{BP} = \overrightarrow{BO} + \overrightarrow{OP}$
$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$
$$= \begin{pmatrix} 1+\cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} \cos\theta + 1 \\ \sin\theta \end{pmatrix}$$

$\overrightarrow{AP} = \overrightarrow{AO} + \overrightarrow{OP}$
$$= \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$
$$= \begin{pmatrix} -1+\cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} \cos\theta - 1 \\ \sin\theta \end{pmatrix}$$

c $\overrightarrow{AP} \bullet \overrightarrow{BP} = (\cos\theta + 1)(\cos\theta - 1) + \sin^2\theta$
$$= \cos^2\theta - 1 + \sin^2\theta$$
$$= 1 - 1 \qquad \{\cos^2\theta + \sin^2\theta = 1\}$$
$$= 0$$

d $\overrightarrow{AP} \bullet \overrightarrow{BP} = 0$, which means $\overrightarrow{AP}$ and $\overrightarrow{BP}$ are perpendicular.
Now, triangle APB is in a semi-circle, and the angle at P is 90°.
So, we have deduced that the angle in a semi-circle is a right angle.

9
$$\begin{aligned}
&\mathbf{a} \bullet (\mathbf{b}+\mathbf{c})\\
&= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \bullet \left[\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}\right]\\
&= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \bullet \begin{pmatrix} b_1+c_1 \\ b_2+c_2 \\ b_3+c_3 \end{pmatrix}\\
&= a_1(b_1+c_1)+a_2(b_2+c_2)+a_3(b_3+c_3)\\
&= a_1b_1+a_1c_1+a_2b_2+a_2c_2+a_3b_3+a_3c_3\\
&= (a_1b_1+a_2b_2+a_3b_3)+(a_1c_1+a_2c_2+a_3c_3)\\
&= \mathbf{a}\bullet\mathbf{b}+\mathbf{a}\bullet\mathbf{c}
\end{aligned}$$

$\therefore\ \mathbf{p} \bullet (\mathbf{c}+\mathbf{d}) = \mathbf{p}\bullet\mathbf{c}+\mathbf{p}\bullet\mathbf{d}$

If we let $\mathbf{p} = \mathbf{a}+\mathbf{b}$,

then
$$\begin{aligned}
&(\mathbf{a}+\mathbf{b}) \bullet (\mathbf{c}+\mathbf{d})\\
&= \mathbf{p} \bullet (\mathbf{c}+\mathbf{d})\\
&= \mathbf{p}\bullet\mathbf{c}+\mathbf{p}\bullet\mathbf{d}\\
&= (\mathbf{a}+\mathbf{b})\bullet\mathbf{c}+(\mathbf{a}+\mathbf{b})\bullet\mathbf{d}\\
&= \mathbf{c}\bullet(\mathbf{a}+\mathbf{b})+\mathbf{d}\bullet(\mathbf{a}+\mathbf{b})\\
&= \mathbf{c}\bullet\mathbf{a}+\mathbf{c}\bullet\mathbf{b}+\mathbf{d}\bullet\mathbf{a}+\mathbf{d}\bullet\mathbf{b}\\
&= \mathbf{a}\bullet\mathbf{c}+\mathbf{a}\bullet\mathbf{d}+\mathbf{b}\bullet\mathbf{c}+\mathbf{b}\bullet\mathbf{d}
\end{aligned}$$

10 **a** **i**
$$\begin{aligned}
\begin{pmatrix} 3 \\ t \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 1 \end{pmatrix} &= 0\\
\therefore\ -6+t &= 0\\
\therefore\ t &= 6
\end{aligned}$$

ii If $\mathbf{p} \parallel \mathbf{q}$ then $\begin{pmatrix} 3 \\ t \end{pmatrix} = k\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

where $k \neq 0$

$\therefore\ 3 = -2k$ and $t = k$

$\therefore\ k = -\frac{3}{2}$ and $t = -\frac{3}{2}$

b **i**
$$\begin{aligned}
\begin{pmatrix} t \\ t+2 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -4 \end{pmatrix} &= 0\\
\therefore\ 3t-4(t+2) &= 0\\
\therefore\ 3t-4t-8 &= 0\\
\therefore\ -t &= 8\\
\therefore\ t &= -8
\end{aligned}$$

ii If $\mathbf{r} \parallel \mathbf{s}$ then $\begin{pmatrix} t \\ t+2 \end{pmatrix} = k\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ where $k \neq 0$

$\therefore\ t = 3k$ and $t+2 = -4k$
$$\begin{aligned}
\therefore\ t+2 &= -4\left(\tfrac{t}{3}\right)\\
\therefore\ 3t+6 &= -4t\\
\therefore\ 7t &= -6\\
\therefore\ t &= -\tfrac{6}{7}
\end{aligned}$$

c **i**
$$\begin{aligned}
\begin{pmatrix} t \\ t+2 \end{pmatrix} \bullet \begin{pmatrix} 2-3t \\ t \end{pmatrix} &= 0\\
\therefore\ 2t-3t^2+t^2+2t &= 0\\
\therefore\ -2t^2+4t &= 0\\
\therefore\ t^2-2t &= 0\\
\therefore\ t(t-2) &= 0\\
\therefore\ t &= 0 \text{ or } 2
\end{aligned}$$

ii If $\mathbf{a} \parallel \mathbf{b}$ then $\begin{pmatrix} t \\ t+2 \end{pmatrix} = k\begin{pmatrix} 2-3t \\ t \end{pmatrix}$

$\therefore\ t = k(2-3t)$ and $t+2 = kt$

$\therefore\ \dfrac{t}{2-3t} = \dfrac{t+2}{t}$ {equating ks}
$$\begin{aligned}
\therefore\ t^2 &= (t+2)(2-3t)\\
\therefore\ t^2 &= 2t-3t^2+4-6t\\
\therefore\ 4t^2+4t-4 &= 0\\
\therefore\ t^2+t-1 &= 0
\end{aligned}$$

which has $\Delta = 1^2-4(1)(-1) = 5$

$$\therefore\ t = \frac{-1\pm\sqrt{5}}{2}$$

11 **a** $\begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = 1(2)+1(3)+5(-1) = 0$

$\therefore\ \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ are perpendicular.

b $\mathbf{a} \bullet \mathbf{b}$
$= \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$
$= 3(-1) + 1(1) + 2(1)$
$= 0$

$\mathbf{b} \bullet \mathbf{c}$
$= \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 5 \\ -4 \end{pmatrix}$
$= (-1)(1) + 1(5) + 1(-4)$
$= 0$

$\mathbf{a} \bullet \mathbf{c}$
$= \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 5 \\ -4 \end{pmatrix}$
$= (3)(1) + 1(5) + 2(-4)$
$= 0$

$\therefore$ **a**, **b**, and **c** are mutually perpendicular.

c **i**
$$\begin{pmatrix} 3 \\ -1 \\ t \end{pmatrix} \bullet \begin{pmatrix} 2t \\ -3 \\ -4 \end{pmatrix} = 0$$
$$\therefore\ 3(2t) + (-1)(-3) + t(-4) = 0$$
$$\therefore\ 6t + 3 - 4t = 0$$
$$\therefore\ 2t + 3 = 0$$
$$\therefore\ t = -\tfrac{3}{2}$$

ii
$$\begin{pmatrix} 3 \\ t \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 1-t \\ -3 \\ 4 \end{pmatrix} = 0$$
$$\therefore\ 3(1-t) + t(-3) + (-2)4 = 0$$
$$\therefore\ 3 - 3t - 3t - 8 = 0$$
$$\therefore\ -6t = 5$$
$$\therefore\ t = -\tfrac{5}{6}$$

12 **a** We have three points: A$(-2, 1)$, B$(-2, 5)$, C$(3, 1)$.

Then $\overrightarrow{AB} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$, and $\overrightarrow{BC} = \begin{pmatrix} 5 \\ -4 \end{pmatrix}$

Now $\overrightarrow{AB} \bullet \overrightarrow{AC} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 5 \\ 0 \end{pmatrix} = 0 + 0 = 0$

$\therefore$ $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{AC}$ and so $\triangle$ABC is right angled at A.

b We have three points: A$(4, 7)$, B$(1, 2)$, C$(-1, 6)$

Then $\overrightarrow{AB} = \begin{pmatrix} -3 \\ -5 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} -5 \\ -1 \end{pmatrix}$, and $\overrightarrow{BC} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$

Now $\overrightarrow{AB} \bullet \overrightarrow{AC} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} \bullet \begin{pmatrix} -5 \\ -1 \end{pmatrix} = 15 + 5 = 20$

$\overrightarrow{AB} \bullet \overrightarrow{BC} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 4 \end{pmatrix} = 6 + (-20) = -14$

$\overrightarrow{AC} \bullet \overrightarrow{BC} = \begin{pmatrix} -5 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 4 \end{pmatrix} = 10 + (-4) = 6$

$\therefore$ none of the sides are perpendicular to each other and so $\triangle$ABC is not right angled.

c We have three points: A$(2, -2)$, B$(5, 7)$, C$(-1, -1)$

Then $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 9 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$, and $\overrightarrow{BC} = \begin{pmatrix} -6 \\ -8 \end{pmatrix}$

Now $\overrightarrow{AB} \bullet \overrightarrow{AC} = \begin{pmatrix} 3 \\ 9 \end{pmatrix} \bullet \begin{pmatrix} -3 \\ 1 \end{pmatrix} = -9 + 9 = 0$

$\therefore$ $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{AC}$ and so $\triangle$ABC is right angled at A.

d We have three points: A$(10, 1)$, B$(5, 2)$, C$(7, 4)$

Then $\overrightarrow{AB} = \begin{pmatrix} -5 \\ 1 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$, and $\overrightarrow{BC} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

Now $\overrightarrow{AC} \bullet \overrightarrow{BC} = \begin{pmatrix} -3 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 2 \end{pmatrix} = -6 + 6 = 0$

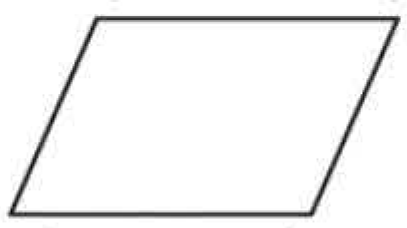
$\therefore$ $\overrightarrow{AC}$ is perpendicular to $\overrightarrow{BC}$ and so $\triangle$ABC is right angled at C.

13 We have three points: A(5, 1, 2), B(6, −1, 0), C(3, 2, 0)

Then $\overrightarrow{AB} = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix}$, and $\overrightarrow{BC} = \begin{pmatrix} -3 \\ 3 \\ 0 \end{pmatrix}$

Now $\overrightarrow{AB} \bullet \overrightarrow{AC} = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix} = (-2) + (-2) + 4 = 0$

$\therefore$ $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{AC}$ and so $\triangle$ABC is right angled at A.

14 **a** 

$\overrightarrow{AB} = \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}$, $\overrightarrow{BC} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$

$\overrightarrow{DC} = \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}$, $\overrightarrow{AD} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$

$\therefore$ $\overrightarrow{AB}$ is parallel to $\overrightarrow{DC}$ and $\overrightarrow{BC}$ is parallel to $\overrightarrow{AD}$.

$\therefore$ ABCD is a parallelogram.

b $|\,\overrightarrow{AB}\,| = \sqrt{(-3)^2 + (-2)^2 + 1^2} = \sqrt{9+4+1} = \sqrt{14}$ units

and $|\,\overrightarrow{BC}\,| = \sqrt{(-2)^2 + 1^2 + 3^2} = \sqrt{4+1+9} = \sqrt{14}$ units

$\therefore$ ABCD is a rhombus.

c $\overrightarrow{AC} \bullet \overrightarrow{BD} = \begin{pmatrix} -5 \\ -1 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = (-5)(1) + (-1)(3) + 4(2) = 0$

$\therefore$ $\overrightarrow{AC}$ is perpendicular to $\overrightarrow{BD}$ which illustrates that the diagonals of a rhombus are perpendicular.

15 **a** $\begin{pmatrix} 5 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 5 \end{pmatrix} = -10 + 10 = 0$, so $\begin{pmatrix} -2 \\ 5 \end{pmatrix}$ is one such vector.

$\therefore$ required vectors have form $k\begin{pmatrix} -2 \\ 5 \end{pmatrix}$, $k \neq 0$.

b $\begin{pmatrix} -1 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -1 \end{pmatrix} = -2 + 2 = 0$, so $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ is one such vector.

$\therefore$ required vectors have form $k\begin{pmatrix} 2 \\ -1 \end{pmatrix}$, $k \neq 0$.

c $\begin{pmatrix} 3 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 3 \end{pmatrix} = 3 - 3 = 0$, so $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ is one such vector.

$\therefore$ required vectors have form $k\begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $k \neq 0$.

d $\begin{pmatrix} -4 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 4 \end{pmatrix} = -12 + 12 = 0$, so $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ is one such vector.

$\therefore$ required vectors have form $k\begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $k \neq 0$.

e $\begin{pmatrix} 2 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 + 0 = 0$, so $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is one such vector.

$\therefore$ required vectors have form $k\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $k \neq 0$.

16 Suppose $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is perpendicular to $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$. $\quad \therefore \ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 0$

So, to find a vector perpendicular to $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, we pick two non-zero integer values for a and b, then solve for c.

For example, if $a = 1,\ b = 2$

then $\begin{pmatrix} 1 \\ 2 \\ c \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 0$

$\therefore \ 1 + 4 - c = 0$

$\therefore \ 5 - c = 0$

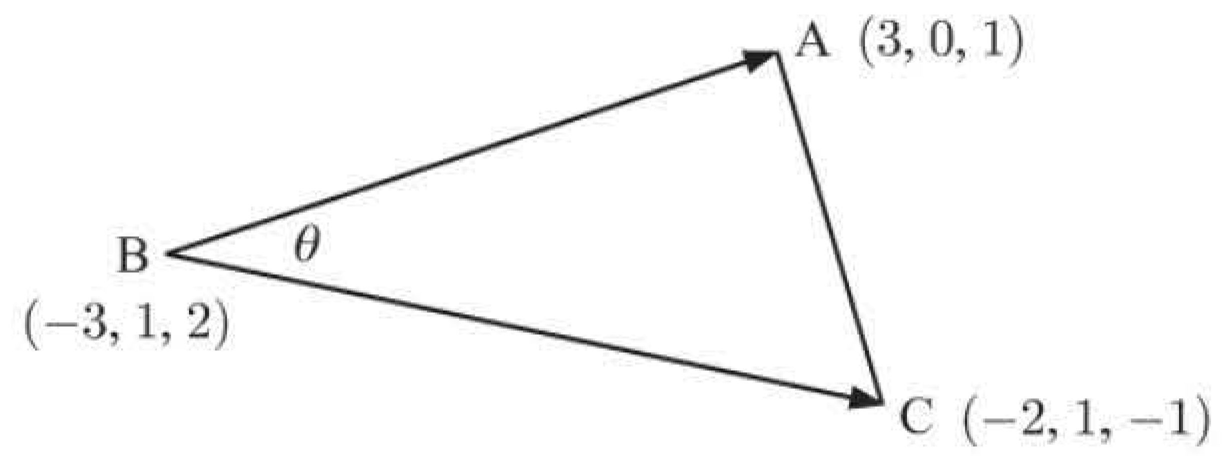
$\therefore \ c = 5$

So, the vector $\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$ is perpendicular to $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

Repeating this process with a different value of a (or b) will give another vector which is perpendicular to $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

17 Given $A(3, 0, 1)$, $B(-3, 1, 2)$, and $C(-2, 1, -1)$,

$\overrightarrow{BC} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$ and $\overrightarrow{BA} = \begin{pmatrix} 6 \\ -1 \\ -1 \end{pmatrix}$



$$\therefore \ \cos\theta = \frac{\overrightarrow{BC} \bullet \overrightarrow{BA}}{|\overrightarrow{BC}||\overrightarrow{BA}|}$$

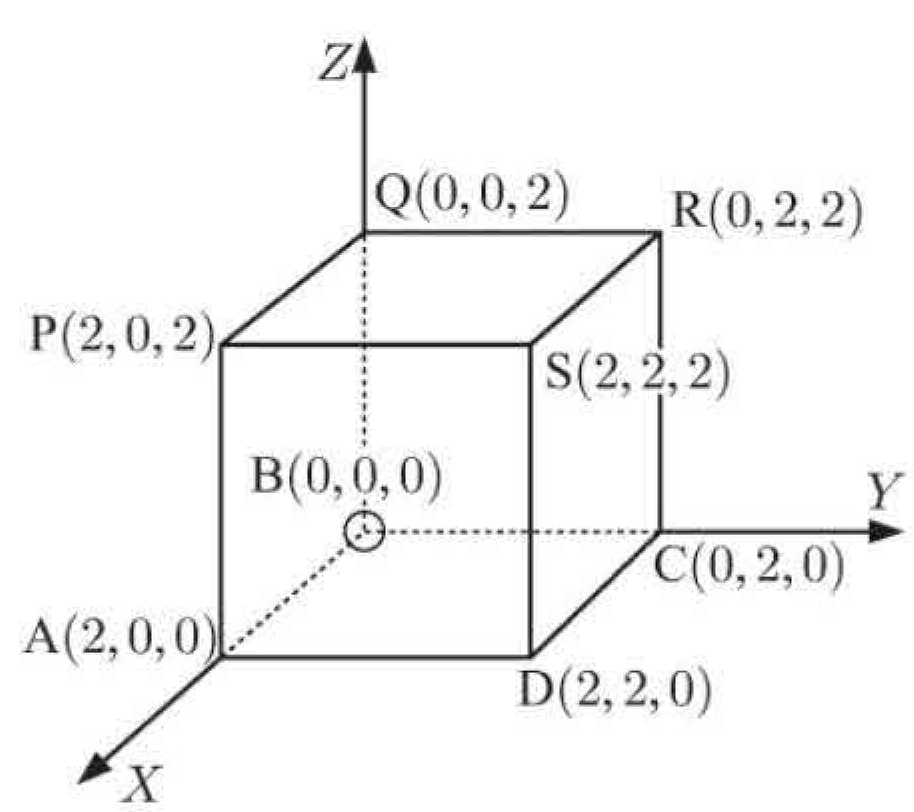
$$= \frac{\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} \bullet \begin{pmatrix} 6 \\ -1 \\ -1 \end{pmatrix}}{\sqrt{1+9}\sqrt{36+1+1}}$$

$$= \frac{6+0+3}{\sqrt{10}\sqrt{38}} = \frac{9}{\sqrt{380}}$$

$$\therefore \ \theta \approx 62.5^\circ$$

If $\overrightarrow{BA}$ and $\overrightarrow{CB}$ are used we would find the exterior angle of the triangle at B, which is 117.5°.

18



a Suppose the origin is at B.

Now $\overrightarrow{BA} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ and $\overrightarrow{BS} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$

$$\therefore \ \overrightarrow{BA} \bullet \overrightarrow{BS} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = 4 + 0 + 0 = 4$$

$$\therefore \ \cos A\widehat{B}S = \frac{4}{\sqrt{4+0+0}\sqrt{4+4+4}}$$

$$= \frac{4}{2 \times 2\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\therefore \ A\widehat{B}S \approx 54.7^\circ$$

b Consider vectors away from B.

$\overrightarrow{BR} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$ and $\overrightarrow{BP} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$

$$\therefore \ \overrightarrow{BR} \bullet \overrightarrow{BP} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} = 0 + 0 + 4 = 4$$

$$\therefore \ \cos R\widehat{B}P = \frac{4}{\sqrt{0+4+4}\sqrt{4+0+4}}$$

$$= \frac{4}{\sqrt{8} \times \sqrt{8}}$$

$$= \tfrac{1}{2} \text{ and so } R\widehat{B}P = 60^\circ$$

c $\overrightarrow{BP} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$ and $\overrightarrow{BS} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$

$$\therefore \ \overrightarrow{BP} \bullet \overrightarrow{BS} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

$$= 4 + 0 + 4 = 8$$

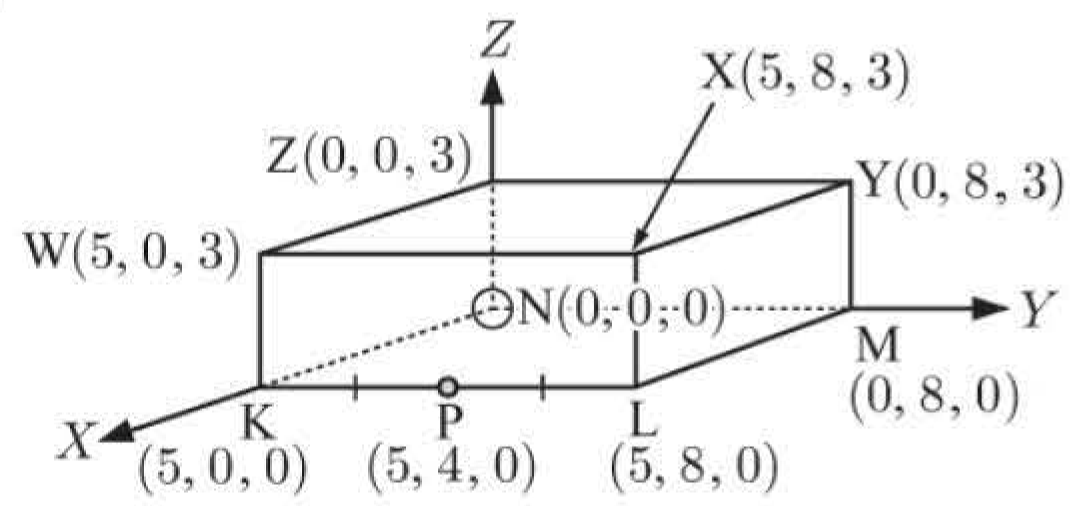
$$\therefore \ \cos P\widehat{B}S = \frac{8}{\sqrt{4+4}\sqrt{4+4+4}}$$

$$= \frac{8}{\sqrt{96}}$$

$$\therefore \ P\widehat{B}S \approx 35.3^\circ$$

19 Suppose the origin is at N.

a



$$\overrightarrow{NY} = \begin{pmatrix} 0 \\ 8 \\ 3 \end{pmatrix} \text{ and } \overrightarrow{NX} = \begin{pmatrix} 5 \\ 8 \\ 3 \end{pmatrix}$$

$$\overrightarrow{NY} \bullet \overrightarrow{NX} = \begin{pmatrix} 0 \\ 8 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 5 \\ 8 \\ 3 \end{pmatrix} = 0 + 64 + 9 = 73$$

$$\therefore \cos Y\widehat{N}X = \frac{73}{\sqrt{64+9}\sqrt{25+64+9}} = \frac{73}{\sqrt{73}\sqrt{98}} = \sqrt{\tfrac{73}{98}}$$

$$\therefore Y\widehat{N}X \approx 30.3^\circ$$

b $\overrightarrow{NY} = \begin{pmatrix} 0 \\ 8 \\ 3 \end{pmatrix}$ and $\overrightarrow{NP} = \begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix}$

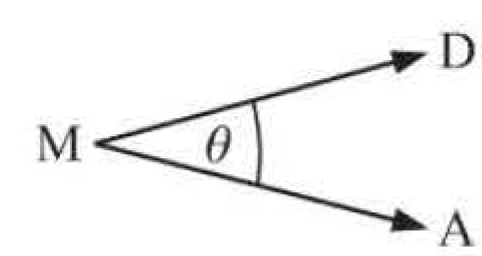
$$\overrightarrow{NY} \bullet \overrightarrow{NP} = \begin{pmatrix} 0 \\ 8 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix} = 0 + 32 + 0 = 32$$

$$\therefore \cos Y\widehat{N}P = \frac{32}{\sqrt{64+9}\sqrt{25+16}} = \frac{32}{\sqrt{73}\sqrt{41}}$$

$$\therefore Y\widehat{N}P \approx 54.2^\circ$$

20 **a** M is the midpoint of [BC]. $\therefore$ M is at $\left(\frac{2+1}{2}, \frac{2+3}{2}, \frac{2+1}{2}\right)$, which is $\left(\frac{3}{2}, \frac{5}{2}, \frac{3}{2}\right)$.

b Now $\overrightarrow{MD} = \begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ -\frac{3}{2} \end{pmatrix}$ and $\overrightarrow{MA} = \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \\ -\frac{1}{2} \end{pmatrix}$



$$\therefore \cos\theta = \frac{\overrightarrow{MD} \bullet \overrightarrow{MA}}{|\overrightarrow{MD}||\overrightarrow{MA}|} = \frac{\begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \bullet \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \\ -\frac{1}{2} \end{pmatrix}}{\sqrt{\frac{9}{4}+\frac{1}{4}+\frac{9}{4}}\sqrt{\frac{1}{4}+\frac{9}{4}+\frac{1}{4}}}$$

$$\therefore \cos\theta = \frac{\frac{3}{4}+\frac{3}{4}+\frac{3}{4}}{\sqrt{\frac{19}{4}}\sqrt{\frac{11}{4}}} = \frac{\frac{9}{4}}{\frac{\sqrt{209}}{4}} = \frac{9}{\sqrt{209}} \quad \text{and so} \quad \theta \approx 51.5^\circ$$

21 **a** $\begin{pmatrix} 2 \\ t \\ t-2 \end{pmatrix} \bullet \begin{pmatrix} t \\ 3 \\ t \end{pmatrix} = 0$

$$\therefore 2t + 3t + t(t-2) = 0$$
$$\therefore 5t + t^2 - 2t = 0$$
$$\therefore t^2 + 3t = 0$$
$$\therefore t(t+3) = 0 \text{ and so } t = 0 \text{ or } t = -3$$

b Given that $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ 2 \\ r \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} s \\ t \\ 1 \end{pmatrix}$ are mutually perpendicular,

$\mathbf{a} \bullet \mathbf{b} = 0$, $\mathbf{b} \bullet \mathbf{c} = 0$, and $\mathbf{a} \bullet \mathbf{c} = 0$

$$\therefore \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 2 \\ r \end{pmatrix} = 0 \qquad \therefore 2 + 4 + 3r = 0$$
$$\therefore 3r = -6$$
$$\therefore r = -2$$

and $\begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} s \\ t \\ 1 \end{pmatrix} = 0 \qquad \therefore 2s + 2t - 2 = 0$

$$\therefore s + t = 1 \qquad .... (1)$$

and $\begin{pmatrix}1\\2\\3\end{pmatrix} \bullet \begin{pmatrix}s\\t\\1\end{pmatrix} = 0 \qquad \therefore \quad s + 2t + 3 = 0$

$\therefore \quad s + 2t = -3 \quad .... (2)$

$(2) - (1)$ gives $t = -4$ and so $s = 5$

$\therefore \quad r = -2, \quad s = 5, \quad$ and $\quad t = -4$

22 **a** Let θ be the angle between the vectors $\mathbf{i} = \begin{pmatrix}1\\0\\0\end{pmatrix}$ and $\begin{pmatrix}1\\2\\3\end{pmatrix}$.

Then $$\cos\theta = \frac{\begin{pmatrix}1\\0\\0\end{pmatrix} \bullet \begin{pmatrix}1\\2\\3\end{pmatrix}}{\left|\begin{pmatrix}1\\0\\0\end{pmatrix}\right|\left|\begin{pmatrix}1\\2\\3\end{pmatrix}\right|} = \frac{1}{\sqrt{1}\sqrt{1+4+9}} = \frac{1}{\sqrt{14}}$$ and so $\theta \approx 74.5^\circ$.

b Let θ be the angle between the vectors $\mathbf{j} = \begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\begin{pmatrix}-1\\1\\3\end{pmatrix}$.

Then $$\cos\theta = \frac{\begin{pmatrix}0\\1\\0\end{pmatrix} \bullet \begin{pmatrix}-1\\1\\3\end{pmatrix}}{\left|\begin{pmatrix}0\\1\\0\end{pmatrix}\right|\left|\begin{pmatrix}-1\\1\\3\end{pmatrix}\right|} = \frac{1}{\sqrt{1}\sqrt{1+1+9}} = \frac{1}{\sqrt{11}}$$ and so $\theta \approx 72.5^\circ$.

23 For example let $\mathbf{a} = \mathbf{i}$, $\mathbf{b} = \mathbf{j}$, and $\mathbf{c} = \mathbf{k}$

$\mathbf{i} \bullet \mathbf{j} = \mathbf{i} \bullet \mathbf{k} = 0$ and $\mathbf{j} \neq \mathbf{k}$

24 **a** Show $|\mathbf{a}+\mathbf{b}|^2 + |\mathbf{a}-\mathbf{b}|^2 = 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2$ using $|\mathbf{x}|^2 = \mathbf{x} \bullet \mathbf{x}$

$$\begin{aligned}
\text{LHS} &= (\mathbf{a}+\mathbf{b}) \bullet (\mathbf{a}+\mathbf{b}) + (\mathbf{a}-\mathbf{b}) \bullet (\mathbf{a}-\mathbf{b}) \\
&= \mathbf{a} \bullet \mathbf{a} + \cancel{\mathbf{a} \bullet \mathbf{b}} + \cancel{\mathbf{b} \bullet \mathbf{a}} + \mathbf{b} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{a} - \cancel{\mathbf{a} \bullet \mathbf{b}} - \cancel{\mathbf{b} \bullet \mathbf{a}} + \mathbf{b} \bullet \mathbf{b} \\
&= 2\mathbf{a} \bullet \mathbf{a} + 2\mathbf{b} \bullet \mathbf{b} \\
&= 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2 \\
&= \text{RHS}
\end{aligned}$$

b Show $|\mathbf{a}+\mathbf{b}|^2 - |\mathbf{a}-\mathbf{b}|^2 = 4\mathbf{a} \bullet \mathbf{b}$ using $|\mathbf{x}|^2 = \mathbf{x} \bullet \mathbf{x}$

$$\begin{aligned}
\text{LHS} &= (\mathbf{a}+\mathbf{b}) \bullet (\mathbf{a}+\mathbf{b}) - (\mathbf{a}-\mathbf{b}) \bullet (\mathbf{a}-\mathbf{b}) \\
&= \mathbf{a} \bullet \mathbf{a} + \mathbf{a} \bullet \mathbf{b} + \mathbf{b} \bullet \mathbf{a} + \mathbf{b} \bullet \mathbf{b} - (\mathbf{a} \bullet \mathbf{a} - \mathbf{a} \bullet \mathbf{b} - \mathbf{b} \bullet \mathbf{a} + \mathbf{b} \bullet \mathbf{b}) \\
&= \cancel{\mathbf{a} \bullet \mathbf{a}} + \mathbf{a} \bullet \mathbf{b} + \mathbf{b} \bullet \mathbf{a} + \cancel{\mathbf{b} \bullet \mathbf{b}} - \cancel{\mathbf{a} \bullet \mathbf{a}} + \mathbf{a} \bullet \mathbf{b} + \mathbf{b} \bullet \mathbf{a} - \cancel{\mathbf{b} \bullet \mathbf{b}} \\
&= \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{b} \\
&= 4\mathbf{a} \bullet \mathbf{b} \\
&= \text{RHS}
\end{aligned}$$

25 **a**

If $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}-\mathbf{b}|$

then $|\mathbf{a}+\mathbf{b}|^2 = |\mathbf{a}-\mathbf{b}|^2$

$\therefore \quad (\mathbf{a}+\mathbf{b}) \bullet (\mathbf{a}+\mathbf{b}) = (\mathbf{a}-\mathbf{b}) \bullet (\mathbf{a}-\mathbf{b})$

$\therefore \quad \cancel{\mathbf{a} \bullet \mathbf{a}} + \mathbf{a} \bullet \mathbf{b} + \mathbf{b} \bullet \mathbf{a} + \cancel{\mathbf{b} \bullet \mathbf{b}} = \cancel{\mathbf{a} \bullet \mathbf{a}} - \mathbf{a} \bullet \mathbf{b} - \mathbf{b} \bullet \mathbf{a} + \cancel{\mathbf{b} \bullet \mathbf{b}}$

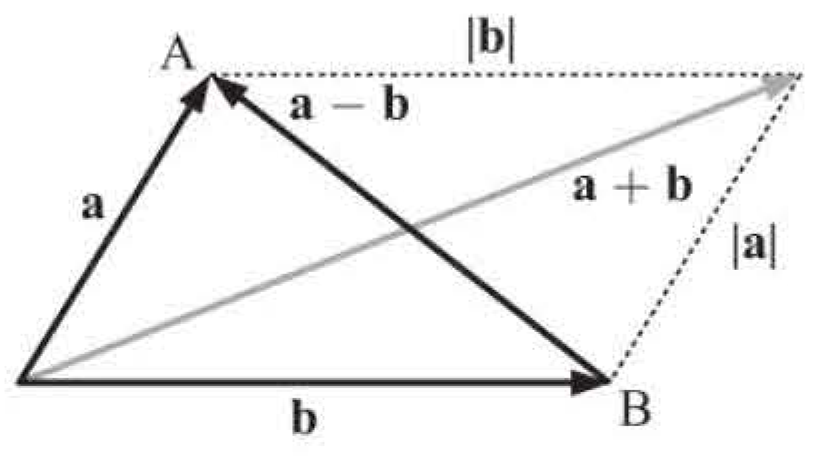
$\therefore \quad \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{b} = 0$

$\therefore \quad 4\mathbf{a} \bullet \mathbf{b} = 0$

$\therefore \quad \mathbf{a} \bullet \mathbf{b} = 0$

$\therefore$ **a** and **b** are perpendicular.

b



As shown $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ are the diagonals of a parallelogram with side lengths $|\mathbf{a}|$ and $|\mathbf{b}|$.

Now if $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}-\mathbf{b}|$ then the parallelogram must be a square, and **a** is perpendicular to **b**.

26
$$\begin{aligned}&(\mathbf{a}+\mathbf{b})\bullet(\mathbf{a}-\mathbf{b})\\ &=\mathbf{a}\bullet\mathbf{a}-\cancel{\mathbf{a}\bullet\mathbf{b}}+\cancel{\mathbf{b}\bullet\mathbf{a}}-\mathbf{b}\bullet\mathbf{b}\\ &=|\mathbf{a}|^2-|\mathbf{b}|^2\\ &=3^2-4^2\\ &=9-16\\ &=-7\end{aligned}$$

27 The dot product is only defined for two vectors. For $\mathbf{a}\bullet\mathbf{b}\bullet\mathbf{c}$, the result of $\mathbf{a}\bullet\mathbf{b}$ is a scalar, and the dot product of the scalar and $\mathbf{c}$ is meaningless.

EXERCISE 14K.1

1 **a**
$$\begin{aligned}\begin{pmatrix}2\\-3\\1\end{pmatrix}\times\begin{pmatrix}1\\4\\-2\end{pmatrix}&=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&-3&1\\1&4&-2\end{vmatrix}\\ &=\begin{vmatrix}-3&1\\4&-2\end{vmatrix}\mathbf{i}-\begin{vmatrix}2&1\\1&-2\end{vmatrix}\mathbf{j}+\begin{vmatrix}2&-3\\1&4\end{vmatrix}\mathbf{k}\\ &=((-3)\times(-2)-1\times4)\mathbf{i}-(2\times(-2)-1\times1)\mathbf{j}+(2\times4-(-3)\times1)\mathbf{k}\\ &=(6-4)\mathbf{i}-(-4-1)\mathbf{j}+(8+3)\mathbf{k}\\ &=2\mathbf{i}-(-5)\mathbf{j}+11\mathbf{k}\\ &=\begin{pmatrix}2\\5\\11\end{pmatrix}\end{aligned}$$

b
$$\begin{aligned}\begin{pmatrix}-1\\0\\2\end{pmatrix}\times\begin{pmatrix}3\\-1\\-2\end{pmatrix}&=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-1&0&2\\3&-1&-2\end{vmatrix}\\ &=\begin{vmatrix}0&2\\-1&-2\end{vmatrix}\mathbf{i}-\begin{vmatrix}-1&2\\3&-2\end{vmatrix}\mathbf{j}+\begin{vmatrix}-1&0\\3&-1\end{vmatrix}\mathbf{k}\\ &=(0\times(-2)-2\times(-1))\mathbf{i}-((-1)\times(-2)-2\times3)\mathbf{j}\\ &\qquad+((-1)\times(-1)-0\times3)\mathbf{k}\\ &=(0+2)\mathbf{i}-(2-6)\mathbf{j}+(1-0)\mathbf{k}\\ &=2\mathbf{i}-(-4)\mathbf{j}+\mathbf{k}\\ &=\begin{pmatrix}2\\4\\1\end{pmatrix}\end{aligned}$$

c
$$\begin{aligned}(\mathbf{i}+\mathbf{j}-2\mathbf{k})\times(\mathbf{i}-\mathbf{k})&=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&1&-2\\1&0&-1\end{vmatrix}\\ &=\begin{vmatrix}1&-2\\0&-1\end{vmatrix}\mathbf{i}-\begin{vmatrix}1&-2\\1&-1\end{vmatrix}\mathbf{j}+\begin{vmatrix}1&1\\1&0\end{vmatrix}\mathbf{k}\\ &=(1\times(-1)-(-2)\times0)\mathbf{i}-(1\times(-1)-(-2)\times1)\mathbf{j}\\ &\qquad+(1\times0-1\times1)\mathbf{k}\\ &=(-1-0)\mathbf{i}-(-1+2)\mathbf{j}+(0-1)\mathbf{k}\\ &=-\mathbf{i}-\mathbf{j}-\mathbf{k}\end{aligned}$$

d
$$\begin{aligned}(2\mathbf{i}-\mathbf{k})\times(\mathbf{j}+3\mathbf{k})&=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&0&-1\\0&1&3\end{vmatrix}\\ &=\begin{vmatrix}0&-1\\1&3\end{vmatrix}\mathbf{i}-\begin{vmatrix}2&-1\\0&3\end{vmatrix}\mathbf{j}+\begin{vmatrix}2&0\\0&1\end{vmatrix}\mathbf{k}\\ &=(0\times3-(-1)\times1)\mathbf{i}-(2\times3-(-1)\times0)\mathbf{j}+(2\times1-0\times0)\mathbf{k}\\ &=(0+1)\mathbf{i}-(6-0)\mathbf{j}+(2-0)\mathbf{k}\\ &=\mathbf{i}-6\mathbf{j}+2\mathbf{k}\end{aligned}$$

2 **a** $\begin{pmatrix}1\\2\\3\end{pmatrix}\times\begin{pmatrix}-1\\3\\-1\end{pmatrix}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&2&3\\-1&3&-1\end{vmatrix}$

$$\begin{aligned}&=\begin{vmatrix}2&3\\3&-1\end{vmatrix}\mathbf{i}-\begin{vmatrix}1&3\\-1&-1\end{vmatrix}\mathbf{j}+\begin{vmatrix}1&2\\-1&3\end{vmatrix}\mathbf{k}\\&=(2\times(-1)-3\times3)\mathbf{i}-(1\times(-1)-(-1)\times3)\mathbf{j}+(1\times3-(-1)\times2)\mathbf{k}\\&=(-2-9)\mathbf{i}-(-1+3)\mathbf{j}+(3+2)\mathbf{k}\\&=-11\mathbf{i}-2\mathbf{j}+5\mathbf{k}\\&=\begin{pmatrix}-11\\-2\\5\end{pmatrix}\end{aligned}$$

b $\mathbf{a}\bullet(\mathbf{a}\times\mathbf{b})=\begin{pmatrix}1\\2\\3\end{pmatrix}\bullet\begin{pmatrix}-11\\-2\\5\end{pmatrix}$
$=-11-4+15$
$=0$

$\mathbf{b}\bullet(\mathbf{a}\times\mathbf{b})=\begin{pmatrix}-1\\3\\-1\end{pmatrix}\bullet\begin{pmatrix}-11\\-2\\5\end{pmatrix}$
$=11-6-5$
$=0$

$\therefore\ \mathbf{a}\bullet(\mathbf{a}\times\mathbf{b})=0=\mathbf{b}\bullet(\mathbf{a}\times\mathbf{b})$

c $\mathbf{a}\times\mathbf{b}$ is a vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

3 **a** $\mathbf{i}\times\mathbf{i}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&0&0\\1&0&0\end{vmatrix}$

$$\begin{aligned}&=\begin{vmatrix}0&0\\0&0\end{vmatrix}\mathbf{i}-\begin{vmatrix}1&0\\1&0\end{vmatrix}\mathbf{j}+\begin{vmatrix}1&0\\1&0\end{vmatrix}\mathbf{k}\\&=(0\times0-0\times0)\mathbf{i}-(1\times0-0\times1)\mathbf{j}\\&\quad+(1\times0-0\times1)\mathbf{k}\\&=\mathbf{0}\end{aligned}$$

$\mathbf{j}\times\mathbf{j}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&1&0\\0&1&0\end{vmatrix}$

$$\begin{aligned}&=\begin{vmatrix}1&0\\1&0\end{vmatrix}\mathbf{i}-\begin{vmatrix}0&0\\0&0\end{vmatrix}\mathbf{j}+\begin{vmatrix}0&1\\0&1\end{vmatrix}\mathbf{k}\\&=(1\times0-0\times1)\mathbf{i}-(0\times0-0\times0)\mathbf{j}\\&\quad+(0\times1-1\times0)\mathbf{k}\\&=\mathbf{0}\end{aligned}$$

$\mathbf{k}\times\mathbf{k}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&0&1\\0&0&1\end{vmatrix}$

$$\begin{aligned}&=\begin{vmatrix}0&1\\0&1\end{vmatrix}\mathbf{i}-\begin{vmatrix}0&1\\0&1\end{vmatrix}\mathbf{j}+\begin{vmatrix}0&0\\0&0\end{vmatrix}\mathbf{k}\\&=(0\times1-1\times0)\mathbf{i}-(0\times1-1\times0)\mathbf{j}+(0\times0-0\times0)\mathbf{k}\\&=\mathbf{0}\end{aligned}$$

$\mathbf{a}\times\mathbf{a}=\mathbf{0}$ for all vectors $\mathbf{a}$.

b **i** $\mathbf{i}\times\mathbf{j}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&0&0\\0&1&0\end{vmatrix}$

$$\begin{aligned}&=\begin{vmatrix}0&0\\1&0\end{vmatrix}\mathbf{i}-\begin{vmatrix}1&0\\0&0\end{vmatrix}\mathbf{j}+\begin{vmatrix}1&0\\0&1\end{vmatrix}\mathbf{k}\\&=(0\times0-0\times1)\mathbf{i}-(1\times0-0\times0)\mathbf{j}\\&\quad+(1\times1-0\times0)\mathbf{k}\\&=\mathbf{k}\end{aligned}$$

$\mathbf{j}\times\mathbf{i}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&1&0\\1&0&0\end{vmatrix}$

$$\begin{aligned}&=\begin{vmatrix}1&0\\0&0\end{vmatrix}\mathbf{i}-\begin{vmatrix}0&0\\1&0\end{vmatrix}\mathbf{j}+\begin{vmatrix}0&1\\1&0\end{vmatrix}\mathbf{k}\\&=(1\times0-0\times0)\mathbf{i}-(0\times0-0\times1)\mathbf{j}\\&\quad+(0\times0-1\times1)\mathbf{k}\\&=-\mathbf{k}\end{aligned}$$

Notice that we observe $\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$.

ii $\mathbf{j}\times\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}\mathbf{k}$

$= (1\times 1 - 0\times 0)\mathbf{i} - (0\times 1 - 0\times 0)\mathbf{j} + (0\times 0 - 1\times 0)\mathbf{k}$

$= \mathbf{i}$

$\mathbf{k}\times\mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}$

$= \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix}\mathbf{k}$

$= (0\times 0 - 1\times 1)\mathbf{i} - (0\times 0 - 1\times 0)\mathbf{j} + (0\times 1 - 0\times 0)\mathbf{k}$

$= -\mathbf{i}$

iii $\mathbf{i}\times\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$= \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}\mathbf{k}$

$= (0\times 1 - 0\times 0)\mathbf{i} - (1\times 1 - 0\times 0)\mathbf{j} + (1\times 0 - 0\times 0)\mathbf{k}$

$= -\mathbf{j}$

$\mathbf{k}\times\mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix}$

$= \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}\mathbf{k}$

$= (0\times 0 - 1\times 0)\mathbf{i} - (0\times 0 - 1\times 1)\mathbf{j} + (0\times 0 - 0\times 1)\mathbf{k}$

$= \mathbf{j}$

$\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$ for all vectors $\mathbf{a}$ and $\mathbf{b}$.

4 **a** $\mathbf{a}\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$

$= \begin{vmatrix} a_2 & a_3 \\ a_2 & a_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ a_1 & a_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ a_1 & a_2 \end{vmatrix}\mathbf{k}$

$= (a_2a_3 - a_2a_3)\mathbf{i} - (a_1a_3 - a_1a_3)\mathbf{j} + (a_1a_2 - a_1a_2)\mathbf{k}$

$= \mathbf{0}$

Hence $\mathbf{a}\times\mathbf{a} = \mathbf{0}$ for all 3-dimensional vectors $\mathbf{a}$.

b $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}$

$= (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$

$-(\mathbf{b}\times\mathbf{a}) = -\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$

$= -\left[\begin{vmatrix} b_2 & b_3 \\ a_2 & a_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} b_1 & b_3 \\ a_1 & a_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} b_1 & b_2 \\ a_1 & a_2 \end{vmatrix}\mathbf{k}\right]$

$= -\left[(b_2a_3 - b_3a_2)\mathbf{i} - (b_1a_3 - b_3a_1)\mathbf{j} + (b_1a_2 - b_2a_1)\mathbf{k}\right]$

$= (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$

$= \mathbf{a}\times\mathbf{b}$

5 **a** $\mathbf{b}\times\mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 0 & 1 & -2 \end{vmatrix}$

$$= \begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 1 \\ 0 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix}\mathbf{k}$$
$$= (2-1)\mathbf{i} - (-4)\mathbf{j} + 2\mathbf{k}$$
$$= \mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$$
$$= \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$$

b $\mathbf{a}\bullet(\mathbf{b}\times\mathbf{c}) = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}\bullet\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$

$$= 1 + 12 + 4$$
$$= 17$$

6 **a** $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ 0 & -1 & 1 \end{vmatrix}$

$$= \begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}\mathbf{k}$$
$$= 2\mathbf{i} - \mathbf{j} - \mathbf{k}$$
$$= \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$$

b $\mathbf{a}\times\mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ 2 & 0 & -1 \end{vmatrix}$

$$= \begin{vmatrix} 0 & 2 \\ 0 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix}\mathbf{k}$$
$$= -(-1-4)\mathbf{j}$$
$$= 5\mathbf{j}$$
$$= \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix}$$

c $(\mathbf{a}\times\mathbf{b}) + (\mathbf{a}\times\mathbf{c}) = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix}$

$$= \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$$

d $(\mathbf{b}+\mathbf{c}) = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$

$$= \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

$\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ 2 & -1 & 0 \end{vmatrix}$

$$= \begin{vmatrix} 0 & 2 \\ -1 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix}\mathbf{k}$$
$$= (0-(-2))\mathbf{i} - (0-4)\mathbf{j} + (-1-0)\mathbf{k}$$
$$= 2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$$
$$= \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$$

7 $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1+c_1 & b_2+c_2 & b_3+c_3 \end{vmatrix}$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2+c_2 & b_3+c_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1+c_1 & b_3+c_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1+c_1 & b_2+c_2 \end{vmatrix}\mathbf{k}$$
$$= (a_2(b_3+c_3) - a_3(b_2+c_2))\mathbf{i} - (a_1(b_3+c_3) - a_3(b_1+c_1))\mathbf{j} + (a_1(b_2+c_2) - a_2(b_1+c_1))\mathbf{k}$$
$$= (a_2b_3 + a_2c_3 - a_3b_2 - a_3c_2)\mathbf{i} - (a_1b_3 + a_1c_3 - a_3b_1 - a_3c_1)\mathbf{j} + (a_1b_2 + a_1c_2 - a_2b_1 - a_2c_1)\mathbf{k}$$

$$\mathbf{a}\times\mathbf{b} + \mathbf{a}\times\mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\therefore\ \mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k} + \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}\mathbf{k}$$

$$\begin{aligned} &= (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k} \\ &\qquad + (a_2c_3 - a_3c_2)\mathbf{i} - (a_1c_3 - a_3c_1)\mathbf{j} + (a_1c_2 - a_2c_1)\mathbf{k} \\ &= (a_2b_3 + a_2c_3 - a_3b_2 - a_3c_2)\mathbf{i} - (a_1b_3 + a_1c_3 - a_3b_1 - a_3c_1)\mathbf{j} \\ &\qquad + (a_1b_2 + a_1c_2 - a_2b_1 - a_2c_1)\mathbf{k} \\ &= \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) \end{aligned}$$

8 $$\begin{aligned} (\mathbf{a}+\mathbf{b})\times(\mathbf{c}+\mathbf{d}) &= ((\mathbf{a}+\mathbf{b})\times\mathbf{c}) + ((\mathbf{a}+\mathbf{b})\times\mathbf{d}) \quad \{\text{using } \mathbf{a}\times(\mathbf{b}+\mathbf{c}) = (\mathbf{a}\times\mathbf{b}) + (\mathbf{a}\times\mathbf{c})\} \\ &= ((\mathbf{a}\times\mathbf{c}) + (\mathbf{b}\times\mathbf{c})) + ((\mathbf{a}\times\mathbf{d}) + (\mathbf{b}\times\mathbf{d})) \\ &= (\mathbf{a}\times\mathbf{c}) + (\mathbf{a}\times\mathbf{d}) + (\mathbf{b}\times\mathbf{c}) + (\mathbf{b}\times\mathbf{d}) \end{aligned}$$

9 **a** $$\begin{aligned} \mathbf{a}\times(\mathbf{a}+\mathbf{b}) &= (\mathbf{a}\times\mathbf{a}) + (\mathbf{a}\times\mathbf{b}) \\ &= (\mathbf{a}\times\mathbf{b}) \quad \{\text{since } \mathbf{a}\times\mathbf{a} = \mathbf{0}\} \end{aligned}$$

b $$\begin{aligned} (\mathbf{a}+\mathbf{b})\times(\mathbf{a}+\mathbf{b}) &= (\mathbf{a}\times\mathbf{a}) + (\mathbf{a}\times\mathbf{b}) + (\mathbf{b}\times\mathbf{a}) + (\mathbf{b}\times\mathbf{b}) && \{\text{from } \mathbf{8}\} \\ &= (\mathbf{a}\times\mathbf{b}) + (\mathbf{b}\times\mathbf{a}) && \{\text{since } (\mathbf{a}\times\mathbf{a}) = \mathbf{0}\} \\ &= (\mathbf{a}\times\mathbf{b}) - (\mathbf{a}\times\mathbf{b}) && \{\text{since } \mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}\} \\ &= \mathbf{0} \end{aligned}$$

c $$\begin{aligned} (\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b}) &= (\mathbf{a}\times\mathbf{a}) + (\mathbf{a}\times(-\mathbf{b})) + (\mathbf{b}\times\mathbf{a}) + (\mathbf{b}\times(-\mathbf{b})) \\ &= (\mathbf{a}\times(-\mathbf{b})) + (\mathbf{b}\times\mathbf{a}) + (\mathbf{b}\times(-\mathbf{b})) && \{\text{since } (\mathbf{a}\times\mathbf{a}) = \mathbf{0}\} \\ &= -(-\mathbf{b}\times\mathbf{a}) + (\mathbf{b}\times\mathbf{a}) - (-\mathbf{b}\times\mathbf{b}) && \{\text{since } \mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}\} \\ &= (\mathbf{b}\times\mathbf{a}) + (\mathbf{b}\times\mathbf{a}) + (\mathbf{b}\times\mathbf{b}) \\ &= 2(\mathbf{b}\times\mathbf{a}) && \{\text{since } (\mathbf{b}\times\mathbf{b}) = \mathbf{0}\} \end{aligned}$$

d $$\begin{aligned} 2\mathbf{a}\bullet(\mathbf{a}\times\mathbf{b}) &= \begin{pmatrix} 2a_1 \\ 2a_2 \\ 2a_3 \end{pmatrix} \bullet \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} \\ &= 2a_1(a_2b_3 - a_3b_2) + 2a_2(a_3b_1 - a_1b_3) + 2a_3(a_1b_2 - a_2b_1) \\ &= \cancel{2a_1a_2b_3} - \cancel{2a_1a_3b_2} + \cancel{2a_2a_3b_1} - \cancel{2a_1a_2b_3} + \cancel{2a_1a_3b_2} - \cancel{2a_2a_3b_1} \\ &= 0 \end{aligned}$$

or $\mathbf{a}\times\mathbf{b}$ is perpendicular to $2a$. The dot product of two vectors which are perpendicular is 0.

10 **a** $$\begin{aligned} \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 1 & 1 & 1 \end{vmatrix} \\ &= \begin{vmatrix} -1 & 3 \\ 1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}\mathbf{k} \\ &= (-1-3)\mathbf{i} - (2-3)\mathbf{j} + (2+1)\mathbf{k} \\ &= -4\mathbf{i} + \mathbf{j} + 3\mathbf{k} \end{aligned}$$

$\therefore$ perpendicular vectors have the form $k\begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix}$, $k \neq 0,\ k \in \mathbb{R}$.

b $$\begin{aligned} \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix} \times \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & 4 \\ 5 & 0 & 2 \end{vmatrix} \\ &= \begin{vmatrix} 3 & 4 \\ 0 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & 4 \\ 5 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & 3 \\ 5 & 0 \end{vmatrix}\mathbf{k} \\ &= 6\mathbf{i} - (-2-20)\mathbf{j} - 15\mathbf{k} \\ &= 6\mathbf{i} + 22\mathbf{j} - 15\mathbf{k} \end{aligned}$$

$\therefore$ perpendicular vectors have the form $k\begin{pmatrix} 6 \\ 22 \\ -15 \end{pmatrix}$, $k \neq 0,\ k \in \mathbb{R}$.

c $(\mathbf{i}+\mathbf{j})\times(\mathbf{i}-\mathbf{j}-\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & -1 & -1 \end{vmatrix}$

$$= \begin{vmatrix} 1 & 0 \\ -1 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}\mathbf{k}$$

$$= -\mathbf{i}+\mathbf{j}-2\mathbf{k}$$

$\therefore$ perpendicular vectors have the form $(-\mathbf{i}+\mathbf{j}-2\mathbf{k})n,\ n \neq 0,\ n \in \mathbb{R}$.

d $(\mathbf{i}-\mathbf{j}-\mathbf{k})\times(2\mathbf{i}+2\mathbf{j}-3\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -1 \\ 2 & 2 & -3 \end{vmatrix}$

$$= \begin{vmatrix} -1 & -1 \\ 2 & -3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -1 \\ 2 & -3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix}\mathbf{k}$$

$$= (3+2)\mathbf{i} - (-3+2)\mathbf{j} + (2+2)\mathbf{k}$$

$$= 5\mathbf{i}+\mathbf{j}+4\mathbf{k}$$

$\therefore$ perpendicular vectors have the form $(5\mathbf{i}+\mathbf{j}+4\mathbf{k})n,\ n \neq 0,\ n \in \mathbb{R}$.

11 $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -1 \\ 1 & -2 & 2 \end{vmatrix}$

$$= \begin{vmatrix} 3 & -1 \\ -2 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix}\mathbf{k}$$

$$= (6-2)\mathbf{i} - (4+1)\mathbf{j} + (-4-3)\mathbf{k}$$

$$= 4\mathbf{i} - 5\mathbf{j} - 7\mathbf{k}$$

Now $|\mathbf{a}\times\mathbf{b}| = \sqrt{4^2+(-5)^2+(-7)^2}$

$$= \sqrt{90}$$

$$= 3\sqrt{10}$$

$\therefore$ the two vectors of length 5 units which are perpendicular to both **a** and **b** are $\pm\frac{5}{3\sqrt{10}}\begin{pmatrix} 4 \\ -5 \\ -7 \end{pmatrix}$

$$= \pm\tfrac{5\sqrt{10}}{30}\begin{pmatrix} 4 \\ -5 \\ -7 \end{pmatrix}$$

$$= \pm\tfrac{10}{\sqrt{6}}\begin{pmatrix} 4 \\ -5 \\ -7 \end{pmatrix}$$

12 **a** A(1, 3, 2), B(0, 2, −5), C(3, 1, −4)

$$\overrightarrow{AB} = \begin{pmatrix} 0-1 \\ 2-3 \\ -5-2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -7 \end{pmatrix} \qquad \overrightarrow{AC} = \begin{pmatrix} 3-1 \\ 1-3 \\ -4-2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -6 \end{pmatrix}$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -1 & -7 \\ 2 & -2 & -6 \end{vmatrix} = \begin{vmatrix} -1 & -7 \\ -2 & -6 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & -7 \\ 2 & -6 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & -1 \\ 2 & -2 \end{vmatrix}\mathbf{k}$$

$$= (6-14)\mathbf{i} - (6+14)\mathbf{j} + (2+2)\mathbf{k}$$

$$= -8\mathbf{i} - 20\mathbf{j} + 4\mathbf{k}$$

$$= -4(2\mathbf{i}+5\mathbf{j}-\mathbf{k})$$

$\therefore$ vectors of the form $k\begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix},\ k \neq 0,\ k \in \mathbb{R}$ will be perpendicular to the plane.

b $P(2, 0, -1),\quad Q(0, 1, 3),\quad R(1, -1, 1)$

$$\overrightarrow{PQ} = \begin{pmatrix} 0-2 \\ 1-0 \\ 3-(-1) \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} \qquad \overrightarrow{PR} = \begin{pmatrix} 1-2 \\ -1-0 \\ 1-(-1) \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$$

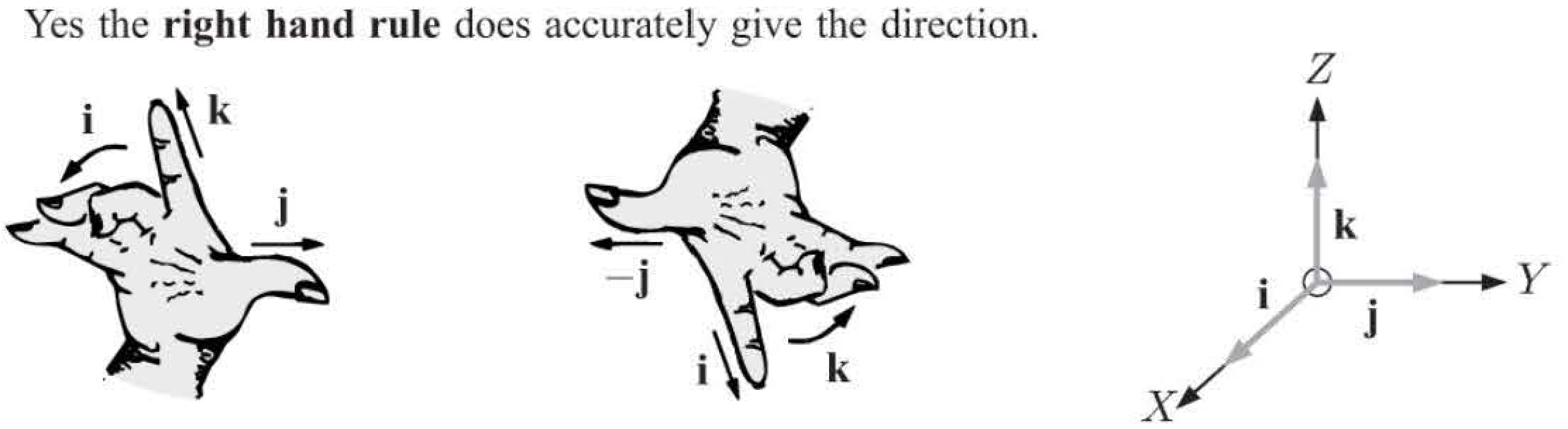
$$\begin{aligned} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 4 \\ -1 & -1 & 2 \end{vmatrix} &= \begin{vmatrix} 1 & 4 \\ -1 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -2 & 4 \\ -1 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -2 & 1 \\ -1 & -1 \end{vmatrix}\mathbf{k} \\ &= (2+4)\mathbf{i} - (-4+4)\mathbf{j} + (2+1)\mathbf{k} \\ &= 6\mathbf{i} + 3\mathbf{k} \\ &= 3(2\mathbf{i} + \mathbf{k}) \end{aligned}$$

$\therefore$ vectors of the form $k\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$, $k \neq 0$, $k \in \mathbb{R}$ will be perpendicular to the plane.

EXERCISE 14K.2

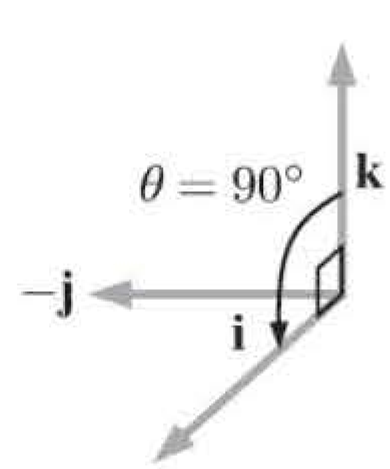
1 **a** $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$ {from **Exercise 14K.1**, question **3**, part **a**}

b Yes the **right hand rule** does accurately give the direction.



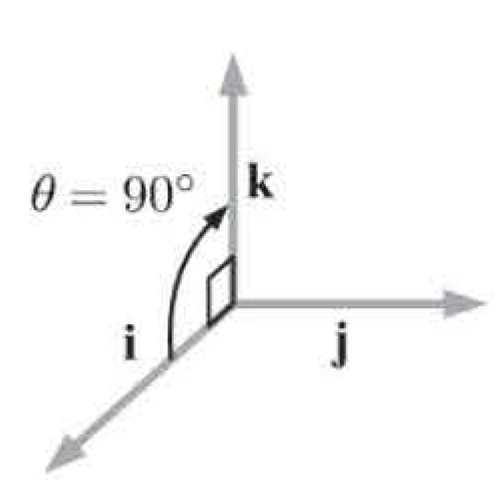
c If $\mathbf{u}$ is the unit vector in the direction $\mathbf{i} \times \mathbf{k}$, then by using the right hand rule, $\mathbf{u} = -\mathbf{j}$.

$$\begin{aligned} |\,\mathbf{i}\,|\,|\,\mathbf{j}\,|\sin\theta\,\mathbf{u} &= 1 \times 1 \times \sin 90^\circ \times (-\mathbf{j}) \\ &= -\mathbf{j} \qquad \{\text{since } \sin 90^\circ = 1\} \end{aligned}$$



If $\mathbf{u}$ is the unit vector in the direction $\mathbf{k} \times \mathbf{i}$, then by using the right hand rule, $\mathbf{u} = \mathbf{j}$.

$$\begin{aligned} |\,\mathbf{i}\,|\,|\,\mathbf{j}\,|\sin\theta\,\mathbf{u} &= 1 \times 1 \times \sin 90^\circ \times \mathbf{j} \\ &= \mathbf{j} \qquad \{\text{since } \sin 90^\circ = 1\} \end{aligned}$$



2 **a** $\mathbf{a} \bullet \mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$$\begin{aligned} &= 2 + 0 - 3 \\ &= -1 \end{aligned}$$

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 1 & 0 & -1 \end{vmatrix} \\ &= \begin{vmatrix} -1 & 3 \\ 0 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix}\mathbf{k} \\ &= \mathbf{i} - (-2-3)\mathbf{j} + \mathbf{k} \\ &= \mathbf{i} + 5\mathbf{j} + \mathbf{k} \end{aligned}$$

b $|\mathbf{a}| = \sqrt{2^2+(-1)^2+3^2}$
$= \sqrt{4+1+9}$
$= \sqrt{14}$

$|\mathbf{b}| = \sqrt{1^2+0^2+(-1)^2}$
$= \sqrt{1+0+1}$
$= \sqrt{2}$

$$\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$$

$$\begin{aligned}\cos\theta &= \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}\\ &= \frac{-1}{\sqrt{14}\sqrt{2}} \qquad \{\mathbf{a} \bullet \mathbf{b} = -1 \text{ from part } \mathbf{a}\}\\ &= -\tfrac{1}{\sqrt{28}}\\ &= -\tfrac{1}{2\sqrt{7}}\\ &= -\tfrac{\sqrt{7}}{14}\end{aligned}$$

c $\sin^2\theta + \cos^2\theta = 1$

$$\begin{aligned}\therefore\ \sin\theta &= \pm\sqrt{1-\cos^2\theta}\\ &= \pm\sqrt{1-\left(\tfrac{1}{\sqrt{28}}\right)^2}\\ &= \pm\sqrt{\tfrac{27}{28}}\end{aligned}$$

But since θ is the angle between two vectors, $0^\circ \leqslant \theta \leqslant 180^\circ$.

$\therefore\ \sin\theta \geqslant \theta$

$\therefore\ \sin\theta = \sqrt{\tfrac{27}{28}}$

d $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\sin\theta$

$$\begin{aligned}\sin\theta &= \frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}|\,|\mathbf{b}|}\\ &= \frac{\left|\begin{pmatrix}1\\5\\1\end{pmatrix}\right|}{\sqrt{14}\sqrt{2}} \qquad \{\text{from parts } \mathbf{a} \text{ and } \mathbf{b}\}\\ &= \frac{\sqrt{1^2+5^2+1^2}}{\sqrt{28}}\\ &= \frac{\sqrt{1+25+1}}{\sqrt{28}}\\ &= \frac{\sqrt{27}}{\sqrt{28}}\end{aligned}$$

3 $(\Rightarrow)$ $\mathbf{a}\times\mathbf{b} = \mathbf{0}$

$$\therefore\ \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\times\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} = \mathbf{0}$$

$$\begin{aligned}\therefore\ a_2b_3 - a_3b_2 &= 0\\ a_1b_3 - a_3b_1 &= 0\\ a_1b_2 - a_2b_1 &= 0\end{aligned}$$

$$\therefore\ \frac{a_1b_3}{a_3} = b_1 \qquad \{\text{assuming that } a_3 \neq 0 \text{ since } \mathbf{a} \neq 0\}$$

$$\frac{a_2b_3}{a_3} = b_2$$

$$\frac{a_3b_3}{a_3} = b_3$$

$\therefore\ \mathbf{b} = \dfrac{b_3}{a_3}\mathbf{a}$ which implies $\mathbf{b}$ is parallel to $\mathbf{a}$.

$(\Leftarrow)$ If $\mathbf{a} \parallel \mathbf{b}$ then $\mathbf{b} = k\mathbf{a}$, $k \neq 0$, $k \in \mathbb{R}$

$$\begin{aligned}\therefore\ \mathbf{a}\times\mathbf{b} &= \mathbf{a}\times k\mathbf{a}\\ &= k(\mathbf{a}\times\mathbf{a})\\ &= k\times\mathbf{0}\\ &= \mathbf{0}\end{aligned}$$

4 O(0, 0, 0), A(2, 3, -1), B(-1, 1, 2)

a **i** $\overrightarrow{OA} = \begin{pmatrix}2-0\\3-0\\-1-0\end{pmatrix} = \begin{pmatrix}2\\3\\-1\end{pmatrix}$ $\qquad \overrightarrow{OB} = \begin{pmatrix}-1-0\\1-0\\2-0\end{pmatrix} = \begin{pmatrix}-1\\1\\2\end{pmatrix}$

ii $\overrightarrow{OA} \times \overrightarrow{OB} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -1 \\ -1 & 1 & 2 \end{vmatrix}$

$$= \begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} \mathbf{k}$$

$$= (6+1)\mathbf{i} - (4-1)\mathbf{j} + (2+3)\mathbf{k}$$

$$= 7\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$$

$$= \begin{pmatrix} 7 \\ -3 \\ 5 \end{pmatrix}$$

iii $|\, \overrightarrow{OA} \times \overrightarrow{OB}\, | = \sqrt{7^2 + (-3)^2 + 5^2}$

$$= \sqrt{49+9+25}$$

$$= \sqrt{83}$$

b Area of $\triangle OAB = \frac{1}{2}\, |\, \overrightarrow{OA}\, ||\, \overrightarrow{OB}\, |\, \sin\theta$

$$= \frac{1}{2}\, |\, \overrightarrow{OA} \times \overrightarrow{OB}\, |$$

$$= \frac{\sqrt{83}}{2} \text{ units}^2$$

5 **a** $\mathbf{a} \times \mathbf{c} = \mathbf{b} \times \mathbf{c}$

$\therefore\ \mathbf{0} = \mathbf{b} \times \mathbf{c} - \mathbf{a} \times \mathbf{c}$

$\therefore\ \mathbf{0} = (\mathbf{b} - \mathbf{a}) \times \mathbf{c}$

$\therefore\ \overrightarrow{OC}$ is parallel to $\overrightarrow{AB}$.

b $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$

$\therefore\ \mathbf{b} \times (\mathbf{a} + \mathbf{b} + \mathbf{c}) = \mathbf{0}$

$\therefore\ \mathbf{b} \times \mathbf{a} + \cancel{\mathbf{b} \times \mathbf{b}} + \mathbf{b} \times \mathbf{c} = \mathbf{0}$ $\quad \{\text{since } \mathbf{b} \times \mathbf{b} = \mathbf{0}\}$

$\therefore\ -\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} = \mathbf{0}$ $\quad \{\text{since } \mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}\}$

$\therefore\ \mathbf{b} \times \mathbf{c} = \mathbf{a} \times \mathbf{b}$

c $\mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a}, \quad \mathbf{c} \neq \mathbf{0}$

$\therefore\ \mathbf{b} \times \mathbf{c} - \mathbf{c} \times \mathbf{a} = \mathbf{0}$

$\therefore\ \mathbf{b} \times \mathbf{c} + \mathbf{a} \times \mathbf{c} = \mathbf{0}$ $\quad \{\text{since } -\mathbf{c} \times \mathbf{a} = \mathbf{a} \times \mathbf{c}\}$

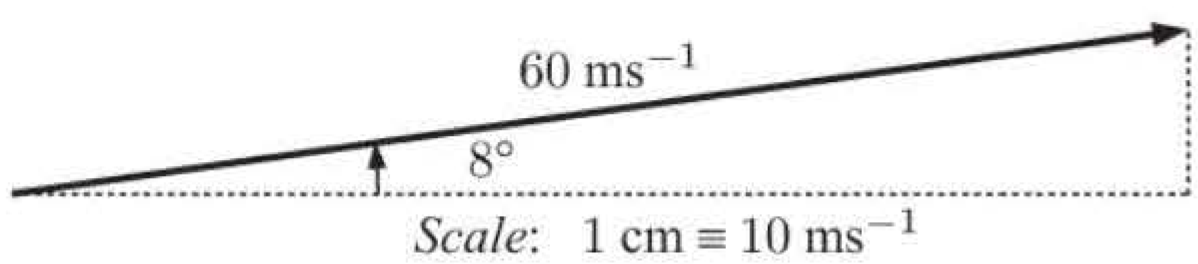
$\therefore\ (\mathbf{b} + \mathbf{a}) \times \mathbf{c} = \mathbf{0}$

$\therefore$ since $\mathbf{c} \neq \mathbf{0}$, $\mathbf{b} + \mathbf{a}$ and $\mathbf{c}$ must be parallel.

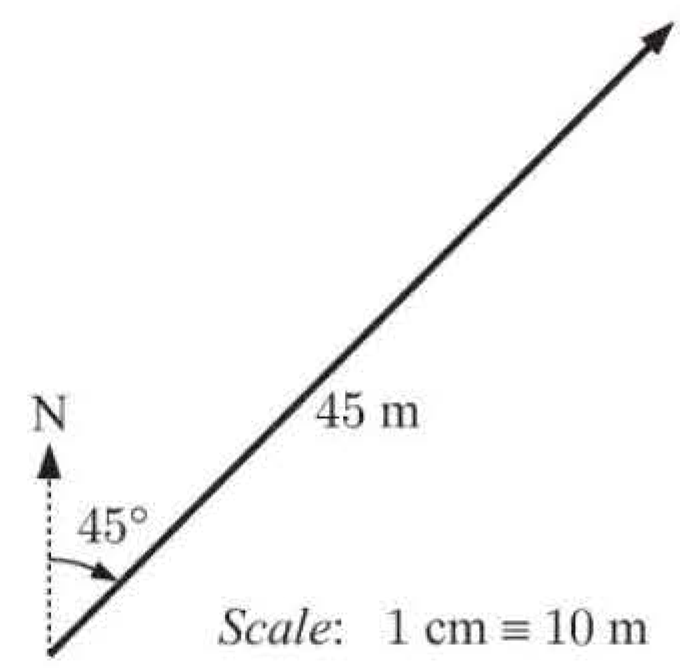
$\therefore\ \mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a} = k\mathbf{c}, \ k \neq 0, \ k \in \mathbb{R}$.

REVIEW SET 14A

1 **a**



b



2 **a** $\overrightarrow{AB} - \overrightarrow{CB} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

b $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{DC} = \overrightarrow{AC} + \overrightarrow{CD} = \overrightarrow{AD}$

3 **a** $\mathbf{q} = \mathbf{p} + \mathbf{r}$

b $\mathbf{l} = \mathbf{k} - \mathbf{j} + \mathbf{n} - \mathbf{m}$

4 $\overrightarrow{SP} = \overrightarrow{SR} + \overrightarrow{RQ} + \overrightarrow{QP}$
$= -\overrightarrow{RS} + \overrightarrow{RQ} - \overrightarrow{PQ}$
$= -\begin{pmatrix} 2 \\ -3 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \end{pmatrix} - \begin{pmatrix} -4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

5 **a** $\overrightarrow{BC} = 2\overrightarrow{OA} = 2\mathbf{p}$

Now $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OB} + \overrightarrow{BC}$
$= -\mathbf{p} + \mathbf{q} + 2\mathbf{p}$
$= \mathbf{p} + \mathbf{q}$

b $\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$
$= \mathbf{p} + \frac{1}{2}\overrightarrow{AC}$
$= \mathbf{p} + \frac{1}{2}(\mathbf{p} + \mathbf{q})$
$= \frac{3}{2}\mathbf{p} + \frac{1}{2}\mathbf{q}$

6 The vectors are parallel, so $\begin{pmatrix} -12 \\ -20 \\ 2 \end{pmatrix} = k\begin{pmatrix} 3 \\ m \\ n \end{pmatrix}$ $\quad \therefore\ 3k = -12,\quad km = -20,\quad kn = 2$
$\therefore\ k = -4,\quad m = 5,\quad n = -\frac{1}{2}$

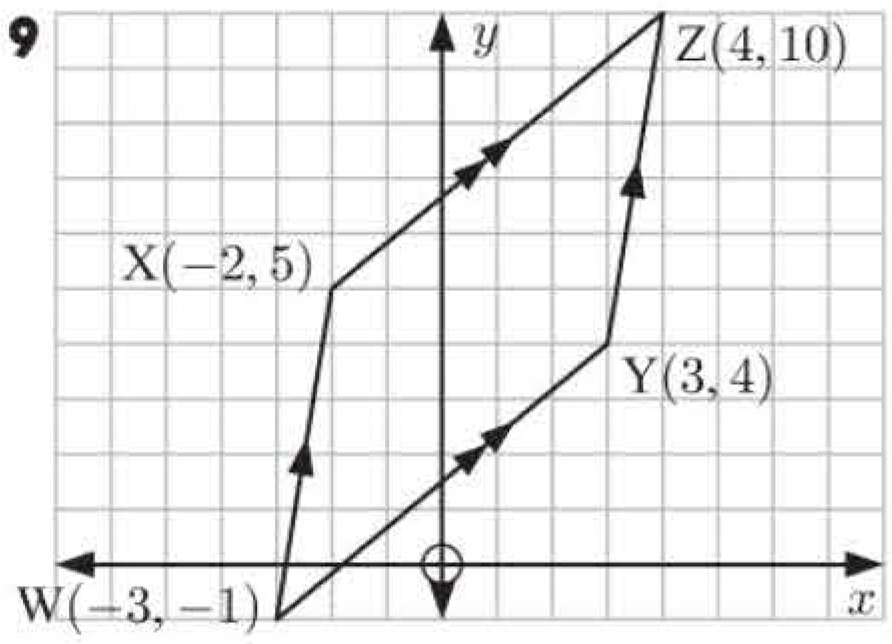
7 $\overrightarrow{CB} = \overrightarrow{CA} + \overrightarrow{AB} = -\overrightarrow{AC} + \overrightarrow{AB} = \begin{pmatrix} 6 \\ -1 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ -7 \\ 4 \end{pmatrix} = \begin{pmatrix} 8 \\ -8 \\ 7 \end{pmatrix}$

8 **a** $\mathbf{p} \bullet \mathbf{q} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 5 \end{pmatrix}$
$= -3 + (-10)$
$= -13$

b $\mathbf{p} - \mathbf{r} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} - \begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \end{pmatrix}$

$\therefore\ \mathbf{q} \bullet (\mathbf{p} - \mathbf{r}) = \begin{pmatrix} -1 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 6 \\ -6 \end{pmatrix} = -6 - 30$
$= -36$

9



$\overrightarrow{WY} = \begin{pmatrix} 3 - -3 \\ 4 - -1 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$

$\overrightarrow{XZ} = \begin{pmatrix} 4 - -2 \\ 10 - 5 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$

So, $\overrightarrow{WY} = \overrightarrow{XZ}$
$\therefore$ [WY] is parallel to [XZ] and they are equal in length. This is sufficient to deduce that WYZX is a parallelogram.

10 $\overrightarrow{AB} = \begin{pmatrix} -1 - 2 \\ 4 - 3 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

$\overrightarrow{AC} = \begin{pmatrix} 3 - 2 \\ k - 3 \end{pmatrix} = \begin{pmatrix} 1 \\ k - 3 \end{pmatrix}$

Now $\overrightarrow{AB} \bullet \overrightarrow{AC} = 0$ $\{$as $B\widehat{A}C = 90°\}$

$\therefore\ \begin{pmatrix} -3 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ k - 3 \end{pmatrix} = 0$
$\therefore\ -3 + k - 3 = 0$
$\therefore\ k = 6$

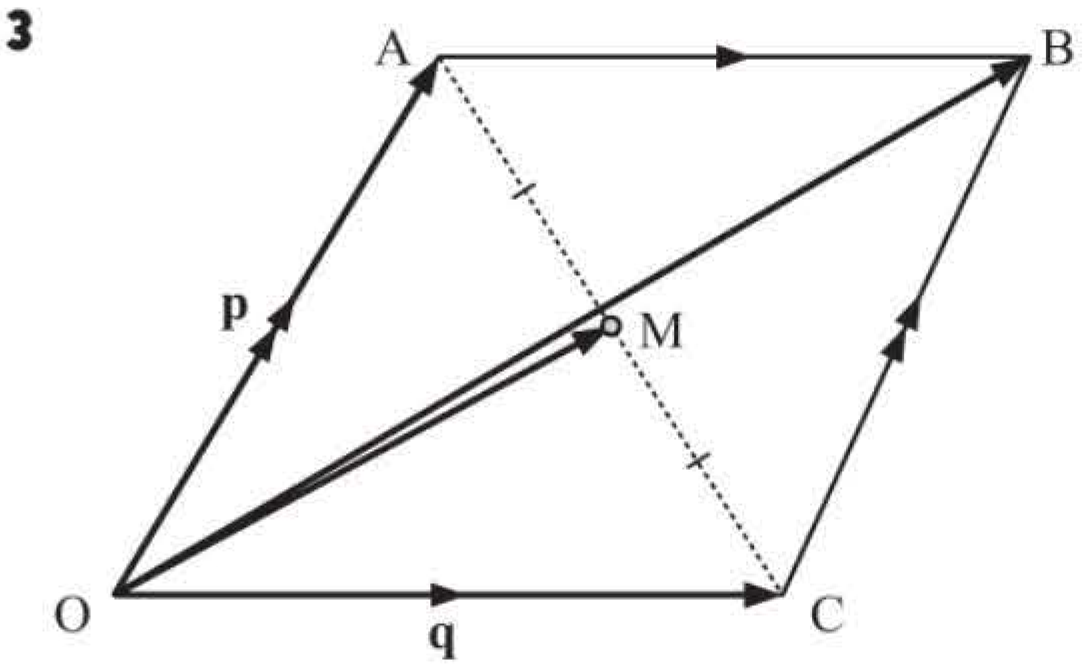
11 **a** $\mathbf{a} \bullet \mathbf{b}$ is a scalar, so $\mathbf{a} \bullet \mathbf{b} \bullet \mathbf{c}$ is a scalar dotted with a vector, which is meaningless.

b $\mathbf{b} \times \mathbf{c}$ must be done first otherwise we have the cross product of a scalar with a vector, which is meaningless.

12 One vector perpendicular to $\begin{pmatrix} -4 \\ 5 \end{pmatrix}$ is $\begin{pmatrix} 5 \\ 4 \end{pmatrix}$ as the dot product $= -20 + 20 = 0$

$\therefore$ all vectors have the form $t\begin{pmatrix} 5 \\ 4 \end{pmatrix}$, $t \neq 0$.

13



a **i** $\overrightarrow{\mathrm{OB}}$
$= \overrightarrow{\mathrm{OA}} + \overrightarrow{\mathrm{AB}}$
$= \overrightarrow{\mathrm{OA}} + \overrightarrow{\mathrm{OC}}$
$= \mathbf{p} + \mathbf{q}$

ii $\overrightarrow{\mathrm{OM}}$
$= \overrightarrow{\mathrm{OA}} + \overrightarrow{\mathrm{AM}}$
$= \overrightarrow{\mathrm{OA}} + \frac{1}{2}\overrightarrow{\mathrm{AC}}$
$= \mathbf{p} + \frac{1}{2}(\overrightarrow{\mathrm{AO}} + \overrightarrow{\mathrm{OC}})$
$= \mathbf{p} + \frac{1}{2}(-\mathbf{p} + \mathbf{q})$
$= \mathbf{p} - \frac{1}{2}\mathbf{p} + \frac{1}{2}\mathbf{q}$
$= \frac{1}{2}\mathbf{p} + \frac{1}{2}\mathbf{q}$

b We notice that $\overrightarrow{\mathrm{OM}} = \frac{1}{2}\overrightarrow{\mathrm{OB}}$

$\therefore$ [OM] $\parallel$ [OB] and $\mathrm{OM} = \frac{1}{2}\mathrm{OB}$

So, O, M, and B are collinear (as O is common) and hence M is the midpoint of [OB].

14 **a**

$$\begin{aligned} 1^2 &= \left(\tfrac{4}{7}\right)^2 + \left(\frac{1}{k}\right)^2 \\ &= \tfrac{16}{49} + \frac{1}{k^2} \\ \therefore\ \frac{1}{k^2} &= 1 - \tfrac{16}{49} \\ &= \tfrac{33}{49} \\ \therefore\ k^2 &= \tfrac{49}{33} \\ \therefore\ k &= \pm\sqrt{\tfrac{49}{33}} \\ &= \pm\tfrac{7}{\sqrt{33}} \end{aligned}$$

b

$$\begin{aligned} 1^2 &= k^2 + k^2 \\ \therefore\ 1 &= 2k^2 \\ \therefore\ k^2 &= \tfrac{1}{2} \\ \therefore\ k &= \pm\sqrt{\tfrac{1}{2}} \\ &= \pm\tfrac{1}{\sqrt{2}} \end{aligned}$$

15 **a**

$$\begin{aligned} \mathbf{a} \bullet \mathbf{b} &= |\,\mathbf{a}\,|\,|\,\mathbf{b}\,|\cos\theta \\ &= 2 \times 4 \times \cos 120^\circ \\ &= 8(-\tfrac{1}{2}) \\ &= -4 \end{aligned}$$

b

$$\begin{aligned} \mathbf{b} \bullet \mathbf{c} &= |\,\mathbf{b}\,|\,|\,\mathbf{c}\,|\cos\theta \\ &= 4 \times 5 \times \cos 60^\circ \\ &= 20(\tfrac{1}{2}) \\ &= 10 \end{aligned}$$

c

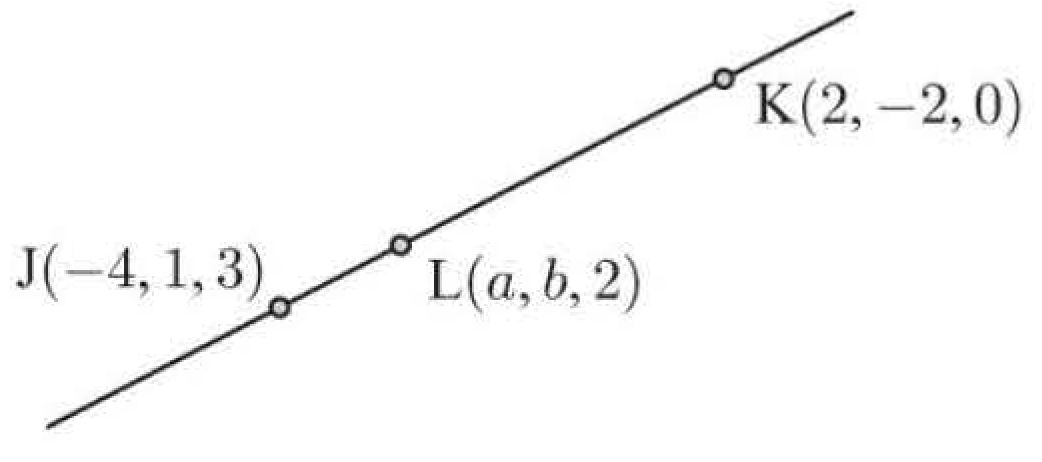
$$\begin{aligned} \mathbf{a} \bullet \mathbf{c} &= |\,\mathbf{a}\,|\,|\,\mathbf{c}\,|\cos\theta \\ &= 2 \times 5 \times \cos 180^\circ \\ &= 10(-1) \\ &= -10 \end{aligned}$$

16

$$\overrightarrow{\mathrm{JK}} = \begin{pmatrix} 2-(-4) \\ -2-1 \\ 0-3 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ -3 \end{pmatrix}$$

$$\overrightarrow{\mathrm{JL}} = \begin{pmatrix} a-(-4) \\ b-1 \\ 2-3 \end{pmatrix} = \begin{pmatrix} a+4 \\ b-1 \\ -1 \end{pmatrix}$$

If J, K, and L are collinear then $\overrightarrow{\mathrm{JK}} \parallel \overrightarrow{\mathrm{JL}}$



$\therefore \begin{pmatrix} 6 \\ -3 \\ -3 \end{pmatrix} = k\begin{pmatrix} a+4 \\ b-1 \\ -1 \end{pmatrix}$ for some $k \neq 0$, $k \in \mathbb{R}$

$\therefore\ -3 = k(-1)$

$\therefore\ k = 3$

$\therefore\ -3 = k(b-1)$ and $6 = k(a+4)$

$\therefore\ -3 = 3(b-1)$ and $6 = 3(a+4)$

$\therefore\ -1 = b-1$ and $2 = a+4$

$\therefore\ b = 0$ and $a = -2$

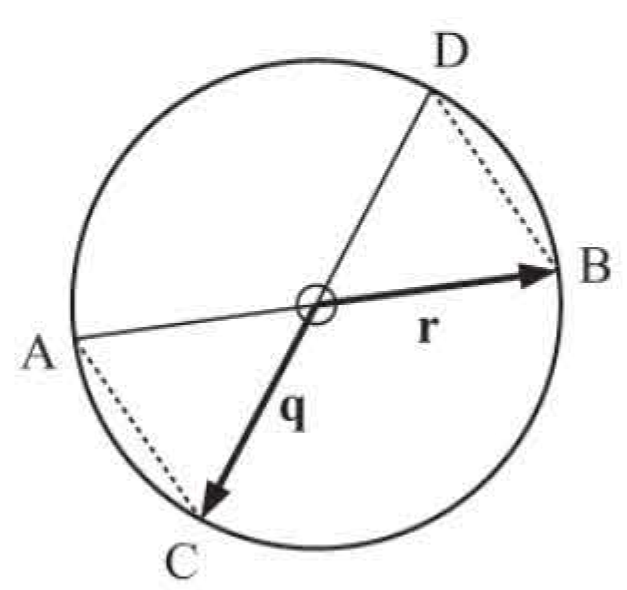
17 $$|\mathbf{u}\times\mathbf{v}| = \sqrt{1^2+(-3)^2+(-4)^2} = \sqrt{1+9+16} = \sqrt{26}$$

$$\sin\theta = \frac{|\mathbf{u}\times\mathbf{v}|}{|\mathbf{u}|\,|\mathbf{v}|} = \frac{\sqrt{26}}{3\times 5} = \tfrac{\sqrt{26}}{15}$$

$$\sin^2\theta+\cos^2\theta = 1$$
$$\cos\theta = \pm\sqrt{1-\sin^2\theta} = \pm\sqrt{1-\tfrac{26}{225}} = \pm\sqrt{\tfrac{199}{225}} = \pm\tfrac{\sqrt{199}}{15}$$
$$\mathbf{u}\bullet\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos\theta = 3\times 5\times\left(\pm\tfrac{\sqrt{199}}{15}\right) = \pm\sqrt{199}$$

So, if θ is acute, $\mathbf{u}\bullet\mathbf{v} = \sqrt{199}$
and if θ is obtuse, $\mathbf{u}\bullet\mathbf{v} = -\sqrt{199}$

18



a **i** $$\overrightarrow{DB} = \overrightarrow{DO}+\overrightarrow{OB} = \overrightarrow{OC}+\overrightarrow{OB} = \mathbf{q}+\mathbf{r}$$

ii $$\overrightarrow{AC} = \overrightarrow{AO}+\overrightarrow{OC} = \overrightarrow{OB}+\overrightarrow{OC} = \mathbf{r}+\mathbf{q}$$

b We see that $\overrightarrow{DB} = \overrightarrow{AC}$
$\therefore$ [DB] is parallel to [AC] and equal in length.

19 **a** $$\begin{pmatrix}2-t\\3\\t\end{pmatrix}\bullet\begin{pmatrix}t\\4\\t+1\end{pmatrix} = 0$$
$$\therefore\ (2-t)t+12+t(t+1) = 0$$
$$\therefore\ 2t-\cancel{t^2}+12+\cancel{t^2}+t = 0$$
$$\therefore\ 3t+12 = 0$$
$$\therefore\ t = -4$$

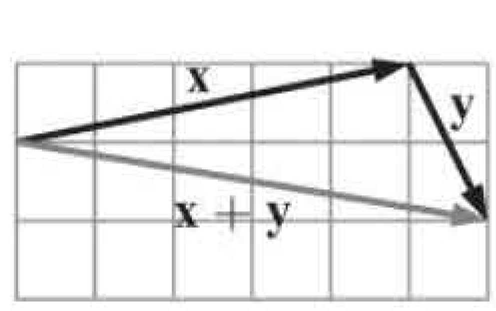
b $$\overrightarrow{KL} = \begin{pmatrix}-3-4\\4-3\\2-(-1)\end{pmatrix} = \begin{pmatrix}-7\\1\\3\end{pmatrix},\qquad \overrightarrow{LM} = \begin{pmatrix}2-(-3)\\1-4\\-2-2\end{pmatrix} = \begin{pmatrix}5\\-3\\-4\end{pmatrix},$$
$$\overrightarrow{MK} = \begin{pmatrix}2-4\\1-3\\-2-(-1)\end{pmatrix} = \begin{pmatrix}-2\\-2\\-1\end{pmatrix}$$

Now $$\overrightarrow{LM}\bullet\overrightarrow{MK} = \begin{pmatrix}5\\-3\\-4\end{pmatrix}\bullet\begin{pmatrix}-2\\-2\\-1\end{pmatrix} = 5\times(-2)+(-3)\times(-2)+(-4)\times(-1) = -10+6+4 = 0$$

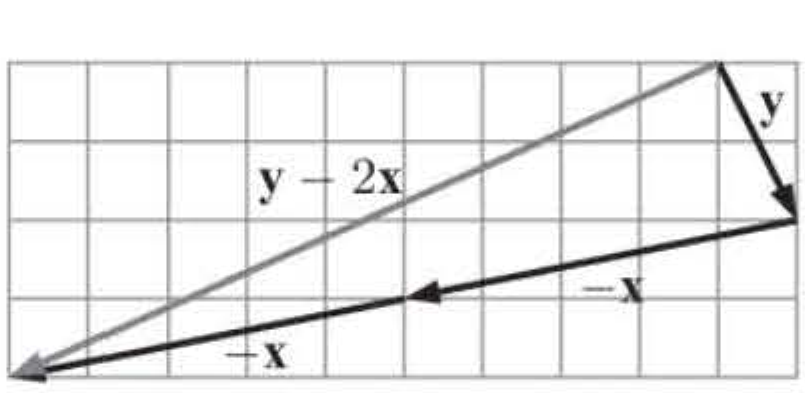
($\overrightarrow{KL}\bullet\overrightarrow{LM} = -50$, and $\overrightarrow{MK}\bullet\overrightarrow{KL} = 9$)
$\therefore$ [LM] and [MK] are perpendicular.
$\therefore$ $\triangle$KLM is right angled at M.

REVIEW SET 14B

1 **a**



b



2 $\overrightarrow{AB} = \begin{pmatrix} 4-(-2) \\ 0-(-1) \\ -1-3 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ -4 \end{pmatrix}$, $\quad \overrightarrow{AC} = \begin{pmatrix} -2-(-2) \\ 1-(-1) \\ -4-3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -7 \end{pmatrix}$,

$\overrightarrow{BC} = \begin{pmatrix} -2-4 \\ 1-0 \\ -4-(-1) \end{pmatrix} = \begin{pmatrix} -6 \\ 1 \\ -3 \end{pmatrix}$

$\therefore$ $AB = \sqrt{6^2+1^2+(-4)^2}$ $\quad AC = \sqrt{0^2+2^2+(-7)^2}$ $\quad BC = \sqrt{(-6)^2+1^2+(-3)^2}$

$= \sqrt{53}$ units $\quad = \sqrt{53}$ units $\quad = \sqrt{46}$ units

$\therefore$ AB = AC, so ABC is an isosceles triangle.

3 **a** $|\mathbf{s}| = \sqrt{(-3)^2+2^2} = \sqrt{13}$ units

b $\mathbf{r}+\mathbf{s} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$\therefore$ $|\mathbf{r}+\mathbf{s}| = \sqrt{1^2+3^2} = \sqrt{10}$ units

c $2\mathbf{s}-\mathbf{r} = 2\begin{pmatrix} -3 \\ 2 \end{pmatrix} - \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -10 \\ 3 \end{pmatrix}$

$\therefore$ $|2\mathbf{s}-\mathbf{r}| = \sqrt{(-10)^2+3^2} = \sqrt{109}$ units

4 $r\begin{pmatrix} -2 \\ 1 \end{pmatrix} + s\begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 13 \\ -24 \end{pmatrix}$

$\therefore$ $\begin{pmatrix} -2r+3s \\ r-4s \end{pmatrix} = \begin{pmatrix} 13 \\ -24 \end{pmatrix}$

$\therefore$ $-2r+3s = 13$

$r-4s = -24$ .... (1)

$\therefore$ $-2r+3s = 13$

$2r-8s = -48$ $\quad \{2\times(1)\}$

adding $-5s = -35$

$\therefore$ $s = 7$

Now using (1), $r-4(7) = -24$

$\therefore$ $r = -24+28$

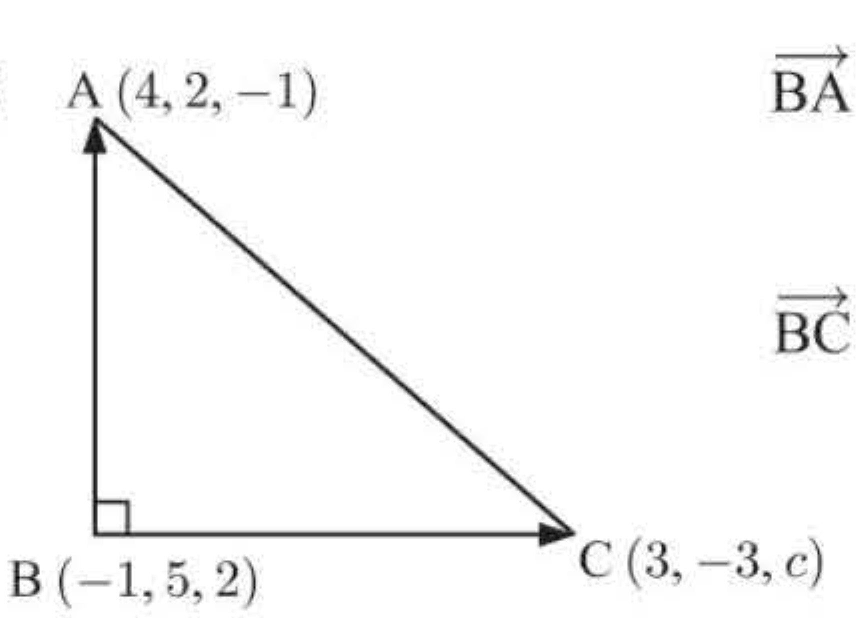
$\therefore$ $r = 4$ and $s = 7$

5 **a** $\overrightarrow{PQ} = \begin{pmatrix} -4-2 \\ 4-3 \\ 2--1 \end{pmatrix} = \begin{pmatrix} -6 \\ 1 \\ 3 \end{pmatrix}$

b $PQ = |\overrightarrow{PQ}| = \sqrt{36+1+9} = \sqrt{46}$ units

c The midpoint is at $\left(\dfrac{2+(-4)}{2}, \dfrac{3+4}{2}, \dfrac{-1+2}{2}\right)$ which is $(-1, \frac{7}{2}, \frac{1}{2})$ or $(-1, 3\frac{1}{2}, \frac{1}{2})$.

6



$\overrightarrow{BA} = \begin{pmatrix} 4--1 \\ 2-5 \\ -1-2 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \\ -3 \end{pmatrix}$

$\overrightarrow{BC} = \begin{pmatrix} 3--1 \\ -3-5 \\ c-2 \end{pmatrix} = \begin{pmatrix} 4 \\ -8 \\ c-2 \end{pmatrix}$

But $\overrightarrow{BA} \bullet \overrightarrow{BC} = 0$

$\therefore$ $20+24-3(c-2) = 0$

$\therefore$ $44 = 3(c-2)$

$\therefore$ $3c-6 = 44$

$\therefore$ $3c = 50$

$\therefore$ $c = \frac{50}{3}$

7 $\mathbf{a}-3\mathbf{x} = \mathbf{b}$

$\therefore$ $\mathbf{a}-\mathbf{b} = 3\mathbf{x}$

$\therefore$ $\mathbf{x} = \frac{1}{3}(\mathbf{a}-\mathbf{b}) = \frac{1}{3}\begin{pmatrix} 2-(-1) \\ -3-2 \\ 1-3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 3 \\ -5 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{5}{3} \\ -\frac{2}{3} \end{pmatrix}$

8 If the angle is θ then using $\mathbf{a} \bullet \mathbf{b} = |\,\mathbf{a}\,|\,|\,\mathbf{b}\,|\cos\theta$,

$$\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} = \sqrt{9+1+4}\sqrt{4+25+1}\cos\theta$$

$$\therefore\ 6+5-2 = \sqrt{14}\sqrt{30}\cos\theta$$

$$\therefore\ \frac{9}{\sqrt{14 \times 30}} = \cos\theta$$

$$\therefore\ \theta \approx 64.0^\circ$$

9 Let $Q(0, 0, z)$ be a point on the Z-axis.

$$\overrightarrow{PQ} = \begin{pmatrix} 0-(-4) \\ 0-2 \\ z-5 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ z-5 \end{pmatrix}$$

$$PQ = \sqrt{4^2+(-2)^2+(z-5)^2} = 6$$

$$\therefore\ 16+4+(z-5)^2 = 36$$

$$\therefore\ (z-5)^2 = 16$$

$$\therefore\ z-5 = \pm 4$$

$$\therefore\ z = 1 \text{ or } 9$$

$\therefore$ Q is $(0, 0, 1)$ or $(0, 0, 9)$.

10 Since they are perpendicular,

$$\begin{pmatrix} 3 \\ 3-2t \end{pmatrix} \bullet \begin{pmatrix} t^2+t \\ -2 \end{pmatrix} = 0$$

$$\therefore\ 3(t^2+t) - 2(3-2t) = 0$$

$$\therefore\ 3t^2+3t-6+4t = 0$$

$$\therefore\ 3t^2+7t-6 = 0$$

$$\therefore\ (3t-2)(t+3) = 0$$

$$\therefore\ t = \tfrac{2}{3} \text{ or } -3$$

11

$$\overrightarrow{PQ} = \begin{pmatrix} 4-(-6) \\ 6-8 \\ 8-2 \end{pmatrix} = \begin{pmatrix} 10 \\ -2 \\ 6 \end{pmatrix} = 2\begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}$$

$$\overrightarrow{QR} = \begin{pmatrix} 19-4 \\ 3-6 \\ 17-8 \end{pmatrix} = \begin{pmatrix} 15 \\ -3 \\ 9 \end{pmatrix} = 3\begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}$$

So, $\overrightarrow{PQ} = \frac{2}{3}\overrightarrow{QR}$

$\therefore$ $\overrightarrow{QR}$ is a scalar multiple of $\overrightarrow{PQ}$.

$\therefore$ P, Q, and R are collinear.

12 **a**

$$\mathbf{u} \bullet \mathbf{v} = \begin{pmatrix} -4 \\ 2 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix}$$

$$= -4(-1)+2(3)+1(-2)$$

$$= 4+6-2$$

$$= 8$$

b If θ is the angle between $\mathbf{u}$ and $\mathbf{v}$ then

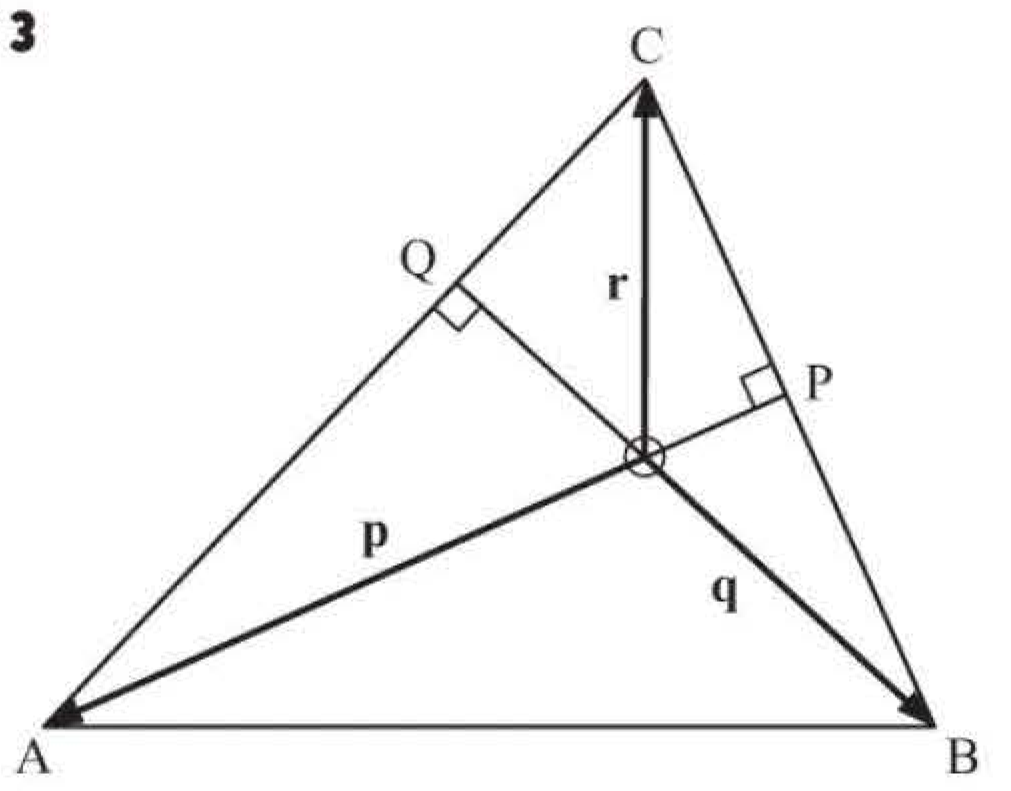
$$\cos\theta = \frac{\mathbf{u} \bullet \mathbf{v}}{|\,\mathbf{u}\,|\,|\,\mathbf{v}\,|}$$

$$= \frac{8}{\sqrt{(-4)^2+2^2+1^2}\sqrt{(-1)^2+3^2+(-2)^2}}$$

$$= \frac{8}{\sqrt{21}\sqrt{14}}$$

$$\therefore\ \theta \approx 62.2^\circ$$

13



a

$$\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC} = -\mathbf{p} + \mathbf{r} = \mathbf{r} - \mathbf{p}$$

$$\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC} = -\mathbf{q} + \mathbf{r} = \mathbf{r} - \mathbf{q}$$

b [AP] $\perp$ [BC] and [BQ] $\perp$ [AC]

$$\therefore\ \mathbf{p} \perp \mathbf{r}-\mathbf{q} \qquad \therefore\ \mathbf{q} \perp (\mathbf{r}-\mathbf{p})$$

$$\therefore\ \mathbf{p} \bullet (\mathbf{r}-\mathbf{q}) = 0 \qquad \therefore\ \mathbf{q} \bullet (\mathbf{r}-\mathbf{p}) = 0$$

$$\therefore\ \mathbf{p} \bullet \mathbf{r} - \mathbf{p} \bullet \mathbf{q} = 0 \qquad \therefore\ \mathbf{q} \bullet \mathbf{r} - \mathbf{q} \bullet \mathbf{p} = 0$$

$$\therefore\ \mathbf{p} \bullet \mathbf{r} = \mathbf{p} \bullet \mathbf{q} \qquad \therefore\ \mathbf{q} \bullet \mathbf{r} = \mathbf{p} \bullet \mathbf{q}$$

$$\therefore\ \mathbf{q} \bullet \mathbf{r} = \mathbf{p} \bullet \mathbf{q} = \mathbf{p} \bullet \mathbf{r}$$

c

$$\mathbf{r} \bullet \overrightarrow{AB} = \mathbf{r} \bullet (-\mathbf{p}+\mathbf{q})$$

$$= -\mathbf{r} \bullet \mathbf{p} + \mathbf{r} \bullet \mathbf{q}$$

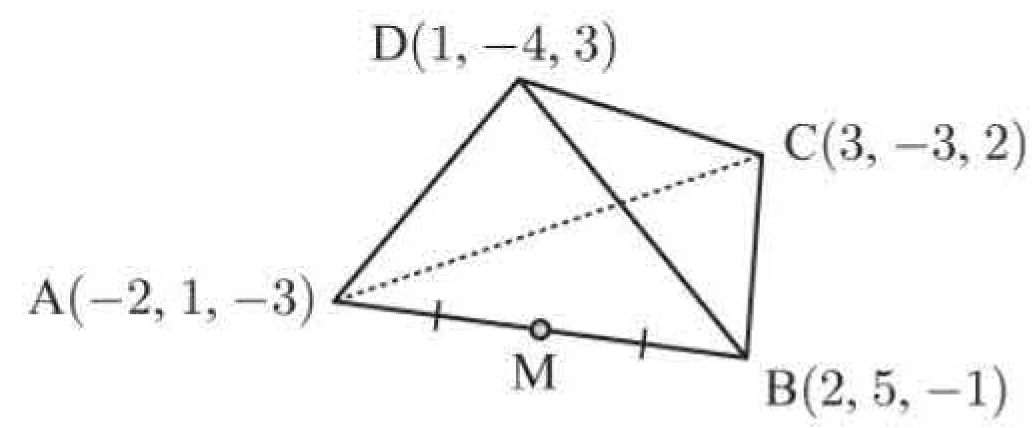
$$= -\mathbf{p} \bullet \mathbf{q} + \mathbf{p} \bullet \mathbf{q} \qquad \{\text{from } \mathbf{b}\}$$

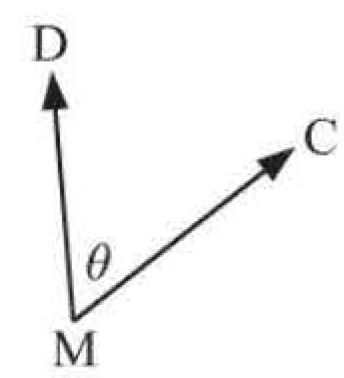
$= 0$ and so $\mathbf{r} \perp \overrightarrow{AB}$ $\therefore$ [OC] $\perp$ [AB]

14 $|\,3\mathbf{i}-2\mathbf{j}+\mathbf{k}\,| = \sqrt{3^2+(-2)^2+1^2} = \sqrt{14}$

$\therefore$ a unit vector in the direction $3\mathbf{i}-2\mathbf{j}+\mathbf{k}$ is $\frac{1}{\sqrt{14}}(3\mathbf{i}-2\mathbf{j}+\mathbf{k})$

$\therefore$ two vectors 4 units long and parallel to $3\mathbf{i}-2\mathbf{j}+\mathbf{k}$ are $\pm\frac{4}{\sqrt{14}}(3\mathbf{i}-2\mathbf{j}+\mathbf{k})$.





15 M is $\left(\frac{-2+2}{2}, \frac{1+5}{2}, \frac{-3-1}{2}\right)$ or $(0, 3, -2)$.

$$\therefore \ \overrightarrow{MD} = \begin{pmatrix} 1-0 \\ -4-3 \\ 3-(-2) \end{pmatrix} = \begin{pmatrix} 1 \\ -7 \\ 5 \end{pmatrix},$$

$$\overrightarrow{MC} = \begin{pmatrix} 3-0 \\ -3-3 \\ 2-(-2) \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \\ 4 \end{pmatrix}$$

$$\therefore \ \overrightarrow{MD} \bullet \overrightarrow{MC} = |\overrightarrow{MD}||\overrightarrow{MC}|\cos\theta$$

$$\therefore \ 3+42+20 = \sqrt{1+49+25}\sqrt{9+36+16}\cos\theta$$

$$\therefore \ 65 = \sqrt{75}\sqrt{61}\cos\theta$$

$$\therefore \ \theta \approx 16.1^\circ$$

16 $$\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & -2 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & -2 \\ 1 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & -2 \\ 1 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}\mathbf{k}$$

$$= (-2+2)\mathbf{i} - (4+2)\mathbf{j} + (2+1)\mathbf{k}$$

$$= -6\mathbf{j} + 3\mathbf{k}$$

$\therefore$ perpendicular vectors have the form $k\begin{pmatrix} 0 \\ -6 \\ 3 \end{pmatrix}$, $k \neq 0$, $k \in \mathbb{R}$.

17 **a** $$\sqrt{k^2 + (\tfrac{1}{\sqrt{2}})^2 + (-k)^2} = 1$$

$$\therefore \ k^2 + \tfrac{1}{2} + k^2 = 1$$

$$\therefore \ 2k^2 = \tfrac{1}{2}$$

$$\therefore \ k^2 = \tfrac{1}{4}$$

$$\therefore \ k = \pm\tfrac{1}{2}$$

b $\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ has length $\sqrt{3^2+2^2+(-1)^2} = \sqrt{14}$ units

$\therefore$ a unit vector in the opposite direction is $-\frac{1}{\sqrt{14}}\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$

$\therefore$ a vector of length 5 units in the opposite direction is $-\frac{5}{\sqrt{14}}\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$.

18 Using $\theta = \cos^{-1}\left(\frac{\mathbf{u} \bullet \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right)$,

$$\mathbf{u} \bullet \mathbf{v} = \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} = -2-4+9 = 3$$

$$|\mathbf{u}| = \sqrt{2^2+(-4)^2+3^2} = \sqrt{4+16+9} = \sqrt{29}$$

$$|\mathbf{v}| = \sqrt{(-1)^2+1^2+3^2} = \sqrt{1+1+9} = \sqrt{11}$$

$$\therefore \ \theta = \cos^{-1}\left(\frac{3}{\sqrt{29}\sqrt{11}}\right) \approx 80.3^\circ$$

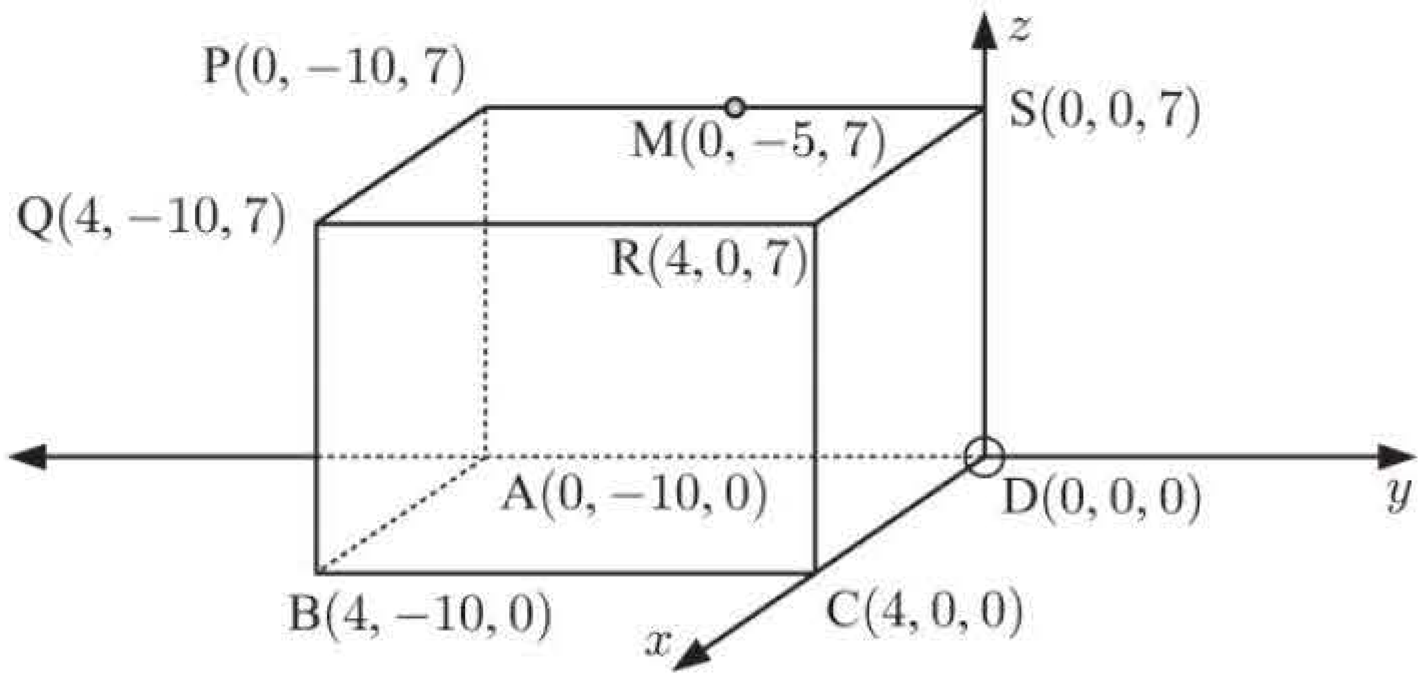
or using $\theta = \sin^{-1}\left(\frac{|\mathbf{u} \times \mathbf{v}|}{|\mathbf{u}||\mathbf{v}|}\right)$,

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & 3 \\ -1 & 1 & 3 \end{vmatrix} = -15\mathbf{i} - 9\mathbf{j} - 2\mathbf{k}$$

$$|\mathbf{u} \times \mathbf{v}| = \sqrt{(-15)^2+(-9)^2+(-2)^2} = \sqrt{310}$$

$$\therefore \ \theta = \sin^{-1}\left(\frac{\sqrt{310}}{\sqrt{29}\sqrt{11}}\right) \approx 80.3^\circ$$

19 Let D be the origin.



$$\overrightarrow{DM} = \begin{pmatrix} 0 \\ -5 \\ 7 \end{pmatrix}, \qquad \overrightarrow{DQ} = \begin{pmatrix} 4 \\ -10 \\ 7 \end{pmatrix}$$

$$\begin{aligned} DM \bullet DQ &= \begin{pmatrix} 0 \\ -5 \\ 7 \end{pmatrix} \bullet \begin{pmatrix} 4 \\ -10 \\ 7 \end{pmatrix} \\ &= 0 + 50 + 49 \\ &= 99 \end{aligned}$$

$$\begin{aligned} |\,DM\,| &= \sqrt{0^2 + (-5)^2 + 7^2} \\ &= \sqrt{25 + 49} \\ &= \sqrt{74} \end{aligned} \qquad \begin{aligned} |\,DQ\,| &= \sqrt{4^2 + (-10)^2 + 7^2} \\ &= \sqrt{16 + 100 + 49} \\ &= \sqrt{165} \end{aligned}$$

$$\begin{aligned} D\widehat{Q}M &= \cos^{-1}\left(\frac{DM \bullet DQ}{|\,DM\,||\,DQ\,|}\right) \\ &= \cos^{-1}\left(\frac{99}{\sqrt{74}\sqrt{165}}\right) \\ &\approx 26.4^\circ \end{aligned}$$

or

$$\begin{aligned} \overrightarrow{DM} \times \overrightarrow{DQ} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -5 & 7 \\ 4 & -10 & 7 \end{vmatrix} \\ &= 35\mathbf{i} + 28\mathbf{j} + 20\mathbf{k} \end{aligned}$$

$$\begin{aligned} |\,\overrightarrow{DM} \times \overrightarrow{DQ}\,| &= \sqrt{35^2 + 28^2 + 20^2} \\ &= \sqrt{2409} \end{aligned}$$

$$\begin{aligned} D\widehat{Q}M &= \sin^{-1}\left(\frac{|\,\overrightarrow{DM} \times \overrightarrow{DQ}\,|}{|\,DM\,||\,DQ\,|}\right) \\ &= \sin^{-1}\left(\frac{\sqrt{2409}}{\sqrt{74}\sqrt{165}}\right) \\ &\approx 26.4^\circ \end{aligned}$$

REVIEW SET 14C

1 **a** $\overrightarrow{PR} + \overrightarrow{RQ} = \overrightarrow{PQ}$

b $\overrightarrow{PS} + \overrightarrow{SQ} + \overrightarrow{QR} = \overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$

2 **a** $\mathbf{m} - \mathbf{n} + \mathbf{p} = \begin{pmatrix} 6 \\ -3 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 11 \end{pmatrix}$

b $2\mathbf{n} - 3\mathbf{p} = 2\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} - 3\begin{pmatrix} -1 \\ 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ -8 \end{pmatrix} - \begin{pmatrix} -3 \\ 9 \\ 18 \end{pmatrix} = \begin{pmatrix} 7 \\ -3 \\ -26 \end{pmatrix}$

c $\mathbf{m} + \mathbf{p} = \begin{pmatrix} 6 \\ -3 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 7 \end{pmatrix}$ $\quad \therefore \ |\,\mathbf{m} + \mathbf{p}\,| = \sqrt{25 + 0 + 49} = \sqrt{74}$ units

3 **a** If $\overrightarrow{AB} = \frac{1}{2}\overrightarrow{CD}$ then [AB] $\parallel$ [CD] and [AB] is half the length of [CD].

b If $\overrightarrow{AB} = 2\overrightarrow{AC}$ then [AB] $\parallel$ [AC] and AB $= 2$AC

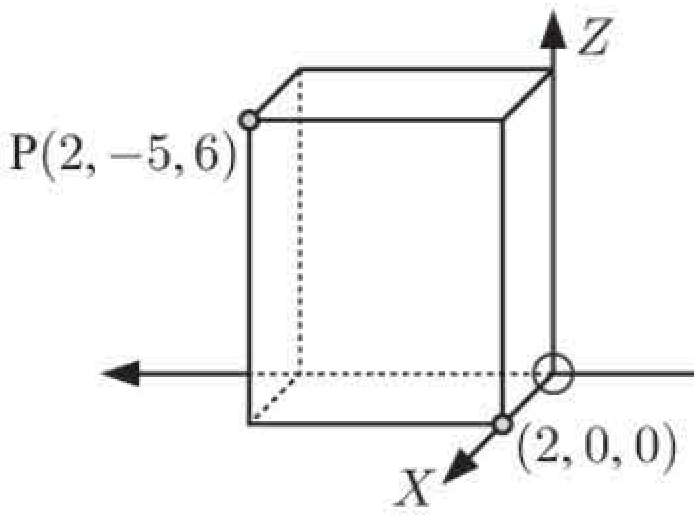
$\therefore$ A, B, and C are collinear and AB $= 2$AC.

So, C is the midpoint of [AB].

4 **a** $\overrightarrow{PQ} = \begin{pmatrix} -1-2 \\ 7--5 \\ 9-6 \end{pmatrix} = \begin{pmatrix} -3 \\ 12 \\ 3 \end{pmatrix}$

b $PQ = \sqrt{(-3)^2 + 12^2 + 3^2}$
$= \sqrt{9+144+9}$
$= \sqrt{162}$ units

c



$\therefore$ the *shortest* distance from P to the x-axis
$= \sqrt{(2-2)^2 + (0--5)^2 + (0-6)^2}$
$= \sqrt{0+25+36}$
$= \sqrt{61}$ units

5 **a** $\overrightarrow{OQ} = \overrightarrow{OR} + \overrightarrow{RQ} = \mathbf{r} + \mathbf{q}$

b $\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OR} + \overrightarrow{RQ} = -\mathbf{p} + \mathbf{r} + \mathbf{q}$

c $\overrightarrow{ON} = \overrightarrow{OR} + \overrightarrow{RN} = \mathbf{r} + \frac{1}{2}\mathbf{q}$

d $\overrightarrow{MN} = \overrightarrow{MQ} + \overrightarrow{QN}$
$= \frac{1}{2}\overrightarrow{PQ} + \frac{1}{2}\overrightarrow{QR}$
$= \frac{1}{2}(-\mathbf{p} + \mathbf{r} + \mathbf{q}) + \frac{1}{2}(-\mathbf{q})$ {from part **b**}
$= -\frac{1}{2}\mathbf{p} + \frac{1}{2}\mathbf{r} + \frac{1}{2}\mathbf{q} - \frac{1}{2}\mathbf{q}$
$= -\frac{1}{2}\mathbf{p} + \frac{1}{2}\mathbf{r}$

6 **a** $\mathbf{p} - 3\mathbf{x} = \mathbf{0}$
$\therefore\ 3\mathbf{x} = \mathbf{p}$
$\therefore\ \mathbf{x} = \frac{1}{3}\mathbf{p}$
$= \frac{1}{3}\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$
$\therefore\ \mathbf{x} = \begin{pmatrix} -1 \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$

b $2\mathbf{q} - \mathbf{x} = \mathbf{r}$
$\therefore\ \mathbf{r} + \mathbf{x} = 2\mathbf{q}$
$\therefore\ \mathbf{x} = 2\mathbf{q} - \mathbf{r}$
$= 2\begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$
$\therefore\ \mathbf{x} = \begin{pmatrix} 4 \\ -8 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 4-3 \\ -8-2 \\ 2-0 \end{pmatrix} = \begin{pmatrix} 1 \\ -10 \\ 2 \end{pmatrix}$

7 Since $\mathbf{v}$ is parallel to $\mathbf{w}$, the angle θ between $\mathbf{v}$ and $\mathbf{w}$ is either $0°$ or $180°$.

Now, $\mathbf{v} \bullet \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\cos\theta$
$= 3 \times 2 \times \cos 0°$ or $3 \times 2 \times \cos 180°$
$= 6(1)$ or $6(-1)$
$= \pm 6$

8 Vectors parallel to $\mathbf{i} + r\mathbf{j} + 2\mathbf{k}$ have form $k\begin{pmatrix} 1 \\ r \\ 2 \end{pmatrix}$, $k \neq 0$, $k \in \mathbb{R}$. If these vectors are perpendicular to $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ then $k\begin{pmatrix} 1 \\ r \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} = 0$

$k(2 + 2r - 2) = 0$
$2kr = 0$
but $k \neq 0$ $\therefore\ r = 0$

length of $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \sqrt{1^2 + 0^2 + 2^2} = \sqrt{1+4}$
$= \sqrt{5}$

$\therefore$ the unit vector is $\frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ or $\begin{pmatrix} \frac{1}{\sqrt{5}} \\ 0 \\ \frac{2}{\sqrt{5}} \end{pmatrix}$

or $\frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{k}$.

9 As the vectors are perpendicular,

$\begin{pmatrix} -4 \\ t+2 \\ t \end{pmatrix} \bullet \begin{pmatrix} t \\ 1+t \\ -3 \end{pmatrix} = 0$
$\therefore\ -4t + (t+2)(1+t) - 3t = 0$
$\therefore\ -4t + t + t^2 + 2 + 2t - 3t = 0$
$\therefore\ t^2 - 4t + 2 = 0$
$\therefore\ t = \dfrac{4 \pm \sqrt{16 - 4(1)(2)}}{2}$
$\therefore\ t = \dfrac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2}$

10 $\overrightarrow{MK} = \begin{pmatrix} 3-4 \\ 1-1 \\ 4-3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ $\quad \overrightarrow{LK} = \begin{pmatrix} 3--2 \\ 1-1 \\ 4-3 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix}$

$\overrightarrow{ML} = \begin{pmatrix} -2-4 \\ 1-1 \\ 3-3 \end{pmatrix} = \begin{pmatrix} -6 \\ 0 \\ 0 \end{pmatrix}$ $\quad \overrightarrow{LM} = \begin{pmatrix} 4--2 \\ 1-1 \\ 3-3 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$

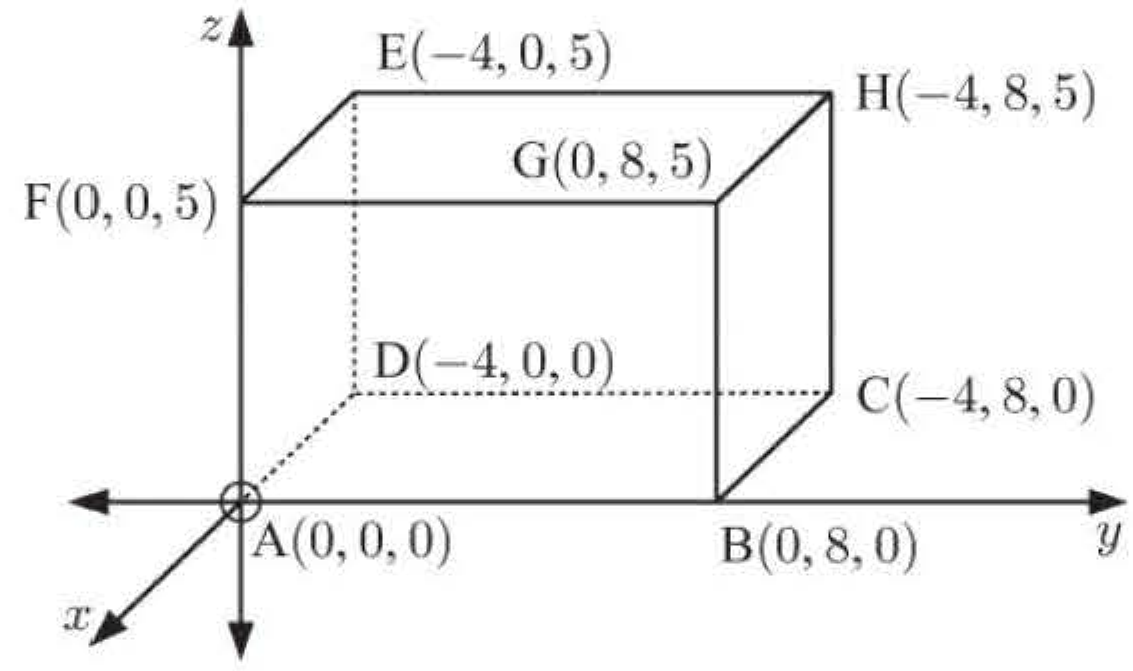
$$\overrightarrow{MK} \bullet \overrightarrow{ML} = |\overrightarrow{MK}||\overrightarrow{ML}| \cos\widehat{M}$$
$$\therefore\ 6+0+0 = \sqrt{1+0+1}\sqrt{36+0+0}\cos\widehat{M}$$
$$\therefore\ 6 = \sqrt{2}\times 6\cos\widehat{M}$$
$$\therefore\ \cos\widehat{M} = \tfrac{1}{\sqrt{2}}$$
$$\therefore\ \widehat{M} = 45^\circ$$
and $\widehat{K} \approx 180^\circ - 45^\circ - 11.3^\circ \approx 123.7^\circ$

$$\therefore\ \overrightarrow{LK} \bullet \overrightarrow{LM} = |\overrightarrow{LK}||\overrightarrow{LM}| \cos\widehat{L}$$
$$\therefore\ 30+0+0 = \sqrt{25+0+1}\sqrt{36+0+0}\cos\widehat{L}$$
$$\therefore\ 30 = \sqrt{26}\times 6\cos\widehat{L}$$
$$\therefore\ \tfrac{5}{\sqrt{26}} = \cos\widehat{L}$$
$$\therefore\ \widehat{L} \approx 11.3^\circ$$

11 **a** $\begin{pmatrix} \frac{5}{13} \\ k \end{pmatrix}$ is a unit vector if
$$\sqrt{\left(\tfrac{5}{13}\right)^2 + k^2} = 1$$
$$\therefore\ \tfrac{25}{169} + k^2 = 1$$
$$\therefore\ k^2 = \tfrac{144}{169}$$
$$\therefore\ k = \pm\tfrac{12}{13}$$

b $\begin{pmatrix} k \\ k \\ k \end{pmatrix}$ is a unit vector if
$$\sqrt{k^2+k^2+k^2} = 1$$
$$\therefore\ 3k^2 = 1$$
$$\therefore\ k^2 = \tfrac{1}{3}$$
$$\therefore\ k = \pm\tfrac{1}{\sqrt{3}}$$

12 Let A be the origin.



$\overrightarrow{AG} = \begin{pmatrix} 0 \\ 8 \\ 5 \end{pmatrix}$, $\quad \overrightarrow{AC} = \begin{pmatrix} -4 \\ 8 \\ 0 \end{pmatrix}$

$$AG \bullet AC = \begin{pmatrix} 0 \\ 8 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} -4 \\ 8 \\ 0 \end{pmatrix}$$
$$= 0\times(-4) + 8\times 8 + 5\times 0$$
$$= 64$$

$$|AG| = \sqrt{0^2+8^2+5^2} = \sqrt{64+25} = \sqrt{89}$$
$$|AC| = \sqrt{(-4)^2+8^2+0^2} = \sqrt{16+64} = \sqrt{80}$$

$$G\widehat{A}C = \cos^{-1}\left(\frac{AG \bullet AC}{|AG||AC|}\right)$$
$$= \cos^{-1}\left(\frac{64}{\sqrt{89}\sqrt{80}}\right)$$
$$\approx 40.7^\circ$$

or $\overrightarrow{AG} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 8 & 5 \\ -4 & 8 & 0 \end{vmatrix}$
$$= 40\mathbf{i} + 20\mathbf{j} + 32\mathbf{k}$$

$$|\overrightarrow{AG} \times \overrightarrow{AC}| = \sqrt{40^2+20^2+32^2} = \sqrt{1600+400+1024} = \sqrt{3024}$$

$$G\widehat{A}C = \sin^{-1}\left(\frac{|\overrightarrow{AG} \times \overrightarrow{AC}|}{|AG||AC|}\right)$$
$$= \sin^{-1}\left(\frac{\sqrt{3024}}{\sqrt{89}\sqrt{80}}\right)$$
$$\approx 40.7^\circ$$

13 LHS $= \mathbf{p} \bullet (\mathbf{q} - \mathbf{r})$

$$= \begin{pmatrix} 3 \\ -2 \end{pmatrix} \bullet \left[\begin{pmatrix} -2 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ -3 \end{pmatrix} \right]$$

$$= \begin{pmatrix} 3 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} -3 \\ 8 \end{pmatrix}$$

$$= -9 - 16$$

$$= -25$$

RHS $= \mathbf{p} \bullet \mathbf{q} - \mathbf{p} \bullet \mathbf{r}$

$$= \begin{pmatrix} 3 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

$$= (-6 - 10) - (3 + 6)$$

$$= -16 - 9$$

$$= -25$$

$\therefore$ LHS = RHS ✓

14 **a** $\overrightarrow{PQ} = \begin{pmatrix} 4-(-1) \\ 0-2 \\ -1-3 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ -4 \end{pmatrix}$

b Let X be the point $(1, 0, 0)$ on the x-axis.

$\overrightarrow{OX} = (1, 0, 0)$, $\quad \theta = \cos^{-1}\left(\dfrac{\overrightarrow{PQ} \bullet \overrightarrow{OX}}{|\overrightarrow{PQ}||\overrightarrow{OX}|}\right)$

$$= \cos^{-1}\left(\frac{5+0+0}{\sqrt{25+4+16}\sqrt{1}}\right)$$

$$= \cos^{-1}\left(\frac{5}{\sqrt{45}}\right)$$

$$\approx 41.8^\circ$$

15 $\overrightarrow{MP} \bullet \overrightarrow{PT} = 0$

Now, $\begin{pmatrix} 5 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 5 \end{pmatrix} = 5 + (-5) = 0$

$\therefore$ $\overrightarrow{PT}$ has the form $k\begin{pmatrix} 1 \\ 5 \end{pmatrix}$, $k \neq 0$, $k \in \mathbb{R}$.

Also, $|\overrightarrow{MP}| = |\overrightarrow{PT}|$

$\therefore$ $\sqrt{5^2 + (-1)^2} = \sqrt{k^2 + (5k)^2}$

$\therefore$ $\sqrt{26} = \sqrt{26k^2}$

$= |k|\sqrt{26}$

$\therefore$ $k = \pm 1$

So $\overrightarrow{PT} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$ or $\begin{pmatrix} -1 \\ -5 \end{pmatrix}$

Finally, $\overrightarrow{OT} = \overrightarrow{OM} + \overrightarrow{MP} + \overrightarrow{PT}$

$$= \begin{pmatrix} -2 \\ 4 \end{pmatrix} + \begin{pmatrix} 5 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

or $\begin{pmatrix} -2 \\ 4 \end{pmatrix} + \begin{pmatrix} 5 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ -5 \end{pmatrix}$

$\therefore$ $\overrightarrow{OT} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$ or $\begin{pmatrix} 2 \\ -2 \end{pmatrix}$

16 $\mathbf{p} \bullet \mathbf{q} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ -4 \\ 2 \end{pmatrix}$

$$= 2(-1) + (-1)(-4) + 4 \times 2$$

$$= -2 + 4 + 8$$

$$= 10$$

$|\mathbf{p}| = \sqrt{2^2 + (-1)^2 + 4^2}$

$= \sqrt{4+1+16}$

$= \sqrt{21}$

$|\mathbf{q}| = \sqrt{(-1)^2 + (-4)^2 + 2^2}$

$= \sqrt{1+16+4}$

$= \sqrt{21}$

If θ is the angle between $\mathbf{p}$ and $\mathbf{q}$ then $\theta = \cos^{-1}\left(\dfrac{\mathbf{p} \bullet \mathbf{q}}{|\mathbf{p}||\mathbf{q}|}\right)$

$$= \cos^{-1}\left(\frac{10}{\sqrt{21}\sqrt{21}}\right)$$

$$\approx 61.6^\circ$$

17 $\mathbf{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$

$$\cos\theta = \frac{\mathbf{u} \bullet \mathbf{v}}{|\,\mathbf{u}\,|\,|\,\mathbf{v}\,|} = \frac{\begin{pmatrix} 2 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 3 \end{pmatrix}}{\sqrt{2^2+1^2}\sqrt{3^2}}$$
$$= \frac{3}{\sqrt{5}\sqrt{9}} = \tfrac{1}{\sqrt{5}}$$

Now $\sin^2\theta + \cos^2\theta = 1$

$\therefore\ \sin^2\theta + \frac{1}{5} = 1$

$\therefore\ \sin^2\theta = \frac{4}{5}$

$\therefore\ \sin\theta = \pm\frac{2}{\sqrt{5}}$

But θ is acute, so $\sin\theta = \frac{2}{\sqrt{5}}$.

or $\sin\theta = \dfrac{|\,\mathbf{u}\times\mathbf{v}\,|}{|\,\mathbf{u}\,|\,|\,\mathbf{v}\,|}$

$$\begin{aligned} |\,\mathbf{u}\times\mathbf{v}\,| &= (2\times 3 - 1\times 0) \\ &= 6 - 0 \\ &= 6 \end{aligned}$$

$\therefore\ \sin\theta = \dfrac{6}{\sqrt{5}\sqrt{9}} = \frac{6}{3\sqrt{5}} = \frac{2}{\sqrt{5}}$

18
$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 3 \\ 2 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 3 \\ -1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & 3 \\ 2 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & 0 \\ 2 & -1 \end{vmatrix}\mathbf{k}$$
$$= (0\times 1 - (-1)\times 3)\mathbf{i} - ((-1)\times 1 - 2\times 3)\mathbf{j} + ((-1)\times(-1) - 2\times 0)\mathbf{k}$$
$$= 3\mathbf{i} + 7\mathbf{j} + \mathbf{k}$$
$$= \begin{pmatrix} 3 \\ 7 \\ 1 \end{pmatrix} \quad \text{is a perpendicular vector}$$

$$\begin{aligned} \sqrt{3^2+7^2+1^2} &= \sqrt{9+49+1} \\ &= \sqrt{59} \end{aligned}$$

$\therefore\ \frac{1}{\sqrt{59}}\begin{pmatrix} 3 \\ 7 \\ 1 \end{pmatrix}$ is a perpendicular unit vector.

$\therefore\ \pm\frac{3}{\sqrt{59}}\begin{pmatrix} 3 \\ 7 \\ 1 \end{pmatrix}$ are the two 3 unit length perpendicular vectors.