Chapter 18

RULES OF DIFFERENTIATION

EXERCISE 18A

1 **a** $f(x) = x^3$
$\therefore\ f'(x) = 3x^2$

b $f(x) = 2x^3$
$\therefore\ f'(x) = 2(3x^2)$
$= 6x^2$

c $f(x) = 7x^2$
$\therefore\ f'(x) = 7(2x)$
$= 14x$

d $f(x) = 6\sqrt{x} = 6x^{\frac{1}{2}}$
$\therefore\ f'(x) = 6\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$
$= \dfrac{3}{\sqrt{x}}$

e $f(x) = 3\sqrt[3]{x} = 3x^{\frac{1}{3}}$
$\therefore\ f'(x) = 3\left(\frac{1}{3}x^{-\frac{2}{3}}\right)$
$= \dfrac{1}{\sqrt[3]{x^2}}$

f $f(x) = x^2 + x$
$\therefore\ f'(x) = 2x + 1$

g $f(x) = 4 - 2x^2$
$\therefore\ f'(x) = 0 - 2(2x)$
$= -4x$

h $f(x) = x^2 + 3x - 5$
$\therefore\ f'(x) = 2x + 3 - 0$
$= 2x + 3$

i $f(x) = \frac{1}{2}x^4 - 6x^2$
$\therefore\ f'(x) = \frac{1}{2}(4x^3) - 6(2x)$
$= 2x^3 - 12x$

j $f(x) = \dfrac{3x - 6}{x} = 3 - 6x^{-1}$
$\therefore\ f'(x) = 0 - 6(-1x^{-2})$
$= \dfrac{6}{x^2}$

k $f(x) = \dfrac{2x - 3}{x^2} = \dfrac{2x}{x^2} - \dfrac{3}{x^2}$
$= 2x^{-1} - 3x^{-2}$
$\therefore\ f'(x) = -2x^{-2} + 6x^{-3} = -\dfrac{2}{x^2} + \dfrac{6}{x^3}$

l $f(x) = \dfrac{x^3 + 5}{x} = x^2 + 5x^{-1}$
$\therefore\ f'(x) = 2x - 5x^{-2}$
$= 2x - \dfrac{5}{x^2}$

m $f(x) = \dfrac{x^3 + x - 3}{x}$
$= x^2 + 1 - 3x^{-1}$
$\therefore\ f'(x) = 2x + 0 + 3x^{-2}$
$= 2x + \dfrac{3}{x^2}$

n $f(x) = \dfrac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$
$\therefore\ f'(x) = -\frac{1}{2}x^{-\frac{3}{2}} = -\dfrac{1}{2x\sqrt{x}}$

o $f(x) = (2x - 1)^2 = 4x^2 - 4x + 1$
$\therefore\ f'(x) = 8x - 4$

p $f(x) = (x + 2)^3$
$= x^3 + 3x^2(2) + 3x(2^2) + 2^3$
$= x^3 + 6x^2 + 12x + 8$
$\therefore\ f'(x) = 3x^2 + 12x + 12$

2 **a** $y = 2.5x^3 - 1.4x^2 - 1.3$
$\therefore\ \dfrac{dy}{dx} = 7.5x^2 - 2.8x$

b $y = \pi x^2$
$\therefore\ \dfrac{dy}{dx} = 2\pi x$

c $y = \dfrac{1}{5x^2} = \frac{1}{5}x^{-2}$
$\therefore\ \dfrac{dy}{dx} = -\frac{2}{5}x^{-3} = -\dfrac{2}{5x^3}$

d $y = 100x$
$\therefore\ \dfrac{dy}{dx} = 100$

e $y = 10(x + 1)$
$= 10x + 10$
$\therefore\ \dfrac{dy}{dx} = 10$

f $y = 4\pi x^3$
$\therefore\ \dfrac{dy}{dx} = 12\pi x^2$

3 **a** $\frac{d}{dx}(6x+2)$
$= 6$

b $\frac{d}{dx}(x\sqrt{x})$
$= \frac{d}{dx}(x^{\frac{3}{2}})$
$= \frac{3}{2}x^{\frac{1}{2}}$
$= \frac{3\sqrt{x}}{2}$

c $\frac{d}{dx}(5-x)^2$
$= \frac{d}{dx}(25-10x+x^2)$
$= -10+2x$
$= 2x-10$

d $\frac{d}{dx}\left(\frac{6x^2-9x^4}{3x}\right)$
$= \frac{d}{dx}(2x-3x^3)$
$= 2-9x^2$

e $\frac{d}{dx}((x+1)(x-2))$
$= \frac{d}{dx}(x^2-x-2)$
$= 2x-1$

f $\frac{d}{dx}\left(\frac{1}{x^2}+6\sqrt{x}\right)$
$= \frac{d}{dx}\left(x^{-2}+6x^{\frac{1}{2}}\right)$
$= -2x^{-3}+3x^{-\frac{1}{2}}$
$= -\frac{2}{x^3}+\frac{3}{\sqrt{x}}$

g $\frac{d}{dx}\left(4x-\frac{1}{4x}\right)$
$= \frac{d}{dx}\left(4x-\frac{1}{4}x^{-1}\right)$
$= 4+\frac{1}{4}x^{-2}$
$= 4+\frac{1}{4x^2}$

h $\frac{d}{dx}(x(x+1)(2x-5))$
$= \frac{d}{dx}\left(x(2x^2-3x-5)\right)$
$= \frac{d}{dx}\left(2x^3-3x^2-5x\right)$
$= 6x^2-6x-5$

4 **a** Consider $y=x^2$ when $x=2$
Now $\frac{dy}{dx}=2x$
$\therefore$ when $x=2$,
$\frac{dy}{dx}=2(2)=4$
$\therefore$ the tangent has gradient 4.

b Consider $y=\frac{8}{x^2}$ at the point $(9, \frac{8}{81})$
Now $y=8x^{-2}$
$\therefore \frac{dy}{dx}=-16x^{-3}=-\frac{16}{x^3}$
$\therefore$ at $(9, \frac{8}{81})$, $x=9$ and so $\frac{dy}{dx}=-\frac{16}{729}$
$\therefore$ the tangent has gradient $-\frac{16}{729}$.

c Consider $y=2x^2-3x+7$ when $x=-1$
Now $\frac{dy}{dx}=4x-3$
$\therefore$ when $x=-1$,
$\frac{dy}{dx}=4(-1)-3=-7$
$\therefore$ the tangent has gradient -7.

d Consider $y=\frac{2x^2-5}{x}$ at the point $(2, \frac{3}{2})$
Now $y=2x-5x^{-1}$
$\therefore \frac{dy}{dx}=2+5x^{-2}=2+\frac{5}{x^2}$
$\therefore$ at $(2, \frac{3}{2})$, $x=2$ and so $\frac{dy}{dx}=2+\frac{5}{4}$
$=\frac{13}{4}$
$\therefore$ the tangent has gradient $\frac{13}{4}$.

e Consider $y=\frac{x^2-4}{x^2}$ at the point $(4, \frac{3}{4})$
Now $y=1-4x^{-2}$
$\therefore \frac{dy}{dx}=0+8x^{-3}=\frac{8}{x^3}$
$\therefore$ at $(4, \frac{3}{4})$, $x=4$ and so
$\frac{dy}{dx}=\frac{8}{4^3}=\frac{1}{8}$
$\therefore$ the tangent has gradient $\frac{1}{8}$.

f Consider $y=\frac{x^3-4x-8}{x^2}$ when $x=-1$
Now $y=x-4x^{-1}-8x^{-2}$
$\therefore \frac{dy}{dx}=1+4x^{-2}+16x^{-3}$
$=1+\frac{4}{x^2}+\frac{16}{x^3}$
$\therefore$ when $x=-1$,
$\frac{dy}{dx}=1+4-16=-11$
$\therefore$ the tangent has gradient -11.

5 $f(x) = x^2 + (b+1)x + 2c, \quad f(2) = 4, \quad \text{and} \quad f'(-1) = 2$

$\therefore \ f'(x) = 2x + (b+1)$

But $f'(-1) = 2$, so $2(-1) + b + 1 = 2$

$\therefore \ -1 + b = 2$

$\therefore \ b = 3$

So, $f(x) = x^2 + (3+1)x + 2c$

$= x^2 + 4x + 2c$

But $f(2) = 4$, so $2^2 + 4(2) + 2c = 4$

$\therefore \ 2c = -8$

$\therefore \ c = -4$

6 **a** $f(x) = 4\sqrt{x} + x = 4x^{\frac{1}{2}} + x$

$\therefore \ f'(x) = 4\left(\frac{1}{2}x^{-\frac{1}{2}}\right) + 1$

$= \dfrac{2}{\sqrt{x}} + 1$

b $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$

$\therefore \ f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$

$= \dfrac{1}{3\sqrt[3]{x^2}}$

c $f(x) = -\dfrac{2}{\sqrt{x}} = -2x^{-\frac{1}{2}}$

$\therefore \ f'(x) = -2\left(-\frac{1}{2}x^{-\frac{3}{2}}\right)$

$= x^{-\frac{3}{2}}$

$= \dfrac{1}{x\sqrt{x}}$

d $f(x) = 2x - \sqrt{x} = 2x - x^{\frac{1}{2}}$

$\therefore \ f'(x) = 2 - \frac{1}{2}x^{-\frac{1}{2}}$

$= 2 - \dfrac{1}{2\sqrt{x}}$

e $f(x) = \dfrac{4}{\sqrt{x}} - 5 = 4x^{-\frac{1}{2}} - 5$

$\therefore \ f'(x) = 4\left(-\frac{1}{2}x^{-\frac{3}{2}}\right)$

$= -2x^{-\frac{3}{2}} = -\dfrac{2}{x\sqrt{x}}$

f $f(x) = 3x^2 - x\sqrt{x} = 3x^2 - x^{\frac{3}{2}}$

$\therefore \ f'(x) = 6x - \frac{3}{2}x^{\frac{1}{2}}$

$= 6x - \frac{3}{2}\sqrt{x}$

g $f(x) = \dfrac{5}{x^2\sqrt{x}} = 5x^{-\frac{5}{2}}$

$\therefore \ f'(x) = 5\left(-\frac{5}{2}x^{-\frac{7}{2}}\right)$

$= -\frac{25}{2}x^{-\frac{7}{2}}$

$= \dfrac{-25}{2x^3\sqrt{x}}$

h $f(x) = 2x - \dfrac{3}{x\sqrt{x}} = 2x - 3x^{-\frac{3}{2}}$

$\therefore \ f'(x) = 2 - 3\left(-\frac{3}{2}x^{-\frac{5}{2}}\right)$

$= 2 + \frac{9}{2}x^{-\frac{5}{2}}$

$= 2 + \dfrac{9}{2x^2\sqrt{x}}$

7 **a** $y = 4x - \dfrac{3}{x} = 4x - 3x^{-1} \quad \therefore \ \dfrac{dy}{dx} = 4 + 3x^{-2} = 4 + \dfrac{3}{x^2}$

$\dfrac{dy}{dx}$ is the gradient function of $y = 4x - \dfrac{3}{x}$ from which the gradient at any point can be found.

b $S = 2t^2 + 4t$ m $\quad \therefore \ \dfrac{dS}{dt} = 4t + 4 \ \text{m s}^{-1}$

$\dfrac{dS}{dt}$ is the instantaneous rate of change in position at time t. It is the velocity function.

c $C = 1785 + 3x + 0.002x^2$ dollars.

$\dfrac{dC}{dx} = 3 + 0.002(2x) = 3 + 0.004x$ dollars per toaster

$\dfrac{dC}{dx}$ is the instantaneous rate of change in cost as the number of toasters changes.

EXERCISE 18B.1

1 **a** $g(x)=x^2, \quad f(x)=2x+7$

$\therefore \quad g(f(x))=g(2x+7)=(2x+7)^2$

b $g(x)=2x+7, \quad f(x)=x^2$

$g(f(x))=g(x^2)=2x^2+7$

c $g(x)=\sqrt{x}, \quad f(x)=3-4x$

$g(f(x))=g(3-4x)=\sqrt{3-4x}$

d $g(x)=3-4x, \quad f(x)=\sqrt{x}$

$g(f(x))=g(\sqrt{x})=3-4\sqrt{x}$

e $g(x)=\frac{2}{x}, \quad f(x)=x^2+3$

$g(f(x))=g(x^2+3)=\frac{2}{x^2+3}$

f $g(x)=x^2+3, \quad f(x)=\frac{2}{x}$

$g(f(x))=g\left(\frac{2}{x}\right)=\left(\frac{2}{x}\right)^2+3=\frac{4}{x^2}+3$

2 **a** $g(f(x))=(3x+10)^3 \qquad \therefore \quad g(x)=x^3, \quad f(x)=3x+10$

b $g(f(x))=\frac{1}{2x+4} \qquad \therefore \quad g(x)=\frac{1}{x}, \quad f(x)=2x+4$

c $g(f(x))=\sqrt{x^2-3x} \qquad \therefore \quad g(x)=\sqrt{x}, \quad f(x)=x^2-3x$

d $g(f(x))=\frac{10}{(3x-x^2)^3} \qquad \therefore \quad g(x)=\frac{10}{x^3}, \quad f(x)=3x-x^2$

{other answers are possible for **2**}

EXERCISE 18B.2

1 **a** $\frac{1}{(2x-1)^2}$

$=(2x-1)^{-2}$

$=u^{-2}$

where $\quad u=2x-1$

b $\sqrt{x^2-3x}$

$=(x^2-3x)^{\frac{1}{2}}$

$=u^{\frac{1}{2}}$

where $\quad u=x^2-3x$

c $\frac{2}{\sqrt{2-x^2}}$

$=2(2-x^2)^{-\frac{1}{2}}$

$=2u^{-\frac{1}{2}}$

where $\quad u=2-x^2$

d $\sqrt[3]{x^3-x^2}$

$=(x^3-x^2)^{\frac{1}{3}}$

$=u^{\frac{1}{3}}$

where $\quad u=x^3-x^2$

e $\frac{4}{(3-x)^3}$

$=4(3-x)^{-3}$

$=4u^{-3}$

where $\quad u=3-x$

f $\frac{10}{x^2-3}$

$=10(x^2-3)^{-1}$

$=10u^{-1}$

where $\quad u=x^2-3$

2 **a** $y=(4x-5)^2$

$\therefore \quad y=u^2 \quad$ where $\quad u=4x-5$

Now $\quad \frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$

$=2u(4)$

$=8u$

$=8(4x-5)$

b $y=\frac{1}{5-2x}$

$\therefore \quad y=u^{-1} \quad$ where $\quad u=5-2x$

Now $\quad \frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$

$=-u^{-2}(-2)$

$=2u^{-2}$

$=2\,(5-2x)^{-2}$

c $y=\sqrt{3x-x^2}$

$\therefore \quad y=u^{\frac{1}{2}} \quad$ where $\quad u=3x-x^2$

Now $\quad \frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$

$=\frac{1}{2}u^{-\frac{1}{2}}(3-2x)$

$=\frac{1}{2}\left(3x-x^2\right)^{-\frac{1}{2}}(3-2x)$

d $y=(1-3x)^4$

$\therefore \quad y=u^4 \quad$ where $\quad u=1-3x$

Now $\quad \frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$

$=4u^3(-3)$

$=-12u^3$

$=-12(1-3x)^3$

e
$$y = 6(5-x)^3$$
$$\therefore\ y = 6u^3 \quad \text{where} \quad u = 5-x$$
$$\text{Now}\ \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 18u^2(-1) = -18u^2 = -18(5-x)^2$$

f
$$y = \sqrt[3]{2x^3 - x^2}$$
$$\therefore\ y = u^{\frac{1}{3}} \quad \text{where} \quad u = 2x^3 - x^2$$
$$\text{Now}\ \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \tfrac{1}{3}u^{-\frac{2}{3}}(6x^2 - 2x) = \tfrac{1}{3}\left(2x^3 - x^2\right)^{-\frac{2}{3}}(6x^2 - 2x)$$

g
$$y = \frac{6}{(5x-4)^2}$$
$$\therefore\ y = 6u^{-2} \quad \text{where} \quad u = 5x-4$$
$$\text{Now}\ \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -12u^{-3}(5) = -60\,(5x-4)^{-3}$$

h
$$y = \frac{4}{3x - x^2}$$
$$\therefore\ y = 4u^{-1} \quad \text{where} \quad u = 3x - x^2$$
$$\text{Now}\ \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -4u^{-2}(3-2x) = -4\left(3x - x^2\right)^{-2}(3-2x)$$

i
$$y = 2\left(x^2 - \frac{2}{x}\right)^3$$
$$\therefore\ y = 2u^3 \quad \text{where} \quad u = x^2 - 2x^{-1}$$
$$\text{Now}\ \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 6u^2(2x + 2x^{-2}) = 6\left(x^2 - \frac{2}{x}\right)^2\left(2x + \frac{2}{x^2}\right)$$

3 **a** $y = \sqrt{1-x^2}$ at $x = \frac{1}{2}$

$$\therefore\ y = \sqrt{u} \quad \text{where} \quad u = 1 - x^2$$
$$\text{Now}\ \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \tfrac{1}{2}u^{-\frac{1}{2}}(-2x) = \frac{-x}{\sqrt{u}} = \frac{-x}{\sqrt{1-x^2}}$$
$$\text{At}\ x = \tfrac{1}{2},\ \frac{dy}{dx} = \frac{-\frac{1}{2}}{\sqrt{1-\frac{1}{4}}} = -\tfrac{1}{2}\left(\tfrac{2}{\sqrt{3}}\right)$$
$\therefore$ gradient of tangent $= -\frac{1}{\sqrt{3}}$

b $y = (3x+2)^6$ at $x = -1$

$$\therefore\ y = u^6 \quad \text{where} \quad u = 3x+2$$
$$\text{Now}\ \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 6u^5(3) = 18u^5 = 18(3x+2)^5$$
$$\text{At}\ x = -1,\ \frac{dy}{dx} = 18(-1)^5$$
$\therefore$ gradient of tangent $= -18$

c $y = \dfrac{1}{(2x-1)^4}$ at $x = 1$

$$\therefore\ y = u^{-4} \quad \text{where} \quad u = 2x - 1$$
$$\text{Now}\ \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -4u^{-5}(2) = \frac{-8}{u^5} = \frac{-8}{(2x-1)^5}$$
$$\text{At}\ x = 1,\ \frac{dy}{dx} = \frac{-8}{1^5}$$
$\therefore$ gradient of tangent $= -8$

d $y = 6 \times \sqrt[3]{1-2x}$ at $x = 0$

$$\therefore\ y = 6u^{\frac{1}{3}} \quad \text{where} \quad u = 1 - 2x$$
$$\text{Now}\ \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 6(\tfrac{1}{3})u^{-\frac{2}{3}}(-2) = 2u^{-\frac{2}{3}}(-2) = \frac{-4}{\sqrt[3]{u^2}} = \frac{-4}{\sqrt[3]{(1-2x)^2}}$$
$$\text{At}\ x = 0,\ \frac{dy}{dx} = \frac{-4}{\sqrt[3]{1^2}}$$
$\therefore$ gradient of tangent $= -4$

e $y = \dfrac{4}{x+2\sqrt{x}}$ at $x = 4$

$\therefore\ y = 4u^{-1}$ where $u = x + 2x^{\frac{1}{2}}$

$$\text{Now } \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$
$$= -4u^{-2}(1 + x^{-\frac{1}{2}})$$
$$= -\frac{4}{u^2}\left(1 + \frac{1}{\sqrt{x}}\right)$$
$$= \frac{-4}{(x+2\sqrt{x})^2}\left(1 + \frac{1}{\sqrt{x}}\right)$$

At $x = 4$, $\dfrac{dy}{dx} = \dfrac{-4}{(4+4)^2}(1+\frac{1}{2}) = -\frac{6}{64}$

$\therefore$ gradient of tangent $= -\frac{3}{32}$

f $y = \left(x + \dfrac{1}{x}\right)^3$ at $x = 1$

$\therefore\ y = u^3$ where $u = x + x^{-1}$

$$\text{Now } \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$
$$= 3u^2(1 - x^{-2})$$
$$= 3\left(x + \frac{1}{x}\right)^2\left(1 - \frac{1}{x^2}\right)$$

At $x = 1$, $\dfrac{dy}{dx} = 3(1+1)^2(1-1)$

$\therefore$ gradient of tangent $= 0$

4 If $f(x) = (2x - b)^a$

then $f'(x) = a(2x-b)^{a-1} \times 2$
$\quad = 2a(2x-b)^{a-1}$

but $f'(x) = 24x^2 - 24x + 6$
$\quad = 6(4x^2 - 4x + 1)$
$\quad = 6(2x-1)^2$

Solving by inspection, $a - 1 = 2$ and $b = 1$
$\therefore\ a = 3$ and $b = 1$

(**Note:** We can also use $2a = 6$ when solving for a.)

5 $$y = \frac{a}{\sqrt{1+bx}} = a\,(1+bx)^{-\frac{1}{2}}$$

When $x = 3$, $y = 1$

$\therefore\ 1 = \dfrac{a}{\sqrt{1+3b}}$

$\therefore\ a = \sqrt{1+3b}$ .... (1)

$$\frac{dy}{dx} = -\tfrac{1}{2}a\,(1+bx)^{-\frac{3}{2}} \times b$$

When $x = 3$, $\dfrac{dy}{dx} = -\frac{1}{8}$

$\therefore\ -\frac{1}{8} = -\frac{1}{2}ab\,(1+3b)^{-\frac{3}{2}}$

$\therefore\ a = \dfrac{1}{4b}(1+3b)^{\frac{3}{2}}$ .... (2)
$\quad = \dfrac{1}{4b}(1+3b)\sqrt{1+3b}$

Equating the RHS of (1) and (2) gives:

$$\sqrt{1+3b} = \frac{1}{4b}(1+3b)\sqrt{1+3b}$$

$\therefore\ 1 = \dfrac{1}{4b}(1+3b)$ $\quad$ {since $b \neq 0$, $\sqrt{1+3b} \neq 0$}

$\therefore\ 4b = 1 + 3b$

$\therefore\ b = 1$

$\therefore\ a = \sqrt{1+3(1)}$
$\quad = \sqrt{4}$
$\quad = 2$

So, $a = 2$ and $b = 1$.

6 **a** $y = x^3 \quad \therefore \quad \dfrac{dy}{dx} = 3x^2$

$x = y^{\frac{1}{3}} \quad \therefore \quad \dfrac{dx}{dy} = \frac{1}{3}y^{-\frac{2}{3}}$

$$\begin{aligned}\frac{dy}{dx}\frac{dx}{dy} &= 3x^2\left(\tfrac{1}{3}\right)y^{-\frac{2}{3}} \\ &= x^2(y)^{-\frac{2}{3}} \\ &= x^2(x^3)^{-\frac{2}{3}} \quad \{\text{substituting } y = x^3\} \\ &= x^2(x^{-2}) \\ &= x^0 \\ &= 1 \quad \text{as required}\end{aligned}$$

b We know that $\dfrac{dy}{du}\dfrac{du}{dx} = \dfrac{dy}{dx}$ {chain rule}

Letting $x = y$, $\dfrac{dy}{du}\dfrac{du}{dy} = \dfrac{dy}{dy}$

$\therefore \quad \dfrac{dy}{du}\dfrac{du}{dy} = 1$

Letting $u = x$, $\dfrac{dy}{dx}\dfrac{dx}{dy} = 1$

EXERCISE 18C

1 **a** $f(x) = x(x-1)$ is the product of

$u = x$ and $v = x - 1$

$\therefore \; u' = 1$ and $v' = 1$

Now $$\begin{aligned} f'(x) &= u'v + uv' \quad \{\text{product rule}\} \\ &= 1 \times (x-1) + x \times 1 \\ &= x - 1 + x \\ &= 2x - 1\end{aligned}$$

b $f(x) = 2x(x+1)$ is the product of

$u = 2x$ and $v = x + 1$

$\therefore \; u' = 2$ and $v' = 1$

Now $$\begin{aligned} f'(x) &= u'v + uv' \quad \{\text{product rule}\} \\ &= 2(x+1) + 2x \times 1 \\ &= 2x + 2 + 2x \\ &= 4x + 2\end{aligned}$$

c $f(x) = x^2\sqrt{x+1}$ is the product of $u = x^2$ and $v = (x+1)^{\frac{1}{2}}$

$\therefore \; u' = 2x$ and $v' = \frac{1}{2}(x+1)^{-\frac{1}{2}}$

Now $$\begin{aligned} f'(x) &= u'v + uv' \quad \{\text{product rule}\} \\ &= 2x(x+1)^{\frac{1}{2}} + x^2 \times \tfrac{1}{2}(x+1)^{-\frac{1}{2}} \\ &= 2x(x+1)^{\frac{1}{2}} + \tfrac{1}{2}x^2(x+1)^{-\frac{1}{2}}\end{aligned}$$

2 **a** $y = x^2(2x-1)$ is the product of

$u = x^2$ and $v = 2x - 1$

$\therefore \; u' = 2x$ and $v' = 2$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$$\begin{aligned}\therefore \; \frac{dy}{dx} &= 2x(2x-1) + x^2(2) \\ &= 2x(2x-1) + 2x^2\end{aligned}$$

b $y = 4x(2x+1)^3$ is the product of

$u = 4x$ and $v = (2x+1)^3$

$\therefore \; u' = 4$ and $v' = 3(2x+1)^2 \times 2$

$= 6(2x+1)^2$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$\therefore \; \dfrac{dy}{dx} = 4(2x+1)^3 + 24x(2x+1)^2$

c $y = x^2\sqrt{3-x}$ is the product of

$u = x^2$ and $v = (3-x)^{\frac{1}{2}}$

$\therefore \; u' = 2x$ and $v' = \frac{1}{2}(3-x)^{-\frac{1}{2}}(-1)$

$= -\frac{1}{2}(3-x)^{-\frac{1}{2}}$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$$\begin{aligned}\therefore \; \frac{dy}{dx} &= 2x(3-x)^{\frac{1}{2}} + x^2\left[-\tfrac{1}{2}(3-x)^{-\frac{1}{2}}\right] \\ &= 2x(3-x)^{\frac{1}{2}} - \tfrac{1}{2}x^2(3-x)^{-\frac{1}{2}}\end{aligned}$$

d $y = \sqrt{x}(x-3)^2$ is the product of

$u = x^{\frac{1}{2}}$ and $v = (x-3)^2$

$\therefore \; u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = 2(x-3)^1$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$\therefore \; \dfrac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}(x-3)^2 + 2\sqrt{x}(x-3)$

e $y = 5x^2(3x^2-1)^2$ is the product of $u = 5x^2$ and $v = (3x^2-1)^2$

$\therefore\ u' = 10x$ and $v' = 2(3x^2-1)^1(6x) = 12x(3x^2-1)$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$$\therefore\ \frac{dy}{dx} = 10x(3x^2-1)^2 + 5x^2(12x)(3x^2-1) = 10x(3x^2-1)^2 + 60x^3(3x^2-1)$$

f $y = \sqrt{x}(x-x^2)^3$ is the product of $u = x^{\frac{1}{2}}$ and $v = (x-x^2)^3$

$\therefore\ u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = 3(x-x^2)^2(1-2x)$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$$\therefore\ \frac{dy}{dx} = \tfrac{1}{2}x^{-\frac{1}{2}}(x-x^2)^3 + 3\sqrt{x}(x-x^2)^2(1-2x)$$

3 **a** $y = x^4(1-2x)^2$ is the product of $u = x^4$ and $v = (1-2x)^2$

$\therefore\ u' = 4x^3$ and $v' = 2(1-2x)^1(-2) = -4(1-2x)$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$$\therefore\ \frac{dy}{dx} = 4x^3(1-2x)^2 - 4x^4(1-2x)$$

At $x = -1$, $\dfrac{dy}{dx} = 4(-1)^3(3)^2 - 4(-1)^4(3) = -48$ $\therefore$ gradient of tangent $= -48$

b $y = \sqrt{x}(x^2-x+1)^2$ is the product of $u = x^{\frac{1}{2}}$ and $v = (x^2-x+1)^2$

$\therefore\ u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = 2(x^2-x+1)(2x-1)$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$$\therefore\ \frac{dy}{dx} = \tfrac{1}{2}x^{-\frac{1}{2}}(x^2-x+1)^2 + 2\sqrt{x}(x^2-x+1)(2x-1)$$

At $x = 4$, $\dfrac{dy}{dx} = \frac{1}{2}(4)^{-\frac{1}{2}}(13)^2 + 2\sqrt{4}(13)(7) = 406\frac{1}{4}$ $\therefore$ gradient of tangent $= 406\frac{1}{4}$

c $y = x\sqrt{1-2x}$ is the product of $u = x$ and $v = (1-2x)^{\frac{1}{2}}$

$\therefore\ u' = 1$ and $v' = \frac{1}{2}(1-2x)^{-\frac{1}{2}}(-2) = -(1-2x)^{-\frac{1}{2}}$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$$\therefore\ \frac{dy}{dx} = \sqrt{1-2x} - \frac{x}{\sqrt{1-2x}}$$

At $x = -4$, $\dfrac{dy}{dx} = \sqrt{9} - \dfrac{(-4)}{\sqrt{9}} = 3 + \frac{4}{3} = \frac{13}{3}$ $\therefore$ gradient of tangent $= \frac{13}{3}$

d $y = x^3\sqrt{5-x^2}$ is the product of $u = x^3$ and $v = (5-x^2)^{\frac{1}{2}}$

$\therefore\ u' = 3x^2$ and $v' = \frac{1}{2}(5-x^2)^{-\frac{1}{2}}(-2x) = -x(5-x^2)^{-\frac{1}{2}}$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$$\therefore\ \frac{dy}{dx} = 3x^2\sqrt{5-x^2} - \frac{x^4}{\sqrt{5-x^2}}$$

At $x = 1$, $\dfrac{dy}{dx} = 3(1)^2\sqrt{4} - \dfrac{1}{\sqrt{4}} = 6 - \frac{1}{2} = \frac{11}{2}$ $\therefore$ gradient of tangent $= \frac{11}{2}$

4 **a** $y = \sqrt{x}(3-x)^2$ is the product of

$u = x^{\frac{1}{2}}$ and $v = (3-x)^2$

$\therefore\ u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = 2(3-x)^1(-1)$

$= -2(3-x)$

Now $\frac{dy}{dx} = u'v + uv'$ {product rule}

$$\therefore\ \frac{dy}{dx} = \frac{1}{2\sqrt{x}}(3-x)^2 - 2\sqrt{x}(3-x)$$

$$= \frac{(3-x)^2 - (2\sqrt{x})(2\sqrt{x})(3-x)}{2\sqrt{x}}$$

$$= \frac{(3-x)\left[(3-x) - 4x\right]}{2\sqrt{x}}$$

$$= \frac{(3-x)(3-5x)}{2\sqrt{x}} \quad \text{as required}$$

b Tangents are horizontal when their gradients are 0.

$\frac{dy}{dx} = 0$ when $(3-x)(3-5x) = 0$

$\therefore\ 3 - x = 0$ or $3 - 5x = 0$

$\therefore\ x = 3$ or $x = \frac{3}{5}$

c $\frac{dy}{dx}$ is undefined when $2\sqrt{x} \leqslant 0$

$\therefore\ \sqrt{x} \leqslant 0$

$\therefore\ x \leqslant 0$

d $\frac{dy}{dx}$ is undefined when $x \leqslant 0$ {from **c**}

But when $x = 0$, $y = \sqrt{0}(3-0)^2$

$= 0$

So, when $x = 0$, y is defined but $\frac{dy}{dx}$ is not.

e As we approach the point $x = 0$ from the right, the curve has steeper and steeper gradient and approaches vertical.

5 $y = -2x^2(x+4)$ is the product of $u = -2x^2$ and $v = x + 4$

$\therefore\ u' = -4x$ and $v' = 1$

Now $\frac{dy}{dx} = u'v + uv'$ {product rule}

$= -4x(x+4) - 2x^2 \times 1$

$= -4x^2 - 16x - 2x^2$

$= -6x^2 - 16x$

If $\frac{dy}{dx} = 10$, $-6x^2 - 16x = 10$

$\therefore\ 6x^2 + 16x + 10 = 0$

$\therefore\ 3x^2 + 8x + 5 = 0$

$\therefore\ (3x+5)(x+1) = 0$

$\therefore\ x = -\frac{5}{3}$ and $x = -1$

EXERCISE 18D

1 **a** $y = \frac{1+3x}{2-x}$ is a quotient where

$u = 1 + 3x$ and $v = 2 - x$

$\therefore\ u' = 3$ and $v' = -1$

Now $\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$ {quotient rule}

$$\therefore\ \frac{dy}{dx} = \frac{3(2-x) - (1+3x)(-1)}{(2-x)^2}$$

$$= \frac{7}{(2-x)^2}$$

b $y = \frac{x^2}{2x+1}$ is a quotient where

$u = x^2$ and $v = 2x + 1$

$\therefore\ u' = 2x$ and $v' = 2$

Now $\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$ {quotient rule}

$$\therefore\ \frac{dy}{dx} = \frac{2x(2x+1) - x^2(2)}{(2x+1)^2}$$

$$= \frac{2x(2x+1) - 2x^2}{(2x+1)^2}$$

c $y = \frac{x}{x^2-3}$ is a quotient where

$u = x$ and $v = x^2 - 3$

$\therefore\ u' = 1$ and $v' = 2x$

Now $\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$ {quotient rule}

$$\therefore\ \frac{dy}{dx} = \frac{1(x^2-3) - x(2x)}{(x^2-3)^2}$$

$$= \frac{(x^2-3) - 2x^2}{(x^2-3)^2}$$

d $y = \frac{\sqrt{x}}{1-2x}$ is a quotient where

$u = x^{\frac{1}{2}}$ and $v = 1 - 2x$

$\therefore\ u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = -2$

Now $\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$ {quotient rule}

$$\therefore\ \frac{dy}{dx} = \frac{\frac{1}{2}x^{-\frac{1}{2}}(1-2x) - \sqrt{x}(-2)}{(1-2x)^2}$$

$$= \frac{\frac{1}{2}x^{-\frac{1}{2}}(1-2x) + 2\sqrt{x}}{(1-2x)^2}$$

e $y = \dfrac{x^2-3}{3x-x^2}$ is a quotient where $u = x^2 - 3$ and $v = 3x - x^2$
$\therefore\ u' = 2x$ and $v' = 3 - 2x$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$\therefore\ \dfrac{dy}{dx} = \dfrac{2x(3x-x^2)-(x^2-3)(3-2x)}{(3x-x^2)^2}$

f $y = \dfrac{x}{\sqrt{1-3x}}$ is a quotient where $u = x$ and $v = (1-3x)^{\frac{1}{2}}$
$\therefore\ u' = 1$ and $v' = -\frac{3}{2}(1-3x)^{-\frac{1}{2}}$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$\therefore\ \dfrac{dy}{dx} = \dfrac{(1-3x)^{\frac{1}{2}} - x\left(-\frac{3}{2}(1-3x)^{-\frac{1}{2}}\right)}{1-3x}$

$= \dfrac{(1-3x)^{\frac{1}{2}} + \frac{3}{2}x(1-3x)^{-\frac{1}{2}}}{1-3x}$

2 a $y = \dfrac{x}{1-2x}$ is a quotient where

$u = x$ and $v = 1 - 2x$
$\therefore\ u' = 1$ and $v' = -2$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$\therefore\ \dfrac{dy}{dx} = \dfrac{1(1-2x)-x(-2)}{(1-2x)^2}$

$= \dfrac{1}{(1-2x)^2}$

At $x = 1$, $\dfrac{dy}{dx} = \dfrac{1}{(1-2)^2} = \dfrac{1}{(-1)^2} = 1$

$\therefore$ the gradient of the tangent $= 1$

b $y = \dfrac{x^3}{x^2+1}$ is a quotient where

$u = x^3$ and $v = x^2 + 1$
$\therefore\ u' = 3x^2$ and $v' = 2x$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$\therefore\ \dfrac{dy}{dx} = \dfrac{3x^2(x^2+1)-x^3(2x)}{(x^2+1)^2}$

$= \dfrac{x^4+3x^2}{(x^2+1)^2}$

At $x = -1$, $\dfrac{dy}{dx} = \dfrac{1+3}{(1+1)^2} = \dfrac{4}{4} = 1$

$\therefore$ the gradient of the tangent $= 1$

c $y = \dfrac{\sqrt{x}}{2x+1}$ is a quotient where

$u = x^{\frac{1}{2}}$ and $v = 2x + 1$
$\therefore\ u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = 2$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$= \dfrac{\frac{1}{2\sqrt{x}}(2x+1) - \sqrt{x}(2)}{(2x+1)^2}$

At $x = 4$, $\dfrac{dy}{dx} = \dfrac{\frac{9}{4}-4}{81} = \dfrac{\left(\frac{9}{4}-4\right)}{81} \times \dfrac{4}{4}$

$= \dfrac{9-16}{324}$

$\therefore$ the gradient of the tangent $= -\frac{7}{324}$

d $y = \dfrac{x^2}{\sqrt{x^2+5}}$ is a quotient where

$u = x^2$ and $v = (x^2+5)^{\frac{1}{2}}$
$\therefore\ u' = 2x$ and $v' = \frac{1}{2}(x^2+5)^{-\frac{1}{2}}(2x)$
$= x(x^2+5)^{-\frac{1}{2}}$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$= \dfrac{2x\sqrt{x^2+5} - x^2\left(\dfrac{x}{\sqrt{x^2+5}}\right)}{(x^2+5)}$

At $x = -2$, $\dfrac{dy}{dx} = \dfrac{-4(3) - 4\left(\frac{-2}{3}\right)}{9}$

$= \dfrac{\left(-12+\frac{8}{3}\right)}{9} \times \dfrac{3}{3}$

$= \dfrac{-36+8}{27}$

$\therefore$ the gradient of the tangent $= -\frac{28}{27}$

3 **a** $y = \dfrac{2\sqrt{x}}{1-x}$ is a quotient where $u = 2x^{\frac{1}{2}}$ and $v = 1 - x$

$\therefore\ u' = x^{-\frac{1}{2}}$ and $v' = -1$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$$\therefore\ \frac{dy}{dx} = \frac{\frac{1}{\sqrt{x}}(1-x) - 2\sqrt{x}(-1)}{(1-x)^2} \times \left(\frac{\sqrt{x}}{\sqrt{x}}\right)$$

$$= \frac{(1-x) + 2x}{\sqrt{x}(1-x)^2}$$

$$= \frac{x+1}{\sqrt{x}(1-x)^2} \quad \text{as required}$$

b **i** $\dfrac{dy}{dx} = 0$ when $x + 1 = 0$ $\therefore\ x = -1$.

However $\dfrac{dy}{dx}$ is not defined for $x \leqslant 0$ because of the $\sqrt{x}$ term. Hence $\dfrac{dy}{dx}$ never equals 0.

ii $\dfrac{dy}{dx}$ is undefined when $x \leqslant 0$ and when $x = 1$.

4 **a** $y = \dfrac{x^2 - 3x + 1}{x + 2}$ is a quotient where $u = x^2 - 3x + 1$ and $v = x + 2$

$\therefore\ u' = 2x - 3$ and $v' = 1$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$$\therefore\ \frac{dy}{dx} = \frac{(2x-3)(x+2) - (x^2 - 3x + 1)(1)}{(x+2)^2}$$

$$= \frac{2x^2 + 4x - 3x - 6 - x^2 + 3x - 1}{(x+2)^2}$$

$$= \frac{x^2 + 4x - 7}{(x+2)^2} \quad \text{as required}$$

b **i** $\dfrac{dy}{dx} = 0$ when $x^2 + 4x - 7 = 0$ $\quad \therefore\ x = \dfrac{-4 \pm \sqrt{44}}{2} = -2 \pm \sqrt{11}$

ii $\dfrac{dy}{dx}$ is undefined when $(x+2)^2 = 0$

$\therefore\ x = -2$

c $\dfrac{dy}{dx}$ is zero when the tangent to the function is horizontal. This occurs at the function's turning points or points of horizontal inflection.

$\dfrac{dy}{dx}$ is undefined at vertical asymptotes of the function.

EXERCISE 18E

1 **a** $\dfrac{d}{dx}(2y) = 2\,\dfrac{dy}{dx}$ **b** $\dfrac{d}{dx}(-3y) = -3\,\dfrac{dy}{dx}$ **c** $\dfrac{d}{dx}(y^3) = 3y^2\,\dfrac{dy}{dx}$

d $\dfrac{d}{dx}\left(\dfrac{1}{y}\right) = \dfrac{d}{dx}(y^{-1}) = -y^{-2}\,\dfrac{dy}{dx}$ **e** $\dfrac{d}{dx}(y^4) = 4y^3\,\dfrac{dy}{dx}$

f $\dfrac{d}{dx}(\sqrt{y}) = \dfrac{d}{dx}(y^{\frac{1}{2}}) = \tfrac{1}{2}y^{-\frac{1}{2}}\,\dfrac{dy}{dx}$ **g** $\dfrac{d}{dx}\left(\dfrac{1}{y^2}\right) = \dfrac{d}{dx}(y^{-2}) = -2y^{-3}\,\dfrac{dy}{dx}$

h $\dfrac{d}{dx}(xy) = y + x\dfrac{dy}{dx}$ {product rule}

i $\dfrac{d}{dx}(x^2y) = 2xy + x^2\dfrac{dy}{dx}$ {product rule}

j $\dfrac{d}{dx}(xy^2) = y^2 + x(2y)\dfrac{dy}{dx} = y^2 + 2xy\dfrac{dy}{dx}$ {product rule}

2 **a** Differentiating both sides of $x^2 + y^2 = 25$ with respect to x,

$$2x + 2y\frac{dy}{dx} = 0$$
$$\therefore\ 2y\frac{dy}{dx} = -2x$$
$$\therefore\ \frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$$

b Differentiating both sides of $x^2 + 3y^2 = 9$ with respect to x,

$$2x + 6y\frac{dy}{dx} = 0$$
$$\therefore\ 6y\frac{dy}{dx} = -2x$$
$$\therefore\ \frac{dy}{dx} = -\frac{2x}{6y} = -\frac{x}{3y}$$

c Differentiating both sides of $y^2 - x^2 = 8$ with respect to x,

$$2y\frac{dy}{dx} - 2x = 0$$
$$\therefore\ 2y\frac{dy}{dx} = 2x$$
$$\therefore\ \frac{dy}{dx} = \frac{2x}{2y} = \frac{x}{y}$$

d Differentiating both sides of $x^2 - y^3 = 10$ with respect to x,

$$2x - 3y^2\frac{dy}{dx} = 0$$
$$\therefore\ 3y^2\frac{dy}{dx} = 2x$$
$$\therefore\ \frac{dy}{dx} = \frac{2x}{3y^2}$$

e Differentiating both sides of $x^2 + xy = 4$ with respect to x,

$$2x + \left(y + x\frac{dy}{dx}\right) = 0 \quad \{\text{product rule}\}$$
$$\therefore\ x\frac{dy}{dx} = -2x - y$$
$$\therefore\ \frac{dy}{dx} = \frac{-2x - y}{x}$$

f Differentiating both sides of $x^3 - 2xy = 5$ with respect to x,

$$3x^2 - \left(2y + 2x\frac{dy}{dx}\right) = 0 \quad \{\text{product rule}\}$$
$$\therefore\ 3x^2 - 2y = 2x\frac{dy}{dx}$$
$$\therefore\ \frac{dy}{dx} = \frac{3x^2 - 2y}{2x}$$

3 **a** Differentiating both sides of $x + y^3 = 4y$ with respect to x,

$$1 + 3y^2\frac{dy}{dx} = 4\frac{dy}{dx}$$

When $y = 1$, $1 + 3\dfrac{dy}{dx} = 4\dfrac{dy}{dx}$

$\therefore\ \dfrac{dy}{dx} = 1$ at this point, and the gradient of the tangent is 1.

b $x + y = 8xy$. Now when $x = \frac{1}{2}$, $\frac{1}{2} + y = 4y$ $\therefore$ $y = \frac{1}{6}$

Thus the point of contact is $(\frac{1}{2}, \frac{1}{6})$.

Differentiating both sides of $x + y = 8xy$ with respect to x,

$$1 + \frac{dy}{dx} = 8y + 8x\frac{dy}{dx}$$

So, at the point $(\frac{1}{2}, \frac{1}{6})$, $1 + \dfrac{dy}{dx} = 8(\frac{1}{6}) + 8(\frac{1}{2})\dfrac{dy}{dx}$

$$\therefore\ 1 + \frac{dy}{dx} = \tfrac{4}{3} + 4\frac{dy}{dx}$$
$$\therefore\ -\tfrac{1}{3} = 3\frac{dy}{dx}$$
$$\therefore\ \frac{dy}{dx} = -\tfrac{1}{9}$$

$\therefore$ the gradient of the tangent is $-\frac{1}{9}$.

EXERCISE 18F

1 **a** $f(x) = e^{4x}$
$\therefore\ f'(x) = 4e^{4x}$

b $f(x) = e^x + 3$
$\therefore\ f'(x) = e^x + 0$
$= e^x$

c $f(x) = e^{-2x}$
$\therefore\ f'(x) = -2e^{-2x}$

d $f(x) = e^{\frac{x}{2}}$
$\therefore\ f'(x) = \frac{1}{2}e^{\frac{x}{2}}$

e $f(x) = 2e^{-\frac{x}{2}}$
$\therefore\ f'(x) = 2e^{-\frac{x}{2}}\left(-\frac{1}{2}\right)$
$= -e^{-\frac{x}{2}}$

f $f(x) = 1 - 2e^{-x}$
$\therefore\ f'(x) = 0 - 2e^{-x}(-1)$
$= 2e^{-x}$

g $f(x) = 4e^{\frac{x}{2}} - 3e^{-x}$
$\therefore\ f'(x) = 4e^{\frac{x}{2}}\left(\frac{1}{2}\right) - 3e^{-x}(-1)$
$= 2e^{\frac{x}{2}} + 3e^{-x}$

h $f(x) = \dfrac{e^x + e^{-x}}{2} = \frac{1}{2}(e^x + e^{-x})$
$\therefore\ f'(x) = \frac{1}{2}(e^x + e^{-x}(-1))$
$= \dfrac{e^x - e^{-x}}{2}$

i $f(x) = e^{-x^2}$
$\therefore\ f'(x) = e^{-x^2}(-2x)$
$= -2xe^{-x^2}$

j $f(x) = e^{\frac{1}{x}}$
$\therefore\ f'(x) = e^{\frac{1}{x}}\left(-\dfrac{1}{x^2}\right)$

k $f(x) = 10\left(1 + e^{2x}\right)$
$= 10 + 10e^{2x}$
$\therefore\ f'(x) = 0 + 10e^{2x}\,(2)$
$= 20e^{2x}$

l $f(x) = 20\left(1 - e^{-2x}\right)$
$= 20 - 20e^{-2x}$
$\therefore\ f'(x) = 0 - 20e^{-2x}\,(-2)$
$= 40e^{-2x}$

m $f(x) = e^{2x+1}$
$\therefore\ f'(x) = e^{2x+1}(2)$
$= 2e^{2x+1}$

n $f(x) = e^{\frac{x}{4}}$
$\therefore\ f'(x) = e^{\frac{x}{4}}\left(\frac{1}{4}\right)$
$= \frac{1}{4}e^{\frac{x}{4}}$

o $f(x) = e^{1-2x^2}$
$\therefore\ f'(x) = e^{1-2x^2}(-4x)$
$= -4xe^{1-2x^2}$

p $f(x) = e^{-0.02x}$
$\therefore\ f'(x) = e^{-0.02x} \times (-0.02)$
$= -0.02e^{-0.02x}$

2 **a** $f(x) = xe^x$
$\therefore\ f'(x) = 1e^x + xe^x$ {product rule}
$= e^x + xe^x$

b $f(x) = x^3e^{-x}$
$\therefore\ f'(x) = 3x^2e^{-x} + x^3(-e^{-x})$
{product rule}
$= 3x^2e^{-x} - x^3e^{-x}$

c $f(x) = \dfrac{e^x}{x}$
$\therefore\ f'(x) = \dfrac{e^x x - e^x\,(1)}{x^2}$ {quotient rule}
$= \dfrac{xe^x - e^x}{x^2}$

d $f(x) = \dfrac{x}{e^x}$
$\therefore\ f'(x) = \dfrac{1e^x - xe^x}{(e^x)^2}$ {quotient rule}
$= \dfrac{e^x(1-x)}{(e^x)^2} = \dfrac{1-x}{e^x}$

e $f(x) = x^2e^{3x}$
$\therefore\ f'(x) = 2xe^{3x} + 3x^2e^{3x}$ {product rule}

f $f(x) = \dfrac{e^x}{\sqrt{x}}$
$\therefore\ f'(x) = \dfrac{e^x\sqrt{x} - \dfrac{e^x}{2\sqrt{x}}}{(\sqrt{x})^2}$ {quotient rule}
$= \dfrac{xe^x - \frac{1}{2}e^x}{x\sqrt{x}}$

g
$$f(x) = \sqrt{x}e^{-x}$$
$$\therefore\ f'(x) = \tfrac{1}{2}x^{-\frac{1}{2}}e^{-x} - x^{\frac{1}{2}}e^{-x}$$
{product rule}

h
$$f(x) = \frac{e^x + 2}{e^{-x}+1}$$
$$\therefore\ f'(x) = \frac{e^x(e^{-x}+1) - (e^x+2)\left(-e^{-x}\right)}{(e^{-x}+1)^2}$$
{quotient rule}
$$= \frac{1 + e^x + 1 + 2e^{-x}}{(e^{-x}+1)^2}$$
$$= \frac{e^x + 2 + 2e^{-x}}{(e^{-x}+1)^2}$$

3 **a**
$$f(x) = (e^x+2)^4$$
$$= u^4 \text{ where } u = e^x + 2$$
$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \quad \text{\{chain rule\}}$$
$$= 4u^3(e^x)$$
$$\therefore\ f'(x) = 4(e^x+2)^3(e^x)$$
$$= 4e^x(e^x+2)^3$$
$$\therefore\ f'(0) = 4\left(e^0+2\right)^3\left(e^0\right)$$
$$= 108$$
$\therefore$ gradient of tangent $= 108$

b
$$f(x) = \frac{1}{2-e^{-x}}$$
$$= u^{-1} \text{ where } u = 2 - e^{-x}$$
$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \quad \text{\{chain rule\}}$$
$$= -u^{-2}(e^{-x})$$
$$\therefore\ f'(x) = -\frac{e^{-x}}{(2-e^{-x})^2}$$
$$\therefore\ f'(0) = -\frac{e^0}{(2-e^0)^2}$$
$$= -1$$
$\therefore$ gradient of tangent $= -1$

c
$$f(x) = \sqrt{e^{2x}+10}$$
$$= u^{\frac{1}{2}} \text{ where } u = e^{2x}+10$$
$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \quad \text{\{chain rule\}}$$
$$= \tfrac{1}{2}u^{-\frac{1}{2}}(2e^{2x})$$
$$\therefore\ f'(x) = \frac{e^{2x}}{\sqrt{e^{2x}+10}}$$
$$\therefore\ f'(\ln 3) = \frac{e^{2\ln 3}}{\sqrt{e^{2\ln 3}+10}} = \tfrac{9}{\sqrt{19}}$$
$\therefore$ gradient of tangent $= \frac{9}{\sqrt{19}}$

4 **a**
$$f(x) = \frac{1}{(1-e^{3x})^2}$$
$$= u^{-2} \text{ where } u = 1 - e^{3x}$$
$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \quad \text{\{chain rule\}}$$
$$= -2u^{-3}(-3e^{3x}) = \frac{6e^{3x}}{u^3}$$
$$\therefore\ f'(x) = \frac{6e^{3x}}{(1-e^{3x})^3}$$

b
$$f(x) = \frac{1}{\sqrt{1-e^{-x}}}$$
$$= u^{-\frac{1}{2}} \text{ where } u = 1 - e^{-x}$$
$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \quad \text{\{chain rule\}}$$
$$= -\tfrac{1}{2}u^{-\frac{3}{2}}(e^{-x})$$
$$= \frac{-e^{-x}}{2u^{\frac{3}{2}}}$$
$$\therefore\ f'(x) = \frac{-e^{-x}}{2(1-e^{-x})^{\frac{3}{2}}}$$

c $f(x) = x\sqrt{1-2e^{-x}} = xu^{\frac{1}{2}}$ where $u = 1-2e^{-x}$

$\therefore\ f'(x) = 1u^{\frac{1}{2}} + x \times \frac{1}{2}u^{-\frac{1}{2}}\dfrac{du}{dx}$ {product rule and chain rule}

$= 1\sqrt{u} + x\,\frac{1}{2}u^{-\frac{1}{2}}2e^{-x}$

$= \dfrac{\sqrt{1-2e^{-x}}}{1} + \dfrac{xe^{-x}}{\sqrt{1-2e^{-x}}}$

$\therefore\ f'(x) = \dfrac{1-2e^{-x}+xe^{-x}}{\sqrt{1-2e^{-x}}}$

5 $f(x) = e^{kx} + x \quad \therefore\ f'(x) = ke^{kx} + 1$

Now $f'(0) = -8$, so $ke^0 + 1 = -8$

$\therefore\ k \times 1 = -9$

$\therefore\ k = -9$

6 **a** $y = 2^x$

$= (e^{\ln 2})^x$

$= e^{x\ln 2}$

$\therefore\ \dfrac{dy}{dx} = e^{x\ln 2} \times \ln 2$

$= 2^x \ln 2$

b $y = b^x$

$= (e^{\ln b})^x$

$= e^{x\ln b}$

$\therefore\ \dfrac{dy}{dx} = e^{x\ln b} \times \ln b$

$= b^x \times \ln b$

7 $f(x) = x^2e^{-x}$ is the product of $u = x^2$ and $v = e^{-x}$

$\therefore\ u' = 2x$ and $v' = -e^{-x}$

$\therefore\ f'(x) = u'v + uv'$

$= 2x\left(e^{-x}\right) + x^2\left(-e^{-x}\right)$

$= 2xe^{-x} - x^2e^{-x}$

Now $f'(x) = 0$ when $2xe^{-x} - x^2e^{-x} = 0$

$\therefore\ xe^{-x}(2-x) = 0$

$\therefore\ x = 0$ or $2 - x = 0$ $\{e^{-x} > 0$ for all $x\}$

$\therefore\ x = 0$ or $x = 2$

$f(0) = 0^2e^0$ and $f(2) = 2^2e^{-2}$

$= 0 \qquad\qquad = 4e^{-2} \quad \left(\text{or } \dfrac{4}{e^2}\right)$

So, the possible coordinates of P are $(0, 0)$ and $\left(2, \dfrac{4}{e^2}\right)$.

8 **a** $y = 2^x$

$\therefore\ \dfrac{dy}{dx} = 2^x \ln 2$

b $y = 5^x$

$\therefore\ \dfrac{dy}{dx} = 5^x \ln 5$

c $y = x\,2^x$

$\therefore\ \dfrac{dy}{dx} = \dfrac{d}{dx}(x)2^x + x\,\dfrac{d}{dx}(2^x)$

{product rule}

$= 2^x + x\,2^x \ln 2$

d $y = x^3 6^{-x} = \dfrac{x^3}{6^x}$

$\therefore\ \dfrac{dy}{dx} = \dfrac{\frac{d}{dx}(x^3)6^x - x^3\frac{d}{dx}(6^x)}{6^{2x}}$

{quotient rule}

$= \dfrac{3x^2 6^x - x^3 \times 6^x \ln 6}{6^{2x}}$

$= \dfrac{x^2(3 - x\ln 6)}{6^x}$

e $y = \dfrac{2^x}{x}$

$\therefore \dfrac{dy}{dx} = \dfrac{\frac{d}{dx}(2^x)x - 2^x\frac{d}{dx}(x)}{x^2}$ {quotient rule}

$= \dfrac{2^x \ln 2 \times x - 2^x}{x^2}$

$= \dfrac{2^x(x\ln 2 - 1)}{x^2}$

f $y = \dfrac{x}{3^x}$

$\therefore \dfrac{dy}{dx} = \dfrac{\frac{d}{dx}(x)3^x - x\frac{d}{dx}(3^x)}{3^{2x}}$ {quotient rule}

$= \dfrac{3^x - x \times 3^x \ln 3}{3^{2x}}$

$= \dfrac{1 - x\ln 3}{3^x}$

9 $x^3e^{3y} + 4x^2y^3 = 27e^{-2x}$

$\therefore 3x^2e^{3y} + x^3\left(3e^{3y}\dfrac{dy}{dx}\right) + 8xy^3 + 4x^2\left(3y^2\dfrac{dy}{dx}\right) = 27(-2e^{-2x})$ {product rule}

$\therefore \dfrac{dy}{dx}(3x^3e^{3y} + 12x^2y^2) = -54e^{-2x} - 3x^2e^{3y} - 8xy^3$

$\therefore \dfrac{dy}{dx} = \dfrac{-54e^{-2x} - 3x^2e^{3y} - 8xy^3}{3x^3e^{3y} + 12x^2y^2}$

$= \dfrac{-(54e^{-2x} + 3x^2e^{3y} + 8xy^3)}{3x^2(xe^{3y} + 4y^2)}$

EXERCISE 18G

1 **a** $y = \ln(7x)$

$\therefore y = \ln 7 + \ln x$

$\therefore \dfrac{dy}{dx} = 0 + \dfrac{1}{x} = \dfrac{1}{x}$

or $y = \ln(7x)$

$\therefore \dfrac{dy}{dx} = \dfrac{7}{7x} \begin{matrix}\leftarrow f'(x)\\ \leftarrow f(x)\end{matrix}$

$= \dfrac{1}{x}$

b $y = \ln(2x+1)$

$\therefore \dfrac{dy}{dx} = \dfrac{2}{2x+1} \begin{matrix}\leftarrow f'(x)\\ \leftarrow f(x)\end{matrix}$

c $y = \ln(x - x^2)$

$\therefore \dfrac{dy}{dx} = \dfrac{1-2x}{x-x^2} \begin{matrix}\leftarrow f'(x)\\ \leftarrow f(x)\end{matrix}$

d $y = 3 - 2\ln x$

$\therefore \dfrac{dy}{dx} = 0 - 2\left(\dfrac{1}{x}\right)$

$= -\dfrac{2}{x}$

e $y = x^2 \ln x$

$\therefore \dfrac{dy}{dx} = 2x\ln x + x^2\left(\dfrac{1}{x}\right)$

$= 2x\ln x + x$

f $y = \dfrac{\ln x}{2x}$

$\therefore \dfrac{dy}{dx} = \dfrac{\left(\frac{1}{x}\right)2x - \ln x \times 2}{(2x)^2}$

$= \dfrac{2 - 2\ln x}{4x^2}$

$= \dfrac{1 - \ln x}{2x^2}$

g $y = e^x \ln x$

$\therefore \dfrac{dy}{dx} = e^x \ln x + \dfrac{e^x}{x}$

h $y = (\ln x)^2$

$\therefore \dfrac{dy}{dx} = 2(\ln x)^1\left(\dfrac{1}{x}\right)$

$= \dfrac{2\ln x}{x}$

i $y = \sqrt{\ln x} = (\ln x)^{\frac{1}{2}}$

$\therefore \dfrac{dy}{dx} = \frac{1}{2}(\ln x)^{-\frac{1}{2}}\left(\dfrac{1}{x}\right)$

$= \dfrac{1}{2x\sqrt{\ln x}}$

j $y = e^{-x}\ln x$

$\therefore \dfrac{dy}{dx} = -e^{-x}\ln x + e^{-x}\left(\dfrac{1}{x}\right)$

$= \dfrac{e^{-x}}{x} - e^{-x}\ln x$

k
$$y = \sqrt{x}\ln(2x)$$
$$\therefore \frac{dy}{dx} = \frac{1}{2\sqrt{x}}\ln(2x) + \sqrt{x}\left(\frac{1}{x}\right)$$
$$= \frac{\ln(2x)}{2\sqrt{x}} + \frac{1}{\sqrt{x}}$$

l
$$y = \frac{2\sqrt{x}}{\ln x}$$
$$\therefore \frac{dy}{dx} = \frac{\frac{1}{\sqrt{x}}\ln x - 2\sqrt{x}\left(\frac{1}{x}\right)}{(\ln x)^2}$$
$$= \frac{\frac{1}{\sqrt{x}}\ln x - \frac{2}{\sqrt{x}}}{(\ln x)^2}$$
$$= \frac{\ln x - 2}{\sqrt{x}(\ln x)^2}$$

m
$$y = 3 - 4\ln(1-x)$$
$$\therefore \frac{dy}{dx} = -\frac{4}{1-x} \times -1$$
$$= \frac{4}{1-x}$$

n
$$y = x\ln(x^2+1)$$
$$\therefore \frac{dy}{dx} = \ln(x^2+1) + x\frac{2x}{x^2+1}$$
$$= \ln(x^2+1) + \frac{2x^2}{x^2+1}$$

2 **a**
$$y = x\ln 5$$
$$\therefore \frac{dy}{dx} = \ln 5$$

b
$$y = \ln(x^3) = 3\ln x$$
$$\therefore \frac{dy}{dx} = 3\left(\frac{1}{x}\right) = \frac{3}{x}$$

c
$$y = \ln(x^4 + x)$$
$$\therefore \frac{dy}{dx} = \frac{4x^3+1}{x^4+x}$$

d
$$y = \ln(10-5x)$$
$$\therefore \frac{dy}{dx} = \frac{-5}{10-5x} = \frac{1}{x-2}$$

e
$$y = [\ln(2x+1)]^3$$
$$\therefore \frac{dy}{dx} = 3\,[\ln(2x+1)]^2 \times \frac{2}{2x+1}$$
$$= \frac{6}{2x+1}\,[\ln(2x+1)]^2$$

f
$$y = \frac{\ln(4x)}{x}$$
$$\therefore \frac{dy}{dx} = \frac{\left(\frac{4}{4x}\right)x - \ln(4x)\times 1}{x^2}$$
$$= \frac{1-\ln(4x)}{x^2}$$

g
$$y = \ln\left(\frac{1}{x}\right)$$
$$= -\ln x$$
$$\therefore \frac{dy}{dx} = -\frac{1}{x}$$

h
$$y = \ln(\ln x)$$
$$\therefore \frac{dy}{dx} = \frac{\frac{1}{x}}{\ln x}$$
$$= \frac{1}{x\ln x}$$

i
$$y = \frac{1}{\ln x} = (\ln x)^{-1}$$
$$\therefore \frac{dy}{dx} = -1(\ln x)^{-2} \times \frac{1}{x}$$
$$= \frac{-1}{x(\ln x)^2}$$

3 **a**
$$y = \ln\sqrt{1-2x}$$
$$= \ln(1-2x)^{\frac{1}{2}}$$
$$= \tfrac{1}{2}\ln(1-2x)$$
$$\therefore \frac{dy}{dx} = \tfrac{1}{2} \times \frac{-2}{1-2x}$$
$$= \frac{-1}{1-2x}$$

b
$$y = \ln\left(\frac{1}{2x+3}\right)$$
$$= -\ln(2x+3)$$
$$\therefore \frac{dy}{dx} = -\frac{2}{2x+3}$$

c
$$y = \ln(e^x\sqrt{x})$$
$$= \ln e^x + \ln x^{\frac{1}{2}}$$
$$= \ln e^x + \tfrac{1}{2}\ln x$$
$$= x + \tfrac{1}{2}\ln x$$
$$\therefore \frac{dy}{dx} = 1 + \tfrac{1}{2}\left(\frac{1}{x}\right)$$
$$= 1 + \frac{1}{2x}$$

d
$$\begin{aligned} y &= \ln(x\sqrt{2-x}) \\ &= \ln x + \ln(2-x)^{\frac{1}{2}} \\ &= \ln x + \tfrac{1}{2}\ln(2-x) \\ \therefore\ \frac{dy}{dx} &= \frac{1}{x} + \tfrac{1}{2}\left(\frac{-1}{2-x}\right) \\ &= \frac{1}{x} - \frac{1}{2(2-x)} \end{aligned}$$

e
$$\begin{aligned} y &= \ln\left(\frac{x+3}{x-1}\right) \\ &= \ln(x+3) - \ln(x-1) \\ \therefore\ \frac{dy}{dx} &= \frac{1}{x+3} - \frac{1}{x-1} \end{aligned}$$

f
$$\begin{aligned} y &= \ln\left(\frac{x^2}{3-x}\right) \\ &= \ln x^2 - \ln(3-x) \\ &= 2\ln x - \ln(3-x) \\ \therefore\ \frac{dy}{dx} &= \frac{2}{x} - \frac{-1}{3-x} = \frac{2}{x} + \frac{1}{3-x} \end{aligned}$$

g
$$\begin{aligned} f(x) &= \ln\left((3x-4)^3\right) \\ &= 3\ln(3x-4) \\ \therefore\ f'(x) &= 3 \times \frac{3}{3x-4} = \frac{9}{3x-4} \end{aligned}$$

h
$$\begin{aligned} f(x) &= \ln\left(x(x^2+1)\right) \\ &= \ln x + \ln(x^2+1) \\ \therefore\ f'(x) &= \frac{1}{x} + \frac{2x}{x^2+1} \end{aligned}$$

i
$$\begin{aligned} f(x) &= \ln\left(\frac{x^2+2x}{x-5}\right) \\ &= \ln(x^2+2x) - \ln(x-5) \\ \therefore\ f'(x) &= \frac{2x+2}{x^2+2x} - \frac{1}{x-5} \end{aligned}$$

4 $y = x\ln x$ is the product of $u = x$ and $v = \ln x$

$\therefore\ u' = 1$ and $v' = \dfrac{1}{x}$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$$\begin{aligned} &= 1\ln x + x \times \frac{1}{x} \\ &= \ln x + 1 \end{aligned}$$

At $x = e$, $\dfrac{dy}{dx} = \ln e + 1 = 1 + 1 = 2$

$\therefore$ gradient of tangent $= 2$

5 $f(x) = a\ln(2x+b)$

Now $f(e) = 3$, $\therefore\ 3 = a\ln(2e+b)$

$\therefore\ a = \dfrac{3}{\ln(2e+b)}$

$f'(x) = a \times \dfrac{2}{2x+b}$

Now $f'(e) = \dfrac{6}{e}$ $\therefore\ \dfrac{6}{e} = \dfrac{2a}{2e+b}$

$$\begin{aligned} \therefore\ 6(2e+b) &= 2ae \\ \therefore\ 3(2e+b) &= ae \\ \therefore\ a &= \frac{3(2e+b)}{e} \\ &= \frac{6e+3b}{e} \\ &= 6 + \frac{3b}{e} \end{aligned}$$

Using technology we find $a = 3$, $b = -e$.

6 For this question, we remember that $\log_a x = \dfrac{\log_e x}{\log_e a} = \dfrac{\ln x}{\ln a}$

a $y = \log_2 x = \dfrac{\ln x}{\ln 2}$

$\therefore\ \dfrac{dy}{dx} = \dfrac{1}{x\ln 2}$

b $y = \log_{10} x = \dfrac{\ln x}{\ln 10}$

$\therefore\ \dfrac{dy}{dx} = \dfrac{1}{x\ln 10}$

c $y = x\log_3 x = \dfrac{x\ln x}{\ln 3}$

Since $\ln 3$ is a constant, $\dfrac{dy}{dx} = \dfrac{\frac{d}{dx}(x)\ln x + x\frac{d}{dx}(\ln x)}{\ln 3}$

$$= \frac{\ln x + x\left(\frac{1}{x}\right)}{\ln 3}$$

$$= \frac{1+\ln x}{\ln 3} = \frac{1}{\ln 3} + \log_3 x$$

7 $e^{2a}\ln b^2 - a^3 b + \ln(ab) = 21$

$$\therefore\ \left(2e^{2a}\,\frac{da}{db}\right)\ln b^2 + e^{2a}\left(\frac{2b}{b^2}\right) - \left(3a^2\frac{da}{db}\right)b - a^3 + \frac{\frac{da}{db}\times b + a}{ab} = 0$$

$$\therefore\ 4abe^{2a}\ln b\,\frac{da}{db} + 2ae^{2a} - 3a^3b^2\,\frac{da}{db} - a^4 b + b\,\frac{da}{db} + a = 0 \qquad \{\times\, ab\}$$

$$\therefore\ \frac{da}{db}\left(4abe^{2a}\ln b - 3a^3b^2 + b\right) = a^4 b - 2ae^{2a} - a$$

$$\therefore\ \frac{da}{db} = \frac{a^4 b - 2ae^{2a} - a}{4abe^{2a}\ln b - 3a^3b^2 + b}$$

EXERCISE 18H

1 **a** $y = \sin(2x)$

$\therefore\ \dfrac{dy}{dx} = \cos(2x)\,\dfrac{d}{dx}(2x)$

$= 2\cos(2x)$

b $y = \sin x + \cos x$

$\therefore\ \dfrac{dy}{dx} = \cos x - \sin x$

c $y = \cos(3x) - \sin x$

$\therefore\ \dfrac{dy}{dx} = -\sin(3x)\times 3 - \cos x$

$= -3\sin(3x) - \cos x$

d $y = \sin(x+1)$

$\therefore\ \dfrac{dy}{dx} = \cos(x+1)\,\dfrac{d}{dx}(x+1)$

$= 1\cos(x+1)$

$= \cos(x+1)$

e $y = \cos(3-2x)$

$\therefore\ \dfrac{dy}{dx} = -\sin(3-2x)\times -2$

$= 2\sin(3-2x)$

f $y = \tan(5x)$

$\therefore\ \dfrac{dy}{dx} = \dfrac{1}{\cos^2(5x)}\times 5$

$= \dfrac{5}{\cos^2(5x)}$

g $y = \sin\left(\frac{x}{2}\right) - 3\cos x$

$\therefore\ \dfrac{dy}{dx} = \frac{1}{2}\cos\left(\frac{x}{2}\right) + 3\sin x$

h $y = 3\tan(\pi x)$

$\therefore\ \dfrac{dy}{dx} = 3\times\dfrac{1}{\cos^2(\pi x)}\times\pi$

$= \dfrac{3\pi}{\cos^2(\pi x)}$

i $y = 4\sin x - \cos(2x)$

$\therefore\ \dfrac{dy}{dx} = 4\cos x + \sin(2x)\times 2$

$= 4\cos x + 2\sin(2x)$

2 **a** $y = x^2 + \cos x$

$\therefore\ \dfrac{dy}{dx} = 2x - \sin x$

b $y = \tan x - 3\sin x$

$\therefore\ \dfrac{dy}{dx} = \dfrac{1}{\cos^2 x} - 3\cos x$

c $y = e^x \cos x$
$\therefore \dfrac{dy}{dx} = e^x \cos x + e^x(-\sin x)$
$= e^x \cos x - e^x \sin x$

d $y = e^{-x} \sin x$
$\therefore \dfrac{dy}{dx} = -e^{-x} \sin x + e^{-x} \cos x$

e $y = \ln(\sin x)$
$\therefore \dfrac{dy}{dx} = \dfrac{\cos x}{\sin x}$

f $y = e^{2x} \tan x$
$\therefore \dfrac{dy}{dx} = 2e^{2x} \tan x + \dfrac{e^{2x}}{\cos^2 x}$

g $y = \sin(3x)$
$\therefore \dfrac{dy}{dx} = 3\cos(3x)$

h $y = \cos(\frac{x}{2})$
$\therefore \dfrac{dy}{dx} = -\frac{1}{2}\sin(\frac{x}{2})$

i $y = 3\tan(2x)$
$\therefore \dfrac{dy}{dx} = \dfrac{3}{\cos^2(2x)} \times 2$
$= \dfrac{6}{\cos^2(2x)}$

j $y = x \cos x$
$\therefore \dfrac{dy}{dx} = 1 \times \cos x + x(-\sin x)$
$= \cos x - x \sin x$

k $y = \dfrac{\sin x}{x}$
$\therefore \dfrac{dy}{dx} = \dfrac{(\cos x)(x) - \sin x \times 1}{x^2}$
$= \dfrac{x \cos x - \sin x}{x^2}$

l $y = x \tan x$
$\therefore \dfrac{dy}{dx} = 1 \times \tan x + x \times \dfrac{1}{\cos^2 x}$
$= \tan x + \dfrac{x}{\cos^2 x}$

3 **a** $\dfrac{d}{dx}(\sec x) = \dfrac{d}{dx}\left(\dfrac{1}{\cos x}\right)$
$= \dfrac{d}{dx}(\cos x)^{-1}$
$= -(\cos x)^{-2} \times (-\sin x)$
$= \dfrac{\sin x}{(\cos x)^2}$
$= \dfrac{1}{\cos x} \times \dfrac{\sin x}{\cos x}$
$= \sec x \tan x$

b $\dfrac{d}{dx}(\cot x) = \dfrac{d}{dx}\left(\dfrac{1}{\tan x}\right)$
$= \dfrac{d}{dx}\left(\dfrac{\cos x}{\sin x}\right)$
$= \dfrac{(-\sin x) \times \sin x - \cos x \times \cos x}{(\sin x)^2}$
$= \dfrac{-(\sin^2 x + \cos^2 x)}{(\sin x)^2}$
$= \dfrac{-1}{(\sin x)^2}$ {since $\sin^2 x + \cos^2 x = 1$}
$= -\left(\dfrac{1}{\sin x}\right)^2$
$= -\csc^2 x$

4 **a** $y = \sin(x^2)$
$\therefore \dfrac{dy}{dx} = 2x\cos(x^2)$

b $y = \cos\left(\sqrt{x}\right) = \cos(x^{\frac{1}{2}})$
$\therefore \dfrac{dy}{dx} = -\sin(x^{\frac{1}{2}}) \times \frac{1}{2}x^{-\frac{1}{2}}$
$= -\dfrac{1}{2\sqrt{x}}\sin(\sqrt{x})$

c $y = \sqrt{\cos x} = (\cos x)^{\frac{1}{2}}$
$\therefore \dfrac{dy}{dx} = \frac{1}{2}(\cos x)^{-\frac{1}{2}} \times (-\sin x)$
$= -\dfrac{\sin x}{2\sqrt{\cos x}}$

d $y = \sin^2 x = (\sin x)^2$
$\therefore \dfrac{dy}{dx} = 2\sin x \cos x$

e $y = \cos^3 x = (\cos x)^3$
$\therefore \dfrac{dy}{dx} = 3\cos^2 x \times (-\sin x)$
$= -3\sin x \cos^2 x$

f $y = \cos x \sin(2x)$
$\therefore \dfrac{dy}{dx} = (-\sin x)\sin(2x) + \cos x(2\cos(2x))$
$= -\sin x \sin(2x) + 2\cos x \cos(2x)$

g
$$y = \cos(\cos x)$$
$$\therefore \frac{dy}{dx} = -\sin(\cos x) \times (-\sin x)$$
$$= \sin x \sin(\cos x)$$

h
$$y = \cos^3(4x) = (\cos(4x))^3$$
$$\therefore \frac{dy}{dx} = 3(\cos(4x))^2 \times (-4\sin(4x))$$
$$= -12\sin(4x)\cos^2(4x)$$

i
$$y = \csc(4x)$$
$$\therefore \frac{dy}{dx} = -\csc(4x)\cot(4x)\frac{d}{dx}(4x)$$
$$= -4\csc(4x)\cot(4x)$$

j
$$y = \sec(2x)$$
$$\therefore \frac{dy}{dx} = \sec(2x)\tan(2x)\frac{d}{dx}(2x)$$
$$= 2\sec(2x)\tan(2x)$$

k
$$y = \frac{2}{\sin^2(2x)} = 2(\sin(2x))^{-2}$$
$$\therefore \frac{dy}{dx} = -4(\sin(2x))^{-3} \times 2\cos(2x)$$
$$= -\frac{8\cos(2x)}{\sin^3(2x)}$$
or $-8\csc^2(2x)\cot(2x)$

l
$$y = 8\cot^3(\tfrac{x}{2}) = 8(\cot(\tfrac{x}{2}))^3$$
$$\therefore \frac{dy}{dx} = 24(\cot(\tfrac{x}{2}))^2 \times -\csc^2(\tfrac{x}{2}) \times \tfrac{1}{2}$$
$$= -12\cot^2(\tfrac{x}{2})\csc^2(\tfrac{x}{2})$$

5 **a**
$$f(x) = \sin^3 x$$
$$= (\sin x)^3$$
$$= u^3 \quad \text{where} \quad u = \sin x$$
$$f'(x) = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \qquad \{\text{chain rule}\}$$
$$\therefore f'(x) = 3u^2 \times \frac{du}{dx}$$
$$= 3\sin^2 x \cos x$$
$$\therefore f'\left(\tfrac{2\pi}{3}\right) = 3\sin^2\left(\tfrac{2\pi}{3}\right)\cos\left(\tfrac{2\pi}{3}\right)$$
$$= -\tfrac{9}{8}$$

b $f(x) = \cos x \sin x$ is the product of
$$u = \cos x \quad \text{and} \quad v = \sin x$$
$$\therefore u' = -\sin x \quad \text{and} \quad v' = \cos x$$
Now $f'(x) = u'v + uv'$
$$= -\sin x \sin x + \cos x \cos x$$
$$= \cos^2 x - \sin^2 x$$
$$\therefore f'\left(\tfrac{\pi}{4}\right) = \cos^2\left(\tfrac{\pi}{4}\right) - \sin^2\left(\tfrac{\pi}{4}\right)$$
$$= 0$$

6 **a**
$$y = x \sec x$$
$$\therefore \frac{dy}{dx} = \sec x + x\sec x \tan x$$
{product rule}
$$= \sec x(x\tan x + 1)$$

b
$$y = e^x \cot x$$
$$\therefore \frac{dy}{dx} = e^x \cot x + e^x(-\csc^2 x)$$
{product rule}
$$= e^x(\cot x - \csc^2 x)$$

c
$$y = 4\sec(2x)$$
$$\therefore \frac{dy}{dx} = 4\sec(2x)\tan(2x)(2)$$
$$= 8\sec(2x)\tan(2x)$$

d
$$y = e^{-x}\cot\left(\tfrac{x}{2}\right)$$
$$\therefore \frac{dy}{dx} = -e^{-x}\cot\left(\tfrac{x}{2}\right) + e^{-x}\left(-\csc^2\left(\tfrac{x}{2}\right)\right)\left(\tfrac{1}{2}\right)$$
{product rule}
$$= -e^{-x}\left(\cot\left(\tfrac{x}{2}\right) + \tfrac{1}{2}\csc^2\left(\tfrac{x}{2}\right)\right)$$

e
$$y = x^2 \csc x$$
$$\therefore \frac{dy}{dx} = 2x\csc x + x^2(-\csc x \cot x) \quad \{\text{product rule}\}$$
$$= x\csc x\,(2 - x\cot x)$$

f
$$y = x\sqrt{\csc x} = x(\csc x)^{\frac{1}{2}}$$
$$\therefore \frac{dy}{dx} = (\csc x)^{\frac{1}{2}} + \tfrac{1}{2}x(\csc x)^{-\frac{1}{2}}(-\csc x \cot x)$$
$$= (\csc x)^{\frac{1}{2}} - \tfrac{1}{2}x(\csc x)^{\frac{1}{2}}\cot x$$
$$= \sqrt{\csc x}\left(1 - \tfrac{1}{2}x\cot x\right)$$

g
$$y = \ln(\sec x)$$
$$\therefore \quad \frac{dy}{dx} = \frac{\frac{d}{dx}(\sec x)}{\sec x}$$
$$= \frac{\sec x \tan x}{\sec x} = \tan x$$

h
$$y = x\csc(x^2)$$
$$\therefore \quad \frac{dy}{dx} = \csc(x^2) + x\left[-\csc(x^2)\cot(x^2)\right](2x)$$
{product rule}
$$= \csc(x^2)(1 - 2x^2\cot(x^2))$$

i $y = \dfrac{\cot x}{\sqrt{x}} = x^{-\frac{1}{2}}\cot x$
$$\therefore \quad \frac{dy}{dx} = -\tfrac{1}{2}x^{-\frac{3}{2}}\cot x + x^{-\frac{1}{2}}(-\csc^2 x) \qquad \text{\{product rule\}}$$
$$= -\frac{\cot x + 2x\csc^2 x}{2x\sqrt{x}}$$
$$= -\frac{\cos x\sin x + 2x}{2x\sqrt{x}\sin^2 x}$$

EXERCISE 18I

1 If $y = \arccos x$ then $x = \cos y$
$$\therefore \quad \frac{dx}{dy} = -\sin y = -\sqrt{1-\cos^2 y} = -\sqrt{1-x^2}$$
$$\therefore \quad \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}, \quad x \in \,]-1, 1[$$

2 If $y = \arctan x$ then $x = \tan y$
$$\therefore \quad 1 = \sec^2 y\,\frac{dy}{dx} \qquad \text{\{differentiating with respect to } x\text{\}}$$
$$\therefore \quad 1 = (1+\tan^2 y)\,\frac{dy}{dx}$$
$$\therefore \quad 1 = (1+x^2)\,\frac{dy}{dx} \quad \text{and so} \quad \frac{dy}{dx} = \frac{1}{1+x^2}, \quad x \in \mathbb{R}$$

3 **a**
$$y = \arctan(2x)$$
$$\therefore \quad \frac{dy}{dx} = 2 \times \frac{1}{1+(2x)^2} = \frac{2}{1+4x^2}$$

b
$$y = \arccos(3x)$$
$$\therefore \quad \frac{dy}{dx} = 3 \times \left(\frac{-1}{\sqrt{1-(3x)^2}}\right) = -\frac{3}{\sqrt{1-9x^2}}$$

c
$$y = \arcsin\left(\tfrac{x}{4}\right)$$
$$\therefore \quad \frac{dy}{dx} = \tfrac{1}{4} \times \frac{1}{\sqrt{1-\left(\frac{x}{4}\right)^2}}$$
$$= \frac{1}{4\sqrt{1-\frac{x^2}{16}}}$$
$$= \frac{1}{\sqrt{16-x^2}}$$

d
$$y = \arccos\left(\tfrac{x}{5}\right)$$
$$\therefore \quad \frac{dy}{dx} = \tfrac{1}{5} \times \left(\frac{-1}{\sqrt{1-\left(\frac{x}{5}\right)^2}}\right)$$
$$= -\frac{1}{5\sqrt{1-\frac{x^2}{25}}}$$
$$= -\frac{1}{\sqrt{25-x^2}}$$

e
$$y = \arctan(x^2)$$
$$\therefore \quad \frac{dy}{dx} = 2x \times \frac{1}{1+(x^2)^2}$$
$$= \frac{2x}{1+x^4}$$

f
$$y = \arccos(\sin x)$$
$$\therefore \quad \frac{dy}{dx} = \cos x \times \left(-\frac{1}{\sqrt{1-\sin^2 x}}\right)$$
$$= -\frac{\cos x}{\cos x}$$
$$= -1$$

4 **a** $y = x \arcsin x$ $\quad \therefore \dfrac{dy}{dx} = \arcsin x + x\left(\dfrac{1}{\sqrt{1-x^2}}\right)$ {product rule}

$$= \arcsin x + \frac{x}{\sqrt{1-x^2}}$$

b $y = e^x \arccos x$ $\quad \therefore \dfrac{dy}{dx} = e^x \arccos x + e^x\left(-\dfrac{1}{\sqrt{1-x^2}}\right)$ {product rule}

$$= e^x \arccos x - \frac{e^x}{\sqrt{1-x^2}}$$

c $y = e^{-x} \arctan x$ $\quad \therefore \dfrac{dy}{dx} = -e^{-x} \arctan x + e^{-x}\left(\dfrac{1}{1+x^2}\right)$ {product rule}

$$= -e^{-x} \arctan x + \frac{e^{-x}}{1+x^2}$$

5 **a**
$$y = \arcsin\left(\tfrac{x}{a}\right)$$
$$\therefore \frac{dy}{dx} = \frac{1}{a}\frac{1}{\sqrt{1-\left(\dfrac{x}{a}\right)^2}}$$
$$= \frac{1}{a\sqrt{1-\dfrac{x^2}{a^2}}}$$
$$= \frac{1}{\sqrt{a^2-x^2}} \quad \text{as required,}$$
and this is defined for $x \in \,]-a, a[$.

b
$$y = \arctan\left(\tfrac{x}{a}\right)$$
$$\therefore \frac{dy}{dx} = \frac{1}{a} \times \frac{1}{1+\left(\dfrac{x}{a}\right)^2}$$
$$= \frac{a}{a^2} \times \frac{1}{1+\dfrac{x^2}{a^2}}$$
$$= \frac{a}{a^2+x^2} \quad \text{as required,}$$
and this is defined for $x \in \mathbb{R}$.

c
$$y = \arccos\left(\tfrac{x}{a}\right)$$
$$\therefore \frac{dy}{dx} = \frac{1}{a} \times -\frac{1}{\sqrt{1-\left(\dfrac{x}{a}\right)^2}} = -\frac{1}{a\sqrt{1-\dfrac{x^2}{a^2}}}$$
$$= -\frac{1}{\sqrt{a^2-x^2}}$$
and this is defined for $x \in \,]-a, a[$.

EXERCISE 18J

1 **a**
$$f(x) = 3x^2 - 6x + 2$$
$$\therefore f'(x) = 6x - 6$$
$$\therefore f''(x) = 6$$

b
$$f(x) = \frac{2}{\sqrt{x}} - 1 = 2x^{-\frac{1}{2}} - 1$$
$$\therefore f'(x) = -x^{-\frac{3}{2}}$$
$$\therefore f''(x) = \tfrac{3}{2}x^{-\frac{5}{2}}$$
$$= \frac{3}{2x^{\frac{5}{2}}}$$

c
$$f(x) = 2x^3 - 3x^2 - x + 5$$
$$\therefore f'(x) = 6x^2 - 6x - 1$$
$$\therefore f''(x) = 12x - 6$$

d
$$f(x) = \frac{2-3x}{x^2} = 2x^{-2} - 3x^{-1}$$
$$\therefore f'(x) = -4x^{-3} + 3x^{-2}$$
$$\therefore f''(x) = 12x^{-4} - 6x^{-3}$$
$$= \frac{12-6x}{x^4}$$

e $$\begin{aligned} f(x) &= (1-2x)^3 \\ \therefore\ f'(x) &= 3(1-2x)^2(-2) \\ &= -6(1-2x)^2 \\ \therefore\ f''(x) &= -12(1-2x)^1(-2) \\ &= 24(1-2x) = 24-48x \end{aligned}$$

f $f(x) = \dfrac{x+2}{2x-1}$ is a quotient with $u = x+2$ and $v = 2x-1$

$\therefore\ u' = 1$ and $v' = 2$

$$\begin{aligned} \therefore\ f'(x) &= \frac{1(2x-1)-2(x+2)}{(2x-1)^2} \quad \{\text{quotient rule}\} \\ &= \frac{-5}{(2x-1)^2} \\ &= -5(2x-1)^{-2} \\ \therefore\ f''(x) &= 10(2x-1)^{-3}(2) = \frac{20}{(2x-1)^3} \end{aligned}$$

2 **a** $$\begin{aligned} y &= x - x^3 \\ \therefore\ \frac{dy}{dx} &= 1-3x^2 \\ \therefore\ \frac{d^2y}{dx^2} &= -6x \end{aligned}$$

b $$\begin{aligned} y &= x^2 - \frac{5}{x^2} \\ &= x^2 - 5x^{-2} \\ \therefore\ \frac{dy}{dx} &= 2x + 10x^{-3} \\ \therefore\ \frac{d^2y}{dx^2} &= 2 - 30x^{-4} \\ &= 2 - \frac{30}{x^4} \end{aligned}$$

c $$\begin{aligned} y &= 2 - \frac{3}{\sqrt{x}} \\ &= 2 - 3x^{-\frac{1}{2}} \\ \therefore\ \frac{dy}{dx} &= \tfrac{3}{2}x^{-\frac{3}{2}} \\ \therefore\ \frac{d^2y}{dx^2} &= -\tfrac{9}{4}x^{-\frac{5}{2}} \end{aligned}$$

d $$\begin{aligned} y &= \frac{4-x}{x} \\ &= 4x^{-1} - 1 \\ \therefore\ \frac{dy}{dx} &= -4x^{-2} \\ \therefore\ \frac{d^2y}{dx^2} &= 8x^{-3} \\ &= \frac{8}{x^3} \end{aligned}$$

e $$\begin{aligned} y &= (x^2-3x)^3 \\ \therefore\ \frac{dy}{dx} &= 3(x^2-3x)^2(2x-3) \\ &= (6x-9)(x^2-3x)^2 \quad \text{which is a product where} \end{aligned}$$

$u = 6x-9$ and $v = (x^2-3x)^2$

$\therefore\ u' = 6$ and $v' = 2(x^2-3x)^1(2x-3)$

$$\begin{aligned} \therefore\ \frac{d^2y}{dx^2} &= 6(x^2-3x)^2 + (6x-9)(2)(x^2-3x)(2x-3) \\ &= 6(x^2-3x)\left[(x^2-3x)+(2x-3)^2\right] \\ &= 6(x^2-3x)(x^2-3x+4x^2-12x+9) \\ &= 6(x^2-3x)(5x^2-15x+9) \end{aligned}$$

f $$\begin{aligned} y &= x^2 - x + \frac{1}{1-x} \\ &= x^2 - x + (1-x)^{-1} \\ \therefore\ \frac{dy}{dx} &= 2x - 1 + (-1)(1-x)^{-2}(-1) \\ &= 2x - 1 + (1-x)^{-2} \\ \therefore\ \frac{d^2y}{dx^2} &= 2 - 2(1-x)^{-3}(-1) \\ &= 2 + \frac{2}{(1-x)^3} \end{aligned}$$

3 **a** $$\begin{aligned} f(x) &= x^3 - 2x + 5 \\ \therefore\ f(2) &= (2)^3 - 2(2) + 5 \\ &= 9 \end{aligned}$$

b $$\begin{aligned} f(x) &= x^3 - 2x + 5 \\ \therefore\ f'(x) &= 3x^2 - 2 \\ \therefore\ f'(2) &= 3(2)^2 - 2 \\ &= 10 \end{aligned}$$

c $f'(x) = 3x^2 - 2$ {from **b**}
$\therefore\ f''(x) = 6x$
$\therefore\ f''(2) = 6(2)$
$= 12$

d $f''(x) = 6x$ {from **c**}
$\therefore\ f^{(3)}(x) = 6$
$\therefore\ f^{(3)}(2) = 6$

4 P_n is "if $y = Ae^{bx}$ where A and b are constants, then $\dfrac{d^n y}{dx^n} = b^n y$, for $n \in \mathbb{Z}^+$".

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= \dfrac{dy}{dx} = Ae^{bx} \times b = bAe^{bx} = by$, RHS $= by$ $\therefore$ P_1 is true.

(2) If P_k is true, then $\dfrac{d^k y}{dx^k} = b^k y$ where $k \geqslant 1$, $k \in \mathbb{Z}^+$.

Now,
$$\begin{aligned}\frac{d^{k+1}y}{dx^{k+1}} &= \frac{d}{dx}\left(\frac{d^k y}{dx^k}\right)\\ &= \frac{d}{dx}(b^k y)\\ &= b^k \times \frac{d}{dx}(y) \qquad \{\text{since } b^k \text{ is a constant}\}\\ &= b^k \times bAe^{bx}\\ &= b^{k+1}Ae^{bx}\\ &= b^{k+1}y\end{aligned}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true
$\therefore$ P_n is true for all $n \geqslant 1$, $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

5 **a** $f(x) = 2x^3 - 6x^2 + 5x + 1$
$\therefore\ f'(x) = 6x^2 - 12x + 5$
$\therefore\ f''(x) = 12x - 12$

So, $f''(x) = 0$ when $12x - 12 = 0$
$\therefore\ 12x = 12$
$\therefore\ x = 1$

b $f(x) = \dfrac{x}{x^2+2}$ is a quotient where $u = x$ and $v = x^2 + 2$
$\therefore\ u' = 1$ and $v' = 2x$

$$\begin{aligned}\therefore\ f'(x) &= \frac{1(x^2+2) - 2x^2}{(x^2+2)^2} \qquad \{\text{quotient rule}\}\\ &= \frac{2 - x^2}{(x^2+2)^2}\end{aligned}$$

This is another quotient, this time with $u = 2 - x^2$ and $v = (x^2+2)^2$
$\therefore\ u' = -2x$ and $v' = 2(x^2+2)(2x)$

$$\begin{aligned}\therefore\ f''(x) &= \frac{-2x(x^2+2)^2 - 4x(x^2+2)(2-x^2)}{(x^2+2)^4}\\ &= \frac{-2x(x^2+2)[x^2+2+2(2-x^2)]}{(x^2+2)^4}\\ &= \frac{-2x(-x^2+6)}{(x^2+2)^3}\\ &= \frac{2x(x^2-6)}{(x^2+2)^3}\end{aligned}$$

So, $f''(x) = 0$ when $2x(x^2 - 6) = 0$
$\therefore\ x = 0$ or $x^2 - 6 = 0$
$\therefore\ x = 0$ or $x = \pm\sqrt{6}$

6 $f(x) = 2x^3 - x$

$\therefore\ f'(x) = 6x^2 - 1$

$\therefore\ f''(x) = 12x$

By substituting the various values of x into these three functions, we can fill in the table as follows:

x	-1	0	1
$f(x)$	$-$	0	$+$
$f'(x)$	$+$	$-$	$+$
$f''(x)$	$-$	0	$+$

7
$$\begin{aligned} f(x) &= \tfrac{2}{3}\sin 3x \\ \therefore\ f'(x) &= \tfrac{2}{3} \times (\cos 3x) \times 3 \\ &= 2\cos 3x \\ \therefore\ f''(x) &= 2 \times (-\sin 3x) \times 3 \\ &= -6\sin 3x \\ \therefore\ f^{(3)}(x) &= -6 \times (\cos 3x) \times 3 \\ &= -18\cos 3x \\ \therefore\ f^{(3)}\left(\tfrac{2\pi}{9}\right) &= -18\cos 3\left(\tfrac{2\pi}{9}\right) \\ &= -18\cos\left(\tfrac{2\pi}{3}\right) \\ &= 9 \end{aligned}$$

8 **a**
$$\begin{aligned} f(x) &= 2\sin^3 x - 3\sin x \\ &= 2(\sin x)^3 - 3\sin x \\ \therefore\ f'(x) &= 2 \times 3(\sin x)^2 \times (\cos x) - 3\cos x \\ &= -3\cos x(1 - 2\sin^2 x) \\ &= -3\cos x\cos 2x \end{aligned}$$

b
$$\begin{aligned} f''(x) &= -3(-\sin x \times \cos 2x + \cos x \times (-2)\sin 2x) \\ &= 3\sin x\cos 2x + 6\cos x\sin 2x \end{aligned}$$

9 **a**
$$\begin{aligned} y &= -\ln x \\ \therefore\ \frac{dy}{dx} &= -1 \times \frac{1}{x} \\ &= -x^{-1} \\ \therefore\ \frac{d^2y}{dx^2} &= -\left(-x^{-2}\right) \\ &= x^{-2} = \frac{1}{x^2} \end{aligned}$$

b $y = x\ln x$ is the product of $u = x$ and $v = \ln x$

$\therefore\ u' = 1$ and $v' = \dfrac{1}{x}$

$$\begin{aligned} \text{Now}\ \frac{dy}{dx} &= u'v + uv' \quad \{\text{product rule}\} \\ &= 1 \times \ln x + x \times \frac{1}{x} \\ &= \ln x + 1 \\ \therefore\ \frac{d^2y}{dx^2} &= \frac{1}{x} \end{aligned}$$

c
$y = (\ln x)^2$

$\therefore\ \dfrac{dy}{dx} = 2(\ln x)\left(\dfrac{1}{x}\right) = \dfrac{2\ln x}{x}$ which is a quotient with $u = 2\ln x$ and $v = x$

$\therefore\ u' = \dfrac{2}{x}$ and $v' = 1$

$$\begin{aligned} \therefore\ \frac{d^2y}{dx^2} &= \frac{u'v - uv'}{v^2} \quad \{\text{quotient rule}\} \\ &= \frac{\frac{2}{x} \times x - 2\ln x \times 1}{x^2} \\ &= \frac{2 - 2\ln x}{x^2} = \frac{2}{x^2}(1 - \ln x) \end{aligned}$$

10 **a**
$$\begin{aligned} f(x) &= x^2 - \frac{1}{x} \\ \therefore\ f(1) &= (1)^2 - \frac{1}{1} \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

b
$$\begin{aligned} f(x) &= x^2 - \frac{1}{x} \\ &= x^2 - x^{-1} \\ \therefore\ f'(x) &= 2x - \left(-x^{-2}\right) \\ &= 2x + x^{-2} \\ &= 2x + \frac{1}{x^2} \\ \therefore\ f'(1) &= 2(1) + \frac{1}{1^2} \\ &= 2 + 1 = 3 \end{aligned}$$

c $f'(x) = 2x + x^{-2}$ {from **b**}

$\therefore\ f''(x) = 2 - 2x^{-3}$

$= 2 - \dfrac{2}{x^3}$

$\therefore\ f''(1) = 2 - \dfrac{2}{1^3}$

$= 2 - 2 = 0$

d $f''(x) = 2 - 2x^{-3}$ {from **c**}

$\therefore\ f^{(3)}(x) = 6x^{-4}$

$= \dfrac{6}{x^4}$

$\therefore\ f^{(3)}(1) = \dfrac{6}{1^4} = 6$

11 $y = 2e^{3x} + 5e^{4x}$ $\quad\therefore\ \dfrac{dy}{dx} = 6e^{3x} + 20e^{4x}$ and $\dfrac{d^2y}{dx^2} = 18e^{3x} + 80e^{4x}$

Now $\dfrac{d^2y}{dx^2} - 7\,\dfrac{dy}{dx} + 12y = \left(18e^{3x} + 80e^{4x}\right) - 7\left(6e^{3x} + 20e^{4x}\right) + 12\left(2e^{3x} + 5e^{4x}\right)$

$$\begin{aligned}
&= 18e^{3x} + 80e^{4x} - 42e^{3x} - 140e^{4x} + 24e^{3x} + 60e^{4x}\\
&= e^{3x}\,[18 - 42 + 24] + e^{4x}\,[80 - 140 + 60]\\
&= e^{3x}(0) + e^{4x}(0)\\
&= 0
\end{aligned}$$

$\therefore\ \dfrac{d^2y}{dx^2} - 7\,\dfrac{dy}{dx} + 12y = 0$

12 If $y = \sin(2x+3)$, then $\dfrac{dy}{dx} = 2\cos(2x+3)$ and $\dfrac{d^2y}{dx^2} = -4\sin(2x+3)$

$\therefore\ \dfrac{d^2y}{dx^2} + 4y = -4\sin(2x+3) + 4\sin(2x+3) = 0$

13 $y = \sin x$

$\therefore\ \dfrac{dy}{dx} = \cos x$

$\therefore\ \dfrac{d^2y}{dx^2} = -\sin x$

$\therefore\ \dfrac{d^3y}{dx^3} = -\cos x$

$\therefore\ \dfrac{d^4y}{dx^4} = -(-\sin x)$

$= \sin x$

$= y$

14 If $y = 2\sin x + 3\cos x$, then $y' = 2\cos x - 3\sin x$ and $y'' = -2\sin x - 3\cos x$

$\therefore\ y'' + y = -2\sin x - 3\cos x + 2\sin x + 3\cos x = 0$

15 **a** $x^2 + y^2 = 25$

$\therefore\ 2x + 2y\,\dfrac{dy}{dx} = 0$

$\therefore\ \dfrac{dy}{dx} = -\dfrac{x}{y}$

$$\therefore\ \frac{d^2y}{dx^2} = \frac{(-1)y - (-x)(1)\,\frac{dy}{dx}}{y^2} = \frac{-y + x\left(-\frac{x}{y}\right)}{y^2} = \frac{-y - \frac{x^2}{y}}{y^2} = \frac{-y^2 - x^2}{y^3}$$

b $x^2 - y^2 = 10$

$\therefore\ 2x - 2y\,\dfrac{dy}{dx} = 0$

$\therefore\ \dfrac{dy}{dx} = \dfrac{x}{y}$

$$\therefore\ \frac{d^2y}{dx^2} = \frac{(1)y - x(1)\,\frac{dy}{dx}}{y^2} = \frac{y - x\left(\frac{x}{y}\right)}{y^2} = \frac{y - \frac{x^2}{y}}{y^2} = \frac{y^2 - x^2}{y^3}$$

c
$$x^3 + 2xy = 4$$
$$\therefore\ 3x^2 + 2y + 2x(1)\frac{dy}{dx} = 0$$
$$\therefore\ \frac{dy}{dx} = \frac{-3x^2 - 2y}{2x}$$
$$\therefore\ \frac{d^2y}{dx^2} = \frac{(-6x - 2\frac{dy}{dx})(2x) - (-3x^2 - 2y)(2)}{(2x)^2}$$
$$= \frac{-12x^2 - 2\left(\frac{-3x^2-2y}{\cancel{2x}}\right)(\cancel{2x}) - 2(-3x^2 - 2y)}{4x^2}$$
$$= \frac{-12x^2 + 6x^2 + 4y + 6x^2 + 4y}{4x^2}$$
$$= \frac{8y}{4x^2} = \frac{2y}{x^2}$$

16 **a**
$$3V^2 + 2q = 2Vq$$
Differentiating with respect to q,
$$\therefore\ 6V\frac{dV}{dq} + 2 = 2\frac{dV}{dq}q + 2V$$
$$\therefore\ \frac{dV}{dq}(6V - 2q) = 2V - 2$$
$$\therefore\ \frac{dV}{dq} = \frac{2V-2}{6V-2q} = \frac{V-1}{3V-q}$$

b
$$3V^2 + 2q = 2Vq$$
Differentiating with respect to V,
$$6V + 2\frac{dq}{dV} = 2q + 2V\frac{dq}{dV} \quad .... (1)$$
$$\therefore\ (2-2V)\frac{dq}{dV} = 2q - 6V$$
$$\therefore\ \frac{dq}{dV} = \frac{2q-6V}{2-2V} = \frac{q-3V}{1-V} \quad .... (2)$$
Now differentiating (1) with respect to V,
$$6 + 2\frac{d^2q}{dV^2} = 2\frac{dq}{dV} + 2\frac{dq}{dV} + 2V\frac{d^2q}{dV^2}$$
$$\therefore\ (2-2V)\frac{d^2q}{dV^2} = 4\frac{dq}{dV} - 6$$
$$= 4\left(\frac{q-3V}{1-V}\right) - 6 \quad \{\text{using (2)}\}$$
$$= \frac{4q - 12V - 6(1-V)}{1-V}$$
$$= \frac{4q - 6V - 6}{1-V}$$
$$\therefore\ \frac{d^2q}{dV^2} = \frac{4q-6V-6}{(2-2V)(1-V)}$$
$$= \frac{2q - 3V - 3}{(1-V)^2}$$

17 $f(x) = e^{ax}(x+1),\quad a \in \mathbb{R}$

a $f'(x) = ae^{ax}(x+1) + e^{ax}(1)$ {product rule}
$= e^{ax}(a[x+1]+1)$

b $f''(x) = ae^{ax}(a[x+1]+1) + e^{ax}(a)$ {product rule}
$= ae^{ax}(a[x+1]+1+1)$
$= ae^{ax}(a[x+1]+2)$

c If $f^{(k)}(x) = a^{k-1}e^{ax}(a[x+1]+k)$
then $f^{(k+1)}(x) = a^{k-1}ae^{ax}(a[x+1]+k) + a^{k-1}e^{ax}(a)$ {product rule}
$= a^k e^{ax}(a[x+1]+k) + a^k e^{ax}$
$= a^k e^{ax}(a[x+1]+[k+1])$

18 $f(x) = e^{-x}(x+2)$

a **i** $f'(x) = -e^{-x}(x+2) + e^{-x}(1)$
$= -e^{-x}(x+2-1)$
$= -e^{-x}(x+1)$

ii $f''(x) = e^{-x}(x+1) - e^{-x}(1)$
$= e^{-x}(x+1-1)$
$= e^{-x}(x)$

iii $f'''(x) = -e^{-x}(x) + e^{-x}(1)$
$= -e^{-x}(x-1)$

iv $f^{(4)}(x) = e^{-x}(x-1) - e^{-x}(1)$
$= e^{-x}(x-1-1)$
$= e^{-x}(x-2)$

b $f^{(n)}(x) = (-1)^n e^{-x}(x-(n-2))$
$= (-1)^n e^{-x}(x-n+2)$

c P_n is "for $f(x) = e^{-x}(x+2)$, $f^{(n)}(x) = (-1)^n e^{-x}(x-n+2)$", $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) For $n = 1$, $f'(x) = -e^{-x}(x+1)$ {using **a i**}
$= (-1)^1 e^{-x}(x-1+2)$ $\therefore$ P_1 is true.

(2) If P_k is true then $f^{(k)}(x) = (-1)^k e^{-x}(x-k+2)$

$$\begin{aligned}\therefore\ f^{(k+1)}(x) &= (-1)^k(-1)e^{-x}(x-k+2) + (-1)^k e^{-x}(1) \quad \{\text{product rule}\}\\ &= (-1)^{k+1}e^{-x}(x-k+2) + (-1)(-1)^k e^{-x}(-1)\\ &= (-1)^{k+1}e^{-x}(x-k+2-1)\\ &= (-1)^{k+1}e^{-x}(x-(k+1)+2)\end{aligned}$$

Thus P_{k+1} is true whenever P_k is true

$\therefore$ since P_1 is true, P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

REVIEW SET 18A

1 **a** $f(x) = 7 + x - 3x^2$
$\therefore$ $f(3) = 7 + 3 - 3(3)^2 = -17$

b $f'(x) = 1 - 6x$
$\therefore$ $f'(3) = 1 - 6(3) = -17$

c $f''(x) = -6$
$\therefore$ $f''(3) = -6$

2 **a** $y = 3x^2 - x^4$
$\therefore$ $\dfrac{dy}{dx} = 6x - 4x^3$

b $y = \dfrac{x^3 - x}{x^2} = x - x^{-1}$
$\therefore$ $\dfrac{dy}{dx} = 1 + x^{-2} = 1 + \dfrac{1}{x^2}$

3 $f(x) = \dfrac{x}{\sqrt{x^2+1}}$ is a quotient with $u = x$ and $v = \left(x^2+1\right)^{\frac{1}{2}}$

$\therefore$ $u' = 1$ and $v' = \frac{1}{2}\left(x^2+1\right)^{-\frac{1}{2}} \times 2x$
$= x\left(x^2+1\right)^{-\frac{1}{2}}$

Now $f'(x) = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$$\begin{aligned}&= \frac{1 \times \left(x^2+1\right)^{\frac{1}{2}} - x \times x\left(x^2+1\right)^{-\frac{1}{2}}}{x^2+1}\\ &= \frac{\sqrt{x^2+1} - \frac{x^2}{\sqrt{x^2+1}}}{x^2+1}\\ &= \frac{(x^2+1) - x^2}{(x^2+1)\sqrt{x^2+1}}\\ &= \frac{1}{(x^2+1)\sqrt{x^2+1}}\end{aligned}$$

The tangent to $f(x)$ has gradient 1 when $f'(x) = 1$

$$\therefore \frac{1}{(x^2+1)\sqrt{x^2+1}} = 1$$

$$\therefore (x^2+1)^{\frac{3}{2}} = 1$$

$$\therefore x^2+1 = 1$$

$$\therefore x^2 = 0$$

$$\therefore x = 0$$

$$\text{and } f(0) = \frac{0}{\sqrt{0^2+1}} = 0$$

$\therefore$ the tangent to $f(x) = \dfrac{x}{\sqrt{x^2+1}}$ has gradient 1 at the point (0, 0).

4 **a**

$$\begin{aligned} y &= e^{x^3+2} \\ &= e^u \quad \text{where } u = x^3+2 \\ \therefore \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \quad \{\text{chain rule}\} \\ &= e^u \times 3x^2 \\ &= 3x^2 e^u \\ &= 3x^2 e^{x^3+2} \end{aligned}$$

b

$$\begin{aligned} y &= \ln\left(\frac{x+3}{x^2}\right) \\ &= \ln(x+3) - \ln(x^2) \\ \therefore \frac{dy}{dx} &= \frac{1}{x+3} - \frac{2x}{x^2} \\ &= \frac{1}{x+3} - \frac{2}{x} \end{aligned}$$

c Consider $\ln(2y+1) = xe^y$

Differentiating with respect to x, $\dfrac{d}{dx}(\ln(2y+1)) = \dfrac{d}{dx}(xe^y)$

$$\therefore \frac{2}{2y+1}\frac{dy}{dx} = \frac{d}{dx}(x)e^y + x\frac{d}{dx}(e^y)$$

$$\therefore \frac{2}{2y+1}\frac{dy}{dx} = e^y + xe^y\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx}\left(\frac{2}{2y+1} - xe^y\right) = e^y$$

$$\therefore \frac{dy}{dx}(2 - xe^y(2y+1)) = e^y(2y+1)$$

$$\therefore \frac{dy}{dx} = \frac{e^y(2y+1)}{2 - xe^y(2y+1)}$$

5

$$\begin{aligned} y &= 3e^x - e^{-x} \\ \therefore \frac{dy}{dx} &= 3e^x - \left(-e^{-x}\right) \\ &= 3e^x + e^{-x} \end{aligned}$$

$$\begin{aligned} \therefore \frac{d^2y}{dx^2} &= 3e^x + \left(-e^{-x}\right) \\ &= 3e^x - e^{-x} \\ &= y \end{aligned}$$

6 **a**

$$\begin{aligned} \frac{d}{dx}(\sin(5x)\ln x) &= \frac{d}{dx}(\sin(5x))\ln x + \sin(5x)\frac{d}{dx}(\ln x) \quad \{\text{product rule}\} \\ &= 5\cos(5x)\ln x + \frac{\sin(5x)}{x} \end{aligned}$$

b

$$\begin{aligned} \frac{d}{dx}(\sin x\cos(2x)) &= \frac{d}{dx}(\sin x)\cos(2x) + \sin x\frac{d}{dx}(\cos(2x)) \quad \{\text{product rule}\} \\ &= \cos x\cos(2x) + \sin x(-2\sin(2x)) \\ &= \cos x\cos(2x) - 2\sin x\sin(2x) \end{aligned}$$

c

$$\begin{aligned} \frac{d}{dx}\left(e^{-2x}\tan x\right) &= \frac{d}{dx}\left(e^{-2x}\right)\tan x + e^{-2x}\frac{d}{dx}(\tan x) \quad \{\text{product rule}\} \\ &= -2e^{-2x}\tan x + \frac{e^{-2x}}{\cos^2 x} \end{aligned}$$

7 $y = \sin^2 x$

$= (\sin x)^2 = u^2$ where $u = \sin x$

$\therefore \dfrac{du}{dx} = \cos x$

Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ {chain rule}

$= 2u\dfrac{du}{dx}$

$= 2\sin x\cos x$

When $x = \frac{\pi}{3}$, $\dfrac{dy}{dx} = 2\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right)$

$= \frac{\sqrt{3}}{2}$

$\therefore$ gradient of tangent $= \frac{\sqrt{3}}{2}$

8
$$x^2 + 2xy + y^2 = 4$$
$$\therefore\ 2x + 2y + 2x(1)\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$
$$\therefore\ \frac{dy}{dx} = \frac{-x-y}{x+y}$$
$$= \frac{-(x+y)}{x+y}$$
$$= -1$$
$$\therefore\ \frac{d^2y}{dx^2} = 0$$

9 **a**
$$M = (t^2+3)^4$$
$$\therefore\ \frac{dM}{dt} = 4(t^2+3)^3(2t)$$
$$= 8t(t^2+3)^3$$

b $A = \dfrac{\sqrt{t+5}}{t^2}$ is a quotient with

$u = (t+5)^{\frac{1}{2}}$ and $v = t^2$

$\therefore\ u' = \frac{1}{2}(t+5)^{-\frac{1}{2}},\ \ v' = 2t$

$$\therefore\ \frac{dA}{dt} = \frac{\frac{1}{2}(t+5)^{-\frac{1}{2}}(t^2) - (t+5)^{\frac{1}{2}}(2t)}{t^4}$$
$$= \frac{\frac{1}{2}t(t+5)^{-\frac{1}{2}} - 2(t+5)^{\frac{1}{2}}}{t^3}$$

10 **a**
$$y = \frac{4}{\sqrt{x}} - 3x = 4x^{-\frac{1}{2}} - 3x$$
$$\therefore\ \frac{dy}{dx} = -2x^{-\frac{3}{2}} - 3$$
$$= \frac{-2}{x\sqrt{x}} - 3$$

b
$$y = \sqrt{x^2-3x} = (x^2-3x)^{\frac{1}{2}}$$
$$\therefore\ \frac{dy}{dx} = \tfrac{1}{2}(x^2-3x)^{-\frac{1}{2}}(2x-3)$$
$$= \frac{2x-3}{2\sqrt{x^2-3x}}$$

11 **a**
$$f(x) = 3x^2 - \frac{1}{x} = 3x^2 - x^{-1}$$
$$\therefore\ f'(x) = 6x + x^{-2}$$
$$\therefore\ f''(x) = 6 - 2x^{-3} = 6 - \frac{2}{x^3}$$
$$\therefore\ f''(2) = 6 - \frac{2}{2^3} = \frac{23}{4}$$

b
$$f(x) = \sqrt{x} = x^{\frac{1}{2}}$$
$$\therefore\ f'(x) = \tfrac{1}{2}x^{-\frac{1}{2}}$$
$$\therefore\ f''(x) = -\tfrac{1}{4}x^{-\frac{3}{2}}$$
$$\therefore\ f''(2) = -\tfrac{1}{4}(2^{-\frac{3}{2}})$$
$$= -\frac{1}{4\sqrt{2^3}} = -\frac{1}{8\sqrt{2}}$$

12
$$y = \left(1 - \tfrac{1}{3}x\right)^3$$
$$\therefore\ \frac{dy}{dx} = 3\left(1-\tfrac{1}{3}x\right)^2\left(-\tfrac{1}{3}\right)$$
$$= -\left(1-\tfrac{1}{3}x\right)^2$$
$$\therefore\ \frac{d^2y}{dx^2} = -2\left(1-\tfrac{1}{3}x\right)\left(-\tfrac{1}{3}\right)$$
$$= \tfrac{2}{3}\left(1-\tfrac{1}{3}x\right)$$
$$= \tfrac{2}{3} - \tfrac{2}{9}x$$
$$\therefore\ \frac{d^3y}{dx^3} = -\tfrac{2}{9}$$

13 P_n is "for $y = \dfrac{1}{2x+1}$, $\dfrac{d^n y}{dx^n} = \dfrac{(-2)^n n!}{(2x+1)^{n+1}}$", $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) $y = \dfrac{1}{2x+1} = (2x+1)^{-1}$

$$\therefore \frac{dy}{dx} = -(2x+1)^{-2}(2) = \frac{-2}{(2x+1)^2}$$

$$= \frac{(-2)^1 1!}{(2x+1)^{1+1}} \qquad \therefore P_1 \text{ is true}$$

(2) If P_k is true then $\dfrac{d^k y}{dx^k} = \dfrac{(-2)^k k!}{(2x+1)^{k+1}}$

$$\text{Now } \frac{d^{k+1}y}{dx^{k+1}} = \frac{d}{dx}\left(\frac{d^k y}{dx^k}\right)$$

$$= \frac{d}{dx}\left(\frac{(-2)^k k!}{(2x+1)^{k+1}}\right) \qquad \{\text{using } P_k\}$$

$$= \frac{d}{dx}\left[(-2)^k k!\,(2x+1)^{-(k+1)}\right]$$

$$= (-2)^k k!\,[-(k+1)]\,(2x+1)^{-(k+1)-1}\,(2)$$

$$= -2(-2)^k(k+1)!\,(2x+1)^{-(k+2)}$$

$$= \frac{(-2)^{k+1}(k+1)!}{(2x+1)^{(k+1)+1}}$$

Thus P_{k+1} is true whenever P_k is true, and P_1 is true

$\therefore$ P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

REVIEW SET 18B

1 **a** $y = 5x - 3x^{-1}$

$\therefore \dfrac{dy}{dx} = 5 + 3x^{-2}$

b $y = (3x^2 + x)^4$

$\therefore \dfrac{dy}{dx} = 4(3x^2+x)^3(6x+1)$

c $y = (x^2+1)(1-x^2)^3$ is a product with $u = x^2+1$ and $v = (1-x^2)^3$

$\therefore u' = 2x$ and $v' = 3(1-x^2)^2(-2x)$

$= -6x(1-x^2)^2$

$$\therefore \frac{dy}{dx} = 2x(1-x^2)^3 + (x^2+1) \times -6x(1-x^2)^2 \qquad \{\text{product rule}\}$$

$$= 2x(1-x^2)^3 - 6x(x^2+1)(1-x^2)^2$$

2 $y = 2x^3 + 3x^2 - 10x + 3$

$\therefore \dfrac{dy}{dx} = 6x^2 + 6x - 10$

The gradient of the tangent is 2 where $6x^2 + 6x - 10 = 2$

$\therefore 6x^2 + 6x - 12 = 0$

$\therefore 6(x^2 + x - 2) = 0$

$\therefore 6(x+2)(x-1) = 0$

$\therefore x = -2$ or 1

$\therefore y = 2(-2)^3 + 3(-2)^2 - 10(-2) + 3$ *or* $y = 2(1)^3 + 3(1)^2 - 10(1) + 3$

$= -16 + 12 + 20 + 3$ $\qquad$ $= 2 + 3 - 10 + 3$

$= 19$ $\qquad$ $= -2$

So, the gradient of the tangent is 2 at $(-2, 19)$ and $(1, -2)$.

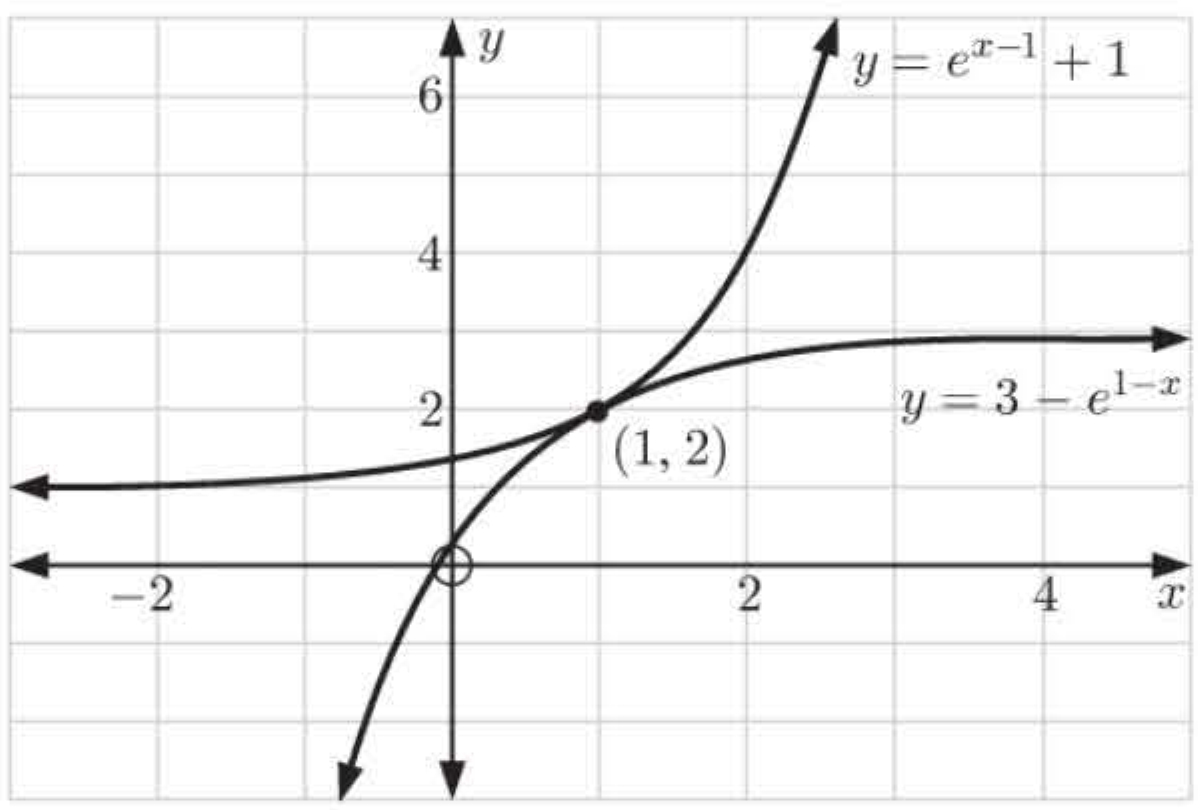
3 $y = \sqrt{5-4x} = (5-4x)^{\frac{1}{2}}$

a $\dfrac{dy}{dx} = \frac{1}{2}(5-4x)^{-\frac{1}{2}}(-4)$
$= -2(5-4x)^{-\frac{1}{2}}$

b $\dfrac{d^2y}{dx^2} = -2(-\frac{1}{2})(5-4x)^{-\frac{3}{2}}(-4)$
$= -4(5-4x)^{-\frac{3}{2}}$

c $\dfrac{d^3y}{dx^3} = -4(-\frac{3}{2})(5-4x)^{-\frac{5}{2}}(-4)$
$= -24(5-4x)^{-\frac{5}{2}}$

4 **a**



b Using technology, the point of intersection is $(1, 2)$.

c If $y = e^{x-1} + 1$
then $\dfrac{dy}{dx} = e^{x-1}$
When $x = 1$, $\dfrac{dy}{dx} = e^{1-1}$
$= e^0$
$= 1$
$\therefore$ gradient of tangent $= 1$

If $y = 3 - e^{1-x}$
then $\dfrac{dy}{dx} = -e^{1-x} \times -1$
$= e^{1-x}$
When $x = 1$, $\dfrac{dy}{dx} = e^{1-1}$
$= e^0$
$= 1$
$\therefore$ gradient of tangent $= 1$

So, the tangents to each curve at $(1, 2)$ both have a gradient of 1.

d If the tangents of each curve at $(1, 2)$ have the same gradient, then they are in fact the same line. That is, the two curves have a *common tangent* at $(1, 2)$.

5 **a** $y = \ln(x^3 - 3x)$
$\therefore \dfrac{dy}{dx} = \dfrac{3x^2-3}{x^3-3x}$

b $y = \dfrac{e^x}{x^2}$
$\therefore \dfrac{dy}{dx} = \dfrac{e^x(x^2) - e^x(2x)}{x^4}$ {quotient rule}
$= \dfrac{e^x(x-2)}{x^3}$

c Consider $e^{x+y} = \ln(y^2+1)$.

Differentiating with respect to x, $\left(1 + \dfrac{dy}{dx}\right)e^{x+y} = \dfrac{2y}{y^2+1}\dfrac{dy}{dx}$

$$\therefore \left(1 + \frac{dy}{dx}\right)e^{x+y}(y^2+1) = 2y\frac{dy}{dx}$$

$$\therefore e^{x+y}(y^2+1) = \frac{dy}{dx}(2y - e^{x+y}(y^2+1))$$

$$\therefore \frac{dy}{dx} = \frac{e^{x+y}(y^2+1)}{2y - e^{x+y}(y^2+1)}$$

6 $f(x) = 2x^4 - 4x^3 - 9x^2 + 4x + 7$
$\therefore f'(x) = 8x^3 - 12x^2 - 18x + 4$
$\therefore f''(x) = 24x^2 - 24x - 18$

So, $f''(x) = 0$ where $24x^2 - 24x - 18 = 0$
$\therefore 4x^2 - 4x - 3 = 0$
$\therefore (2x+1)(2x-3) = 0$
$\therefore x = -\frac{1}{2}$ or $x = \frac{3}{2}$

7 **a** $f(x) = x - \cos x$
$\therefore\ f(\pi) = \pi - \cos\pi$
$= \pi - (-1)$
$= \pi + 1$

b $f(x) = x - \cos x$
$\therefore\ f'(x) = 1 - (-\sin x)$
$= 1 + \sin x$
$\therefore\ f'\left(\frac{\pi}{2}\right) = 1 + \sin\left(\frac{\pi}{2}\right)$
$= 1 + 1 = 2$

c $f'(x) = 1 + \sin x$ {from **b**}
$\therefore\ f''(x) = \cos x$
$\therefore\ f''\left(\frac{3\pi}{4}\right) = \cos\left(\frac{3\pi}{4}\right)$
$= -\frac{1}{\sqrt{2}}$ $\left(\text{or } -\frac{\sqrt{2}}{2}\right)$

8 $y = 3\sin bx - a\cos 2x$

$\therefore\ \dfrac{dy}{dx} = 3 \times (\cos bx) \times b - a \times (-\sin 2x) \times 2$
$= 3b\cos bx + 2a\sin 2x$

$\therefore\ \dfrac{d^2y}{dx^2} = 3b \times (-\sin bx) \times b + 2a \times (\cos 2x) \times 2$
$= -3b^2 \sin bx + 4a\cos 2x$

Now $y + \dfrac{d^2y}{dx^2} = 6\cos 2x$

$\therefore\ 3\sin bx - a\cos 2x + 4a\cos 2x - 3b^2\sin bx = 6\cos 2x$
$\therefore\ 3\sin bx - 3b^2\sin bx + 3a\cos 2x = 6\cos 2x$
$\therefore\ (3 - 3b^2)\sin bx + 3a\cos 2x = 6\cos 2x$

$\therefore\ 3 - 3b^2 = 0$ and $3a = 6$
$\therefore\ 3(1 - b^2) = 0$ and $a = 2$
$\therefore\ 3(1+b)(1-b) = 0$
$\therefore\ 1 + b = 0$ or $1 - b = 0$
$\therefore\ b = -1$ or 1

So, $a = 2$ and $b = -1$ or 1.

9 **a** $\dfrac{d}{dx}(10x - \sin(10x)) = 10 - 10\cos(10x)$

b $\dfrac{d}{dx}\left(\ln\left(\dfrac{1}{\cos x}\right)\right) = \dfrac{1}{\left(\frac{1}{\cos x}\right)} \times \dfrac{d}{dx}\left(\dfrac{1}{\cos x}\right)$ {chain rule}
$= \cos x \times \dfrac{d}{dx}\left((\cos x)^{-1}\right)$
$= \cos x \times \left(-(\cos x)^{-2} \times (-\sin x)\right)$
$= \dfrac{\cos x \sin x}{\cos^2 x} = \tan x$

c $\dfrac{d}{dx}(\sin(5x)\ln(2x)) = \dfrac{d}{dx}(\sin(5x))\ln(2x) + \sin(5x)\dfrac{d}{dx}(\ln(2x))$ {product rule}
$= 5\cos(5x)\ln(2x) + \sin(5x) \times \dfrac{2}{2x}$
$= 5\cos(5x)\ln(2x) + \dfrac{\sin(5x)}{x}$

10 **a** $f(x) = \dfrac{(x+3)^3}{\sqrt{x}}$ is a quotient with $u = (x+3)^3$ and $v = x^{\frac{1}{2}}$

$\therefore\ u' = 3(x+3)^2$ and $v' = \frac{1}{2}x^{-\frac{1}{2}}$

$\therefore\ f'(x) = \dfrac{3(x+3)^2\sqrt{x} - \frac{1}{2}x^{-\frac{1}{2}}(x+3)^3}{x}$ {quotient rule}

b $f(x) = x^4\sqrt{x^2+3}$ is a product with $u = x^4$ and $v = (x^2+3)^{\frac{1}{2}}$

$\therefore\ u' = 4x^3$ and $v' = \frac{1}{2}(x^2+3)^{-\frac{1}{2}}(2x) = x(x^2+3)^{-\frac{1}{2}}$

$\therefore\ f'(x) = 4x^3\sqrt{x^2+3} + \dfrac{x^5}{\sqrt{x^2+3}}$ {product rule}

11 **a** $y = \dfrac{x}{\sqrt{\sec x}} = x(\cos x)^{\frac{1}{2}} \quad \therefore \quad \dfrac{dy}{dx} = (\cos x)^{\frac{1}{2}} + x \times \frac{1}{2}(\cos x)^{-\frac{1}{2}}(-\sin x)$ {product rule}

$$= \sqrt{\cos x} - \frac{x \sin x}{2\sqrt{\cos x}}$$

b $y = e^x \cot(2x) \quad \therefore \quad \dfrac{dy}{dx} = e^x \cot(2x) + e^x \times 2\left(-\csc^2(2x)\right)$ {product rule}

$$= e^x\left(\cot(2x) - 2\csc^2(2x)\right)$$

c $y = \arccos\left(\frac{x}{2}\right) \quad \therefore \quad \dfrac{dy}{dx} = \frac{1}{2} \times \left(\dfrac{-1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\right) = -\dfrac{1}{2\sqrt{1-\frac{x^2}{4}}} = -\dfrac{1}{\sqrt{4-x^2}}$

12 $f(x) = 2x^3 + Ax + B \quad \therefore \quad f'(x) = 6x^2 + A$

Now as the gradient at $(-2, 33)$ is 10,

$\therefore \; f'(-2) = 10$

$\therefore \; 10 = 6(-2)^2 + A$

$\therefore \; A = -14$

$\therefore \; f(x) = 2x^3 - 14x + B$

Then, since $(-2, 33)$ lies on the curve,

$f(-2) = 33$

$\therefore \; 2(-2)^3 - 14(-2) + B = 33$

$\therefore \; -16 + 28 + B = 33$

$\therefore \; B = 21$

13
$$x^2 - 3y^2 = 0$$
$$\therefore \; 2x - 6y\frac{dy}{dx} = 0$$
$$\therefore \; \frac{dy}{dx} = \frac{x}{3y}$$

For any point (x, y) on $x^2 - 3y^2 = 0$, $\dfrac{dy}{dx}$ gives us the gradient of the tangent at that point.

REVIEW SET 18C

1 **a** $y = x^3\sqrt{1-x^2}$ is a product where $u = x^3$ and $v = \left(1-x^2\right)^{\frac{1}{2}}$

$\therefore \; u' = 3x^2$ and $v' = \frac{1}{2}\left(1-x^2\right)^{-\frac{1}{2}}(-2x)$

$= -x\left(1-x^2\right)^{-\frac{1}{2}}$

$\therefore \; \dfrac{dy}{dx} = 3x^2\left(1-x^2\right)^{\frac{1}{2}} + x^3 \times -x\left(1-x^2\right)^{-\frac{1}{2}}$ {product rule}

$= 3x^2\left(1-x^2\right)^{\frac{1}{2}} - x^4\left(1-x^2\right)^{-\frac{1}{2}}$

b $y = \dfrac{x^2-3x}{\sqrt{x+1}}$ is a quotient where $u = x^2 - 3x$ and $v = (x+1)^{\frac{1}{2}}$

$\therefore \; u' = 2x - 3$ and $v' = \frac{1}{2}(x+1)^{-\frac{1}{2}}$

$\therefore \; \dfrac{dy}{dx} = \dfrac{(2x-3)(x+1)^{\frac{1}{2}} - \frac{1}{2}(x^2-3x)(x+1)^{-\frac{1}{2}}}{x+1}$ {quotient rule}

2 **a**
$$y = 3x^4 - \frac{2}{x} = 3x^4 - 2x^{-1}$$
$$\therefore \; \frac{dy}{dx} = 12x^3 + 2x^{-2}$$
$$\therefore \; \frac{d^2y}{dx^2} = 36x^2 - 4x^{-3}$$
$$= 36x^2 - \frac{4}{x^3}$$

b
$$y = x^3 - x + \frac{1}{\sqrt{x}} = x^3 - x + x^{-\frac{1}{2}}$$
$$\therefore \; \frac{dy}{dx} = 3x^2 - 1 - \tfrac{1}{2}x^{-\frac{3}{2}}$$
$$\therefore \; \frac{d^2y}{dx^2} = 6x + \tfrac{3}{4}x^{-\frac{5}{2}}$$

3 $y = xe^x$ is the product of $u = x$ and $v = e^x$

$\therefore\ u' = 1$ and $v' = e^x$

Now
$$\begin{aligned}\frac{dy}{dx} &= u'v + uv' \quad \{\text{product rule}\}\\ &= 1 \times e^x + x \times e^x\\ &= e^x + xe^x\\ &= (1+x)e^x\end{aligned}$$

If $\dfrac{dy}{dx} = 2e$, then $(1+x)e^x = 2e$

Solving by inspection, we find $x = 1$. When $x = 1$, $y = 1 \times e^1 = e$.

$\therefore$ the gradient of $y = xe^x$ is $2e$ at the point $(1, e)$.

4 **a**
$$\begin{aligned}f(x) &= \ln(e^x + 3)\\ \therefore\ f'(x) &= \frac{e^x}{e^x + 3}\end{aligned}$$

b
$$\begin{aligned}f(x) &= \ln\left[\frac{(x+2)^3}{x}\right]\\ &= \ln(x+2)^3 - \ln x\\ &= 3\ln(x+2) - \ln x\\ \therefore\ f'(x) &= \frac{3}{x+2} - \frac{1}{x}\end{aligned}$$

c
$$\begin{aligned}f(x) &= x^{x^2}\\ \therefore\ \ln y &= \ln x^{x^2}\\ \therefore\ \ln y &= x^2 \ln x\end{aligned}$$

Differentiating with respect to x,
$$\begin{aligned}\frac{1}{y}\frac{dy}{dx} &= 2x\ln x + x^2\left(\frac{1}{x}\right)\\ \therefore\ \frac{dy}{dx} &= y(2x\ln x + x)\\ &= x^{x^2}(2x\ln x + x)\\ &= x^{x^2+1}(2\ln x + 1)\end{aligned}$$

5
$$\begin{aligned}y &= \left(x - \frac{1}{x}\right)^4\\ &= \left(x - x^{-1}\right)^4\\ \therefore\ \frac{dy}{dx} &= 4\left(x - x^{-1}\right)^3(1 + x^{-2})\\ &= 4\left(x - \frac{1}{x}\right)^3\left(1 + \frac{1}{x^2}\right)\end{aligned}$$

When $x = 1$,
$$\begin{aligned}\frac{dy}{dx} &= 4\left(1 - \frac{1}{1}\right)^3\left(1 + \frac{1}{1^2}\right)\\ &= 4 \times 0 \times 2\\ &= 0\end{aligned}$$

6 **a**
$$\begin{aligned}f(x) &= x^{\frac{1}{2}}\cos(4x)\\ \therefore\ f'(x) &= \tfrac{1}{2}x^{-\frac{1}{2}}\cos(4x) + x^{\frac{1}{2}}(-4\sin(4x)) \qquad \{\text{product rule}\}\\ &= \tfrac{1}{2}x^{-\frac{1}{2}}\cos(4x) - 4x^{\frac{1}{2}}\sin(4x)\end{aligned}$$

and
$$\begin{aligned}f''(x) &= -\tfrac{1}{4}x^{-\frac{3}{2}}\cos(4x) + \tfrac{1}{2}x^{-\frac{1}{2}}(-4\sin(4x)) - \left[2x^{-\frac{1}{2}}\sin(4x) + 4x^{\frac{1}{2}} \times 4\cos(4x)\right]\\ &= -\tfrac{1}{4}x^{-\frac{3}{2}}\cos(4x) - 4x^{-\frac{1}{2}}\sin(4x) - 16x^{\frac{1}{2}}\cos(4x)\end{aligned}$$

b
$$\begin{aligned}f'(x) &= \tfrac{1}{2}x^{-\frac{1}{2}}\cos(4x) - 4x^{\frac{1}{2}}\sin(4x) \qquad \{\text{from } \mathbf{a}\}\\ \therefore\ f'\left(\tfrac{\pi}{16}\right) &= \tfrac{1}{2}\left(\tfrac{\pi}{16}\right)^{-\frac{1}{2}}\cos\left(\tfrac{\pi}{4}\right) - 4\left(\tfrac{\pi}{16}\right)^{\frac{1}{2}}\sin\left(\tfrac{\pi}{4}\right) \approx -0.455\\ f''(x) &= -\tfrac{1}{4}x^{-\frac{3}{2}}\cos(4x) - 4x^{-\frac{1}{2}}\sin(4x) - 16x^{\frac{1}{2}}\cos(4x)\\ \therefore\ f''\left(\tfrac{\pi}{8}\right) &= -\tfrac{1}{4}\left(\tfrac{\pi}{8}\right)^{-\frac{3}{2}}\cos\left(\tfrac{\pi}{2}\right) - 4\left(\tfrac{\pi}{8}\right)^{-\frac{1}{2}}\sin\left(\tfrac{\pi}{2}\right) - 16\left(\tfrac{\pi}{8}\right)^{\frac{1}{2}}\cos\left(\tfrac{\pi}{2}\right)\\ &= 0 - 4\left(\tfrac{\pi}{8}\right)^{-\frac{1}{2}}(1) - 0 \qquad \{\text{since } \cos\left(\tfrac{\pi}{2}\right) = 0\}\\ &\approx -6.38\end{aligned}$$

7
$$\begin{aligned} y &= 3\sin 2x + 2\cos 2x \\ \therefore\ \frac{dy}{dx} &= 3\times(\cos 2x)\times 2 + 2\times(-\sin 2x)\times 2 \\ &= 6\cos 2x - 4\sin 2x \\ \therefore\ \frac{d^2y}{dx^2} &= 6\times(-\sin 2x)\times 2 - 4\times(\cos 2x)\times 2 \\ &= -12\sin 2x - 8\cos 2x \end{aligned}$$

$$\begin{aligned} \therefore\ 4y + \frac{d^2y}{dx^2} &= 4(3\sin 2x + 2\cos 2x) + (-12\sin 2x - 8\cos 2x) \\ &= 12\sin 2x + 8\cos 2x - 12\sin 2x - 8\cos 2x \\ &= 0 \end{aligned}$$

8 **a** $f(x) = \dfrac{6x}{3+x^2}$ Now, $f(x) = -\frac{1}{2}$ when $\dfrac{6x}{3+x^2} = -\frac{1}{2}$

$$\begin{aligned} \therefore\ 12x &= -(3+x^2) \\ \therefore\ x^2 + 12x + 3 &= 0 \\ \therefore\ x &= \frac{-12 \pm \sqrt{12^2 - 4\times 1\times 3}}{2} \\ &= \frac{-12\pm\sqrt{132}}{2} \approx -11.7 \text{ or } -0.255 \end{aligned}$$

b $f(x) = \dfrac{6x}{3+x^2}$ is a quotient where $u = 6x$ and $v = 3+x^2$

$\therefore\ u' = 6$ and $v' = 2x$

$$\begin{aligned} \therefore\ f'(x) &= \frac{u'v - uv'}{v^2} \quad \{\text{quotient rule}\} \\ &= \frac{6(3+x^2) - 6x(2x)}{(3+x^2)^2} \\ &= \frac{18 + 6x^2 - 12x^2}{(3+x)^2} \\ &= \frac{18 - 6x^2}{(3+x^2)^2} \end{aligned}$$

Now, $f'(x) = 0$ when $\dfrac{18-6x^2}{(3+x^2)^2} = 0$

$$\begin{aligned} \therefore\ 18 - 6x^2 &= 0 \quad \{\text{since } (3+x^2)^2 > 0 \text{ for all } x\in\mathbb{R}\} \\ \therefore\ 6(3-x^2) &= 0 \\ \therefore\ x^2 &= 3 \\ \therefore\ x &= \pm\sqrt{3} \approx -1.73 \text{ or } 1.73 \end{aligned}$$

c $f'(x) = \dfrac{18-6x^2}{(3+x^2)^2}$ is a quotient where $u = 18 - 6x^2$ and $v = (3+x^2)^2$

$\therefore\ u' = -12x$ and $v' = 2(3+x^2)\times 2x$

$= 4x(3+x^2)$

$$\begin{aligned} f''(x) &= \frac{u'v - uv'}{v^2} \quad \{\text{quotient rule}\} \\ &= \frac{-12x(3+x^2)^2 - (18-6x^2)\times 4x(3+x^2)}{(3+x^2)^4} \\ &= \frac{(3+x^2)[-12x(3+x^2) - 4x(18-6x^2)]}{(3+x^2)^4} \\ &= \frac{-36x - 12x^3 - 72x + 24x^3}{(3+x^2)^3} \\ &= \frac{12x^3 - 108x}{(3+x^2)^3} \end{aligned}$$

Now, $f''(x) = 0$ when $\dfrac{12x^3 - 108x}{(3+x^2)^3} = 0$

$\therefore\ 12x^3 - 108x = 0$ $\{$since $3 + x^2 > 0$ for all $x \in \mathbb{R}\}$

$\therefore\ 12x(x^2 - 9) = 0$

$\therefore\ 12x(x+3)(x-3) = 0$

$\therefore\ x = 0,\ -3,$ or 3

9 **a** $f(x) = -10\sin 2x \cos 2x,\quad 0 \leqslant x \leqslant \pi$

$\therefore\ f(x) = -5\sin 4x \quad \{2\sin A\cos A = \sin 2A\}$

b $f'(x) = -20\cos 4x$

If $f'(x) = 0,\quad -20\cos 4x = 0$

$\therefore\ \cos 4x = 0$

$\therefore\ 4x = \frac{\pi}{2} + n\pi,\quad n$ any integer

$\therefore\ x = \frac{\pi}{8} + \frac{n\pi}{4}$

So, for the domain $0 \leqslant x \leqslant \pi,\quad x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$

10 **a** $e^x y - xy^2 = 1$

$$\therefore\ e^x y + e^x(1)\frac{dy}{dx} - \left((1)y^2 + x(2y)\frac{dy}{dx}\right) = 0$$

$$\therefore\ e^x y + e^x\frac{dy}{dx} - y^2 - 2xy\frac{dy}{dx} = 0$$

$$\therefore\ \frac{dy}{dx} = \frac{y^2 - e^x y}{e^x - 2xy}$$

b When $x = 0,\quad e^0 y - 0y^2 = 1$

$\therefore\ y = 1$

When $x = 0$ and $y = 1$, $\dfrac{dy}{dx} = \dfrac{1^2 - e^0(1)}{e^0 - 2(0)(1)}$

$= \frac{0}{1}$

$= 0$

So, the gradient of the curve at $x = 0$ is 0.

11 P_n is "If $y = x^n$, then $\dfrac{dy}{dx} = nx^{n-1}$", $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, then $y = x$. This has gradient 1, so $\dfrac{dy}{dx} = 1 = 1x^0$. $\therefore$ P_1 is true.

(2) If P_k is true, then $y = x^k$ implies that $\dfrac{dy}{dx} = kx^{k-1}$

If $y = x^{k+1} = x^k x$,

then $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(x^k\right)x + x^k\dfrac{d}{dx}(x)$ $\{$product rule$\}$

$= kx^{k-1}x + x^k \times 1$

$= kx^k + x^k$

$= (k+1)x^k$

Hence P_{k+1} is true whenever P_k is true, and P_1 is true

$\therefore$ P_n is true for all $n \in \mathbb{Z}^+$ $\{$Principle of mathematical induction$\}$

12 **a** $y = uv$

$\therefore \quad \dfrac{dy}{dx} = \left(\dfrac{du}{dx}\right)v + u\left(\dfrac{dv}{dx}\right)$ {product rule}

$\therefore \quad \dfrac{d^2y}{dx^2} = \left[\left(\dfrac{d^2u}{dx^2}\right)v + \dfrac{du}{dx}\dfrac{dv}{dx}\right] + \left[\dfrac{du}{dx}\dfrac{dv}{dx} + u\left(\dfrac{d^2v}{dx^2}\right)\right]$ {product rule}

$= \left(\dfrac{d^2u}{dx^2}\right)v + 2\dfrac{du}{dx}\dfrac{dv}{dx} + u\left(\dfrac{d^2v}{dx^2}\right)$

b $y = uvw = u(vw)$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{du}{dx}(vw) + u\left[\dfrac{d}{dx}(vw)\right]$ {product rule}

$= \dfrac{du}{dx}vw + u\left(\dfrac{dv}{dx}w + v\dfrac{dw}{dx}\right)$ {product rule}

$= \dfrac{du}{dx}vw + u\dfrac{dv}{dx}w + uv\dfrac{dw}{dx}$

13 **a** $f(x) = xe^{ax}$

$f'(x) = e^{ax} + x(ae^{ax})$
$= e^{ax}(ax+1)$

$f''(x) = ae^{ax}(ax+1) + e^{ax}(a)$
$= ae^{ax}(ax+1+1)$
$= ae^{ax}(ax+2)$

$f'''(x) = a^2e^{ax}(ax+2) + ae^{ax}(a)$
$= a^2e^{ax}(ax+2+1)$
$= a^2e^{ax}(ax+3)$

$f^{(4)}(x) = a^3e^{ax}(ax+3) + a^2e^{ax}(a)$
$= a^3e^{ax}(ax+3+1)$
$= a^3e^{ax}(ax+4)$

b $f^{(n)}(x) = a^{n-1}e^{ax}(ax+n)$

c P_n is "for $f(x) = xe^{ax}$, $f^{(n)}(x) = a^{n-1}e^{ax}(ax+n)$", $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) For $n = 1$, $f'(x) = e^{ax}(ax+1)$ {using **a**}
$= a^{1-1}e^{ax}(ax+1)$ $\therefore$ P_1 is true.

(2) If P_k is true then $f^{(k)}(x) = a^{k-1}e^{ax}(ax+k)$

$\therefore \quad f^{(k+1)}(x) = a^{k-1}(a)e^{ax}(ax+k) + a^{k-1}e^{ax}(a)$ {product rule}
$= a^ke^{ax}(ax+k) + a^ke^{ax}$
$= a^{(k+1)-1}e^{ax}(ax+[k+1])$

Thus P_{k+1} is true whenever P_k is true

$\therefore$ since P_1 is true, P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}