Chapter 5

TRANSFORMING FUNCTIONS

EXERCISE 5A

1 $f(x) = x$

a $f(2x) = 2x$

b $f(x) + 2$
$= x + 2$

c $\frac{1}{2} f(x) = \dfrac{x}{2}$

d $2\,f(x) + 3$
$= 2x + 3$

2 $f(x) = x^2$

a $f(3x) = (3x)^2$
$= 9x^2$

b $f\left(\dfrac{x}{2}\right) = \left(\dfrac{x}{2}\right)^2$
$= \dfrac{x^2}{4}$

c $3\,f(x) = 3x^2$

d $2\,f(x-1) + 5$
$= 2(x-1)^2 + 5$
$= 2(x^2 - 2x + 1) + 5$
$= 2x^2 - 4x + 7$

3 $f(x) = x^3$

a $f(4x)$
$= (4x)^3$
$= 64x^3$

b $\frac{1}{2} f(2x)$
$= \frac{1}{2}(2x)^3$
$= \frac{1}{2} \times 8x^3$
$= 4x^3$

c $f(x+1)$
$= (x+1)^3$
$= x^3 + 3x^2 + 3x + 1$

d $2\,f(x+1) - 3$
$= 2(x+1)^3 - 3$
$= 2(x^3 + 3x^2 + 3x + 1) - 3$
$= 2x^3 + 6x^2 + 6x - 1$

4 $f(x) = 2^x$

a $f(2x) = 2^{2x}$
$= 4^x$

b $f(-x) + 1$
$= 2^{-x} + 1$

c $f(x-2) + 3$
$= 2^{x-2} + 3$

d $2\,f(x) + 3$
$= 2 \times 2^x + 3$
$= 2^{x+1} + 3$

5 $f(x) = \dfrac{1}{x}$

a $f(-x)$
$= \dfrac{1}{(-x)}$
$= -\dfrac{1}{x}$

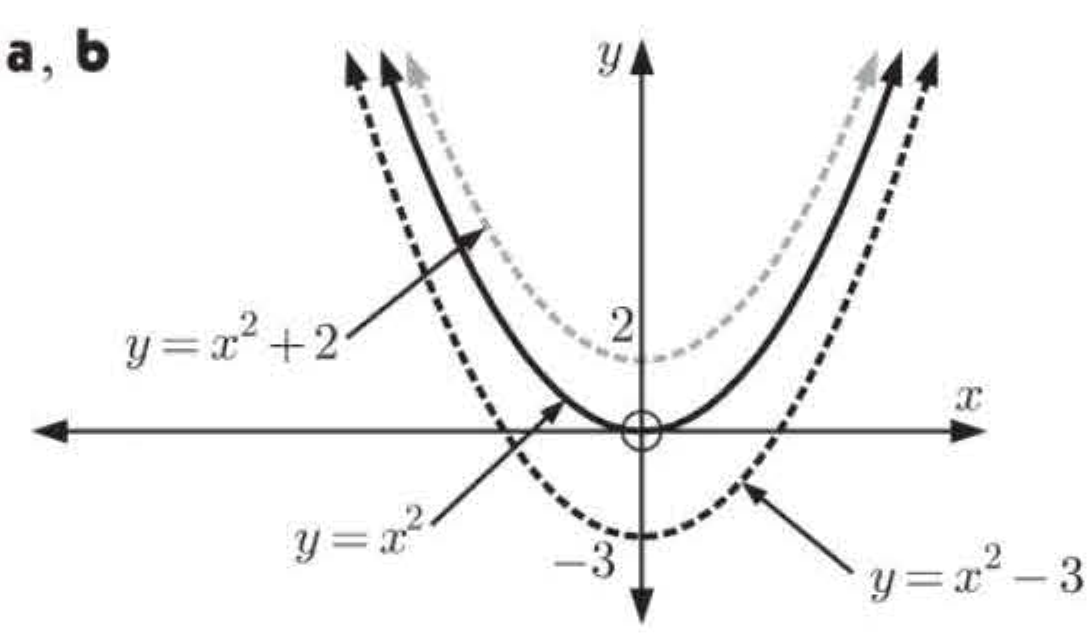
b $f(\frac{1}{2}x)$
$= \dfrac{1}{\frac{1}{2}x}$
$= \dfrac{2}{x}$

c $2\,f(x) + 3$
$= 2\left(\dfrac{1}{x}\right) + 3$
$= \dfrac{2}{x} + 3$
$= \dfrac{2+3x}{x}$

d $3\,f(x-1) + 2$
$= 3\left(\dfrac{1}{x-1}\right) + 2$
$= \dfrac{3 + 2(x-1)}{x-1}$
$= \dfrac{2x+1}{x-1}$

EXERCISE 5B

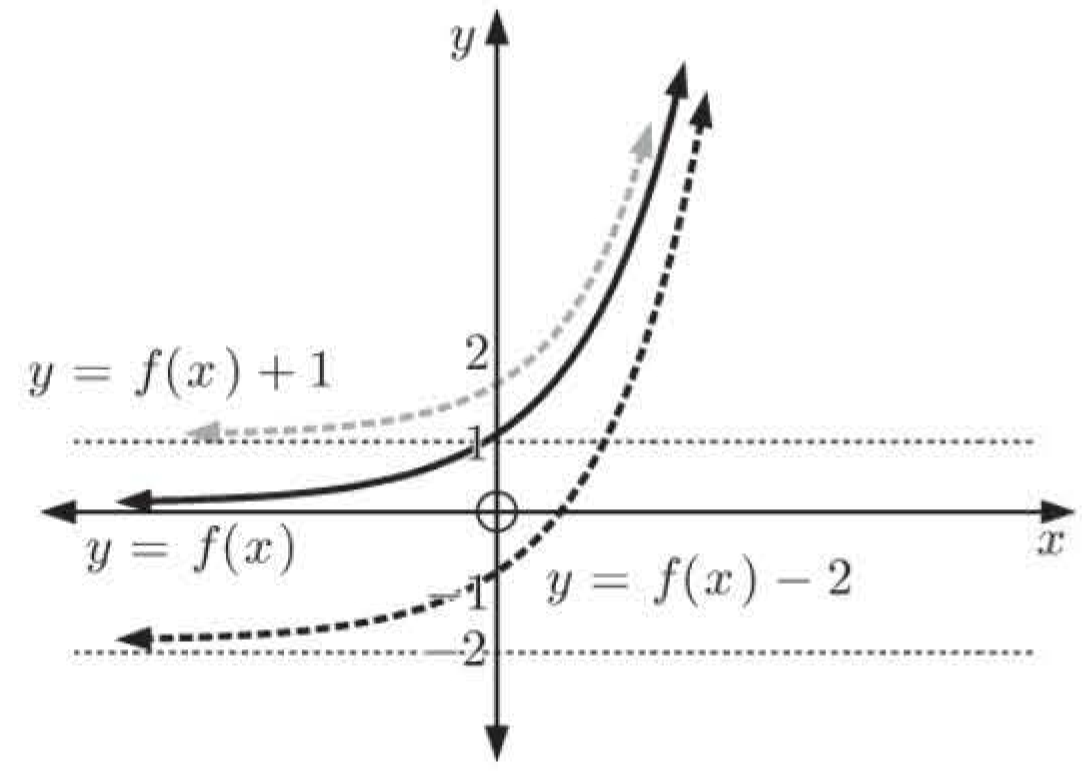
1 **a, b**



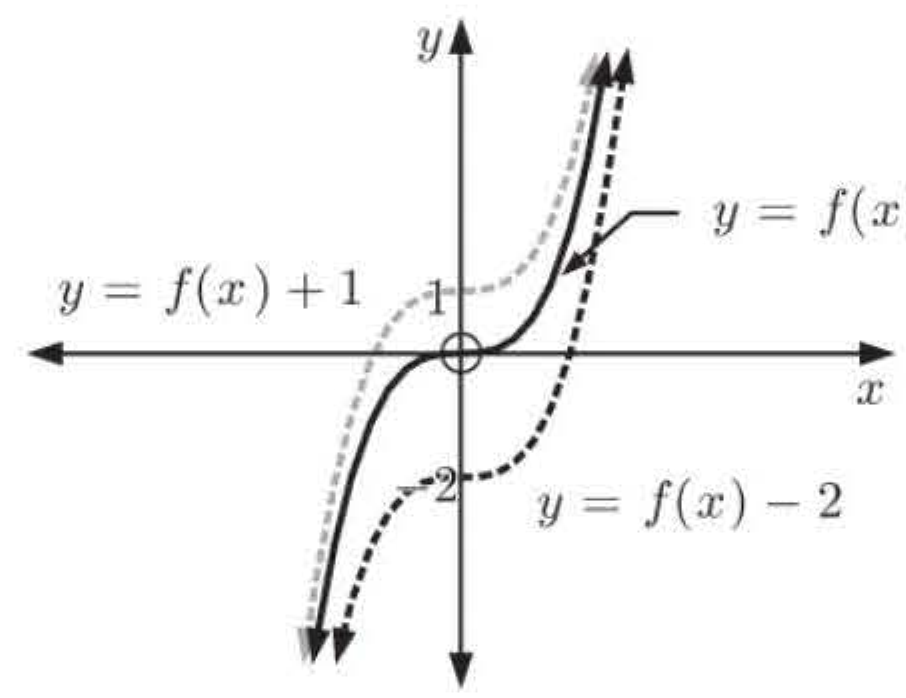
c **i** If $b > 0$, the function is translated vertically upwards through b units.

ii If $b < 0$, the function is translated vertically downwards $|b|$ units.

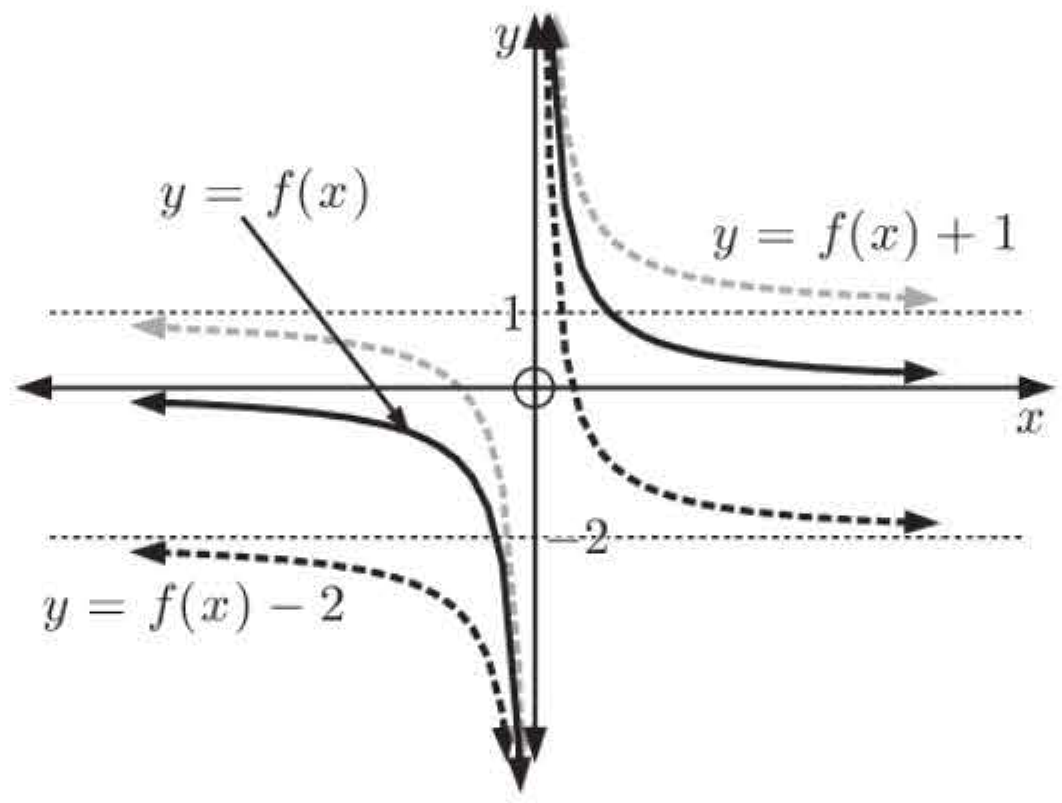
2 **a**



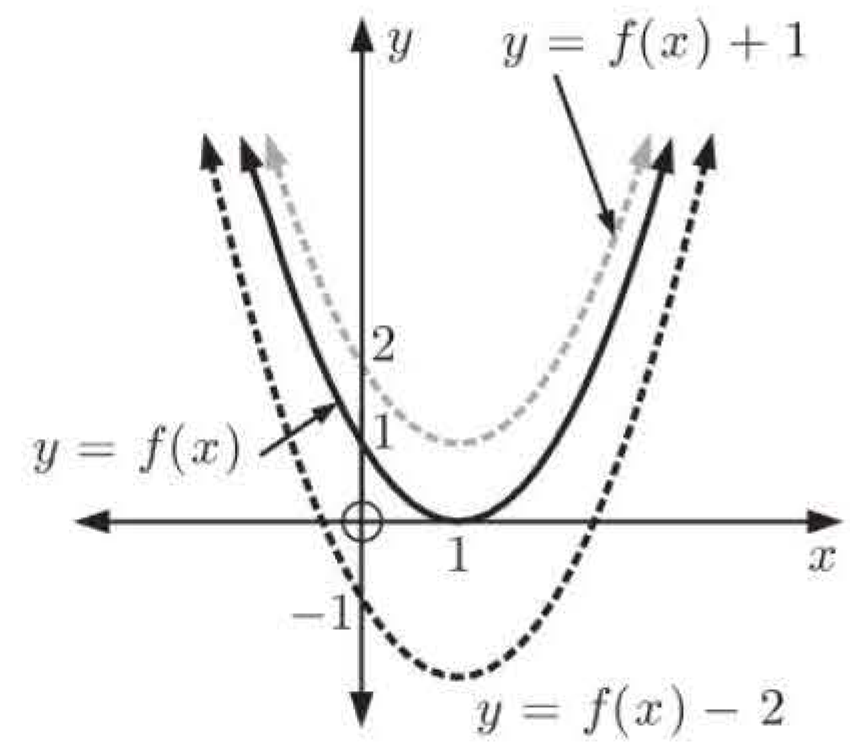
b



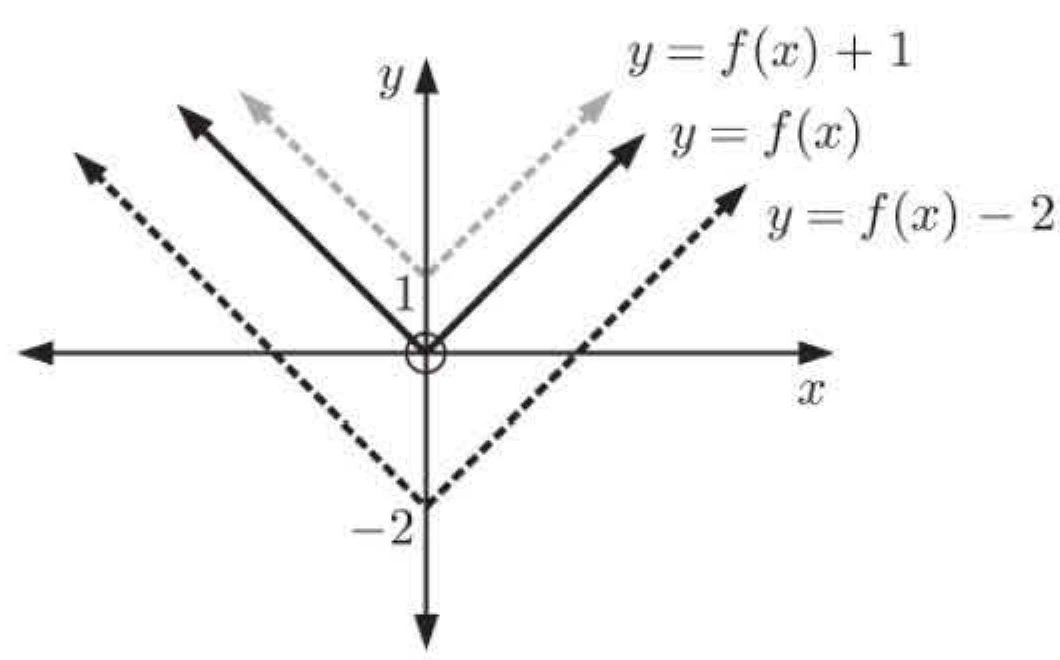
c



d

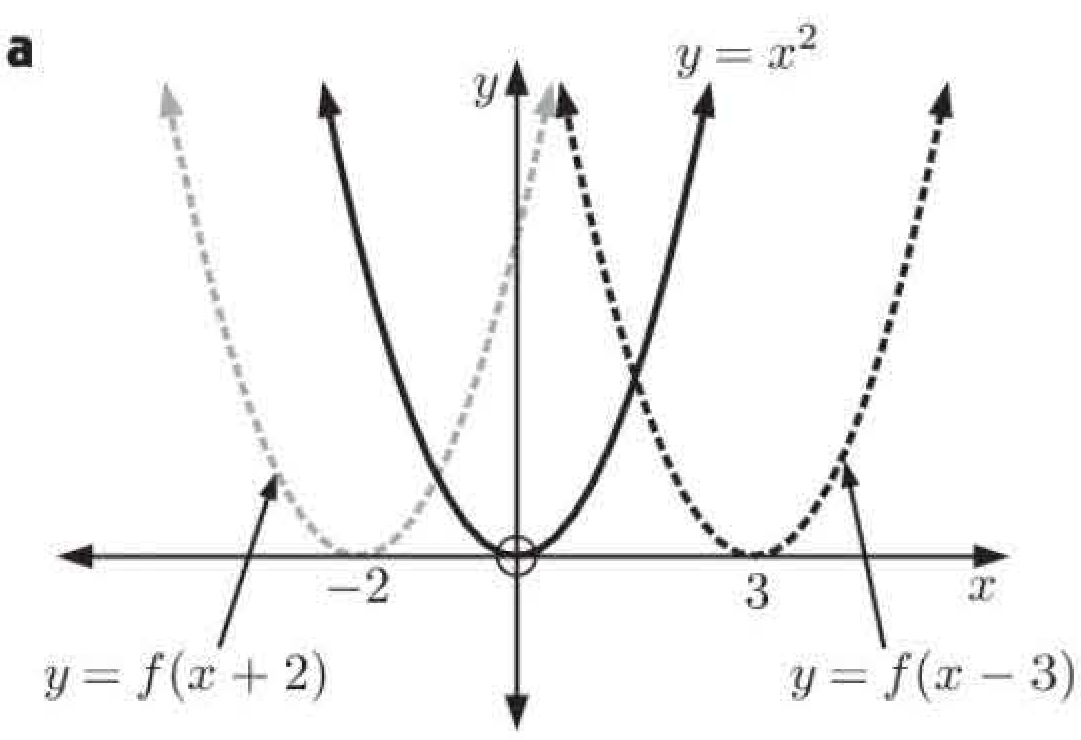


e



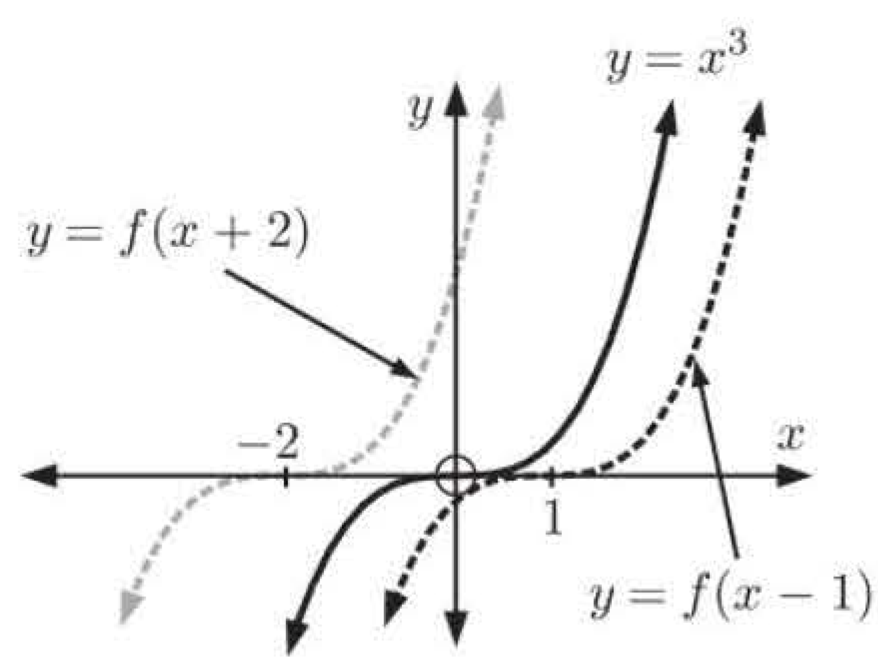
Summary: For $y = f(x) + b$, $y = f(x)$ is translated vertically through b units.
If $b > 0$ movement is vertically upwards b units.
If $b < 0$ movement is vertically downwards $|b|$ units.

3 **a**

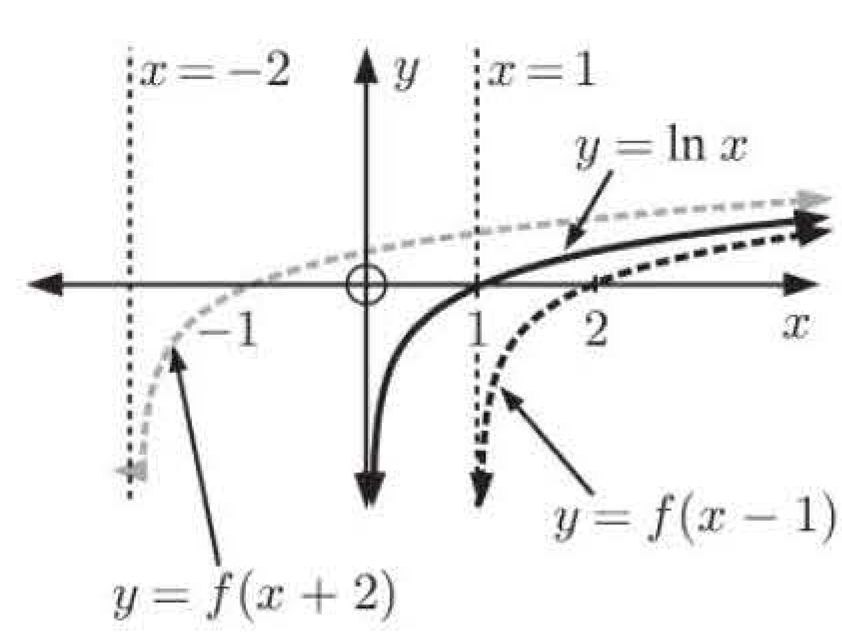


b **i** If $a > 0$, the graph is translated a units right.
ii If $a < 0$, the graph is translated $|a|$ units left.

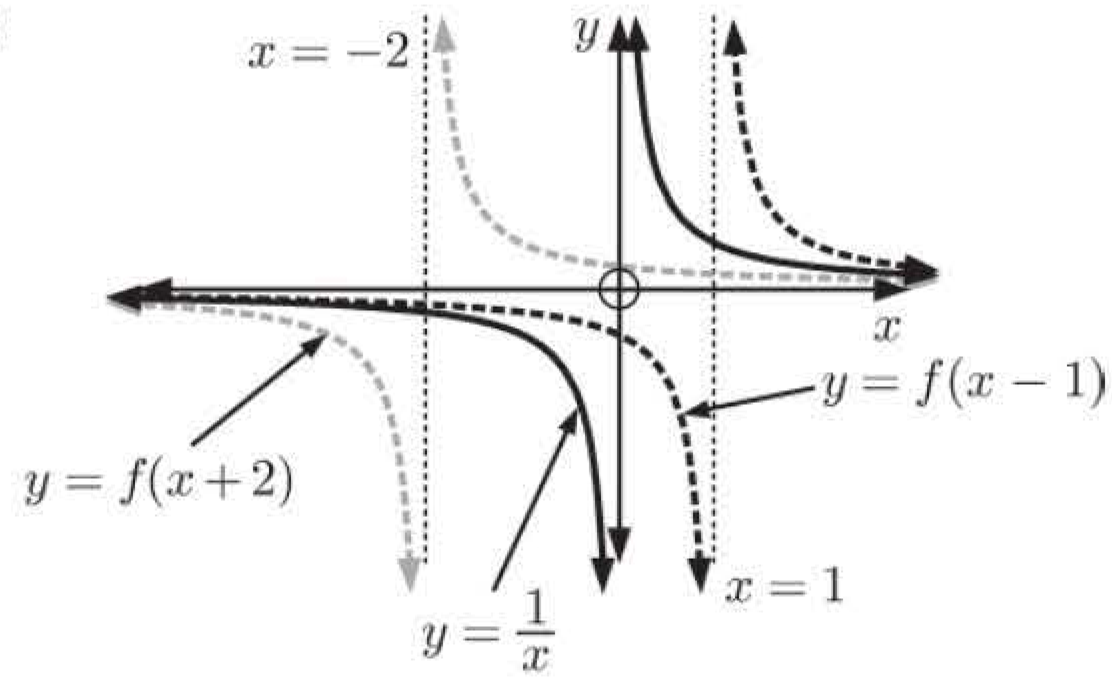
4 **a**



b



c



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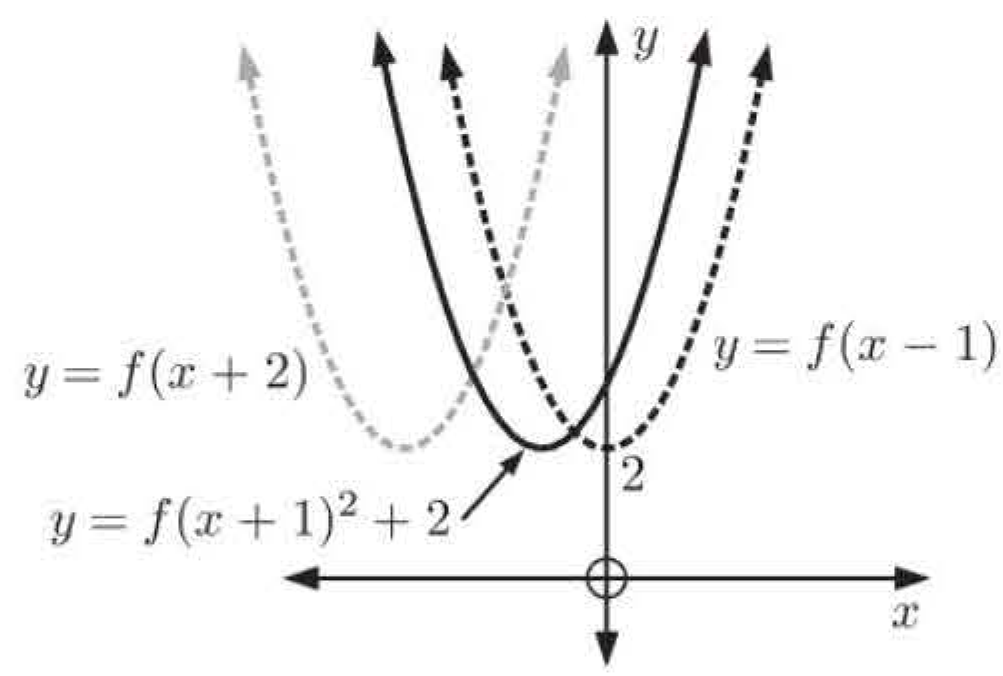

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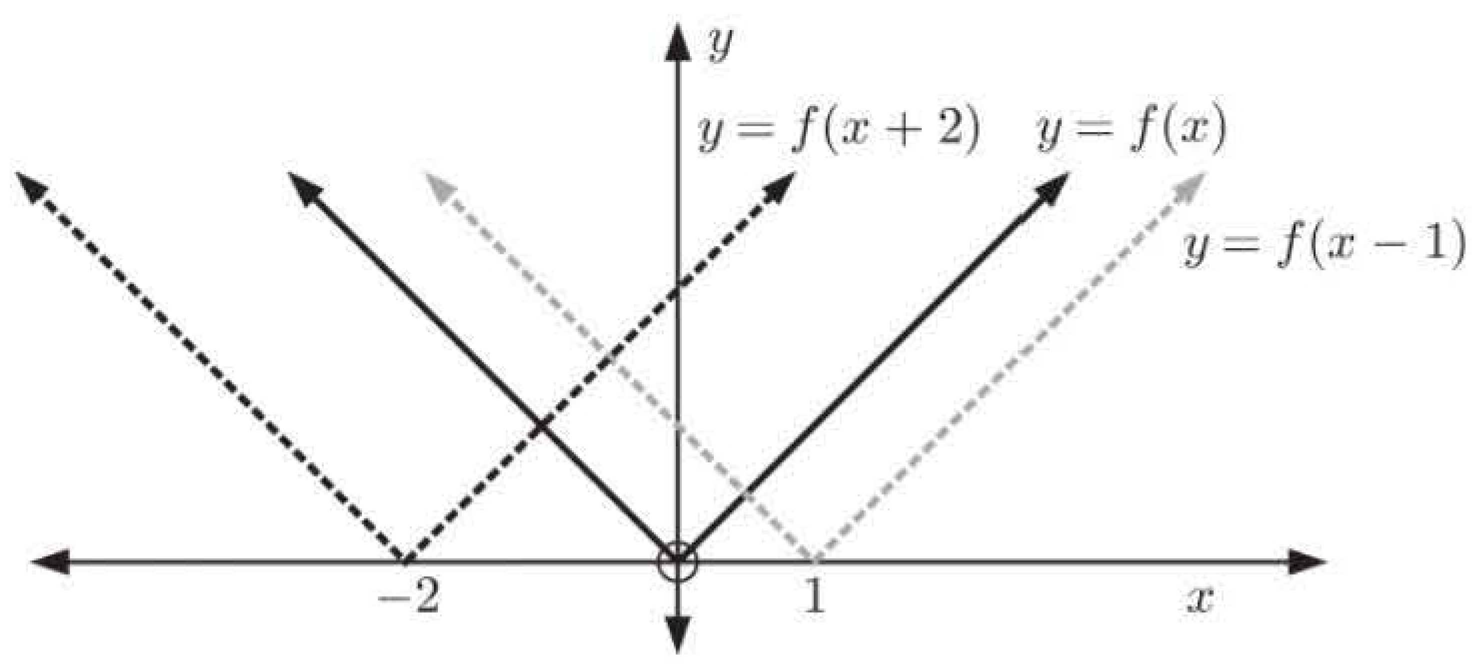
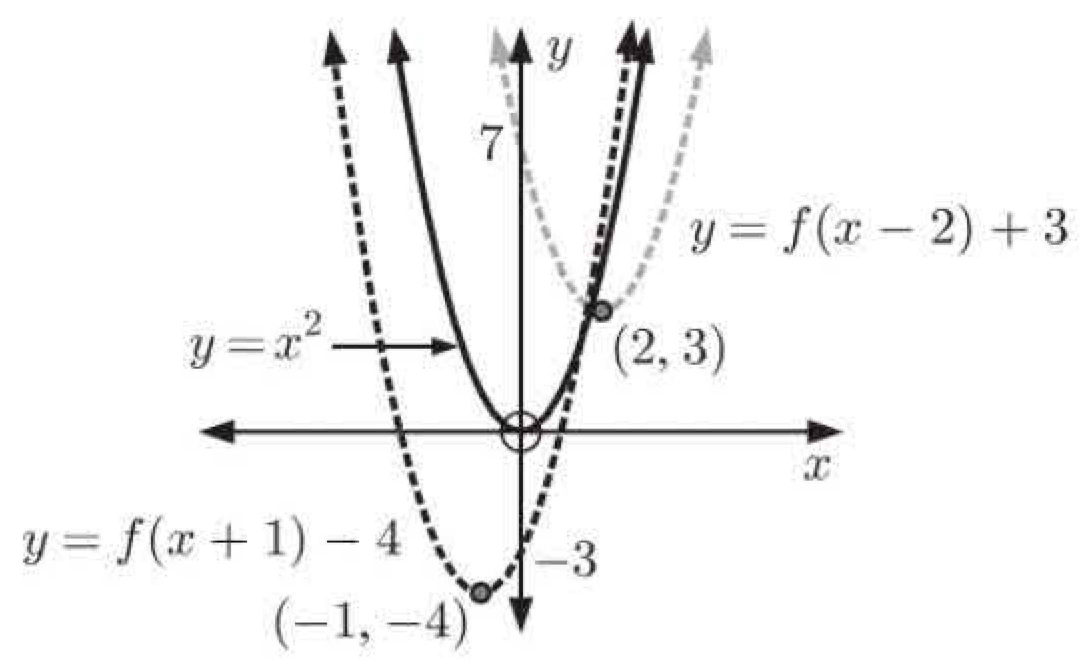


Summary: For $y=f(x-a)$, $y=f(x)$ is translated horizontally through a units. If $a>0$ movement is to the right. If $a<0$ movement is to the left.

5 **a**



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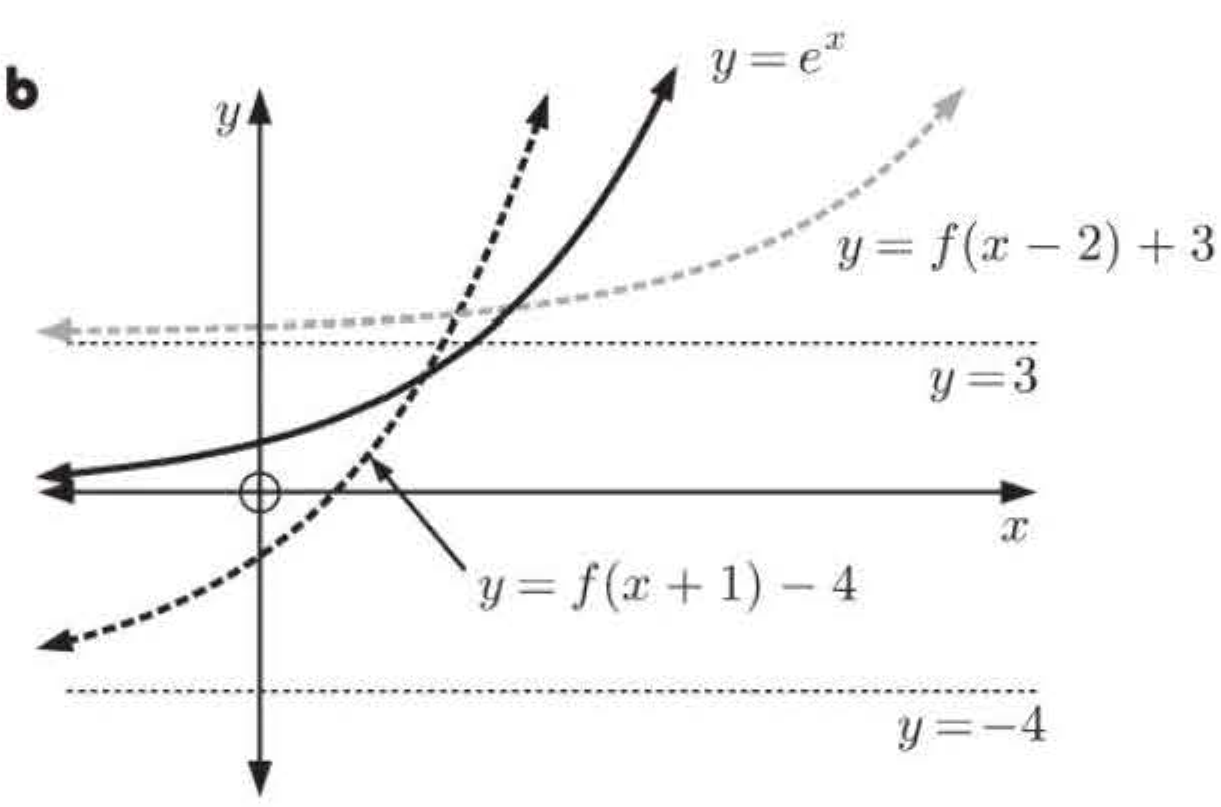

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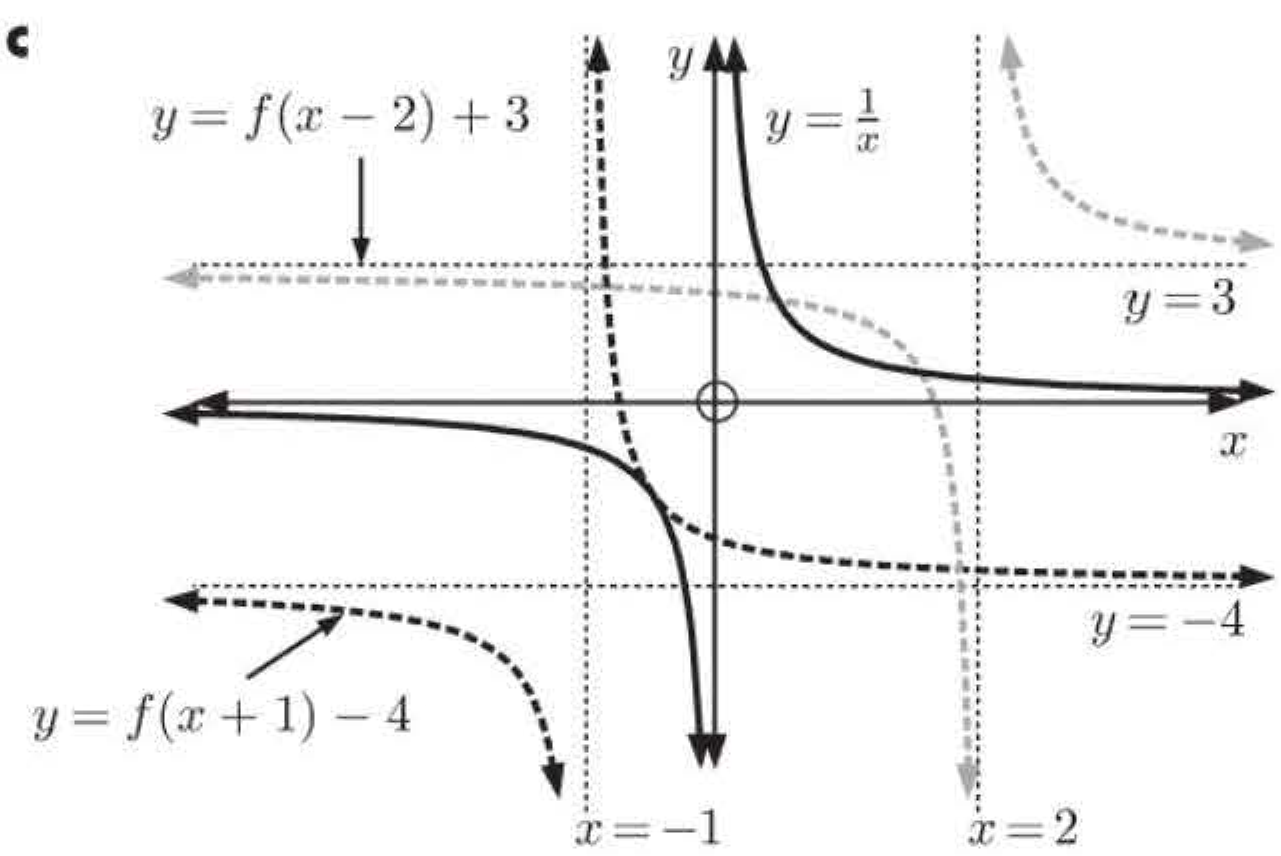
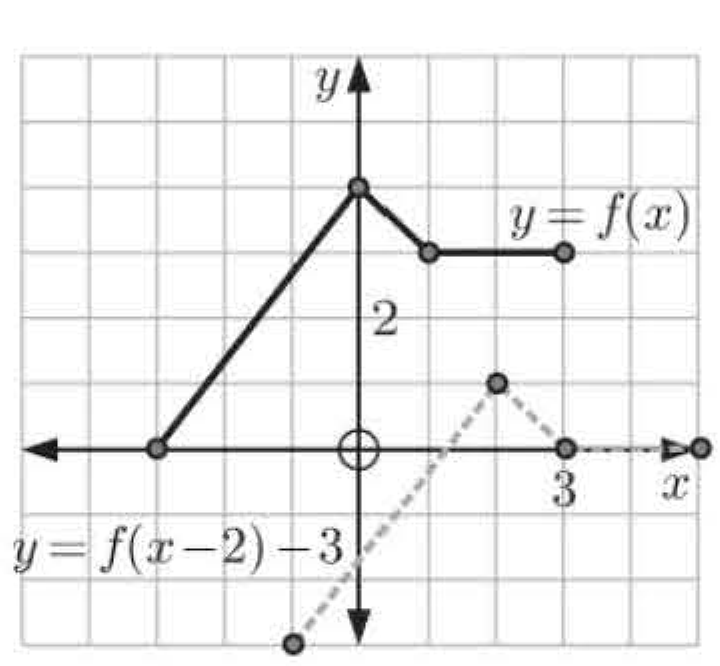


6 A translation of 2 units right and 3 units down or $\binom{2}{-3}$.

a



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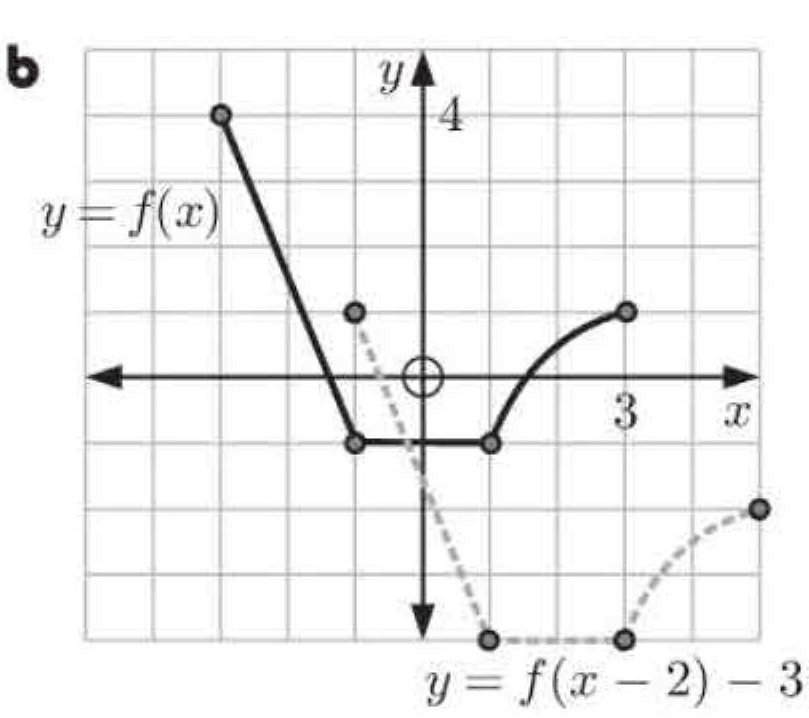


7 To translate $f(x)$ 3 units right, we need to find $f(x-3)$.

$$\begin{aligned}\therefore\ g(x) &= f(x-3)\\ &= (x-3)^2 - 2(x-3) + 2\\ &= x^2 - 6x + 9 - 2x + 6 + 2\\ \therefore\ g(x) &= x^2 - 8x + 17\end{aligned}$$

8 **a** The transformation from $f(x) = x^2$ to $g(x) = (x-3)^2 + 2$ is a translation of 3 units right and 2 units up.

i $(0, 0)$ is translated to $(3, 2)$. **ii** $(-3, 9)$ is translated to $(0, 11)$.

iii $f(2) = 2^2 = 4$

$\therefore$ $(2, 4)$ is translated to $(5, 6)$.

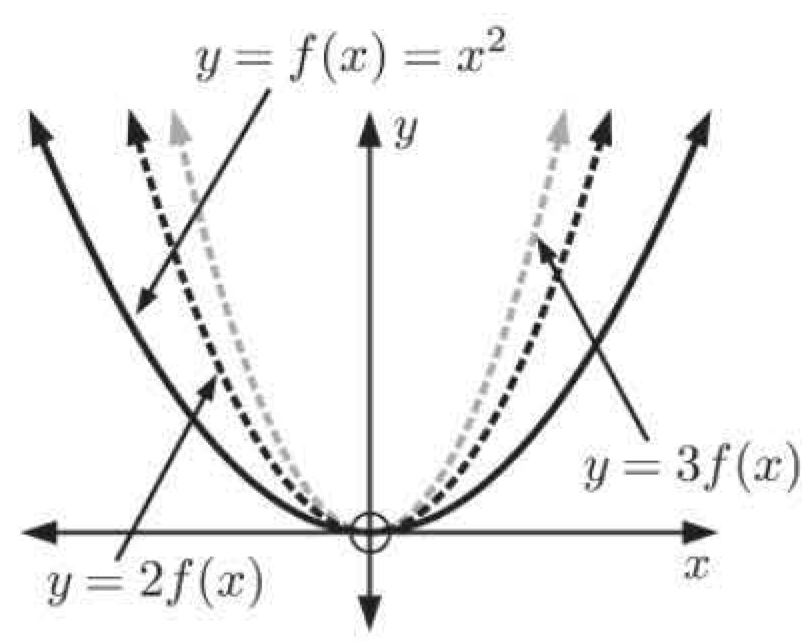
b The transformation from $g(x)$ back to $f(x)$ is a translation of $\binom{-3}{-2}$.

i $(1, 6)$ is translated to $(-2, 4)$. **ii** $(-2, 27)$ is translated to $(-5, 25)$.

iii $(1\frac{1}{2}, 4\frac{1}{4})$ is translated to $(-1\frac{1}{2}, 2\frac{1}{4})$.

EXERCISE 5C

1 **a**



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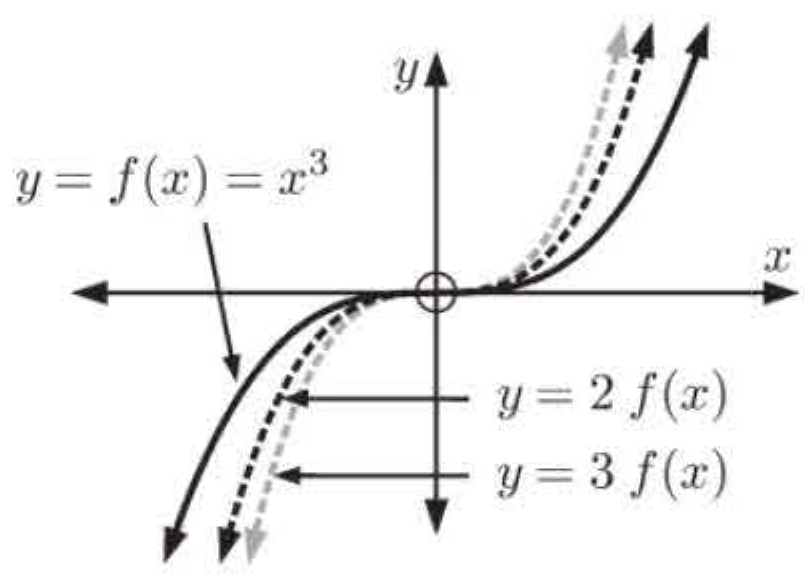

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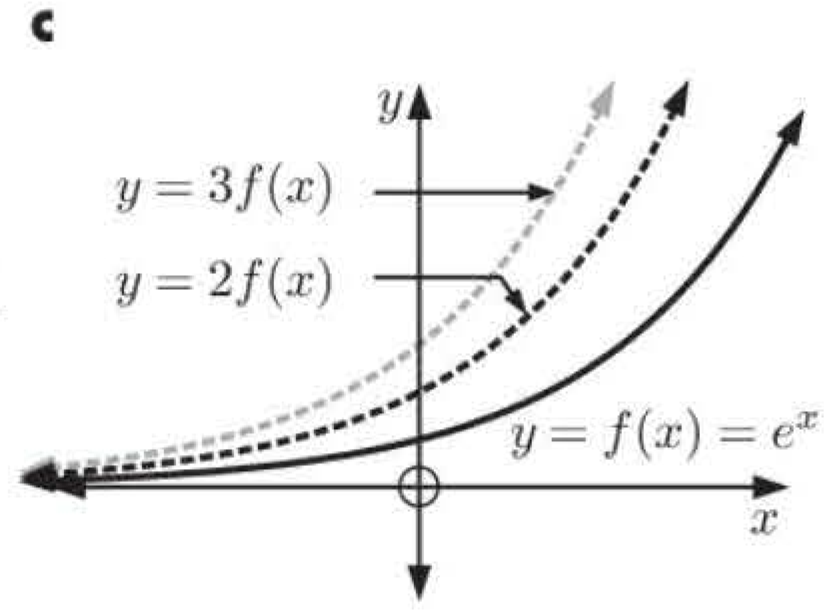

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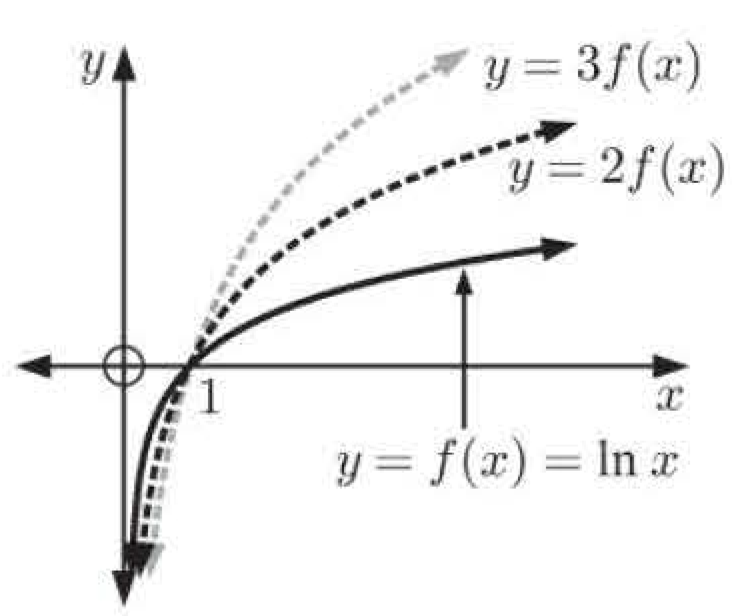
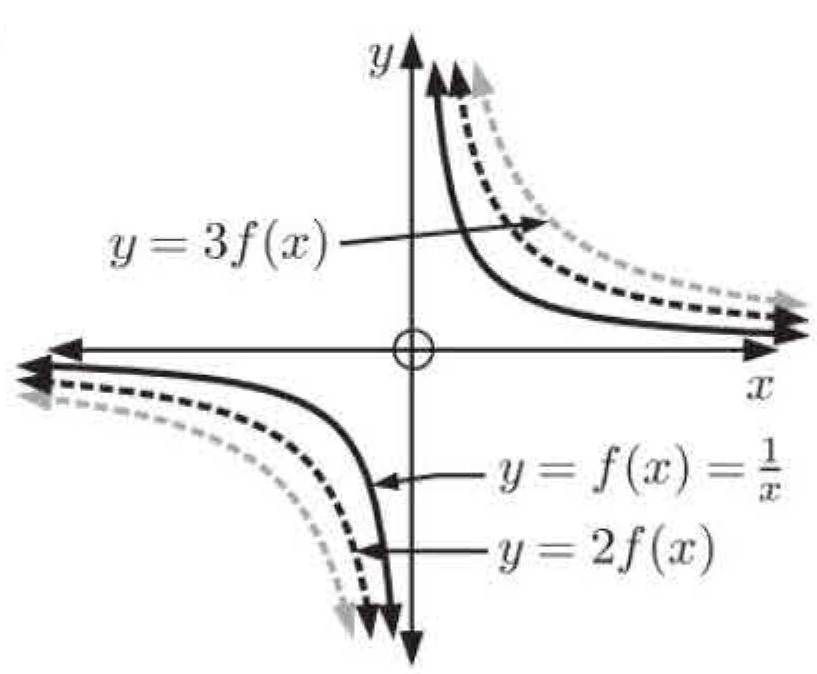
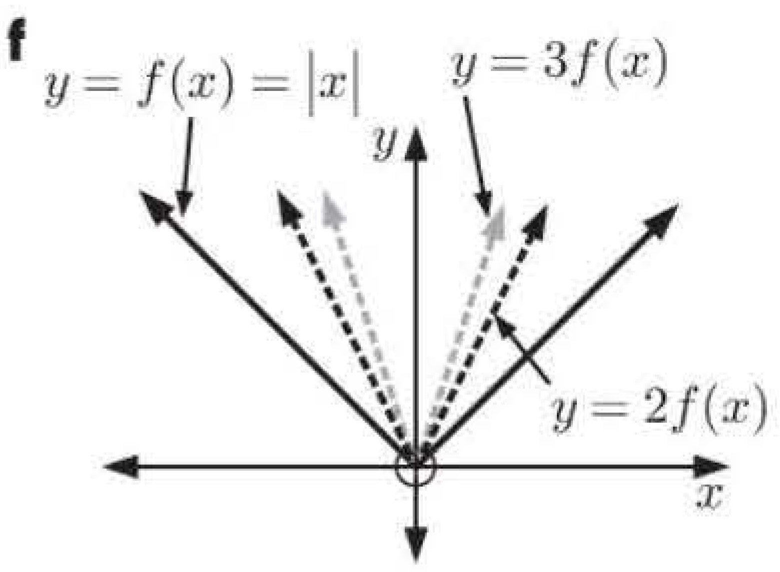


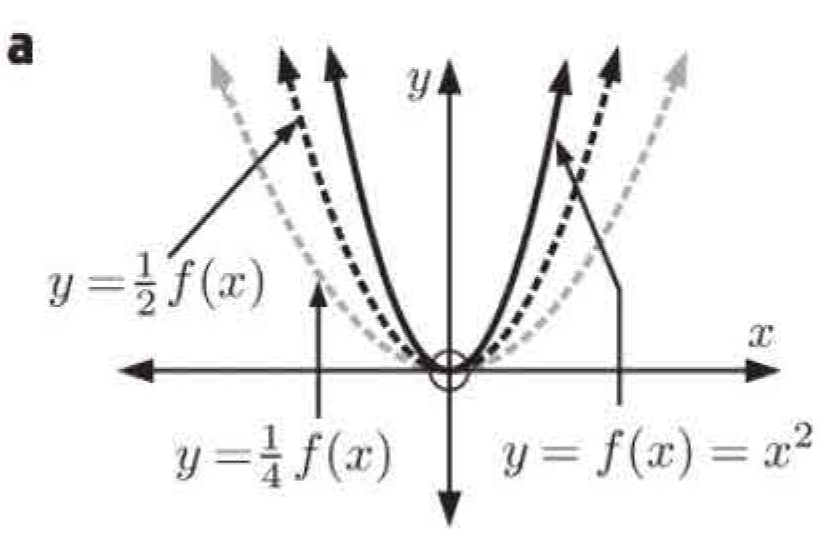
e



f



2 **a**



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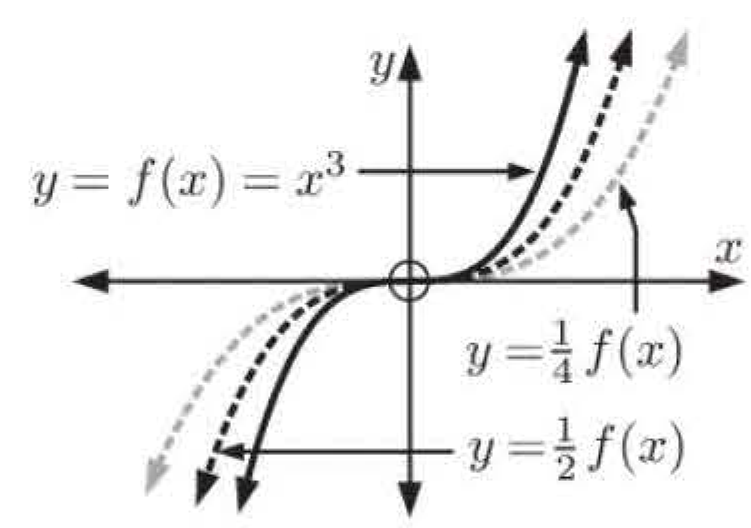

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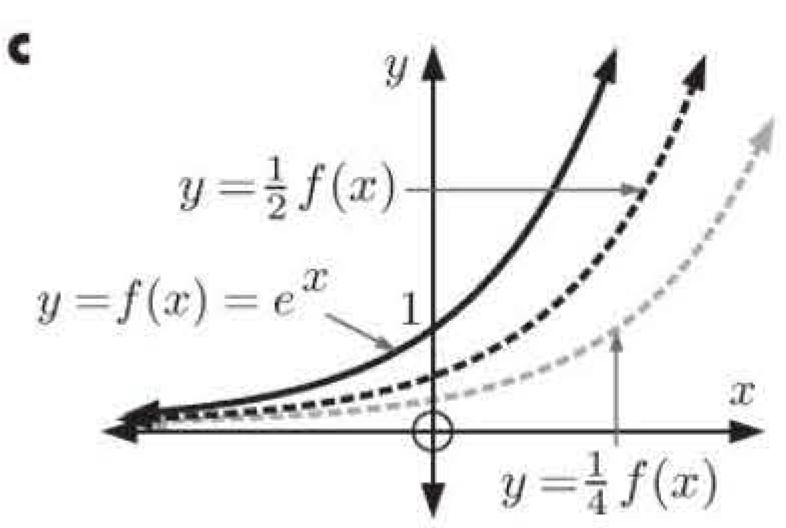
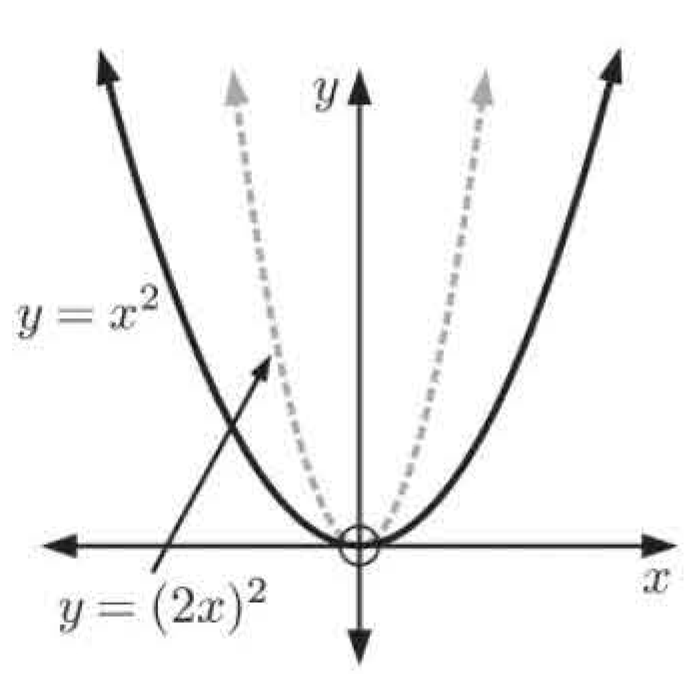


3 **a**



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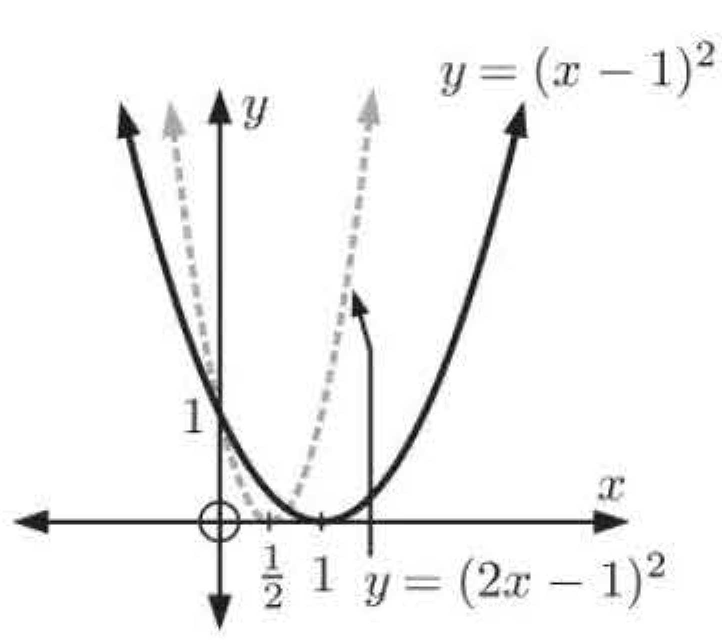

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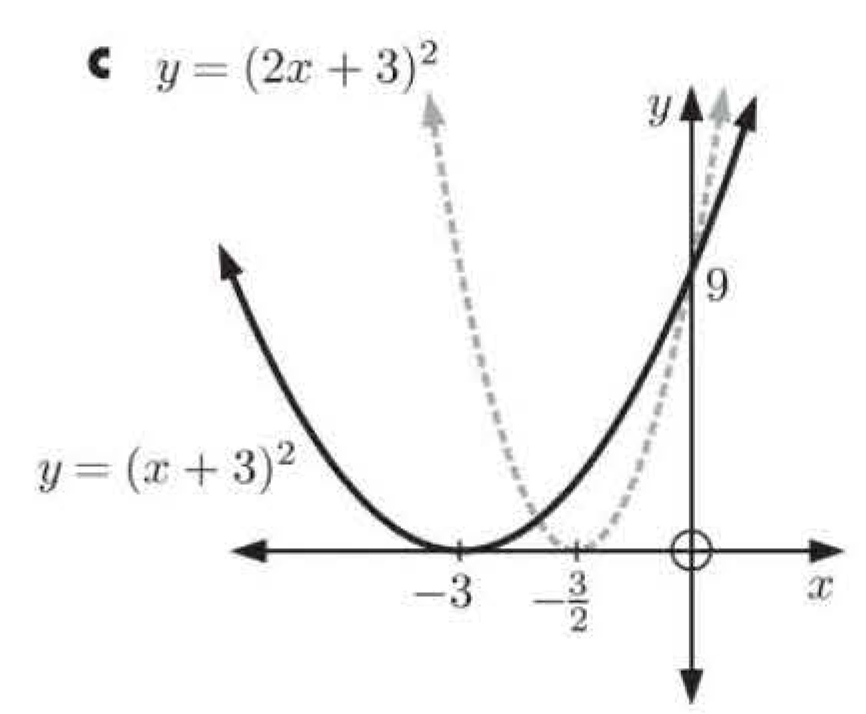
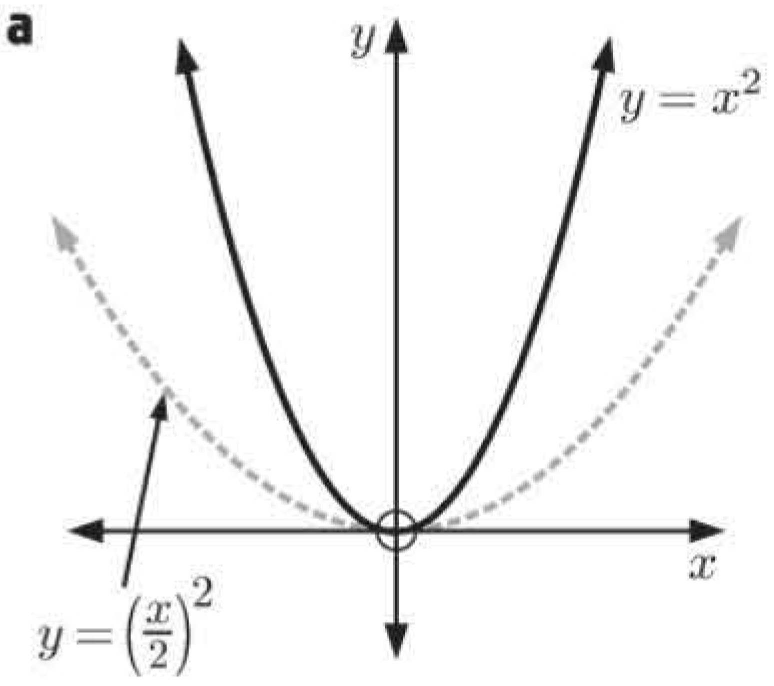


4 **a**


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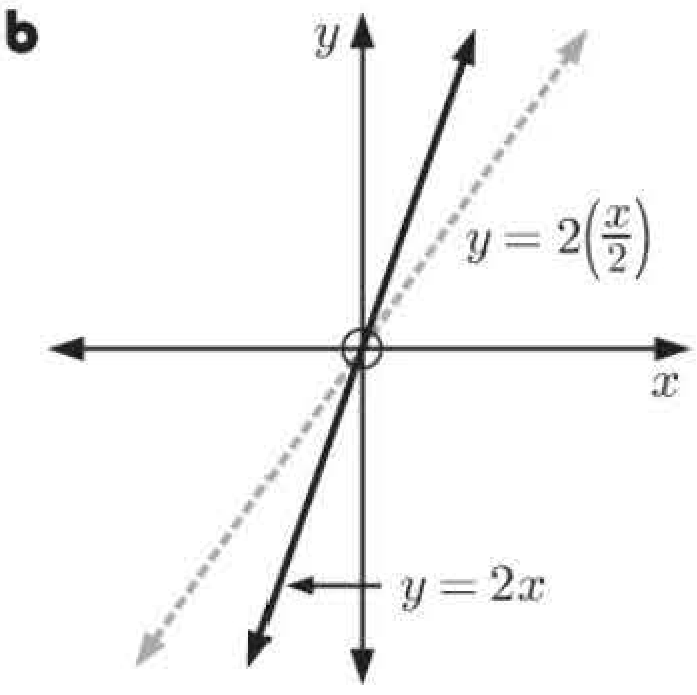

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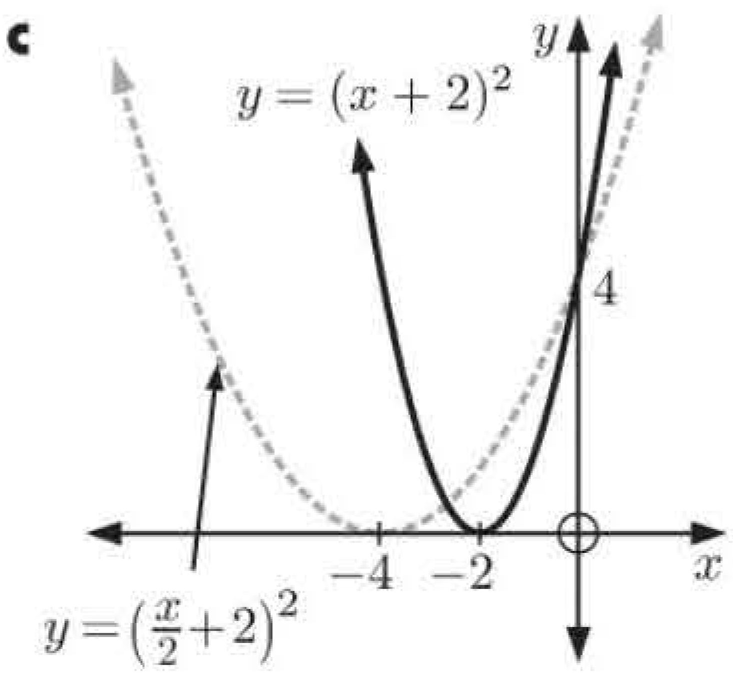
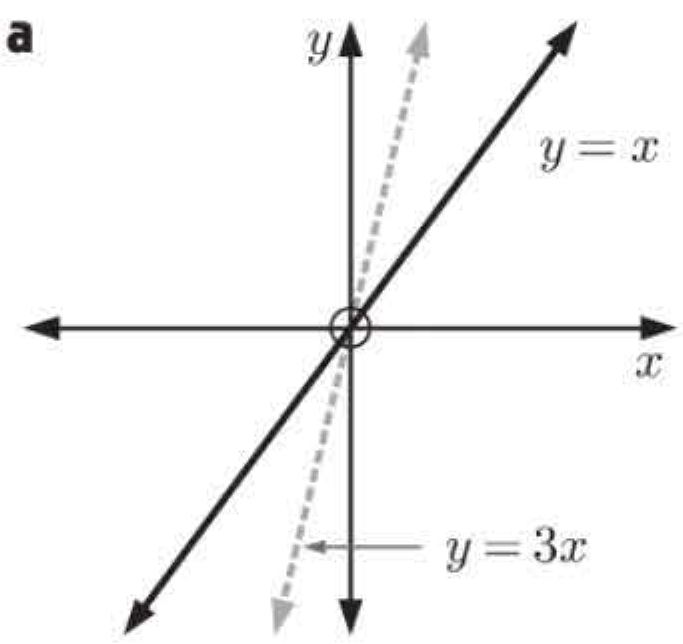


5 **a**


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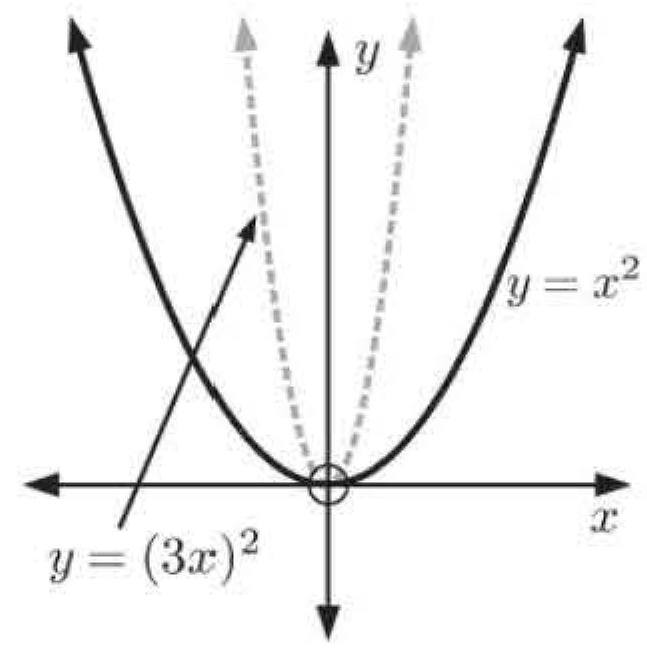

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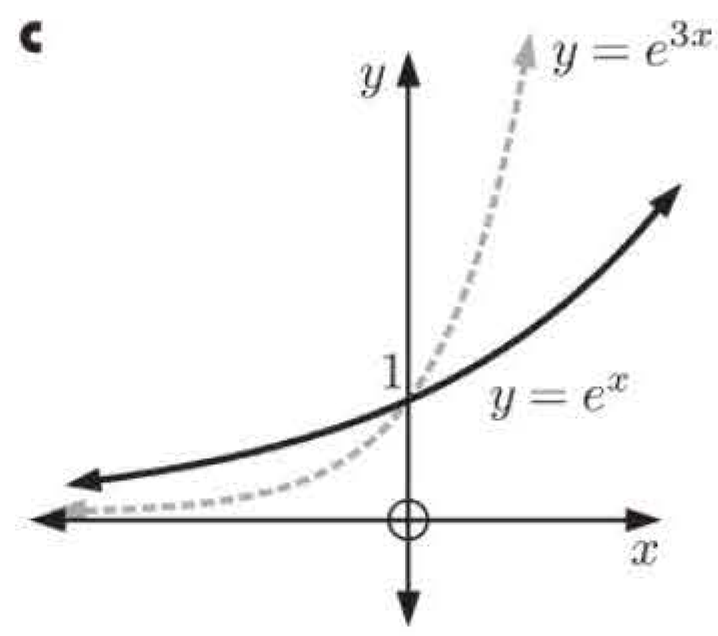
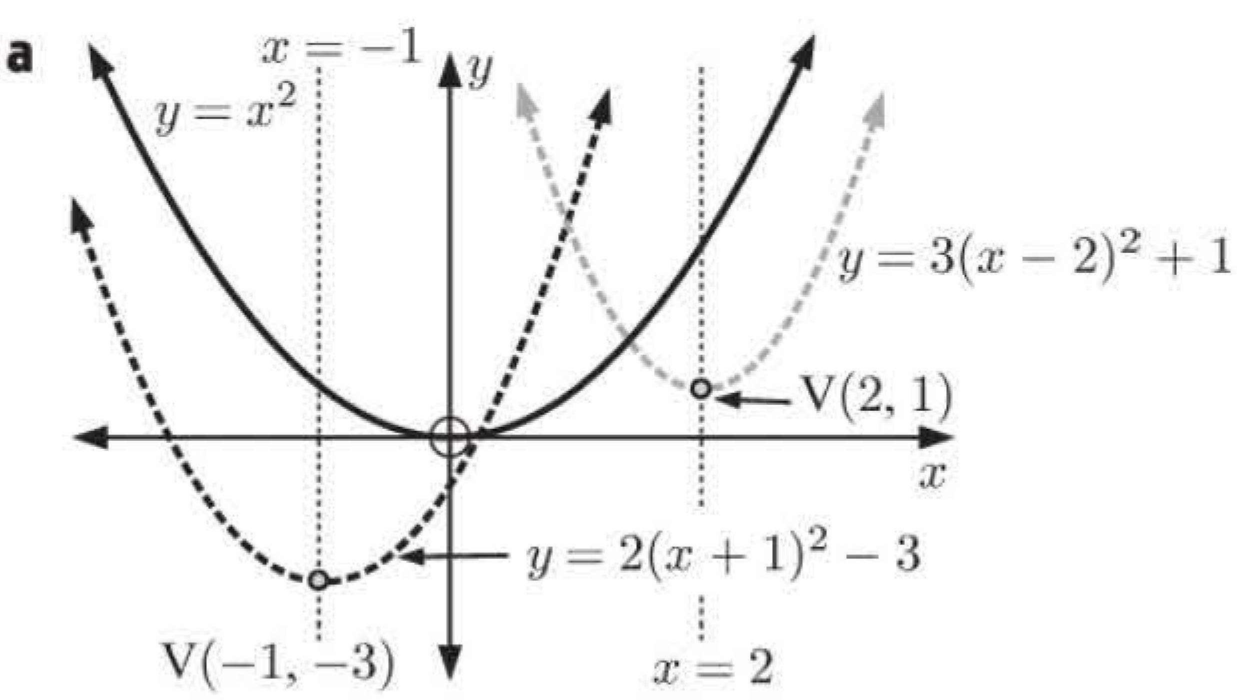


6 **a**


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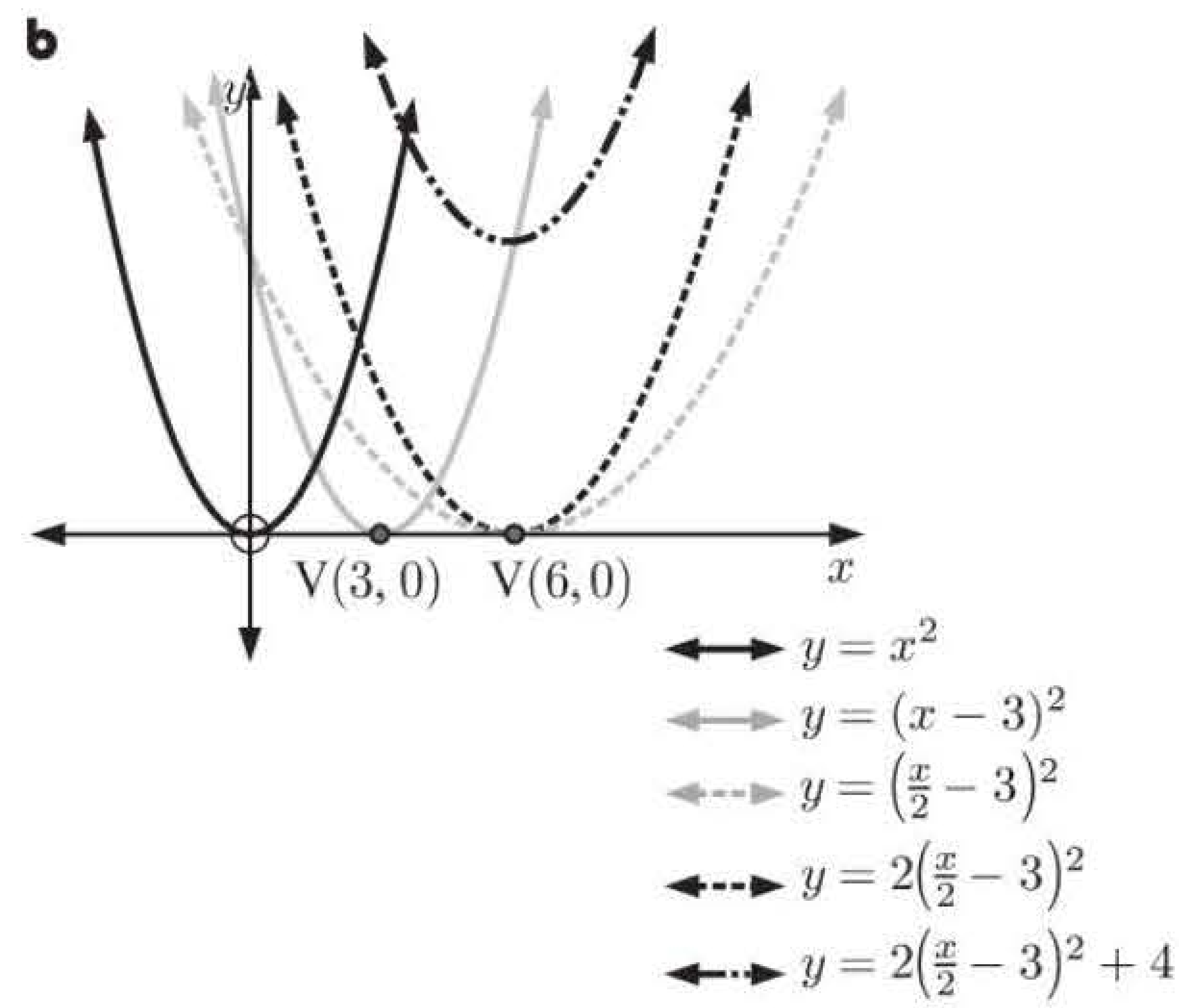

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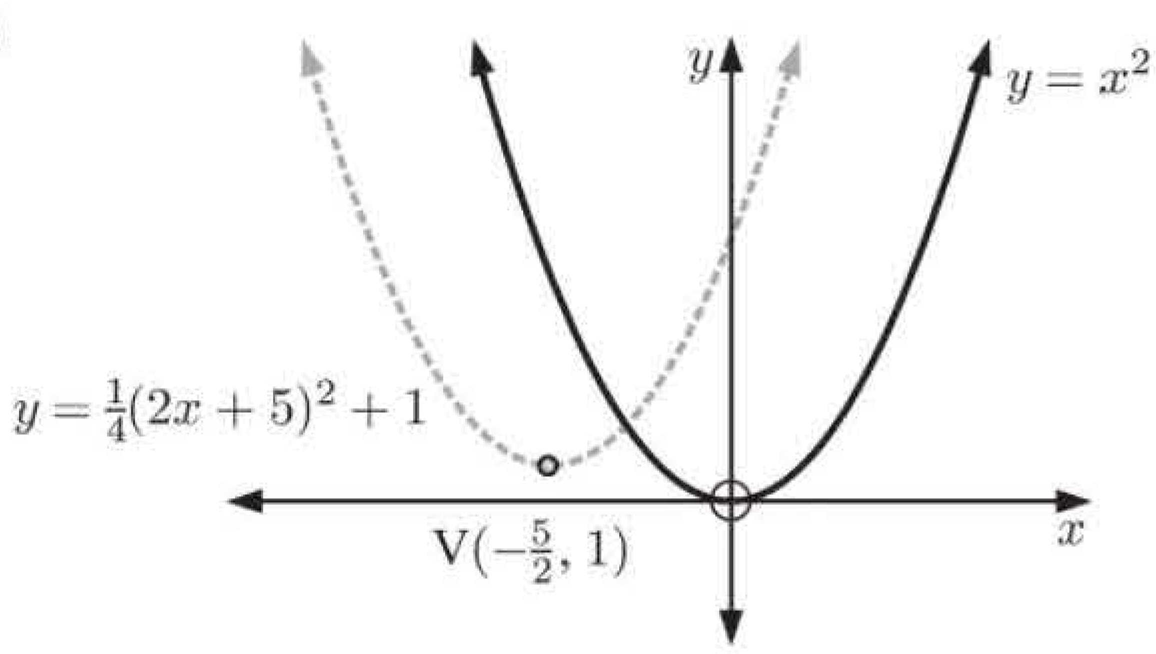


7 **a** The transformation from $y = f(x)$ to $y = 3\,f(2x)$ is a horizontal stretch of factor $\frac{1}{2}$ followed by a vertical stretch of factor 3.

i $(3, -5) \to (\frac{3}{2}, -5) \to (\frac{3}{2}, -15)$ $\quad\therefore$ $(3, -5)$ is transformed to $(\frac{3}{2}, -15)$

ii $(1, 2) \to (\frac{1}{2}, 2) \to (\frac{1}{2}, 6)$ $\quad\therefore$ $(1, 2)$ is transformed to $(\frac{1}{2}, 6)$

iii $(-2, 1) \to (-1, 1) \to (-1, 3)$ $\quad\therefore$ $(-2, 1)$ is transformed to $(-1, 3)$

b The transformation from $y = 3\,f(2x)$ back to $y = f(x)$ is a vertical stretch of factor $\frac{1}{3}$ followed by a horizontal stretch of factor 2.

i $(2, 1) \to (2, \frac{1}{3}) \to (4, \frac{1}{3})$ $\quad\therefore$ $(4, \frac{1}{3})$ is the point on $y = f(x)$

ii $(-3, 2) \to (-3, \frac{2}{3}) \to (-6, \frac{2}{3})$ $\quad\therefore$ $(-6, \frac{2}{3})$ is the point on $y = f(x)$

iii $(-7, 3) \to (-7, 1) \to (-14, 1)$ $\quad\therefore$ $(-14, 1)$ is the point on $y = f(x)$

8 **a** $f(x) \underset{\substack{\text{horizontal}\\ \text{translation}}}{\to} f(x+1) \underset{\substack{\text{horizontal}\\ \text{stretch}}}{\to} f(\frac{1}{2}x+1) \underset{\substack{\text{vertical}\\ \text{stretch}}}{\to} 2\,f(\frac{1}{2}x+1) \underset{\substack{\text{vertical}\\ \text{translation}}}{\to} 3 + 2\,f(\frac{1}{2}x+1)$

$\therefore$ $f(x)$ is translated horizontally 1 unit left, then horizontally stretched with scale factor 2, then vertically stretched with scale factor 2, then translated 3 units upwards.

b **i** $(1, -3) \to (0, -3) \to (0, -3) \to (0, -6) \to (0, -3)$ $\therefore$ $(1, -3)$ is translated to $(0, -3)$

ii $(2, 1) \to (1, 1) \to (2, 1) \to (2, 2) \to (2, 5)$ $\therefore$ $(2, 1)$ is translated to $(2, 5)$

iii $(-1, -2) \to (-2, -2) \to (-4, -2) \to (-4, -4) \to (-4, -1)$
$\therefore$ $(-1, -2)$ is translated to $(-4, -1)$

c To transform points on $y = 3 + 2f(\frac{1}{2}x + 1)$ back to points on $y = f(x)$, we translate 3 units downwards, then vertically stretch with scale factor $\frac{1}{2}$, then horizontally stretch with scale factor $\frac{1}{2}$, then translate 1 unit right.

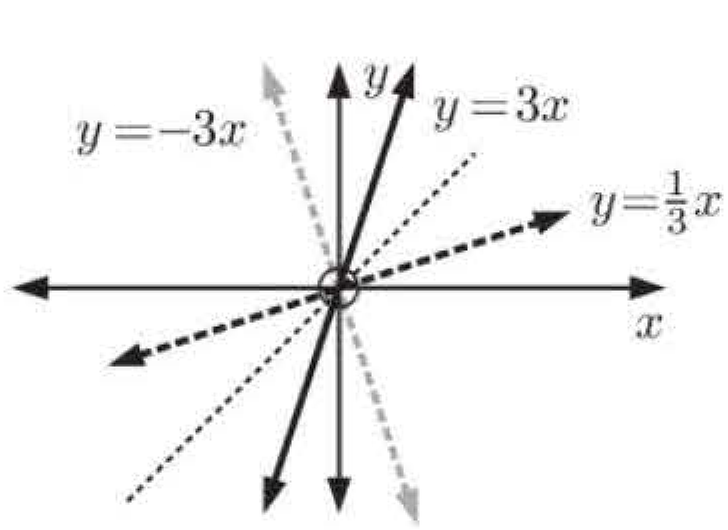
i $(-2, -5) \to (-2, -8) \to (-2, -4) \to (-1, -4) \to (0, -4)$
$\therefore$ the point on $f(x)$ is $(0, -4)$.

ii $(1, -1) \to (1, -4) \to (1, -2) \to (\frac{1}{2}, -2) \to (\frac{3}{2}, -2)$ $\therefore$ the point on $f(x)$ is $(\frac{3}{2}, -2)$.

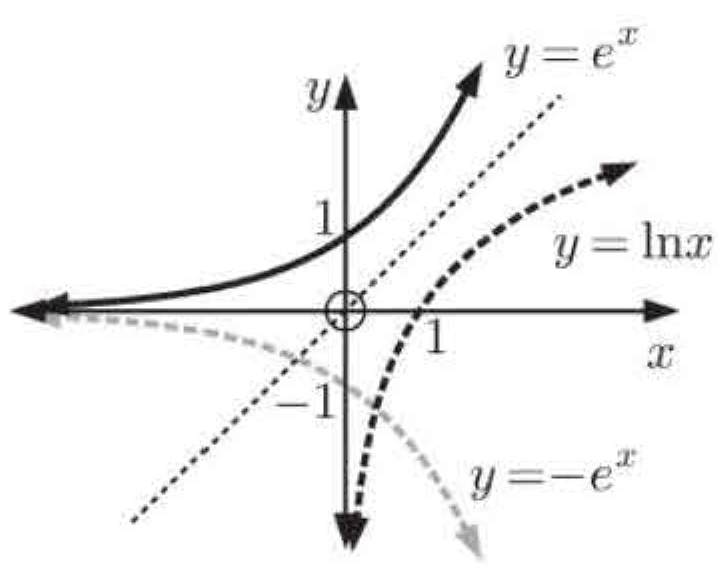
iii $(5, 0) \to (5, -3) \to (5, -\frac{3}{2}) \to (\frac{5}{2}, -\frac{3}{2}) \to (\frac{7}{2}, -\frac{3}{2})$ $\therefore$ the point on $f(x)$ is $(\frac{7}{2}, -\frac{3}{2})$.

EXERCISE 5D

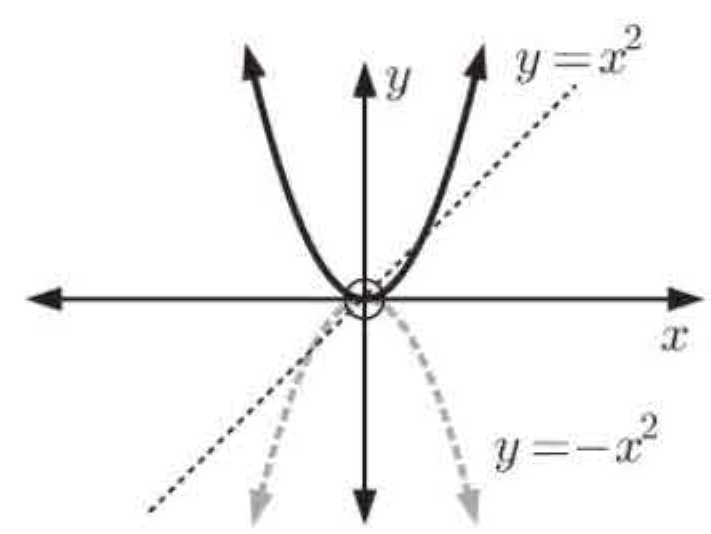
1 **a** If $f(x) = 3x$
then $-f(x) = -(3x)$
$= -3x$
And $f(x) = 3x$
has inverse function
$x = 3y$
$\therefore$ $y = \frac{1}{3}x$
$\therefore$ $f^{-1}(x) = \frac{1}{3}x$



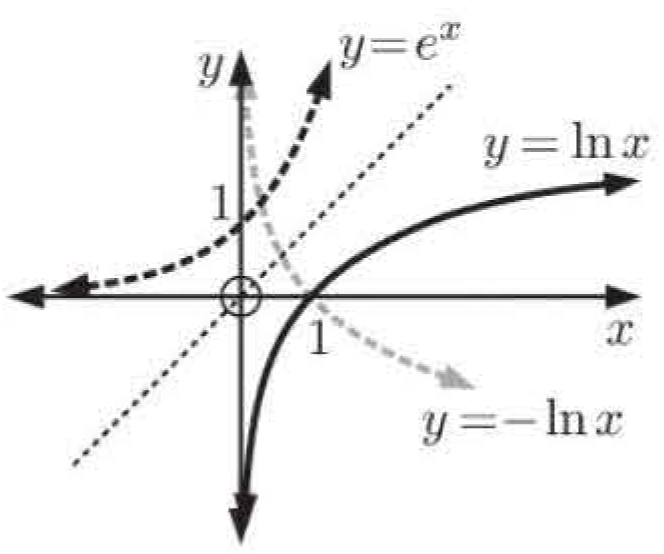
b If $f(x) = e^x$
then $-f(x) = -(e^x)$
$= -e^x$
And $f(x) = e^x$
has inverse function
$x = e^y$
$\therefore$ $y = \ln x$
$\therefore$ $f^{-1}(x) = \ln x$



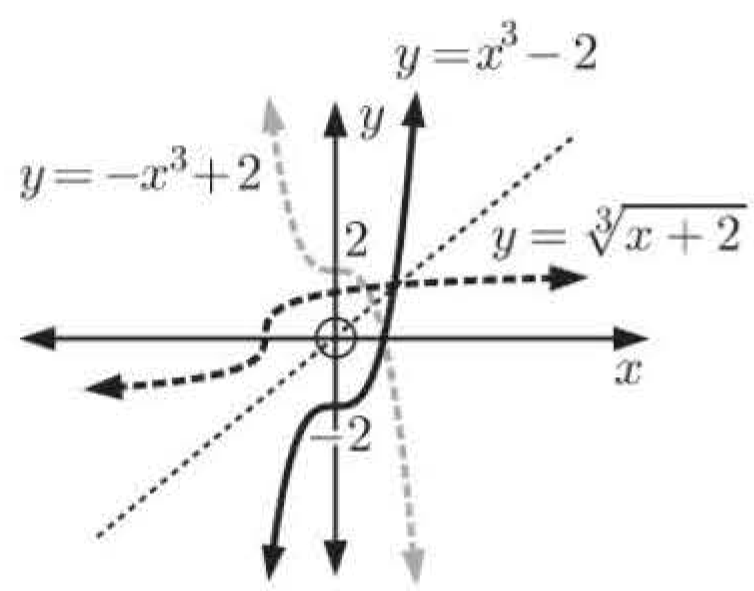
c If $f(x) = x^2$
then $-f(x) = -(x^2)$
$= -x^2$
$f(x) = x^2$ does not have an inverse function as it does not satisfy the horizontal line test.



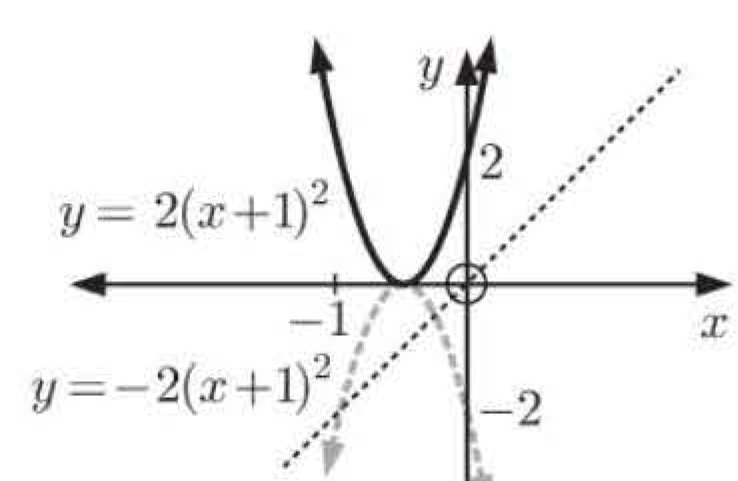
d If $f(x) = \ln x$
then $-f(x) = -(\ln x)$
$= -\ln x$
And $f(x) = \ln x$
has inverse function
$x = \ln y$
$\therefore$ $y = e^x$
$\therefore$ $f^{-1}(x) = e^x$



e If $f(x) = x^3 - 2$
then $-f(x) = -(x^3 - 2)$
$= -x^3 + 2$
And $f(x) = x^3 - 2$
has inverse function
$x = y^3 - 2$
$\therefore$ $y^3 = x + 2$
$\therefore$ $y = \sqrt[3]{x+2}$
$\therefore$ $f^{-1}(x) = \sqrt[3]{x+2}$

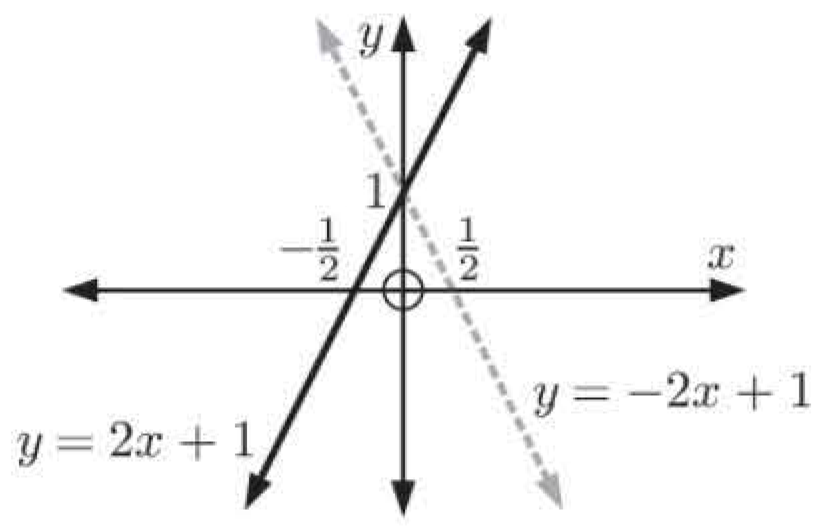


f If $f(x) = 2(x+1)^2$
then $-f(x) = -(2(x+1)^2)$
$= -2(x+1)^2$
$f(x) = 2(x+1)^2$ does not have an inverse function as it does not satisfy the horizontal line test.



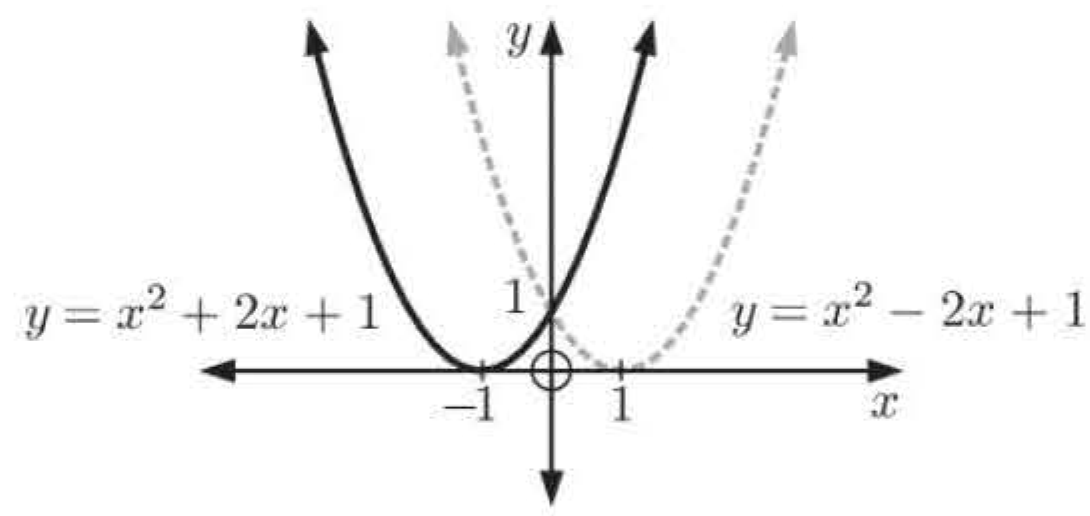
2 **a** $f(x) = 2x + 1$

$\therefore\ f(-x) = 2(-x) + 1$
$= -2x + 1$



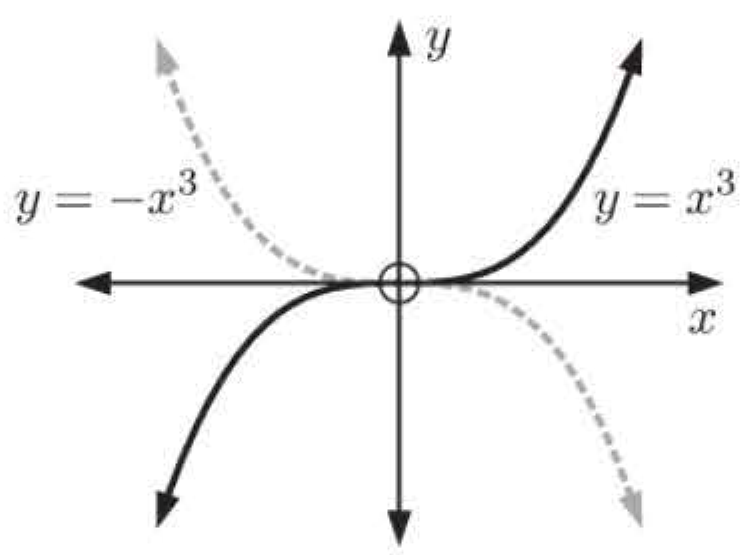
b $f(x) = x^2 + 2x + 1$

$\therefore\ f(-x) = (-x)^2 + 2(-x) + 1$
$= x^2 - 2x + 1$



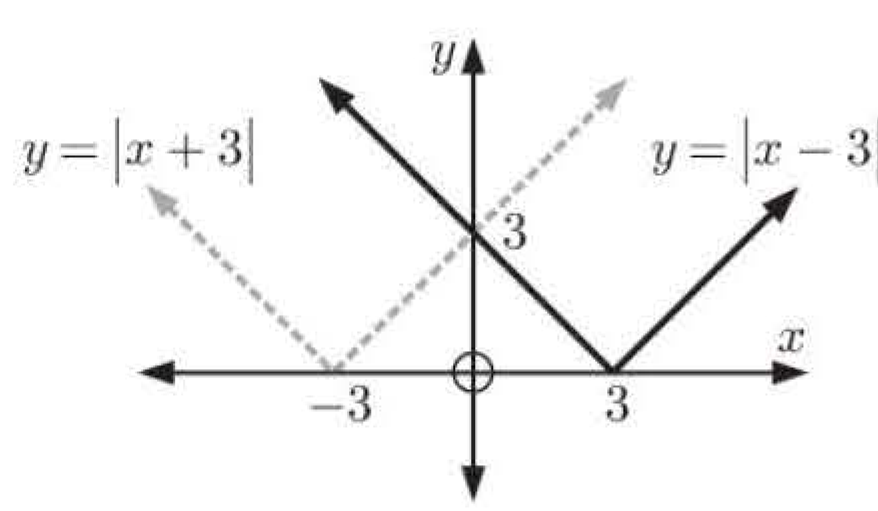
c $f(x) = x^3$

$\therefore\ f(-x) = (-x)^3$
$= -x^3$



d $f(x) = |x - 3|$

$\therefore\ f(-x) = |-x - 3|$
$= |x + 3|$



3 If $g(x)$ is the reflection of $f(x)$ in the x-axis, then $g(x) = -f(x)$

$\therefore\ g(x) = -(x^3 - \ln x)$
$= -x^3 + \ln x$

4 If $g(x)$ is the reflection of $f(x)$ in the y-axis, then $g(x) = f(-x)$

$\therefore\ g(x) = (-x)^4 - 2(-x)^3 - 3(-x)^2 + 5(-x) - 7$
$= x^4 + 2x^3 - 3x^2 - 5x - 7$

5 **a** To transform $y = f(x)$ to $g(x) = -f(x)$, we reflect $y = f(x)$ in the x-axis. To do this we keep the x-coordinates the same and take the negative of the y-coordinates.

i $(3, 0)$ is transformed to $(3, 0)$ **ii** $(2, -1)$ is transformed to $(2, 1)$
iii $(-3, 2)$ is transformed to $(-3, -2)$

b To find the points on $f(x)$ corresponding to $g(x)$, we again take the negative of the y-coordinates.

i The point transformed to $(7, -1)$ is $(7, 1)$.
ii The point transformed to $(-5, 0)$ is $(-5, 0)$.
iii The point transformed to $(-3, -2)$ is $(-3, 2)$.

6 **a** To transform $y = f(x)$ to $h(x) = f(-x)$, we reflect $y = f(x)$ in the y-axis.
To do this we keep the y-coordinates the same and take the negative of the x-coordinates.

i $(2, -1)$ is transformed to $(-2, -1)$. **ii** $(0, 3)$ is transformed to $(0, 3)$.
iii $(-1, 2)$ is transformed to $(1, 2)$. **iv** $(3, 0)$ is transformed to $(-3, 0)$.

b To find the points on $f(x)$ corresponding to $h(x)$, we again take the negative of the x-coordinates.

i The point transformed to $(5, -4)$ is $(-5, -4)$.
ii The point transformed to $(0, 3)$ is $(0, 3)$.
iii The point transformed to $(2, 3)$ is $(-2, 3)$.
iv The point transformed to $(3, 0)$ is $(-3, 0)$.

7 **a** To transform $y = f(x)$ to $m(x) = f^{-1}(x)$, we reflect $y = f(x)$ in the line $y = x$.
To do this we swap the x and y-coordinates.

i $(3, 1)$ is transformed to $(1, 3)$. **ii** $(-2, 4)$ is transformed to $(4, -2)$.
iii $(0, -5)$ is transformed to $(-5, 0)$.

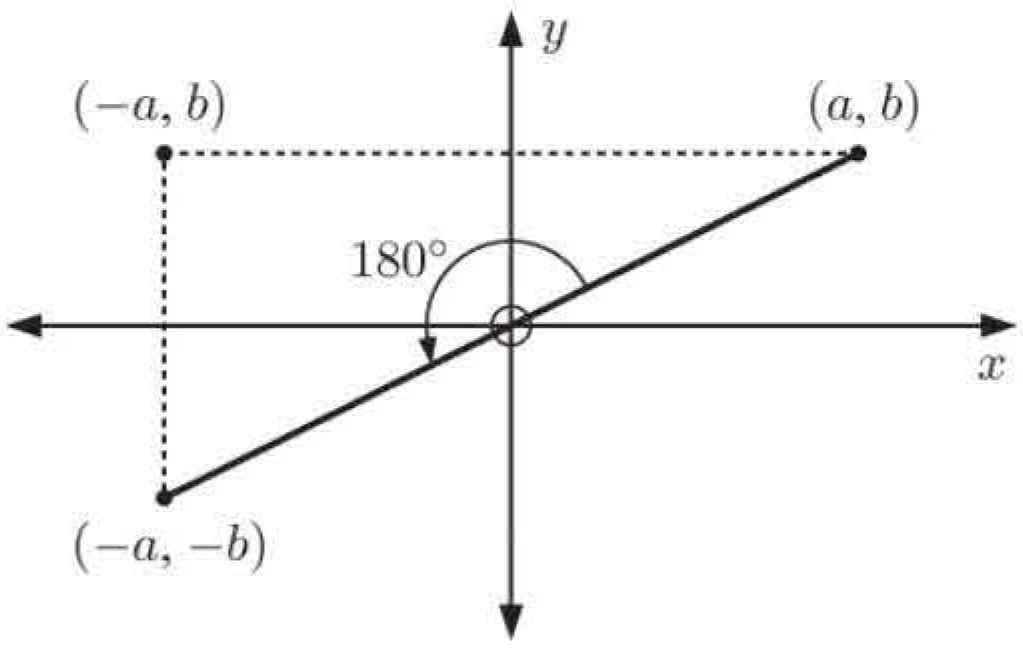
b To find the points on $f(x)$ corresponding to $m(x)$, we again swap the x and y-coordinates.

i The point transformed to $(-1, 1)$ is $(1, -1)$.

ii The point transformed to $(6, 0)$ is $(0, 6)$.

iii The point transformed to $(3, -2)$ is $(-2, 3)$.

8 **a** $f(x)$ is reflected in the y-axis to give $y = f(-x)$, then reflected in the x-axis to give $y = -f(-x)$. This has the effect of rotating the point about the origin through $180°$.

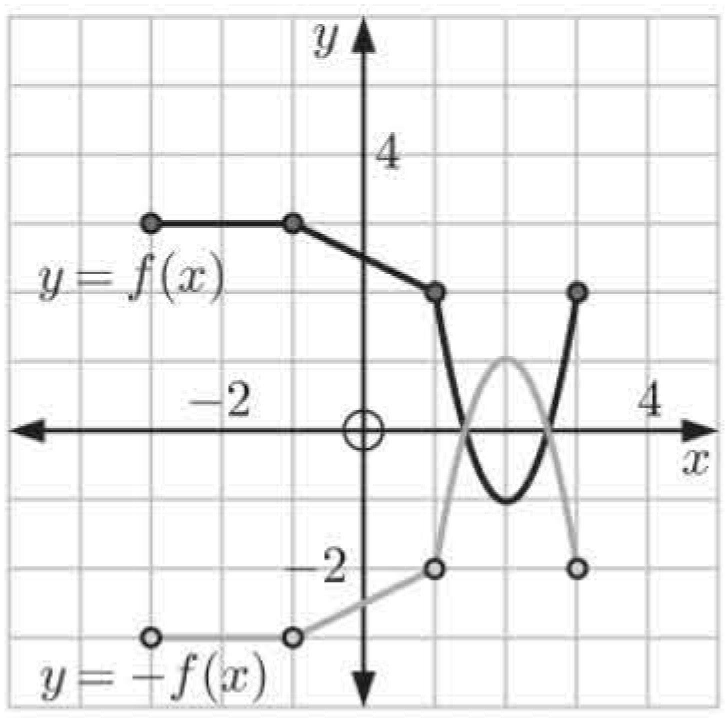


b The point (a, b) is transformed to the point $(-a, -b)$.

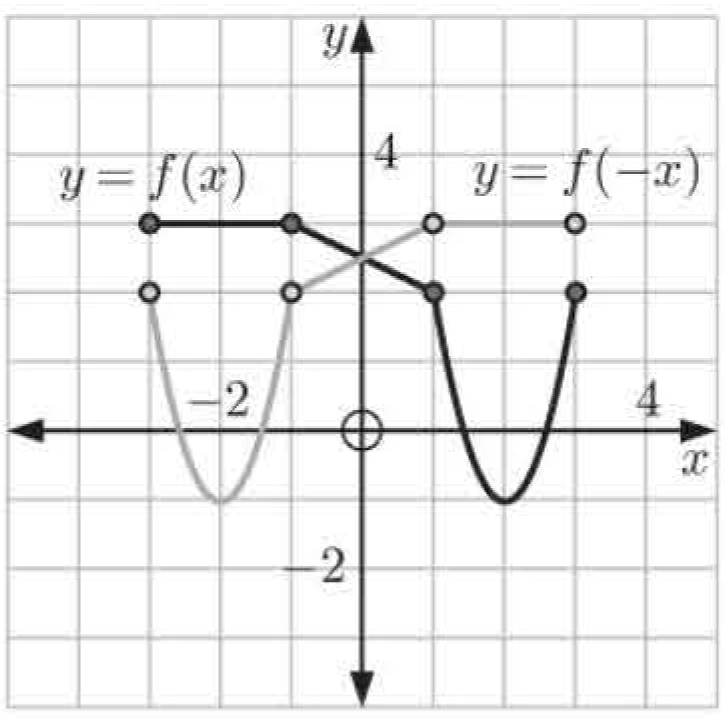
$\therefore$ $(3, -7)$ is transformed to $(-3, 7)$

c The point that transforms to $(-5, -1)$ is $(5, 1)$.

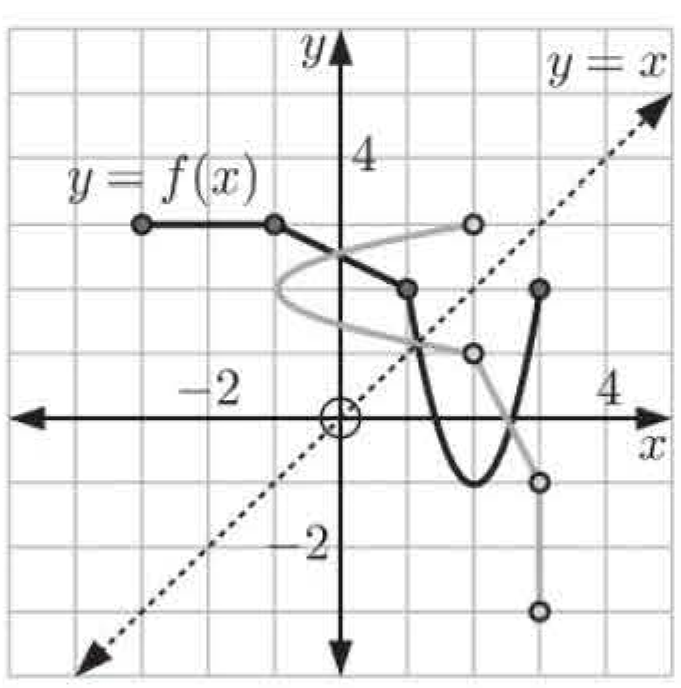
9 **a** **i**



ii



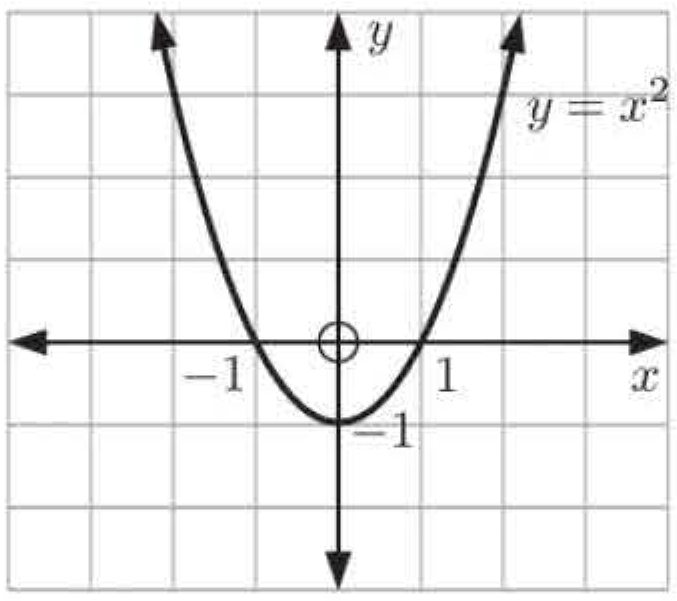
b **i**



ii The reflection of $y = f(x)$ in the line $y = x$ is not $y = f^{-1}(x)$ as $y = f(x)$ is not a function.

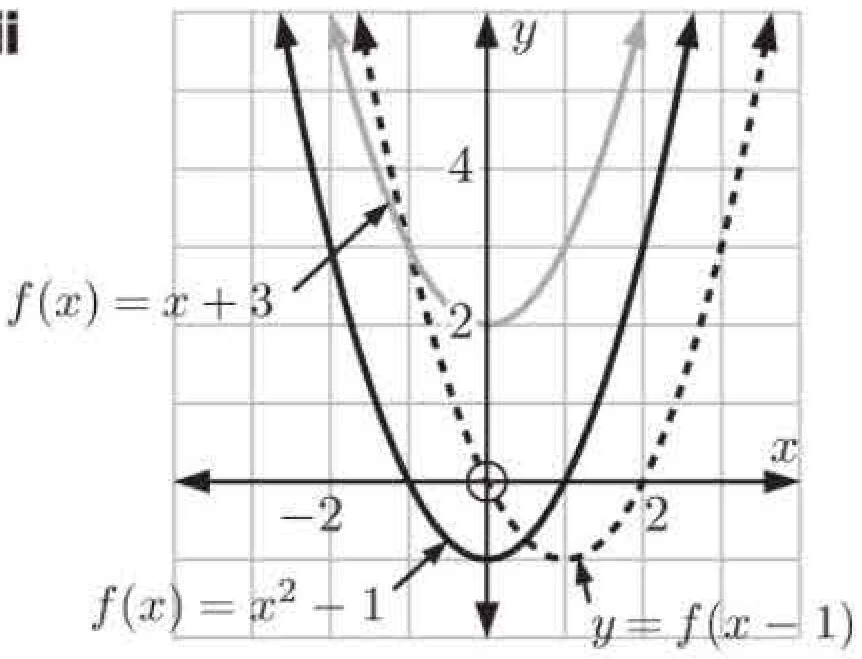
EXERCISE 5E

1 **a**

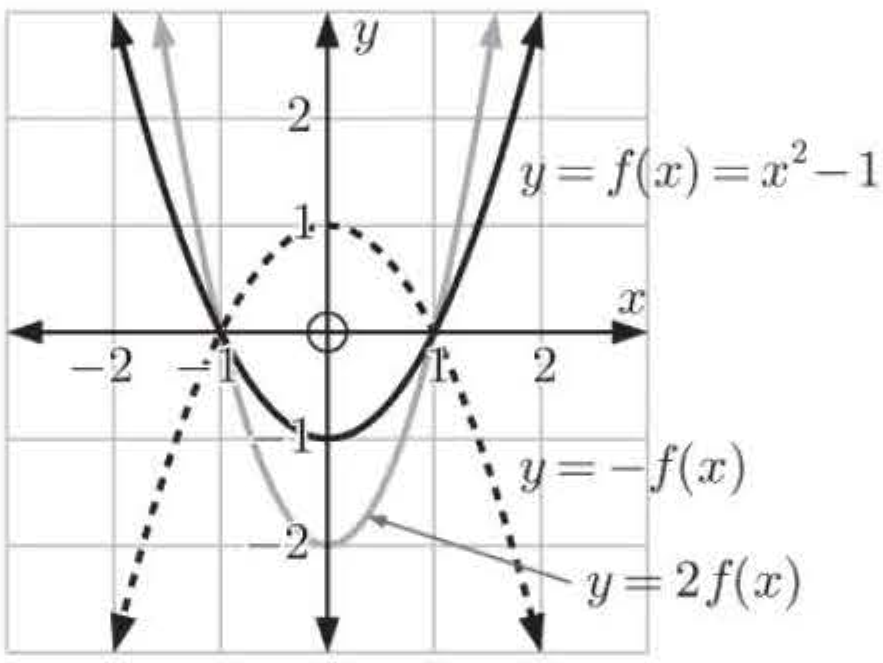


$y = x^2 - 1$ has x-intercepts -1 and 1, and y-intercept -1.

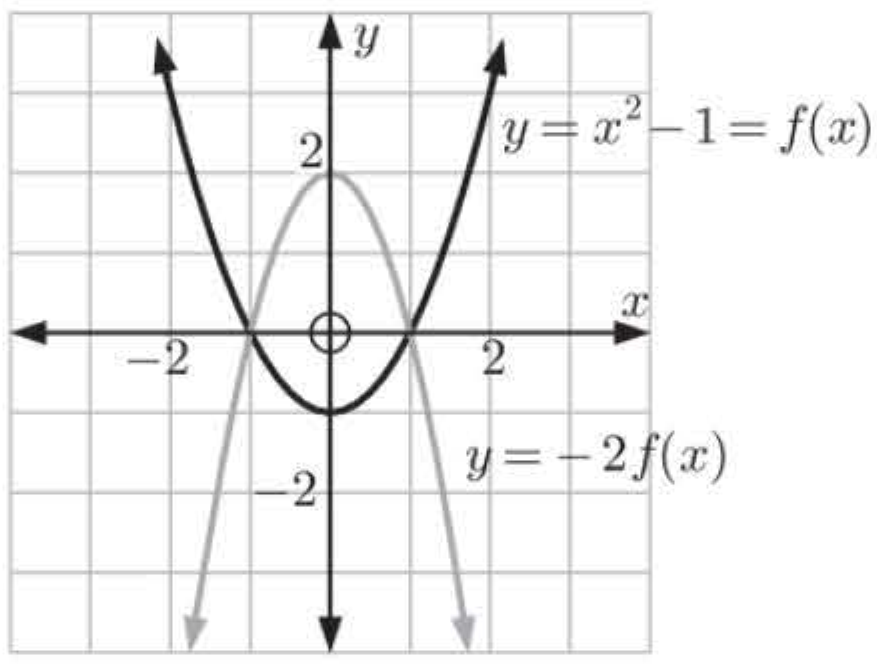
b **i, ii**



iii, iv



c **i** a vertical translation of 3 units upwards **ii** a horizontal translation of 1 unit to the right
iii a vertical stretch with scale factor 2 **iv** a reflection in the x-axis

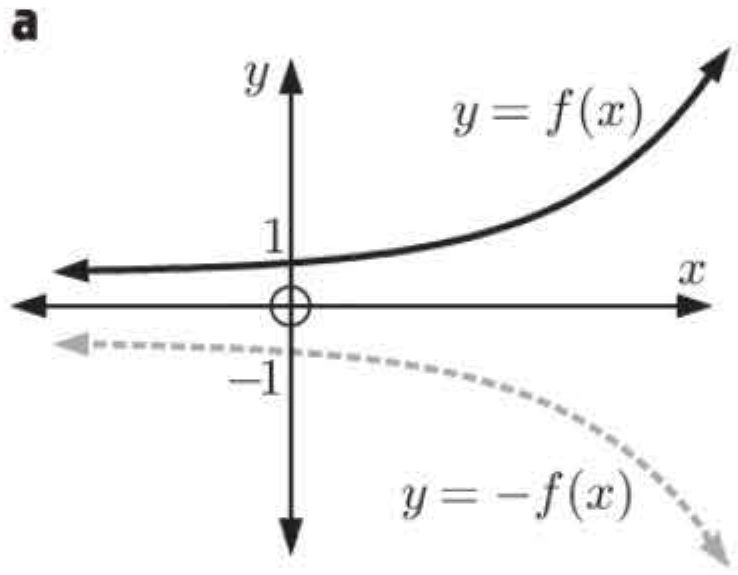
d 

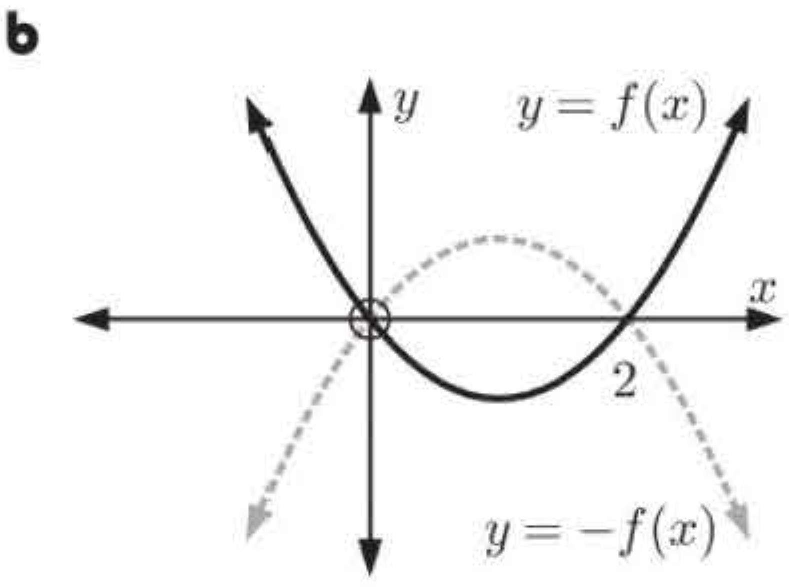
A reflection in the x-axis, followed by a vertical stretch with scale factor 2.

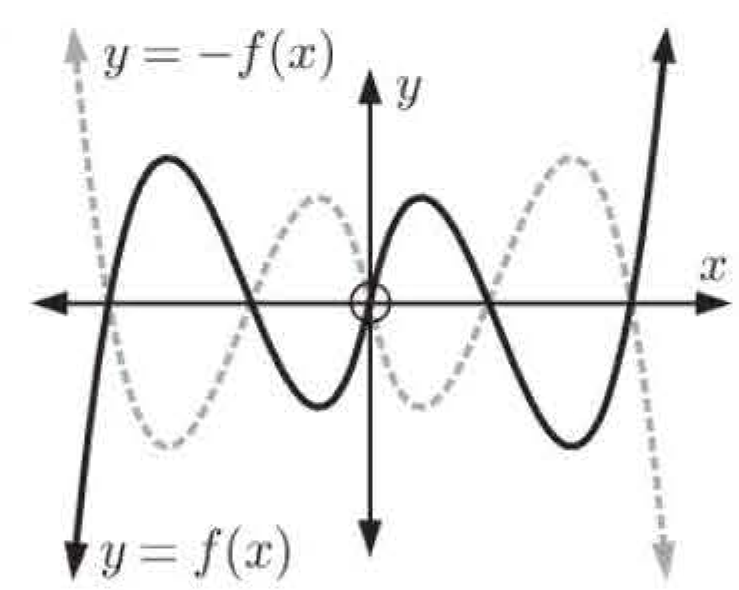
e $(-1, 0)$ and $(1, 0)$

2 **a** **i** A vertical stretch with scale factor 3. **ii** $g(x) = 3f(x)$
b **i** A translation of 2 units downwards. **ii** $g(x) = f(x) - 2$
c **i** A vertical stretch with scale factor $\frac{1}{2}$. **ii** $g(x) = \frac{1}{2}f(x)$
d **i** A reflection in the y-axis. **ii** $g(x) = f(-x)$

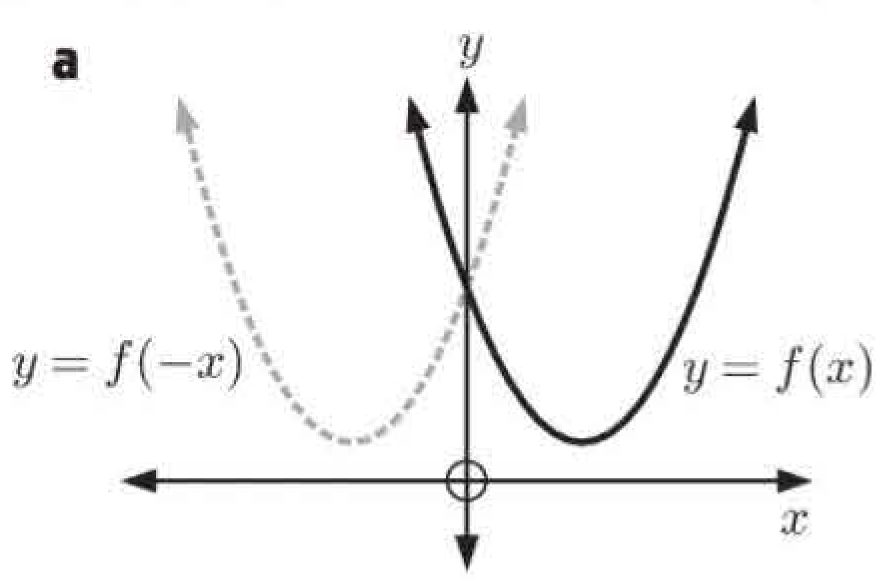
3 $y = -f(x)$ is obtained from $y = f(x)$ by reflecting it in the x-axis.

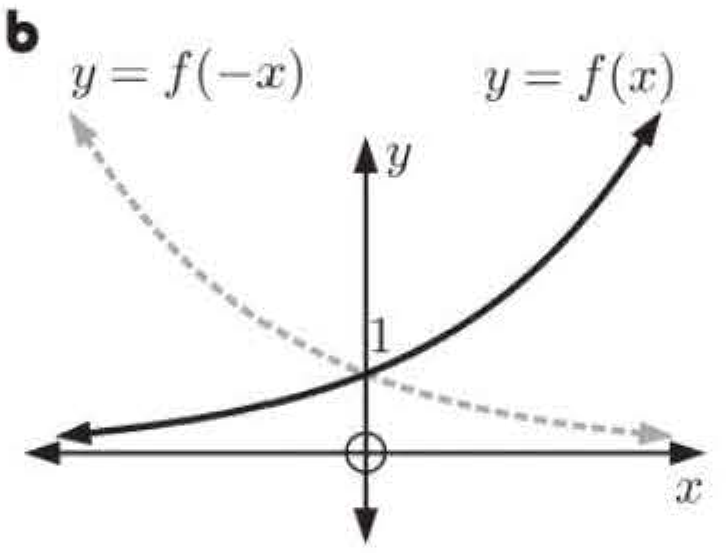
a 

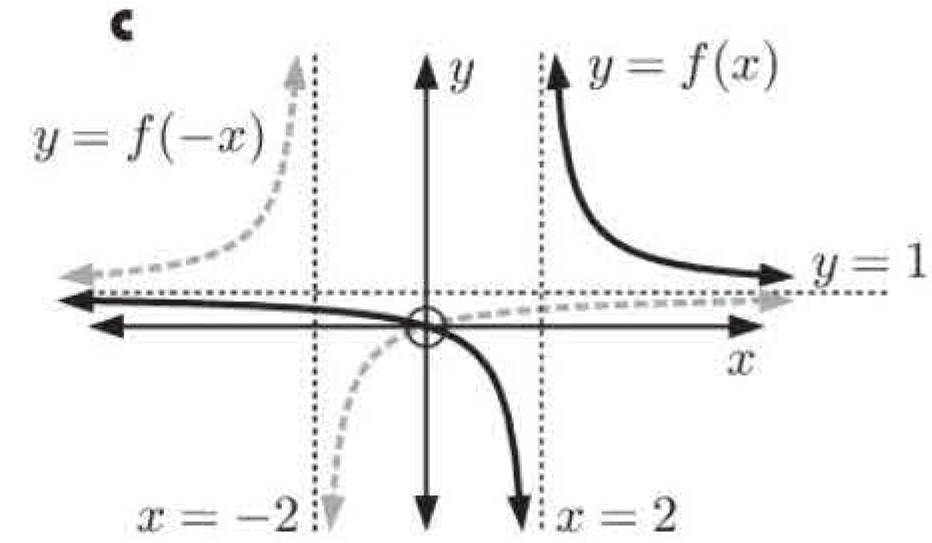
b 

c 

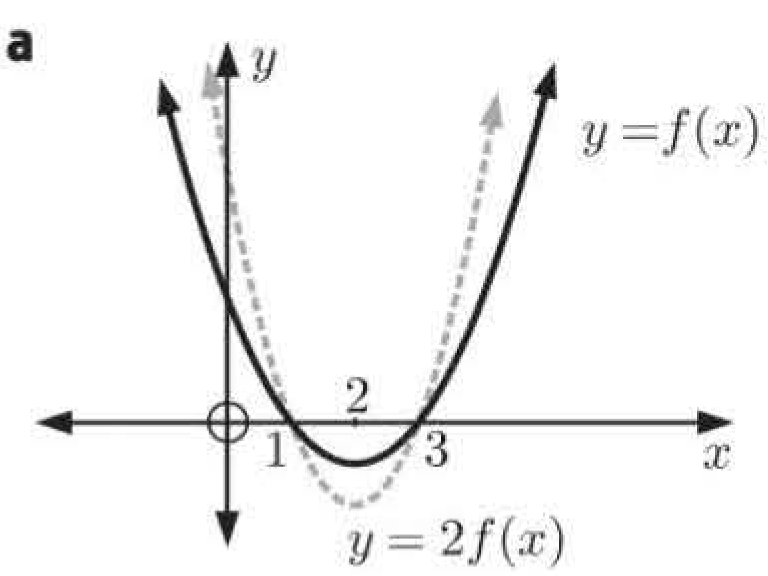
4 $y = f(-x)$ is obtained from $y = f(x)$ by reflecting it in the y-axis.

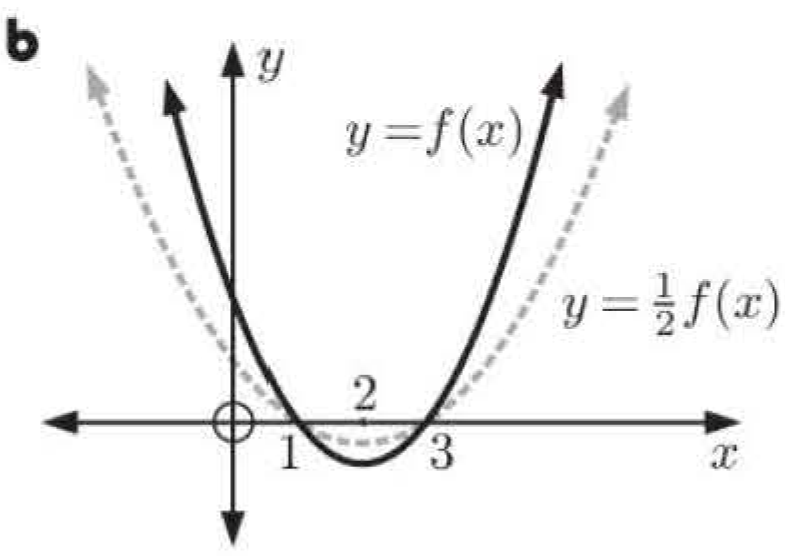
a 

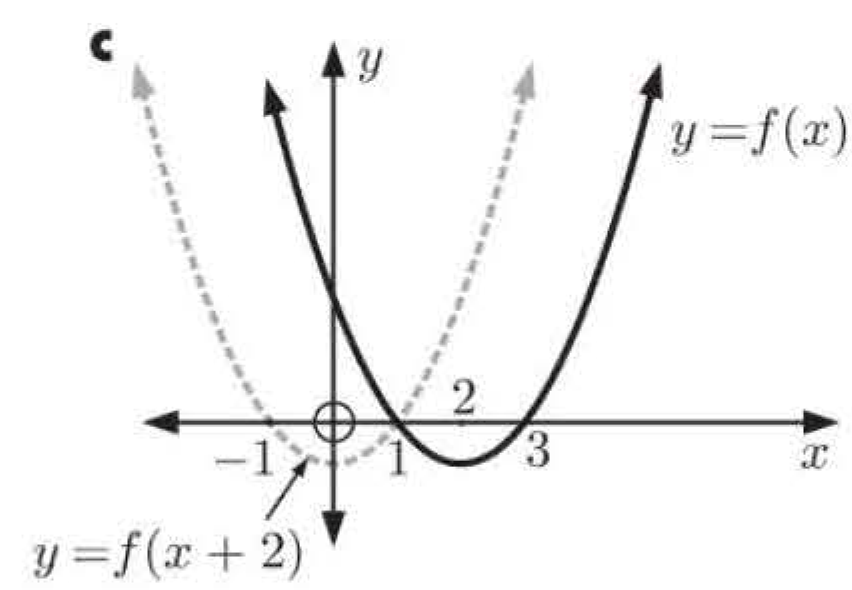
b 

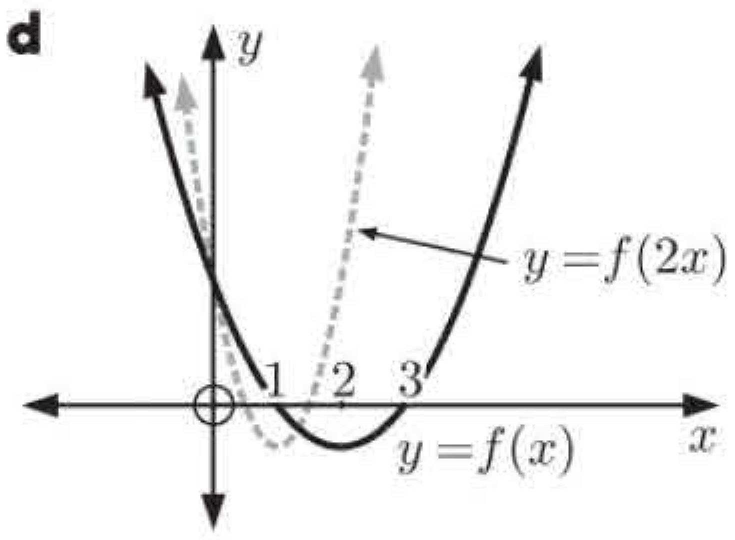
c 

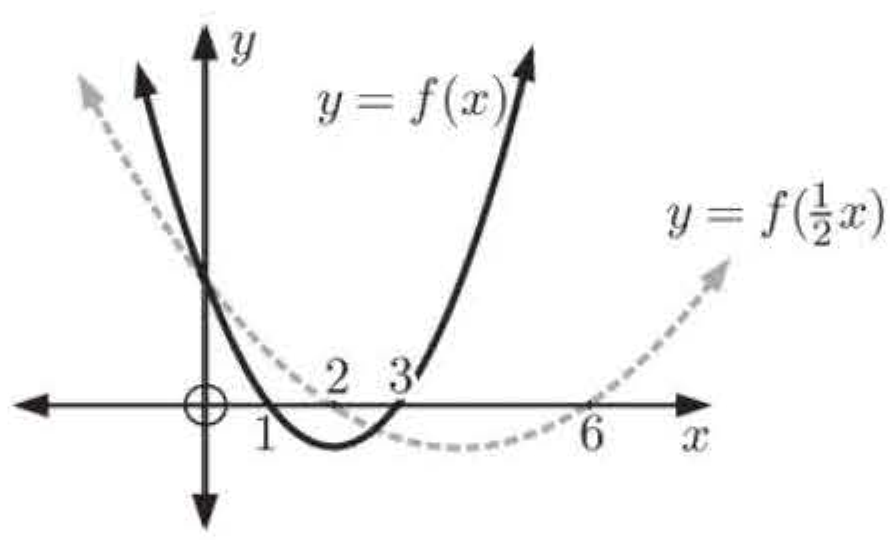
5 $y = 2x^4$ and $y = 6x^4$ are 'thinner' than $y = x^4$ and $y = \frac{1}{2}x^4$ is 'fatter'.
$\therefore$ **a** is **A**, **b** is **B**, **c** is **D**, and **d** is **C**.

6 **a** 

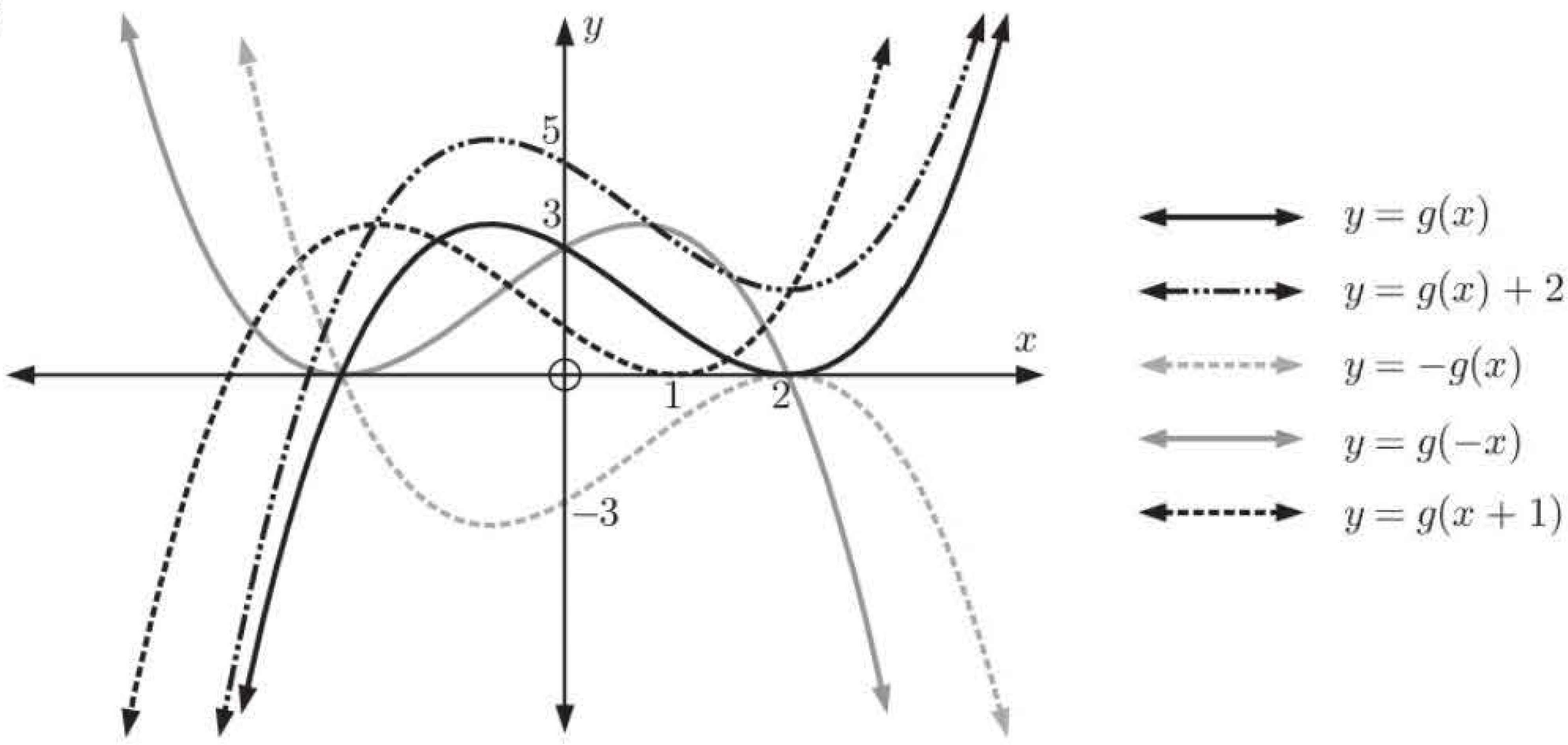
b 

c 

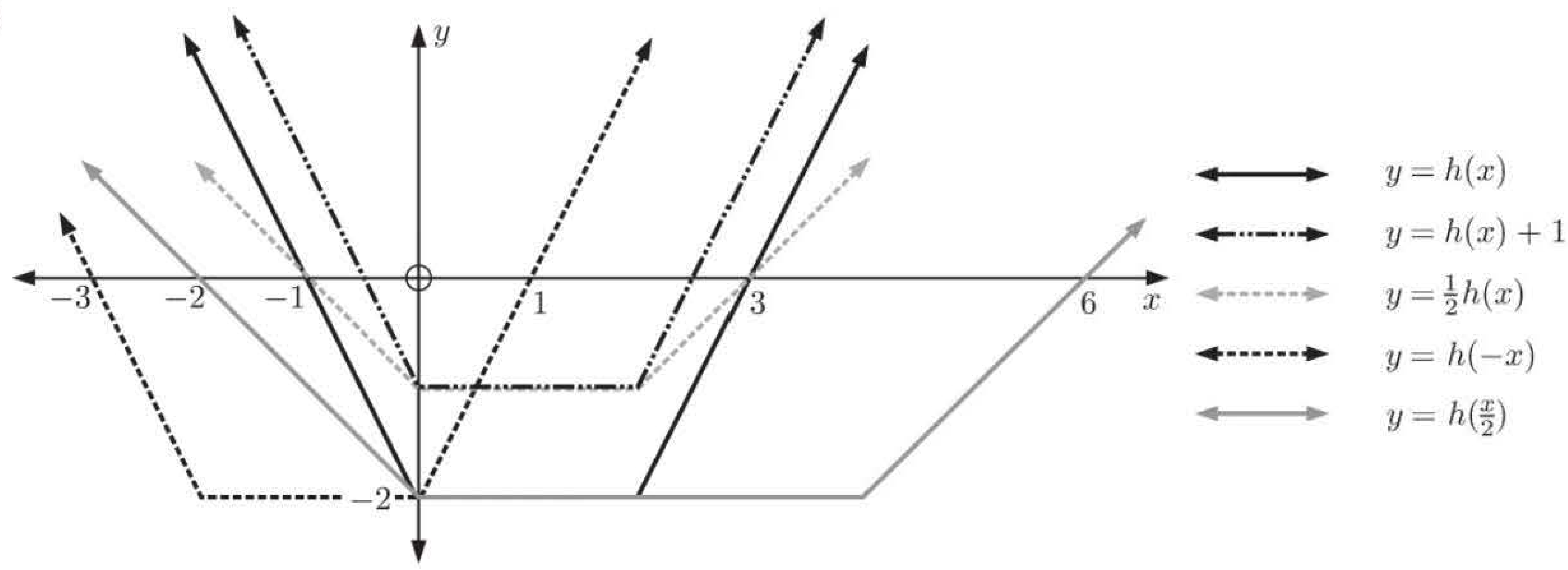
d 

e 

7



8



EXERCISE 5F

1 **a** Under a vertical stretch with scale factor $\frac{1}{2}$, $y = \dfrac{1}{x}$ becomes $y = \frac{1}{2}\left(\dfrac{1}{x}\right)$. $\therefore$ $y = \dfrac{1}{2x}$

b Under a horizontal stretch with scale factor 3, $y = \dfrac{1}{x}$ becomes $y = \dfrac{1}{\left(\frac{x}{3}\right)}$. $\therefore$ $y = \dfrac{3}{x}$

c Under a horizontal translation of -3, $y = \dfrac{1}{x}$ becomes $y = \dfrac{1}{x+3}$.

d Under a vertical translation of 4, $y = \dfrac{1}{x}$ becomes $y = \dfrac{1}{x} + 4$. $\therefore$ $y = \dfrac{4x+1}{x}$

2 **a** Under a vertical stretch with scale factor 3, $f(x)$ becomes $3f(x)$.

$\therefore$ $\dfrac{1}{x}$ becomes $3\left(\dfrac{1}{x}\right) = \dfrac{3}{x}$.

Under a translation of $\binom{1}{-1}$, $f(x)$ becomes $f(x-1) - 1$.

$\therefore$ $\dfrac{3}{x}$ becomes $\dfrac{3}{x-1} - 1$.

So, $y = \dfrac{1}{x}$ becomes
$$g(x) = \frac{3}{x-1} - 1 = \frac{3-(x-1)}{x-1} = \frac{-x+4}{x-1}$$

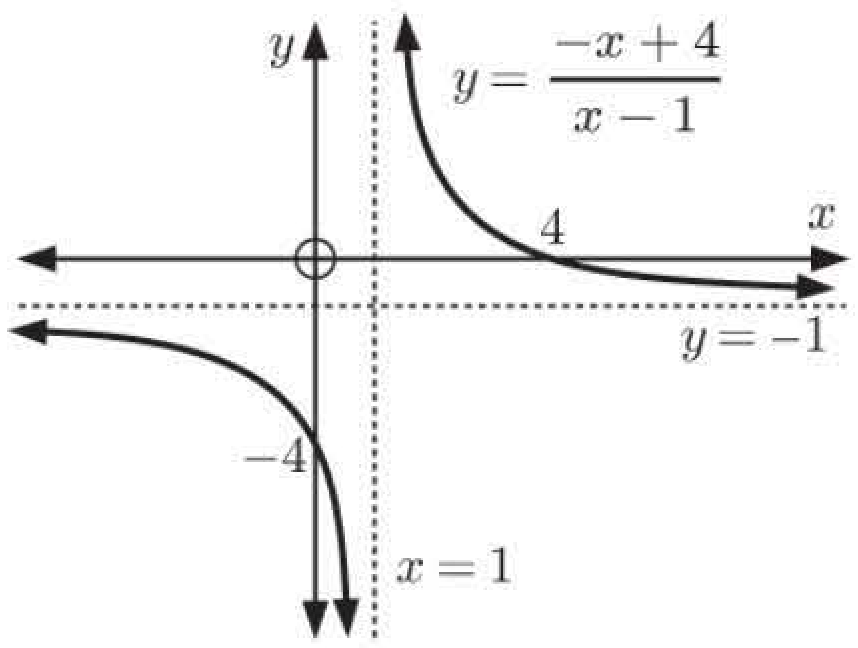
b The asymptotes of $y = \dfrac{1}{x}$ are $x = 0$ and $y = 0$.

These are unchanged by the stretch, and shifted $\binom{1}{-1}$ by the translation.

$\therefore$ the vertical asymptote is $x = 1$ and the horizontal asymptote is $y = -1$.

c Domain is $\{x \mid x \neq 1\}$, range is $\{y \mid y \neq -1\}$.

d



e The graph is not symmetric about $y = x$, so $g(x)$ is not a self-inverse function.

3 **a** **i** $f(x) = \frac{2x+4}{x-1} = \frac{2(x-1)+6}{x-1} = \frac{6}{x-1} + 2$

$y = f(x)$ is a translation of $y = \frac{6}{x}$ through $\binom{1}{2}$.

Now $y = \frac{6}{x}$ has asymptotes $x = 0$ and $y = 0$.

$\therefore$ $y = f(x)$ has vertical asymptote $x = 1$ and horizontal asymptote $y = 2$.

ii $\frac{1}{x}$ becomes $\frac{6}{x}$ under a vertical stretch with scale factor 6.

$\frac{6}{x}$ becomes $\frac{6}{x-1} + 2$ under a translation through $\binom{1}{2}$.

So, $y = \frac{1}{x}$ is transformed to $y = f(x)$ under a vertical stretch with scale factor 6, followed by a translation through $\binom{1}{2}$.

b **i** $f(x) = \frac{3x-2}{x+1} = \frac{3(x+1)-5}{x+1} = -\frac{5}{x+1} + 3$

$y = f(x)$ is a translation of $y = -\frac{5}{x}$ through $\binom{-1}{3}$.

Now $y = -\frac{5}{x}$ has asymptotes $x = 0$ and $y = 0$.

$\therefore$ $y = f(x)$ has vertical asymptote $x = -1$ and horizontal asymptote $y = 3$.

ii $\frac{1}{x}$ becomes $\frac{5}{x}$ under a vertical stretch with scale factor 5.

$\frac{5}{x}$ becomes $-\frac{5}{x}$ under a reflection in the x-axis.

$-\frac{5}{x}$ becomes $-\frac{5}{x+1} + 3$ under a translation through $\binom{-1}{3}$.

So, $y = \frac{1}{x}$ is transformed to $y = f(x)$ under a vertical stretch with scale factor 5, followed by a reflection in the x-axis, followed by a translation through $\binom{-1}{3}$.

c **i** $f(x) = \frac{2x+1}{2-x} = \frac{-2(2-x)+5}{2-x} = \frac{5}{2-x} - 2 = -\frac{5}{x-2} - 2$

$y = f(x)$ is a translation of $y = -\frac{5}{x}$ through $\binom{2}{-2}$.

Now $y = -\frac{5}{x}$ has asymptotes $x = 0$ and $y = 0$.

$\therefore$ $y = f(x)$ has vertical asymptote $x = 2$ and horizontal asymptote $y = -2$.

ii $\frac{1}{x}$ becomes $\frac{5}{x}$ under a vertical stretch with scale factor 5.

$\frac{5}{x}$ becomes $-\frac{5}{x}$ under a reflection in the x-axis.

$-\frac{5}{x}$ becomes $-\frac{5}{x-2} - 2$ under a translation through $\binom{2}{-2}$.

So, $y = \frac{1}{x}$ is transformed to $y = f(x)$ under a vertical stretch with scale factor 5, followed by a reflection in the x-axis, followed by a translation through $\binom{2}{-2}$.

4 **a** **i** $f(x) = \dfrac{2x+3}{x+1}$

$= \dfrac{2(x+1)+1}{x+1}$

$= \dfrac{1}{x+1} + 2$

$y = f(x)$ is a translation of $y = \dfrac{1}{x}$ through $\binom{-1}{2}$.

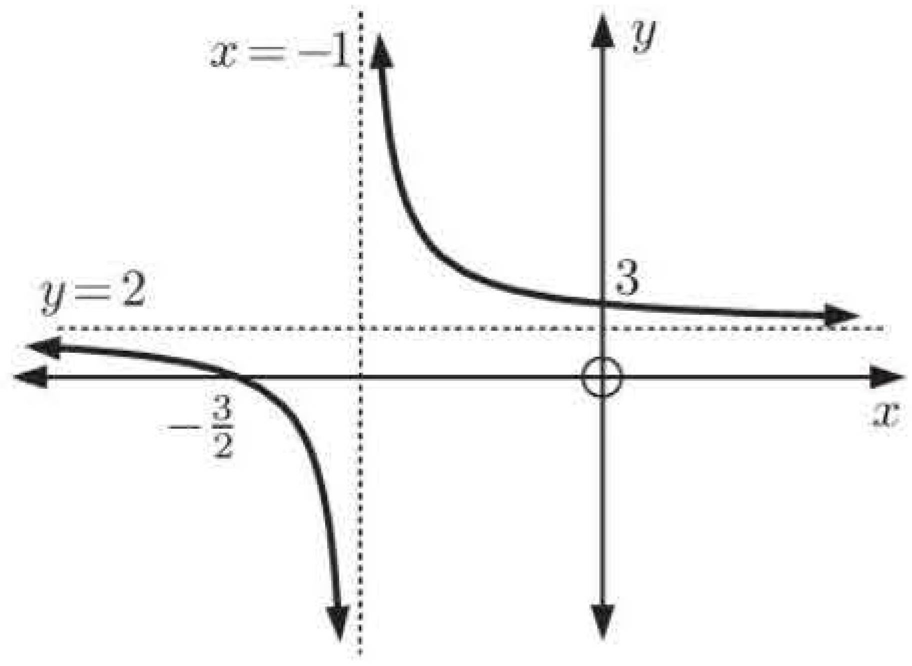
Now $y = \dfrac{1}{x}$ has asymptotes $x = 0$ and $y = 0$.

$\therefore$ $y = f(x)$ has vertical asymptote $x = -1$ and horizontal asymptote $y = 2$.

ii As $x \to -1^-$, $y \to -\infty$.
As $x \to -1^+$, $y \to \infty$.
As $x \to -\infty$, $y \to 2^-$.
As $x \to \infty$, $y \to 2^+$.

iii When $x = 0$, $y = \frac{1}{1} + 2 = 3$.
$\therefore$ the y-intercept is 3.
When $y = 0$, $2x + 3 = 0$
$\therefore$ $x = \frac{-3}{2}$
$\therefore$ the x-intercept is $\frac{-3}{2}$.

iv



v $\dfrac{1}{x}$ becomes $\dfrac{1}{x+1} + 2$ under a translation through $\binom{-1}{2}$.

So, $y = \dfrac{1}{x}$ is transformed to $y = f(x)$ under a translation through $\binom{-1}{2}$.

vi To transform $y = f(x)$ into $y = \dfrac{1}{x}$, we need to reverse the process in **v**.

We need a translation through $\binom{1}{-2}$.

b **i** $f(x) = \dfrac{3}{x-2}$

$y = f(x)$ is a translation of $y = \dfrac{3}{x}$ through $\binom{2}{0}$.

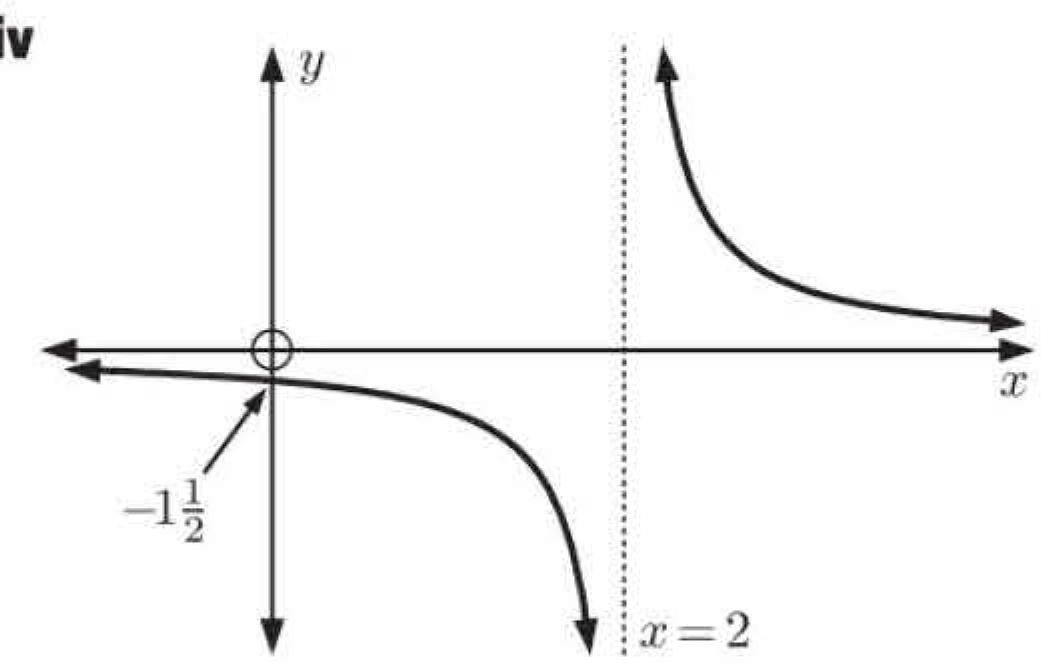
Now $y = \dfrac{3}{x}$ has asymptotes $x = 0$ and $y = 0$.

$\therefore$ $y = f(x)$ has vertical asymptote $x = 2$ and horizontal asymptote $y = 0$.

ii As $x \to 2^-$, $y \to -\infty$.
As $x \to 2^+$, $y \to \infty$.
As $x \to -\infty$, $y \to 0^-$.
As $x \to \infty$, $y \to 0^+$.

iii When $x = 0$, $y = \frac{3}{-2} = -1\frac{1}{2}$.
$\therefore$ the y-intercept is $-1\frac{1}{2}$.
When $y = 0$, $\dfrac{3}{x-2} = 0$
which is not possible
$\therefore$ no x-intercept.

iv



v $\dfrac{1}{x}$ becomes $\dfrac{3}{x}$ under a vertical stretch with scale factor 3.

$\dfrac{3}{x}$ becomes $\dfrac{3}{x-2}$ under a translation through $\binom{2}{0}$.

So, $y = \dfrac{1}{x}$ is transformed to $y = f(x)$ under a vertical stretch with scale factor 3, followed by a translation through $\binom{2}{0}$.

vi To transform $y = f(x)$ into $y = \dfrac{1}{x}$, we need to reverse the process in **v**.

We need a translation through $\binom{-2}{0}$, followed by a vertical stretch with scale factor $\frac{1}{3}$.

c **i** $f(x) = \dfrac{2x-1}{3-x}$
$= \dfrac{-2(3-x)+5}{3-x}$
$= \dfrac{5}{3-x} - 2$
$= -\dfrac{5}{x-3} - 2$

$y = f(x)$ is a translation of $y = -\dfrac{5}{x}$ through $\binom{3}{-2}$.

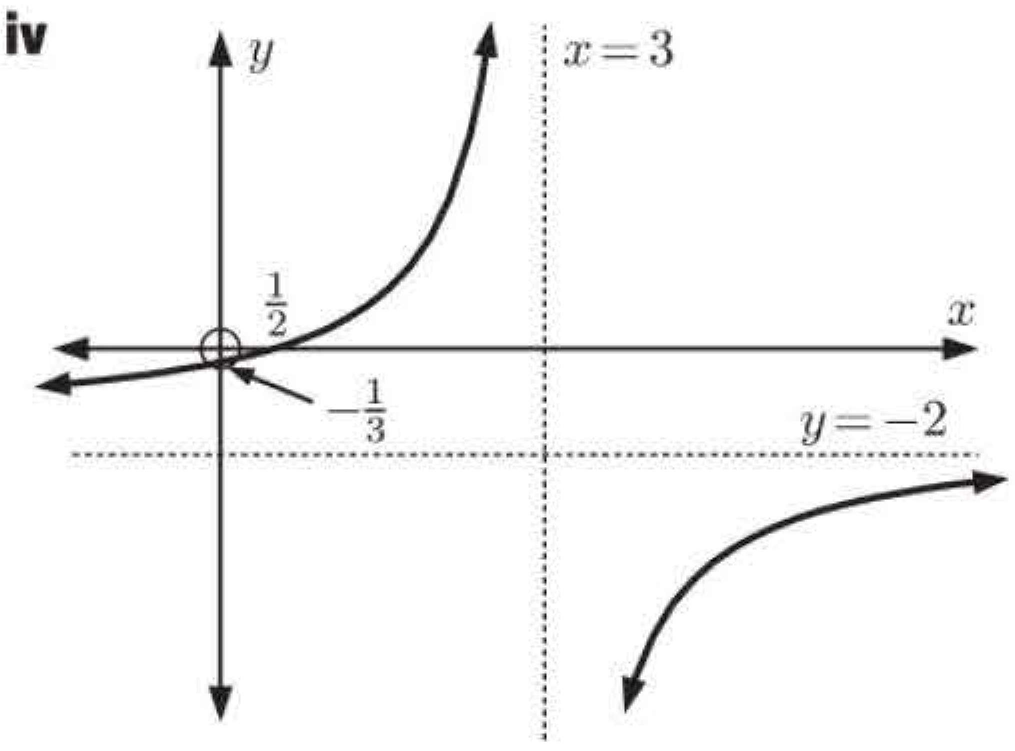
Now $y = -\dfrac{5}{x}$ has asymptotes $x = 0$ and $y = 0$.

$\therefore$ $y = f(x)$ has vertical asymptote $x = 3$ and horizontal asymptote $y = -2$.

ii As $x \to 3^-$, $y \to \infty$.
As $x \to 3^+$, $y \to -\infty$.
As $x \to -\infty$, $y \to -2^+$.
As $x \to \infty$, $y \to -2^-$.

iii When $x = 0$, $y = \frac{-5}{-3} - 2 = -\frac{1}{3}$.
$\therefore$ the y-intercept is $-\frac{1}{3}$.
When $y = 0$, $2x - 1 = 0$
$\therefore$ $x = \frac{1}{2}$
$\therefore$ the x-intercept is $\frac{1}{2}$.

iv



v $\dfrac{1}{x}$ becomes $\dfrac{5}{x}$ under a vertical stretch with scale factor 5.
$\dfrac{5}{x}$ becomes $-\dfrac{5}{x}$ under a reflection in the x-axis.
$-\dfrac{5}{x}$ becomes $-\dfrac{5}{x-3} - 2$ under a translation through $\binom{3}{-2}$.
So, $y = \dfrac{1}{x}$ is transformed to $y = f(x)$ under a vertical stretch with scale factor 5, followed by a reflection in the x-axis, followed by a translation through $\binom{3}{-2}$.

vi To transform $y = f(x)$ into $y = \dfrac{1}{x}$, we need to reverse the process in **v**.
We need a translation through $\binom{-3}{2}$, followed by a reflection in the x-axis, followed by a vertical stretch with scale factor $\frac{1}{5}$.

d **i** $f(x) = \dfrac{5x-1}{2x+1}$
$= \dfrac{\frac{5}{2}x - \frac{1}{2}}{x + \frac{1}{2}}$
$= \dfrac{\frac{5}{2}(x+\frac{1}{2}) - \frac{7}{4}}{x + \frac{1}{2}}$
$= -\dfrac{\frac{7}{4}}{x+\frac{1}{2}} + \frac{5}{2}$

$y = f(x)$ is a translation of $y = -\dfrac{\frac{7}{4}}{x}$ through $\begin{pmatrix} -\frac{1}{2} \\ \frac{5}{2} \end{pmatrix}$.

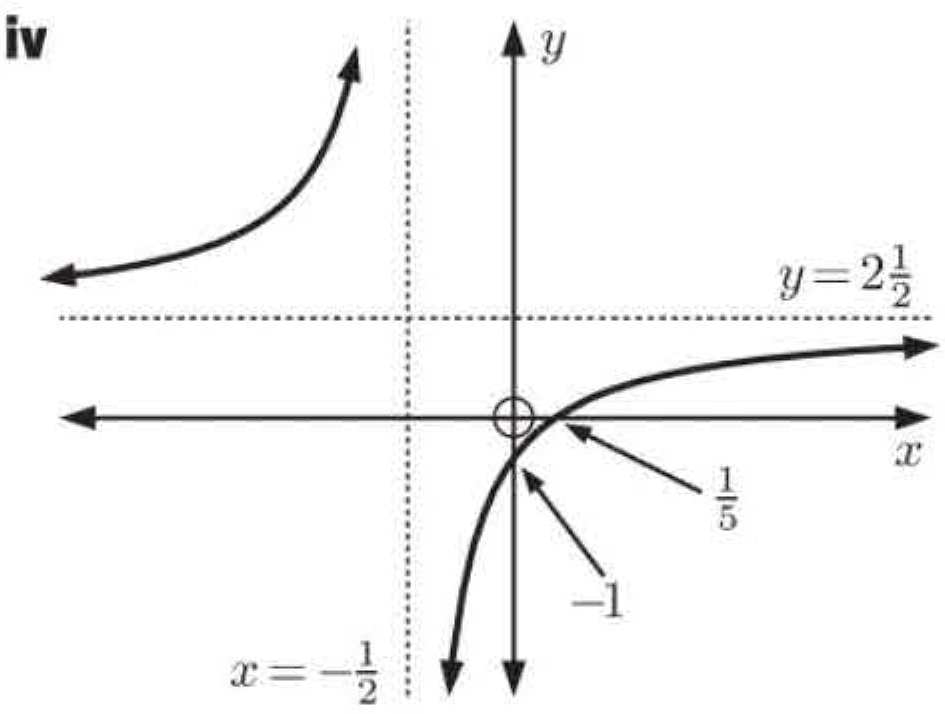
Now $y = -\dfrac{\frac{7}{4}}{x}$ has asymptotes $x = 0$ and $y = 0$.

$\therefore$ $y = f(x)$ has vertical asymptote $x = -\frac{1}{2}$ and horizontal asymptote $y = 2\frac{1}{2}$.

ii As $x \to -\frac{1}{2}^-$, $y \to \infty$.
As $x \to -\frac{1}{2}^+$, $y \to -\infty$.
As $x \to -\infty$, $y \to 2\frac{1}{2}^+$.
As $x \to \infty$, $y \to 2\frac{1}{2}^-$.

iii When $x = 0$, $y = \frac{-1}{1} = -1$.
$\therefore$ the y-intercept is -1.
When $y = 0$, $5x - 1 = 0$
$\therefore$ $x = \frac{1}{5}$
$\therefore$ the x-intercept is $\frac{1}{5}$.

iv



v $\dfrac{1}{x}$ becomes $\dfrac{\frac{7}{4}}{x}$ under a vertical stretch with scale factor $\frac{7}{4}$.

$\dfrac{\frac{7}{4}}{x}$ becomes $-\dfrac{\frac{7}{4}}{x}$ under a reflection in the x-axis.

$-\dfrac{\frac{7}{4}}{x}$ becomes $-\dfrac{\frac{7}{4}}{x+\frac{1}{2}}+\frac{5}{2}$ under a translation through $\begin{pmatrix} -\frac{1}{2} \\ \frac{5}{2} \end{pmatrix}$.

So, $y=\dfrac{1}{x}$ is transformed to $y=f(x)$ under a vertical stretch with scale factor $\frac{7}{4}$, followed by a reflection in the x-axis, followed by a translation through $\begin{pmatrix} -\frac{1}{2} \\ \frac{5}{2} \end{pmatrix}$.

vi To transform $y=f(x)$ into $y=\dfrac{1}{x}$, we need to reverse the process in **v**.

We need a translation through $\begin{pmatrix} \frac{1}{2} \\ -\frac{5}{2} \end{pmatrix}$, followed by a reflection in the x-axis, followed by a vertical stretch with scale factor $\frac{4}{7}$.

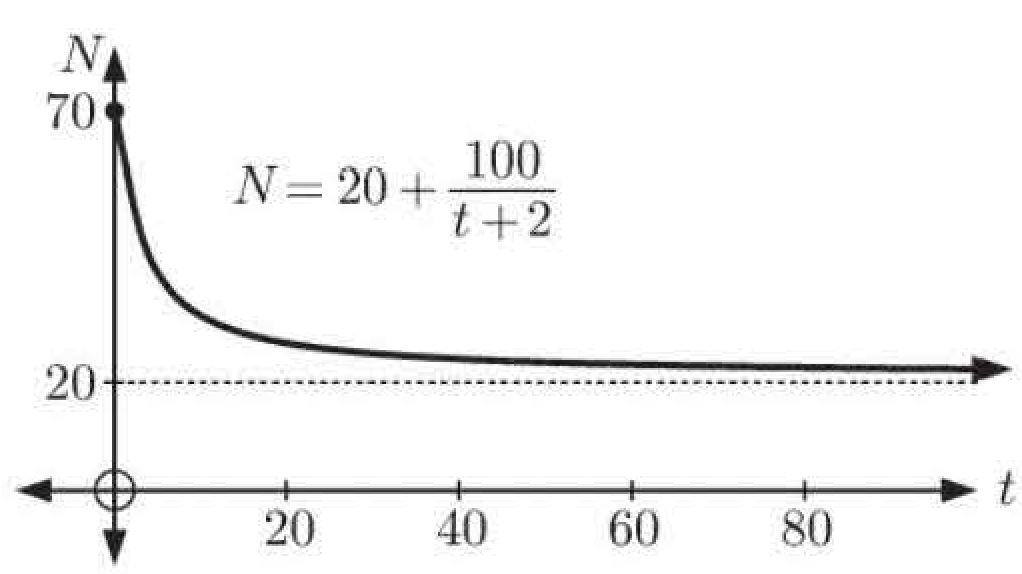
5 $N=20+\dfrac{100}{t+2}$ weeds per hectare

a When $t=0$,
$$\begin{aligned} N &= 20+\tfrac{100}{2} \\ &= 20+50 \\ &= 70 \text{ weeds/ha} \end{aligned}$$

b When $t=8$,
$$\begin{aligned} N &= 20+\tfrac{100}{10} \\ &= 20+10 \\ &= 30 \text{ weeds/ha} \end{aligned}$$

c When $N=40$,
$$\begin{aligned} 20+\frac{100}{t+2} &= 40 \\ \therefore\ \frac{100}{t+2} &= 20 \\ \therefore\ t+2 &= 5 \\ \therefore\ t &= 3 \text{ days} \end{aligned}$$

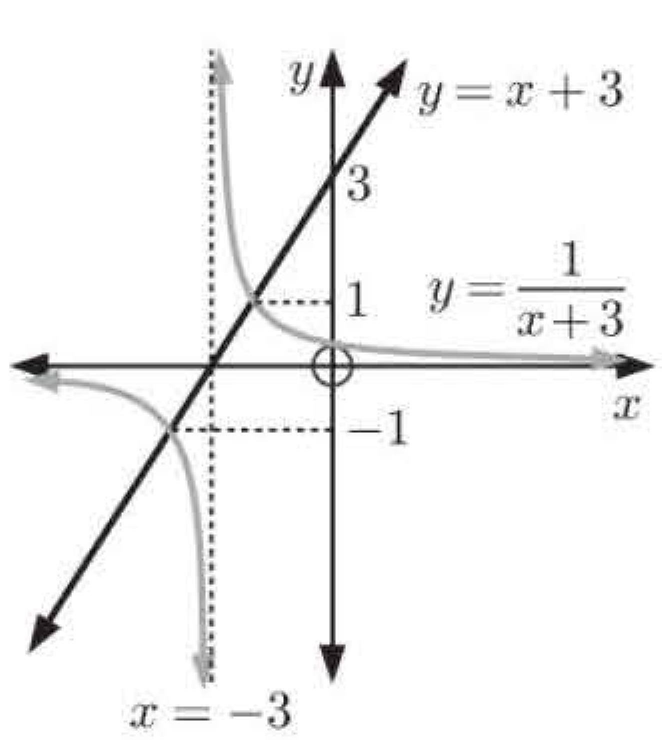
d



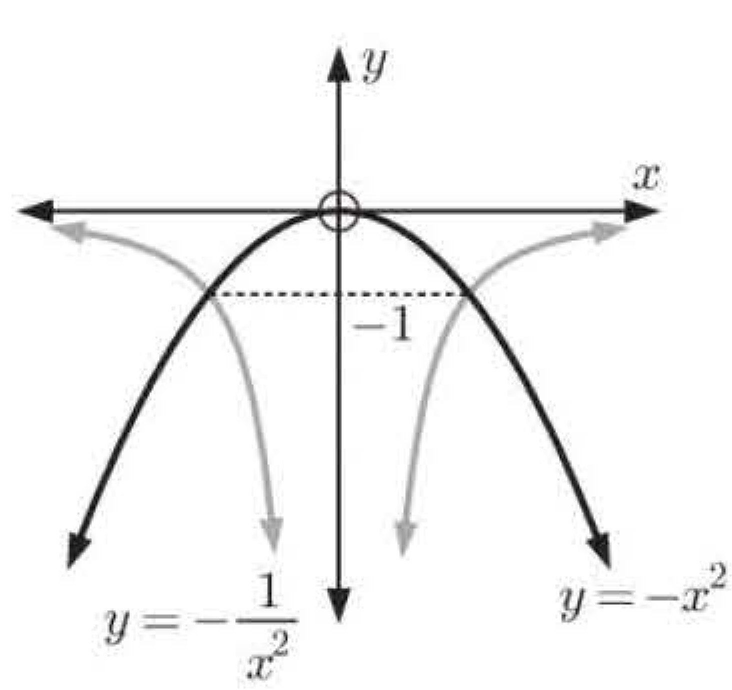
e No, the number of weeds per hectare will approach 20 (from above), so at least 20 weeds will remain..

EXERCISE 5G

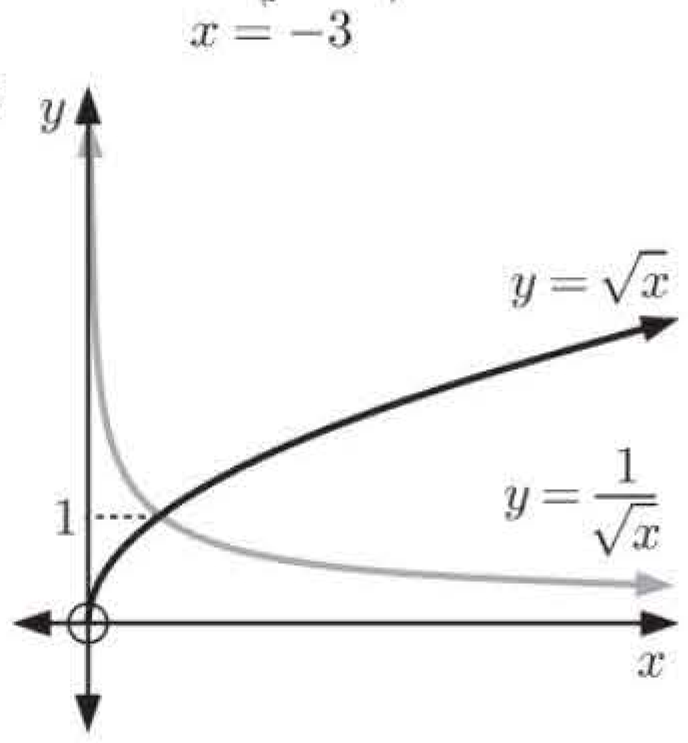
1 **a**



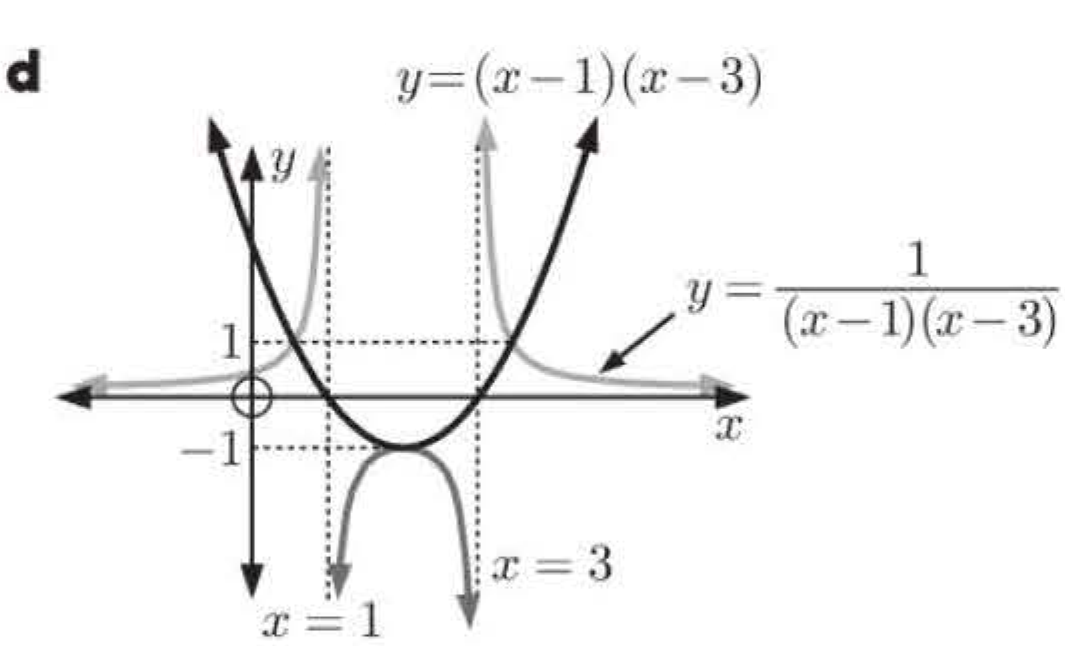
b



c



d



2 If $f(x) = \dfrac{1}{f(x)}$ then $y = \dfrac{1}{y}$

$\therefore\ y^2 = 1$

$\therefore\ y = \pm 1$

For **1 a**: When $y = 1$, $x + 3 = 1$ $\quad\therefore\ x = -2$

When $y = -1$, $x + 3 = -1$ $\quad\therefore\ x = -4$

So, the invariant points are $(-2, 1)$ and $(-4, -1)$.

For **1 b**: When $y = 1$, $-x^2 = 1$ which has no real solutions

When $y = -1$, $-x^2 = -1$ $\quad\therefore\ x^2 = 1$ $\quad\therefore\ x = \pm 1$

So, the invariant points are $(1, -1)$ and $(-1, -1)$.

For **1 c**: When $y = 1$, $\sqrt{x} = 1$ $\quad\therefore\ x = 1$

When $y = -1$, $\sqrt{x} = -1$ which has no real solutions.

So, the invariant point is $(1, 1)$.

For **1 d**: When $y = 1$, $(x-1)(x-3) = 1$

$\therefore\ x^2 - 4x + 3 = 1$

$\therefore\ x^2 - 4x + 2 = 0$

$\therefore\ x = \dfrac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2}$

$= 3.41$ or 0.586

When $y = -1$, $(x-1)(x-3) = -1$

$\therefore\ x^2 - 4x + 3 = -1$

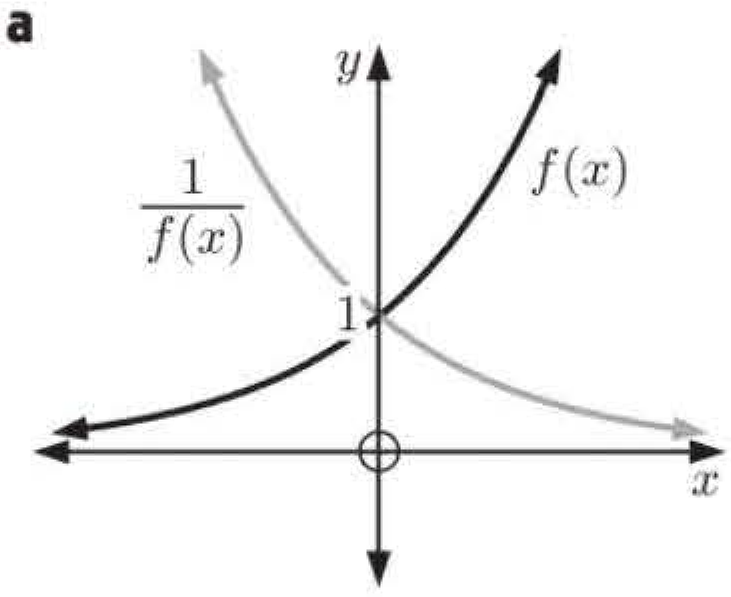
$\therefore\ x^2 - 4x + 4 = 0$

$\therefore\ (x-2)^2 = 0$

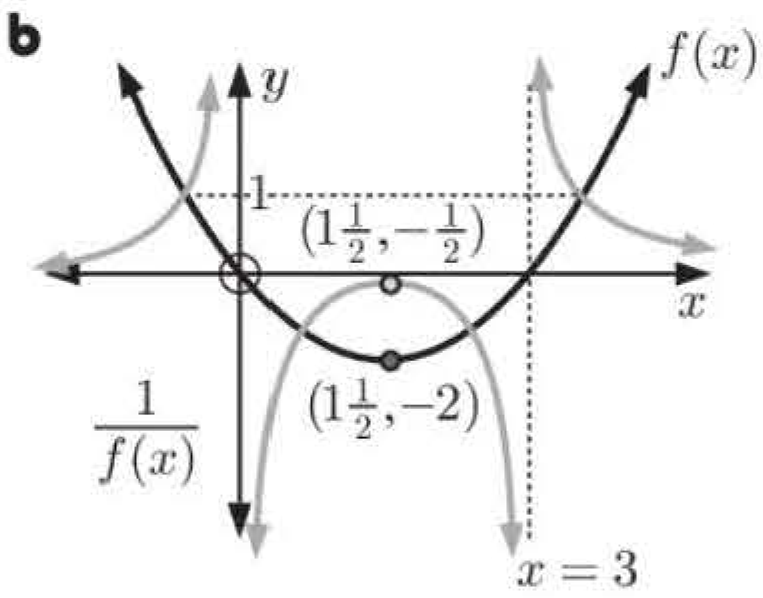
$\therefore\ x = 2$

So, the invariant points are $(3.41, 1)$, $(0.586, 1)$, and $(2, -1)$.

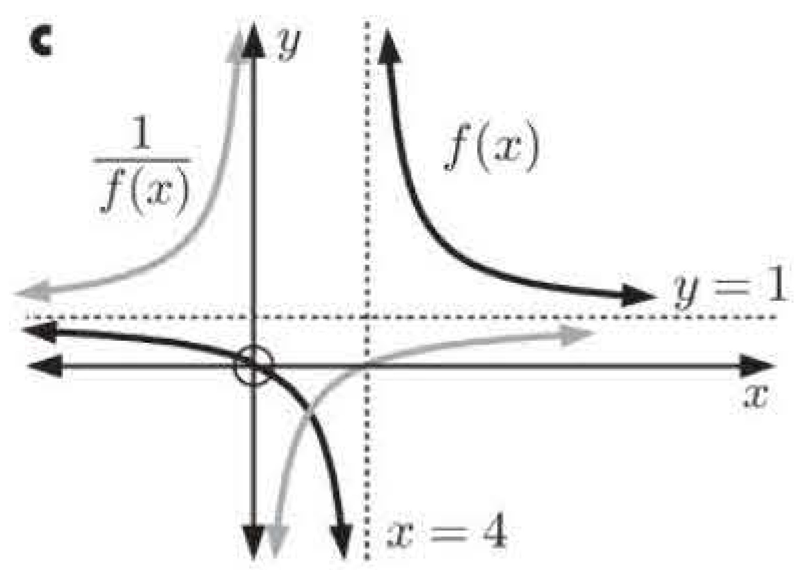
3 **a**



b

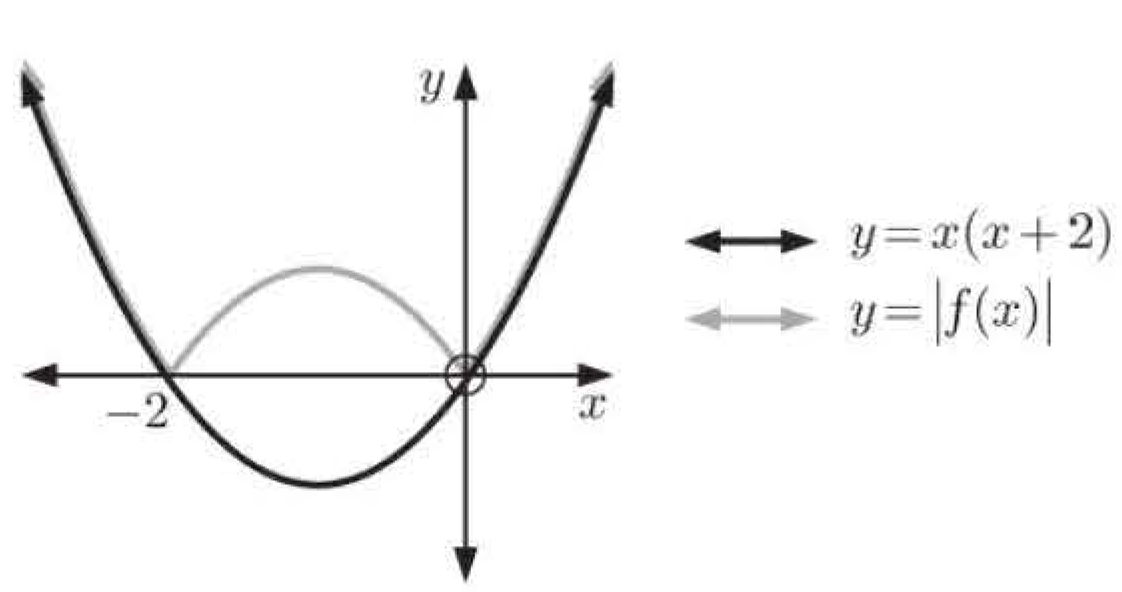


c

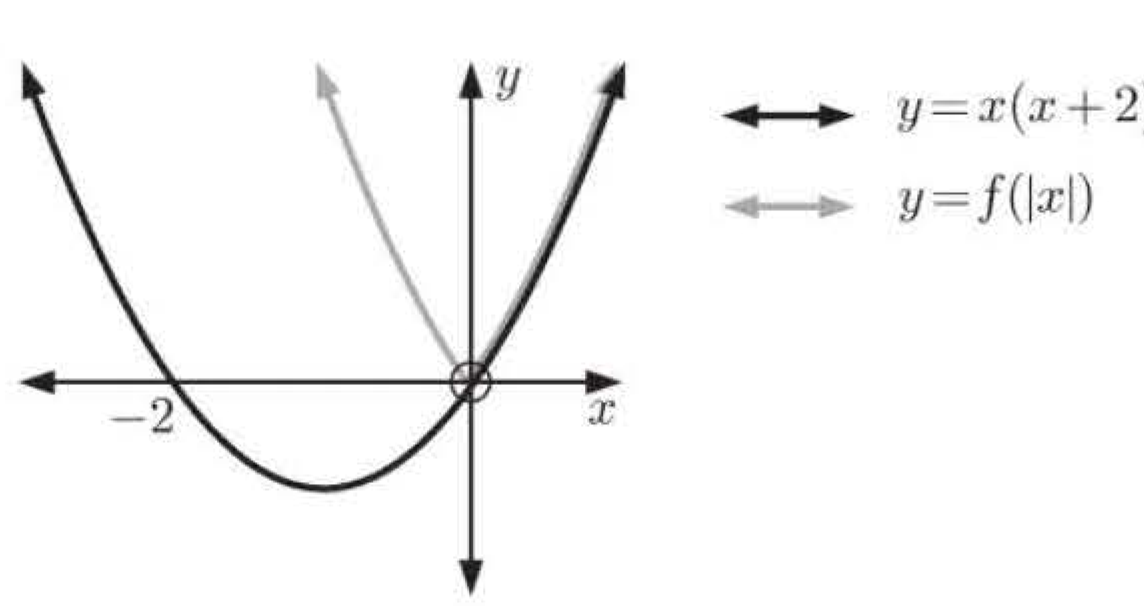


EXERCISE 5H

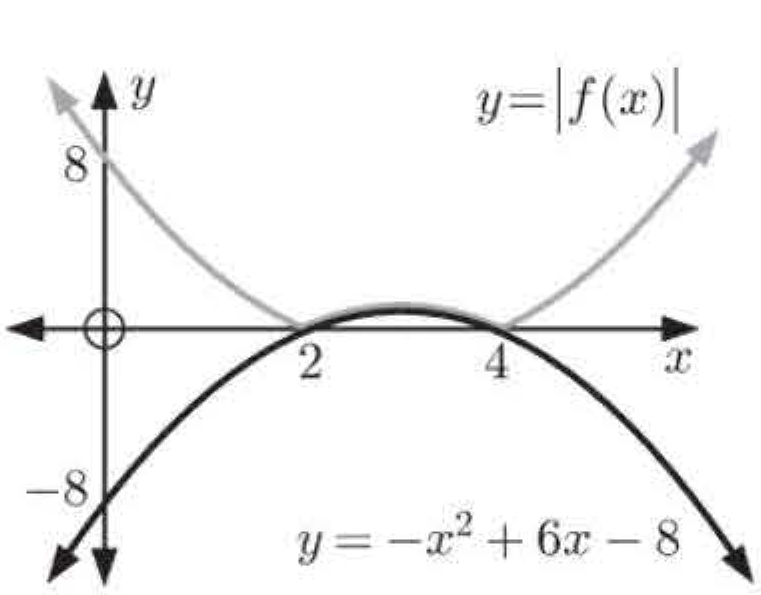
1 **a**



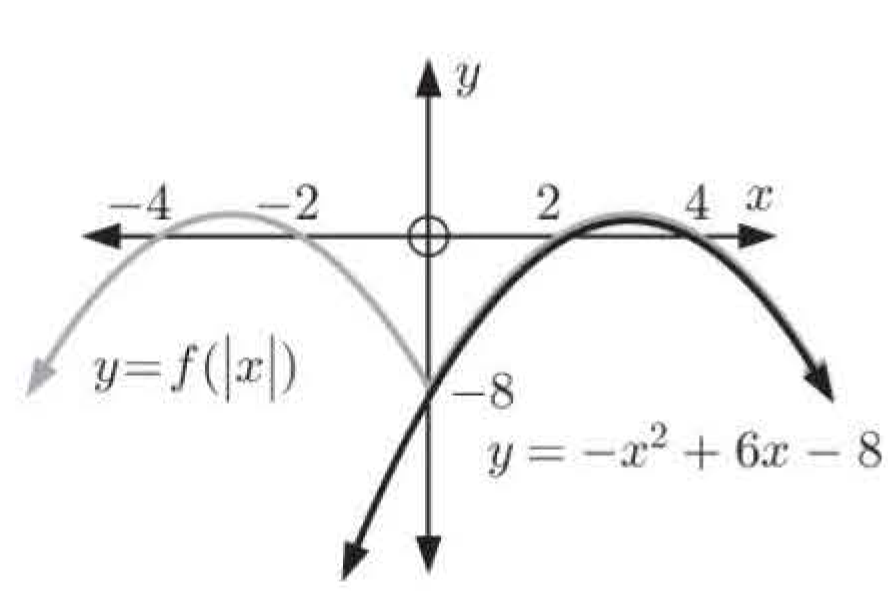
b



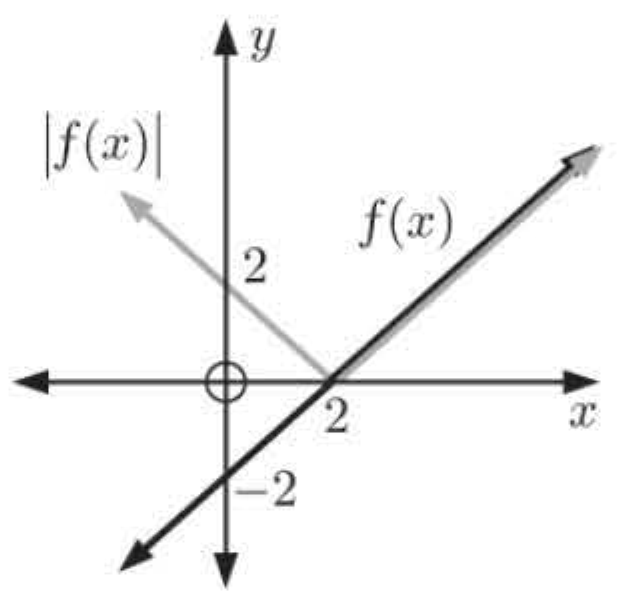
2 **a**



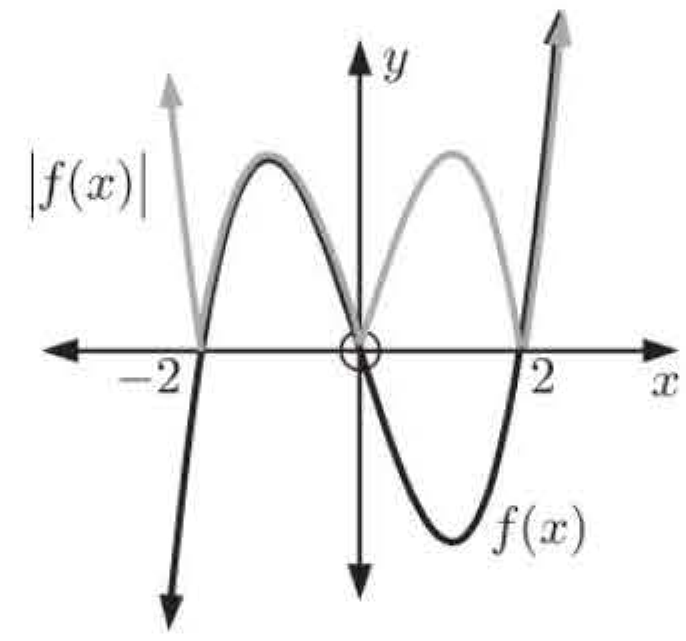
b



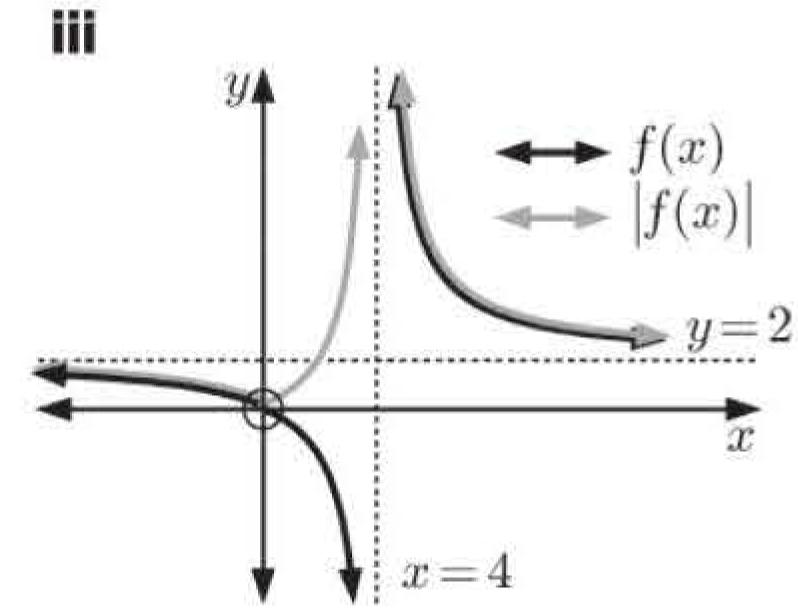
3 **a** **i**



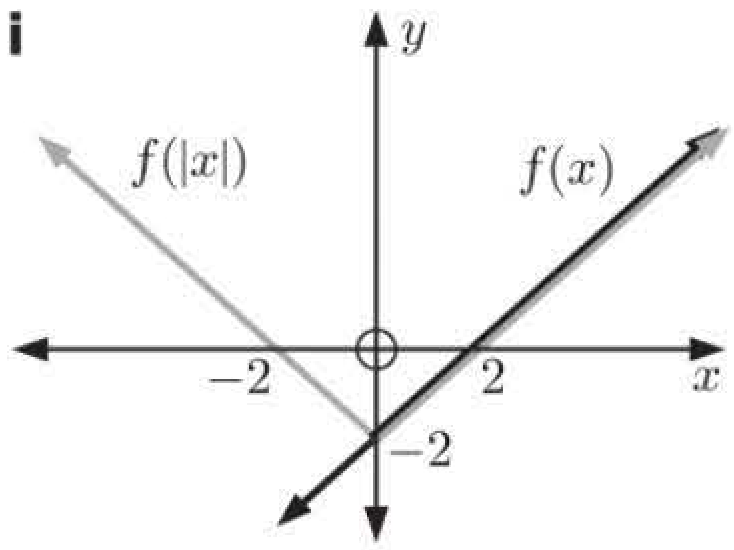
ii



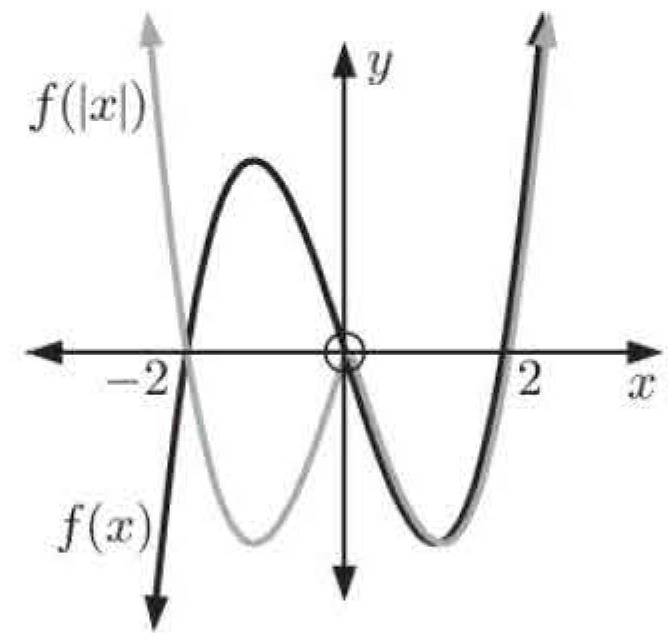
iii



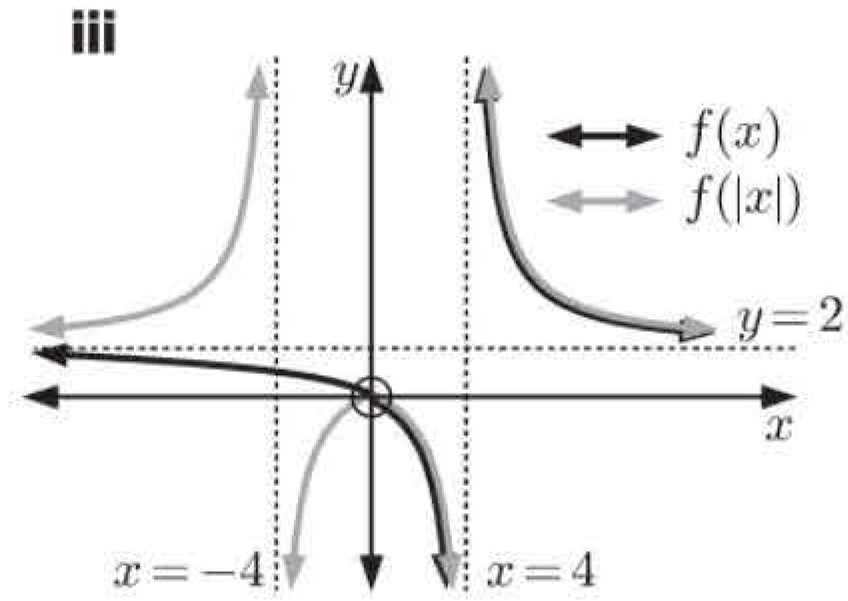
b **i**



ii



iii



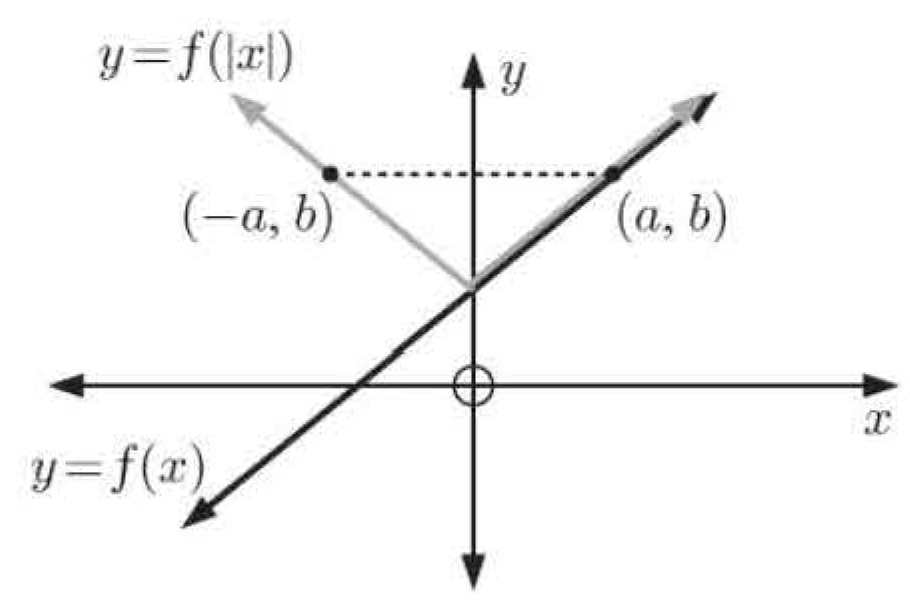
4 To transform $f(x)$ to $|f(x)|$, the point (a, b) on $f(x)$ is transformed to $(a, |b|)$.

a $(3, 0)$ is transformed to $(3, 0)$ **b** $(5, -2)$ is transformed to $(5, 2)$

c $(0, 7)$ is transformed to $(0, 7)$ **d** $(2, 2)$ is transformed to $(2, 2)$

5 **a** For any point (a, b), $a \geqslant 0$, on $f(x)$, (a, b) is also a point on $f(|x|)$, and (a, b) is also transformed to $(-a, b)$.

- **i** $(0, 3)$ is transformed to $(0, 3)$
- **ii** $(1, 3)$ is transformed to $(1, 3)$ and $(-1, 3)$
- **iii** $(7, -4)$ is transformed to $(7, -4)$ and $(-7, -4)$



b The point (a, b) on $f(|x|)$ has been transformed by the point $(|a|, b)$ on $f(x)$.

- **i** $(0, 3)$ has been transformed from $(0, 3)$
- **ii** $(-1, 3)$ has been transformed from $(1, 3)$
- **iii** $(10, -8)$ has been transformed from $(10, -8)$

REVIEW SET 5A

1 $f(x) = x^2 - 2x$

a $f(3)$
$= 3^2 - 2(3)$
$= 9 - 6$
$= 3$

b $f(2x)$
$= (2x)^2 - 2(2x)$
$= 4x^2 - 4x$

c $f(-x)$
$= (-x)^2 - 2(-x)$
$= x^2 + 2x$

d $3\,f(x) - 2$
$= 3(x^2 - 2x) - 2$
$= 3x^2 - 6x - 2$

2 $f(x) = 5 - x - x^2$

a $f(-1) = 5 - (-1) - (-1)^2$
$= 5 + 1 - 1$
$= 5$

b $f(x-1) = 5 - (x-1) - (x-1)^2$
$= 5 - x + 1 - [x^2 - 2x + 1]$
$= 6 - x - x^2 + 2x - 1$
$= -x^2 + x + 5$

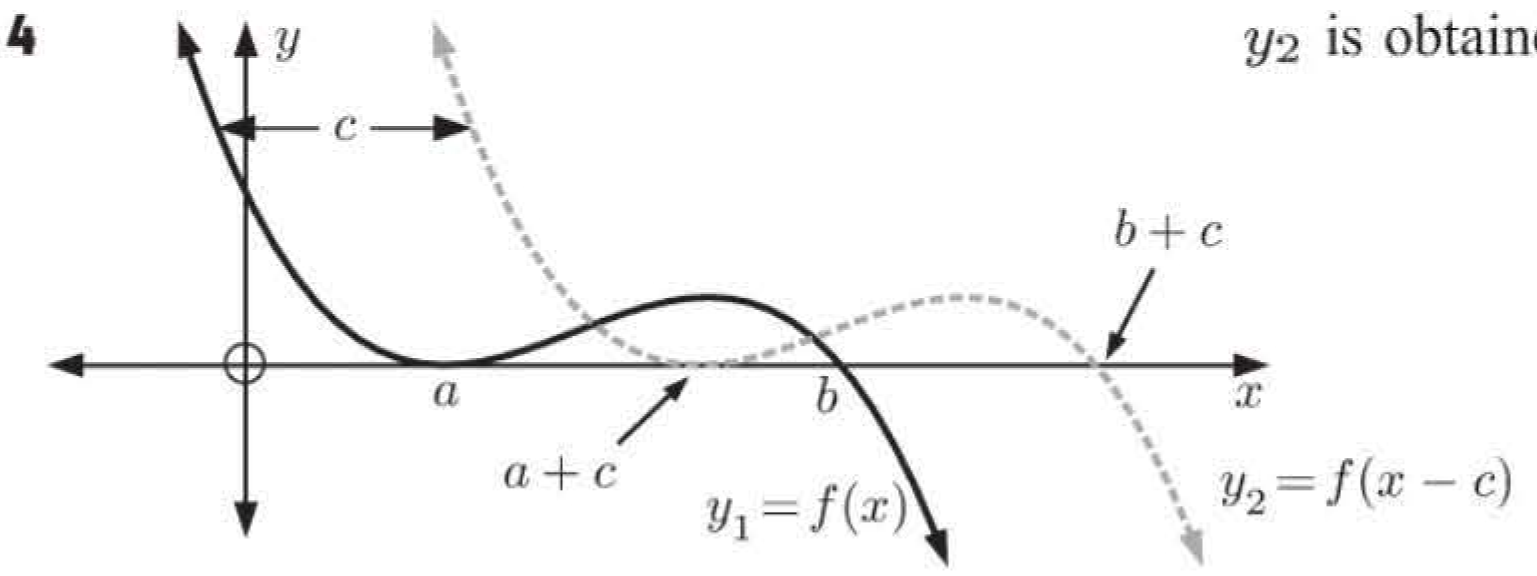
c $f\left(\frac{x}{2}\right) = 5 - \left(\frac{x}{2}\right) - \left(\frac{x}{2}\right)^2$
$= 5 - \frac{1}{2}x - \frac{1}{4}x^2$

d $2\,f(x) - f(-x)$
$= 2(5 - x - x^2) - \left[5 - (-x) - (-x)^2\right]$
$= 10 - 2x - 2x^2 - [5 + x - x^2]$
$= 10 - 2x - 2x^2 - 5 - x + x^2$
$= -x^2 - 3x + 5$

3 $f(x) = 3x^3 - 2x^2 + x + 2$

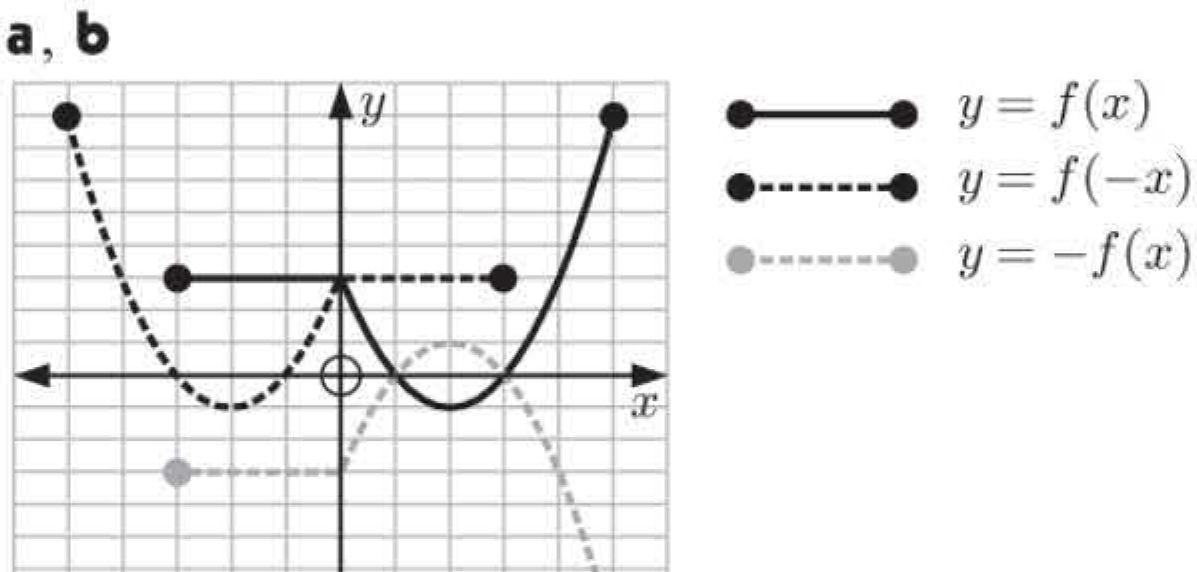
If $g(x)$ is $f(x)$ translated $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$, then
$$\begin{aligned} g(x) &= f(x-1) - 2 \\ &= 3(x-1)^3 - 2(x-1)^2 + (x-1) + 2 - 2 \\ &= 3(x^3 - 3x^2 + 3x - 1) - 2(x^2 - 2x + 1) + x - 1 \\ &= 3x^3 - 9x^2 + 9x - 3 - 2x^2 + 4x - 2 + x - 1 \\ &= 3x^3 - 11x^2 + 14x - 6 \end{aligned}$$

4



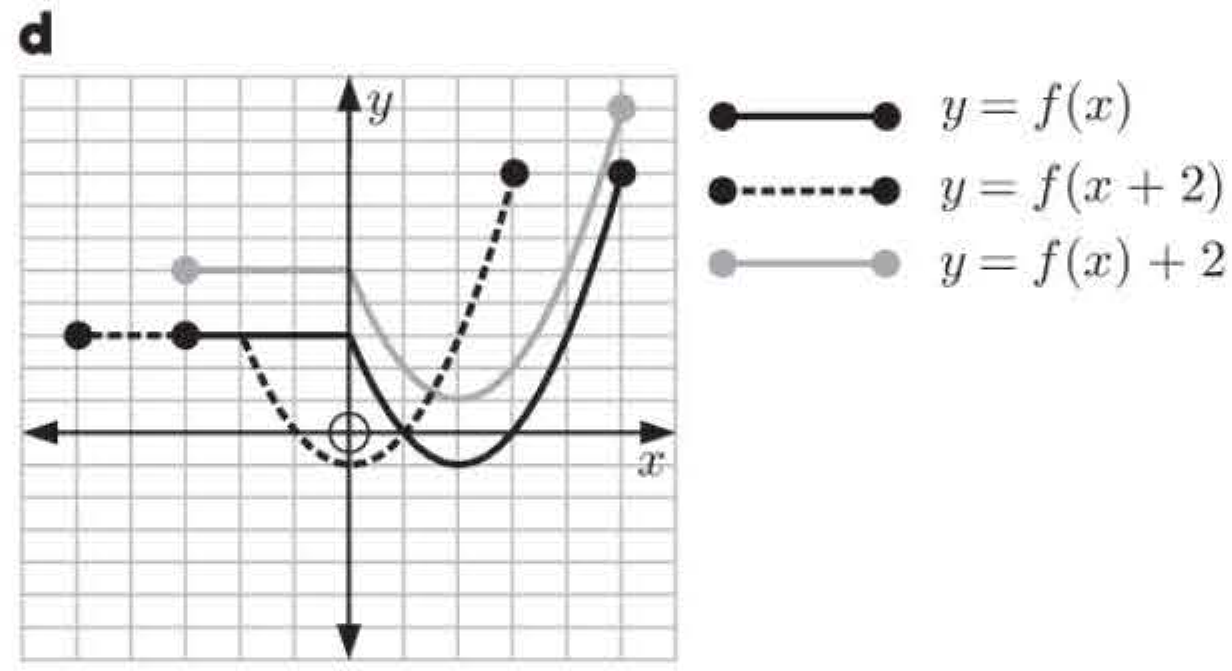
y_2 is obtained by translating y_1 c units to the right.

5 **a, b**

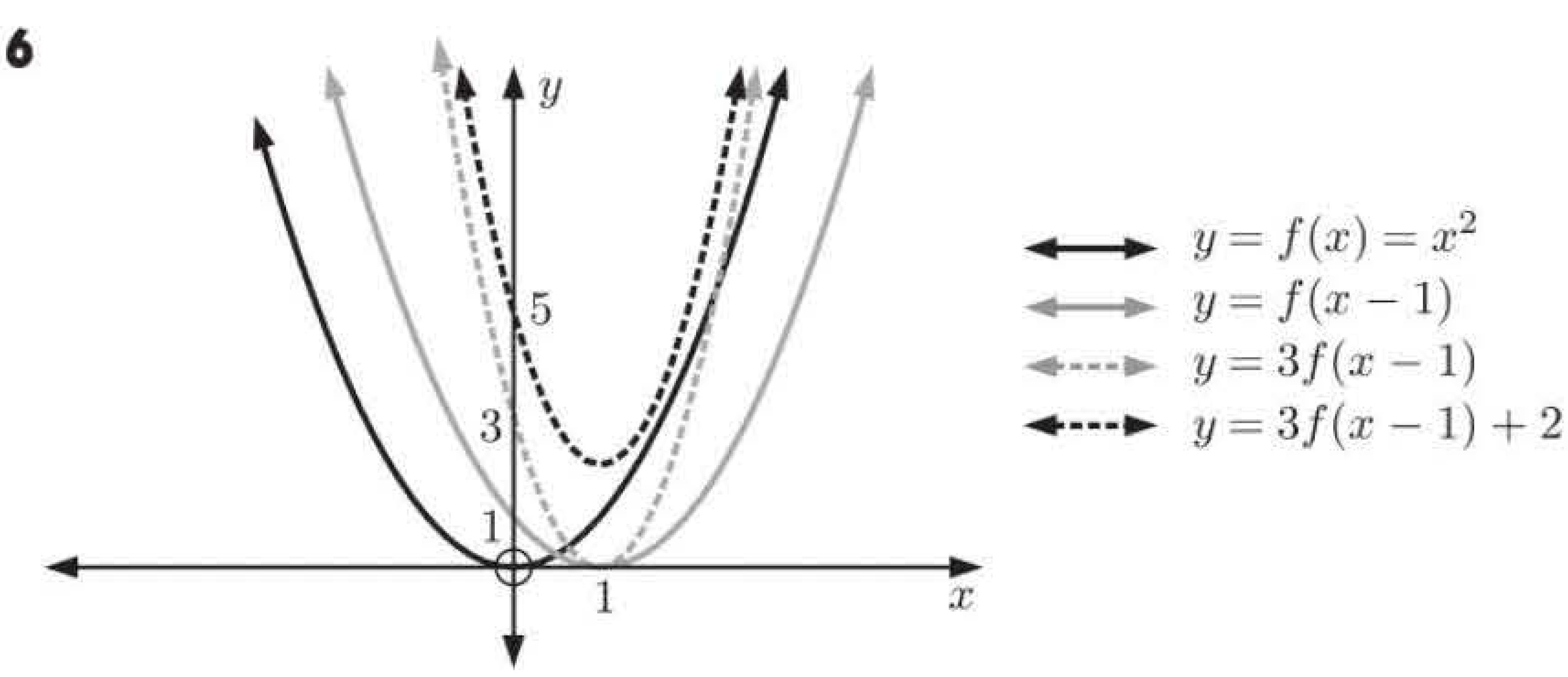


(drawn on two graphs)

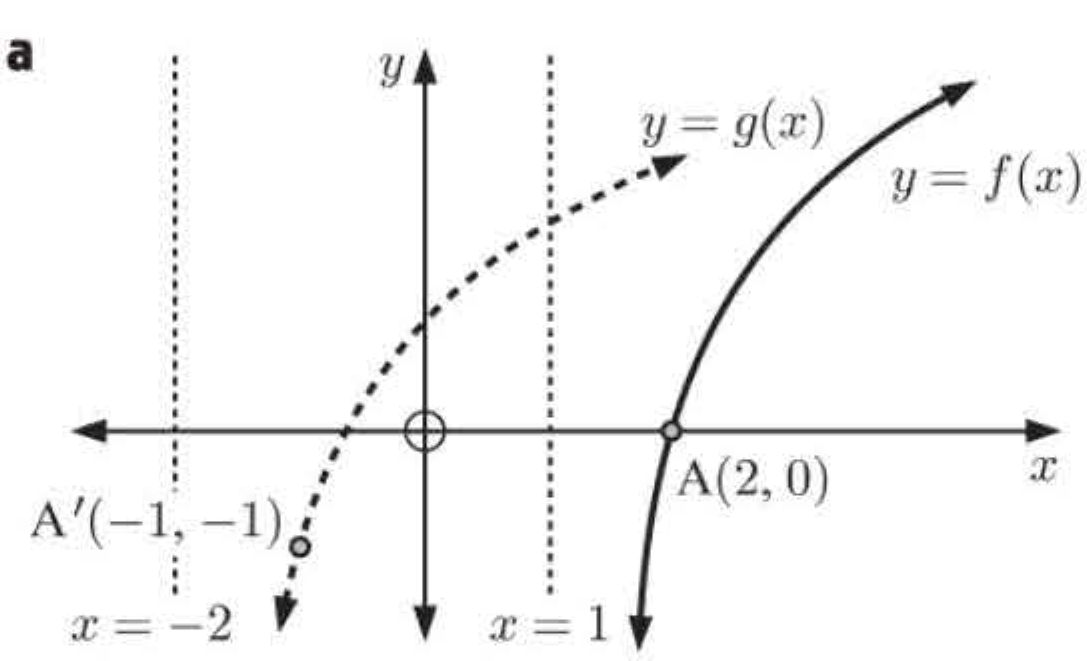
c, d



6



7 **a**



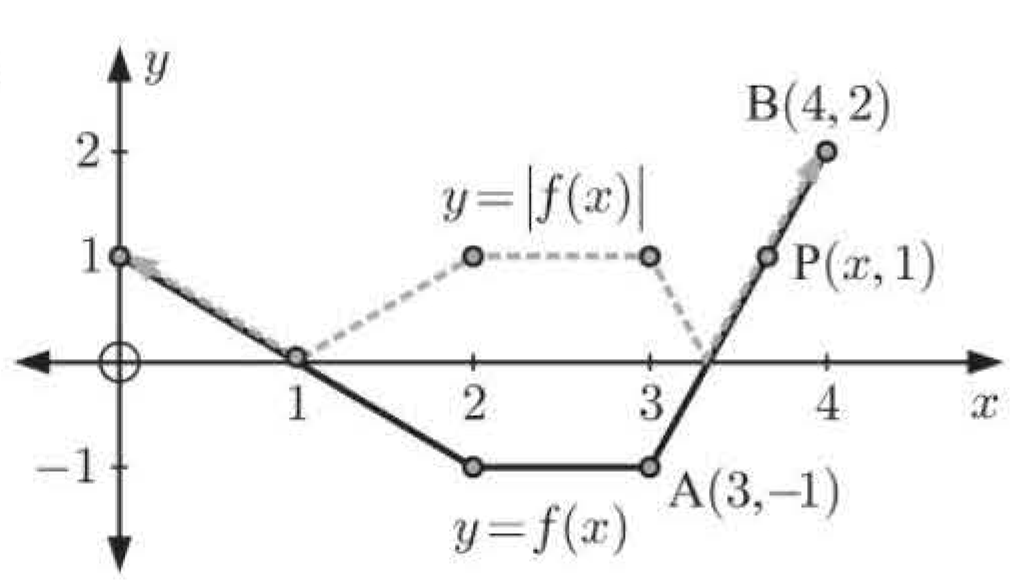
b $f(x+3) - 1$ is a translation of $f(x)$ by $\begin{pmatrix} -3 \\ -1 \end{pmatrix}$.

$\therefore$ vertical asymptote is at $x = 1 - 3 = -2$.

c A(2, 0) translated by $\begin{pmatrix} -3 \\ -1 \end{pmatrix}$ gives

$(2-3,\ 0-1)$ which is $A'(-1, -1)$.

8 **a**



b When $x = 0$,

$$\begin{aligned} \frac{1}{f(x)} &= \frac{1}{f(0)} \\ &= \tfrac{1}{1} \\ &= 1 \end{aligned}$$

$\therefore$ the y-intercept of $\dfrac{1}{f(x)}$ is 1.

c Invariant points for $\dfrac{1}{f(x)}$ occur when $f(x) = \pm 1$.

$f(x) = -1$ for all $x \in [2, 3]$

$f(x) = 1$ when $x = 0$ and at point P.

To find the point P where $f(x) = 1$,

note that the gradient of $[AB] = \dfrac{2-(-1)}{4-3} = 3$,

so $\dfrac{2-1}{4-x} = 3$

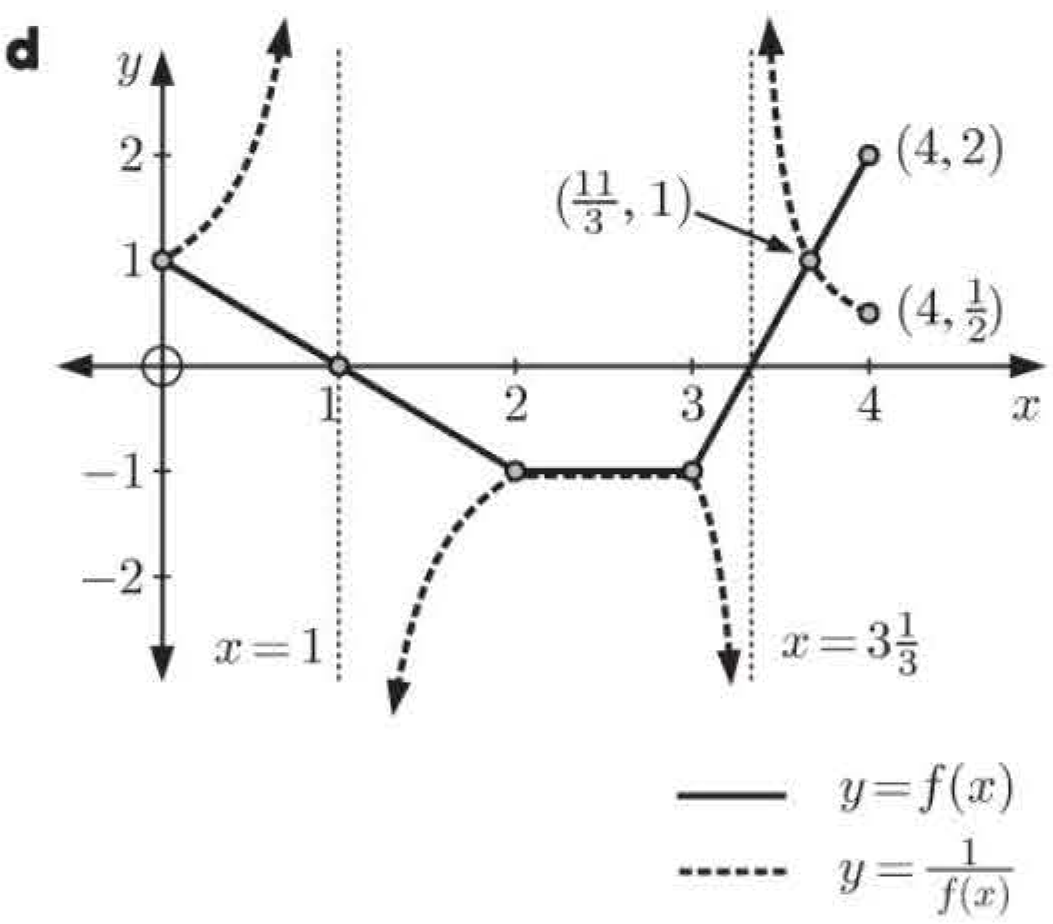
$\therefore \quad 1 = 12 - 3x$

$\therefore \quad 3x = 11$

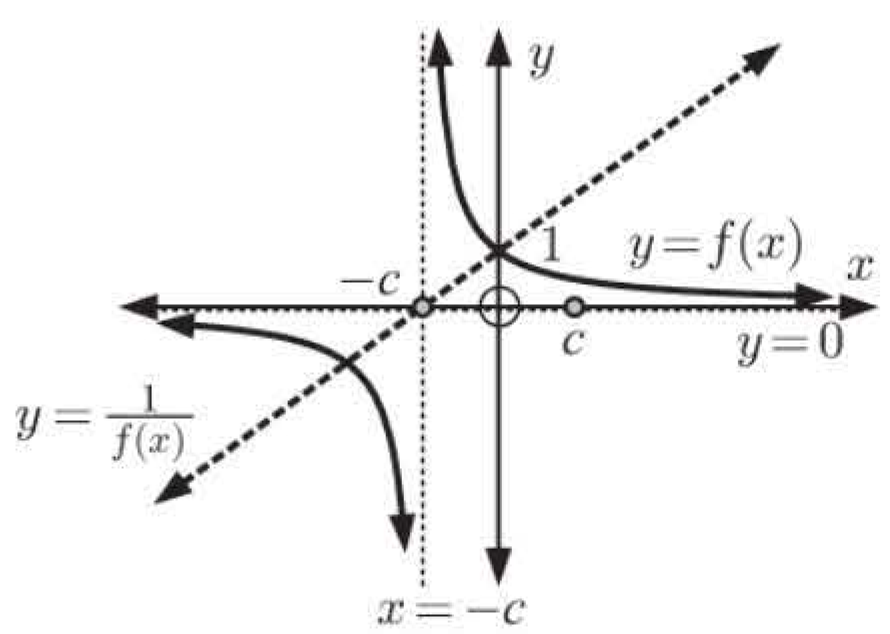
$\therefore \quad x = \frac{11}{3}$

$\therefore$ $f(x)$ is invariant for $\dfrac{1}{f(x)}$ at $(0, 1)$, $(\frac{11}{3}, 1)$, and all the points on $y = -1$, $x \in [2, 3]$.

d



9



a $f(x) = \dfrac{c}{x+c}$ has a VA $x = -c$ $\{f(x)$ is undefined$\}$ and a HA $y = 0$ $\{$as $|x| \to \infty$, $f(x) \to 0\}$

$f(0) = \dfrac{c}{0+c} = 1 \quad \therefore$ the y-intercept is 1

There are no x-intercepts $\{\dfrac{c}{x+c} \neq 0$ for $c > 0\}$

b $\dfrac{1}{f(x)} = \dfrac{x+c}{c}$

$\dfrac{1}{f(0)} = \dfrac{0+c}{c} = 1 \quad \therefore$ the y-intercept of $\dfrac{1}{f(x)}$ is 1

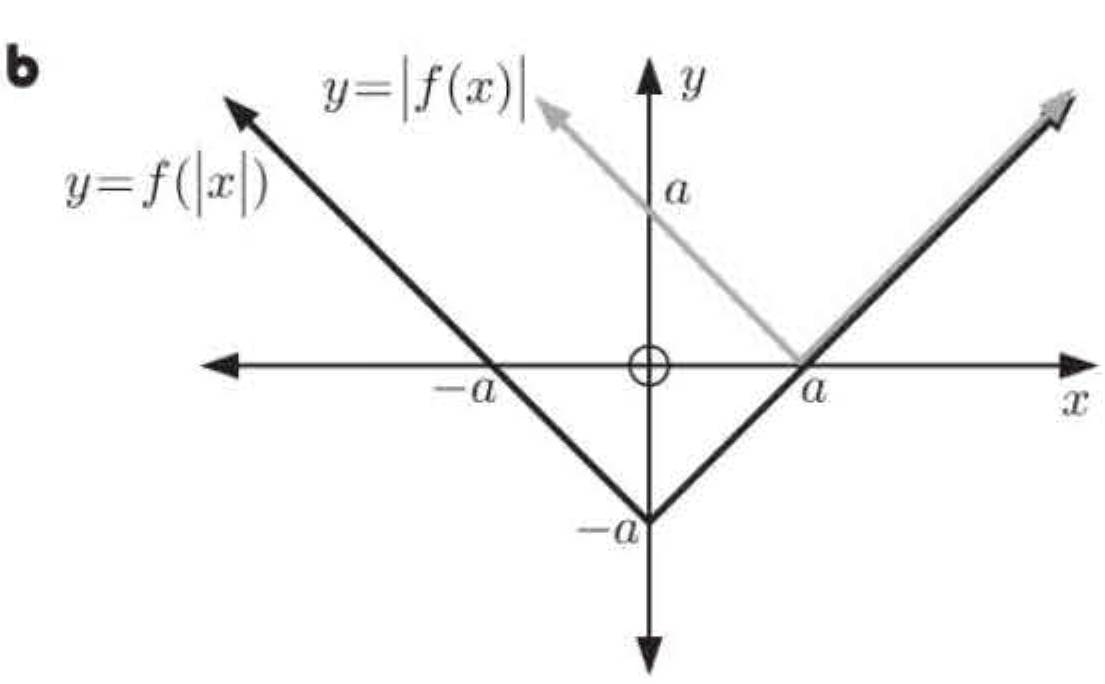
$\dfrac{1}{f(x)} = 0$ when $x + c = 0$ or $x = -c$

$\therefore$ the x-intercept of $\dfrac{1}{f(x)}$ is $-c$.

10 **a** $f(x) = x - a, \quad a > 0$

$\therefore \quad |f(x)| = |x - a|$ and $f(|x|) = |x| - a$

b



c Using the graph in **b**, $|x - a| = |x| - a$ when $x \geqslant a$.

or solving algebraically:

For $x < 0$ and $a > 0$, $|x - a| = a - x$ and $|x| = -x$

If $|x - a| = |x| - a$ then $a - x = -x - a$

$\therefore \quad 2a = 0$

$\therefore \quad a = 0$ which is not true.

For $0 \leqslant x < a$ and $a > 0$, $|x-a| = a-x$ and $|x| = x$

If $|x-a| = |x| - a$ then $a - x = x - a$

$\therefore\ 2x = 2a$

$\therefore\ x = a$ which is not true.

For $x \geqslant a$ and $a > 0$, $|x-a| = x-a$ and $|x| = x$

If $|x-a| = |x| - a$ then $x - a = x - a$ which is true.

So, for $a > 0$, $|x-a| = |x| - a$ is true for all $x \geqslant a$.

REVIEW SET 5B

1 When $y = 0$, $(x+1)^2 - 4 = 0$

$\therefore\ (x+1)^2 = 4$

$\therefore\ x + 1 = \pm 2$

$\therefore\ x = 2 - 1$ or $-2 - 1$

$\therefore\ x = 1$ or -3

When $x = 0$, $y = 1^2 - 4$

$= -3$

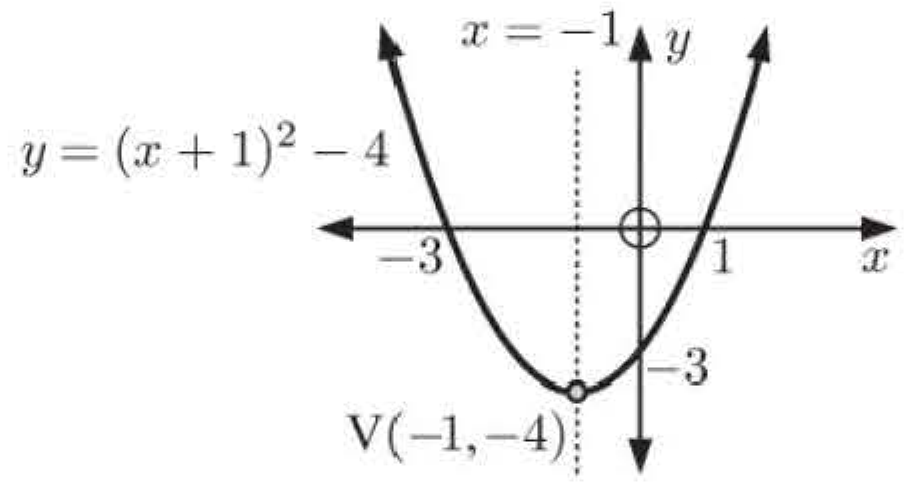
$\therefore$ y-intercept is -3

$\therefore$ x-intercepts are $1, -3$

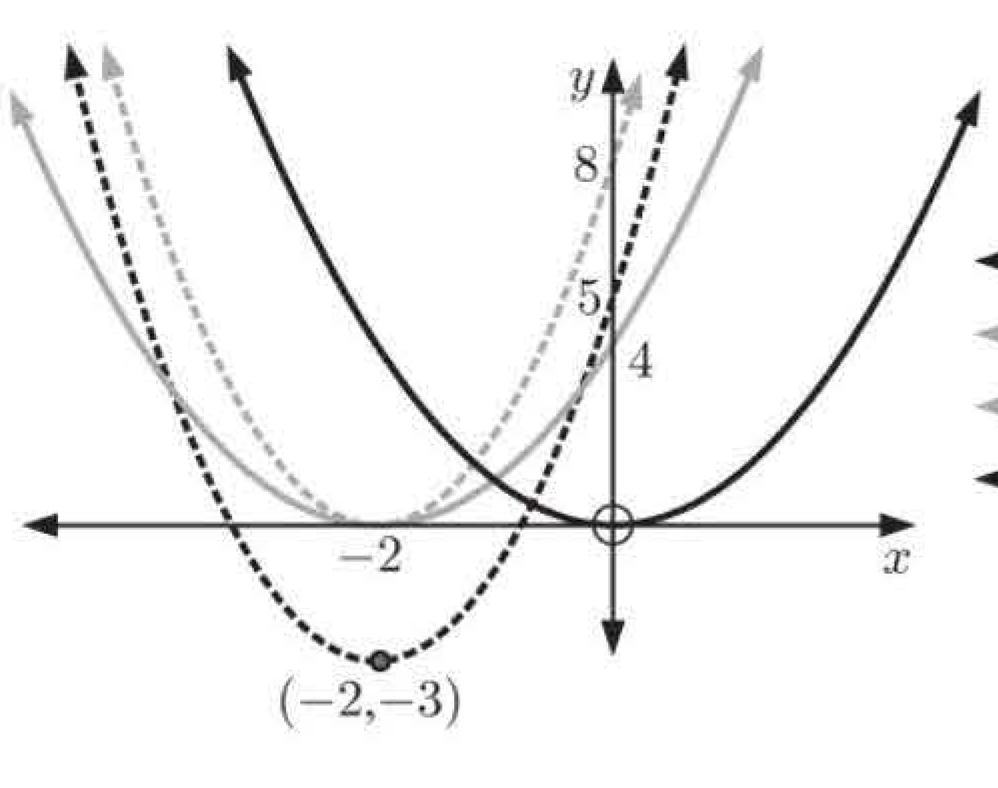
$y = (x+1)^2 - 4$ is obtained from $y = x^2$ under a translation of $\binom{-1}{-4}$.

$y = x^2$ has its vertex at $(0, 0)$, so the vertex of $y = (x+1)^2 - 4$ must be $(-1, -4)$.

So, the graph of $y = f(x)$ is:



2



3 **a** The function does not have any axes intercepts.

b As $x \to 0^-$, $y \to -\infty$

As $x \to 0^+$, $y \to \infty$

$\therefore$ the vertical asymptote is $x = 0$.

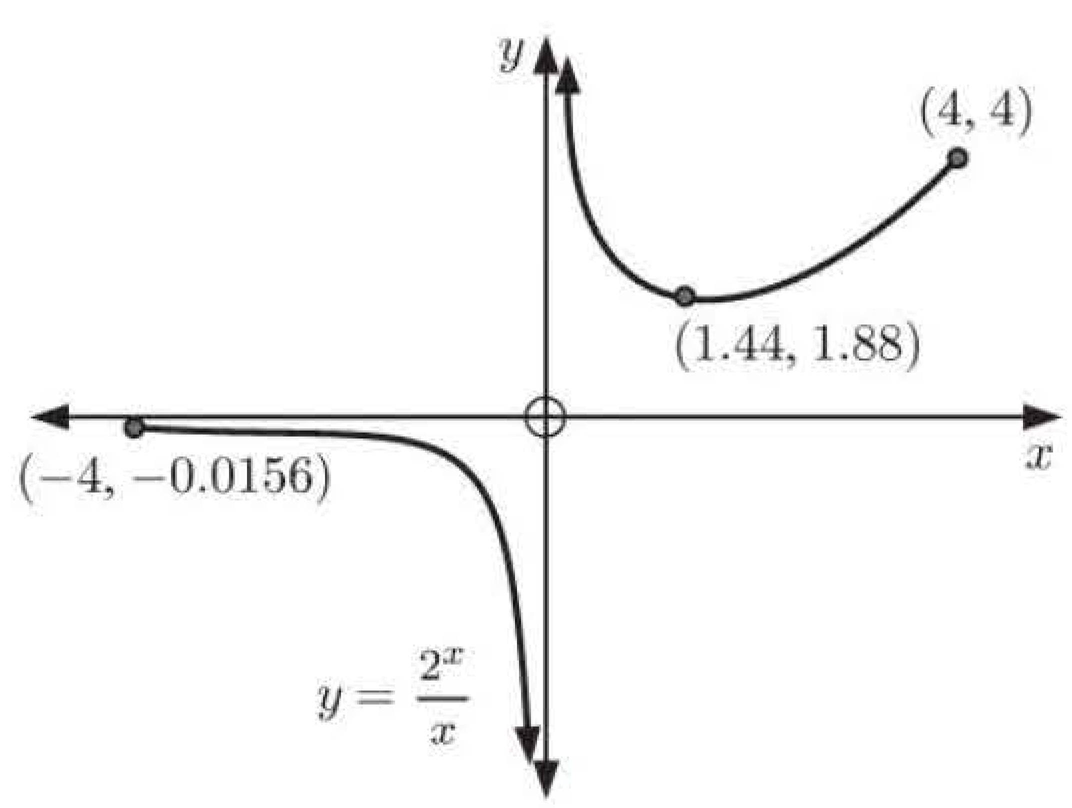
As $x \to \infty$, $y \to \infty$

As $x \to -\infty$, $y \to 0^-$

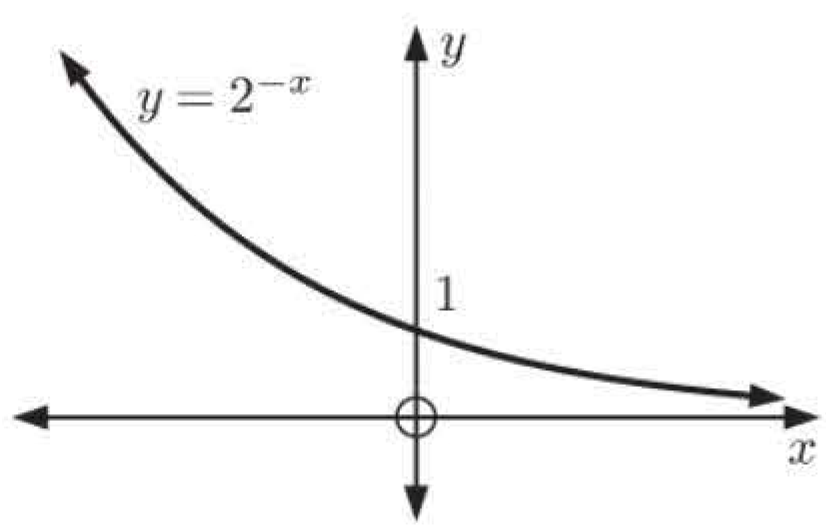
$\therefore$ the horizontal asymptote is $y = 0$.

c There is a local minimum at $(1.44, 1.88)$.

d



4 **a** Graph $y = f(x)$ using a graphics calculator:



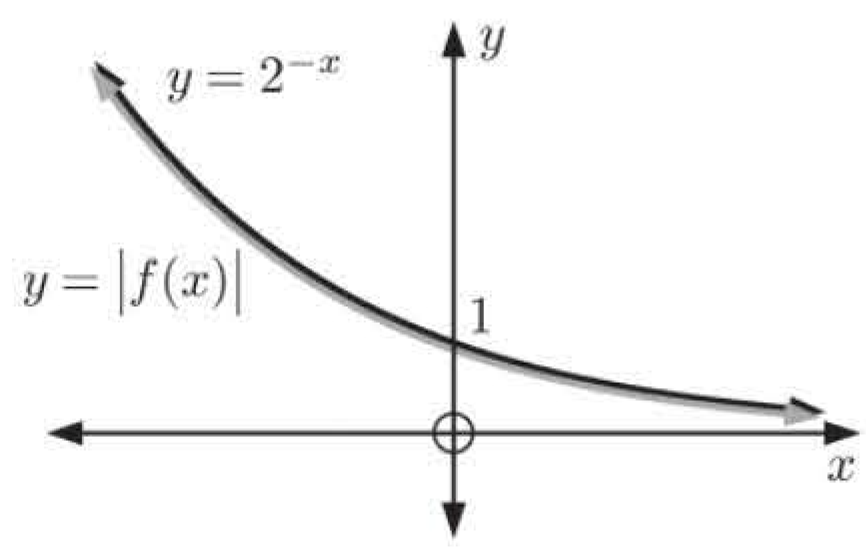
i $x \to \infty$ means x is very large and positive. We see the graph approaching the x-axis.
$\therefore$ $y \to 0$ $\therefore$ **true**.

ii $x \to -\infty$ means x is very large and negative. We see the graph heading for ∞. $\therefore$ **false**.

iii When $x = 0$, $y = 2^0 = 1 \neq \frac{1}{2}$ $\therefore$ **false**.

iv The graph is above the x-axis for all x.
$\therefore$ $2^{-x} > 0$ for all x $\therefore$ **true**.

b

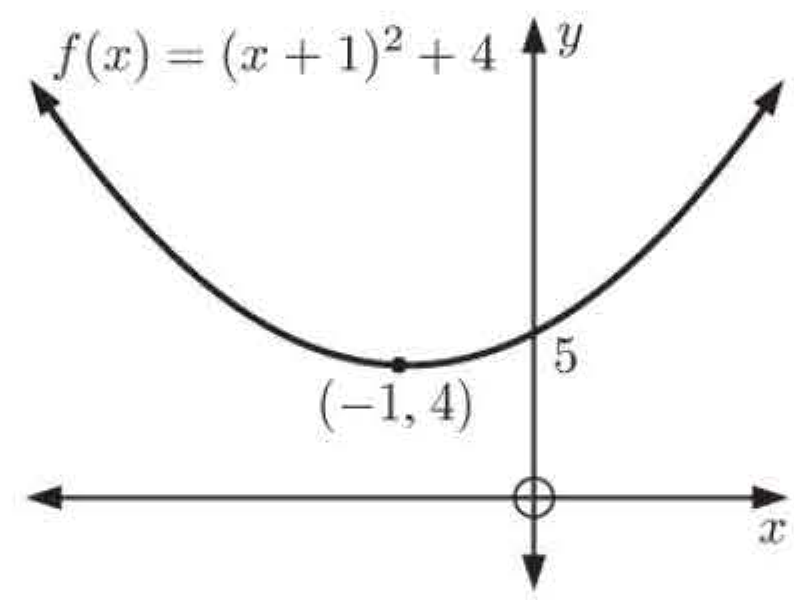


c $y = |f(x)|$ has horizontal asymptote $y = 0$.

5 **a** $f(x) = (x+1)^2 + 4$ is translated by $\binom{2}{4}$ to get $g(x)$.

$$\begin{aligned}\therefore\ g(x) &= f(x-2) + 4 \\ &= \left[((x-2)+1)^2 + 4\right] + 4 \\ &= (x-1)^2 + 8\end{aligned}$$

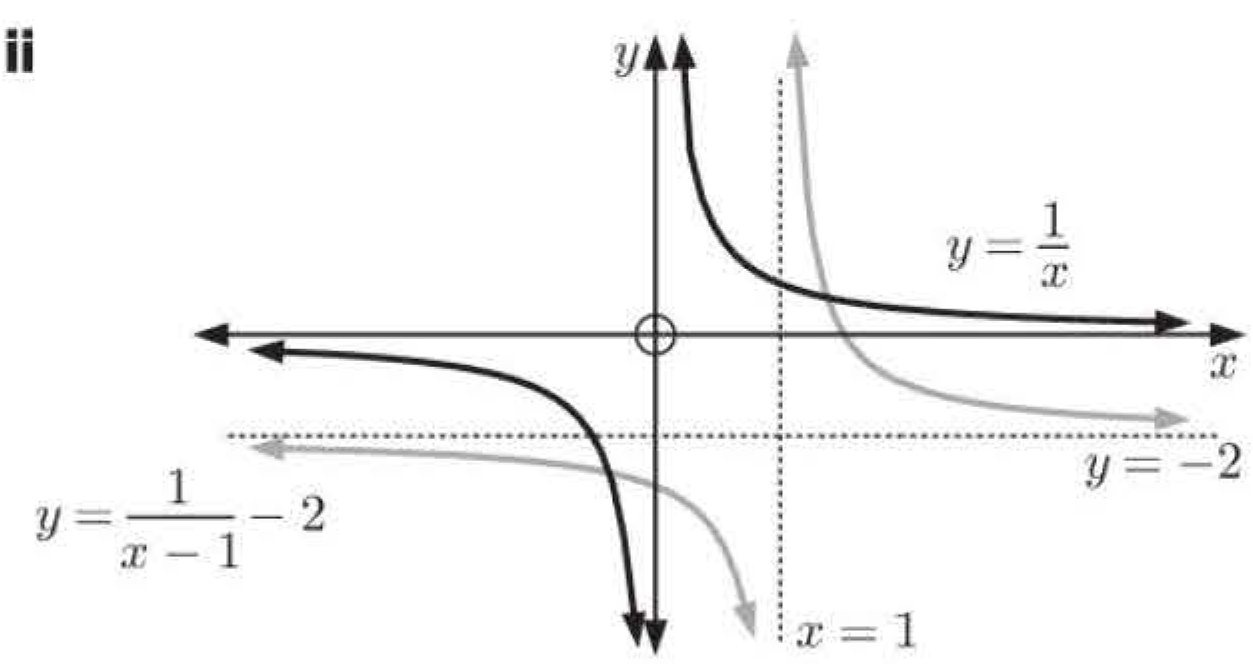
b We graph the function using technology, and from this we can see that the range is $\{y \mid y \geqslant 4\}$.



c $g(x)$ is $f(x)$ translated by $\binom{2}{4}$, so the minimum value of $g(x)$ is $4 + 4 = 8$.
$\therefore$ the range of $g(x)$ is $\{y \mid y \geqslant 8\}$.

6 **a** **i** Under translation $\binom{1}{-2}$, $y = \dfrac{1}{x}$ becomes $y = \dfrac{1}{x-1} - 2$

ii



For $y = \dfrac{1}{x}$, V.A. is $x = 0$, H.A. is $y = 0$.

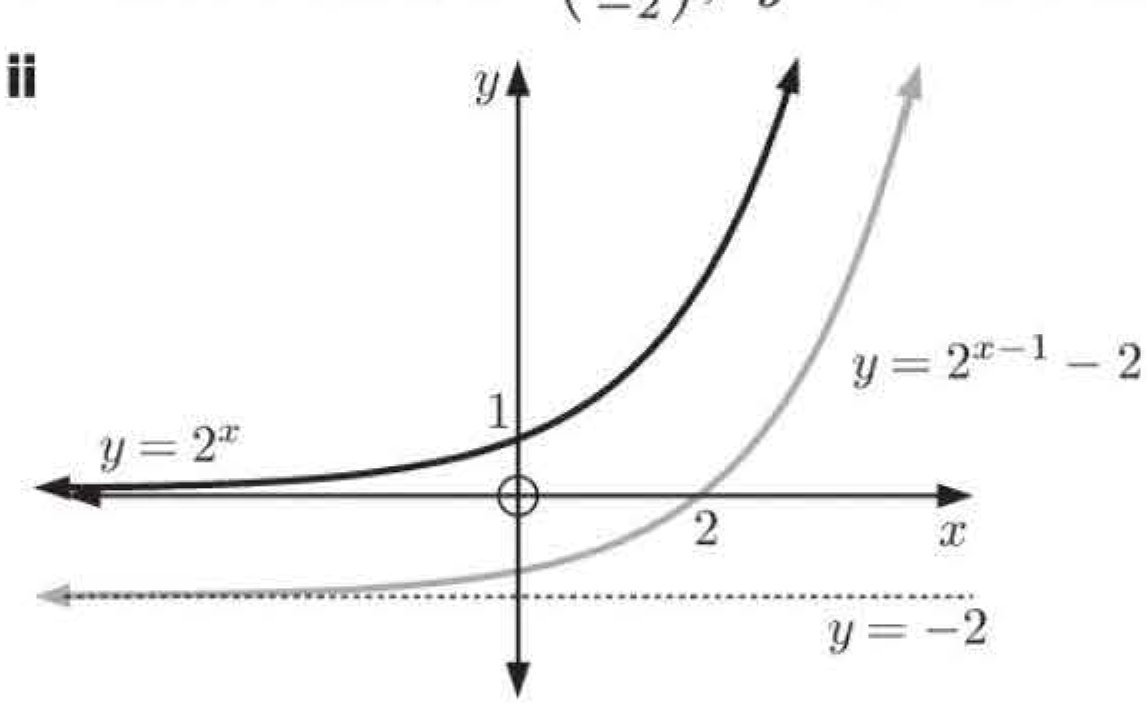
For $y = \dfrac{1}{x-1} - 2$, V.A. is $x = 1$, H.A. is $y = -2$.

iii For $y = \dfrac{1}{x}$, domain is $\{x \mid x \neq 0\}$, range is $\{y \mid y \neq 0\}$.

For $y = \dfrac{1}{x-1} - 2$, domain is $\{x \mid x \neq 1\}$, range is $\{y \mid y \neq -2\}$.

b **i** Under translation $\binom{1}{-2}$, $y = 2^x$ becomes $y = 2^{x-1} - 2$

ii



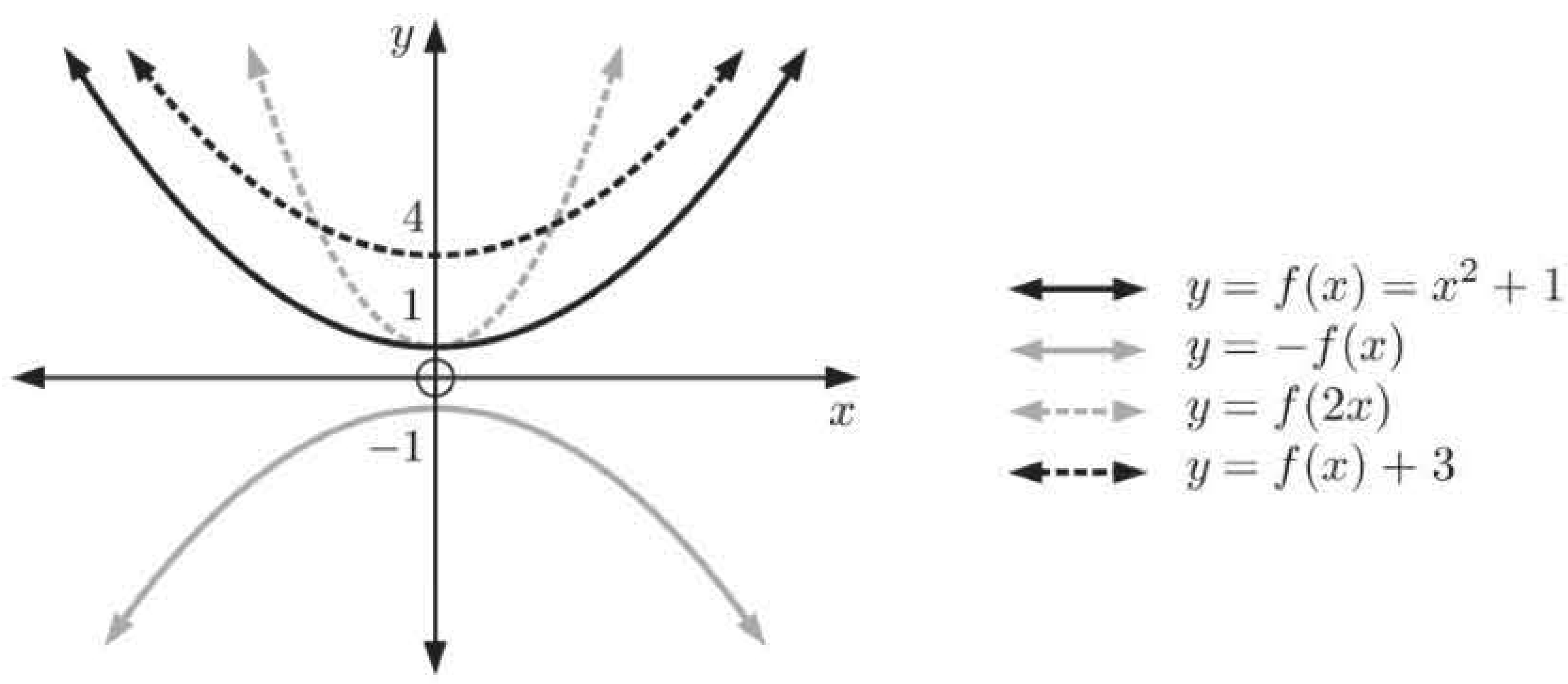
For $y = 2^x$, H.A. is $y = 0$, no V.A.

For $y = 2^{x-1} - 2$, H.A. is $y = -2$, no V.A.

iii For $y = 2^x$, domain is $\{x \mid x \in \mathbb{R}\}$, range is $\{y \mid y > 0\}$.

For $y = 2^{x-1} - 2$, domain is $\{x \mid x \in \mathbb{R}\}$, range is $\{y \mid y > -2\}$.

7



8 **a** $f(x) = x + 2$

stretching $f(x)$ vertically with scale factor 2 becomes $2\,f(x) = 2x + 4$

stretching the function horizontally with scale factor $\frac{1}{2}$ becomes $2\,f(2x) = 2(2x) + 4$
$$= 4x + 4$$

translating $\frac{1}{2}$ horizontally and -3 vertically, the function becomes $4(x - \frac{1}{2}) + 4 - 3$
$$= 4x - 2 + 1$$
$$\therefore \quad F(x) = 4x - 1$$

b $(1, 3) \to (1, 6) \to (\frac{1}{2}, 6) \to (1, 6) \to (1, 3)$

$\therefore$ $(1, 3)$ is an invariant point under the transformation.

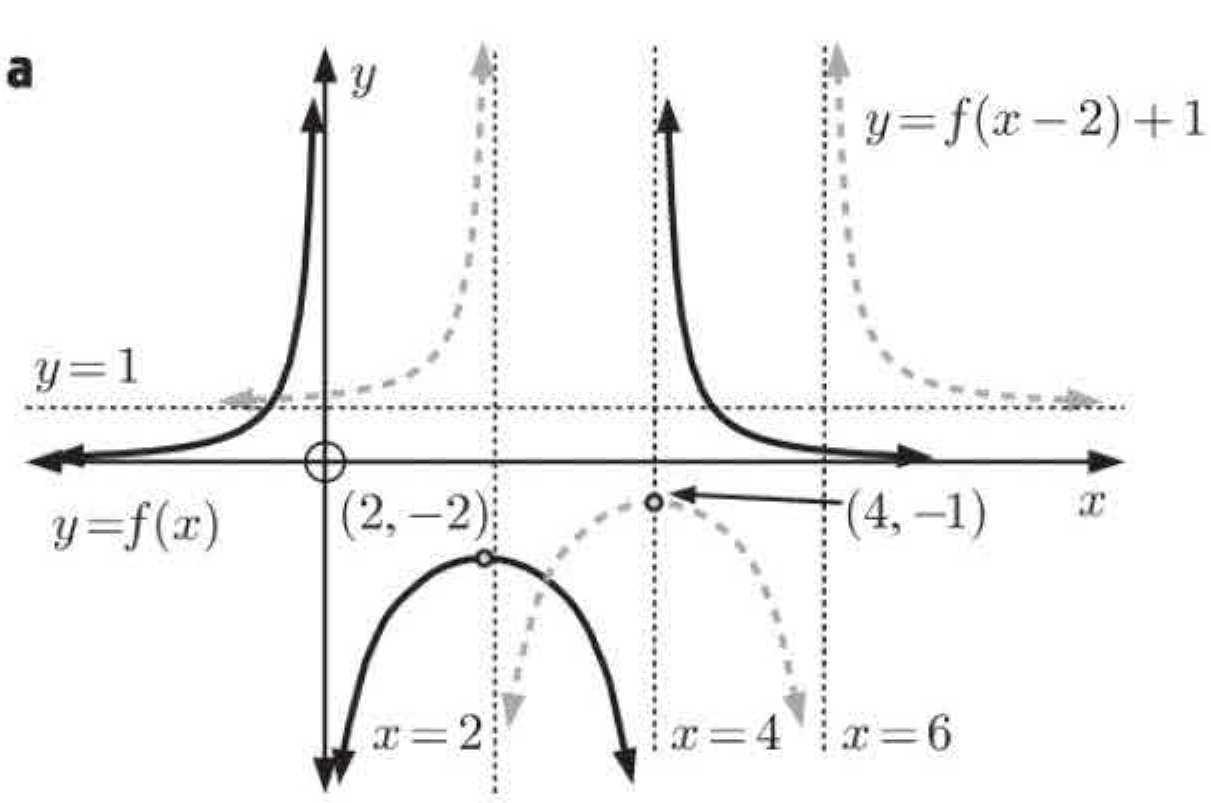
c $(0, 2) \to (0, 4) \to (0, 4) \to (\frac{1}{2}, 4) \to (\frac{1}{2}, 1)$ $\therefore$ $(0, 2)$ transforms to $(\frac{1}{2}, 1)$.

$(-1, 1) \to (-1, 2) \to (-\frac{1}{2}, 2) \to (0, 2) \to (0, -1)$ $\therefore$ $(-1, 1)$ transforms to $(0, -1)$.

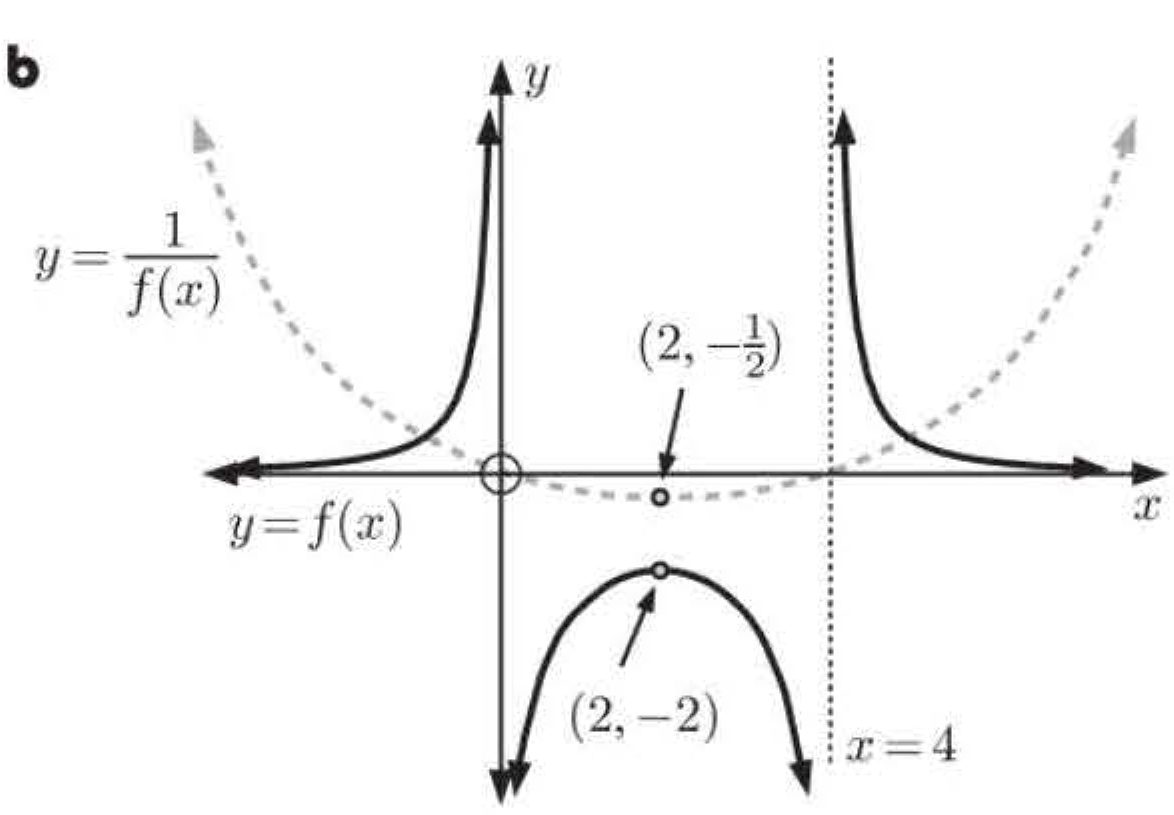
d When $x = \frac{1}{2}$, $F(x) = 4(\frac{1}{2}) - 1 = 1$ $\therefore$ $(\frac{1}{2}, 1)$ lies on $F(x)$.

When $x = 0$, $F(x) = 4(0) - 1 = -1$ $\therefore$ $(0, -1)$ lies on $F(x)$.

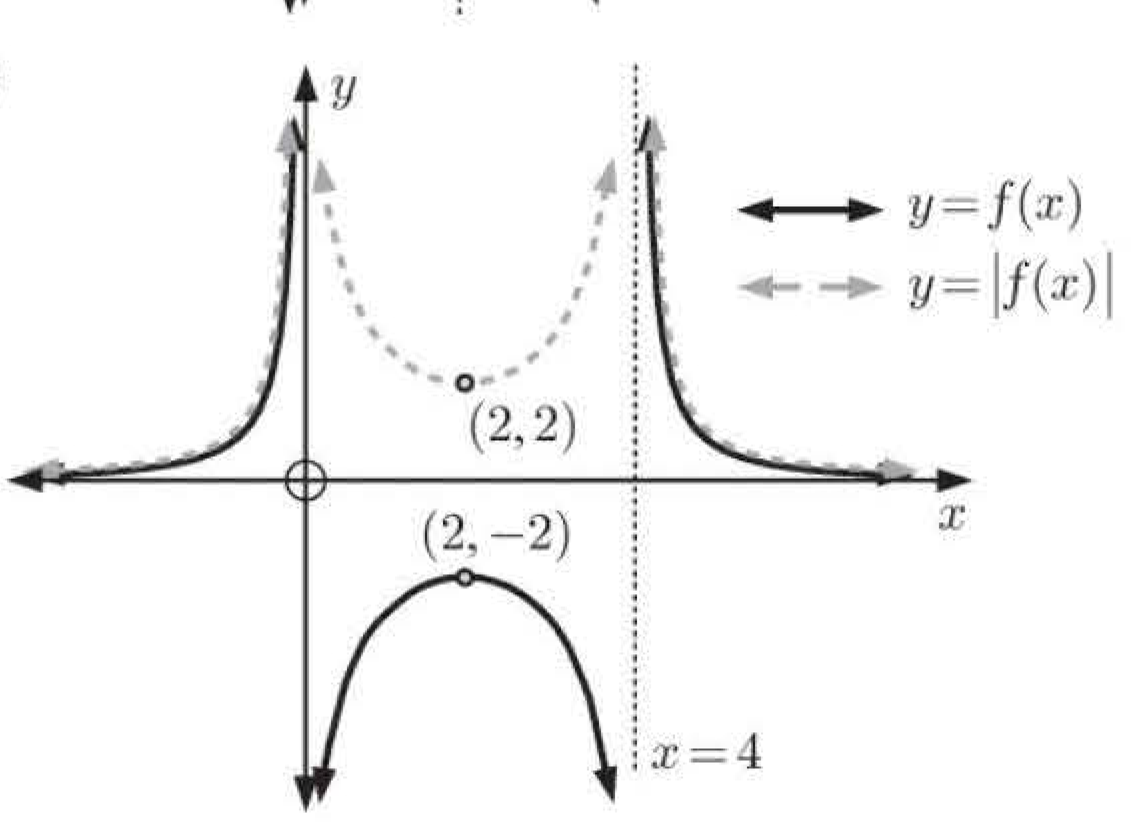
9 **a**



b



c



10 **a** $f(x)=\dfrac{2x-3}{3x+5}$
$=\dfrac{\frac{2}{3}x-1}{x+\frac{5}{3}}$
$=\dfrac{\frac{2}{3}(x+\frac{5}{3})-\frac{19}{9}}{x+\frac{5}{3}}$
$=-\dfrac{\frac{19}{9}}{x+\frac{5}{3}}+\frac{2}{3}$

$y=f(x)$ is a translation of $y=-\dfrac{\frac{19}{9}}{x}$ through $\begin{pmatrix}-\frac{5}{3}\\ \frac{2}{3}\end{pmatrix}$.

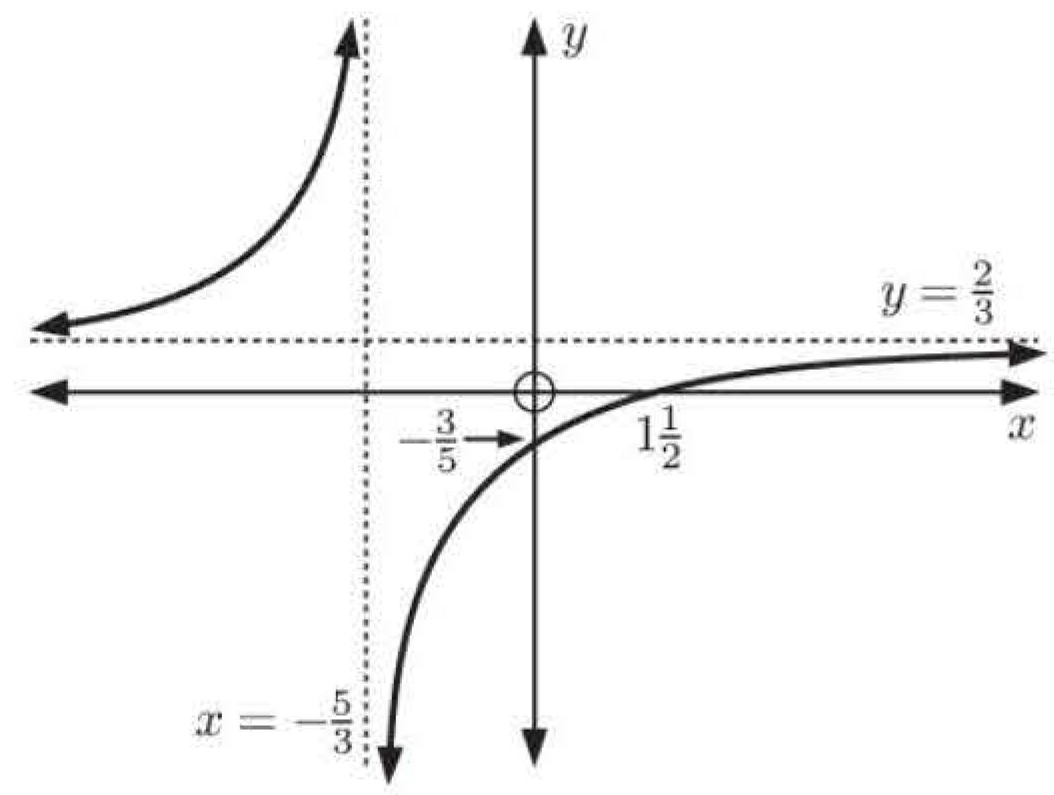
Now $y=-\dfrac{\frac{19}{9}}{x}$ has asymptotes $x=0$ and $y=0$.

$\therefore$ $y=f(x)$ has vertical asymptote $x=-\frac{5}{3}$ and horizontal asymptote $y=\frac{2}{3}$.

b As $x\to-\frac{5}{3}^-$, $y\to\infty$.
As $x\to-\frac{5}{3}^+$, $y\to-\infty$.
As $x\to-\infty$, $y\to\frac{2}{3}^+$.
As $x\to\infty$, $y\to\frac{2}{3}^-$.

c When $x=0$, $y=-\frac{3}{5}$.
$\therefore$ the y-intercept is $-\frac{3}{5}$.
When $y=0$, $2x-3=0$
$\therefore$ $x=\frac{3}{2}$
$\therefore$ the x-intercept is $\frac{3}{2}$.

d



e $\dfrac{1}{x}$ becomes $\dfrac{\frac{19}{9}}{x}$ under a vertical stretch with scale factor $\frac{19}{9}$.

$\dfrac{\frac{19}{9}}{x}$ becomes $-\dfrac{\frac{19}{9}}{x}$ under a reflection in the x-axis.

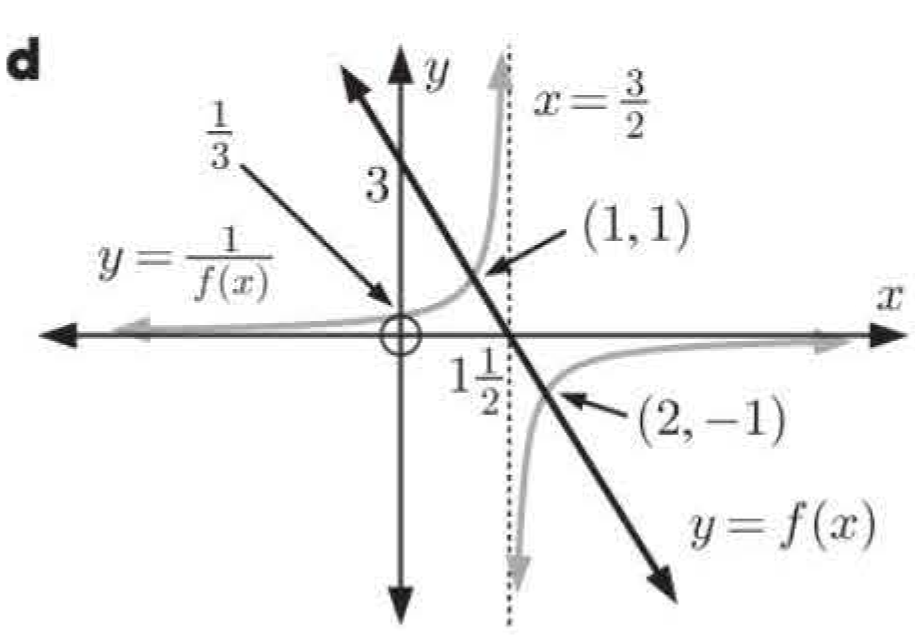
$-\dfrac{\frac{19}{9}}{x}$ becomes $-\dfrac{\frac{19}{9}}{x+\frac{5}{3}}+\frac{2}{3}$ under a translation through $\begin{pmatrix}-\frac{5}{3}\\ \frac{2}{3}\end{pmatrix}$.

So, $y=\dfrac{1}{x}$ is transformed to $y=f(x)$ under a vertical stretch with scale factor $\frac{19}{9}$, followed by a reflection in the x-axis, followed by a translation through $\begin{pmatrix}-\frac{5}{3}\\ \frac{2}{3}\end{pmatrix}$.

f To transform $y=f(x)$ into $y=\dfrac{1}{x}$, we need to reverse the process in **v**.

We need a translation through $\begin{pmatrix}\frac{5}{3}\\ -\frac{2}{3}\end{pmatrix}$, followed by a reflection in the x-axis, followed by a vertical stretch with scale factor $\frac{9}{19}$.

11 **a, d**



$f(0)=-2(0)+3=3$ $\therefore$ the y-intercept of $f(x)$ is 3
$f(x)=0$ when $-2x+3=0$
$\therefore$ $x=\frac{3}{2}$
$\therefore$ the x-intercept of $f(x)$ is $\frac{3}{2}$.

b The invariant points for $y=\dfrac{1}{f(x)}$ occur when $f(x)=\pm1$

$f(x)=1$ when $-2x+3=1$
$\therefore$ $2x=2$
$\therefore$ $x=1$

$f(x)=-1$ when $-2x+3=-1$
$\therefore$ $2x=4$
$\therefore$ $x=2$

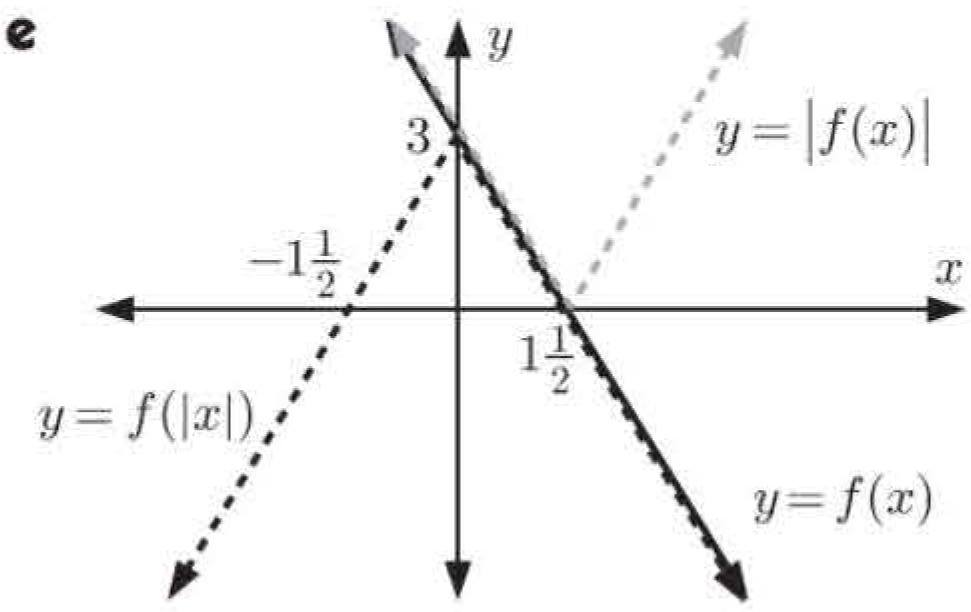
$\therefore$ the invariant points are $(1,\ 1)$ and $(2,\ -1)$.

c $\dfrac{1}{f(x)}$ is undefined when $f(x) = 0$

$\therefore$ the vertical asymptote of $y = \dfrac{1}{f(x)}$ is $x = \frac{3}{2}$

The y-intercept of $y = \dfrac{1}{f(x)}$ is $\dfrac{1}{f(0)} = \frac{1}{3}$

e



REVIEW SET 5C

1 $f(x) = \dfrac{4}{x}$

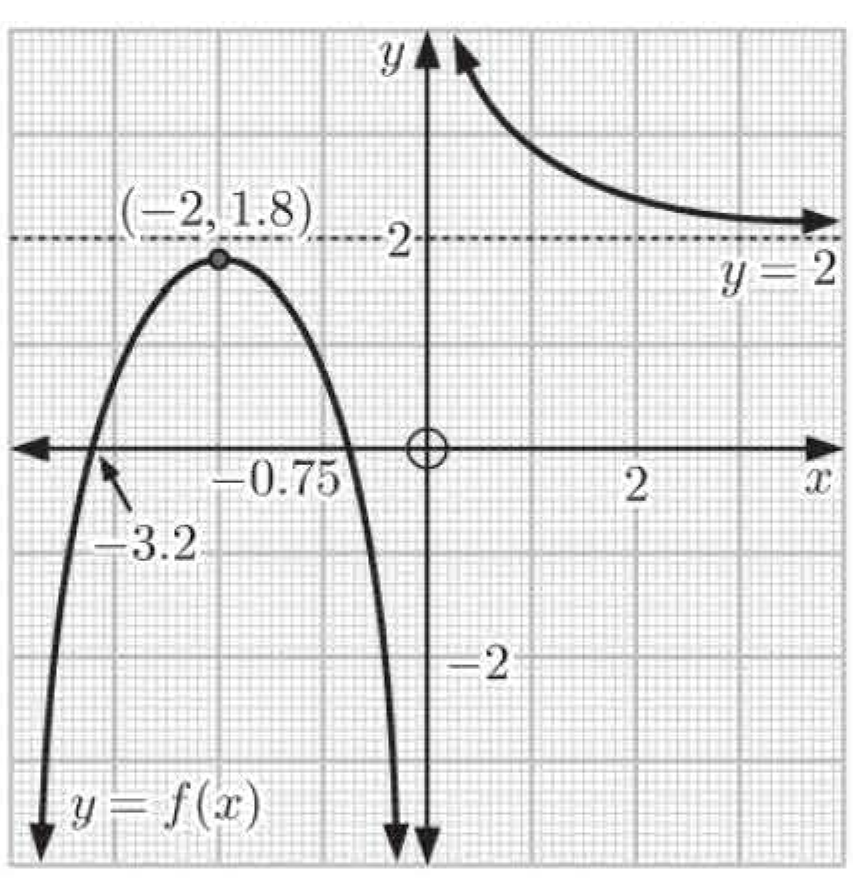
a $f(-4)$
$= \dfrac{4}{-4}$
$= -1$

b $f(2x)$
$= \dfrac{4}{2x}$
$= \dfrac{2}{x}$

c $f\left(\frac{x}{2}\right)$
$= \dfrac{4}{\frac{x}{2}}$
$= 4 \times \dfrac{2}{x}$
$= \dfrac{8}{x}$

d $4\,f(x+2) - 3$
$= 4\left(\dfrac{4}{x+2}\right) - 3$
$= \dfrac{16}{x+2} - 3$
$= \dfrac{16 - 3(x+2)}{x+2} = \dfrac{10 - 3x}{x+2}$

2 **a**



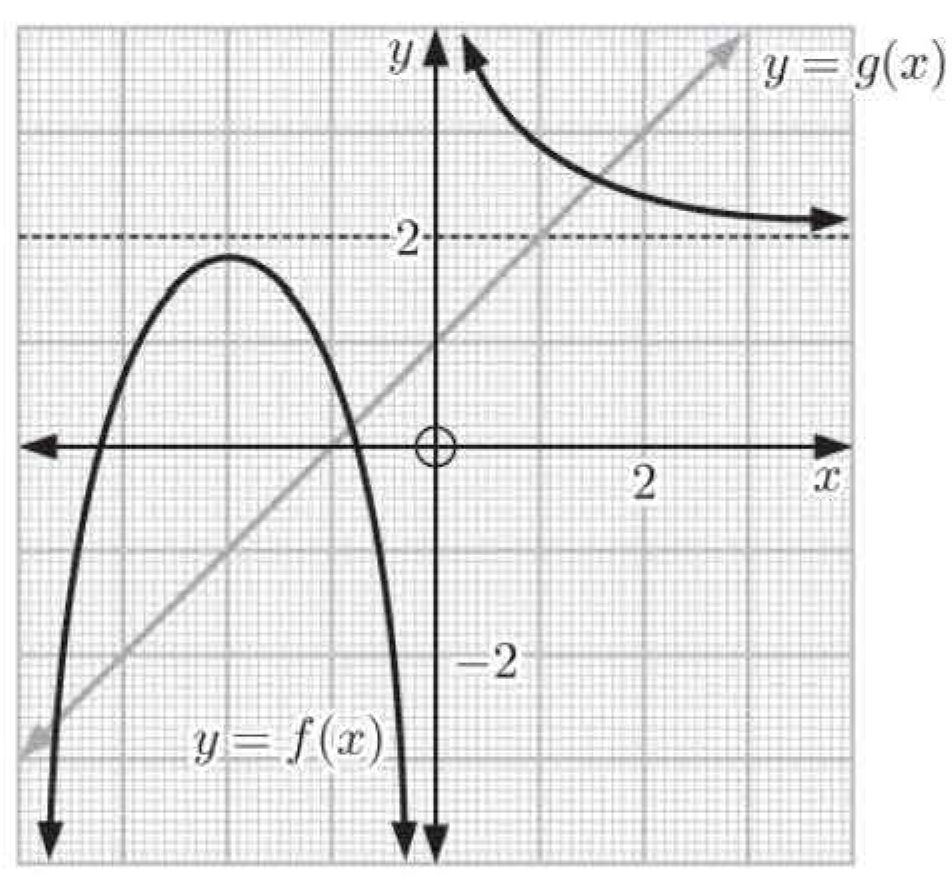
i The coordinates of the turning point are $(-2, 1.8)$.

ii The equation of the vertical asymptote is $x = 0$.

iii The equation of the horizontal asymptote is $y = 2$.

iv The x-intercepts are -3.2 and -0.75.

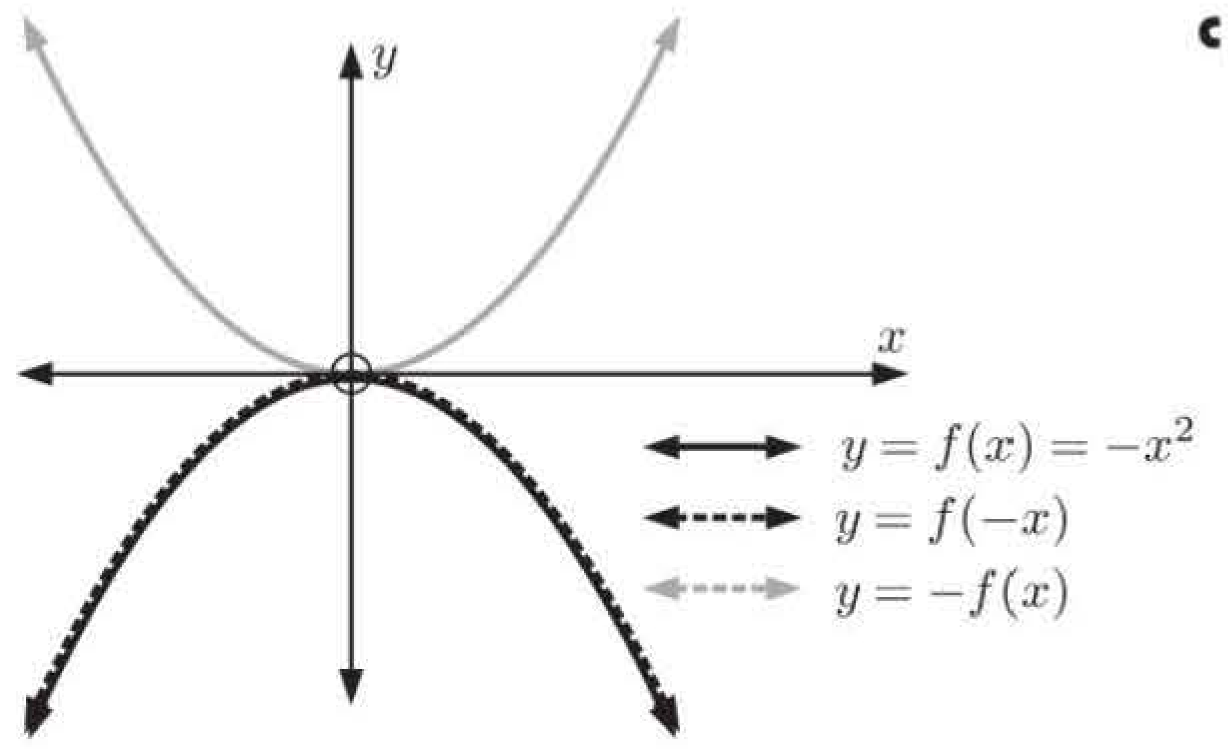
b



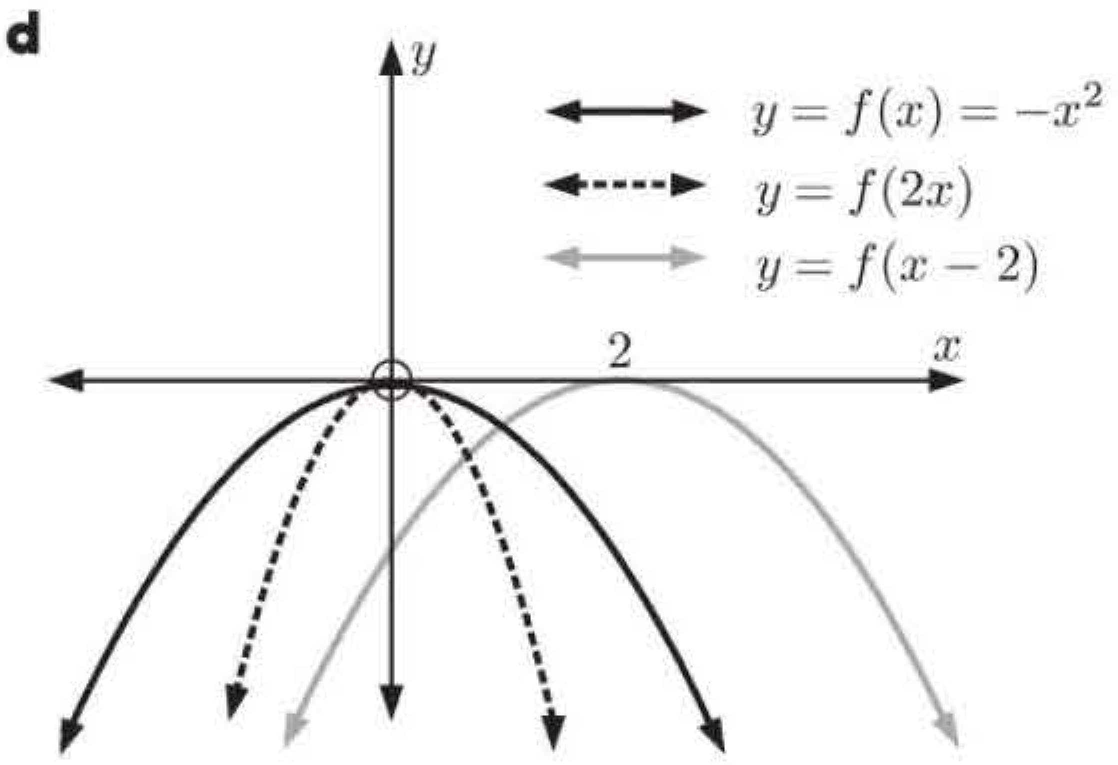
The coordinates of the points of intersection are $(-3.65, -2.65)$, $(-0.8, 0.2)$, and $(1.55, 2.55)$.

3 So that you can see the answers more easily, they have been drawn on two graphs.

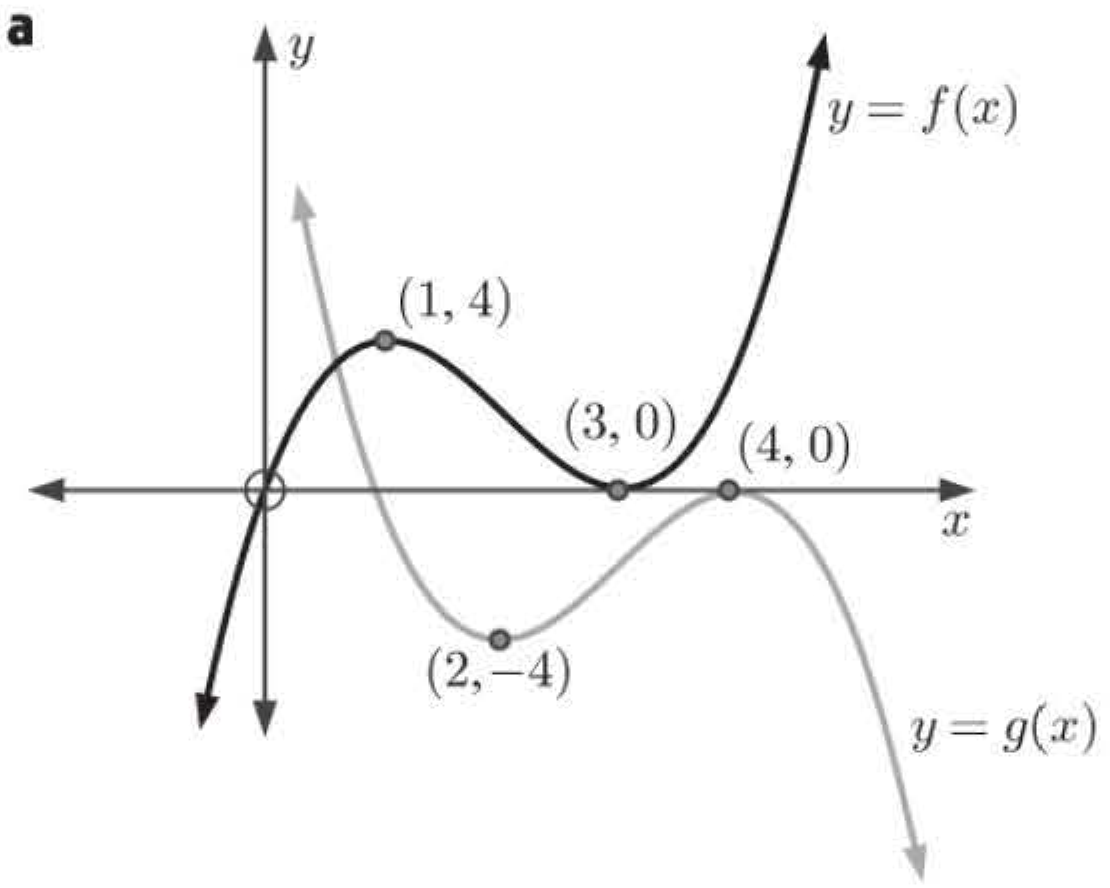
a, b



c, d



4 **a**



b $g(x)$ is obtained from $f(x)$ by a translation of $\binom{1}{0}$ and then a reflection in the x-axis. So, to get the turning point coordinates we add 1 to the x-coordinate and find the negative of the y-coordinate.

$(1, 4) \mapsto (2, -4)$ and $(3, 0) \mapsto (4, 0)$.

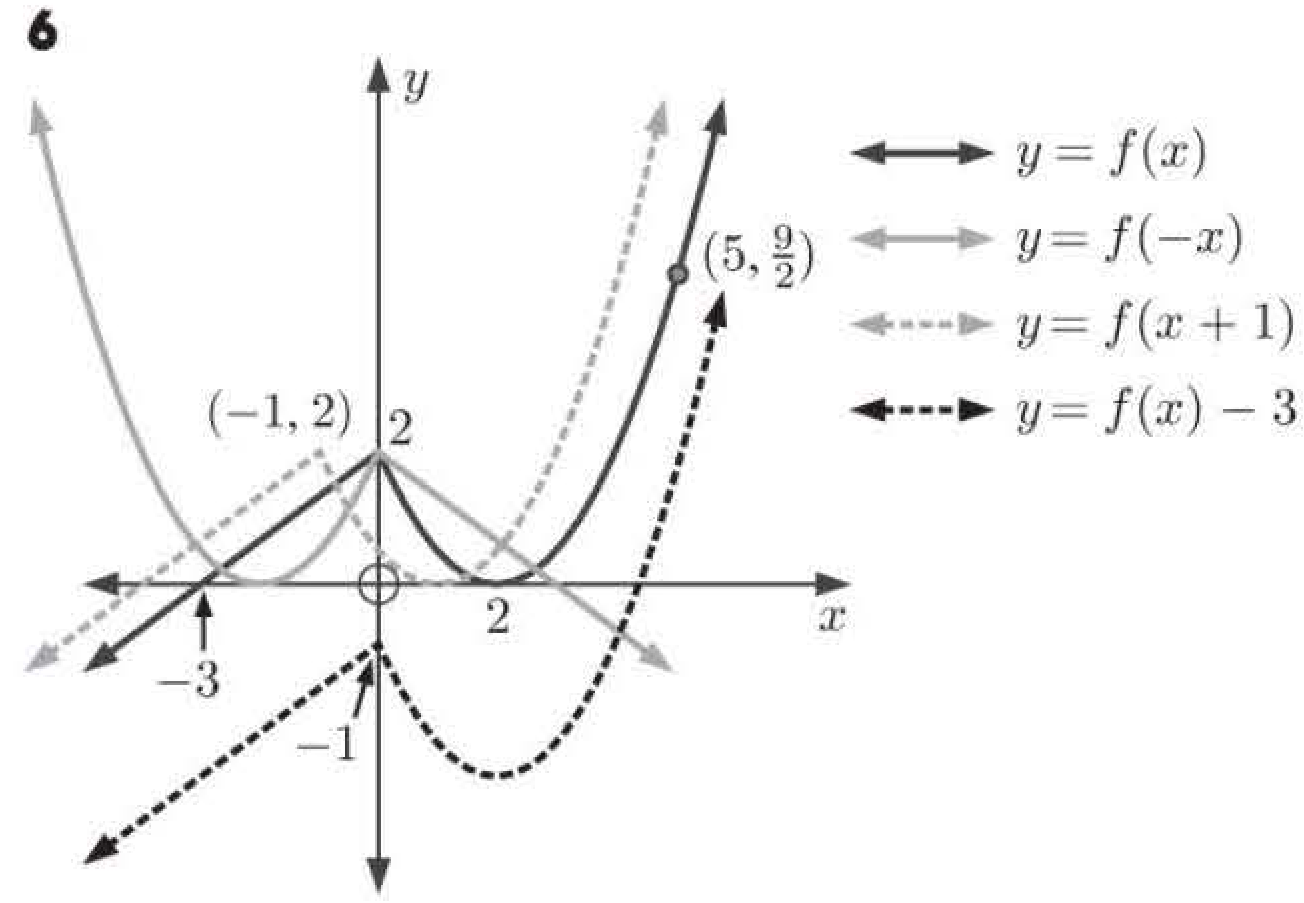
So, the turning points of $g(x)$ are $(2, -4)$ and $(4, 0)$.

5 $f(x) = x^2$ is first reflected in the x-axis to become $-f(x) = -x^2$

The function is then translated by $\binom{-3}{2}$ to become

$$\begin{aligned}-f(x+3)+2 &= -(x+3)^2+2\\ &= -(x^2+6x+9)+2\\ \therefore \quad g(x) &= -x^2-6x-7\end{aligned}$$

6



7 $f(x) = x^3 + 3x^2 - x + 4$

$$\begin{aligned}g(x) &= f(x+1)+3\\ &= \left[(x+1)^3+3(x+1)^2-(x+1)+4\right]+3\\ &= x^3+3x^2+3x+1+3(x^2+2x+1)-x-1+4+3\\ &= x^3+3x^2+3x+1+3x^2+6x+3-x-1+4+3\\ &= x^3+6x^2+8x+10\end{aligned}$$

8 **a** $f(x) = 3x + 2$

i A translation of 2 units to the left gives

$$\begin{aligned}y &= f(x+2)\\ &= 3(x+2)+2\\ &= 3x+8\end{aligned}$$

ii A translation of 6 units upwards gives

$$\begin{aligned}y &= f(x)+6\\ &= 3x+2+6\\ &= 3x+8\end{aligned}$$

b $f(x) = ax + b$ translated k units to the left gives

$$\begin{aligned}y &= f(x+k)\\ &= a(x+k)+b\\ &= ax+ak+b\\ &= (ax+b)+ka\\ &= f(x)+ka\end{aligned}$$

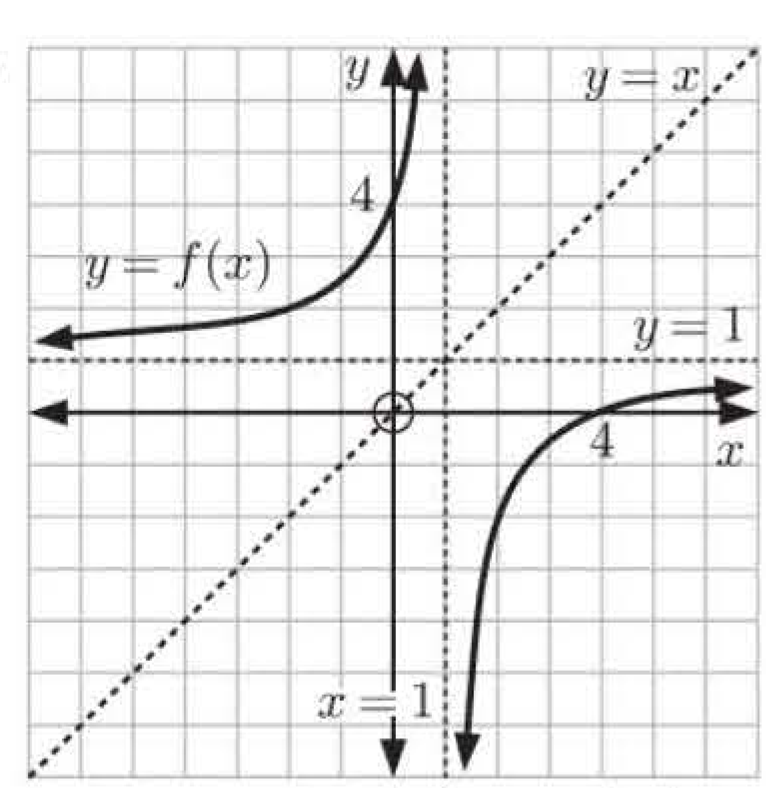
which is $f(x)$ translated ka units upwards.

9 **a** $y = \dfrac{1}{x}$ under a reflection in the y-axis becomes $y = \dfrac{1}{(-x)} = -\dfrac{1}{x}$

$y = -\dfrac{1}{x}$ under a vertical stretch with scale factor 3 becomes $y = 3\left(-\dfrac{1}{x}\right) = -\dfrac{3}{x}$

$y = -\dfrac{3}{x}$ under a translation of $\binom{1}{1}$ becomes $y = \dfrac{-3}{x-1} + 1$

b



domain of $f(x)$ is $\{x \mid x \neq 1\}$

range of $f(x)$ is $\{y \mid y \neq 1\}$

c Yes, since it is a one-to-one function (passes both the vertical and horizontal line tests).

d
$$f(x) = y = \frac{-3}{x-1} + 1$$
$$\therefore \text{ inverse function is } \quad x = \frac{-3}{y-1} + 1$$
$$\therefore \quad x - 1 = \frac{-3}{y-1}$$
$$\therefore \quad y - 1 = \frac{-3}{x-1}$$
$$\therefore \quad y = \frac{-3}{x-1} + 1$$
$$\therefore \quad f^{-1}(x) = f(x) = \frac{-3}{x-1} + 1$$

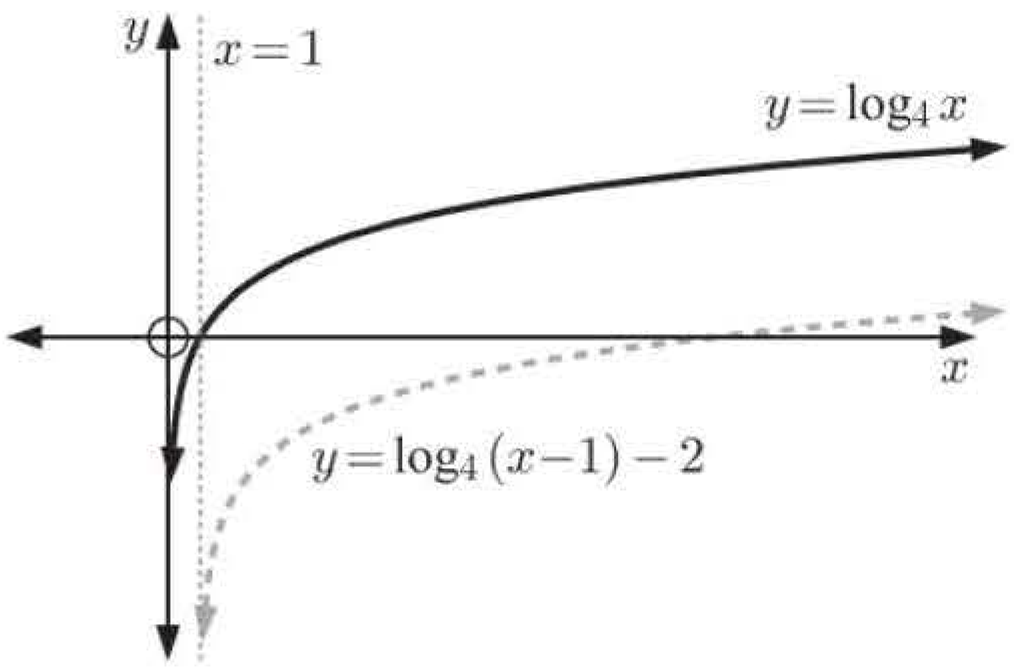
$\therefore$ it is a self-inverse function.

Also, the graph of $f(x)$ is symmetrical about the line $y = x$.

10 **a** Under translation $\binom{1}{-2}$,

$$y = \log_4 x$$
becomes $\quad y = \log_4(x-1) - 2$

b



c For $y = \log_4 x$, VA is $x = 0$, no HA.
For $y = \log_4(x-1) - 2$, VA is $x = 1$, no HA.

d For $y = \log_4 x$,
domain is $\{x \mid x > 0\}$,
range is $\{y \mid y \in \mathbb{R}\}$.

For $y = \log_4(x-1) - 2$,
domain is $\{x \mid x > 1\}$,
range is $\{y \mid y \in \mathbb{R}\}$.

11 **a** Under a vertical stretch with scale factor $\frac{1}{3}$, $f(x)$ becomes $\frac{1}{3}f(x)$.

$$\therefore \quad \frac{1}{x} \text{ becomes } \frac{1}{3}\left(\frac{1}{x}\right) = \frac{1}{3x}$$

Under a reflection in the y-axis, $f(x)$ becomes $f(-x)$.

$$\therefore \quad \frac{1}{3x} \text{ becomes } \frac{1}{3(-x)} = \frac{-1}{3x}$$

Under a translation of 2 units to the right, $f(x)$ becomes $f(x-2)$.

$$\therefore \quad \frac{-1}{3x} \text{ becomes } \frac{-1}{3(x-2)} = \frac{-1}{3x-6}$$

So, $y = \dfrac{1}{x}$ becomes $g(x) = \dfrac{-1}{3x-6}$.

b The asymptotes of $y = \dfrac{1}{x}$ are $x = 0$ and $y = 0$.
These are unchanged by the stretch and the reflection, and shifted 2 units to the right by the translation.
$\therefore$ the vertical asymptote is $x = 2$ and the horizontal asymptote is $y = 0$.

c Domain is $\{x \mid x \neq 2\}$,
range is $\{y \mid y \neq 0\}$.

d