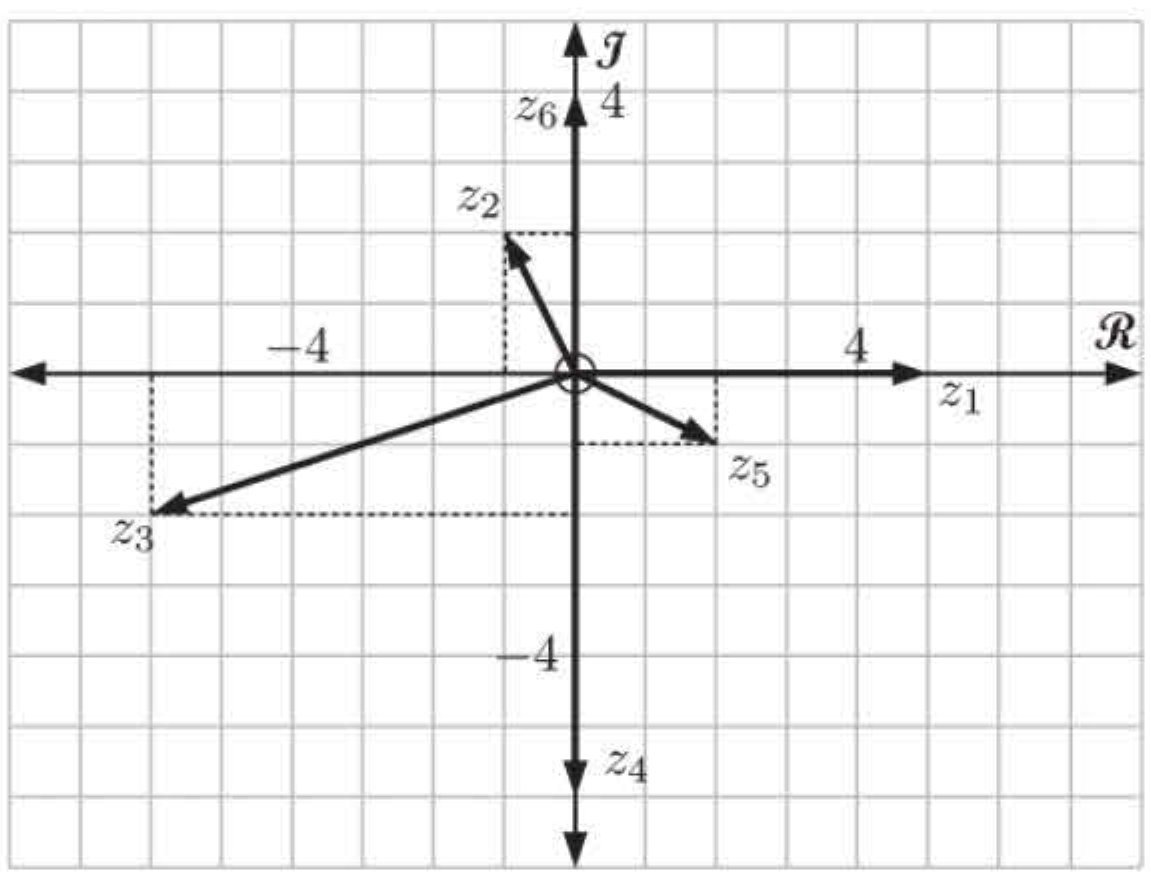
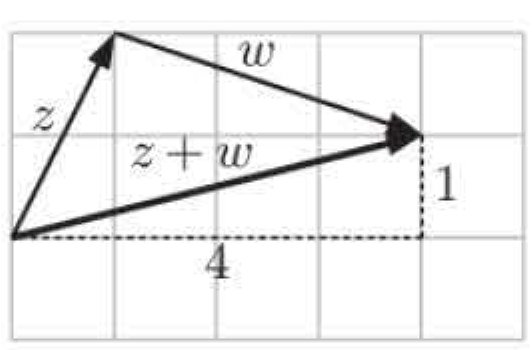
Chapter 16
COMPLEX NUMBERS

EXERCISE 16A

1

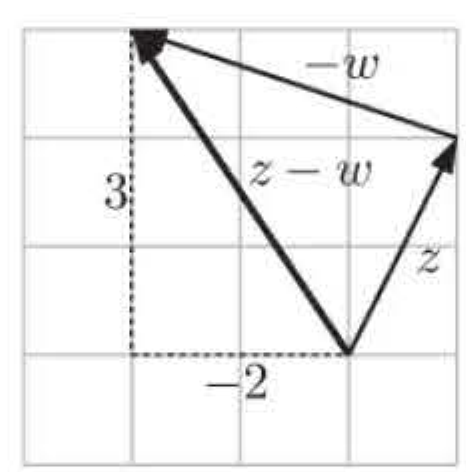


2 **a**



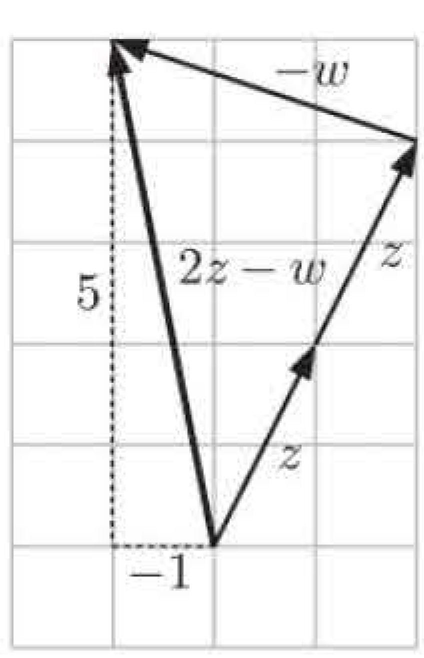
$z + w$
$= (1 + 2i) + (3 - i)$
$= 4 + i$

b



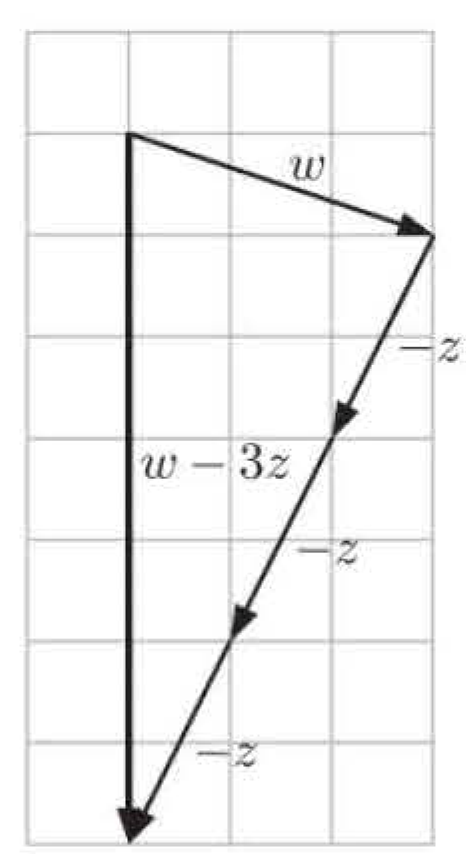
$z - w$
$= (1 + 2i) - (3 - i)$
$= 1 + 2i - 3 + i$
$= -2 + 3i$

c



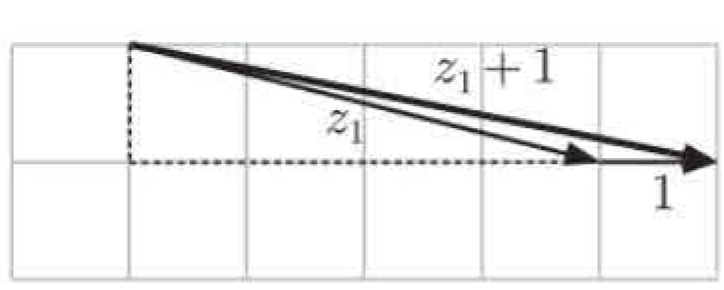
$2z - w$
$= 2(1 + 2i) - (3 - i)$
$= 2 + 4i - 3 + i$
$= -1 + 5i$

d



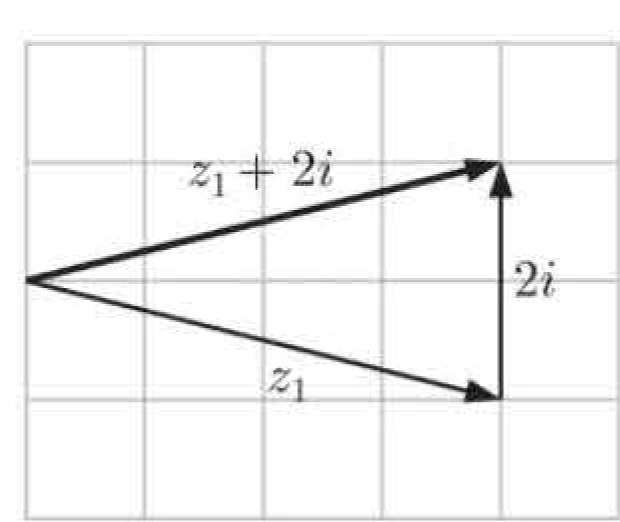
$w - 3z$
$= (3 - i) - 3(1 + 2i)$
$= 3 - i - 3 - 6i$
$= -7i$

3 **a**



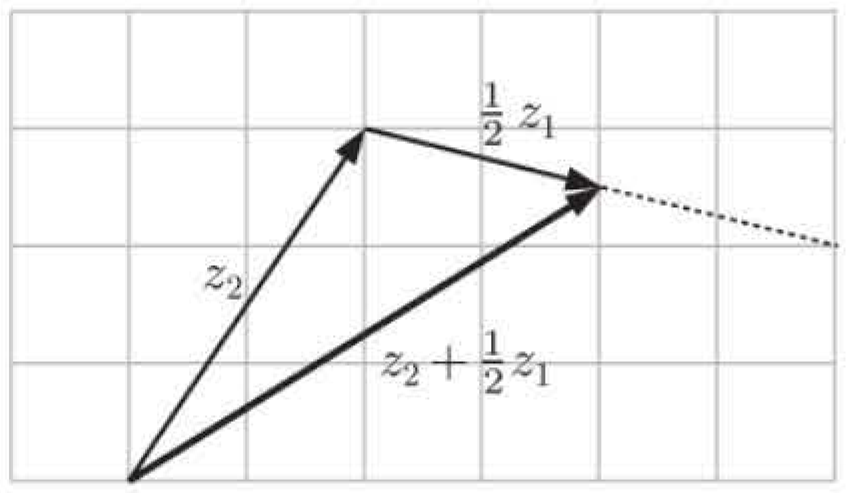
$z_1 + 1$
$= 4 - i + 1$
$= 5 - i$

b



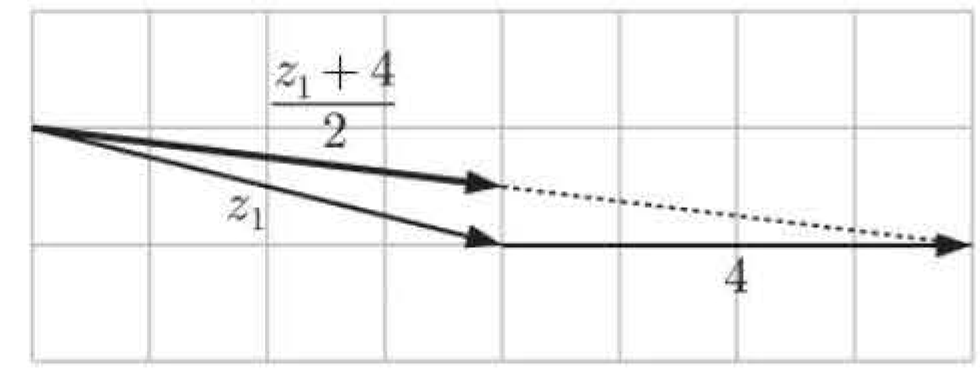
$z_1 + 2i$
$= 4 - i + 2i$
$= 4 + i$

c



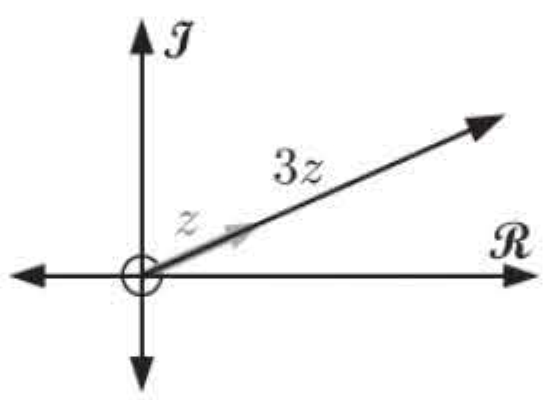
$$\begin{aligned} z_2 + \tfrac{1}{2}z_1 &= (2+3i) + \tfrac{1}{2}(4-i) \\ &= 2 + 3i + 2 - \tfrac{1}{2}i \\ &= 4 + \tfrac{5}{2}i \end{aligned}$$

d



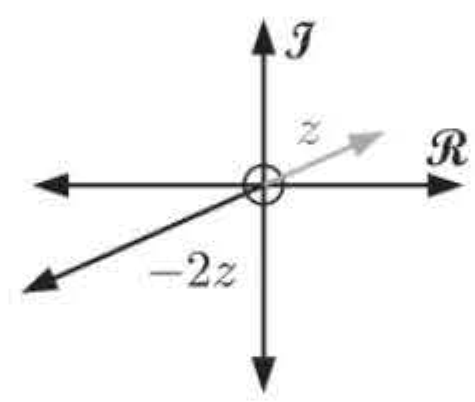
$$\begin{aligned} \frac{z_1+4}{2} &= \frac{4-i+4}{2} \\ &= \frac{8-i}{2} \\ &= 4 - \tfrac{1}{2}i \end{aligned}$$

4 **a**



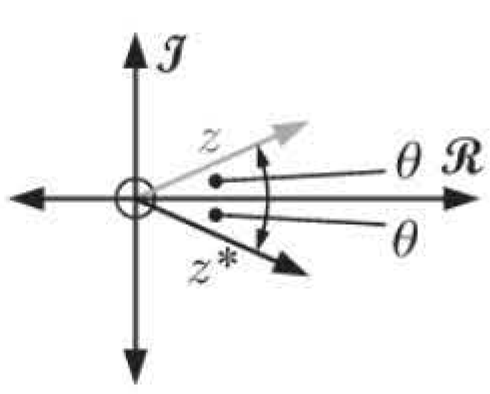
$3z$ is parallel to z and 3 times its length.

b



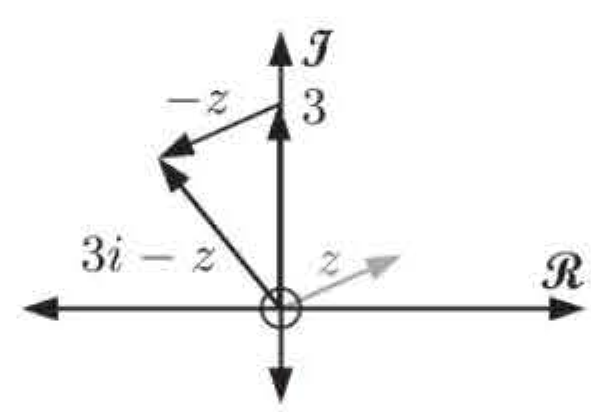
$-2z$ is parallel to z, in the opposite direction and twice its length.

c



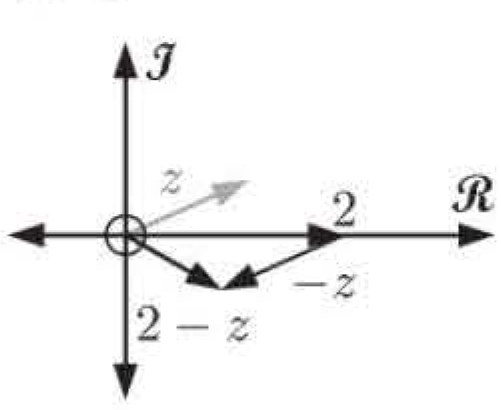
Reflect z in a horizontal line through the start of z.

d



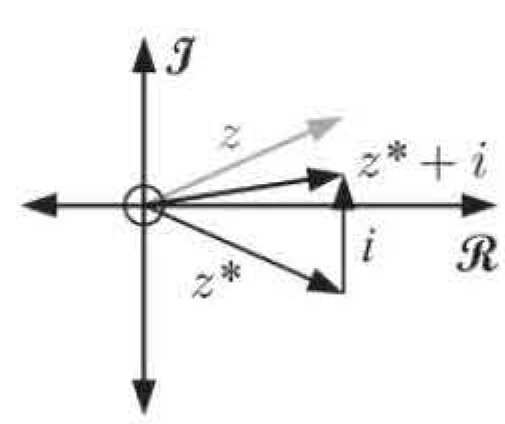
Add $-z$ to $3i$.

e



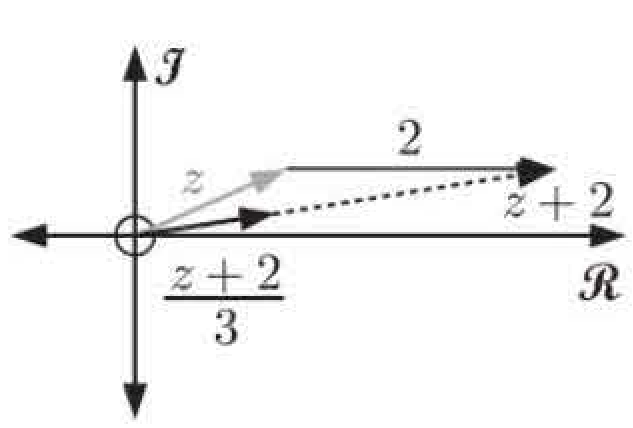
Add $-z$ to 2.

f



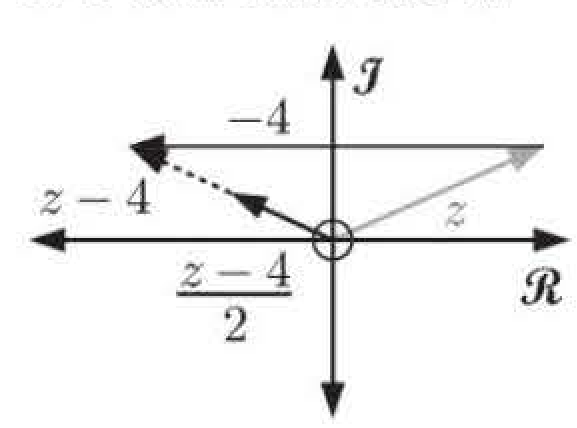
Reflect z in a horizontal line through the start of z and then add i.

g



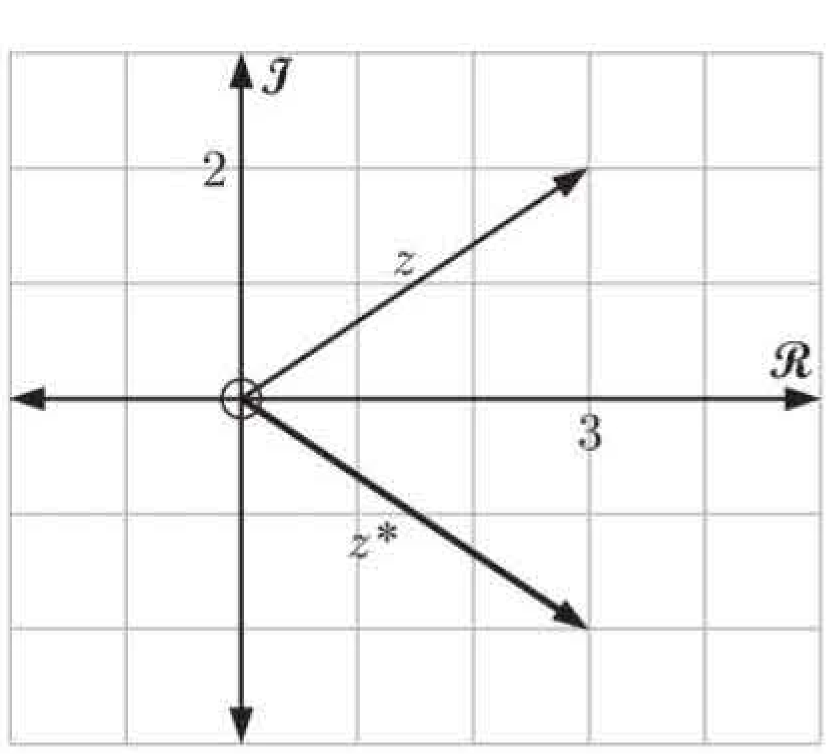
Add z and 2 and find the vector $\frac{1}{3}$ of the length of the result.

h

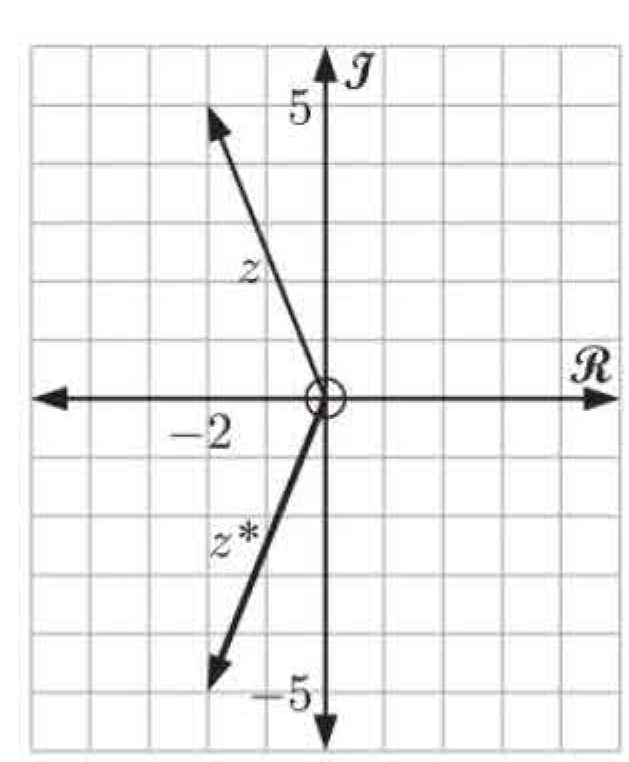


Add z and -4 and find the vector $\frac{1}{2}$ of the length of the result.

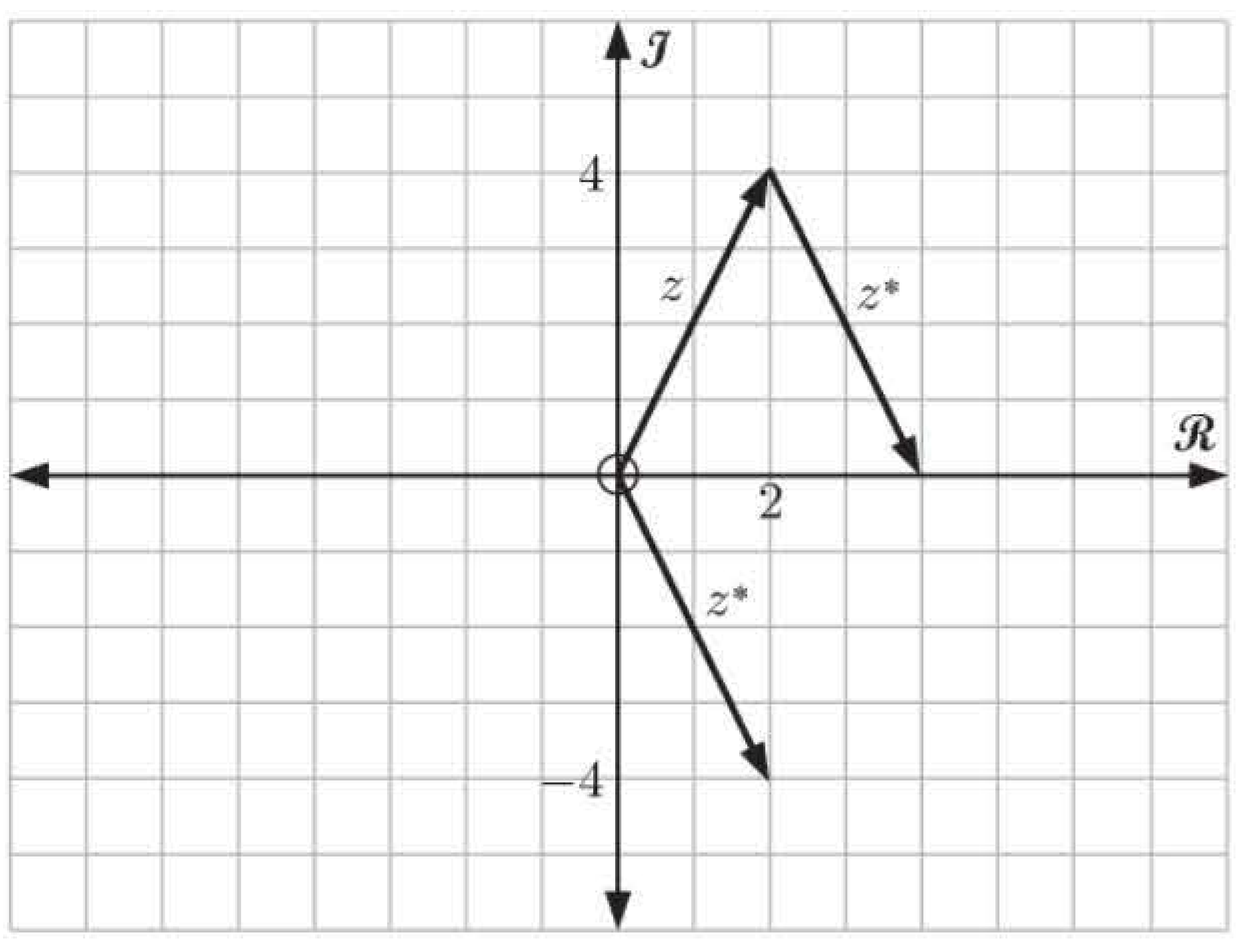
5 **a**



b



6



Suppose $z = a + bi$, where $a, b \in \mathbb{R}$

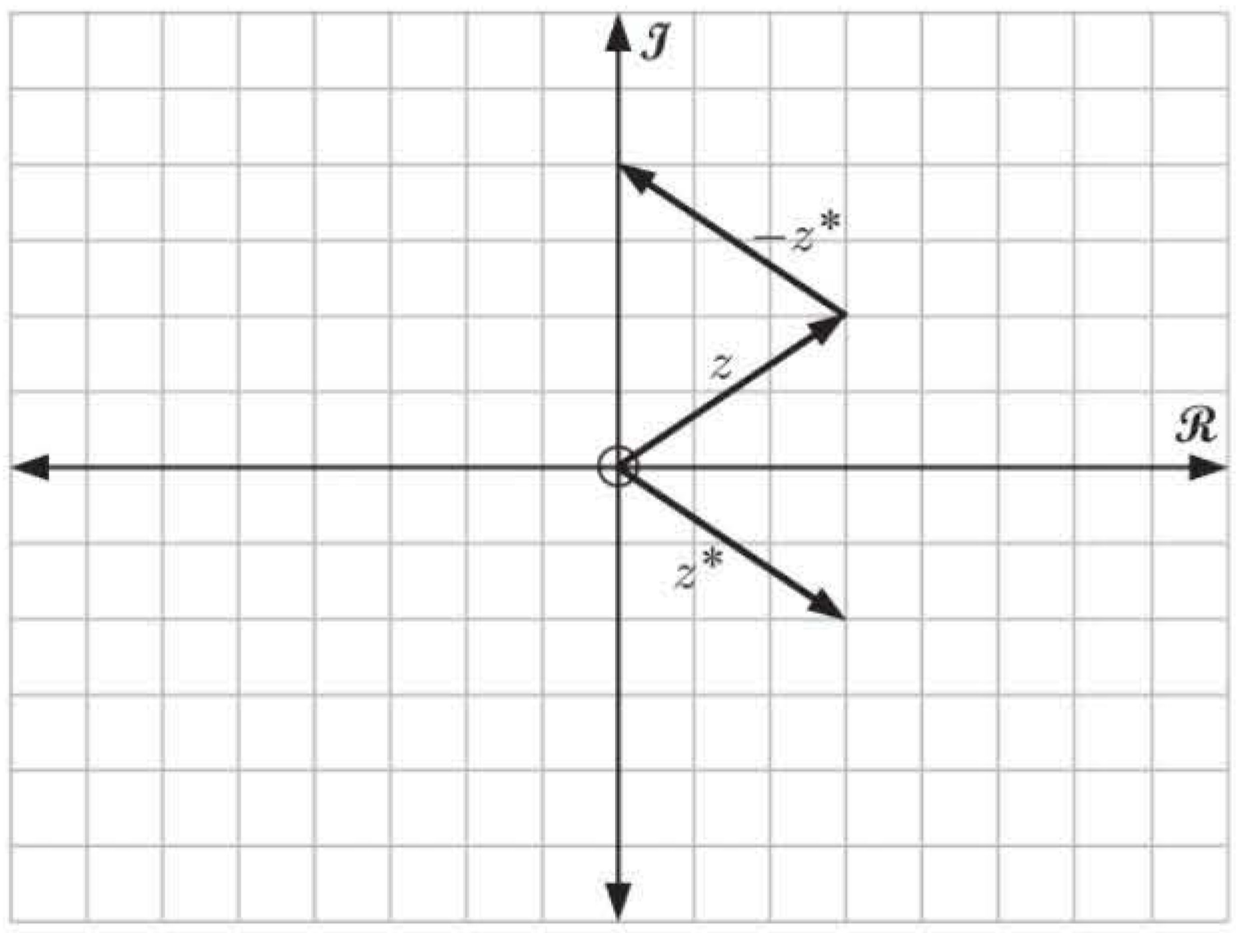
$\therefore$ $z^* = a - bi$

and $z + z^* = a + \cancel{bi} + a - \cancel{bi}$

$= 2a$, which is real (since a is real)

$\therefore$ $z + z^*$ is always real for any complex number z.

7



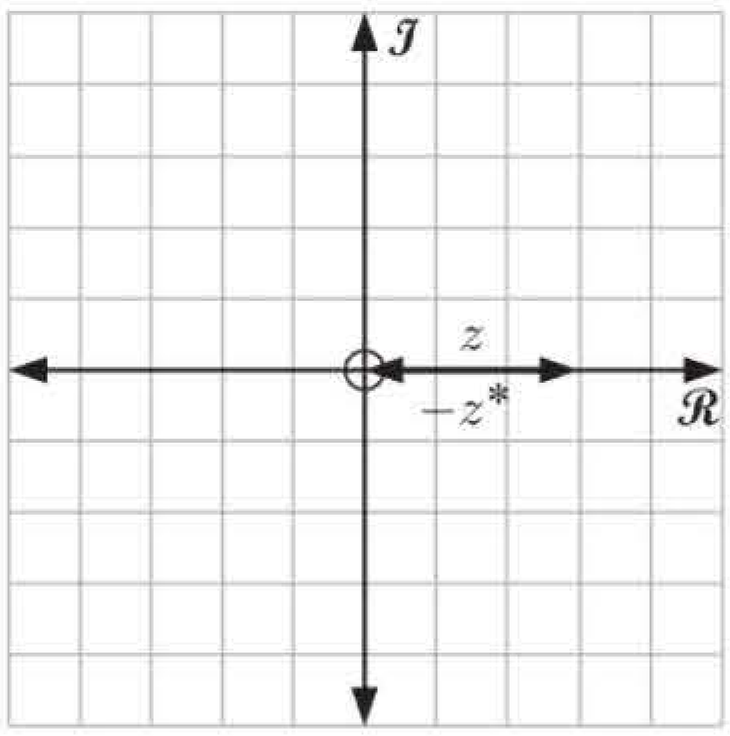
Suppose $z = a + bi$, where $a, b \in \mathbb{R}$

$\therefore$ $z^* = a - bi$

and $z - z^* = (a + bi) - (a - bi)$

$= a + bi - a + bi$

$= 2bi$

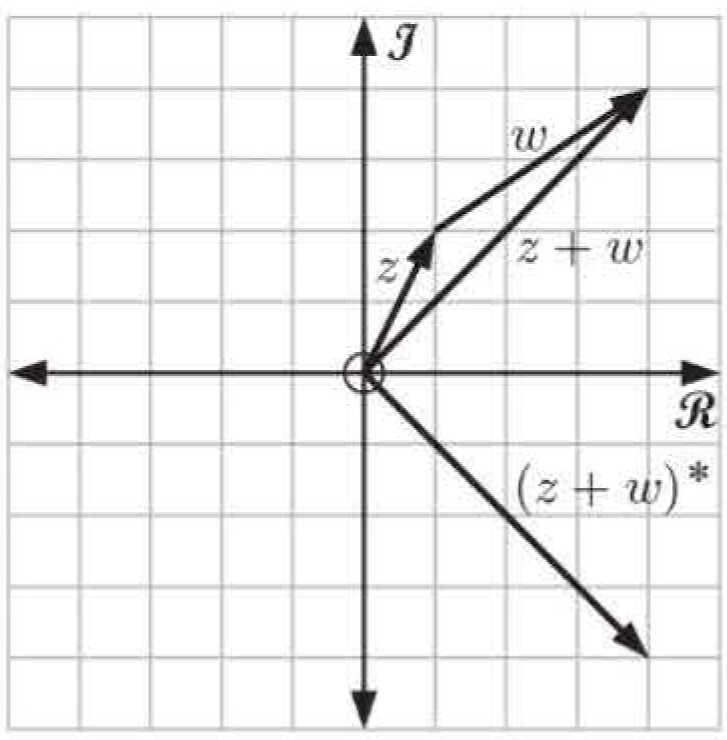


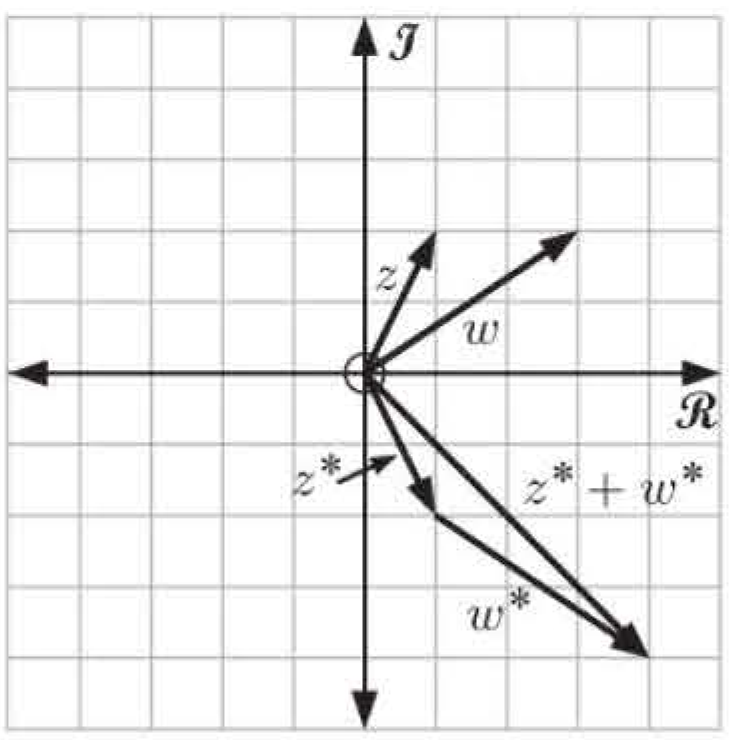
Since b is real, $z - z^*$ is purely imaginary for $b \neq 0$.

If $b = 0$ then $z - z^* = 0$.

$\therefore$ $z - z^*$ is purely imaginary, unless z is real, then $z - z^* = 0$.

8 **a**



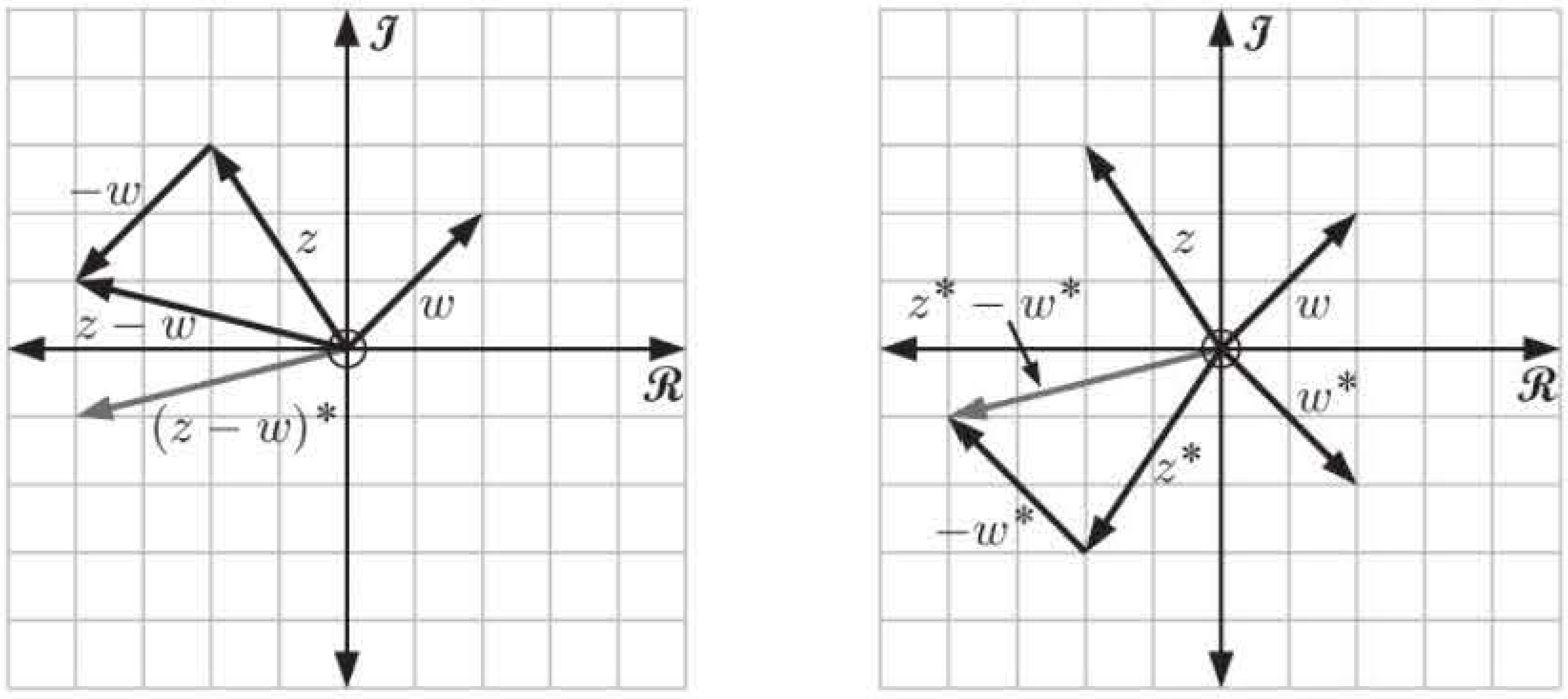


Let $z = a + bi$ and $w = c + di$

$$\begin{aligned}
\therefore\ (z+w)^* &= (a + bi + c + di)^* \\
&= ((a+c) + (b+d)i)^* \\
&= (a+c) - (b+d)i \\
&= a + c - bi - di \\
&= (a - bi) + (c - di) \\
&= z^* + w^*
\end{aligned}$$

$\therefore$ $(z+w)^* = z^* + w^*$ for all complex z, w

b



Let $z = a + bi$ and $w = c + di$

$$\begin{aligned}
\therefore\ (z-w)^* &= ((a+bi)-(c+di))^* \\
&= (a+bi-c-di)^* \\
&= ((a-c)+(b-d)i)^* \\
&= (a-c)-(b-d)i \\
&= a-c-bi+di \\
&= (a-bi)-(c-di) \\
&= z^*-w^*
\end{aligned}$$

$\therefore\ (z-w)^* = z^* - w^*$ for all complex z, w

EXERCISE 16B.1

1 **a** $|3-4i|$
$= \sqrt{3^2+(-4)^2}$
$= \sqrt{9+16}$
$= \sqrt{25}$
$= 5$

b $|5+12i|$
$= \sqrt{5^2+12^2}$
$= \sqrt{25+144}$
$= \sqrt{169}$
$= 13$

c $|-8+2i|$
$= \sqrt{(-8)^2+2^2}$
$= \sqrt{64+4}$
$= \sqrt{68}$
$= 2\sqrt{17}$

d $|3i|$
$= \sqrt{0^2+3^2}$
$= \sqrt{9}$
$= 3$

e $|-4|$
$= \sqrt{(-4)^2+0^2}$
$= \sqrt{16}$
$= 4$

2 **a** $|z|$
$= |2+i|$
$= \sqrt{2^2+1^2}$
$= \sqrt{5}$

b $|z^*|$
$= |2-i|$
$= \sqrt{2^2+(-1)^2}$
$= \sqrt{5}$

c $|z^*|^2$
$= (\sqrt{5})^2$ {from **b**}
$= 5$

d zz^*
$= (2+i)(2-i)$
$= 4-\cancel{2i}+\cancel{2i}-i^2$
$= 4+1$
$= 5$

e $|zw|$
$= |(2+i)(-1+3i)|$
$= \left|-2+6i-i+3i^2\right|$
$= |-5+5i|$
$= \sqrt{(-5)^2+5^2}$
$= \sqrt{50}$ or $5\sqrt{2}$

f $|z|\,|w|$
$= |2+i|\,|-1+3i|$
$= \sqrt{2^2+1^2}\sqrt{(-1)^2+3^2}$
$= \sqrt{5}\times\sqrt{10}$
$= \sqrt{50}$ or $5\sqrt{2}$

g
$$\begin{aligned}&\left|\frac{z}{w}\right|\\ &=\left|\frac{2+i}{-1+3i}\right|\\ &=\left|\frac{(2+i)}{(-1+3i)}\times\frac{(-1-3i)}{(-1-3i)}\right|\\ &=\left|\frac{-2-6i-i-3i^2}{(-1)^2-(3i)^2}\right|\\ &=\left|\frac{-2+3-7i}{10}\right|\\ &=\left|\tfrac{1}{10}-\tfrac{7}{10}i\right|\\ &=\sqrt{\left(\tfrac{1}{10}\right)^2+\left(\tfrac{-7}{10}\right)^2}\\ &=\sqrt{\tfrac{1+49}{100}}\\ &=\sqrt{\tfrac{50}{100}}\\ &=\tfrac{1}{\sqrt{2}}\end{aligned}$$

h
$$\begin{aligned}&\frac{|z|}{|w|}\\ &=\frac{|2+i|}{|-1+3i|}\\ &=\frac{\sqrt{2^2+1^2}}{\sqrt{(-1)^2+3^2}}\\ &=\tfrac{\sqrt{5}}{\sqrt{10}}\\ &=\sqrt{\tfrac{5}{10}}\\ &=\tfrac{1}{\sqrt{2}}\end{aligned}$$

i
$$\begin{aligned}z^2&=(2+i)^2\\ &=4+4i+i^2\\ &=3+4i\\ \therefore\ \left|z^2\right|&=\sqrt{3^2+4^2}\\ &=\sqrt{25}\\ &=5\end{aligned}$$

j
$$\begin{aligned}&|z|^2\\ &=(\sqrt{5})^2\quad\{\text{from } \mathbf{a}\}\\ &=5\end{aligned}$$

k
$$\begin{aligned}z^3&=z^2\times z\\ &=(3+4i)(2+i)\qquad\{\text{from } \mathbf{i}\}\\ &=6+3i+8i+4i^2\\ &=2+11i\\ \therefore\ \left|z^3\right|&=\sqrt{2^2+11^2}\\ &=\sqrt{4+121}\\ &=\sqrt{125}\\ &=5\sqrt{5}\end{aligned}$$

l
$$\begin{aligned}&|z|^3\\ &=(\sqrt{5})^3\quad\{\text{from } \mathbf{a}\}\\ &=\sqrt{125}\\ &=5\sqrt{5}\end{aligned}$$

3 **a** Let $z=a+bi$ where $a, b\in\mathbb{R}$
$$\begin{aligned}\therefore\ |z^*|&=|a-bi|\\ &=\sqrt{a^2+(-b)^2}\\ &=\sqrt{a^2+b^2}\\ &=|a+bi|\\ &=|z|\quad\text{as required}\end{aligned}$$

b Let $z=a+bi$ where $a, b\in\mathbb{R}$
$$\begin{aligned}\therefore\ zz^*&=(a+bi)(a-bi)\\ &=a^2-\cancel{abi}+\cancel{abi}-b^2i^2\\ &=a^2+b^2\\ &=\left(\sqrt{a^2+b^2}\right)^2\\ &=|z|^2\quad\text{as required}\end{aligned}$$

4 **a**
$$\begin{aligned}z&=\cos\theta+i\sin\theta\\ \therefore\ |z|&=\sqrt{\cos^2\theta+\sin^2\theta}\\ &=\sqrt{1}\\ &=1\end{aligned}$$

b
$$\begin{aligned}z&=r(\cos\theta+i\sin\theta),\quad r\in\mathbb{R}\\ &=r\cos\theta+ri\sin\theta\\ \therefore\ |z|&=\sqrt{r^2\cos^2\theta+r^2\sin^2\theta}\\ &=\sqrt{r^2(\cos^2\theta+\sin^2\theta)}\\ &=\sqrt{r^2}\\ &=|r|\end{aligned}$$

5
$$\left|\frac{z}{w}\right|\times|w|=\left|\frac{z}{w}\times w\right|\qquad\{\text{using }\ |z_1|\,|z_2|=|z_1z_2|\,\}$$
$$\therefore\ \left|\frac{z}{w}\right|\times|w|=|z|$$
$$\therefore\ \left|\frac{z}{w}\right|=\frac{|z|}{|w|}\quad\text{provided }\ w\neq 0\quad\{\text{dividing both sides by } |w|\,\}$$

6 **a** **i** $|z_1 z_2 z_3| = |(z_1 z_2) z_3|$
$= |z_1 z_2|\,|z_3|$ {as $|zw| = |z|\,|w|$}
$= |z_1|\,|z_2|\,|z_3|$ {$|zw| = |z|\,|w|$ again}

ii Now extending this result by the same argument,
$|z_1 z_2 z_3 z_4| = |(z_1 z_2 z_3) z_4|$
$= |z_1 z_2 z_3|\,|z_4|$
$= |z_1|\,|z_2|\,|z_3|\,|z_4|$

b The generalisation of **a** is: $|z_1 z_2 z_3 .... z_n| = |z_1|\,|z_2|\,|z_3| .... |z_n|$

c P_n is "$|z_1 z_2 .... z_n| = |z_1|\,|z_2| .... |z_n|$" for $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$ then LHS $= |z_1|$, RHS $= |z_1|$ $\therefore$ P_1 is true.

(2) If P_k is true then $|z_1 z_2 .... z_k| = |z_1|\,|z_2| .... |z_k|$

$|z_1 z_2 .... z_k z_{k+1}| = |(z_1 z_2 .... z_k) z_{k+1}|$
$= |z_1 z_2 .... z_k|\,|z_{k+1}|$ {$|z|\,|w| = |zw|$}
$= |z_1|\,|z_2| .... |z_k|\,|z_{k+1}|$ {P_k}

$\therefore$ P_{k+1} is true whenever P_k is true, and P_1 is true.
$\therefore$ P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

d $|z_1 z_2 .... z_n| = |z_1|\,|z_2| .... |z_n|$
Letting $z_1 = z_2 = ... = z_n = z$ we have $|zz....z| = |z|\,|z| .... |z|$
$\therefore$ $|z^n| = |z|^n$

e If $z = 1 - i\sqrt{3}$ then $|z| = \sqrt{1^2 + (-\sqrt{3})^2}$
$= \sqrt{4} = 2$
$\therefore$ $\left|z^{20}\right| = |z|^{20}$
$= 2^{20} = 1\,048\,576$

7 **a** $|2z|$
$= |2|\,|z|$
$= 2 \times 3$
$= 6$

b $|-3z|$
$= |-3|\,|z|$
$= 3 \times 3$
$= 9$

c $|(1+2i)z|$
$= |1+2i| \times |z|$
$= \sqrt{1+4} \times 3$
$= 3\sqrt{5}$

d $|iz|$
$= |i|\,|z|$
$= 1 \times 3$
$= 3$

e $\left|\dfrac{1}{z}\right|$
$= \dfrac{|1|}{|z|}$
$= \frac{1}{3}$

f $\left|\dfrac{2i}{z^2}\right|$
$= \dfrac{|2i|}{|z^2|}$
$= \dfrac{|2i|}{|z|^2}$
$= \dfrac{2}{3^2} = \frac{2}{9}$

8 **a** $w = \dfrac{z+1}{z-1}$. Let $z = a + bi$ where $a, b \in \mathbb{R}$.

$$\therefore\ w = \frac{a+bi+1}{a+bi-1}$$
$$= \frac{(a+1)+bi}{(a-1)+bi}$$
$$= \frac{(a+1)+bi}{(a-1)+bi} \times \frac{(a-1)-bi}{(a-1)-bi}$$
$$= \frac{(a+1)(a-1) - b(a+1)i + b(a-1)i - b^2 i^2}{(a-1)^2 - (bi)^2}$$
$$= \frac{a^2 - 1 - \cancel{abi} - bi + \cancel{abi} - bi + b^2}{(a-1)^2 + b^2}$$
$$= \left(\frac{a^2 + b^2 - 1}{(a-1)^2 + b^2}\right) + \left(\frac{-2b}{(a-1)^2 + b^2}\right) i$$

b $$\mathcal{Re}\,(w) = \frac{a^2+b^2-1}{(a-1)^2+b^2} = \frac{a^2+b^2-1}{a^2-2a+1+b^2} = \frac{a^2+b^2-1}{a^2+b^2-2a+1}$$

Since $|z| = 1$ $\sqrt{a^2+b^2} = 1$ $\therefore$ $a^2+b^2 = 1$

$\therefore$ $\mathcal{Re}\,(w) = \dfrac{1-1}{1-2a+1} = 0$ provided $a \neq 1$

If $a = 1$, then $\mathcal{Re}\,(w)$ is undefined.

9 **a**
$$\begin{aligned}
|z+9| &= 3\,|z+1| \\
\therefore\ |z+9|^2 &= 9\,|z+1|^2 \\
\therefore\ (z+9)(z+9)^* &= 9(z+1)(z+1)^* \qquad \{zz^* = |z|^2\} \\
\therefore\ (z+9)(z^*+9) &= 9(z+1)(z^*+1) \qquad \{(z \pm w)^* = z^* \pm w^*\} \\
\therefore\ zz^* + \cancel{9z} + \cancel{9z^*} + 81 &= 9zz^* + \cancel{9z} + \cancel{9z^*} + 9 \\
\therefore\ 72 &= 8zz^* \\
\therefore\ zz^* &= 9 \\
\therefore\ |z|^2 &= 9 \\
\therefore\ |z| &= 3 \qquad \{|z| > 0\}
\end{aligned}$$

b
$$\begin{aligned}
\left|\frac{z+4}{z+1}\right| &= 2 \\
\therefore\ \frac{|z+4|}{|z+1|} &= 2 \qquad \left\{\left|\frac{z}{w}\right| = \frac{|z|}{|w|}\right\} \\
\therefore\ |z+4| &= 2\,|z+1| \\
\therefore\ |z+4|^2 &= 4\,|z+1|^2 \\
\therefore\ (z+4)(z+4)^* &= 4(z+1)(z+1)^* \qquad \{zz^* = |z|^2\} \\
\therefore\ (z+4)(z^*+4) &= 4(z+1)(z^*+1) \qquad \{(z \pm w)^* = z^* \pm w^*\} \\
\therefore\ zz^* + \cancel{4z} + \cancel{4z^*} + 16 &= 4zz^* + \cancel{4z} + \cancel{4z^*} + 4 \\
\therefore\ 12 &= 3zz^* \\
\therefore\ zz^* &= 4 \\
\therefore\ |z|^2 &= 4 \\
\therefore\ |z| &= 2 \qquad \{|z| > 0\}
\end{aligned}$$

10
$$\begin{aligned}
|z+w| &= |z-w| \\
\therefore\ |z+w|^2 &= |z-w|^2 \\
\therefore\ (z+w)(z+w)^* &= (z-w)(z-w)^* \qquad \{zz^* = |z|^2\} \\
\therefore\ (z+w)(z^*+w^*) &= (z-w)(z^*-w^*) \qquad \{(z \pm w)^* = z^* \pm w^*\} \\
\therefore\ \cancel{zz^*} + zw^* + wz^* + \cancel{ww^*} &= \cancel{zz^*} - zw^* - wz^* + \cancel{ww^*} \\
\therefore\ zw^* + wz^* &= -zw^* - wz^* \\
\therefore\ 2zw^* &= -2wz^* \\
\therefore\ \frac{z}{z^*} &= -\frac{w}{w^*}
\end{aligned}$$

EXERCISE 16B.2

1 **a** A(3, 6), B(−1, 2), $z = 3+6i$, $w = -1+2i$

i
$$\begin{aligned}
z - w &= (3+6i) - (-1+2i) \\
&= 4+4i \\
|z-w| &= \sqrt{4^2+4^2} \\
&= \sqrt{32} \\
&= 4\sqrt{2}
\end{aligned}$$
$\therefore$ AB $= 4\sqrt{2}$ units

ii
$$\begin{aligned}
\frac{z+w}{2} &= \frac{(3+6i)+(-1+2i)}{2} \\
&= \frac{2+8i}{2} \\
&= 1+4i
\end{aligned}$$
and so M is at (1, 4)

b A$(-4, 7)$, B$(1, -3)$, $z = -4 + 7i$, $w = 1 - 3i$

i
$$\begin{aligned} z - w &= (-4 + 7i) - (1 - 3i) \\ &= -5 + 10i \\ |z - w| &= \sqrt{(-5)^2 + 10^2} \\ &= \sqrt{125} \\ &= 5\sqrt{5} \end{aligned}$$
$\therefore$ AB $= 5\sqrt{5}$ units

ii
$$\begin{aligned} \frac{z+w}{2} &= \frac{(-4+7i)+(1-3i)}{2} \\ &= \frac{-3+4i}{2} \\ &= -\tfrac{3}{2} + 2i \end{aligned}$$
and so M is at $\left(-\frac{3}{2}, 2\right)$

2 **a** **i** $\overrightarrow{\text{OQ}} = z + w$ **ii** $\overrightarrow{\text{PR}} = w - z$

b In $\triangle$OPQ, $|z + w|$ represents the length of OQ, $|z| =$ length of OP, and $|w|$ the length of PQ. Now if w, z are not parallel, we will form the $\triangle$OPQ and this means OQ $<$ OP $+$ PQ.

$\therefore$ $|z + w| < |z| + |w|$

If w and z are parallel then we form a straight line and OQ $=$ OP $+$ PQ

$$\therefore \quad |z + w| = |z| + |w|$$

Consequently $|z + w| \leqslant |z| + |w|$

c In $\triangle$OPR, the length of RP is represented by $|z - w|$. If w and z are not parallel, we form a triangle and RP $+$ OP $>$ OR. $\therefore$ $|z - w| + |z| > |w|$

$$\therefore \quad |z - w| > |w| - |z|$$

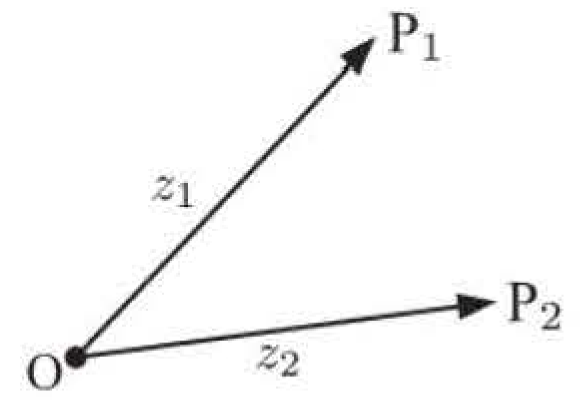
Equality will occur when w is parallel to z, in which case a straight line OPR is formed and

$$|z - w| = |w| - |z|$$

Consequently $|z - w| \geqslant |w| - |z|$

3 $z_1 \equiv \overrightarrow{\text{OP}_1}$ and $z_2 \equiv \overrightarrow{\text{OP}_2}$

$$\begin{aligned} \therefore \quad z_1 - z_2 &= \overrightarrow{\text{OP}_1} - \overrightarrow{\text{OP}_2} \\ &= \overrightarrow{\text{OP}_1} + \overrightarrow{\text{P}_2\text{O}} \qquad \{\text{since } \overrightarrow{\text{P}_2\text{O}} = -\overrightarrow{\text{OP}_2}\} \\ &= \overrightarrow{\text{P}_2\text{O}} + \overrightarrow{\text{OP}_1} \\ &= \overrightarrow{\text{P}_2\text{P}_1} \end{aligned}$$

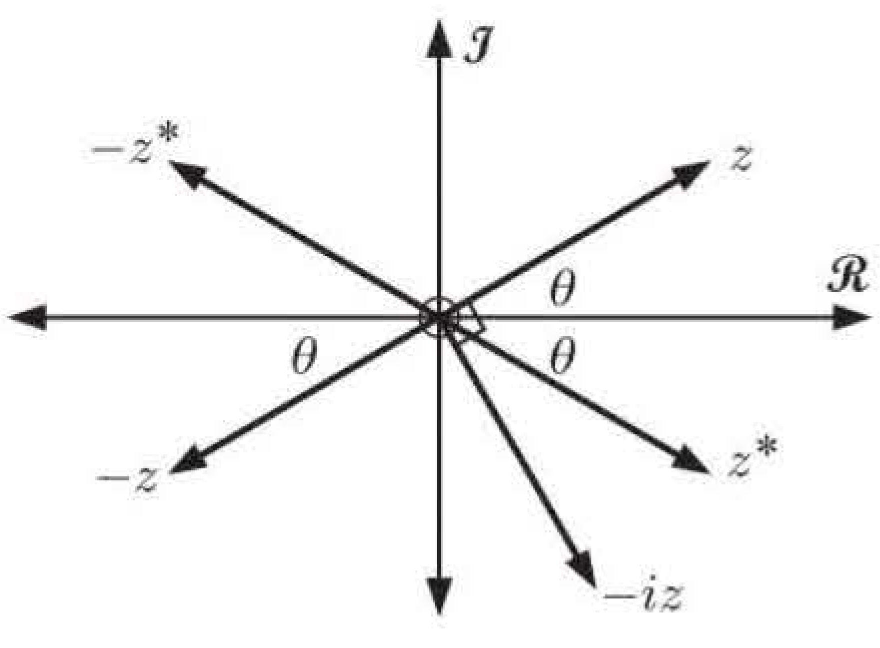


4 **a** $z \mapsto z^*$. Reflection in the real axis.

b $z \mapsto -z$. Rotation of π about O.

c $z \mapsto -z^*$. Reflection in the imaginary axis.

d $z \mapsto -iz$. Clockwise rotation of $\frac{\pi}{2}$ about O.



5 $\dfrac{50}{z^*} - \dfrac{10}{z} = 2 + 9i$ where $z = a + bi$, $a, b \in \mathbb{R}$

$$\therefore \quad 50z - 10z^* = (2 + 9i)(|z|^2) \qquad \{\text{multiply both sides by } zz^* = |z|^2\}$$
$$\therefore \quad 50(a + bi) - 10(a - bi) = (2 + 9i)(40) \qquad \{|z| = 2\sqrt{10} \quad \therefore \ |z|^2 = 40\}$$
$$\therefore \quad 50a + 50bi - 10a + 10bi = 80 + 360i$$
$$\therefore \quad 40a + 60bi = 80 + 360i$$

Equating real and imaginary parts, $40a = 80$ and $60b = 360$

$$\therefore \quad a = 2 \quad \text{and} \quad b = 6$$
$$\therefore \quad z = 2 + 6i$$

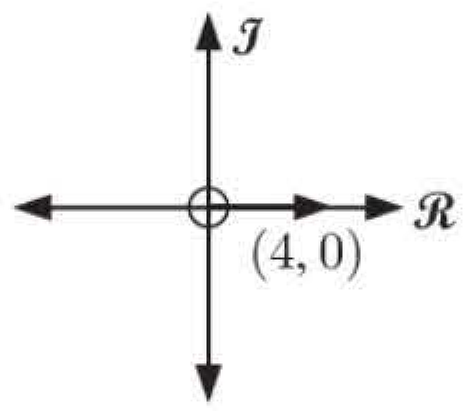
EXERCISE 16C.1

1 **a** $z = 4$

$\arg z = 0$

$|z| = 4$

$\therefore \ z = 4 \text{ cis } 0$

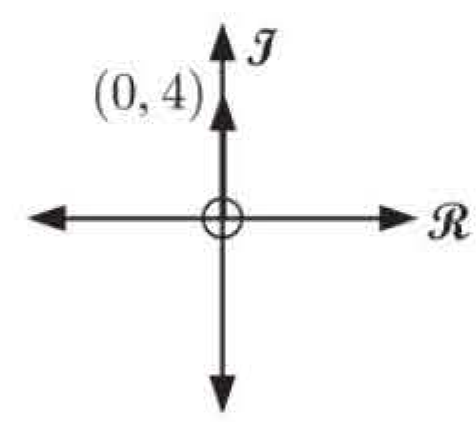


b $z = 4i$

$\arg z = \frac{\pi}{2}$

$|z| = 4$

$\therefore \ z = 4 \text{ cis}\left(\frac{\pi}{2}\right)$

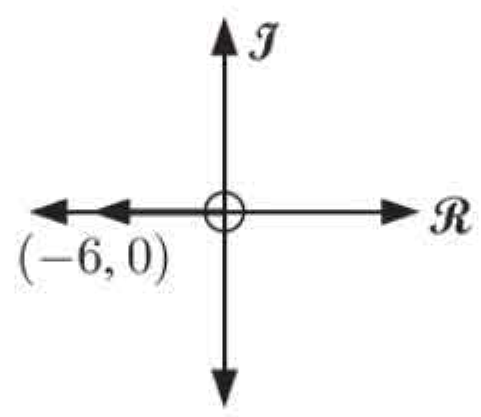


c $z = -6$

$\arg z = \pi$

$|z| = 6$

$\therefore \ z = 6 \text{ cis } \pi$

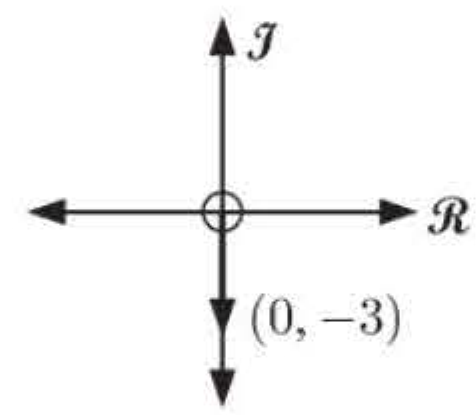


d $z = -3i$

$\arg z = -\frac{\pi}{2}$

$|z| = 3$

$\therefore \ z = 3 \text{ cis}\left(-\frac{\pi}{2}\right)$

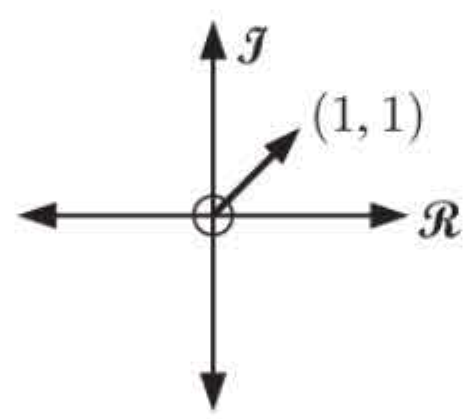


e $z = 1 + i$

$\arg z = \frac{\pi}{4}$

$$|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$\therefore \ z = \sqrt{2} \text{ cis}\left(\frac{\pi}{4}\right)$

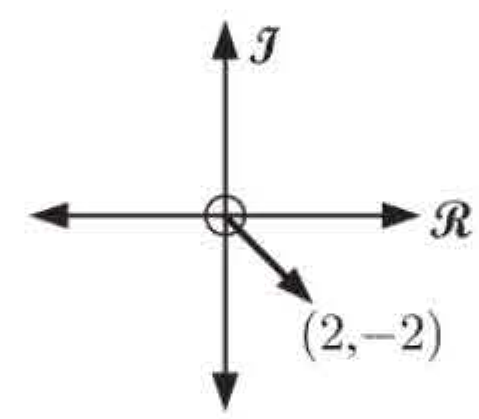


f $z = 2 - 2i$

$\arg z = -\frac{\pi}{4}$

$$|z| = \sqrt{2^2 + 2^2} = 2\sqrt{2}$$

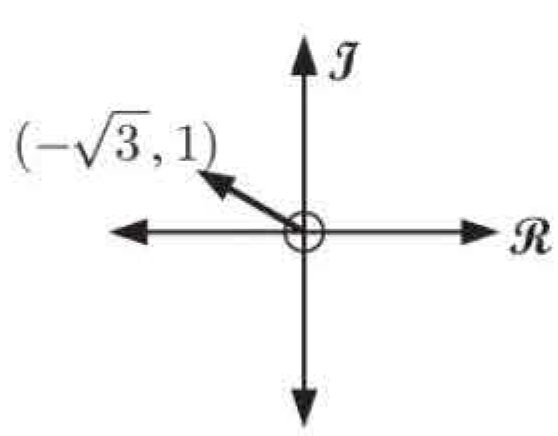
$\therefore \ z = 2\sqrt{2} \text{ cis}\left(-\frac{\pi}{4}\right)$



g $z = -\sqrt{3} + i$, $\arg z = \frac{5\pi}{6}$

$$|z| = \sqrt{(-\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$$

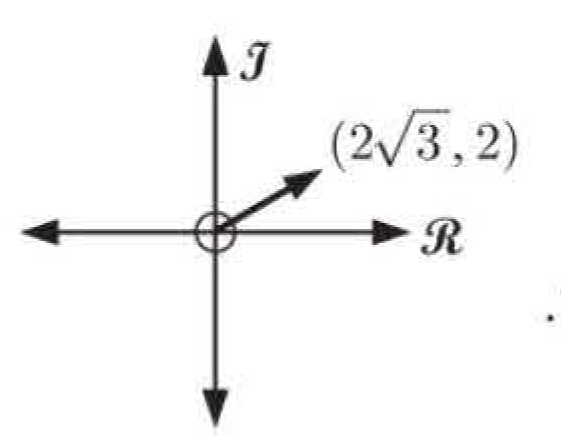
$\therefore \ z = 2 \text{ cis}\left(\frac{5\pi}{6}\right)$



h $z = 2\sqrt{3} + 2i$ $\arg z = \frac{\pi}{6}$

$$|z| = \sqrt{12 + 4} = \sqrt{16} = 4$$

$\therefore \ z = 4 \text{ cis}\left(\frac{\pi}{6}\right)$



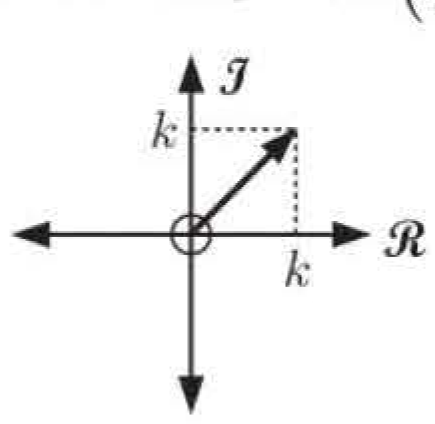
2 $z = 0 = 0 + 0i$ cannot be written in polar form. The vector representing $\overrightarrow{\text{OP}}$ has length zero, and an argument is not defined (no angle can be formed with the positive x-axis).

3 If $k = 0$ it is not possible.

If $k > 0$, $|z| = \sqrt{k^2 + k^2} = k\sqrt{2}$

$\arg z = \frac{\pi}{4}$

$\therefore \ z = k\sqrt{2} \text{ cis}\left(\frac{\pi}{4}\right)$



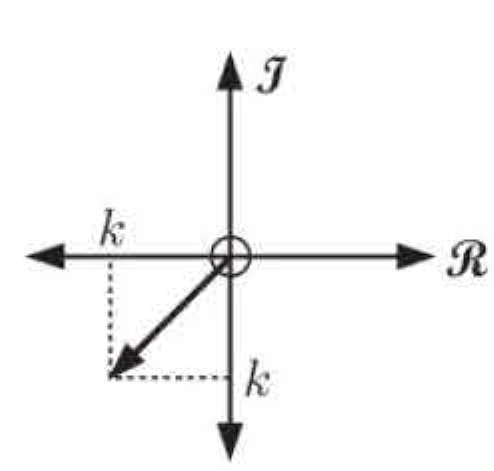
If $k < 0$, $|z| = \sqrt{k^2 + k^2} = |k|\sqrt{2}$

Since $k < 0$

$|z| = -k\sqrt{2}$

$\arg z = -\frac{3\pi}{4}$

$\therefore \ z = -k\sqrt{2} \text{ cis}\left(-\frac{3\pi}{4}\right)$



4 **a**

$$2 \text{ cis}\left(\tfrac{\pi}{2}\right) = 2\left(\cos\left(\tfrac{\pi}{2}\right) + i\sin\left(\tfrac{\pi}{2}\right)\right) = 2(0 + i) = 2i$$

b

$$8 \text{ cis}\left(\tfrac{\pi}{4}\right) = 8\left(\cos\left(\tfrac{\pi}{4}\right) + i\sin\left(\tfrac{\pi}{4}\right)\right) = 8\left(\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}i\right) = 4\sqrt{2} + 4\sqrt{2}i$$

c $4 \text{ cis}\left(\frac{\pi}{6}\right)$
$= 4\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$
$= 4\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$
$= 2\sqrt{3} + 2i$

d $\sqrt{2} \text{ cis}\left(-\frac{\pi}{4}\right)$
$= \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$
$= \sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$
$= 1 - i$

e $\sqrt{3} \text{ cis}\left(\frac{2\pi}{3}\right)$
$= \sqrt{3}\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$
$= \sqrt{3}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$
$= -\frac{\sqrt{3}}{2} + \frac{3}{2}i$

f $5 \text{ cis } \pi$
$= 5(\cos\pi + i\sin\pi)$
$= 5(-1)$
$= -5$

5 **a** $\text{cis } 0$
$= \cos 0 + i\sin 0$
$= 1$

b $|\text{cis } \theta|$
$= |\cos\theta + i\sin\theta|$
$= \sqrt{\cos^2\theta + \sin^2\theta}$
$= \sqrt{1} = 1$

c $\text{cis } \alpha \times \text{cis } \beta = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$
$= \cos\alpha\cos\beta + i\cos\alpha\sin\beta + i\sin\alpha\cos\beta + i^2\sin\alpha\sin\beta$
$= [\cos\alpha\cos\beta - \sin\alpha\sin\beta] + i[\sin\alpha\cos\beta + \sin\beta\cos\alpha]$
$= \cos(\alpha+\beta) + i\sin(\alpha+\beta)$
$= \text{cis}(\alpha+\beta)$

EXERCISE 16C.2

1 **a** $\text{cis } \theta \text{ cis } 2\theta$
$= \text{cis}(\theta + 2\theta)$
$= \text{cis } 3\theta$

b $\frac{\text{cis } 3\theta}{\text{cis } \theta}$
$= \text{cis}(3\theta - \theta)$
$= \text{cis } 2\theta$

c $[\text{cis } \theta]^3$
$= (\text{cis } \theta)(\text{cis } \theta)(\text{cis } \theta)$
$= (\text{cis } 2\theta)(\text{cis } \theta)$
$= \text{cis } 3\theta$

d $\text{cis}\left(\frac{\pi}{18}\right) \times \text{cis}\left(\frac{\pi}{9}\right)$
$= \text{cis}\left(\frac{\pi}{18} + \frac{\pi}{9}\right)$
$= \text{cis}\left(\frac{3\pi}{18}\right)$
$= \text{cis}\left(\frac{\pi}{6}\right)$
$= \cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)$
$= \frac{\sqrt{3}}{2} + \frac{1}{2}i$

e $2 \text{ cis}\left(\frac{\pi}{12}\right) \text{cis}\left(\frac{\pi}{6}\right)$
$= 2 \text{ cis}\left(\frac{\pi}{12} + \frac{\pi}{6}\right)$
$= 2 \text{ cis}\left(\frac{3\pi}{12}\right)$
$= 2 \text{ cis}\left(\frac{\pi}{4}\right)$
$= 2(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right))$
$= 2\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$
$= \sqrt{2} + i\sqrt{2}$

f $2 \text{ cis}\left(\frac{2\pi}{5}\right) \times 4 \text{ cis}\left(\frac{8\pi}{5}\right)$
$= 8 \text{ cis}\left(\frac{2\pi}{5} + \frac{8\pi}{5}\right)$
$= 8 \text{ cis}\left(\frac{10\pi}{5}\right)$
$= 8 \text{ cis } 2\pi$
$= 8(\cos 2\pi + i\sin 2\pi)$
$= 8(1)$
$= 8$

g $\frac{4 \text{ cis}\left(\frac{\pi}{12}\right)}{2 \text{ cis}\left(\frac{7\pi}{12}\right)}$
$= 2 \text{ cis}\left(\frac{\pi}{12} - \frac{7\pi}{12}\right)$
$= 2 \text{ cis}\left(-\frac{6\pi}{12}\right)$
$= 2 \text{ cis}\left(-\frac{\pi}{2}\right)$
$= 2(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right))$
$= 2(-i)$
$= -2i$

h $\frac{\sqrt{32} \text{ cis}\left(\frac{\pi}{8}\right)}{\sqrt{2} \text{ cis}\left(-\frac{7\pi}{8}\right)}$
$= \frac{\sqrt{32}}{\sqrt{2}} \text{ cis}\left(\frac{\pi}{8} - \left(-\frac{7\pi}{8}\right)\right)$
$= \sqrt{16} \text{ cis}\left(\frac{8\pi}{8}\right)$
$= 4 \text{ cis } \pi$
$= 4(\cos\pi + i\sin\pi)$
$= 4(-1)$
$= -4$

i $\left[\sqrt{2} \text{ cis}\left(\frac{\pi}{8}\right)\right]^4$
$= \sqrt{2} \text{ cis}\left(\frac{\pi}{8}\right) \times \sqrt{2} \text{ cis}\left(\frac{\pi}{8}\right)$
$\times \sqrt{2} \text{ cis}\left(\frac{\pi}{8}\right) \times \sqrt{2} \text{ cis}\left(\frac{\pi}{8}\right)$
$= (\sqrt{2})^4 \text{ cis}\left(\frac{\pi}{8} + \frac{\pi}{8} + \frac{\pi}{8} + \frac{\pi}{8}\right)$
$= 4 \text{ cis}\left(\frac{\pi}{2}\right)$
$= 4(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right))$
$= 4i$

2 **a** $\text{cis } 17\pi$
$= \text{cis } (\pi + 8(2\pi))$
$= \text{cis } \pi$
$= -1$

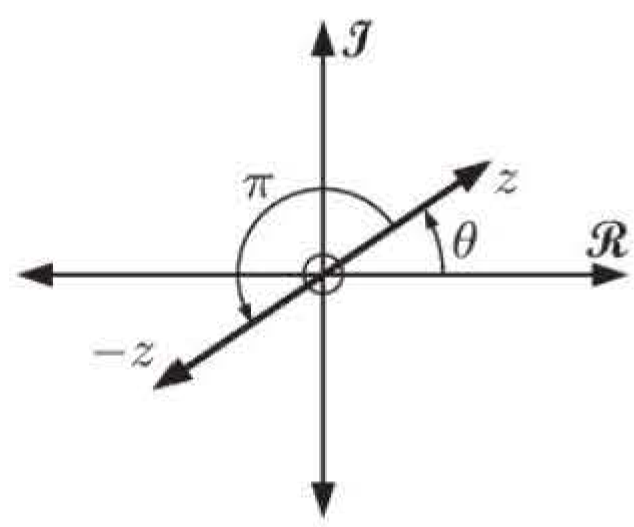
b $\text{cis } (-37\pi)$
$= \text{cis } (\pi - 19(2\pi))$
$= \text{cis } \pi$
$= -1$

c $\text{cis } \left(\frac{91\pi}{3}\right)$
$= \text{cis } \left(\frac{\pi}{3} + 15(2\pi)\right)$
$= \text{cis } \left(\frac{\pi}{3}\right)$
$= \frac{1}{2} + \frac{\sqrt{3}}{2}i$

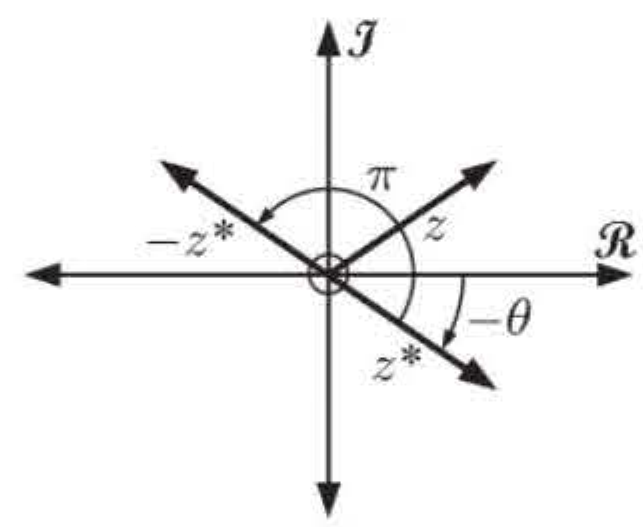
3 **a** $z = 2 \text{ cis } \theta$
$|z| = 2$
$\arg z = \theta$

b $z^* = 2 \text{ cis } (-\theta)$

c $-z = 2 \text{ cis } (\theta + \pi)$



d $-z^* = 2 \text{ cis } (\pi - \theta)$



4 **a** $i = 1 \text{ cis } \left(\frac{\pi}{2}\right) = \text{cis } \left(\frac{\pi}{2}\right)$

b $iz = \text{cis } \left(\frac{\pi}{2}\right) \times r \text{ cis } \theta$
$= r \text{ cis } \left(\theta + \frac{\pi}{2}\right)$

c $z \mapsto -iz$ $\quad -iz = \text{cis } \left(-\frac{\pi}{2}\right) \times r \text{ cis } \theta$
$= r \text{ cis } \left(\theta - \frac{\pi}{2}\right)$

So, a clockwise rotation of $\frac{\pi}{2}$ about O maps z onto $-iz$.

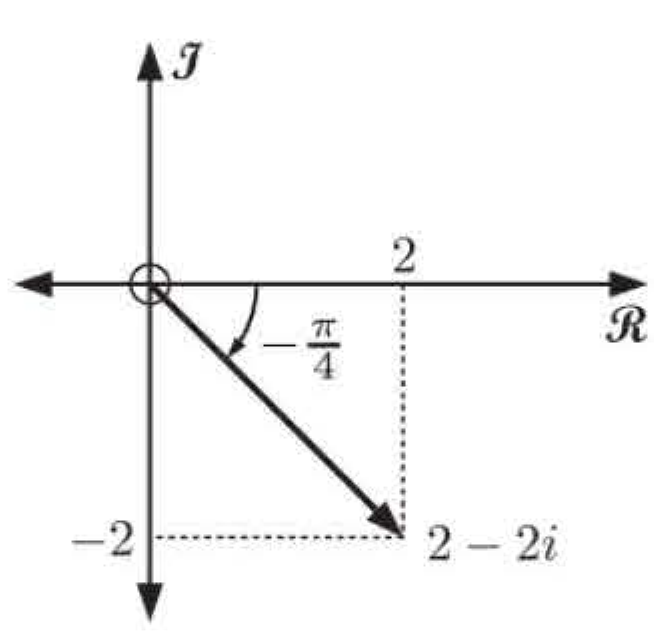
5 **a** **i** $\cos\theta - i\sin\theta$
$= \cos(-\theta) + i\sin(-\theta)$
$= \text{cis}(-\theta)$

ii $\sin\theta - i\cos\theta$
$= -i\cos\theta - i^2\sin\theta$
$= -i(\cos\theta + i\sin\theta)$
$= \text{cis } \left(-\frac{\pi}{2}\right) \text{ cis } \theta$
$= \text{cis } \left(\theta - \frac{\pi}{2}\right)$

b If $z = r \text{ cis } \theta$ then $z^* = r \text{ cis}(-\theta)$ in polar form.

EXERCISE 16C.3

1 **a**

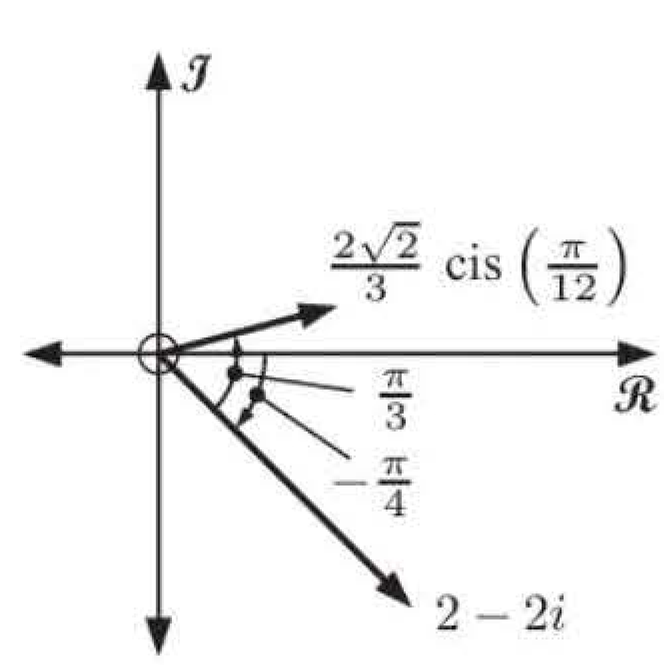


$z = 2 - 2i$
$\therefore |z| = \sqrt{2^2 + (-2)^2}$
$= 2\sqrt{2}$

$\therefore z = 2\sqrt{2}\left(\frac{2}{2\sqrt{2}} - \frac{2}{2\sqrt{2}}i\right)$
$= 2\sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$
$= 2\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$
$= 2\sqrt{2} \text{ cis } \left(-\frac{\pi}{4}\right)$

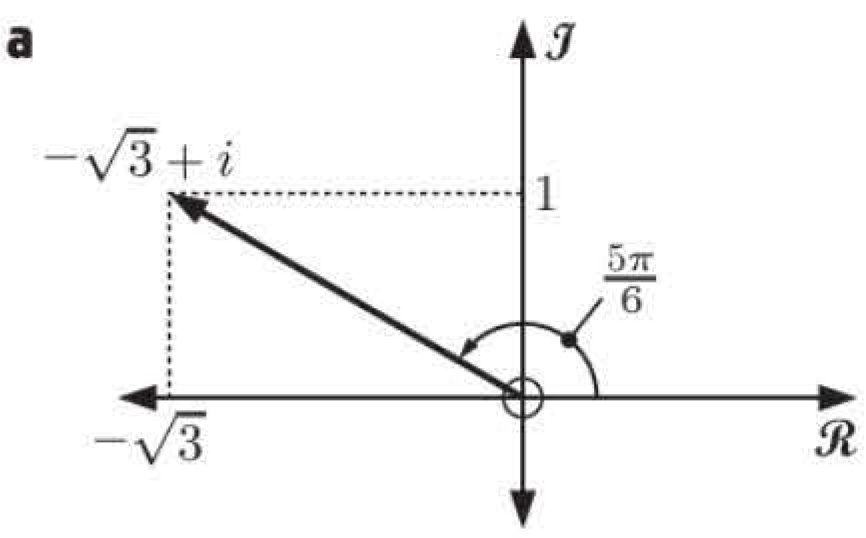
b $z \times \frac{1}{3} \text{ cis } \left(\frac{\pi}{3}\right) = 2\sqrt{2} \text{ cis } \left(-\frac{\pi}{4}\right) \times \frac{1}{3} \text{ cis } \left(\frac{\pi}{3}\right)$
$= \frac{2\sqrt{2}}{3} \text{ cis } \left(\frac{\pi}{3} - \frac{\pi}{4}\right)$
$= \frac{2\sqrt{2}}{3} \text{ cis } \left(\frac{4\pi - 3\pi}{12}\right)$
$= \frac{2\sqrt{2}}{3} \text{ cis } \left(\frac{\pi}{12}\right)$

c



d When z was multiplied by $\frac{1}{3} \text{ cis } \left(\frac{\pi}{3}\right)$, its modulus was multiplied by $\frac{1}{3}$, and it was rotated anti-clockwise through $\frac{\pi}{3}$ about the origin.

2 **a**

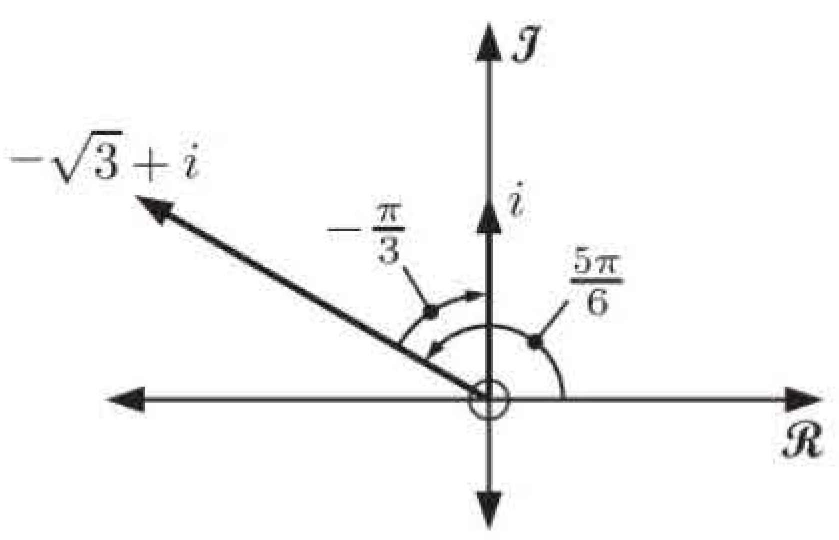


$$z = -\sqrt{3}+i$$
$$\therefore\ |z| = \sqrt{(-\sqrt{3})^2+1^2} = \sqrt{3+1} = 2$$

$$\therefore\ z = 2\left(-\tfrac{\sqrt{3}}{2}+\tfrac{1}{2}i\right) = 2\left(\cos\left(\tfrac{5\pi}{6}\right)+i\sin\left(\tfrac{5\pi}{6}\right)\right) = 2\text{ cis}\left(\tfrac{5\pi}{6}\right)$$

b $$z\times\tfrac{1}{2}\text{ cis}\left(-\tfrac{\pi}{3}\right) = 2\text{ cis}\left(\tfrac{5\pi}{6}\right)\times\tfrac{1}{2}\text{ cis}\left(-\tfrac{\pi}{3}\right) = \text{cis}\left(\tfrac{5\pi}{6}-\tfrac{\pi}{3}\right) = \text{cis}\left(\tfrac{3\pi}{6}\right) = \text{cis}\left(\tfrac{\pi}{2}\right) = i$$

c



d When z was multiplied by $\frac{1}{2}$ cis $\left(-\frac{\pi}{3}\right)$, its modulus was halved, and it was rotated clockwise through $\frac{\pi}{3}$ about the origin.

3 **a** Consider $$\cos\left(\tfrac{\pi}{12}\right)+i\sin\left(\tfrac{\pi}{12}\right) = \text{cis}\left(\tfrac{\pi}{12}\right) = \text{cis}\left(\tfrac{4\pi}{12}-\tfrac{3\pi}{12}\right) = \text{cis}\left(\tfrac{\pi}{3}-\tfrac{\pi}{4}\right) = \text{cis}\left(\tfrac{\pi}{3}\right)\times\text{cis}\left(-\tfrac{\pi}{4}\right)$$
$$= \left(\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right)\left(\tfrac{1}{\sqrt{2}}-\tfrac{1}{\sqrt{2}}i\right) = \left(\tfrac{1}{2\sqrt{2}}+\tfrac{\sqrt{3}}{2\sqrt{2}}\right)+i\left(\tfrac{\sqrt{3}}{2\sqrt{2}}-\tfrac{1}{2\sqrt{2}}\right)$$

Equating real parts: $\cos\left(\frac{\pi}{12}\right) = \dfrac{1+\sqrt{3}}{2\sqrt{2}} = \dfrac{\sqrt{2}+\sqrt{6}}{4}$

Equating imaginary parts: $\sin\left(\frac{\pi}{12}\right) = \dfrac{\sqrt{3}-1}{2\sqrt{2}} = \dfrac{\sqrt{6}-\sqrt{2}}{4}$

b Consider $$\cos\left(\tfrac{11\pi}{12}\right)+i\sin\left(\tfrac{11\pi}{12}\right) = \text{cis}\left(\tfrac{11\pi}{12}\right) = \text{cis}\left(\tfrac{3\pi}{12}+\tfrac{8\pi}{12}\right) = \text{cis}\left(\tfrac{\pi}{4}+\tfrac{2\pi}{3}\right) = \text{cis}\left(\tfrac{\pi}{4}\right)\times\text{cis}\left(\tfrac{2\pi}{3}\right)$$
$$= \left(\tfrac{1}{\sqrt{2}}+\tfrac{1}{\sqrt{2}}i\right)\left(-\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right) = \left(-\tfrac{1}{2\sqrt{2}}-\tfrac{\sqrt{3}}{2\sqrt{2}}\right)+i\left(\tfrac{\sqrt{3}}{2\sqrt{2}}-\tfrac{1}{2\sqrt{2}}\right)$$

Equating real parts: $\cos\left(\frac{11\pi}{12}\right) = \dfrac{-1-\sqrt{3}}{2\sqrt{2}} = \dfrac{-\sqrt{2}-\sqrt{6}}{4}$

Equating imaginary parts: $\sin\left(\frac{11\pi}{12}\right) = \dfrac{\sqrt{3}-1}{2\sqrt{2}} = \dfrac{\sqrt{6}-\sqrt{2}}{4}$

4 P_n is "$\arg(z^n) = n\arg(z)$", $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$ then $\text{LHS} = \arg(z^1)$, $\text{RHS} = 1 \times \arg(z)$ $\quad\therefore\ P_1$ is true.

$\qquad\qquad = \arg(z) \qquad\qquad = \arg(z)$

(2) If P_k is true then:

$$\begin{aligned} \arg(z^k) &= k\arg(z) \\ \therefore\ \arg(z^{k+1}) &= \arg(z^k z^1) \\ &= \arg(z^k) + \arg(z) \quad \{\text{since } \arg(zw) = \arg(z) + \arg(w)\} \\ &= k\arg(z) + \arg(z) \quad \{P_k\} \\ &= (k+1)\arg(z) \end{aligned}$$

$\therefore$ P_{k+1} is true whenever P_k is true, and P_1 is true.

$\therefore$ P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

5 Let $z = R \text{ cis } \theta$ and $w = r \text{ cis } \phi$, $w \neq 0$

$$\frac{z}{w} = \frac{R \text{ cis } \theta}{r \text{ cis } \phi} = \frac{R}{r} \text{ cis}\,(\theta - \phi)$$

$$\therefore \left|\frac{z}{w}\right| = \frac{R}{r} = \frac{|z|}{|w|}$$

and $\arg\left(\dfrac{z}{w}\right) = \theta - \phi = \arg z - \arg w$ if $w \neq 0$

6 **a**

$$\begin{aligned} z &= 3 \text{ cis } \theta \\ \therefore\ -z &= -1 \times 3 \text{ cis } \theta \\ &= \text{cis}\,(\pi) \times 3 \text{ cis } \theta \\ &= 3 \text{ cis}\,(\theta + \pi) \end{aligned}$$

$\therefore$ $|-z| = 3$, but θ is acute, so $\theta + \pi \notin\ ]-\pi, \pi]$

$$\begin{aligned} \therefore\ \arg(-z) &= (\theta + \pi) - 2\pi \\ &= \theta - \pi \end{aligned}$$

b $z^* = 3 \text{ cis}\,(-\theta)$

$\therefore$ $|z^*| = 3$ and $\arg z^* = -\theta$

c

$$\begin{aligned} iz &= \text{cis}\left(\tfrac{\pi}{2}\right) \times 3 \text{ cis } \theta \\ &= 3 \text{ cis}\left(\tfrac{\pi}{2} + \theta\right) \end{aligned}$$

$\therefore$ $|iz| = 3$ and $\arg(iz) = \theta + \frac{\pi}{2}$

d

$$\begin{aligned} (1+i)z &= \sqrt{2} \text{ cis}\left(\tfrac{\pi}{4}\right) \times 3 \text{ cis } \theta \\ &= 3\sqrt{2} \text{ cis}\left(\theta + \tfrac{\pi}{4}\right) \end{aligned}$$

$\therefore$ $|(1+i)\,z| = 3\sqrt{2}$ and $\arg\,[(1+i)\,z] = \theta + \frac{\pi}{4}$

e

$$\begin{aligned} \frac{z}{i} &= \frac{3 \text{ cis } \theta}{\text{cis}\left(\frac{\pi}{2}\right)} \\ &= 3 \text{ cis}\left(\theta - \tfrac{\pi}{2}\right) \end{aligned}$$

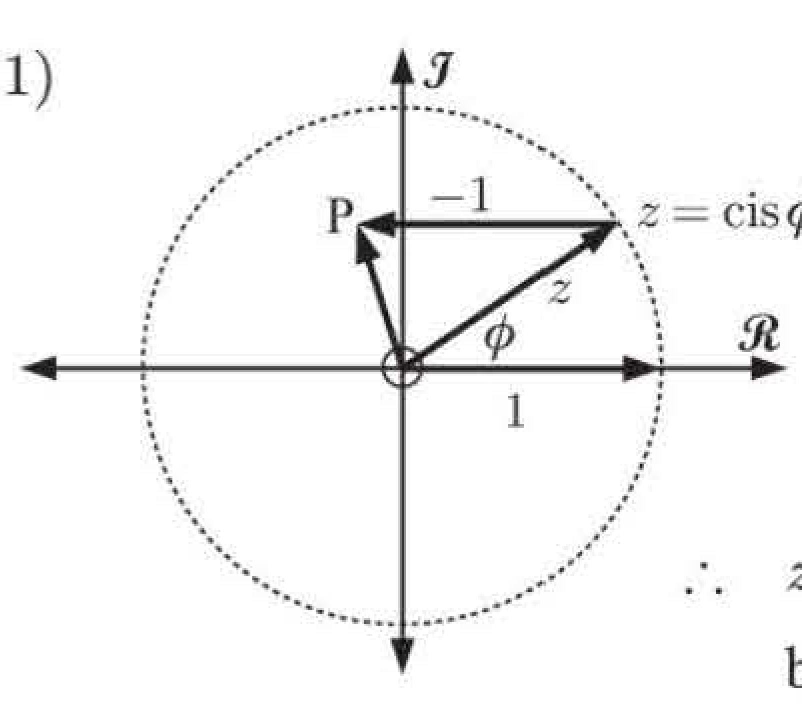
$\therefore$ $\left|\dfrac{z}{i}\right| = 3$ and $\arg\left(\dfrac{z}{i}\right) = \theta - \frac{\pi}{2}$

f

$$\begin{aligned} \frac{z}{1-i} &= \frac{3 \text{ cis } \theta}{\sqrt{2} \text{ cis}\left(-\frac{\pi}{4}\right)} \\ &= \tfrac{3}{\sqrt{2}} \text{ cis}\left(\theta - \left(-\tfrac{\pi}{4}\right)\right) \\ &= \tfrac{3}{\sqrt{2}} \text{ cis}\left(\theta + \tfrac{\pi}{4}\right) \end{aligned}$$

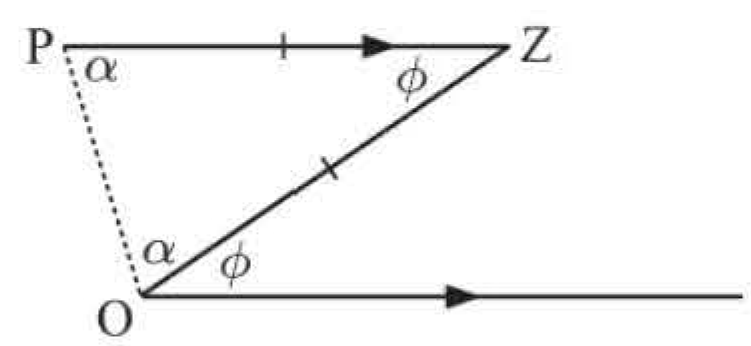
$\therefore$ $\left|\dfrac{z}{1-i}\right| = \frac{3}{\sqrt{2}}$ and $\arg\left(\dfrac{z}{1-i}\right) = \theta + \frac{\pi}{4}$

7 **a** $z - 1 = z + (-1)$



$\therefore$ $z - 1$ is represented by the vector $\overrightarrow{\text{OP}}$

Considering the $\triangle$OZP



$\widehat{PZO} = \phi$ {alternate angles}
$OZ = ZP = 1$
$\therefore$ $\triangle$OPZ is isosceles
$\therefore$ $\widehat{POZ} = \widehat{OPZ} = \alpha$

$\therefore\ 2\alpha + \phi = \pi$ and so $\alpha = \dfrac{\pi - \phi}{2}$

$$\therefore\ \arg(z-1) = \frac{\pi-\phi}{2} + \phi$$
$$= \tfrac{\pi}{2} - \tfrac{\phi}{2} + \phi$$
$$= \tfrac{\pi}{2} + \tfrac{\phi}{2} \quad .... (*)$$

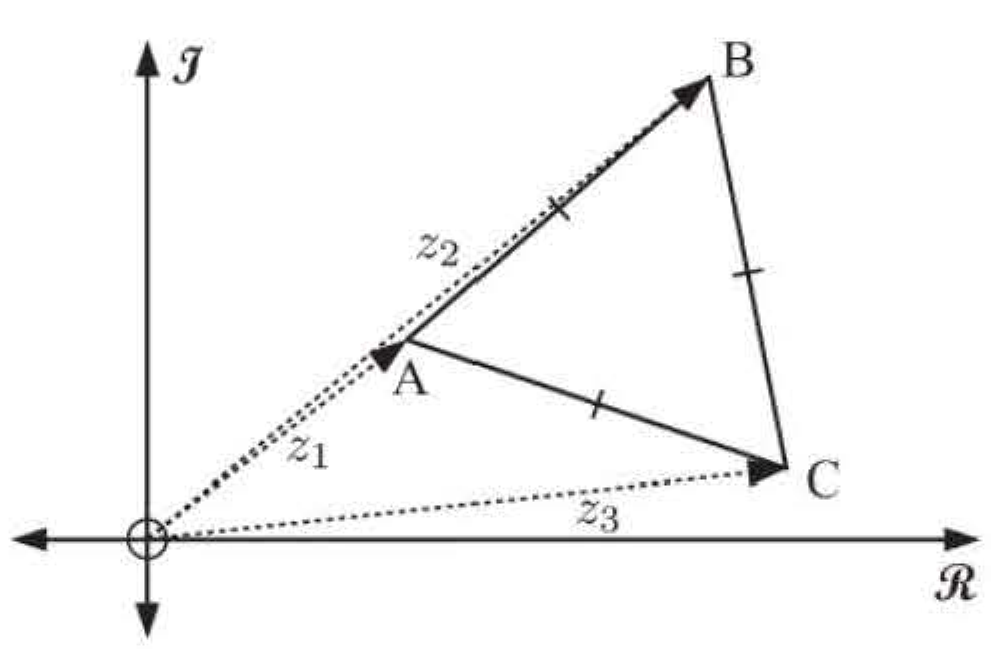
Using the cosine rule in $\triangle$OZP:

$$OP^2 = 1^2 + 1^2 - 2(1)(1)\cos\phi$$
$$\therefore\ OP^2 = 2 - 2\cos\phi$$
$$\therefore\ OP^2 = 2 - 2\left(1 - 2\sin^2\left(\tfrac{\phi}{2}\right)\right)$$
$$\therefore\ OP^2 = 2 - 2 + 4\sin^2\left(\tfrac{\phi}{2}\right)$$
$$\therefore\ OP^2 = 4\sin^2\left(\tfrac{\phi}{2}\right)$$
$$\therefore\ |z-1| = 2\sin\left(\tfrac{\phi}{2}\right) \quad .... (**)$$

b $z - 1 = 2\sin\left(\frac{\phi}{2}\right)\operatorname{cis}\left(\frac{\pi}{2} + \frac{\phi}{2}\right)$ {using $(*)$ and $(**)$}

c $(z-1)^* = 2\sin\left(\frac{\phi}{2}\right)\operatorname{cis}\left(-\frac{\pi}{2} - \frac{\phi}{2}\right)$

8 **a** Now $z_2 - z_1 = \overrightarrow{AB}$
$z_3 - z_2 = \overrightarrow{BC}$

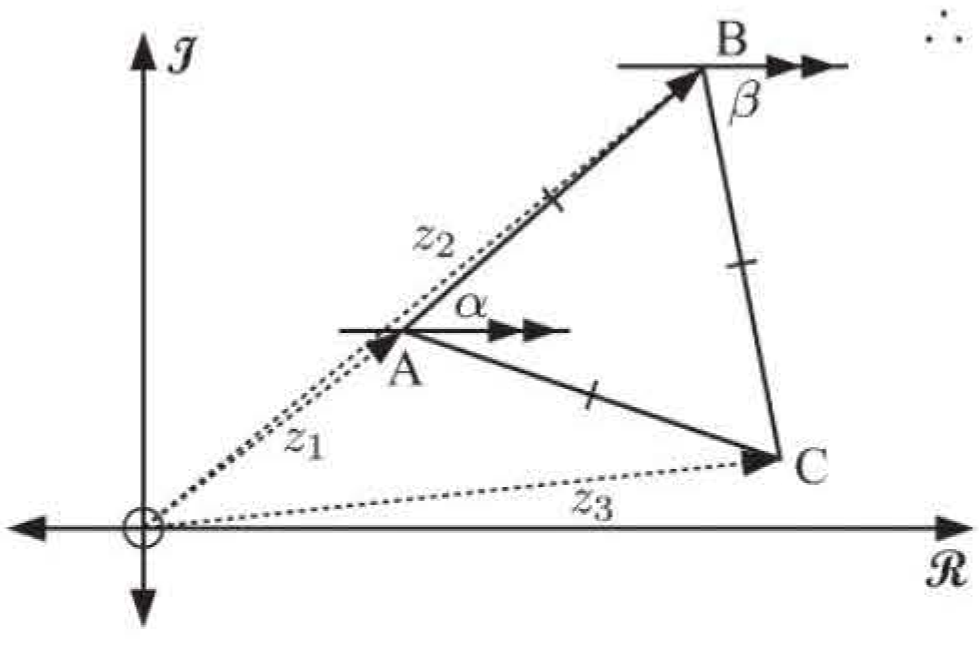


b
$$\left|\frac{z_2 - z_1}{z_3 - z_2}\right| = \frac{|z_2 - z_1|}{|z_3 - z_2|} = \frac{|\overrightarrow{AB}|}{|\overrightarrow{BC}|}$$

But $\triangle$ABC is equilateral

$$\therefore\ |\overrightarrow{AB}| = |\overrightarrow{BC}|$$
$$\therefore\ \left|\frac{z_2 - z_1}{z_3 - z_2}\right| = 1$$

c



$z_2 - z_1 = \overrightarrow{AB}$ and $z_3 - z_2 = \overrightarrow{BC}$ {from **a**}

$\therefore$ let $\arg(z_2 - z_1) = \alpha$
and $\arg(z_3 - z_2) = -\beta$ as shown

$$\therefore\ \arg\left(\frac{z_2 - z_1}{z_3 - z_2}\right) = \arg(z_2 - z_1) - \arg(z_3 - z_2)$$
$$= \alpha - (-\beta)$$
$$= \alpha + \beta$$

But $\widehat{ABC} = \frac{\pi}{3}$ since the triangle is equilateral

$\therefore\ \alpha + \beta + \frac{\pi}{3} = \pi$ {co-interior angles}

$$\therefore\ \alpha + \beta = \tfrac{2\pi}{3}$$
$$\therefore\ \arg\left(\frac{z_2 - z_1}{z_3 - z_2}\right) = \tfrac{2\pi}{3}$$

d From **b** and **c**, $\dfrac{z_2 - z_1}{z_3 - z_2} = 1\operatorname{cis}\left(\frac{2\pi}{3}\right)$

$$\therefore\ \left(\frac{z_2 - z_1}{z_3 - z_2}\right)^3 = \left(\operatorname{cis}\left(\tfrac{2\pi}{3}\right)\right)^3$$
$$= \operatorname{cis}\left(\tfrac{2\pi}{3} + \tfrac{2\pi}{3} + \tfrac{2\pi}{3}\right)$$
$$= \operatorname{cis} 2\pi$$
$$= 1$$

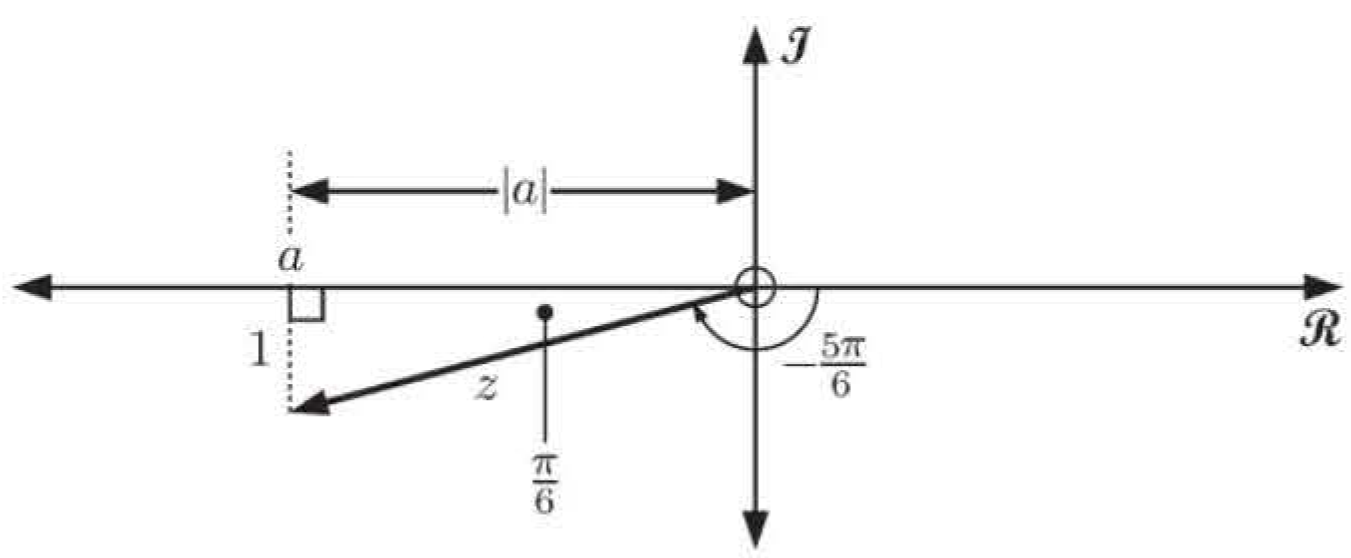
9 $\tan\frac{\pi}{6} = \dfrac{1}{|a|}$

$\therefore\ |a| = \dfrac{1}{\tan\frac{\pi}{6}}$

$\therefore\ a = \pm\sqrt{3}$

but from the graph, a is negative

$\therefore\ a = -\sqrt{3}$



EXERCISE 16C.4

1 Using technology

a $\sqrt{3}\,\mathrm{cis}(2.5187)$
$= \sqrt{3}(\cos(2.5187) + i\sin(2.5187))$
$\approx -1.41 + 1.01i$

b $\sqrt{11}\,\mathrm{cis}\left(-\frac{3\pi}{8}\right)$
$= \sqrt{11}\left(\cos\left(-\frac{3\pi}{8}\right) + i\sin\left(-\frac{3\pi}{8}\right)\right)$
$\approx 1.27 - 3.06i$

c $2.836\,49\ \mathrm{cis}(-2.684\,32) = 2.836\,49(\cos(-2.684\,32) + i\sin(-2.684\,32))$
$\approx -2.55 - 1.25i$

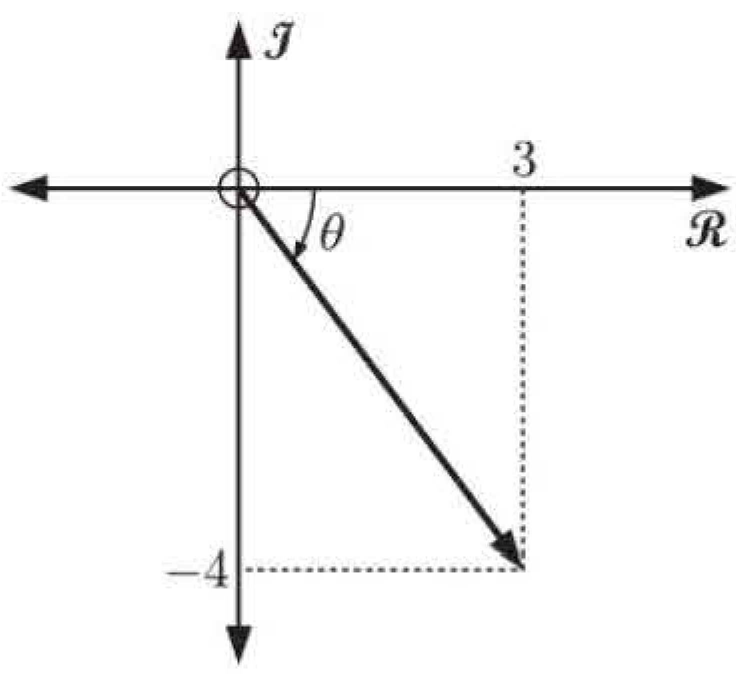
2 Using technology

a $3 - 4i$ has $r = \sqrt{3^2 + (-4)^2}$
$= \sqrt{9+16} = \sqrt{25} = 5$

$\therefore\ \cos\theta = \frac{3}{5}$ and $\sin\theta = -\frac{4}{5}$

$\therefore\ \theta = \arcsin(-\frac{4}{5}) \approx -0.927\,295\,218$

$\therefore\ 3 - 4i \approx 5\ \mathrm{cis}(-0.927)$

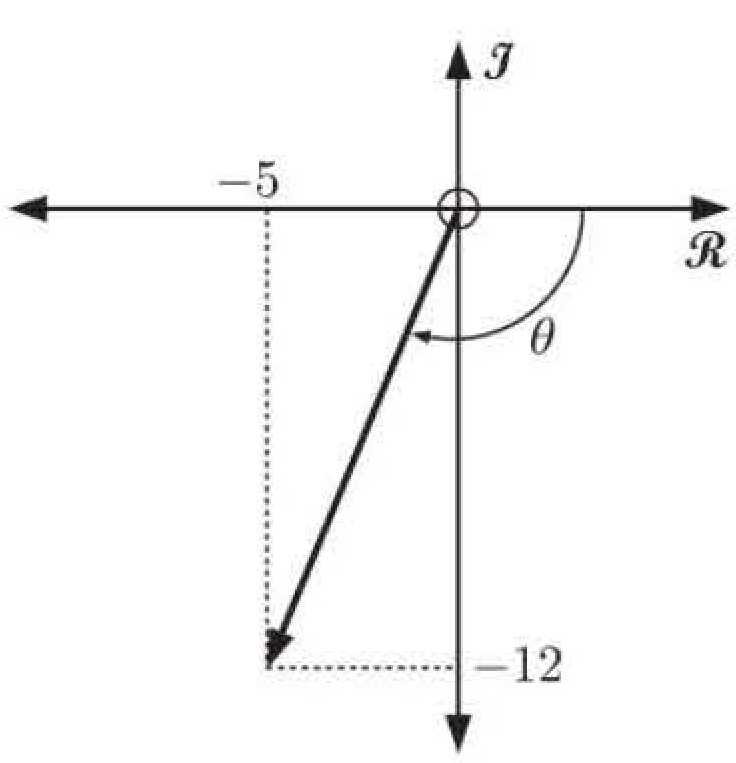


b $-5 - 12i$ has $r = \sqrt{(-5)^2 + (-12)^2}$
$= \sqrt{25+144} = \sqrt{169} = 13$

$\therefore\ \cos\theta = -\frac{5}{13}$ and $\sin\theta = -\frac{12}{13}$

$\therefore\ \theta = -\arccos(-\frac{5}{13})$ {quadrant 4}
$\approx -1.965\,587\,446$

$\therefore\ -5 - 12i \approx 13\ \mathrm{cis}(-1.97)$



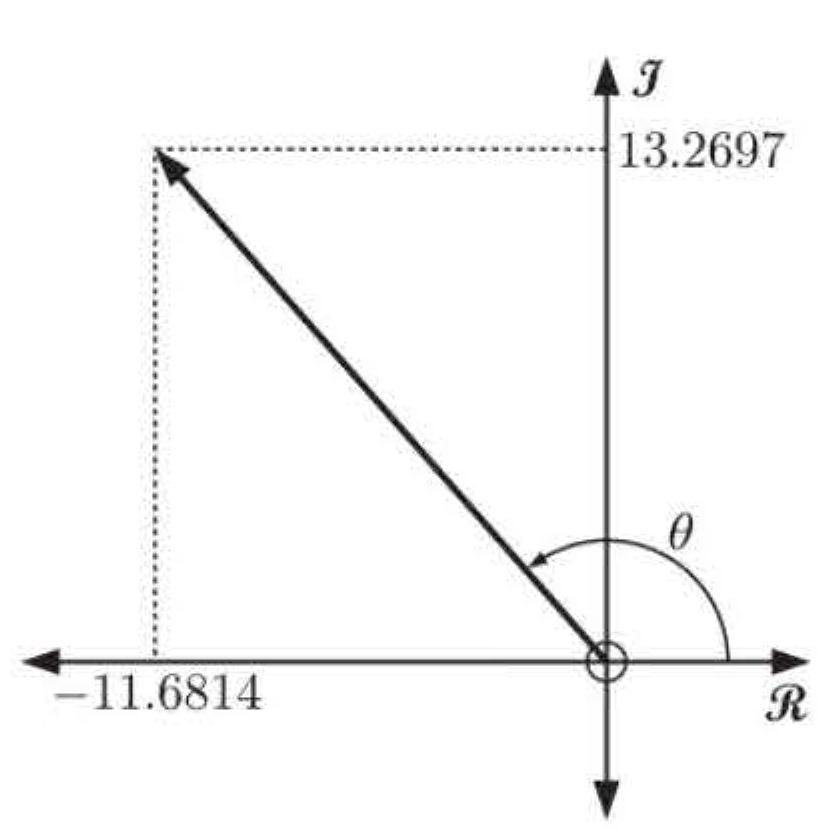
c $-11.6814 + 13.2697i$

has $r = \sqrt{(-11.6814)^2 + 13.2697^2}$
$\approx \sqrt{136.455\,106 + 176.084\,938\,1}$
$\approx \sqrt{312.540\,044\,1}$
$\approx 17.678\,802\,11$

$\therefore\ \cos\theta = -\dfrac{11.6814}{17.678\,802\,11}$ and $\sin\theta = \dfrac{13.2697}{17.678\,802\,11}$

$\therefore\ \theta = \pi - \arcsin\left(\dfrac{13.2697}{17.678\,802\,11}\right)$ {quadrant 2}
$\approx 2.292\,623\,752$

$\therefore\ -11.6814 + 13.2697i \approx 17.7\ \mathrm{cis}(2.29)$



3 **a** $3 \text{ cis}\left(\frac{\pi}{4}\right) + \text{cis}\left(-\frac{3\pi}{4}\right)$

$= 3\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right) + \left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right)$

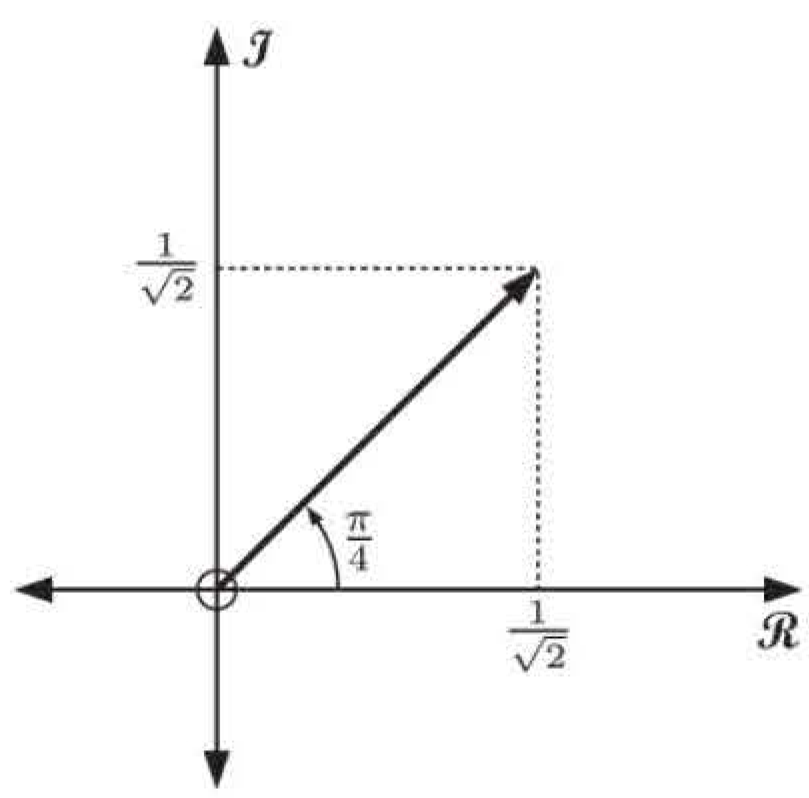
$= 3\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) + \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$

$= \frac{3}{\sqrt{2}} + \frac{3}{\sqrt{2}}i - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$

$= \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}}i$

$= 2\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$

$= 2 \text{ cis}\left(\frac{\pi}{4}\right)$



b $2 \text{ cis}\left(\frac{2\pi}{3}\right) + 5 \text{ cis}\left(-\frac{2\pi}{3}\right)$

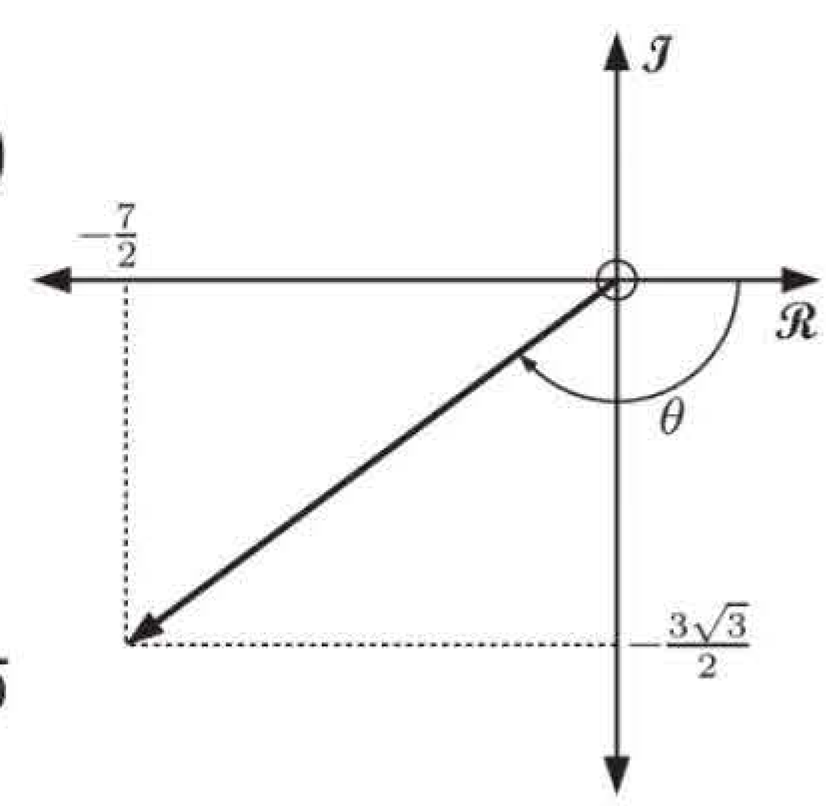
$= 2\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right) + 5\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right)$

$= 2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) + 5\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$

$= -1 + \sqrt{3}i - \frac{5}{2} - \frac{5\sqrt{3}}{2}i$

$= -\frac{7}{2} - \frac{3\sqrt{3}}{2}i$

Now $r = \sqrt{\left(-\frac{7}{2}\right)^2 + \left(-\frac{3\sqrt{3}}{2}\right)^2} = \sqrt{\frac{49+27}{4}} = \sqrt{\frac{76}{4}} = \sqrt{19}$



$\therefore \cos\theta = -\frac{7}{2\sqrt{19}}$ and $\sin\theta = -\frac{3\sqrt{3}}{2\sqrt{19}}$

$\therefore \theta = -\arccos\left(-\frac{7}{2\sqrt{19}}\right)$ {quadrant 4}

$\approx -2.503\,032\,957$

$\therefore 2 \text{ cis}\left(\frac{2\pi}{3}\right) + 5 \text{ cis}\left(-\frac{2\pi}{3}\right) \approx \sqrt{19} \text{ cis}(-2.50)$

4 **a** Sum of roots

$= 2 \text{ cis}\left(\frac{2\pi}{3}\right) + 2 \text{ cis}\left(\frac{4\pi}{3}\right)$

$= 2\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right) + 2\left(\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right)$

$= 2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + 2\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

$= -1 + \sqrt{3}i - 1 - \sqrt{3}i$

$= -2$

Product of roots

$= 2 \text{ cis}\left(\frac{2\pi}{3}\right) \times 2 \text{ cis}\left(\frac{4\pi}{3}\right)$

$= 4 \text{ cis}\left(\frac{2\pi}{3} + \frac{4\pi}{3}\right)$

$= 4 \text{ cis}(2\pi)$

$= 4(1 + 0i)$

$= 4$

$\therefore$ the equations are

$a(x^2 - (-2)x + 4) = 0$

$\therefore a(x^2 + 2x + 4) = 0, \; a \neq 0, \; a \in \mathbb{R}.$

b Sum of roots

$= \sqrt{2} \text{ cis}\left(\frac{\pi}{4}\right) + \sqrt{2} \text{ cis}\left(-\frac{\pi}{4}\right)$

$= \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right) + \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$

$= \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) + \sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$

$= 1 + i + 1 - i$

$= 2$

Product of roots

$= \sqrt{2}\text{ cis}\left(\frac{\pi}{4}\right) \times \sqrt{2}\text{ cis}\left(-\frac{\pi}{4}\right)$

$= 2\text{ cis}\left(\frac{\pi}{4} - \frac{\pi}{4}\right)$

$= 2\text{ cis}(0)$

$= 2(1 + 0i)$

$= 2$

$\therefore$ the equations are

$a(x^2 - 2x + 2) = 0,\ a \neq 0,\ a \in \mathbb{R}.$

EXERCISE 16D

1 **a** $e^{i\pi} = \cos\pi + i\sin\pi$
$= -1 + i(0)$
$= -1$

b $e^{i\frac{\pi}{3}} = \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)$
$= \frac{1}{2} + \frac{\sqrt{3}}{2}i$

c $e^{-i\frac{\pi}{2}} = \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)$
$= 0 + i(-1)$
$= -i$

2 **a** $\text{cis }\theta\text{ cis }\phi = e^{i\theta}e^{i\phi}$
$= e^{i(\theta+\phi)}$
$= \text{cis}(\theta + \phi)$

b $\dfrac{\text{cis }\theta}{\text{cis }\phi} = \dfrac{e^{i\theta}}{e^{i\phi}}$
$= e^{i(\theta-\phi)}$
$= \text{cis}(\theta - \phi)$

3 **a** $\sqrt{z} = (\text{cis }\theta)^{\frac{1}{2}}$
$= (e^{i\theta})^{\frac{1}{2}}$
$= e^{\frac{i\theta}{2}}$
$= \text{cis}\left(\frac{\theta}{2}\right)$
$\therefore\ \arg(\sqrt{z}) = \frac{\theta}{2}$

b $iz = \text{cis}\left(\frac{\pi}{2}\right)\text{ cis }\theta$
$= \text{cis}\left(\frac{\pi}{2} + \theta\right)$
$\therefore\ \arg(iz) = \frac{\pi}{2} + \theta$

c $-iz^2 = \text{cis}\left(-\frac{\pi}{2}\right)\text{cis}^2\theta$
$= \text{cis}\left(-\frac{\pi}{2}\right)\text{ cis }\theta\text{ cis }\theta$
$= \text{cis}\left(-\frac{\pi}{2} + \theta + \theta\right)$
$= \text{cis}\left(2\theta - \frac{\pi}{2}\right)$
$\therefore\ \arg(-iz^2) = 2\theta - \frac{\pi}{2}$

d $\dfrac{i}{z} = \dfrac{\text{cis}\left(\frac{\pi}{2}\right)}{\text{cis }\theta}$
$= \text{cis}\left(\frac{\pi}{2} - \theta\right)$
$\therefore\ \arg\left(\dfrac{i}{z}\right) = \frac{\pi}{2} - \theta$

4 **a** $e^{i} = \cos(1) + i\sin(1)$ $\{\theta = 1\}$
$\approx 0.540 + 0.841i$

b $3^i = (e^{\ln 3})^i$
$= e^{i\ln 3}$
$= \cos(\ln 3) + i\sin(\ln 3)$
$\approx 0.455 + 0.891i$

c $i = 0 + 1i$
$= \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)$
$= e^{i\frac{\pi}{2}}$

$\therefore\ i^i = (e^{i\frac{\pi}{2}})^i$
$= e^{i^2\frac{\pi}{2}}$
$= e^{-\frac{\pi}{2}}$

d $i^i = e^{-\frac{\pi}{2}}$ {from **c**}
$\therefore\ (i^i)^i = (e^{-\frac{\pi}{2}})^i$
$= e^{-i\frac{\pi}{2}}$

$\therefore\ \left((i^i)^i\right)^2 = (e^{-i\frac{\pi}{2}})^2$
$= e^{-i\pi}$
$= \cos(-\pi) + i\sin(-\pi)$
$= -1 + i(0)$
$= -1$

EXERCISE 16E

1 $(|z| \text{ cis } \theta)^n = (|z|\, e^{i\theta})^n$ {Euler form}
$= |z|^n\, (e^{i\theta})^n$
$= |z|^n\, e^{in\theta}$
$= |z|^n \text{ cis } n\theta$ {polar form}

2 **a** $\left(\sqrt{2} \text{ cis}\left(\frac{\pi}{5}\right)\right)^{10}$
$= (\sqrt{2})^{10} \text{ cis}\left(\frac{10\pi}{5}\right)$
$= 2^5 \text{ cis } 2\pi$
$= 2^5$
$= 32$

b $\left(\text{cis}\left(\frac{\pi}{12}\right)\right)^{36}$
$= \text{cis}\left(\frac{36\pi}{12}\right)$
$= \text{cis } 3\pi$
$= -1$

c $\left(\sqrt{2} \text{ cis}\left(\frac{\pi}{8}\right)\right)^{12}$
$= \left(\sqrt{2}\right)^{12} \text{ cis}\left(\frac{12\pi}{8}\right)$
$= 2^6 \text{ cis}\left(\frac{3\pi}{2}\right)$
$= 64(-i)$
$= -64i$

d $\sqrt{5 \text{ cis}\left(\frac{\pi}{7}\right)}$
$= \left(5 \text{ cis}\left(\frac{\pi}{7}\right)\right)^{\frac{1}{2}}$
$= \sqrt{5} \text{ cis}\left(\frac{1}{2} \times \frac{\pi}{7}\right)$
$= \sqrt{5} \text{ cis}\left(\frac{\pi}{14}\right)$
(or $\approx 2.180 + 0.498i$)

e $\sqrt[3]{8 \text{ cis}\left(\frac{\pi}{2}\right)}$
$= \left(8 \text{ cis}\left(\frac{\pi}{2}\right)\right)^{\frac{1}{3}}$
$= 2 \text{ cis}\left(\frac{\pi}{6}\right)$
$= 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$
$= \sqrt{3} + i$

f $\left(8 \text{ cis}\left(\frac{\pi}{5}\right)\right)^{\frac{5}{3}}$
$= 8^{\frac{5}{3}} \text{ cis}\left(\frac{5}{3} \times \frac{\pi}{5}\right)$
$= 2^5 \text{ cis}\left(\frac{\pi}{3}\right)$
$= 32\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$
$= 16 + 16\sqrt{3}i$

3 **a** $(1+i)^{15}$
$= \left(\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)\right)^{15}$
$= \left(\sqrt{2} \text{ cis}\left(\frac{\pi}{4}\right)\right)^{15}$
$= \sqrt{2}^{14}\sqrt{2} \text{ cis}\left(\frac{15\pi}{4}\right)$
$= 2^7\sqrt{2} \text{ cis}\left(\frac{7\pi}{4}\right)$
$= 128\sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$
$= 128 - 128i$

b $(1 - i\sqrt{3})^{11}$
$= \left(2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right)^{11}$
$= \left(2 \text{ cis}\left(-\frac{\pi}{3}\right)\right)^{11}$
$= 2^{11} \text{ cis}\left(-\frac{11\pi}{3}\right)$
$= 2048 \text{ cis}\left(\frac{\pi}{3}\right)$
$= 2048\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$
$= 1024 + 1024\sqrt{3}i$

c $(\sqrt{2} - i\sqrt{2})^{-19}$
$= \left(2\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)\right)^{-19}$
$= \left(2 \text{ cis}\left(-\frac{\pi}{4}\right)\right)^{-19}$
$= 2^{-19} \text{ cis}\left(\frac{19\pi}{4}\right)$
$= 2^{-19} \text{ cis}\left(\frac{3\pi}{4}\right)$
$= 2^{-19}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$
$= \frac{1}{524\,288}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$

d $(-1+i)^{-11}$
$= \left(\sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)\right)^{-11}$
$= \left(\sqrt{2} \text{ cis}\left(\frac{3\pi}{4}\right)\right)^{-11}$
$= (\sqrt{2})^{-11} \text{ cis}\left(-\frac{33\pi}{4}\right)$
$= (\sqrt{2})^{-11} \text{ cis}\left(-\frac{\pi}{4}\right)$
$= (\sqrt{2})^{-11}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$
$= (\sqrt{2})^{-12}(1-i)$
$= \frac{1}{64}(1-i)$

e $(\sqrt{3}-i)^{\frac{1}{2}}$
$= \left(2\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)\right)^{\frac{1}{2}}$
$= \left(2 \text{ cis}\left(-\frac{\pi}{6}\right)\right)^{\frac{1}{2}}$
$= 2^{\frac{1}{2}} \text{ cis}\left(-\frac{\pi}{12}\right)$
$= \sqrt{2} \text{ cis}\left(-\frac{\pi}{12}\right)$
$= \sqrt{2}\cos\left(-\frac{\pi}{12}\right) + i\sqrt{2}\sin\left(-\frac{\pi}{12}\right)$

f $(2+2i\sqrt{3})^{-\frac{5}{2}}$
$= \left(4\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\right)^{-\frac{5}{2}}$
$= \left(4 \text{ cis}\left(\frac{\pi}{3}\right)\right)^{-\frac{5}{2}}$
$= 2^{-\frac{5}{2}\times 2} \text{ cis}\left(-\frac{5}{2}\times\frac{\pi}{3}\right)$
$= 2^{-5} \text{ cis}\left(-\frac{5\pi}{6}\right)$
$= \frac{1}{32}\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)$
$= -\frac{1}{64}\left(\sqrt{3}+i\right)$

4 **a** $z = |z| \text{ cis } \theta$
$\sqrt{z} = (|z| \text{ cis } \theta)^{\frac{1}{2}}$
$\sqrt{z} = |z|^{\frac{1}{2}} \text{ cis}\left(\frac{\theta}{2}\right)$ {De Moivre}

b $-\frac{\pi}{2} < \phi \leqslant \frac{\pi}{2}$

c True: $\cos\phi \geqslant 0$ for all $-\frac{\pi}{2} < \phi \leqslant \frac{\pi}{2}$

5 $\text{cis}(-\theta) = \cos(-\theta) + i\sin(-\theta)$
$= \cos\theta - i\sin\theta$

$\therefore$ $(\cos\theta - i\sin\theta)^{-3} = [\text{cis}(-\theta)]^{-3}$
$= \text{cis } 3\theta$ {De Moivre}

6 **a** $z = 1+i = \sqrt{2} \text{ cis}\left(\frac{\pi}{4}\right)$
$\therefore$ $z^n = (\sqrt{2})^n \text{ cis}\left(\frac{n\pi}{4}\right)$ {De Moivre}

b **i** If z^n is real then $\sin\left(\frac{n\pi}{4}\right) = 0$
$\therefore$ $\frac{n\pi}{4} = 0 + k\pi$
$\therefore$ $n = 4k$ where k is any integer

ii If z^n is purely imaginary then $\cos\left(\frac{n\pi}{4}\right) = 0$
$\therefore$ $\frac{n\pi}{4} = \frac{\pi}{2} + k\pi$
$\therefore$ $n = 2+4k$ where k is any integer

7 **a** $z = 2 \text{ cis } \theta$
$\therefore$ $z^3 = 2^3 \text{ cis } 3\theta$
$\therefore$ $\left|z^3\right| = 8$
and $\arg z^3 = 3\theta$

b $z = 2 \text{ cis } \theta$
$\therefore$ $iz^2 = i(2 \text{ cis } \theta)^2$
$= \text{cis}\left(\frac{\pi}{2}\right)(4 \text{ cis } 2\theta)$
$= 4 \text{ cis}\left(\frac{\pi}{2}+2\theta\right)$
$\therefore$ $\left|iz^2\right| = 4$
and $\arg(iz^2) = \frac{\pi}{2} + 2\theta$

c $z = 2 \text{ cis } \theta$
$\therefore$ $\frac{1}{z} = (2 \text{ cis } \theta)^{-1}$
$= \frac{1}{2} \text{ cis}(-\theta)$
$\therefore$ $\left|\frac{1}{z}\right| = \frac{1}{2}$
and $\arg\left(\frac{1}{z}\right) = -\theta$

d $z = 2 \text{ cis } \theta$
$\therefore$ $-\frac{i}{z^2} = -i \times z^{-2}$
$= \text{cis}\left(-\frac{\pi}{2}\right) \times (2 \text{ cis } \theta)^{-2}$
$= 2^{-2} \text{ cis}\left(-\frac{\pi}{2}\right) \text{cis}(-2\theta)$
$= \frac{1}{4} \text{ cis}\left(-\frac{\pi}{2}-2\theta\right)$
$\therefore$ $\left|-\frac{i}{z^2}\right| = \frac{1}{4}$ and $\arg\left(-\frac{i}{z^2}\right) = -\frac{\pi}{2} - 2\theta$

8 If $z = \text{cis}\,\theta$, then
$$\begin{aligned}\frac{z^2-1}{z^2+1} &= \frac{(\text{cis}\,\theta)^2-1}{(\text{cis}\,\theta)^2+1}\\ &= \frac{\text{cis}\,2\theta-1}{\text{cis}\,2\theta+1} \quad \{\text{De Moivre}\}\\ &= \frac{\cos 2\theta + i\sin 2\theta - 1}{\cos 2\theta + i\sin 2\theta + 1}\\ &= \frac{(\cancel{1} - 2\sin^2\theta) + i\sin 2\theta - \cancel{1}}{(2\cos^2\theta - \cancel{1}) + i\sin 2\theta + \cancel{1}}\\ &= \frac{-2\sin^2\theta + 2i\cos\theta\sin\theta}{2\cos^2\theta + 2i\cos\theta\sin\theta}\\ &= \frac{2\sin\theta(i\cos\theta + i^2\sin\theta)}{2\cos\theta(\cos\theta + i\sin\theta)}\\ &= \frac{i\sin\theta\ \text{cis}\,\theta}{\cos\theta\ \text{cis}\,\theta}\\ &= i\tan\theta\end{aligned}$$

9 **a**
$$\begin{aligned}\cos 3\theta + i\sin 3\theta &= \text{cis}\,3\theta\\ &= (\text{cis}\,\theta)^3 \qquad \{\text{De Moivre's theorem}\}\\ &= (\cos\theta + i\sin\theta)^3\\ &= \cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3\\ &= \left[\cos^3\theta - 3\cos\theta\sin^2\theta\right] + i\left[3\cos^2\theta\sin\theta - \sin^3\theta\right]\end{aligned}$$

i
$$\begin{aligned}\cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta\\ &\quad \{\text{equating real parts}\}\\ &= \cos^3\theta - 3\cos\theta(1-\cos^2\theta)\\ &= \cos^3\theta - 3\cos\theta + 3\cos^3\theta\\ &= 4\cos^3\theta - 3\cos\theta\end{aligned}$$

ii
$$\begin{aligned}\sin 3\theta &= 3\cos^2\theta\sin\theta - \sin^3\theta\\ &\quad \{\text{equating imaginary parts}\}\\ &= 3(1-\sin^2\theta)\sin\theta - \sin^3\theta\\ &= 3\sin\theta - 3\sin^3\theta - \sin^3\theta\\ &= 3\sin\theta - 4\sin^3\theta\end{aligned}$$

b
$$\begin{aligned}\tan 3\theta = \frac{\sin 3\theta}{\cos 3\theta} &= \frac{3\sin\theta - 4\sin^3\theta}{4\cos^3\theta - 3\cos\theta} \qquad \{\text{using } \mathbf{a}\}\\ &= \frac{\sin\theta(3-4\sin^2\theta)}{\cos\theta(4\cos^2\theta - 3)}\\ &= \frac{\sin\theta\left[3(\cos^2\theta+\sin^2\theta) - 4\sin^2\theta\right]}{\cos\theta\left[4\cos^2\theta - 3(\cos^2\theta+\sin^2\theta)\right]} \qquad \{\cos^2\theta+\sin^2\theta=1\}\\ &= \frac{\sin\theta(3\cos^2\theta - \sin^2\theta)}{\cos\theta(\cos^2\theta - 3\sin^2\theta)}\\ &= \frac{3\sin\theta\cos^2\theta - \sin^3\theta}{\cos^3\theta - 3\sin^2\theta\cos\theta}\\ &= \frac{3\dfrac{\sin\theta}{\cos\theta} - \dfrac{\sin^3\theta}{\cos^3\theta}}{1 - 3\dfrac{\sin^2\theta}{\cos^2\theta}} \qquad \{\div \text{ top and bottom by } \cos^3\theta\}\\ &= \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}\end{aligned}$$

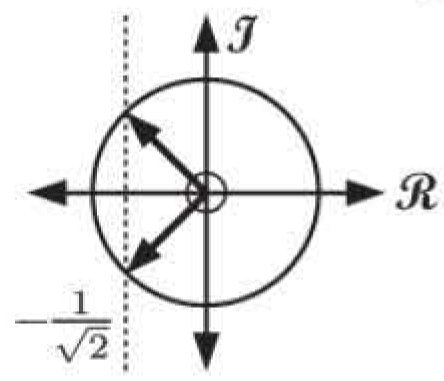
c **i**

$$4x^3 - 3x = -\tfrac{1}{\sqrt{2}}$$

Let $x = \cos\theta$

$$\therefore\ 4\cos^3\theta - 3\cos\theta = -\tfrac{1}{\sqrt{2}}$$

$$\therefore\ \cos 3\theta = -\tfrac{1}{\sqrt{2}}$$



$$3\theta = \tfrac{3\pi}{4}, \tfrac{5\pi}{4}, \tfrac{11\pi}{4}, \tfrac{13\pi}{4}, \tfrac{19\pi}{4}, \tfrac{21\pi}{4}$$

$\{0 \leqslant 3\theta \leqslant 6\pi\}$

$$\therefore\ \theta = \tfrac{\pi}{4}, \tfrac{5\pi}{12}, \tfrac{11\pi}{12}, \tfrac{13\pi}{12}, \tfrac{19\pi}{12}, \tfrac{7\pi}{4}$$

$\{0 \leqslant \theta \leqslant 2\pi\}$

$\therefore\ x = \cos\theta = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \quad \{= \cos\frac{7\pi}{4}\}$

or $\cos\frac{5\pi}{12} \quad \{= \cos\frac{19\pi}{12}\}$

or $\cos\frac{11\pi}{12} \quad \{= \cos\frac{13\pi}{12}\}$

ii

$$x^3 - 3\sqrt{3}x^2 - 3x + \sqrt{3} = 0$$

$$\therefore\ \sqrt{3} - 3\sqrt{3}x^2 = 3x - x^3$$

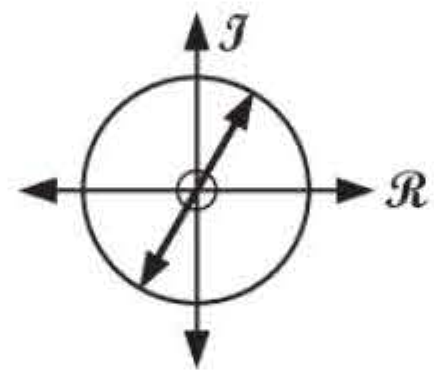
$$\therefore\ \sqrt{3}(1 - 3x^2) = 3x - x^3$$

$$\therefore\ \sqrt{3} = \frac{3x - x^3}{1 - 3x^2}$$

Let $x = \tan\theta$

$$\therefore\ \sqrt{3} = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

$$\therefore\ \tan 3\theta = \sqrt{3}$$



$$3\theta = \tfrac{\pi}{3}, \tfrac{4\pi}{3}, \tfrac{7\pi}{3}, \tfrac{10\pi}{3}, \tfrac{13\pi}{3}, \tfrac{16\pi}{3}$$

$\{0 \leqslant 3\theta \leqslant 6\pi\}$

$$\therefore\ \theta = \tfrac{\pi}{9}, \tfrac{4\pi}{9}, \tfrac{7\pi}{9}, \tfrac{10\pi}{9}, \tfrac{13\pi}{9}, \tfrac{16\pi}{9}$$

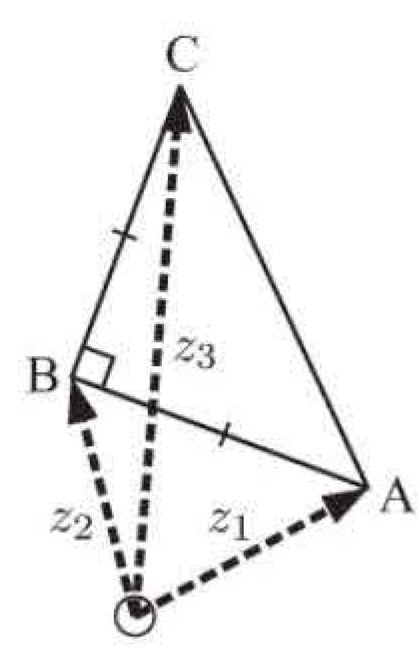
$\{0 \leqslant \theta \leqslant 2\pi\}$

$\therefore\ x = \tan\theta = \tan\frac{\pi}{9} \quad \{= \tan\frac{10\pi}{9}\}$

or $\tan\frac{4\pi}{9} \quad \{= \tan\frac{13\pi}{9}\}$

or $\tan\frac{7\pi}{9} \quad \{= \tan\frac{16\pi}{9}\}$

10 **a**



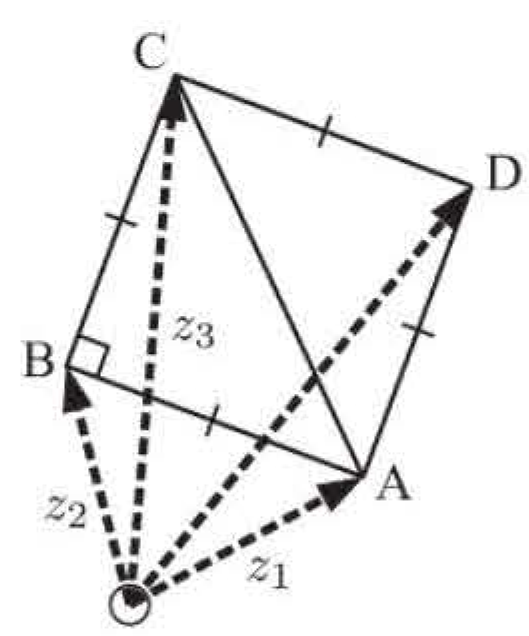
$$\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC} = -z_2 + z_3 = z_3 - z_2$$
$$\overrightarrow{BA} = \overrightarrow{BO} + \overrightarrow{OA} = -z_2 + z_1 = z_1 - z_2$$

Suppose $\overrightarrow{BA}$ has length r and argument θ.

$\therefore\ \overrightarrow{BA} = r \text{ cis } \theta$ and $\overrightarrow{BC} = r \text{ cis } (\theta + \frac{\pi}{2})$

$$\begin{aligned}\therefore\ -(z_3 - z_2)^2 &= -1 \times (\overrightarrow{BC})^2 \\ &= \text{cis } \pi \times r^2 \text{ cis } (2\theta + \pi) \\ &= r^2 \text{ cis } (2\theta + 2\pi) \\ &= r^2 \text{ cis } 2\theta \\ &= (r \text{ cis } \theta)^2 \\ &= (z_1 - z_2)^2\end{aligned}$$

b



$\overrightarrow{CD} = \overrightarrow{BA} = z_1 - z_2$

Now $\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD}$

$\therefore\ \overrightarrow{OD} = z_3 + z_1 - z_2$

$\therefore\ z_3 + z_1 - z_2$ represents D

11

$$\begin{aligned}\cos 4\theta + i\sin 4\theta &= \text{cis } 4\theta \\ &= (\text{cis } \theta)^4 \qquad \{\text{De Moivre's theorem}\} \\ &= (\cos\theta + i\sin\theta)^4 \\ &= \cos^4\theta + 4\cos^3\theta(i\sin\theta) + 6\cos^2\theta(i\sin\theta)^2 + 4\cos\theta(i\sin\theta)^3 + (i\sin\theta)^4 \\ &= \cos^4\theta + [4\cos^3\theta\sin\theta]i - 6\cos^2\theta\sin^2\theta - [4\cos\theta\sin^3\theta]i + \sin^4\theta \\ &= [\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta] + [4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta]i\end{aligned}$$

a Equating real parts gives $\cos 4\theta = \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$
$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta$
$= 8\cos^4\theta - 8\cos^2\theta + 1$

b Equating imaginary parts gives $\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$

12 **a** **i** If $z = \text{cis}\,\theta$, then
$$\begin{aligned} z^n + \frac{1}{z^n} &= z^n + z^{-n} \\ &= (\text{cis}\,\theta)^n + (\text{cis}\,\theta)^{-n} \\ &= \text{cis}\,(n\theta) + \text{cis}\,(-n\theta) \qquad \{\text{De Moivre}\} \\ &= (\cos n\theta + i\sin n\theta) + (\cos(-n\theta) + i\sin(-n\theta)) \\ &= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta \\ &= 2\cos n\theta \end{aligned}$$

ii In **a i** if we let $n = 1$ we get $z + \dfrac{1}{z} = 2\cos\theta$.

iii
$$\begin{aligned} \left(z+\frac{1}{z}\right)^3 &= z^3 + 3z^2\left(\frac{1}{z}\right) + 3z\left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 \\ &= z^3 + 3z + \frac{3}{z} + \frac{1}{z^3} \end{aligned}$$

iv From **a iii**, $\left(z+\dfrac{1}{z}\right)^3 = \left(z^3 + \dfrac{1}{z^3}\right) + 3\left(z + \dfrac{1}{z}\right)$

Using **a i** and **a ii**, $(2\cos\theta)^3 = 2\cos 3\theta + 3(2\cos\theta)$
$\therefore\ 8\cos^3\theta = 2\cos 3\theta + 6\cos\theta$
$\therefore\ \cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$

v If we let $\theta = \frac{13\pi}{12}$ in **a iv** we get
$$\begin{aligned} \cos^3\left(\tfrac{13\pi}{12}\right) &= \tfrac{1}{4}\cos\left(\tfrac{39\pi}{12}\right) + \tfrac{3}{4}\cos\left(\tfrac{13\pi}{12}\right) \\ &= \tfrac{1}{4}\cos\left(\tfrac{13\pi}{4}\right) + \tfrac{3}{4}\cos\left(\tfrac{3\pi}{4} + \tfrac{\pi}{3}\right) \\ &= \tfrac{1}{4}\cos\left(\tfrac{5\pi}{4}\right) + \tfrac{3}{4}\left[\cos\left(\tfrac{3\pi}{4}\right)\cos\left(\tfrac{\pi}{3}\right) - \sin\left(\tfrac{3\pi}{4}\right)\sin\left(\tfrac{\pi}{3}\right)\right] \\ &= \tfrac{1}{4}\left(-\tfrac{1}{\sqrt{2}}\right) + \tfrac{3}{4}\left[\left(-\tfrac{1}{\sqrt{2}}\right)\left(\tfrac{1}{2}\right) - \left(\tfrac{1}{\sqrt{2}}\right)\left(\tfrac{\sqrt{3}}{2}\right)\right] \\ &= -\tfrac{1}{4\sqrt{2}} - \tfrac{3}{8\sqrt{2}} - \tfrac{3\sqrt{3}}{8\sqrt{2}} \\ &= \tfrac{-2-3-3\sqrt{3}}{8\sqrt{2}} \times \tfrac{\sqrt{2}}{\sqrt{2}} \\ &= \tfrac{-5\sqrt{2}-3\sqrt{6}}{16} \end{aligned}$$

b If $z = \text{cis}\,\theta$, then
$$\begin{aligned} z^n - \frac{1}{z^n} &= z^n - z^{-n} \\ &= (\text{cis}\,\theta)^n - (\text{cis}\,\theta)^{-n} \\ &= \text{cis}\ n\theta - \text{cis}\,(-n\theta) \qquad \{\text{De Moivre}\} \\ &= \cos n\theta + i\sin n\theta - [\cos(-n\theta) + i\sin(-n\theta)] \\ &= \cos n\theta + i\sin n\theta - \cos(-n\theta) - i\sin(-n\theta) \\ &= \cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta \\ &= 2i\sin n\theta \quad .... \ (*) \end{aligned}$$

If we let $n = 1$, $z - \dfrac{1}{z} = 2i\sin\theta$

$$\begin{aligned} \therefore\ [2i\sin\theta]^3 &= \left(z - \frac{1}{z}\right)^3 \\ \therefore\ 8i^3\sin^3\theta &= z^3 + 3z^2\left(-\frac{1}{z}\right) + 3z\left(-\frac{1}{z}\right)^2 + \left(-\frac{1}{z}\right)^3 \\ &= z^3 - \frac{1}{z^3} - 3\left(z - \frac{1}{z}\right) \end{aligned}$$

$\therefore\ 8i^3\sin^3\theta = 2i\sin 3\theta - 3\times 2i\sin\theta$ {using $(*)$}

$\therefore\ -8i\sin^3\theta = 2i\sin 3\theta - 6i\sin\theta$

$\therefore\ \sin^3\theta = -\frac{1}{4}\sin 3\theta + \frac{3}{4}\sin\theta$

$\therefore\ \sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$

c $\sin^3\theta\cos^3\theta = (\frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta)(\frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta)$ {using **12 a iv** and **b**}

$= \frac{3}{16}\sin\theta\cos 3\theta + \frac{9}{16}\sin\theta\cos\theta - \frac{1}{16}\sin 3\theta\cos 3\theta - \frac{3}{16}\sin 3\theta\cos\theta$

$= \frac{3}{16}(\sin\theta\cos 3\theta - \sin 3\theta\cos\theta) + \frac{9}{32}(2\sin\theta\cos\theta) - \frac{1}{32}(2\sin 3\theta\cos 3\theta)$

$= \frac{3}{16}(\sin(\theta - 3\theta)) + \frac{9}{32}\sin 2\theta - \frac{1}{32}\sin 6\theta$

$= -\frac{3}{16}\sin 2\theta + \frac{9}{32}\sin 2\theta - \frac{1}{32}\sin 6\theta$

$= \frac{3}{32}\sin 2\theta - \frac{1}{32}\sin 6\theta$

$= \frac{1}{32}(3\sin 2\theta - \sin 6\theta)$

EXERCISE 16F.1

1 **a** The cube roots of 1 are solutions to $z^3 = 1$, or $z^3 - 1 = 0$

Now $z = 1$ is a solution, so $z - 1$ is a factor.

$$\begin{array}{r|rrr|r} 1 & 1 & 0 & 0 & -1 \\ & 0 & 1 & 1 & 1 \\ \hline & 1 & 1 & 1 & 0 \end{array}$$

$\therefore\ (z-1)(z^2+z+1) = 0$

$\therefore\ z = 1$ or $\dfrac{-1\pm\sqrt{1-4}}{2}$

$\therefore\ z = 1$ or $-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

b $z^3 = 1$

$\therefore\ z^3 = 1\ \text{cis}\,(0 + k2\pi)$

$\therefore\ z = (1\ \text{cis}\,(k2\pi))^{\frac{1}{3}}$

$\therefore\ z = 1\ \text{cis}\left(\frac{k2\pi}{3}\right)$ {De Moivre}

$\therefore\ z = \text{cis}\,0,\ \text{cis}\left(\frac{2\pi}{3}\right)$ or $\text{cis}\left(\frac{4\pi}{3}\right)$ {when $k = 0, 1, 2$}

$\therefore\ z = 1,\ -\frac{1}{2} + i\frac{\sqrt{3}}{2},\ -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

2 **a** $z^3 = -8i$

$\therefore\ z^3 = 8\ \text{cis}\left(-\frac{\pi}{2} + k2\pi\right)$

$\therefore\ z = \left(8\ \text{cis}\left(-\frac{\pi}{2} + k2\pi\right)\right)^{\frac{1}{3}}$

$\therefore\ z = 8^{\frac{1}{3}}\ \text{cis}\left(-\frac{\pi}{6} + \frac{k2\pi}{3}\right)$ {De Moivre}

$\therefore\ z = 2\ \text{cis}\left(-\frac{\pi}{6}\right),\ 2\ \text{cis}\left(\frac{3\pi}{6}\right)$ or $2\ \text{cis}\left(\frac{7\pi}{6}\right)$ {when $k = 0, 1, 2$}

$\therefore\ z = 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right),\ 2\ \text{cis}\,\frac{\pi}{2}$ or $2\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$

$\therefore\ z = \sqrt{3} - i,\ 2i,\ -\sqrt{3} - i$

b $z^3 = -27i$ $\therefore\ z^3 = 27\ \text{cis}\left(-\frac{\pi}{2} + k2\pi\right)$

$\therefore\ z = \left(27\ \text{cis}\left(-\frac{\pi}{2} + k2\pi\right)\right)^{\frac{1}{3}}$

$\therefore\ z = 27^{\frac{1}{3}}\ \text{cis}\left(-\frac{\pi}{6} + \frac{k4\pi}{6}\right)$ {De Moivre}

$\therefore\ z = 3\ \text{cis}\left(-\frac{\pi}{6}\right),\ 3\ \text{cis}\left(\frac{3\pi}{6}\right)$ or $3\ \text{cis}\left(\frac{7\pi}{6}\right)$ {when $k = 0, 1, 2$}

$\therefore\ z = 3\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right),\ 3\ \text{cis}\,\frac{\pi}{2}$ or $3\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$

$\therefore\ z = \frac{3\sqrt{3}}{2} - \frac{3}{2}i,\ 3i,\ \frac{-3\sqrt{3}}{2} - \frac{3}{2}i$

3 $z^3 = -1$

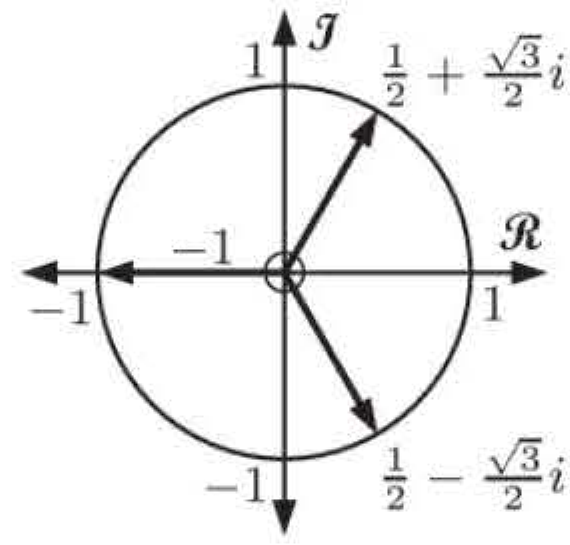
$\therefore\ z^3 = 1 \text{ cis}\,(\pi + k2\pi)$

$\therefore\ z = (\text{cis}\,(\pi + k2\pi))^{\frac{1}{3}}$

$\therefore\ z = \text{cis}\left(\frac{\pi}{3} + \frac{k2\pi}{3}\right)$ {De Moivre}

$\therefore\ z = \text{cis}\left(\frac{\pi}{3}\right),\ \text{cis}\,\pi,\ \text{cis}\left(\frac{5\pi}{3}\right)$ {when $k = 0, 1, 2$}

$\therefore\ z = \frac{1}{2} + i\frac{\sqrt{3}}{2},\ -1,\ \frac{1}{2} - i\frac{\sqrt{3}}{2}$



4 **a** $z^4 = 16 \quad \therefore\ z^4 = 16 \text{ cis}\,(0 + k2\pi)$

$\therefore\ z = (16 \text{ cis}\,(k2\pi))^{\frac{1}{4}}$

$\therefore\ z = 16^{\frac{1}{4}} \text{ cis}\left(\frac{k\pi}{2}\right)$ {De Moivre}

$\therefore\ z = 2 \text{ cis } 0,\ 2 \text{ cis}\left(\frac{\pi}{2}\right),\ 2 \text{ cis}\,\pi,\ 2 \text{ cis}\left(\frac{3\pi}{2}\right)$ {when $k = 0, 1, 2,$ or 3}

$\therefore\ z = \pm 2 \text{ or } \pm 2i$

b $z^4 = -16 \quad \therefore\ z^4 = 16 \text{ cis}\,(\pi + k2\pi)$

$\therefore\ z = (16 \text{ cis}\,(\pi + k2\pi))^{\frac{1}{4}}$

$\therefore\ z = 16^{\frac{1}{4}} \text{ cis}\left(\frac{\pi}{4} + \frac{k2\pi}{4}\right)$ {De Moivre}

$\therefore\ z = 2 \text{ cis}\left(\frac{\pi}{4}\right), 2 \text{ cis}\left(\frac{3\pi}{4}\right), 2 \text{ cis}\left(\frac{5\pi}{4}\right), 2 \text{ cis}\left(\frac{7\pi}{4}\right)$ {when $k = 0, 1, 2,$ or 3}

$\therefore\ z = 2\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right),\ 2\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right),\ 2\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right),\ 2\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$

$\therefore\ z = \sqrt{2} \pm i\sqrt{2},\ -\sqrt{2} \pm i\sqrt{2}$

5 $z^4 = -i$

$\therefore\ z^4 = \text{cis}\left(-\frac{\pi}{2} + k2\pi\right)$

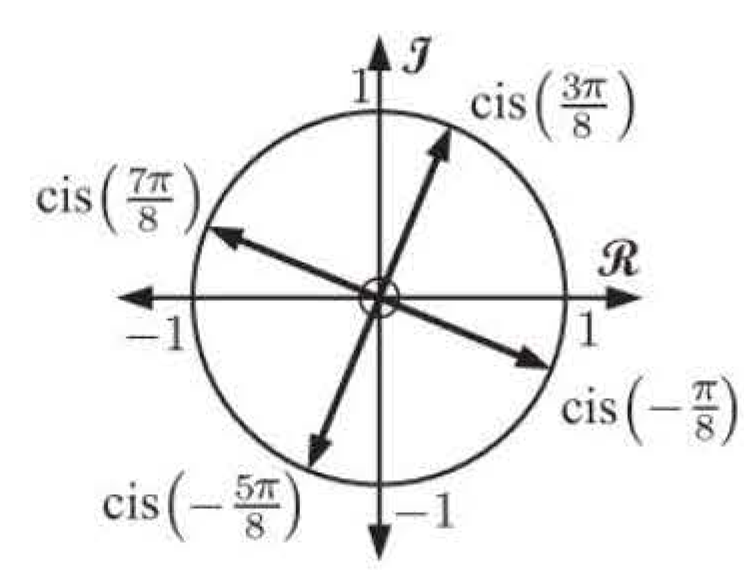
$\therefore\ z = \left(\text{cis}\left(-\frac{\pi}{2} + k2\pi\right)\right)^{\frac{1}{4}}$

$\therefore\ z = \text{cis}\left(-\frac{\pi}{8} + \frac{k\pi}{2}\right)$ {De Moivre}

$\therefore\ z = \text{cis}\left(-\frac{\pi}{8} + \frac{k4\pi}{8}\right)$

$\therefore\ z = \text{cis}\left(-\frac{5\pi}{8}\right),\ \text{cis}\left(-\frac{\pi}{8}\right),\ \text{cis}\left(\frac{3\pi}{8}\right),\ \text{cis}\left(\frac{7\pi}{8}\right)$

{when $k = -1, 0, 1, 2$}



6 **a** $z^3 = 2 + 2i$

$\therefore\ z^3 = 2\sqrt{2} \text{ cis}\left(\frac{\pi}{4} + k2\pi\right)$

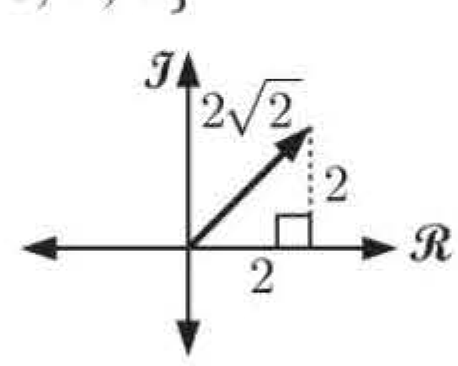
$\therefore\ z = \left[2\sqrt{2} \text{ cis}\left(\frac{\pi}{4} + k2\pi\right)\right]^{\frac{1}{3}}$

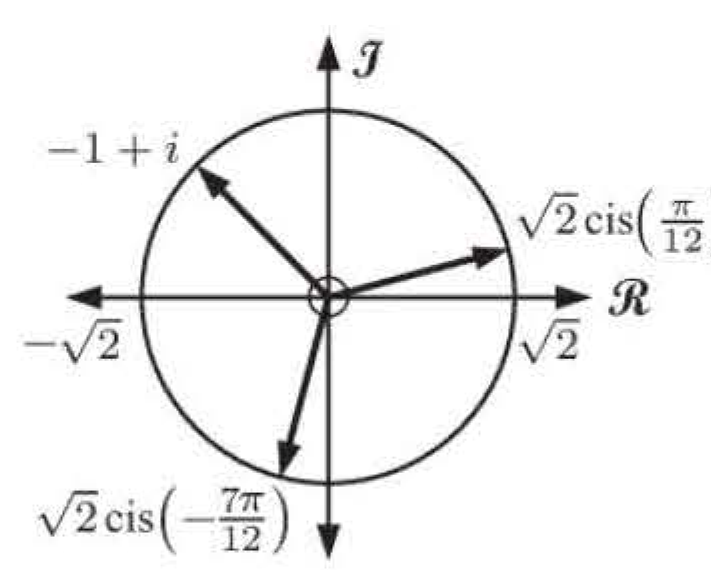
$\therefore\ z = \left(2\sqrt{2}\right)^{\frac{1}{3}} \text{ cis}\left(\frac{\pi}{12} + \frac{k2\pi}{3}\right)$ {De Moivre}

$\therefore\ z = \sqrt{2} \text{ cis}\left(\frac{\pi}{12} + \frac{k8\pi}{12}\right)$

$\therefore\ z = \sqrt{2} \text{ cis}\left(-\frac{7\pi}{12}\right),\ \sqrt{2} \text{ cis}\left(\frac{\pi}{12}\right),\ \sqrt{2} \text{ cis}\left(\frac{3\pi}{4}\right) = -1 + i$

{when $k = -1, 0, 1$}





b $z^3 = -2 + 2i$

$\therefore\ z^3 = 2\sqrt{2} \text{ cis}\left(\frac{3\pi}{4} + k2\pi\right)$

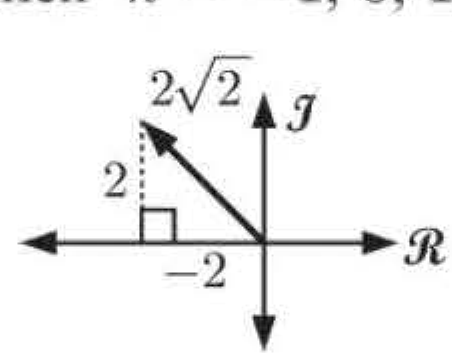
$\therefore\ z = \left[2\sqrt{2} \text{ cis}\left(\frac{3\pi}{4} + k2\pi\right)\right]^{\frac{1}{3}}$

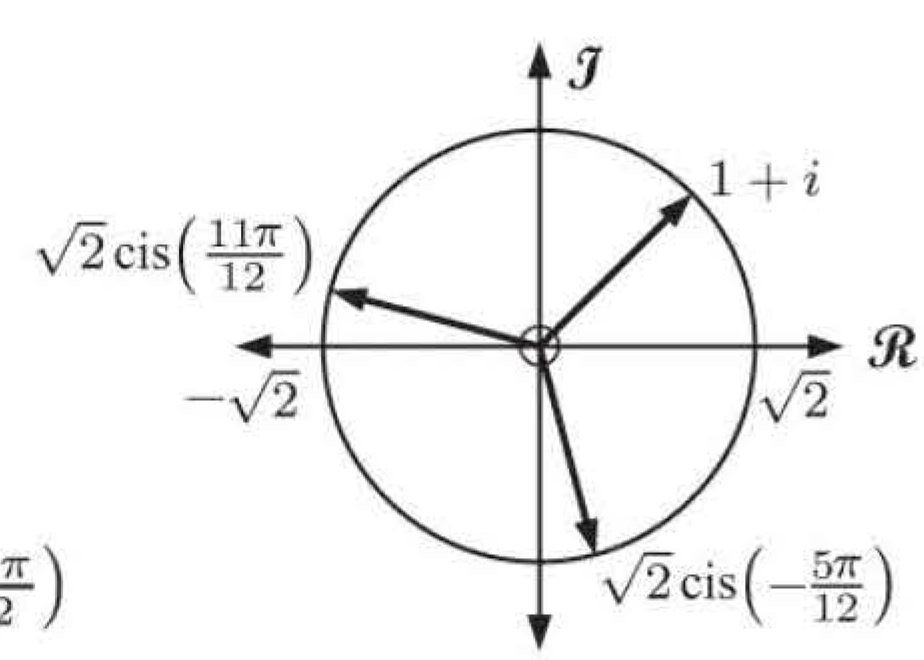
$\therefore\ z = \left(2\sqrt{2}\right)^{\frac{1}{3}} \text{ cis}\left(\frac{\pi}{4} + \frac{k2\pi}{3}\right)$ {De Moivre}

$\therefore\ z = \sqrt{2} \text{ cis}\left(\frac{3\pi}{12} + \frac{k8\pi}{12}\right)$

$\therefore\ z = \sqrt{2} \text{ cis}\left(-\frac{5\pi}{12}\right),\ \sqrt{2} \text{ cis}\left(\frac{\pi}{4}\right) = 1 + i,\ \sqrt{2} \text{ cis}\left(\frac{11\pi}{12}\right)$

{when $k = -1, 0, 1$}





c

$$z^2 = \tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i$$

$$\therefore\ z^2 = \text{cis}\left(\tfrac{\pi}{3} + k2\pi\right)$$

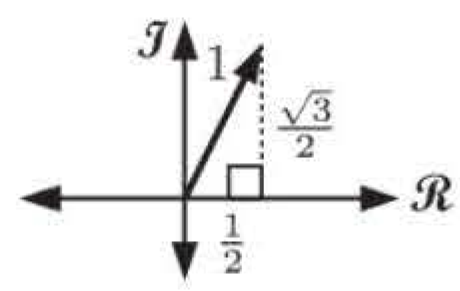
$$\therefore\ z = \left[\text{cis}\left(\tfrac{\pi}{3} + k2\pi\right)\right]^{\frac{1}{2}}$$

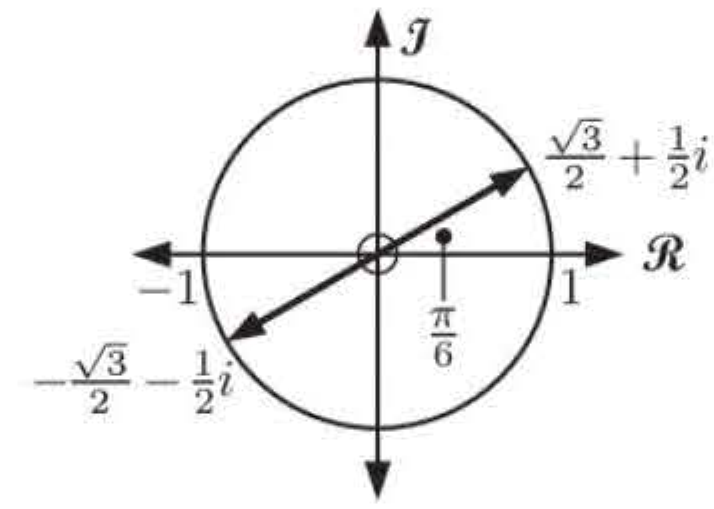
$$\therefore\ z = \text{cis}\left(\tfrac{\pi}{6} + k\pi\right) \quad \{\text{De Moivre}\}$$

$$\therefore\ z = \text{cis}\left(\tfrac{\pi}{6} + \tfrac{k6\pi}{6}\right)$$

$$\therefore\ z = \text{cis}\left(-\tfrac{5\pi}{6}\right) = -\tfrac{\sqrt{3}}{2} - \tfrac{1}{2}i, \quad \text{cis}\left(\tfrac{\pi}{6}\right) = \tfrac{\sqrt{3}}{2} + \tfrac{1}{2}i$$

$$\{\text{when } k = -1, 0\}$$





d

$$z^4 = \sqrt{3} + i$$

$$\therefore\ z^4 = 2\ \text{cis}\left(\tfrac{\pi}{6} + k2\pi\right)$$

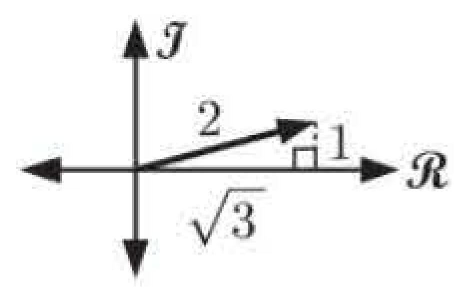
$$\therefore\ z = \left[2\ \text{cis}\left(\tfrac{\pi}{6} + k2\pi\right)\right]^{\frac{1}{4}}$$

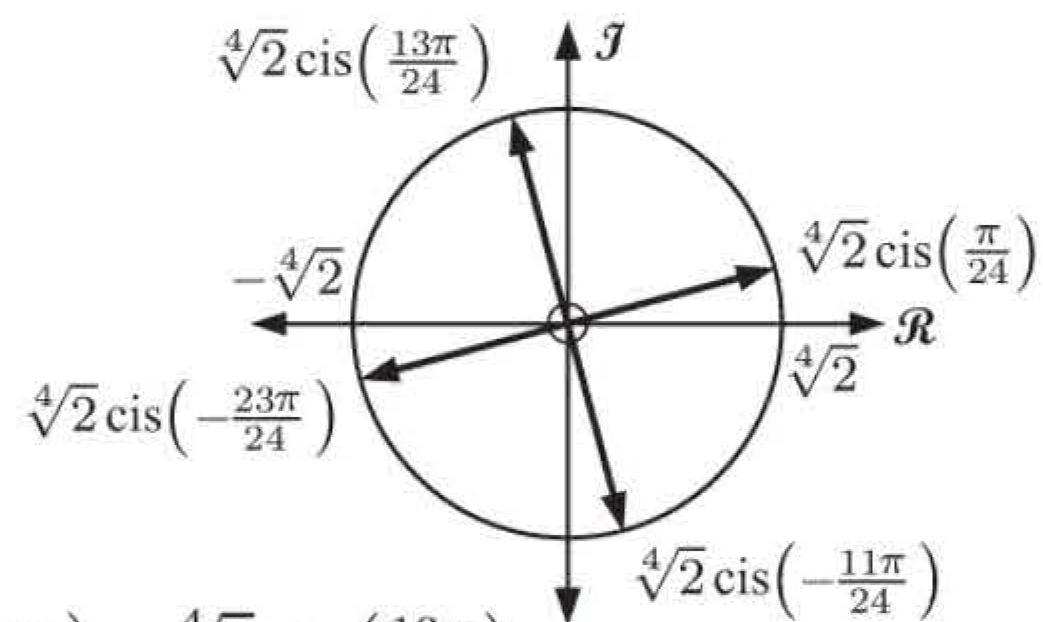
$$\therefore\ z = 2^{\frac{1}{4}}\ \text{cis}\left(\tfrac{\pi}{24} + \tfrac{k\pi}{2}\right) \quad \{\text{De Moivre}\}$$

$$\therefore\ z = \sqrt[4]{2}\ \text{cis}\left(\tfrac{\pi}{24} + \tfrac{k12\pi}{24}\right)$$

$$\therefore\ z = \sqrt[4]{2}\ \text{cis}\left(-\tfrac{23\pi}{24}\right),\ \sqrt[4]{2}\ \text{cis}\left(-\tfrac{11\pi}{24}\right),\ \sqrt[4]{2}\ \text{cis}\left(\tfrac{\pi}{24}\right),\ \sqrt[4]{2}\ \text{cis}\left(\tfrac{13\pi}{24}\right)$$

$$\{\text{when } k = -2, -1, 0, 1\}$$





e

$$z^5 = -4 - 4i$$

$$\therefore\ z^5 = 4\sqrt{2}\ \text{cis}\left(-\tfrac{3\pi}{4} + k2\pi\right)$$

$$\therefore\ z = \left[4\sqrt{2}\ \text{cis}\left(-\tfrac{3\pi}{4} + k2\pi\right)\right]^{\frac{1}{5}}$$

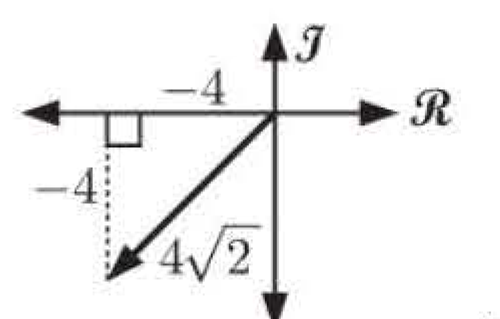
$$\therefore\ z = \left(4\sqrt{2}\right)^{\frac{1}{5}}\ \text{cis}\left(-\tfrac{3\pi}{20} + \tfrac{k2\pi}{5}\right) \quad \{\text{De Moivre}\}$$

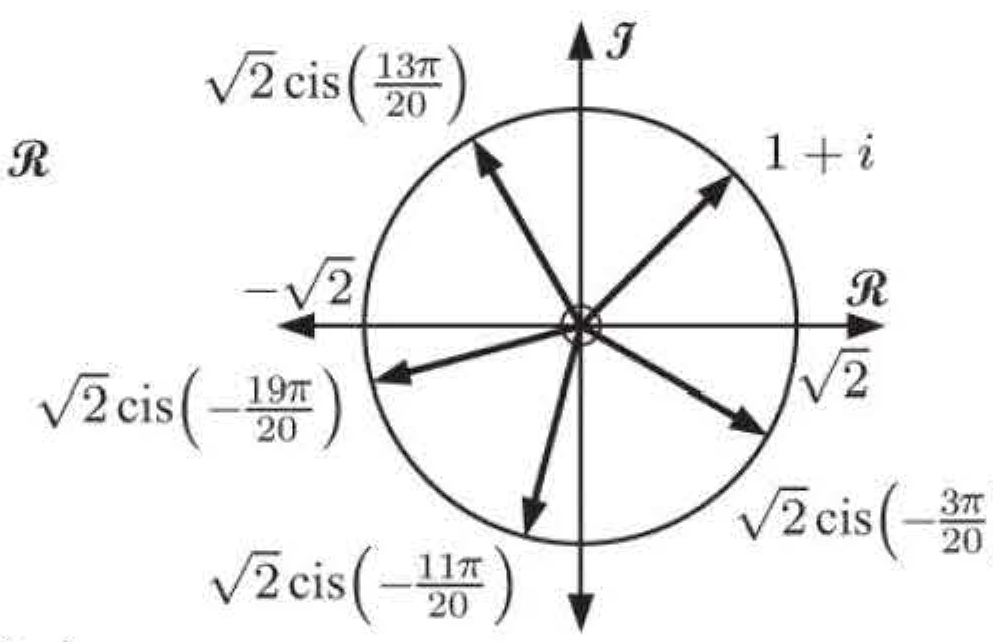
$$\therefore\ z = \sqrt{2}\ \text{cis}\left(-\tfrac{3\pi}{20} + \tfrac{k8\pi}{20}\right)$$

$$\therefore\ z = \sqrt{2}\ \text{cis}\left(-\tfrac{19\pi}{20}\right),\ \sqrt{2}\ \text{cis}\left(-\tfrac{11\pi}{20}\right),\ \sqrt{2}\ \text{cis}\left(-\tfrac{3\pi}{20}\right),$$

$$\sqrt{2}\ \text{cis}\left(\tfrac{\pi}{4}\right) = 1 + i,\ \sqrt{2}\ \text{cis}\left(\tfrac{13\pi}{20}\right)$$

$$\{\text{when } k = -2, -1, 0, 1, 2\}$$





f

$$z^3 = -2\sqrt{3} - 2i$$

$$\therefore\ z^3 = 4\ \text{cis}\left(-\tfrac{5\pi}{6} + k2\pi\right)$$

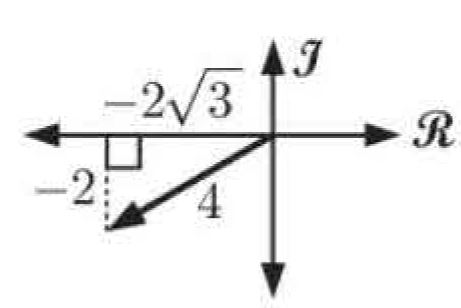
$$\therefore\ z = \left[4\ \text{cis}\left(-\tfrac{5\pi}{6} + k2\pi\right)\right]^{\frac{1}{3}}$$

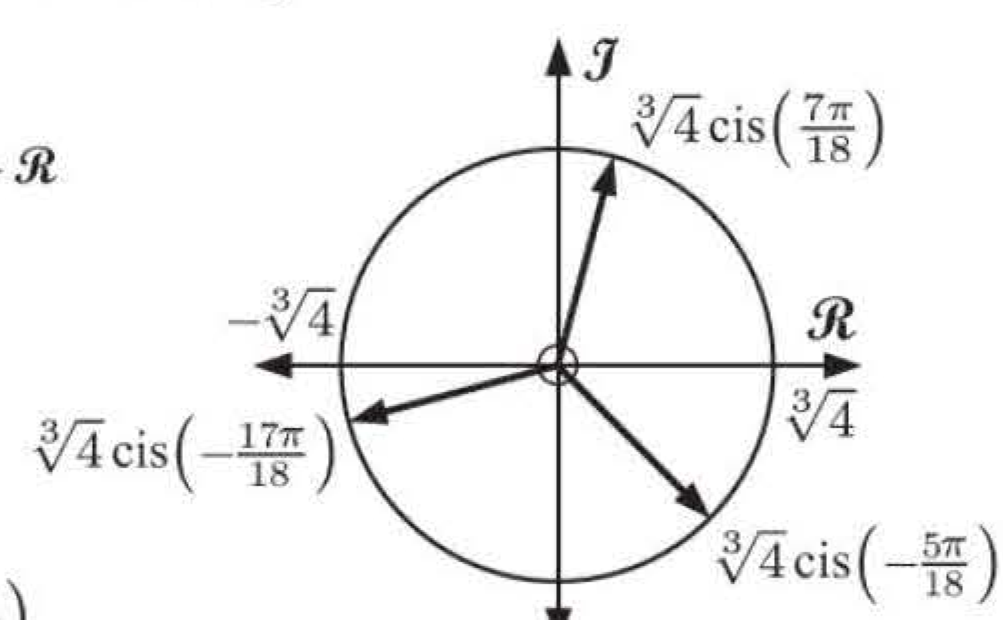
$$\therefore\ z = 4^{\frac{1}{3}}\ \text{cis}\left(-\tfrac{5\pi}{18} + \tfrac{k2\pi}{3}\right) \quad \{\text{De Moivre}\}$$

$$\therefore\ z = \sqrt[3]{4}\ \text{cis}\left(-\tfrac{5\pi}{18} + \tfrac{k12\pi}{18}\right)$$

$$\therefore\ z = \sqrt[3]{4}\ \text{cis}\left(-\tfrac{17\pi}{18}\right),\ \sqrt[3]{4}\ \text{cis}\left(-\tfrac{5\pi}{18}\right),\ \sqrt[3]{4}\ \text{cis}\left(\tfrac{7\pi}{18}\right)$$

$$\{\text{when } k = -1, 0, 1\}$$





7 **a**

$$z^4 + 1 = 0$$

$$\therefore\ z^4 = -1$$

$$\therefore\ z^4 = \text{cis}\,(\pi + k2\pi)$$

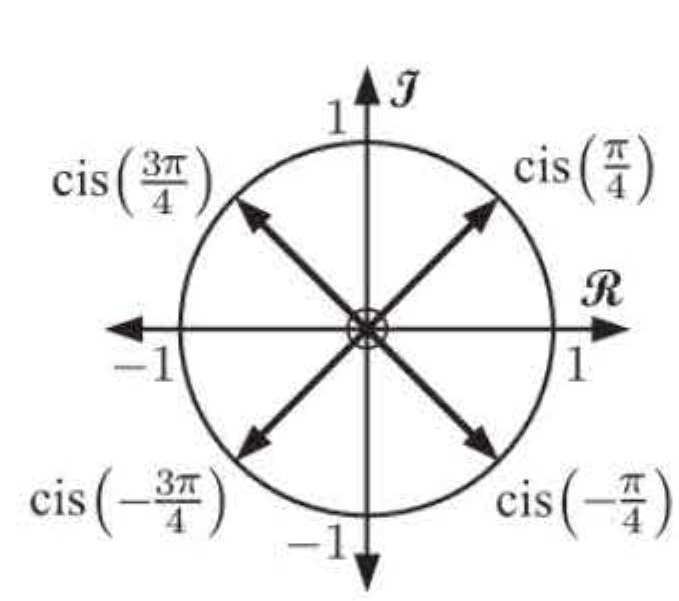
$$\therefore\ z = [\text{cis}\,(\pi + k2\pi)]^{\frac{1}{4}}$$

$$\therefore\ z = \text{cis}\left(\tfrac{\pi}{4} + \tfrac{k2\pi}{4}\right) \qquad \{\text{De Moivre}\}$$

$$\therefore\ z = \text{cis}\left(-\tfrac{3\pi}{4}\right),\ \text{cis}\left(-\tfrac{\pi}{4}\right),\ \text{cis}\left(\tfrac{\pi}{4}\right),\ \text{cis}\left(\tfrac{3\pi}{4}\right)$$

$$\{\text{when } k = -2, -1, 0, \text{ or } 1\}$$

$$\therefore\ z = \tfrac{1}{\sqrt{2}} \pm \tfrac{1}{\sqrt{2}}i,\ -\tfrac{1}{\sqrt{2}} \pm \tfrac{1}{\sqrt{2}}i$$



b For the pair of roots $\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i$,

$\text{sum} = \frac{1}{\sqrt{2}} + \cancel{\frac{1}{\sqrt{2}}i} + \frac{1}{\sqrt{2}} - \cancel{\frac{1}{\sqrt{2}}i}$
$= \frac{2}{\sqrt{2}} = \sqrt{2}$

$\text{product} = \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$
$= \frac{1}{2} + \frac{1}{2} = 1$

and so we have a quadratic factor of $z^2 - \sqrt{2}z + 1$

For the pair of roots $-\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i$,

$\text{sum} = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$
$= -\frac{2}{\sqrt{2}} = -\sqrt{2}$

$\text{product} = \left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$
$= \frac{1}{2} + \frac{1}{2} = 1$

and so we have a quadratic factor of $z^2 + \sqrt{2}z + 1$

$\therefore \; z^4 + 1 = \left(z^2 - \sqrt{2}z + 1\right)\left(z^2 + \sqrt{2}z + 1\right)$

8 **a**

$$\begin{aligned}
z &= \frac{\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)^2}{\left(\cos\frac{\pi}{10} - i\sin\frac{\pi}{10}\right)^5\left(\cos\frac{\pi}{30} + i\sin\frac{\pi}{30}\right)^{25}} \\
&= \frac{\text{cis}\left(-\frac{\pi}{6}\right)^2}{\left(\cos\left(-\frac{\pi}{10}\right) + i\sin\left(-\frac{\pi}{10}\right)\right)^5\left(\cos\frac{\pi}{30} + i\sin\frac{\pi}{30}\right)^{25}} \\
&= \frac{\text{cis}\left(-\frac{\pi}{3}\right)}{\left(\text{cis}\left(-\frac{\pi}{10}\right)\right)^5\left(\text{cis}\left(\frac{\pi}{30}\right)\right)^{25}} \\
&= \frac{\text{cis}\left(-\frac{\pi}{3}\right)}{\text{cis}\left(-\frac{\pi}{2}\right)\text{cis}\left(\frac{5\pi}{6}\right)} \\
&= \frac{\text{cis}\left(-\frac{\pi}{3}\right)}{\text{cis}\left(\frac{\pi}{3}\right)} \\
&= \text{cis}\left(-\frac{2\pi}{3}\right)
\end{aligned}$$

$\therefore \; |z| = 1, \quad \arg z = -\frac{2\pi}{3}$

b

$$\begin{aligned}
z^3 &= \left[\text{cis}\left(-\frac{2\pi}{3}\right)\right]^3 \\
&= \text{cis}\,(-2\pi) \quad \{\text{De Moivre}\} \\
&= 1
\end{aligned}$$

$\therefore \; z$ is a cube root of 1.

c

$$\begin{aligned}
(1-2z)(2z^2-1) &= 2z^2 - 1 - 4z^3 + 2z \\
&= 2z^2 - 1 - 4(1) + 2z \quad \{z^3 = 1\} \\
&= 2z^2 + 2z - 5
\end{aligned}$$

Now $z^3 = 1 \;\; \therefore \; z^2 = z^{-1}, \;\; z \neq 0$

$$\begin{aligned}
&= \left[\text{cis}\left(-\frac{2\pi}{3}\right)\right]^{-1} \\
&= \text{cis}\left(\frac{2\pi}{3}\right) \quad \{\text{De Moivre}\} \\
&= z^*
\end{aligned}$$

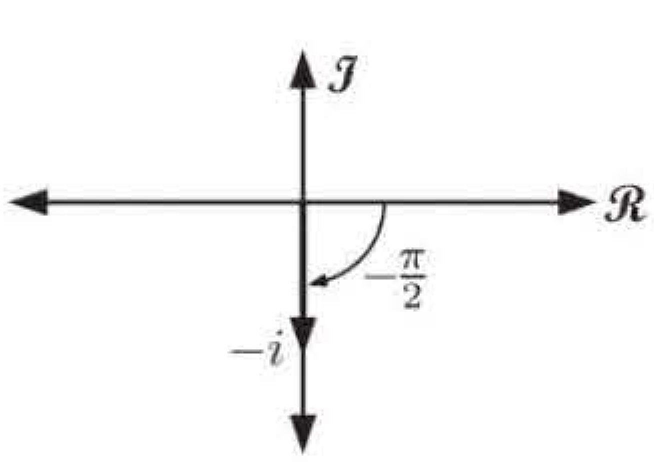
$$\begin{aligned}
\therefore \; 2z^2 + 2z - 5 &= 2z^* + 2z - 5 \\
&= 2(z + z^*) - 5
\end{aligned}$$

which is real as $z + z^*$ is always real.

9 **a** $-16i = 16 \text{ cis}\left(-\frac{\pi}{2}\right)$

b

$$\begin{aligned}
z^4 &= -16i \\
\therefore \; z^4 &= 16 \text{ cis}\left(-\frac{\pi}{2} + k2\pi\right) \\
\therefore \; z &= \left[16 \text{ cis}\left(-\frac{\pi}{2} + k2\pi\right)\right]^{\frac{1}{4}} \\
\therefore \; z &= 16^{\frac{1}{4}} \text{ cis}\left(-\frac{\pi}{8} + \frac{k\pi}{2}\right) \quad \{\text{De Moivre}\} \\
\therefore \; z &= 2 \text{ cis}\left(-\frac{\pi}{8} + \frac{k4\pi}{8}\right)
\end{aligned}$$



$\therefore \; z = 2 \text{ cis}\left(-\frac{5\pi}{8}\right), \; 2 \text{ cis}\left(-\frac{\pi}{8}\right), \; 2 \text{ cis}\left(\frac{3\pi}{8}\right), \; 2 \text{ cis}\left(\frac{7\pi}{8}\right) \quad \{\text{when } k = -1, 0, 1, 2\}$

i The 4th root in the second quadrant is $z = 2\text{ cis}\left(\frac{7\pi}{8}\right)$ $\quad \{\frac{\pi}{2} < \frac{7\pi}{8} < \pi\}$

ii In Cartesian form, $z = 2\left[\cos\left(\frac{7\pi}{8}\right) + i\sin\left(\frac{7\pi}{8}\right)\right]$

$$= 2\cos\left(\tfrac{7\pi}{8}\right) + \left[2\sin\left(\tfrac{7\pi}{8}\right)\right] i$$

EXERCISE 16F.2

1 **a** **i** $(z+3)^3 = 1$

$\therefore\ z+3 = 1,\ w,\text{ or } w^2$ where $w = \text{cis}\left(\frac{2\pi}{3}\right)$

$\therefore\ z+3 = w^n$ where $n = 0, 1, 2$

$\therefore\ z = w^n - 3$ where $n = 0, 1, 2$ and $w = \text{cis}\left(\frac{2\pi}{3}\right)$

ii $(z-1)^3 = 8$

$\therefore\ \left[\dfrac{z-1}{2}\right]^3 = 1$

$\therefore\ \dfrac{z-1}{2} = 1,\ w,\text{ or } w^2$ where $w = \text{cis}\left(\frac{2\pi}{3}\right)$

$\therefore\ \dfrac{z-1}{2} = w^n$ where $n = 0, 1, 2$

$\therefore\ z = 2w^n + 1$ where $n = 0, 1, 2$ and $w = \text{cis}\left(\frac{2\pi}{3}\right)$

iii $(2z-1)^3 = -1$

$\therefore\ (2z-1)^3 = (-1)^3$

$\therefore\ \left(\dfrac{2z-1}{-1}\right)^3 = 1$

$\therefore\ (1-2z)^3 = 1$

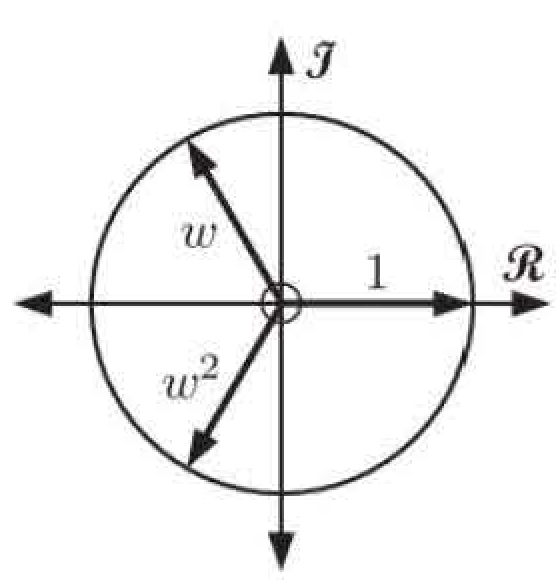
$\therefore\ 1 - 2z = 1,\ w,\ w^2$ where $w = \text{cis}\left(\frac{2\pi}{3}\right)$

$\therefore\ 1 - 2z = w^n$ where $n = 0, 1, 2$

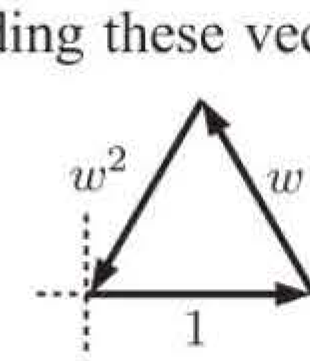
$\therefore\ 2z = 1 - w^n$

$\therefore\ z = \dfrac{1-w^n}{2}$ where $n = 0, 1, 2$ and $w = \text{cis}\left(\frac{2\pi}{3}\right)$

b The following represents the cube roots of unity:



Adding these vectorially



the resultant vector is $\mathbf{0}$

$\therefore\ 1 + w + w^2 = 0$

2 **a** If $w = \text{cis}\left(\frac{\pi}{2}\right)$, $w^2 = \text{cis}\,\pi$

and $w^3 = \text{cis}\left(\frac{3\pi}{2}\right)$ {De Moivre}

$\therefore\ w = i,\ w^2 = -1,$ and $w^3 = -i$

$\therefore\ 1, i, -1, -i$ can be written as

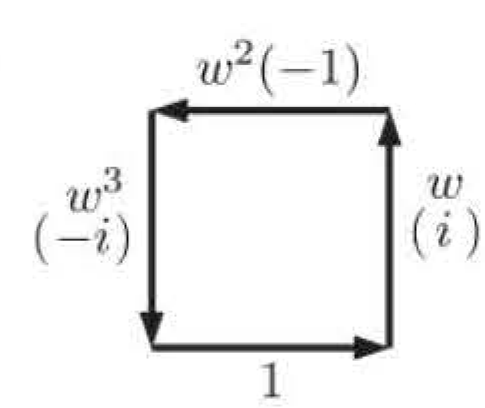
$1, w, w^2, w^3,$ where $w = \text{cis}\left(\frac{\pi}{2}\right)$

b $1 + w + w^2 + w^3 = 1 + (i) + (-1) + (-i)$

$= 1 + i - 1 - i$

$= 0$

c Adding these vectorially



3 **a** The 5th roots of unity are the solutions to $z^5 = 1$.

$\therefore\ z^5 = \text{cis}\,(0 + k2\pi)$

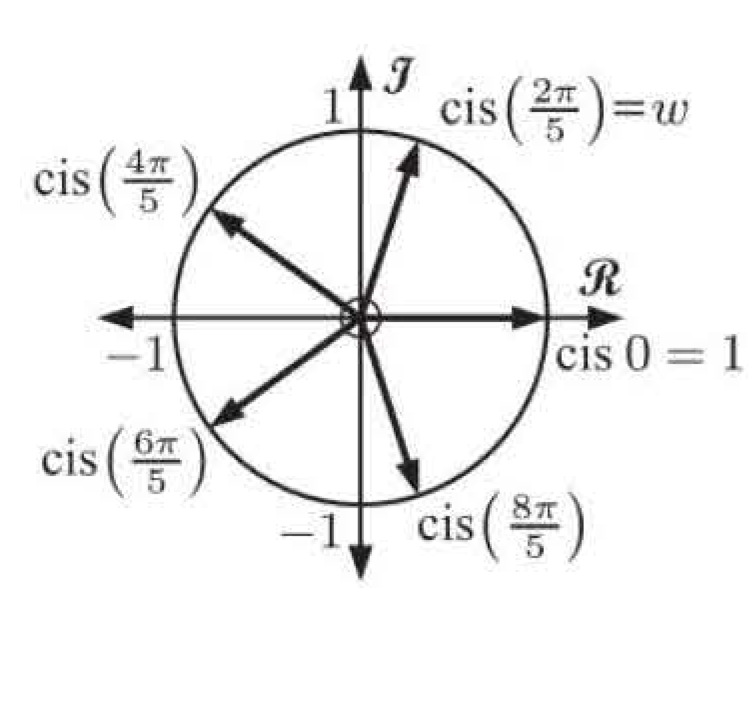
$\therefore\ z^5 = \text{cis}\,(k2\pi)$

$\therefore\ z = [\text{cis}\,(k2\pi)]^{\frac{1}{5}}$

$\therefore\ z = \text{cis}\left(\frac{k2\pi}{5}\right)$ {De Moivre}

$\therefore\ z = \text{cis}\,0 = 1,\ \text{cis}\left(\frac{2\pi}{5}\right),\ \text{cis}\left(\frac{4\pi}{5}\right),\ \text{cis}\left(\frac{6\pi}{5}\right),\ \text{cis}\left(\frac{8\pi}{5}\right)$

{when $k = 0, 1, 2, 3, 4$}



b If $\text{cis}\left(\frac{2\pi}{5}\right) = w$, then $\text{cis}\left(\frac{4\pi}{5}\right) = \left(\text{cis}\ \frac{2\pi}{5}\right)^2 = w^2$

$$\text{cis}\left(\tfrac{6\pi}{5}\right) = \left(\text{cis}\ \tfrac{2\pi}{5}\right)^3 = w^3$$

$$\text{cis}\left(\tfrac{8\pi}{5}\right) = \left(\text{cis}\ \tfrac{2\pi}{5}\right)^4 = w^4$$

Hence the five roots can be expressed as $1, w, w^2, w^3, w^4$ where $w = \text{cis}\left(\frac{2\pi}{5}\right)$

c

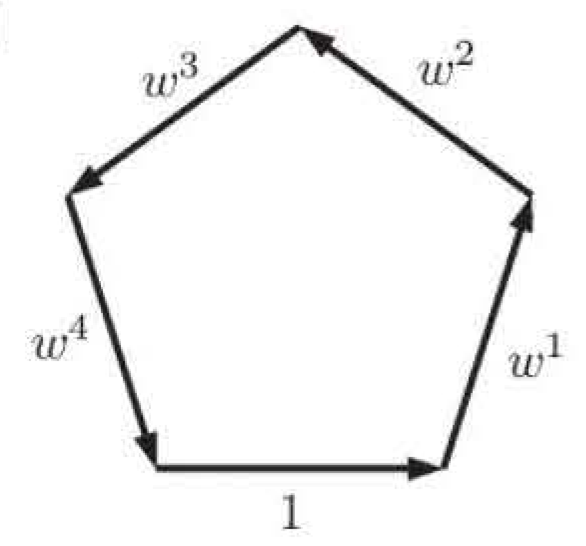
$$(1 + w + w^2 + w^3 + w^4)(1 - w)$$
$$= 1 + w + w^2 + w^3 + w^4 - w - w^2 - w^3 - w^4 - w^5$$
$$= 1 - w^5$$

Since $w^5 = 1$, $1 - w^5 = 0$

$\therefore\ (1 + w + w^2 + w^3 + w^4)(1 - w) = 0$

But $w \neq 1$, so $1 + w + w^2 + w^3 + w^4 = 0$

d



4 **a** The nth roots of unity are the solutions to $z^n = 1$

$$z^n = 1$$
$$\therefore\ z^n = \text{cis}\,(0 + k2\pi)$$
$$\therefore\ z^n = \text{cis}\,(k2\pi)$$
$$\therefore\ z = [\text{cis}\,(k2\pi)]^{\frac{1}{n}}$$
$$\therefore\ z = \text{cis}\left(\tfrac{k2\pi}{n}\right) \qquad \{\text{De Moivre}\}$$

The nth root of unity with smallest positive argument is $w = \text{cis}\left(\frac{2\pi}{n}\right)$, when $k = 1$.

b **i** The nth roots of unity are

$$\text{cis}\ 0 = 1 \quad \text{or}$$
$$\text{cis}\left(\tfrac{2\pi}{n}\right) = w \quad \text{or}$$
$$\text{cis}\left(\tfrac{4\pi}{n}\right) = \left(\text{cis}\left(\tfrac{2\pi}{n}\right)\right)^2 = w^2 \quad \text{or}$$
$$\vdots$$
$$\text{cis}\left(\tfrac{2\pi}{n}(n-1)\right) = \left[\text{cis}\left(\tfrac{2\pi}{n}\right)\right]^{n-1} = w^{n-1} \quad \{\text{letting } k = 0, 1, 2, 3, ..., n-1\}$$

$\therefore$ the n roots of $z^n = 1$ are $1, w, w^2, w^3, ..., w^{n-1}$ where $w = \text{cis}\left(\frac{2\pi}{n}\right)$

ii Now $(1 + w + w^2 + ... + w^{n-1})(w - 1) = w^n - 1$

But w is a solution to $z^n - 1 = 0$ so $w^n - 1 = 0$

$\therefore\ (1 + w + w^2 + ... + w^{n-1})(w - 1) = 0$

$\therefore$ since $w \neq 1$, $1 + w + w^2 + ... + w^{n-1} = 0$

5 Let $\alpha = r\ \text{cis}\ \theta$

$$\therefore\ z^n = r\ \text{cis}\,(\theta + k2\pi)$$
$$\therefore\ z = [r\ \text{cis}\ (\theta + k2\pi)]^{\frac{1}{n}}$$
$$\therefore\ z = r^{\frac{1}{n}}\ \text{cis}\left(\tfrac{\theta + k2\pi}{n}\right) \qquad \{\text{De Moivre}\}$$
$$\therefore\ z = r^{\frac{1}{n}}\ \text{cis}\left(\tfrac{\theta}{n}\right)\ \text{cis}\left(\tfrac{k2\pi}{n}\right)$$

$\therefore$ the n zeros of $z^n = \alpha$ are $r^{\frac{1}{n}}\ \text{cis}\left(\frac{\theta}{n}\right)$, $r^{\frac{1}{n}}\ \text{cis}\left(\frac{\theta}{n}\right)\ \text{cis}\left(\frac{2\pi}{n}\right)$, $r^{\frac{1}{n}}\ \text{cis}\left(\frac{\theta}{n}\right)\ \text{cis}\left(\frac{4\pi}{n}\right)$, ...,

$r^{\frac{1}{n}}\ \text{cis}\left(\frac{\theta}{n}\right)\text{cis}\left(\frac{2\pi}{n}(n-1)\right)$ {letting $k = 0, 1, 2, ..., n-1$}

$\therefore$ the sum of the n zeros of z is

$$r^{\frac{1}{n}}\ \text{cis}\left(\tfrac{\theta}{n}\right) + r^{\frac{1}{n}}\ \text{cis}\left(\tfrac{\theta}{n}\right)\ \text{cis}\left(\tfrac{2\pi}{n}\right) + r^{\frac{1}{n}}\ \text{cis}\left(\tfrac{\theta}{n}\right)\ \text{cis}\left(\tfrac{4\pi}{n}\right) + ... + r^{\frac{1}{n}}\ \text{cis}\left(\tfrac{\theta}{n}\right)\ \text{cis}\left(\tfrac{2\pi}{n}(n-1)\right)$$

$$= r^{\frac{1}{n}}\ \text{cis}\left(\tfrac{\theta}{n}\right) \underbrace{\left[1 + \text{cis}\left(\tfrac{2\pi}{n}\right) + \text{cis}\left(\tfrac{4\pi}{n}\right) + ... + \text{cis}\left(\tfrac{2\pi}{n}(n-1)\right)\right]}_{\text{these are the } n\text{th roots of unity, whose sum } = 0 \quad \{\text{using } \mathbf{4}\}}$$

$$= 0$$

EXERCISE 16G

1 $z^2-(2+i)z+(3+i)=0$

$$\begin{aligned}\therefore\ z &= \frac{2+i\pm\sqrt{(2+i)^2-4(1)(3+i)}}{2}\\ &= \frac{2+i\pm\sqrt{4+\cancel{4i}-1-12-\cancel{4i}}}{2}\\ &= \frac{2+i\pm\sqrt{-9}}{2}\\ &= \frac{2+i\pm 3i}{2}\\ &= \frac{2+4i}{2}\ \text{ or }\ \frac{2-2i}{2}\\ &= 1+2i\ \text{ or }\ 1-i\end{aligned}$$

2 **a**

$$\begin{aligned} z^* &= -iz\\ \therefore\ x-iy &= -i(x+iy)\\ \therefore\ x-iy &= -ix+y\end{aligned}$$

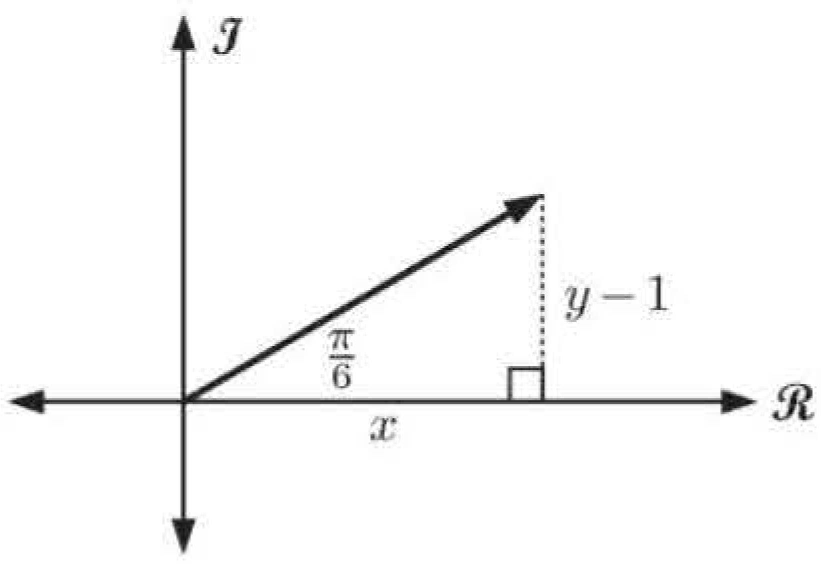
Equating real and imaginary parts,

$x=y$ and $-y=-x$

$\therefore\ y=x$

b

$$\begin{aligned}\arg(z-i) &= \tfrac{\pi}{6}\\ \therefore\ \arg(x+iy-i) &= \tfrac{\pi}{6}\\ \therefore\ \arg(x+(y-1)i) &= \tfrac{\pi}{6}\\ \therefore\ \tan\tfrac{\pi}{6} = \tfrac{1}{\sqrt3} &= \frac{y-1}{x}\\ \therefore\ y-1 &= \frac{x}{\sqrt3}\\ \therefore\ y &= \frac{x}{\sqrt3}+1,\quad x>0\end{aligned}$$

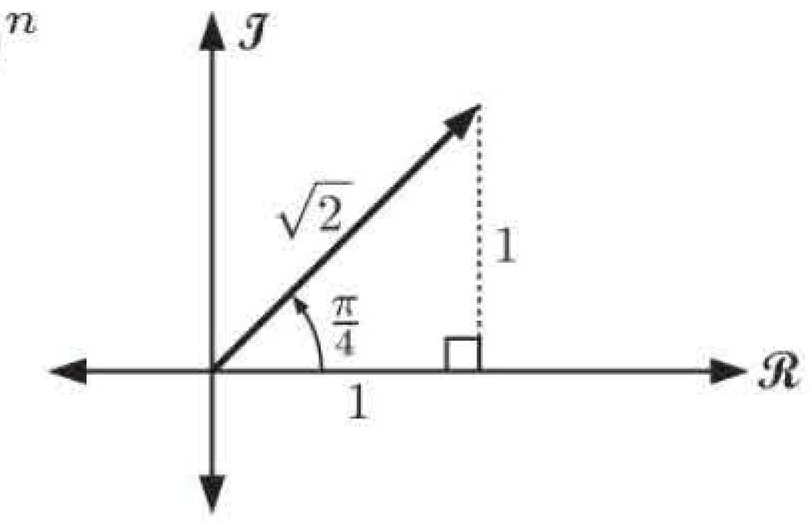


c

$$\begin{aligned}|z+3|+|z-3| &= 8\\ \therefore\ |z+3| &= 8-|z-3|\\ \therefore\ |z+3|^2 &= (8-|z-3|)^2\\ \therefore\ |z+3|^2 &= 64-16\,|z-3|+|z-3|^2\\ \therefore\ (z+3)(z+3)^* &= 64-16\,|z-3|+(z-3)(z-3)^* \qquad \{|z|^2=zz^*\}\\ \therefore\ (z+3)(z^*+3) &= 64-16\,|z-3|+(z-3)(z^*-3) \qquad \{(z\pm w)^*=z^*\pm w^*\}\\ \therefore\ \cancel{zz^*}+3z+3z^*+\cancel{9} &= 64-16\,|(x-3)+yi|+\cancel{zz^*}-3z-3z^*+\cancel{9}\\ \therefore\ 6z+6z^* &= 64-16\sqrt{(x-3)^2+y^2}\\ \therefore\ 6(x+yi)+6(x-yi) &= 64-16\sqrt{(x-3)^2+y^2}\\ \therefore\ 12x-64 &= -16\sqrt{(x-3)^2+y^2}\\ \therefore\ 3x-16 &= -4\sqrt{(x-3)^2+y^2}\\ \therefore\ (3x-16)^2 &= 16\left[(x-3)^2+y^2\right]\\ \therefore\ 9x^2-96x+256 &= 16(x^2-6x+9+y^2)\\ \therefore\ 9x^2-\cancel{96x}+256 &= 16x^2-\cancel{96x}+144+16y^2\\ \therefore\ 112 &= 7x^2+16y^2\end{aligned}$$

3 $$(1+i)^{2n} = \binom{2n}{0}1^{2n}i^0 + \binom{2n}{1}1^{2n-1}i^1 + \binom{2n}{2}1^{2n-2}i^2 + \binom{2n}{3}1^{2n-3}i^3 + ... + \binom{2n}{2n}1^0 i^{2n}$$
$$= \binom{2n}{0} + \binom{2n}{1}i - \binom{2n}{2} - \binom{2n}{3}i + ... + \binom{2n}{2n}(-1)^n$$

But $$(1+i)^{2n} = \left(\sqrt{2}\text{ cis }\tfrac{\pi}{4}\right)^{2n}$$
$$= 2^n \text{ cis}\left(\tfrac{n\pi}{2}\right)$$
$$= 2^n\left[\cos\left(\tfrac{n\pi}{2}\right) + i\sin\left(\tfrac{n\pi}{2}\right)\right]$$



So, $\binom{2n}{0} + \binom{2n}{1}i - \binom{2n}{2} - \binom{2n}{3}i + ... + \binom{2n}{2n}(-1)^n = 2^n\left[\cos\left(\frac{n\pi}{2}\right) + i\sin\left(\frac{n\pi}{2}\right)\right]$

Equating real parts,
$$\binom{2n}{0} - \binom{2n}{2} + \binom{2n}{4} - \binom{2n}{6} + ... + (-1)^n\binom{2n}{2n} = 2^n\cos\left(\tfrac{n\pi}{2}\right), \quad n \in \mathbb{Z}^+$$

4 $$1 + \text{cis }\theta + \text{cis }2\theta + \text{cis }3\theta + ... + \text{cis }n\theta$$
$$= 1 + (\cos\theta + i\sin\theta) + (\cos 2\theta + i\sin 2\theta) + (\cos 3\theta + i\sin 3\theta) + ... + (\cos n\theta + i\sin n\theta)$$
$$= (1 + \cos\theta + \cos 2\theta + \cos 3\theta + ... + \cos n\theta) + i(\sin\theta + \sin 2\theta + \sin 3\theta + ... + \sin n\theta)$$
$$= \sum_{r=0}^{n}\cos r\theta + i\sum_{r=1}^{n}\sin r\theta$$

$\therefore$ $\mathcal{Re}\,(1 + \text{cis }\theta + \text{cis }2\theta + \text{cis }3\theta + ... + \text{cis }n\theta) = \displaystyle\sum_{r=0}^{n}\cos r\theta$ .... (1)

Now $1 + \text{cis }\theta + \text{cis }2\theta + \text{cis }3\theta + ...+ \text{cis }n\theta = 1 + \text{cis }\theta + (\text{cis }\theta)^2 + (\text{cis }\theta)^3 + ... + (\text{cis }\theta)^n$
which is a geometric series with $u_1 = 1$, $r = \text{cis }\theta$

$\therefore$ it has sum $S_n = \dfrac{u_1(r^n - 1)}{r-1}$

$$\underbrace{1 + \text{cis }\theta + \text{cis }2\theta + ... + \text{cis }n\theta}_{n+1\text{ terms}} = S_{n+1}$$
$$= \frac{1((\text{cis }\theta)^{n+1} - 1)}{\text{cis }\theta - 1}$$
$$= \frac{\text{cis }(n+1)\theta - 1}{\text{cis }\theta - 1}$$
$$= \frac{\cos(n+1)\theta + i\sin(n+1)\theta - 1}{\cos\theta + i\sin\theta - 1}$$
$$= \left(\frac{\cos(n+1)\theta + i\sin(n+1)\theta - 1}{\cos\theta - 1 + i\sin\theta}\right)\left(\frac{\cos\theta - 1 - i\sin\theta}{\cos\theta - 1 - i\sin\theta}\right)$$

$\therefore$ $\mathcal{Re}\,(1 + \text{cis }\theta + \text{cis }2\theta + ... + \text{cis }n\theta)$
$$= \frac{\cos(n+1)\theta\cos\theta - \cos(n+1)\theta + \sin(n+1)\theta\sin\theta - \cos\theta + 1}{(\cos\theta - 1)^2 + \sin^2\theta}$$
$$= \frac{[\cos(n+1)\theta\cos\theta + \sin(n+1)\theta\sin\theta] - \cos(n+1)\theta - \cos\theta + 1}{\cos^2\theta - 2\cos\theta + 1 + \sin^2\theta}$$
$$= \frac{\cos n\theta - \cos(n+1)\theta - \cos\theta + 1}{2 - 2\cos\theta} \quad .... (2)$$

Equating (1) and (2) gives $\displaystyle\sum_{r=0}^{n}\cos r\theta = \frac{\cos n\theta - \cos(n+1)\theta - \cos\theta + 1}{2 - 2\cos\theta}$

5 $$2\cos\left(\tfrac{\theta}{2}\right)\text{ cis}\left(\tfrac{\theta}{2}\right) = 2\cos\left(\tfrac{\theta}{2}\right)\left[\cos\left(\tfrac{\theta}{2}\right) + i\sin\left(\tfrac{\theta}{2}\right)\right]$$
$$= 2\cos^2\left(\tfrac{\theta}{2}\right) + 2i\cos\left(\tfrac{\theta}{2}\right)\sin\left(\tfrac{\theta}{2}\right)$$
$$= \cos\theta + 1 + i\left(2\cos\left(\tfrac{\theta}{2}\right)\sin\left(\tfrac{\theta}{2}\right)\right)$$
$$= \cos\theta + 1 + i\sin\theta$$
$$= 1 + \text{cis }\theta$$

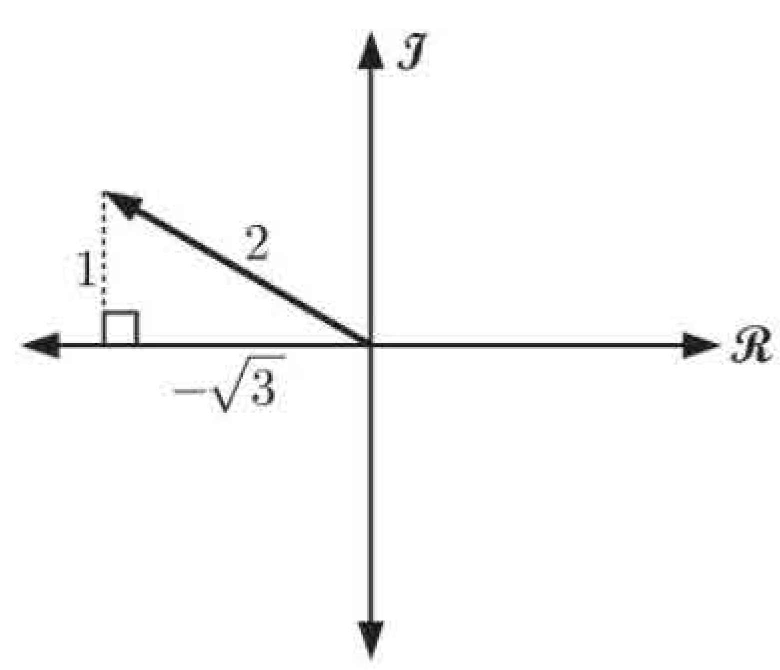
$$\left\{\begin{array}{r} \cos 2X = 2\cos^2 X - 1 \\ \therefore\ 2\cos^2 X = \cos 2X + 1 \end{array}\right\}$$

$$\begin{aligned}\text{Consider}\quad (1+\text{cis}\,\theta)^n &= \tbinom{n}{0}1^n(\text{cis}\,\theta)^0+\tbinom{n}{1}1^{n-1}(\text{cis}\,\theta)^1+\tbinom{n}{2}1^{n-2}(\text{cis}\,\theta)^2+\dots+\tbinom{n}{n}1^0(\text{cis}\,\theta)^n\\ &= \tbinom{n}{0}+\tbinom{n}{1}\text{cis}\,\theta+\tbinom{n}{2}\text{cis}\,2\theta+\dots+\tbinom{n}{n}\text{cis}\,n\theta\\ \therefore\ \mathcal{R}e\,[(1+\text{cis}\,\theta)^n] &= \tbinom{n}{0}+\tbinom{n}{1}\cos\theta+\tbinom{n}{2}\cos 2\theta+\dots+\tbinom{n}{n}\cos n\theta\\ &= \sum_{r=0}^{n}\tbinom{n}{r}\cos(r\theta)\end{aligned}$$

$$\begin{aligned}\text{So}\quad \sum_{r=0}^{n}\tbinom{n}{r}\cos(r\theta) &= \mathcal{R}e\,[(1+\text{cis}\,\theta)^n]\\ &= \mathcal{R}e\left[\left(2\cos\left(\tfrac{\theta}{2}\right)\text{cis}\left(\tfrac{\theta}{2}\right)\right)^n\right]\\ &= \mathcal{R}e\left[2^n\cos^n\left(\tfrac{\theta}{2}\right)\text{cis}\left(\tfrac{n\theta}{2}\right)\right]\\ &= 2^n\cos^n\left(\tfrac{\theta}{2}\right)\cos\left(\tfrac{n\theta}{2}\right)\end{aligned}$$

REVIEW SET 16A

1
$$\begin{aligned}(i-\sqrt{3})^5 &= \left[2\,\text{cis}\left(\tfrac{5\pi}{6}\right)\right]^5\\ &= 2^5\,\text{cis}\left(\tfrac{25\pi}{6}\right)\\ &= 32\,\text{cis}\left(\tfrac{\pi}{6}\right)\\ &= 32\left(\tfrac{\sqrt{3}}{2}+\tfrac{1}{2}i\right)\\ &= 16\sqrt{3}+16i\end{aligned}$$



$\therefore$ real part is $16\sqrt{3}$, imaginary part is 16.

2 **a**
$$\begin{aligned}|z-i| &= |z+1+i|\\ \text{Since } z=x+iy,\quad |x+iy-i| &= |x+iy+1+i|\\ \therefore\ |x+i(y-1)| &= |(x+1)+i(y+1)|\\ \therefore\ \sqrt{x^2+(y-1)^2} &= \sqrt{(x+1)^2+(y+1)^2}\\ \therefore\ x^2+(y-1)^2 &= (x+1)^2+(y+1)^2\\ \therefore\ \cancel{x^2}+\cancel{y^2}-2y+1 &= \cancel{x^2}+2x+1+\cancel{y^2}+2y+1\\ \therefore\ 1-2y &= 2x+2y+2\\ \therefore\ 2x+4y &= -1\end{aligned}$$

b
$$\begin{aligned}z^*-iz &= 0\\ \text{Since}\quad z &= x+iy,\\ x-iy-i(x+iy) &= 0\\ \therefore\ x-iy-ix+y &= 0\\ \therefore\ (x+y)-i(x+y) &= 0\\ \therefore\ x+y &= 0\\ \therefore\ y &= -x\end{aligned}$$

3
$$\begin{aligned}|z+16| &= 4\,|z+1|\\ \therefore\ |z+16|^2 &= 16\,|z+1|^2\\ \therefore\ (z+16)(z+16)^* &= 16(z+1)(z+1)^* && \{|w|^2=ww^*\}\\ \therefore\ (z+16)(z^*+16) &= 16(z+1)(z^*+1) && \{(z\pm w)^*=z^*\pm w^*\}\\ \therefore\ zz^*+16z+16z^*+256 &= 16(zz^*+z+z^*+1)\\ \therefore\ |z|^2+\cancel{16z}+\cancel{16z^*}+256 &= 16\,|z|^2+\cancel{16z}+\cancel{16z^*}+16\\ 240 &= 15\,|z|^2\\ \therefore\ |z|^2 &= 16\\ \therefore\ |z| &= 4\ \text{ as }\ |z|\geqslant 0\end{aligned}$$

4 **a**
$$\begin{aligned}z &= r\,\text{cis}\,\theta\\ z^* &= r\,\text{cis}\,(-\theta)\end{aligned}$$
$\therefore$ reflection in the real axis.

b
$$\begin{aligned}z &= r\,\text{cis}\,\theta\\ -z &= -r\,\text{cis}\,\theta\\ &= \text{cis}\,\pi\times r\,\text{cis}\,\theta\\ &= r\,\text{cis}\,(\theta+\pi)\end{aligned}$$
$\therefore$ rotation of π about O.

c
$$\begin{aligned}z &= r\,\text{cis}\,\theta\\ iz &= ir\,\text{cis}\,\theta\\ &= \text{cis}\left(\tfrac{\pi}{2}\right)r\,\text{cis}\,\theta\\ &= r\,\text{cis}\left(\theta+\tfrac{\pi}{2}\right)\end{aligned}$$
$\therefore$ anticlockwise rotation of $\frac{\pi}{2}$ about O.

5 Let $z = a + bi$ and $w = c + di$ $\quad \therefore \quad z + w = (a + c) + i(b + d)$, so $b + d = 0$ .... (1)

$$\text{and} \quad zw = (a + bi)(c + di) = [ac - bd] + i[ad + bc]$$

so, $ad + bc = 0$ .... (2)

From (1), $d = -b$ and in (2), $a(-b) + bc = 0$

$$\therefore \quad b(c - a) = 0$$

$$\therefore \quad a = c \text{ as } b \neq 0$$

So $z = a + bi$ and $w = a - bi$

$$\therefore \quad z^* = a - bi = w$$

6

$$(x + iy)^n = X + Yi$$

$$\therefore \quad |(x + yi)^n| = |X + Yi|$$

$$\therefore \quad |x + iy|^n = |X + Yi|$$

$$\therefore \quad \left(\sqrt{x^2 + y^2}\right)^n = \sqrt{X^2 + Y^2}$$

Squaring both sides, $X^2 + Y^2 = (x^2 + y^2)^n$

7

$$\begin{aligned} |z - w|^2 + |z + w|^2 &= (z - w)(z - w)^* + (z + w)(z + w)^* \\ &= (z - w)(z^* - w^*) + (z + w)(z^* + w^*) \\ &= zz^* - \cancel{zw^*} - \cancel{wz^*} + ww^* + zz^* + \cancel{zw^*} + \cancel{wz^*} + ww^* \\ &= 2zz^* + 2ww^* \\ &= 2\,|z|^2 + 2\,|w|^2 \\ &= 2\left(|z|^2 + |w|^2\right) \end{aligned}$$

8 **a** Since 1 is a root of $z^5 - 1 = 0$, we find that

$z^5 - 1 = (z - 1)(1 + z + z^2 + z^3 + z^4)$

$\therefore$ since α is a root,

$$(\alpha - 1)(1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4) = 0$$

But $\alpha \neq 1$, so $1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 = 0$

$$\begin{array}{r|rrrrr|r} 1 & 1 & 0 & 0 & 0 & 0 & -1 \\ & 0 & 1 & 1 & 1 & 1 & 1 \\ \hline & 1 & 1 & 1 & 1 & 1 & 0 \end{array}$$

or Note that $1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4$ is the sum of a geometric series.

Then $S_n = \dfrac{u_1(1 - r^n)}{1 - r}$ where $u_1 = 1$, $r = \alpha$, $n = 5$

$\therefore \quad S_5 = \dfrac{1(1 - \alpha^5)}{1 - \alpha}$ and $\alpha^5 = \text{cis}\,(2\pi) = 1$

$$\therefore \quad 1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 = \frac{1(1 - 1)}{1 - \alpha} = 0$$

b Now if $\left(\dfrac{z + 2}{z - 1}\right)^5 = 1$ then $\dfrac{z + 2}{z - 1} = 1, \alpha, \alpha^2, \alpha^3$, or α^4 where $\alpha = \text{cis}\left(\frac{2\pi}{5}\right)$

$$\therefore \quad \frac{z + 2}{z - 1} = \alpha^n \quad \text{where } n = 0, 1, 2, 3, 4$$

$$\therefore \quad z + 2 = \alpha^n(z - 1)$$

$$\therefore \quad z + 2 = \alpha^n z - \alpha^n$$

$$\therefore \quad z(\alpha^n - 1) = \alpha^n + 2 \quad \text{and so} \quad z = \frac{\alpha^n + 2}{\alpha^n - 1}$$

$\therefore$ roots of $\left(\dfrac{z + 2}{z - 1}\right)^5 = 1$ are $\dfrac{\alpha + 2}{\alpha - 1}, \dfrac{\alpha^2 + 2}{\alpha^2 - 1}, \dfrac{\alpha^3 + 2}{\alpha^3 - 1}, \dfrac{\alpha^4 + 2}{\alpha^4 - 1}$, where $\alpha = \text{cis}\left(\frac{2\pi}{5}\right)$

Note: The case where $n = 0$ requires $\dfrac{z + 2}{z - 1} = 1$, and so $z + 2 = z - 1$, which has no solution.

9 Since $\left|\dfrac{z+1}{z-1}\right| = \dfrac{|z+1|}{|z-1|} = 1$, then $|z+1| = |z-1|$

Letting $z = x + iy$, $\therefore\ |(x+1)+iy| = |(x-1)+iy|$

$\therefore\ \sqrt{(x+1)^2+y^2} = \sqrt{(x-1)^2+y^2}$

Squaring both sides, we get $(x+1)^2 + \cancel{y^2} = (x-1)^2 + \cancel{y^2}$

$\therefore\ x^2 + 2x + 1 = x^2 - 2x + 1$

$\therefore\ 4x = 0$

$\therefore\ x = 0$

Therefore since $z \neq 0$, z is purely imaginary.

10
$$\begin{aligned} w &= \frac{1+z}{1+z^*} \\ &= \frac{1+\text{cis}\,\phi}{1+\text{cis}\,(-\phi)} \times \frac{\text{cis}\,\phi}{\text{cis}\,\phi} \\ &= \frac{(1+\text{cis}\,\phi)\ \text{cis}\,\phi}{\text{cis}\,\phi + \text{cis}\,0} \\ &= \frac{(1+\text{cis}\,\phi)\ \text{cis}\,\phi}{1+\text{cis}\,\phi} \\ &= \text{cis}\,\phi \end{aligned}$$

11 The 3 cube roots of $-64i$ are the solutions to $z^3 = -64i$

$\therefore\ z^3 = 64\ \text{cis}\,(-\frac{\pi}{2} + k2\pi)$ for integer k

$\therefore\ z = \left[64\ \text{cis}\,(-\frac{\pi}{2} + k2\pi)\right]^{\frac{1}{3}}$

$\therefore\ z = 64^{\frac{1}{3}}\ \text{cis}\left(-\frac{\pi}{6} + \frac{k2\pi}{3}\right)$

$\therefore\ z = 4\ \text{cis}\left(-\frac{\pi}{6} + \frac{k4\pi}{6}\right)$

$\therefore\ z = 4\ \text{cis}\left(-\frac{5\pi}{6}\right),\ 4\ \text{cis}\left(-\frac{\pi}{6}\right),\ 4\ \text{cis}\left(\frac{\pi}{2}\right)$ {letting $k = -1, 0, 1$}

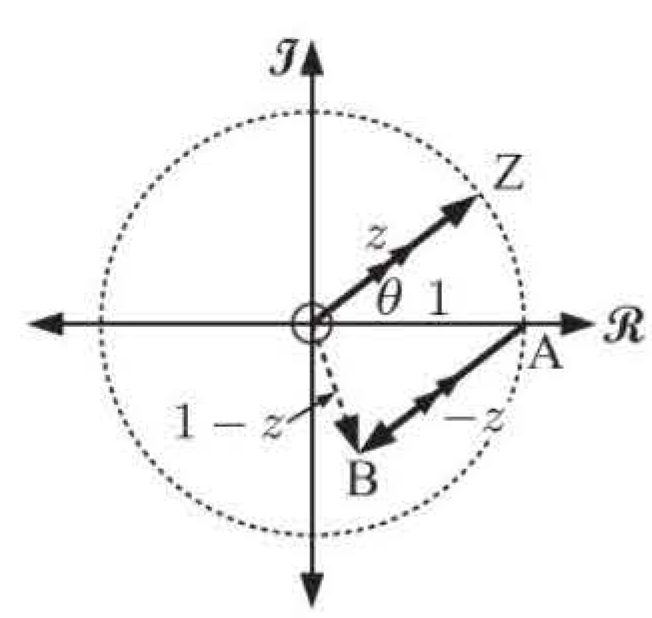
$\therefore\ z = 4(-\frac{\sqrt{3}}{2} - \frac{1}{2}i),\ 4(\frac{\sqrt{3}}{2} - \frac{1}{2}i),\ 4i$

$\therefore\ z = -2\sqrt{3} - 2i,\ 2\sqrt{3} - 2i,\ 4i$

12 **a**
$$\begin{aligned} (2z)^{-1} &= (2\ \text{cis}\,\theta)^{-1} \\ &= 2^{-1}\ \text{cis}\,(-\theta) \end{aligned}$$

$\therefore\ \left|(2z)^{-1}\right| = \frac{1}{2}$ and $\arg\left[(2z)^{-1}\right] = -\theta$

b
$$\begin{aligned} 1 - z &= 1 - \text{cis}\,\theta \\ &= (1-\cos\theta) - i\sin\theta \\ \therefore\ |1-z| &= \sqrt{(1-\cos\theta)^2 + \sin^2\theta} \\ &= \sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta} \\ &= \sqrt{2 - 2\cos\theta} \\ &= \sqrt{2 - 2\left(1 - 2\sin^2\left(\tfrac{\theta}{2}\right)\right)} \\ &= \sqrt{4\sin^2\left(\tfrac{\theta}{2}\right)} \\ &= 2\sin\left(\tfrac{\theta}{2}\right) \end{aligned}$$



$\triangle$OAB is isosceles since $|z| = 1$,

so we let $\widehat{AOB} = \widehat{ABO} = \phi$

Since [OZ] $\parallel$ [AB], $\widehat{OAB} = \theta$ {alternate $\measuredangle$s}

$\therefore\ \phi + \phi + \theta = \pi$

$\therefore\ 2\phi = \pi - \theta$

$\phi = \frac{\pi}{2} - \frac{\theta}{2}$

But $\arg(1-z) = -\phi$,

so $\arg(1-z) = -\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = \frac{\theta}{2} - \frac{\pi}{2}$

13
$$-1 + i\sqrt{3} = 2\ \text{cis}\left(\tfrac{2\pi}{3}\right)$$

$\therefore\ (-1 + i\sqrt{3})^m = 2^m\ \text{cis}\left(\frac{m2\pi}{3}\right)$ {De Moivre}

$= 2^m\left[\cos\left(\frac{m2\pi}{3}\right) + i\sin\left(\frac{m2\pi}{3}\right)\right]$

This is real provided $\sin\left(\frac{m2\pi}{3}\right) = 0$

$\therefore\ \frac{m2\pi}{3} = 0 + k\pi$

$\therefore\ m = \frac{3k}{2}$ where k is any integer

14 We first note that $\cos(A+B)+\cos(A-B)=2\cos A\cos B$

and $\sin(A+B)+\sin(A-B)=2\sin A\cos B$ .... $(*)$

Now $\text{cis}\,\theta+\text{cis}\,\phi=(\cos\theta+i\sin\theta)+(\cos\phi+i\sin\phi)$

$$=(\cos\theta+\cos\phi)+i(\sin\theta+\sin\phi)$$

$$=\left[\cos\left(\frac{\theta+\phi}{2}+\frac{\theta-\phi}{2}\right)+\cos\left(\frac{\theta+\phi}{2}-\frac{\theta-\phi}{2}\right)\right]$$

$$+i\left[\sin\left(\frac{\theta+\phi}{2}+\frac{\theta-\phi}{2}\right)+\sin\left(\frac{\theta+\phi}{2}-\frac{\theta-\phi}{2}\right)\right]$$

$$=2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)+2i\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)\quad\{\text{using }(*)\}$$

$$=2\cos\left(\frac{\theta-\phi}{2}\right)\text{cis}\left(\frac{\theta+\phi}{2}\right)\quad\text{as required}$$

Now $Z^5=1$ has solutions $Z=1,\ \alpha,\ \alpha^2,\ \alpha^3,\ \alpha^4$ where $\alpha=\text{cis}\left(\frac{2\pi}{5}\right)$

$\therefore\ Z=\text{cis}\left(\frac{2n\pi}{5}\right)$ where $n=0,\ 1,\ 2,\ 3,\ 4$

$\therefore$ if $\left(\dfrac{z+1}{z-1}\right)^5=1$, then $\dfrac{z+1}{z-1}=\text{cis}\left(\frac{2n\pi}{5}\right)$

$$\therefore\ z+1=\text{cis}\left(\tfrac{2n\pi}{5}\right)(z-1)$$

$$\therefore\ z\left(\text{cis}\left(\tfrac{2n\pi}{5}\right)-1\right)=\text{cis}\left(\tfrac{2n\pi}{5}\right)+1$$

$$\therefore\ z=\frac{\text{cis}\left(\frac{2n\pi}{5}\right)+1}{\text{cis}\left(\frac{2n\pi}{5}\right)-1}$$

$$\therefore\ z=\frac{\text{cis}\left(\frac{2n\pi}{5}\right)+\text{cis}\,0}{\text{cis}\left(\frac{2n\pi}{5}\right)+\text{cis}\,\pi}$$

$$\therefore\ z=\frac{2\cos\left(\dfrac{\frac{2n\pi}{5}-0}{2}\right)\text{cis}\left[\dfrac{\frac{2n\pi}{5}+0}{2}\right]}{2\cos\left(\dfrac{\frac{2n\pi}{5}-\pi}{2}\right)\text{cis}\left[\dfrac{\frac{2n\pi}{5}+\pi}{2}\right]}\quad\{\text{using the above identity}\}$$

$$\therefore\ z=\frac{\cos\frac{n\pi}{5}}{\cos\left(\frac{n\pi}{5}-\frac{\pi}{2}\right)}\,\text{cis}\left[\tfrac{n\pi}{5}-\left(\tfrac{n\pi}{5}+\tfrac{\pi}{2}\right)\right]$$

$$\therefore\ z=\frac{\cos\frac{n\pi}{5}}{\cos\left(\frac{\pi}{2}-\frac{n\pi}{5}\right)}\,\text{cis}\left(-\tfrac{\pi}{2}\right)$$

$$\therefore\ z=\frac{\cos\frac{n\pi}{5}}{\sin\frac{n\pi}{5}}(-i)$$

$\therefore\ z=-i\cot\left(\frac{n\pi}{5}\right),\quad n=1,\ 2,\ 3,\ 4$ {excluding $n=0$ since $\cot 0$ is undefined}

Now $\cot\left(\frac{n\pi}{5}\right)=-\cot\left(\pi-\frac{n\pi}{5}\right)$

$$=-\cot\left(\tfrac{(5-n)\pi}{5}\right),\quad n=1,\ 2,\ 3,\ 4$$

$$=-\cot\left(\tfrac{n\pi}{5}\right),\quad n=4,\ 3,\ 2,\ 1$$

$\therefore\ z=i\cot\left(\frac{n\pi}{5}\right),\quad n=1,\ 2,\ 3,\ 4$

REVIEW SET 16B

1 $z_1 = \text{cis}\left(\frac{\pi}{6}\right)$ and $z_2 = \text{cis}\left(\frac{\pi}{4}\right)$

$$\begin{aligned}\therefore \left(\frac{z_1}{z_2}\right)^3 &= \left[\frac{\text{cis}\left(\frac{\pi}{6}\right)}{\text{cis}\left(\frac{\pi}{4}\right)}\right]^3 \\ &= \left[\text{cis}\left(\tfrac{\pi}{6} - \tfrac{\pi}{4}\right)\right]^3 \\ &= \left[\text{cis}\left(-\tfrac{\pi}{12}\right)\right]^3 \\ &= \text{cis}\left(-\tfrac{3\pi}{12}\right) \qquad \{\text{De Moivre}\} \\ &= \text{cis}\left(-\tfrac{\pi}{4}\right) \\ &= \cos\left(-\tfrac{\pi}{4}\right) + i\sin\left(-\tfrac{\pi}{4}\right) \\ &= \tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}i\end{aligned}$$

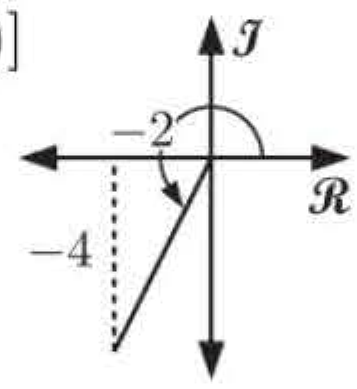
2 $z = 4 + i, \quad w = 2 - 3i$

a
$$\begin{aligned}&2w^* - iz \\ &= 2(2+3i) - i(4+i) \\ &= 4 + 6i - 4i - i^2 \\ &= 5 + 2i\end{aligned}$$

b
$$\begin{aligned}|w - z^*| &= |(2-3i) - (4-i)| \\ &= |2 - 3i - 4 + i| \\ &= |-2 - 2i| \\ &= \sqrt{(-2)^2 + (-2)^2} \\ &= \sqrt{8} \\ &= 2\sqrt{2}\end{aligned}$$

c
$$\begin{aligned}\left|z^{10}\right| &= |z|^{10} \\ &= |4+i|^{10} \\ &= \left(\sqrt{16+1}\right)^{10} \\ &= \sqrt{17}^{10} \\ &= 17^5\end{aligned}$$

d
$$\begin{aligned}\arg(w - z) &= \arg[(2-3i) - (4+i)] \\ &= \arg[-2 - 4i] \\ &\approx -2.03\end{aligned}$$



3 If $\dfrac{2-3i}{2a+bi} = 3 + 2i$, then $\dfrac{2-3i}{3+2i} = 2a + bi$

$$\begin{aligned}\therefore\ 2a + bi &= \left(\frac{2-3i}{3+2i}\right)\left(\frac{3-2i}{3-2i}\right) \\ &= \frac{6 - 4i - 9i - 6}{9+4} \\ &= \frac{0 - 13i}{13} \\ &= 0 - i\end{aligned}$$

$\therefore\ 2a = 0$ and $b = -1 \quad \therefore\ a = 0,\ b = -1$

4 **a** If $\arg z = \frac{\pi}{2}$, then we have a ray vertically upwards beginning at the origin.

If $\arg(z - i) = \frac{\pi}{2}$, the graph is translated $\binom{0}{1}$, and we have a ray vertically upwards beginning at i.

$\therefore\ x = 0$, and geometrically we require $y > 1$.

b $\left|\dfrac{z+2}{z-2}\right| = 2, \quad \therefore\ \dfrac{|z+2|}{|z-2|} = 2$

$\therefore\ |z+2| = 2\,|z-2|$

If $z = x + iy$, then $\sqrt{(x+2)^2 + y^2} = 2\sqrt{(x-2)^2 + y^2}$

$$\therefore \quad (x+2)^2 + y^2 = 4(x-2)^2 + 4y^2$$
$$\therefore \quad x^2 + 4x + 4 + y^2 = 4x^2 - 16x + 16 + 4y^2$$
$$\therefore \quad 3x^2 + 3y^2 - 20x + 12 = 0, \quad \text{which is a circle}$$

5
$$\begin{aligned} 2 - 2\sqrt{3}i &= 4\left(\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i\right) \\ &= 4 \text{ cis}\left(-\tfrac{\pi}{3}\right) \end{aligned}$$
$$\therefore \quad (2 - 2\sqrt{3}i)^n = 4^n \text{ cis}\left(-\tfrac{n\pi}{3}\right) \qquad \{\text{De Moivre}\}$$

This is real if $\sin\left(-\frac{n\pi}{3}\right) = 0$

$$\therefore \quad -\tfrac{n\pi}{3} = k\pi, \quad k \text{ an integer}$$
$$\therefore \quad n = 3k \quad \text{where } k \text{ is an integer}$$

6 The cube roots of -27 are the solutions to $z^3 = -27$.

$$\therefore \quad z^3 = 27 \text{ cis}(\pi + k2\pi)$$
$$\therefore \quad z = [27 \text{ cis}(\pi + k2\pi)]^{\frac{1}{3}}$$
$$\therefore \quad z = 27^{\frac{1}{3}} \text{ cis}\left(\tfrac{\pi + k2\pi}{3}\right)$$
$$\therefore \quad z = 3 \text{ cis}\left(\tfrac{\pi + k2\pi}{3}\right)$$
$$\therefore \quad z = 3 \text{ cis}\left(-\tfrac{\pi}{3}\right),\ 3 \text{ cis}\left(\tfrac{\pi}{3}\right),\ 3 \text{ cis}\,\pi \qquad \{\text{letting } k = -1, 0, 1\}$$
$$\therefore \quad z = 3\left(\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i\right),\ 3\left(\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i\right),\ -3$$
$$\therefore \quad z = \tfrac{3}{2} \pm \tfrac{3\sqrt{3}}{2}i \text{ or } -3$$

7 **a**
$$\begin{aligned} z &= 4 \text{ cis}\,\theta \\ z^3 &= (4 \text{ cis}\,\theta)^3 \\ &= 4^3 \text{ cis}\,3\theta \end{aligned}$$
$$\therefore \quad \left|z^3\right| = 64$$
$$\text{and} \quad \arg\left(z^3\right) = 3\theta$$

b
$$\begin{aligned} \frac{1}{z} &= z^{-1} \\ &= (4 \text{ cis}\,\theta)^{-1} \\ &= 4^{-1} \text{ cis}(-\theta) \end{aligned}$$
$$\therefore \quad \left|\frac{1}{z}\right| = \tfrac{1}{4} \quad \text{and} \quad \arg\left(\frac{1}{z}\right) = -\theta$$

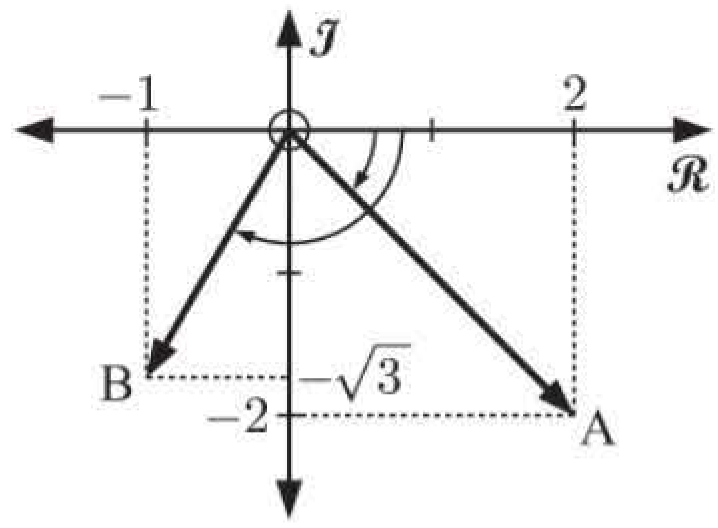
c
$$z = 4 \text{ cis}\,\theta$$
$$\begin{aligned} \therefore \quad iz^* &= \left(\text{cis}\,\tfrac{\pi}{2}\right)(4 \text{ cis}(-\theta)) \\ &= 4 \text{ cis}\left(\tfrac{\pi}{2} - \theta\right) \end{aligned}$$
$$\therefore \quad |iz^*| = 4 \quad \text{and} \quad \arg(iz^*) = \tfrac{\pi}{2} - \theta$$

8 **a**
$$\text{Let} \quad z = r \text{ cis}\,\theta$$
$$\therefore \quad z^n = r^n \text{ cis}\,n\theta \quad \{\text{De Moivre}\}$$
$$\text{and so} \quad \arg z^n = n\theta$$
$$\therefore \quad \arg z^n = n \arg z \quad \text{as required}$$

b
$$\begin{aligned} \left(\frac{z}{w}\right)^* &= \left(\frac{a+bi}{c+di}\right)^*, \quad w \neq 0 \\ &= \left(\frac{(a+bi)(c-di)}{(c+di)(c-di)}\right)^* \\ &= \left(\frac{(ac+bd) + i(bc-ad)}{c^2+d^2}\right)^* \\ &= \frac{(ac+bd) - i(bc-ad)}{c^2+d^2} \end{aligned}$$

and also
$$\begin{aligned} \frac{z^*}{w^*} &= \frac{a-bi}{c-di}, \quad w \neq 0 \\ &= \frac{(a-bi)(c+di)}{(c-di)(c+di)} \\ &= \frac{(ac+bd) - i(bc-ad)}{c^2+d^2} \\ &= \left(\frac{z}{w}\right)^* \end{aligned}$$

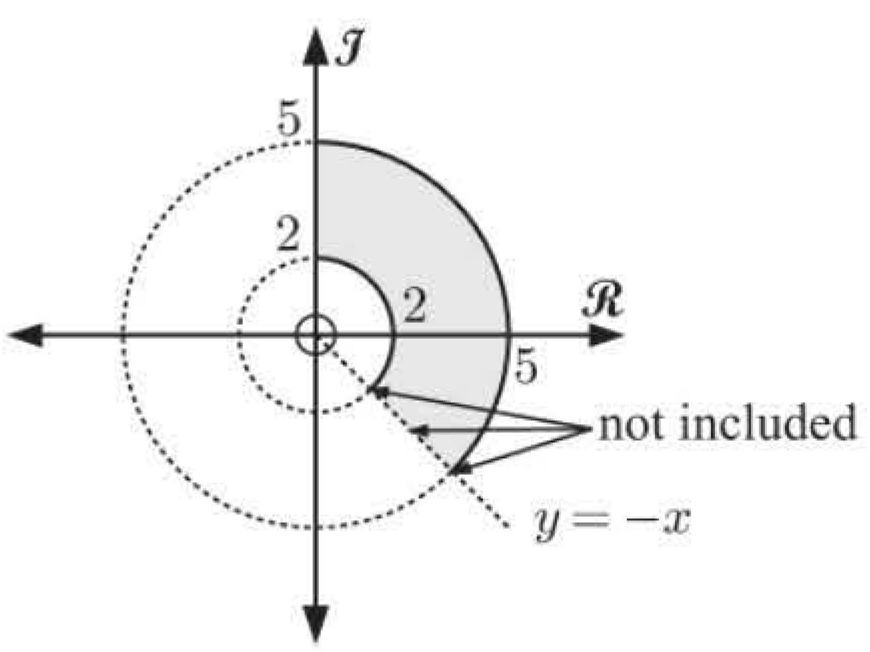
9



a
$$\arg \overrightarrow{\mathrm{OA}} = -\tfrac{\pi}{4}$$
$$\arg \overrightarrow{\mathrm{OB}} = -\tfrac{2\pi}{3}$$
$$\therefore\ \mathrm{A\widehat{O}B} = \tfrac{2\pi}{3} - \tfrac{\pi}{4} = \tfrac{5\pi}{12}$$

b
$$\begin{aligned} &zw \\ &= 2\sqrt{2}\ \mathrm{cis}\left(\tfrac{-\pi}{4}\right) \times 2\ \mathrm{cis}\left(\tfrac{-2\pi}{3}\right) \\ &= 4\sqrt{2}\ \mathrm{cis}\left(\tfrac{-\pi}{4} + \tfrac{-2\pi}{3}\right) \\ &= 4\sqrt{2}\ \mathrm{cis}\left(-\tfrac{11\pi}{12}\right) \\ \therefore\ &\arg(zw) = -\tfrac{11\pi}{12} \end{aligned}$$

10 $\{z \mid 2 \leqslant |z| \leqslant 5 \quad \text{and} \quad -\frac{\pi}{4} < \arg z \leqslant \frac{\pi}{2}\}$



11 If $z = r\ \mathrm{cis}\ \theta$, then $|z| = r$ and $\arg z = \theta$

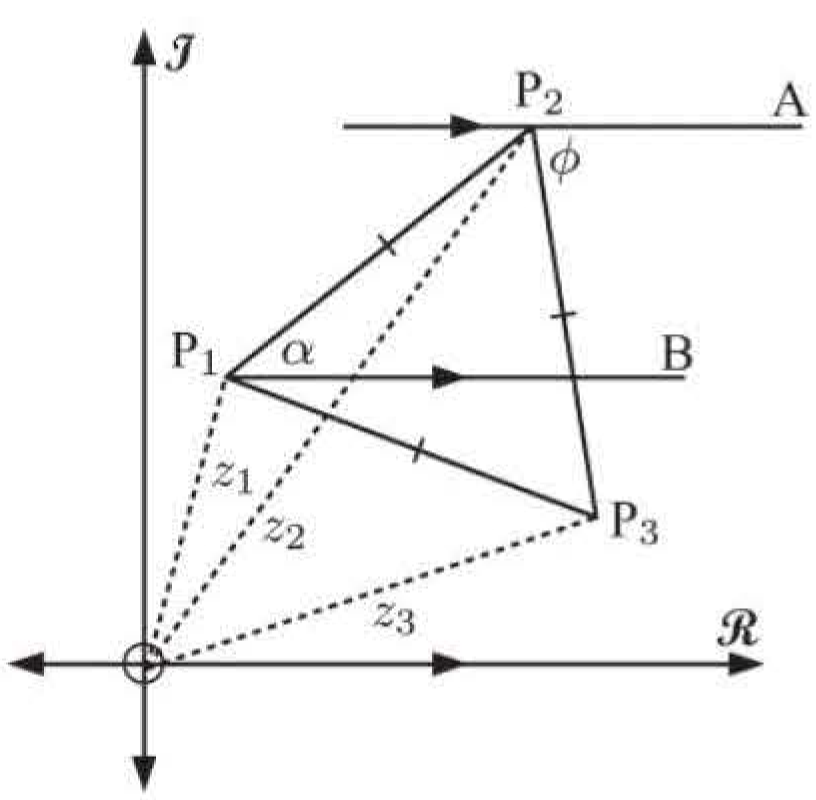
Now $\dfrac{1}{z} = (r\ \mathrm{cis}\ \theta)^{-1} = r^{-1}\ \mathrm{cis}\,(-\theta) = \dfrac{1}{r}\ \mathrm{cis}\,(-\theta)$

$\therefore\ \left|\dfrac{1}{z}\right| = \dfrac{1}{r} = \dfrac{1}{|z|}$ (if $z \neq 0$),

and $\arg\left(\dfrac{1}{z}\right) = -\theta = -\arg z$

12
$$\begin{aligned} z &= \mathrm{cis}\ \alpha \\ \therefore\ 1 + z &= 1 + \mathrm{cis}\ \alpha \\ &= 1 + \cos\alpha + i\sin\alpha \\ &= \left[1 + 2\cos^2\left(\tfrac{\alpha}{2}\right) - 1\right] + i\left[2\sin\left(\tfrac{\alpha}{2}\right)\cos\left(\tfrac{\alpha}{2}\right)\right] \\ &= 2\cos^2\left(\tfrac{\alpha}{2}\right) + i\left[2\sin\left(\tfrac{\alpha}{2}\right)\cos\left(\tfrac{\alpha}{2}\right)\right] \\ &= 2\cos\left(\tfrac{\alpha}{2}\right)\left[\cos\left(\tfrac{\alpha}{2}\right) + i\sin\left(\tfrac{\alpha}{2}\right)\right] \\ &= 2\cos\left(\tfrac{\alpha}{2}\right)\mathrm{cis}\left(\tfrac{\alpha}{2}\right) \end{aligned}$$
$\therefore\ |1 + z| = 2\cos\left(\frac{\alpha}{2}\right)$ and $\arg(1 + z) = \frac{\alpha}{2}$

13 **a**



Now $z_2 - z_1 = \overrightarrow{\mathrm{P_1P_2}}$

so $\arg(z_2 - z_1) = \alpha$ as shown on the diagram alongside

$z_3 - z_2 = \overrightarrow{\mathrm{P_2P_3}}$

so $\arg(z_3 - z_2) = -\phi$

Now $\mathrm{P_1\widehat{P_2}P_3} = \frac{\pi}{3}$ since the $\triangle$ is equilateral

$\therefore\ \alpha + \frac{\pi}{3} + \phi = \pi$ $\{(\mathrm{P_1B}) \parallel (\mathrm{P_2A})$, co-interior angles$\}$

$\therefore\ \phi = -\alpha + \frac{2\pi}{3}$

$\therefore\ \arg(z_3 - z_2) = \alpha - \frac{2\pi}{3}$ as required

b
$$\begin{aligned} \left|\frac{z_2 - z_1}{z_3 - z_2}\right| &= \left|\frac{\overrightarrow{\mathrm{P_1P_2}}}{\overrightarrow{\mathrm{P_2P_3}}}\right| \\ &= \frac{|\,\overrightarrow{\mathrm{P_1P_2}}\,|}{|\,\overrightarrow{\mathrm{P_2P_3}}\,|} \\ &= 1 \quad \text{since the } \triangle \text{ is equilateral} \\ \arg\left(\frac{z_2 - z_1}{z_3 - z_2}\right) &= \arg(z_2 - z_1) - \arg(z_3 - z_2) \\ &= \alpha - \left(\alpha - \tfrac{2\pi}{3}\right) \quad \{\text{from } \mathbf{a}\} \\ &= \alpha - \alpha + \tfrac{2\pi}{3} \\ &= \tfrac{2\pi}{3} \end{aligned}$$

14 The fifth roots of unity are $1, w, w^2, w^3$, and w^4, where $w = \text{cis}\left(\frac{2\pi}{5}\right)$.

a
$$(2z-1)^5 = 32$$
$$\therefore \frac{(2z-1)^5}{2^5} = 1$$
$$\therefore \left(\frac{2z-1}{2}\right)^5 = 1$$
$$\therefore \left(z-\tfrac{1}{2}\right)^5 = 1$$
$$\therefore z - \tfrac{1}{2} = 1, w, w^2, w^3, \text{ or } w^4$$
$$\therefore z = \tfrac{3}{2}, w+\tfrac{1}{2}, w^2+\tfrac{1}{2}, w^3+\tfrac{1}{2}, \text{ or } w^4+\tfrac{1}{2}$$
where $w = \text{cis}\left(\frac{2\pi}{5}\right)$

b
$$z^5+5z^4+10z^3+10z^2+5z = 0$$
$$\therefore z^5+5z^4+10z^3+10z^2+5z+1 = 1$$
$$\therefore (z+1)^5 = 1$$
$$\therefore z+1 = 1, w, w^2, w^3, w^4$$
$$\therefore z = 0, w-1, w^2-1, w^3-1, \text{ or } w^4-1$$
where $w = \text{cis}\left(\frac{2\pi}{5}\right)$

c
$$(z+1)^5 = (z-1)^5$$
$$\therefore \frac{(z+1)^5}{(z-1)^5} = 1, \quad z \neq 1$$
$$\therefore \left(\frac{z+1}{z-1}\right)^5 = 1$$
$$\therefore \frac{z+1}{z-1} = 1, w, w^2, w^3, \text{ or } w^4$$
$\dfrac{z+1}{z-1} = 1$ has no solutions as $z+1 \neq z-1$ for any z

If $\dfrac{z+1}{z-1} = w^k$, $k = 1, 2, 3, 4$
$$z+1 = w^k z - w^k$$
$$\therefore z(1-w^k) = -w^k - 1$$
$$\therefore z = \frac{-w^k-1}{1-w^k} = \frac{w^k+1}{w^k-1}$$
$$\therefore z = \frac{w+1}{w-1}, \frac{w^2+1}{w^2-1}, \frac{w^3+1}{w^3-1}, \frac{w^4+1}{w^4-1}$$
where $w = \text{cis}\left(\frac{2\pi}{5}\right)$

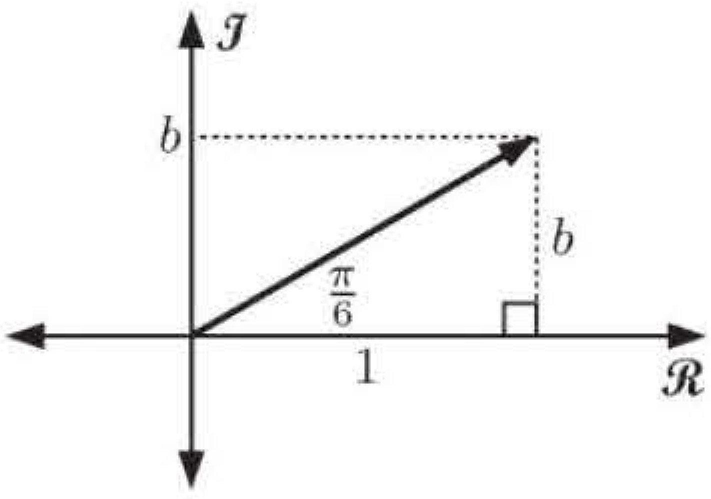
REVIEW SET 16C

1 **a** $-5i = 5 \text{ cis}\left(-\frac{\pi}{2}\right)$

b $2 - 2i\sqrt{3} = 4\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$
$= 4 \text{ cis}\left(-\frac{\pi}{3}\right)$

c $k - ki = -k\sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$
$= -k\sqrt{2} \text{ cis}\left(\frac{3\pi}{4}\right)$ which is in polar form since $k < 0$

2 $z = (1+bi)^2$ has argument $\frac{\pi}{3}$
$\therefore$ $1 + bi$ has argument $\frac{\pi}{6}$ $(b > 0)$
$$\therefore \tan\left(\tfrac{\pi}{6}\right) = \frac{b}{1}$$
$$\therefore b = \tfrac{1}{\sqrt{3}}$$



3 **a** $\text{cis }\theta \times \text{cis }\phi = (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)$
$= (\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\sin\theta\cos\phi + \cos\theta\sin\phi)$
$= \cos(\theta+\phi) + i\sin(\theta+\phi)$
$= \text{cis}(\theta+\phi)$ as required

b
$$1 - i = \sqrt{2}\left(\tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}i\right) = \sqrt{2} \text{ cis}\left(-\tfrac{\pi}{4}\right)$$
$$\therefore (1-i)z = \sqrt{2} \text{ cis}\left(-\tfrac{\pi}{4}\right) \times 2\sqrt{2} \text{ cis } \alpha$$
$$= 4 \text{ cis}\left(\alpha - \tfrac{\pi}{4}\right) \quad \{\text{using } \mathbf{a}\}$$
$$\therefore \arg[(1-i)z] = \alpha - \tfrac{\pi}{4}$$

4 **a** $\left|\dfrac{{z_1}^2}{{z_2}^2}\right| = \dfrac{|z_1|^2}{|z_2|^2}$ But $|z_1| = |z_2|$ since the triangle is isosceles

$\therefore \left|\dfrac{{z_1}^2}{{z_2}^2}\right| = 1$

Also, $\arg\left(\dfrac{{z_1}^2}{{z_2}^2}\right) = \arg({z_1}^2) - \arg({z_2}^2)$

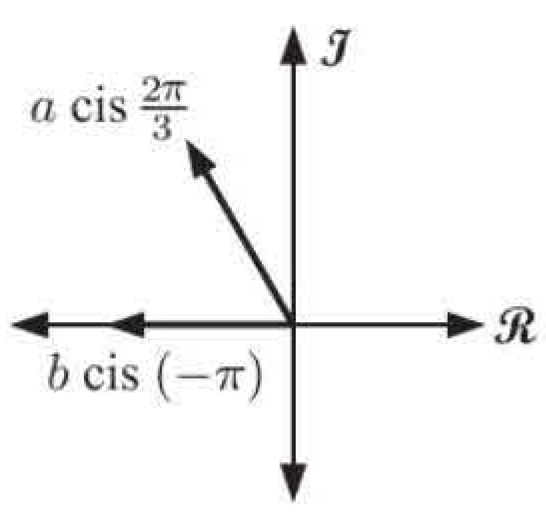
$$= 2\arg z_1 - 2\arg z_2$$
$$= 2(\arg z_1 - \arg z_2)$$
$$= 2 \times \tfrac{\pi}{2} \quad \text{since } z_1 \text{ and } z_2 \text{ are perpendicular}$$
$$= \pi$$

b $\dfrac{{z_1}^2}{{z_2}^2} = \text{cis}\,\pi = -1 \qquad \therefore\ {z_1}^2 = -{z_2}^2 \qquad \therefore\ {z_1}^2 + {z_2}^2 = 0$

5 $z = \sqrt[4]{a}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \sqrt[4]{a}\ \text{cis}\ \frac{\pi}{6}$

$w = \sqrt[4]{b}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right) = \sqrt[4]{b}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right) = \sqrt[4]{b}\ \text{cis}\left(-\frac{\pi}{4}\right)$

$$\therefore\ \left(\frac{z}{w}\right)^4 = \frac{z^4}{w^4} = \frac{\left(\sqrt[4]{a}\ \text{cis}\ \frac{\pi}{6}\right)^4}{\left(\sqrt[4]{b}\ \text{cis}\left(-\frac{\pi}{4}\right)\right)^4}$$
$$= \frac{a\ \text{cis}\ \frac{2\pi}{3}}{b\ \text{cis}\ (-\pi)}$$
$$= \frac{a\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)}{b(-1)}$$
$$= \frac{a}{2b} - \frac{a\sqrt{3}}{2b}i$$



$\therefore\ \mathcal{Re}\left(\left(\frac{z}{w}\right)^4\right) = \frac{a}{2b}, \quad \mathcal{Im}\left(\left(\frac{z}{w}\right)^4\right) = -\frac{a\sqrt{3}}{2b}$

6 **a** The 5th roots of unity are the solutions to $z^5 = 1$.

$\therefore\ z^5 = \text{cis}\,(0 + k2\pi)$

$\therefore\ z^5 = \text{cis}\,(k2\pi)$

$\therefore\ z = [\text{cis}\,(k2\pi)]^{\frac{1}{5}}$

$\therefore\ z = \text{cis}\left(\frac{k2\pi}{5}\right)$ {De Moivre}

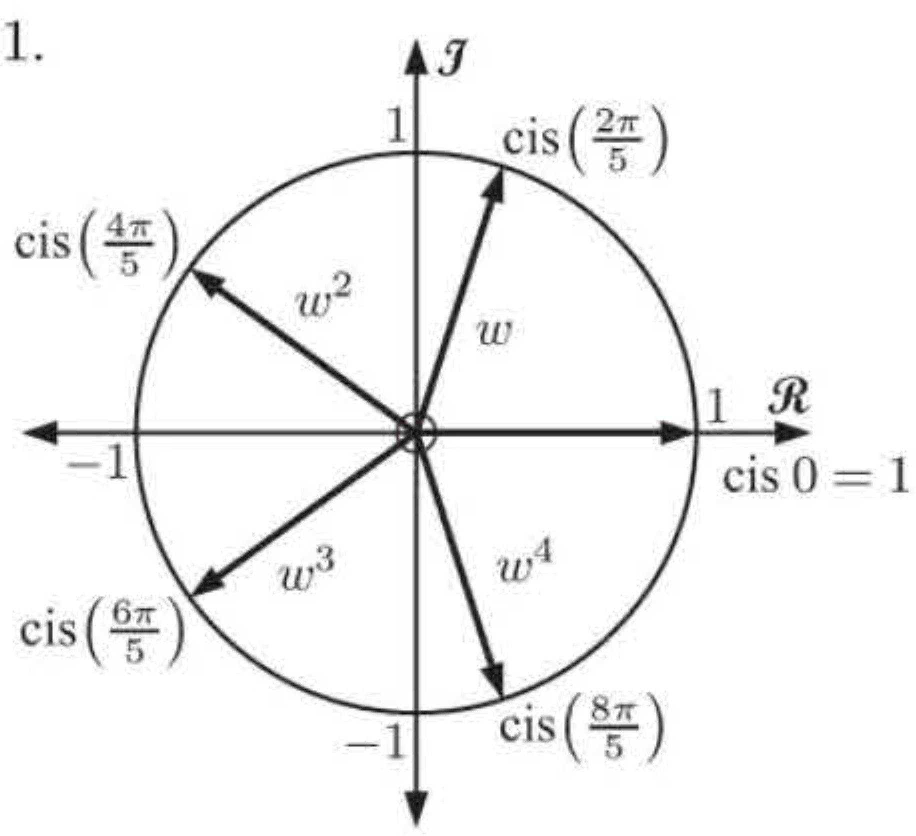
$\therefore\ z = \text{cis}\ 0 = 1$ or

$\text{cis}\left(\frac{2\pi}{5}\right) = w$ or

$\text{cis}\left(\frac{4\pi}{5}\right) = \left(\text{cis}\ \frac{2\pi}{5}\right)^2 = w^2$ or

$\text{cis}\left(\frac{6\pi}{5}\right) = \left(\text{cis}\ \frac{2\pi}{5}\right)^3 = w^3$ or

$\text{cis}\left(\frac{8\pi}{5}\right) = \left(\text{cis}\ \frac{2\pi}{5}\right)^4 = w^4$ {when $k = 0, 1, 2, 3, 4$}



Hence the five roots can be expressed as $1, w, w^2, w^3, w^4$ where $w = \text{cis}\left(\frac{2\pi}{5}\right)$

b $z^5 - 1 = (z-1)(z^4 + z^3 + z^2 + z + 1)$ .... (1)

Also, since $1, w, w^2, w^3$ and w^4 are the solutions to $z^5 = 1$,

$z^5 - 1 = (z-1)(z-w)(z-w^2)(z-w^3)(z-w^4)$ .... (2)

Equating (1) and (2),

$(z-1)(z^4 + z^3 + z^2 + z + 1) = (z-1)(z-w)(z-w^2)(z-w^3)(z-w^4)$

$\therefore\ z^4 + z^3 + z^2 + z + 1 = (z-w)(z-w^2)(z-w^3)(z-w^4)$

c $(2-w)(2-w^2)(2-w^3)(2-w^4) = 2^4+2^3+2^2+2+1$ {letting $z=2$}
$= 16+8+4+2+1$
$= 31$

7 The cube roots of $-8i$ are solutions to $z^3 = -8i$

$z^3 = 8 \text{ cis}\left(-\frac{\pi}{2}+k2\pi\right)$ where k is an integer

$\therefore \ z = 8^{\frac{1}{3}} \text{ cis}\left(-\frac{\pi}{6}+\frac{k2\pi}{3}\right)$

$\therefore \ z = 2 \text{ cis}\left(-\frac{\pi}{6}+\frac{k4\pi}{6}\right)$

$\therefore \ z = 2 \text{ cis}\left(-\frac{5\pi}{6}\right),\ 2 \text{ cis}\left(-\frac{\pi}{6}\right),\ 2 \text{ cis}\left(\frac{\pi}{2}\right)$ {letting $k=-1, 0, 1$}

$\therefore \ z = -\sqrt{3}-i,\ \sqrt{3}-i,\ 2i$

8 **a** $\cos 3\theta + i\sin 3\theta = \text{cis } 3\theta = (\text{cis } \theta)^3$

b
$$\frac{1}{\cos 2\theta + i\sin 2\theta} = \frac{1}{\text{cis } 2\theta} = (\text{cis } 2\theta)^{-1} = [(\text{cis } \theta)^2]^{-1} = (\text{cis } \theta)^{-2}$$

c
$$\cos\theta - i\sin\theta = \cos(-\theta)+i\sin(-\theta) = \text{cis}\,(-\theta) = (\text{cis } \theta)^{-1}$$

9 The fifth roots of $2+2i$ are the solutions to $z^5 = 2+2i$

$\therefore \ z^5 = 2\sqrt{2} \text{ cis}\left(\frac{\pi}{4}+k2\pi\right)$

$\therefore \ z = \left[2^{\frac{3}{2}} \text{ cis}\left(\frac{\pi}{4}+k2\pi\right)\right]^{\frac{1}{5}}$

$\therefore \ z = 2^{0.3} \text{ cis}\left(\frac{\pi}{20}+\frac{k2\pi}{5}\right)$ {De Moivre}

$\therefore \ z = 2^{0.3} \text{ cis}\left(\frac{\pi}{20}+\frac{k8\pi}{20}\right)$

$\therefore \ z = 2^{0.3} \text{ cis}\left(-\frac{3\pi}{4}\right),\ 2^{0.3} \text{ cis}\left(-\frac{7\pi}{20}\right),\ 2^{0.3} \text{ cis}\left(\frac{\pi}{20}\right),\ 2^{0.3} \text{ cis}\left(\frac{9\pi}{20}\right),\ 2^{0.3} \text{ cis}\left(\frac{17\pi}{20}\right)$

{letting $k=-2, -1, 0, 1, 2$}

10 Let $z = x+iy$ $\quad \therefore \ z+\frac{1}{z} = (x+iy)+\frac{1}{x+iy}\times\frac{x-iy}{x-iy}$

$$= (x+iy)+\frac{(x-iy)}{x^2+y^2}$$

$$= \frac{\left(x^2+y^2\right)(x+iy)+(x-iy)}{x^2+y^2}$$

$$= \frac{x^3+ix^2y+xy^2+y^3i+x-iy}{x^2+y^2}$$

$$= \frac{x(x^2+y^2+1)+i(x^2+y^2-1)y}{x^2+y^2}$$

which is real if $\frac{(x^2+y^2-1)y}{x^2+y^2} = 0$ $\quad \therefore \ x^2+y^2-1=0$ or $y=0$

$\therefore \ x^2+y^2=1$ or $y=0$

$\therefore \ |z|^2=1$ or $y=0$

$\therefore \ |z|=1$ or z is real

or Let $z = r \text{ cis } \theta$ $\therefore\ z + \dfrac{1}{z} = r \text{ cis } \theta + \dfrac{1}{r} \text{ cis }(-\theta)$

$$= r\cos\theta + ir\sin\theta + \frac{1}{r}\cos(-\theta) + i \times \frac{1}{r}\sin(-\theta)$$

This is real if $r\sin\theta + \dfrac{1}{r}\sin(-\theta) = 0$

$$\therefore\ r\sin\theta - \frac{1}{r}\sin\theta = 0$$

$$\therefore\ \sin\theta\left(r - \frac{1}{r}\right) = 0$$

$$\therefore\ \sin\theta = 0 \quad \text{or} \quad r - \frac{1}{r} = 0$$

$$\therefore\ \theta = 0 \quad \text{or} \quad r = 1 \quad \{r \geqslant 0\}$$

$$\therefore\ z = r \text{ (which is real) or } |z| = 1$$

11 **a** If $z = \text{cis } \theta$

$$= \cos\theta + i\sin\theta$$

$$\therefore\ |z| = \sqrt{\cos^2\theta + \sin^2\theta} = \sqrt{1} = 1$$

b If $z = \text{cis } \theta$

then $z^* = \text{cis }(-\theta) = (\text{cis } \theta)^{-1} = z^{-1} = \dfrac{1}{z}$

c $z = \text{cis } \theta$

$\therefore\ z^4 = (\text{cis } \theta)^4$

$\therefore\ z^4 = \text{cis } 4\theta$ {De Moivre}

$\therefore\ z^4 = \cos 4\theta + i\sin 4\theta$ .... (1)

Also, $z^4 = (\cos\theta + i\sin\theta)^4$

$\therefore\ z^4 = \cos^4\theta + 4\cos^3\theta i\sin\theta + 6\cos^2\theta\, i^2\sin^2\theta + 4\cos\theta\, i^3\sin^3\theta + i^4\sin^4\theta$

$\therefore\ z^4 = \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta$ .... (2)

Equating real parts in (1) and (2) gives

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$

$$\begin{aligned}\therefore\ \sin^4\theta &= \cos 4\theta - \cos^4\theta + 6\cos^2\theta\sin^2\theta\\ &= \cos 4\theta - (1-\sin^2\theta)^2 + 6(1-\sin^2\theta)\sin^2\theta\\ &= \cos 4\theta - (1 - 2\sin^2\theta + \sin^4\theta) + 6\sin^2\theta - 6\sin^4\theta\\ &= \cos 4\theta - 1 + 2\sin^2\theta - \sin^4\theta + 6\sin^2\theta - 6\sin^4\theta\end{aligned}$$

$$\begin{aligned}\therefore\ 8\sin^4\theta &= \cos 4\theta - 1 + 8\sin^2\theta\\ &= \cos 4\theta - 1 + 8\left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta\right)\\ &= \cos 4\theta - 1 + 4 - 4\cos 2\theta\\ &= \cos 4\theta - 4\cos 2\theta + 3\end{aligned}$$

$$\therefore\ \sin^4\theta = \tfrac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3)$$

12 **a** If w is the root of $z^5 = 1$ with smallest positive argument, then $w = \text{cis}\left(\frac{2\pi}{5}\right)$ and $w^4 = \text{cis}\left(\frac{8\pi}{5}\right)$.

$$\begin{aligned}\text{These have sum } &= \cos\left(\tfrac{2\pi}{5}\right) + i\sin\left(\tfrac{2\pi}{5}\right) + \cos\left(\tfrac{8\pi}{5}\right) + i\sin\left(\tfrac{8\pi}{5}\right)\\ &= \cos\left(\tfrac{2\pi}{5}\right) + \cancel{i\sin\left(\tfrac{2\pi}{5}\right)} + \cos\left(\tfrac{2\pi}{5}\right) - \cancel{i\sin\left(\tfrac{2\pi}{5}\right)}\\ &= 2\cos\left(\tfrac{2\pi}{5}\right)\end{aligned}$$

and product $= \text{cis}\left(\frac{2\pi}{5}\right) \times \text{cis}\left(\frac{8\pi}{5}\right) = \text{cis}\left(\frac{10\pi}{5}\right) = \text{cis}2\pi = 1$

$\therefore$ a real quadratic with roots $w,\ w^4$ is $a\left(z^2 - 2\cos\left(\frac{2\pi}{5}\right)z + 1\right) = 0,\ a \neq 0$

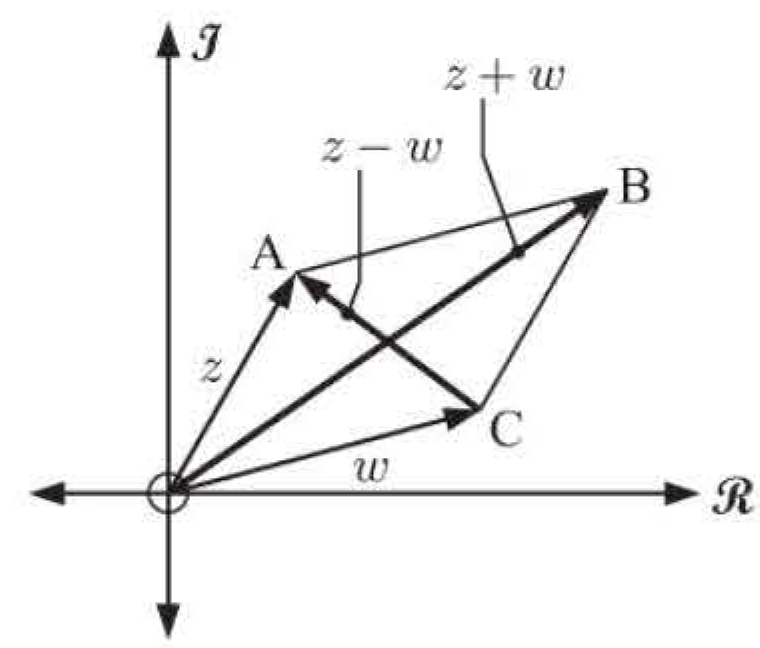
b Let $\alpha = w + w^4$ and $\beta = w^2 + w^3$

Now we know that $1 + w + w^2 + w^3 + w^4 = 0$ .... $(*)$

$$\begin{aligned} 1 + (w + w^4) + (w^2 + w^3) &= 0 \\ 1 + \alpha + \beta &= 0 \\ \alpha + \beta &= -1 \\ \text{and} \quad \alpha\beta &= (w + w^4)(w^2 + w^3) \\ &= w^3 + w^4 + w^6 + w^7 \\ &= w^3 + w^4 + w + w^2 \quad \{\text{as } w^5 = 1\} \\ &= w + w^2 + w^3 + w^4 \\ &= -1 \qquad \{\text{from } (*)\} \end{aligned}$$

$\therefore$ the quadratic equation is $a(z^2 + z - 1) = 0, \quad a \neq 0$

13 Consider the diagram which shows vectors w, z, $z + w$, and $z - w$.



Clearly OABC is a parallelogram with $\overrightarrow{OB} = z + w$ and $\overrightarrow{CA} = z - w$

If $|z + w| = |z - w|$, the diagonals are equal in length.

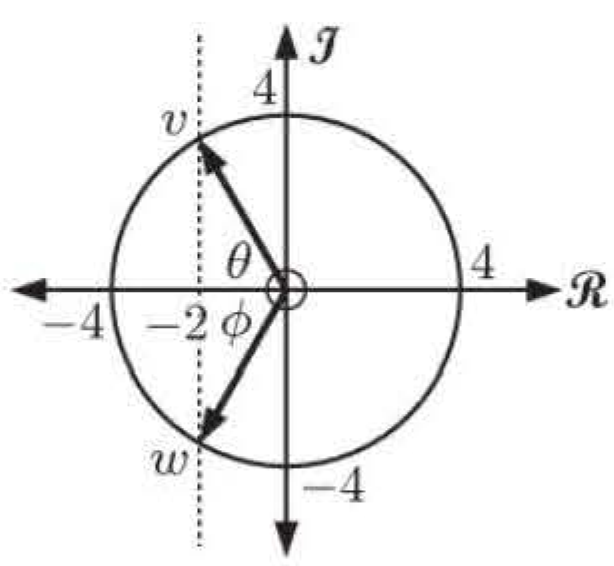
Hence, OABC is actually a rectangle and so $\widehat{COA}$ is a right angle $\therefore$ $\arg z$ and $\arg w$ differ by $\frac{\pi}{2}$.

14 **a**

$$\begin{aligned} |z| &= |z + 4| \\ \therefore \quad |z|^2 &= |z + 4|^2 \\ \text{Let} \quad z &= x + yi \\ \therefore \quad |x + yi|^2 &= |(x + 4) + yi|^2 \\ \therefore \quad x^2 + y^2 &= (x + 4)^2 + y^2 \\ \therefore \quad \cancel{x^2} + \cancel{y^2} &= \cancel{x^2} + 8x + 16 + \cancel{y^2} \\ \therefore \quad -8x &= 16 \\ \therefore \quad x &= -2 \end{aligned}$$

$\therefore$ the real part of z is -2.

b **i**



ii In the diagram above, $\cos\theta = \frac{2}{4} = \frac{1}{2}$

$\therefore \quad \theta = \arccos(\frac{1}{2}) = \frac{\pi}{3}$

$\therefore \quad \arg v = \pi - \theta = \frac{2\pi}{3}$

iii $\cos\phi = \frac{2}{4} = \frac{1}{2}$

$\therefore \quad \phi = \arccos(\frac{1}{2}) = \frac{\pi}{3}$

$\therefore \quad \arg w = -\pi + \phi = -\frac{2\pi}{3}$

iv $v = 4 \text{ cis}\left(\frac{2\pi}{3}\right)$, $\quad w = 4 \text{ cis}\left(-\frac{2\pi}{3}\right)$

$$\begin{aligned} \therefore \quad \frac{v^m w}{i} &= \frac{\left(4 \text{ cis}\left(\frac{2\pi}{3}\right)\right)^m 4 \text{ cis}\left(-\frac{2\pi}{3}\right)}{\text{cis}\left(\frac{\pi}{2}\right)} \\ &= \frac{4^m \text{ cis}\left(\frac{2m\pi}{3}\right) 4 \text{ cis}\left(-\frac{2\pi}{3}\right)}{\text{cis}\left(\frac{\pi}{2}\right)} \\ &= 4^{m+1} \text{ cis}\left(\frac{2m\pi}{3} - \frac{2\pi}{3} - \frac{\pi}{2}\right) \end{aligned}$$

$$\begin{aligned} \therefore \quad \arg\left(\frac{v^m w}{i}\right) &= \frac{2m\pi}{3} - \frac{2\pi}{3} - \frac{\pi}{2} \\ &= \frac{4m\pi}{6} - \frac{4\pi}{6} - \frac{3\pi}{6} \\ &= \frac{\pi(4m - 7)}{6} \end{aligned}$$

v $\dfrac{v^m w}{i}$ is real when

$$\begin{aligned} \arg\left(\frac{v^m w}{i}\right) &= 0 + k\pi, \quad k \in \mathbb{Z} \\ \therefore \quad \frac{\cancel{\pi}(4m - 7)}{6} &= k\cancel{\pi} \\ \therefore \quad 4m - 7 &= 6k \\ \therefore \quad m &= \frac{7 + 6k}{4} \end{aligned}$$

One such value of m is

$m = \frac{7}{4}$ $\quad \{\text{when } k = 0\}$