Chapter 8

COUNTING AND THE BINOMIAL EXPANSION

EXERCISE 8A

1 There are 3 paths from P to Q
3 paths from Q to R
2 paths from R to S

$\therefore$ number of routes possible
$= 3 \times 3 \times 2$ {product principle}
$= 18$

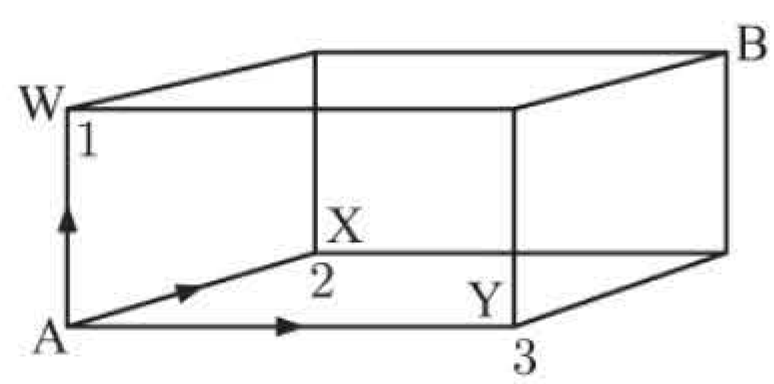
2 **a** There are 4 choices for A. But once A is located, there is 1 choice for B, 1 for C, and 1 for D.
$\therefore$ there are $4 \times 1 \times 1 \times 1 = 4$ ways.

b There are 4 choices for A. But once A is located there are 2 choices for B.

Once B is located there is 1 choice for C and 1 for D.
$\therefore$ there are $4 \times 2 \times 1 \times 1 = 8$ ways.

c There are 4 choices for A. Once A is located there are 3 choices for B. Once B is located there are 2 choices for C and then 1 for D.
$\therefore$ there are $4 \times 3 \times 2 \times 1 = 24$ ways.

3



From A there are 3 possible first leg paths, to W, X, or Y. Then there are 2 second leg paths to B.
$\therefore$ total number $= 3 \times 2 = 6$ paths.

4 Any of the 7 teams could be in 'top' position. Then there are 6 left which could be in the 'second' position.
So, there are $7 \times 6 = 42$ possible ways.

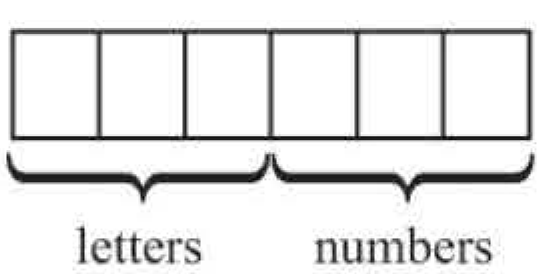
5 Any of the 8 teams could be 'top'.
Any of the remaining 7 could be second.
Any of the remaining 6 could be third.
Any of the remaining 5 could be fourth.
$\therefore$ there are $8 \times 7 \times 6 \times 5$
$= 1680$ ways.

6 There are 5 digits to choose from.

a Number of ways $= 5 \times 5 \times 5 = 125$

b Number of ways $= 5 \times 4 \times 3 = 60$

7



Repetitions are allowed.
$\therefore$ total number of ways
$= 26 \times 26 \times 26 \times 10 \times 10 \times 10$
$= 17\,576\,000$

8 **a** The 1st letter could go into either of the 2 boxes, and the second could go into either of the 2 boxes,
$\therefore$ there are $2 \times 2 = 4$ ways.

These are:

Box X	Box Y
A, B	-
A	B
B	A
-	A, B

b There are $3 \times 3 = 9$ ways.

c There are $3 \times 3 \times 3 \times 3 = 81$ ways.

EXERCISE 8B

1 **a** There are $2 \times 2 + 3 \times 3$
$= 13$ different paths

b There are $4 \times 2 + 3 \times 2 \times 2$
$= 20$ different paths

c There are $2 + 4 \times 2 + 3 \times 3$
$= 19$ different paths

d There are $2 \times 2 + 2 \times 2 + 2 \times 3 \times 4$
$= 32$ different paths

2 There are $3 \times 2 + 3 \times 1 + 2 \times 2$
$= 13$ different train journeys

EXERCISE 8C.1

1 $0! = 1$

$1! = 1$

$2! = 2 \times 1 = 2$

$3! = 3 \times 2 \times 1 = 6$

$4! = 4 \times 3 \times 2 \times 1 = 24$

$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

$6! = 6 \times 5! = 6 \times 120 = 720$

$7! = 7 \times 6! = 7 \times 720 = 5040$

$8! = 8 \times 7! = 8 \times 5040 = 40\,320$

$9! = 9 \times 8! = 9 \times 40\,320 = 362\,880$

$10! = 10 \times 9! = 10 \times 362\,880 = 3\,628\,800$

2 **a** $\dfrac{6!}{5!} = \dfrac{6 \times \cancel{5!}}{\cancel{5!}_1} = 6$ **b** $\dfrac{6!}{4!} = \dfrac{6 \times 5 \times \cancel{4!}}{\cancel{4!}_1} = 30$ **c** $\dfrac{6!}{7!} = \dfrac{\cancel{6!}^1}{7 \times \cancel{6!}} = \frac{1}{7}$

d $\dfrac{4!}{6!} = \dfrac{\cancel{4!}^1}{6 \times 5 \times \cancel{4!}} = \frac{1}{30}$ **e** $\dfrac{100!}{99!} = \dfrac{100 \times \cancel{99!}}{\cancel{99!}_1} = 100$ **f** $\dfrac{7!}{5! \times 2!} = \dfrac{7 \times 6 \times \cancel{5!}}{\cancel{5!} \times 2}$ $= 21$

3 **a** $\dfrac{n!}{(n-1)!}$ $= \dfrac{n \times \cancel{(n-1)!}}{\cancel{(n-1)!}_1}$ $= n, \quad n \geqslant 1$

b $\dfrac{(n+2)!}{n!}$ $= \dfrac{(n+2)(n+1)\cancel{n!}}{\cancel{n!}_1}$ $= (n+2)(n+1), \quad n \geqslant 0$

c $\dfrac{(n+1)!}{(n-1)!}$ $= \dfrac{(n+1)(n)\cancel{(n-1)!}}{\cancel{(n-1)!}_1}$ $= n(n+1), \quad n \geqslant 1$

4 **a** $7 \times 6 \times 5$ $= \dfrac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1}$ $= \dfrac{7!}{4!}$

b 10×9 $= \dfrac{10 \times 9 \times 8!}{8!}$ $= \dfrac{10!}{8!}$

c $11 \times 10 \times 9 \times 8 \times 7$ $= \dfrac{11 \times 10 \times 9 \times 8 \times 7 \times 6!}{6!}$ $= \dfrac{11!}{6!}$

d $\dfrac{13 \times 12 \times 11}{3 \times 2 \times 1}$ $= \dfrac{13 \times 12 \times 11 \times 10!}{10! \times 3 \times 2 \times 1}$ $= \dfrac{13!}{10! \times 3!}$

e $\dfrac{1}{6 \times 5 \times 4}$ $= \dfrac{3!}{6 \times 5 \times 4 \times 3!}$ $= \dfrac{3!}{6!}$

f $\dfrac{4 \times 3 \times 2 \times 1}{20 \times 19 \times 18 \times 17}$ $= \dfrac{4! \times 16!}{20 \times 19 \times 18 \times 17 \times 16!}$ $= \dfrac{4! \times 16!}{20!}$

5 **a** $5! + 4!$ $= 5 \times 4! + 4!$ $= 4!(5+1)$ $= 6 \times 4!$

b $11! - 10!$ $= 11 \times 10! - 10!$ $= 10!(11-1)$ $= 10 \times 10!$

c $6! + 8!$ $= 6! + 8 \times 7 \times 6!$ $= 6!(1 + 8 \times 7)$ $= 57 \times 6!$

d $12! - 10!$ $= 12 \times 11 \times 10! - 10!$ $= 10!(12 \times 11 - 1)$ $= 131 \times 10!$

e $9! + 8! + 7!$ $= 9 \times 8 \times 7! + 8 \times 7! + 7!$ $= 7!(72 + 8 + 1)$ $= 81 \times 7!$

f $7! - 6! + 8!$ $= 7 \times 6! - 6! + 8 \times 7 \times 6!$ $= 6!(7 - 1 + 56)$ $= 62 \times 6!$

g $12! - 2 \times 11!$ $= 12 \times 11! - 2 \times 11!$ $= 11!(12-2)$ $= 10 \times 11!$

h $3 \times 9! + 5 \times 8!$ $= 3 \times 9 \times 8! + 5 \times 8!$ $= 8!(27+5)$ $= 32 \times 8!$

6 **a** $$\begin{aligned}&\frac{12!-11!}{11}\\&=\frac{12\times 11!-11!}{11}\\&=\frac{11!(12-1)}{11}\\&=\frac{11!\times 11}{11}\\&=11!\end{aligned}$$

b $$\begin{aligned}&\frac{10!+9!}{11}\\&=\frac{10\times 9!+9!}{11}\\&=\frac{9!(10+1)}{11}\\&=\frac{9!\times 11}{11}\\&=9!\end{aligned}$$

c $$\begin{aligned}&\frac{10!-8!}{89}\\&=\frac{10\times 9\times 8!-8!}{89}\\&=\frac{8!(90-1)}{89}\\&=\frac{8!\times 89}{89}\\&=8!\end{aligned}$$

d $$\begin{aligned}&\frac{10!-9!}{9!}\\&=\frac{10\times 9!-9!}{9!}\\&=\frac{9!(10-1)}{9!}\\&=9\end{aligned}$$

e $$\begin{aligned}&\frac{6!+5!-4!}{4!}\\&=\frac{6\times 5\times 4!+5\times 4!-4!}{4!}\\&=\frac{4!(30+5-1)}{4!}\\&=34\end{aligned}$$

f $$\begin{aligned}&\frac{n!+(n-1)!}{(n-1)!}\\&=\frac{n\times(n-1)!+(n-1)!}{(n-1)!}\\&=\frac{(n-1)!(n+1)}{(n-1)!}\\&=n+1\end{aligned}$$

g $$\begin{aligned}&\frac{n!-(n-1)!}{n-1}\\&=\frac{n\times(n-1)!-(n-1)!}{n-1}\\&=\frac{(n-1)!(n-1)}{n-1}\\&=(n-1)!\end{aligned}$$

h $$\begin{aligned}&\frac{(n+2)!+(n+1)!}{n+3}\\&=\frac{(n+2)(n+1)!+(n+1)!}{n+3}\\&=\frac{(n+1)!(n+2+1)}{n+3}\\&=(n+1)!\end{aligned}$$

EXERCISE 8C.2

1 **a** $$\begin{aligned}\binom{3}{1}&=\frac{3!}{1!(3-1)!}\\&=\frac{3!}{1!\times 2!}\\&=\frac{3\times\cancel{2\times 1}}{1\times\cancel{2\times 1}}\\&=3\end{aligned}$$

b $$\begin{aligned}\binom{4}{2}&=\frac{4!}{2!(4-2)!}\\&=\frac{4!}{2!\times 2!}\\&=\frac{4\times 3\times\cancel{2\times 1}}{2\times 1\times\cancel{2\times 1}}\\&=\frac{12}{2}\\&=6\end{aligned}$$

c $$\begin{aligned}\binom{7}{3}&=\frac{7!}{3!(7-3)!}\\&=\frac{7!}{3!\times 4!}\\&=\frac{7\times 6\times 5\times\cancel{4\times 3\times 2\times 1}}{3\times 2\times 1\times\cancel{4\times 3\times 2\times 1}}\\&=\frac{210}{6}\\&=35\end{aligned}$$

d $$\begin{aligned}\binom{10}{4}&=\frac{10!}{4!(10-4)!}\\&=\frac{10!}{4!\times 6!}\\&=\frac{10\times 9\times 8\times 7\times\cancel{6\times 5\times 4\times 3\times 2\times 1}}{4\times 3\times 2\times 1\times\cancel{6\times 5\times 4\times 3\times 2\times 1}}\\&=\frac{5040}{24}\\&=210\end{aligned}$$

2 **a** **i** $\binom{8}{2} = \dfrac{8!}{2!(8-2)!}$
$= \dfrac{8!}{2! \times 6!}$
$= \dfrac{8 \times 7 \times \cancel{6 \times 5 \times 4 \times 3 \times 2 \times 1}}{2 \times 1 \times \cancel{6 \times 5 \times 4 \times 3 \times 2 \times 1}}$
$= \dfrac{56}{2}$
$= 28$

ii $\binom{8}{6} = \dfrac{8!}{6!(8-6)!}$
$= \dfrac{8!}{6! \times 2!}$
$= \dfrac{8 \times 7 \times \cancel{6 \times 5 \times 4 \times 3 \times 2 \times 1}}{\cancel{6 \times 5 \times 4 \times 3 \times 2 \times 1} \times 2 \times 1}$
$= \dfrac{56}{2}$
$= 28$

b $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$ for all $n \in \mathbb{Z}^+$, $r = 0, 1, 2, ..., n$.

$\binom{n}{n-r} = \dfrac{n!}{(n-r)!(n-(n-r))!}$ for all $n \in \mathbb{Z}^+$, $r = 0, 1, 2, ..., n$.
$= \dfrac{n!}{(n-r)!(n-n+r)!}$
$= \dfrac{n!}{(n-r)!r!}$
$= \dfrac{n!}{r!(n-r)!}$

$\therefore \binom{n}{r} = \binom{n}{n-r}$ for all $n \in \mathbb{Z}^+$, $r = 0, 1, 2, ..., n$.

3 $\binom{9}{k} = 4\binom{7}{k-1}$

$\therefore \dfrac{9!}{k!(9-k)!} = 4\left(\dfrac{7!}{(k-1)!(7-[k-1])!}\right)$

$\therefore \dfrac{9!}{k!(9-k)!} = \dfrac{4 \times 7!}{(k-1)!(8-k)!}$

$\therefore \dfrac{9!}{4 \times 7!} = \dfrac{k!(9-k)!}{(k-1)!(8-k)!}$

$\therefore \dfrac{9 \times 8 \times \cancel{7!}}{4 \times \cancel{7!}} = \dfrac{k\cancel{(k-1)!} \times (9-k)\cancel{(8-k)!}}{\cancel{(k-1)!(8-k)!}_1}$

$\therefore \dfrac{9 \times 8}{4} = k(9-k)$

$\therefore 18 = 9k - k^2$

$\therefore k^2 - 9k + 18 = 0$

$\therefore (k-3)(k-6) = 0$

$\therefore k = 3$ or 6

EXERCISE 8D

1 **a** W, X, Y, Z

b WX, WY, WZ, XW, XY, XZ, YW, YX, YZ, ZW, ZX, ZY

c WXY, WXZ, WYX, WYZ, WZX, WZY, XWY, XWZ, XYW, XYZ, XZW, XZY, YWX, YWZ, YXW, YXZ, YZW, YZX, ZWX, ZWY, ZXW, ZXY, ZYW, ZYX

2 **a** AB, AC, AD, AE, BA, BC, BD, BE, CA, CB, CD, CE, DA, DB, DC, DE, EA, EB, EC, ED

b ABC, ABD, ABE, ACB, ACD, ACE, ADB, ADC, ADE, AEB, AEC, AED, BAC, BAD, BAE, BCA, BCD, BCE, BDA, BDC, BDE, BEA, BEC, BED, CAB, CAD, CAE, CBA, CBD, CBE, CDA, CDB, CDE, CEA, CEB, CED, DAB, DAC, DAE, DBA, DBC, DBE, DCA, DCB, DCE, DEA, DEB, DEC, EAB, EAC, EAD, EBA, EBC, EBD, ECA, ECB, ECD, EDA, EDB, EDC

(2 at a time: 20   3 at a time: 60)

3 **a** There are $5! = 120$ different orderings.

b There are $8 \times 7 \times 6 = 336$ different orderings.

c There are $10 \times 9 \times 8 \times 7 = 5040$ different signals.

4 **a** $\begin{array}{|c|c|}\hline 4 & 3 \\ \hline\end{array}$ $\therefore$ there are $4 \times 3 = 12$ different signals

b $\begin{array}{|c|c|c|}\hline 4 & 3 & 2 \\ \hline\end{array}$ $\therefore$ there are $4 \times 3 \times 2 = 24$ different signals

c $12 + 24 = 36$ different signals {using **a** and **b**}

5 There are 6 different letters $\therefore$ $6! = 720$ permutations.

a $\begin{array}{|c|c|c|c|c|c|}\hline 4 & 3 & 2 & 1 & 1 & 1 \\ \hline\end{array}$ $\therefore$ there are $4 \times 3 \times 2 \times 1 \times 1 \times 1 = 24$ permutations

E D

b $\begin{array}{|c|c|c|c|c|c|}\hline 1 & 4 & 3 & 2 & 1 & 1 \\ \hline\end{array}$ $\therefore$ there are $1 \times 4 \times 3 \times 2 \times 1 \times 1 = 24$ permutations

F A

c $\begin{array}{|c|c|c|c|c|c|}\hline 2 & 4 & 3 & 2 & 1 & 1 \\ \hline\end{array}$ $\therefore$ there are $2 \times 4 \times 3 \times 2 \times 1 \times 1 = 48$ permutations

A or E the other one

6 **a** $\begin{array}{|c|c|c|}\hline 7 & 7 & 7 \\ \hline\end{array}$ So, there are $7^3 = 343$ different numbers.

b $\begin{array}{|c|c|c|}\hline 7 & 6 & 5 \\ \hline\end{array}$ So, there are $7 \times 6 \times 5 = 210$ different numbers.

c $\begin{array}{|c|c|c|}\hline 6 & 5 & 4 \\ \hline\end{array}$ So, there are $6 \times 5 \times 4 = 120$ different numbers.

6 remain fill 1st with any of 4 odds

7 **a** $\begin{array}{|c|c|c|}\hline 9 & 9 & 8 \\ \hline\end{array}$

cannot use 0

So, there are
$9 \times 9 \times 8 = 648$ numbers.

b $\begin{array}{|c|c|c|}\hline 8 & 8 & 1 \\ \hline\end{array}$

cannot use 0 or 5 a 5

So, there are
$8 \times 8 \times 1 = 64$ numbers.

c $\begin{array}{|c|c|c|}\hline 9 & 8 & 1 \\ \hline\end{array}$

0

So, there are
$9 \times 8 \times 1 = 72$
different numbers.

d $64 + 72 = 136$ different numbers.
{using **b** and **c**}

8 **a** As there are no restrictions, the number of ways is $5! = 120$.

b X and Y are together in 2! ways {XY or YX}
They, together with the other three books can be permutated in 4! ways.
$\therefore$ total number $= 2! \times 4! = 48$ ways.

c $120 - 48 = 72$ ways {using **a** and **b**}

9 **a** As there are no restrictions, the number of ways is $10! = 3\,628\,800$.

b A, B and C are together in 3! ways {ABC, ACB, BAC, BCA, CAB, CBA}
They, together with the other 7 can be permutated in 8! ways.
$\therefore$ total number is $3! \times 8! = 241\,920$ ways.

10 **a** $\begin{array}{|c|c|c|}\hline 4 & 4 & 3 \\ \hline \end{array}$ So, there are $4 \times 4 \times 3 = 48$ different numbers.

$\uparrow$ not 0

b $\begin{array}{|c|c|c|}\hline 2 & 4 & 3 \\ \hline \end{array}$ So, there are $2 \times 4 \times 3 = 24$ different numbers.

$\uparrow$ 1 or 3

c The last digit must be a 0 or 8.

If it is 0: $\begin{array}{|c|c|c|}\hline 3 & 3 & 1 \\ \hline \end{array}$ $\therefore$ $3 \times 3 \times 1 = 9$ different numbers

$\uparrow$ 3, 5, or 8 $\quad\uparrow$ 0

If it is 8: $\begin{array}{|c|c|c|}\hline 2 & 3 & 1 \\ \hline \end{array}$ $\therefore$ $2 \times 3 \times 1 = 6$ different numbers

$\uparrow$ 3 or 5 $\quad\uparrow$ 8

$\therefore$ in total there are $9 + 6 = 15$ different numbers.

11 **a** $\begin{array}{|c|c|c|c|}\hline 6 & 5 & 4 & 3 \\ \hline \end{array}$ So, there are $6 \times 5 \times 4 \times 3 = 360$ different arrangements.

b If no vowels are used, there are 4 letters to choose from.

$\therefore$ $\begin{array}{|c|c|c|c|}\hline 4 & 3 & 2 & 1 \\ \hline \end{array}$ So, there are $4! = 24$ different arrangements.

$\therefore$ if at least one vowel must be used, there are $360 - 24$ {from **a**}

$= 336$ different arrangements

c We first count the number of ways two vowels are adjacent.

A and O can be put together in $2!$ ways {AO or OA}

These vowels can be placed in any one of 3 positions {1st and 2nd, 2nd and 3rd, or 3rd and 4th}

The remaining 2 places can be filled from the other 4 letters in 4×3 different ways.

$\therefore$ two vowels are adjacent in $2! \times 3 \times 4 \times 3 = 72$ ways

$\therefore$ no two vowels are adjacent in $360 - 72 = 288$ ways

12 **a** $\begin{array}{|c|c|c|c|c|}\hline 9 & 8 & 7 & 6 & 5 \\ \hline \end{array}$ So, there are $9 \times 8 \times 7 \times 6 \times 5 = 15\,120$ different ways.

b $\begin{array}{|c|c|c|c|c|}\hline 4 & 3 & 2 & 6 & 5 \\ \hline \end{array}$ So, there are $4 \times 3 \times 2 \times 6 \times 5 = 720$ different ways.

$\uparrow$ 2, 4, 6, or 8

13 **a** $\begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ \hline \end{array}$ $\therefore$ $10! = 3\,628\,800$ different ways.

b **i** $\begin{array}{|c|c|c|c|c|c|}\hline 10 & 9 & 8 & 7 & 6 & 5 \\ \hline \end{array}$ $\therefore$ $10 \times 9 \times 8 \times 7 \times 6 \times 5 = 151\,200$ different ways

$\uparrow$ Alice can sit in any of the 10 seats

$\uparrow$ Her friend can sit in any of the remaining 9 seats

ii Alice can sit in any of the 8 middle seats.

She can choose the two friends to sit next to her in 5×4 different ways.

The remaining 3 friends can occupy the other 7 seats in $7 \times 6 \times 5$ different ways.

$\therefore$ there are $8 \times 5 \times 4 \times 7 \times 6 \times 5 = 33\,600$ different ways.

EXERCISE 8E

1 ABCD, ABCE, ABCF, ABDE, ABDF, ABEF, ACDE, ACDF, ACEF, ADEF, BCDE, BCDF, BCEF, BDEF, CDEF, and $\binom{6}{4} = 15$ ✓

2 There are $\binom{17}{11} = 12\,376$ different teams.

3 **a** There are $\binom{9}{5} = 126$ different possible selections.

b If question 1 is compulsory there are $\binom{1}{1}\binom{8}{4} = 1 \times 70 = 70$ possible selections.

4 **a** If no restrictions, there are $\binom{13}{3} = 286$ different committees.

b $\binom{1}{1}\binom{12}{2} = 66$ of them consist of the president and two others.

5 **a** If no restrictions, there are $\binom{12}{5} = 792$ different teams.

b **i** Those containing the captain and vice-captain number $\binom{2}{2}\binom{10}{3} = 1 \times 120 = 120$.

ii Those containing exactly one of the captain and vice captain number $\binom{2}{1}\binom{10}{4} = 2 \times 210 = 420$.

6 Number of different teams $= \binom{3}{3}\binom{1}{0}\binom{11}{6} = 1 \times 1 \times 462 = 462$.

7 **a** If 1 person must be selected, number of ways $= \binom{1}{1}\binom{9}{3} = 84$

b If 2 are always excluded, the number of ways $= \binom{2}{0}\binom{8}{4} = 70$

c If 1 is always 'in' and 2 are always 'out', the number of ways is $\binom{1}{1}\binom{2}{0}\binom{7}{3} = 35$

8 **a** If there are no restrictions the number of ways $= \binom{16}{5} = 4368$

b The three men can be chosen in $\binom{10}{3}$ ways and the 2 women in $\binom{6}{2}$ ways.

$\therefore$ total number of ways $= \binom{10}{3} \times \binom{6}{2} = 120 \times 15 = 1800$ ways.

c If it contains all men, the number of ways $= \binom{10}{5} \times \binom{6}{0} = 252$

d If it contains at least 3 men it would contain
3 men and 2 women or 4 men and 1 woman or 5 men and 0 women
and this can be done in $\binom{10}{3}\binom{6}{2} + \binom{10}{4}\binom{6}{1} + \binom{10}{5}\binom{6}{0}$ ways $= 3312$ ways.

e If it contains at least one of each sex, the total number of ways
$= \binom{10}{1}\binom{6}{4} + \binom{10}{2}\binom{6}{3} + \binom{10}{3}\binom{6}{2} + \binom{10}{4}\binom{6}{1} = 4110$
or $\binom{16}{5} - \binom{10}{0}\binom{6}{5} - \binom{10}{5}\binom{6}{0} = 4110$

9 **a** The 2 doctors can be chosen in $\binom{6}{2}$ ways

The 1 dentist can be chosen in $\binom{3}{1}$ ways

The 2 others can be chosen in $\binom{7}{2}$ ways

$\therefore$ the total number of ways $= \binom{6}{2} \times \binom{3}{1} \times \binom{7}{2} = 945$

b If it contains 2 doctors, 3 must be chosen from the other 10,

$\therefore$ there are $\binom{6}{2}\binom{10}{3} = 1800$ ways.

c If it contains at least one person from each of the two professions this can be done in
$\binom{9}{1}\binom{7}{4} + \binom{9}{2}\binom{7}{3} + \binom{9}{3}\binom{7}{2} + \binom{9}{4}\binom{7}{1} = 4347$ *or* $\binom{16}{5} - \binom{9}{0}\binom{7}{5} = 4347$

10 There are 20 points (for vertices) to choose from and any 2 form a line.
This can be done in $\binom{20}{2}$ ways. But this count includes the 20 lines joining the vertices.
$\therefore$ the number of diagonals $= \binom{20}{2} - 20 = 190 - 20 = 170$

11 **a** **i** $\binom{12}{2} = 66$ lines can be determined.

ii Of the lines in **a i** $\binom{1}{1}\binom{11}{1} = 11$ pass through B.

b **i** $\binom{12}{3} = 220$ triangles can be determined.

ii Of the triangles in **b i** $\binom{1}{1}\binom{11}{2} = 55$ have one vertex B.

12 The digits must be from 1 to 9. So, there are 9 of them, and we want any 4.

This can be done in $\binom{9}{4} = 126$ ways.

Once they have been selected they can be put in one ascending order

$\therefore$ total number $= 126 \times 1 = 126$.

13 **a** The different committees of 4, consisting of selections from 5 men and 6 women *in all possible ways* are

(4 men, 0 women) or (3 men, 1 woman) or (2 men, 2 women) or (1 man, 3 women) or (0 men, 4 women)

$\therefore \binom{5}{4}\binom{6}{0} + \binom{5}{3}\binom{6}{1} + \binom{5}{2}\binom{6}{2} + \binom{5}{1}\binom{6}{3} + \binom{5}{0}\binom{6}{4} = \binom{11}{4} \longleftarrow$ total number unrestricted

b The generalisation is:

$$\binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \binom{m}{2}\binom{n}{r-2} + \ldots + \binom{m}{r-2}\binom{n}{2} + \binom{m}{r-1}\binom{n}{1} + \binom{m}{r}\binom{n}{0} = \binom{m+n}{r}$$

14 **a** Consider a simpler case of 4 people (A, B, C, and D) going into two equal groups.

AB	CD	(1)	(1) and (6) are the same division.
AC	BD	(2)	(2) and (5) are the same division.
AD	BC	(3)	(3) and (4) are the same division.
BC	AD	(4)	
BD	AC	(5)	So, the number of ways is $\frac{1}{2}$ of $\binom{4}{2}$.
CD	AB	(6)	

So, for 2 equal groups of 6, the number of ways $\frac{1}{2}$ of $\binom{12}{6}\binom{6}{6}$

$= \frac{1}{2} \times 924$

$= 462$

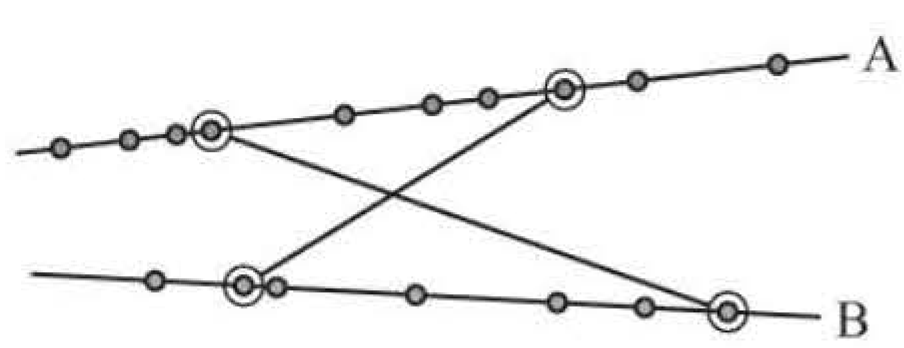
b For 3 equal groups of 4, the number of ways $= \dfrac{1}{3!} \times \binom{12}{4} \times \binom{8}{4} \times \binom{4}{4}$

$= 5775$

15 There is one point of intersection for every combination of 4 points (2 from A, 2 from B) as shown.

There are $\binom{10}{2} \times \binom{7}{2}$ ways to choose these points.

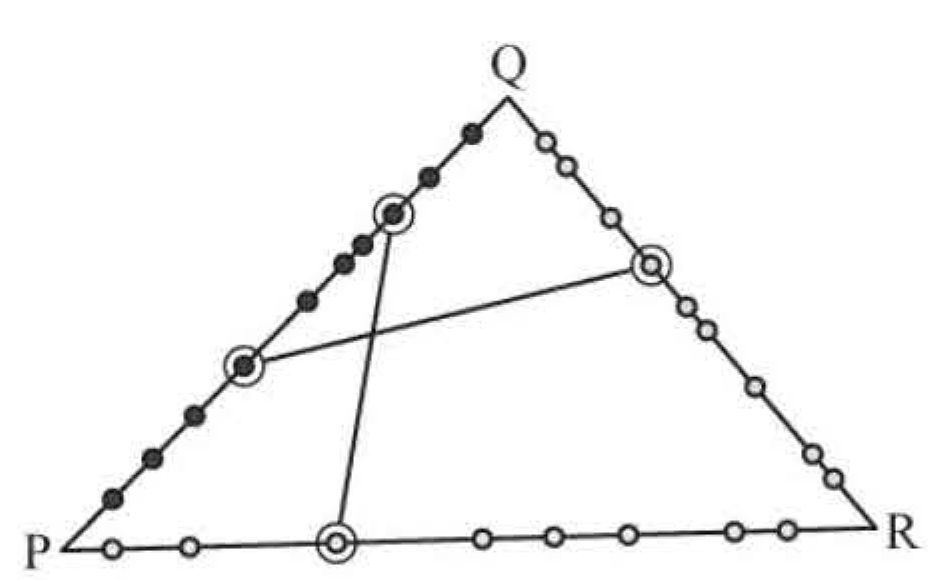
$\therefore$ the maximum number of points of intersection (when none of the intersection points coincide) is $\binom{10}{2} \times \binom{7}{2} = 945$



16 There is one point of intersection for every combination of 4 points (no more than 2 from any one line) as shown.

$\therefore$ the maximum number of points of intersection (when none of the intersection points coincide) is

$\binom{10}{2}\binom{9}{2}\binom{8}{0} + \binom{10}{2}\binom{9}{0}\binom{8}{2} + \binom{10}{0}\binom{9}{2}\binom{8}{2}$
$+ \binom{10}{2}\binom{9}{1}\binom{8}{1} + \binom{10}{1}\binom{9}{2}\binom{8}{1} + \binom{10}{1}\binom{9}{1}\binom{8}{2}$
$= 12\,528$



17 **a** Each handshake occurs between 2 people. So we need the number of ways 2 people can be selected from 10.

$\therefore$ the number of handshakes between committee members is $\binom{10}{2} = 45$.

A 10-sided polygon can be used to solve this problem by counting the total number of lines (including edges) between each pair of vertices. Each line represents a handshake between 2 of the 10 committee members.

b The number of handshakes between 273 delegates is $\binom{273}{2} = 37\,128$.

c The number of different orders in which the committee members can line up on stage is $10! = 3\,628\,800$.

EXERCISE 8F

1 **a** $(p+q)^3$
$= p^3 + 3p^2q + 3pq^2 + q^3$

b $(x+1)^3$
$= x^3 + 3x^2(1)^1 + 3x(1)^2 + (1)^3$
$= x^3 + 3x^2 + 3x + 1$

c $(x-3)^3$
$= x^3 + 3x^2(-3) + 3x(-3)^2 + (-3)^3$
$= x^3 - 9x^2 + 27x - 27$

d $(2+x)^3$
$= 2^3 + 3(2)^2x + 3(2)x^2 + x^3$
$= 8 + 12x + 6x^2 + x^3$

e $(3x-1)^3$
$= (3x)^3 + 3(3x)^2(-1) + 3(3x)(-1)^2 + (-1)^3$
$= 27x^3 - 27x^2 + 9x - 1$

f $(2x+5)^3$
$= (2x)^3 + 3(2x)^2(5) + 3(2x)(5)^2 + (5)^3$
$= 8x^3 + 60x^2 + 150x + 125$

g $(2a-b)^3 = (2a)^3 + 3(2a)^2(-b) + 3(2a)(-b)^2 + (-b)^3$
$= 8a^3 - 12a^2b + 6ab^2 - b^3$

h $(3x - \frac{1}{3})^3 = (3x)^3 + 3(3x)^2(-\frac{1}{3}) + 3(3x)(-\frac{1}{3})^2 + (-\frac{1}{3})^3$
$= 27x^3 - 9x^2 + x - \frac{1}{27}$

i $\left(2x + \dfrac{1}{x}\right)^3 = (2x)^3 + 3(2x)^2\left(\dfrac{1}{x}\right) + 3(2x)\left(\dfrac{1}{x}\right)^2 + \left(\dfrac{1}{x}\right)^3$
$= 8x^3 + 12x + \dfrac{6}{x} + \dfrac{1}{x^3}$

2 **a** $(1+x)^4 = 1^4 + 4(1)^3x + 6(1)^2x^2 + 4(1)x^3 + x^4$
$= 1 + 4x + 6x^2 + 4x^3 + x^4$

b $(p-q)^4 = p^4 + 4p^3(-q) + 6p^2(-q)^2 + 4p(-q)^3 + (-q)^4$
$= p^4 - 4p^3q + 6p^2q^2 - 4pq^3 + q^4$

c $(x-2)^4 = x^4 + 4x^3(-2)^1 + 6x^2(-2)^2 + 4x(-2)^3 + (-2)^4$
$= x^4 - 8x^3 + 24x^2 - 32x + 16$

d $(3-x)^4 = 3^4 + 4(3)^3(-x) + 6(3)^2(-x)^2 + 4(3)(-x)^3 + (-x)^4$
$= 81 - 108x + 54x^2 - 12x^3 + x^4$

e $(1+2x)^4 = 1^4 + 4(1)^3(2x) + 6(1)^2(2x)^2 + 4(1)(2x)^3 + (2x)^4$
$= 1 + 8x + 24x^2 + 32x^3 + 16x^4$

f $(2x-3)^4 = (2x)^4 + 4(2x)^3(-3)^1 + 6(2x)^2(-3)^2 + 4(2x)(-3)^3 + (-3)^4$
$= 16x^4 - 12 \times 8x^3 + 54 \times 4x^2 - 108 \times 2x + 81$
$= 16x^4 - 96x^3 + 216x^2 - 216x + 81$

g $(2x+b)^4 = (2x)^4 + 4(2x)^3b + 6(2x)^2b^2 + 4(2x)b^3 + b^4$
$= 16x^4 + 32x^3b + 24x^2b^2 + 8xb^3 + b^4$

h $\left(x + \dfrac{1}{x}\right)^4 = x^4 + 4x^3\left(\dfrac{1}{x}\right) + 6x^2\left(\dfrac{1}{x}\right)^2 + 4x\left(\dfrac{1}{x}\right)^3 + \left(\dfrac{1}{x}\right)^4$
$= x^4 + 4x^2 + 6 + \dfrac{4}{x^2} + \dfrac{1}{x^4}$

i $\left(2x - \dfrac{1}{x}\right)^4 = (2x)^4 + 4(2x)^3\left(-\dfrac{1}{x}\right) + 6(2x)^2\left(-\dfrac{1}{x}\right)^2 + 4(2x)\left(-\dfrac{1}{x}\right)^3 + \left(-\dfrac{1}{x}\right)^4$
$= 16x^4 - 32x^2 + 24 - \dfrac{8}{x^2} + \dfrac{1}{x^4}$

3 **a** $(x+2)^5 = x^5 + 5x^4(2) + 10x^3(2)^2 + 10x^2(2)^3 + 5x(2)^4 + 2^5$
$= x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32$

b $(x-2y)^5 = x^5 + 5x^4(-2y) + 10x^3(-2y)^2 + 10x^2(-2y)^3 + 5x(-2y)^4 + (-2y)^5$
$= x^5 - 10x^4y + 40x^3y^2 - 80x^2y^3 + 80xy^4 - 32y^5$

c $(1+2x)^5 = 1^5 + 5(1)^4(2x) + 10(1)^3(2x)^2 + 10(1)^2(2x)^3 + 5(1)(2x)^4 + (2x)^5$
$= 1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5$

d $\left(x - \frac{1}{x}\right)^5 = x^5 + 5x^4\left(-\frac{1}{x}\right) + 10x^3\left(-\frac{1}{x}\right)^2 + 10x^2\left(-\frac{1}{x}\right)^3 + 5x\left(-\frac{1}{x}\right)^4 + \left(-\frac{1}{x}\right)^5$
$= x^5 - 5x^3 + 10x - \frac{10}{x} + \frac{5}{x^3} - \frac{1}{x^5}$

4 **a**
1 5 10 10 5 1 ⟵ the 5th row
1 6 15 20 15 6 1 ⟵ the 6th row

b **i** $(x+2)^6 = x^6 + 6x^5(2) + 15x^4(2)^2 + 20x^3(2)^3 + 15x^2(2)^4 + 6x(2)^5 + (2)^6$
$= x^6 + 12x^5 + 60x^4 + 160x^3 + 240x^2 + 192x + 64$

ii $(2x-1)^6 = (2x)^6 + 6(2x)^5(-1) + 15(2x)^4(-1)^2 + 20(2x)^3(-1)^3 + 15(2x)^2(-1)^4$
$+ 6(2x)(-1)^5 + (-1)^6$
$= 64x^6 - 6 \times 32x^5 + 15 \times 16x^4 - 20 \times 8x^3 + 15 \times 4x^2 - 6 \times 2x + 1$
$= 64x^6 - 192x^5 + 240x^4 - 160x^3 + 60x^2 - 12x + 1$

iii $\left(x + \frac{1}{x}\right)^6 = x^6 + 6x^5\left(\frac{1}{x}\right) + 15x^4\left(\frac{1}{x}\right)^2 + 20x^3\left(\frac{1}{x}\right)^3 + 15x^2\left(\frac{1}{x}\right)^4 + 6x\left(\frac{1}{x}\right)^5 + \left(\frac{1}{x}\right)^6$
$= x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}$

5 **a** $\left(1+\sqrt{2}\right)^3 = (1)^3 + 3(1)^2\left(\sqrt{2}\right) + 3(1)\left(\sqrt{2}\right)^2 + \left(\sqrt{2}\right)^3$
$= 1 + 3\sqrt{2} + 3 \times 2 + 2 \times \sqrt{2}$
$= 1 + 3\sqrt{2} + 6 + 2\sqrt{2}$
$= 7 + 5\sqrt{2}$

b $\left(\sqrt{5}+2\right)^4 = \left(\sqrt{5}\right)^4 + 4\left(\sqrt{5}\right)^3(2) + 6\left(\sqrt{5}\right)^2(2)^2 + 4\left(\sqrt{5}\right)(2)^3 + 2^4$
$= 25 + 8 \times 5\sqrt{5} + 24 \times 5 + 32\sqrt{5} + 16$
$= 25 + 40\sqrt{5} + 120 + 32\sqrt{5} + 16$
$= 161 + 72\sqrt{5}$

c $\left(2-\sqrt{2}\right)^5$
$= (2)^5 + 5(2)^4\left(-\sqrt{2}\right) + 10(2)^3\left(-\sqrt{2}\right)^2 + 10(2)^2\left(-\sqrt{2}\right)^3 + 5(2)^1\left(-\sqrt{2}\right)^4 + \left(-\sqrt{2}\right)^5$
$= 32 - 80\sqrt{2} + 160 - 80\sqrt{2} + 40 - 4\sqrt{2}$
$= 232 - 164\sqrt{2}$

6 **a** $(2+x)^6 = (2)^6 + 6(2)^5x + 15(2)^4x^2 + 20(2)^3x^3 + 15(2)^2x^4 + 6(2)x^5 + x^6$
$= 64 + 192x + 240x^2 + 160x^3 + 60x^4 + 12x^5 + x^6$

b $(2.01)^6$ is obtained by letting $x = 0.01$
$\therefore \quad (2.01)^6 = 64 + 192 \times (0.01) + 240 \times (0.01)^2 + 160 \times (0.01)^3$
$+ 60 \times (0.01)^4 + 12 \times (0.01)^5 + (0.01)^6$
$= 65.944\,160\,601\,201$

$$\begin{array}{rl} & 64 \\ & 1.92 \\ & 0.024 \\ & 0.000\,16 \\ & 0.000\,000\,6 \\ & 0.000\,000\,001\,2 \\ + & 0.000\,000\,000\,001 \\ \hline & 65.944\,160\,601\,201 \end{array}$$

7 **a** $(a+b)^3 = 8 + 12e^x + ...$

and $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$\therefore\ a^3 = 8$

$\therefore\ a = 2$

and $3a^2b = 12e^x$

$\therefore\ 3(2)^2b = 12e^x$

$\therefore\ 12b = e^x$

$\therefore\ b = e^x$

So, $a = 2,\ b = e^x$.

b $(a+b)^3 = (2+e^x)^3$

$= 2^3 + 3(2)^2e^x + 3(2)(e^x)^2 + (e^x)^3$

$= 8 + 12e^x + 6e^{2x} + e^{3x}$

So, the remaining two terms are $6e^{2x}$ and e^{3x}.

8 $(2x+3)(x+1)^4$

$= (2x+3)(x^4 + 4x^3 + 6x^2 + 4x + 1)$

$= 2x^5 + 8x^4 + 12x^3 + 8x^2 + 2x + 3x^4 + 12x^3 + 18x^2 + 12x + 3$

$= 2x^5 + 11x^4 + 24x^3 + 26x^2 + 14x + 3$

9 **a** $(3a+b)^5 = (3a)^5 + 5(3a)^4b + 10(3a)^3b^2 + ...$

$\therefore$ the coefficient of a^3b^2 is $10 \times 3^3 = 270$

b $(2a+3b)^6 = (2a)^6 + 6(2a)^5(3b) + 15(2a)^4(3b)^2 + 20(2a)^3(3b)^3 + ...$

$\therefore$ the coefficient of a^3b^3 is $20 \times 2^3 \times 3^3 = 4320$

EXERCISE 8G

1 **a** $(1+2x)^{11} = 1^{11} + \binom{11}{1}(2x)^1 + \binom{11}{2}(2x)^2 + ... + \binom{11}{10}(2x)^{10} + (2x)^{11}$

b $\left(3x + \dfrac{2}{x}\right)^{15}$

$= (3x)^{15} + \binom{15}{1}(3x)^{14}\left(\dfrac{2}{x}\right)^1 + \binom{15}{2}(3x)^{13}\left(\dfrac{2}{x}\right)^2 + ... + \binom{15}{14}(3x)^1\left(\dfrac{2}{x}\right)^{14} + \left(\dfrac{2}{x}\right)^{15}$

c $\left(2x - \dfrac{3}{x}\right)^{20}$

$= (2x)^{20} + \binom{20}{1}(2x)^{19}\left(-\dfrac{3}{x}\right)^1 + \binom{20}{2}(2x)^{18}\left(-\dfrac{3}{x}\right)^2 + ... + \binom{20}{19}(2x)^1\left(-\dfrac{3}{x}\right)^{19} + \left(-\dfrac{3}{x}\right)^{20}$

2 **a** For $(2x+5)^{15}$, $a = (2x)$, $b = 5$, and $n = 15$

Now $T_{r+1} = \binom{n}{r}a^{n-r}b^r$ and letting $r = 5$ gives $T_6 = \binom{15}{5}(2x)^{10}5^5$.

b For $(x^2+y)^9$, $a = (x^2)$, $b = y$, and $n = 9$

Now $T_{r+1} = \binom{n}{r}a^{n-r}b^r$ and letting $r = 3$ gives $T_4 = \binom{9}{3}(x^2)^6(y)^3$.

c For $\left(x - \dfrac{2}{x}\right)^{17}$, $a = x$, $b = \left(-\dfrac{2}{x}\right)$, and $n = 17$

Now $T_{r+1} = \binom{n}{r}a^{n-r}b^r$ and letting $r = 9$ gives $T_{10} = \binom{17}{9}x^8\left(-\dfrac{2}{x}\right)^9$.

d For $\left(2x^2 - \dfrac{1}{x}\right)^{21}$, $a = (2x^2)$, $b = \left(-\dfrac{1}{x}\right)$, and $n = 21$

Now $T_{r+1} = \binom{n}{r}a^{n-r}b^r$ and letting $r = 8$ gives $T_9 = \binom{21}{8}(2x^2)^{13}\left(-\dfrac{1}{x}\right)^8$.

3 **a** For $(x+b)^7$, $a = x$, $b = b$, and $n = 7$

$\therefore$ the general term $T_{r+1} = \binom{7}{r}x^{7-r}b^r$

b If $x^{7-r} = x^4$ then $7 - r = 4$

$\therefore\ r = 3$

Now $T_4 = \binom{7}{3} x^4 b^3$

$\therefore$ the coefficient of x^4 is $\binom{7}{3} b^3 = 35b^3$

But the coefficient of x^4 is -280

So, $35b^3 = -280$

$\therefore\ b^3 = -8$

$\therefore\ b = \sqrt[3]{-8}$

$\therefore\ b = -2$

4 **a** For $\left(x + \dfrac{2}{x^2}\right)^{15}$, $a = x$, $b = \dfrac{2}{x^2}$, and $n = 15$

Now
$$\begin{aligned} T_{r+1} &= \tbinom{n}{r} a^{n-r} b^r \\ &= \tbinom{15}{r} x^{15-r} \left(\frac{2}{x^2}\right)^r \\ &= \tbinom{15}{r} x^{15-r} \frac{2^r}{x^{2r}} \\ &= \tbinom{15}{r} 2^r x^{15-3r} \end{aligned}$$

The constant term does not contain x.

$\therefore\ 15 - 3r = 0$

$\therefore\ r = 5$

so $T_6 = \binom{15}{5} 2^5 x^0$

$\therefore$ the constant term is $\binom{15}{5} 2^5$.

b For $\left(x - \dfrac{3}{x^2}\right)^9$, $a = x$, $b = \left(-\dfrac{3}{x^2}\right)$, and $n = 9$

Now
$$\begin{aligned} T_{r+1} &= \tbinom{n}{r} a^{n-r} b^r \\ &= \tbinom{9}{r} x^{9-r} \left(-\frac{3}{x^2}\right)^r \\ &= \tbinom{9}{r} x^{9-r} \frac{(-3)^r}{x^{2r}} \\ &= \tbinom{9}{r} (-3)^r x^{9-3r} \end{aligned}$$

The constant term does not contain x.

$\therefore\ 9 - 3r = 0$

$\therefore\ r = 3$

so $T_4 = \binom{9}{3} (-3)^3 x^0$

$\therefore$ the constant term is $\binom{9}{3} (-3)^3$.

5 **a**

Row 1	1 1	←	**b** **i**	sum $= 1+1 = 2 = 2^1$
Row 2	1 2 1	←	**ii**	sum $= 1+2+1 = 4 = 2^2$
Row 3	1 3 3 1	←	**iii**	sum $= 1+3+3+1 = 8 = 2^3$
Row 4	1 4 6 4 1	←	**iv**	sum $= 1+4+6+4+1 = 16 = 2^4$
Row 5	1 5 10 10 5 1	←	**v**	sum $= 1+5+10+10+5+1 = 32 = 2^5$

c The sum of the numbers in row n of Pascal's triangle is 2^n.

d $(1+x)^n$

$= \binom{n}{0} 1^n + \binom{n}{1} 1^{n-1} x + \binom{n}{2} 1^{n-2} x^2 + \binom{n}{3} 1^{n-3} x^3 + ... + \binom{n}{n-1} 1^1 x^{n-1} + \binom{n}{n} x^n$

$= \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \binom{n}{3} x^3 + ... + \binom{n}{n-1} x^{n-1} + \binom{n}{n} x^n$ {as all powers of 1 are 1}

Now letting $x = 1$ gives LHS $= (1+1)^n = 2^n$

and RHS $= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + ... + \binom{n}{n-1} + \binom{n}{n}$

$\therefore\ \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + ... + \binom{n}{n-1} + \binom{n}{n} = 2^n$

6 **a** In $(3 + 2x^2)^{10}$, $a = 3$, $b = (2x^2)$, and $n = 10$

Now
$$\begin{aligned} T_{r+1} &= \tbinom{n}{r} a^{n-r} b^r \\ &= \tbinom{10}{r} 3^{10-r} (2x^2)^r \\ &= \tbinom{10}{r} 3^{10-r} 2^r x^{2r} \end{aligned}$$

We now let $2r = 10$

$\therefore\ r = 5$

So, $T_6 = \underbrace{\tbinom{10}{5} 3^5 2^5}\, x^{10}$

$\therefore$ the coefficient is $\binom{10}{5} 3^5 2^5$.

b In $\left(2x^2-\frac{3}{x}\right)^6$, $a=(2x^2)$, $b=\left(-\frac{3}{x}\right)$, and $n=6$

Now
$$\begin{aligned}T_{r+1}&=\binom{n}{r}a^{n-r}b^r\\&=\binom{6}{r}\left(2x^2\right)^{6-r}\left(-\frac{3}{x}\right)^r\\&=\binom{6}{r}2^{6-r}x^{12-2r}\frac{(-3)^r}{x^r}\\&=\binom{6}{r}2^{6-r}(-3)^rx^{12-3r}\end{aligned}$$

We now let $12-3r=3$
$\therefore\ 3r=9$
$\therefore\ r=3$

So, $T_4=\underbrace{\binom{6}{3}2^3(-3)^3}\,x^3$

$\therefore$ the coefficient is $\binom{6}{3}2^3(-3)^3$.

c In $\left(2x^2-3y\right)^6$, $a=(2x^2)$, $b=(-3y)$, and $n=6$

Now
$$\begin{aligned}T_{r+1}&=\binom{n}{r}a^{n-r}b^r\\&=\binom{6}{r}\left(2x^2\right)^{6-r}(-3y)^r\\&=\binom{6}{r}2^{6-r}x^{12-2r}(-3)^ry^r\\&=\binom{6}{r}2^{6-r}(-3)^rx^{12-2r}y^r\end{aligned}$$

We find r such that
$12-2r=6$ and $r=3$
$\therefore\ r=3$ is the solution

So, $T_4=\underbrace{\binom{6}{3}2^3(-3)^3}\,x^6y^3$

$\therefore$ the coefficient is $\binom{6}{3}2^3(-3)^3$.

d In $\left(2x^2-\frac{1}{x}\right)^{12}$, $a=(2x^2)$, $b=\left(-\frac{1}{x}\right)$, and $n=12$

Now
$$\begin{aligned}T_{r+1}&=\binom{n}{r}a^{n-r}b^r\\&=\binom{12}{r}\left(2x^2\right)^{12-r}\left(-\frac{1}{x}\right)^r\\&=\binom{12}{r}2^{12-r}x^{24-2r}\frac{(-1)^r}{x^r}\\&=\binom{12}{r}2^{12-r}(-1)^rx^{24-3r}\end{aligned}$$

We now let $24-3r=12$
$\therefore\ 3r=12$
$\therefore\ r=4$

So, $T_5=\underbrace{\binom{12}{4}2^8(-1)^4}\,x^{12}$

$\therefore$ the coefficient is $\binom{12}{4}2^8(-1)^4$.

7 **a** $(x+2)(x^2+1)^8$

$$=(x+2)\left[(x^2)^8+\binom{8}{1}(x^2)^7+\binom{8}{2}(x^2)^6+\ldots.+\binom{8}{6}(x^2)^2+\binom{8}{7}(x^2)^1+\binom{8}{8}\right]$$

only terms which when multiplied give an x^5

$\therefore$ coefficient of x^5 is $1\times\binom{8}{6}=\binom{8}{6}=28$.

b $(2-x)(3x+1)^9$

$$=(2-x)\left[(3x)^9+\binom{9}{1}(3x)^8+\binom{9}{2}(3x)^7+\binom{9}{3}(3x)^6+\binom{9}{4}(3x)^5+\ldots.\right]$$

$\therefore$ term containing x^6 is $2\times\binom{9}{3}3^6x^6+(-x)\times\binom{9}{4}3^5x^5=2\binom{9}{3}3^6x^6-\binom{9}{4}3^5x^6=91\,854x^6$

8 In $(x^2y-2y^2)^6$, $a=(x^2y)$, $b=(-2y^2)$, and $n=6$.

Now
$$\begin{aligned}T_{r+1}&=\binom{n}{r}a^{n-r}b^r\\&=\binom{6}{r}(x^2y)^{6-r}(-2y^2)^r\\&=\binom{6}{r}x^{12-2r}y^{6-r}(-2)^ry^{2r}\\&=\binom{6}{r}(-2)^rx^{12-2r}y^{6+r}\end{aligned}$$

Since x and y are raised to the same power,
$12-2r=6+r$
$\therefore\ 3r=6$
$\therefore\ r=2$

$T_3=\binom{6}{2}(-2)^2x^8y^8=60x^8y^8$

9 **a** $(1+x)^n$ has $T_3 = \binom{n}{2}1^{n-2}x^2 = \binom{n}{2}x^2$ and $n \geqslant 2$

But this term is $36x^2$ $\therefore$ $\binom{n}{2} = 36$

$$\therefore \frac{n(n-1)}{2} = 36$$

$$\therefore n(n-1) = 72$$

$$\therefore n^2 - n - 72 = 0$$

$$\therefore (n-9)(n+8) = 0$$

$$\therefore n = 9 \text{ or } -8$$

But $n \geqslant 2$, so $n = 9$

and $T_4 = \binom{n}{3}1^{n-3}x^3$

$\therefore T_4 = \binom{9}{3}x^3$

$= 84x^3$

b $(1+kx)^n = 1^n + \binom{n}{1}1^{n-1}(kx)^1 + \binom{n}{2}1^{n-2}(kx)^2 + ...$

$= 1 + \binom{n}{1}kx + \binom{n}{2}k^2x^2 + ...$

$\therefore$ $\binom{n}{1}k = -12$ and $\binom{n}{2}k^2 = 60$

$\therefore$ $nk = -12$ and $\dfrac{n(n-1)}{2}k^2 = 60$

$$\therefore n(n-1)k^2 = 120$$

But $k = -\dfrac{12}{n}$ $\therefore$ $n(n-1)\dfrac{144}{n^2} = 120$

$$\therefore 144(n-1) = 120n \quad \{n \geqslant 2\}$$

$$\therefore 144n - 120n = 144$$

$$\therefore 24n = 144$$

$$\therefore n = 6 \quad \text{and so } k = -2$$

10 $T_{r+1} = \binom{n}{r}a^{n-r}b^r$ where $n = 10$, $a = (x^2)$, $b = \left(\dfrac{1}{ax}\right)$

$= \binom{10}{r}(x^2)^{10-r}\left(\dfrac{1}{ax}\right)^r$

$= \binom{10}{r}x^{20-2r} \times \dfrac{1}{a^r x^r}$

$= \binom{10}{r}x^{20-3r} \times \dfrac{1}{a^r}$

We let $20 - 3r = 11$

$\therefore$ $3r = 9$

$\therefore$ $r = 3$

and $T_4 = \binom{10}{3}x^{11} \times \dfrac{1}{a^3}$

$= \dfrac{\binom{10}{3}}{a^3}x^{11}$

So, $\dfrac{\binom{10}{3}}{a^3} = 15$

$\therefore$ $\dfrac{120}{a^3} = 15$

$\therefore$ $a^3 = 8$

$\therefore$ $a = 2$

11 $(1+2x-x^2)^5$

$= \left([1+2x] - x^2\right)^5$

$= (1+2x)^5 + 5(1+2x)^4(-x^2) + 10(1+2x)^3(-x^2)^2 + ...$

{all further terms contain higher powers of x than x^4}

$= 1^5 + 5(1^4)(2x) + 10(1^3)(2x)^2 + 10(1^2)(2x)^3 + 5(1)(2x)^4 + ...$

$\quad - 5x^2(1^4 + 4(1^3)(2x) + 6(1^2)(2x)^2 + ...) + 10x^4(1^3 + ...) + ...$

$= 1 + 10x + 40x^2 + 80x^3 + 80x^4 - 5x^2 - 40x^3 - 120x^4 + 10x^4 + ...$

$= 1 + 10x + 35x^2 + 40x^3 - 30x^4 + ...$

12 $\binom{n}{1} = \dfrac{n!}{1!(n-1)!} = \dfrac{n\cancel{(n-1)!}}{1\cancel{(n-1)!}} = \dfrac{n}{1} = n$

and $\binom{n}{2} = \dfrac{n!}{2!(n-2)!} = \dfrac{n(n-1)\cancel{(n-2)!}}{2\cancel{(n-2)!}} = \dfrac{n(n-1)}{2}$ for $n \in \mathbb{Z}^+,\ n \geqslant 2$.

13 **a** From **5 d**, $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + ... + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n$

Now, letting $x=-1$ gives LHS $= (1+(-1))^n = 0$

and $\text{RHS} = \binom{n}{0} + \binom{n}{1}(-1) + \binom{n}{2}(-1)^2 + \binom{n}{3}(-1)^3 + ...$

$+ \binom{n}{n-1}(-1)^{n-1} + \binom{n}{n}(-1)^n$

$= \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + ... + (-1)^n\binom{n}{n}$

$\therefore\ \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + ... + (-1)^n\binom{n}{n} = 0$

b As $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + ... + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n$,

$(1+x)^{2n+1} = \binom{2n+1}{0} + \binom{2n+1}{1}x + \binom{2n+1}{2}x^2 + ... + \binom{2n+1}{2n}x^{2n} + \binom{2n+1}{2n+1}x^{2n+1}$

Now letting $x=1$ gives LHS $= 2^{2n+1} = 2^{2n} \times 2^1 = 4^n \times 2$

and $\text{RHS} = \binom{2n+1}{0} + \binom{2n+1}{1} + \binom{2n+1}{2} + ... + \binom{2n+1}{2n} + \binom{2n+1}{2n+1}$

$= 2\left[\binom{2n+1}{0} + \binom{2n+1}{1} + \binom{2n+1}{2} + ... + \binom{2n+1}{n}\right]$

$\left\{\binom{2n+1}{2n+1} = \binom{2n+1}{0},\ \binom{2n+1}{2n} = \binom{2n+1}{1},\ ....,\ \binom{2n+1}{n+1} = \binom{2n+1}{n}\right\}$

$\therefore\ 2\left[\binom{2n+1}{0} + \binom{2n+1}{1} + \binom{2n+1}{2} + ... + \binom{2n+1}{n}\right] = 4^n \times 2$

$\therefore\ \binom{2n+1}{0} + \binom{2n+1}{1} + \binom{2n+1}{2} + ... + \binom{2n+1}{n} = 4^n$

14 $$\sum_{r=0}^{n} 2^r\binom{n}{r} = 2^0\binom{n}{0} + 2^1\binom{n}{1} + 2^2\binom{n}{2} + ... + 2^{n-1}\binom{n}{n-1} + 2^n\binom{n}{n}$$

Using **5 d**, $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + ... + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n$

$\therefore$ letting $x=2$, $(1+2)^n = \binom{n}{0} + \binom{n}{1}2 + \binom{n}{2}2^2 + ... + \binom{n}{n-1}2^{n-1} + \binom{n}{n}2^n$

$\therefore\ 3^n = 2^0\binom{n}{0} + 2^1\binom{n}{1} + 2^2\binom{n}{2} + ... + 2^{n-1}\binom{n}{n-1} + 2^n\binom{n}{n}$

$$\therefore\ \sum_{r=0}^{n} 2^r\binom{n}{r} = 3^n$$

15 For any polynomial $f(x)$, the sum of its coefficients is $f(1)$.

Let $f(x) = x^3 + 2x^2 + 3x - 7$

$\therefore$ the sum of the coefficients of $f(x) = f(1)$

$= 1^3 + 2(1)^2 + 3(1) - 7$

$= 1 + 2 + 3 - 7 = -1$

Now consider the function $g(x) = (x^3 + 2x^2 + 3x - 7)^{100}$

$= [f(x)]^{100}$

The sum of the coefficients of $g(x) = g(1)$

$= [f(1)]^{100}$

$= (-1)^{100} = 1$

$\therefore$ the sum of the coefficients of $(x^3 + 2x^2 + 3x - 7)^{100}$ is 1.

16 From **5 d**, $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + ... + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n$

$\therefore\ (1+x)^{2n} = \binom{2n}{0} + \binom{2n}{1}x + ... + \binom{2n}{n-1}x^{n-1} + \binom{2n}{n}x^n + ... + \binom{2n}{2n-1}x^{2n-1} + \binom{2n}{2n}x^{2n}$

Now $(1+x)^n(1+x)^n = (1+x)^{2n}$

$$\therefore\ \left[\binom{n}{0} + \binom{n}{1}x + ... + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n\right]\left[\binom{n}{0} + \binom{n}{1}x + ... + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n\right]$$
$$= \binom{2n}{0} + \binom{2n}{1}x + ... + \binom{2n}{n-1}x^{n-1} + \binom{2n}{n}x^n + ... + \binom{2n}{2n-1}x^{2n-1} + \binom{2n}{2n}x^{2n}$$

Equating coefficients of x^n,

$\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + ... + \binom{n}{n-1}\binom{n}{1} + \binom{n}{n}\binom{n}{0} = \binom{2n}{n}$

But $\binom{n}{n} = \binom{n}{0}$, $\binom{n}{n-1} = \binom{n}{1}$, and so on.

$\therefore\ \binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + ... + \binom{n}{n-1}^2 + \binom{n}{n}^2 = \binom{2n}{n}$

17 **a** $(3+x)^n = 3^n + \binom{n}{1}3^{n-1}x + \binom{n}{2}3^{n-2}x^2 + \binom{n}{3}3^{n-3}x^3 + ... + \binom{n}{n-1}3x^{n-1} + x^n$

b Now letting $x = 1$ in **a**,

$\text{LHS} = (3+1)^n = 4^n$

and $\text{RHS} = 3^n + \binom{n}{1}3^{n-1} + \binom{n}{2}3^{n-2} + \binom{n}{3}3^{n-3} + ... + 3n + 1$

$\therefore\ 3^n + \binom{n}{1}3^{n-1} + \binom{n}{2}3^{n-2} + \binom{n}{3}3^{n-3} + ... + 3n + 1 = 4^n.$

18 **a**
$$\begin{aligned} n\binom{n-1}{r-1} &= n\frac{(n-1)!}{(r-1)!(n-1-[r-1])!} \\ &= \frac{n\times(n-1)!}{(r-1)!(n-r)!} \\ &= r\times\frac{n!}{r!(n-r)!} \\ &= r\binom{n}{r} \end{aligned}$$

b
$$\begin{aligned} &\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + ... + n\binom{n}{n} \\ &= n\binom{n-1}{0} + n\binom{n-1}{1} + n\binom{n-1}{2} + ... + n\binom{n-1}{n-1} \quad \{\text{using } \mathbf{a}\} \\ &= n\left[\binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2} + ... + \binom{n-1}{n-1}\right] \\ &= n\,2^{n-1} \quad \{\text{second part of } \mathbf{5\ d}\} \end{aligned}$$

c **i**
$$\begin{aligned} \sum_{r=0}^{n} P_r &= P_0 + P_1 + P_2 + ... + P_n \\ &= \binom{n}{0}p^0(1-p)^n + \binom{n}{1}p^1(1-p)^{n-1} + \binom{n}{2}p^2(1-p)^{n-2} + ... + \binom{n}{n}p^n(1-p)^0 \\ &= (p+[1-p])^n \quad \{\text{binomial expansion}\} \\ &= 1^n = 1 \end{aligned}$$

ii
$$\begin{aligned} \sum_{r=1}^{n} r\,P_r &= 1P_1 + 2P_2 + 3P_3 + ... + nP_n \\ &= 1\binom{n}{1}p^1(1-p)^{n-1} + 2\binom{n}{2}p^2(1-p)^{n-2} + 3\binom{n}{3}p^3(1-p)^{n-3} + ... \\ &\quad + n\binom{n}{n}p^n(1-p)^0 \\ &= n\binom{n-1}{0}p^1(1-p)^{n-1} + n\binom{n-1}{1}p^2(1-p)^{n-2} + n\binom{n-1}{2}p^3(1-p)^{n-3} + ... \\ &\quad + n\binom{n-1}{n-1}p^n \quad \{\text{using } \mathbf{a}\} \\ &= np[\binom{n-1}{0}p^0(1-p)^{n-1} + \binom{n-1}{1}p^1(1-p)^{n-2} + \binom{n-1}{2}p^2(1-p)^{n-3} + ... \\ &\quad + \binom{n-1}{n-1}p^{n-1}] \\ &= np\left[(p+(1-p))^{n-1}\right] \\ &= np\times 1^{n-1} \\ &= np \end{aligned}$$

REVIEW SET 8A

1 **a** $\dfrac{n!}{(n-2)!}=\dfrac{n(n-1)(n-2)!}{(n-2)!}$
$\quad = n(n-1), \quad n \geqslant 2$

b $\dfrac{n!+(n+1)!}{n!}=\dfrac{n!+(n+1)n!}{n!}$
$\quad =\dfrac{n!(1+n+1)}{n!}$
$\quad = n+2$

2 There are $\binom{8}{2}=28$ handshakes made.

3 **a** $\begin{array}{|c|c|c|c|c|}\hline 4 & 3 & 2 & 1 & 1 \\ \hline \multicolumn{4}{c}{} & \uparrow \\ \multicolumn{4}{c}{} & \text{T}\end{array}$ $\quad \therefore\ 4\times 3\times 2\times 1\times 1=24$ arrangements end in T

b $\begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 2 & 1 & 1 \\ \hline \uparrow & \multicolumn{3}{c}{} & \uparrow \\ \text{P} & \multicolumn{3}{c}{} & \text{T}\end{array}$ $\quad \therefore\ 1\times 3\times 2\times 1\times 1=6$ arrangements begin with P and end with T

4 **a** $\begin{array}{|c|c|c|}\hline 9 & 10 & 10 \\ \hline \uparrow & \multicolumn{2}{c}{} \end{array}$ $\quad$ So, $9\times 10\times 10=900$ three digit numbers can be formed.
cannot use 0

b To be divisible by 5 the last digit must be 0 or 5.

either $\begin{array}{|c|c|c|}\hline 9 & 10 & 1 \\ \hline \multicolumn{2}{c}{} & \uparrow \\ \multicolumn{2}{c}{} & 0\end{array}$ or $\begin{array}{|c|c|c|}\hline 9 & 10 & 1 \\ \hline \multicolumn{2}{c}{} & \uparrow \\ \multicolumn{2}{c}{} & 5\end{array}$ $\quad \therefore\ 9\times 10\times 1+9\times 10\times 1=180$ of them are divisible by 5.

5 **a** $\quad (a+b)^4=e^{4x}-4e^{2x}+\ldots.$
and $(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4$

$\therefore\ a^4=e^{4x}$ and $4a^3b=-4e^{2x}$
$\therefore\ a^4=(e^x)^4 \qquad \therefore\ 4e^{3x}b=-4e^{2x}$
$\therefore\ a=e^x \qquad \therefore\ b=-e^{-x}$

b $(a+b)^4=e^{4x}-4e^{2x}+6(e^x)^2(-e^{-x})^2+4e^x(-e^{-x})^3+(-e^{-x})^4$
$\quad =e^{4x}-4e^{2x}+6e^{2x}e^{-2x}-4e^xe^{-3x}+e^{-4x}$
$\quad =e^{4x}-4e^{2x}+6-4e^{-2x}+e^{-4x}$

6 $\left(\sqrt{3}+2\right)^5=\left(\sqrt{3}\right)^5+5\left(\sqrt{3}\right)^4(2)+10\left(\sqrt{3}\right)^3(2)^2+10\left(\sqrt{3}\right)^2(2)^3+5\left(\sqrt{3}\right)^1(2)^4+2^5$
$\quad =9\sqrt{3}+90+120\sqrt{3}+240+80\sqrt{3}+32$
$\quad =362+209\sqrt{3}$

7 For $\left(3x^2+\dfrac{1}{x}\right)^8$, $\ a=(3x^2)$, $\ b=\left(\dfrac{1}{x}\right)$, $\ n=8$

$T_{r+1}=\binom{n}{r}a^{n-r}b^r$
$\quad =\binom{8}{r}\left(3x^2\right)^{8-r}\left(\dfrac{1}{x}\right)^r$
$\quad =\binom{8}{r}3^{8-r}x^{16-2r-r}$
$\quad =\binom{8}{r}3^{8-r}x^{16-3r}$

Now a constant term does not contain x
$\therefore\ 16-3r=0$
$\therefore\ 3r=16$
$\therefore\ r=5\frac{1}{3}$
which is impossible as r is in $\mathbb{Z}$
$\therefore$ no constant term exists.

8 $(1+cx)(1+x)^4 = (1+cx)(1^4 + \binom{4}{1}1^3x + \binom{4}{2}1^2x^2 + \binom{4}{3}1x^3 + x^4)$

$\therefore$ coefficient of x^3 is $1 \times \binom{4}{3} \times 1 + c \times \binom{4}{2} \times 1^2 = 4 + 6c$

But the coefficient of x^3 is 22, so $4 + 6c = 22$

$$\therefore \ 6c = 18$$
$$\therefore \ c = 3$$

9 **a** $(2+x)^n = 2^n + \binom{n}{1}2^{n-1}x^1 + \binom{n}{2}2^{n-2}x^2 + \binom{n}{3}2^{n-3}x^3 + ... + \binom{n}{n-1}2^1x^{n-1} + x^n$

b $2^n + \binom{n}{1}2^{n-1} + \binom{n}{2}2^{n-2} + \binom{n}{3}2^{n-3} + ... + 2n + 1$

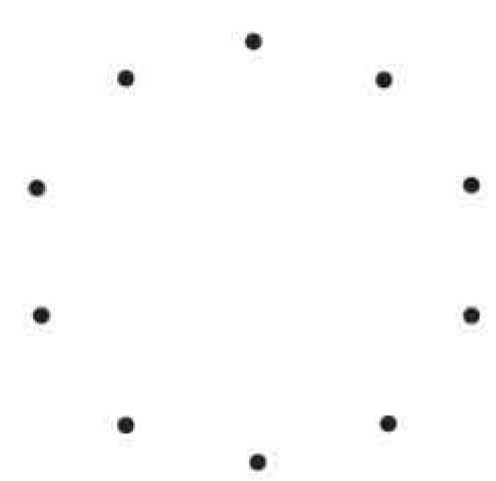
$= 2^n + \binom{n}{1}2^{n-1}x^1 + \binom{n}{2}2^{n-2}x^2 + \binom{n}{3}2^{n-3}x^3 + ... + \binom{n}{n-1}2^1x^{n-1} + x^n$, where $x = 1$

$= (2+1)^n$

$= 3^n$

REVIEW SET 8B

1



a To form a line we need to select any two points from the 10.

$\therefore$ total is $\binom{10}{2} = 45$ lines.

b To form a triangle we need to select any three points from the 10.

$\therefore$ total is $\binom{10}{3} = 120$ triangles.

2 $(4+x)^3 = 4^3 + 3(4)^2x^1 + 3(4)^1x^2 + x^3$

$= 64 + 48x + 12x^2 + x^3$

Letting $x = 0.02$ gives $(4.02)^3 = 64 + 48(0.02) + 12(0.02)^2 + (0.02)^3$

$= 64 + 0.96 + 0.0048 + 0.000\,008$

$= 64.964\,808$

3
$$T_{r+1} = \binom{6}{r}(3x)^{6-r}\left(\frac{-2}{x^2}\right)^r$$
$$= \binom{6}{r}3^{6-r}x^{6-r}(-2)^r x^{-2r}$$
$$= \binom{6}{r}3^{6-r}x^{6-3r}(-2)^r$$

If we let $6 - 3r = 0$ then $r = 2$

$\therefore \ T_3 = \underbrace{\binom{6}{2}3^4(-2)^2}\,x^0$

$\therefore$ constant term $= \binom{6}{2}3^4(-2)^2 = 4860$

4 In the expansion of $(x+5)^6$, $a = x$, $b = 5$, $n = 6$

$T_{r+1} = \binom{n}{r}a^{n-r}b^r$

$= \binom{6}{r}x^{6-r}5^r$

For the coefficient of x^3 we let $6 - r = 3$

$\therefore \ r = 3$

and $T_4 = \underbrace{\binom{6}{3}5^3}\,x^3$

$\therefore$ the coefficient is $\binom{6}{3}5^3 = 2500.$

5 **a** With no restrictions there are $\binom{10}{5} = 252$ different teams.

b Those consisting of at least one of each sex

$= \binom{10}{5} - \binom{6}{5}\binom{4}{0}$ {there are no teams consisting of 5 women}

$= 246$

6 **a**

9	9	8	7

cannot use 0

So, $9 \times 9 \times 8 \times 7 = 4536$ numbers are possible.

b Either they end in 0 or in 5

9	8	7	1

0

or

8	8	7	1

cannot use 0 5

So, $9 \times 8 \times 7 \times 1 + 8 \times 8 \times 7 \times 1$
$= 952$ numbers are possible.

7 The sixth row of Pascal's triangle is 1 6 15 20 15 6 1

$\therefore\ (a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$

a
$$\begin{aligned}(x-3)^6 &= x^6 + 6x^5(-3) + 15x^4(-3)^2 + 20x^3(-3)^3 + 15x^2(-3)^4 + 6x(-3)^5 + (-3)^6 \\ &= x^6 - 18x^5 + 135x^4 - 540x^3 + 1215x^2 - 1458x + 729\end{aligned}$$

b
$$\begin{aligned}&\left(1+\frac{1}{x}\right)^6 \\ &= (1)^6 + 6(1)^5\left(\frac{1}{x}\right) + 15(1)^4\left(\frac{1}{x}\right)^2 + 20(1)^3\left(\frac{1}{x}\right)^3 + 15(1)^2\left(\frac{1}{x}\right)^4 + 6(1)\left(\frac{1}{x}\right)^5 + \left(\frac{1}{x}\right)^6 \\ &= 1 + \frac{6}{x} + \frac{15}{x^2} + \frac{20}{x^3} + \frac{15}{x^4} + \frac{6}{x^5} + \frac{1}{x^6}\end{aligned}$$

8 In $\left(2x - \frac{3}{x^2}\right)^{12}$, $a = (2x)$, $b = \left(-\frac{3}{x^2}\right)$, $n = 12$

$$\begin{aligned}T_{r+1} &= \binom{n}{r}a^{n-r}b^r \\ &= \binom{12}{r}(2x)^{12-r}\left(-\frac{3}{x^2}\right)^r \\ &= \binom{12}{r}2^{12-r}x^{12-r}\frac{(-3)^r}{x^{2r}} \\ &= \binom{12}{r}2^{12-r}(-3)^r x^{12-3r}\end{aligned}$$

For the coefficient of x^{-6} we let $12 - 3r = -6$
$\therefore\ 3r = 18$
$\therefore\ r = 6$

So, $T_7 = \underbrace{\binom{12}{6}2^6(-3)^6}\, x^{-6}$

$\therefore$ the coefficient is $\binom{12}{6}2^6(-3)^6 = 43\,110\,144.$

9
$$\begin{aligned}&(2x+3)(x-2)^6 \\ &= (2x+3)\left[x^6 + \binom{6}{1}x^5(-2) + \binom{6}{2}x^4(-2)^2 + ...\right]\end{aligned}$$

$\therefore$ coefficient of x^5 is $2 \times \binom{6}{2} \times (-2)^2 + 3 \times \binom{6}{1} \times (-2) = 8\binom{6}{2} - 6\binom{6}{1} = 84$

10
$$\begin{aligned}T_{r+1} &= \binom{9}{r}(2x)^{9-r}\left(\frac{1}{ax^2}\right)^r \\ &= \binom{9}{r}2^{9-r}x^{9-r} \times \frac{1}{a^r x^{2r}} \\ &= \binom{9}{r}2^{9-r}a^{-r}x^{9-3r}\end{aligned}$$

Letting $r = 2$, $T_3 = \underbrace{\binom{9}{2}2^7a^{-2}}\, x^3$

$\therefore\ \frac{\binom{9}{2}2^7}{a^2} = 288$

$\therefore\ a^2 = \frac{\binom{9}{2}2^7}{288} = 16$

$\therefore\ a = \pm 4$

REVIEW SET 8C

1 **a**

26	26	10	10	10	10

$\therefore$ there are $26^2 \times 10^4 = 6\,760\,000$ if there are no restrictions

b

5	26	10	10	10	10

a vowel

$\therefore$ there are $5 \times 26 \times 10^4 = 1\,300\,000$ possibilities if the first letter is a vowel

c

26	25	10	9	8	7

$\therefore$ there are $26 \times 25 \times 10 \times 9 \times 8 \times 7 = 3\,276\,000$, if there are no repetitions

2 **a** Total number $= \binom{8+7}{5} = \binom{15}{5} = 3003$ committees

b Total with 2 men and 3 women $= \binom{8}{2}\binom{7}{3} = 980$ committees

c Total with at least one man = total unrestricted − total with all women

$$= 3003 - \binom{8}{0}\binom{7}{5} = 2982 \text{ committees}$$

3 **a** $(x-2y)^3 = x^3 + 3x^2(-2y) + 3x(-2y)^2 + (-2y)^3$
$= x^3 - 6x^2y + 12xy^2 - 8y^3$

b $(3x+2)^4 = (3x)^4 + 4(3x)^3(2) + 6(3x)^2(2)^2 + 4(3x)(2)^3 + (2)^4$
$= 81x^4 + 216x^3 + 216x^2 + 96x + 16$

4 In the expansion of $(2x+5)^6$, $a = (2x)$, $b = 5$, $n = 6$

$T_{r+1} = \binom{n}{r}a^{n-r}b^r$
$= \binom{6}{r}(2x)^{6-r}5^r$
$= \binom{6}{r}2^{6-r}x^{6-r}5^r$

For the coefficient of x^3 we let $6 - r = 3$

$\therefore\ r = 3$

and $T_4 = \underbrace{\binom{6}{3}2^3 5^3}\, x^3$

$\therefore$ the coefficient is $\binom{6}{3}2^3 5^3 = 20\,000$.

5 In the expansion of $\left(2x^2 - \frac{1}{x}\right)^6$, $a = 2x^2$, $b = -\frac{1}{x}$, $n = 6$

$T_{r+1} = \binom{n}{r}a^{n-r}b^r$
$= \binom{6}{r}(2x^2)^{6-r}\left(-\frac{1}{x}\right)^r$
$= \binom{6}{r}2^{6-r}x^{12-2r}(-1)^r x^{-r}$
$= \binom{6}{r}2^{6-r}(-1)^r x^{12-3r}$

For the constant term we let $12 - 2r = 0$

$\therefore\ r = 4$

and $T_5 = \underbrace{\binom{6}{4}2^2(-1)^4}\, x^0$

$\therefore$ the constant term is $\binom{6}{4}2^2(-1)^4 = 60$.

6 $(1+kx)^n = 1^n + \binom{n}{1}1^{n-1}(kx)^1 + \binom{n}{2}1^{n-2}(kx)^2 + ...$
$= 1 + nkx + \binom{n}{2}k^2x^2 + ...$

$\therefore\ nk = -4$ and $\binom{n}{2}k^2 = \frac{15}{2}$

$\therefore\ k = -\frac{4}{n}$ and $\frac{n(n-1)}{2}k^2 = \frac{15}{2}$

$\therefore\ n(n-1)k^2 = 15$

Using $k = -\frac{4}{n}$, $n(n-1)\left(\frac{16}{n^2}\right) = 15$

$\therefore\ 16(n-1) = 15n \quad \{n \geqslant 2\}$

$\therefore\ 16n - 16 = 15n$

$\therefore\ n = 16$ and so $k = \frac{-4}{16} = -\frac{1}{4}$

7 The three sisters can sit together in 3! ways. They as a group, plus the other 5 people, make 6 items which can be permutated in 6! ways.

$\therefore$ total number of orderings $= 3! \times 6! = 4320$

8 **a** If no restrictions there are $\binom{18}{8} = 43\,758$ different teams possible.

b If 4 of each sex are needed there are $\binom{11}{4} \times \binom{7}{4} = 11\,550$ different teams.

c If at least 2 women are needed, total number $= \binom{18}{8} - \binom{11}{8}\binom{7}{0} - \binom{11}{7}\binom{7}{1}$

$= 41\,283$ different teams

9 $(m-2n)^{10} = m^{10} + \binom{10}{1}m^9(-2n) + \binom{10}{2}m^8(-2n)^2 + ... + (-2n)^{10}$

$= m^{10} - 20m^9n + 45m^8(4n^2) + ... + 1024n^{10}$

$= m^{10} - 20m^9n + 180m^8n^2 + ... + 1024n^{10}$

$\therefore\ k = 180$

10 $\left(x^3 + \frac{q}{x^3}\right)^8$ has $T_{r+1} = \binom{8}{r}(x^3)^{8-r}\left(\frac{q}{x^3}\right)^r$

$= \binom{8}{r}x^{24-3r}\frac{q^r}{x^{3r}}$

$= \binom{8}{r}x^{24-6r}q^r$

which has constant term $\binom{8}{4}q^4$ $\{24-6r=0$ when $r=4\}$

$\left(x^3 + \frac{q}{x^3}\right)^4$ has $T_{r+1} = \binom{4}{r}(x^3)^{4-r}\left(\frac{q}{x^3}\right)^r$

$= \binom{4}{r}x^{12-3r}q^r x^{-3r}$

$= \binom{4}{r}x^{12-6r}q^r$

which has constant term $\binom{4}{2}q^2$ $\{12-6r=0$ when $r=2\}$

$\therefore\ \binom{8}{4}q^4 = \binom{4}{2}q^2$

$\therefore\ 70q^4 - 6q^2 = 0$

$\therefore\ q^2(70q^2-6) = 0$

$\therefore\ 70q^2 - 6 = 0$ $\{q=0$ gives a trivial solution$\}$

$\therefore\ q^2 = \frac{6}{70} = \frac{3}{35}$

$\therefore\ q = \pm\sqrt{\frac{3}{35}}$