Chapter 27
MISCELLANEOUS QUESTIONS

EXERCISE 27A

1 **a** $(1-i)^2 = 1 - 2i + i^2 = -2i$

$\therefore\ (1-i)^{4n} = \left[(1-i)^2\right]^{2n} = (-2i)^{2n} = \left[(-2i)^2\right]^n = (-4)^n$

b $(1-i)^{16} = (1-i)^{4\times4} = (-4)^4 = 256$ $\{$using $(1-i)^{4n} = (-4)^n\}$

c If $z^{16} = 256$ we have a polynomial with real coefficients.

From **b**, $1-i$ is one solution and so $1+i$ must also be a solution {Theorem of real polynomials}

Thus $z = 1 \pm i$ are two solutions of $z^{16} = 256$.

2 $S_n = n^3 + 2n - 1$

Now $u_n = S_n - S_{n-1},\quad n > 1$

$$\begin{aligned}
&= n^3 + 2n - 1 - \left[(n-1)^3 + 2(n-1) - 1\right] \\
&= n^3 + 2n - 1 - \left[n^3 - 3n^2 + 3n - 1\right] - 2n + 2 + 1 \\
&= n^3 + 2n - 1 - n^3 + 3n^2 - 3n + 1 - 2n + 3 \\
&= 3n^2 - 3n + 3,\quad n > 1
\end{aligned}$$

and $u_1 = S_1 = 2$

$\therefore\ u_1 = 2,\ u_n = 3n^2 - 3n + 3,\ n > 1$

3 Consider $\dfrac{3x-1}{|x+1|} > 2$.

If $x = -1$, LHS is undefined, so $x = -1$ is not a solution.

If $x \neq -1$, $|x+1| > 0$ and so $3x - 1 > 2|x+1|$.

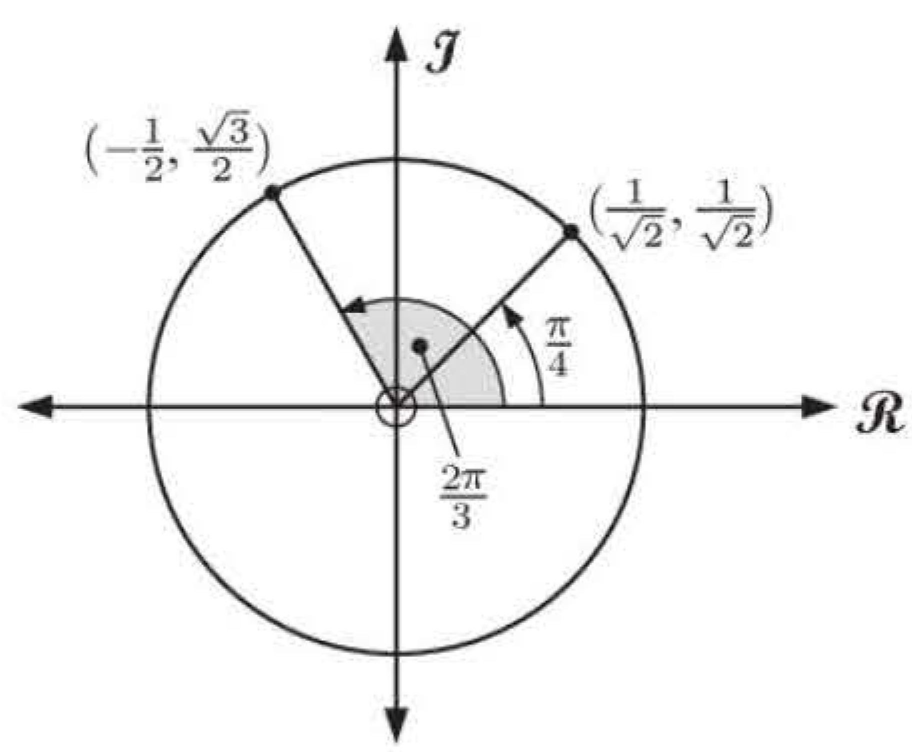
So, if $x > -1$, $3x - 1 > 2x + 2$ $\therefore\ x > 3$

if $x < -1$, $3x - 1 > -2x - 2$ $\therefore\ 5x > -1$ and so $x > -\frac{1}{5}$, which is impossible.

Thus, $x > 3$ is the solution.

4 **a**
$$\begin{aligned}
z &= \frac{-1 + i\sqrt{3}}{4} \\
&= \tfrac{1}{2}\left(-\tfrac{1}{2} + i\tfrac{\sqrt{3}}{2}\right) \\
&= \tfrac{1}{2}\operatorname{cis}\left(\tfrac{2\pi}{3}\right) \\
&= \tfrac{1}{2}\left(\cos\tfrac{2\pi}{3} + i\sin\tfrac{2\pi}{3}\right)
\end{aligned}$$

$$\begin{aligned}
w &= \frac{\sqrt{2} + i\sqrt{2}}{4} \\
&= \tfrac{1}{2}\left(\tfrac{1}{\sqrt{2}} + i\tfrac{1}{\sqrt{2}}\right) \\
&= \tfrac{1}{2}\operatorname{cis}\left(\tfrac{\pi}{4}\right) \\
&= \tfrac{1}{2}(\cos\tfrac{\pi}{4} + i\sin\tfrac{\pi}{4})
\end{aligned}$$



b
$$\begin{aligned}
zw &= \tfrac{1}{2}\operatorname{cis}\left(\tfrac{2\pi}{3}\right) \times \tfrac{1}{2}\operatorname{cis}\left(\tfrac{\pi}{4}\right) \\
&= \tfrac{1}{4}\operatorname{cis}\left(\tfrac{2\pi}{3} + \tfrac{\pi}{4}\right) \\
&= \tfrac{1}{4}\operatorname{cis}\left(\tfrac{11\pi}{12}\right) \\
&= \tfrac{1}{4}\left(\cos\left(\tfrac{11\pi}{12}\right) + i\sin\left(\tfrac{11\pi}{12}\right)\right)
\end{aligned}$$

c $zw = \frac{1}{16}(-1 + i\sqrt{3})(\sqrt{2} + i\sqrt{2}) = \frac{1}{16}\left(\left[-\sqrt{2} - \sqrt{6}\right] + i\left[\sqrt{6} - \sqrt{2}\right]\right)$

Equating real and imaginary parts of zw gives $\cos\left(\frac{11\pi}{12}\right) = \frac{-\sqrt{2}-\sqrt{6}}{4}$ and $\sin\left(\frac{11\pi}{12}\right) = \frac{\sqrt{6}-\sqrt{2}}{4}$.

5 As $y^2 = 4x$, $2y\dfrac{dy}{dx} = 4$ and so $\dfrac{dy}{dx} = \dfrac{2}{y}$.

But the tangent has gradient m, so at the point of contact, $m = \dfrac{2}{y}$ or $y = \dfrac{2}{m}$.

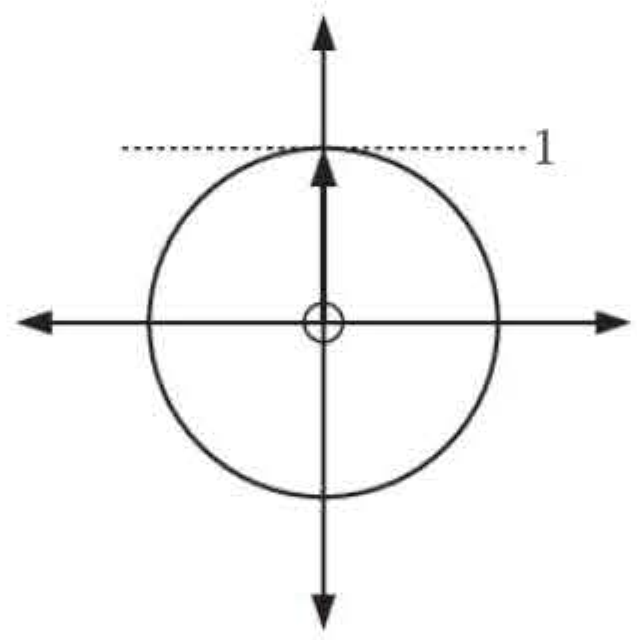
So, the y-coordinate of the point of contact is $\dfrac{2}{m}$.

The x-coordinate of the point of contact is $\dfrac{y^2}{4} = \dfrac{4}{m^2} \div 4 = \dfrac{1}{m^2}$

$\therefore$ the point of contact is $\left(\dfrac{1}{m^2}, \dfrac{2}{m}\right)$.

This point lies on $y = mx + c$, so
$$\frac{2}{m} = m\left(\frac{1}{m^2}\right) + c$$
$$\therefore \frac{2}{m} = \frac{1}{m} + c$$
$$\therefore c = \frac{1}{m}$$

6
$$\sin^2 x + \sin x - 2 = 0, \quad -2\pi \leqslant x \leqslant 2\pi$$
$$\therefore (\sin x + 2)(\sin x - 1) = 0$$
$$\therefore \sin x = -2 \text{ or } 1$$
$$\therefore \sin x = 1 \qquad \{\text{as } -1 \leqslant \sin x \leqslant 1\}$$
$$\therefore x = \tfrac{\pi}{2} + k2\pi, \ k \in \mathbb{Z}$$
$$\therefore x = -\tfrac{3\pi}{2} \text{ or } \tfrac{\pi}{2}$$

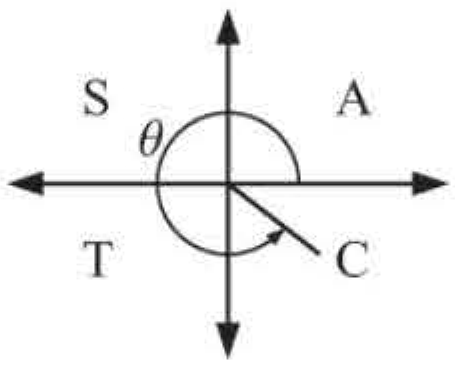


7 $f(x) = \ln x$ has inverse $f^{-1}(x) = e^x$.
$g(x) = 3 + x$ has inverse given by $x = 3 + y$
$\therefore y = x - 3$ so $g^{-1}(x) = x - 3$.

a
$$f^{-1}(2) \times g^{-1}(2)$$
$$= e^2 \times -1$$
$$= -e^2$$

b
$$(f \circ g)(x) = f(g(x)) = f(3 + x)$$
$$= \ln(3 + x)$$
$\therefore$ the inverse of $(f \circ g)(x)$ is $x = \ln(3 + y)$
$$\therefore 3 + y = e^x$$
$$\therefore y = e^x - 3$$
So, $(f \circ g)^{-1}(x) = e^x - 3$
and $(f \circ g)^{-1}(2) = e^2 - 3$

8 $\sin\theta = -\frac{5}{13}$



a
$$\cos\theta$$
$$= \sqrt{1 - \sin^2\theta}$$
$$= \sqrt{1 - \tfrac{25}{169}}$$
$$= \sqrt{\tfrac{144}{169}}$$
$$= \tfrac{12}{13}$$

b
$$\tan\theta$$
$$= \frac{\sin\theta}{\cos\theta}$$
$$= -\tfrac{5}{13} \div \tfrac{12}{13}$$
$$= -\tfrac{5}{12}$$

c
$$\sin 2\theta$$
$$= 2\sin\theta\cos\theta$$
$$= 2\left(\tfrac{-5}{13}\right)\left(\tfrac{12}{13}\right)$$
$$= -\tfrac{120}{169}$$

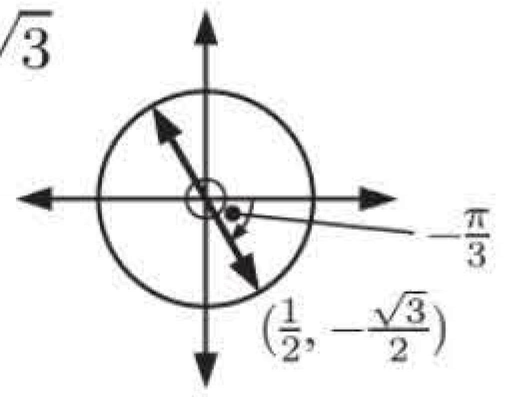
d
$$\sec 2\theta$$
$$= \frac{1}{\cos 2\theta}$$
$$= \frac{1}{2\cos^2\theta - 1}$$
$$= \frac{1}{2\left(\frac{12}{13}\right)^2 - 1}$$
$$= \frac{1}{2\left(\frac{144}{169}\right) - 1} \times \frac{169}{169}$$
$$= \frac{169}{288 - 169}$$
$$= \frac{169}{119}$$

9 $\sqrt{3}\cos x \csc x + 1 = 0, \quad 0 \leqslant x \leqslant 2\pi$

$\therefore \sqrt{3}\cos x \left(\dfrac{1}{\sin x}\right) = -1$

$\therefore \dfrac{\cos x}{\sin x} = -\frac{1}{\sqrt{3}}$

$\therefore \tan x = -\sqrt{3}$



$\therefore x = \frac{2\pi}{3}$ or $\frac{5\pi}{3}$

10 Let X be the number of snails.

$\mu = \sigma^2 = m$ and $\sigma = d \quad \therefore m = d^2$

Now $\text{P}(X = 8) = \frac{1}{2}\text{P}(X = 7)$

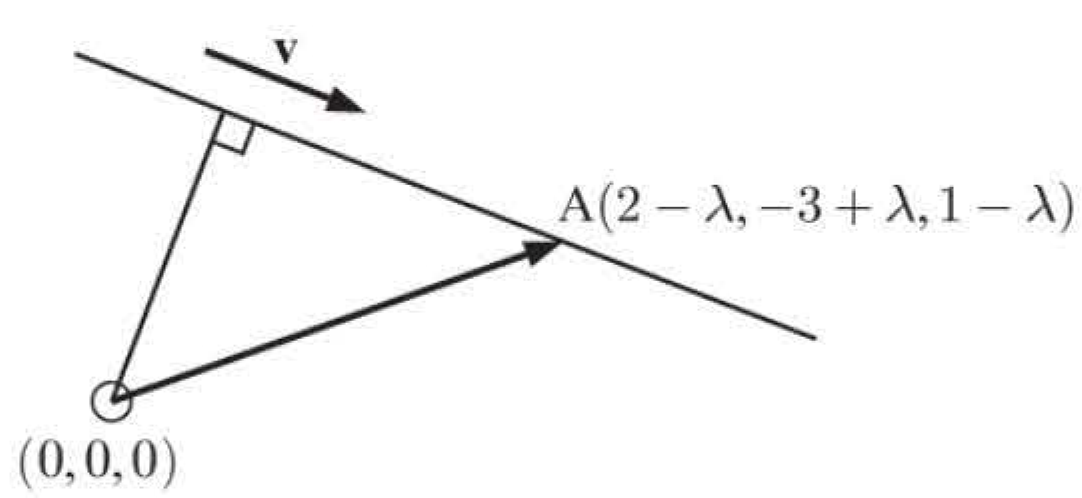
where $\text{P}(X = x) = \dfrac{m^x e^{-m}}{x!} = \dfrac{d^{2x}e^{-d^2}}{x!}$

$\therefore \dfrac{d^{16}e^{-d^2}}{8!} = \frac{1}{2}\dfrac{d^{14}e^{-d^2}}{7!}$

$\dfrac{d^2}{8} = \frac{1}{2}$

$\therefore d = 2 \ \{\text{as } d > 0\}$

11



$\overrightarrow{\text{OA}} = \begin{pmatrix} 2-\lambda \\ \lambda-3 \\ 1-\lambda \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$

The shortest distance occurs when $\overrightarrow{\text{OA}} \bullet \mathbf{v} = 0$

$\therefore -(2-\lambda) + \lambda - 3 + (-1)(1-\lambda) = 0$

$\therefore \lambda - 2 + \lambda - 3 - 1 + \lambda = 0$

$\therefore 3\lambda = 6 \quad \therefore \lambda = 2$

So, the point on L that is nearest the origin is $(0, -1, -1)$.

12 $f'(x) > 0$ and $f''(x) < 0$ for all x

$\therefore f(x)$ is increasing and concave downwards for all x.

a $f(2) = 1$ and $f'(2) = 2$

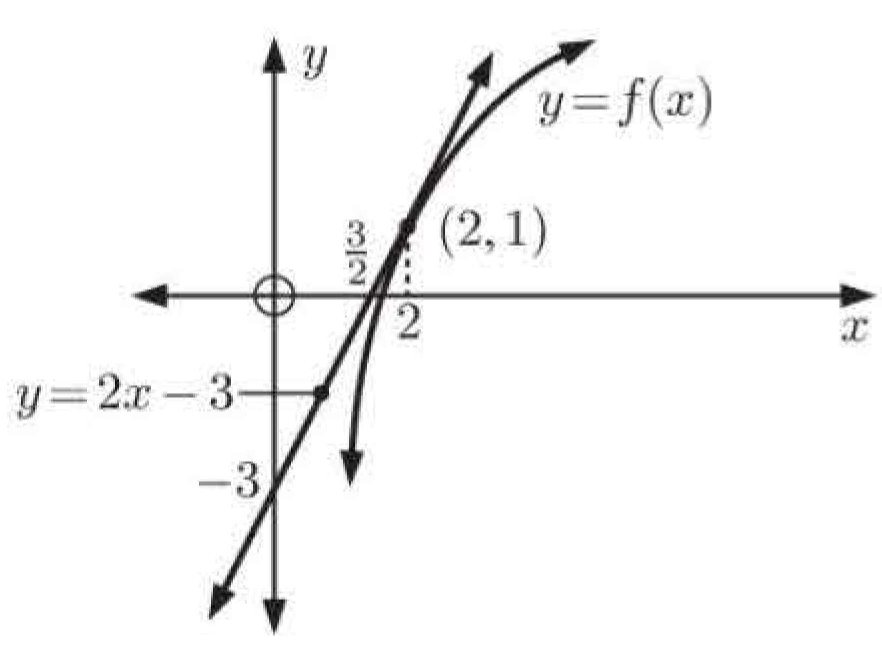
$\therefore$ (2, 1) lies on the curve and the tangent at this point has gradient 2

$\therefore$ the equation of the tangent is $y = 2x + c$

and $1 = 2(2) + c$, so $c = -3$

$\therefore$ the tangent has equation $y = 2x - 3$.

b



c As $f(x)$ is increasing it has *at most one* zero. But $f(x)$ is also concave downwards for all x, so it always lies below the tangent shown. So, for $x < \frac{3}{2}$, the tangent's y-values are negative and so $f(x)$ is also negative. Thus $f(x)$ has *exactly one* zero.

d From the graph, the x-intercept of $y = f(x)$ lies inside $]\frac{3}{2}, 2[$.

13 P_n is "$2n^3 - 3n^2 + n + 31 \geqslant 0$" for $n \in \mathbb{Z}, \ n \geqslant -2$.

Proof: (By the principle of mathematical induction)

(1) If $n = -2$, $2(-2)^3 - 3(-2)^2 + (-2) + 31 = -16 - 12 - 2 + 31 = 1$ which is $\geqslant 0$

$\therefore P_{-2}$ is true.

(2) If P_k is assumed true then $2k^3 - 3k^2 + k + 31 \geqslant 0$

Thus $2(k+1)^3 - 3(k+1)^2 + (k+1) + 31$

$$\begin{aligned}
&= 2(k^3 + 3k^2 + 3k + 1) - 3(k^2 + 2k + 1) + k + 32 \\
&= \left[2k^3 - 3k^2 + k + 31\right] + 6k^2 + 6k + 2 - 6k - 3 + 1 \\
&= \underbrace{\left[2k^3 - 3k^2 + k + 31\right]}_{\geqslant 0 \ \{\text{using } P_k\}} + \underbrace{6k^2}_{\geqslant 0 \text{ as } k^2 \geqslant 0 \text{ for all } k} \\
&\geqslant 0
\end{aligned}$$

Thus P_{k+1} is true whenever P_k is true.

$\therefore$ since P_{-2} is true, P_n is true for all $n \in \mathbb{Z}, \ n \geqslant -2$ {Principle of mathematical induction}

14 We integrate by parts with $u = \arctan x$ $\quad v' = x$

$$u' = \frac{1}{1+x^2} \qquad v = \frac{x^2}{2}$$

$\therefore \quad \int x \arctan x \, dx$

$$= \arctan x \left(\frac{x^2}{2}\right) - \int \frac{x^2}{2(1+x^2)}\, dx$$

$$= \tfrac{1}{2}x^2 \arctan x - \tfrac{1}{2}\int \frac{1+x^2-1}{1+x^2}\, dx$$

$$= \tfrac{1}{2}x^2 \arctan x - \tfrac{1}{2}\int \left(1 - \frac{1}{1+x^2}\right) dx$$

$$= \tfrac{1}{2}x^2 \arctan x - \frac{x}{2} + \tfrac{1}{2}\arctan x + c$$

Check:

$$\frac{d}{dx}\left(\tfrac{1}{2}x^2 \arctan x - \tfrac{1}{2}x + \tfrac{1}{2}\arctan x + c\right)$$

$$= x \arctan x + \tfrac{1}{2}x^2\left(\frac{1}{1+x^2}\right) - \tfrac{1}{2} + \tfrac{1}{2}\left(\frac{1}{1+x^2}\right) + 0$$

$$= x \arctan x + \frac{\tfrac{1}{2}x^2 - \tfrac{1}{2}(1+x^2) + \tfrac{1}{2}}{1+x^2}$$

$$= x \arctan x + \frac{0}{1+x^2}$$

$$= x \arctan x \quad \checkmark$$

15 **a** $\log_2(x^2 - 2x + 1) = 1 + \log_2(x-1)$

$\therefore \log_2(x-1)^2 - \log_2(x-1) = 1$

$\therefore 2\log_2(x-1) - \log_2(x-1) = 1$

$\therefore \log_2(x-1) = 1$

$\therefore x - 1 = 2^1$

$\therefore x = 3$

b $3^{2x+1} = 5(3^x) + 2$

$\therefore 3(3^x)^2 - 5(3^x) - 2 = 0$

$\therefore 3m^2 - 5m - 2 = 0 \quad \{m = 3^x\}$

$\therefore (3m+1)(m-2) = 0$

$\therefore m = -\tfrac{1}{3}$ or 2

$\therefore 3^x = -\tfrac{1}{3}$ or $3^x = 2$

The first equation is impossible as $3^x > 0$ for all x.

$\therefore 3^x = 2$

$\therefore x = \dfrac{\ln 2}{\ln 3}$

(or $x = \log_3 2$)

16 P_n is "$\sum_{r=1}^{n} r3^r = \tfrac{3}{4}[(2n-1)3^n + 1]$" for $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= \sum_{r=1}^{1} r3^r = 1(3)^1 = 3$

RHS $= \tfrac{3}{4}\left[(1)3^1 + 1\right] = \tfrac{3}{4} \times 4 = 3$ $\quad \therefore P_1$ is true.

(2) If P_k is assumed true, then $\sum_{r=1}^{k} (r3^r) = \tfrac{3}{4}\left[(2k-1)3^k + 1\right]$

Thus $\sum_{r=1}^{k+1} r3^r = \sum_{r=1}^{k} r3^r + (k+1)3^{k+1}$

$$= \tfrac{3}{4}\left[(2k-1)3^k + 1\right] + (k+1)3^{k+1} \qquad \{\text{using } P_k\}$$

$$= \tfrac{3}{4}\left[(2k-1)3^k + 1 + \tfrac{4}{3}(k+1)3^{k+1}\right]$$

$$= \tfrac{3}{4}\left[(2k-1)3^k + 1 + (4k+4)3^k\right]$$

$$= \tfrac{3}{4}\left[(6k+3)3^k + 1\right]$$

$$= \tfrac{3}{4}\left[(2k+1)3^{k+1} + 1\right]$$

$$= \tfrac{3}{4}\left[(2(k+1)-1)3^{k+1} + 1\right]$$

Thus P_{k+1} is true whenever P_k is true.

$\therefore$ since P_1 is true, P_n is true for all $n \in \mathbb{Z}^+$ $\quad$ {Principle of mathematical induction}

17 Since $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1, \quad b^2x^2+a^2y^2=a^2b^2$ .... $(*)$

$$\therefore\ b^2(2x)+a^2(2y)\,\frac{dy}{dx}=0$$

$$\therefore\ \text{at the point } (x_1,\, y_1),\quad \frac{dy}{dx}=-\frac{b^2x}{a^2y}=-\frac{b^2x_1}{a^2y_1}$$

$$\therefore\ \text{the equation of the tangent is } \frac{y-y_1}{x-x_1}=\frac{-b^2x_1}{a^2y_1}$$

$$\therefore\ a^2y_1y-a^2{y_1}^2=-b^2x_1x+b^2{x_1}^2$$

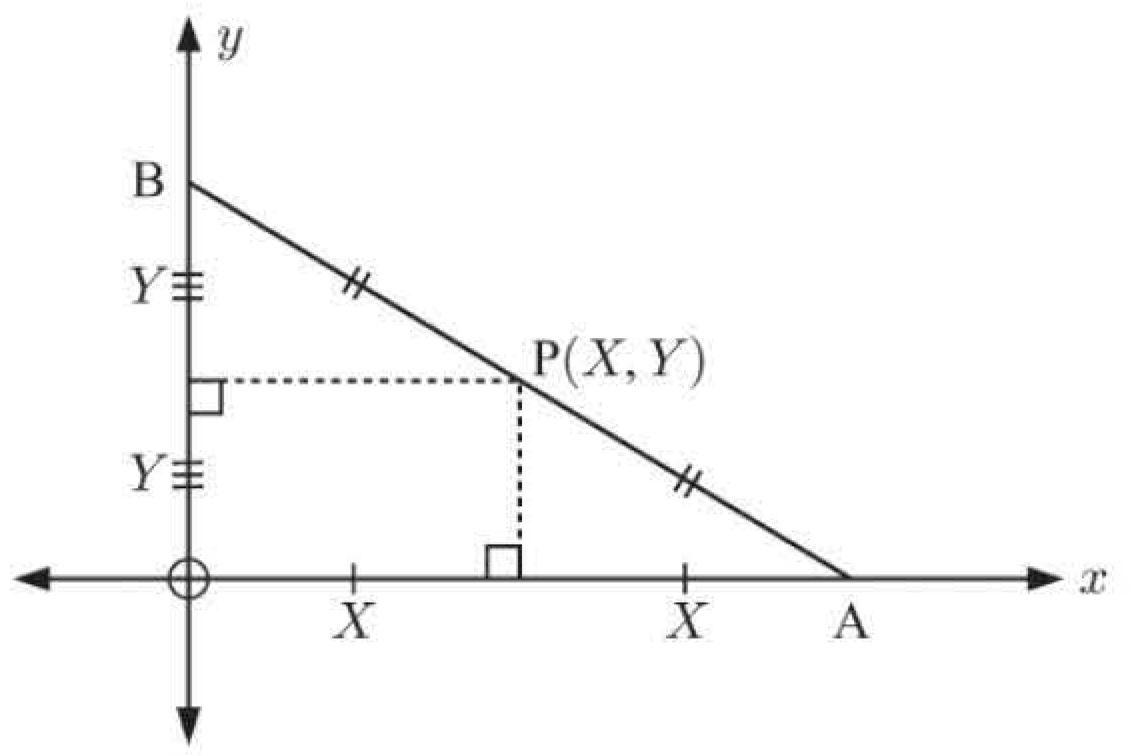
$$\therefore\ a^2y_1y+b^2x_1x=b^2{x_1}^2+a^2{y_1}^2$$

As $(x_1,\, y_1)$ lies on the curve, $b^2{x_1}^2+a^2{y_1}^2=a^2b^2$ {from $(*)$}

$\therefore$ the equation of the tangent is $b^2x_1x+a^2y_1y=a^2b^2$

or $\left(\dfrac{x_1}{a^2}\right)x+\left(\dfrac{y_1}{b^2}\right)y=1$ {dividing throughout by a^2b^2}

18



Let $\text{P}(X,\, Y)$ be the midpoint of [AB].

$\therefore$ A is at $(2X,\, 0)$ and B is at $(0,\, 2Y)$.

Let [AB] have fixed length l units.

$$\therefore\ (2X)^2+(2Y)^2=l^2$$

$$\therefore\ X^2+Y^2=\left(\frac{l}{2}\right)^2$$

which is the equation of a circle, centre $(0,\, 0)$ and radius $\dfrac{l}{2}$ units.

19 **a**

$$\frac{Ax+B}{x^2+5}+\frac{C}{x-1}=\frac{Ax+B}{x^2+5}-\frac{C}{1-x}$$

$$=\frac{(Ax+B)(1-x)-C(x^2+5)}{(x^2+5)(1-x)}$$

$$=\frac{-(A+C)x^2+(A-B)x+(B-5C)}{(x^2+5)(1-x)}$$

So, if $\dfrac{x+5}{(x^2+5)(1-x)}=\dfrac{Ax+B}{x^2+5}+\dfrac{C}{x-1}$ for all x then

$$-(A+C)=0,\ A-B=1, \text{ and } B-5C=5$$

Solving these simultaneously gives $A=1,\ B=0,\ C=-1$

b

$$\int_2^4\frac{x+5}{(x^2+5)(1-x)}\,dx=\int_2^4\left(\frac{x}{x^2+5}-\frac{1}{x-1}\right)dx$$

$$=\tfrac{1}{2}\int_2^4\frac{2x}{x^2+5}\,dx-\int_2^4\frac{1}{x-1}\,dx$$

$$=\tfrac{1}{2}\left[\ln\left|x^2+5\right|\right]_2^4-[\ln|x-1|]_2^4$$

$$=\tfrac{1}{2}(\ln 21-\ln 9)-(\ln 3-\ln 1)$$

$$=\tfrac{1}{2}\ln\left(\tfrac{7}{3}\right)-\ln 3$$

$$=\tfrac{1}{2}(\ln 7-\ln 3)-\ln 3$$

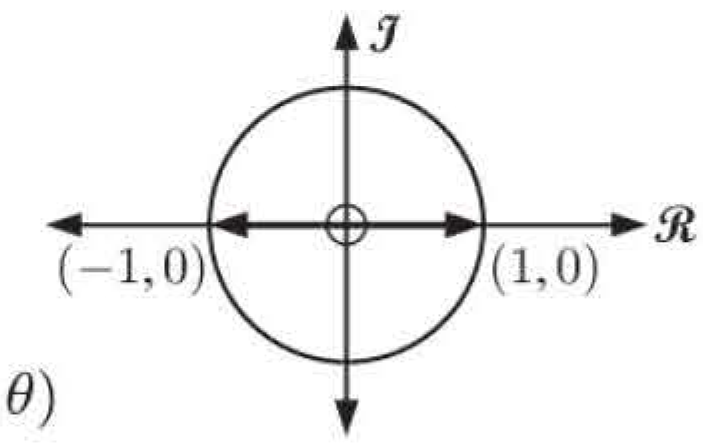
$$=\tfrac{1}{2}\ln 7-\tfrac{3}{2}\ln 3$$

20 **a** Let $z = |z| \operatorname{cis}\theta$, $\quad \therefore \quad \dfrac{1}{z} = \dfrac{\operatorname{cis} 0}{|z| \operatorname{cis}\theta} = \dfrac{1}{|z|} \operatorname{cis}(-\theta)$

$$\therefore \quad z + \frac{1}{z} = |z| \operatorname{cis}\theta + \frac{1}{|z|}\operatorname{cis}(-\theta)$$

$$= |z|(\cos\theta + i\sin\theta) + \frac{1}{|z|}(\cos\theta - i\sin\theta)$$

$$= \cos\theta\left(|z| + \frac{1}{|z|}\right) + i\sin\theta\left(|z| - \frac{1}{|z|}\right)$$

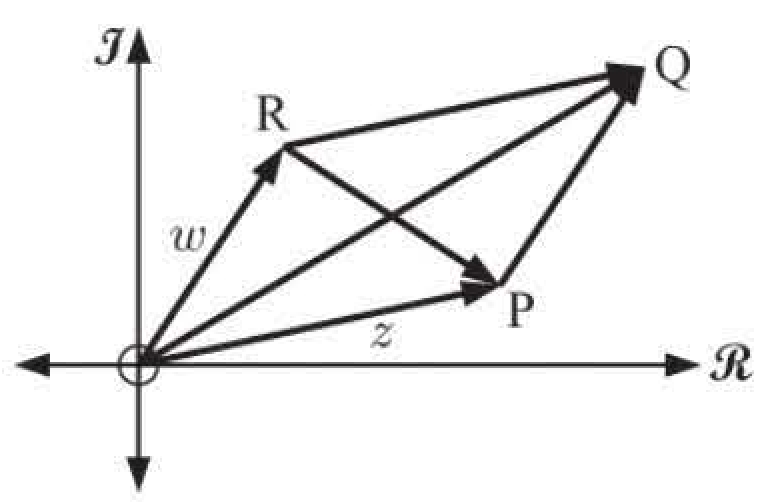


Now $z + \dfrac{1}{z}$ is real, $\quad \therefore \quad \sin\theta = 0$ or $|z| - \dfrac{1}{|z|} = 0$

$$\therefore \quad \theta = k\pi \text{ or } |z| = 1$$

$$\therefore \quad z \text{ is real or } |z| = 1$$

b



If $z = \overrightarrow{\text{OP}}$ and $w = \overrightarrow{\text{OR}}$ then $\overrightarrow{\text{RP}} = z - w$ and $\overrightarrow{\text{OQ}} = z + w$ in parallelogram OPQR. So, if $|z + w| = |z - w|$ the diagonals are equal in length, which is only possible when the parallelogram is a rectangle.

Thus, $\widehat{\text{POR}} = \frac{\pi}{2}$, so $\arg w - \arg z = \frac{\pi}{2}$

If P and R were interchanged then $\arg z - \arg w = \frac{\pi}{2}$

Thus, the arguments differ by $\frac{\pi}{2}$.

c If $z = r \operatorname{cis}\theta$

$$z^4 = [r \operatorname{cis}\theta]^4$$
$$= r^4 \operatorname{cis} 4\theta$$
{De Moivre}

$$\frac{1}{z} = \frac{1}{r\operatorname{cis}\theta}$$
$$= \frac{1}{r\operatorname{cis}\theta}\left(\frac{\operatorname{cis}(-\theta)}{\operatorname{cis}(-\theta)}\right)$$
$$= \frac{1}{r}\,\frac{\operatorname{cis}(-\theta)}{\operatorname{cis} 0}$$
$$= \frac{1}{r}\operatorname{cis}(-\theta)$$

$$iz^* = \operatorname{cis}\left(\tfrac{\pi}{2}\right)(r\operatorname{cis}(-\theta))$$
$$= r\operatorname{cis}\left(\tfrac{\pi}{2} - \theta\right)$$

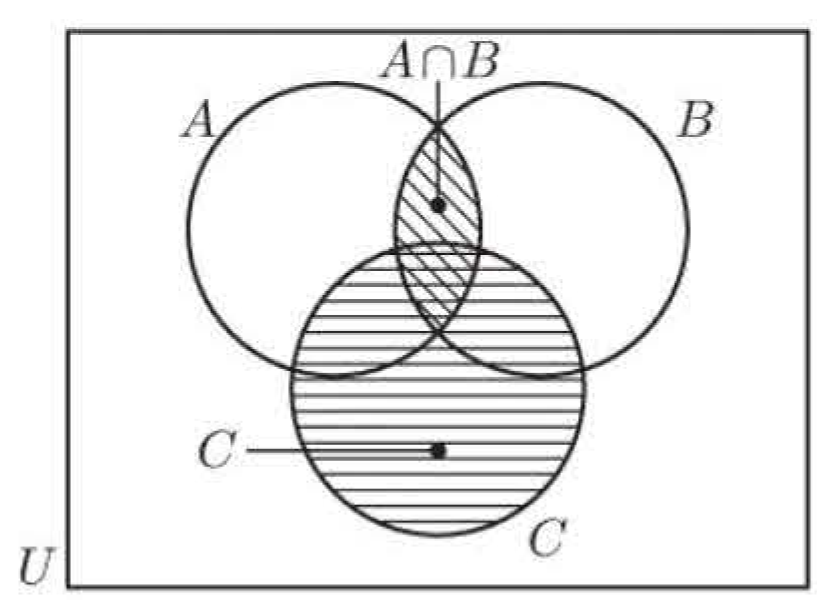
21 **a** **i**

$$(A \cup B) \cap A'$$
$$= A' \cap (A \cup B)$$
$$= (A' \cap A) \cup (A' \cap B)$$
$$= \varnothing \cup (A' \cap B)$$
$$= A' \cap B$$

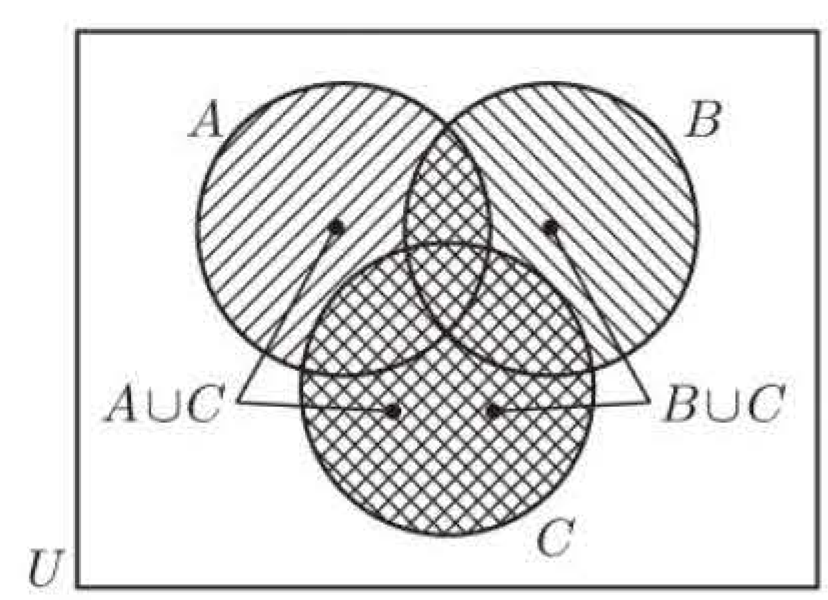
ii

$$(A \cap B) \cup (A' \cap B)$$
$$= (A \cup A') \cap B$$
$$= U \cap B$$
$$= B$$

b



$(A \cap B) \cup C$ includes all that is shaded.

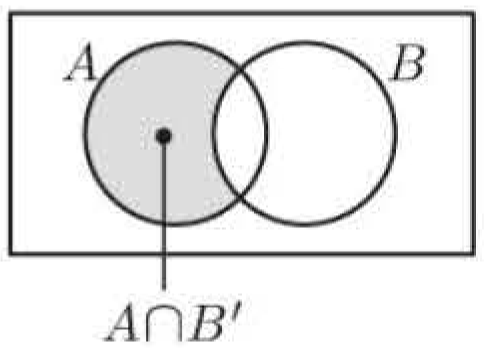


$(A \cup C) \cap (B \cup C)$ includes all parts double shaded.

As the shadings match, the identity is verified.

c **i** $\mathrm{P}(A' \cap B')$
$= \mathrm{P}((A \cup B)')$
$= 1 - \mathrm{P}(A \cup B)$
$= 1 - [\mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)]$
$= 1 - \mathrm{P}(A) - \mathrm{P}(B) + \mathrm{P}(A)\mathrm{P}(B)$
$= [1 - \mathrm{P}(A)]\,[1 - \mathrm{P}(B)]$
$= \mathrm{P}(A')\,\mathrm{P}(B')$
$\therefore$ A' and B' are independent.

ii $\mathrm{P}(A \cap B')$
$= \mathrm{P}(A) - \mathrm{P}(A \cap B)$
$= \mathrm{P}(A) - \mathrm{P}(A)\mathrm{P}(B)$
$= \mathrm{P}(A)\,[1 - \mathrm{P}(B)]$
$= \mathrm{P}(A)\,\mathrm{P}(B')$
$\therefore$ A and B' are independent.



22 **a** Let $x = \sin\theta$, $\dfrac{dx}{d\theta} = \cos\theta$

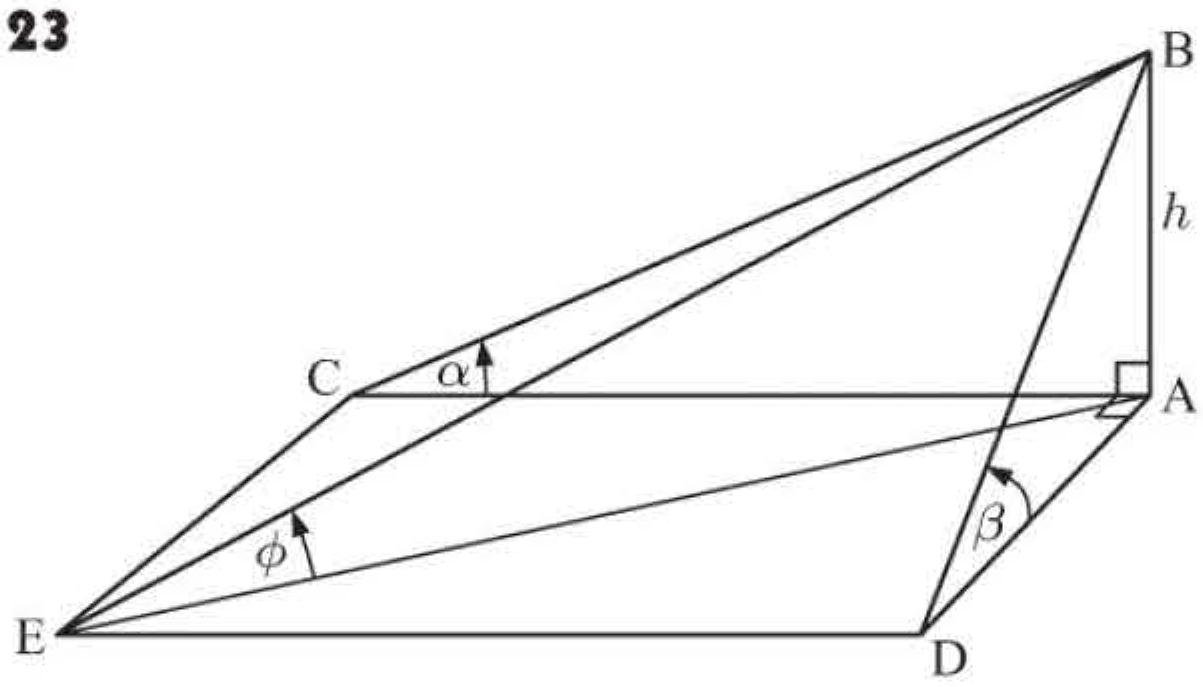
$$\therefore \int \frac{x}{\sqrt{1-x^2}}\,dx = \int \frac{\sin\theta}{\cos\theta}\cos\theta\,d\theta$$
$$= \int \sin\theta\,d\theta$$
$$= -\cos\theta + c$$
$$= -\sqrt{1-\sin^2\theta} + c$$
$$= -\sqrt{1-x^2} + c$$

b
$$\int \frac{1+x}{1+x^2}\,dx$$
$$= \int \frac{1}{1+x^2}\,dx + \int \frac{x}{1+x^2}\,dx$$
$$= \int \frac{1}{1+x^2}\,dx + \tfrac{1}{2}\int \frac{2x}{1+x^2}\,dx$$
$$= \arctan x + \tfrac{1}{2}\ln\left|1+x^2\right| + c$$
$$= \arctan x + \tfrac{1}{2}\ln\left(1+x^2\right) + c \quad \{x^2+1>0\}$$

c Let $x = \sin\theta$, $\dfrac{dx}{d\theta} = \cos\theta$

$$\therefore \int \frac{1}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{1-\sin^2\theta}}\cos\theta\,d\theta$$
$$= \int \frac{1}{\cos\theta}\cos\theta\,d\theta$$
$$= \int 1\,d\theta = \theta + c$$
$$= \arcsin x + c$$

23



Suppose the pole has height h and the angle of elevation of B from E is ϕ.

$$\tan\alpha = \frac{h}{\mathrm{AC}},\quad \tan\beta = \frac{h}{\mathrm{AD}},\quad \tan\phi = \frac{h}{\mathrm{AE}}$$
$\therefore$ $\mathrm{AC} = h\cot\alpha$, $\mathrm{AD} = h\cot\beta$, $\mathrm{AE} = h\cot\phi$

But $\mathrm{AE}^2 = \mathrm{AC}^2 + \mathrm{AD}^2$ {Pythagoras}

$$\therefore\ h^2\cot^2\phi = h^2\cot^2\alpha + h^2\cot^2\beta$$
$$\therefore\ \cot^2\phi = \cot^2\alpha + \cot^2\beta$$
$$\therefore\ \cot\phi = \sqrt{\cot^2\alpha + \cot^2\beta} \quad \{\cot\phi > 0\}$$
$$\therefore\ \phi = \operatorname{arccot}\left(\sqrt{\cot^2\alpha + \cot^2\beta}\right)$$

24 **a**
$$\frac{1}{1+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \dots + \frac{1}{\sqrt{99}+\sqrt{100}}$$
$$= \frac{1}{1+\sqrt{2}}\left(\frac{1-\sqrt{2}}{1-\sqrt{2}}\right) + \frac{1}{\sqrt{2}+\sqrt{3}}\left(\frac{\sqrt{2}-\sqrt{3}}{\sqrt{2}-\sqrt{3}}\right) + \frac{1}{\sqrt{3}+\sqrt{4}}\left(\frac{\sqrt{3}-\sqrt{4}}{\sqrt{3}-\sqrt{4}}\right) + \dots$$
$$+ \frac{1}{\sqrt{99}+\sqrt{100}}\left(\frac{\sqrt{99}-\sqrt{100}}{\sqrt{99}-\sqrt{100}}\right)$$
$$= \frac{1-\sqrt{2}}{1-2} + \frac{\sqrt{2}-\sqrt{3}}{2-3} + \frac{\sqrt{3}-\sqrt{4}}{3-4} + \dots + \frac{\sqrt{99}-\sqrt{100}}{99-100}$$
$$= -\left(1-\sqrt{2}+\sqrt{2}-\sqrt{3}+\sqrt{3}-\sqrt{4}+\dots+\sqrt{99}-\sqrt{100}\right)$$
$$= -(1-10)$$
$$= 9$$

b Using the same technique, $\dfrac{1}{1+\sqrt{2}}+\dfrac{1}{\sqrt{2}+\sqrt{3}}+\dfrac{1}{\sqrt{3}+\sqrt{4}}+....+\dfrac{1}{\sqrt{n}+\sqrt{n+1}}$

$$= -\left(1-\sqrt{n+1}\right)$$
$$= \sqrt{n+1}-1$$

25 Since $x > y > z > 0$, $\dfrac{1}{x} < \dfrac{1}{y} < \dfrac{1}{z}$

$\therefore \ \dfrac{1}{z}-\dfrac{1}{y}=\dfrac{1}{y}-\dfrac{1}{x}$ and so $\dfrac{1}{x}+\dfrac{1}{z}=\dfrac{2}{y}$ $\left\{\dfrac{1}{x}, \dfrac{1}{y}, \dfrac{1}{z} \text{ in arithmetic progression}\right\}$

$\therefore \ \dfrac{x+z}{xz}=\dfrac{2}{y}$

$\therefore \ xy+yz=2xz \ .... \ (*)$

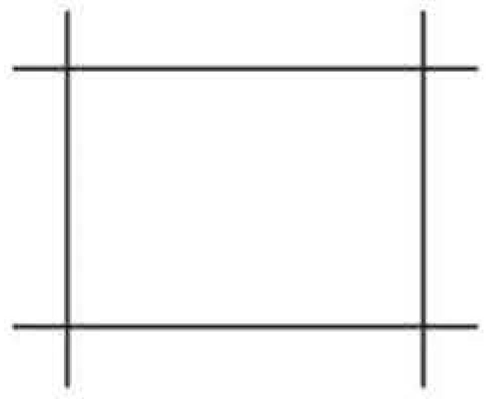
Now $(x-y+z)^2 = x^2+y^2+z^2-2xy+2xz-2yz$
$\qquad = x^2+y^2+z^2-2(xy+yz)+2xz$

$\therefore \ (x-y+z)^2 = x^2+y^2+z^2-4xz+2xz \ \{\text{using } (*)\}$
$\qquad = x^2-2xz+z^2+y^2$
$\qquad = (x-z)^2+y^2$

Hence $x-z$, y, and $x-y+z$ form the sides of a right angled triangle.

26 Each rectangle is determined by choosing the two pairs of opposite sides.

This can be done in $\binom{m+2}{2}\times\binom{n+2}{2}=\dfrac{(m+2)(m+1)(n+2)(n+1)}{4}$ ways.



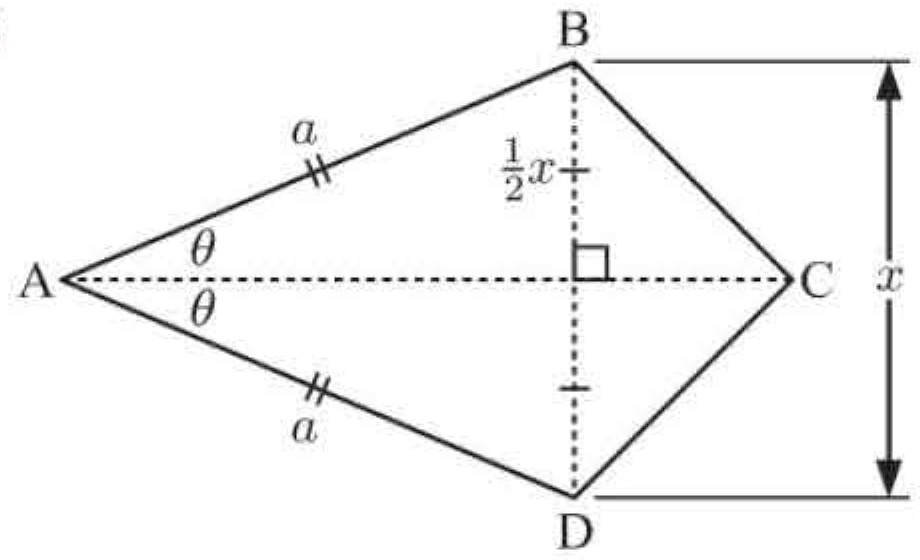
27 The two numbers selected are different and there are $\binom{n}{2}$ different selections.

Of these $(1, 4)$, $(2, 8)$, $(3, 12)$, ..., $(\frac{n}{4}, n)$ are the different outcomes where one is 4 times the other.

There are $\frac{n}{4}$ of these.

$\therefore$ P(one is 4 times the other) $= \dfrac{\frac{n}{4}}{\binom{n}{2}} = \dfrac{\frac{n}{4}}{\frac{n(n-1)}{2}} = \dfrac{1}{2(n-1)}$

28



The diagonals [AC] and [BD] intersect at right angles.

Let $\text{BD} = x$. Using the cosine rule,

$x^2 = a^2+a^2-2aa\cos 2\theta$

$\therefore \ x^2 = 2a^2(1-\cos 2\theta) \ .... \ (1)$

But $\sin\theta = \dfrac{\frac{x}{2}}{a} = \dfrac{x}{2a}$

$\therefore \ x^2 = 4a^2\sin^2\theta \ .... \ (2)$

From (1) and (2), $2a^2(1-\cos 2\theta) = 4a^2\sin^2\theta$

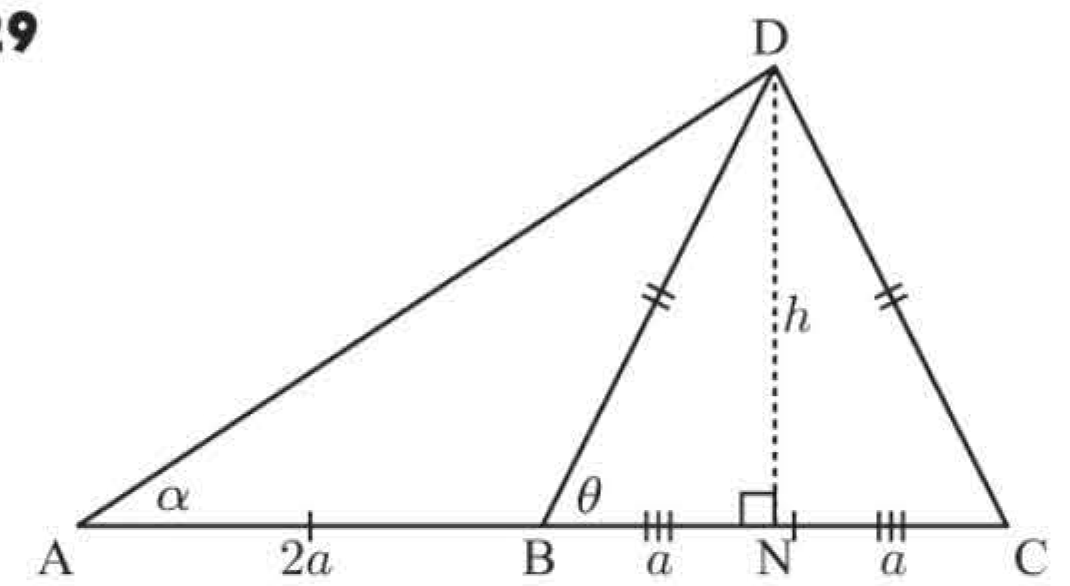
$\therefore \ 1-\cos 2\theta = 2\sin^2\theta$

$\therefore \ \sin^2\theta = \frac{1}{2}-\frac{1}{2}\cos 2\theta$

and $\cos^2\theta = 1-\sin^2\theta$
$\qquad = 1-\left(\frac{1}{2}-\frac{1}{2}\cos 2\theta\right)$

$\therefore \ \cos^2\theta = \frac{1}{2}+\frac{1}{2}\cos 2\theta$

29



Draw [DN] $\perp$ [BC] in isosceles $\triangle$BCD.

$\therefore$ BN $=$ NC $= a$ say and so AB $=$ BC $= 2a$

If DN $= h$, then $\tan\alpha = \dfrac{\text{DN}}{\text{AN}} = \dfrac{h}{3a}$

and $\tan\theta = \dfrac{h}{a}$

$\therefore\ 3\tan\alpha = \tan\theta$

30 **a**

$$\begin{aligned} 8^{2x+3} &= 4\sqrt[3]{2} \\ \therefore\ 2^{3(2x+3)} &= 2^2 \times 2^{\frac{1}{3}} \\ \therefore\ 6x+9 &= \tfrac{7}{3} \\ \therefore\ 18x+27 &= 7 \\ \therefore\ 18x &= -20 \\ \therefore\ x &= -1\tfrac{1}{9} \end{aligned}$$

b

$$\begin{aligned} 3^{2x+1} + 8(3^x) &= 3 \\ \therefore\ 3(3^x)^2 + 8(3^x) - 3 &= 0 \\ \therefore\ 3m^2 + 8m - 3 &= 0 \quad \text{where } m = 3^x \\ \therefore\ (3m-1)(m+3) &= 0 \quad \text{where } m = 3^x \\ \therefore\ m &= \tfrac{1}{3} \text{ or } -3 \end{aligned}$$

But $m = 3^x > 0$, so $3^x = \frac{1}{3} = 3^{-1}$

$\therefore\ x = -1$

c

$$\begin{aligned} \ln(\ln x) &= 1 \\ \therefore\ \ln x &= e^1 = e \\ \therefore\ x &= e^e \end{aligned}$$

d

$$\begin{aligned} \log_{\frac{1}{9}} x &= \log_9 5 \\ \therefore\ \frac{\log x}{\log\left(\frac{1}{9}\right)} &= \frac{\log 5}{\log 9} \\ \therefore\ \frac{\log x}{-\log 9} &= \frac{\log 5}{\log 9} \\ \therefore\ \log x &= -\log 5 = \log(5^{-1}) \\ \therefore\ x &= 5^{-1} = \tfrac{1}{5} \end{aligned}$$

31

$$\begin{aligned} \frac{d}{dx}(\tan^3 x) &= 3\,(\tan x)^2 \times \sec^2 x \\ &= 3\,\frac{\sin^2 x}{\cos^2 x}\,\frac{1}{\cos^2 x} \\ &= 3(1-\cos^2 x)\sec^4 x \\ &= 3\sec^4 x - 3\sec^2 x \end{aligned}$$

$$\begin{aligned} \therefore\ \int(3\sec^4 x - 3\sec^2 x)\,dx &= \tan^3 x + c \\ \therefore\ 3\int\sec^4 x\,dx - 3\tan x + c &= \tan^3 x + c \\ \therefore\ 3\int\sec^4 x\,dx &= 3\tan x + \tan^3 x + c \\ \therefore\ \int\sec^4 x\,dx &= \tan x + \tfrac{1}{3}\tan^3 x + c \end{aligned}$$

32 If X and Y are independent events then $\mathrm{P}(X\cap Y) = \mathrm{P}(X)\,\mathrm{P}(Y)$.

Thus $\mathrm{P}((A\cap B)\cap(A\cup B)) = \mathrm{P}(A\cap B)\,\mathrm{P}(A\cup B)$

$\therefore\ \mathrm{P}(A\cap B) = \mathrm{P}(A\cap B)\,\mathrm{P}(A\cup B)$ {since $A\cap B \subseteq A\cup B$}

$\therefore\ \mathrm{P}(A\cap B) = 0$ or $\mathrm{P}(A\cup B) = 1$

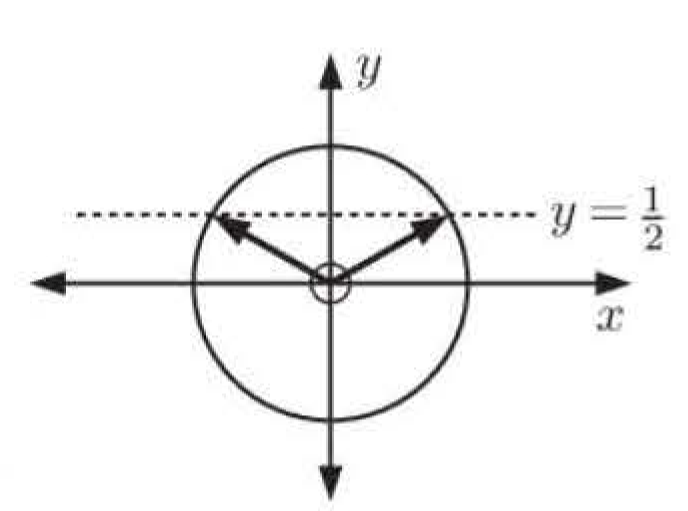
$\therefore$ A and B are disjoint or either A or B must occur.

33

$$\begin{aligned} (3-i\sqrt{2})^4 &= 3^4 + 4(3^3)(-i\sqrt{2}) + 6(3^2)(-i\sqrt{2})^2 + 4(3)(-i\sqrt{2})^3 + (-i\sqrt{2})^4 \\ &= 81 - 108\sqrt{2}i - 108 + 24\sqrt{2}i + 4 \\ &= -23 - 84\sqrt{2}i \end{aligned}$$

34

$$\begin{aligned} \sin\theta\cos\theta &= \tfrac{1}{4}, \quad \theta\in[-\pi,\,\pi] \\ \therefore\ \tfrac{1}{2}\sin 2\theta &= \tfrac{1}{4} \\ \therefore\ \sin 2\theta &= \tfrac{1}{2} \\ \therefore\ 2\theta &= \tfrac{\pi}{6} + k2\pi \text{ or } \tfrac{5\pi}{6} + k2\pi \\ \therefore\ 2\theta &= \tfrac{\pi}{6}, -\tfrac{11\pi}{6}, \tfrac{5\pi}{6}, -\tfrac{7\pi}{6} \qquad \{\text{as } -2\pi \leqslant 2\theta \leqslant 2\pi\} \\ \therefore\ \theta &= -\tfrac{11\pi}{12}, -\tfrac{7\pi}{12}, \tfrac{\pi}{12}, \tfrac{5\pi}{12} \end{aligned}$$



35 As the cubic has real coefficients, both $2+i$ and $2-i$ are roots.

These have sum 4 and product $4+1=5$, and so come from the quadratic factor z^2-4z+5.

Thus $z^3+az^2+bz+15=(z^2-4z+5)(z+3)$

$\therefore\ z^3+az^2+bz+15=z^3-z^2-7z+15$

$\therefore\ a=-1$ and $b=-7$ {equating coefficients}

36 **a**

$(0.5)^{x+1}>0.125$

$\therefore\ (0.5)^{x+1}>(0.5)^3$

But $y=(0.5)^{x+1}$ is decreasing,

so $x+1<3$

$\therefore\ x<2$

b

$\left(\frac{2}{3}\right)^x>\left(\frac{3}{2}\right)^{x-1}$

$\therefore\ \left(\frac{2}{3}\right)^x>\left(\frac{2}{3}\right)^{1-x}$

But $y=\left(\frac{2}{3}\right)^x$ is decreasing,

$\therefore\ x<1-x$

$\therefore\ 2x<1$

$\therefore\ x<\frac{1}{2}$

c

$4^x+2^{x+3}<48$

$\therefore\ (2^x)^2+8(2^x)-48<0$

$(2^x+12)(2^x-4)<0$

where $2^x+12>0$ for all x

$\therefore\ 2^x-4<0$

$\therefore\ 2^x<2^2$

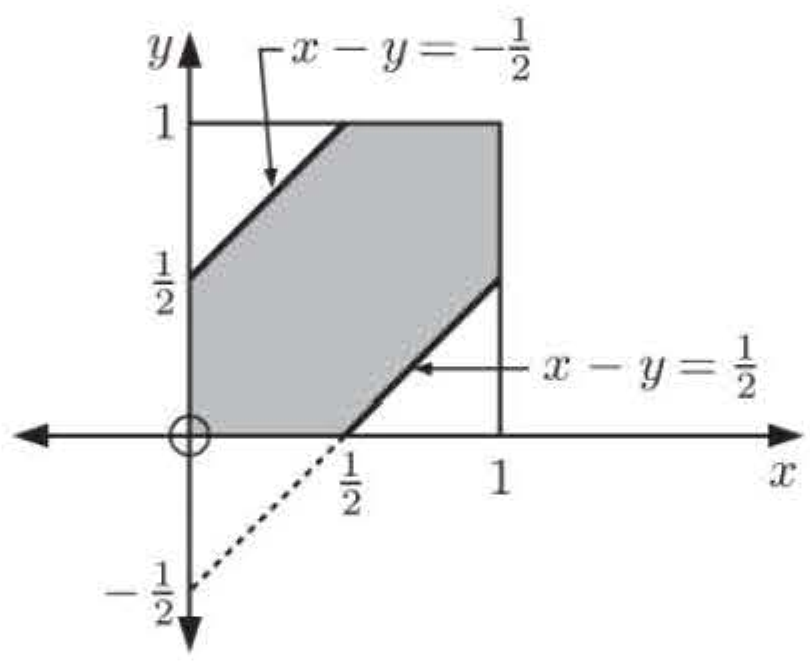
But $y=2^x$ is increasing,

$\therefore\ x<2$

37 Let X arrive x hours after 1 pm, $0\leqslant x\leqslant 1$ and Y arrive y hours after 1 pm, $0\leqslant y\leqslant 1$.

They meet provided $-\frac{1}{2}<x-y<\frac{1}{2}$ as the difference between their arrival times is not more than half an hour.

$\therefore\ \text{P(they meet)}=\dfrac{\text{shaded area}}{\text{area of square}}=\dfrac{\frac{3}{4}}{1}=\frac{3}{4}.$



38 We draw the smaller circle in another position, and deliberately put P close to the x-axis, but not on it.

We let $\widehat{\text{XOP}}=\alpha$ and M be the centre of the smaller circle.

We join [MP] and let $\widehat{\text{MOP}}=\theta$

$\therefore\ \widehat{\text{QMP}}=2\theta$ {angle at the centre theorem}

Now arc QX = arc QP, since both represent the distance the smaller circle has been rolled.

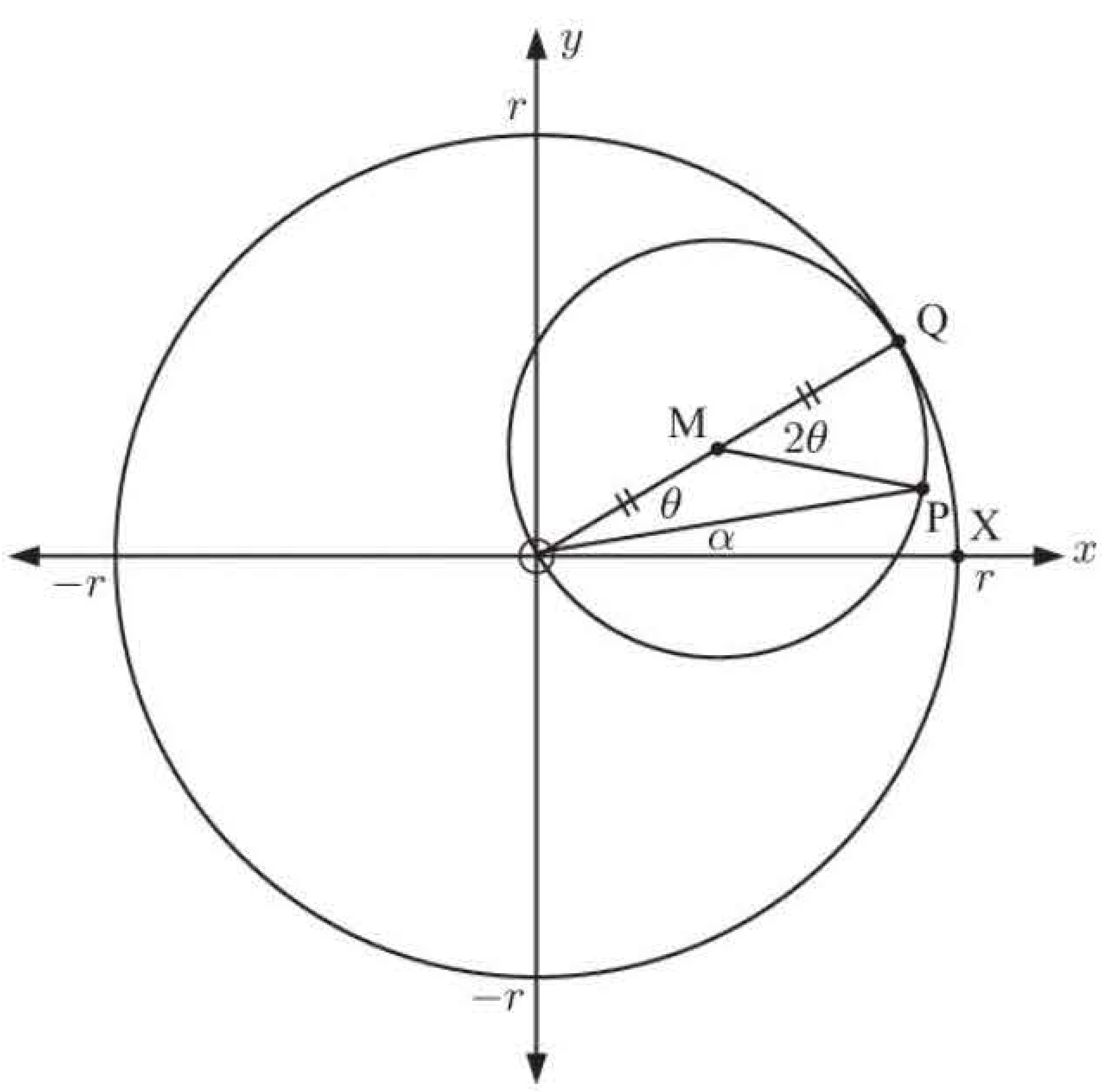
$\therefore\ r(\theta+\alpha)=\dfrac{r}{2}(2\theta)$

$\therefore\ r\theta+r\alpha=r\theta$

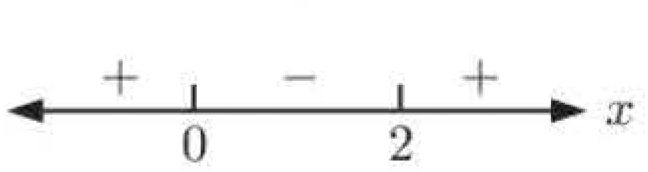
$\therefore\ r\alpha=0$

$\therefore\ \alpha=0$ as $r\neq 0$

$\therefore$ P lies on the x-axis.



39 **a** $f(x)=\ln(x(x-2))$ is defined when $x(x-2)>0$



$\therefore\ x<0$ or $x>2$

So the domain is $x\in\,]-\infty,0[\,\cup\,]2,\infty[$

b $f(x) = \ln x + \ln(x-2)$ {log law}

$\therefore\ f'(x) = \dfrac{1}{x} + \dfrac{1}{x-2} = \dfrac{2x-2}{x(x-2)}$

c $f'(3) = \frac{1}{3} + 1 = \frac{4}{3}$ at $(3, \ln 3)$

$\therefore$ the tangent has equation

$$\frac{y - \ln 3}{x-3} = \frac{4}{3}$$

$\therefore\ 4x - 12 = 3y - 3\ln 3$

$\therefore\ 4x - 3y = 12 - 3\ln 3$

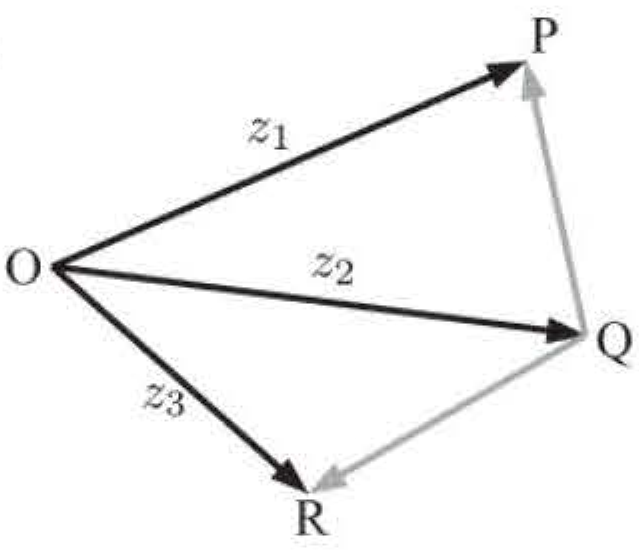
40 **a** Let $z = R\,\mathrm{cis}\,\theta$

$\therefore\ iz = \mathrm{cis}\,\frac{\pi}{2} \times R\,\mathrm{cis}\,\theta$

$\therefore\ iz = R\,\mathrm{cis}\left(\frac{\pi}{2} + \theta\right)$

$\therefore\ \arg(iz) = \frac{\pi}{2} + \theta$

b



$z_3 - z_2 = \overrightarrow{QR}$

$z_1 - z_2 = \overrightarrow{QP}$

Now $i(z_3 - z_2) = z_1 - z_2$

$\therefore\ i\,\overrightarrow{QR} = \overrightarrow{QP}$

$\therefore\ \overrightarrow{QR} \perp \overrightarrow{QP}$ {from **a**}

$\therefore\ P\widehat{Q}R$ is a right angle.

Also $|\overrightarrow{QP}| = |i\,\overrightarrow{QR}| = |i|\,|\overrightarrow{QR}|$

$\therefore\ QP = 1 \times QR = QR$

$\therefore\ \triangle PQR$ is right angled and isosceles, right angled at Q.

41 $(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \dots + \binom{n}{n}x^n$ .... (1)

$(1+x)^2(1+x)^{n-2} = (1 + 2x + 1x^2)\left[1 + \binom{n-2}{1}x + \binom{n-2}{2}x^2 + \dots + \binom{n-2}{n-2}x^{n-2}\right]$ .... (2)

In (1), the coefficient of x^r is $\binom{n}{r}$

In (2), the coefficient of x^r is $1 \times \binom{n-2}{r} + 2\binom{n-2}{r-1} + 1\binom{n-2}{r-2}$

Equating these gives $\binom{n}{r} = \binom{n-2}{r} + 2\binom{n-2}{r-1} + \binom{n-2}{r-2}$

42

$$x^2 + y^2 = 52xy$$

$\therefore\ x^2 - 2xy + y^2 = 50xy$

$\therefore\ (x-y)^2 = 50xy$

$\therefore\ \dfrac{(x-y)^2}{25} = 2xy$

$\therefore\ \left(\dfrac{x-y}{5}\right)^2 = 2xy$

$\therefore\ \log\left(\dfrac{x-y}{5}\right)^2 = \log(2xy)$

$\therefore\ 2\log\left(\dfrac{x-y}{5}\right) = \log x + \log 2y$

$\therefore\ \log\left(\dfrac{x-y}{5}\right) = \frac{1}{2}(\log x + \log 2y)$

43

$$\sin(xy) + y^2 = x$$

$\therefore\ \cos(xy)\left[1y + x\dfrac{dy}{dx}\right] + 2y\dfrac{dy}{dx} = 1$

$\therefore\ \dfrac{dy}{dx}(x\cos(xy) + 2y) = 1 - y\cos(xy)$

$\therefore\ \dfrac{dy}{dx} = \dfrac{1 - y\cos(xy)}{x\cos(xy) + 2y}$

44 After labelling the triangles,

$\triangle$ABC and $\triangle$ABD are isosceles {equal base angles}

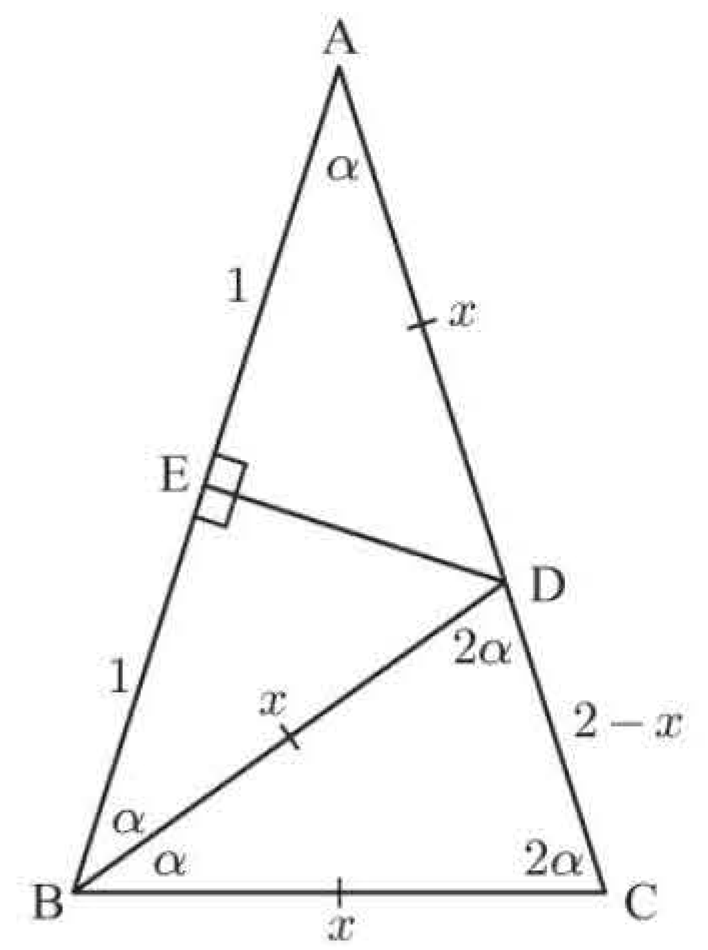
and $\widehat{\text{BDC}} = 2\alpha$ {exterior angle of $\triangle$ABD}

$\therefore$ $\triangle$BDC is isosceles {equal base angles}

Thus AD = BD = BC = x, say.

If we let AE = EB = 1 unit, then AC = 2 and DC = $2 - x$.

Now $\triangle$ABC and $\triangle$BCD are similar {equiangular}



$$\therefore \frac{\text{AB}}{\text{BC}} = \frac{\text{BC}}{\text{CD}} \text{ and so } \frac{2}{x} = \frac{x}{2-x}$$

$$\therefore x^2 + 2x - 4 = 0$$

$$\therefore x = -1 \pm \sqrt{5}$$

$$\therefore x = \sqrt{5} - 1 \quad \{\text{as } x \text{ must be } > 0\}$$

$$\text{But } 5\alpha = 180^\circ \quad \{\text{angle sum of a triangle}\}$$

$$\therefore \alpha = 36^\circ$$

and in $\triangle$BED, $$\cos 36^\circ = \frac{1}{\sqrt{5}-1} = \frac{1}{\sqrt{5}-1}\left(\frac{\sqrt{5}+1}{\sqrt{5}+1}\right) = \frac{1+\sqrt{5}}{4}.$$

45 By the Remainder theorem, $P(1) = -3$ and $P(-3) = -15$ where $P(x) = x^n + ax^2 - 6$

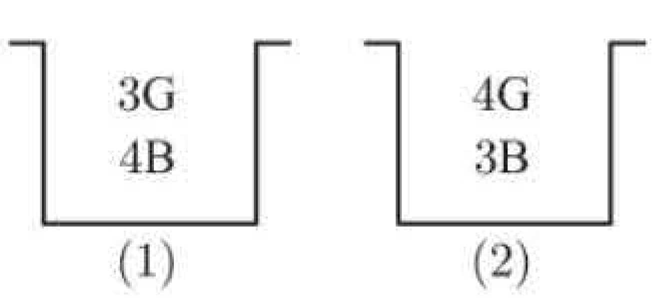
$\therefore (1)^n + a(1)^2 - 6 = -3$ and $(-3)^n + a(-3)^2 - 6 = -15$

$\therefore 1 + a - 6 = -3$ and $(-3)^n + 9a = -9$

$\therefore a = 2$ and $(-3)^n = -27$

$\therefore n = 3$

46 **a**



P(both same colour)

$= \text{P(GG or BB)}$

$= \frac{3}{7} \times \frac{4}{7} + \frac{4}{7} \times \frac{3}{7}$

$= \frac{24}{49}$

b P(G from (2) | both different)

$= \dfrac{\text{P(G from (2) } \cap \text{ both different)}}{\text{P(both different)}}$

$= \dfrac{\text{P(G from (2) \textbf{and} B from (1))}}{1 - \frac{24}{49}}$

$= \dfrac{\frac{4}{7} \times \frac{4}{7}}{\frac{25}{49}}$

$= \frac{16}{25}$

47 **a** $y = f(x-2) + 1$ is a translation of $y = f(x)$ through $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. $\therefore$ A$(-2, 3) \to$ A$'(0, 4)$.

b $y = 2f(x-2)$ is obtained from $y = f(x)$ by a translation through $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ followed by a vertical stretch with scale factor $k = 2$. $\therefore$ A$(-2, 3) \to$ A$'(0, 3) \to$ A$''(0, 6)$.

c Consider $y = -|f(x)| - 2$. When $x = -2$, $y = -|f(-2)| - 2 = -3 - 2 = -5$

$\therefore$ A$(-2, 3) \to$ A$'(-2, -5)$.

d $y = f(2x-3) = f\left(2(x - \frac{3}{2})\right)$.

It is obtained from $y = f(x)$ by a horizontal stretch with scale factor $k = \frac{1}{2}$ followed by a translation through $\begin{pmatrix} \frac{3}{2} \\ 0 \end{pmatrix}$. $\therefore$ A$(-2, 3) \to$ A$'(-1, 3) \to$ A$''(\frac{1}{2}, 3)$.

e Consider $y = \dfrac{1}{f(x)}$. When $x = -2$, $y = \dfrac{1}{f(-2)} = \frac{1}{3}$. $\therefore$ A$(-2, 3) \to$ A$'(-2, \frac{1}{3})$.

f Consider $y = f^{-1}(x)$. For an inverse function, the point is reflected in the line $y = x$.

$\therefore$ A$(-2, 3) \to$ A$'(3, -2)$.

48 **a** $z = re^{i\theta} = r\operatorname{cis}\theta$

$$\therefore\ z + \frac{1}{z} = r\operatorname{cis}\theta + \frac{1}{r\operatorname{cis}\theta}$$
$$= r\operatorname{cis}\theta + \frac{1}{r}\operatorname{cis}(-\theta)$$
$$= r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)$$
$$= \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$$

Thus $a = \left(r + \frac{1}{r}\right)\cos\theta$ and $b = \left(r - \frac{1}{r}\right)\sin\theta$

b If $z + \frac{1}{z}$ is real then

$$\left(r - \frac{1}{r}\right)\sin\theta = 0$$
$$\therefore\ r - \frac{1}{r} = 0 \text{ or } \sin\theta = 0$$
$$\therefore\ r^2 = 1 \text{ or } \sin\theta = 0$$
$$\therefore\ r = 1 \text{ or } \theta = k\pi,\ k \in \mathbb{Z}\ \{r > 0\}$$
$$\therefore\ r = 1 \text{ or } z \text{ is real and non-zero}$$

49 P_n is "If u_n is defined by $u_1 = u_2 = 1$ and $u_{n+2} = u_{n+1} + u_n$ then $u_n \leqslant 2^n$" for $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$ or 2, $u_1 = u_2 = 1$ and so $u_1, u_2 \leqslant 2^1$ $\quad\therefore\ P_1$ and P_2 are true.

(2) If P_k is assumed true then $u_k \leqslant 2^k$

Now $u_{k+1} = u_k + u_{k-1},\quad k \geqslant 2$

$\therefore\ u_{k+1} \leqslant u_k + u_k$ {the sequence $\{u_n\}$ is increasing, so $u_{k-1} \leqslant u_k$ for all $k \in \mathbb{Z}$}

$\therefore\ u_{k+1} \leqslant 2u_k$

$\therefore\ u_{k+1} \leqslant 2(2^k)$ {by P_k}

$\therefore\ u_{k+1} \leqslant 2^{k+1}$

Thus P_{k+1} is true whenever P_k is true, for $k \geqslant 2$.

$\therefore$ since P_1 and P_2 are true, P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

50 P_n is "$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\ldots\left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}$" for $n \in \mathbb{Z}^+,\ n \geqslant 2$.

Proof: (By the principle of mathematical induction)

(1) If $n = 2$, LHS $= 1 - \frac{1}{2^2} = \frac{3}{4}$ and RHS $= \frac{2+1}{2(2)} = \frac{3}{4}$ $\quad\therefore\ P_2$ is true.

(2) If P_k is assumed true then

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\ldots\left(1 - \frac{1}{k^2}\right) = \frac{k+1}{2k}$$

$$\therefore\ \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\ldots\left(1 - \frac{1}{k^2}\right)\left(1 - \frac{1}{(k+1)^2}\right) = \frac{k+1}{2k}\left(1 - \frac{1}{(k+1)^2}\right)$$
$$= \frac{k+1}{2k}\left(\frac{(k+1)^2 - 1}{(k+1)^2}\right)$$
$$= \frac{k+1}{2k} \times \frac{k^2 + 2k}{(k+1)^2}$$
$$= \frac{k(k+2)}{2k(k+1)}$$
$$= \frac{(k+1)+1}{2(k+1)}$$

Thus P_{k+1} is true whenever P_k is true.

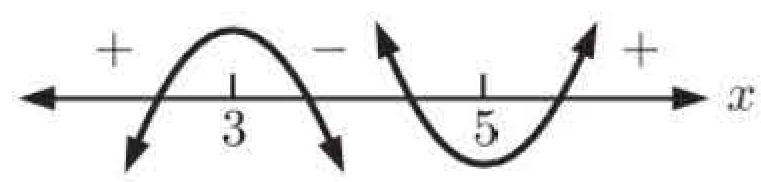
$\therefore$ since P_2 is true, P_n is true for all $n \geqslant 2$ {Principle of mathematical induction}

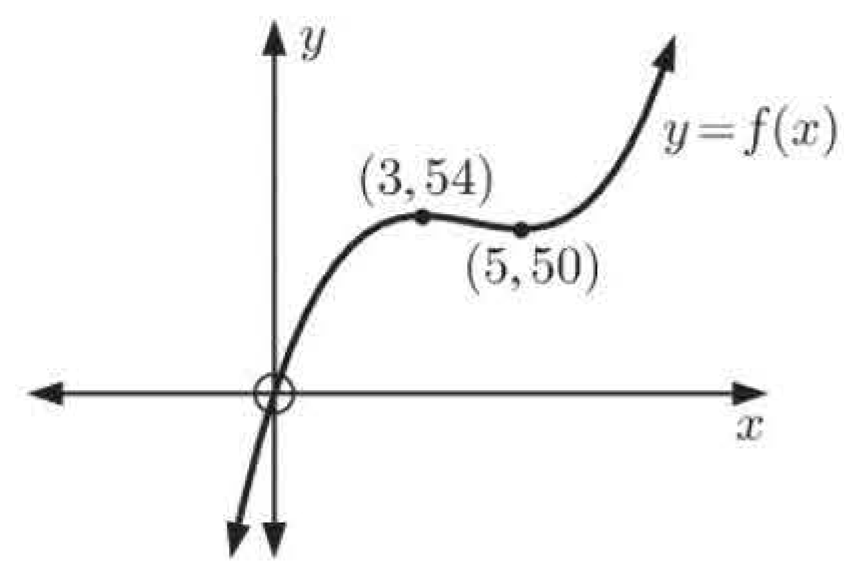
51 **a** $y = x^3 - 12x^2 + 45x$

$$\therefore \quad \frac{dy}{dx} = 3x^2 - 24x + 45$$
$$= 3(x^2 - 8x + 15)$$
$$= 3(x-3)(x-5)$$

which is 0 when $x = 3$ or 5

The sign diagram is:





local maximum at (3, 54)

local minimum at (5, 50)

b $y = x^3 - 12x^2 + 45x$ meets $y = k$ where $x^3 - 12x^2 + 45x = k$.

Now $y = k$ is a horizontal line, so for 3 real roots we need to observe where $y = k$ meets the curve in 3 places.

Thus $50 < k < 54$ or $k \in \,]\,50, 54\,[$.

52 **a** Using the cosine rule,

$$y^2 = x^2 + 8^2 - 2(x)(8)\cos\theta$$
$$\therefore \quad y^2 = x^2 + 64 - 16x\cos\theta$$

But $x + y + 8 = 20$

$$\therefore \quad y = 12 - x$$

Hence, $(12 - x)^2 = x^2 + 64 - 16x\cos\theta$

$$\therefore \quad 144 - 24x + x^2 = x^2 + 64 - 16x\cos\theta$$
$$\therefore \quad 16x\cos\theta = 24x - 80$$
$$\therefore \quad \cos\theta = \frac{3x - 10}{2x}$$

b The area, $A = \frac{1}{2}(8x)\sin\theta = 4x\sin\theta$

$$\therefore \quad A^2 = 16x^2\sin^2\theta$$
$$= 16x^2(1 - \cos^2\theta)$$
$$= 16x^2\left(1 - \left(\frac{3x-10}{2x}\right)^2\right)$$
$$= 16x^2\left(1 - \frac{9x^2 - 60x + 100}{4x^2}\right)$$
$$= 16x^2 - 36x^2 + 240x - 400$$
$$= -20x^2 + 240x - 400$$
$$= -20(x^2 - 12x + 20)$$

c A^2 is a quadratic in x with $a = -20$ and so the shape is 

Thus it has a maximum value when $x = \dfrac{-b}{2a} = \frac{-240}{-40} = 6$

$$\therefore \quad A^2_{\max} = -20(36 - 72 + 20) = 320$$

So, $A_{\max} = \sqrt{320} = 8\sqrt{5}\text{ cm}^2$ when $x = y = 6$

$\therefore$ the triangle is isosceles.

53 **a** $u_3 = u_1 + 2d = \dfrac{1}{k}$ .... (1)

$u_4 = u_1 + 3d = k$

$u_6 = u_1 + 5d = k^2 + 1$

So, $u_1 + 3d - u_1 - 2d = k - \dfrac{1}{k}$

$$\therefore \quad d = k - \frac{1}{k} \quad .... \text{ (2)}$$

and $u_1 + 5d - u_1 - 3d = k^2 + 1 - k$

$$\therefore \quad 2d = k^2 + 1 - k \quad .... \text{ (3)}$$

From (2) and (3), $k^2 - k + 1 = 2k - \dfrac{2}{k}$

$$\therefore \quad k^3 - 3k^2 + k + 2 = 0$$
$$\therefore \quad (k-2)(k^2 - k - 1) = 0$$
$$\therefore \quad k = 2 \text{ or } \frac{1 \pm \sqrt{1 - 4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

But $k \in \mathbb{Q}$, so $k = 2$

b Using (2), $d = 2 - \frac{1}{2} = \frac{3}{2}$

$\therefore$ using (1), $u_1 + 2\left(\frac{3}{2}\right) = \frac{1}{2}$

$\therefore \; u_1 = -\frac{5}{2}$

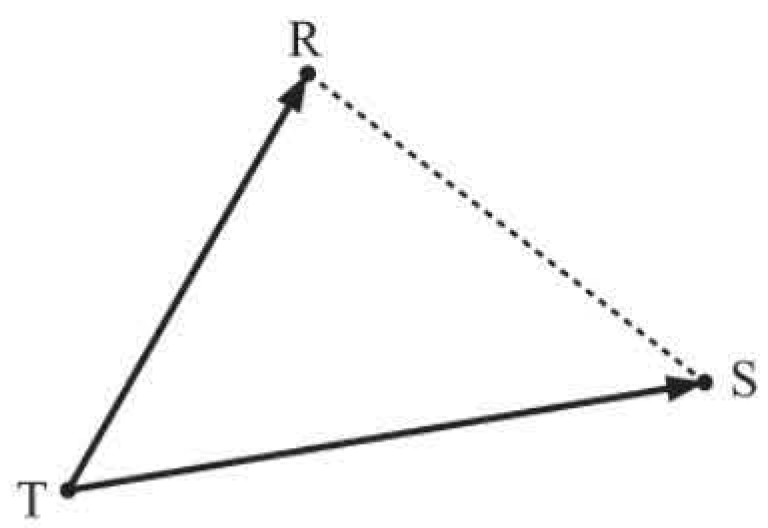
Now $u_n = u_1 + (n-1)d$

$\therefore \; u_n = -\frac{5}{2} + (n-1)\frac{3}{2}$

$\therefore \; u_n = \frac{3}{2}n - 4$

$\therefore \; u_n = \dfrac{3n-8}{2}$ for all $n \in \mathbb{Z}^+$

54



$$\begin{aligned}\overrightarrow{\mathrm{TR}} &= \mathbf{r} - \mathbf{t}\\ &= 2\mathbf{i} - 2\mathbf{j} + \mathbf{k} - \mathbf{i} - 2\mathbf{j} + \mathbf{k}\\ &= \mathbf{i} - 4\mathbf{j} + 2\mathbf{k}\\ \overrightarrow{\mathrm{TS}} &= \mathbf{s} - \mathbf{t}\\ &= 3\mathbf{i} + \mathbf{j} + 2\mathbf{k} - \mathbf{i} - 2\mathbf{j} + \mathbf{k}\\ &= 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}\end{aligned}$$

$\therefore$ area $\triangle$RST

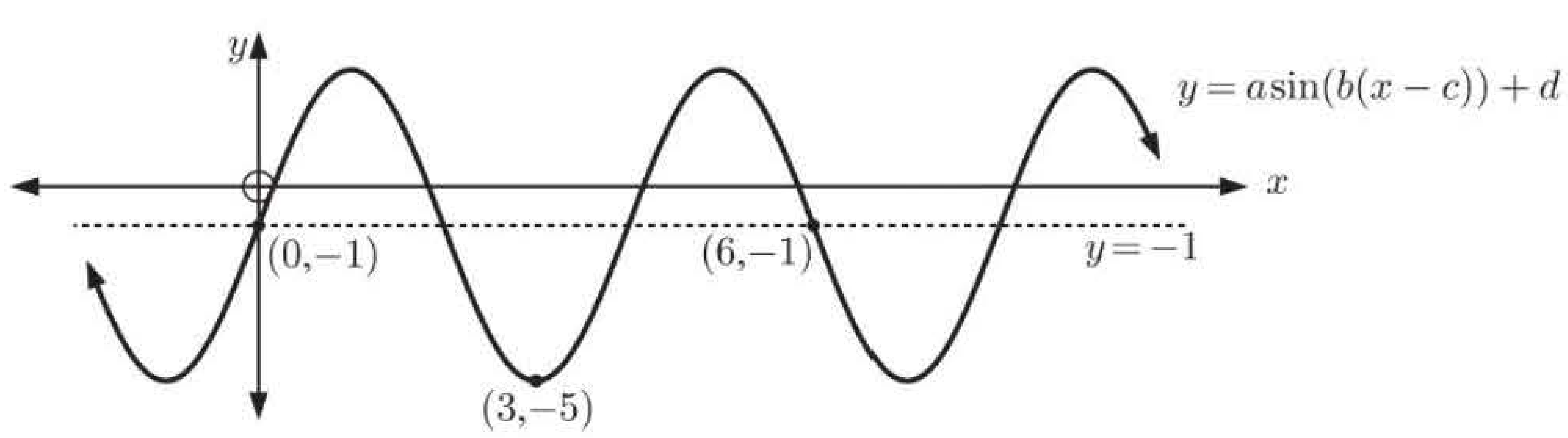
$$\begin{aligned}&= \tfrac{1}{2} \,|\, \overrightarrow{\mathrm{TR}} \times \overrightarrow{\mathrm{TS}} \,|\\ &= \tfrac{1}{2}\left\|\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & -4 & 2\\ 2 & -1 & 3\end{vmatrix}\right\|\\ &= \tfrac{1}{2}\,|-10\mathbf{i} + \mathbf{j} + 7\mathbf{k}|\\ &= \tfrac{1}{2}\sqrt{100+1+49}\\ &= \tfrac{1}{2}\sqrt{150}\\ &= \tfrac{5}{2}\sqrt{6} \text{ units}^2\end{aligned}$$

55 $x = \log_3 y^2 \quad \therefore \; y^2 = 3^x = (81^{\frac{1}{4}})^x = 81^{\frac{x}{4}}$

$\therefore \; y = (81^{\frac{x}{4}})^{\frac{1}{2}} = 81^{\frac{x}{8}}$

$\therefore \; 81 = y^{\frac{8}{x}}$ and so $\log_y 81 = \dfrac{8}{x}$

56



The amplitude $= a = 4$. The period $= 4 = \dfrac{2\pi}{b} \quad \therefore \; b = \frac{\pi}{2}$

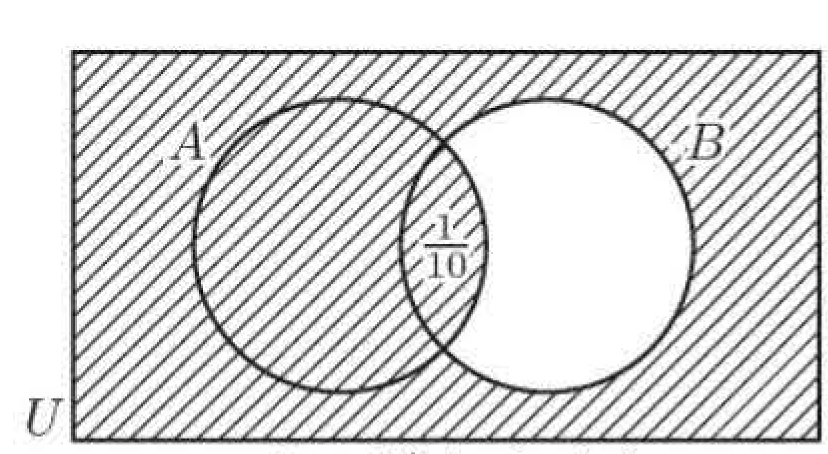
The basic sine curve has been translated through $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$. $\quad \therefore \; c = 0,\; d = -1$

Thus $\; y = 4\sin\left(\frac{\pi}{2}x\right) - 1$

Check: $y(3) = 4\sin\left(\frac{3\pi}{2}\right) - 1 = 4(-1) - 1 = -5$ ✓

$y(6) = 4\sin(3\pi) - 1 = 4(0) - 1 = -1$ ✓

57 As A and B are independent, $\mathrm{P}(A \mid B) = \mathrm{P}(A)$ and $\mathrm{P}(B \mid A) = \mathrm{P}(B)$



$A \cup B'$ is shaded

$\therefore \; \mathrm{P}(A) = \frac{1}{4}$ and $\mathrm{P}(B) = \frac{2}{5}$

and $\mathrm{P}(A \cap B) = \mathrm{P}(A)\,\mathrm{P}(B) = \frac{1}{4} \times \frac{2}{5} = \frac{1}{10}$

So, $\mathrm{P}(A \cup B') = 1 - \mathrm{P}(B) + \mathrm{P}(A \cap B)$

$= 1 - \frac{2}{5} + \frac{1}{10}$

$= \frac{7}{10}$

58 $\dfrac{58}{9(3-7i)} = \dfrac{58}{9(3-7i)}\left(\dfrac{3+7i}{3+7i}\right)$

$= \dfrac{58(3+7i)}{9(9+49)}$

$= \dfrac{3+7i}{9}$

Thus, $z^2 = 1 + i + \dfrac{3+7i}{9} = \dfrac{12+16i}{9}$

$\therefore\ z = \pm\dfrac{\sqrt{12+16i}}{3}$

We now let $\sqrt{12+16i} = c + di,\ c > 0$

$\therefore\ 12 + 16i = c^2 - d^2 + 2cdi$

$\therefore\ c^2 - d^2 = 12$ and $cd = 8$

$\therefore\ c = 4,\ d = 2$ or $\cancel{c = -4,\ d = -2}$ {as $c > 0$}

$\therefore\ z = \pm\dfrac{4+2i}{3}$

$\therefore\ z = \frac{4}{3} + \frac{2}{3}i$ or $-\frac{4}{3} - \frac{2}{3}i$

59 $4^x = 8^y$ and $9^y = \dfrac{243}{3^x}$

$\therefore\ (2^2)^x = (2^3)^y$ and $(3^2)^y \times 3^x = 3^5$

$\therefore\ 2x = 3y$ and $2y + x = 5$

{equating indices}

Solving these simultaneously gives

$2(5 - 2y) = 3y$

$\therefore\ 10 = 7y$

$\therefore\ y = \frac{10}{7}$

So, $2x = 3\left(\frac{10}{7}\right)$

$\therefore\ x = \frac{15}{7}$

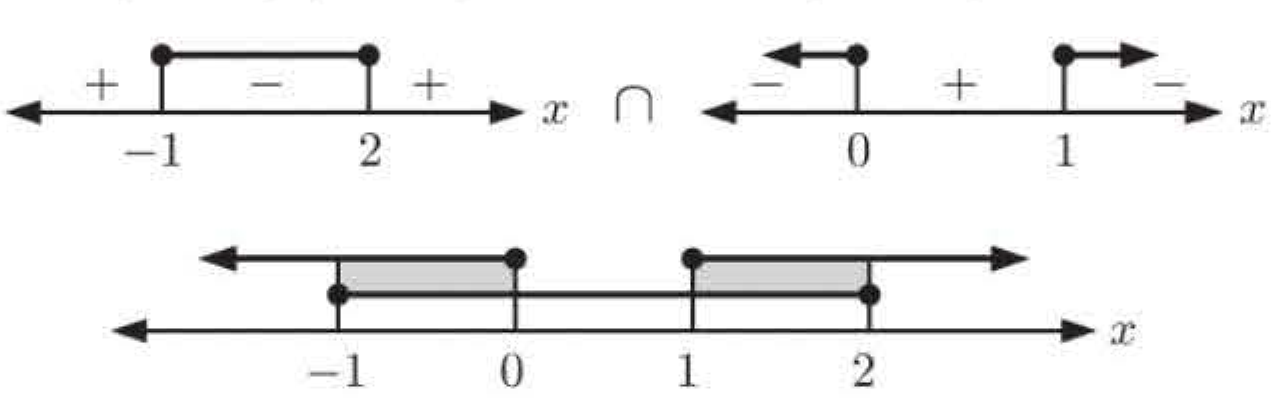
So, $x = 2\frac{1}{7},\ y = 1\frac{3}{7}$

60 For $f(x)$ to be defined we require that

$-1 \leqslant 1 + x - x^2 \leqslant 1$

$\therefore\ x^2 - x - 2 \leqslant 0$ and $x - x^2 \leqslant 0$

$\therefore\ (x-2)(x+1) \leqslant 0$ and $x(1-x) \leqslant 0$



$\therefore\ x \in [-1, 0] \cup [1, 2]$

$f(x) = \arccos u$ where $u = 1 + x - x^2$

$f'(x) = f'(u)\,\dfrac{du}{dx}$

$= \dfrac{-1}{\sqrt{1-u^2}}\,(1 - 2x)$

$= \dfrac{2x-1}{\sqrt{1-(1+x-x^2)^2}}$

61 $\displaystyle\sum_{n=1}^{m} f(n) = m^3 + 3m$

$\therefore\ f(1) + f(2) + f(3) + ... + f(m) = m^3 + 3m$

$\therefore\ f(1) + f(2) + f(3) + ... + f(m-1) = (m-1)^3 + 3(m-1)$

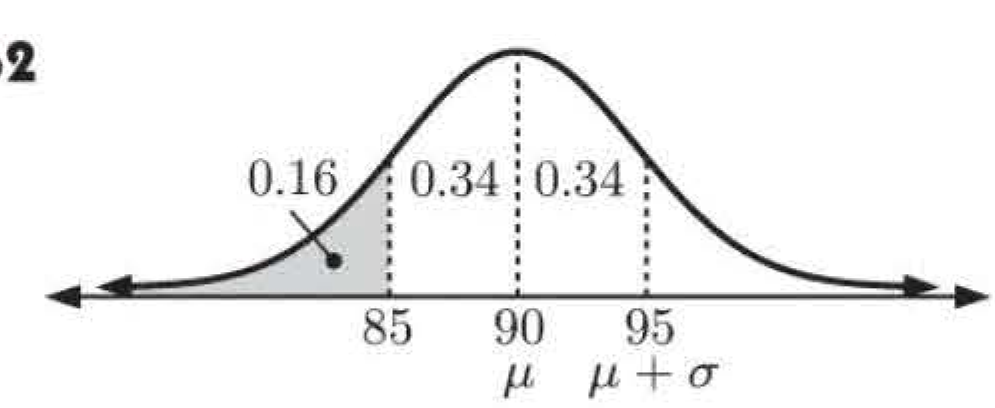
Thus $(m-1)^3 + 3(m-1) + f(m) = m^3 + 3m$

Hence $\cancel{m^3} - 3m^2 + \cancel{3m} - 1 + 3m - 3 + f(m) = \cancel{m^3} + \cancel{3m}$

$\therefore\ f(m) = 3m^2 - 3m + 4$

Hence $f(n) = 3n^2 - 3n + 4$ {replacing m by n}

62



a $\mathrm{P}(X < 85) \approx 0.16$
From the diagram,
$\mathrm{P}(90 < X < 95) \approx 0.34$

b As roughly 34% of scores lie between μ and $\mu + \sigma$ for the normal distribution then $\sigma \approx 5$.

63 $y = \dfrac{\tan x}{\sin(2x) + 1} = \dfrac{\sin x}{\cos x(\sin 2x + 1)}$ is undefined if $\cos x = 0$ or $\sin 2x = -1$

$\therefore\ x = \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$ or $2x = -\frac{\pi}{2} + k2\pi,\ k \in \mathbb{Z}$

$\therefore\ x = -\frac{\pi}{2}, -\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$ for $x \in [-\pi, \pi]$

$\therefore$ the vertical asymptotes are: $x = -\frac{\pi}{2},\ x = -\frac{\pi}{4}, x = \frac{\pi}{2},\ x = \frac{3\pi}{4}$.

There are no horizontal asymptotes.

64 **a** $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -3 \\ 0 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -3 \\ 1 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -3 \\ 0 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}\mathbf{k}$

$= 5\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

b $|\mathbf{a}\times\mathbf{b}| = \sqrt{25+4+1} = \sqrt{30}$ units

$\therefore$ $\frac{1}{\sqrt{30}}(5\mathbf{i} - 2\mathbf{j} + \mathbf{k})$ is a unit vector perpendicular to both **a** and **b**.

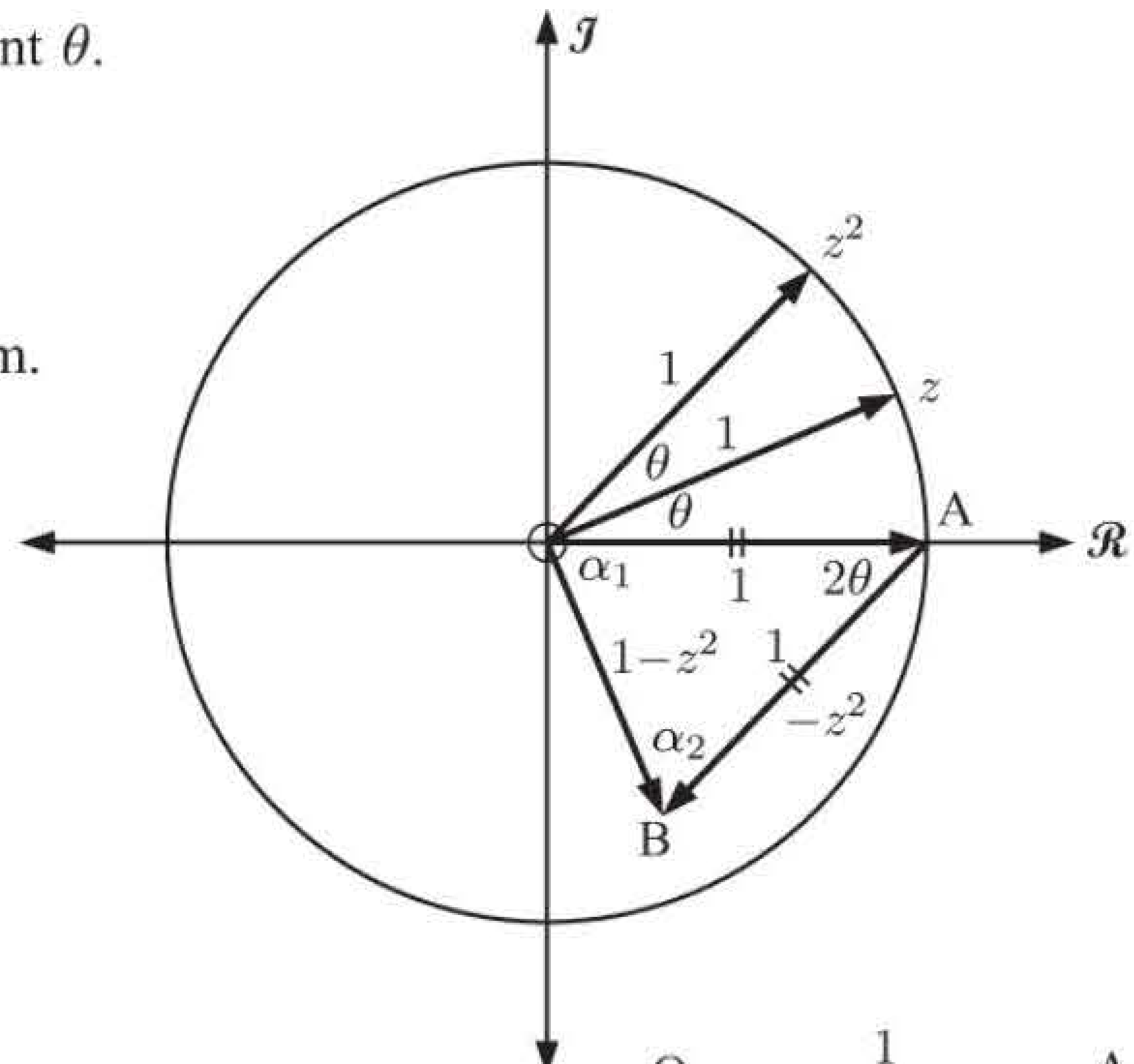
$\therefore$ the required vector is $\frac{5}{\sqrt{30}}(5\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = \frac{\sqrt{30}}{6}(5\mathbf{i} - 2\mathbf{j} + \mathbf{k})$

65 $z = \cos\theta + i\sin\theta = \operatorname{cis}\theta$ has modulus 1 and argument θ.

$z^2 = (\operatorname{cis}\theta)^2 = \operatorname{cis}2\theta$ {De Moivre}

$\therefore$ z^2 has modulus 1 and argument 2θ

$1 - z^2$ is found using vector subtraction on the diagram.



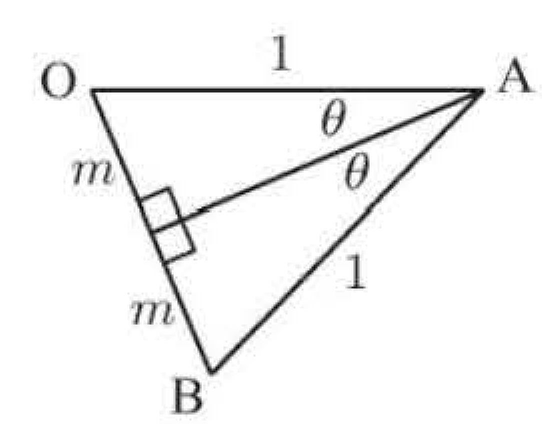
$\triangle$OAB is isosceles as OA $=$ BA $= 1$

Hence $\alpha_1 = \alpha_2$ {isosceles $\triangle$ theorem}

As z^2 and $-z^2$ are parallel,

$\widehat{OAB} = 2\theta$ {equal alternate angles}

Thus $2\alpha + 2\theta = \pi$ and so $\alpha = \frac{\pi}{2} - \theta$



Now $\arg(1 - z^2) = -\alpha_1 = \theta - \frac{\pi}{2}$

and $\sin\theta = \dfrac{m}{1} = m$

$\therefore$ $|1 - z^2| = 2m = 2\sin\theta$

{$\sin\theta > 0$, as $0 < \theta < \frac{\pi}{4}$}

So, $1 - z^2$ has argument $\theta - \frac{\pi}{2}$ and modulus $2\sin\theta$.

66 We integrate by parts with $u = x^2$ $\quad v' = \sin x$

$u' = 2x \quad v = -\cos x$

$\therefore$ $\int x^2 \sin x\, dx = x^2(-\cos x) - \int -\cos x\,(2x)\, dx$

$= -x^2\cos x + 2\int x\cos x\, dx$

We again integrate by parts, this time with $u = x$ $\quad v' = \cos x$

$u' = 1 \quad v = \sin x$

$\therefore$ $\int x^2 \sin x\, dx = -x^2\cos x + 2\left[x\sin x - \int \sin x\, dx\right]$

$= -x^2\cos x + 2x\sin x - 2(-\cos x) + c$

$= -x^2\cos x + 2x\sin x + 2\cos x + c$

67 Let $f(x) = ax + b$

$\therefore$ $f(2x+3) = a(2x+3) + b$

$= 2ax + [3a + b]$

So, $2a = 5$ and $3a + b = -7$

$\therefore$ $a = \frac{5}{2}$ and $\frac{15}{2} + b = -7$

$\therefore$ $b = -\frac{29}{2}$

$\therefore$ $f(x) = \frac{5}{2}x - \frac{29}{2}$ or $\dfrac{5x-29}{2}$

To obtain $f^{-1}(x)$ we use $x = \dfrac{5y-29}{2}$

$\therefore$ $2x = 5y - 29$

$y = \dfrac{2x+29}{5}$

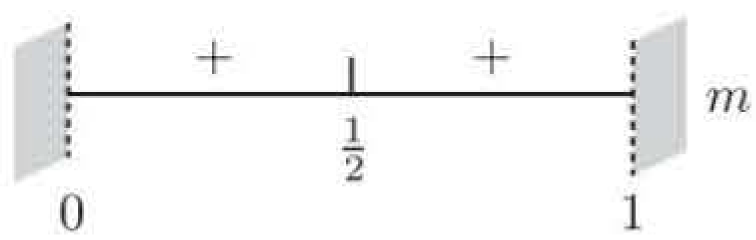
So, $f^{-1}(x) = \dfrac{2x+29}{5}$

68

$$\begin{aligned}
\log_x 4 + \log_2 x &= 3\\
\therefore\ \frac{\log 4}{\log x} + \frac{\log x}{\log 2} - 3 &= 0\\
\therefore\ \log 4 \log 2 + (\log x)^2 - 3\log 2 \log x &= 0\\
\therefore\ (\log x)^2 - 3\log 2(\log x) + \log 4 \log 2 &= 0\\
\therefore\ (\log x)^2 - 3\log 2(\log x) + 2(\log 2)^2 &= 0\\
\therefore\ (\log x - 2\log 2)(\log x - \log 2) &= 0\\
\therefore\ \log x = 2\log 2 \text{ or } \log 2\\
\therefore\ x = 4 \text{ or } 2
\end{aligned}$$

69 The discriminant,

$$\begin{aligned}
\Delta &= 1^2 - 4(m-1)(-m)\\
&= 1 + 4m(m-1)\\
&= 4m^2 - 4m + 1\\
&= (2m-1)^2\\
&\geqslant 0 \text{ for all } 0 < m < 1
\end{aligned}$$



So, the roots are always real.

sum of roots $= \dfrac{-b}{a} = \dfrac{-1}{m-1}$ which is positive since $m - 1 < 0$ for all $0 < m < 1$

product of roots $= \dfrac{c}{a} = \dfrac{-m}{m-1}$ which is positive since $-m < 0$ for all $0 < m < 1$.

As the sum and product of the roots are both positive, both roots are positive.

70 **a**

$$\begin{aligned}
&\sin 15^\circ\\
&= \sin(45^\circ - 30^\circ)\\
&= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ\\
&= \left(\tfrac{1}{\sqrt{2}}\right)\left(\tfrac{\sqrt{3}}{2}\right) - \left(\tfrac{1}{\sqrt{2}}\right)\left(\tfrac{1}{2}\right)\\
&= \left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)\frac{\sqrt{2}}{\sqrt{2}}\\
&= \frac{\sqrt{6}-\sqrt{2}}{4}
\end{aligned}$$

b

$$\begin{aligned}
&\cos^2 165^\circ + \cos^2 285^\circ\\
&= [\cos(180^\circ - 15^\circ)]^2 + [\cos(270^\circ + 15^\circ)]^2\\
&= (-\cos 15^\circ)^2 + (\sin 15^\circ)^2\\
&\quad \{\cos(\pi - \theta) = -\cos\theta \text{ and } \cos\left(\tfrac{3\pi}{2} + \theta\right) = \sin\theta\}\\
&= \cos^2 15^\circ + \sin^2 15^\circ\\
&= 1
\end{aligned}$$

71 **a**

$$\begin{aligned}
x^2 - 3xy + y^2 &= 7\\
\therefore\ 2x - \left[3y + 3x\frac{dy}{dx}\right] + 2y\frac{dy}{dx} &= 0\\
\therefore\ 2x - 3y + (2y - 3x)\frac{dy}{dx} &= 0\\
\therefore\ \frac{dy}{dx} &= \frac{3y - 2x}{2y - 3x}
\end{aligned}$$

b We need to find where $\dfrac{3y-2x}{2y-3x} = \frac{2}{3}$

$$\begin{aligned}
\therefore\ 9y - 6x &= 4y - 6x\\
\therefore\ 5y &= 0\\
\therefore\ y &= 0\\
\therefore\ x^2 = 7 &\text{ and so } x = \pm\sqrt{7}
\end{aligned}$$

$\therefore$ the points are $(\sqrt{7}, 0)$ and $(-\sqrt{7}, 0)$.

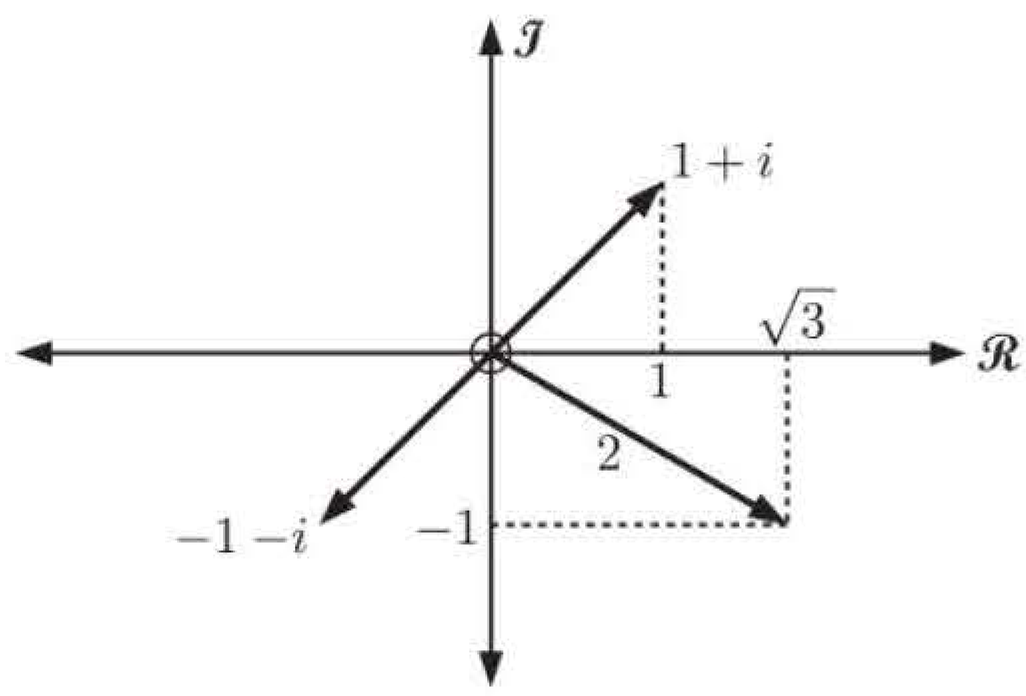
72 **a** **i** If A and B are mutually exclusive then
$P(A \cup B) = P(A) + P(B) = \frac{1}{3} + \frac{2}{7} = \frac{13}{21}$

ii If A and B are independent then

$$\begin{aligned}
P(A \cup B) &= P(A) + P(B) - P(A)P(B)\\
&= \tfrac{1}{3} + \tfrac{2}{7} - \tfrac{1}{3} \times \tfrac{2}{7}\\
&= \tfrac{11}{21}
\end{aligned}$$

b $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{P(A) + P(B) - P(A \cup B)}{P(B)} = \left(\dfrac{\frac{13}{21} - \frac{3}{7}}{\frac{2}{7}}\right)\frac{21}{21} = \frac{4}{6} = \frac{2}{3}$

73 **a**
$$1+i=\sqrt{2}\operatorname{cis}\tfrac{\pi}{4}=\sqrt{2}e^{i\frac{\pi}{4}}$$
$$\sqrt{3}-i=2\operatorname{cis}\left(-\tfrac{\pi}{6}\right)=2e^{i\left(-\frac{\pi}{6}\right)}$$
$$\therefore\ \frac{-1-i}{\sqrt{3}-i}=\frac{\sqrt{2}e^{i\left(-\frac{3\pi}{4}\right)}}{2e^{i\left(-\frac{\pi}{6}\right)}}$$
$$=\tfrac{1}{\sqrt{2}}e^{i\left(-\frac{3\pi}{4}+\frac{\pi}{6}\right)}$$
$$=\tfrac{1}{\sqrt{2}}e^{i\left(-\frac{7\pi}{12}\right)}$$



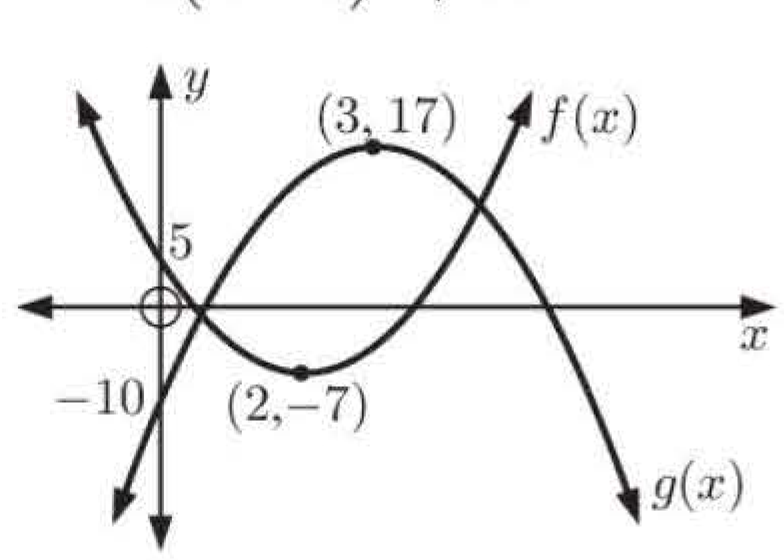
b $z^n=\left(\frac{1}{\sqrt{2}}\right)^n e^{i\left(-\frac{7\pi n}{12}\right)}$ which is real when $-\dfrac{7\pi n}{12}=0+k\pi$
$$\therefore\ n=0-\frac{12k}{7},\ k\in\mathbb{Z}$$

$\therefore$ the smallest positive integer is $n=12$ when $k=-7$.

74 **a** Since **p** and **q** perpendicular,
$$\mathbf{p}\bullet\mathbf{q}=0$$
$$\therefore\ -t+2+2t-4t=0$$
$$\therefore\ 3t=2$$
$$\therefore\ t=\tfrac{2}{3}$$

b Since **p** and **q** parallel,
$$\mathbf{q}=k\mathbf{p}\ \text{ for some scalar } k$$
$$\therefore\ \begin{pmatrix}-t\\1+t\\2t\end{pmatrix}=\begin{pmatrix}k\\2k\\-2k\end{pmatrix}$$
$$\therefore\ k=-t\ \text{ and }\ 2k=1+t=-2t$$
$$\therefore\ 3t=-1$$
$$\therefore\ t=-\tfrac{1}{3}$$

75
$$f(x)=3x^2-12x+5$$
$$=3(x^2-4x+4)+5-12$$
$$=3(x-2)^2-7$$
$$g(x)=-3x^2+18x-10$$
$$=-3(x^2-6x+9)-10+27$$
$$=-3(x-3)^2+17$$



$$g(x)=-3(x-3)^2+17$$
$$=-(3(x-1-2)^2-7-10)$$
$$=-(f(x-1)^2-10)$$

So, we translate $y=f(x)$ through $\binom{1}{-10}$ and then reflect the result in the x-axis.

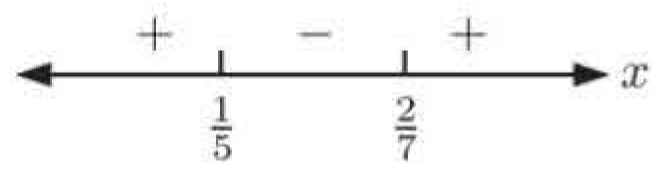
76
$$\text{Area}=\int_0^{\frac{\pi}{4}}\tan^2 x+2\sin^2 x\,dx$$
$$=\int_0^{\frac{\pi}{4}}\sec^2 x-1+2\left(\tfrac{1}{2}-\tfrac{1}{2}\cos 2x\right)dx$$
$$=\int_0^{\frac{\pi}{4}}\sec^2 x-\cos 2x\,dx$$
$$=\left[\tan x-\tfrac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}}$$
$$=\left(1-\tfrac{1}{2}\right)-(0-0)$$
$$=\tfrac{1}{2}\text{ unit}^2$$

77 The x-intercepts -3 and $-\frac{1}{4}$ indicate that $(x+3)$ and $(4x+1)$ are factors of $f(x)$.
$f(x)$ touches the x-axis at $\frac{3}{2}$, so $(2x-3)^2$ is also a factor of $f(x)$.
Thus, the quartic has the form $f(x)=a(x+3)(4x+1)(2x-3)^2$, where $a\neq 0$.
But $f(0)=9$, so $a(3)(1)(-3)^2=9$
$$\therefore\ a=\tfrac{1}{3}$$
$$\therefore\ f(x)=\tfrac{1}{3}(x+3)(4x+1)(2x-3)^2$$

78 **a**

$$|1-4x| > \tfrac{1}{3}|2x-1|$$
$$\therefore\ 3|1-4x| > |2x-1|$$
$$\therefore\ 9(1-4x)^2 > (2x-1)^2$$
$$\therefore\ 9(1-4x)^2-(2x-1)^2 > 0$$
$$\therefore\ [3(1-4x)+(2x-1)]\,[3(1-4x)-(2x-1)] > 0$$
$$\therefore\ (-10x+2)(-14x+4) > 0$$

Sign diagram:

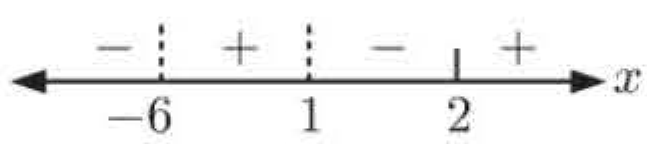


$\therefore\ x < \frac{1}{5}$ or $x > \frac{2}{7}$

b

$$\frac{x-2}{6-5x-x^2} \leqslant 0$$
$$\therefore\ \frac{x-2}{x^2+5x-6} \geqslant 0$$
$$\therefore\ \frac{x-2}{(x-1)(x+6)} \geqslant 0$$

Sign diagram:



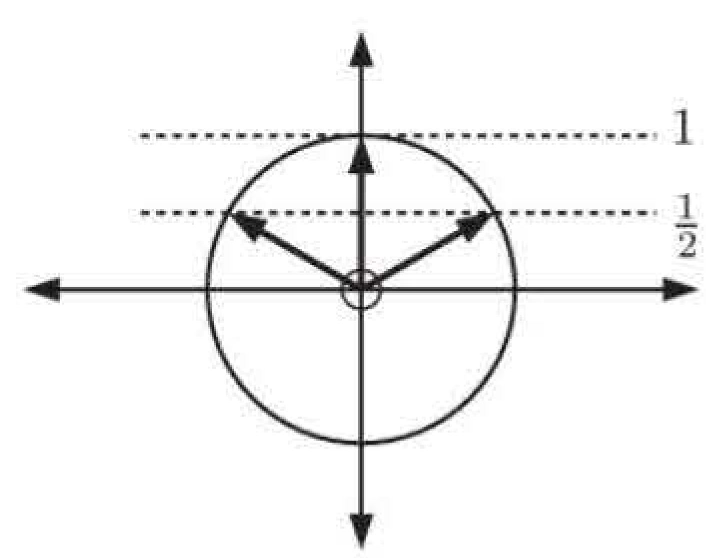
$\therefore\ -6 < x < 1$ or $x \geqslant 2$

79 $3\sec 2x = \cot 2x + 3\tan 2x, \quad -\pi \leqslant x \leqslant \pi$

$$\therefore\ \frac{3}{\cos 2x} = \frac{\cos 2x}{\sin 2x} + 3\,\frac{\sin 2x}{\cos 2x}, \quad -2\pi \leqslant 2x \leqslant 2\pi$$

Multiplying all terms by $\sin 2x\cos 2x$ gives:

$$3\sin 2x = \cos^2 2x + 3\sin^2 2x$$
$$\therefore\ 3\sin 2x = 1-\sin^2 2x + 3\sin^2 2x$$
$$\therefore\ 2\sin^2 2x - 3\sin 2x + 1 = 0$$
$$\therefore\ (2\sin 2x - 1)(\sin 2x - 1) = 0$$
$$\therefore\ \sin 2x = \tfrac{1}{2} \text{ or } 1$$
$$\therefore\ 2x = \tfrac{-11\pi}{6}, \tfrac{-3\pi}{2}, \tfrac{-7\pi}{6}, \tfrac{\pi}{6}, \tfrac{\pi}{2}, \text{ or } \tfrac{5\pi}{6}$$
$$\therefore\ x = \tfrac{-11\pi}{12}, \tfrac{-3\pi}{4}, \tfrac{-7\pi}{12}, \tfrac{\pi}{12}, \tfrac{\pi}{4}, \text{ or } \tfrac{5\pi}{12}$$



80

$$e^{xy} + xy^2 - \sin y = 2$$
$$e^{xy}\left(1y + x\,\frac{dy}{dx}\right) + 1y^2 + x\left(2y\,\frac{dy}{dx}\right) - \cos y\,\frac{dy}{dx} = 0$$
$$\frac{dy}{dx}\,(xe^{xy} + 2xy - \cos y) = -y^2 - ye^{xy}$$
$$\therefore\ \frac{dy}{dx} = \frac{-y^2-ye^{xy}}{xe^{xy}+2xy-\cos y}$$

81

$$\frac{z+u}{z-u} = \frac{x+2i+3+iy}{x+2i-3-iy}$$
$$= \frac{(x+3)+i(y+2)}{(x-3)-i(y-2)} \times \frac{(x-3)+i(y-2)}{(x-3)+i(y-2)}$$
$$= \frac{\left[(x^2-9)-(y^2-4)\right] + i\left[(x+3)(y-2)+(y+2)(x-3)\right]}{(x-3)^2+(y-2)^2}$$

This is purely imaginary when

$$x^2-9-y^2+4=0 \quad \text{and} \quad (x+3)(y-2)+(y+2)(x-3) \neq 0$$
$$\therefore\ x^2-y^2=5 \quad \text{and} \quad xy-2x+3y-6+xy-3y+2x-6 \neq 0$$
$$\therefore\ 2xy \neq 12$$
$$\therefore\ xy \neq 6$$

Since $x^2 = 5+y^2$ where $y^2 \geqslant 0$, $x^2 \geqslant 5$

So, $x \geqslant \sqrt{5}$ or $x \leqslant -\sqrt{5}$, and the smallest positive x is $\sqrt{5}$.

82

$$\text{Area} = \int_0^{\frac{\pi}{3}} \frac{\tan x}{\cos 2x + 1}\,dx$$
$$= \int_0^{\frac{\pi}{3}} \frac{\tan x}{2\cos^2 x - 1 + 1}\,dx$$
$$= \tfrac{1}{2}\int_0^{\frac{\pi}{3}} \tan x \sec^2 x\,dx$$
$$= \tfrac{1}{2}\left[\frac{(\tan x)^2}{2}\right]_0^{\frac{\pi}{3}} \quad \{\frac{d}{dx}\tan x = \sec^2 x\}$$
$$= \tfrac{1}{4}\left((\sqrt{3})^2 - 0^2\right)$$
$$= \tfrac{3}{4}\text{ unit}^2$$

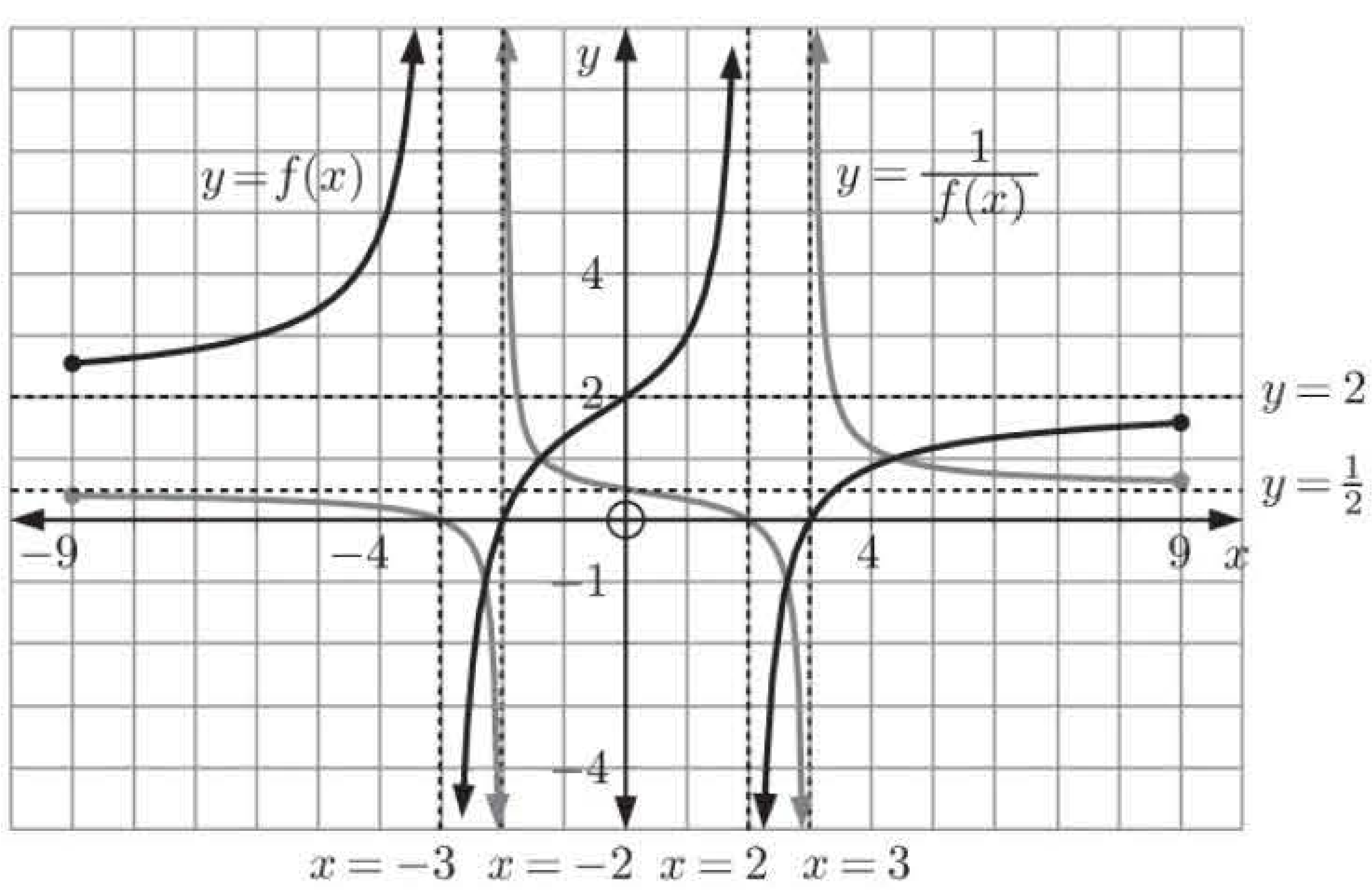
83

$$\frac{a+b+26}{5} = 8 \quad \{\text{as the mean is } 8\}$$
$$\therefore\ a + b = 14 \quad .... (1)$$
But $\dfrac{\sum(x_i - \overline{x})^2}{n} = 8$ also,

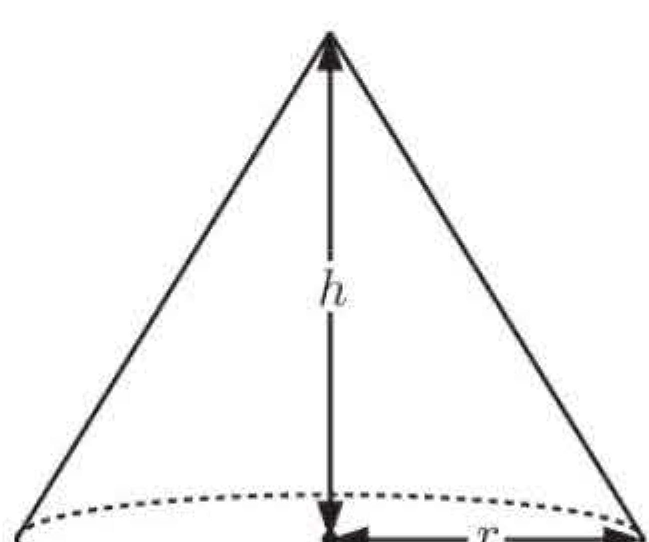
so $\dfrac{(a-8)^2 + (b-8)^2 + 4 + 25 + 1}{5} = 8$

Using (1), $(a-8)^2 + (6-a)^2 + 30 = 40$
$$\therefore\ a^2 - 16a + 64 + 36 - 12a + a^2 - 10 = 0$$
$$\therefore\ 2a^2 - 28a + 90 = 0$$
$$\therefore\ a^2 - 14a + 45 = 0$$
$$\therefore\ (a-9)(a-5) = 0$$
$$\therefore\ a = 5 \text{ or } 9$$
When $a = 5,\ b = 9,$ and when $a = 9,\ b = 5.$
As $b > a,\ a = 5,\ b = 9$

84



85



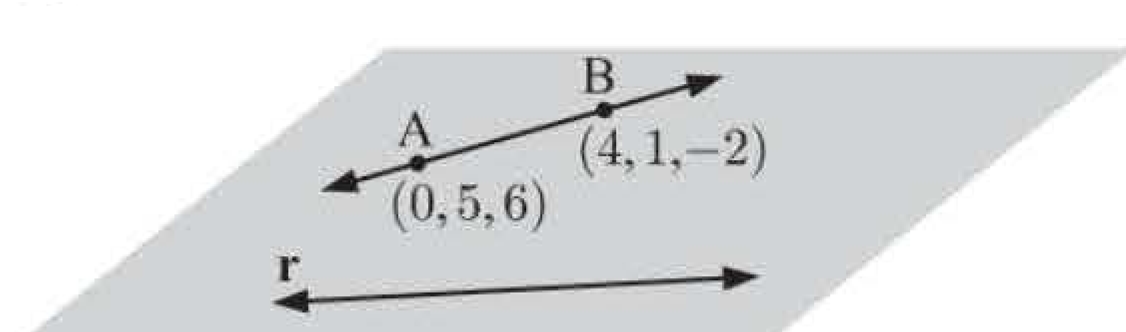
$$h = 2r$$
$$V = \tfrac{1}{3}\pi r^2 h$$
$$\therefore\ V = \tfrac{\pi}{3} r^2 (2r)$$
$$\therefore\ V = \tfrac{2\pi}{3} r^3$$
So, $\dfrac{dV}{dt} = 2\pi r^2 \dfrac{dr}{dt}$

Particular case:

When $h = 20$ cm, $r = 10$ cm

and $\dfrac{dV}{dt} = 5\ \text{cm}^3\ \text{s}^{-1}$
$$\therefore\ 5 = 2\pi(10^2)\,\frac{dr}{dt}$$
$$\therefore\ 5 = 200\pi\,\frac{dr}{dt}$$
$$\therefore\ \frac{dr}{dt} = \frac{1}{40\pi}\ \text{cm s}^{-1}$$

86



$$\overrightarrow{\text{AB}} = \begin{pmatrix} 4 \\ -4 \\ -8 \end{pmatrix} = 4\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$$
$\therefore$ the first line has equation $x = t,\ y = 5 - t,\ z = 6 - 2t$

The lines are not parallel as the direction vectors of the lines are not multiples of each other.

For the lines to be coplanar, they must intersect.

The first line meets the second line where $\begin{pmatrix} 0 \\ 5 \\ 6 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} a \\ 3 \\ 2 \end{pmatrix} + s\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

$$\therefore \quad t = a + 2s, \quad \underbrace{5 - t = 3 - s}_{\therefore\ s = t-2}, \quad \text{and} \quad \underbrace{6 - 2t = 2 + s}_{\therefore\ s + 2t = 4}$$

$$\begin{aligned} \text{Thus } t - 2 + 2t &= 4 \\ \therefore\ 3t &= 6 \\ \therefore\ t &= 2 \text{ and } s = 0 \end{aligned}$$

$\therefore$ the lines are coplanar if $2 = a + 2(0)$

$$\therefore\ a = 2$$

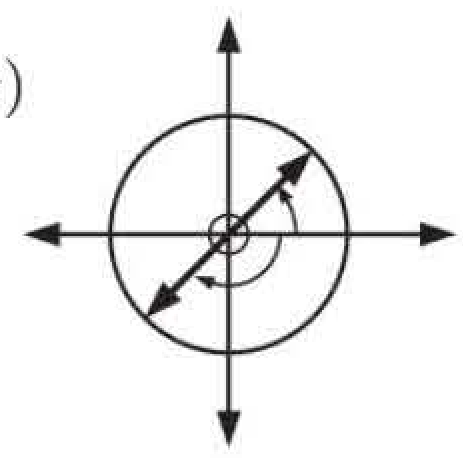
87 $y = \dfrac{\sin x}{\tan x + 1}, \quad -\pi \leqslant x \leqslant \frac{\pi}{2}$

$$\therefore\ \frac{dy}{dx} = \frac{\cos x(\tan x + 1) - \sin x \sec^2 x}{(\tan x + 1)^2}$$

$$\therefore\ \frac{dy}{dx} = \frac{\sin x + \cos x - \dfrac{\sin x}{\cos^2 x}}{(\tan x + 1)^2}$$

which is 0 when $\sin x + \cos x = \dfrac{\sin x}{\cos^2 x}$

$$\begin{aligned} \therefore\ \sin x \cos^2 x + \cos^3 x &= \sin x \\ \therefore\ \cos^3 x &= \sin x(1 - \cos^2 x) \\ \therefore\ \cos^3 x &= \sin^3 x \\ \therefore\ \tan^3 x &= 1 \\ \therefore\ \tan x &= 1 \\ \therefore\ x &= \tfrac{\pi}{4}, -\tfrac{3\pi}{4} \end{aligned}$$



$\therefore$ the stationary points are at $\left(\frac{\pi}{4}, \frac{\sqrt{2}}{4}\right)$ and $\left(-\frac{3\pi}{4}, -\frac{\sqrt{2}}{4}\right)$.

88 $P(X = x) = a\left(\frac{2}{5}\right)^x$ where $x = 0, 1, 2, 3, 4, 5, ...$

$$\therefore\ a\left(\tfrac{2}{5}\right)^0 + a\left(\tfrac{2}{5}\right)^1 + a\left(\tfrac{2}{5}\right)^2 + ... = 1 \qquad \{\textstyle\sum P(x) = 1\}$$

$$\therefore\ a\left(1 + \tfrac{2}{5} + \left(\tfrac{2}{5}\right)^2 + ...\right) = 1$$

$$\therefore\ a\left(\frac{1}{1 - \frac{2}{5}}\right) = 1 \qquad \{\text{sum of an infinite geometric series}\}$$

$$\therefore\ \frac{a}{\frac{3}{5}} = 1$$

$$\therefore\ a = \tfrac{3}{5}$$

89 $4\sin x = \sqrt{3}\csc x + 2 - 2\sqrt{3}$ where $0 \leqslant x \leqslant 2\pi$

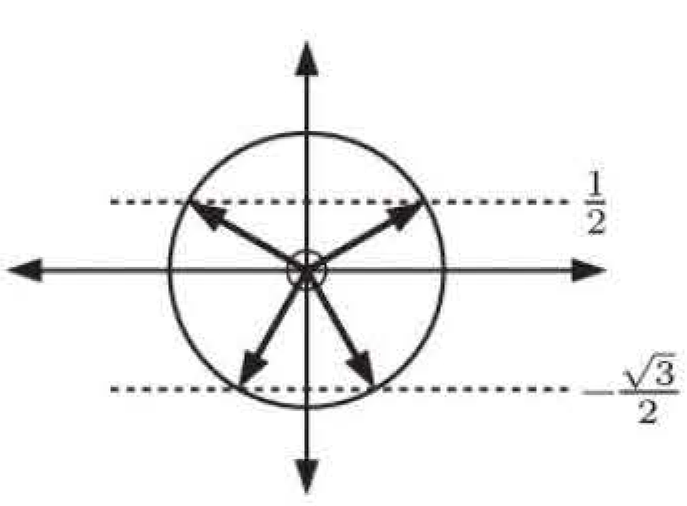
$$\therefore\ 4\sin x = \frac{\sqrt{3}}{\sin x} + 2 - 2\sqrt{3}$$

$$\therefore\ 4\sin^2 x + (2\sqrt{3} - 2)\sin x - \sqrt{3} = 0$$

$$\therefore\ (2\sin x + \sqrt{3})(2\sin x - 1) = 0$$

$$\therefore\ \sin x = -\tfrac{\sqrt{3}}{2} \text{ or } \tfrac{1}{2}$$

$$\therefore\ x = \tfrac{\pi}{6}, \tfrac{5\pi}{6}, \tfrac{4\pi}{3}, \tfrac{5\pi}{3}$$



90 **a** $\left(\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 1 & p & 2 & 0 \\ -2 & p^2 & -4 & q \end{array}\right)$

b $\sim \left(\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & p+2 & -1 & -1 \\ 0 & p^2-4 & 2 & q+2 \end{array}\right) \begin{array}{l} \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 + 2R_1 \end{array}$

$\sim \left(\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & p+2 & -1 & -1 \\ 0 & 0 & p & p+q \end{array}\right) \quad R_3 \to R_3 - (p-2)R_2$

$$\begin{array}{cccc} 0 & p^2-4 & 2 & q+2 \\ 0 & -(p^2-4) & p-2 & p-2 \\ \hline 0 & 0 & p & p+q \\ \hline \end{array}$$

c **i** For a unique solution, $p \neq 0$. The planes meet at one point only.

ii There are no solutions if $p = 0,\ q \neq 0$. The three planes have no common point of intersection as planes 2 and 3 are parallel.

iii There are infinitely many solutions when $p = q = 0$. Planes 2 and 3 are coincident.

d When $p = q = 0$ the augmented matrix is $\left(\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$ ← this equation is $2y - z = -1$.

Letting $y = t,\quad z = 1 + 2t$.

Using $x - 2y + 3z = 1$, we find $x - 2(t) + 3(1 + 2t) = 1$

$\therefore\ x - 2t + 3 + 6t = 1$

$\therefore\ x = -2 - 4t$

So, $x = -2 - 4t,\ y = t,\ z = 1 + 2t,\ t \in \mathbb{R}$.

91 Since the polynomial is real, $1 \pm 2i$ and $\pm ai$ are zeros, $a \neq 0$.

$1 \pm 2i$ have sum 2 and product $1 + 4 = 5$ and so come from $z^2 - 2z + 5$

$\pm ai$ have sum 0 and product a^2 and so come from $z^2 + a^2$

$\therefore\ P(z) = k(z^2 - 2z + 5)(z^2 + a^2)$

But $k = 1$ and $P(0) = 1(5)(a^2) = 10$ so $a^2 = 2$

$\therefore\ P(z) = (z^2 - 2z + 5)(z^2 + 2)$

92

$\tan 2A = \frac{3}{2}$

$\therefore\ \dfrac{2\tan A}{1 - \tan^2 A} = \frac{3}{2}$

$\therefore\ 4\tan A = 3 - 3\tan^2 A$

$\therefore\ 3\tan^2 A + 4\tan A - 3 = 0$

$\therefore\ \tan A = \dfrac{-4 \pm \sqrt{16 - 4(3)(-3)}}{6}$

$= -\dfrac{2}{3} \pm \dfrac{\sqrt{13}}{3}$

But A is acute, so $\tan A > 0$

$\therefore\ \tan A = \dfrac{\sqrt{13} - 2}{3}$

93 **a** $\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ 2 & -1 & 1 \end{vmatrix}$

$= \mathbf{i}(0) - \mathbf{j}(3) + \mathbf{k}(-3) = -3\mathbf{j} - 3\mathbf{k}$

b $\text{LHS} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & -1 \\ 0 & -3 & -3 \end{vmatrix}$

$= \mathbf{i}(-9) - \mathbf{j}(-9) + \mathbf{k}(-9)$

$= -9\mathbf{i} + 9\mathbf{j} - 9\mathbf{k}$

$\text{RHS} = \mathbf{b}(\mathbf{a} \bullet \mathbf{c}) - \mathbf{c}(\mathbf{a} \bullet \mathbf{b})$

$= 3\mathbf{b} - 6\mathbf{c}$

$= 3\mathbf{i} + 3\mathbf{j} - 3\mathbf{k} - 12\mathbf{i} + 6\mathbf{j} - 6\mathbf{k}$

$= -9\mathbf{i} + 9\mathbf{j} - 9\mathbf{k}$

94 **a** $(-1 + i\sqrt{2})^3 = (-1)^3 + 3(-1)^2 i\sqrt{2} + 3(-1)(i\sqrt{2})^2 + (i\sqrt{2})^3$

$= -1 + 3\sqrt{2}i + 6 - 2\sqrt{2}i$

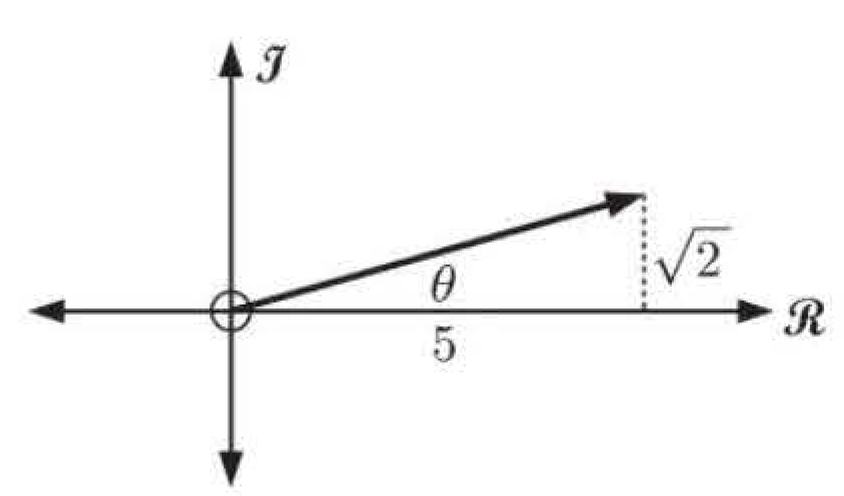
$= 5 + i\sqrt{2}$

b $\left|5 + i\sqrt{2}\right| = \sqrt{25 + 2} = \sqrt{27} = (\sqrt{3})^3$

$\arg(5 + i\sqrt{2}) = \theta = \arctan\left(\frac{\sqrt{2}}{5}\right)$

$\therefore\ 5 + i\sqrt{2} = \left(\sqrt{3}\right)^3 \operatorname{cis}\left[\arctan\left(\frac{\sqrt{2}}{5}\right)\right]$

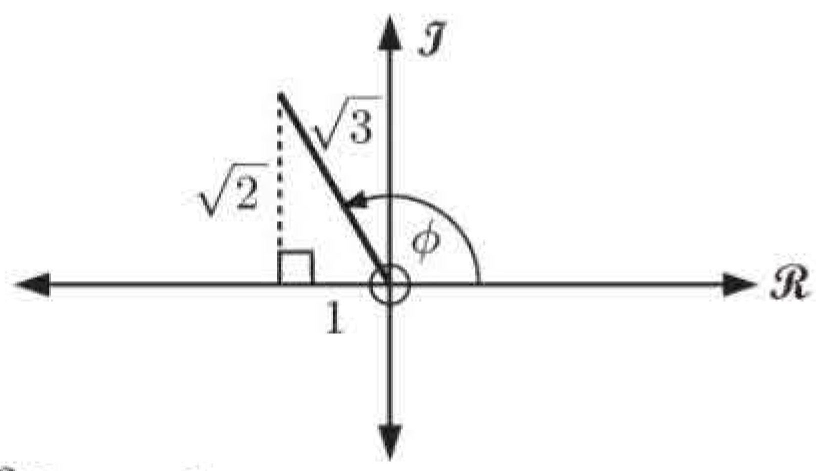
$\therefore\ a = \sqrt{3},\ \theta = \arctan\left(\frac{\sqrt{2}}{5}\right)$



c $z^3 = 5 + i\sqrt{2} = \left(\sqrt{3}\right)^3 \text{cis}\left[\arctan\left(\frac{\sqrt{2}}{5}\right)\right]$

$\therefore\ z = \sqrt{3}\ \text{cis}\left[\dfrac{\arctan\left(\frac{\sqrt{2}}{5}\right) + k2\pi}{3}\right]$ where $k = 0, 1, 2$ {De Moivre}

d From **a**, one of the solutions to $z^3 = 5 + i\sqrt{2}$ is



$$z = -1 + i\sqrt{2}$$

$$\therefore\ z = \sqrt{3}\ \text{cis}\ \phi$$

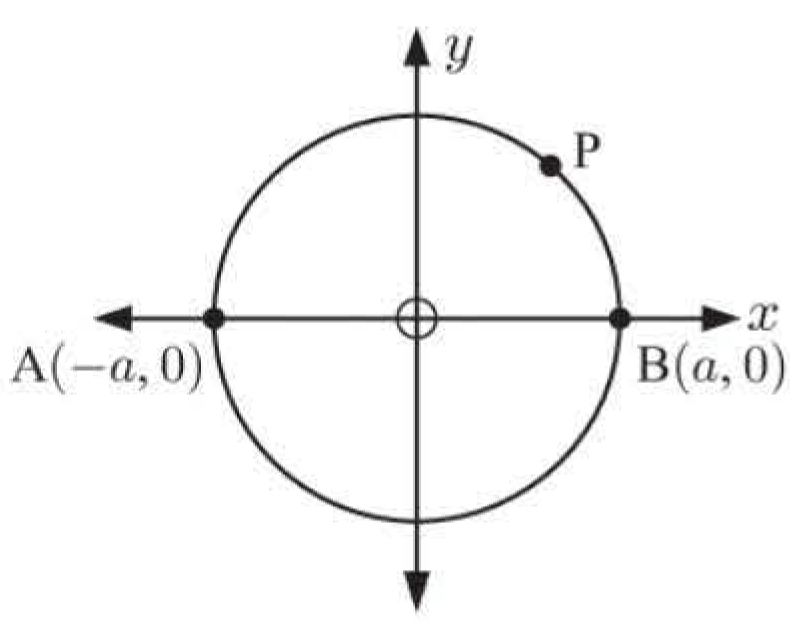
$$\therefore\ z = \sqrt{3}\ \text{cis}\left[\arccos\left(\frac{-1}{\sqrt{3}}\right)\right]$$

This corresponds to the solution in **c** where $k = 1$ $\{\frac{\theta}{3} + \frac{2\pi}{3} = \phi\}$

Equating arguments gives: $\dfrac{\arctan\left(\frac{\sqrt{2}}{5}\right) + 2\pi}{3} = \arccos\left(\frac{-1}{\sqrt{3}}\right)$

$$\therefore\ \arctan\left(\frac{\sqrt{2}}{5}\right) + 2\pi = 3\arccos\left(\frac{-1}{\sqrt{3}}\right)$$

95 **a**



As $x^2 + y^2 = a^2$,

$$y^2 = a^2 - x^2$$

$$\therefore\ y = \sqrt{a^2 - x^2} \quad \text{as } y \text{ is } > 0$$

$$\therefore\ \text{P is } (x, \sqrt{a^2 - x^2})$$

b **i** $\overrightarrow{\text{AP}} = \begin{pmatrix} x+a \\ \sqrt{a^2-x^2} \end{pmatrix}$, $\overrightarrow{\text{AB}} = \begin{pmatrix} 2a \\ 0 \end{pmatrix}$, $\overrightarrow{\text{OP}} = \begin{pmatrix} x \\ \sqrt{a^2-x^2} \end{pmatrix}$

ii

$$\begin{aligned}
&\cos(\text{P}\widehat{\text{A}}\text{B}) \\
&= \frac{\overrightarrow{\text{AP}} \bullet \overrightarrow{\text{AB}}}{|\overrightarrow{\text{AP}}||\overrightarrow{\text{AB}}|} \\
&= \frac{2a(x+a) + 0}{\sqrt{(x+a)^2 + a^2 - x^2} \times 2a} \\
&= \frac{x+a}{\sqrt{x^2 + 2ax + a^2 + a^2 - x^2}} \\
&= \frac{x+a}{\sqrt{2a(x+a)}} \\
&= \sqrt{\frac{x+a}{2a}}
\end{aligned}$$

$$\begin{aligned}
&\cos(\text{P}\widehat{\text{O}}\text{B}) \\
&= \frac{\overrightarrow{\text{OP}} \bullet \overrightarrow{\text{OB}}}{|\overrightarrow{\text{OP}}||\overrightarrow{\text{OB}}|} \quad \text{where } \overrightarrow{\text{OB}} = \begin{pmatrix} a \\ 0 \end{pmatrix} \\
&= \frac{ax}{\sqrt{x^2 + a^2 - x^2} \times a} \\
&= \frac{ax}{a^2} \\
&= \frac{x}{a}
\end{aligned}$$

c Thus,

$$\begin{aligned}
2\cos^2(\text{P}\widehat{\text{A}}\text{B}) - 1 &= \frac{2(x+a)}{2a} - 1 \\
&= \frac{x}{a} + 1 - 1 \\
&= \frac{x}{a} \\
&= \cos(\text{P}\widehat{\text{O}}\text{B})
\end{aligned}$$

Hence, $\text{P}\widehat{\text{O}}\text{B} = 2 \times \text{P}\widehat{\text{A}}\text{B}$

The angle at the centre of a circle is twice the angle on the circle subtended by the same arc.

96 P_n is "$x^n - y^n$ has factor $x-y$" for $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n=1$, $x^1 - y^1$ has factor $x-y$ ✓ $\therefore$ P_1 is true.

(2) If P_k is assumed true then $x^k - y^k = (x-y)f_k(x, y)$, where $f_k(x, y)$ is another factor.

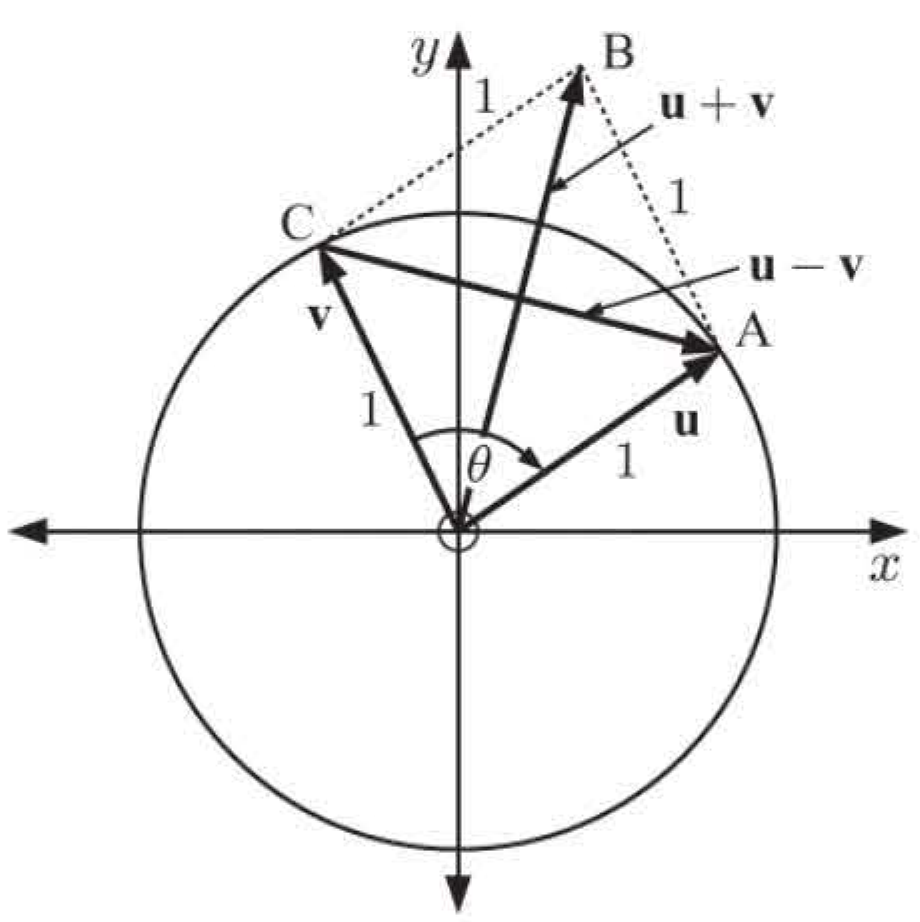
Now
$$\begin{aligned} x^{k+1} - y^{k+1} &= x(x^k - y^k) + xy^k - y^{k+1} \\ &= x(x-y)f_k(x, y) + y^k(x-y) \\ &= (x-y)\left[xf_k(x, y) + y^k\right] \\ &\equiv (x-y)\,f_{k+1}(x, y) \end{aligned}$$

$\therefore$ $x^{k+1} - y^{k+1}$ has factor $x-y$.

Thus P_{k+1} is true whenever P_k is true.

$\therefore$ since P_1 is true, P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

97 **a**



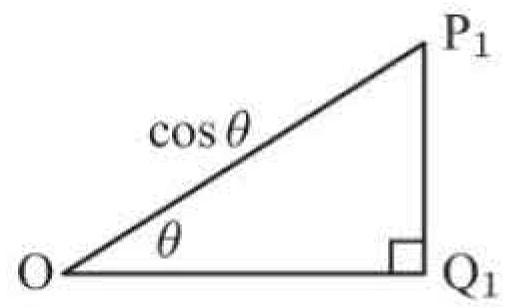
We complete the rhombus OABC (each side has length 1).
Using the cosine rule in triangle OAC,

$$|\mathbf{u}-\mathbf{v}|^2 = 1^2 + 1^2 - 2(1)(1)\cos\theta$$
$$\therefore\ |\mathbf{u}-\mathbf{v}|^2 = 2 - 2\cos\theta$$
$$\therefore\ |\mathbf{u}-\mathbf{v}| = \sqrt{2-2\cos\theta}$$

Now in $\triangle$OAB, $\widehat{\text{OAB}} = 180° - \theta$, and using the cosine rule again,
$$\begin{aligned} |\mathbf{u}+\mathbf{v}|^2 &= 1^2 + 1^2 - 2(1)(1)\cos(180° - \theta) \\ &= 2 - 2(-\cos\theta) \\ &= 2 + 2\cos\theta \end{aligned}$$
$$\therefore\ |\mathbf{u}+\mathbf{v}| = \sqrt{2+2\cos\theta}$$

b Now, if $|\mathbf{u}+\mathbf{v}| = 5\,|\mathbf{u}-\mathbf{v}|$ then
$$\sqrt{2+2\cos\theta} = 5\sqrt{2-2\cos\theta}$$
$$\therefore\ 2+2\cos\theta = 25(2-2\cos\theta)$$
$$\therefore\ 2+2\cos\theta = 50 - 50\cos\theta$$
$$\therefore\ 52\cos\theta = 48$$
$$\therefore\ \cos\theta = \tfrac{12}{13}$$

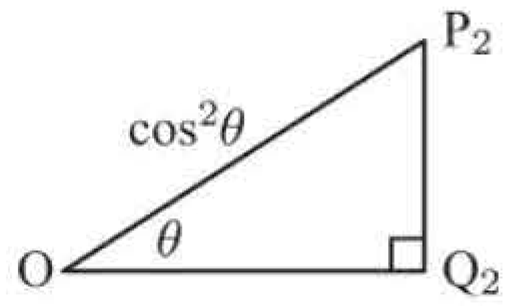
98 P lies on the unit circle, so OQ $= \cos\theta$ and PQ $= \sin\theta$



In $\triangle \text{OP}_1\text{Q}_1$, $\text{OP}_1 = \text{OQ} = \cos\theta$

So, $\sin\theta = \dfrac{\text{P}_1\text{Q}_1}{\cos\theta}$ and $\cos\theta = \dfrac{\text{OQ}_1}{\cos\theta}$

$\therefore$ $\text{OQ}_1 = \cos^2\theta$ and $\text{P}_1\text{Q}_1 = \sin\theta\cos\theta$



Likewise in $\triangle \text{OP}_2\text{Q}_2$, $\sin\theta = \dfrac{\text{P}_2\text{Q}_2}{\cos^2\theta}$ and $\cos\theta = \dfrac{\text{OQ}_2}{\cos^2\theta}$

$\therefore$ $\text{OQ}_2 = \cos^3\theta$ and $\text{P}_2\text{Q}_2 = \sin\theta\cos^2\theta$

Thus
$$\begin{aligned} &\text{PQ} + \text{P}_1\text{Q}_1 + \text{P}_2\text{Q}_2 + \text{P}_3\text{Q}_3 + ... \\ &= \sin\theta + \sin\theta\cos\theta + \sin\theta\cos^2\theta + \sin\theta\cos^3\theta + ... \\ &= \sin\theta(1 + \cos\theta + \cos^2\theta + \cos^3\theta + ...) \\ &= \sin\theta\left(\frac{1}{1-\cos\theta}\right) \quad \{\text{sum of an infinite geometric series with } |r| = |\cos\theta| \leqslant 1\} \\ &= \frac{\sin\theta}{1-\cos\theta} \\ &= \frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{2\sin^2\left(\frac{\theta}{2}\right)} \\ &= \cot\left(\frac{\theta}{2}\right) \end{aligned}$$

99 Suppose the common root is α and the other roots are β and γ.

$\therefore\ x^2+ax+bc=(x-\alpha)(x-\beta)$ and $x^2+bx+ca=(x-\alpha)(x-\gamma)$

Thus $\alpha+\beta=-a$ and $\alpha+\gamma=-b$ so $(\alpha+\beta)(\alpha+\gamma)=ab$ .... (1)

Also, $\alpha\beta=bc$ and $\alpha\gamma=ca$ so $\alpha^2\beta\gamma=abc^2$ .... (2)

Now α is a common root of both equations $\therefore\ \alpha^2+a\alpha+bc=0$ .... (3) and

$\alpha^2+b\alpha+ca=0$ .... (4)

Subtracting (4) from (3), $(a-b)\alpha-(a-b)c=0$

$\therefore\ (a-b)(\alpha-c)=0$

$\therefore\ a=b$ or $\alpha=c$

But if $a=b$, both equations are the same, and so the equations would have two common roots

$\therefore\ \alpha=c$ .... (5)

Using (2), $\alpha^2\beta\gamma=abc^2$

$\therefore\ c^2\beta\gamma=abc^2$

$\therefore\ \beta\gamma=ab$ .... (6)

Using (1), $\alpha^2+\alpha(\beta+\gamma)+\beta\gamma=ab$

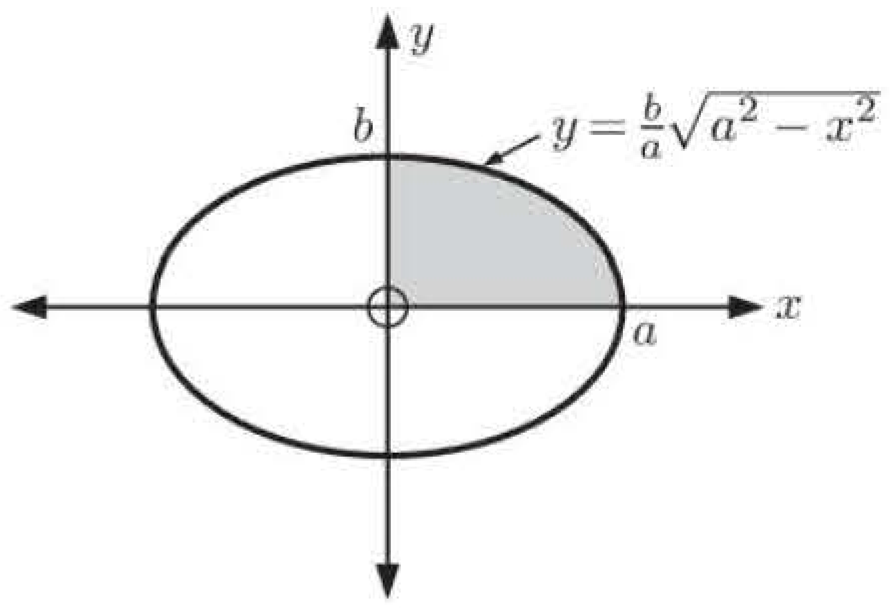
$\therefore\ c^2+c(\beta+\gamma)=0$ {using (5), (6)}

$\therefore\ \beta+\gamma=-c$ .... (7)

From (6) and (7), β and γ are the roots of $x^2+cx+ab=0$.

100 **a** Since $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, $y^2=b^2\left(1-\dfrac{x^2}{a^2}\right)=\dfrac{b^2}{a^2}(a^2-x^2)$

$$\therefore\ y=\pm\frac{b}{a}\sqrt{a^2-x^2}$$



$y=\dfrac{b}{a}\sqrt{a^2-x^2}$ is the top half of the ellipse

$$\therefore\ \text{shaded area}=\int_0^a y\,dx=\int_0^a\frac{b}{a}\sqrt{a^2-x^2}\,dx$$

$$=\frac{b}{a}\int_0^a\sqrt{a^2-x^2}\,dx$$

b Let $x=a\sin\theta$, $\dfrac{dx}{d\theta}=a\cos\theta$

when $x=0$, $\sin\theta=0$, $\theta=0$

when $x=a$, $\sin\theta=1$, $\theta=\frac{\pi}{2}$

$$\therefore\ \text{shaded area}=\frac{b}{a}\int_0^{\frac{\pi}{2}}\sqrt{a^2-a^2\sin^2\theta}\,a\cos\theta\,d\theta$$

$$=b\int_0^{\frac{\pi}{2}}a\sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta$$

$$=ab\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta$$

$$=ab\int_0^{\frac{\pi}{2}}\left(\tfrac{1}{2}+\tfrac{1}{2}\cos2\theta\right)d\theta$$

$$=ab\left[\tfrac{1}{2}\theta+\tfrac{1}{4}\sin2\theta\right]_0^{\frac{\pi}{2}}$$

$$=ab\left(\tfrac{\pi}{4}+\tfrac{1}{4}\sin\pi-0-0\right)$$

$$=\frac{\pi ab}{4}$$

$\therefore$ area of ellipse $=4\times$ shaded area $=\pi ab$

c Volume $=2\pi\int_0^a y^2\,dx$

$$=2\pi\frac{b^2}{a^2}\int_0^a(a^2-x^2)\,dx$$

$$=2\pi\frac{b^2}{a^2}\left[a^2x-\frac{x^3}{3}\right]_0^a$$

$$=2\pi\frac{b^2}{a^2}\left(a^3-\frac{a^3}{3}-0\right)$$

$$=2\pi\times\frac{b^2}{a^2}\times\frac{2a^3}{3}$$

$$=\tfrac{4}{3}\pi ab^2$$

(**Note:** When $a=b$, $V=\frac{4}{3}\pi a^3$, which is the volume of a sphere of radius a.)

101 Now $f(x) = (a_1x - b_1)^2 + (a_2x - b_2)^2 + ... + (a_nx - b_n)^2$
is the sum of squares, so $f(x) \geqslant 0$ for all x.

But $f(x) = \sum_{i=1}^{n}(a_ix - b_i)^2$

$$= \left(\sum_{i=1}^{n} a_i^{\,2}\right)x^2 - 2\left(\sum_{i=1}^{n} a_ib_i\right)x + \left(\sum_{i=1}^{n} b_i^{\,2}\right)$$

Since $f(x) \geqslant 0$ for all x, the discriminant must be non-positive.

Hence $\left(2\sum_{i=1}^{n} a_ib_i\right)^2 - 4\left(\sum_{i=1}^{n} a_i^{\,2}\right)\left(\sum_{i=1}^{n} b_i^{\,2}\right) \leqslant 0$

$$\therefore\ 4\left(\sum_{i=1}^{n} a_i^{\,2}\right)\left(\sum_{i=1}^{n} b_i^{\,2}\right) \geqslant 4\left(\sum_{i=1}^{n} a_ib_i\right)^2$$

$$\therefore\ \left(\sum_{i=1}^{n} a_i^{\,2}\right)\left(\sum_{i=1}^{n} b_i^{\,2}\right) \geqslant \left(\sum_{i=1}^{n} a_ib_i\right)^2$$

102 **a** P_n is: "$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + ... + \binom{n}{n}x^n$" for $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= (1+x)^1 = 1 + x$

RHS $= 1 + \binom{1}{1}x = 1 + x$ $\therefore$ P_1 is true.

(2) If P_k is assumed true, then

$$(1+x)^k = 1 + \binom{k}{1}x + \binom{k}{2}x^2 + ... + \binom{k}{k}x^k$$

Thus
$$\begin{aligned}(1+x)^{k+1} &= \left[1 + \binom{k}{1}x + \binom{k}{2}x^2 + ... + \binom{k}{k}x^k\right][1+x] \\ &= 1 + \binom{k}{1}x + \binom{k}{2}x^2 + \binom{k}{3}x^3 + ... + \binom{k}{k}x^k \\ &\qquad + x + \binom{k}{1}x^2 + \binom{k}{2}x^3 + ... + \binom{k}{k-1}x^k + \binom{k}{k}x^{k+1} \\ &= 1 + \left[\binom{k}{1} + \binom{k}{0}\right]x + \left[\binom{k}{2} + \binom{k}{1}\right]x^2 + ... \\ &\qquad + \left[\binom{k}{k} + \binom{k}{k-1}\right]x^k + \binom{k}{k}x^{k+1} \\ &= 1 + \binom{k+1}{1}x + \binom{k+1}{2}x^2 + ... + \binom{k+1}{k}x^k + \binom{k+1}{k+1}x^{k+1}\end{aligned}$$

{Using Pascal's Rule and $\binom{k+1}{k+1} = \binom{k}{k} = 1$}

Thus P_{k+1} is true whenever P_k is true.

$\therefore$ since P_1 is true, P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

b Letting $x = \dfrac{b}{a}$, $\left(1 + \dfrac{b}{a}\right)^n = 1 + \binom{n}{1}\dfrac{b}{a} + \binom{n}{2}\dfrac{b^2}{a^2} + ... + \binom{n}{n}\dfrac{b^n}{a^n}$, $n \in \mathbb{Z}^+$

$$\therefore\ a^n\left(1 + \frac{b}{a}\right)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + ... + \binom{n}{n}b^n$$

$$\therefore\ \left[a\left(1 + \frac{b}{a}\right)\right]^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + ... + \binom{n}{n}b^n$$

$$\therefore\ (a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + ... + \binom{n}{n}b^n$$

103 $\log_y 16y - \log_{16y} y = \frac{8}{3}$ {as $x = 16y$}

$\therefore$ $\dfrac{\log 16y}{\log y} - \dfrac{\log y}{\log 16y} - \frac{8}{3} = 0$

$\therefore$ $m - \dfrac{1}{m} - \frac{8}{3} = 0$ {letting $\dfrac{\log 16y}{\log y} = m$}

$\therefore$ $3m^2 - 8m - 3 = 0$ {$\times 3m$}

$\therefore$ $(3m+1)(m-3) = 0$

$$\therefore\ m = \frac{\log 16y}{\log y} = -\tfrac{1}{3} \text{ or } 3$$

$$\therefore\ \log 16y = -\tfrac{1}{3}\log y \quad\text{or}\quad \log 16y = 3\log y$$

$$\therefore\ 16y = y^{-\frac{1}{3}} \quad\text{or}\quad 16y = y^3$$

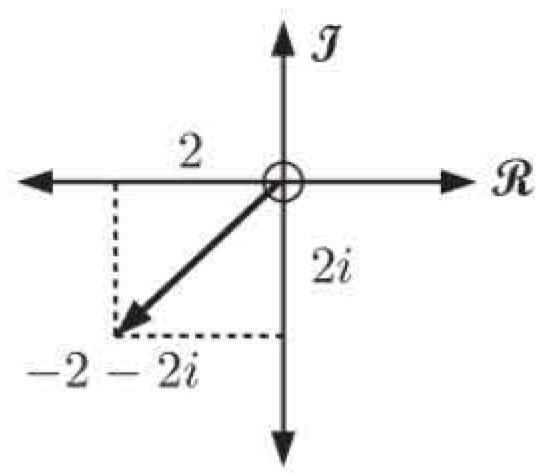
$$\therefore\ y^{\frac{4}{3}} = \tfrac{1}{16} \quad\text{or}\quad y(y^2-16)=0$$

$$\therefore\ y = \left(\pm\tfrac{1}{2}\right)^3 \quad\text{or}\quad y=0 \text{ or } \pm 4$$

$$\therefore\ y = \tfrac{1}{8} \text{ or } 4 \qquad \{\text{as } y \text{ is a base, } y>0\}$$

$$\therefore\ y = \tfrac{1}{8},\ x=2 \quad\text{or}\quad y=4,\ x=64$$

104 **a**



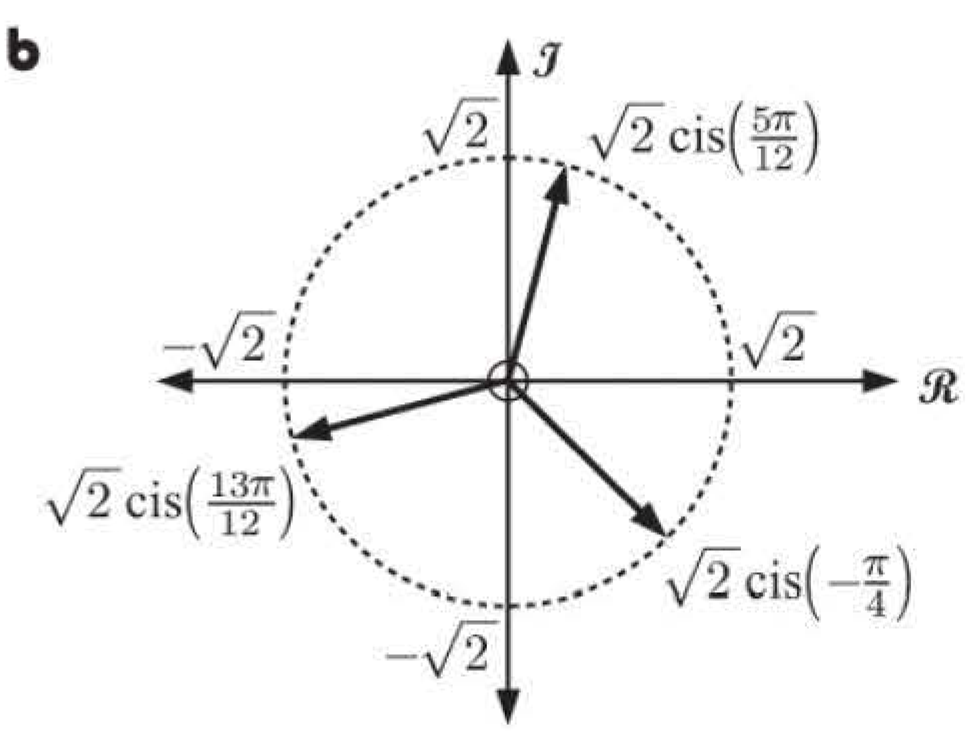
$|-2-2i| = \sqrt{4+4} = \sqrt{8} = (\sqrt{2})^3$ and $\arg(-2-2i) = \frac{-3\pi}{4}$

So, when $z^3 = -2-2i$, $z^3 = (\sqrt{2})^3 \operatorname{cis}\left(\frac{-3\pi}{4} + k2\pi\right)$

$\therefore\ z = \sqrt{2}\operatorname{cis}\left(\frac{-\pi}{4} + k\frac{2\pi}{3}\right)$, $k = 0, 1, 2$ {De Moivre}

$\therefore$ the cube roots are: $\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)$, $\sqrt{2}\operatorname{cis}\left(\frac{5\pi}{12}\right)$, $\sqrt{2}\operatorname{cis}\left(\frac{13\pi}{12}\right)$

b



c

$$\begin{aligned}
&\alpha_1+\alpha_2+\alpha_3\\
&= \sqrt{2}\left[\operatorname{cis}\left(\tfrac{-\pi}{4}\right) + \operatorname{cis}\left(\tfrac{5\pi}{12}\right) + \operatorname{cis}\left(\tfrac{13\pi}{12}\right)\right]\\
&= \sqrt{2}\left[a^{-3} + a^5 + a^{13}\right] \quad \text{where } a = \operatorname{cis}\left(\tfrac{\pi}{12}\right)\\
&= \sqrt{2}a^{-3}\left[1 + a^8 + a^{16}\right]\\
&= \sqrt{2}a^{-3}\left[\frac{(a^8)^3-1}{a^8-1}\right] \quad \{\text{sum of a geometric series}\}\\
&= \frac{\sqrt{2}}{a^3}\left[\frac{a^{24}-1}{a^8-1}\right]\\
&= \frac{\sqrt{2}}{a^3}\left[\frac{\left[\operatorname{cis}\left(\frac{\pi}{12}\right)\right]^{24}-1}{a^8-1}\right]\\
&= \frac{\sqrt{2}}{a^3}\left[\frac{\operatorname{cis}(2\pi)-1}{a^8-1}\right]\\
&= 0
\end{aligned}$$

d Let $z^n = \beta\operatorname{cis}(0+k2\pi)$ {as $\operatorname{cis}(0+k2\pi)=1$}

$$\therefore\ z = \beta^{\frac{1}{n}}\left[\operatorname{cis}\left(\frac{k2\pi}{n}\right)\right],\quad k = 0, 1, 2, 3, \ldots, n-1 \quad \{\text{De Moivre}\}$$

$$\therefore\ z = \beta^{\frac{1}{n}}\operatorname{cis}0,\ \beta^{\frac{1}{n}}\operatorname{cis}\left(\frac{2\pi}{n}\right),\ \beta^{\frac{1}{n}}\operatorname{cis}\left(\frac{4\pi}{n}\right),\ \ldots,\ \beta^{\frac{1}{n}}\operatorname{cis}\left(\frac{(n-1)2\pi}{n}\right)$$

$$\therefore\ z = \beta^{\frac{1}{n}},\ \beta^{\frac{1}{n}}\alpha,\ \beta^{\frac{1}{n}}\alpha^2,\ \beta^{\frac{1}{n}}\alpha^3,\ \ldots,\ \beta^{\frac{1}{n}}\alpha^{n-1} \quad\text{where } \alpha = \operatorname{cis}\left(\frac{2\pi}{n}\right)$$

The sum of these zeros is

$$\begin{aligned}
&\beta^{\frac{1}{n}}\left[1+\alpha+\alpha^2+\alpha^3+\ldots+\alpha^{n-1}\right]\\
&= \beta^{\frac{1}{n}}\left[\tfrac{1-\alpha^n}{1-\alpha}\right] \quad \{\text{sum of a geometric series with } u_1 = 1,\ r=\alpha,\ \text{“}n\text{”} = n\}\\
&= \beta^{\frac{1}{n}}\left[\tfrac{1-\operatorname{cis}2\pi}{1-\alpha}\right] \quad \left\{\text{since } \alpha^n = \left[\operatorname{cis}\left(\frac{2\pi}{n}\right)\right]^n = \operatorname{cis}2\pi\right\}\\
&= \beta^{\frac{1}{n}}(0) \quad \{\text{as } \operatorname{cis}2\pi = 1\}\\
&= 0
\end{aligned}$$

105 Let the roots be $a-3b,\ a-b,\ a+b,\ a+3b$

$a-b$ and $a+b$ have sum $2a$ and product a^2-b^2

$a-3b$ and $a+3b$ have sum $2a$ and product a^2-9b^2

$$\therefore\ x^4-(3m+2)x^2+m^3=\left[x^2-2ax+(a^2-b^2)\right]\left[x^2-2ax+(a^2-9b^2)\right]$$

Equating coefficients of x^3 gives $0=-2a-2a=-4a \qquad \therefore\ a=0$

$$\begin{aligned}\text{Thus }\ x^4-(3m+2)x^2+m^2&=(x^2-b^2)(x^2-9b^2)\\&=x^4-10b^2x^2+9b^4\end{aligned}$$

Equating coefficients, $3m+2=10b^2$ and $m^2=9b^4$

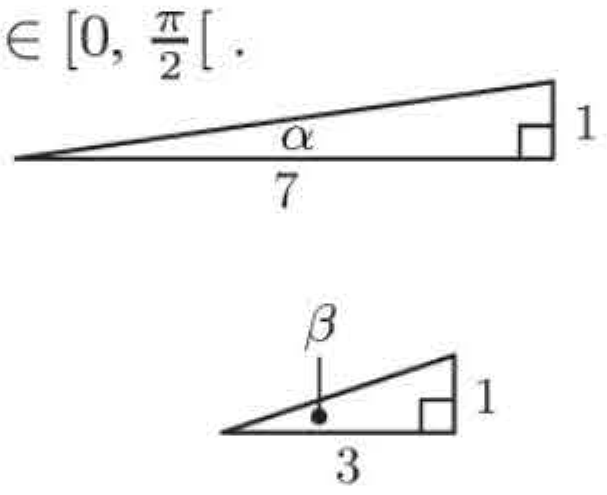
$$\begin{aligned}\therefore\ m&=\pm 3b^2\\ \therefore\ 3m+2&=10\left(\pm\frac{m}{3}\right)\\ \therefore\ 9m+6&=\pm 10m\\ \therefore\ m&=6\ \text{ or }\ -\tfrac{6}{19}\end{aligned}$$

106 Let $\arctan\left(\frac{1}{7}\right)=\alpha$ and $\arctan\left(\frac{1}{3}\right)=\beta$ where $\alpha,\beta\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$

$\therefore\ \tan\alpha=\frac{1}{7}$ and $\tan\beta=\frac{1}{3}$

Since $\tan\alpha,\ \tan\beta>0$ and $\alpha,\beta\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$, we know that actually $\alpha,\beta\in\left[0,\frac{\pi}{2}\right[$.

From the diagrams we see that $\alpha,\beta\in\left]0,\frac{\pi}{4}\right[$.



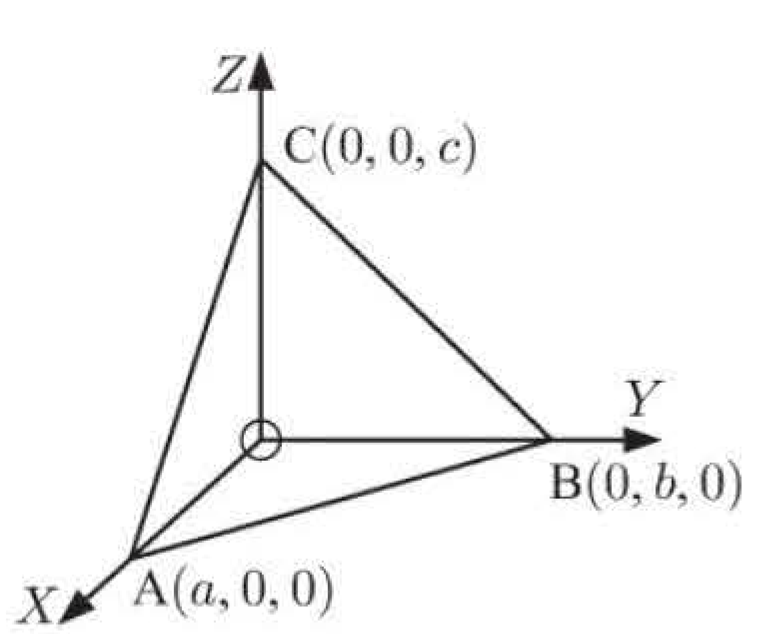
We need to find $\tan(\alpha+2\beta)$ where $\tan 2\beta=\dfrac{2\tan\beta}{1-\tan^2\beta}=\dfrac{\frac{2}{3}}{1-\frac{1}{9}}=\frac{3}{4}$

Now $\tan(\alpha+2\beta)=\dfrac{\tan\alpha+\tan 2\beta}{1-\tan\alpha\tan 2\beta}=\dfrac{\frac{1}{7}+\frac{3}{4}}{1-\frac{3}{28}}=1$

$\therefore\ \alpha+2\beta=\frac{\pi}{4}+k\pi,\ k\in\mathbb{Z}$

$\therefore\ \alpha+2\beta=\frac{\pi}{4}$, the only solution satisfying $\alpha,\beta\in\left]0,\frac{\pi}{4}\right[$.

107



$$\overrightarrow{AB}=\begin{pmatrix}-a\\b\\0\end{pmatrix}\ \text{ and }\ \overrightarrow{BC}=\begin{pmatrix}0\\-b\\c\end{pmatrix}$$

$$\begin{aligned}\therefore\ \overrightarrow{AB}\times\overrightarrow{BC}&=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-a&b&0\\0&-b&c\end{vmatrix}\\&=\mathbf{i}\begin{vmatrix}b&0\\-b&c\end{vmatrix}-\mathbf{j}\begin{vmatrix}-a&0\\0&c\end{vmatrix}+\mathbf{k}\begin{vmatrix}-a&b\\0&-b\end{vmatrix}\\&=bc\mathbf{i}+ac\mathbf{j}+ab\mathbf{k}\end{aligned}$$

$\therefore$ the plane has equation $bcx+acy+abz=bc(a)+0+0$ {using point A}

Dividing through by abc gives $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=1$.

108 Given: $(1+x)^n=\binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\binom{n}{3}x^3+\ldots.+\binom{n}{n}x^n,\ n\in\mathbb{Z}^+ \quad \ldots.\ (*)$

a Differentiating both sides gives:

$$n(1+x)^{n-1}=\binom{n}{1}+2\binom{n}{2}x+3\binom{n}{3}x^2+\ldots.+n\binom{n}{n}x^{n-1}$$

Letting $x=1$, $\binom{n}{1}+2\binom{n}{2}+3\binom{n}{3}+\ldots.+n\binom{n}{n}=n\,2^{n-1}$

b
$$\begin{aligned}&\binom{n}{0}+2\binom{n}{1}+3\binom{n}{2}+\ldots.+(n+1)\binom{n}{n}\\&=\left[\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\ldots.+\binom{n}{n}\right]+\left[\binom{n}{1}+2\binom{n}{2}+\ldots.+n\binom{n}{n}\right]\\&=2^n+n2^{n-1}\quad\{\text{letting } x=1 \text{ in } (*)\text{, and using } \textbf{a}\}\\&=(n+2)2^{n-1}\end{aligned}$$

c We notice that $\dfrac{1}{r+1}\dbinom{n}{r} = \dfrac{1}{r+1}\left(\dfrac{n!}{r!(n-r!)}\right) = \dfrac{n!}{(r+1)!(n-r)!}$

$$= \left[\frac{n!}{(r+1)!(n-r)!}\right]\frac{n+1}{n+1}$$

$$= \frac{1}{n+1}\left[\frac{(n+1)!}{(r+1)!(n-r)!}\right]$$

$$= \frac{1}{n+1}\binom{n+1}{r+1}$$

$$\therefore \quad \tfrac{1}{1}\tbinom{n}{0} + \tfrac{1}{2}\tbinom{n}{1} + \tfrac{1}{3}\tbinom{n}{2} + \dots + \tfrac{1}{n+1}\tbinom{n}{n}$$

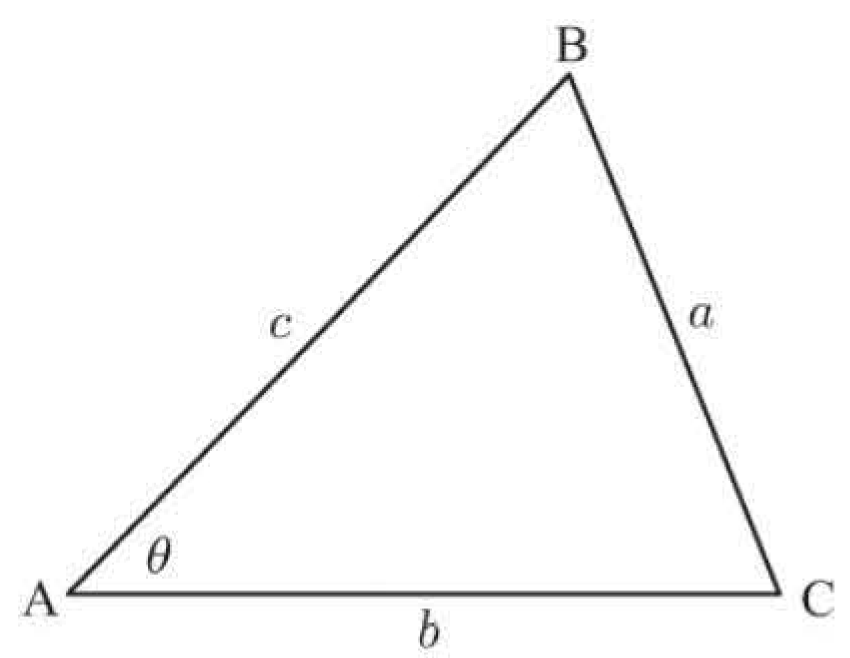
$$= \tfrac{1}{n+1}\tbinom{n+1}{1} + \tfrac{1}{n+1}\tbinom{n+1}{2} + \tfrac{1}{n+1}\tbinom{n+1}{3} + \dots + \tfrac{1}{n+1}\tbinom{n+1}{n+1}$$

$$= \tfrac{1}{n+1}\left[\tbinom{n+1}{1} + \tbinom{n+1}{2} + \tbinom{n+1}{3} + \dots + \tbinom{n+1}{n+1}\right]$$

$$= \tfrac{1}{n+1}\left[(1+1)^{n+1} - \tbinom{n+1}{0}\right] \quad \{\text{using } (*) \text{ with } n \text{ replaced by } n+1,\ x=1\}$$

$$= \frac{2^{n+1}-1}{n+1}$$

109



Area of $\triangle$ABC, $A = \frac{1}{2}bc\sin A$

$$\therefore \quad A^2 = \tfrac{1}{4}b^2c^2\sin^2\theta$$

$$= \tfrac{1}{4}b^2c^2\left(1-\cos^2\theta\right)$$

$$= \frac{b^2c^2}{4}\left(1-\left[\frac{b^2+c^2-a^2}{2bc}\right]^2\right)$$

$$= \frac{b^2c^2}{4}\left(1+\frac{b^2+c^2-a^2}{2bc}\right)\left(1-\frac{b^2+c^2-a^2}{2bc}\right)$$

$$= \frac{b^2c^2}{4}\left(\frac{2bc+b^2+c^2-a^2}{2bc}\right)\left(\frac{2bc-b^2-c^2+a^2}{2bc}\right)$$

$$= \tfrac{1}{16}\left((b+c)^2-a^2\right)\left(a^2-(b-c)^2\right)$$

$$= \tfrac{1}{16}(b+c-a)(b+c+a)(a-b+c)(a+b-c)$$

$$= \left(\frac{a+b+c}{2}\right)\left(\frac{b+c-a}{2}\right)\left(\frac{a+c-b}{2}\right)\left(\frac{a+b-c}{2}\right)$$

$$= s(s-a)(s-b)(s-c) \quad \text{where} \quad s = \frac{a+b+c}{2}$$

Thus $A = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \dfrac{a+b+c}{2}$.

110 $P(X=x) = \dfrac{m^x e^{-m}}{x!}$ for $x = 0, 1, 2, 3, 4, \dots$ where m = mean = variance = σ^2

Thus $P(X=x) = \dfrac{\sigma^{2x}e^{-\sigma^2}}{x!}$

But $P(X=2) - P(X=1) = 3P(X=0)$

$$\therefore \quad \frac{\sigma^4 e^{-\sigma^2}}{2!} - \frac{\sigma^2 e^{-\sigma^2}}{1!} = \frac{3\sigma^0 e^{-\sigma^2}}{0!}$$

$$\therefore \quad \frac{\sigma^4}{2} - \sigma^2 = 3$$

$$\therefore \quad \sigma^4 - 2\sigma^2 - 6 = 0$$

$$\therefore \quad \sigma^2 = \frac{2 \pm \sqrt{4-4(1)(-6)}}{2} = \frac{2\pm\sqrt{28}}{2} = 1 \pm \sqrt{7}$$

But $\sigma^2 > 0$, so $\sigma^2 = 1+\sqrt{7}$ and $\sigma > 0$ so $\sigma = \sqrt{1+\sqrt{7}}$

111 **a** Let $\dfrac{1}{n(n+2)} = \dfrac{A}{n} + \dfrac{B}{n+2} = \dfrac{A(n+2)+Bn}{n(n+2)}$ for all n

$\therefore \dfrac{1}{n(n+2)} = \dfrac{(A+B)n+2A}{n(n+2)}$ for all n

$\therefore A+B=0$ and $2A=1$ $\quad \therefore A=\frac{1}{2},\ B=-\frac{1}{2}$

b $$\frac{1}{1\times 3}+\frac{1}{2\times 4}+\frac{1}{3\times 5}+....+\frac{1}{n(n+2)}$$

$$=\frac{\frac{1}{2}}{1}-\frac{\frac{1}{2}}{3}+\frac{\frac{1}{2}}{2}-\frac{\frac{1}{2}}{4}+\frac{\frac{1}{2}}{3}-\frac{\frac{1}{2}}{5}+\frac{\frac{1}{2}}{4}-\frac{\frac{1}{2}}{6}+....+\frac{\frac{1}{2}}{n-1}-\frac{\frac{1}{2}}{n+1}+\frac{\frac{1}{2}}{n}-\frac{\frac{1}{2}}{n+2}$$

$$=\frac{1}{2}+\frac{1}{4}-\frac{1}{2n+2}-\frac{1}{2n+4}$$ {as all other terms cancel}

$$=\frac{3}{4}-\frac{1}{2n+2}-\frac{1}{2n+4}$$

c $$\sum_{r=1}^{\infty}\frac{1}{r(r+2)}=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{r(r+2)}=\lim_{n\to\infty}\left(\tfrac{3}{4}-\tfrac{1}{2n+2}-\tfrac{1}{2n+4}\right)=\tfrac{3}{4}$$

d P_n is "$\dfrac{1}{1\times 3}+\dfrac{1}{2\times 4}+\dfrac{1}{3\times 5}+....+\dfrac{1}{n(n+2)}=\dfrac{3}{4}-\dfrac{1}{2n+2}-\dfrac{1}{2n+4}$" for $n\in\mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n=1$, LHS $=\frac{1}{1\times 3}=\frac{1}{3}$, RHS $=\frac{3}{4}-\frac{1}{4}-\frac{1}{6}=\frac{1}{3}$ ✓

(2) If P_k is assumed true then

$$\frac{1}{1\times 3}+\frac{1}{2\times 4}+\frac{1}{3\times 5}+....+\frac{1}{k(k+2)}=\frac{3}{4}-\frac{1}{2k+2}-\frac{1}{2k+4}$$

So, $$\frac{1}{1\times 3}+\frac{1}{2\times 4}+\frac{1}{3\times 5}+....+\frac{1}{k(k+2)}+\frac{1}{(k+1)(k+3)}$$

$$=\frac{3}{4}-\frac{1}{2k+2}-\frac{1}{2k+4}+\frac{1}{(k+1)(k+3)}$$

$$=\frac{3}{4}-\frac{1}{2k+4}+\frac{1}{(k+1)(k+3)}-\frac{1}{2(k+1)}$$

$$=\frac{3}{4}-\frac{1}{2k+4}+\frac{2-(k+3)}{2(k+1)(k+3)}$$

$$=\frac{3}{4}-\frac{1}{2k+4}+\frac{{}^{-1}\cancel{-(k+1)}}{2\cancel{(k+1)}(k+3)}$$

$$=\frac{3}{4}-\frac{1}{2k+4}-\frac{1}{2k+6}$$

$$=\frac{3}{4}-\frac{1}{2(k+1)+2}-\frac{1}{2(k+1)+4}$$

Thus P_{k+1} is true whenever P_k is true.

Since P_1 is true, P_n is true for all $n\in\mathbb{Z}^+$ {Principle of mathematical induction}

112 **a** $$y=\ln(\tan x)$$

$$\therefore \frac{dy}{dx}=\frac{\sec^2 x}{\tan x}$$

$$=\frac{1}{\cos^2 x}\times\frac{\cos x}{\sin x}$$

$$=\frac{1}{\sin x\cos x}\left(\frac{2}{2}\right)$$

$$=\frac{2}{\sin 2x}=2\csc(2x)$$

$\therefore k=2$

b As $\dfrac{d\,[\ln(\tan x)]}{dx}=2\csc(2x)$,

$\int \csc(2x)\,dx=\frac{1}{2}\ln(\tan x)+c$

$$\therefore \text{ area}=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\csc(2x)\,dx=\left[\tfrac{1}{2}\ln(\tan x)\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

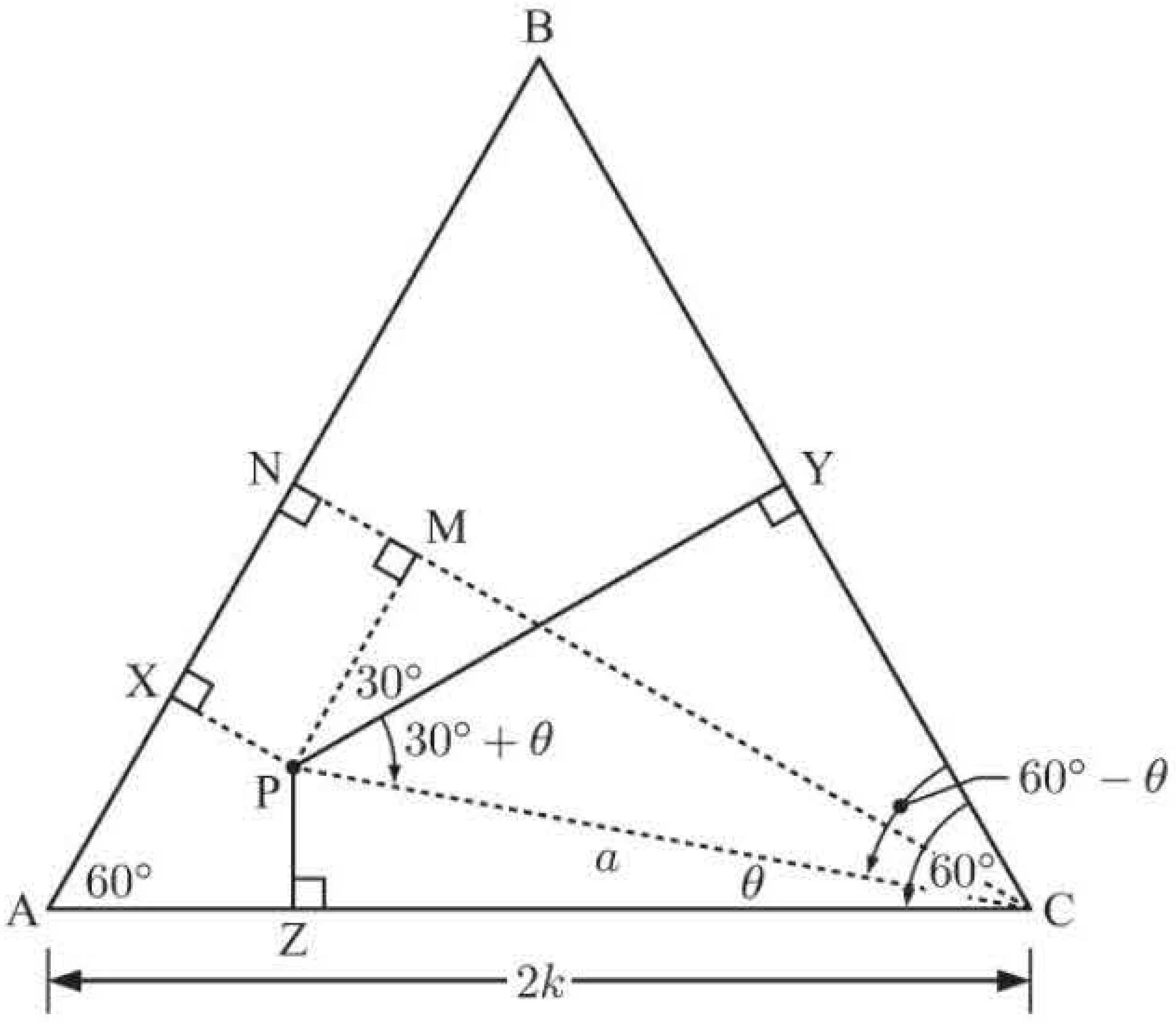
$$=\tfrac{1}{2}\ln(\sqrt{3})-\tfrac{1}{2}\ln\left(\tfrac{1}{\sqrt{3}}\right)$$

$$=\tfrac{1}{2}\ln(3^{\frac{1}{2}})-\tfrac{1}{2}\ln(3^{-\frac{1}{2}})$$

$$=\tfrac{1}{4}\ln 3+\tfrac{1}{4}\ln 3$$

$$=\tfrac{1}{2}\ln 3\text{ units}^2$$

113 **a** Let $\text{PC} = a$ units



So, $\sin\theta = \dfrac{\text{PZ}}{a}$

$\therefore \quad \text{PZ} = a\sin\theta \quad .... \ (1)$

and $\sin(60^\circ - \theta) = \dfrac{\text{PY}}{a}$

$\therefore \quad \text{PY} = a\sin(60^\circ - \theta) \quad .... \ (2)$

Now $\widehat{\text{MPC}} = 30^\circ + 30^\circ + \theta$
$= 60^\circ + \theta$

so in $\triangle\text{CMP}$, $\sin(60^\circ + \theta) = \dfrac{\text{CM}}{a}$

$\therefore \quad \text{CM} = a\sin(60^\circ + \theta) \quad .... \ (3)$

Also, in $\triangle\text{ACN}$, $\sin 60^\circ = \dfrac{\text{CN}}{2k}$

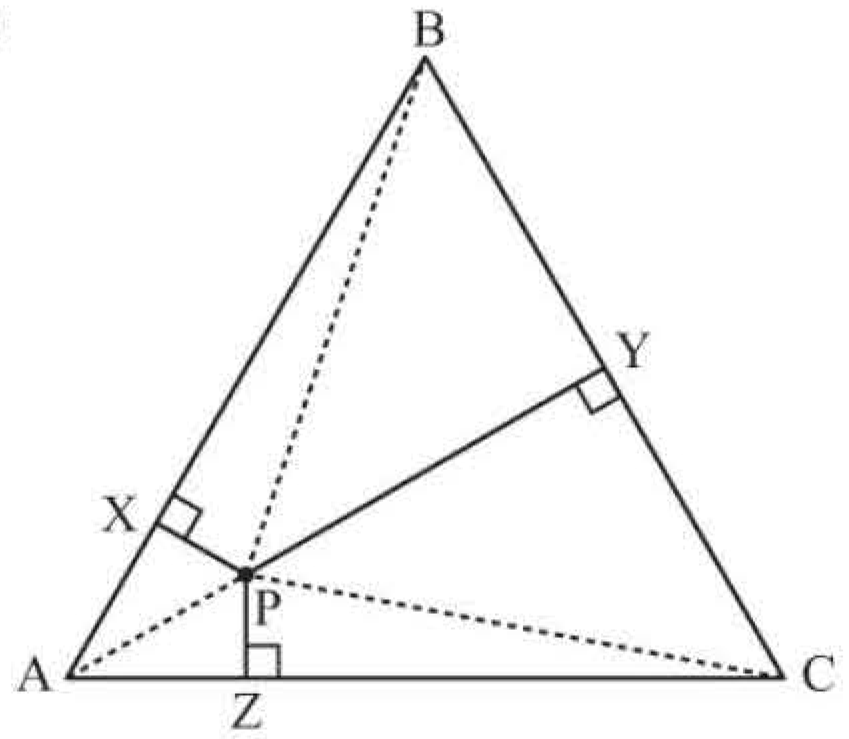
$\therefore \quad \text{CN} = 2k\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}k \quad .... \ (4)$

$\therefore \quad \text{PX} = \text{NM} = \text{CN} - \text{CM}$
$= k\sqrt{3} - a\sin(60^\circ + \theta) \qquad$ {using (3) and (4)}

Thus $\text{PX} + \text{PY} + \text{PZ} = k\sqrt{3} - a\sin(60^\circ + \theta) + a\sin(60^\circ - \theta) + a\sin\theta \qquad$ {using (1) and (2)}
$= k\sqrt{3} + a[\sin\theta + \sin(60^\circ - \theta) - \sin(60^\circ + \theta)]$
$= k\sqrt{3} + a[\sin\theta - \sin\theta]$
$= k\sqrt{3}$ for all θ, which is a constant.

b If P is at A, $\text{PX} + \text{PY} + \text{PZ} = \text{altitude from A to [BC]} + 0 + 0$
$= \text{AC}\sin 60^\circ = k\sqrt{3}$ ✓

c



Area $\triangle\text{ABC}$ = area $\triangle\text{ABP}$ + area $\triangle\text{BCP}$ + area $\triangle\text{ACP}$
$= \frac{1}{2}(2k\text{PX}) + \frac{1}{2}(2k\text{PY}) + \frac{1}{2}(2k\text{PZ})$
$= k(\text{PX} + \text{PY} + \text{PZ}) \quad .... \ (5)$

But area $\triangle\text{ABC} = \frac{1}{2}(2k)(\sqrt{3}k) = k^2\sqrt{3} \quad .... \ (6)$

From (5) and (6), $\text{PX} + \text{PY} + \text{PZ} = k\sqrt{3}$.

114 **a** Consider $1 + a\operatorname{cis}\theta + a^2\operatorname{cis}2\theta + a^3\operatorname{cis}3\theta + ... + a^n\operatorname{cis}n\theta \quad .... \ (1)$
$= 1 + a\,[\operatorname{cis}\theta]^1 + a^2\,[\operatorname{cis}\theta]^2 + a^3\,[\operatorname{cis}\theta]^3 + ... + a^n\,[\operatorname{cis}\theta]^n$

which is a geometric series with $u_1 = 1$, $r = a\operatorname{cis}\theta$ and has $n+1$ terms. So, its sum is:

$$1\left[\frac{(a\operatorname{cis}\theta)^{n+1} - 1}{a\operatorname{cis}\theta - 1}\right]$$

$$= \frac{a^{n+1}\operatorname{cis}(n+1)\theta - 1}{a\operatorname{cis}\theta - 1}$$

$$= \frac{a^{n+1}\,[\cos(n+1)\theta + i\sin(n+1)\theta] - 1}{a\cos\theta + ai\sin\theta - 1}$$

$$= \left[\frac{a^{n+1}\cos(n+1)\theta - 1 + ia^{n+1}\sin(n+1)\theta}{a\cos\theta - 1 + ia\sin\theta}\right]\left[\frac{a\cos\theta - 1 - ia\sin\theta}{a\cos\theta - 1 - ia\sin\theta}\right]$$

$$= \frac{(a^{n+1}\cos(n+1)\theta - 1 + ia^{n+1}\sin(n+1)\theta)(a\cos\theta - 1 - ia\sin\theta)}{(a\cos\theta - 1)^2 + a^2\sin^2\theta} \quad .... \ (2)$$

Equating the real parts of (1) and (2) gives

$$\begin{aligned}
&1 + a\cos\theta + a^2\cos 2\theta + a^3\cos 3\theta + ... + a^n(\cos n\theta)\\
&= \frac{(a^{n+1}\cos(n+1)\theta - 1)(a\cos\theta - 1) + a^{n+2}\sin(n+1)\theta\sin\theta}{a^2\cos^2\theta - 2a\cos\theta + 1 + a^2\sin^2\theta}\\
&= \frac{a^{n+2}\cos(n+1)\theta\cos\theta - a\cos\theta - a^{n+1}\cos(n+1)\theta + 1 + a^{n+2}\sin(n+1)\theta\sin\theta}{a^2 - 2a\cos\theta + 1}\\
&= \frac{a^{n+2}\left[\cos(n+1)\theta\cos\theta + \sin(n+1)\theta\sin\theta\right] - a\cos\theta - a^{n+1}\cos(n+1)\theta + 1}{a^2 - 2a\cos\theta + 1}\\
&= \frac{a^{n+2}\cos n\theta - a\cos\theta - a^{n+1}\cos(n+1)\theta + 1}{a^2 - 2a\cos\theta + 1}\\
&= \frac{a^{n+1}(a\cos n\theta - \cos(n+1)\theta) - a\cos\theta + 1}{a^2 - 2a\cos\theta + 1}
\end{aligned}$$

b Equating the imaginary parts of (1) and (2) gives

$$\begin{aligned}
&a\sin\theta + a^2\sin 2\theta + a^3\sin 3\theta + ... + a^n\sin n\theta\\
&= \frac{(a^{n+1}\sin(n+1)\theta)(a\cos\theta - 1) - a\sin\theta(a^{n+1}\cos(n+1)\theta - 1)}{a^2 - 2a\cos\theta + 1}\\
&= \frac{a^{n+2}\sin(n+1)\theta\cos\theta - a^{n+1}\sin(n+1)\theta - a^{n+2}\cos(n+1)\theta\sin\theta + a\sin\theta}{a^2 - 2a\cos\theta + 1}\\
&= \frac{a^{n+2}(\sin(n+1)\theta\cos\theta - \cos(n+1)\theta\sin\theta) - a^{n+1}\sin(n+1)\theta + a\sin\theta}{a^2 - 2a\cos\theta + 1}\\
&= \frac{a^{n+2}\sin n\theta - a^{n+1}\sin(n+1)\theta + a\sin\theta}{a^2 - 2a\cos\theta + 1}\\
&= \frac{a^{n+1}(a\sin n\theta - \sin(n+1)\theta) + a\sin\theta}{a^2 - 2a\cos\theta + 1}
\end{aligned}$$

115 Let $u = \sqrt{x+2}$, so $u^2 = x + 2$

$$\therefore\ 2u\,\frac{du}{dx} = 1$$

$$\begin{aligned}
\therefore\ \int \frac{x}{1+\sqrt{x+2}}\,dx &= \int \left(\frac{u^2-2}{1+u}\right) 2u\,du\\
&= \int \frac{2u^3 - 4u}{u+1}\,du\\
&= \int \left(2u^2 - 2u - 2 + \frac{2}{u+1}\right) du\\
&= \frac{2u^3}{3} - \frac{2u^2}{2} - 2u + 2\ln|u+1| + c\\
&= \tfrac{2}{3}(x+2)^{\frac{3}{2}} - x - 2 - 2\sqrt{x+2} + 2\ln(\sqrt{x+2}+1) + c
\end{aligned}$$

$$\begin{array}{r|l}
 & 2u^2 - 2u - 2\\ \hline
u+1 & 2u^3 + 0u^2 - 4u + 0\\
 & -\,(2u^3 + 2u^2)\\ \hline
 & -\,2u^2 - 4u + 0\\
 & -\,(-\,2u^2 - 2u)\\ \hline
 & -\,2u + 0\\
 & -\,(-\,2u - 2)\\ \hline
 & 2
\end{array}$$

116 **a** P(stopped at least once)
$= 1 -$ P(never stopped in n intersections)
$= 1 - (1-p)^n$

b
$$\begin{aligned}
P(A_k \mid B_k) &= \frac{P(A_k \cap B_k)}{P(B_k)}\\
&= \frac{P(A_k)}{P(B_k)}\\
&= \frac{\binom{n}{k} p^k (1-p)^{n-k}}{\sum\limits_{r=k}^{n} \binom{n}{r} p^r (1-p)^{n-r}}
\end{aligned}$$

c **i** If A_1 and B_1 are independent then

$$P(A_1 \mid B_1) = P(A_1)$$

$$\therefore \quad \frac{\binom{n}{1} p(1-p)^{n-1}}{1-(1-p)^n} = \binom{n}{1} p(1-p)^{n-1}$$

$$\therefore \quad 1-(1-p)^n = 1$$

$$\therefore \quad (1-p)^n = 0$$

$$\therefore \quad p = 1$$

ii If $P(A_2 \mid B_2) = P(A_1)$ and $n = 2$,

$$\frac{\binom{2}{2} p^2}{\binom{2}{2} p^2} = \binom{2}{1} p(1-p)$$

$$\therefore \quad 1 = 2p(1-p)$$

$$\therefore \quad 2p^2 - 2p + 1 = 0$$

where $\Delta = 4 - 4(2)(1) < 0$

$\therefore$ there are no real solutions

117 **a** $\cos 4\theta + i \sin 4\theta = \text{cis}\, 4\theta$

$$= (\text{cis}\,\theta)^4 \quad \{\text{De Moivre}\}$$

$$= (C + iS)^4 \quad \text{where } C = \cos\theta \text{ and } S = \sin\theta$$

$$= C^4 + 4C^3(iS) + 6C^2(iS)^2 + 4C(iS)^3 + (iS)^4$$

$$= (C^4 - 6C^2S^2 + S^4) + i(4C^3S - 4CS^3)$$

Equating real and imaginary parts, $\sin 4\theta = 4C^3S - 4CS^3$ and $\cos 4\theta = C^4 - 6C^2S^2 + S^4$.

Hence, $\tan 4\theta = \dfrac{\sin 4\theta}{\cos 4\theta} = \dfrac{4C^3S - 4CS^3}{C^4 - 6C^2S^2 + S^4}$

$$= \frac{4\left(\frac{S}{C}\right) - 4\left(\frac{S}{C}\right)^3}{1 - 6\left(\frac{S}{C}\right)^2 + \left(\frac{S}{C}\right)^4} \quad \{\text{dividing all terms by } C^4\}$$

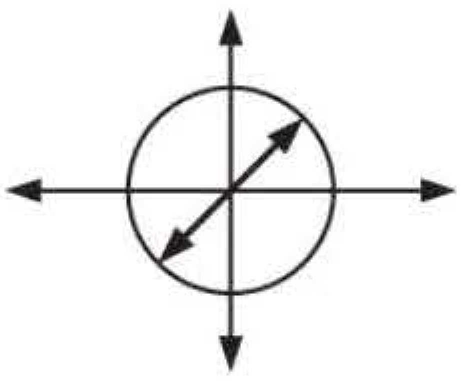
$$= \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$$

b $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$

If we let $x = \tan\theta$ then $\tan^4\theta + 4\tan^3\theta - 6\tan^2\theta - 4\tan\theta + 1 = 0$

$\therefore \quad 1 - 6\tan^2\theta + \tan^4\theta = 4\tan\theta - 4\tan^3\theta$

$$\therefore \quad \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta} = 1$$



$$\therefore \quad \tan 4\theta = 1 \quad \{\text{by } \mathbf{a}\}$$

$$\therefore \quad 4\theta = \tfrac{\pi}{4} + k\pi, \quad k \in \mathbb{Z}$$

$$\therefore \quad \theta = \tfrac{\pi}{16} + \tfrac{k\pi}{4}$$

Thus $x = \tan\left(\frac{\pi}{16}\right), \ \tan\left(\frac{5\pi}{16}\right), \ \tan\left(\frac{9\pi}{16}\right), \ \tan\left(\frac{13\pi}{16}\right)$

118 **a** The system has augmented matrix

$$\left(\begin{array}{ccc|c} 2 & -1 & 3 & 4 \\ 2 & 1 & a+3 & 10-a \\ 4 & 6 & a^2+6 & a^2 \end{array}\right) = \left(\begin{array}{ccc|c} 2 & -1 & 3 & 4 \\ 0 & 2 & a & 6-a \\ 0 & 8 & a^2 & a^2-8 \end{array}\right) \begin{array}{l} \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$= \left(\begin{array}{ccc|c} 2 & -1 & 3 & 4 \\ 0 & 2 & a & 6-a \\ 0 & 0 & a^2-4a & a^2+4a-32 \end{array}\right) \begin{array}{l} \\ \\ R_3 \to R_3 - 4R_2 \end{array}$$

b The last equation is $a(a-4)z = (a+8)(a-4)$

i The system has no solutions if the LHS $= 0$ and the RHS $\neq .0$.
This occurs when $a = 0$, since we get $0z = -32$.
The line of intersection of any two planes is parallel to the third.

ii The system has infinitely many solutions if the last equation has the form $0z = 0$, which is true for any real number z. This occurs when $a = 4$.

The augmented matrix becomes $\left(\begin{array}{ccc|c} 2 & -1 & 3 & 4 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$

Letting $z = t$, $2y + 4t = 2$

$$\therefore \quad y + 2t = 1$$
$$\therefore \quad y = 1 - 2t$$

and so $2x - (1 - 2t) + 3t = 4$

$$\therefore \quad 2x - 1 + 2t + 3t = 4$$
$$\therefore \quad x = \frac{5-5t}{2}$$

$\therefore$ when $a = 4$, there are infinitely many solutions of the form

$$x = \frac{5-5t}{2},\ y = 1 - 2t,\ z = t,\ t \in \mathbb{R}.$$

The planes meet in a common line of intersection.

iii The system has a unique solution for all other values of a. So, $a \neq 0$ or 4.

In this case $z = \dfrac{a+8}{a}$

$$\therefore \quad 2y + a\left(\frac{a+8}{a}\right) = 6 - a$$
$$\therefore \quad 2y = 6 - a - a - 8$$
$$= -2a - 2$$
$$\therefore \quad y = -a - 1$$

and

$$2x + a + 1 + 3\left(\frac{a+8}{a}\right) = 4$$
$$\therefore \quad 2x = 4 - a - 1 - 3 - \frac{24}{a}$$
$$= -a - \frac{24}{a}$$
$$\therefore \quad x = -\frac{a}{2} - \frac{12}{a}$$

$\therefore$ when $a \neq 0$ or 4, there is a unique solution of the form

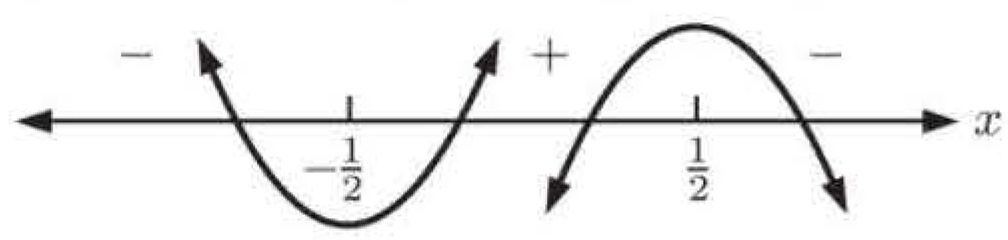
$$x = -\frac{a}{2} - \frac{12}{a}, \quad y = -a - 1, \quad z = \frac{a+8}{a}$$

When $a = 2$, the solution is $x = -7$, $y = -3$, $z = 5$.
This means that the 3 planes meet at one point only.

119 $f(x) = xe^{1-2x^2}$

a
$$f'(x) = 1e^{1-2x^2} + xe^{1-2x^2}(-4x)$$
$$= e^{1-2x^2}(1 - 4x^2)$$
$$f''(x) = -4xe^{1-2x^2}(1 - 4x^2) + e^{1-2x^2}(-8x)$$
$$= e^{1-2x^2}(-4x + 16x^3 - 8x)$$
$$= e^{1-2x^2}(16x^3 - 12x)$$

b $f'(x) = e^{1-2x^2}(1 + 2x)(1 - 2x)$



$\therefore$ there is a local minimum at $\left(-\frac{1}{2}, -\frac{\sqrt{e}}{2}\right)$

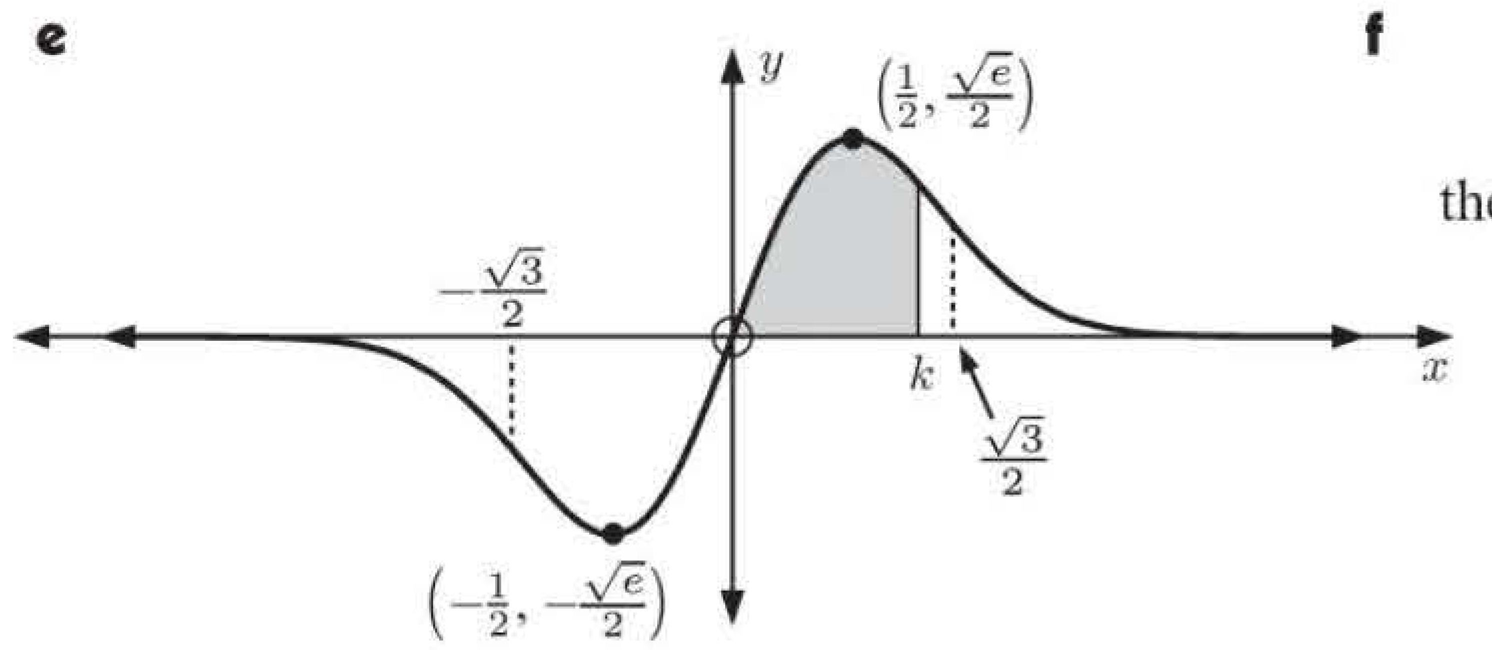
and a local maximum at $\left(\frac{1}{2}, \frac{\sqrt{e}}{2}\right)$.

c $f''(x) = 0$ when $16x^3 - 12x = 0$

$$\therefore \quad 4x(4x^2 - 3) = 0$$
$$\therefore \quad x = 0 \text{ or } \pm\tfrac{\sqrt{3}}{2}$$

d As $x \to \infty$, $f(x) \to 0$ (above).
As $x \to -\infty$, $f(x) \to 0$ (below).

e



f If $\int_0^k xe^{1-2x^2}\,dx = \dfrac{e-1}{4}$

then $\frac{1}{-4}\int_0^k e^{1-2x^2}(-4x)\,dx = \dfrac{e-1}{4}$

$$\therefore \quad \left[e^{1-2x^2}\right]_0^k = 1 - e$$
$$\therefore \quad e^{1-2k^2} - e = 1 - e$$
$$\therefore \quad e^{1-2k^2} = 1$$
$$\therefore \quad 1 - 2k^2 = 0$$
$$\therefore \quad k = \pm\tfrac{1}{\sqrt{2}}$$

But $k > 0$, so $k = \frac{1}{\sqrt{2}}$

120 **a** $\text{cis}\,\theta + \text{cis}\,\phi = \cos\theta + i\sin\theta + \cos\phi + i\sin\phi$

$$= [\cos\theta + \cos\phi] + i[\sin\theta + \sin\phi]$$

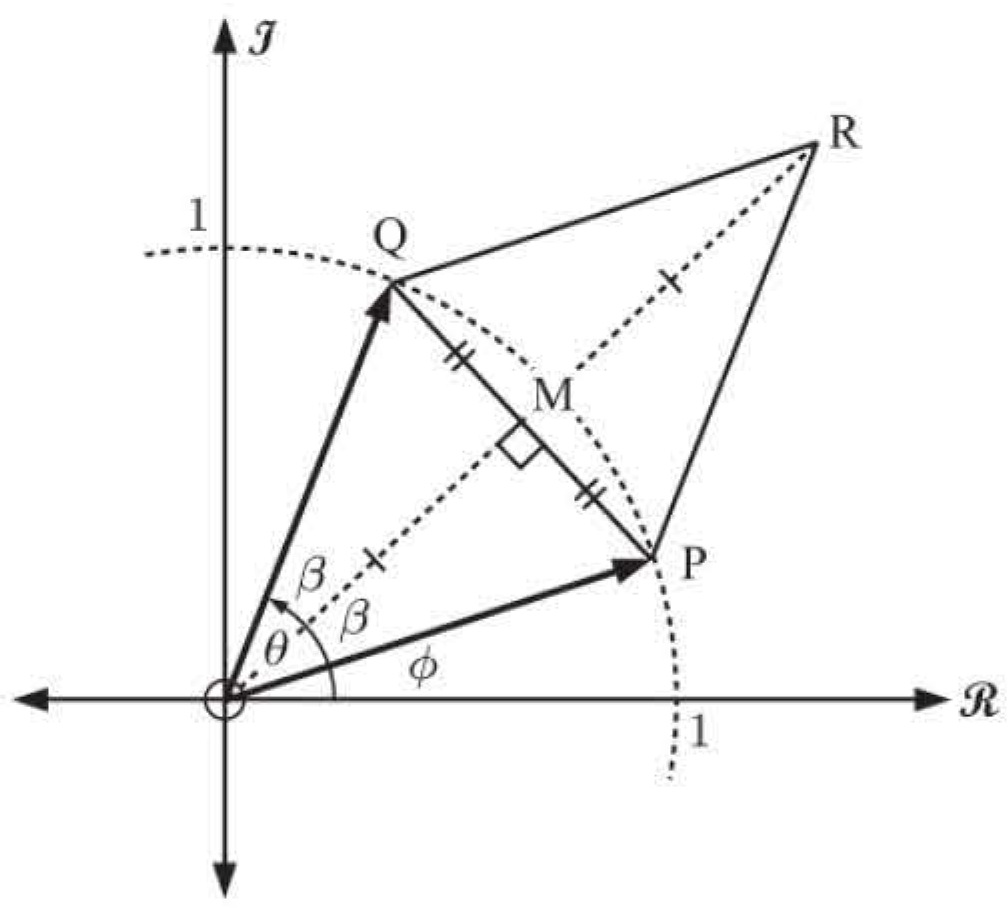
$$= 2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right) + i2\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$$

$$= 2\cos\left(\frac{\theta-\phi}{2}\right)\left[\cos\left(\frac{\theta+\phi}{2}\right) + i\sin\left(\frac{\theta+\phi}{2}\right)\right]$$

$$= 2\cos\left(\frac{\theta-\phi}{2}\right)\text{cis}\left(\frac{\theta+\phi}{2}\right)$$

b $|\text{cis}\,\theta + \text{cis}\,\phi| = 2\left|\cos\left(\frac{\theta-\phi}{2}\right)\right|$ and $\arg(\text{cis}\,\theta + \text{cis}\,\phi) = \frac{\theta+\phi}{2}$

c



Let $\overrightarrow{\text{OP}} = \text{cis}\,\phi$ and $\overrightarrow{\text{OQ}} = \text{cis}\,\theta$, $\theta > \phi$

We complete rhombus OPRQ, where the diagonals bisect each other at right angles.

$\overrightarrow{\text{OR}} = \overrightarrow{\text{OP}} + \overrightarrow{\text{PR}} = \overrightarrow{\text{OP}} + \overrightarrow{\text{OQ}} = \text{cis}\,\phi + \text{cis}\,\theta$

Since $\theta = \phi + 2\beta$, $\beta = \dfrac{\theta-\phi}{2}$

Now $\arg\overrightarrow{\text{OR}} = \phi + \beta = \dfrac{\theta+\phi}{2}$

and $|\overrightarrow{\text{OR}}| = 2(\text{OM})$ where $\cos\beta = \dfrac{\text{OM}}{\text{OP}} = \text{OM}$

$\therefore$ $|\overrightarrow{\text{OR}}| = 2\cos\left(\dfrac{\theta-\phi}{2}\right)$

Note: If θ and ϕ are interchanged, $|\overrightarrow{\text{OR}}| = 2\cos\left(\dfrac{\phi-\theta}{2}\right)$.

Thus $|\overrightarrow{\text{OR}}|$ is actually $2\left|\cos\left(\dfrac{\theta-\phi}{2}\right)\right|$.

d $\left(\dfrac{z+1}{z-1}\right)^5 = 1$ $\quad\therefore\quad \dfrac{z+1}{z-1} = 1, \alpha, \alpha^2, \alpha^3$, and α^4 where $\alpha = \text{cis}\left(\frac{2\pi}{5}\right)$

$$\therefore\quad \frac{z+1}{z-1} = \alpha^k \text{ where } k = 0, 1, 2, 3, 4 \text{ and } \alpha = \text{cis}\left(\tfrac{2\pi}{5}\right)$$

$$\therefore\quad z = \frac{1+\alpha^k}{\alpha^k - 1} \quad \{\text{making } z \text{ the subject}\}$$

$$\therefore\quad z = \frac{\text{cis}\,0 + \left[\text{cis}\left(\frac{2\pi}{5}\right)\right]^k}{\left[\text{cis}\left(\frac{2\pi}{5}\right)\right]^k + \text{cis}\,\pi} \qquad \{\text{as } \text{cis}\,\pi = -1\}$$

$$\therefore\quad z = \frac{\text{cis}\left(\frac{2\pi k}{5}\right) + \text{cis}\,0}{\text{cis}\left(\frac{2\pi k}{5}\right) + \text{cis}\,\pi}$$

$$\therefore\quad z = \frac{\cancel{2}\cos\left(\frac{\pi k}{5}\right)\text{cis}\left(\frac{\pi k}{5}\right)}{\cancel{2}\cos\left(\frac{\left(\frac{2\pi k}{5}\right)-\pi}{2}\right)\text{cis}\left(\frac{\left(\frac{2\pi k}{5}\right)+\pi}{2}\right)}$$

$$\therefore\quad z = \frac{\cos\left(\frac{\pi k}{5}\right)}{\cos\left(\frac{\pi k}{5} - \frac{\pi}{2}\right)}\text{cis}\left(\tfrac{\pi k}{5} - \tfrac{\pi k}{5} - \tfrac{\pi}{2}\right)$$

$$\therefore\quad z = \frac{\cos\left(\frac{\pi k}{5}\right)}{\sin\left(\frac{\pi k}{5}\right)}\text{cis}\left(-\tfrac{\pi}{2}\right) \qquad \{\cos(\theta - \tfrac{\pi}{2}) = \sin\theta\}$$

$$\therefore\quad z = \cot\left(\tfrac{\pi k}{5}\right) \times -i$$

$$\therefore\quad z = -i\cot\left(\tfrac{\pi k}{5}\right) \quad \text{for } k = 0, 1, 2, 3, 4$$

However $\cot 0$ is undefined, so we reject the solution when $k = 0$

$\therefore\quad z = -i\cot\left(\frac{k\pi}{5}\right)$ where $k = 1, 2, 3, 4$.

121 If $u = \ln x$, $\dfrac{du}{dx} = \dfrac{1}{x}$ and $\therefore$ $\displaystyle\int \sin(\ln x)\,dx = \int \sin u(x\,du)$

$$= \int \sin u e^u\,du$$

$$= \int e^u \sin u\,du \quad .... \ (*)$$

Now consider $\int e^x \sin x\,dx$ using integration by parts

$$\therefore \int e^x \sin x\,dx = e^x \sin x - \int e^x \cos x\,dx$$

$$= e^x \sin x - [e^x \cos x - \int e^x(-\sin x)\,dx]$$

Thus $\int e^x \sin x\,dx = e^x(\sin x - \cos x) - \int e^x \sin x\,dx$

$$\therefore\ 2\int e^x \sin x\,dx = e^x(\sin x - \cos x)$$

$$\therefore \int e^x \sin x\,dx = \frac{e^x}{2}(\sin x - \cos x)$$

Hence, in $(*)$, $\displaystyle\int \sin(\ln x)\,dx = \frac{e^u}{2}(\sin u - \cos u)$

$$= \frac{x}{2}(\sin(\ln x) - \cos(\ln x))$$

122 **a** Since $P(x)$ is a real polynomial, both $1 + ki$ and $1 - ki$ are zeros, $k \in \mathbb{Z}$.
These have sum 2 and product $1 + k^2$ and therefore come from the quadratic factor $x^2 - 2x + 1 + k^2$.

b By comparison with $P(x)$, $1 + k^2$ is a factor of -10.

$\therefore\ 1 + k^2 = \pm1, \pm2, \pm5, \pm10$ where $k \in \mathbb{Z}$

$\therefore\ k^2 = 0, 1, 4, 9$

$\therefore\ k = 0, \pm1, \pm2, \pm3$

c As p and q are integer zeros, they come from
$(x - p)(x - q) = x^2 - (p + q)x + pq$
$\therefore$ pq is a factor of -10
The possibilities are as shown in the table:

$1 + k^2$	pq
1	-10
2	-5
5	-2
10	-1

d Without loss of generality we assume $p > q$. Then as $p + q = -1$, the only possibility is $p = 1$, $q = -2$, and $1 + k^2 = 5$ $\quad\therefore\ k = \pm2$

$\therefore\ P(x) = (x - 1)(x + 2)(x - 1 - 2i)(x - 1 + 2i)$

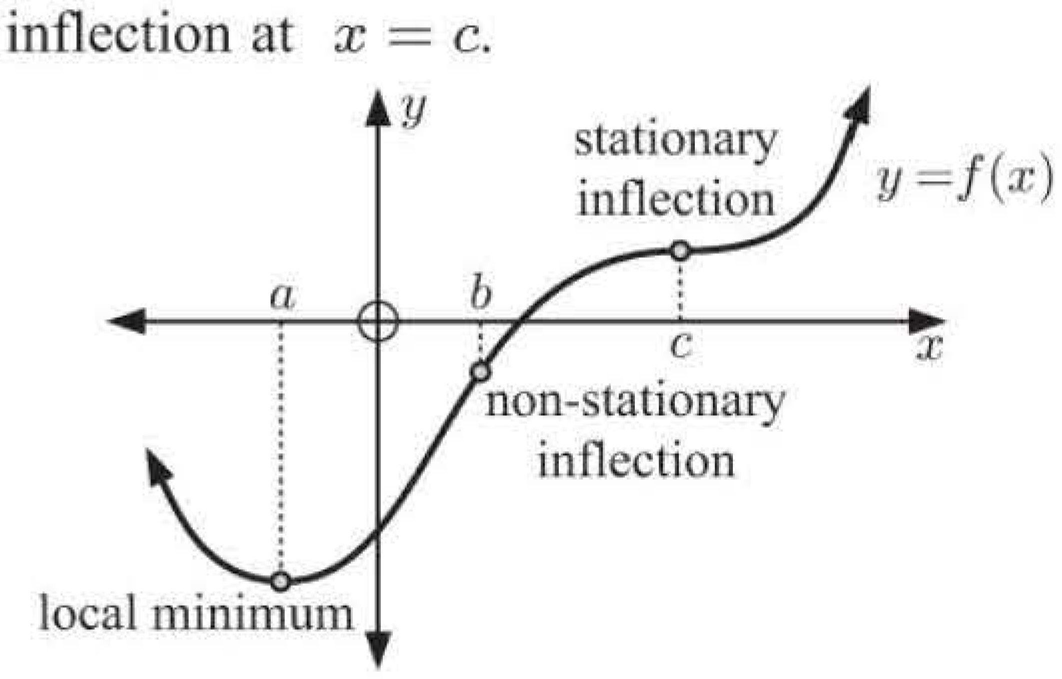
So, the zeros of $P(x)$ are 1, -2, $1 + 2i$, and $1 - 2i$.

123 **a** $f'(x)$ near a has sign diagram

$f'(x)$ near b has sign diagram , but $f'(b) \neq 0$

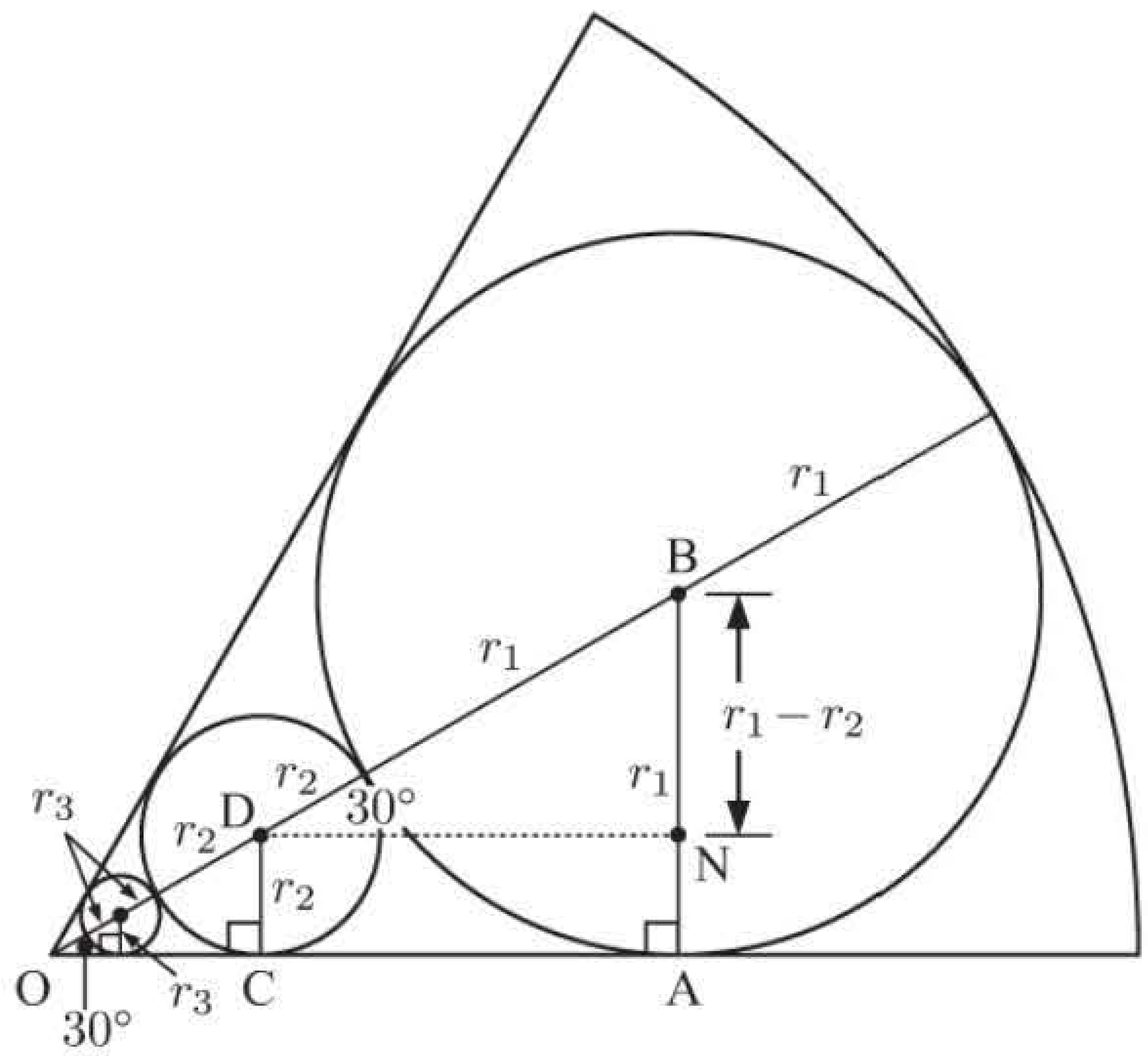
$f'(x)$ near c has sign diagram , but $f'(c) = 0$

$\therefore$ there is a local minimum at $x = a$, a non-stationary inflection at $x = b$, and a stationary inflection at $x = c$.



b $f(x) = f_1(x) + k$, k a constant

124 **a**



$$\sin 30^\circ = \frac{r_1}{10 - r_1} \quad \{\text{in } \triangle\text{OAB}\}$$

$$\therefore \ \tfrac{1}{2} = \frac{r_1}{10 - r_1} \quad \therefore \ r_1 = \tfrac{10}{3}$$

In $\triangle$DBN, $\sin 30^\circ = \dfrac{r_1 - r_2}{r_1 + r_2} = \tfrac{1}{2}$

$$\therefore \ 2r_1 - 2r_2 = r_1 + r_2$$

$$\therefore \ r_1 = 3r_2$$

$$\therefore \ r_2 = \tfrac{1}{3} r_1$$

So, in successive circles, radii are reduced by a factor of 3.

$\therefore \ r_2 = \frac{10}{9}, \ r_3 = \frac{10}{27}, \ r_4 = \frac{10}{81},$ and so on.

Thus the total area of the circles $= \pi {r_1}^2 + \pi {r_2}^2 + \pi {r_3}^2 + \pi {r_4}^2 + ...$

$$= \pi \left(\left(\tfrac{10}{3}\right)^2 + \left(\tfrac{10}{9}\right)^2 + \left(\tfrac{10}{27}\right)^2 + \left(\tfrac{10}{81}\right)^2 + ... \right)$$

$$= \pi \times \left(\tfrac{10}{3}\right)^2 \left[1 + \left(\tfrac{1}{3}\right)^2 + \left(\tfrac{1}{3}\right)^4 + \left(\tfrac{1}{3}\right)^6 + ... \right]$$

$$= \pi \times \tfrac{100}{9} \times \left(\frac{1}{1 - \frac{1}{9}} \right)$$

$$= \tfrac{25\pi}{2} \text{ units}^2$$

b We let $\frac{\alpha}{2}$ replace 30° in the calculations of **a**, and let $a = \sin\left(\frac{\alpha}{2}\right)$.

In this case $a = \dfrac{r_1}{10 - r_1}$, and so $r_1 = \dfrac{10a}{1 + a}$

Now $a = \dfrac{r_1 - r_2}{r_1 + r_2}$ {using $\triangle$DBN}

$$\therefore \ a(r_1 + r_2) = r_1 - r_2$$

$$\therefore \ r_2(a + 1) = r_1(1 - a)$$

$$\therefore \ r_2 = r_1 \left(\frac{1 - a}{a + 1} \right) = \frac{10a}{1 + a} \left(\frac{1 - a}{a + 1} \right) = \frac{10a(1 - a)}{(a + 1)^2}$$

Thus $r_3 = r_2 \left(\dfrac{1 - a}{a + 1} \right) = \dfrac{10a(1 - a)^2}{(a + 1)^3}$, and so on.

$$\therefore \ \text{total area} = \pi({r_1}^2 + {r_2}^2 + {r_3}^2 + {r_4}^2 + ...)$$

$$= \pi \left[\left(\frac{10a}{1 + a} \right)^2 + \left(\frac{10a(1 - a)}{(1 + a)^2} \right)^2 + \left(\frac{10a(1 - a)^2}{(1 + a)^3} \right)^2 + ... \right]$$

$$= \pi \frac{100a^2}{(1 + a)^2} \left[1 + \left(\frac{1 - a}{1 + a} \right)^2 + \left(\frac{1 - a}{1 + a} \right)^4 + ... \right]$$

This is the sum of an infinite geometric series which converges since $\left| \dfrac{1 - \sin\left(\frac{\alpha}{2}\right)}{1 + \sin\left(\frac{\alpha}{2}\right)} \right| < 1.$

$$\therefore \ \text{total area} = \pi \times \frac{100a^2}{(1 + a)^2} \times \frac{1}{1 - \left(\frac{1 - a}{1 + a} \right)^2}$$

$$= \pi \times \frac{100a^2}{(1 + a)^2 - (1 - a)^2}$$

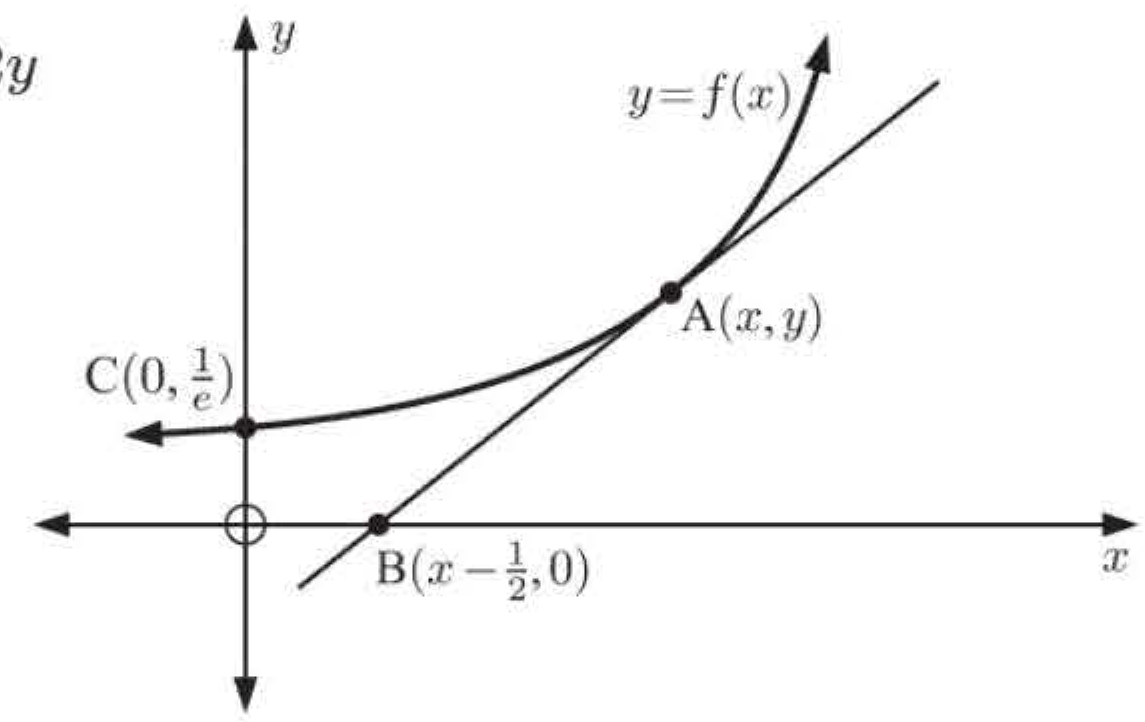
$$= \pi \times \frac{100a^2}{4a}$$

$$= 25\pi a$$

$$= 25\pi \sin\left(\tfrac{\alpha}{2}\right) \text{ units}^2$$

125 **a** At any point $A(x, y)$, $\dfrac{dy}{dx} = \dfrac{y-0}{x-\left(x-\frac{1}{2}\right)} = 2y$

$\therefore \dfrac{dy}{dx} = \dfrac{1}{2y}$



b $\therefore x = \displaystyle\int \frac{1}{2y}\, dy$

$\therefore x = \frac{1}{2}\ln|y| + c$

But when $x = 0$, $y = e^{-1}$

$\therefore 0 = \frac{1}{2}(-1) + c$

$\therefore c = \frac{1}{2}$

$\therefore x = \frac{1}{2}\ln|y| + \frac{1}{2}$

$\therefore 2x = \ln|y| + 1$

$\therefore \ln|y| = 2x - 1$

$|y| = e^{2x-1}$

$\therefore y = \pm e^{2x-1}$

$\therefore y = e^{2x-1}$ (If $y = -e^{2x-1}$ and $x = 0$, $y = -\dfrac{1}{e} \neq \dfrac{1}{e}$)

126 **a** **i** $\sin 2A + \sin 2B + \sin 2C$

$= \sin 2A + \sin 2B + \sin(2\pi - 2A - 2B)$ $\{$as $A + B + C = \pi\}$

$= \sin 2A + \sin 2B - \sin(2A + 2B)$ $\{\sin(2\pi - \theta) = -\sin\theta\}$

$= \sin 2A + \sin 2B - [\sin 2A \cos 2B + \cos 2A \sin 2B]$

$= \sin 2A(1 - \cos 2B) + \sin 2B(1 - \cos 2A)$

$= 2\sin A\cos A(2\sin^2 B) + 2\sin B\cos B\left(2\sin^2 A\right)$ $\{\cos 2\theta = 1 - 2\sin^2\theta\}$

$= 4\sin A\cos A\sin^2 B + 4\sin^2 A\sin B\cos B$

$= 4\sin A\sin B\,[\sin B\cos A + \cos B\sin A]$

$= 4\sin A\sin B\sin(A + B)$

$= 4\sin A\sin B\sin(\pi - C)$

$= 4\sin A\sin B\sin C$ $\{\sin(\pi - \theta) = \sin\theta\}$

ii $\tan A + \tan B + \tan C$

$= \tan A + \tan B + \tan(\pi - (A + B))$

$= \tan A + \tan B - \tan(A + B)$ $\{\tan(\pi - \theta) = -\tan\theta\}$

$= \dfrac{\tan A + \tan B}{1} - \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

$= \dfrac{(\tan A + \tan B)(1 - \tan A\tan B) - (\tan A + \tan B)}{1 - \tan A\tan B}$

$= \dfrac{\cancel{\tan A} + \cancel{\tan B} - \tan^2 A\tan B - \tan A\tan^2 B - \cancel{\tan A} - \cancel{\tan B}}{1 - \tan A\tan B}$

$= -\tan A\tan B\,\dfrac{(\tan A + \tan B)}{1 - \tan A\tan B}$

$= -\tan A\tan B\tan(A + B)$

$= \tan A\tan B\tan(\pi - (A + B))$ $\{\tan(\pi - \theta) = -\tan\theta\}$

$= \tan A\tan B\tan C$

b $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

$\therefore \tan A + \tan B = \tan C(\tan A \tan B - 1)$ .... (1)

Suppose $\tan A \tan B = 1$ .... (2)

Then $\tan A + \tan B + \tan C = \tan C$

$\therefore \tan A + \tan B = 0$

$\therefore \tan A = -\tan B$

$\therefore \tan A = -\dfrac{1}{\tan A}$ {using (2)}

$\therefore \tan^2 A = -1$ which is impossible

$\therefore \tan A \tan B \neq 1$, and the supposition (2) is false

So, using (1), $\dfrac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C$

$\therefore \tan(A+B) = \tan(-C)$

$\therefore A + B = -C + k\pi, \ k \in \mathbb{Z}$ {equal $\tan$s are π apart}

$\therefore A + B + C = k\pi, \quad k \in \mathbb{Z}$

127 **a** Suppose $\sqrt{14 - 4\sqrt{6}} = a + b\sqrt{6}$ where $a, b \in \mathbb{Z}$ .... $(*)$

$\therefore a^2 + 2ab\sqrt{6} + 6b^2 = 14 - 4\sqrt{6}$ {squaring both sides}

$\therefore a^2 + 6b^2 = 14$ and $ab = -2$

$\therefore a^2 + 6\left(\dfrac{-2}{a}\right)^2 = 14$

$\therefore a^2 + \dfrac{24}{a^2} - 14 = 0$

$\therefore a^4 - 14a^2 + 24 = 0$

$\therefore (a^2 - 2)(a^2 - 12) = 0$

$\therefore a^2 = 2$ or 12

$\therefore a = \pm\sqrt{2}$ or $\pm 2\sqrt{3}$ which is a contradiction to $(*)$

Hence the supposition is false, and so $\sqrt{14 - 4\sqrt{6}}$ cannot be written in the form $a + b\sqrt{6}$ with $a, b \in \mathbb{Z}$.

b As $\sqrt{6} = \sqrt{2}\sqrt{3}$ we try $\sqrt{14 - 4\sqrt{6}} = a\sqrt{3} + b\sqrt{2}$ where $a, b \in \mathbb{Z}$.

$\therefore 3a^2 + 2b^2 + 2ab\sqrt{6} = 14 - 4\sqrt{6}$

$\therefore 3a^2 + 2b^2 = 14$ and $ab = -2$

$\therefore 3a^2 + 2\left(\dfrac{-2}{a}\right)^2 - 14 = 0$

$\therefore 3a^2 + \dfrac{8}{a^2} - 14 = 0$

$\therefore 3a^4 - 14a^2 + 8 = 0$

$\therefore (a^2 - 4)(3a^2 - 2) = 0$

$\therefore a^2 = 4$ or $\frac{2}{3}$

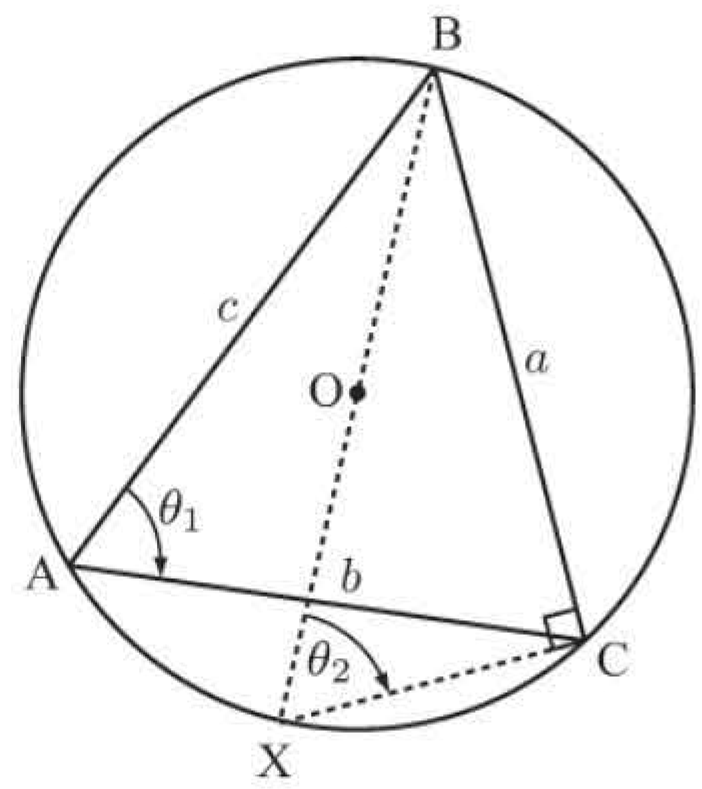
Since $a \in \mathbb{Z}$, $a = \pm 2$ and so $b = -\dfrac{2}{a} = \mp 1$

$\therefore \sqrt{14 - 4\sqrt{6}} = 2\sqrt{3} - \sqrt{2}$ or $-2\sqrt{3} + \sqrt{2}$

We reject the second one as it is negative.

So, $\sqrt{14 - 4\sqrt{6}} = 2\sqrt{3} - \sqrt{2}$.

128 **a**



Draw diameter BOX and join [CX].

Now $\theta_1 = \theta_2$ {angles in the same segment theorem}

and $\widehat{BCX} = 90°$ {angle in a semi-circle theorem}

Now the area is $\frac{1}{2}bc\sin\theta_1 = \frac{1}{2}bc\sin\theta_2$

$$= \tfrac{1}{2}bc\,\frac{a}{\text{BX}}$$
$$= \tfrac{1}{2}bc \times \frac{a}{2r}$$
$$= \frac{abc}{4r}$$

b

$$\sin A = \cos B + \cos C$$
$$\therefore\ \sin(\pi - [B+C]) = \cos B + \cos C$$
$$\therefore\ \sin[B+C] = \cos B + \cos C$$
$$\therefore\ 2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B+C}{2}\right) = 2\cos\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)$$
$$\therefore\ 2\cos\left(\frac{B+C}{2}\right)\left[\sin\left(\frac{B+C}{2}\right) - \cos\left(\frac{B-C}{2}\right)\right] = 0$$
$$\therefore\ \cos\left(\frac{B+C}{2}\right) = 0 \ \ .... \ (1) \quad \text{or} \quad \sin\left(\frac{B+C}{2}\right) = \cos\left(\frac{B-C}{2}\right) \ \ .... \ (2)$$

In (1), $\dfrac{B+C}{2} = \dfrac{\pi}{2} + k\pi$

$\therefore\ B + C = \pi + k2\pi,\ k \in \mathbb{Z}$

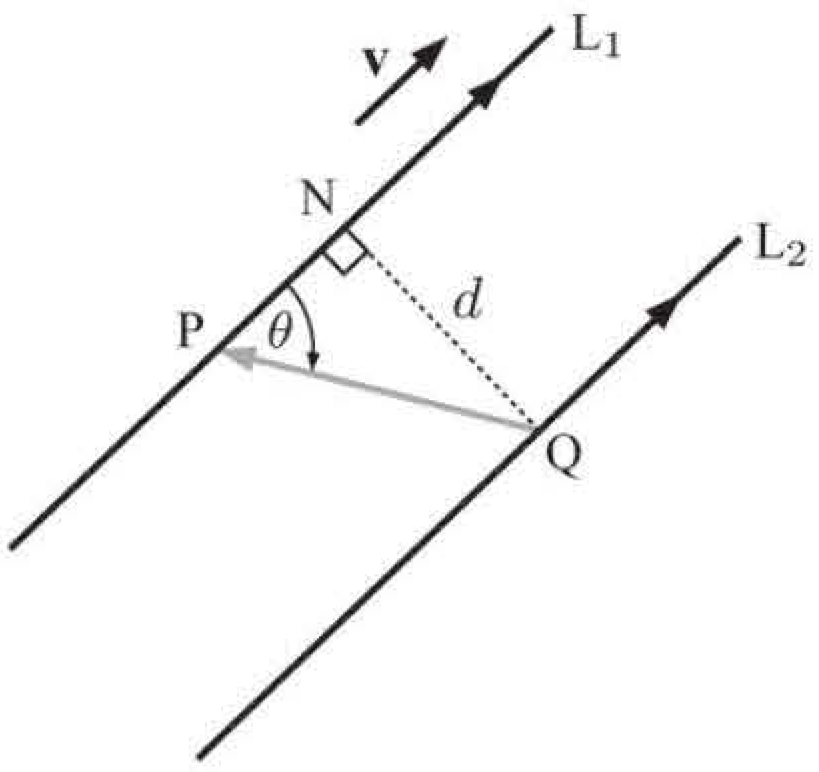
$\therefore\ B + C = \pi$, 3π, $-\pi$, and so on, all of which are impossible.

In (2), $\sin\left(\dfrac{B+C}{2}\right) = \sin\left(\frac{\pi}{2} - \dfrac{B-C}{2}\right)$ $\{\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)\}$

$$\therefore\ \frac{B+C}{2} = \frac{\pi - B + C}{2} \quad \text{or} \quad \frac{B+C}{2} = \pi - \left(\frac{\pi - B + C}{2}\right)$$
$$\therefore\ B + \cancel{C} = \pi - B + \cancel{C} \quad \text{or} \quad \cancel{B} + C = 2\pi - \pi + \cancel{B} - C$$
$$\therefore\ B = \tfrac{\pi}{2} \quad \text{or} \quad C = \tfrac{\pi}{2}$$

$\therefore$ the triangle is right angled at B or C.

129



Let θ be the angle between L_1 and $\overrightarrow{QP}$.

Now in $\triangle$PQN, $\sin\theta = \dfrac{d}{|\overrightarrow{QP}|}$

$$\therefore\ d = |\overrightarrow{QP}|\sin\theta$$
$$= \frac{|\overrightarrow{QP}|\,|\mathbf{v}|\sin\theta}{|\mathbf{v}|}$$
$$= \frac{|\overrightarrow{QP} \times \mathbf{v}|}{|\mathbf{v}|}$$

130 **a** $\dfrac{P}{a-x} + \dfrac{Q}{a+x} = \dfrac{P(a+x) + Q(a-x)}{(a-x)(a+x)} = \dfrac{[P-Q]x + [P+Q]a}{a^2 - x^2}$

So, if $\dfrac{1}{a^2 - x^2} = \dfrac{P}{a-x} + \dfrac{Q}{a+x}$ for all x, then $P - Q = 0$ and $[P+Q]a = 1$

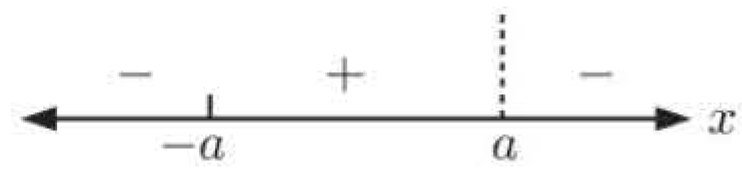
Thus $P = Q$ and so $P = Q = \dfrac{1}{2a}$.

b $\displaystyle\int \frac{1}{a^2-x^2}\,dx = \int \left(\frac{\frac{1}{2a}}{a-x}+\frac{\frac{1}{2a}}{a+x}\right)dx = \frac{1}{2a}\int\left(\frac{1}{a-x}+\frac{1}{a+x}\right)dx$

$$= \frac{1}{2a}\left(-\ln|a-x|+\ln|a+x|\right)+c$$

$$= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|+c, \quad \text{provided } x \neq a \text{ or } -a$$

c $\dfrac{a+x}{a-x}$ has sign diagram



So, $$\frac{d}{dx}\left[\frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|+c\right] = \begin{cases} \dfrac{d}{dx}\left[\dfrac{1}{2a}\ln\left(\dfrac{a+x}{a-x}\right)\right] & \text{if } -a<x<a \\ \dfrac{d}{dx}\left[\dfrac{1}{2a}\ln\left(-\dfrac{a+x}{a-x}\right)\right] & \text{if } x<-a \text{ or } x>a \end{cases}$$

$$= \begin{cases} \dfrac{1}{2a}\dfrac{d}{dx}\left[\ln(a+x)-\ln(a-x)\right] & \text{if } -a<x<a \\ \dfrac{1}{2a}\dfrac{d}{dx}\left[\ln(a+x)-\ln(x-a)\right] & \text{if } x>a \\ \dfrac{1}{2a}\dfrac{d}{dx}\left[\ln(-a-x)-\ln(a-x)\right] & \text{if } x<-a \end{cases}$$

$$= \begin{cases} \dfrac{1}{2a}\left[\dfrac{1}{a+x}-\dfrac{-1}{a-x}\right] & \text{if } -a<x<a \\ \dfrac{1}{2a}\left[\dfrac{1}{a+x}-\dfrac{1}{x-a}\right] & \text{if } x>a \\ \dfrac{1}{2a}\left[\dfrac{-1}{-a-x}-\dfrac{-1}{a-x}\right] & \text{if } x<-a \end{cases}$$

$$= \frac{1}{2a}\left(\frac{1}{a+x}+\frac{1}{a-x}\right)$$

$$= \frac{1}{2a}\times\frac{a-x+a+x}{(a+x)(a-x)}$$

$$= \frac{1}{2a}\times\frac{2a}{a^2-x^2}$$

$$= \frac{1}{a^2-x^2} \quad \text{as required}$$

131 If $(2-x)\mathbf{a}+y\mathbf{b} = y\mathbf{a}+(x-3)\mathbf{b}$

then $(2-x-y)\mathbf{a} = (x-3-y)\mathbf{b}$

which are parallel unless $2-x-y=0$ and $x-3-y=0$

$\therefore\ x+y=2$ and $x-y=3$

Solving simultaneously, $x=\frac{5}{2},\ y=-\frac{1}{2}$.

132 **a** $\theta_1=\theta_2$ and $\phi_1=\phi_2$ {equal alternate angles}

Now $\tan\theta = \dfrac{\text{PN}}{b}$ and $\tan\phi = \dfrac{\text{PM}}{a}$

$\therefore$ PN $= b\tan\theta$ and PM $= a\tan\phi$

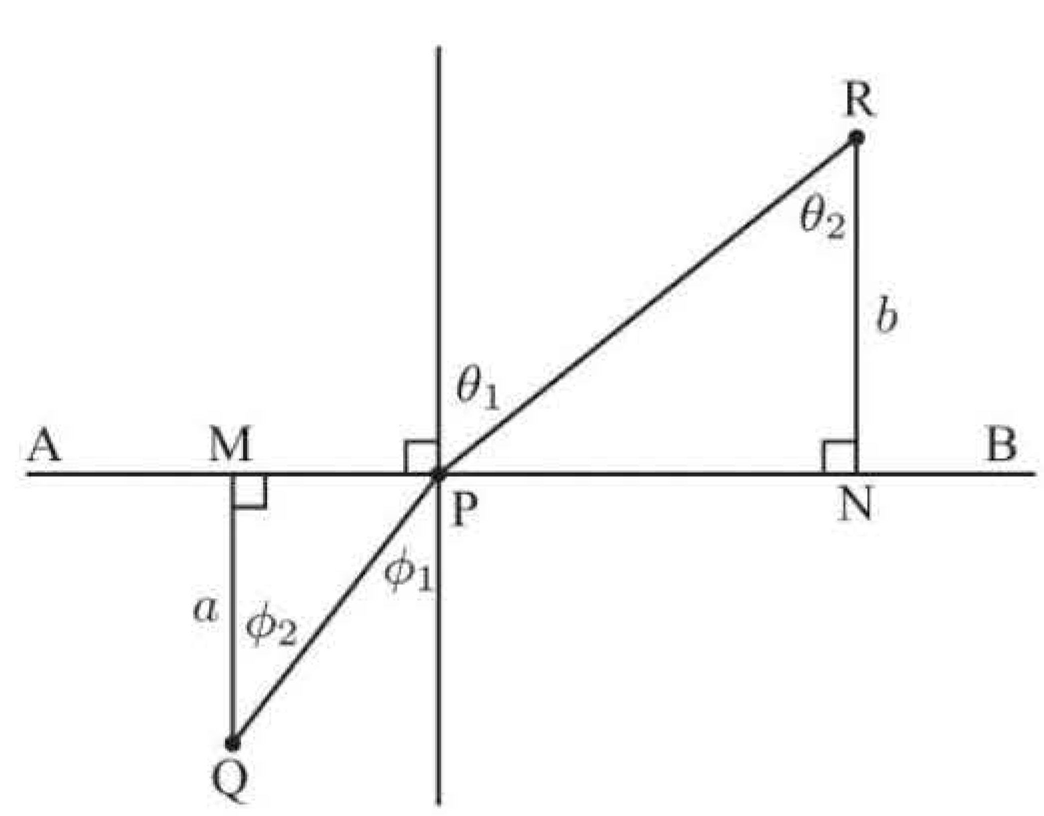
But MN is constant, so we let

$b\tan\theta + a\tan\phi = k$

Differentiating with respect to θ gives

$$b\sec^2\theta + a\sec^2\phi\frac{d\phi}{d\theta} = 0$$

$$\therefore\ \frac{d\phi}{d\theta} = \frac{-b\sec^2\theta}{a\sec^2\phi} = \frac{-b\cos^2\phi}{a\cos^2\theta}$$



b Now $\text{speed} = \dfrac{\text{distance}}{\text{time}}$ $\quad\therefore\quad$ $\text{time} = \dfrac{\text{distance}}{\text{speed}}$

$\therefore$ the total time T to travel from R to P to Q is given by

$$\begin{aligned} T &= \frac{\text{PR}}{v_1} + \frac{\text{PQ}}{v_2} \\ &= \frac{b}{v_1\cos\theta} + \frac{a}{v_2\cos\phi} \qquad \left\{\text{since } \cos\theta = \frac{b}{\text{PR}},\ \cos\phi = \frac{a}{\text{PQ}}\right\} \\ &= \frac{b}{v_1}[\cos\theta]^{-1} + \frac{a}{v_2}[\cos\phi]^{-1} \end{aligned}$$

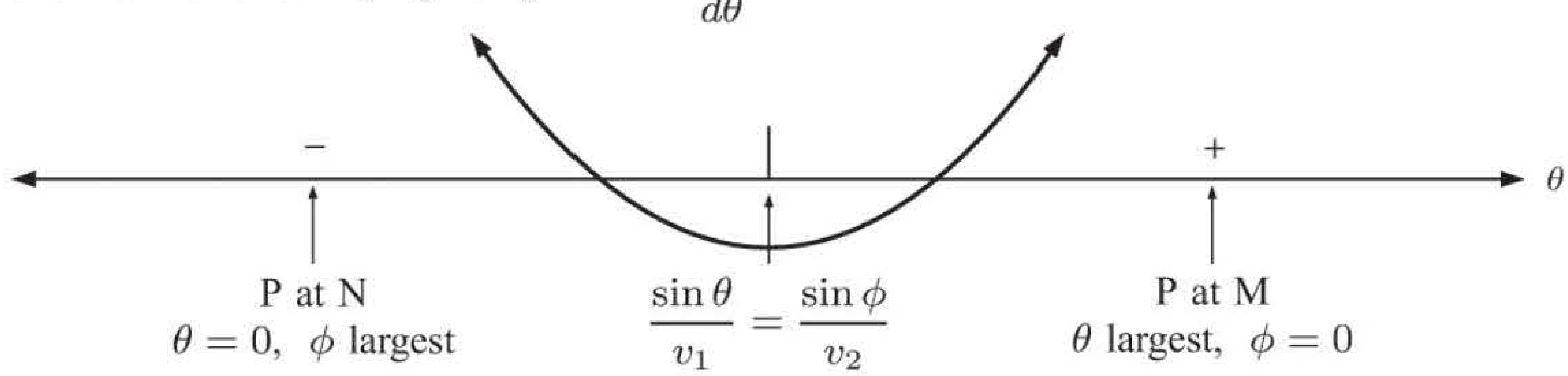
$$\begin{aligned} \therefore\ \frac{dT}{d\theta} &= -\frac{b}{v_1}[\cos\theta]^{-2} \times -\sin\theta - \frac{a}{v_2}[\cos\phi]^{-2} \times -\sin\phi\,\frac{d\phi}{d\theta} \\ &= \frac{b\sin\theta}{v_1\cos^2\theta} + \frac{a\sin\phi}{v_2\cos^2\phi}\left(\frac{-b\cos^2\phi}{a\cos^2\theta}\right) \qquad \{\text{from } \mathbf{a}\} \\ &= \frac{b\sin\theta}{v_1\cos^2\theta} - \frac{b\sin\phi}{v_2\cos^2\theta} \\ &= \frac{b}{\cos^2\theta}\left[\frac{\sin\theta}{v_1} - \frac{\sin\phi}{v_2}\right] \end{aligned}$$

$\therefore\ \dfrac{dT}{d\theta} = 0$ when $\dfrac{\sin\theta}{v_1} = \dfrac{\sin\phi}{v_2}$, or when $\dfrac{\sin\theta}{\sin\phi} = \dfrac{v_1}{v_2}$.

So, there is only one case when $\dfrac{dT}{d\theta} = 0$.

The question is whether we have a minimum or a maximum T in this case.

We construct the following sign diagram for $\dfrac{dT}{d\theta}$:



$\dfrac{dT}{d\theta} = \dfrac{b}{\cos^2\theta}\left[-\dfrac{\sin\phi}{v_2}\right]$ $\qquad\qquad$ $\dfrac{dT}{d\theta} = \dfrac{b}{\cos^2\theta}\left(\dfrac{\sin\theta}{v_1} - 0\right)$

$\therefore\ \dfrac{dT}{d\theta} < 0$ $\qquad\qquad$ $\therefore\ \dfrac{dT}{d\theta} > 0$ as $b, v_1, \sin\theta, \cos\theta > 0$

$\therefore$ the minimum value of T occurs when $\dfrac{dT}{d\theta} = 0$.

133 Let X be the outcome of a spin.

x	1	2	4
$P(X = x)$	$\frac{1}{2}$	$\frac{3}{8}$	$\frac{1}{8}$

Since spins are independent,

$\quad$ P(player spins x_1 and operator spins x_2)

$= P(x_1) \times P(x_2)$

For example, P(player spins 2 and operator spins 4) $= \frac{3}{8} \times \frac{1}{8} = \frac{3}{64}$

Results table:

		Operator		
		1	2	4
Player	1	$\frac{1}{4}$	$\frac{3}{16}$	$\frac{1}{16}$
	2	$\frac{3}{16}$	$\frac{9}{64}$	$\frac{3}{64}$
	4	$\frac{1}{16}$	$\frac{3}{64}$	$\frac{1}{64}$

In these 3 the player wins $\therefore$ gaining $a - k$ dollars

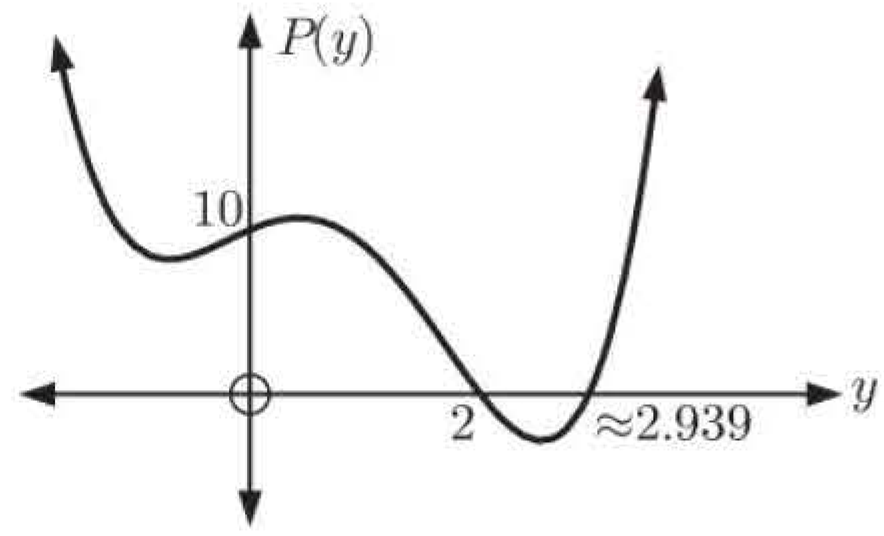
In these 6 the player loses $\therefore$ gaining $-k$ dollars

$$\therefore \text{ expected gain} = (a-k)\left[\tfrac{3}{16}+\tfrac{1}{16}+\tfrac{3}{64}\right]+(-k)\left[\tfrac{1}{4}+\tfrac{3}{16}+\tfrac{1}{16}+\tfrac{9}{64}+\tfrac{3}{64}+\tfrac{1}{64}\right]$$
$$= (a-k)\left(\tfrac{19}{64}\right) - k\left(\tfrac{45}{64}\right)$$
$$= \tfrac{19}{64}a - k \quad \text{which is zero in a 'fair' game}$$
$$\therefore \quad k = \frac{19a}{64}$$

EXERCISE 27B

1 Since $x = 1-y^2$, $\quad (1-y^2)^2 + 3(1-y^2)y + 9 = 0$

$$\therefore \quad 1 - 2y^2 + y^4 + 3y - 3y^3 + 9 = 0$$
$$\therefore \quad y^4 - 3y^3 - 2y^2 + 3y + 10 = 0$$



Using technology there are two real solutions,
$y = 2$ and $y \approx 2.939$
When $y=2$, $x=-3$ and when $y \approx 2.939$, $x \approx -7.64$.
So, $x=-3$ and $x \approx -7.64$ are the solutions.

2 Let n be the number of years after winter 1969 and let u_n be the number of trees at time n.
Each summer, 10% die out and 100 new ones are planted, so $u_{n+1} = 0.9u_n + 100$.
We also know that $u_{11} = 1200$, since there were 1200 trees in 1980.
We hence have a sequence of the form $u_{n+1} = au_n + b$, $n = 1, 2, 3, 4, 5, ...$

Now $\quad u_2 = au_1 + b$

$$u_3 = au_2 + b = a(au_1+b)+b = a^2u_1 + ab + b$$
$$u_4 = au_3 + b = a(a^2u_1+ab+b)+b = a^3u_1 + a^2b + ab + b \quad \text{and so on}$$

This suggests: $\quad u_{n+1} = a^n u_1 + b(1 + a + a^2 + ... + a^{n-1})$

$$\therefore \quad u_{n+1} = a^n u_1 + b\left(\frac{1-a^n}{1-a}\right) \qquad \{\text{sum of a geometric series}\}$$

a In this case $a = 0.9$ and $b = 100$

$$\therefore \quad u_{n+1} = (0.9)^n u_1 + 100\left(\frac{1-(0.9)^n}{1-0.9}\right)$$
$$\therefore \quad u_{n+1} = (0.9)^n u_1 + 1000\,(1-(0.9)^n)$$

But $\quad u_{11} = (0.9)^{10}u_1 + 1000(1-(0.9)^{10}) = 1200$

$$\therefore \quad 0.348\,68u_1 + 651.32 \approx 1200$$
$$\therefore \quad u_1 \approx \frac{548.68}{0.348\,68} \approx 1574 \text{ trees}$$

$\therefore$ there were about 1574 trees at the end of winter in 1970.

b As n gets large,

$$(0.9)^n \to 0$$

and $\quad 1-(0.9)^n \to 1$

$$\therefore \quad u_{n+1} \to 1000$$

This indicates a stable number of trees at 1000.

3 Let $\text{P}(A \text{ hits}) = 2p$ and $\text{P}(B \text{ hits}) = p$

Now $\text{P(at least one hits)} = \frac{1}{2}$, so $\text{P(both miss)} = \frac{1}{2}$

$$\therefore \quad (1-2p)(1-p) = \tfrac{1}{2}$$
$$\therefore \quad 4p^2 - 6p + 1 = 0$$
$$\therefore \quad p = \frac{3 \pm \sqrt{5}}{4}$$
$$\therefore \quad p = \frac{3-\sqrt{5}}{4} \quad \left\{\text{since } \frac{3+\sqrt{5}}{4} > 1\right\}$$
$$\therefore \quad 2p = \frac{3-\sqrt{5}}{2} \approx 0.382$$
$$\therefore \quad \text{P}(A \text{ hits}) \approx 0.382$$

4 P_n is "$\dfrac{1}{a(a+1)} + \dfrac{1}{(a+1)(a+2)} + \dfrac{1}{(a+2)(a+3)} + ... + \dfrac{1}{(a+n-1)(a+n)} = \dfrac{n}{a(a+n)}$" for $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= \dfrac{1}{a(a+1)}$ and RHS $= \dfrac{1}{a(a+1)}$ $\quad\therefore$ P_1 is true.

(2) If P_k is assumed true then

$$\frac{1}{a(a+1)} + \frac{1}{(a+1)(a+2)} + ... + \frac{1}{(a+k-1)(a+k)} = \frac{k}{a(a+k)}$$

Thus $\dfrac{1}{a(a+1)} + \dfrac{1}{(a+1)(a+2)} + ... + \dfrac{1}{(a+k-1)(a+k)} + \dfrac{1}{(a+k)(a+k+1)}$

$$= \frac{k}{a(a+k)} + \frac{1}{(a+k)(a+k+1)} \quad \{\text{using } P_k\}$$

$$= \frac{k}{a(a+k)}\left(\frac{a+k+1}{a+k+1}\right) + \frac{1}{(a+k)(a+k+1)}\left(\frac{a}{a}\right)$$

$$= \frac{ak + k^2 + k + a}{a(a+k)(a+k+1)}$$

$$= \frac{\cancel{(a+k)}(k+1)}{a\cancel{(a+k)}(a+k+1)}$$

$$= \frac{k+1}{a(a+[k+1])}$$

Thus P_{k+1} is true whenever P_k is true.

$\therefore$ since P_1 is true, P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

5 Now $2z + w = i$,

so $6z + 3w = 3i$

Also, $z - 3w = 7 - 10i$

$\therefore$ $7z = 7 - 7i$

$\therefore$ $z = 1 - i$

$\therefore$ $w = i - 2z$

$= i - 2 + 2i$

$= -2 + 3i$

Thus $z + w = -1 + 2i$

6 **Proof:** (By contradiction)

Suppose neither equation has real roots

$\therefore$ $b_1^2 - 4c_1 < 0$ and $b_2^2 - 4c_2 < 0$

$\therefore$ $b_1^2 + b_2^2 < 4c_1 + 4c_2$

$\therefore$ $b_1^2 + b_2^2 < 4(c_1 + c_2)$

$\therefore$ $b_1^2 + b_2^2 < 2b_1b_2$ {given $b_1b_2 = 2(c_1 + c_2)$}

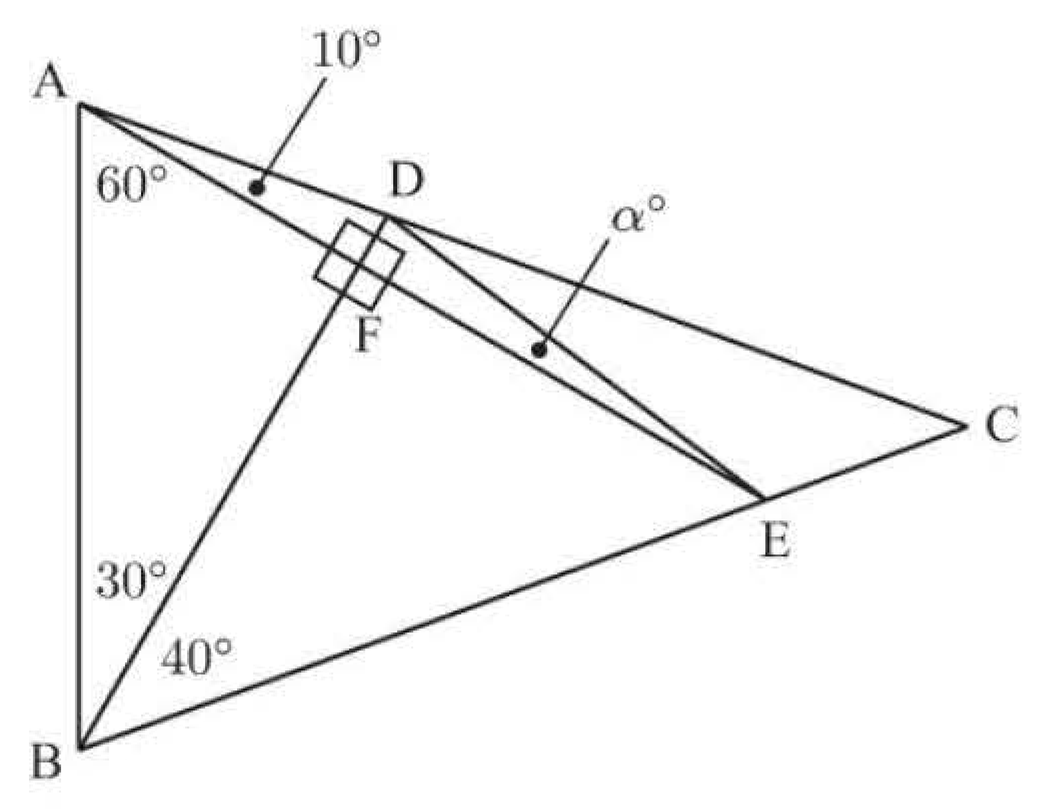
$\therefore$ $b_1^2 - 2b_1b_2 + b_2^2 < 0$

$\therefore$ $(b_1 - b_2)^2 < 0$

which is a contradiction as no perfect square of real numbers can be negative.

Thus the supposition is false and so at least one of the equations has real roots.

7



We notice that $\widehat{\text{AFB}} = 90°$ {angles in a $\triangle$}

$\therefore$ $\tan 30° = \dfrac{\text{AF}}{\text{BF}}$, $\tan 40° = \dfrac{\text{EF}}{\text{BF}}$, $\tan 10° = \dfrac{\text{DF}}{\text{AF}}$

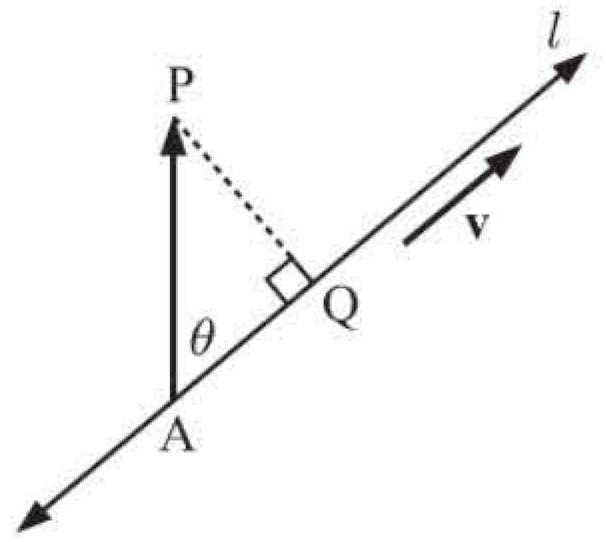
Now $\tan\alpha = \dfrac{\text{DF}}{\text{EF}} = \dfrac{\text{DF}}{\text{AF}} \times \dfrac{\text{AF}}{\text{BF}} \times \dfrac{\text{BF}}{\text{EF}}$

$$= \frac{\tan 10° \times \tan 30°}{\tan 40°}$$

$\therefore$ $\alpha = \arctan\left(\dfrac{\tan 10° \times \tan 30°}{\tan 40°}\right)$

$\therefore$ $\alpha \approx 6.92°$

8 **a**



Let angle PAQ $= \theta$

$$\therefore \sin\theta = \frac{\text{PQ}}{|\overrightarrow{\text{AP}}|}$$

$$\therefore \text{PQ} = |\overrightarrow{\text{AP}}|\sin\theta$$

$$= \frac{|\overrightarrow{\text{AP}}|\,|\mathbf{v}|\sin\theta}{|\mathbf{v}|}$$

$$= \frac{|\overrightarrow{\text{AP}}\times\mathbf{v}|}{|\mathbf{v}|}$$

$\{$as $|\mathbf{a}|\,|\mathbf{b}|\sin\theta = |\mathbf{a}\times\mathbf{b}|\,\}$

b P is at $(2, -1, 3)$.

A is at $(-1, 1, 2)$ and $\mathbf{v} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$

$$\overrightarrow{\text{AP}}\times\mathbf{v} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ 3 & -1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} -2 & 1 \\ -1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 1 \\ 3 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & -2 \\ 3 & -1 \end{vmatrix}\mathbf{k}$$

$$= -\mathbf{i} + 3\mathbf{k}$$

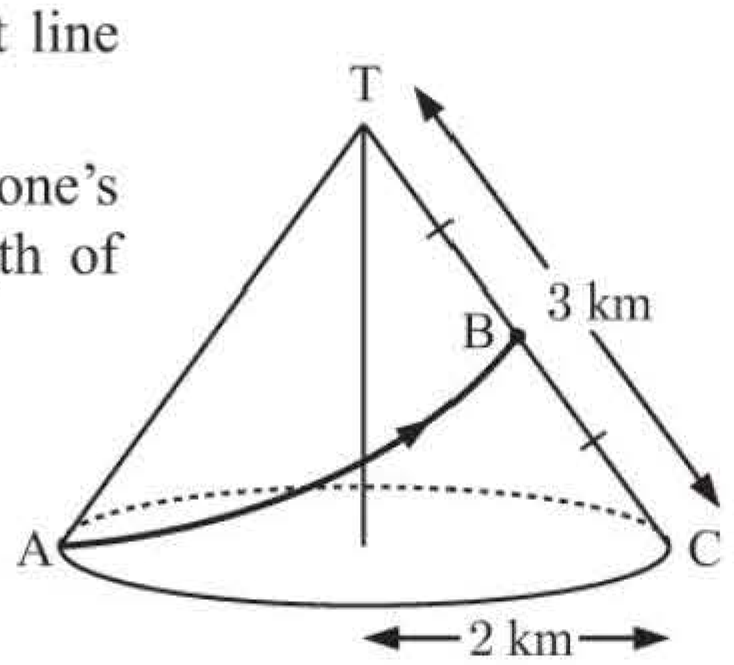
$$\therefore |\overrightarrow{\text{AP}}\times\mathbf{v}| = \sqrt{1+0+9} = \sqrt{10} \text{ units}$$

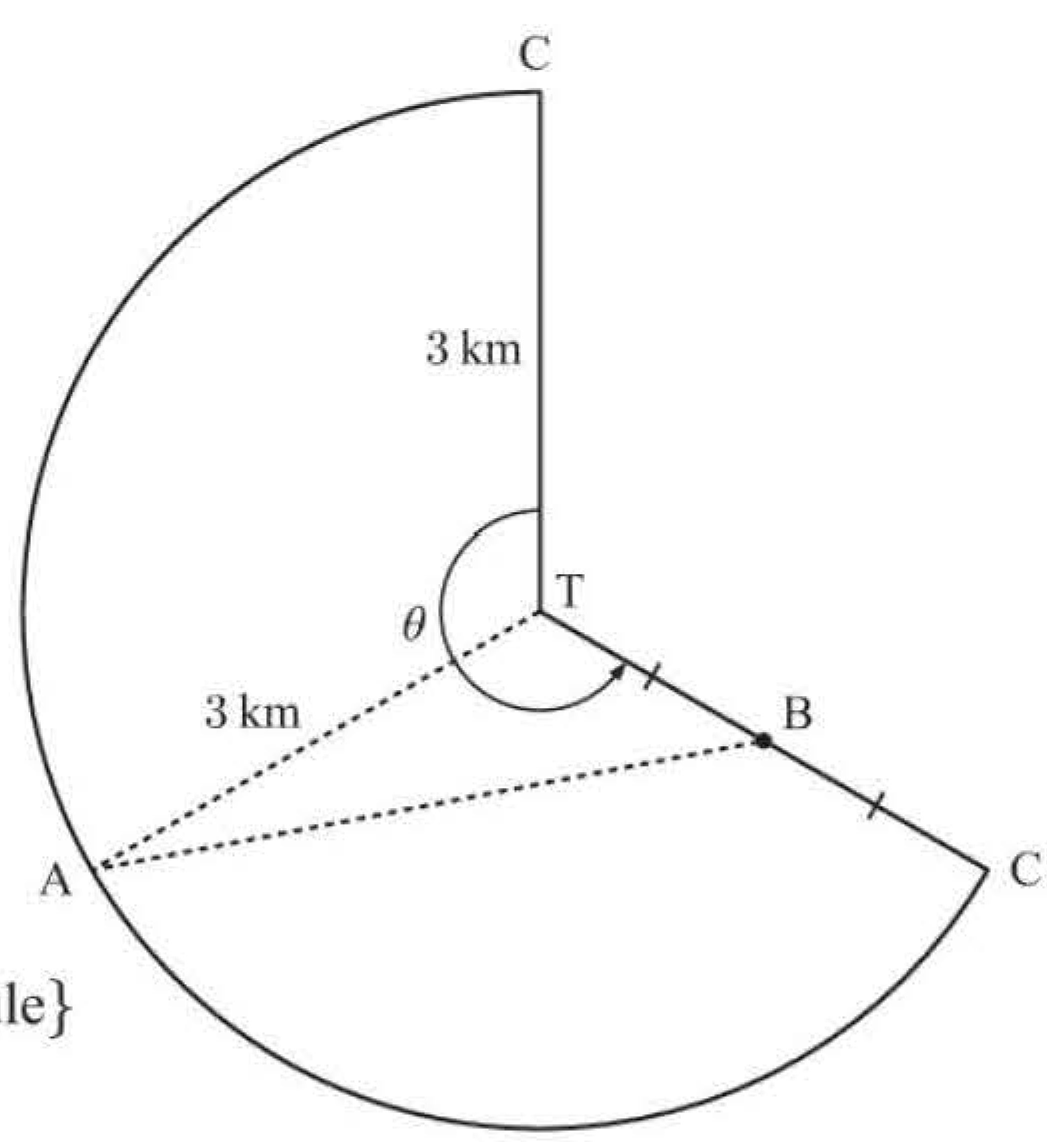
$$\therefore \text{PQ} = \frac{\sqrt{10}}{\sqrt{9+1+1}}$$

$$\approx 0.953 \text{ units}$$

9 Consider a model of the mountain. We cut the model along [CT] and flatten it out. To make AB as short as possible, [AB] is a straight line on a sector of a circle.

The circumference of the cone's base is equal to the arc length of the sector, so





$$2\pi \times 2 = \theta \times 3$$

$$\therefore \theta = \tfrac{4\pi}{3}$$

Thus $\widehat{\text{ATB}} = \frac{1}{2}\theta = \frac{2\pi}{3}$

and $\text{AB}^2 = 3^2 + \left(\frac{3}{2}\right)^2 - 2\times 3\times\frac{3}{2}\cos\left(\frac{2\pi}{3}\right)$ $\{$cosine rule$\}$

$$\therefore \text{AB} = \sqrt{3^2 + (1.5)^2 - 9\cos\left(\tfrac{2\pi}{3}\right)}$$

$$\therefore \text{AB} \approx 3.97 \text{ km}$$

10 $\log_3(x-k) + \log_3(x+2) = 1$

$$\therefore \log_3(x-k)(x+2) = 1$$

$$\therefore (x-k)(x+2) = 3^1 = 3$$

$$\therefore x^2 + [2-k]x - 2k - 3 = 0$$

This quadratic in x has $\Delta = (2-k)^2 - 4(1)(-2k-3)$

$$= 4 - 4k + k^2 + 8k + 12$$

$$= k^2 + 4k + 16$$

$$= (k+2)^2 + 12$$

Since $(k+2)^2 \geqslant 0$, $\Delta \geqslant 0$ for all k

$\therefore$ the original equation has a real solution for all real k.

11 **a** $u_1 = \dfrac{1}{\sin\theta} - \sin\theta = \dfrac{1-\sin^2\theta}{\sin\theta} = \dfrac{\cos^2\theta}{\sin\theta}$

$$u_4 = \frac{1}{\cos\theta} - \cos\theta = \frac{1-\cos^2\theta}{\cos\theta} = \frac{\sin^2\theta}{\cos\theta}$$

$$\therefore \frac{u_2}{u_1} = \frac{\cos\theta}{\left(\dfrac{\cos^2\theta}{\sin\theta}\right)} = \tan\theta, \quad \frac{u_3}{u_2} = \frac{\sin\theta}{\cos\theta} = \tan\theta, \quad \text{and} \quad \frac{u_4}{u_3} = \frac{\left(\dfrac{\sin^2\theta}{\cos\theta}\right)}{\sin\theta} = \tan\theta$$

So, the sequence is geometric with $u_1 = \dfrac{\cos^2\theta}{\sin\theta}$ and $r = \tan\theta$

$\therefore\ u_n = u_1 r^{n-1} = \dfrac{\cos^2\theta}{\sin\theta} \times (\tan\theta)^{n-1} = \dfrac{\cos\theta}{\tan\theta} \times \tan^{n-1}\theta$

$\therefore\ u_n = \cos\theta\tan^{n-2}\theta$

b $u_1 = 1,\ u_2 = \cos^1\theta,\ u_3 = \cos^3\theta,\ u_4 = \cos^7\theta,\ u_5 = \cos^{15}\theta$

We notice that $u_5 = u_4^2\cos\theta,\ u_4 = u_3^2\cos\theta,\ u_3 = u_2^2\cos\theta,\ u_2 = u_1^2\cos\theta,$ suggesting that $u_1 = 1$ and $u_{n+1} = u_n^2\cos\theta$ for all $n \in \mathbb{Z}^+$.

12 If X is the number of seedlings in a selected row, then $X \sim \text{B}(10, \frac{1}{2})$.

P(randomly selected row has *at least* 8 seedlings) $= \text{P}(X = 8, 9, \text{ or } 10)$

$$= \frac{\binom{10}{8} + \binom{10}{9} + \binom{10}{10}}{2^{10}}$$

$= \frac{7}{128}$ {technology}

$\therefore$ P(randomly selected row has *less than* 8 seedlings) $= 1 - \frac{7}{128} = \frac{121}{128}$

$\therefore$ P(*all 10 rows* have *less than* 8 seedlings) $= \left(\frac{121}{128}\right)^{10}$

$\therefore$ P(row with maximum germination contains *at least* 8 seedlings) $= 1 - \left(\frac{121}{128}\right)^{10} \approx 0.430$

13 **a** $L = \int_0^1 \sqrt{1 + (2x)^2}\,dx$

≈ 1.48 units {technology}

b $y = \sin x$, so $\dfrac{dy}{dx} = \cos x$

$\therefore\ L = \int_0^\pi \sqrt{1 + \cos^2 x}\,dx$

≈ 3.82 units {technology}

14 $\dfrac{P(x)}{(x-a)^2} = Q(x) + \dfrac{bx + c}{(x-a)^2}$ {the division process}

$\therefore\ P(x) = Q(x)(x-a)^2 + bx + c$

$\therefore\ P(a) = Q(a) \times 0 + ab + c = ab + c$ .... (1)

Also, $P'(x) = Q'(x)(x-a)^2 + Q(x)2(x-a) + b$

$\therefore\ P'(a) = 0 + 0 + b = b$ .... (2)

So, the remainder is $bx + c = bx + (ab + c) - ab$

$= P'(a)\,x + P(a) - a\,P'(a)$ {using (1) and (2)}

$= P'(a)(x - a) + P(a)$

15 **a** $-8i = 8(-i) = 8\,\text{cis}\left(-\frac{\pi}{2}\right)$

b As $z^3 = -8i$,

$z^3 = 8\ \text{cis}\left(-\frac{\pi}{2} + k2\pi\right),\ k \in \mathbb{Z}$

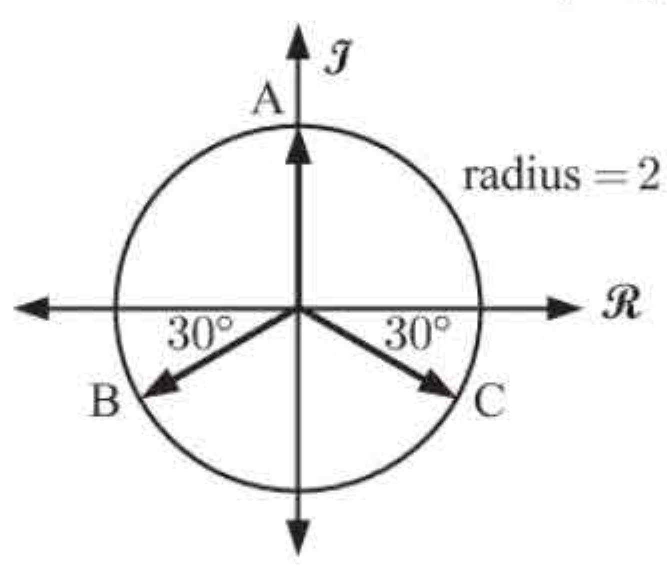
$\therefore\ z = 8^{\frac{1}{3}}\ \text{cis}\left(\dfrac{-\frac{\pi}{2} + k2\pi}{3}\right)$ {De Moivre}

$\therefore\ z = 2\ \text{cis}\left(\dfrac{-\pi + k4\pi}{6}\right)$

$\therefore\ z = 2\ \text{cis}\left(-\frac{5\pi}{6}\right),\ 2\ \text{cis}\left(-\frac{\pi}{6}\right),\ 2\ \text{cis}\left(\frac{\pi}{2}\right)$

$\{k = -1, 0, 1\}$

c



We let A, B, and C be the points corresponding to z_1, z_2, z_3 say.

d If $z_1 = 2\text{ cis}\left(-\frac{\pi}{6}\right)$,

$$\begin{aligned} {z_1}^2 &= 4\text{ cis}\left(-\tfrac{\pi}{3}\right) \quad \text{and} \\ z_2z_3 &= 2\text{ cis}\left(\tfrac{\pi}{2}\right) \times 2\text{ cis}\left(-\tfrac{5\pi}{6}\right) \\ &= 4\text{ cis}\left(\tfrac{\pi}{2} - \tfrac{5\pi}{6}\right) \\ &= 4\text{ cis}\left(-\tfrac{\pi}{3}\right) \\ &= {z_1}^2 \end{aligned}$$

If $z_1 = 2\text{ cis}\left(-\frac{5\pi}{6}\right)$,

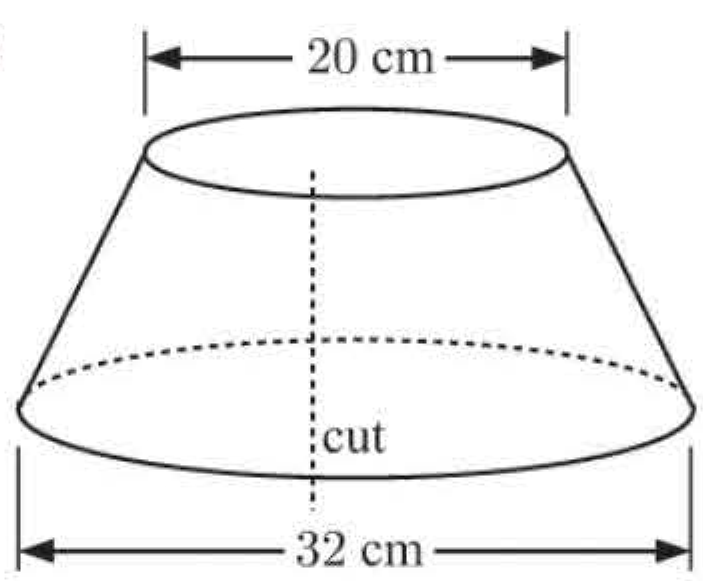
$$\begin{aligned} {z_1}^2 &= 4\text{ cis}\left(-\tfrac{5\pi}{3}\right) \quad \text{and} \\ z_2z_3 &= 2\text{ cis}\left(-\tfrac{\pi}{6}\right) \times 2\text{ cis}\left(\tfrac{\pi}{2}\right) \\ &= 4\text{ cis}\left(\tfrac{\pi}{2} - \tfrac{\pi}{6}\right) \\ &= 4\text{ cis}\,\tfrac{\pi}{3} \\ &= 4\text{ cis}\left(-\tfrac{5\pi}{3}\right) \\ &= {z_1}^2 \end{aligned}$$

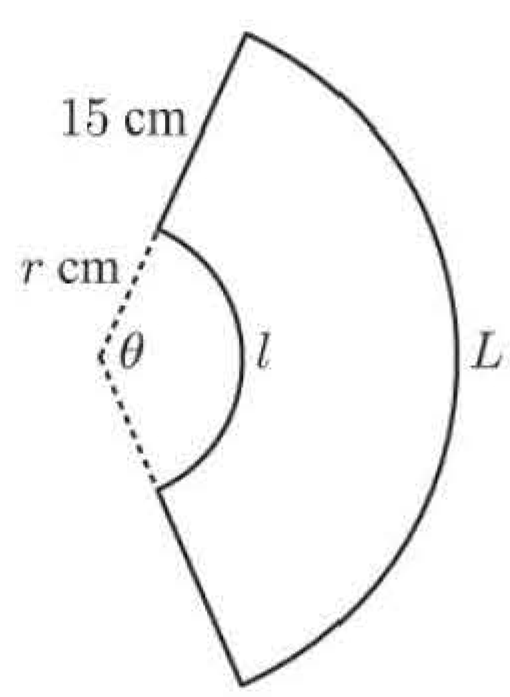
If $z_1 = 2\text{ cis}\left(\frac{\pi}{2}\right)$,

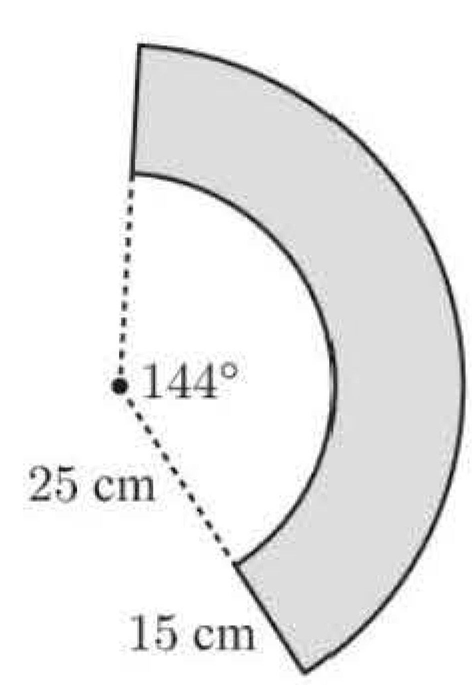
$$\begin{aligned} {z_1}^2 &= (2i)^2 = -4 \quad \text{and} \\ z_2z_3 &= 2\text{ cis}\left(\tfrac{-5\pi}{6}\right) \times 2\text{ cis}\left(\tfrac{-\pi}{6}\right) \\ &= 4\text{ cis}\left(\tfrac{-5\pi}{6} + \tfrac{-\pi}{6}\right) \\ &= 4\text{ cis}(-\pi) \\ &= -4 \\ &= {z_1}^2 \end{aligned}$$

e
$$\begin{aligned} z_1z_2z_3 &= z_1(z_2z_3) \\ &= z_1 \times {z_1}^2 \quad \{\text{from } \mathbf{d}\} \\ &= {z_1}^3 \\ &= -8i \quad \{\text{as } z_1 \text{ is a root of } z^3 = -8i\} \end{aligned}$$

16







$$\begin{aligned} l &= 2\pi(10) = 20\pi \\ L &= 2\pi(16) = 32\pi \end{aligned}$$

But $l = r\theta$ and $L = (r+15)\theta$

$\therefore\ r\theta = 20\pi$ and $(r+15)\theta = 32\pi$

Thus $20\pi + 15\theta = 32\pi$

$\therefore\ 15\theta = 12\pi$

$\therefore\ \theta = \frac{4\pi}{5}^c = 144°$

and $r\left(\frac{4\pi}{5}\right) = 20\pi$

$\therefore\ r = 25$

$\therefore\ r = 25$ and $\theta = 144°$

17 $X \sim \text{N}(90, \sigma^2)$

a
$$\begin{aligned} \text{P}(X < 88) &\approx 0.289\,25 \\ \therefore\ \text{P}\left(\frac{X-90}{\sigma} < \frac{88-90}{\sigma}\right) &\approx 0.289\,25 \\ \therefore\ \text{P}\left(Z < \frac{-2}{\sigma}\right) &\approx 0.289\,25 \\ \therefore\ \frac{-2}{\sigma} &\approx -0.555\,577 \\ \therefore\ \sigma &\approx 3.599\,86 \end{aligned}$$

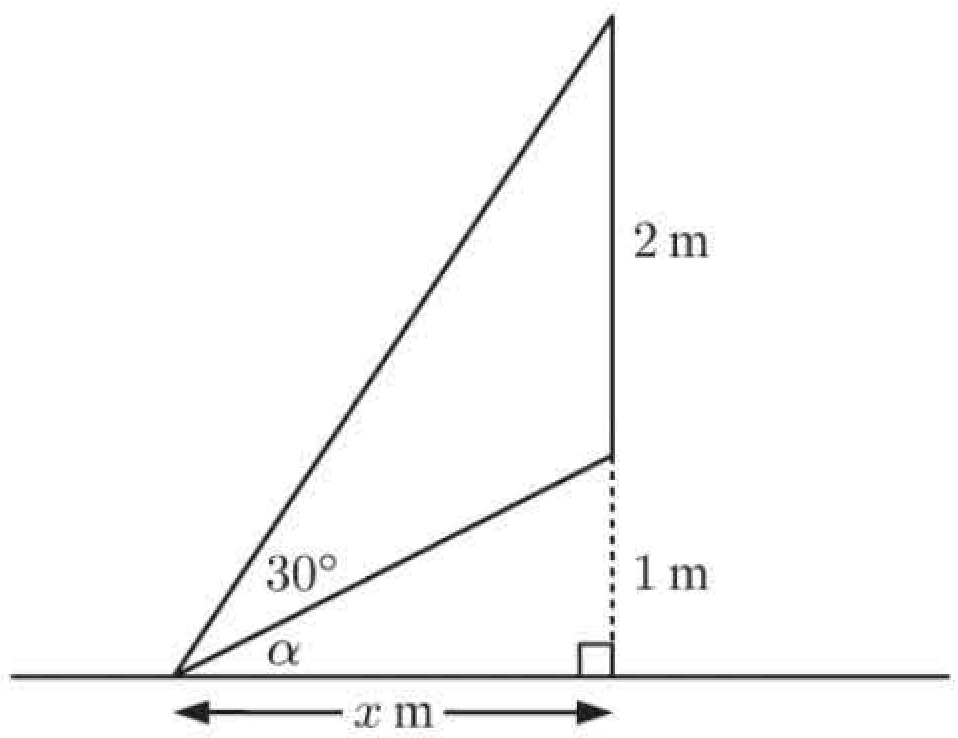
b
$$\begin{aligned} &\text{P}(X < 89 \text{ or } X > 91) \\ &= 1 - \text{P}(89 \leqslant X \leqslant 91) \\ &\approx 1 - 0.219 \\ &\approx 0.781 \end{aligned}$$

18 **a**
$$\begin{aligned}(f\circ g)(x) &= f(g(x))\\ &= f\left(\frac{x+1}{x-2}\right)\\ &= 2\left(\frac{x+1}{x-2}\right)+1\\ &= \frac{2x+2+x-2}{x-2}\\ &= \frac{3x}{x-2}\end{aligned}$$

b $y=\dfrac{x+1}{x-2}$ has inverse $x=\dfrac{y+1}{y-2}$

$$\begin{aligned}\therefore\ xy-2x &= y+1\\ \therefore\ y(x-1) &= 2x+1\\ \therefore\ y &= \frac{2x+1}{x-1}\\ \therefore\ f^{-1}(x) &= \frac{2x+1}{x-1}\end{aligned}$$

19



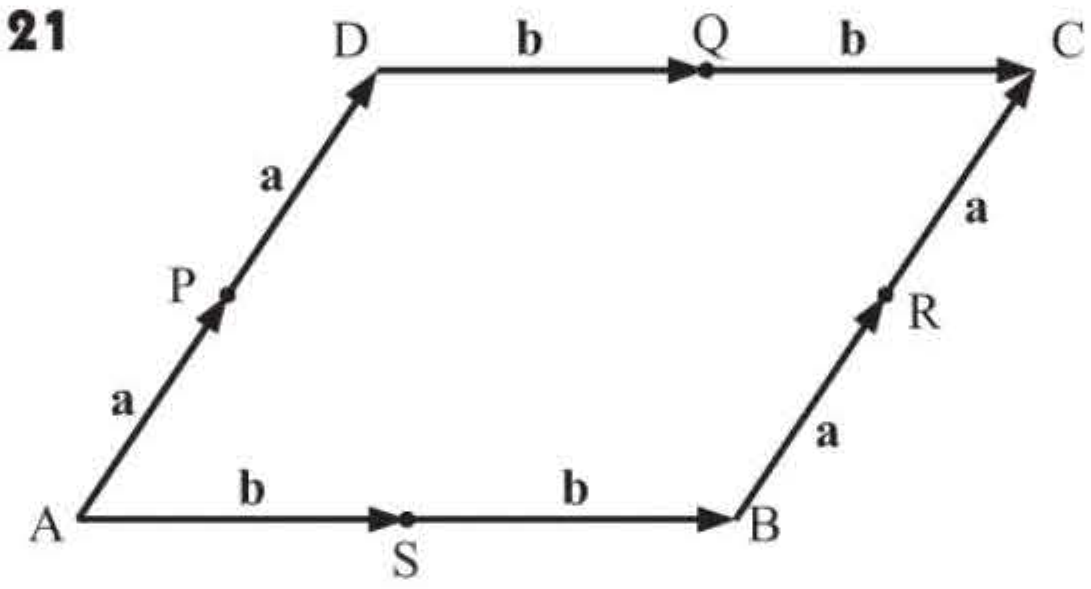
$\tan\alpha=\dfrac{1}{x}$ and $\tan(\alpha+30°)=\dfrac{3}{x}$

$$\begin{aligned}\therefore\ \frac{\tan\alpha+\tan 30°}{1-\tan\alpha\tan 30°} &= \frac{3}{x}\\ \therefore\ \frac{1}{x}+\frac{1}{\sqrt{3}} &= \frac{3}{x}\left(1-\frac{1}{x}\frac{1}{\sqrt{3}}\right)=\frac{3}{x}-\frac{\sqrt{3}}{x^2}\\ \therefore\ \frac{1}{\sqrt{3}}-\frac{2}{x}+\frac{\sqrt{3}}{x^2} &= 0\\ \therefore\ x^2-2\sqrt{3}x+3 &= 0 \quad \{\times\ \sqrt{3}x^2\}\\ \therefore\ (x-\sqrt{3})^2 &= 0\\ \therefore\ x &= \sqrt{3}\approx 1.73\end{aligned}$$

So, she is $\sqrt{3}$ m or 1.73 m from the wall.

20
$$\begin{aligned}\log_a(x+2) &= \log_a x+2\\ \therefore\ \log_a(x+2)-\log_a x &= 2\\ \therefore\ \log_a\left(\frac{x+2}{x}\right) &= 2\\ \therefore\ \frac{x+2}{x} &= a^2\\ \therefore\ 1+\frac{2}{x} &= a^2\\ \therefore\ \frac{2}{x} &= a^2-1\\ \therefore\ x &= \frac{2}{a^2-1},\quad a>1\end{aligned}$$

21



Using the vectors given, $\overrightarrow{PQ}=\mathbf{a}+\mathbf{b}=\overrightarrow{SR}$
$\overrightarrow{QR}=\mathbf{b}-\mathbf{a}=\overrightarrow{PS}$

$\therefore$ [PQ] $\parallel$ [SR] and [QR] $\parallel$ [PS]

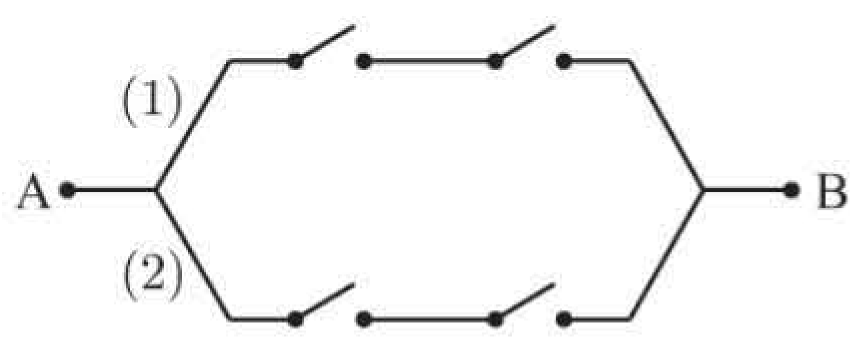
$\therefore$ PQRS is a parallelogram

But
$$\begin{aligned}\overrightarrow{PQ}\bullet\overrightarrow{QR} &= (\mathbf{a}+\mathbf{b})\bullet(\mathbf{b}-\mathbf{a})\\ &= \cancel{\mathbf{a}\bullet\mathbf{b}}-\mathbf{a}\bullet\mathbf{a}+\mathbf{b}\bullet\mathbf{b}-\cancel{\mathbf{b}\bullet\mathbf{a}}\\ &= \mathbf{b}\bullet\mathbf{b}-\mathbf{a}\bullet\mathbf{a} \quad \{\text{as } \mathbf{a}\bullet\mathbf{b}=\mathbf{b}\bullet\mathbf{a}\}\\ &= |\mathbf{b}|^2-|\mathbf{a}|^2 \quad \{\text{as } \mathbf{x}\bullet\mathbf{x}=|\mathbf{x}|^2\}\\ &= 0 \text{ as } |\mathbf{a}|=|\mathbf{b}|\end{aligned}$$

$\therefore$ $\widehat{PQR}$ is a right angle

So, PQRS is a rectangle.

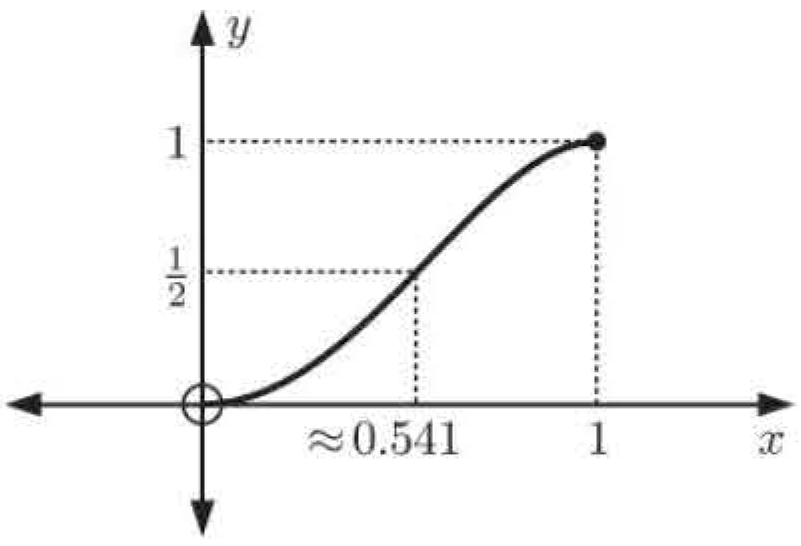
22 **a**



P(current flows)
$= \text{P}((1) \text{ closed } \cup \text{ (2) closed})$
$= \text{P}((1) \text{ closed}) + \text{P}((2) \text{ closed}) - \text{P}((1) \textbf{ and } (2) \text{ closed})$
$= p^2 + p^2 - p^4$
$= 2p^2 - p^4$

b We need to solve $2p^2 - p^4 > \frac{1}{2}$

We graph $y = 2x^2 - x^4$ in $[0, 1]$.



So, for $2p^2 - p^4 > \frac{1}{2}$, $p > 0.541$

$\therefore$ the least value of p is ≈ 0.541

23 $h(x) = x^3 - 6tx^2 + 11t^2x - 6t^3$

a $h(t) = t^3 - 6t^3 + 11t^3 - 6t^3 = 0$

$\therefore$ $x = t$ is a zero of $h(x)$.

b By inspection, $h(x) = (x - t)(x^2 - 5tx + 6t^2)$

$\therefore$ $h(x) = (x - t)(x - 2t)(x - 3t)$

c $y = x^3 + 6x^2$ meets $y = -6 - 11x$ where $x^3 + 6x^2 = -6 - 11x$

$\therefore$ $x^3 + 6x^2 + 11x + 6 = 0$ which is $h(x)$ when $t = -1$

$\therefore$ $(x + 1)(x + 2)(x + 3) = 0$ {using **b**}

$\therefore$ $x = -1, -2$, or -3

So, the graphs meet at $(-1, 5)$, $(-2, 16)$, and $(-3, 27)$.

24 $\sum_{i=1}^{25} x_i = 1650$ and $\sum_{i=1}^{25} x_i^{\,2} = 115\,492$

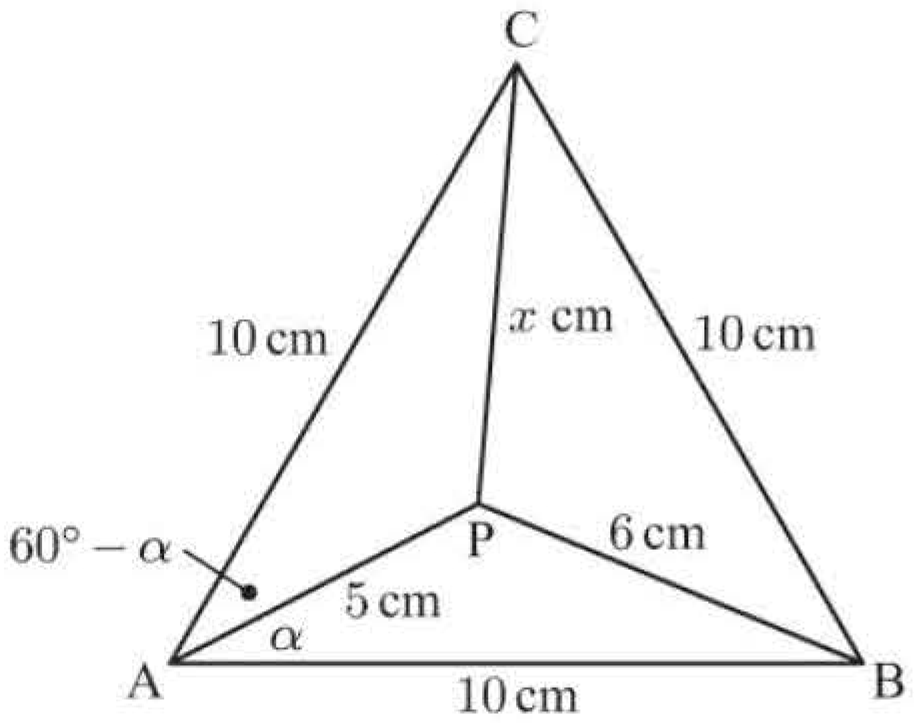
a $$\overline{x} = \frac{\sum_{i=1}^{25} x_i}{25} = \frac{1650}{25} = 66$$

b $$\sum_{i=1}^{25} (x_i - \overline{x})^2 = \sum_{i=1}^{25} x_i^{\,2} - n\overline{x}^2 = 115\,492 - 25 \times 66^2 = 6592$$

Now $$s_n^{\,2} = \frac{\sum_{i=1}^{25} (x_i - \overline{x})^2}{25} = \frac{6592}{25} \approx 264$$

$\therefore$ the variance ≈ 264

25



Using the cosine rule in $\triangle$ABP,

$$\cos\alpha = \frac{5^2 + 10^2 - 6^2}{2(5)(10)} = \frac{89}{100} = 0.89$$

$\therefore$ $\alpha \approx 27.127°$

$\therefore$ $60° - \alpha \approx 32.873°$

So, in $\triangle$APC, $x^2 = 10^2 + 5^2 - 2(10)(5)\cos 32.873°$

$\therefore$ $x^2 \approx 41.0127$

$\therefore$ $x \approx 6.40$

Thus P is about 6.40 cm from C.

26 **a** $$w = \frac{z - 1}{z^* + 1} = \frac{a + bi - 1}{a - bi + 1} = \frac{[a - 1] + bi}{[a + 1] - bi}$$

$$\therefore\ w = \left(\frac{[a - 1] + bi}{[a + 1] - bi}\right)\left(\frac{[a + 1] + bi}{[a + 1] + bi}\right)$$

$$= \frac{\left[a^2 - 1 - b^2\right] + i\,[2ab]}{[a + 1]^2 + b^2}$$

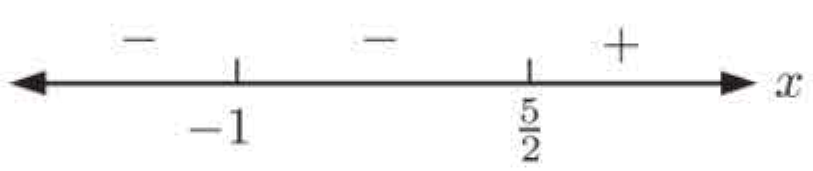
b So, w is purely imaginary when

$a^2 - b^2 - 1 = 0$ and $2ab \neq 0$

$\therefore$ $a^2 - b^2 = 1$ and $ab \neq 0$

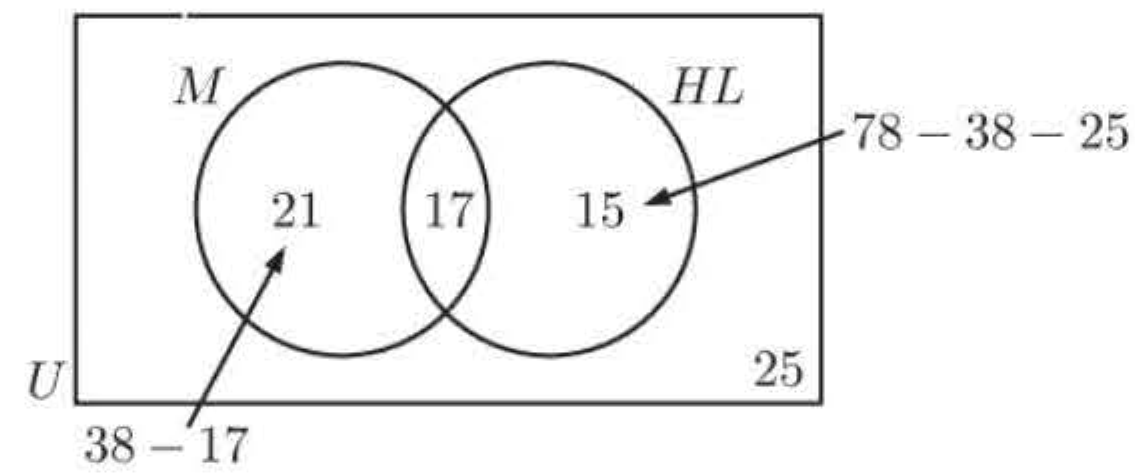
27 $f(x) = 2x^3 - x^2 - 8x - 5$
$= (x+1)(2x^2 - 3x - 5)$
$= (x+1)(2x-5)(x+1)$
$= (x+1)^2(2x-5)$

which has sign diagram:



$\therefore \quad f(x) \geqslant 0$ for $x = -1$ or $x \geqslant \frac{5}{2}$

28



Of the 32 students studying higher level Mathematics, 17 are male.

$\therefore \quad \mathrm{P}(M \mid HL) = \frac{17}{32}$

29 **a** $(2-\sqrt{3})^{n+1} = (2-\sqrt{3})^n(2-\sqrt{3})$
$= (a_n - b_n\sqrt{3})(2-\sqrt{3})$
$= (2a_n + 3b_n) - (a_n + 2b_n)\sqrt{3}$

$\therefore \quad a_{n+1} = 2a_n + 3b_n$ and $b_{n+1} = a_n + 2b_n$

b $(2-\sqrt{3})^1 = 2-\sqrt{3}$ $\quad$ $(2-\sqrt{3})^2 = 7-4\sqrt{3}$ $\quad$ $(2-\sqrt{3})^3 = 26-15\sqrt{3}$

$\therefore \quad a_1 = 2,\ b_1 = 1$ $\quad$ $\therefore \quad a_2 = 7,\ b_2 = 4$ $\quad$ $\therefore \quad a_3 = 26,\ b_3 = 15$

$\therefore \quad a_1^{\,2} - 3b_1^{\,2} = 4 - 3(1) = 1$ $\quad$ $\therefore \quad a_2^{\,2} - 3b_2^{\,2} = 49 - 3(16) = 1$ $\quad$ $\therefore \quad a_3^{\,2} - 3b_3^{\,2} = 676 - 3(225) = 1$

c P_n is: "If $(2-\sqrt{3})^n = a_n - b_n\sqrt{3},\ a_n, b_n \in \mathbb{Z}$, then $a_n^{\,2} - 3b_n^{\,2} = 1$" for $n \in \mathbb{Z}^+$.

d **Proof:** (By the principle of mathematical induction)

(1) If $n = 1$, $a_1^{\,2} - 3b_1^{\,2} = 1$ was shown in **b**. $\quad \therefore \ P_1$ is true.

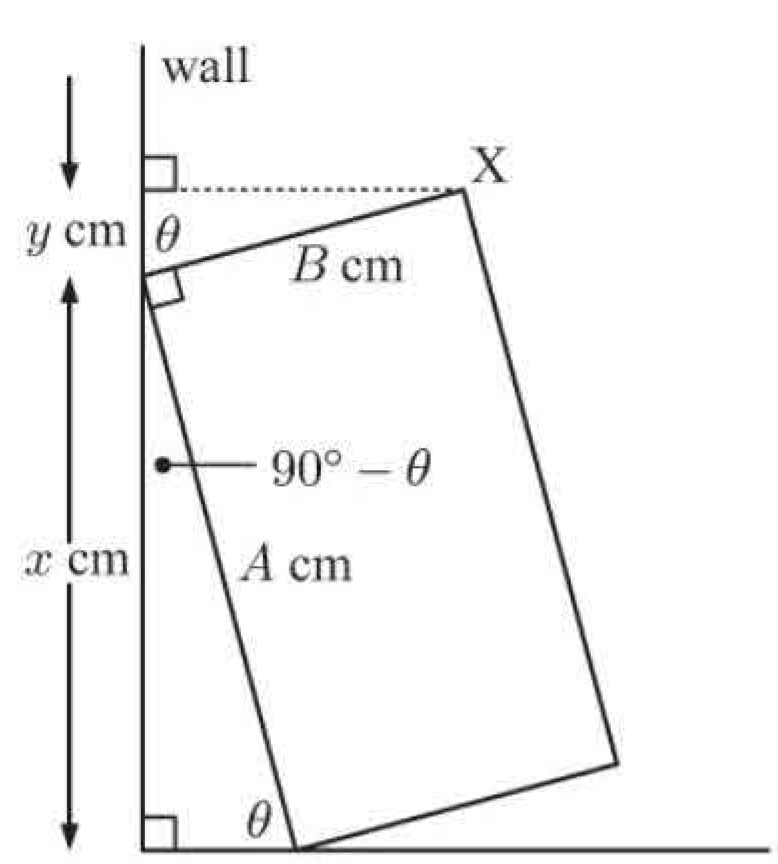
(2) If P_k is assumed true then $a_k^{\,2} - 3b_k^{\,2} = 1$

Now $a_{k+1}^{\,2} - 3b_{k+1}^{\,2} = (2a_k + 3b_k)^2 - 3(a_k + 2b_k)^2$ {from **a**}
$= 4a_k^2 + 12a_kb_k + 9b_k^2 - 3\left(a_k^2 + 4a_kb_k + 4b_k^2\right)$
$= a_k^2 - 3b_k^2$
$= 1$ {using P_k}

Thus P_{k+1} is true whenever P_k is true.

$\therefore$ since P_1 is true, P_n is true for all $n \in \mathbb{Z}$ {Principle of mathematical induction}

30 **a**

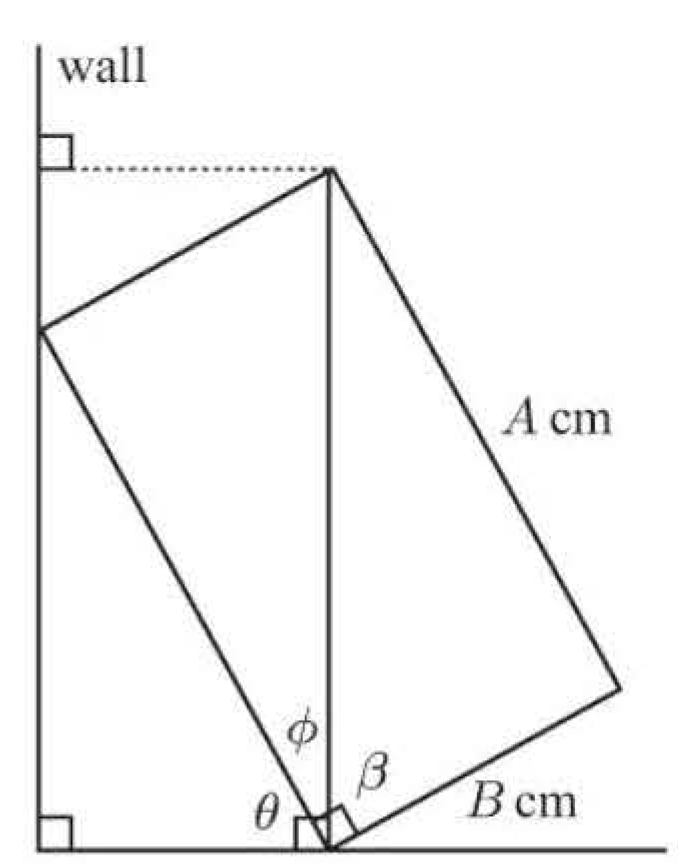


$\sin\theta = \dfrac{x}{A}$ and $\cos\theta = \dfrac{y}{B}$

$\therefore \quad x = A\sin\theta$ and $y = B\cos\theta$

$\therefore \quad H = x + y = A\sin\theta + B\cos\theta$

b



H must be $\leqslant$ diagonal of refrigerator

$\therefore \quad A\sin\theta + B\cos\theta \leqslant \sqrt{A^2 + B^2}$

with equality when H is the diagonal

In this case, $\phi = \frac{\pi}{2} - \theta$

and $\beta = \frac{\pi}{2} - \phi = \theta$

$\therefore \quad \tan\theta = \tan\beta = \dfrac{A}{B}$

31 **a**
$$T_{r+1} = \binom{8}{r}(2x^3)^{8-r}\left(\frac{-1}{2x}\right)^r \quad \text{where } r = 0, 1, 2, 3, ..., 8$$
$$= \binom{8}{r}2^{8-r}x^{24-3r}\left(\tfrac{-1}{2}\right)^r x^{-r}$$
$$= \binom{8}{r}2^{8-r}\left(\tfrac{-1}{2}\right)^r x^{24-4r}$$

We require $24 - 4r = 12$, so $4r = 12$ or $r = 3$

$$\therefore\ T_4 = \binom{8}{3}2^5\left(\tfrac{-1}{2}\right)^3 x^{12}$$

$\therefore$ the coefficient of x^{12} is $-\binom{8}{3}2^2 = -224$

b $(1+2x)^5(2-x)^6 = \left[1+\binom{5}{1}2x+\binom{5}{2}(2x)^2+....\right]\left[2^6-\binom{6}{1}2^5x+\binom{6}{2}2^4x^2-....\right]$

The coefficient of x^2 is $\quad 1\times\binom{6}{2}2^4+\binom{5}{1}2\times(-1)\binom{6}{1}2^5+\binom{5}{2}2^2\times 2^6$
$$= 240 - 1920 + 2560$$
$$= 880$$

c
$$(1+2x-3x^2)^4$$
$$= \left([1+2x]-3x^2\right)^4$$
$$= (1+2x)^4 + 4(1+2x)^3(-3x^2) \underbrace{+\,6(1+2x)^2(-3x^2)^2 + ...}_{\text{all terms have order higher than } x^3}$$
$$= 1 + \binom{4}{1}(2x) + \binom{4}{2}(2x)^2 + \binom{4}{3}(2x)^3 + (2x)^4 - 12x^2\left(1+\binom{3}{1}(2x)+....\right) + ...$$

$\therefore$ the coefficient of x^3 is $4\times 2^3 - 12\times 3\times 2 = -40$

32 Let F be the event of a faulty chip.

$\therefore$ $\text{P}(F) = 0.03$ and $\text{P}(F') = 0.97$

If X is the number which are faulty then $X \sim \text{B}(500, 0.03)$

So, $\text{P}(5 \leqslant X \leqslant 10) = \text{P}(X \leqslant 10) - \text{P}(X \leqslant 4)$ $\quad$ {1% is 5, 2% is 10}
$$\approx 0.114\,787 - 0.000\,754$$
$$\approx 0.114$$

33 **a** As a is real, $p(x)$ has all real coefficients.

By the theorem on real polynomials, $-2+i$ and $-2-i$ are both zeros.

These have sum -4 and product $4+1=5$, so come from the quadratic factor x^2+4x+5.

b Hence, $p(x) = x^3 + (5+4a)x + 5a = (x^2+4x+5)(x+a)$ $\quad$ {comparing constant terms}

$\therefore$ $x^3 + (5+4a)x + 5a = x^3 + [a+4]x^2 + [4a+5]x + 5a$

$\therefore$ $a + 4 = 0$ $\quad$ {equating coefficients of x^2}

$\therefore$ $a = -4$ $\quad$ and the real zero is $-a$ which is 4.

34
$$\binom{n}{3} = 3\binom{n-1}{2} - \binom{n-1}{1}$$
$$\therefore\ \frac{n(n-1)(n-2)}{6} - \frac{3(n-1)(n-2)}{2} + (n-1) = 0$$
$$\therefore\ \frac{n-1}{6}\left(n(n-2) - 9(n-2) + 6\right) = 0$$
$$\therefore\ (n-1)(n^2 - 2n - 9n + 18 + 6) = 0$$
$$\therefore\ (n-1)(n^2 - 11n + 24) = 0$$
$$\therefore\ (n-1)(n-3)(n-8) = 0$$
$$\therefore\ n = 1,\ 3, \text{ or } 8$$

But $n \geqslant 3$, $n-1 \geqslant 2$ and $n-1 \geqslant 1$, so $n \geqslant 3$

$$\therefore\ n = 3 \text{ or } 8$$

35 **a** As the sum of the probabilities must be 1,

$$\tfrac{1}{12} + k + \tfrac{1}{4} + \tfrac{1}{3} = 1$$

$$\therefore\ k + \frac{1+3+4}{12} = 1$$

$$\therefore\ k = 1 - \tfrac{2}{3}$$

$$\therefore\ k = \tfrac{1}{3}$$

b Let X be the number of 2s when the die is rolled 2400 times.

So, $X \sim \mathrm{B}(2400,\ \frac{1}{3})$

mean

$$= np$$
$$= 2400 \times \tfrac{1}{3}$$
$$= 800$$

standard deviation

$$= \sqrt{np(1-p)}$$
$$= \sqrt{2400 \times \tfrac{1}{3} \times \tfrac{2}{3}}$$
$$= \tfrac{\sqrt{4800}}{3}$$
$$= \tfrac{40\sqrt{3}}{3}$$

36 Let the terms of the geometric series be x, rx, and r^2x, where r is the common ratio.

$\therefore\ x + rx + r^2x = 39$ .... (1)

But $x,\ \frac{5}{3}rx,$ and r^2x are arithmetic, so

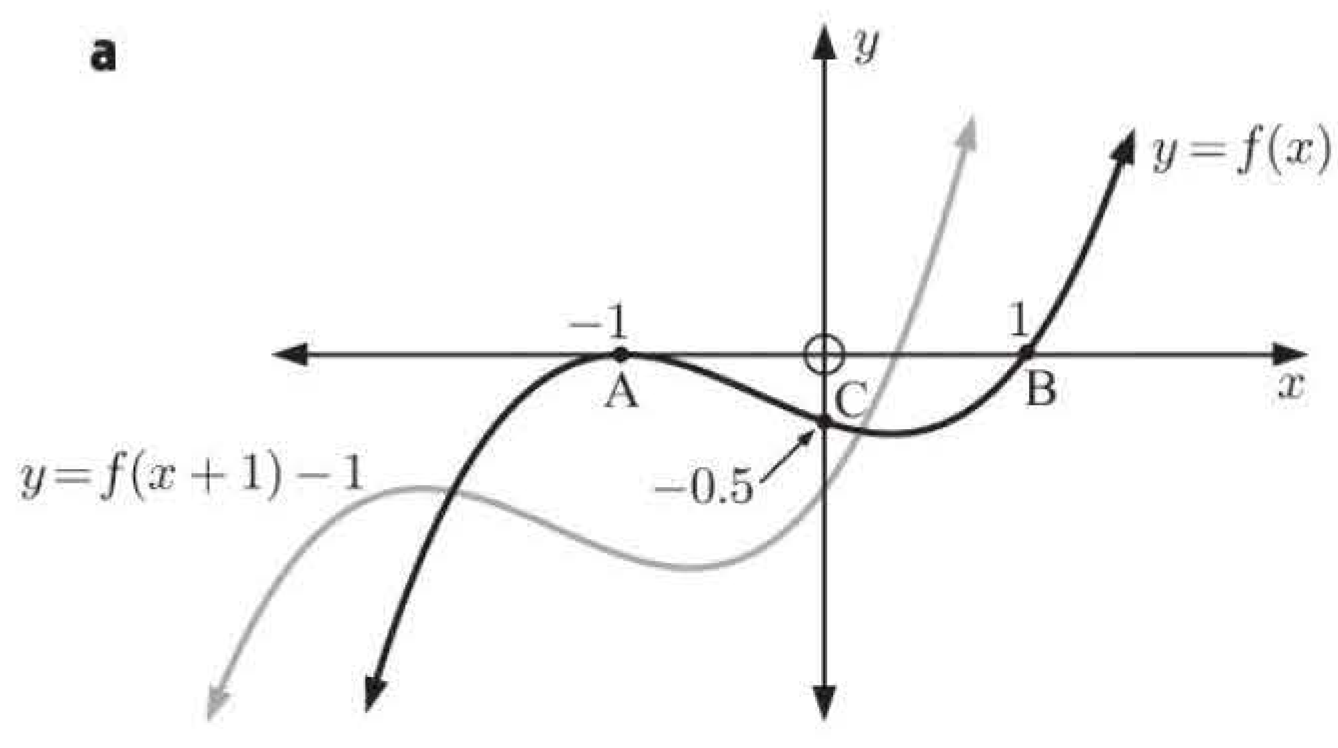
$$\tfrac{5}{3}rx - x = r^2x - \tfrac{5}{3}rx$$
$$\therefore\ \tfrac{10}{3}rx - x = r^2x$$
$$\therefore\ \tfrac{10}{3}r - 1 = r^2$$
$$\therefore\ 3r^2 - 10r + 3 = 0$$
$$\therefore\ (3r-1)(r-3) = 0$$
$$\therefore\ r = \tfrac{1}{3} \text{ or } 3$$

Now from (1), $x = \dfrac{39}{1+r+r^2}$

$\therefore$ when $r = \frac{1}{3}$, $x = \dfrac{39}{1+\frac{1}{3}+\frac{1}{9}} = 27$, and when $r = 3$, $x = \dfrac{39}{1+3+9} = 3$

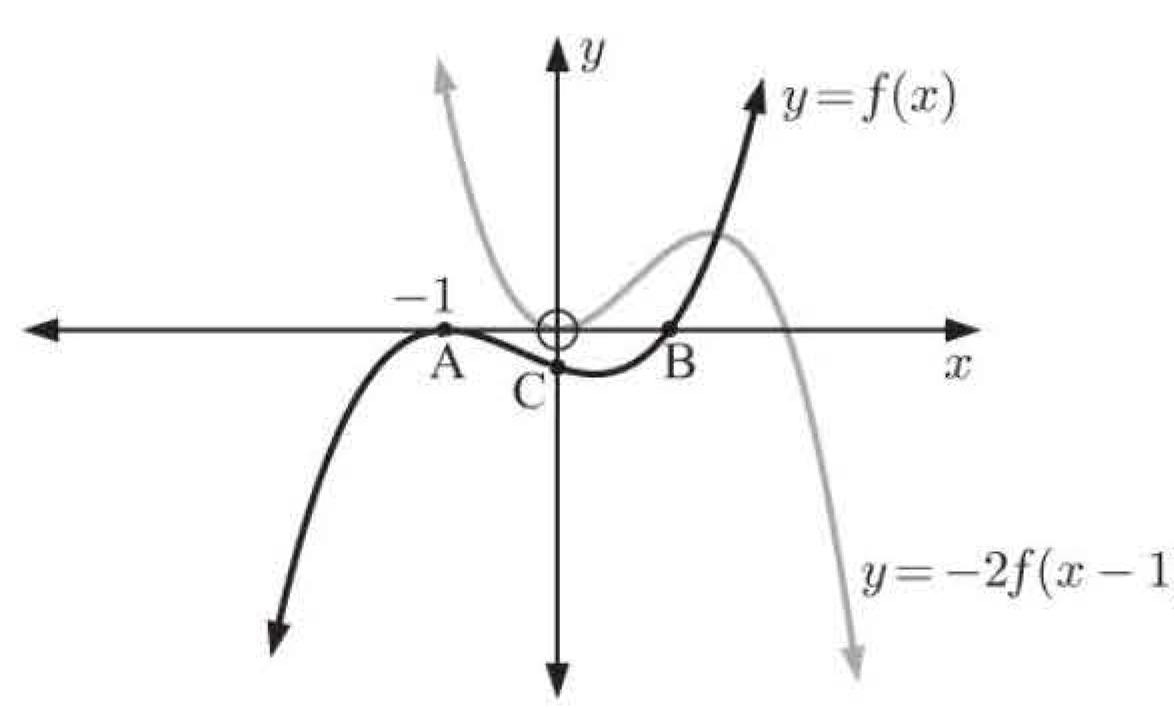
$\therefore$ the smallest possible value of the first term is 3.

37 **a**



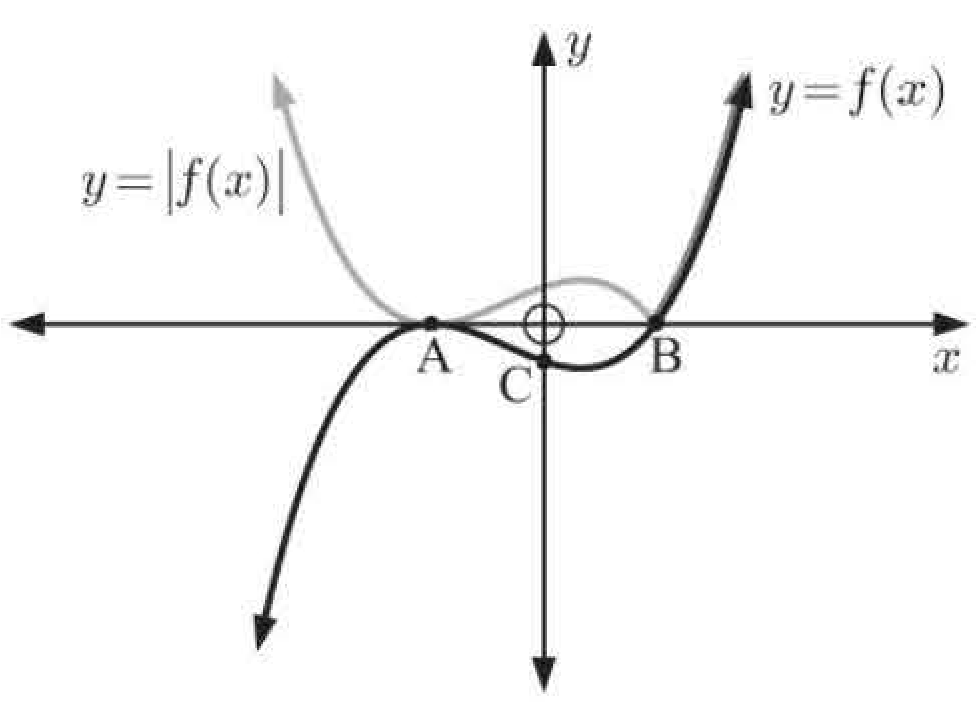
$\mathrm{A}(-1,\ 0) \to \mathrm{A}'(-2,\ -1)$
$\mathrm{B}(1,\ 0) \to \mathrm{B}'(0,\ -1)$
$\mathrm{C}(0,\ -\frac{1}{2}) \to \mathrm{C}'(-1,\ -\frac{3}{2})$

b



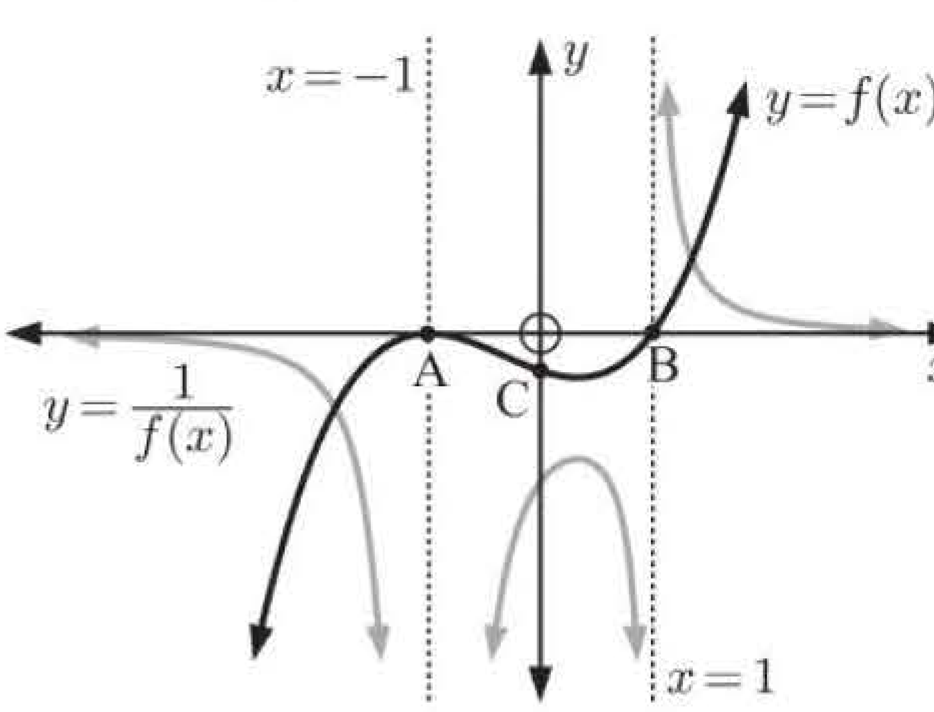
$\mathrm{A}(-1,\ 0) \to \mathrm{A}'(0,\ 0)$
$\mathrm{B}(1,\ 0) \to \mathrm{B}'(2,\ 0)$
$\mathrm{C}(0,\ -\frac{1}{2}) \to \mathrm{C}'(1,\ 1)$

c



$\mathrm{A}(-1,\ 0) \to \mathrm{A}'(-1,\ 0)$
$\mathrm{B}(1,\ 0) \to \mathrm{B}'(1,\ 0)$
$\mathrm{C}(0,\ -\frac{1}{2}) \to \mathrm{C}'(0,\ \frac{1}{2})$

d



$\mathrm{A}(-1,\ 0) \to \text{VA},\ x = -1$
$\mathrm{B}(1,\ 0) \to \text{VA},\ x = 1$
$\mathrm{C}(0,\ -\frac{1}{2}) \to \mathrm{C}'(0,\ -2)$

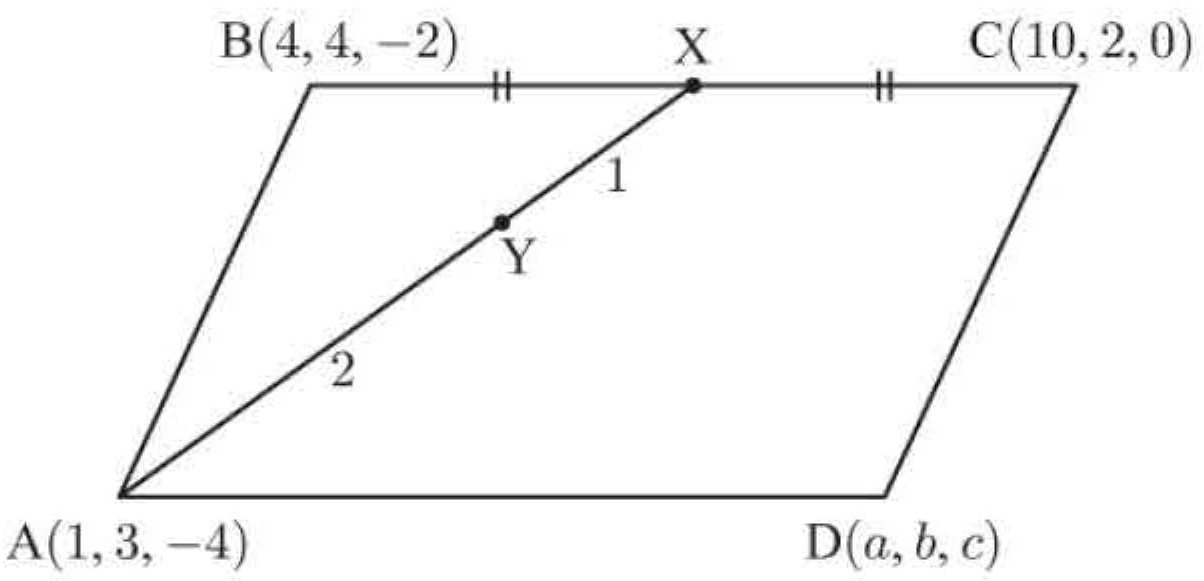
38 **a** For 200 metres of rope, $m = 1.4$

$$\text{P(2 flaws)} = \frac{1.4^2 e^{-1.4}}{2!} \approx 0.242$$

b For 400 metres of rope, $m = 2.8$

$$\begin{aligned}\text{P(at least 2 flaws)} &= 1 - \text{P}(X \leqslant 1)\\ &\approx 1 - 0.231\\ &\approx 0.769\end{aligned}$$

39 **a**



$$\overrightarrow{\text{AD}} = \overrightarrow{\text{BC}}$$

$$\therefore \begin{pmatrix} a-1 \\ b-3 \\ c+4 \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \\ 2 \end{pmatrix}$$

$\therefore$ $a = 7,\ b = 1,\ c = -2$

$\therefore$ D is at $(7, 1, -2)$, X is at $(7, 3, -1)$

Now $\overrightarrow{\text{OY}} = \overrightarrow{\text{OA}} + \frac{2}{3}\overrightarrow{\text{AX}}$

$$= \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} 6 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix}$$

So, Y is at $(5, 3, -2)$.

b $\overrightarrow{\text{BY}} = \begin{pmatrix} 5-4 \\ 3-4 \\ -2+2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\overrightarrow{\text{BD}} = \begin{pmatrix} 7-4 \\ 1-4 \\ -2+2 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix}$

$\therefore$ $\overrightarrow{\text{BD}} = 3\overrightarrow{\text{BY}}$

$\therefore$ [BD] $\parallel$ [BY] and BD = 3(BY)

Since B is common to both [BD] and [BY], B, D, and Y are collinear.

40

$$\frac{dy}{dx} = \cos^2 x = \tfrac{1}{2} + \tfrac{1}{2}\cos 2x$$

$$\therefore\ y = \int \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2x\right) dx$$

$$\therefore\ y = \tfrac{1}{2}x + \tfrac{1}{4}\sin 2x + c$$

But $y(0) = 4$, so $4 = 0 + 0 + c$

Thus $y = \frac{1}{2}x + \frac{1}{4}\sin 2x + 4$

41 $X \sim \text{N}(\mu,\ 2.83^2)$

$$\begin{aligned}\therefore\quad & \text{P}(-4 < X - \mu < 4)\\ &= \text{P}\left(\frac{-4}{2.83} < \frac{X-\mu}{2.83} < \frac{4}{2.83}\right)\\ &= \text{P}(-1.4134 < Z < 1.4134)\\ &\approx 0.842\end{aligned}$$

42

$$\text{Area} = \int_a^{a+2} x^2\, dx = \tfrac{31}{6}$$

$$\therefore\ \left[\frac{x^3}{3}\right]_a^{a+2} = \tfrac{31}{6}$$

$$\therefore\ \frac{(a+2)^3}{3} - \frac{a^3}{3} = \tfrac{31}{6}$$

$$\therefore\ \frac{\cancel{a^3} + 6a^2 + 12a + 8 - \cancel{a^3}}{3} = \tfrac{31}{6}$$

$$\therefore\ 12a^2 + 24a + 16 = 31$$

$$\therefore\ 12a^2 + 24a - 15 = 0$$

$$\therefore\ 4a^2 + 8a - 5 = 0$$

$$\therefore\ (2a-1)(2a+5) = 0$$

$$a = \tfrac{1}{2} \text{ or } -\tfrac{5}{2}$$

But $a > 0$, so $a = \frac{1}{2}$

43 As $P(x)$ is a real polynomial, both $1-2i$ and $1+2i$ are zeros. These have sum 2 and product $1+4=5$.

$\therefore$ $P(x)$ has a quadratic factor x^2-2x+5

$\therefore$ $P(x)=(x^2-2x+5)(x^2+ax+10)$

$\therefore$ the coefficient of x^3 is $a-2$

So, $a-2=0$

$\therefore$ $a=2$

Check:

$$\begin{array}{rrrrr} & & 1 & -2 & 5 \\ & \times & 1 & 2 & 10 \\ \hline & & 10 & -20 & 50 \\ & 2 & -4 & 10 & \\ 1 & -2 & 5 & & \\ \hline 1 & 0 & 11 & -10 & 50 \end{array} \checkmark$$

Thus $P(x)=(x^2-2x+5)(x^2+2x+10)$ where

$x^2+2x+10$ has zeros $\dfrac{-2\pm\sqrt{4-4(1)(10)}}{2}=\dfrac{-2\pm 6i}{2}=-1\pm 3i$

So, the other three zeros are $1+2i$, $-1+3i$, and $-1-3i$.

44 **a** The number of ways of selecting 2 females from n is $\binom{n}{2}$

and the number of ways of selecting 1 male from n is $\binom{n}{1}$.

$\therefore$ there are $\binom{n}{2}\binom{n}{1}=n\binom{n}{2}$ ways of selecting 2 females and 1 male.

b The number of ways of selecting 3 females from n is $\binom{n}{3}$.

c The total number of ways of selecting a committee of 3 from $2n$ people is $\binom{2n}{3}$.

As there are equal numbers of male and female members, exactly half of these committees will have more females than males.

$\therefore$ total number of committees with more females than males $=\frac{1}{2}\binom{2n}{3}$.

$\therefore$ using **a** and **b**, $n\binom{n}{2}+\binom{n}{3}=\frac{1}{2}\binom{2n}{3}$.

d **i** If $n=6$,

P(Mrs Jones is on the committee)

$$=\frac{\binom{1}{1}\binom{5}{1}\binom{6}{1}+\binom{1}{1}\binom{5}{2}\binom{6}{0}}{\frac{1}{2}\binom{12}{3}}$$

$$=\frac{30+10}{110}$$

$$=\tfrac{4}{11}$$

ii P(Mr Jones is on | Mrs Jones is on)

$$=\frac{\text{P(Mr Jones is on} \cap \text{Mrs Jones is on)}}{\text{P(Mrs Jones is on)}}$$

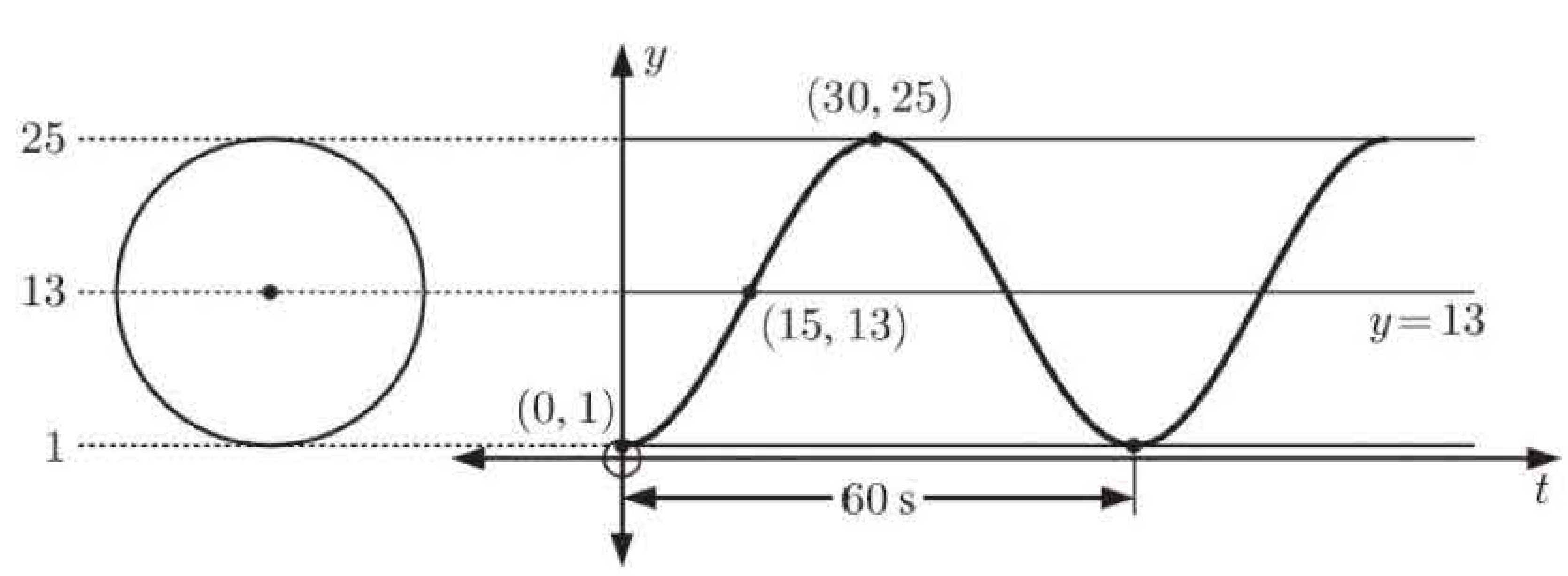
$$=\frac{\text{P(both are on)}}{\text{P(Mrs Jones is on)}}$$

$$=\frac{\binom{2}{2}\binom{5}{1}\binom{5}{0}}{\frac{1}{2}\binom{12}{3}}\div\tfrac{4}{11}$$

$$=\tfrac{5}{110}\times\tfrac{11}{4}$$

$$=\tfrac{1}{8}$$

45 **a**



We model the Ferris wheel using $h(t)=a+b\sin(c(t-d))$.

The amplitude $=b=12$. The period $=\dfrac{2\pi}{c}=60$ $\therefore$ $c=\frac{\pi}{30}$

The basic sine curve has been translated through $\binom{15}{13}$. $\therefore$ $d=15$, $a=13$

Thus $h(t)=12\sin\left(\frac{\pi}{30}(t-15)\right)+13$

Check: $h(0)=12\sin\left(\frac{-\pi}{2}\right)+13=12(-1)+13=1$ ✓

$h(30)=12\sin\left(\frac{\pi}{2}\right)+13=12(1)+13=25$ ✓

b $h(91) = 12\sin\left(\dfrac{\pi \times 76}{30}\right) + 13 \approx 24.9$ m

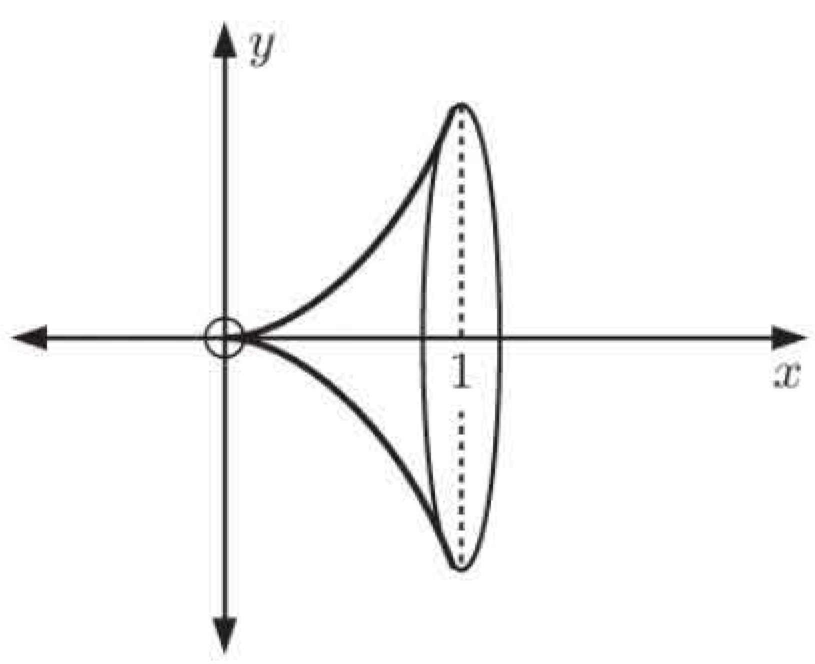
46 Let $x^2 + ax + b$ have zeros α and 2α

$\therefore$ the sum of the zeros $= 3\alpha = \dfrac{-a}{1}$ and the product of the zeros $= 2\alpha^2 = \dfrac{b}{1}$

$$\therefore\ \alpha = \frac{-a}{3} \text{ and } 2\alpha^2 = b$$

$$\therefore\ 2\left(\frac{a^2}{9}\right) = b \text{ and so } 2a^2 = 9b$$

47



$$\begin{aligned}\text{Volume} &= \pi \int_0^1 y^2\,dx \\ &= \pi \int_0^1 x^2 e^{2x^3}\,dx \\ &= \tfrac{\pi}{6} \int_0^1 e^{2x^3}(6x^2)\,dx \\ &= \tfrac{\pi}{6}\left[e^{2x^3}\right]_0^1 \\ &= \tfrac{\pi}{6}(e^2 - 1) \text{ units}^3\end{aligned}$$

48 **a** The line meets the plane where

$$\begin{aligned} 2(-4+3\lambda) + (2+\lambda) - (-1+2\lambda) &= 2 \\ \therefore\ -8 + 6\lambda + 2 + \lambda + 1 - 2\lambda &= 2 \\ \therefore\ 5\lambda - 5 &= 2 \\ \therefore\ \lambda &= \tfrac{7}{5} \end{aligned}$$

$\therefore$ the point of intersection is $\left(\frac{1}{5}, \frac{17}{5}, \frac{9}{5}\right)$.

b L_1 has $\mathbf{v}_1 = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$.

L_2 has $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$.

Since $\mathbf{v}_2$ is **not** a multiple of $\mathbf{v}_1$, L_1 and L_2 are not parallel.

c The lines meet where

$$-4 + 3\lambda = \frac{2 + \lambda - 5}{2} = \frac{-(-1+2\lambda) - 1}{2}$$

$$\therefore\ -4 + 3\lambda = \frac{\lambda - 3}{2} = -\lambda$$

$$\begin{aligned} -8 + 6\lambda &= \lambda - 3 \\ \therefore\ 5\lambda &= 5 \\ \therefore\ \lambda &= 1 \end{aligned}$$

Check:

$$\begin{aligned} \lambda - 3 &= -2\lambda \\ \therefore\ 3\lambda &= 3 \\ \therefore\ \lambda &= 1 \end{aligned}$$

So, they meet at $(-1, 3, 1)$.

d Normal vector $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2$

$$\begin{aligned} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 2 \\ 1 & 2 & -2 \end{vmatrix} \\ &= \mathbf{i}(-6) - \mathbf{j}(-8) + \mathbf{k}(5) \\ &= -6\mathbf{i} + 8\mathbf{j} + 5\mathbf{k} \end{aligned}$$

$\therefore$ the equation of the plane is

$$\begin{aligned} -6x + 8y + 5z &= -6(-1) + 8(3) + 5(1) \\ \therefore\ -6x + 8y + 5z &= 35 \\ \therefore\ 6x - 8y - 5z &= -35 \end{aligned}$$

49 $f(n) = \begin{cases} 0.6e^{-0.6n}, & n \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$

a P(lasts at least a year)

$$\begin{aligned} &= \mathrm{P}(N \geqslant 1) \\ &= 1 - \mathrm{P}(0 \leqslant N < 1) \\ &= 1 - \int_0^1 0.6e^{-0.6n}\,dn \\ &\approx 0.549 \quad \{\text{technology}\} \end{aligned}$$

b P(cell fails within one year)

$$\begin{aligned} &= \mathrm{P}(\text{all components fail in one year}) \\ &\approx (1 - 0.548\,81)^8 \\ &\approx 0.001\,72 \quad \{\text{technology}\} \end{aligned}$$

50 **a** A and B are mutually exclusive if $A \cap B = \varnothing$.

In this case $\mathrm{P}(A \cap B) = 0$, so $x = 0$.

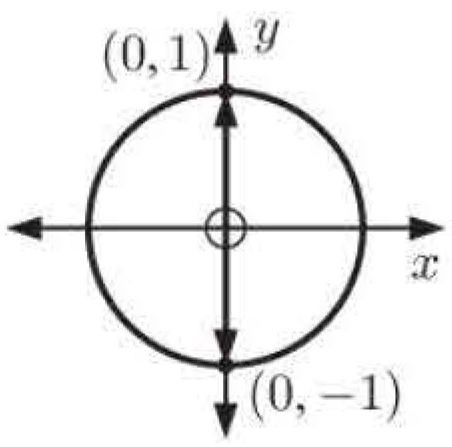
b If A and B are independent, then
$$\begin{aligned} P(A\cap B) &= P(A)P(B) \\ \therefore\ x &= (0.3+x)(0.2+x) \\ \therefore\ x &= 0.06+0.5x+x^2 \\ \therefore\ x^2-0.5x+0.06 &= 0 \\ \therefore\ (x-0.2)(x-0.3) &= 0 \\ \therefore\ x &= 0.2 \text{ or } 0.3 \end{aligned}$$

51 **a** Zeros are $3\pm 2i$ with sum 6 and product $9+4=13$
$\therefore\ f(x)=a(x^2-6x+13),\ a\neq 0$
But $f(0)=-13$, so $a=-1$
$\therefore\ f(x)=-x^2+6x-13$

b $$\begin{aligned} f(x) &= -1(x^2-6x+13) \\ &= -1\left([x-3]^2+13-9\right) \\ &= -(x-3)^2-4 \end{aligned}$$

52 $f(x)=2\tan(3(x-1))+4,\quad x\in[-1,1]$

a $y=\tan nx$ has period $\dfrac{\pi}{n}$, so $f(x)$ has period $=\frac{\pi}{3}$



b Asymptotes are solutions of $\cos(3(x-1))=0$
$$\begin{aligned} \therefore\ 3(x-1) &= \tfrac{\pi}{2}+k\pi \\ \therefore\ x-1 &= \tfrac{\pi}{6}+\frac{k\pi}{3} \\ \therefore\ x &= 1+\tfrac{\pi}{6}+\frac{k\pi}{3} \end{aligned}$$

For the domain $-1\leqslant x\leqslant 1$, asymptotes are $x\approx -0.571$ and $x\approx 0.476$.

c $y=\tan x \ \to\ y=\tan(3x)\ \to\ y=2\tan(3x)\ \to\ y=2\tan(3(x-1))+4$

We have a horizontal stretch with scale factor $\frac{1}{3}$, followed by a vertical stretch with scale factor 2, followed by a translation through $\binom{1}{4}$.

d The domain is $x\in[-1,1]$ but $x\not\approx -0.571$ or 0.476.
The range is $y\in\mathbb{R}$.

53 $v(t)=\cos\left(\frac{1}{3}t\right)\ \text{cm s}^{-1}$

$\therefore$ the total distance travelled in the first 10π seconds $\displaystyle =\int_0^{10\pi}\left|\cos(\tfrac{1}{3}t)\right|\,dt$
≈ 20.6 cm

54 **a** If there are no restrictions, there are $\binom{16}{6}=8008$ possible choices.

b $\binom{7}{2}\binom{6}{2}\binom{3}{2}+\binom{7}{2}\binom{6}{3}\binom{3}{1}+\binom{7}{2}\binom{6}{4}\binom{3}{0}+\binom{7}{3}\binom{6}{2}\binom{3}{1}+\binom{7}{3}\binom{6}{3}\binom{3}{0}+\binom{7}{4}\binom{6}{2}\binom{3}{0}$
$=5320$ possible choices

c $\binom{1}{1}\binom{3}{1}\binom{12}{4}+\binom{1}{1}\binom{3}{2}\binom{12}{3}+\binom{1}{1}\binom{3}{3}\binom{12}{2}=2211$ possible choices

55 **a** Substituting $x=-2t+2,\ y=t$, and $z=3t+1$ into $2x+y+z$, we get:
$$\begin{aligned} &2(-2t+2)+t+3t+1 \\ &=-4t+4+t+3t+1 \\ &=5\ \checkmark \end{aligned}$$
$\therefore$ the line lies in the plane.

b If $x+ky+z=3$ contains L_1 then
$$\begin{aligned} (-2t+2)+k(t)+(3t+1) &= 3 \\ \therefore\ -2t+kt+3t &= 3-2-1 \\ \therefore\ t(k+1) &= 0 \\ \therefore\ k &= -1 \end{aligned}$$
as t is not necessarily equal to 0.

c From **a** and **b**, both $2x+y+z=5$ and $x-y+z=3$ contain L_1.
The solution of the system of 3 equations is where L_1 meets the third plane.

$$\therefore\ 2(-2t+2)+p(t)+2(3t+1)=q$$
$$\therefore\ -4t+4+pt+6t+2=q$$
$$\therefore\ t(p+2)=q-6$$

Now t can be any real number. So, we have infinitely many solutions when $p+2=0$ and $q-6=0$.
$\therefore\ p=-2,\ q=6$

d With $p=-2$, $q=6$, the system has augmented matrix

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 1 & -1 & 1 & 3 \\ 2 & -2 & 2 & 6 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 1 & -1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_3 \to R_3 - 2R_2$$

The row of all zeros and the fact that the first two rows are not multiples indicates there are infinitely many solutions.

56 $\dfrac{u_1}{1-r}=49$ and $u_1r=10$

$$\therefore\ \frac{10}{r}=49(1-r)$$
$$\therefore\ 10=49r-49r^2$$
$$\therefore\ 49r^2-49r+10=0$$
$$\therefore\ (7r-2)(7r-5)=0$$
$$\therefore\ r=\tfrac{2}{7}\text{ or }\tfrac{5}{7}$$

When $r=\frac{2}{7}$, $u_1=35$.

When $r=\frac{5}{7}$, $u_1=14$.

Thus $S_3=\dfrac{35\left(1-\left(\frac{2}{7}\right)^3\right)}{1-\frac{2}{7}}=47\frac{6}{7}$

or $S_3=\dfrac{14\left(1-\left(\frac{5}{7}\right)^3\right)}{1-\frac{5}{7}}=31\frac{1}{7}$

57 The average number of amoebas in 10 mL of water, $m=4$. $\therefore\ X\sim \text{Po}(4)$

a $\text{P}(X\leqslant 5)\approx 0.785$

b If Y is the number of days where no more than 5 amoebas are collected then $Y\sim \text{B}(20,\ 0.785\,13)$.

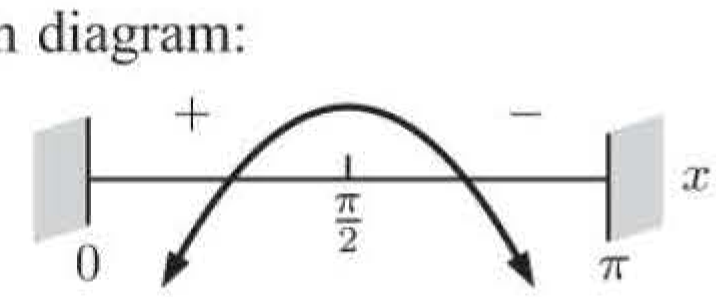
$$\text{P}(Y>10)=1-\text{P}(Y\leqslant 10)$$
$$\approx 1-0.004\,52$$
$$\approx 0.995$$

58 $f(x)=e^{\sin^2 x},\ x\in[0,\ \pi]$

a $f'(x)=e^{\sin^2 x}\times 2\sin x\cos x$
$\quad =e^{\sin^2 x}\sin 2x$

which is 0 when $\sin 2x=0$
$$\therefore\ 2x=0+k\pi$$
$$\therefore\ x=0+\frac{k\pi}{2}$$
$$\therefore\ x=0,\ \tfrac{\pi}{2},\ \pi$$

Sign diagram:



$\therefore\ f(x)$ has maximum value when $x=\frac{\pi}{2}$, and this value is e.

b $f''(x)=e^{\sin^2 x}\times\sin 2x\sin 2x+e^{\sin^2 x}2\cos 2x$
$\quad =e^{\sin^2 x}(\sin^2 2x+2\cos 2x)$

c $\sin^2 2x+2\cos 2x=0$ needs to be solved, as $f''(x)=0$ for points of inflection.
$\sin^2 2x+2\cos 2x=0$
$$\therefore\ x\approx 0.999\ \text{ or }\ 2.14\quad \{\text{technology}\}$$
$\therefore$ the points of inflection are $(0.999,\ 2.03)$ and $(2.14,\ 2.03)$.

59 $\dfrac{P(x)}{x(2x-3)}=Q(x)+\dfrac{ax+b}{x(2x-3)}$ {division process}

$\therefore\ P(x)=Q(x)\,[x(2x-3)]+ax+b$

a If $Q(x)=ax+b$, $\quad P(x)=(ax+b)[x(2x-3)]+ax+b$
$$\therefore\ P(x)=(ax+b)(2x^2-3x+1)$$

b Factorising the quadratic, $P(x) = (ax+b)(2x-1)(x-1)$

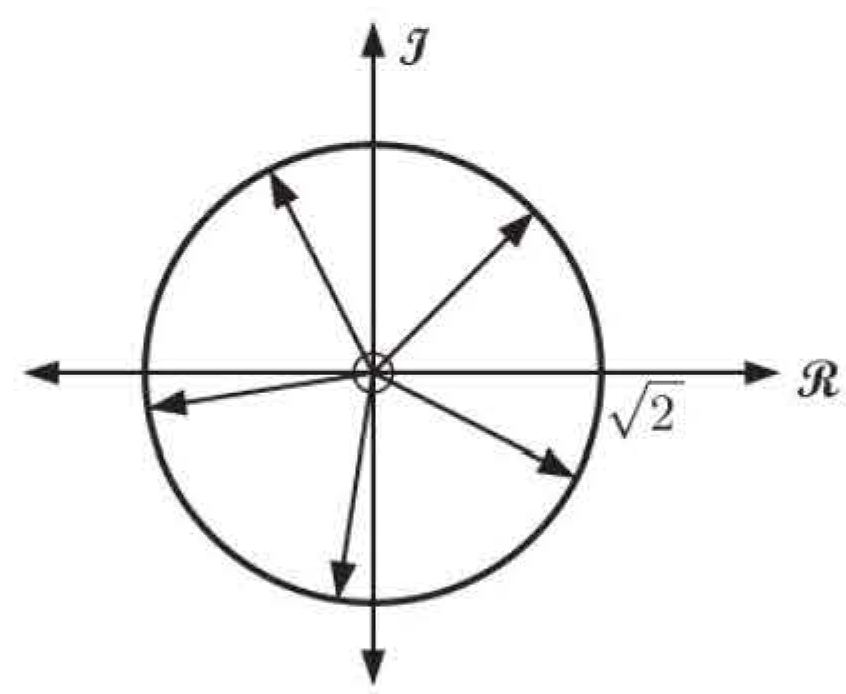
c Now $P(0) = 7$ so $b(-1)(-1) = 7$ $\therefore$ $b = 7$
and $P(2) = 39$ so $(2a+7)(3)(1) = 39$ $\therefore$ $2a+7 = 13$ and so $a = 3$
Thus $P(x) = (3x+7)(2x^2-3x+1)$

60
$$\int_0^a \frac{x}{x^2+1}\,dx = 3$$
$$\therefore\ \tfrac{1}{2}\int_0^a \frac{2x}{x^2+1}\,dx = 3$$
$$\therefore\ \left[\ln|x^2+1|\right]_0^a = 6$$
$$\therefore\ \ln|a^2+1| - \ln 1 = 6$$
$$\therefore\ \ln(a^2+1) = 6 \qquad \{a^2+1>0\}$$
$$\therefore\ a^2+1 = e^6$$
$$\therefore\ a^2 = e^6 - 1$$
$\therefore$ since $a > 0$, $a = \sqrt{e^6-1}$

61 $|1+i| = \sqrt{2}$

$\therefore$ the five fifth roots of $a+bi$ are equally spaced around a circle centre O with radius $\sqrt{2}$.

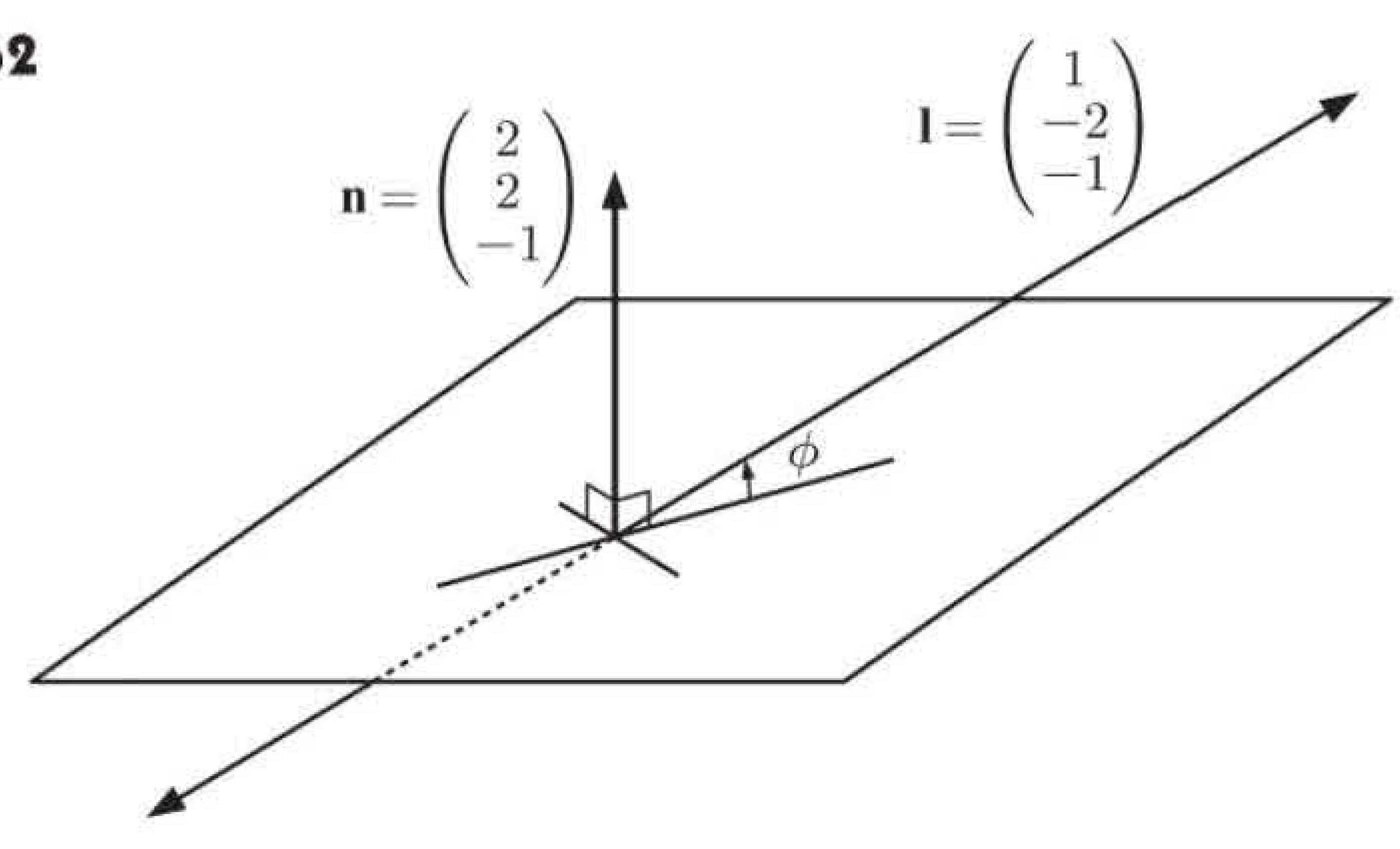
Now $\arg(1+i) = \frac{\pi}{4}$



$$\begin{aligned}\therefore\ \text{the other arguments} &= \tfrac{\pi}{4} + k\,\tfrac{2\pi}{5},\quad k\in\mathbb{Z}\\ &= \tfrac{5\pi}{20} + \tfrac{k8\pi}{20},\quad k\in\mathbb{Z}\\ &= -\tfrac{19\pi}{20}, -\tfrac{11\pi}{20}, -\tfrac{3\pi}{20}, \tfrac{13\pi}{20}\\ &\quad \{\text{when } k = -3, -2, -1, 1\}\end{aligned}$$

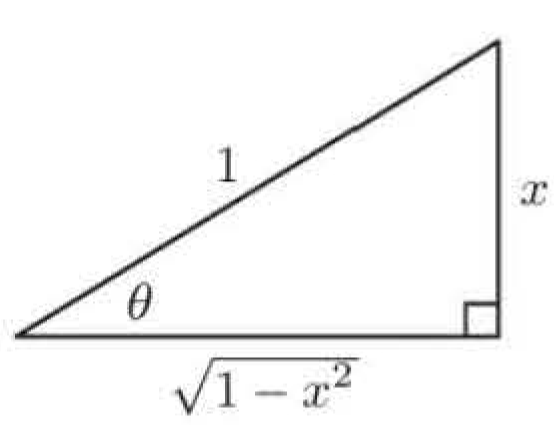
$\therefore$ the other roots are $\sqrt{2}\operatorname{cis}\left(-\frac{19\pi}{20}\right)$, $\sqrt{2}\operatorname{cis}\left(-\frac{11\pi}{20}\right)$, $\sqrt{2}\operatorname{cis}\left(-\frac{3\pi}{20}\right)$, and $\sqrt{2}\operatorname{cis}\left(\frac{13\pi}{20}\right)$.

62



$$\begin{aligned}\phi &= \arcsin\left(\frac{|\mathbf{n}\bullet\mathbf{l}|}{|\mathbf{n}|\,|\mathbf{l}|}\right)\\ &= \arcsin\left(\frac{|2-4+1|}{\sqrt{4+4+1}\sqrt{1+4+1}}\right)\\ &= \arcsin\left(\frac{1}{3\sqrt{6}}\right)\\ &\approx 7.82^\circ\end{aligned}$$

63 **a** Let $\theta = \arcsin x$
$$\begin{aligned}&\therefore\ x = \sin\theta\\ &\therefore\ \sin(2\arcsin x)\\ &\quad = \sin 2\theta\\ &\quad = 2\sin\theta\cos\theta\\ &\quad = 2x\sqrt{1-x^2}\end{aligned}$$



b Let $u = 1-x^2$, $\dfrac{du}{dx} = -2x$
$$\begin{aligned}\therefore\ &\int \sin(2\arcsin x)\,dx\\ &= \int 2x\sqrt{1-x^2}\,dx\\ &= \int \sqrt{u}\left(-\frac{du}{dx}\right)dx\\ &= -\int u^{\frac{1}{2}}\,du\\ &= \frac{-u^{\frac{3}{2}}}{\frac{3}{2}} + c\\ &= -\tfrac{2}{3}(1-x^2)^{\frac{3}{2}} + c\end{aligned}$$

c
$$\begin{aligned}&\int_0^1 \sin(2\arcsin x)\,dx\\ &= \left[-\tfrac{2}{3}(1-x^2)^{\frac{3}{2}}\right]_0^1\\ &= (0) - \left(-\tfrac{2}{3}\right)\\ &= \tfrac{2}{3}\end{aligned}$$

64 **a** $x = 3 + a\lambda,\ y = -2 - \lambda,\ z = 2 + 2\lambda$

meets $\dfrac{x-4}{2} = 1 - y = \dfrac{z+2}{3}$ where

$$\frac{3 + a\lambda - 4}{2} = 1 + 2 + \lambda = \frac{2 + 2\lambda + 2}{3}$$

$$\therefore\ \frac{a\lambda - 1}{2} = \underbrace{\lambda + 3 = \frac{2\lambda + 4}{3}}$$

$$\therefore\ 3\lambda + 9 = 2\lambda + 4$$

$$\therefore\ \lambda = -5$$

Hence $\dfrac{-5a - 1}{2} = -2$

$\therefore\ 5a + 1 = 4$

$\therefore\ a = \frac{3}{5}$

and P is $\left(3 + \frac{3}{5}(-5),\ -2 + 5,\ 2 + 2(-5)\right)$

or $(0, 3, -8)$.

b l_1 has direction vector $\begin{pmatrix} 3 \\ -5 \\ 10 \end{pmatrix} = \mathbf{v}_1$

l_2 has direction vector $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \mathbf{v}_2$

$\left\{\text{as } \dfrac{x-4}{2} = \dfrac{y-1}{-1} = \dfrac{z+2}{3}\right\}$

Now $\cos\theta = \dfrac{|\mathbf{v}_1 \bullet \mathbf{v}_2|}{|\mathbf{v}_1|\,|\mathbf{v}_2|}$ {as θ is acute}

$$= \frac{|6 + 5 + 30|}{\sqrt{9 + 25 + 100}\sqrt{4 + 1 + 9}}$$

$$= \frac{41}{\sqrt{134}\sqrt{14}}$$

$\therefore\ \theta \approx 18.8°$

c $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -5 & 10 \\ 2 & -1 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} -5 & 10 \\ -1 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 10 \\ 2 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & -5 \\ 2 & -1 \end{vmatrix}\mathbf{k}$$

$$= -5\mathbf{i} + 11\mathbf{j} + 7\mathbf{k}$$

$\therefore$ the plane has equation $5x - 11y - 7z = 5(3) - 11(-2) - 7(2)$

$\therefore\ 5x - 11y - 7z = 23$

65 **a** $s(t) = t^3 - 7t^2 + 10t + 14$ m

$\therefore\ v(t) = 3t^2 - 14t + 10$ m s^{-1}

The object is stationary when $v(t) = 0$

$\therefore\ 3t^2 - 14t + 10 = 0$

$\therefore\ t \approx 0.880$ s or 3.79 s {technology}

b Total distance travelled in first 5 seconds

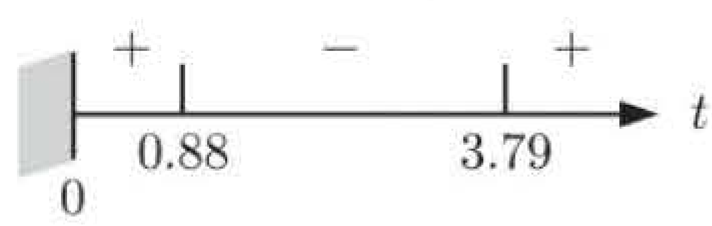
$= \int_0^5 \left|3t^2 - 14t + 10\right|\,dt$

≈ 24.5 m {technology}

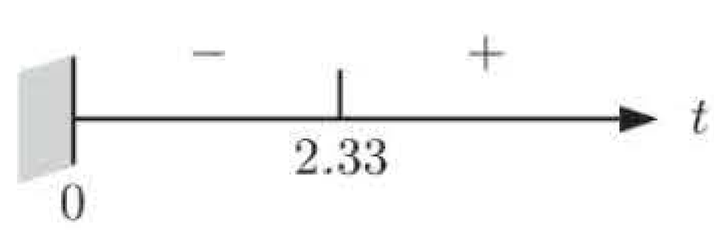
c **i** $a(t) = 6t - 14$ m s^{-2}

$\therefore$ acceleration is zero when $t \approx 2.33$ s

ii $v(t)$ sign diagram is



$a(t)$ sign diagram is

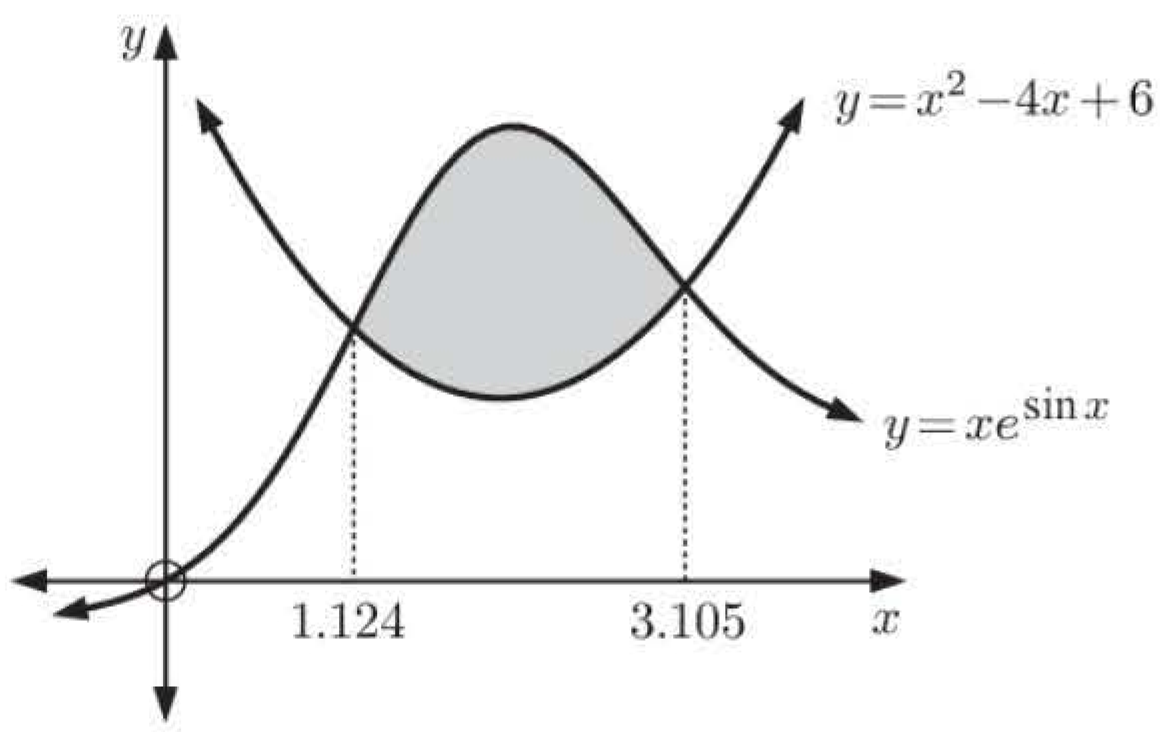


The change of sign about 2.33 on the $a(t)$ sign diagram indicates that velocity is at a minimum at this time.

To the left of 2.33, $v(t)$ and $a(t)$ are the same sign, indicating increasing speed, and to the right they are of opposite sign, indicating decreasing speed.

$\therefore$ speed is at a maximum at $t \approx 2.33$ s.

66



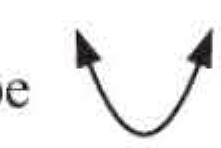
Using technology, the graphs intersect when $x \approx 1.124$ and 3.105

$\therefore$ area $\approx \int_{1.124}^{3.105} (xe^{\sin x} - x^2 + 4x - 6)\, dx$

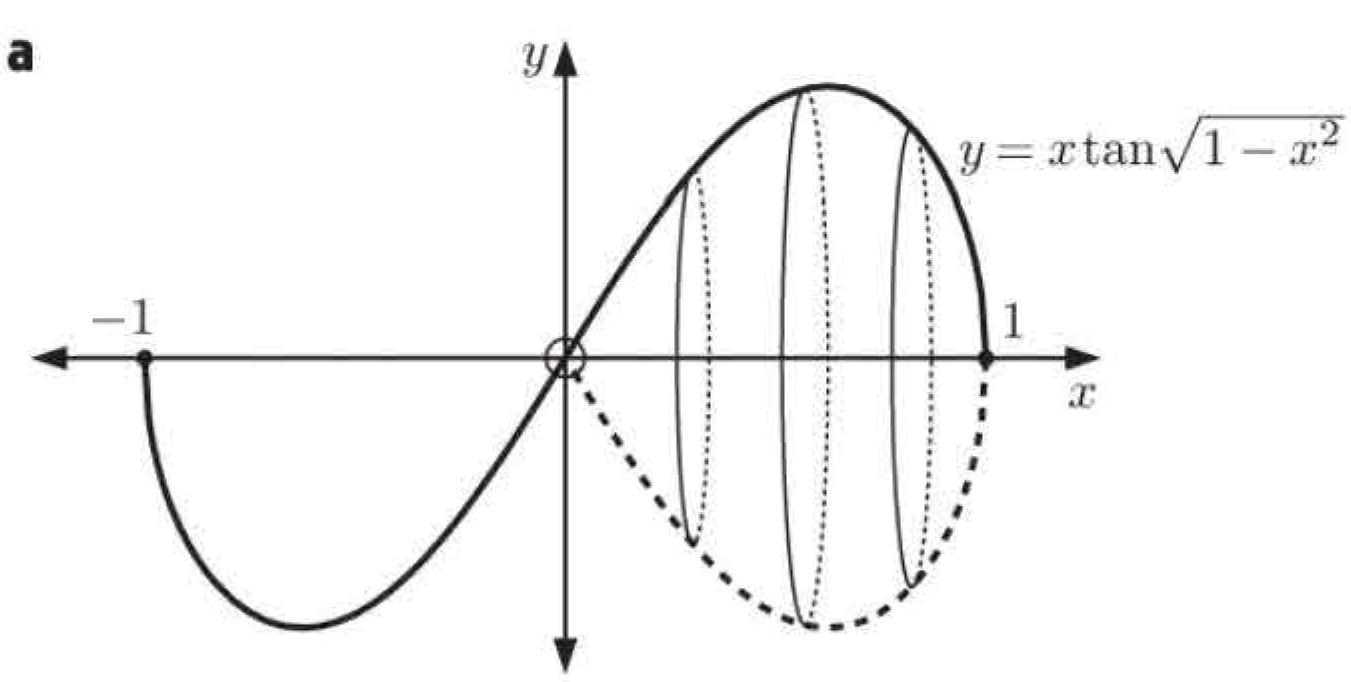
≈ 3.76 units^2 {using technology}

67 **a** As a, b, and c are real, all the coefficients of $P(z)$ are real.
Consequently $-3 + 2i$ and $-3 - 2i$ are both zeros.
They have sum -6 and product $9 + 4 = 13$.
$\therefore$ $z^2 + 6z + 13$ is a factor of $P(z)$.

Thus $P(z) = (z + 2)(z^2 + 6z + 13)$
$= z^3 + 8z^2 + 25z + 26$
$\therefore$ $a = 8$, $b = 25$, and $c = 26$

b If $P(z) \geqslant 0$ then $(z + 2)(z^2 + 6z + 13) \geqslant 0$.
Now $z^2 + 6z + 13 > 0$ for all z, since its roots are complex and it has shape 
$\therefore$ $P(z) \geqslant 0$ provided $z + 2 \geqslant 0$
$\therefore$ $z \geqslant -2$

68 **a**



b $V = \pi \int_0^1 (x \tan \sqrt{1 - x^2})^2\, dx$
≈ 0.676 units^3 {technology}

69 P_n is "$3(5^{2n+1}) + 2^{3n+1}$ is divisible by 17" for $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $3(5^3) + 2^{3+1} = 391 = 17 \times 23$ where $23 \in \mathbb{Z}$ $\therefore$ P_1 is true.

(2) If P_k is assumed true, then $3(5^{2k+1}) + 2^{3k+1} = 17A$ for some $A \in \mathbb{Z}$

Thus $3(5^{2(k+1)+1}) + 2^{3(k+1)+1}$
$= 3(5^{2k+1+2}) + 2^{3k+1+3}$
$= 3 \times 25 \times 5^{2k+1} + 8 \times 2^{3k+1}$
$= 25\left(17A - 2^{3k+1}\right) + 8 \times 2^{3k+1}$ {by rearranging and substituting P_k}
$= 25 \times 17 \times A + (8 - 25)2^{3k+1}$
$= 25 \times 17 \times A - 17 \times 2^{3k+1}$
$= 17\left(25A - 2^{3k+1}\right)$ where $25A - 2^{3k+1} \in \mathbb{Z}$

Thus P_{k+1} is true whenever P_k is true.

$\therefore$ since P_1 is true, P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

70 **a** If $x = a^{\frac{1}{3}} + b^{\frac{1}{3}}$ then $x^3 = (a^{\frac{1}{3}} + b^{\frac{1}{3}})^3$

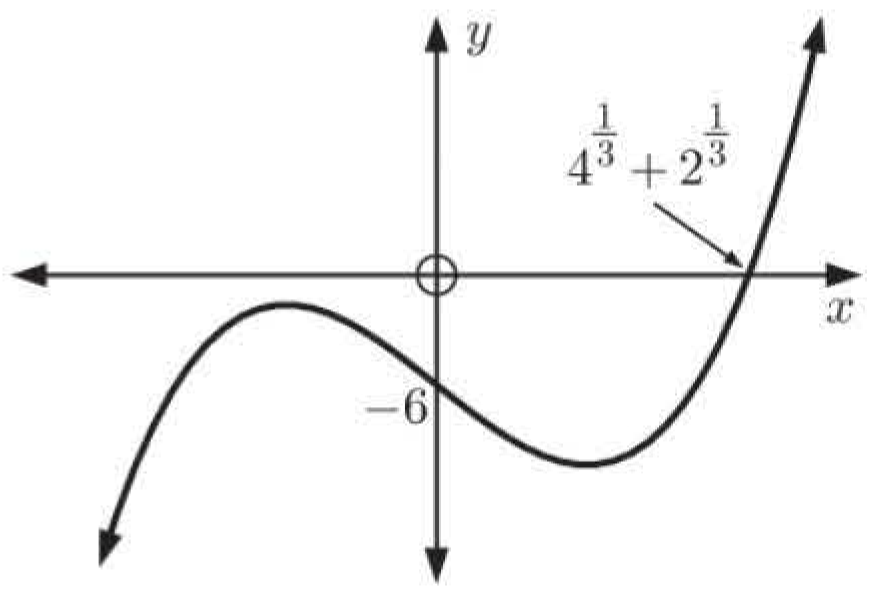
$$\therefore\ x^3 = (a^{\frac{1}{3}})^3 + 3(a^{\frac{1}{3}})^2 b^{\frac{1}{3}} + 3(a^{\frac{1}{3}})(b^{\frac{1}{3}})^2 + (b^{\frac{1}{3}})^3$$
$$\therefore\ x^3 = a + b + 3a^{\frac{1}{3}} b^{\frac{1}{3}} (a^{\frac{1}{3}} + b^{\frac{1}{3}})$$
$$\therefore\ x^3 = 3(ab)^{\frac{1}{3}} x + (a+b)$$

b $x^3 = 6x + 6$ is of the above form where $(ab)^{\frac{1}{3}} = 2$ and $a + b = 6$

$$\therefore\ ab = 8 \text{ and } a + b = 6$$
$$\therefore\ a = 2,\ b = 4 \text{ or } a = 4,\ b = 2$$

Thus $x = 4^{\frac{1}{3}} + 2^{\frac{1}{3}}$ is a root of $x^3 = 6x + 6$

The graph of $f(x) = x^3 - 6x - 6$ is:



As a cubic has at most two turning points, it is clear that $x = \sqrt[3]{4} + \sqrt[3]{2}$ is the only real zero of $f(x) = x^3 - 6x - 6$, and hence is the only real solution of $x^3 = 6x + 6$.

71
$$y = 3\,\frac{dy}{dx}$$
$$\therefore\ \frac{dy}{dx} = \frac{y}{3}$$
$$\therefore\ \frac{dx}{dy} = \frac{3}{y} \qquad \left\{\frac{dy}{dx} \times \frac{dx}{dy} = 1\right\}$$
$$\therefore\ x = \int \frac{3}{y}\, dy$$
$$\therefore\ x = 3\ln|y| + c$$
$$\therefore\ x = 3\ln y + c \qquad \{\text{as } y > 0, \text{ given}\}$$

Now when $x = 0$, $y = 3$

$$\therefore\ 0 = 3\ln 3 + c$$
$$\therefore\ c = -3\ln 3$$

Hence, $x = 3\ln y - 3\ln 3$

$$\therefore\ \frac{x}{3} = \ln y - \ln 3 = \ln\left(\frac{y}{3}\right)$$
$$\therefore\ \frac{y}{3} = e^{\frac{x}{3}}$$
$$\therefore\ y = 3e^{\frac{x}{3}}$$

72 **a** Let u_{n+1} be the amount still owing after n quarters and let R be the repayment each quarter. Each quarter, interest is charged at $\frac{12\%}{4} = 3\%$, so $u_{n+1} = 1.03u_n - R$.

From **2**, $u_{n+1} = (1.03)^n u_1 - R\left(\dfrac{1-(1.03)^n}{1-1.03}\right)$

We want $u_{n+1} = 0$, $u_1 = 20\,000$, and $n = 40$

Thus $(1.03)^{40} \times 20\,000 + \dfrac{100R}{3}\left(1 - (1.03)^{40}\right) = 0$

$$\therefore\ 65\,240.756 \approx R \times 75.401\,26$$
$$\therefore\ R \approx 865.25$$

So, repayments of \$865.25 each quarter are required.

b From **a**, $$0=\left(1+\frac{r}{100m}\right)^{mn}P-R\left(\frac{1-\left(1+\frac{r}{100m}\right)^{mn}}{-\frac{r}{100m}}\right)$$

$$\therefore\ R\left(\frac{1-\left(1+\frac{r}{100m}\right)^{mn}}{-\frac{r}{100m}}\right)=P\left(1+\frac{r}{100m}\right)^{mn}$$

$$\therefore\ R=\frac{P\left(1+\frac{r}{100m}\right)^{mn}\left(-\frac{r}{100m}\right)}{1-\left(1+\frac{r}{100m}\right)^{mn}}$$

$$\therefore\ R=\frac{P\left(\frac{r}{100m}\right)\left(1+\frac{r}{100m}\right)^{mn}}{\left(1+\frac{r}{100m}\right)^{mn}-1}$$

73 **a** We select the captain first and then the other 10 from the remaining 21.

This can be done in $\binom{11}{1}\binom{21}{10}=3\,879\,876$ ways.

b This identity can be established using the general solution of **a**, using n instead of 11. The number of choices is therefore $\binom{n}{1}\binom{2n-1}{n-1}$ .... (1).

We can do this count in a different way.

Suppose we select i members from A and $n-i$ from B. There are i ways of choosing the captain from A.

So, we have $i\times\binom{n}{i}\binom{n}{n-i}$ selections where $i=1,\,2,\,3,\,4,\,....,\,n$

$=i\binom{n}{i}^2$ {as $\binom{n}{i}=\binom{n}{n-i}$ by Pascal's rule}

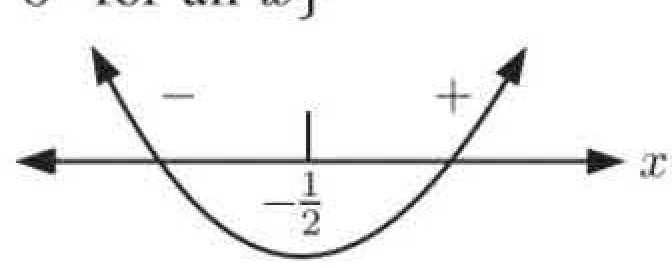
Thus the total number of ways is $1\binom{n}{1}^2+2\binom{n}{2}^2+3\binom{n}{3}^2+....+n\binom{n}{n}^2$ .... (2)

From (1) and (2), $1\binom{n}{1}^2+2\binom{n}{2}^2+3\binom{n}{3}^2+....+n\binom{n}{n}^2=n\binom{2n-1}{n-1}$.

74 **a** $$y=xe^{2x}$$
$$\therefore\ \frac{dy}{dx}=(1)e^{2x}+x(2e^{2x})$$
$$=e^{2x}(1+2x)$$

which is 0 when $x=-\frac{1}{2}$ $\{e^{2x}>0$ for all $x\}$

and the sign diagram of $\frac{dy}{dx}$ is:



$\therefore$ a local minimum occurs at $\left(-\frac{1}{2},\,-\frac{1}{2e}\right)$

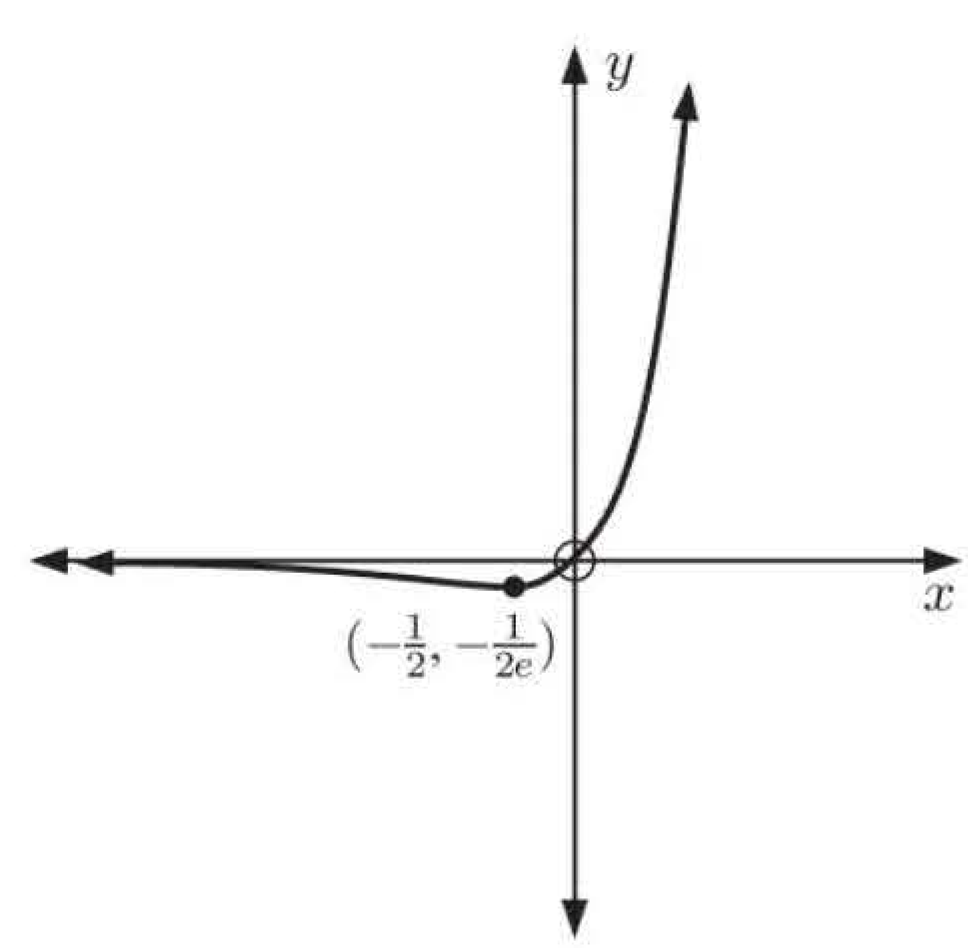
$\therefore$ $k=-\frac{1}{2e}$

b The graph of $y=xe^{2x}$ is shown alongside.

i When $k=-\frac{1}{2e}$ or $k\geqslant 0$, $y=k$ cuts the curve once only.

ii When $-\frac{1}{2e}<k<0$, $y=k$ cuts the curve twice.

iii When $k<-\frac{1}{2e}$, $y=k$ never cuts the graph.



c **i** When $y = xe^{ax}$, $\dfrac{dy}{dx} = e^{ax}(ax+1)$

$y = x$ meets $y = xe^{ax}$ where $x = xe^{ax}$

$$\therefore\ x(1-e^{ax}) = 0$$

$$\therefore\ x = 0,\quad a \in \mathbb{R}$$

When $x = 0$, $y = 0$ and $\dfrac{dy}{dx} = 1$, which is the gradient of $y = x$.

$\therefore$ $y = x$ is a tangent to $y = xe^{ax}$, and the point of contact is $(0, 0)$.

ii The normal at $(0, 0)$ has slope -1, and equation $y = -x$

$\therefore$ it makes an angle of $45°$ to the x-axis.

75 A is the event of a family having at most one boy in n children.
B is the event of a family having every child the same sex in n children.

$$\begin{aligned} \mathrm{P}(A) &= \mathrm{P}(0\text{B and }n\text{G or 1B and }(n-1)\text{G}) \\ &= \binom{n}{0}\left(\tfrac{1}{2}\right)^n + \binom{n}{1}\left(\tfrac{1}{2}\right)^1\left(\tfrac{1}{2}\right)^{n-1} \\ &= \left(\tfrac{1}{2}\right)^n + n\left(\tfrac{1}{2}\right)^n \\ &= \left(\tfrac{1}{2}\right)^n(n+1) \end{aligned}$$

$$\begin{aligned} \mathrm{P}(B) &= \mathrm{P}(0\text{B and }n\text{G or }n\text{B and 0G}) \\ &= \binom{n}{0}\left(\tfrac{1}{2}\right)^n + \binom{n}{n}\left(\tfrac{1}{2}\right)^n \\ &= 2\left(\tfrac{1}{2}\right)^n \end{aligned}$$

$$\mathrm{P}(A \cap B) = \mathrm{P}(0\text{B and }n\text{G}) = \left(\tfrac{1}{2}\right)^n$$

A and B are independent when

$$\mathrm{P}(A \cap B) = \mathrm{P}(A)\,\mathrm{P}(B)$$

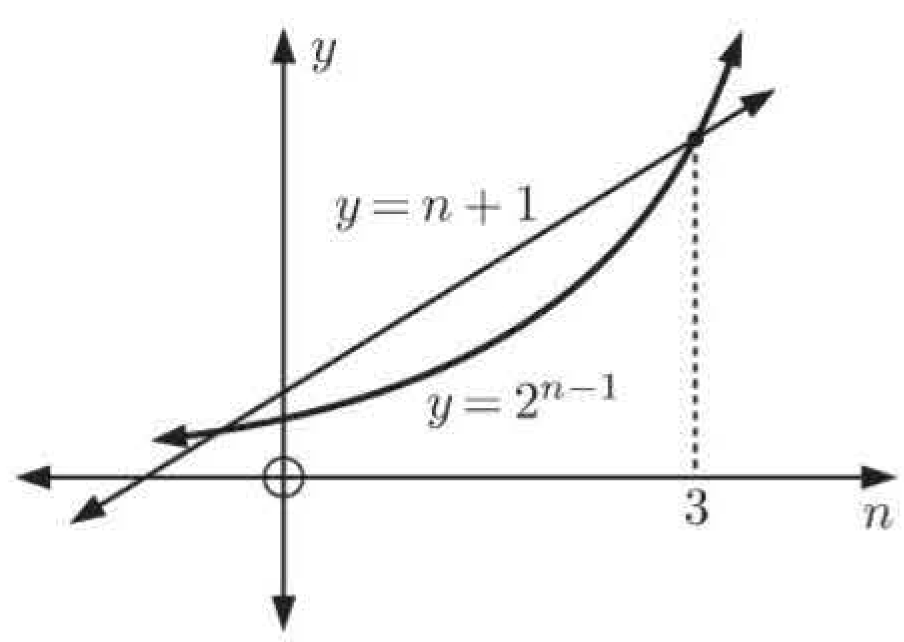
$$\therefore\ \left(\tfrac{1}{2}\right)^n = \left(\tfrac{1}{2}\right)^n(n+1) \times 2\left(\tfrac{1}{2}\right)^n$$

$$\therefore\ 1 = 2(n+1)\left(\tfrac{1}{2}\right)^n$$

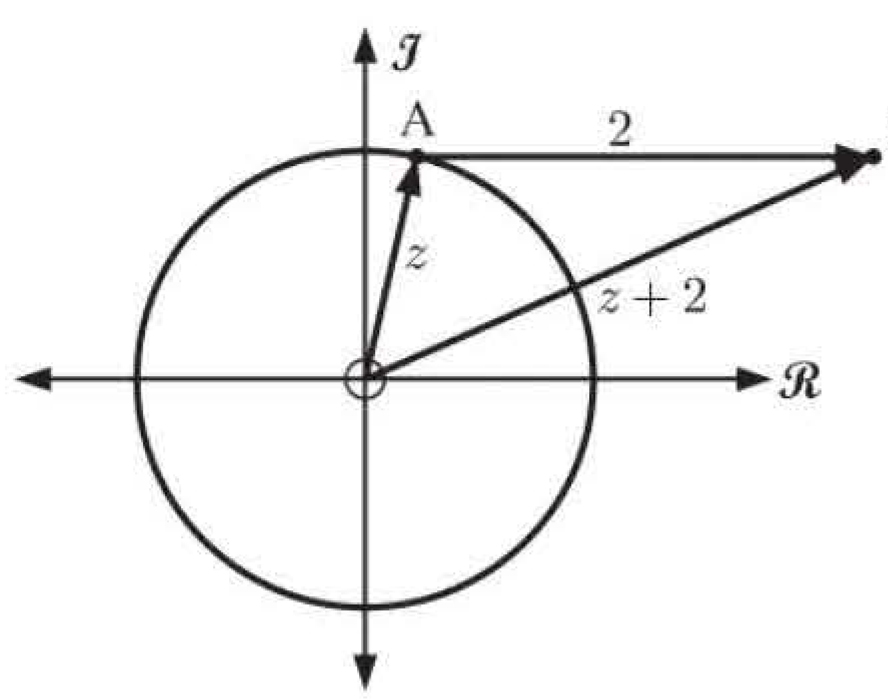
$$\therefore\ 2^n = 2(n+1)$$

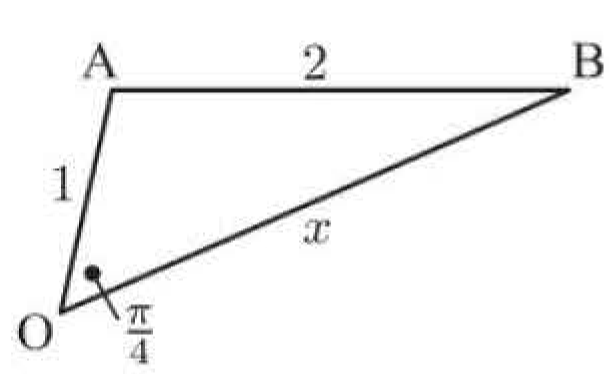
$$\therefore\ 2^{n-1} = n+1$$

$$\therefore\ n = 3\quad \{n > 1\}$$



76





As $|z| = 1$ and $\arg z \in [0, \frac{\pi}{2}]$, z lies on the unit circle in the first quadrant.

$z + 2$ is shown using vector addition.

Now $\arg\left(\dfrac{z}{z+2}\right) = \frac{\pi}{4}$

$$\therefore\ \arg z - \arg(z+2) = \tfrac{\pi}{4}$$

$$\therefore\ \widehat{\text{AOB}} = \tfrac{\pi}{4}$$

Using the cosine rule in $\triangle$OAB gives

$$2^2 = x^2 + 1^2 - 2(x)(1)\cos\tfrac{\pi}{4}$$

$$\therefore\ 4 = x^2 + 1 - \sqrt{2}x$$

$$\therefore\ x^2 - \sqrt{2}x - 3 = 0$$

$$\therefore\ x \approx 2.58\quad \{\text{technology, } x > 0\}$$

$$\therefore\ |z+2| \approx 2.58$$

77 We assume that the typist would know that the coefficients $a = 1$ and $b = 1$ are excluded, so we assume $a \neq 1$ and $b \neq 1$.

Thus a and b are from $\{2, 3, 4, 5, 6, 7, 8, 9\}$, but c is from $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$\therefore$ the total number of guesses is $8 \times 8 \times 9 = 576$.

For real roots, $b^2 - 4ac \geqslant 0$ and so $ac \leqslant \dfrac{b^2}{4}$.

If $b = 2$, $ac \leqslant 1$ which is impossible. 0 solutions

If $b = 3$, $ac \leqslant 2\frac{1}{4}$ $\therefore$ $a = 2$, $c = 1$. 1 solution

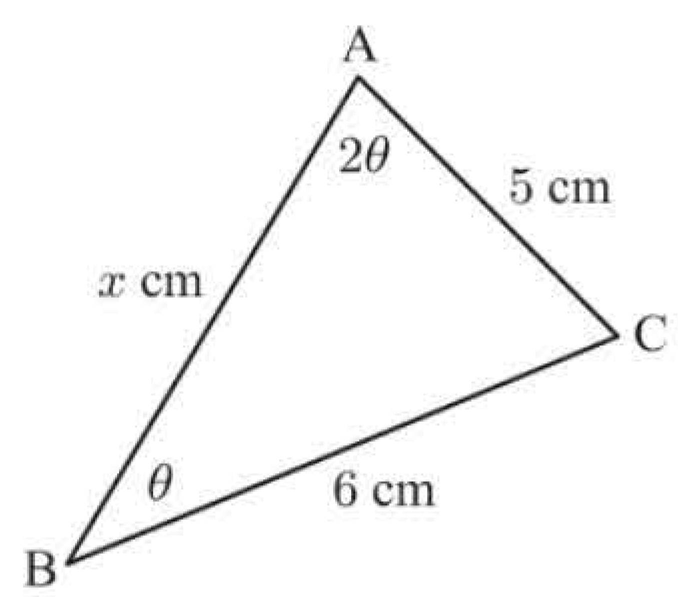
If $b = 4$, $ac \leqslant 4$ $\therefore$ $a = 2$, $c = 1$ or 2; $a = 3$, $c = 1$; $a = 4$, $c = 1$. 4 solutions

The number of solutions is best shown in a table:

b values	max. ac	a values: 2	3	4	5	6	7	8	9	Total
2	1	0	0	0	0	0	0	0	0	0
3	$2\frac{1}{4}$	1	0	0	0	0	0	0	0	1
4	4	2	1	1	0	0	0	0	0	4
5	$6\frac{1}{4}$	3	2	1	1	1	0	0	0	8
6	9	4	3	2	1	1	1	1	1	14
7	$12\frac{1}{4}$	6	4	3	2	2	1	1	1	20
8	16	8	5	4	3	2	2	2	1	27
9	$20\frac{1}{4}$	9	6	5	4	3	2	2	2	33
										107

$\therefore$ P(real roots) $= \frac{107}{576}$

78 **a**



Let the angle at B be θ and at A be 2θ

By the sine rule, $\dfrac{\sin 2\theta}{6} = \dfrac{\sin\theta}{5}$

$\therefore \dfrac{2\sin\theta\cos\theta}{\sin\theta} = \dfrac{6}{5}$

$\therefore \cos\theta = \frac{3}{5}$ .... $(*)$ $\{$as $\sin\theta \neq 0\}$

b Let $\text{AB} = x$ cm.

Using the cosine rule,

$$5^2 = x^2 + 6^2 - 2x(6)\cos\theta$$

$$\therefore\ 25 = x^2 + 36 - 12x\left(\tfrac{3}{5}\right)$$

$$\therefore\ x^2 - \tfrac{36}{5}x + 11 = 0$$

$$5x^2 - 36x + 55 = 0$$

$$\therefore\ (x-5)(5x-11) = 0$$

$$\therefore\ x = 5 \text{ or } \tfrac{11}{5}$$

$\therefore$ AB $= 5$ cm or 2.2 cm

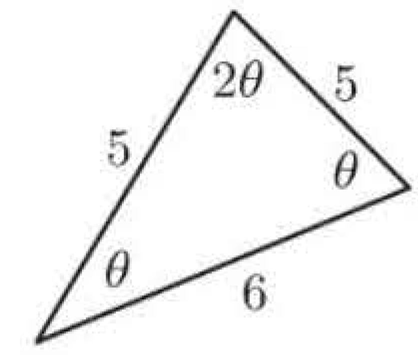
Check: If $\text{AB} = 5$ we have an isosceles triangle

$\therefore\ 4\theta = 180°$

$\therefore\ \theta = 45°$

which contradicts $(*)$

as $\cos 45° = \frac{1}{\sqrt{2}} \neq \frac{3}{5}$.

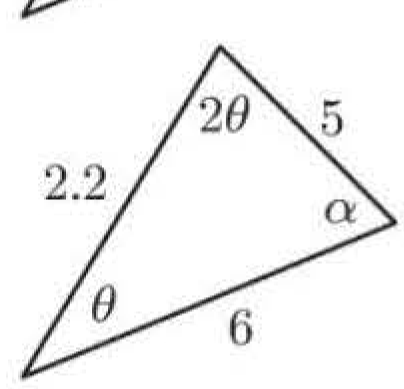


If $\text{AB} = 2.2$ we have

$\theta \approx 53.1°$

$\therefore\ 2\theta \approx 106.3°$

and $\alpha \approx 20.6°$



$\therefore$ AB $= 2.2$ is the only valid solution.

79 P_n is "$\dfrac{1}{\sin 2x} + \dfrac{1}{\sin 4x} + \dfrac{1}{\sin 8x} + ... + \dfrac{1}{\sin(2^n x)} = \cot x - \cot(2^n x)$" for $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= \dfrac{1}{\sin 2x}$ and RHS $= \cot x - \cot 2x$

$$= \frac{\cos x}{\sin x} - \frac{\cos 2x}{\sin 2x}$$

$$= \frac{\cos x}{\sin x}\left(\frac{2\cos x}{2\cos x}\right) - \frac{\cos 2x}{\sin 2x}$$

$$= \frac{2\cos^2 x - \cos 2x}{\sin 2x}$$

$$= \frac{2\cos^2 x - \left[2\cos^2 x - 1\right]}{\sin 2x}$$

$$= \frac{1}{\sin 2x}$$

$\therefore\ P_1$ is true.

(2) If P_k is true, $\dfrac{1}{\sin 2x}+\dfrac{1}{\sin 4x}+....+\dfrac{1}{\sin(2^k x)}=\cot x-\cot(2^k x)$

$$\therefore \quad \frac{1}{\sin 2x}+\frac{1}{\sin 4x}+....+\frac{1}{\sin(2^k x)}+\frac{1}{\sin(2^{k+1}x)}$$
$$=\cot x-\cot(2^k x)+\frac{1}{\sin(2^{k+1}x)}$$
$$=\cot x+\frac{1}{\sin(2^{k+1}x)}-\frac{\cos(2^k x)}{\sin(2^k x)}\left(\frac{2\cos(2^k x)}{2\cos(2^k x)}\right)$$
$$=\cot x+\frac{1}{\sin(2^{k+1}x)}-\frac{2\cos^2(2^k x)}{\sin(2^{k+1}x)}$$
$$=\cot x+\frac{1-2\cos^2(2^k x)}{\sin(2^{k+1}x)}$$
$$=\cot x+\frac{-\cos(2^{k+1}x)}{\sin(2^{k+1}x)} \qquad \{\cos 2\theta=2\cos^2\theta-1\}$$
$$=\cot x-\cot(2^{k+1}x)$$

Thus P_{k+1} is true whenever P_k is true.

$\therefore$ since P_1 is true, P_n is true for all $n\in\mathbb{Z}^+$ {Principle of mathematical induction}

80 **a** $I_0=\displaystyle\int_0^{\frac{\pi}{2}}\cos x\,dx$
$$=\Big[\sin x\Big]_0^{\frac{\pi}{2}}$$
$$=1-0$$
$$=1$$

b $I_1=\int x\cos x\,dx \qquad \left\{\begin{array}{ll} u'=\cos x & v=x \\ u=\sin x & v'=1\end{array}\right.$
$$=x\sin x-\int\sin x\,dx$$
$$=x\sin x+\cos x+c$$
$$\therefore \quad \int_0^{\pi} x\cos x\,dx$$
$$=\tfrac{\pi}{2}\sin(\tfrac{\pi}{2})+\cos(\tfrac{\pi}{2})-(0+\cos 0)$$
$$=\tfrac{\pi}{2}-1$$

c $\int x^n\cos x\,dx \qquad \left\{\begin{array}{ll} u'=\cos x & v=x^n \\ u=\sin x & v'=nx^{n-1}\end{array}\right.$

$$=x^n\sin x-n\int x^{n-1}\sin x\,dx \qquad \left\{\begin{array}{l} u'=\sin x \\ u=-\cos x \\ v=x^{n-1} \\ v'=(n-1)x^{n-2}\end{array}\right.$$
$$=x^n\sin x-n\left[-x^{n-1}\cos x+(n-1)\int x^{n-2}\cos x\,dx\right]$$
$$=x^n\sin x+nx^{n-1}\cos x-n(n-1)\int x^{n-2}\cos x\,dx$$

$$\therefore \quad I_n=[x^n\sin x]_0^{\frac{\pi}{2}}+n\left[x^{n-1}\cos x\right]_0^{\frac{\pi}{2}}-n(n-1)I_{n-2}$$
$$=(\tfrac{\pi}{2})^n\sin(\tfrac{\pi}{2})-0+n(0-0)-n(n-1)I_{n-2}$$
$$=(\tfrac{\pi}{2})^n-n(n-1)I_{n-2}$$

d $\displaystyle\int_0^{\frac{\pi}{2}}x^3\cos x\,dx=I_3$
$$=(\tfrac{\pi}{2})^3-(3)(2)I_1$$
$$=(\tfrac{\pi}{2})^3-6(\tfrac{\pi}{2}-1)$$
$$=(\tfrac{\pi}{2})^3-3\pi+6$$

81 **a** If $z = \text{cis}\,\theta$, then $z^n + \dfrac{1}{z^n} = 2\cos n\theta$, $n \in \mathbb{Z}^+$ {see **Ex 16E 12 a i**}

So, with $n = 1$, $z + \dfrac{1}{z} = 2\cos\theta$

$$\therefore \left(z + \frac{1}{z}\right)^3 = 8\cos^3\theta$$

$$\therefore z^3 + 3z + \frac{3}{z} + \frac{1}{z^3} = 8\cos^3\theta$$

$$\therefore \cos^3\theta = \tfrac{1}{8}\left[z^3 + \frac{1}{z^3} + 3\left(z + \frac{1}{z}\right)\right]$$

$$= \tfrac{1}{8}\left[2\cos 3\theta + 6\cos\theta\right]$$

$$= \tfrac{3}{4}\cos\theta + \tfrac{1}{4}\cos 3\theta$$

b Letting $x = \frac{1}{m}\cos\theta$, we have

$$\frac{\cos^3\theta}{m^3} - \frac{3\cos\theta}{m} + 1 = 0$$

$$\therefore \cos^3\theta - 3m^2\cos\theta + m^3 = 0$$

$$\therefore \tfrac{3}{4}\cos\theta + \tfrac{1}{4}\cos 3\theta - 3m^2\cos\theta + m^3 = 0$$

$$\therefore \tfrac{1}{4}\cos 3\theta + (\tfrac{3}{4} - 3m^2)\cos\theta + m^3 = 0$$

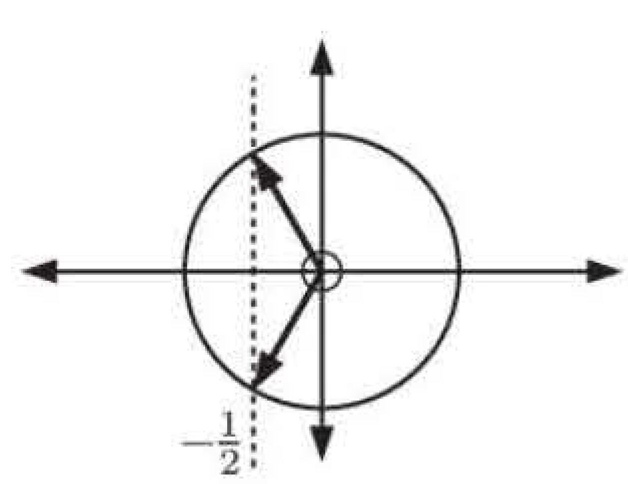
which when choosing $\underline{m = \frac{1}{2}}$ becomes

$\frac{1}{4}\cos 3\theta = -\frac{1}{8}$

$\therefore \cos 3\theta = -\frac{1}{2}$

$\therefore 3\theta = \frac{2\pi}{3} + k2\pi$ or $\frac{-2\pi}{3} + k2\pi$

$\therefore \theta = \frac{2\pi}{9} + \frac{k6\pi}{9}$ or $\frac{-2\pi}{9} + k\frac{6\pi}{9}$



Now $x = \dfrac{\cos\theta}{\frac{1}{2}} = 2\cos\theta$

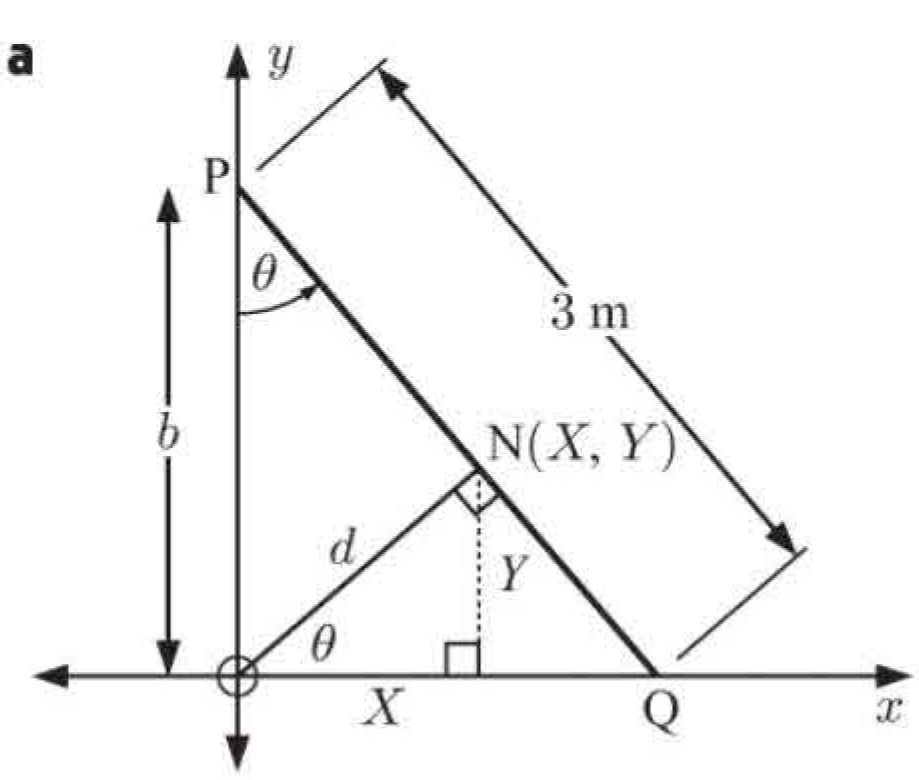
$\therefore x = 2\cos\left(\frac{2\pi}{9}\right),\ 2\cos\left(\frac{8\pi}{9}\right),\ 2\cos\left(\frac{14\pi}{9}\right)$

$\approx 1.53,\ -1.88,\ 0.347$

As these are all different, they are the required roots.

(Any solution of 3 consecutive values of θ will generate these roots when decimalised.)

82 **a**



As $\widehat{OPQ} = \theta$, $\widehat{PON} = 90° - \theta$ and so $\widehat{NOQ} = \theta$

So, $\cos\theta = \dfrac{X}{d}$ and $\sin\theta = \dfrac{Y}{d}$

$\therefore X = d\cos\theta$ and $Y = d\sin\theta$

But $\sin\theta = \dfrac{d}{b}$ and $\cos\theta = \dfrac{b}{3}$

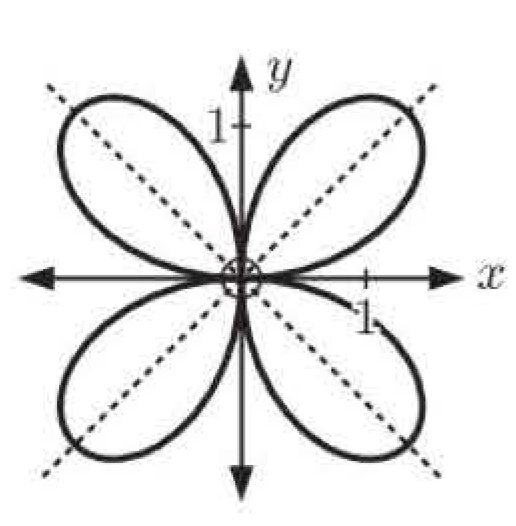
$\therefore d = b\sin\theta$

$= (3\cos\theta)\sin\theta$

Thus $X = 3\sin\theta\cos^2\theta$ and $Y = 3\sin^2\theta\cos\theta$

$\therefore$ N is at $(3\sin\theta\cos^2\theta, 3\sin^2\theta\cos\theta)$.

b



a 'four leaf clover'

83 **a** u_{m+1}, u_{n+1}, and u_{p+1} are consecutive terms of a geometric sequence.

$$\therefore \frac{u_{n+1}}{u_{m+1}} = \frac{u_{p+1}}{u_{n+1}}$$

$$\therefore \frac{u_1+nd}{u_1+md} = \frac{u_1+pd}{u_1+nd}$$

$$\therefore (u_1+nd)^2 = (u_1+md)(u_1+pd)$$

$$\therefore \cancel{u_1^2} + 2ndu_1 + n^2d^2 = \cancel{u_1^2} + mdu_1 + pdu_1 + mpd^2$$

Dividing each term by d^2 gives

$$\frac{2nu_1}{d} + n^2 = \frac{mu_1}{d} + \frac{pu_1}{d} + mp$$

$$\therefore \frac{u_1}{d}(2n-m-p) = mp - n^2$$

$$\therefore \frac{d}{u_1} = \frac{2n-m-p}{mp-n^2}$$

b If $\dfrac{2mp}{m+p} = n$,

then $m+p = \dfrac{2mp}{n}$

$$\therefore \frac{d}{u_1} = \frac{2n - \frac{2mp}{n}}{mp-n^2}\left(\frac{n}{n}\right)$$

$$= \frac{2n^2-2mp}{n(mp-n^2)}$$

$$= \frac{2\cancel{(n^2-mp)}}{-n\cancel{(n^2-mp)}}$$

$$= -\frac{2}{n} \quad \text{provided } n^2 \neq mp$$

< 0 for all values as $n > 0$

84 **a** L_1 can be written in the form $x = 8+3\lambda$, $y = -13-5\lambda$, $z = -3-2\lambda$

Substituting into L_2 gives: $\dfrac{8+3\lambda+10}{6} = \dfrac{-13-5\lambda-7}{-5} = \dfrac{-3-2\lambda-11}{-5}$

$$\therefore \frac{3\lambda+18}{6} = \frac{-20-5\lambda}{-5} = \frac{-2\lambda-14}{-5}$$

$$\therefore \frac{\lambda+6}{2} = \lambda+4 = \frac{2\lambda+14}{5}$$

$\therefore \lambda+6 = 2\lambda+8$ and $5\lambda+20 = 2\lambda+14$

$\therefore \lambda = -2$ and $3\lambda = -6$

So, $\lambda = -2$ is a common solution.

Substituting $\lambda = -2$ gives $x = 2$, $y = -3$, $z = 1$, so they meet at A(2, −3, 1).

b L_1 meets $3x+2y-z=-2$ where $3(8+3\lambda)+2(-13-5\lambda)-(-3-2\lambda) = -2$

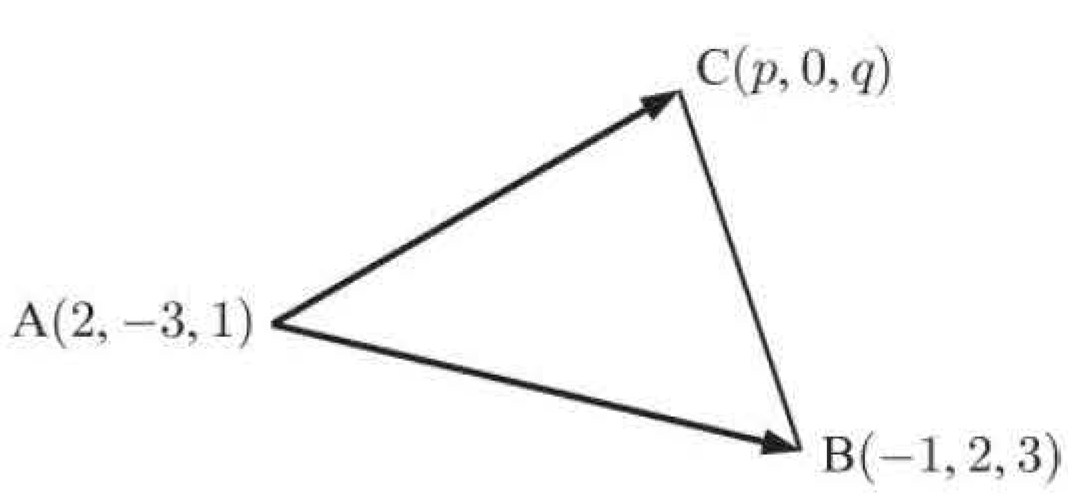
$$\therefore 24+9\lambda-26-10\lambda+3+2\lambda = -2$$

$$\therefore \lambda+1 = -2$$

$$\therefore \lambda = -3$$

So, B is at $(-1, 2, 3)$.

c



$$\overrightarrow{AB} = \begin{pmatrix} -3 \\ 5 \\ 2 \end{pmatrix}, \quad \overrightarrow{AC} = \begin{pmatrix} p-2 \\ 3 \\ q-1 \end{pmatrix}$$

$$\therefore \text{area } \triangle ABC = \tfrac{1}{2}\,|\overrightarrow{AB} \times \overrightarrow{AC}|$$

$$= \tfrac{1}{2}\left\|\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 5 & 2 \\ p-2 & 3 & q-1 \end{vmatrix}\right\| = \tfrac{\sqrt{3}}{2}$$

But C lies on $3x+2y-z=-2$

$$\therefore 3p-q = -2$$

$$\therefore q = 3p+2$$

Thus $\left\|\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 5 & 2 \\ p-2 & 3 & 3p+1 \end{vmatrix}\right\| = \sqrt{3}$

$$\therefore \quad |(15p-1)\mathbf{i}-(1-11p)\mathbf{j}+(1-5p)\mathbf{k}|=\sqrt{3}$$
$$\therefore \quad \sqrt{(15p-1)^2+(1-11p)^2+(1-5p)^2}=\sqrt{3}$$
$$\therefore \quad 225p^2-30p+1+1-22p+121p^2+1-10p+25p^2=3$$
$$\therefore \quad 371p^2-62p=0$$
$$\therefore \quad p(371p-62)=0$$
$$\therefore \quad p=0 \text{ or } \tfrac{62}{371}$$

85 **a** Suppose $e^x=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+....$ .... (1)

By successive differentiation of both sides, we find:

$e^x=a_1+2a_2x+3a_3x^2+4a_4x^3+5a_5x^4+....$ .... (2)

and $e^x=2a_2+6a_3x+12a_4x^2+20a_5x^3+30a_6x^4+....$ .... (3)

and $e^x=6a_3+24a_4x+60a_5x^2+120a_6x^3+210a_7x^4+....$ .... (4)

and $e^x=24a_4+120a_5x+360a_6x^2+....$ .... (5)

Letting $x=0$ in (1) to (5) gives $1=a_0,\ 1=a_1,\ 1=2a_2,\ 1=6a_3,\ 1=24a_4,\ ....$

Thus $a_0=1,\ a_1=1,\ a_2=\frac{1}{2},\ a_3=\frac{1}{6},\ a_4=\frac{1}{24},\ ....$

$$\therefore \quad a_0=\frac{1}{0!},\ a_1=\frac{1}{1!},\ a_2=\frac{1}{2!},\ a_3=\frac{1}{3!},\ a_4=\frac{1}{4!},\ ....$$

b *Conjecture*: $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$

c When $x=1$, $e=\sum_{n=0}^{\infty}\frac{1}{n!}=1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+....$

$\approx 2.718\,281\,828....$ (adding the first 13 terms)

which checks with the calculator result.

86
$$|\mathbf{a}-\mathbf{b}|^2=(\mathbf{a}-\mathbf{b})\bullet(\mathbf{a}-\mathbf{b})$$
$$\therefore \quad |\mathbf{a}-\mathbf{b}|^2=\mathbf{a}\bullet\mathbf{a}-2\mathbf{a}\bullet\mathbf{b}+\mathbf{b}\bullet\mathbf{b}$$
$$\therefore \quad |\mathbf{a}-\mathbf{b}|^2=|\mathbf{a}|^2-2\mathbf{a}\bullet\mathbf{b}+|\mathbf{b}|^2$$
$$\therefore \quad 400=49-2\mathbf{a}\bullet\mathbf{b}+256$$
$$\therefore \quad 2\mathbf{a}\bullet\mathbf{b}=-95$$

But $|\mathbf{a}+\mathbf{b}|^2=|\mathbf{a}|^2+2\mathbf{a}\bullet\mathbf{b}+|\mathbf{b}|^2$ {same reasoning as above}

$=49-95+250$

$=210$

$$\therefore \quad |\mathbf{a}+\mathbf{b}|=\sqrt{210}\approx 14.5$$

87 If k is the third root then $x^3+ax^2+bx+c=(x-\alpha)(x-\beta)(x-k)$

$=x^3-[\alpha+\beta+k]\,x^2+[\alpha\beta+\alpha k+\beta k]\,x-\alpha\beta k$

Equating coefficients: $a=-(\alpha+\beta+k)$ .... (1)

$b=\alpha\beta+\alpha k+\beta k$ .... (2)

$c=-\alpha\beta k$ and so $k=-\dfrac{c}{\alpha\beta}$ .... (3)

So, $(\alpha\beta)^3-b(\alpha\beta)^2+ac(\alpha\beta)-c^2$

$=(\alpha\beta)^3-[\alpha\beta+\alpha k+\beta k]\,(\alpha\beta)^2-\alpha\beta c(\alpha+\beta+k)-c^2$ {using (1) and (2)}

$=(\alpha\beta)^3-(\alpha\beta)^3-(\alpha+\beta)k(\alpha\beta)^2-\alpha\beta c(\alpha+\beta+k)-c^2$

$=-(\alpha+\beta)(-\dfrac{c}{\alpha\beta})(\alpha\beta)^2-\alpha\beta c(\alpha+\beta-\dfrac{c}{\alpha\beta})-c^2$ {using (3)}

$=\cancel{-(\alpha+\beta)(-c\alpha\beta)}-\cancel{\alpha\beta c(\alpha+\beta)}+\cancel{c}^2-\cancel{c}^2$

$=0$

$\therefore$ $\alpha\beta$ is a root of $x^3-bx^2+acx-c^2=0$.

88 **a** It cuts the x-axis when $y = 0$

$$\therefore\ x^4 = x^2$$
$$\therefore\ x^2(x^2-1) = 0$$
$$\therefore\ x^2(x+1)(x-1) = 0$$
$$\therefore\ x = 0, \pm 1$$

$\therefore$ the x-intercepts are -1, 0, and 1.

It cuts the y-axis when $x = 0$

$$\therefore\ y^4 = -y^2$$
$$\therefore\ y^2(y^2+1) = 0$$
$$\therefore\ y = 0$$

$\therefore$ the y-intercept is 0.

b

$$(x^2+y^2)^2 = x^2 - y^2$$
$$\therefore\ 2(x^2+y^2)\left[2x + 2y\,\frac{dy}{dx}\right] = 2x - 2y\,\frac{dy}{dx}$$
$$\therefore\ 4x(x^2+y^2) + 4y(x^2+y^2)\,\frac{dy}{dx} = 2x - 2y\,\frac{dy}{dx}$$
$$\therefore\ \left[4y(x^2+y^2) + 2y\right]\frac{dy}{dx} = 2x - 4x(x^2+y^2)$$
$$\therefore\ \frac{dy}{dx} = \frac{x - 2x(x^2+y^2)}{2y(x^2+y^2)+y}$$

which is $0 \Leftrightarrow x\left[1 - 2(x^2+y^2)\right] = 0$

But x is clearly not 0

$\therefore\ x^2 + y^2 = \frac{1}{2}$ .... (1)

Thus $(\frac{1}{2})^2 = x^2 - y^2$

$\therefore\ x^2 - y^2 = \frac{1}{4}$ .... (2)

From (1) and (2), $2x^2 = \frac{3}{4}$

$$\therefore\ x = \pm\tfrac{\sqrt{3}}{2\sqrt{2}} = \pm\tfrac{\sqrt{6}}{4}$$

and $y^2 = \frac{1}{8}$

$$\therefore\ y = \pm\tfrac{1}{2\sqrt{2}} = \pm\tfrac{\sqrt{2}}{4}$$

$\therefore$ the four points are $\left(\frac{\sqrt{6}}{4}, \frac{\sqrt{2}}{4}\right)$, $\left(\frac{\sqrt{6}}{4}, -\frac{\sqrt{2}}{4}\right)$, $\left(-\frac{\sqrt{6}}{4}, \frac{\sqrt{2}}{4}\right)$, $\left(-\frac{\sqrt{6}}{4}, -\frac{\sqrt{2}}{4}\right)$.

89 **a** $\int \ln x\, dx = \int 1 \ln x\, dx$

We integrate by parts with $u = \ln x$ $\quad v' = 1$
$u' = \dfrac{1}{x}$ $\quad v = x$

$$\therefore\ \int \ln x\, dx = x\ln x - \int\left(\frac{1}{x}\right)x\, dx$$
$$= x\ln x - \int 1\, dx$$
$$= x\ln x - x + c$$

Check:
$$\frac{d}{dx}(x\ln x - x + c)$$
$$= 1\ln x + x\left(\frac{1}{x}\right) - 1 + 0$$
$$= \ln x + 1 - 1$$
$$= \ln x \ \checkmark$$

b For $f(x)$ to be a pdf,

$$\int_1^k f(x)\, dx = 1$$
$$\therefore\ \int_1^k \ln x\, dx = 1$$
$$\therefore\ [x\ln x - x]_1^k = 1 \quad \{\text{using } \mathbf{a}\}$$
$$\therefore\ (k\ln k - k) - (0 - 1) = 1$$
$$\therefore\ k(\ln k - 1) = 0$$
$$\therefore\ k = 0 \text{ or } \ln k = 1$$
$$\therefore\ k = 0 \text{ or } e$$

But $k \geqslant 1$, so $k = e$

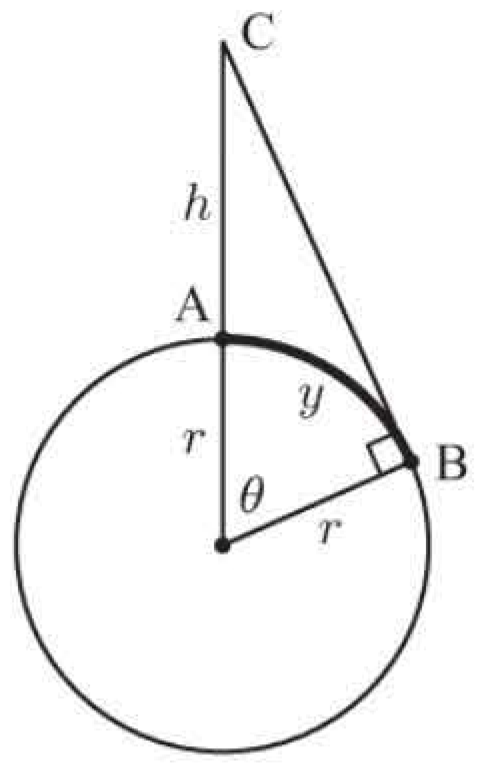
c Let m be the median.

$$\therefore\ \int_1^m \ln x\, dx = \tfrac{1}{2}$$
$$\therefore\ [x\ln x - x]_1^m = \tfrac{1}{2}$$
$$\therefore\ m\ln m - m - (-1) = \tfrac{1}{2}$$
$$\therefore\ m\ln m - m + \tfrac{1}{2} = 0$$
$$\therefore\ m \approx 0.187 \text{ or } 2.16$$

{technology}

But $m \geqslant 1$, so $m \approx 2.16$

90 **a**



Let the arc length AB be y.

$\therefore \quad y = r\theta \quad$ and $\quad \cos\theta = \dfrac{r}{r+h} \quad$ .... (1)

Substituting $\theta = \frac{y}{r}$ into (1) gives

$$\cos\left(\tfrac{y}{r}\right) = r(r+h)^{-1}$$

We differentiate with respect to t:

$$-\sin\left(\tfrac{y}{r}\right)\frac{1}{r}\frac{dy}{dt} = -\frac{r}{(r+h)^2}\frac{dh}{dt} \qquad \{r \text{ is constant}\}$$

$$\therefore \quad \sin\theta\,\frac{dy}{dt} = \left(\frac{r}{r+h}\right)^2\frac{dh}{dt}$$

$$\therefore \quad \frac{dy}{dt} = \frac{\cos^2\theta}{\sin\theta}\frac{dh}{dt} \qquad \{\text{using (1)}\}$$

b The rocket has velocity $\dfrac{dh}{dt}$.

$\therefore \quad \dfrac{dh}{dt} = r\sin t \quad$ for $\quad t \in [0, \pi]$

$\therefore \quad h = \int r\sin t\,dt = -r\cos t + c$

But when $t = 0, \quad h = 0$

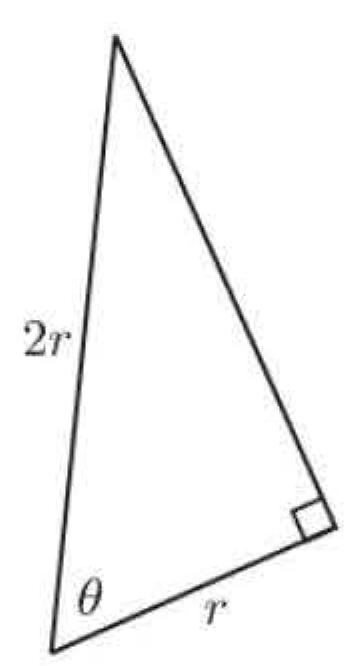
$\therefore \quad c = r$

So, $\quad h = r(1-\cos t)$

Now when $t = \frac{\pi}{2}, \ \cos t = 0$ and $h = r$.

This means that the rocket is r km above the earth's surface.

c At $t = \frac{\pi}{2}$, we have:



$$\cos\theta = \frac{r}{2r} = \tfrac{1}{2}$$

$$\therefore \quad \theta = \tfrac{\pi}{3} \quad \text{and} \quad \sin\theta = \tfrac{\sqrt{3}}{2}$$

$$\therefore \quad \frac{dy}{dt} \approx \frac{\left(\frac{1}{2}\right)^2}{\frac{\sqrt{3}}{2}} \times 6000 \times \sin\left(\tfrac{\pi}{2}\right) \qquad \{\text{using } \mathbf{a},\ \frac{dh}{dt} = r\sin t\}$$

$$\approx \tfrac{1}{4} \times \tfrac{2}{\sqrt{3}} \times 6000 \times 1$$

$$\approx 1730 \text{ km h}^{-1}$$

91 **a**

$$\operatorname{arctanh}(z) = \tfrac{1}{2}\left[\ln(1+z) - \ln(1-z)\right]$$

$$\therefore \quad \frac{d}{dz}\left(\operatorname{arctanh}(z)\right) = \tfrac{1}{2}\left[\frac{1}{1+z} - \frac{-1}{1-z}\right]$$

$$= \tfrac{1}{2}\left[\frac{1}{1+z} + \frac{1}{1-z}\right]$$

$$= \tfrac{1}{2}\left[\frac{1-z+1+z}{1-z^2}\right]$$

$$= \frac{1}{1-z^2}$$

b $f(x) = \int x(1-x^2)^{-\frac{3}{2}}\, dx$ Let $u = 1 - x^2$

$= -\frac{1}{2}\int u^{-\frac{3}{2}}(-2x)\, dx$ $\therefore \dfrac{du}{dx} = -2x$

$$= -\tfrac{1}{2}\int u^{-\frac{3}{2}}\,\frac{du}{dx}\, dx$$

$$= -\tfrac{1}{2}\int u^{-\frac{3}{2}}\, du$$

$$= -\tfrac{1}{2}\left(\frac{u^{-\frac{1}{2}}}{-\frac{1}{2}}\right) + c$$

$$= \frac{1}{\sqrt{u}} + c$$

$$= \frac{1}{\sqrt{1-x^2}} + c$$

Now $V = \pi\int_0^{\frac{1}{2}} [f(x)]^2\, dx$

$$= \pi\int_0^{\frac{1}{2}}\left[\frac{1}{1-x^2} + \frac{2c}{\sqrt{1-x^2}} + c^2\right] dx$$

$$= \pi\left[\operatorname{arctanh}(x) + 2c\arcsin x + c^2x\right]_0^{\frac{1}{2}}$$

$$= \pi\left[\left(\operatorname{arctanh}(\tfrac{1}{2}) + 2c\arcsin(\tfrac{1}{2}) + \frac{c^2}{2}\right) - (\operatorname{arctanh}(0) + 2c\arcsin(0) + 0)\right]$$

$$= \pi\left[\tfrac{1}{2}\left(\ln(\tfrac{3}{2}) - \ln(\tfrac{1}{2})\right) + 2c(\tfrac{\pi}{6}) + \frac{c^2}{2} - 0\right]$$

Thus $\pi\left[\frac{1}{2}\ln 3 + \frac{\pi}{3}c + \dfrac{c^2}{2}\right] \approx 14.589$

$\therefore \dfrac{c^2}{2} + \frac{\pi}{3}c + \frac{1}{2}\ln 3 \approx 4.6438$

$\therefore c \approx -4.09$ or 2.00

$\therefore f(x) \approx \dfrac{1}{\sqrt{1-x^2}} + 2$ or $f(x) \approx \dfrac{1}{\sqrt{1-x^2}} - 4.09$

92 **a** $f(x) = x - 4\sqrt{2}\sqrt{x}$

i The domain is $\{x \mid x \geqslant 0\}$

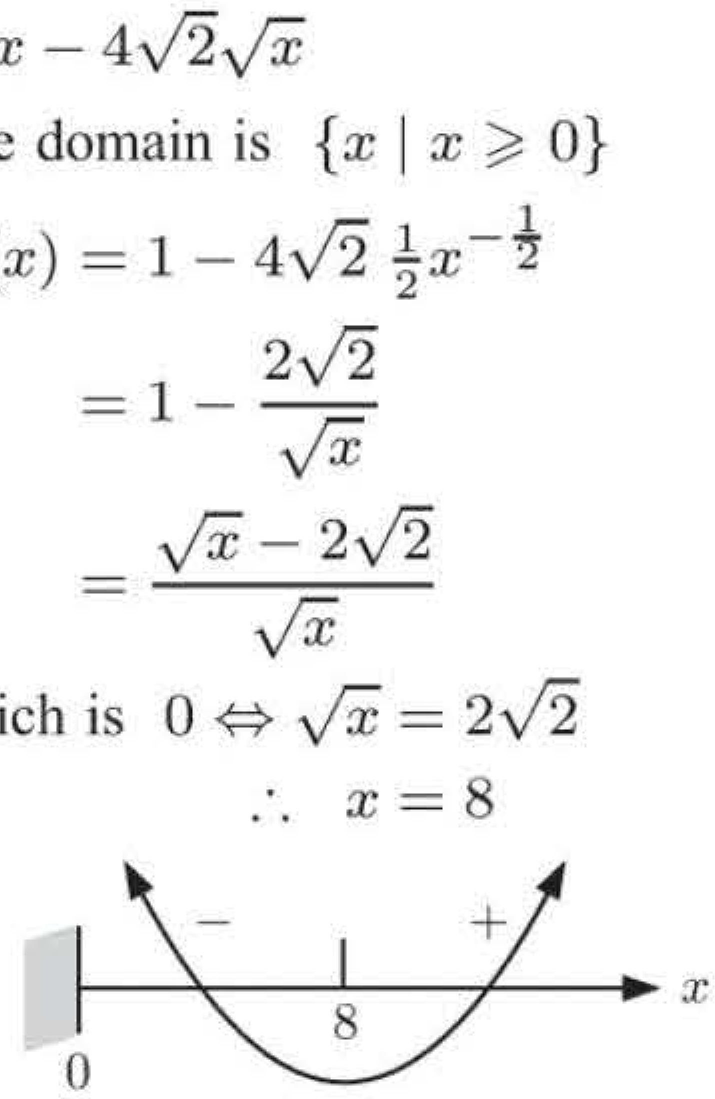
ii $f'(x) = 1 - 4\sqrt{2}\,\frac{1}{2}x^{-\frac{1}{2}}$

$$= 1 - \frac{2\sqrt{2}}{\sqrt{x}}$$

$$= \frac{\sqrt{x} - 2\sqrt{2}}{\sqrt{x}}$$

which is $0 \Leftrightarrow \sqrt{x} = 2\sqrt{2}$

$\therefore x = 8$



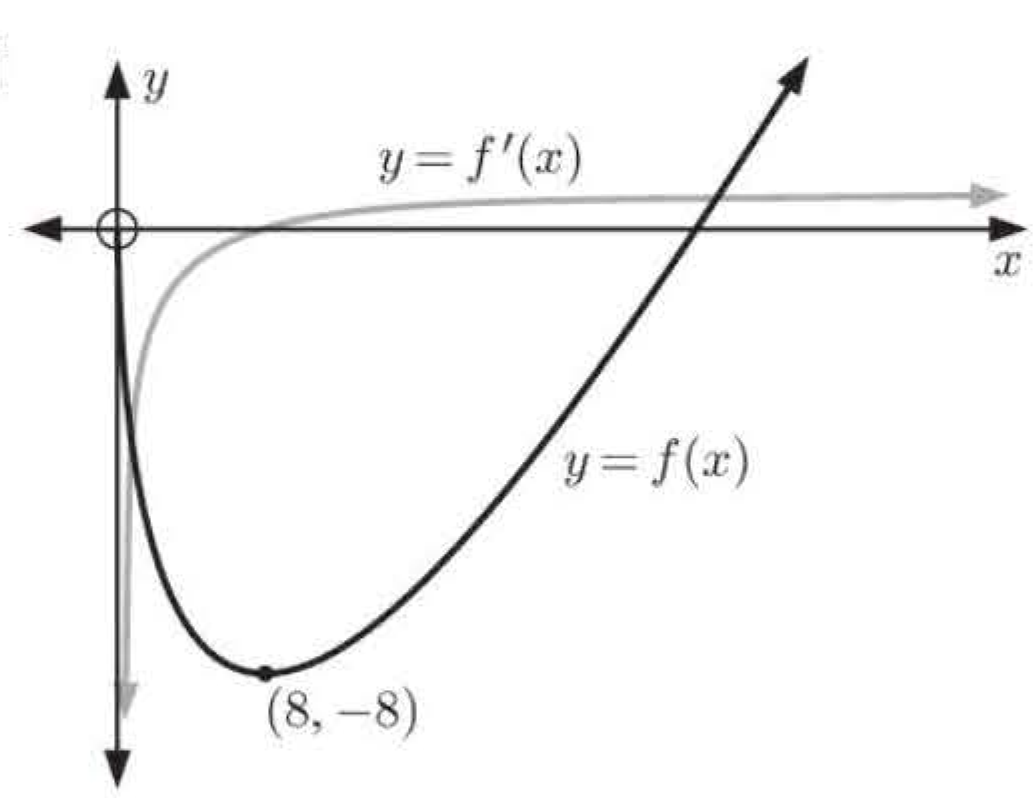
and $f(8) = 8 - 4\sqrt{2}(\sqrt{8})$

$= 8 - 16$

$= -8$

$\therefore$ the minimum value is -8 when $x = 8$.

iii



b **i** $(\sqrt{a}-\sqrt{b})^2 \geqslant 0$ for all $a \geqslant 0,\ b \geqslant 0$

$\therefore\ a - 2\sqrt{ab} + b \geqslant 0$

$\therefore\ a + b \geqslant 2\sqrt{ab}$

$\therefore\ \dfrac{a+b}{2} \geqslant \sqrt{ab}$

Note: Equality occurs when $a = b$.

ii Using $a = x,\ b = 8$:

Given x, 8 are $\geqslant 0$

$\dfrac{x+8}{2} \geqslant \sqrt{8x}$ {AM-GM inequality}

$\therefore\ x + 8 \geqslant 2 \times 2\sqrt{2}\sqrt{x}$

$\therefore\ x + 8 \geqslant 4\sqrt{2}\sqrt{x}$

$\therefore\ x - 4\sqrt{2}\sqrt{x} \geqslant -8$

with equality occuring $\Leftrightarrow\ x = 8$

$\therefore\ -8$ is the minimum value when $x = 8$.

93 **a** If $x = 1+\lambda,\ y = -1 + a\lambda,\ z = 2-\lambda$ lies on $3x - ky + z = 3$, then

$$3(1+\lambda) - k(-1+a\lambda) + 2 - \lambda = 3 \quad \text{for all } \lambda$$
$$\therefore\ 3 + 3\lambda + k - ak\lambda + 2 - \lambda = 3$$
$$\therefore\ (3+k+2) + \lambda(3 - ak - 1) = 3$$
$$\therefore\ k + 5 = 3 \quad \text{and} \quad 2 - ak = 0$$
$$\therefore\ k = -2 \quad \text{and} \quad 2 + 2a = 0$$
$$\therefore\ k = -2 \quad \text{and} \quad a = -1$$

b P_2 has normal vector $\mathbf{n}_2 = \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix}$

P_1 has normal vector $\mathbf{n}_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

Now $\mathbf{n}_1 \bullet \mathbf{n}_2 = 6 - 2 - 4 = 0$

$\therefore\ \mathbf{n}_1 \perp \mathbf{n}_2$ and so $P_1 \perp P_2$.

c We need to solve $\begin{cases} 2x - y - 4z = 9 \\ 3x + 2y + z = 3 \end{cases}$

$$\left[\begin{array}{ccc|c} 2 & -1 & -4 & 9 \\ 3 & 2 & 1 & 3 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 2 & -1 & -4 & 9 \\ 0 & 7 & 14 & -21 \end{array}\right] \quad R_2 \to 2R_2 - 3R_1$$

The second equation simplifies to $y + 2z = -3$

If $z = t$ then $y = -3 - 2t$

and $2x + 3 + 2t - 4t = 9$

$\therefore\ 2x = 6 + 2t$

$\therefore\ x = 3 + t$

So, L_2 has equation

$x = 3 + t,\ y = -3 - 2t,\ z = t,\ t \in \mathbb{R}$.

d L_1 and L_2 meet when $1 + \lambda = 3 + t,\quad -1 - \lambda = -3 - 2t,\quad 2 - \lambda = t$

$\therefore\ \lambda = 2 + t,\quad \lambda = 2 + 2t,\quad \lambda = 2 - t$

$\therefore\ 2 + t = 2 + 2t = 2 - t$

The common solution is $t = 0$, so L_1 and L_2 meet at $(3, -3, 0)$.

e $\mathbf{L}_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ and $\mathbf{L}_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

$$\cos\theta = \frac{|\mathbf{L}_1 \bullet \mathbf{L}_2|}{|\mathbf{L}_1|\,|\mathbf{L}_2|} = \frac{|1+2-1|}{\sqrt{3}\sqrt{6}} = \tfrac{2}{\sqrt{18}}$$

$$\therefore\ \theta = \arccos\left(\tfrac{2}{\sqrt{18}}\right) \approx 61.9^\circ$$

94 **a** There are $12! = 479\,001\,600$ possible orders.

b **i** There are $\binom{4}{2} \times 2! = 12$ ways to place I and E amongst the last 4, and the other 10 are ordered in 10! ways.

$\therefore$ the total number is $12 \times 10! = 43\,545\,600$ ways.

ii The 3 can be together in 2 ways (PIL or LIP) and this group together with the other 9 can be ordered in 10! ways.

$\therefore$ the total number is $2 \times 10! = 7\,257\,600$ ways.

iii Istvan will be between Paul and Laszlo in $\frac{1}{3}$ of all possible cases, since each of them will be the 'middle student' $\frac{1}{3}$ of the time.

$\therefore$ the total number of ways $= \frac{1}{3}$ of $12! = 159\,667\,200$ ways.

iv The students can be arranged in the form

$$\left.\begin{array}{ll} \text{A x x x H x x x x x x x x} & \text{in } 10! \text{ ways} \\ \text{H x x x A x x x x x x x x} & \text{in } 10! \text{ ways} \\ \text{x A x x x H x x x x x x x} & \text{in } 10! \text{ ways} \\ \vdots & \\ \text{x x x x x x x x H x x x A} & \text{in } 10! \text{ ways} \end{array}\right\} 8 \times 2 = 16 \text{ of these}$$

$\therefore$ the total number of ways $= 16 \times 10! = 58\,060\,800$ ways.

c **i** There are $\binom{12}{4}$ ways to choose the first group, $\binom{8}{4}$ ways to choose the second group, and $\binom{4}{4}$ ways to choose the third group. The order of groups is not important, so we divide by 3!.

So, there are $\dfrac{1}{3!}\binom{12}{4}\binom{8}{4}\binom{4}{4} = 5775$ ways.

ii There are $\binom{2}{2}\binom{10}{2}$ ways to choose the group with Ben and Marton. There are then $\binom{8}{4}$ ways to choose the second group, and $\binom{4}{4}$ ways to choose the third group. The order of groups 2 and 3 is not important, so we divide by 2!.

So, there are $\dfrac{1}{2!}\binom{2}{2}\binom{10}{2}\binom{8}{4}\binom{4}{4} = 1575$ ways.

95 **a** $a = \frac{1}{2}v^2$

$\therefore \dfrac{dv}{dt} = \dfrac{v^2}{2}$

$\therefore \dfrac{dt}{dv} = \dfrac{2}{v^2}$

b $t = \int 2v^{-2}\, dv$

$= 2\left(\dfrac{v^{-1}}{-1}\right) + c$

$= -\dfrac{2}{v} + c$

But when $t = 0$, $v = -1$

$\therefore 0 = 2 + c$

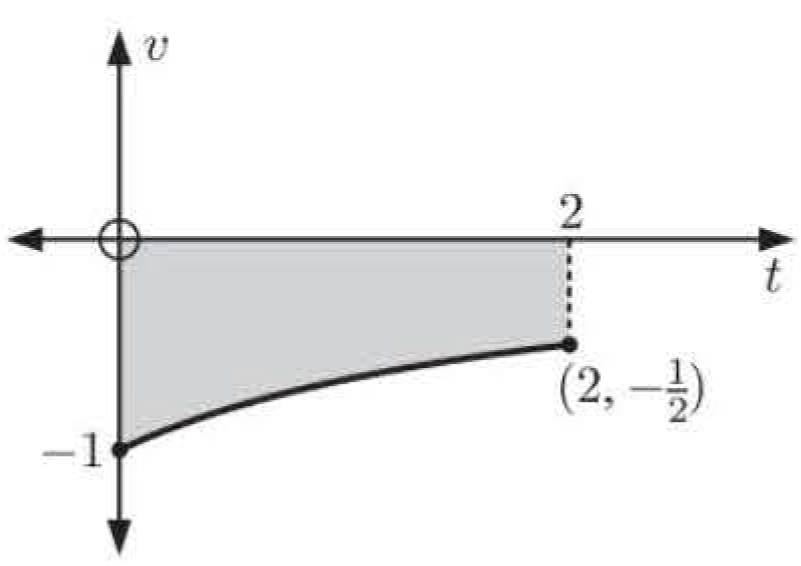
$\therefore c = -2$

Hence, $t = -\dfrac{2}{v} - 2$

$\therefore \dfrac{2}{v} = -t - 2$

$\therefore v = \dfrac{-2}{t+2}$

c



$\therefore$ there is no direction change

total distance travelled = shaded area

$= -\displaystyle\int_0^2 \frac{-2}{t+2}\, dt$

$= 2\left[\ln|t+2|\right]_0^2$

$= 2\ln 4 - 2\ln 2$

$= 2(\ln 4 - \ln 2)$

$= 2\ln 2$

≈ 1.39 units

96 P_n is: "$n^2 \times 1 + (n-1)^2 \times 2 + (n-2)^2 \times 3 + \dots + 2^2 \times (n-1) + 1^2 \times n = \dfrac{n(n+1)^2(n+2)}{12}$" for $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= 1^2 \times 1 = 1$ and RHS $= \dfrac{1 \times 2^2 \times 3}{12} = 1$ Thus P_1 is true.

(2) If P_k is assumed true then

$$k^2 \times 1 + (k-1)^2 \times 2 + (k-2)^2 \times 3 + ... + 2^2 \times (k-1) + 1^2 \times k = \frac{k(k+1)^2(k+2)}{12}$$

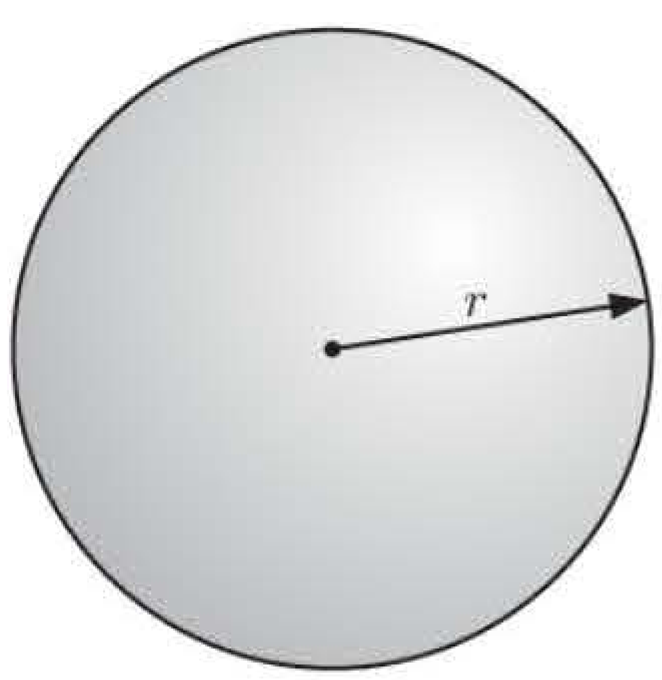
Thus,

$$\begin{aligned}
&(k+1)^2 \times 1 + k^2 \times 2 + (k-1)^2 \times 3 + ... + 3^2(k-1) + 2^2 \times k + 1^2 \times (k+1) \\
&= k^2 \times 1 + (k-1)^2 \times 2 + (k-2)^2 \times 3 + ... + 3^2(k-2) + 2^2 \times (k-1) + 1^2 \times k \\
&\quad + (k+1)^2 + k^2 + (k-1)^2 + (k-2)^2 + ... + 3^2 + 2^2 + 1^2 \\
&= \frac{k(k+1)^2(k+2)}{12} + 1^2 + 2^2 + 3^2 + 4^2 + ... + k^2 + (k+1)^2 \\
&= \frac{k(k+1)^2(k+2)}{12} + \frac{(k+1)(k+2)(2k+3)}{6} \\
&\qquad \left\{1^2 + 2^2 + 3^2 + 4^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6},\ n \in \mathbb{Z}^+\right\} \\
&= \frac{(k+1)(k+2)[k(k+1) + 2(2k+3)]}{12} \\
&= \frac{(k+1)(k+2)(k^2+5k+6)}{12} \\
&= \frac{(k+1)(k+2)(k+2)(k+3)}{12} \\
&= \frac{(k+1)([k+1]+1)^2([k+1]+2)}{12}
\end{aligned}$$

Thus P_{k+1} is true whenever P_k is true

$\therefore$ since P_1 is true, P_n is true for all $n \in \mathbb{Z}^+$. {Principle of mathematical induction}

97 **a**



$$\begin{aligned}
V &= \tfrac{4}{3}\pi r^3 \\
\therefore \frac{dV}{dt} &= 4\pi r^2 \frac{dr}{dt}
\end{aligned}$$

But $\dfrac{dr}{dt} = -\frac{8}{5}$ cm/min {as it is constant}

$$\therefore \frac{dV}{dt} = 4\pi r^2 (\tfrac{-8}{5}) = -6.4\pi r^2$$

At time $t = 2.5$, $r = 8 - \frac{8}{5} \times 2.5 = 4$ cm

$$\begin{aligned}
\therefore \frac{dV}{dt} &= -6.4\pi \times 4^2 \\
&\approx -322
\end{aligned}$$

$\therefore$ the volume is decreasing at 322 cm^3/min.

b The average change in volume $= \dfrac{V(5) - V(1)}{5-1}$

$$\begin{aligned}
&= \frac{0 - \frac{4}{3}\pi(6.4)^3}{4} \qquad \{\text{when } t = 1,\ r = 6.4\} \\
&= -\tfrac{\pi}{3}(6.4)^3 \\
&\approx -275
\end{aligned}$$

$\therefore$ on average, the volume is decreasing at 275 cm^3/min.

98 **a** Let $A = 3\log_y x$, $\quad B = 3\log_z y$, $\quad C = 7\log_x z$

$\therefore \log_y x = \dfrac{A}{3}$, $\quad \log_z y = \dfrac{B}{3}$, $\quad \log_x z = \dfrac{C}{7}$

$\therefore x = y^{\frac{A}{3}}$, $\quad y = z^{\frac{B}{3}}$, $\quad z = x^{\frac{C}{7}}$

Thus $x = y^{\frac{A}{3}} = (z^{\frac{B}{3}})^{\frac{A}{3}} = z^{\frac{AB}{9}} = (x^{\frac{C}{7}})^{\frac{AB}{9}} = x^{\frac{ABC}{63}}$

Consequently $ABC = 63$ .... (1)

But $3, A, B, C$ are in arithmetic sequence

$\therefore\ A = \dfrac{3+B}{2}$ and $B = \dfrac{A+C}{2}$

$\therefore\ B = 2A - 3$ and $C = 2B - A$

$$= 4A - 6 - A$$
$$= 3A - 6$$

Substituting in (1) gives $A(2A-3)(3A-6) = 63$

$$\therefore\ A(2A-3)(A-2) = 21$$
$$\therefore\ 2A^3 - 7A^2 + 6A - 21 = 0$$

$\therefore\ A = \frac{7}{2}$, the only real solution {technology}

Hence, $B = 2(\frac{7}{2}) - 3$ and $C = 3(\frac{7}{2}) - 6$

$= 4$ $\qquad = \frac{9}{2}$

So, $x^{18} = (y^{\frac{A}{3}})^{18}$ and $y^{21} = (z^{\frac{B}{3}})^{21}$

$= (y^{\frac{7}{6}})^{18}$ $\qquad = z^{7B}$

$= y^{21}$ $\qquad = z^{28}$

$\therefore\ x^{18} = y^{21} = z^{28}$

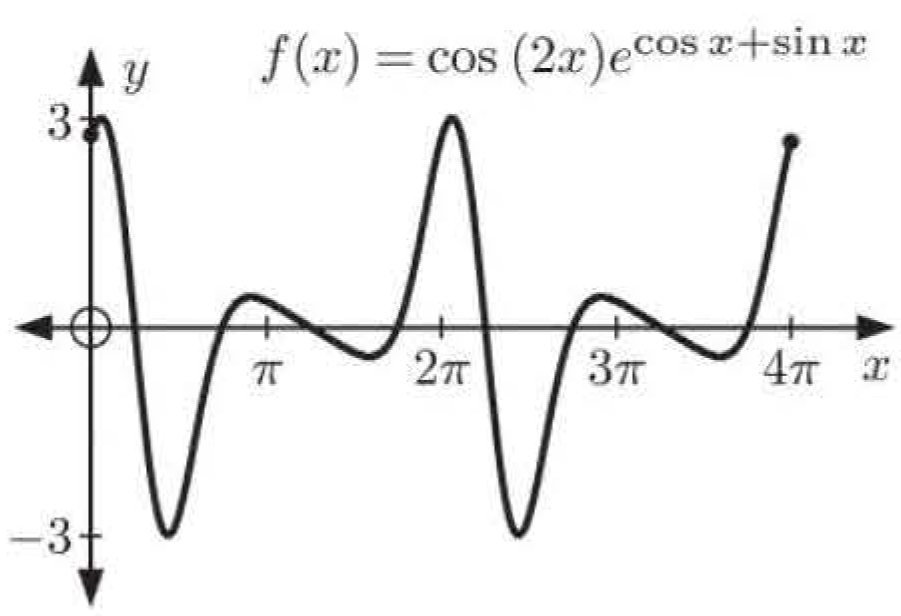
b $x^{18} = y^{21}$ and $x^{18} = z^{28}$

$\therefore\ x = y^{\frac{21}{18}}$ $\qquad \therefore\ x^{18} = z^{\frac{28}{18}}$

$\therefore\ x = y^{\frac{7}{6}}$ $\qquad \therefore\ x = y^{\frac{14}{9}}$

99 **a** The period is 2π.

b



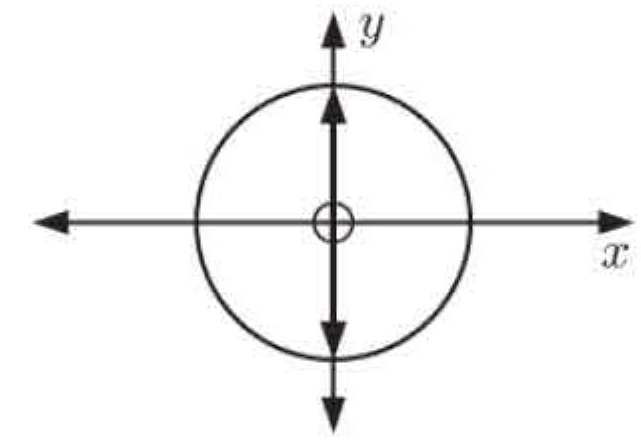
c **i** If $u = e^{\cos x + \sin x}$

then $\dfrac{du}{dx} = e^{\cos x + \sin x}(-\sin x + \cos x)$ and $\ln u = \cos x + \sin x$

$$\therefore\ \int \cos(2x)e^{\cos x+\sin x}\, dx$$
$$= \int (\cos^2 x - \sin^2 x)e^{\cos x + \sin x}\, dx$$
$$= \int (\cos x + \sin x)(\cos x - \sin x)e^{\cos x+\sin x}\, dx$$
$$= \int \ln u \frac{du}{dx}\, dx$$
$$= \int 1 \ln u\, du \qquad \left\{ \begin{array}{ll} a' = 1 & b = \ln u \\ a = u & b' = \dfrac{1}{u} \end{array} \right.$$
$$= u \ln u - \int 1\, du$$
$$= u\ln u - u + c$$
$$= e^{\cos x+\sin x}(\cos x + \sin x) - e^{\cos x + \sin x} + c$$
$$= e^{\cos x+\sin x}(\cos x + \sin x - 1) + c$$

ii $f(x)$ cuts the x-axis when $y = 0$

$$\therefore \cos(2x) = 0$$
$$\therefore 2x = \tfrac{\pi}{2} \quad \{\text{the first positive solution}\}$$
$$\therefore x = \tfrac{\pi}{4}$$

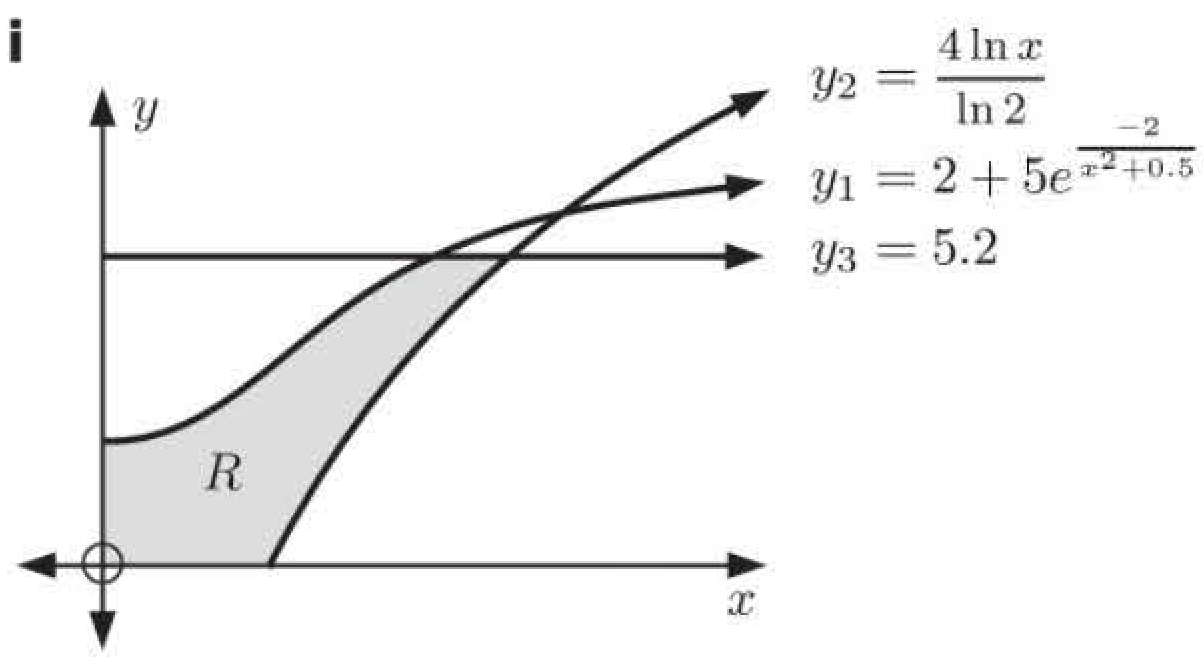


d Area $= \int_0^{\frac{\pi}{4}} \cos(2x) e^{\cos x + \sin x}\, dx$

$$= \left[e^{\cos x + \sin x}(\cos x + \sin x - 1)\right]_0^{\frac{\pi}{4}}$$
$$= e^{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}}\left(\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}} - 1\right) - e^1(1 + 0 - 1)$$
$$= e^{\sqrt{2}}(\sqrt{2} - 1) \text{ units}^2$$

(or ≈ 1.70 units^2)

100 **a** **i**



ii y_1 cuts the y-axis when $x = 0$

$\therefore y_1 = 2 + 5e^{\frac{-2}{0.5}} = 2 + 5e^{-4}$

$\therefore$ the y-intercept is $2 + 5e^{-4}$.

b **i**

$$y = 2 + 5e^{\frac{-2}{x^2+0.5}}$$
$$\therefore y - 2 = 5e^{\frac{-2}{x^2+0.5}}$$
$$\therefore \frac{y-2}{5} = e^{\frac{-2}{x^2+0.5}}$$
$$\therefore \ln\left(\frac{y-2}{5}\right) = \frac{-2}{x^2+0.5}$$
$$\therefore x^2 + 0.5 = \frac{-2}{\ln\left(\frac{y-2}{5}\right)}$$
$$\therefore x^2 = \frac{-2}{\ln\left(\frac{y-2}{5}\right)} - 0.5$$
$$\therefore x = \sqrt{\frac{-2}{\ln\left(\frac{y-2}{5}\right)} - 0.5} \quad \{x \geqslant 0\}$$

ii If $y = \dfrac{4\ln x}{\ln 2}$, then $4\ln x = y\ln 2$

$$\therefore \ln x^4 = \ln 2^y$$
$$\therefore x^4 = 2^y$$
$$\therefore x = 2^{\frac{y}{4}}$$

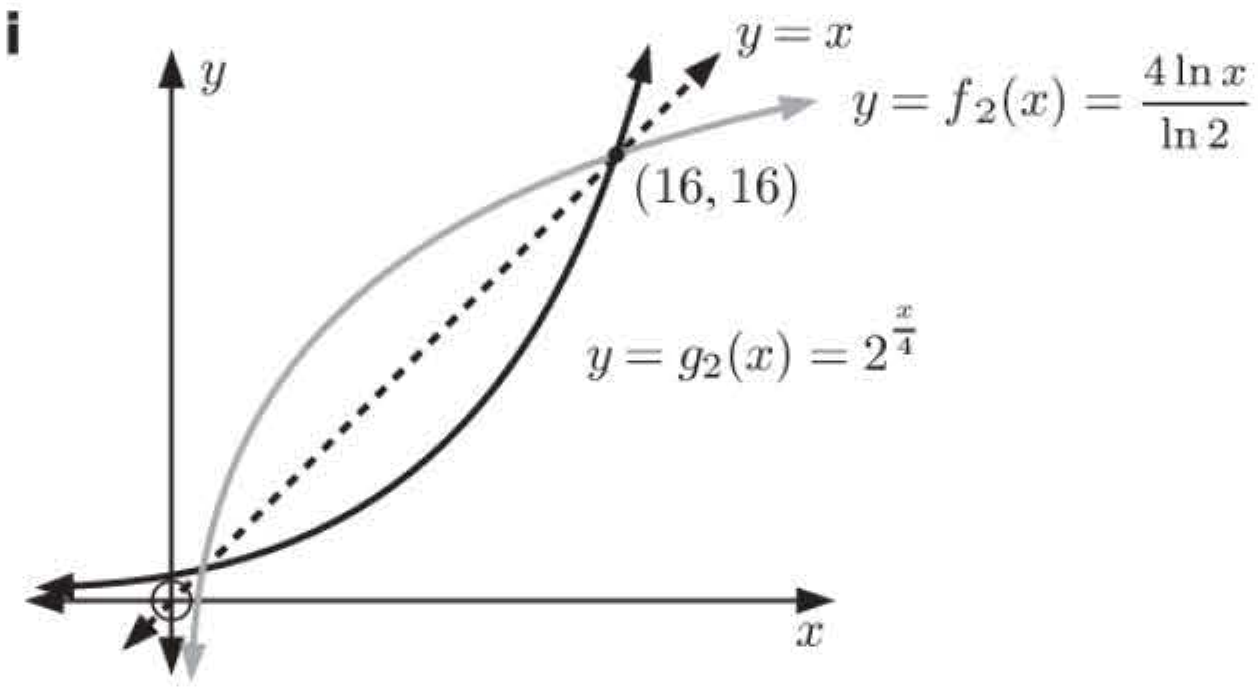
c **i** $$V = \pi\left[\int_0^{5.2} {x_2}^2\, dy - \int_{2+\frac{5}{e^4}}^{5.2} {x_1}^2\, dy\right]$$

$$\therefore V = \pi\left[\int_0^{5.2} 2^{\frac{y}{2}}\, dy - \int_{2+\frac{5}{e^4}}^{5.2}\left(\frac{-2}{\ln\left(\frac{y-2}{5}\right)} - 0.5\right) dy\right] \text{ cm}^3$$

ii Thus, $$V = \pi\left[\int_0^{5.2} 2^{\frac{y}{2}}\, dy + \int_{2+\frac{5}{e^4}}^{5.2} \frac{2}{\ln\left(\frac{y-2}{5}\right)} + 0.5\, dy\right] \text{ cm}^3$$

$$\approx 31.1 \text{ cm}^3$$

d **i**



ii $$\begin{aligned}(f_2 \circ g_2)(x) &= f_2(g_2(x)) \\ &= f_2(2^{\frac{x}{4}}) \\ &= \frac{4\ln 2^{\frac{x}{4}}}{\ln 2} \\ &= \frac{4(\frac{x}{4})\ln 2}{\ln 2} \\ &= x\end{aligned}$$

$$\begin{aligned}(g_2 \circ f_2)(x) &= g_2(f_2(x)) \\ &= g_2\left(\frac{4\ln x}{\ln 2}\right) \\ &= 2^{\frac{\ln x}{\ln 2}} \\ &= 2^{\log_2 x} \\ &= x\end{aligned}$$

iii They are inverse functions.

e **i** $f_1(x) = 2 + 5e^{\frac{-2}{x^2+0.5}}$ has inverse function $x = 2 + 5e^{\frac{-2}{y^2+0.5}}$

From **b i**, the rearrangement making y the subject is $y = \sqrt{\dfrac{-2}{\ln\left(\frac{x-2}{5}\right) - 0.5}}$

$$\therefore\ g_1(x) = \sqrt{\frac{-2}{\ln\left(\frac{x-2}{5}\right) - 0.5}}$$

ii For $f_1(x) = 2 + 5e^{\frac{-2}{x^2+0.5}}$, as $x \to \infty$, $e^{\frac{-2}{x^2+0.5}} \to e^0$

$$\therefore\ f_1(x) \to 7$$
$$\therefore\ x < 7$$

As $f_1(x)$ is increasing for all $x \geqslant 0$, $\min_x = f_1(0) = 2 + 5e^{-4}$

$\therefore$ the range of $f_1(x)$ is $\{y \mid 2 + 5e^{-4} \leqslant y < 7\}$.

$\therefore$ the domain of $g_1(x)$ is $\{x \mid 2 + 5e^{-4} \leqslant x < 7\}$.