Chapter 4
LOGARITHMS

EXERCISE 4A

1 **a** $\log 10\,000 = \log_{10} 10^4 = 4$

b $\log 0.001 = \log_{10} 10^{-3} = -3$

c $\log 10 = \log_{10} 10^1 = 1$

d $\log 1 = \log_{10} 10^0 = 0$

e $\log \sqrt{10} = \log_{10} 10^{\frac{1}{2}} = \frac{1}{2}$

f $\log \sqrt[3]{10} = \log_{10} 10^{\frac{1}{3}} = \frac{1}{3}$

g $\log \left(\dfrac{1}{\sqrt[4]{10}}\right) = \log_{10} 10^{-\frac{1}{4}} = -\frac{1}{4}$

h $\log 10\sqrt{10} = \log_{10} 10^{\frac{3}{2}} = \frac{3}{2} = 1\frac{1}{2}$

i $\log \sqrt[3]{100} = \log_{10}(10^2)^{\frac{1}{3}} = \log_{10} 10^{\frac{2}{3}} = \frac{2}{3}$

j $\log\left(\dfrac{100}{\sqrt{10}}\right) = \log_{10}\left(\dfrac{10^2}{10^{\frac{1}{2}}}\right) = \log_{10} 10^{\frac{3}{2}} = \frac{3}{2} = 1\frac{1}{2}$

k $\log\left(10 \times \sqrt[3]{10}\right) = \log_{10}(10^1 \times 10^{\frac{1}{3}}) = \log_{10} 10^{\frac{4}{3}} = \frac{4}{3} = 1\frac{1}{3}$

l $\log 1000\sqrt{10} = \log_{10}(10^3 \times 10^{\frac{1}{2}}) = \log_{10} 10^{\frac{7}{2}} = \frac{7}{2} = 3\frac{1}{2}$

2 **a** $\log 10^n = \log_{10} 10^n = n$

b $\log(10^a \times 100) = \log_{10}\left(10^a \times 10^2\right) = \log_{10}\left(10^{a+2}\right) = a + 2$

c $\log\left(\dfrac{10}{10^m}\right) = \log_{10}(10^{1-m}) = 1 - m$

d $\log\left(\dfrac{10^a}{10^b}\right) = \log_{10}(10^{a-b}) = a - b$

3 **a** $6 = 10^{\log 6} \approx 10^{0.7782}$

b $60 = 10^{\log 60} \approx 10^{1.7782}$

c $6000 = 10^{\log 6000} \approx 10^{3.7782}$

d $0.6 = 10^{\log(0.6)} \approx 10^{-0.2218}$

e $0.006 = 10^{\log(0.006)} \approx 10^{-2.2218}$

f $15 = 10^{\log 15} \approx 10^{1.1761}$

g $1500 = 10^{\log 1500} \approx 10^{3.1761}$

h $1.5 = 10^{\log 1.5} \approx 10^{0.1761}$

i $0.15 = 10^{\log(0.15)} \approx 10^{-0.8239}$

j $0.000\,15 = 10^{\log(0.000\,15)} \approx 10^{-3.8239}$

4 **a** **i** $\log 3 \approx 0.477$ **ii** $\log 300 \approx 2.477$

b
$$\begin{aligned}\log 300 &= \log(3 \times 10^2)\\ &\approx \log(10^{0.477} \times 10^2)\\ &\approx \log 10^{2.477}\\ &\approx 2.477\\ &\approx \log 3 + 2\end{aligned}$$

5 **a** **i** $\log 5 \approx 0.699$ **ii** $\log 0.05 \approx -1.301$

b
$$\begin{aligned}\log 0.05 &= \log(5 \times 10^{-2})\\ &\approx \log(10^{0.699} \times 10^{-2})\\ &\approx \log 10^{(0.699-2)}\\ &\approx 0.699 - 2\\ &\approx \log 5 - 2\end{aligned}$$

6 **a** $\log x = 2$ $\therefore\ x = 10^2$ $\therefore\ x = 100$

b $\log x = 1$ $\therefore\ x = 10^1$ $\therefore\ x = 10$

c $\log x = 0$ $\therefore\ x = 10^0$ $\therefore\ x = 1$

d $\log x = -1$ $\therefore\ x = 10^{-1}$ $\therefore\ x = \frac{1}{10}$

e $\log x = \frac{1}{2}$ $\therefore\ x = 10^{\frac{1}{2}}$ $(= \sqrt{10})$

f $\log x = -\frac{1}{2}$ $\therefore\ x = 10^{-\frac{1}{2}}$ $\left(= \dfrac{1}{10^{\frac{1}{2}}} = \dfrac{1}{\sqrt{10}}\right)$

g $\log x = 4$ $\therefore\ x = 10^4$ $\therefore\ x = 10\,000$

h $\log x = -5$ $\therefore\ x = 10^{-5}$ $\therefore\ x = 0.000\,01$

i $\log x \approx 0.8351$ $\therefore\ x \approx 10^{0.8351}$ $\therefore\ x \approx 6.84$

j $\log x \approx 2.1457$ $\therefore\ x \approx 10^{2.1457}$ $\therefore\ x \approx 140$

k $\log x \approx -1.378$ $\therefore\ x \approx 10^{-1.378}$ $\therefore\ x \approx 0.0419$

l $\log x \approx -3.1997$ $\therefore\ x \approx 10^{-3.1997}$ $\therefore\ x \approx 0.000\,631$

EXERCISE 4B

1 **a** $10^2 = 100$ **b** $10^4 = 10\,000$ **c** $10^{-1} = 0.1$

d $10^{\frac{1}{2}} = \sqrt{10}$ **e** $2^3 = 8$ **f** $3^2 = 9$

g $2^{-2} = \frac{1}{4}$ **h** $3^{1.5} = \sqrt{27}$ **i** $5^{-\frac{1}{2}} = \frac{1}{\sqrt{5}}$

2 **a** $\log_2 4 = 2$ **b** $\log_4 64 = 3$ **c** $\log_5 25 = 2$

d $\log_7 49 = 2$ **e** $\log_2 64 = 6$ **f** $\log_2\left(\frac{1}{8}\right) = -3$

g $\log_{10}(0.01) = -2$ **h** $\log_2\left(\frac{1}{2}\right) = -1$ **i** $\log_3\left(\frac{1}{27}\right) = -3$

3 **a** $\log_{10} 100\,000$ $= \log_{10} 10^5$ $= 5$

b $\log_{10}(0.01)$ $= \log_{10} 10^{-2}$ $= -2$

c $\log_3 \sqrt{3}$ $= \log_3 3^{\frac{1}{2}}$ $= \frac{1}{2}$

d $\log_2 8$ $= \log_2 2^3$ $= 3$

e $\log_2 64$ $= \log_2 2^6$ $= 6$

f $\log_2 128$ $= \log_2 2^7$ $= 7$

g $\log_5 25$ $= \log_5 5^2$ $= 2$

h $\log_5 125$ $= \log_5 5^3$ $= 3$

i $\log_2(0.125)$ $= \log_2\left(\frac{1}{8}\right)$ $= \log_2\left(2^{-3}\right)$ $= -3$

j $\log_9 3$ $= \log_9 9^{\frac{1}{2}}$ $= \frac{1}{2}$

k $\log_4 16$ $= \log_4 4^2$ $= 2$

l $\log_{36} 6$ $= \log_{36} 36^{\frac{1}{2}}$ $= \frac{1}{2}$

m $\log_3 243$ $= \log_3 3^5$ $= 5$

n $\log_2 \sqrt[3]{2}$ $= \log_2 2^{\frac{1}{3}}$ $= \frac{1}{3}$

o $\log_a a^n$ $= n,\ a > 0$

p $\log_8 2$ $= \log_8 8^{\frac{1}{3}}$ $= \frac{1}{3}$

q $\log_t\left(\frac{1}{t}\right)$ $= \log_t t^{-1}$ $= -1,\ t > 0$

r $\log_6 6\sqrt{6}$ $= \log_6(6^1 \times 6^{\frac{1}{2}})$ $= \log_6 6^{\frac{3}{2}}$ $= \frac{3}{2}$

s $\log_4 1$ $= \log_4 4^0$ $= 0$

t $\log_9 9$ $= \log_9 9^1$ $= 1$

4 **a** $\log_{10} 152 \approx 2.18$ **b** $\log_{10} 25 \approx 1.40$ **c** $\log_{10} 74 \approx 1.87$ **d** $\log_{10} 0.8 \approx -0.0969$

5

a $\log_2 x = 3$
$\therefore\ x = 2^3$
$\therefore\ x = 8$

b $\log_4 x = \frac{1}{2}$
$\therefore\ x = 4^{\frac{1}{2}}$
$\therefore\ x = 2$

c $\log_x 81 = 4$
$\therefore\ 81 = x^4$
$\therefore\ x = \pm\sqrt[4]{81}$
$\therefore\ x = \pm 3$
$\therefore\ x = 3\ \{\text{as } x > 0\}$

d $\log_2(x-6) = 3$
$\therefore\ x - 6 = 2^3$
$\therefore\ x - 6 = 8$
$\therefore\ x = 14$

6

a $\log_4 16$
$= \log_4 4^2$
$= 2$

b $\log_2 4$
$= \log_2 2^2$
$= 2$

c $\log_3(\frac{1}{3})$
$= \log_3 3^{-1}$
$= -1$

d $\log_{10} \sqrt[4]{1000}$
$= \log_{10}(10^3)^{\frac{1}{4}}$
$= \log_{10} 10^{\frac{3}{4}}$
$= \frac{3}{4}$

e $\log_7\left(\frac{1}{\sqrt{7}}\right)$
$= \log_7 7^{-\frac{1}{2}}$
$= -\frac{1}{2}$

f $\log_5(25\sqrt{5})$
$= \log_5(5^2 5^{\frac{1}{2}})$
$= \log_5 5^{\frac{5}{2}}$
$= \frac{5}{2}$

g $\log_3\left(\frac{1}{\sqrt{27}}\right)$
$= \log_3\left(\dfrac{1}{(3^3)^{\frac{1}{2}}}\right)$
$= \log_3 3^{-\frac{3}{2}}$
$= -\frac{3}{2}$

h $\log_4\left(\frac{1}{2\sqrt{2}}\right)$
$= \log_4\left(2^{-\frac{3}{2}}\right)$
$= \log_4\left((2^2)^{-\frac{3}{4}}\right)$
$= \log_4 4^{-\frac{3}{4}}$
$= -\frac{3}{4}$

i $\log_x x^2$
$= 2,\ x > 0$

j $\log_x \sqrt{x}$
$= \log_x x^{\frac{1}{2}}$
$= \frac{1}{2},\ x > 0$

k $\log_m m^3$
$= 3,\ m > 0$

l $\log_x(x\sqrt{x})$
$= \log_x(x^1 \times x^{\frac{1}{2}})$
$= \log_x x^{\frac{3}{2}}$
$= \frac{3}{2},\ x > 0$

m $\log_n\left(\frac{1}{n}\right)$
$= \log_n n^{-1}$
$= -1,\ n > 0$

n $\log_a\left(\frac{1}{a^2}\right)$
$= \log_a a^{-2}$
$= -2,\ a > 0$

o $\log_a\left(\frac{1}{\sqrt{a}}\right)$
$= \log_a a^{-\frac{1}{2}}$
$= -\frac{1}{2},\ a > 0$

p $\log_m \sqrt{m^5}$
$= \log_m (m^5)^{\frac{1}{2}}$
$= \log_m m^{\frac{5}{2}}$
$= \frac{5}{2},\ m > 0$

EXERCISE 4C.1

1

a $\log 8 + \log 2$
$= \log(8 \times 2)$
$= \log 16$

b $\log 4 + \log 5$
$= \log(4 \times 5)$
$= \log 20$

c $\log 40 - \log 5$
$= \log\left(\frac{40}{5}\right)$
$= \log 8$

d $\log p - \log m$
$= \log\left(\frac{p}{m}\right)$

e $\log_4 8 - \log_4 2$
$= \log_4\left(\frac{8}{2}\right)$
$= \log_4 4$
$= 1$

f $\log 5 + \log(0.4)$
$= \log(5 \times 0.4)$
$= \log 2$

g $\log 2 + \log 3 + \log 4$
$= \log(2 \times 3 \times 4)$
$= \log 24$

h $1 + \log_2 3$
$= \log_2 2^1 + \log_2 3$
$= \log_2(2 \times 3)$
$= \log_2 6$

i $\log 4 - 1$
$= \log 4 - \log 10^1$
$= \log\left(\frac{4}{10}\right)$
$= \log 0.4$

j $\log 5+\log 4-\log 2$
$=\log\left(\frac{5\times 4}{2}\right)$
$=\log 10$
$=1$

k $2+\log 2$
$=\log 10^2+\log 2$
$=\log(100\times 2)$
$=\log 200$

l $t+\log w$
$=\log 10^t+\log w$
$=\log\left(10^t\times w\right)$

m $\log_m 40-2$
$=\log_m 40-\log_m m^2$
$=\log_m\left(\frac{40}{m^2}\right)$

n $\log_3 6-\log_3 2-\log_3 3$
$=\log_3(6\div 2\div 3)$
$=\log_3 1$
$=0$

o $\log 50-4$
$=\log 50-\log 10^4$
$=\log\left(\frac{50}{10^4}\right)$
$=\log 0.005$

p $3-\log_5 50$
$=\log_5 5^3-\log_5 50$
$=\log_5\left(\frac{125}{50}\right)$
$=\log_5\left(\frac{5}{2}\right)$

q $\log_5 100-\log_5 4$
$=\log_5\left(\frac{100}{4}\right)$
$=\log_5 25$
$=\log_5 5^2$
$=2$

r $\log\left(\frac{4}{3}\right)+\log 3+\log 7$
$=\log\left(\frac{4}{3}\times 3\times 7\right)$
$=\log 28$

2 **a** $5\log 2+\log 3$
$=\log 2^5+\log 3$
$=\log(2^5\times 3)$
$=\log 96$

b $2\log 3+3\log 2$
$=\log 3^2+\log 2^3$
$=\log(9\times 8)$
$=\log 72$

c $3\log 4-\log 8$
$=\log 4^3-\log 8$
$=\log\left(\frac{64}{8}\right)$
$=\log 8$

d $2\log_3 5-3\log_3 2$
$=\log_3 5^2-\log_3 2^3$
$=\log_3\left(\frac{25}{8}\right)$

e $\frac{1}{2}\log_6 4+\log_6 3$
$=\log_6 4^{\frac{1}{2}}+\log_6 3$
$=\log_6(2\times 3)$
$=\log_6 6$
$=1$

f $\frac{1}{3}\log\left(\frac{1}{8}\right)$
$=\log\left(\frac{1}{8}\right)^{\frac{1}{3}}$
$=\log\left(2^{-3}\right)^{\frac{1}{3}}$
$=\log 2^{-1}$
$=\log\left(\frac{1}{2}\right)$ or $-\log 2$

g $3-\log 2-2\log 5$
$=\log 10^3-\log 2-\log 5^2$
$=\log(1000\div 2\div 25)$
$=\log 20$

h $1-3\log 2+\log 20$
$=\log 10^1-\log 2^3+\log 20$
$=\log(10\div 8\times 20)$
$=\log 25$

i $2-\frac{1}{2}\log_n 4-\log_n 5$
$=\log_n n^2-\log_n 4^{\frac{1}{2}}-\log_n 5$
$=\log_n(n^2\div 2\div 5)$
$=\log_n\left(\frac{n^2}{10}\right)$

3 **a** $\dfrac{\log 4}{\log 2}$
$=\dfrac{\log 2^2}{\log 2}$
$=\dfrac{2\log 2}{\log 2}$
$=2$

b $\dfrac{\log_5 27}{\log_5 9}$
$=\dfrac{\log_5 3^3}{\log_5 3^2}$
$=\dfrac{3\log_5 3}{2\log_5 3}$
$=\frac{3}{2}$

c $\dfrac{\log 8}{\log 2}$
$=\dfrac{\log 2^3}{\log 2}$
$=\dfrac{3\log 2}{\log 2}$
$=3$

d $\dfrac{\log 3}{\log 9}$
$=\dfrac{\log 3}{\log 3^2}$
$=\dfrac{\log 3}{2\log 3}$
$=\frac{1}{2}$

e $\dfrac{\log_3 25}{\log_3(0.2)}$
$=\dfrac{\log_3 5^2}{\log_3 5^{-1}}$
$=\dfrac{2\log_3 5}{-1\log_3 5}$
$=-2$

f $\dfrac{\log_4 8}{\log_4(0.25)}$
$=\dfrac{\log_4 2^3}{\log_4 2^{-2}}$ $\{0.25=\frac{1}{4}=\frac{1}{2^2}\}$
$=\dfrac{3\log_4 2}{-2\log_4 2}$
$=-\frac{3}{2}$

4 **a** $\log 9 = \log 3^2$
$= 2\log 3$

b $\log\sqrt{2} = \log 2^{\frac{1}{2}}$
$= \frac{1}{2}\log 2$

c $\log\left(\frac{1}{8}\right) = \log\left(\frac{1}{2^3}\right)$
$= \log 2^{-3}$
$= -3\log 2$

d $\log\left(\frac{1}{5}\right) = \log 5^{-1}$
$= -1\log 5$
$= -\log 5$

e $\log 5 = \log\left(\frac{10}{2}\right)$
$= \log 10^1 - \log 2$
$= 1 - \log 2$

f $\log 5000 = \log\left(\frac{10\,000}{2}\right)$
$= \log 10^4 - \log 2$
$= 4 - \log 2$

5 **a** $\log_b 6$
$= \log_b(2\times 3)$
$= \log_b 2 + \log_b 3$
$= p + q$

b $\log_b 45$
$= \log_b(3^2 5)$
$= 2\log_b 3 + \log_b 5$
$= 2q + r$

c $\log_b 108$
$= \log_b(2^2 3^3)$
$= 2\log_b 2 + 3\log_b 3$
$= 2p + 3q$

d $\log_b\left(\frac{5\sqrt{3}}{2}\right)$
$= \log_b(5\times 3^{\frac{1}{2}}) - \log_b 2$
$= \log_b 5 + \frac{1}{2}\log_b 3 - \log_b 2$
$= r + \frac{1}{2}q - p$

e $\log_b\left(\frac{5}{32}\right)$
$= \log_b 5 - \log_b 2^5$
$= \log_b 5 - 5\log_b 2$
$= r - 5p$

f $\log_b(0.\overline{2})$
$= \log_b(\frac{2}{9})$
$= \log_b 2 - \log_b 3^2$
$= p - 2q$

6 **a** $\log_2(PR)$
$= \log_2 P + \log_2 R$
$= x + z$

b $\log_2(RQ^2)$
$= \log_2 R + \log_2 Q^2$
$= \log_2 R + 2\log_2 Q$
$= z + 2y$

c $\log_2\left(\frac{PR}{Q}\right)$
$= \log_2(PR) - \log_2 Q$
$= \log_2 P + \log_2 R - \log_2 Q$
$= x + z - y$

d $\log_2\left(P^2\sqrt{Q}\right)$
$= \log_2 P^2 + \log_2 Q^{\frac{1}{2}}$
$= 2\log_2 P + \frac{1}{2}\log_2 Q$
$= 2x + \frac{1}{2}y$

e $\log_2\left(\frac{Q^3}{\sqrt{R}}\right)$
$= \log_2 Q^3 - \log_2 R^{\frac{1}{2}}$
$= 3\log_2 Q - \frac{1}{2}\log_2 R$
$= 3y - \frac{1}{2}z$

f $\log_2\left(\frac{R^2\sqrt{Q}}{P^3}\right)$
$= \log_2 R^2 + \log_2 Q^{\frac{1}{2}} - \log_2 P^3$
$= 2\log_2 R + \frac{1}{2}\log_2 Q - 3\log_2 P$
$= 2z + \frac{1}{2}y - 3x$

7 **a** $\log_t N^2 = 1.72$
$\therefore\ 2\log_t N = 1.72$
$\therefore\ \log_t N = 1.72 \div 2$
$= 0.86$

b $\log_t(MN)$
$= \log_t M + \log_t N$
$= 1.29 + 0.86$
$= 2.15$

c $\log_t\left(\frac{N^2}{\sqrt{M}}\right)$
$= \log_t N^2 - \log_t M^{\frac{1}{2}}$
$= 1.72 - \frac{1}{2}\log_t M$
$= 1.72 - \frac{1}{2}(1.29)$
$= 1.075$

EXERCISE 4C.2

1 **a** $y = 2^x$
$\therefore\ \log y = \log 2^x$
$\therefore\ \log y = x\log 2$

b $y = 20b^3$
$\therefore\ \log y = \log(20b^3)$
$\therefore\ \log y = \log 20 + \log b^3$
$\therefore\ \log y \approx 1.30 + 3\log b$

c $M = ad^4$
$\therefore\ \log M = \log(ad^4)$
$\therefore\ \log M = \log a + \log d^4$
$\therefore\ \log M = \log a + 4\log d$

d $T = 5\sqrt{d} = 5d^{\frac{1}{2}}$
$\therefore\ \log T = \log(5d^{\frac{1}{2}})$
$\therefore\ \log T = \log 5 + \log d^{\frac{1}{2}}$
$\therefore\ \log T \approx 0.699 + \frac{1}{2}\log d$

e $R = b\sqrt{l} = bl^{\frac{1}{2}}$
$\therefore\ \log R = \log(bl^{\frac{1}{2}})$
$\therefore\ \log R = \log b + \log l^{\frac{1}{2}}$
$\therefore\ \log R = \log b + \frac{1}{2}\log l$

f $Q = \frac{a}{b^n}$
$\therefore\ \log Q = \log\left(\frac{a}{b^n}\right)$
$\therefore\ \log Q = \log a - \log b^n$
$\therefore\ \log Q = \log a - n\log b$

g $y = ab^x$
$\therefore\ \log y = \log(ab^x)$
$\therefore\ \log y = \log a + \log b^x$
$\therefore\ \log y = \log a + x\log b$

h $F = \dfrac{20}{\sqrt{n}} = \dfrac{20}{n^{\frac{1}{2}}}$
$\therefore\ \log F = \log\left(\dfrac{20}{n^{\frac{1}{2}}}\right)$
$\therefore\ \log F = \log 20 - \log n^{\frac{1}{2}}$
$\therefore\ \log F \approx 1.30 - \frac{1}{2}\log n$

i $L = \dfrac{ab}{c}$
$\therefore\ \log L = \log\left(\dfrac{ab}{c}\right)$
$\therefore\ \log L = \log ab - \log c$
$\therefore\ \log L = \log a + \log b - \log c$

j $N = \sqrt{\dfrac{a}{b}}$
$\therefore\ N = \left(\dfrac{a}{b}\right)^{\frac{1}{2}}$
$\therefore\ \log N = \log\left(\dfrac{a}{b}\right)^{\frac{1}{2}}$
$\therefore\ \log N = \frac{1}{2}\log\left(\dfrac{a}{b}\right)$
$\therefore\ \log N = \frac{1}{2}\log a - \frac{1}{2}\log b$

k $S = 200 \times 2^t$
$\therefore\ \log S = \log(200 \times 2^t)$
$\therefore\ \log S = \log 200 + \log 2^t$
$\therefore\ \log S = \log 200 + t\log 2$
$\therefore\ \log S \approx 2.30 + t\log 2$

l $y = \dfrac{a^m}{b^n}$
$\therefore\ \log y = \log\left(\dfrac{a^m}{b^n}\right)$
$\therefore\ \log y = \log a^m - \log b^n$
$\therefore\ \log y = m\log a - n\log b$

2 **a** $\log D = \log e + \log 2$
$= \log(e \times 2)$
$\therefore\ D = 2e$

b $\log_a F = \log_a 5 - \log_a t$
$= \log_a\left(\dfrac{5}{t}\right)$
$\therefore\ F = \dfrac{5}{t}$

c $\log P = \frac{1}{2}\log x$
$= \log x^{\frac{1}{2}}$
$\therefore\ P = \sqrt{x}$

d $\log_n M = 2\log_n b + \log_n c$
$= \log_n b^2 + \log_n c$
$= \log_n(b^2c)$
$\therefore\ M = b^2c$

e $\log B = 3\log m - 2\log n$
$= \log m^3 - \log n^2$
$= \log\left(\dfrac{m^3}{n^2}\right)$
$\therefore\ B = \dfrac{m^3}{n^2}$

f $\log N = -\frac{1}{3}\log p$
$= \log p^{-\frac{1}{3}}$
$= \log\left(\dfrac{1}{\sqrt[3]{p}}\right)$
$\therefore\ N = \dfrac{1}{\sqrt[3]{p}}$

g $\log P = 3\log x + 1$
$= \log x^3 + \log 10^1$
$= \log(10x^3)$
$\therefore\ P = 10x^3$

h $\log_a Q = 2 - \log_a x$
$= \log_a a^2 - \log_a x$
$= \log_a\left(\dfrac{a^2}{x}\right)$
$\therefore\ Q = \dfrac{a^2}{x}$

3 **a** $y = 3 \times 2^x$
$\therefore\ \log_2 y = \log_2(3 \times 2^x)$
$\therefore\ \log_2 y = \log_2 3 + \log_2 2^x$
$\therefore\ \log_2 y = \log_2 3 + x$

b $\log_2 y = \log_2 3 + x$ {from **a**}
$\therefore\ x = \log_2 y - \log_2 3$
$\therefore\ x = \log_2\left(\dfrac{y}{3}\right)$

c **i** When $y = 3$,
$x = \log_2\left(\frac{3}{3}\right)$
$\therefore\ x = \log_2 1$
$\therefore\ x = 0$

ii When $y = 12$,
$x = \log_2\left(\frac{12}{3}\right)$
$\therefore\ x = \log_2 4$
$\therefore\ x = \log_2 2^2$
$\therefore\ x = 2$

iii When $y = 30$,
$x = \log_2\left(\frac{30}{3}\right)$
$\therefore\ x = \log_2 10$
$\therefore\ x \approx 3.32$

4 **a** $\log_3 27 + \log_3\left(\frac{1}{3}\right) = \log_3 x$

$\therefore \log_3\left(27 \times \frac{1}{3}\right) = \log_3 x$

$\therefore \log_3 9 = \log_3 x$

$\therefore x = 9$

b $\log_5 x = \log_5 8 - \log_5(6-x)$

$\therefore \log_5 x = \log_5\left(\frac{8}{6-x}\right)$

$\therefore x = \frac{8}{6-x}$

$\therefore 6x - x^2 = 8$

$\therefore x^2 - 6x + 8 = 0$

$\therefore (x-2)(x-4) = 0$

$\therefore x = 2 \text{ or } 4$

Note: $x > 0$ and $6 - x > 0$ so $0 < x < 6$

c $\log_5 125 - \log_5 \sqrt{5} = \log_5 x$

$\therefore \log_5\left(\frac{125}{\sqrt{5}}\right) = \log_5 x$

$\therefore x = \frac{125}{\sqrt{5}} = 25\sqrt{5}$

d $\log_{20} x = 1 + \log_{20} 10$

$\therefore \log_{20} x = \log_{20} 20^1 + \log_{20} 10$

$= \log_{20} 200$

$\therefore x = 200$

e $\log x + \log(x+1) = \log 30$

$\therefore \log[x(x+1)] = \log 30$

$\therefore x^2 + x = 30$

$\therefore x^2 + x - 30 = 0$

$\therefore (x+6)(x-5) = 0$

$\therefore x = -6 \text{ or } 5$

but $x > 0$ for $\log x$ to exist

$\therefore x = 5$

f $\log(x+2) - \log(x-2) = \log 5$

$\therefore \log\left(\frac{x+2}{x-2}\right) = \log 5$

$\therefore \frac{x+2}{x-2} = 5$

$\therefore x + 2 = 5x - 10$

$\therefore -4x = -12$

$\therefore x = 3$

Note: $x + 2 > 0$ and $x - 2 > 0$

$\therefore x > 2$ ✓

EXERCISE 4D.1

1 **a** $\ln e^2$
$= 2 \quad \{\ln e^x = x\}$

b $\ln e^3$
$= 3$

c $\ln \sqrt{e}$
$= \ln e^{\frac{1}{2}}$
$= \frac{1}{2}$

d $\ln 1$
$= \ln e^0$
$= 0$

e $\ln\left(\frac{1}{e}\right)$
$= \ln e^{-1}$
$= -1$

f $\ln \sqrt[3]{e}$
$= \ln e^{\frac{1}{3}}$
$= \frac{1}{3}$

g $\ln\left(\frac{1}{e^2}\right)$
$= \ln e^{-2}$
$= -2$

h $\ln\left(\frac{1}{\sqrt{e}}\right)$
$= \ln e^{-\frac{1}{2}}$
$= -\frac{1}{2}$

2 **a** $e^{\ln 3}$
$= 3$
$\{\text{using } e^{\ln x} = x\}$

b $e^{2\ln 3}$
$= 3^2$
$\{\text{using } e^{x \ln a} = a^x\}$
$= 9$

c $e^{-\ln 5}$
$= e^{-1 \ln 5}$
$= 5^{-1}$
$= \frac{1}{5}$

d $e^{-2\ln 2}$
$= 2^{-2}$
$= \frac{1}{2^2}$
$= \frac{1}{4}$

3 $\ln x$ exists only when $x > 0$.
$\therefore \ln(-2)$ and $\ln(0)$ do not exist.

Note: If $\ln(-2) = a$ then $-2 = e^a$
and $e^a = -2$ has no solutions as $e^a > 0$ for all a.

4 **a** $\ln e^a$
$= a$

b $\ln(e \times e^a)$
$= \ln e^{1+a}$
$= a + 1$

c $\ln(e^a \times e^b)$
$= \ln(e^{a+b})$
$= a + b$

d $\ln(e^a)^b$
$= \ln e^{ab}$
$= ab$

e $\ln\left(\frac{e^a}{e^b}\right)$
$= \ln(e^{a-b})$
$= a - b$

5 **a** $6 = e^{1.7918}$ **b** $60 = e^{4.0943}$ **c** $6000 = e^{8.6995}$ **d** $0.6 = e^{-0.5108}$
e $0.006 = e^{-5.1160}$ **f** $15 = e^{2.7081}$ **g** $1500 = e^{7.3132}$ **h** $1.5 = e^{0.4055}$
i $0.15 = e^{-1.8971}$ **j** $0.000\,15 = e^{-8.8049}$

6 **a** $\ln x = 3$
$\therefore\ x = e^3$
$\therefore\ x \approx 20.1$

b $\ln x = 1$
$\therefore\ x = e^1$
$\therefore\ x = e \approx 2.72$

c $\ln x = 0$
$\therefore\ x = e^0$
$\therefore\ x = 1$

d $\ln x = -1$
$\therefore\ x = e^{-1}$
$\therefore\ x \approx 0.368$

e $\ln x = -5$
$\therefore\ x = e^{-5}$
$\therefore\ x \approx 0.006\,74$

f $\ln x \approx 0.835$
$\therefore\ x \approx e^{0.835}$
$\therefore\ x \approx 2.30$

g $\ln x \approx 2.145$
$\therefore\ x \approx e^{2.145}$
$\therefore\ x \approx 8.54$

h $\ln x \approx -3.2971$
$\therefore\ x \approx e^{-3.2971}$
$\therefore\ x \approx 0.0370$

EXERCISE 4D.2

1 **a** $\ln 15 + \ln 3$
$= \ln(15 \times 3)$
$= \ln 45$

b $\ln 15 - \ln 3$
$= \ln(\frac{15}{3})$
$= \ln 5$

c $\ln 20 - \ln 5$
$= \ln(\frac{20}{5})$
$= \ln 4$

d $\ln 4 + \ln 6$
$= \ln(4 \times 6)$
$= \ln 24$

e $\ln 5 + \ln(0.2)$
$= \ln(5 \times 0.2)$
$= \ln 1$
$= 0$

f $\ln 2 + \ln 3 + \ln 5$
$= \ln(2 \times 3 \times 5)$
$= \ln 30$

g $1 + \ln 4$
$= \ln e^1 + \ln 4$
$= \ln(e \times 4)$
$= \ln(4e)$

h $\ln 6 - 1$
$= \ln 6 - \ln e^1$
$= \ln\left(\frac{6}{e}\right)$

i $\ln 5 + \ln 8 - \ln 2$
$= \ln(5 \times 8 \div 2)$
$= \ln 20$

j $2 + \ln 4$
$= \ln e^2 + \ln 4$
$= \ln(e^2 \times 4)$
$= \ln(4e^2)$

k $\ln 20 - 2$
$= \ln 20 - \ln e^2$
$= \ln\left(\frac{20}{e^2}\right)$

l $\ln 12 - \ln 4 - \ln 3$
$= \ln(12 \div 4 \div 3)$
$= \ln 1$
$= 0$

2 **a** $5\ln 3 + \ln 4$
$= \ln(3^5) + \ln 4$
$= \ln(243 \times 4)$
$= \ln 972$

b $3\ln 2 + 2\ln 5$
$= \ln(2^3) + \ln(5^2)$
$= \ln(8 \times 25)$
$= \ln 200$

c $3\ln 2 - \ln 8$
$= \ln(2^3) - \ln 8$
$= \ln(\frac{8}{8})$
$= \ln 1 = 0$

d $3\ln 4 - 2\ln 2$
$= \ln(4^3) - \ln(2^2)$
$= \ln(\frac{64}{4})$
$= \ln 16$

e $\frac{1}{3}\ln 8 + \ln 3$
$= \ln(8^{\frac{1}{3}}) + \ln 3$
$= \ln(2 \times 3)$
$= \ln 6$

f $\frac{1}{3}\ln(\frac{1}{27})$
$= \ln\left((\frac{1}{27})^{\frac{1}{3}}\right)$
$= \ln\left(\frac{1}{27^{\frac{1}{3}}}\right)$
$= \ln(\frac{1}{3})$

g $-\ln 2$
$= \ln(2^{-1})$
$= \ln(\frac{1}{2})$

h $-\ln(\frac{1}{2})$
$= \ln((\frac{1}{2})^{-1})$
$= \ln 2$

i $-2\ln(\frac{1}{4})$
$= \ln((\frac{1}{4})^{-2})$
$= \ln(4^2)$
$= \ln 16$

3 **a** $\ln 27$
$= \ln 3^3$
$= 3\ln 3$

b $\ln\sqrt{3}$
$= \ln 3^{\frac{1}{2}}$
$= \frac{1}{2}\ln 3$

c $\ln(\frac{1}{16})$
$= \ln\left(\frac{1}{2^4}\right)$
$= \ln(2^{-4})$
$= -4\ln 2$

d $\ln(\frac{1}{6})$
$= \ln 6^{-1}$
$= -1\ln 6$
$= -\ln 6$

e $\ln\left(\frac{1}{\sqrt{2}}\right)$
$= \ln 2^{-\frac{1}{2}}$
$= -\frac{1}{2}\ln 2$

f $\ln\left(\frac{e}{5}\right)$
$= \ln e^1 - \ln 5$
$= 1 - \ln 5$

4 **a** $\ln\sqrt[3]{5}$
$= \ln 5^{\frac{1}{3}}$
$= \frac{1}{3}\ln 5$

b $\ln(\frac{1}{32})$
$= \ln 2^{-5}$
$= -5\ln 2$

c $\ln\left(\frac{1}{\sqrt[5]{2}}\right)$
$= \ln\left(\frac{1}{2^{\frac{1}{5}}}\right)$
$= \ln 2^{-\frac{1}{5}}$
$= -\frac{1}{5}\ln 2$

d $\ln\left(\frac{e^2}{8}\right)$
$= \ln e^2 - \ln 8$
$= 2 - \ln 2^3$
$= 2 - 3\ln 2$

5 **a** $\ln D = \ln x + 1$
$\therefore \ln D - \ln x = 1$
$\therefore \ln\left(\frac{D}{x}\right) = 1$
$\therefore \frac{D}{x} = e^1$
$\therefore D = ex$

b $\ln F = -\ln p + 2$
$\therefore \ln F + \ln p = 2$
$\therefore \ln(Fp) = 2$
$\therefore Fp = e^2$
$\therefore F = \frac{e^2}{p}$

c $\ln P = \frac{1}{2}\ln x$
$\therefore \ln P = \ln x^{\frac{1}{2}}$
$\therefore P = \sqrt{x}$

d $\ln M = 2\ln y + 3$
$\therefore \ln M - 2\ln y = 3$
$\therefore \ln\left(\frac{M}{y^2}\right) = 3$
$\therefore \frac{M}{y^2} = e^3$
$\therefore M = e^3y^2$

e $\ln B = 3\ln t - 1$
$\therefore \ln B - \ln t^3 = -1$
$\therefore \ln\left(\frac{B}{t^3}\right) = -1$
$\therefore \frac{B}{t^3} = e^{-1}$
$\therefore B = \frac{t^3}{e}$

f $\ln N = -\frac{1}{3}\ln g$
$\therefore \ln N = \ln g^{-\frac{1}{3}}$
$\therefore N = g^{-\frac{1}{3}}$
$\therefore N = \frac{1}{\sqrt[3]{g}}$

g $\ln Q \approx 3\ln x + 2.159$
$\therefore \ln Q - 3\ln x \approx 2.159$
$\therefore \ln\left(\frac{Q}{x^3}\right) \approx 2.159$
$\therefore \frac{Q}{x^3} \approx e^{2.159}$
$\therefore \frac{Q}{x^3} \approx 8.66$
$\therefore Q \approx 8.66x^3$

h $\ln D \approx 0.4\ln n - 0.6582$
$\therefore \ln D - \ln n^{0.4} \approx -0.6582$
$\therefore \ln\left(\frac{D}{n^{0.4}}\right) \approx -0.6582$
$\therefore \frac{D}{n^{0.4}} \approx e^{-0.6582}$
$\therefore \frac{D}{n^{0.4}} \approx 0.518$
$\therefore D \approx 0.518n^{0.4}$

EXERCISE 4E

1 **a** $2^x = 10$
$\therefore \log 2^x = \log 10$
$\therefore x\log 2 = \log 10^1$
$\therefore x = \frac{1}{\log 2}$

b $3^x = 20$
$\therefore \log 3^x = \log 20$
$\therefore x\log 3 = \log 20$
$\therefore x = \frac{\log 20}{\log 3}$

c $4^x = 100$
$\therefore \log 4^x = \log 100$
$\therefore x\log 4 = \log 10^2$
$\therefore x = \frac{2}{\log 4}$

d
$$\left(\tfrac{1}{2}\right)^x = 0.0625$$
$$\therefore\ \log(\tfrac{1}{2})^x = \log(\tfrac{1}{16})$$
$$\therefore\ x\log(2^{-1}) = \log(2^{-4})$$
$$\therefore\ x = \frac{-4\log 2}{-\log 2}$$
$$\therefore\ x = 4$$

e
$$\left(\tfrac{3}{4}\right)^x = 0.1$$
$$\therefore\ \log(\tfrac{3}{4})^x = \log 10^{-1}$$
$$\therefore\ x\log(\tfrac{3}{4}) = -1$$
$$\therefore\ x = -\frac{1}{\log(\frac{3}{4})}$$

f
$$10^x = 0.000\,01$$
$$\therefore\ \log 10^x = \log 0.000\,01$$
$$\therefore\ \log 10^x = \log 10^{-5}$$
$$\therefore\ x\log 10 = -5\log 10$$
$$\therefore\ x = -5$$

2 **a**
$$e^x = 10$$
$$\therefore\ x = \ln 10$$

b
$$e^x = 1000$$
$$\therefore\ x = \ln 1000$$

c
$$2e^x = 0.3$$
$$\therefore\ e^x = 0.15$$
$$\therefore\ x = \ln 0.15$$

d
$$e^{\frac{x}{2}} = 5$$
$$\therefore\ \frac{x}{2} = \ln 5$$
$$\therefore\ x = 2\ln 5$$

e
$$e^{2x} = 18$$
$$\therefore\ 2x = \ln 18$$
$$\therefore\ x = \tfrac{1}{2}\ln 18$$

f
$$e^{-\frac{x}{2}} = 1$$
$$\therefore\ -\frac{x}{2} = \ln 1$$
$$\therefore\ -\frac{x}{2} = 0$$
$$\therefore\ x = 0$$

3 **a**
$$R = 200 \times 2^{0.25t}$$
$$\therefore\ 2^{0.25t} = \frac{R}{200}$$
$$\therefore\ \log 2^{0.25t} = \log\left(\frac{R}{200}\right)$$
$$\therefore\ 0.25t\log 2 = \log R - \log 200$$
$$\therefore\ t = \frac{\log R - \log 200}{0.25\log 2}$$

b **i** When $R = 600$,
$$t = \frac{\log 600 - \log 200}{0.25\log 2}$$
$$\therefore\ t \approx 6.34$$

ii When $R = 1425$,
$$t = \frac{\log 1425 - \log 200}{0.25\log 2}$$
$$\therefore\ t \approx 11.3$$

4 **a**
$$M = 20 \times 5^{-0.02x}$$
$$\therefore\ 5^{-0.02x} = \frac{M}{20}$$
$$\therefore\ \log 5^{-0.02x} = \log\left(\frac{M}{20}\right)$$
$$\therefore\ -0.02x\log 5 = \log M - \log 20$$
$$\therefore\ x = \frac{\log M - \log 20}{-0.02\log 5}$$

b **i** When $M = 100$,
$$x = \frac{\log 100 - \log 20}{-0.02\log 5}$$
$$\therefore\ x = -50$$

ii When $M = 232$,
$$x = \frac{\log 232 - \log 20}{-0.02\log 5}$$
$$\therefore\ x \approx -76.1$$

5 **a**
$$4 \times 2^{-x} = 0.12$$
$$\therefore\ 2^{-x} = 0.03$$
$$\therefore\ \log 2^{-x} = \log(0.03)$$
$$\therefore\ -x\log 2 = \log(0.03)$$
$$\therefore\ x = -\frac{\log(0.03)}{\log 2}$$

b
$$300 \times 5^{0.1x} = 1000$$
$$\therefore\ 5^{0.1x} = \tfrac{10}{3}$$
$$\therefore\ \log 5^{0.1x} = \log\left(\tfrac{10}{3}\right)$$
$$\therefore\ 0.1x\log 5 = \log\left(\tfrac{10}{3}\right)$$
$$\therefore\ x\log 5 = 10\log\left(\tfrac{10}{3}\right)$$
$$\therefore\ x = \frac{10\log\left(\frac{10}{3}\right)}{\log 5}$$

c
$$\begin{aligned}32 \times 3^{-0.25x} &= 4\\ \therefore\ 3^{-0.25x} &= \tfrac{1}{8}\\ \therefore\ \log 3^{-0.25x} &= \log\left(\tfrac{1}{8}\right)\\ \therefore\ -0.25x\log 3 &= \log\left(\tfrac{1}{8}\right)\\ \therefore\ x\log 3 &= -4\log\left(\tfrac{1}{8}\right)\\ \therefore\ x &= \frac{-4\log\left(\frac{1}{8}\right)}{\log 3}\end{aligned}$$

d
$$\begin{aligned}20 \times e^{2x} &= 840\\ \therefore\ e^{2x} &= 42\\ \therefore\ \ln e^{2x} &= \ln 42\\ \therefore\ 2x &= \ln 42\\ \therefore\ x &= \tfrac{1}{2}\ln 42\end{aligned}$$

e
$$\begin{aligned}50 \times e^{-0.03x} &= 0.05\\ \therefore\ e^{-0.03x} &= 0.001\\ \therefore\ \ln e^{-0.03x} &= \ln(0.001)\\ \therefore\ -0.03x &= \ln(0.001)\\ \therefore\ -\tfrac{3}{100}x &= \ln(0.001)\\ \therefore\ x &= -\tfrac{100}{3}\ln(0.001)\end{aligned}$$

f
$$\begin{aligned}41e^{0.3x} - 27 &= 0\\ \therefore\ 41e^{0.3x} &= 27\\ \therefore\ e^{0.3x} &= \tfrac{27}{41}\\ \therefore\ \ln e^{0.3x} &= \ln\left(\tfrac{27}{41}\right)\\ \therefore\ 0.3x &= \ln\left(\tfrac{27}{41}\right)\\ \therefore\ \tfrac{3}{10}x &= \ln\left(\tfrac{27}{41}\right)\\ \therefore\ x &= \tfrac{10}{3}\ln\left(\tfrac{27}{41}\right)\end{aligned}$$

6 **a**
$$\begin{aligned}e^{2x} &= 2e^x\\ \therefore\ e^x(e^x-2) &= 0\\ \therefore\ e^x &= 2\quad \{\text{as } e^x > 0\}\\ \therefore\ x &= \ln 2\end{aligned}$$

b
$$\begin{aligned}e^x &= e^{-x}\\ \therefore\ x &= -x\\ \therefore\ 2x &= 0\\ \therefore\ x &= 0\end{aligned}$$

c
$$\begin{aligned}e^{2x} - 5e^x + 6 &= 0\\ \therefore\ (e^x-3)(e^x-2) &= 0\\ \therefore\ e^x &= 3 \text{ or } 2\\ \therefore\ x &= \ln 3 \ \text{ or } \ \ln 2\end{aligned}$$

d
$$\begin{aligned}e^x + 2 &= 3e^{-x}\\ \therefore\ e^{2x} + 2e^x &= 3\quad \{\times e^x\}\\ \therefore\ e^{2x} + 2e^x - 3 &= 0\\ \therefore\ (e^x+3)(e^x-1) &= 0\\ \therefore\ e^x &= -3 \text{ or } 1\\ \therefore\ e^x &= 1\quad \{\text{as } e^x > 0\}\\ \therefore\ x &= \ln 1\\ \therefore\ x &= 0\end{aligned}$$

e
$$\begin{aligned}1 + 12e^{-x} &= e^x\\ \therefore\ e^x + 12 &= e^{2x}\quad \{\times e^x\}\\ \therefore\ e^{2x} - e^x - 12 &= 0\\ \therefore\ (e^x-4)(e^x+3) &= 0\\ \therefore\ e^x &= 4 \text{ or } -3\\ \therefore\ e^x &= 4\quad \{\text{as } e^x > 0\}\\ \therefore\ x &= \ln 4\end{aligned}$$

f
$$\begin{aligned}e^x + e^{-x} &= 3\\ \therefore\ e^{2x} + 1 &= 3e^x\quad \{\times e^x\}\\ \therefore\ e^{2x} - 3e^x + 1 &= 0\\ \therefore\ e^x &= \frac{3 \pm \sqrt{9-4}}{2}\\ \therefore\ e^x &= \frac{3\pm\sqrt{5}}{2}\\ \therefore\ x &= \ln\left(\tfrac{3+\sqrt{5}}{2}\right) \ \text{ or } \ \ln\left(\tfrac{3-\sqrt{5}}{2}\right)\\ &\approx 0.962 \ \text{ or } \ -0.962\end{aligned}$$

7 **a**
$$\begin{aligned}&y = e^x \ \text{ and } \ y = e^{2x} - 6\\ &\text{meet when } \ e^x = e^{2x} - 6\\ \therefore\ & e^{2x} - e^x - 6 = 0\\ \therefore\ & (e^x-3)(e^x+2) = 0\\ \therefore\ & e^x = 3 \ \text{ or } \ -2\\ \therefore\ & e^x = 3 \quad \{\text{as } e^x > 0\}\\ \therefore\ & x = \ln 3 \ \text{ and } \ y = e^x = 3\\ \therefore\ & \text{they meet at } \ (\ln 3, 3).\end{aligned}$$

b
$$\begin{aligned}&y = 2e^x + 1 \ \text{ and } \ y = 7 - e^x\\ &\text{meet when } \ 2e^x + 1 = 7 - e^x\\ \therefore\ & 3e^x = 6\\ \therefore\ & e^x = 2\\ \therefore\ & x = \ln 2 \ \text{ and } \ y = 7 - e^x = 5\\ \therefore\ & \text{they meet at } \ (\ln 2, 5).\end{aligned}$$

c $y = 3 - e^x$ and $y = 5e^{-x} - 3$
meet when $3 - e^x = 5e^{-x} - 3$
$\therefore\ 3e^x - e^{2x} = 5 - 3e^x \quad \{\times e^x\}$
$\therefore\ e^{2x} - 6e^x + 5 = 0$
$\therefore\ (e^x - 5)(e^x - 1) = 0$
$\therefore\ e^x = 1 \text{ or } 5$
$\therefore\ x = 0 \text{ or } \ln 5$

When $x = 0,\ y = 3 - e^0 = 3 - 1 = 2$
When $x = \ln 5,\ y = 3 - e^{\ln 5} = 3 - 5 = -2$
$\therefore$ they meet at $(0, 2)$ and $(\ln 5, -2)$.

EXERCISE 4F

1 **a** $\log_3 12 = \dfrac{\log_{10} 12}{\log_{10} 3} \approx 2.26$

b $\log_{\frac{1}{2}} 1250 = \dfrac{\log_{10} 1250}{\log_{10}(0.5)} \approx -10.3$

c $\log_3(0.067) = \dfrac{\log_{10}(0.067)}{\log_{10} 3} \approx -2.46$

d $\log_{0.4}(0.006\,984) = \dfrac{\log_{10}(0.006\,984)}{\log_{10}(0.4)} \approx 5.42$

2 **a** $2^x = 0.051$
$\therefore\ x = \log_2(0.051)$
$\therefore\ x = \dfrac{\ln(0.051)}{\ln 2}$
$\therefore\ x \approx -4.29$

b $4^x = 213.8$
$\therefore\ x = \log_4 213.8$
$\therefore\ x = \dfrac{\ln(213.8)}{\ln 4}$
$\therefore\ x \approx 3.87$

c $3^{2x+1} = 4.069$
$\therefore\ 2x + 1 = \log_3(4.069)$
$\therefore\ 2x + 1 = \dfrac{\ln(4.069)}{\ln 3}$
$\therefore\ 2x + 1 \approx 1.2774$
$\therefore\ 2x \approx 0.2774$
$\therefore\ x \approx 0.139$

3 **a** $25^x - 3(5^x) = 0$
$\therefore\ 5^{2x} - 3(5^x) = 0$
$\therefore\ 5^x(5^x - 3) = 0$
$\therefore\ 5^x = 3$
$\{\text{as } 5^x > 0 \text{ for all } x\}$
$\therefore\ x = \log_5 3$
$\therefore\ x = \dfrac{\log 3}{\log 5}$

b $8(9^x) - 3^x = 0$
$\therefore\ 8 \times 3^{2x} - 3^x = 0$
$\therefore\ 3^x(8 \times 3^x - 1) = 0$
$\therefore\ 8 \times 3^x - 1 = 0$
$\{\text{as } 3^x > 0 \text{ for all } x\}$
$\therefore\ 3^x = \frac{1}{8}$
$\therefore\ x = \log_3(\frac{1}{8})$
$\therefore\ x = \dfrac{\log(\frac{1}{8})}{\log 3}$

c $2^x - 2(4^x) = 0$
$\therefore\ 2^x - 2 \times 2^{2x} = 0$
$\therefore\ 2^x(1 - 2 \times 2^x) = 0$
$\therefore\ 1 - 2 \times 2^x = 0$
$\{\text{as } 2^x > 0 \text{ for all } x\}$
$\therefore\ 2^x = \frac{1}{2}$
$\therefore\ x = \log_2(\frac{1}{2})$
$\therefore\ x = \dfrac{\log(\frac{1}{2})}{\log 2}$
$\therefore\ x = -1$

4 **a** $\log_4 x^3 + \log_2 \sqrt{x} = 8$
$\therefore\ \dfrac{\log x^3}{\log 4} + \dfrac{\log x^{\frac{1}{2}}}{\log 2} = 8$
$\therefore\ \dfrac{3\log x}{2\log 2} + \dfrac{\frac{1}{2}\log x}{\log 2} = 8$
$\therefore\ \dfrac{3\log x}{2\log 2} + \dfrac{\log x}{2\log 2} = 8$
$\therefore\ \dfrac{4\log x}{2\log 2} = 8$
$\therefore\ \log x = 4\log 2$
$\therefore\ \log x = \log 2^4$
$\therefore\ x = 16$

b $\log_{16} x^5 = \log_{64} 125 - \log_4 \sqrt{x}$
$\therefore\ \dfrac{\log x^5}{\log 16} = \dfrac{\log 125}{\log 64} - \dfrac{\log x^{\frac{1}{2}}}{\log 4}$
$\therefore\ \dfrac{5\log x}{4\log 2} = \dfrac{\log 125}{6\log 2} - \dfrac{\frac{1}{2}\log x}{2\log 2}$
$\therefore\ \dfrac{15\log x}{12\log 2} = \dfrac{2\log 125}{12\log 2} - \dfrac{3\log x}{12\log 2}$
$\therefore\ 15\log x = 2\log 125 - 3\log x$
$\therefore\ 18\log x = 2\log 125$
$\therefore\ \log x = \frac{1}{9}\log 5^3$
$\therefore\ \log x = \log(5^3)^{\frac{1}{9}}$
$\therefore\ x = 5^{\frac{1}{3}} \approx 1.71$

5
$$4^x \times 5^{4x+3} = 10^{2x+3}$$
$$\therefore\ \log(4^x \times 5^{4x+3}) = \log 10^{2x+3}$$
$$\therefore\ x\log 4 + (4x+3)\log 5 = 2x+3$$
$$\therefore\ x\log 4 + 4x\log 5 + 3\log 5 = 2x+3$$
$$\therefore\ x[\log 4 + 4\log 5 - 2] = 3 - 3\log 5$$
$$\therefore\ x = \frac{3-3\log 5}{\log 4 + 4\log 5 - 2}$$
$$\therefore\ x = \frac{\log 10^3 - \log 5^3}{\log 4 + \log 5^4 - \log 10^2}$$
$$\therefore\ x = \frac{\log(\frac{1000}{125})}{\log\left(\frac{4\times 5^4}{10^2}\right)}$$
$$\therefore\ x = \frac{\log 8}{\log 25} \quad \text{or} \quad \log_{25} 8$$

6
$$\log_9 x + \log_{27} x = p$$
$$\therefore\ \frac{\log x}{\log 9} + \frac{\log x}{\log 27} = p$$
$$\therefore\ \frac{\log x}{2\log 3} + \frac{\log x}{3\log 3} = p$$
$$\therefore\ \frac{3\log x}{6\log 3} + \frac{2\log x}{6\log 3} = p$$
$$\therefore\ \frac{5\log x}{6\log 3} = p$$
$$\therefore\ \tfrac{5}{6}\log_3 x = p$$
$$\therefore\ \log_3 x = \tfrac{6}{5}p$$

Now, $\log_{81} x = \dfrac{\log x}{\log 81}$
$$= \frac{\log x}{4\log 3}$$
$$= \tfrac{1}{4}\log_3 x$$
$$= \tfrac{1}{4}\times\tfrac{6}{5}p$$
$$= \tfrac{3}{10}p$$

So, $\log_3 x + \log_{81} x = \tfrac{6}{5}p + \tfrac{3}{10}p$
$$= \tfrac{15}{10}p$$
$$= \tfrac{3}{2}p$$

7
$$\frac{4}{\log_5 4} + \frac{3}{\log_7 8} = \frac{4}{\frac{\log 4}{\log 5}} + \frac{3}{\frac{\log 8}{\log 7}}$$
$$= \frac{4\log 5}{\log 4} + \frac{3\log 7}{\log 8}$$
$$= \frac{4\log 5}{2\log 2} + \frac{3\log 7}{3\log 2}$$
$$= \frac{2\log 5}{\log 2} + \frac{\log 7}{\log 2}$$
$$= \log_2 25 + \log_2 7$$
$$= \log_2(25\times 7)$$
$$= \log_2 175$$

So, $2^{\frac{4}{\log_5 4} + \frac{3}{\log_7 8}} = 2^{\log_2 175} = 175$

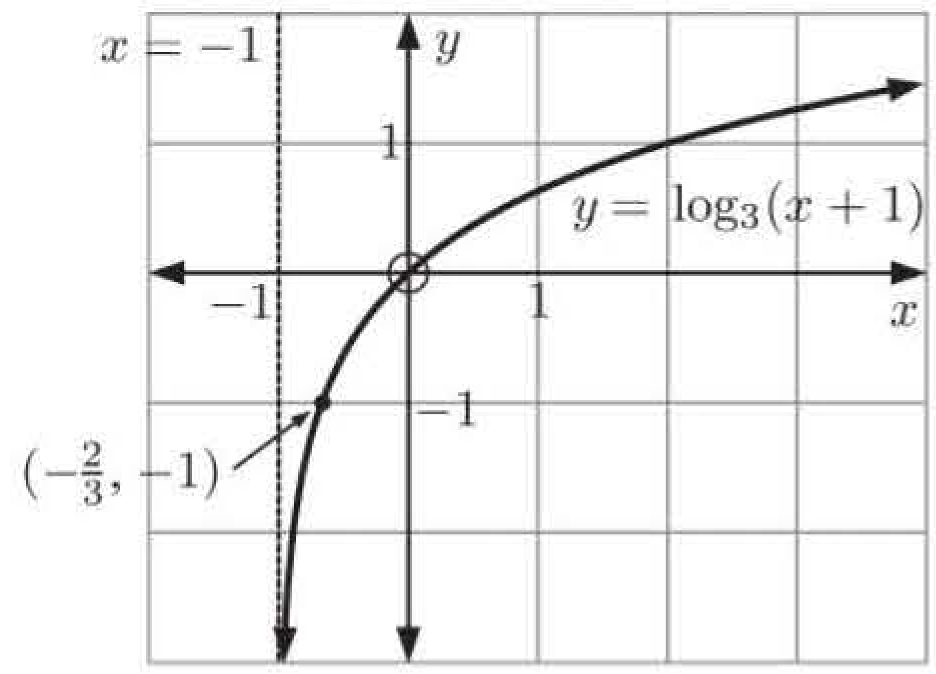
EXERCISE 4G

1 **a** $f(x) = \log_3(x+1)$

i We require $x + 1 > 0 \quad \therefore\ x > -1$
So, the domain is $\{x \mid x > -1\}$ and the range is $\{y \mid y \in \mathbb{R}\}$.

ii As $x \to -1^+$, $y \to -\infty$,
so $x = -1$ is a vertical asymptote.
As $x \to \infty$, $y \to \infty$.
When $x = 0$, $y = \log_3 1 = 0$
So, the y-intercept is 0.
When $y = 0$, $\log_3(x+1) = 0$
$$\therefore\ x + 1 = 3^0$$
$$\therefore\ x + 1 = 1$$
$$\therefore\ x = 0$$
So, the x-intercept is 0.

iii We graph using $y = \dfrac{\log(x+1)}{\log 3}$



iv If $f(x) = -1$
then $\log_3(x+1) = -1$
$\therefore \quad x+1 = 3^{-1}$
$\therefore \quad x = \frac{1}{3} - 1$
$\therefore \quad x = -\frac{2}{3}$
which checks with the graph

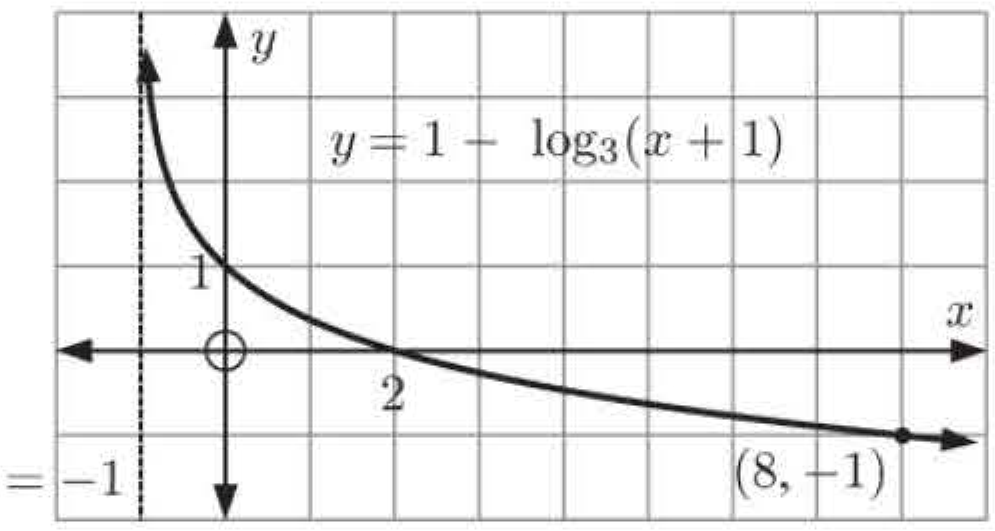
v f is defined by $y = \log_3(x+1)$
$\therefore \ f^{-1}$ is defined by $x = \log_3(y+1)$
$\therefore \quad y+1 = 3^x$
$\therefore \quad y = 3^x - 1$
$\therefore \quad f^{-1}(x) = 3^x - 1$
Horizontal asymptote is $y = -1$.
Domain is $x \in \mathbb{R}$.
Range is $\{y \mid y > -1\}$.

b $f(x) = 1 - \log_3(x+1)$

i We require $x + 1 > 0 \quad \therefore \ x > -1$
So, the domain is $\{x \mid x > -1\}$ and the range is $\{y \mid y \in \mathbb{R}\}$.

ii As $x \to -1^+$, $y \to \infty$,
so $x = -1$ is a vertical asymptote.
As $x \to \infty$, $y \to -\infty$.
When $x = 0$, $y = 1 - \log_3 1$
$= 1 - 0 = 1$
So, the y-intercept is 1.
When $y = 0$, $1 - \log_3(x+1) = 0$
$\therefore \quad \log_3(x+1) = 1$
$\therefore \quad x + 1 = 3^1$
$= 3$
$\therefore \quad x = 2$
So, the x-intercept is 2.

iii We graph using $y = 1 - \dfrac{\log(x+1)}{\log 3}$



iv If $f(x) = -1$
then $1 - \log_3(x+1) = -1$
$\therefore \quad \log_3(x+1) = 2$
$\therefore \quad x+1 = 3^2$
$\therefore \quad x = 8$
which checks with the graph

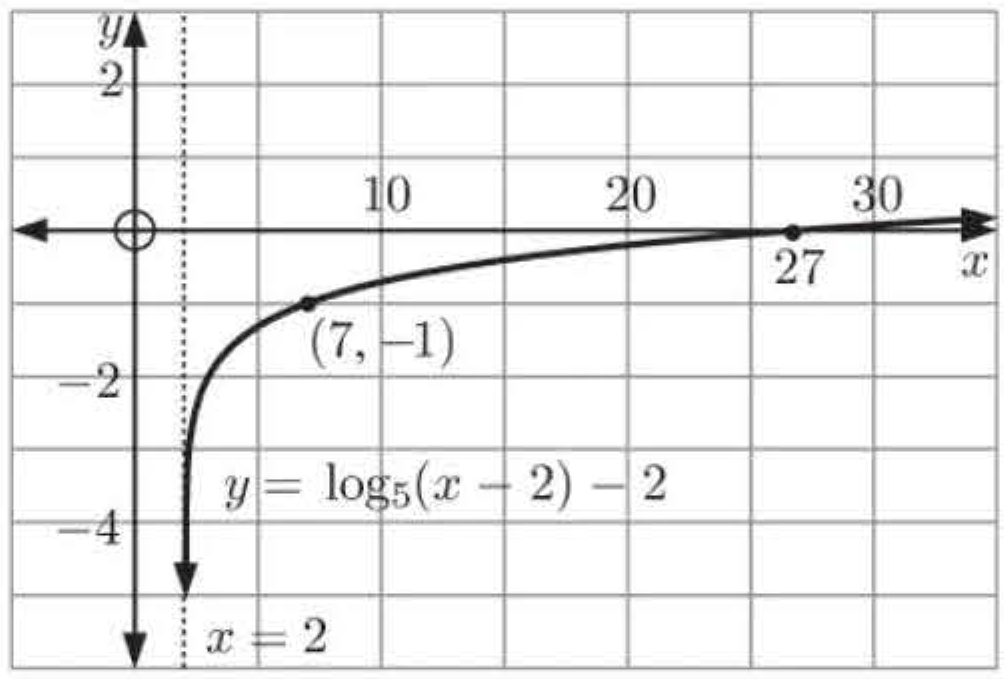
v f is defined by $y = 1 - \log_3(x+1)$
$\therefore \ f^{-1}$ is defined by $x = 1 - \log_3(y+1)$
$\therefore \quad \log_3(y+1) = 1 - x$
$\therefore \quad y + 1 = 3^{1-x}$
$\therefore \quad y = 3^{1-x} - 1$
$\therefore \quad f^{-1}(x) = 3^{1-x} - 1$
Horizontal asymptote is $y = -1$.
Domain is $x \in \mathbb{R}$.
Range is $\{y \mid y > -1\}$.

c $f(x) = \log_5(x-2) - 2$

i We require $x - 2 > 0 \quad \therefore \ x > 2$.
So, the domain is $\{x \mid x > 2\}$ and the range is $\{y \mid y \in \mathbb{R}\}$.

ii As $x \to 2^+$, $y \to -\infty$,
so $x = 2$ is a vertical asymptote.
As $x \to \infty$, $y \to \infty$.
When $x = 0$, y is undefined.
$\therefore$ there is no y-intercept.
When $y = 0$, $\log_5(x-2) = 2$
$\therefore \quad x - 2 = 5^2$
$= 25$
$\therefore \quad x = 27$
So, the x-intercept is 27.

iii We graph using $y = \dfrac{\log(x-2)}{\log 5} - 2$



iv If $f(x) = -1$

then $\log_5(x-2) - 2 = -1$

$\therefore \log_5(x-2) = 1$

$\therefore x - 2 = 5^1$

$\therefore x = 5 + 2$

$\therefore x = 7$

which checks with the graph

v f is defined by $y = \log_5(x-2) - 2$

$\therefore f^{-1}$ is defined by $x = \log_5(y-2) - 2$

$\therefore x + 2 = \log_5(y-2)$

$\therefore y - 2 = 5^{x+2}$

$\therefore y = 5^{x+2} + 2$

$\therefore f^{-1}(x) = 5^{x+2} + 2$

Horizontal asymptote is $y = 2$.
Domain is $x \in \mathbb{R}$.
Range is $\{y \mid y > 2\}$.

d $f(x) = 1 - \log_5(x-2)$

i We require $x - 2 > 0 \quad \therefore x > 2$.
So, the domain is $\{x \mid x > 2\}$ and the range is $\{y \mid y \in \mathbb{R}\}$.

ii As $x \to 2^+$, $y \to \infty$,
so $x = 2$ is a vertical asymptote.
As $x \to \infty$, $y \to -\infty$.
When $x = 0$, y is undefined.
$\therefore$ there is no y-intercept.
When $y = 0$, $1 - \log_5(x-2) = 0$

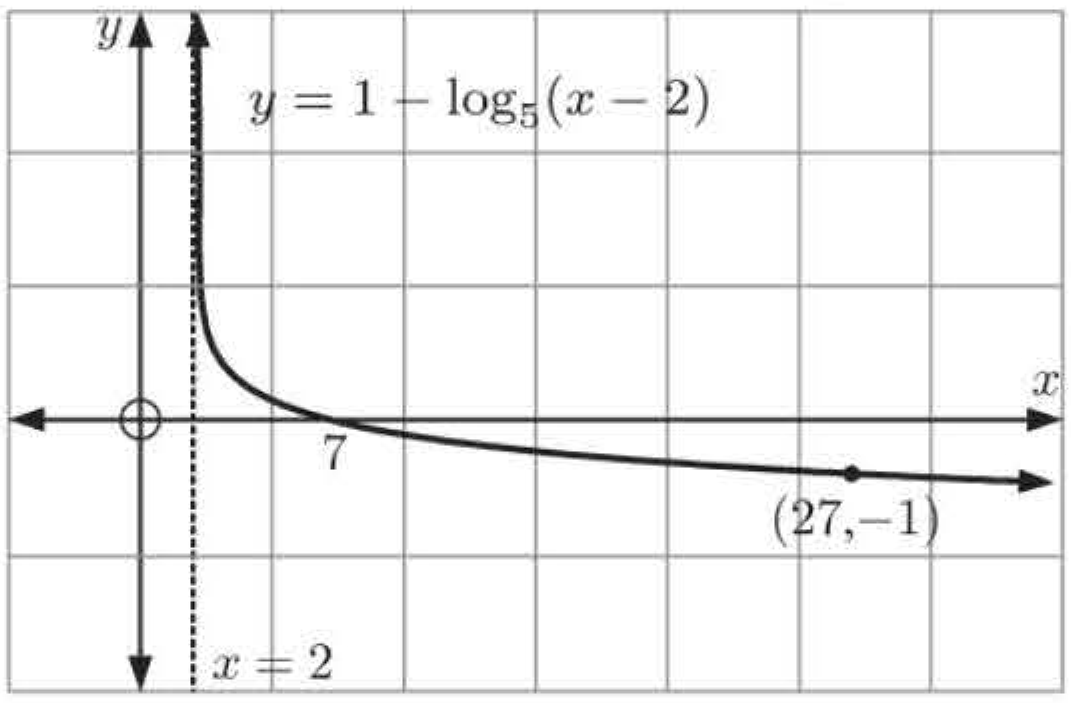
$\therefore \log_5(x-2) = 1$

$\therefore x - 2 = 5^1$

$\therefore x = 7$

So, x-intercept is 7.

iii We graph using $y = 1 - \dfrac{\log(x-2)}{\log 5}$



iv If $f(x) = -1$

then $1 - \log_5(x-2) = -1$

$\therefore \log_5(x-2) = 2$

$\therefore x - 2 = 5^2$

$\therefore x = 27$

which checks with the graph

v f is defined by $y = 1 - \log_5(x-2)$

$\therefore f^{-1}$ is defined by $x = 1 - \log_5(y-2)$

$\therefore \log_5(y-2) = 1 - x$

$\therefore y - 2 = 5^{1-x}$

$\therefore y = 5^{1-x} + 2$

$\therefore f^{-1}(x) = 5^{1-x} + 2$

Horizontal asymptote is $y = 2$.
Domain is $x \in \mathbb{R}$.
Range is $\{y \mid y > 2\}$.

e $f(x) = 1 - 2\log_2 x$

i We require $x > 0$.
So, the domain is $\{x \mid x > 0\}$ and the range is $\{y \mid y \in \mathbb{R}\}$.

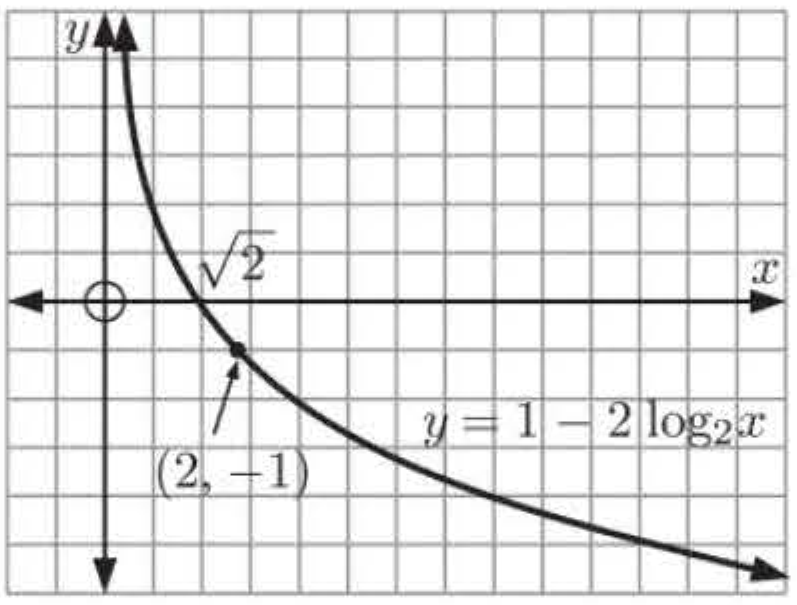
ii As $x \to 0^+$, $y \to \infty$,
so $x = 0$ is a vertical asymptote.
As $x \to \infty$, $y \to -\infty$.
When $x = 0$, y is undefined.
$\therefore$ there is no y-intercept.
When $y = 0$, $\log_2 x = \frac{1}{2}$

$\therefore x = 2^{\frac{1}{2}}$

$\therefore x = \sqrt{2}$

$\therefore$ x-intercept is $\sqrt{2} \approx 1.41$.

iii We graph using $y = 1 - \dfrac{2\log x}{\log 2}$



iv If $f(x) = -1$

then $1 - 2\log_2 x = -1$

$\therefore\ -2\log_2 x = -2$

$\therefore\ \log_2 x = 1$

$\therefore\ x = 2^1$

$\therefore\ x = 2$

which checks with the graph

v f is defined by $y = 1 - 2\log_2 x$

$\therefore\ f^{-1}$ is defined by $x = 1 - 2\log_2 y$

$\therefore\ 2\log_2 y = 1 - x$

$\therefore\ \log_2 y = \dfrac{1-x}{2}$

$\therefore\ y = 2^{\frac{1-x}{2}}$

$\therefore\ f^{-1}(x) = 2^{\frac{1-x}{2}}$

Horizontal asymptote is $y = 0$.

Domain is $x \in \mathbb{R}$. Range is $\{y \mid y > 0\}$.

f $f(x) = \log_2(x^2 - 3x - 4)$

i We require $x^2 - 3x - 4 > 0$ $\therefore$ $(x+1)(x-4) > 0$.

Sign diagram of LHS: $+$ | -1 | $-$ | 4 | $+$ (x) $\therefore$ LHS > 0 when $x < -1$ or $x > 4$.

So, the domain is $\{x \mid x < -1 \text{ or } x > 4\}$ and the range is $\{y \mid y \in \mathbb{R}\}$.

ii As $x \to -1^-$, $y \to -\infty$,

As $x \to 4^+$, $y \to -\infty$

so $x = -1$ and $x = 4$ are vertical asymptotes.

When $x = 0$, y is undefined.

$\therefore$ there is no y-intercept.

When $y = 0$,

$\log_2(x^2 - 3x - 4) = 0$

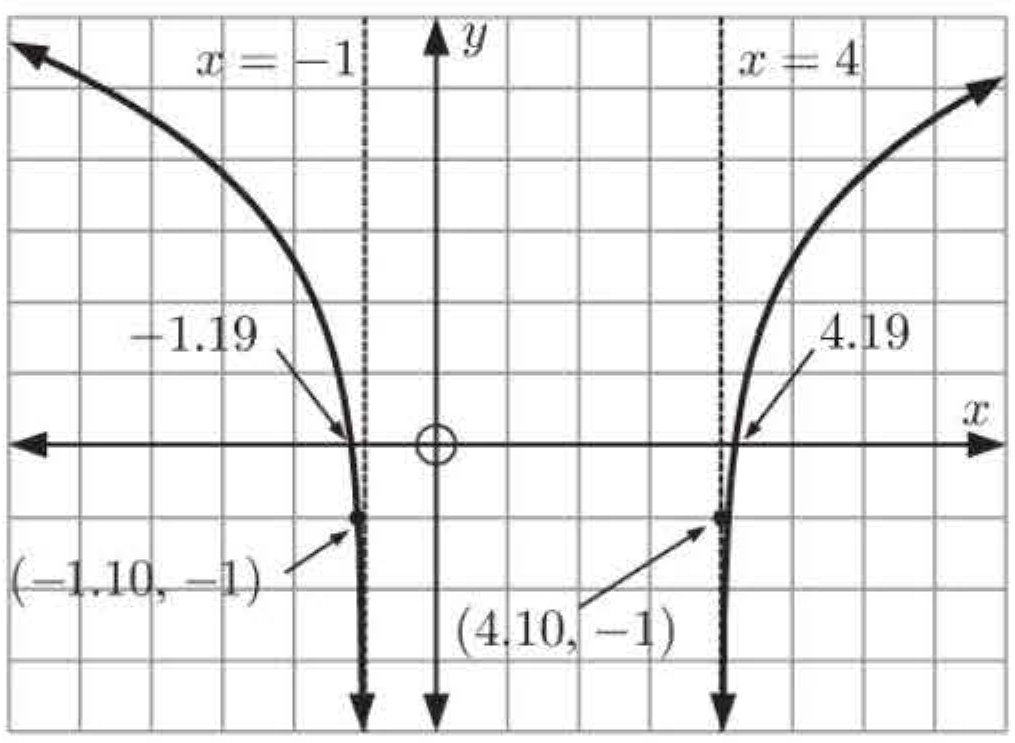
$\therefore\ x^2 - 3x - 4 = 2^0 = 1$

$\therefore\ x^2 - 3x - 5 = 0$

$\therefore\ x \approx -1.19$ or 4.19

{using technology}

iii We graph using $y = \dfrac{\log(x^2 - 3x - 4)}{\log 2}$



iv If $f(x) = -1$

then $\log_2(x^2 - 3x - 4) = -1$

$\therefore\ x^2 - 3x - 4 = 2^{-1}$

$\therefore\ x^2 - 3x - \frac{9}{2} = 0$

$\therefore\ 2x^2 - 6x - 9 = 0$

$\therefore\ x \approx -1.10$ or 4.10 {using technology}

which checks with the graph

v If f is defined by $y = \log_2(x^2 - 3x - 4)$, $x > 4$

then f^{-1} is defined by $x = \log_2(y^2 - 3y - 4)$, $y > 4$

$\therefore\ y^2 - 3y - 4 = 2^x$, $y > 4$

$\therefore\ y^2 - 3y - 4 - 2^x = 0$, $y > 4$

$$\therefore\ y = \frac{3 \pm \sqrt{9 + 4(4 + 2^x)}}{2},\quad y > 4$$

$$\therefore\ y = \frac{3 + \sqrt{25 + 2^{x+2}}}{2}\quad \text{as } y > 4$$

$$\therefore\ f^{-1}(x) = \frac{3 + \sqrt{25 + 2^{x+2}}}{2}$$

If f is defined by $y = \log_2(x^2 - 3x - 4)$, $x < -1$

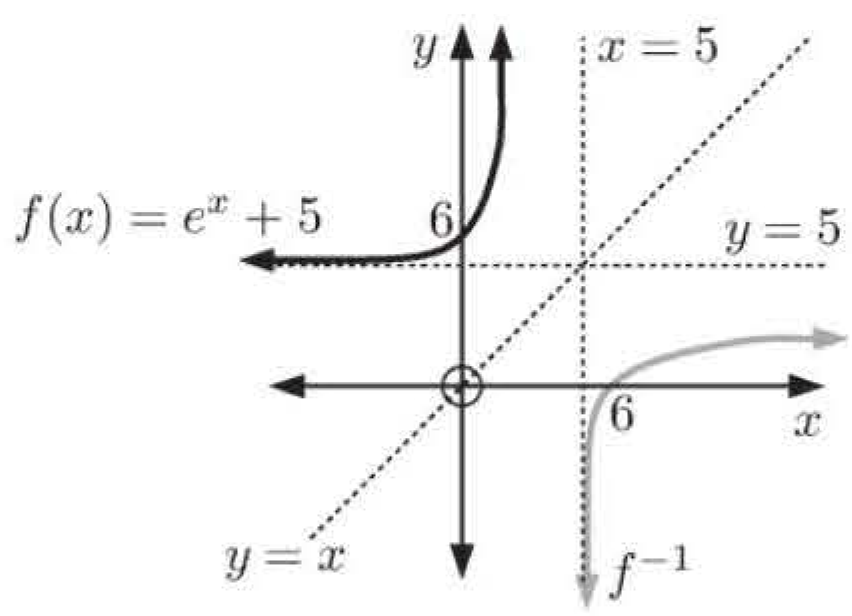
then f^{-1} is defined by $x = \log_2(y^2 - 3y - 4)$, $y < -1$

and by the same working, $y = \dfrac{3-\sqrt{25+2^{x+2}}}{2}$ as $y < -1$

$$\therefore \quad f^{-1}(x) = \frac{3-\sqrt{25+2^{x+2}}}{2}$$

2 **a** **i** $f(x) = e^x + 5$
or $y = e^x + 5$
has inverse function
$x = e^y + 5$
$\therefore \quad x - 5 = e^y$
$\therefore \quad y = \ln(x-5)$
$\therefore \quad f^{-1}(x) = \ln(x-5)$

ii

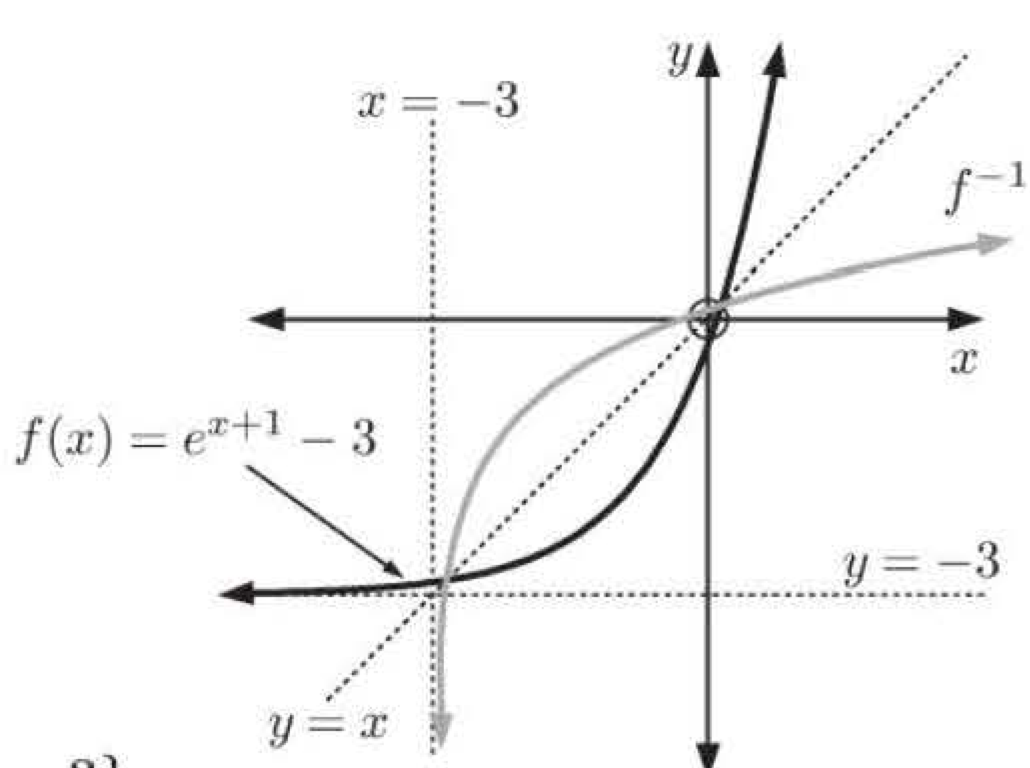


iii Domain of f is $\{x \mid x \in \mathbb{R}\}$, range is $\{y \mid y > 5\}$.
Domain of f^{-1} is $\{x \mid x > 5\}$, range is $\{y \mid y \in \mathbb{R}\}$.

iv f has a H.A. $y = 5$. f^{-1} has a V.A. $x = 5$.
When $x = 0$, $y = e^0 + 5$
$\therefore \quad y = 6$
$\therefore$ y-intercept of f is 6.
When $y = 0$, $e^x + 5$ is undefined
$\therefore$ f has no x-intercept.
$\therefore$ the x-intercept of f^{-1} is 6, and f^{-1} has no y-intercept.

b **i** $f(x) = e^{x+1} - 3$
or $y = e^{x+1} - 3$
has inverse function
$x = e^{y+1} - 3$
$\therefore \quad x + 3 = e^{y+1}$
$\therefore \quad y + 1 = \ln(x+3)$
$\therefore \quad f^{-1}(x) = \ln(x+3) - 1$

ii

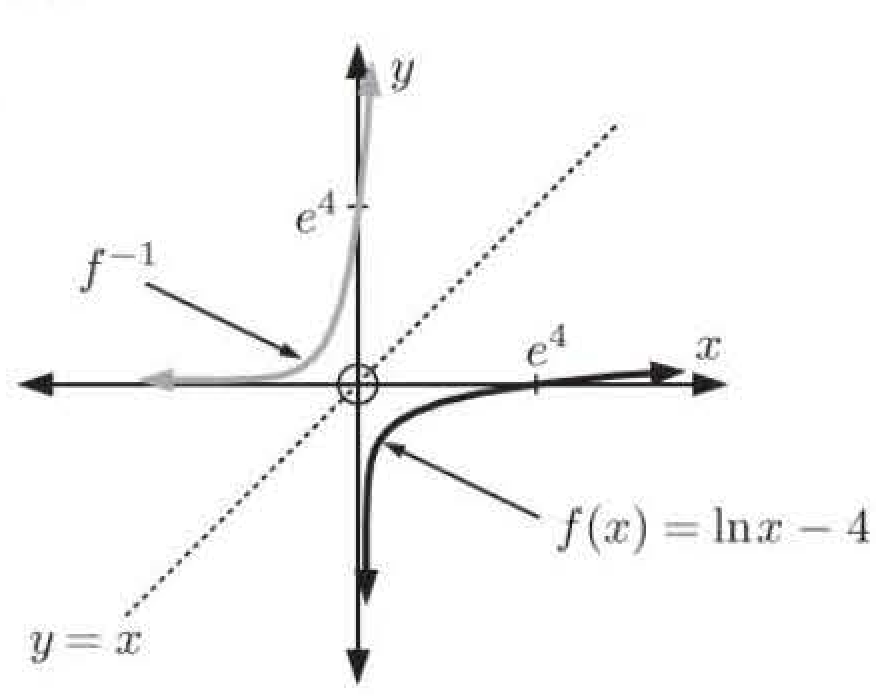


iii Domain of f is $\{x \mid x \in \mathbb{R}\}$, range is $\{y \mid y > -3\}$.
Domain of f^{-1} is $\{x \mid x > -3\}$, range is $\{y \mid y \in \mathbb{R}\}$.

iv f has a H.A. $y = -3$. f^{-1} has a V.A. $x = -3$.
When $x = 0$, $y = e^{0+1} - 3 = e - 3$
$\therefore$ the y-intercept of f is $e - 3 \approx -0.282$
When $y = 0$, $e^{x+1} - 3 = 0$
$\therefore \quad e^{x+1} = 3$
$\therefore \quad x + 1 = \ln 3$
$\therefore \quad x = \ln 3 - 1$
$\therefore$ the x-intercept of f is $\ln 3 - 1 \approx 0.0986$
$\therefore$ the x-intercept of f^{-1} is $e - 3 \approx -0.282$
and the y-intercept of f^{-1} is $\ln 3 - 1 \approx 0.0986$

c **i** $f(x) = \ln x - 4$, $x > 0$
$\therefore \quad y = \ln x - 4$
and has inverse function
$x = \ln y - 4$
$\therefore \quad x + 4 = \ln y$
$\therefore \quad y = e^{x+4}$
$\therefore \quad f^{-1}(x) = e^{x+4}$

ii



iii Domain of f is $\{x \mid x > 0\}$, range is $\{y \mid y \in \mathbb{R}\}$.
Domain of f^{-1} is $\{x \mid x \in \mathbb{R}\}$, range is $\{y \mid y > 0\}$.

iv f has a V.A. $x = 0$. f^{-1} has a H.A. $y = 0$.
When $x = 0$, $\ln x - 4$ is undefined.
$\therefore$ f has no y-intercept.
When $y = 0$, $\ln x - 4 = 0$

$$\therefore \ln x = 4$$
$$\therefore x = e^4 \approx 54.6$$

$\therefore$ the x-intercept of f is e^4.
$\therefore$ f^{-1} has no x-intercept and the y-intercept of f^{-1} is e^4.

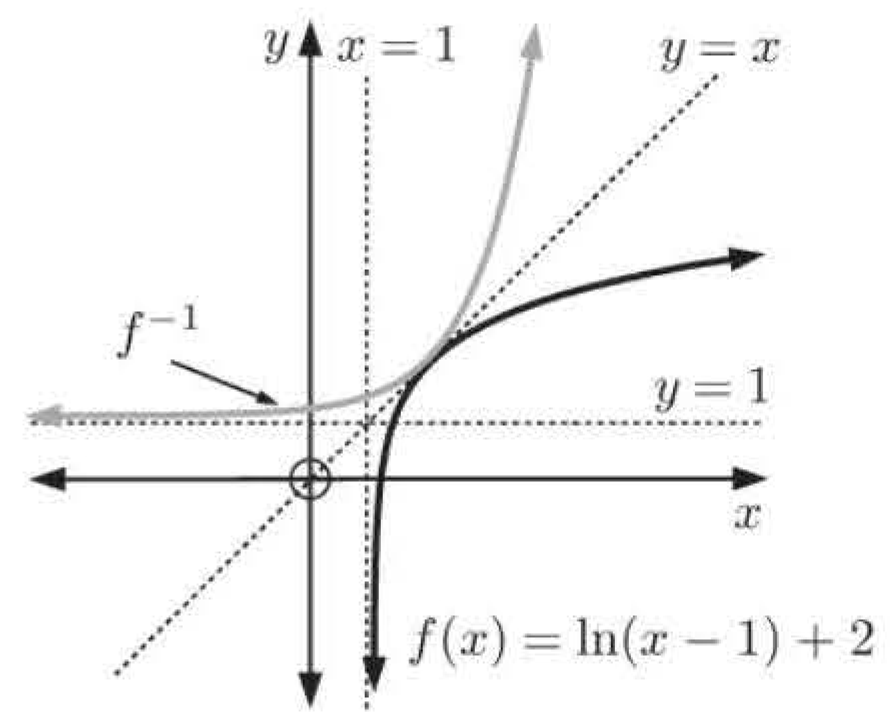
d **i** $f(x) = \ln(x-1) + 2,\ x > 1$

$$\therefore y = \ln(x-1) + 2$$

and has inverse function

$$x = \ln(y-1) + 2$$
$$\therefore \ln(y-1) = x - 2$$
$$\therefore y - 1 = e^{x-2}$$
$$\therefore y = e^{x-2} + 1$$
$$\therefore f^{-1}(x) = e^{x-2} + 1$$

ii



iii Domain of f is $\{x \mid x > 1\}$, range is $\{y \mid y \in \mathbb{R}\}$.
Domain of f^{-1} is $\{x \mid x \in \mathbb{R}\}$, range is $\{y \mid y > 1\}$.

iv f has a V.A. $x = 1$. f^{-1} has a H.A. $y = 1$.
When $x = 0$, $\ln(x-1) + 2$ is undefined.
$\therefore$ f has no y-intercept.
When $y = 0$, $\ln(x-1) + 2 = 0$

$$\therefore \ln(x-1) = -2$$
$$\therefore x - 1 = e^{-2}$$
$$\therefore x = 1 + e^{-2}$$

$\therefore$ the x-intercept of f is $1 + e^{-2}$.
$\therefore$ f^{-1} has no x-intercept and the y-intercept of f^{-1} is $1 + e^{-2}$.

3 **a** f is $y = e^{2x}$
so the inverse function f^{-1} is

$$x = e^{2y}$$
$$\therefore 2y = \ln x$$
$$\therefore y = \tfrac{1}{2}\ln x$$
$$\therefore f^{-1}(x) = \tfrac{1}{2}\ln x$$
$$\therefore (f^{-1} \circ g)(x) = f^{-1}(g(x)) = f^{-1}(2x-1) = \tfrac{1}{2}\ln(2x-1)$$

b

$$(g \circ f)(x) = g(f(x)) = g(e^{2x}) = 2(e^{2x}) - 1$$

So, $y = 2e^{2x} - 1$ which has inverse

$$x = 2e^{2y} - 1$$
$$\therefore x + 1 = 2e^{2y}$$
$$\therefore \tfrac{1}{2}(x+1) = e^{2y}$$
$$\therefore 2y = \ln\left(\frac{x+1}{2}\right)$$
$$\therefore y = \tfrac{1}{2}\ln\left(\frac{x+1}{2}\right)$$
$$\therefore (g \circ f)^{-1}(x) = \tfrac{1}{2}\ln\left(\frac{x+1}{2}\right)$$

4 **a** $y = \ln x$ cuts the x-axis when $y = 0$

$\therefore \quad \ln x = 0$

$\therefore \quad x = e^0 = 1$

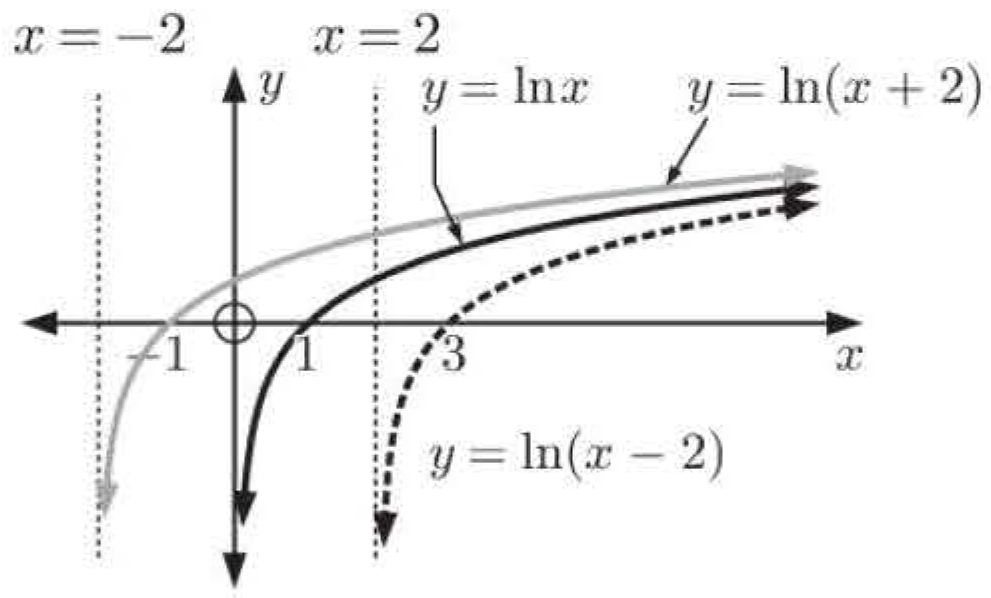
So, graph A is that of $y = \ln x$.

Note: x-intercept of $y = \ln(x-2)$ is when $x - 2 = e^0 = 1$

$\therefore \quad x = 3$

b The x-intercept of $y = \ln(x+2)$ occurs when $x + 2 = e^0 = 1$

$\therefore \quad x = -1$



c $y = \ln x$ has a V.A. of $x = 0$.

$y = \ln(x-2)$ has a V.A. of $x = 2$.

$y = \ln(x+2)$ has a V.A. of $x = -2$.

5 Since $y = \ln(x^2)$, $y = 2\ln x$ {log law}

$\therefore$ the new y-values are twice the old y-values.

$\therefore$ Kelly is correct.

(Note that $y = \ln x^2$ is also defined for $x < 0$. However, we are only concerned with $y = \ln x^2$ for $x > 0$.)

6 **a** $f(x) = e^{x+3} + 2$ or $y = e^{x+3} + 2$ has inverse function

$x = e^{y+3} + 2$

$\therefore \quad x - 2 = e^{y+3}$

$\therefore \quad \ln(x-2) = y + 3$

$\therefore \quad y = \ln(x-2) - 3$

So, $f^{-1}(x) = \ln(x-2) - 3$

b **i** $f(x) < 2.1$ when $e^{x+3} + 2 < 2.1$

$\therefore \quad e^{x+3} < 0.1$

$\therefore \quad x + 3 < \ln(0.1)$

$\therefore \quad x < \ln(0.1) - 3$

$\therefore \quad x < -5.30$

ii Similarly, $f(x) < 2.01$ when

$x < \ln(0.01) - 3$

$\therefore \quad x < -7.61$

iii $f(x) < 2.001$ when

$x < \ln(0.001) - 3$

$\therefore \quad x < -9.91$

iv $f(x) < 2.0001$ when

$x < \ln(0.0001) - 3$

$\therefore \quad x < -12.2$

We conjecture that the H.A. is $y = 2$.

c As $x \to \infty$, $y \to \infty$

As $x \to -\infty$, $e^{x+3} \to 0$ $\therefore$ $y \to 2$

$\therefore$ H.A. is $y = 2$.

d f has a H.A. $y = 2$ and range $\{y \mid y > 2\}$

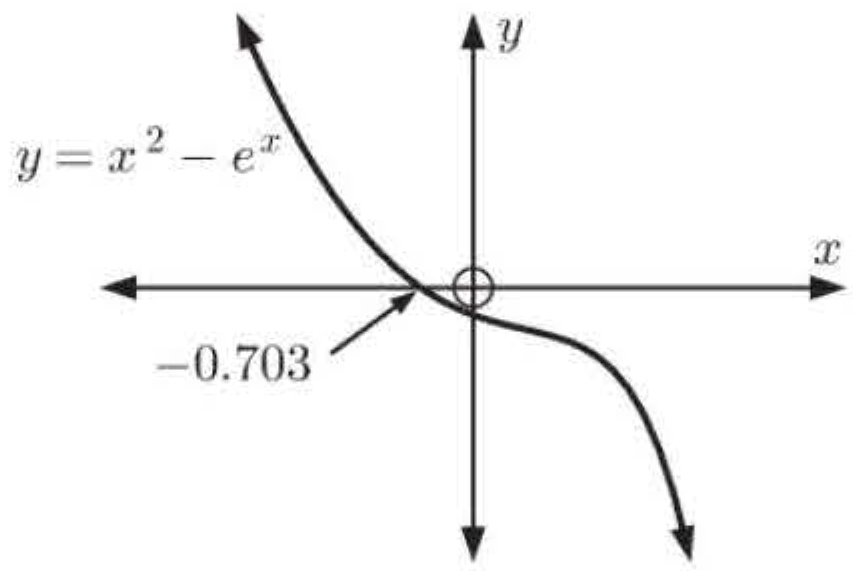
$\therefore$ f^{-1} has a V.A. $x = 2$ and domain $\{x \mid x > 2\}$

7 **a** $x^2 > e^x \;\Rightarrow\; x^2 - e^x > 0$

$\therefore$ to solve $x^2 > e^x$, we find where the graph of $y = x^2 - e^x$ is above the x-axis.

The graph cuts the x-axis when $x \approx -0.703$

$\therefore$ $x^2 > e^x$ when $x < -0.703$

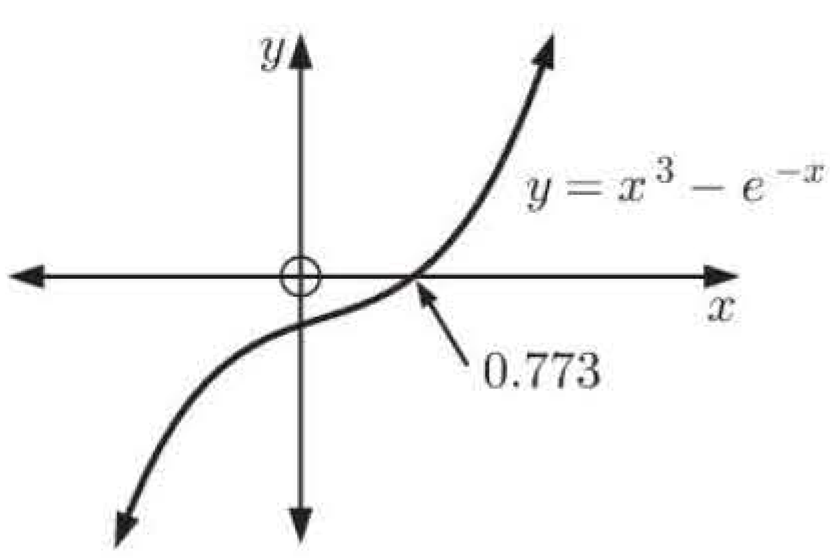


b $x^3 < e^{-x} \;\Rightarrow\; x^3 - e^{-x} < 0$

$\therefore$ to solve $x^3 < e^{-x}$, we find where the graph of $y = x^3 - e^{-x}$ is below the x-axis.

The graph cuts the x-axis when $x \approx 0.773$

$\therefore$ $x^3 < e^{-x}$ when $x < 0.773$

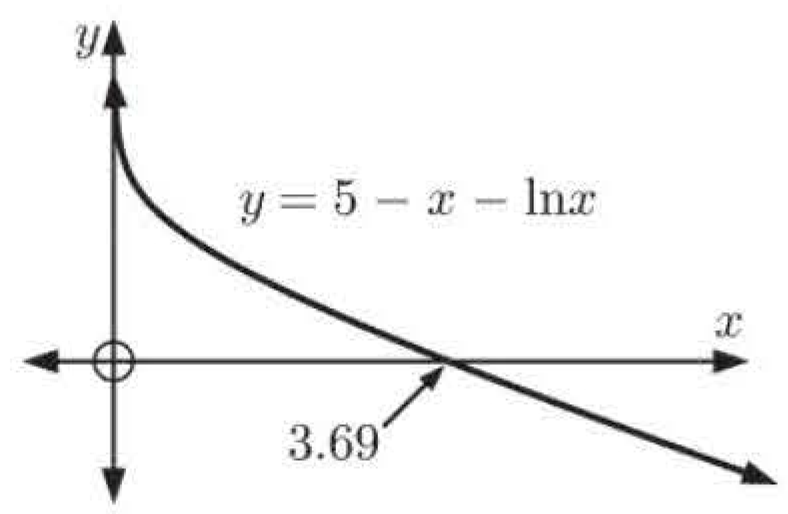


c $5 - x > \ln x \quad \Rightarrow \quad 5 - x - \ln x > 0$

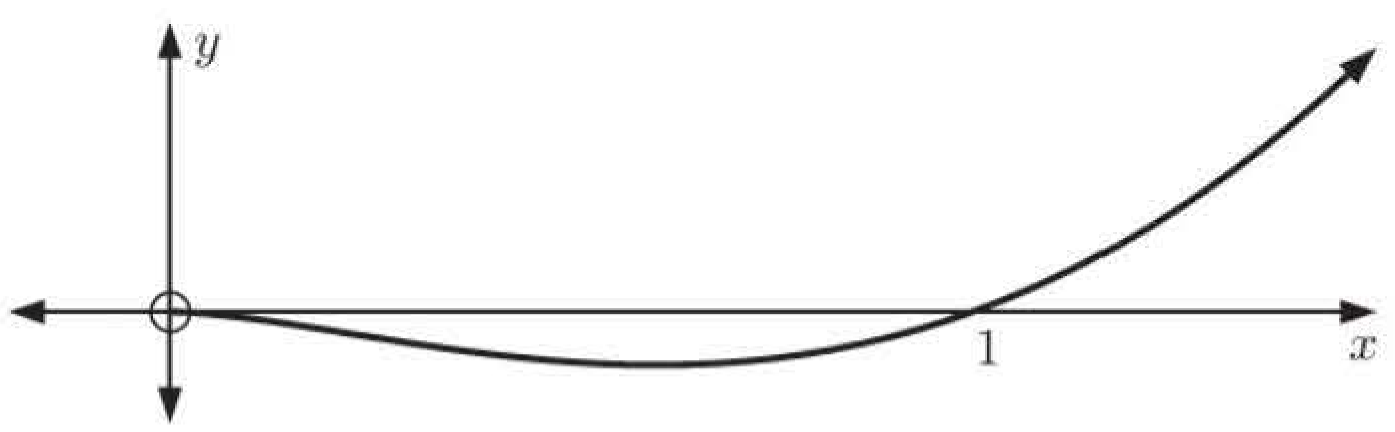
$\therefore$ to solve $5 - x > \ln x$, we find where the graph of $y = 5 - x - \ln x$ is above the x-axis.

The graph cuts the x-axis when $x \approx 3.69$ and $\ln x$ is only defined for $x > 0$

$\therefore \quad 5 - x > \ln x$ when $0 < x < 3.69$



8 f is defined when $\ln x$ is defined. This is when $x > 0$.
So, the domain is $x \in \left]0, \infty\right[$



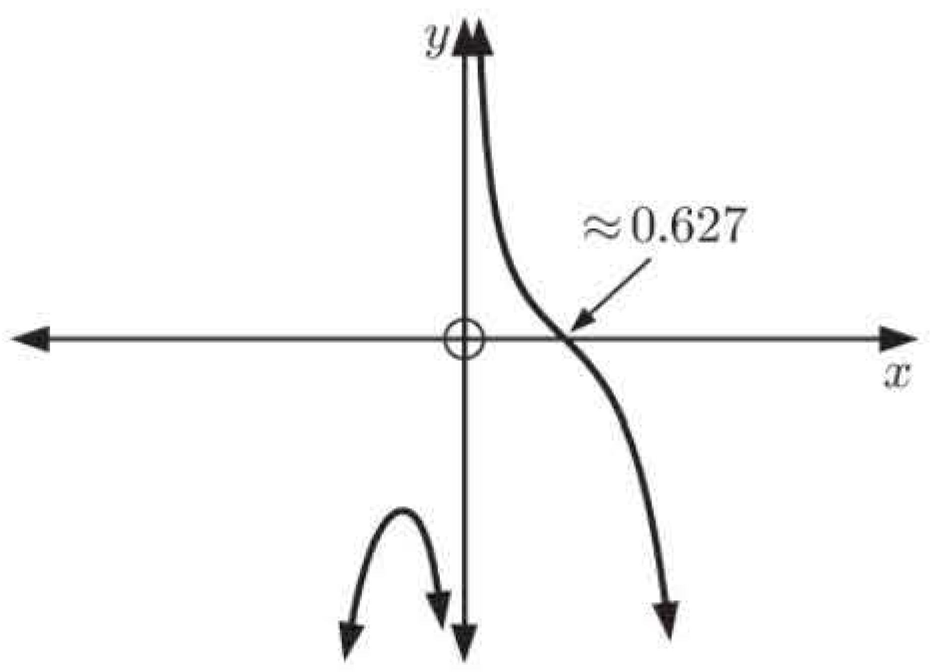
If $f(x) \leqslant 0$ then $x^2 \ln x \leqslant 0$

$\therefore$ the graph of $y = x^2 \ln x$ is on or below $y = 0$.

$\therefore \quad 0 < x \leqslant 1 \quad \therefore \quad x \in \left]0, 1\right]$

9 $f(x) = \dfrac{2}{x} - e^{2x^2 - x + 1}$

a



b domain is $\{x \mid x \in \mathbb{R},\ x \neq 0\}$
range is $\{y \mid y \in \mathbb{R}\}$

c If $e^{2x^2 - x + 1} > \dfrac{2}{x}$

then $\dfrac{2}{x} - e^{2x^2 - x + 1} < 0$

So, we want x such that $f(x) < 0$.
This is for $x < 0$ or $x > 0.627$
$\therefore \quad x \in \left]-\infty, 0\right[$ or $x \in \left]0.627, \infty\right[$

EXERCISE 4H

1 $W_t = 20 \times 2^{0.15t}$ grams

a When $W_t = 30$,

$$\begin{aligned}
20 \times 2^{0.15t} &= 30 \\
\therefore \quad 2^{0.15t} &= 1.5 \\
\therefore \quad \log 2^{0.15t} &= \log(1.5) \\
\therefore \quad 0.15t \log 2 &= \log(1.5) \\
\therefore \quad t &= \frac{\log(1.5)}{0.15 \times \log 2} \\
\therefore \quad t &\approx 3.90 \text{ hours}
\end{aligned}$$

$\therefore$ it takes about 3.90 hours to reach 30 g.

b When $W_t = 100$,

$$\begin{aligned}
20 \times 2^{0.15t} &= 100 \\
\therefore \quad 2^{0.15t} &= 5 \\
\therefore \quad \log 2^{0.15t} &= \log 5 \\
\therefore \quad 0.15t \log 2 &= \log 5 \\
\therefore \quad t &= \frac{\log 5}{0.15 \times \log 2} \\
\therefore \quad t &\approx 15.5 \text{ hours}
\end{aligned}$$

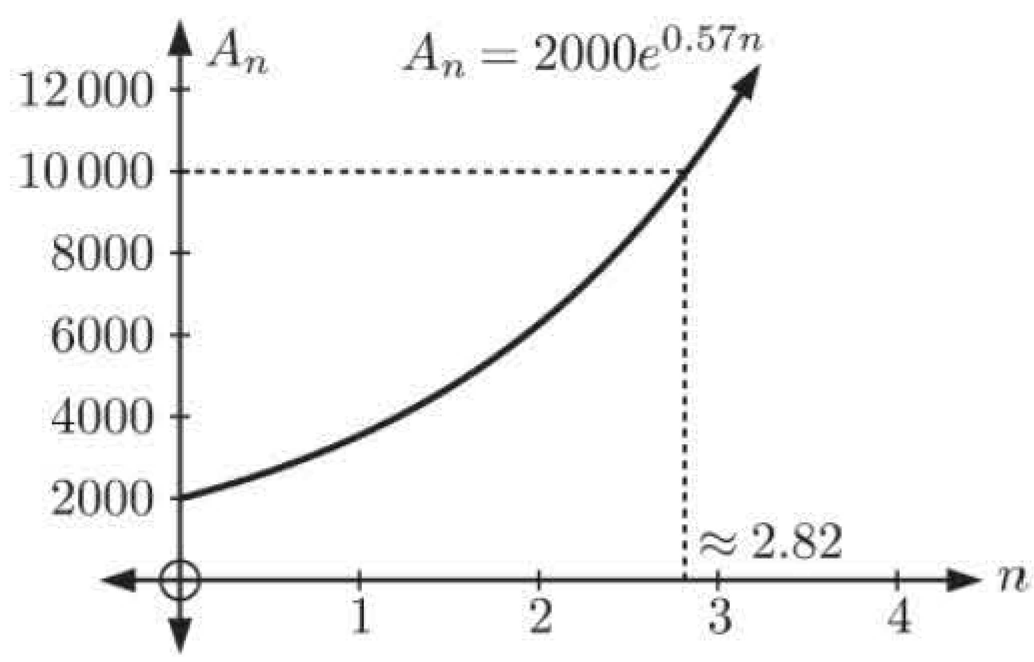
$\therefore$ it takes about 15.5 hours to reach 100 g.

2 When $M_t = 50$, $25 \times e^{0.1t} = 50$

$$\begin{aligned}
\therefore \quad e^{0.1t} &= 2 \\
\therefore \quad \ln e^{0.1t} &= \ln 2 \\
\therefore \quad 0.1t &= \ln 2 \\
\therefore \quad t &= 10 \ln 2
\end{aligned}$$

$\therefore$ it takes $10 \ln 2$ hours to reach 50 g.

3 **a**



b When $A_n = 10\,000$, $t \approx 2.82$

$\therefore$ we estimate that it will take 2.82 weeks for the infested area to reach 10 000 ha.

c **i** When $A_n = 10\,000$,

$$\begin{aligned} 2000 \times e^{0.57n} &= 10\,000 \\ \therefore\ e^{0.57n} &= 5 \\ \therefore\ \ln e^{0.57n} &= \ln 5 \\ \therefore\ 0.57n &= \ln 5 \\ \therefore\ n &= \frac{\ln 5}{0.57} \\ \therefore\ n &\approx 2.82 \end{aligned}$$

$\therefore$ it takes about 2.82 weeks for the infested area to reach 10 000 hectares.

4 $r = 107.5\% = 1.075$, $u_1 = 160\,000$, $u_{n+1} = 250\,000$

$$\begin{aligned} u_{n+1} &= u_1 \times r^n \\ \therefore\ 250\,000 &= 160\,000 \times (1.075)^n \\ \therefore\ (1.075)^n &= \tfrac{25}{16} \\ \therefore\ \log(1.075)^n &= \log(\tfrac{25}{16}) \\ \therefore\ n\log(1.075) &= \log(\tfrac{25}{16}) \\ \therefore\ n &= \frac{\log(\frac{25}{16})}{\log(1.075)} \approx 6.1709 \end{aligned}$$

$\therefore$ it would take 6.17 years or 6 years 62 days.

5 $u_1 = 10\,000$, $u_{n+1} = 15\,000$, $r = 104.8\% = 1.048$

$$\begin{aligned} u_{n+1} &= u_1 \times r^n \\ \therefore\ 15\,000 &= 10\,000 \times (1.048)^n \\ \therefore\ (1.048)^n &= 1.5 \\ \therefore\ \log(1.048)^n &= \log(1.5) \\ \therefore\ n\log(1.048) &= \log(1.5) \\ \therefore\ n &= \frac{\log(1.5)}{\log(1.048)} \\ \therefore\ n &\approx 8.648 \end{aligned}$$

$\therefore$ it would take 9 years. {interest compounded annually}

6 **a** 8.4% p.a. compounded monthly

is $\dfrac{8.4\%}{12} = 0.7\%$ a month

$= 0.007$

So $r = 1 + 0.007$

$\therefore\ r = 1.007$

b $u_1 = 15\,000$ and $u_{n+1} = 25\,000$

$$\begin{aligned} u_{n+1} &= u_1 \times r^n \\ \therefore\ 25\,000 &= 15\,000 \times (1.007)^n \\ \therefore\ (1.007)^n &= \tfrac{25}{15} = \tfrac{5}{3} \\ \therefore\ \log(1.007)^n &= \log(\tfrac{5}{3}) \\ \therefore\ n\log(1.007) &= \log(\tfrac{5}{3}) \\ \therefore\ n &= \frac{\log(\frac{5}{3})}{\log(1.007)} \approx 73.23 \end{aligned}$$

$\therefore$ he will withdraw the money after 74 months.

7 $M_t = 1000e^{-0.04t}$ $\quad\therefore\ M_0 = 1000e^0 = 1000$ g

a For M_t to halve, $M_t = 500$

$$\begin{aligned} \therefore\ 1000e^{-0.04t} &= 500 \\ \therefore\ e^{-0.04t} &= 0.5 \\ \therefore\ -0.04t &= \ln(0.5) \\ \therefore\ t &= \frac{\ln(0.5)}{-0.04} \\ \therefore\ t &\approx 17.3 \text{ years} \end{aligned}$$

b For $M_t = 25$ g,

$$\begin{aligned} \therefore\ 1000e^{-0.04t} &= 25 \\ \therefore\ e^{-0.04t} &= 0.025 \\ \therefore\ -0.04t &= \ln(0.025) \\ \therefore\ t &= \frac{\ln(0.025)}{-0.04} \\ \therefore\ t &\approx 92.2 \text{ years} \end{aligned}$$

c For $M_t = 1\%$ of M_0

$$\therefore\ 1000e^{-0.04t} = 0.01 \times 1000$$
$$\therefore\ e^{-0.04t} = 0.01$$
$$\therefore\ -0.04t = \ln(0.01)$$
$$\therefore\ t = \frac{\ln(0.01)}{-0.04}$$
$$\therefore\ t \approx 115 \text{ years}$$

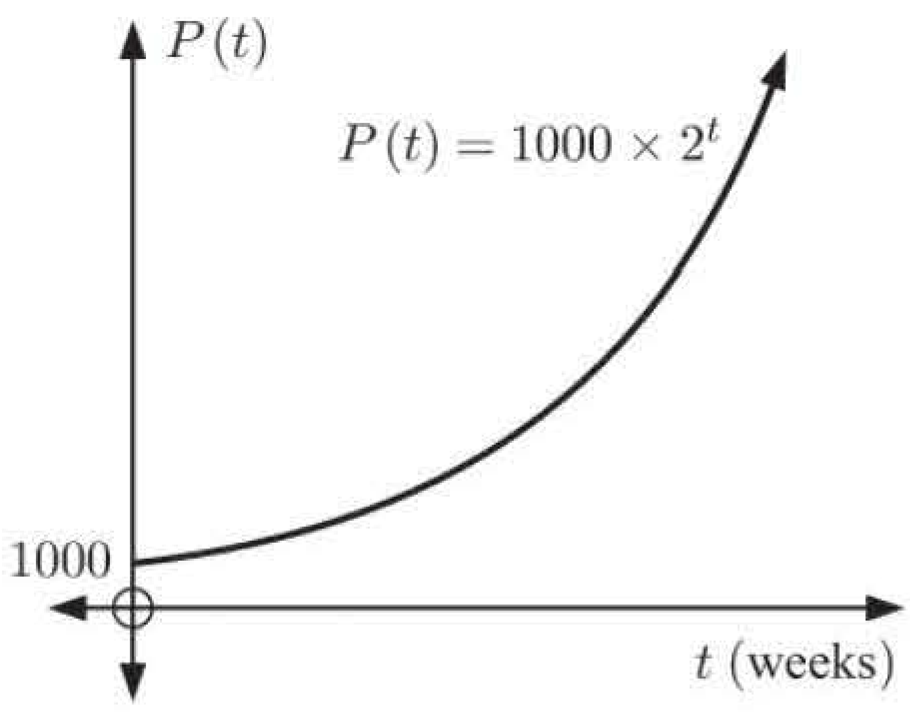
8 $V = 50(1 - e^{-0.2t})\ \text{m s}^{-1}$

So, when $V = 40$, $50(1 - e^{-0.2t}) = 40$

$$\therefore\ 1 - e^{-0.2t} = 0.8$$
$$\therefore\ e^{-0.2t} = 0.2$$
$$\therefore\ -0.2t = \ln(0.2)$$
$$\therefore\ -\tfrac{1}{5}t = \ln(\tfrac{1}{5})$$
$$\therefore\ -\tfrac{1}{5}t = \ln(5^{-1})$$
$$\therefore\ -\tfrac{1}{5}t = -\ln 5$$
$$\therefore\ t = 5\ln 5$$

$\therefore$ it will take $5\ln 5$ seconds for the man's speed to reach $40\ \text{m s}^{-1}$.

9 **a**



b When $P(t) = 20\,000$,

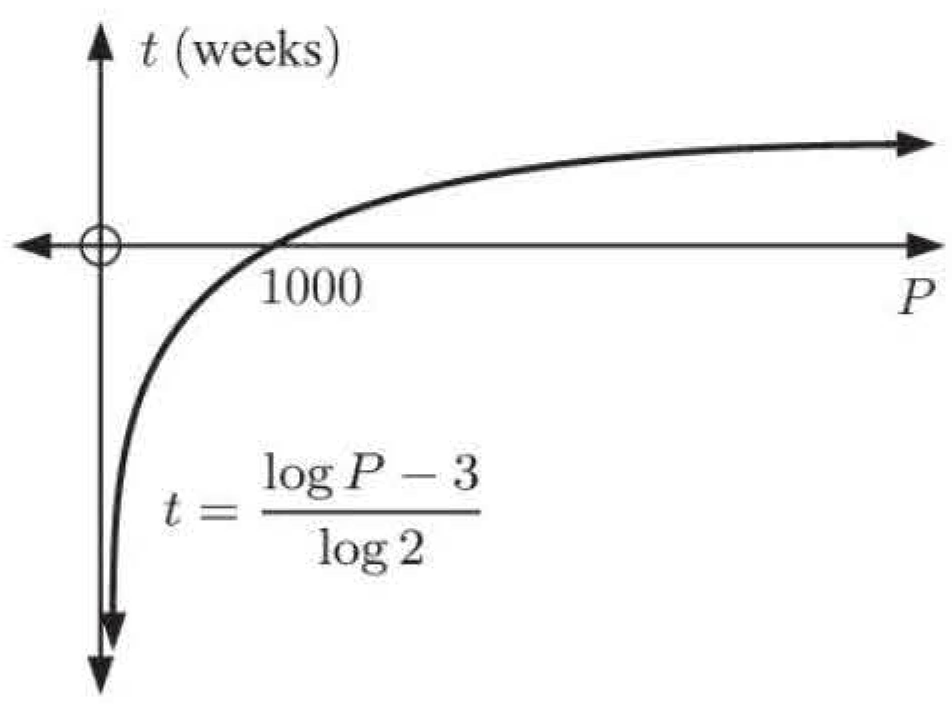
$$1000 \times 2^t = 20\,000$$
$$\therefore\ 2^t = 20$$
$$\therefore\ \log 2^t = \log 20$$
$$\therefore\ t\log 2 = \log 20$$
$$\therefore\ t = \frac{\log 20}{\log 2} \approx 4.32$$

$\therefore$ it will take 4.32 weeks for the population to reach 20 000 mice.

c

$$P = 1000 \times 2^t$$
$$\therefore\ 2^t = \frac{P}{1000}$$
$$\therefore\ \log 2^t = \log\left(\frac{P}{1000}\right)$$
$$\therefore\ t\log 2 = \log P - \log 1000$$
$$\therefore\ t = \frac{\log P - 3}{\log 2}$$

d



10 $T = 4 + 96 \times e^{-0.03t}\ ^\circ\text{C}$

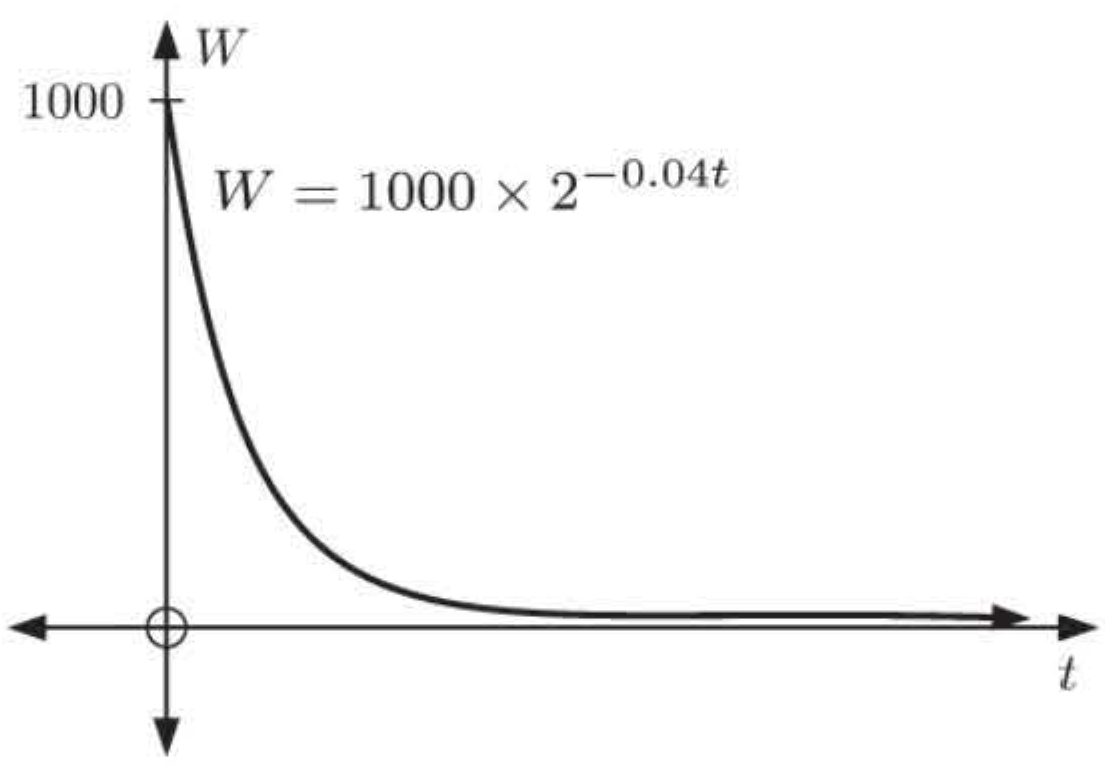
a When $T = 25$,

$$4 + 96 \times e^{-0.03t} = 25$$
$$\therefore\ 96 \times e^{-0.03t} = 21$$
$$\therefore\ e^{-0.03t} = \tfrac{21}{96}$$
$$\therefore\ -0.03t = \ln(\tfrac{21}{96})$$
$$\therefore\ t = \frac{\ln(\frac{21}{96})}{-0.03}$$
$$\therefore\ t \approx 50.7 \text{ minutes}$$

b When $T = 5$,

$$4 + 96 \times e^{-0.03t} = 5$$
$$\therefore\ 96 \times e^{-0.03t} = 1$$
$$\therefore\ e^{-0.03t} = \tfrac{1}{96}$$
$$\therefore\ -0.03t = \ln(\tfrac{1}{96})$$
$$\therefore\ t = \frac{\ln(\frac{1}{96})}{-0.03}$$
$$\therefore\ t \approx 152 \text{ minutes}$$

11 **a**



b
$$W = 1000 \times 2^{-0.04t}$$
$$\therefore\ 2^{-0.04t} = \frac{W}{1000}$$
$$\therefore\ \log 2^{-0.04t} = \log\left(\frac{W}{1000}\right)$$
$$\therefore\ -0.04t \log 2 = \log W - \log 1000$$
$$\therefore\ 0.04t \log 2 = 3 - \log W$$
$$\therefore\ t = \frac{3 - \log W}{0.04 \log 2}$$

c **i** When $W = 20$,
$$t = \frac{3 - \log 20}{0.04 \log 2} \approx 141$$
$\therefore$ it will take about 141 years for the weight to reach 20 grams.

ii When $W = 0.001$,
$$t = \frac{3 - \log(0.001)}{0.04 \log 2} \approx 498$$
$\therefore$ it will take about 498 years for the weight to reach 0.001 grams.

12 $W = W_0 \times 2^{-0.0002t}$ grams

a When W is 25% of original,
$$W = \tfrac{1}{4} \text{ of } W_0$$
$$\therefore\ W_0 \times 2^{-0.0002t} = \tfrac{1}{4} \times W_0$$
$$\therefore\ 2^{-0.0002t} = 2^{-2}$$
$$\therefore\ 0.0002t = 2$$
$$\therefore\ t = \frac{2}{0.0002}$$
$$\therefore\ t = 10\,000$$
$\therefore$ it would take 10 000 years.

b When W is 0.1% of original,
$$W = \tfrac{0.1}{100} \text{ of } W_0$$
$$\therefore\ W_0 \times 2^{-0.0002t} = \tfrac{1}{1000} \times W_0$$
$$\therefore\ \log 2^{-0.0002t} = \log(0.001)$$
$$\therefore\ -0.0002t \log 2 = \log(0.001)$$
$$\therefore\ t = \frac{\log(0.001)}{-0.0002 \times \log 2}$$
$$\therefore\ t \approx 49\,829$$
$\therefore$ it would take about 49 800 years.

13 $I = I_0 \times 2^{-0.02t}$ amps

When I is 10% of its original value,
$$I = 10\% \text{ of } I_0$$
$$\therefore\ I_0 \times 2^{-0.02t} = 0.1 \times I_0$$
$$\therefore\ 2^{-0.02t} = 0.1$$
$$\therefore\ \log 2^{-0.02t} = \log(0.1)$$
$$\therefore\ -0.02t \log 2 = \log(0.1)$$
$$\therefore\ -\tfrac{1}{50} t \log 2 = -1$$
$$\therefore\ t = \frac{50}{\log 2} \text{ seconds}$$

14 $V = 60(1 - 2^{-0.2t})$ m s^{-1}

When $V = 50$, $60(1 - 2^{-0.2t}) = 50$
$$\therefore\ 1 - 2^{-0.2t} = 0.8\overline{3}$$
$$\therefore\ 2^{-0.2t} = 0.1\overline{6}$$
$$\therefore\ \log 2^{-0.2t} = \log 0.1\overline{6}$$
$$\therefore\ -0.2t \log 2 = \log 0.1\overline{6}$$
$$\therefore\ t = \frac{\log 0.1\overline{6}}{-0.2 \times \log 2}$$
$$\therefore\ t \approx 12.9 \text{ seconds}$$

REVIEW SET 4A

1 **a** $\log_4 64$
$= \log_4 4^3$
$= 3$

b $\log_2 256$
$= \log_2 2^8$
$= 8$

c $\log_2(0.25)$
$= \log_2(\frac{1}{4})$
$= \log_2 2^{-2}$
$= -2$

d $\log_{25} 5$
$= \log_{25} 25^{\frac{1}{2}}$
$= \frac{1}{2}$

e $\log_8 1$
$= \log_8 8^0$
$= 0$

f $\log_{81} 3$
$= \log_{81} 81^{\frac{1}{4}}$
$= \frac{1}{4}$

g $\log_9(0.\overline{1})$
$= \log_9(\frac{1}{9})$
$= \log_9 9^{-1}$
$= -1$

h $\log_k \sqrt{k}$
$= \log_k k^{\frac{1}{2}}$
$= \frac{1}{2}$
provided $k > 0$, $k \neq 1$

2 **a** $\log\sqrt{10}$
$= \log 10^{\frac{1}{2}}$
$= \frac{1}{2}$

b $\log\left(\frac{1}{\sqrt[3]{10}}\right)$
$= \log 10^{-\frac{1}{3}}$
$= -\frac{1}{3}$

c $\log(10^a \times 10^{b+1})$
$= \log 10^{a+b+1}$
$= a+b+1$

3 **a** $4\ln 2 + 2\ln 3$
$= \ln 2^4 + \ln 3^2$
$= \ln(16\times 9)$
$= \ln 144$

b $\frac{1}{2}\ln 9 - \ln 2$
$= \ln 9^{\frac{1}{2}} - \ln 2$
$= \ln 3 - \ln 2$
$= \ln(\frac{3}{2})$

c $2\ln 5 - 1$
$= \ln 5^2 - \ln e^1$
$= \ln\left(\frac{25}{e}\right)$

d $\frac{1}{4}\ln 81$
$= \ln\left(3^4\right)^{\frac{1}{4}}$
$= \ln 3^1$
$= \ln 3$

4 **a** $\ln\left(e\sqrt{e}\right)$
$= \ln(e^1 e^{\frac{1}{2}})$
$= \ln e^{\frac{3}{2}}$
$= \frac{3}{2}$

b $\ln\left(\frac{1}{e^3}\right)$
$= \ln e^{-3}$
$= -3$

c $\ln(e^{2x}) = 2x$

d $\ln\left(\frac{e}{e^x}\right)$
$= \ln(e^{1-x})$
$= 1-x$

5 **a** $\log 16 + 2\log 3$
$= \log 16 + \log 3^2$
$= \log(16\times 9)$
$= \log 144$

b $\log_2 16 - 2\log_2 3$
$= \log_2 16 - \log_2 3^2$
$= \log_2(\frac{16}{9})$

c $2 + \log_4 5$
$= \log_4 4^2 + \log_4 5$
$= \log_4(16\times 5)$
$= \log_4 80$

6 **a** $P = 3\times b^x$
$\therefore\ \log P = \log(3\times b^x)$
$\therefore\ \log P = \log 3 + \log b^x$
$\therefore\ \log P = \log 3 + x\log b$

b $m = \dfrac{n^3}{p^2}$
$\therefore\ \log m = \log\left(\dfrac{n^3}{p^2}\right)$
$\therefore\ \log m = \log n^3 - \log p^2$
$\therefore\ \log m = 3\log n - 2\log p$

7 $\log_3 7 \times 2\log_7 x$
$= \cancel{\log_3 7}^{\,1} \times 2 \times \dfrac{\log_3 x}{\cancel{\log_3 7}_{\,1}}$
$= 2\log_3 x$

8 **a** $\log T = 2\log x - \log y$
$\therefore\ \log T = \log x^2 - \log y$
$\therefore\ \log T = \log\left(\dfrac{x^2}{y}\right)$
$\therefore\ T = \dfrac{x^2}{y}$

b $\log_2 K = \log_2 n + \frac{1}{2}\log_2 t$
$\therefore\ \log_2 K = \log_2 n + \log_2 t^{\frac{1}{2}}$
$\therefore\ \log_2 K = \log_2(n\times\sqrt{t})$
$\therefore\ K = n\sqrt{t}$

9 **a** $\ln 32 = \ln 2^5$
$= 5\ln 2$

b $\ln 125 = \ln 5^3$
$= 3\ln 5$

c $\ln 729 = \ln 3^6$
$= 6\ln 3$

10 $\log_2 x$ is defined for all $x > 0$
$\therefore$ the domain is $\{x \mid x > 0\}$
and the range is $y\in\mathbb{R}$.
$\ln(x+5)$ is defined for all $x > -5$
$\therefore$ the domain is $\{x \mid x > -5\}$
and the range is $y\in\mathbb{R}$.

So, the completed table is:

Function	$y=\log_2 x$	$y=\ln(x+5)$
Domain	$x>0$	$x>-5$
Range	$y\in\mathbb{R}$	$y\in\mathbb{R}$

11 **a**
$$\begin{aligned}&\log_5 36\\&= \log_5(2^2 \times 3^2)\\&= \log_5 2^2 + \log_5 3^2\\&= 2\log_5 2 + 2\log_5 3\\&= 2A + 2B\end{aligned}$$

b
$$\begin{aligned}&\log_5 54\\&= \log_5(2 \times 3^3)\\&= \log_5 2 + \log_5 3^3\\&= \log_5 2 + 3\log_5 3\\&= A + 3B\end{aligned}$$

c
$$\begin{aligned}&\log_5(8\sqrt{3})\\&= \log_5(2^3 \times 3^{\frac{1}{2}})\\&= \log_5 2^3 + \log_5 3^{\frac{1}{2}}\\&= 3\log_5 2 + \tfrac{1}{2}\log_5 3\\&= 3A + \tfrac{1}{2}B\end{aligned}$$

d
$$\begin{aligned}&\log_5(20.25)\\&= \log_5(\tfrac{81}{4})\\&= \log_5\left(\frac{3^4}{2^2}\right)\\&= \log_5 3^4 - \log_5 2^2\\&= 4\log_5 3 - 2\log_5 2\\&= 4B - 2A\end{aligned}$$

e
$$\begin{aligned}&\log_5(0.\overline{8})\\&= \log_5(\tfrac{8}{9})\\&= \log_5\left(\frac{2^3}{3^2}\right)\\&= \log_5 2^3 - \log_5 3^2\\&= 3\log_5 2 - 2\log_5 3\\&= 3A - 2B\end{aligned}$$

12 **a**
$$\begin{aligned}3e^x - 5 &= -2e^{-x}\\ \therefore\ 3e^{2x} - 5e^x &= -2\\ &\{\text{multiplying both sides by } e^x\}\\ \therefore\ 3e^{2x} - 5e^x + 2 &= 0\\ \therefore\ (3e^x - 2)(e^x - 1) &= 0\end{aligned}$$
$$\begin{aligned}\therefore\ 3e^x - 2 = 0 \quad &\text{or} \quad e^x - 1 = 0\\ \therefore\ e^x = \tfrac{2}{3} \quad &\text{or} \quad e^x = 1\\ \therefore\ x = \ln(\tfrac{2}{3}) \quad &\text{or} \quad x = 0\end{aligned}$$

b
$$\begin{aligned}2\ln x - 3\ln\left(\frac{1}{x}\right) &= 10\\ \therefore\ \ln x^2 - \ln\left(\frac{1}{x^3}\right) &= 10\\ \therefore\ \ln x^2 + \ln\left(\frac{1}{x^3}\right)^{-1} &= 10\\ \therefore\ \ln(x^2 \times x^3) &= 10\\ \therefore\ \ln x^5 &= 10\\ \therefore\ x^5 &= e^{10}\\ \therefore\ x &= \sqrt[5]{e^{10}}\\ \therefore\ x &= e^2\end{aligned}$$

REVIEW SET 4B

1 **a**
$$\begin{aligned}32 &= 10^{\log 32}\\ &\approx 10^{1.5051}\end{aligned}$$

b
$$\begin{aligned}&0.0013\\&= 10^{\log(0.0013)}\\&\approx 10^{-2.8861}\end{aligned}$$

c
$$\begin{aligned}&8.963 \times 10^{-5}\\&= 10^{\log(8.963)} \times 10^{-5}\\&\approx 10^{0.952} \times 10^{-5}\\&\approx 10^{-4.0475}\end{aligned}$$

2 **a**
$$\begin{aligned}&\log_2 x = -3\\ \therefore\ & x = 2^{-3}\\ \therefore\ & x = \tfrac{1}{8}\end{aligned}$$

b
$$\begin{aligned}&\log_5 x \approx 2.743\\ \therefore\ & x \approx 5^{2.743}\\ \therefore\ & x \approx 82.7\end{aligned}$$

c
$$\begin{aligned}&\log_3 x \approx -3.145\\ \therefore\ & x \approx 3^{-3.145}\\ \therefore\ & x \approx 0.0316\end{aligned}$$

3 **a**
$$\begin{aligned}&\log_2 k \approx 1.699 + x\\ \therefore\ & k \approx 2^{1.699+x}\\ \therefore\ & k \approx 2^{1.699} \times 2^x\\ \therefore\ & k \approx 3.25 \times 2^x\end{aligned}$$

b
$$\begin{aligned}\log_a Q &= 3\log_a P + \log_a R\\ &= \log_a P^3 + \log_a R\\ &= \log_a(P^3 \times R)\\ \therefore\ Q &= P^3 R\end{aligned}$$

c
$$\begin{aligned}&\log A \approx 5\log B - 2.602\\ \therefore\ & \log A - \log B^5 \approx -2.602\\ \therefore\ & \log\left(\frac{A}{B^5}\right) \approx -2.602\\ \therefore\ & \frac{A}{B^5} \approx 10^{-2.602} \approx 0.0025\\ \therefore\ & A \approx \frac{B^5}{400}\end{aligned}$$

4 **a**
$$\begin{aligned} 5^x &= 7 \\ \therefore\ \log 5^x &= \log 7 \\ \therefore\ x\log 5 &= \log 7 \\ \therefore\ x &= \frac{\log 7}{\log 5} \end{aligned}$$

b
$$\begin{aligned} 20\times 2^{2x+1} &= 640 \\ \therefore\ 2^{2x+1} &= 32 \\ \therefore\ \log 2^{2x+1} &= \log 32 \\ \therefore\ (2x+1)\log 2 &= \log 32 \\ \therefore\ 2x+1 &= \frac{\log 32}{\log 2} = 5 \\ \therefore\ 2x &= 4 \\ \therefore\ x &= 2 \end{aligned}$$

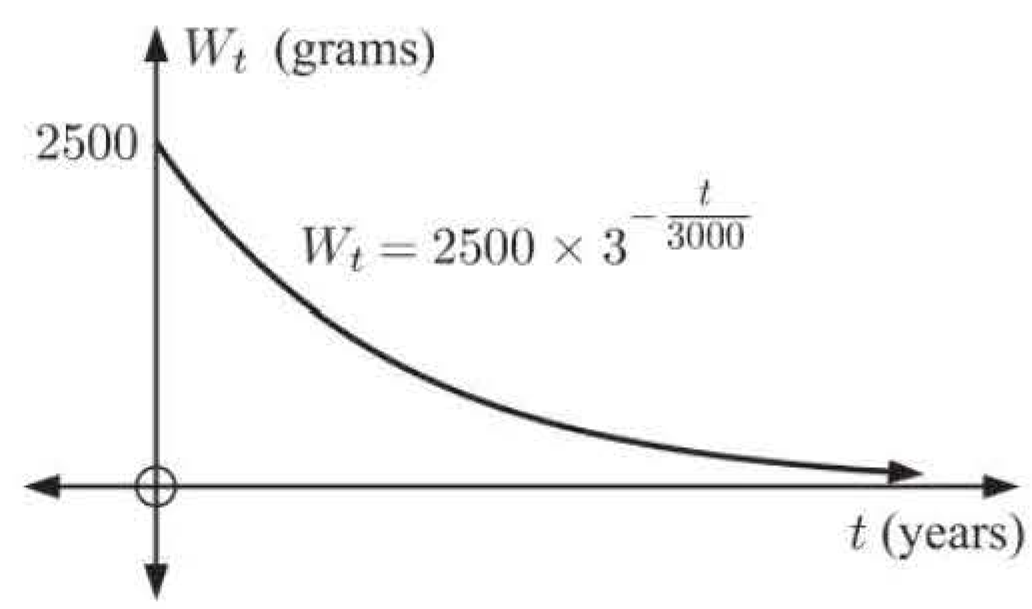
5 $W_t = 2500\times 3^{-\frac{t}{3000}}$ grams

a
$$\begin{aligned} W_0 &= 2500\times 3^0 \\ &= 2500\times 1 \\ &= 2500 \text{ grams} \end{aligned}$$

b We need t when $W_t = 30\%$ of 2500 g
$$\begin{aligned} \therefore\ 2500\times 3^{-\frac{t}{3000}} &= 0.3\times 2500 \\ \therefore\ \log 3^{-\frac{t}{3000}} &= \log(0.3) \\ \therefore\ -\frac{t}{3000}\times\log 3 &= \log(0.3) \\ \therefore\ t &= \frac{-\log(0.3)\times 3000}{\log 3} \\ \therefore\ t &\approx 3287.7 \end{aligned}$$
$\therefore$ about 3290 years

c
$$\begin{aligned} &\%\text{ change} \\ &= \left(\frac{W_{1500}-W_0}{W_0}\right)\times 100\% \\ &= \left(\frac{2500\times 3^{-\frac{1500}{3000}}-2500}{2500}\right)\times 100\% \\ &= (3^{-\frac{1}{2}}-1)\times 100\% \\ &\approx -42.3\% \end{aligned}$$
So, a loss of 42.3%.

d



6
$$\begin{aligned} 16^x - 5\times 8^x &= 0 \\ \therefore\ 2^x\times 8^x - 5\times 8^x &= 0 \\ \therefore\ 8^x(2^x-5) &= 0 \\ \therefore\ 2^x &= 5 \quad \text{as}\quad 8^x>0 \quad\text{for all } x \\ \therefore\ x &= \log_2 5 \quad\text{as required} \end{aligned}$$

7 **a**
$$\begin{aligned} \ln x &= 5 \\ \therefore\ x &= e^5 \end{aligned}$$

b
$$\begin{aligned} 3\ln x + 2 &= 0 \\ \therefore\ 3\ln x &= -2 \\ \therefore\ \ln x &= -\tfrac{2}{3} \\ \therefore\ x &= e^{-\frac{2}{3}} \end{aligned}$$

c
$$\begin{aligned} e^x &= 400 \\ \therefore\ x &= \ln 400 \end{aligned}$$

d
$$\begin{aligned} e^{2x+1} &= 11 \\ \therefore\ 2x+1 &= \ln 11 \\ \therefore\ 2x &= \ln 11 - 1 \\ \therefore\ x &= \frac{\ln 11 - 1}{2} \end{aligned}$$

e
$$\begin{aligned} 25e^{\frac{x}{2}} &= 750 \\ \therefore\ e^{\frac{x}{2}} &= 30 \\ \therefore\ \frac{x}{2} &= \ln 30 \\ \therefore\ x &= 2\ln 30 \end{aligned}$$

8 $P_t = P_0 \times 2^{\frac{t}{3}}, \quad t \geqslant 0$

When $t = 0$, $P_0 = P_0 \times 2^0 = P_0$. So the initial population was P_0.

a If P_t doubles, $P_t = 2P_0$

$$\therefore \quad P_0 2^{\frac{t}{3}} = 2P_0$$
$$\therefore \quad 2^{\frac{t}{3}} = 2^1$$
$$\therefore \quad \frac{t}{3} = 1$$
$$\therefore \quad t = 3$$

So, it would take 3 years.

b $$\% \text{ increase} = \left(\frac{P_4 - P_0}{P_0}\right) \times 100\%$$
$$= \left(\frac{P_0 \times 2^{\frac{4}{3}} - P_0}{P_0}\right) \times 100\%$$
$$= (2^{\frac{4}{3}} - 1) \times 100\%$$
$$\approx 151.98\%$$

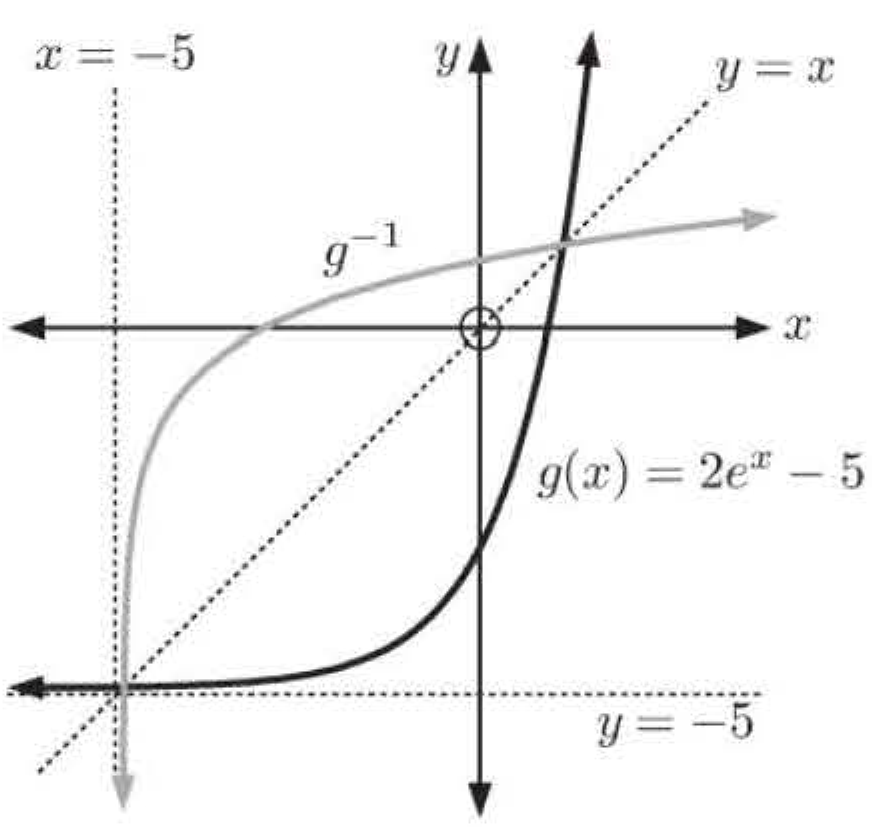
So, the % increase is 152%.

or $$P_4 = P_0 \times 2^{\frac{4}{3}}$$
$$\approx P_0 \times 2.52$$
$$\approx 252\% \text{ of } P_0$$

So, an increase of 152%.

9 **a** $g(x) = 2e^x - 5$ has inverse function $x = 2e^y - 5$

$$\therefore \quad 2e^y = x + 5$$
$$\therefore \quad e^y = \frac{x+5}{2}$$
$$\therefore \quad y = \ln\left(\frac{x+5}{2}\right)$$
$$\therefore \quad g^{-1}(x) = \ln\left(\frac{x+5}{2}\right)$$

b 

c Domain of g is $\{x \mid x \in \mathbb{R}\}$, range is $\{y \mid y > -5\}$

Domain of g^{-1} is $\{x \mid x > -5\}$, range is $\{y \mid y \in \mathbb{R}\}$

d When $x = 0$, $y = 2e^0 - 5 = -3$

$\therefore$ the y-intercept of g is -3.

When $y = 0$, $2e^x - 5 = 0$

$$\therefore \quad e^x = \tfrac{5}{2}$$
$$\therefore \quad x = \ln\left(\tfrac{5}{2}\right)$$

$\therefore$ the x-intercept of g is $\ln\left(\frac{5}{2}\right) \approx 0.916$.

$\therefore$ g^{-1} has x-intercept -3 and y-intercept $\ln\left(\frac{5}{2}\right) \approx 0.916$.

As $x \to -\infty$, $y \to -5^+$

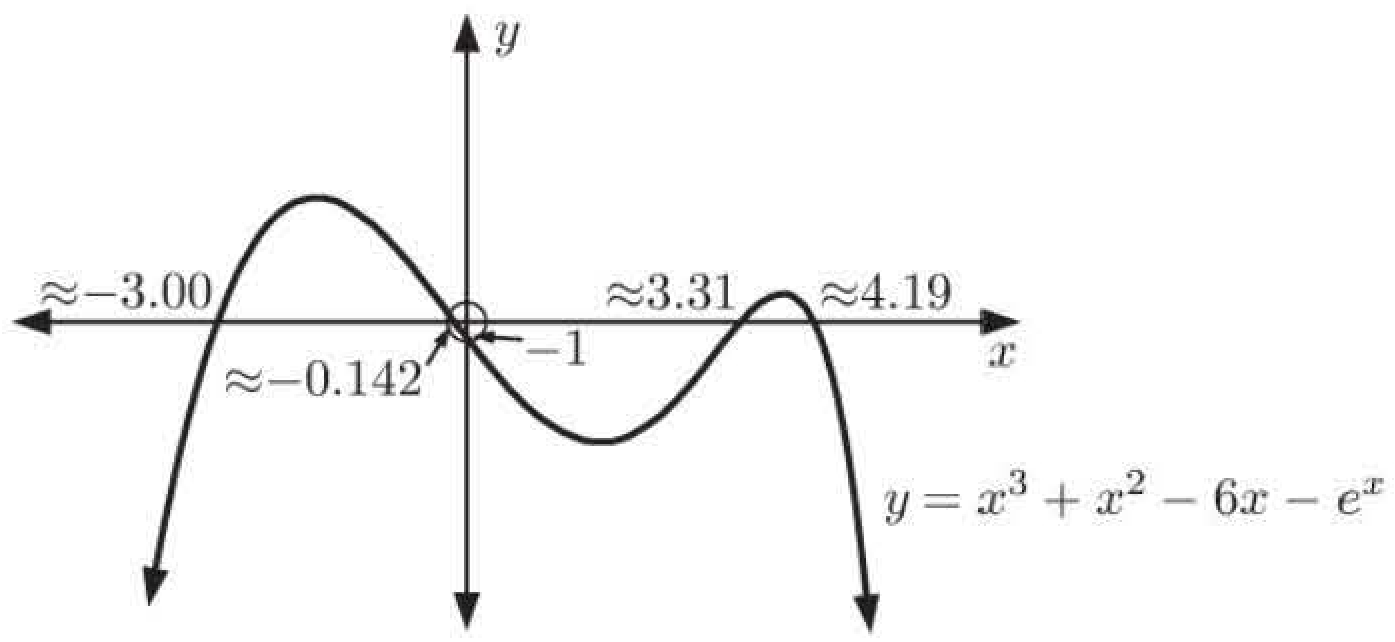
$\therefore$ the H.A. of g is $y = -5$.

$\therefore$ g^{-1} has V.A. $x = -5$.

10 **a** $$g(5) = \ln(5+4)$$
$$= \ln 9$$
$$\therefore \quad (f \circ g)(5) = f(g(5))$$
$$= e^{\ln 9}$$
$$= 9$$

b $$f(0) = e^0$$
$$= 1$$
$$\therefore \quad (g \circ f)(0) = g(f(0))$$
$$= \ln(1+4)$$
$$= \ln 5$$

11 **a**



b $e^x < x^3 + x^2 - 6x$ when $x^3 + x^2 - 6x - e^x > 0$

$\therefore$ from the graph, when $-3.00 < x < -0.142$ and $3.31 < x < 4.19$.

REVIEW SET 4C

1 **a**
$$\log\sqrt{1000} = \log\left(10^3\right)^{\frac{1}{2}} = \log 10^{\frac{3}{2}} = \tfrac{3}{2}$$

b
$$\log\left(\frac{10}{\sqrt[3]{10}}\right) = \log\left(\frac{10^1}{10^{\frac{1}{3}}}\right) = \log 10^{\frac{2}{3}} = \tfrac{2}{3}$$

c
$$\log\left(\frac{10^a}{10^{-b}}\right) = \log\left(10^{a-(-b)}\right) = \log 10^{a+b} = a+b$$

2 **a**
$$e^{4\ln x} = \left(e^{\ln x}\right)^4 = x^4$$

b $\ln(e^5) = 5$ $\{$as $\ln e^a = a\}$

c $\ln\left(\sqrt{e}\right) = \ln e^{\frac{1}{2}} = \tfrac{1}{2}$

d
$$10^{\log x + \log 3} = 10^{\log x} \times 10^{\log 3} = x \times 3 = 3x$$

e $\ln\left(\frac{1}{e^x}\right) = \ln e^{-x} = -x$

f
$$\frac{\log x^2}{\log_3 9} = \frac{\log x^2}{2} = \tfrac{1}{2}\log x^2 = \log\left(x^2\right)^{\frac{1}{2}} = \log x$$

3 **a** $20 = e^{\ln 20} \approx e^{2.9957}$

b $3000 = e^{\ln 3000} \approx e^{8.0064}$

c $0.075 = e^{\ln(0.075)} \approx e^{-2.5903}$

4 **a** $\log x = 3$
$\therefore\ x = 10^3$
$\therefore\ x = 1000$

b $\log_3(x+2) = 1.732$
$\therefore\ x + 2 = 3^{1.732}$
$\therefore\ x + 2 \approx 6.7046$
$\therefore\ x \approx 4.7046$
$\therefore\ x \approx 4.70$

c $\log_2\left(\frac{x}{10}\right) = -0.671$
$\therefore\ \frac{x}{10} = 2^{-0.671}$
$\therefore\ \frac{x}{10} \approx 0.628\,07$
$\therefore\ x \approx 6.28$

5 **a**
$$\ln 60 - \ln 20 = \ln\left(\tfrac{60}{20}\right) = \ln 3$$

b
$$\ln 4 + \ln 1 = \ln 4 + 0 = \ln 4$$

c
$$\ln 200 - \ln 8 + \ln 5 = \ln\left(\tfrac{200}{8}\right) + \ln 5 = \ln\left(\tfrac{200}{8} \times 5\right) = \ln 125$$

6 **a**
$$\begin{aligned} M &= ab^n \\ \therefore\ \log M &= \log(ab^n) \\ \therefore\ \log M &= \log a + \log b^n \\ \therefore\ \log M &= \log a + n\log b \end{aligned}$$

b
$$\begin{aligned} T &= \frac{5}{\sqrt{l}} \\ \therefore\ \log T &= \log\left(\frac{5}{l^{\frac{1}{2}}}\right) \\ \therefore\ \log T &= \log 5 - \log l^{\frac{1}{2}} \\ \therefore\ \log T &= \log 5 - \tfrac{1}{2}\log l \end{aligned}$$

c
$$\begin{aligned} G &= \frac{a^2b}{c} \\ \therefore\ \log G &= \log\left(\frac{a^2b}{c}\right) \\ \therefore\ \log G &= \log(a^2b) - \log c \\ \therefore\ \log G &= \log a^2 + \log b - \log c \\ \therefore\ \log G &= 2\log a + \log b - \log c \end{aligned}$$

7 **a**
$$\begin{aligned} 3^x &= 300 \\ \therefore\ \log 3^x &= \log 300 \\ \therefore\ x\log 3 &= \log 300 \\ \therefore\ x &= \frac{\log 300}{\log 3} \\ \therefore\ x &\approx 5.19 \end{aligned}$$

b
$$\begin{aligned} 30 \times 5^{1-x} &= 0.15 \\ \therefore\ 5^{1-x} &= 0.005 \\ \therefore\ \log 5^{1-x} &= \log(0.005) \\ \therefore\ (1-x)\log 5 &= \log(0.005) \\ \therefore\ 1-x &= \frac{\log(0.005)}{\log 5} \\ \therefore\ 1-x &\approx -3.292 \\ \therefore\ x &\approx 4.29 \end{aligned}$$

c
$$\begin{aligned} 3^{x+2} &= 2^{1-x} \\ \therefore\ \log 3^{x+2} &= \log 2^{1-x} \\ \therefore\ (x+2)\log 3 &= (1-x)\log 2 \\ \therefore\ x\log 3 + 2\log 3 &= \log 2 - x\log 2 \\ \therefore\ x(\log 3 + \log 2) &= \log 2 - 2\log 3 \\ \therefore\ x\log 6 &= \log(\tfrac{2}{9}) \\ \therefore\ x &= \frac{\log(\frac{2}{9})}{\log 6} \\ \therefore\ x &\approx -0.839 \end{aligned}$$

8 **a**
$$\begin{aligned} e^{2x} &= 3e^x \\ \therefore\ e^{2x} - 3e^x &= 0 \\ \therefore\ e^x(e^x - 3) &= 0 \\ \therefore\ e^x - 3 &= 0 \quad \{e^x > 0 \text{ for all } x\} \\ \therefore\ e^x &= 3 \\ \therefore\ x &= \ln 3 \end{aligned}$$

b
$$\begin{aligned} e^{2x} - 7e^x + 12 &= 0 \\ \therefore\ (e^x - 3)(e^x - 4) &= 0 \\ \therefore\ e^x - 3 = 0 \quad &\text{or} \quad e^x - 4 = 0 \\ \therefore\ e^x = 3 \quad &\text{or} \quad e^x = 4 \\ \therefore\ x = \ln 3 \quad &\text{or} \quad \ln 4 \end{aligned}$$

9 **a**
$$\begin{aligned} \ln P &= 1.5\ln Q + \ln T \\ \therefore\ \ln P &= \ln Q^{1.5} + \ln T \\ &= \ln(TQ^{1.5}) \\ \therefore\ P &= TQ^{1.5} \end{aligned}$$

b
$$\begin{aligned} \ln M &= 1.2 - 0.5\ln N \\ \therefore\ \ln M + \ln N^{\frac{1}{2}} &= 1.2 \\ \therefore\ \ln\left(M\sqrt{N}\right) &= 1.2 \\ \therefore\ M\sqrt{N} &= e^{1.2} \\ \therefore\ M &= \frac{e^{1.2}}{\sqrt{N}} \end{aligned}$$

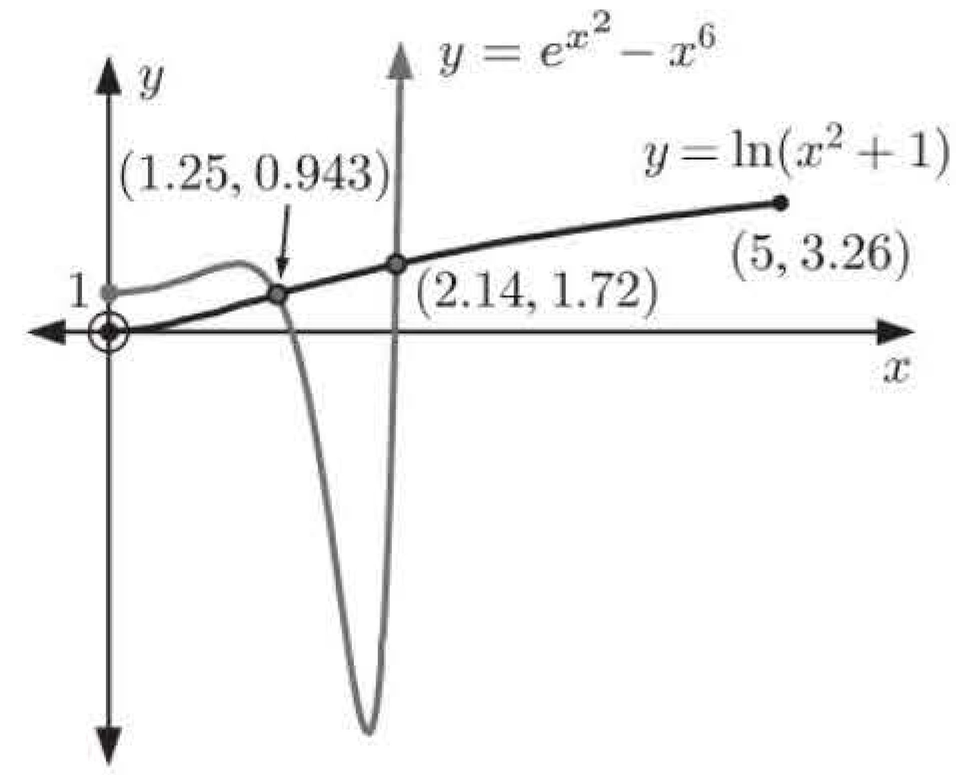
10 **a**
$$f(x) = e^{x^2} - x^6$$
$$\therefore\ f(-x) = e^{(-x)^2} - (-x)^6$$
$$= e^{x^2} - x^6$$
$$= f(x)$$

So, $f(x)$ is an even function.

$$g(x) = \ln(x^2+1)$$
$$\therefore\ g(-x) = \ln((-x)^2+1)$$
$$= \ln(x^2+1)$$
$$= g(x)$$

So, $g(x)$ is an even function.

b



On the domain $0 \leqslant x \leqslant 5$, the graphs intersect at $(1.25, 0.943)$ and $(2.14, 1.72)$.

c $x^6 - e^{x^2} + \ln(x^2+1) > 0$

when $\ln(x^2+1) > e^{x^2} - x^6$

$\therefore$ using **b**, $1.25 < x < 2.14$.

11 $g(x) = \log_3(x+2) - 2$

a We require $x + 2 > 0$, so $x > -2$

$\therefore$ the domain is $\{x \mid x > -2\}$ and the range is $\{y \mid y \in \mathbb{R}\}$.

b If $x \to -2^+$, $y \to -\infty$ $\quad\therefore$ V.A. is $x = -2$.

As $x \to \infty$, $y \to \infty$.

When $x = 0$, $g(0) = \log_3 2 - 2 \approx -1.37$ So, the y-intercept ≈ -1.37

When $y = 0$, $\log_3(x+2) = 2$ $\quad\therefore\ x + 2 = 3^2$

$\therefore\ x = 7$ So, the x-intercept is 7.

c g^{-1} is defined by $x = \log_3(y+2) - 2$

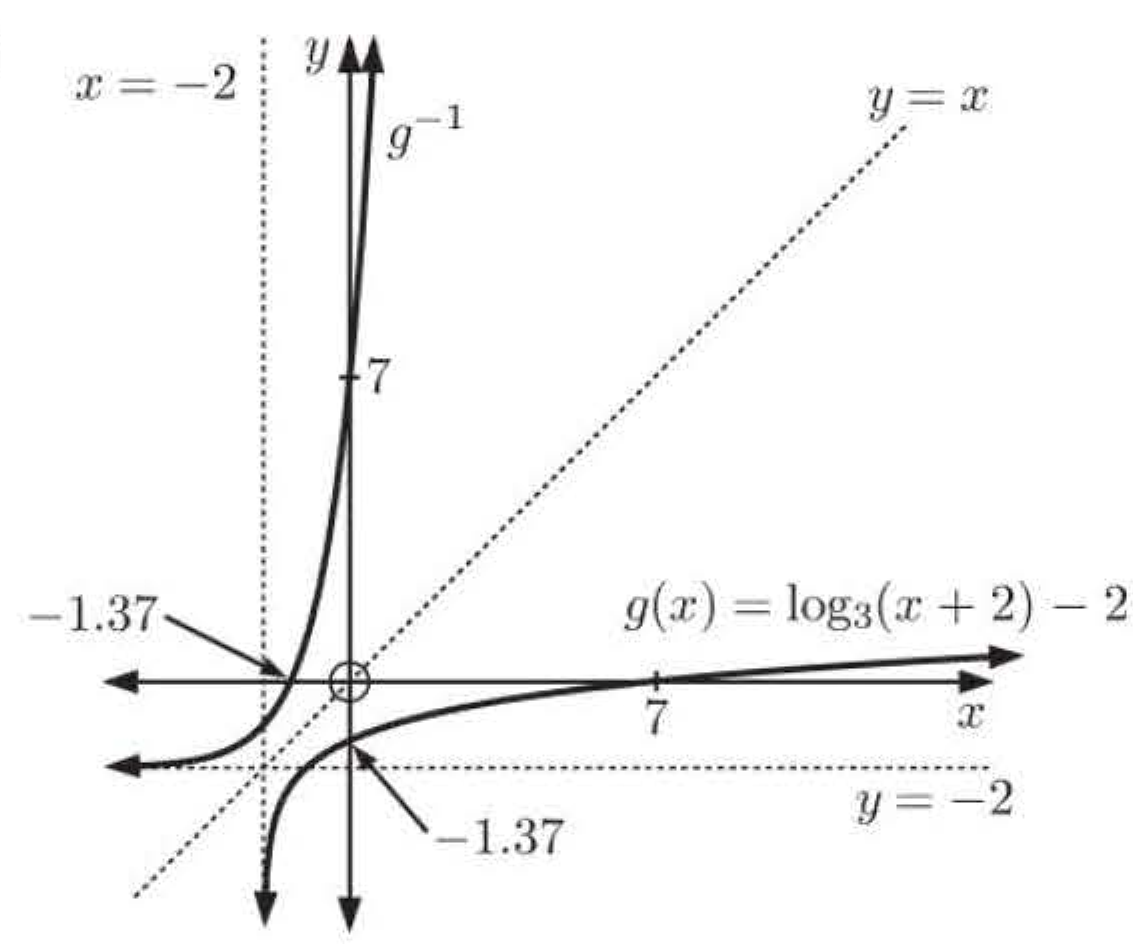
$$\therefore\ \log_3(y+2) = x + 2$$
$$\therefore\ y + 2 = 3^{x+2}$$
$$\therefore\ y = 3^{x+2} - 2$$
$$\therefore\ g^{-1}(x) = 3^{x+2} - 2$$

Horizontal asymptote is $y = -2$.

Domain is $x \in \mathbb{R}$.

Range is $\{y \mid y > -2\}$.

d



12 $W_t = 8000 \times e^{-\frac{t}{20}}$ grams

$$W_0 = 8000e^0$$
$$= 8000 \times 1$$
$$= 8000 \text{ grams}$$

a When $W_t = \frac{1}{2} \times 8000$ grams, $8000e^{-\frac{t}{20}} = 4000$

$$\therefore\ e^{-\frac{t}{20}} = 0.5$$
$$\therefore\ -\frac{t}{20} = \ln(0.5)$$
$$\therefore\ t = -20\ln(0.5) \approx 13.9 \text{ weeks}$$

b When $W_t = 1000$ g,

$$8000e^{-\frac{t}{20}} = 1000$$
$$\therefore\ e^{-\frac{t}{20}} = \tfrac{1}{8}$$
$$\therefore\ -\frac{t}{20} = \ln(\tfrac{1}{8})$$
$$\therefore\ t = -20\ln(\tfrac{1}{8})$$
$$\therefore\ t \approx 41.6 \text{ weeks}$$

c When $W_t = 0.1\%$ of W_0

$$= \tfrac{1}{1000} \times 8000 = 8 \text{ g},$$
$$8000e^{-\frac{t}{20}} = 8$$
$$\therefore\ e^{-\frac{t}{20}} = 0.001$$
$$\therefore\ -\frac{t}{20} = \ln(0.001)$$
$$\therefore\ t = -20\ln(0.001) \approx 138 \text{ weeks}$$