Chapter 2

FUNCTIONS

EXERCISE 2A

1 **a** $\{(1,\,3),\,(2,\,4),\,(3,\,5),\,(4,\,6)\}$ is a function since no two ordered pairs have the same x-coordinate.

b $\{(1,\,3),\,(3,\,2),\,(1,\,7),\,(-1,\,4)\}$ is not a function since two of the ordered pairs, $(1,\,3)$ and $(1,\,7)$, have the same x-coordinate 1.

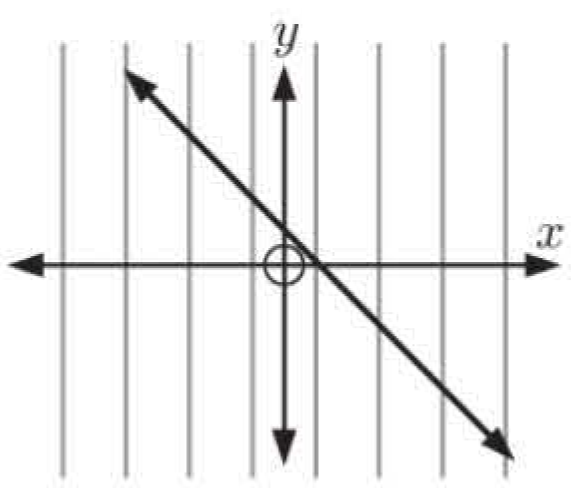
c $\{(2,-1),(2,0),(2,3),(2,11)\}$ is not a function since each ordered pair has the same x-coordinate 2.

d $\{(7,\,6),\,(5,\,6),\,(3,\,6),\,(-4,\,6)\}$ is a function since no two ordered pairs have the same x-coordinate.

e $\{(0,\,0),\,(1,\,0),\,(3,\,0),\,(5,\,0)\}$ is a function since no two ordered pairs have the same x-coordinate.

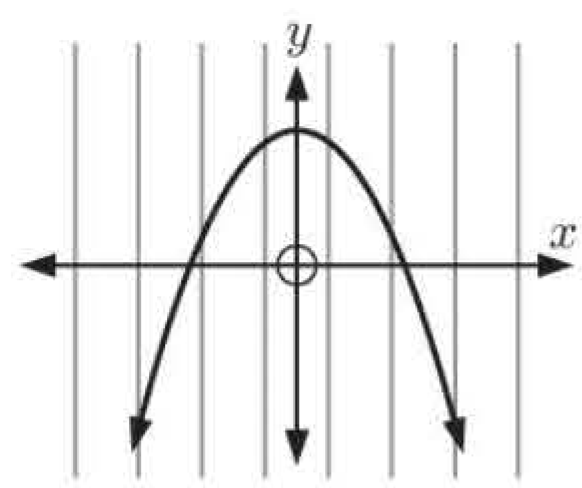
f $\{(0,0),\,(0,-2),\,(0,2),\,(0,4)\}$ is not a function since each ordered pair has the same x-coordinate 0.

2 **a**



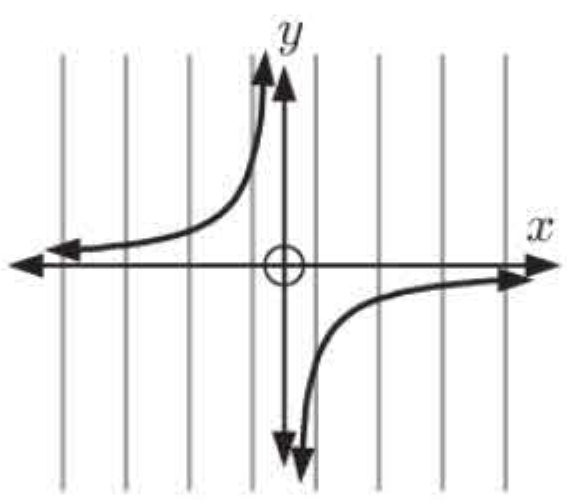
Each line cuts the graph no more than once, so it is a function.

b



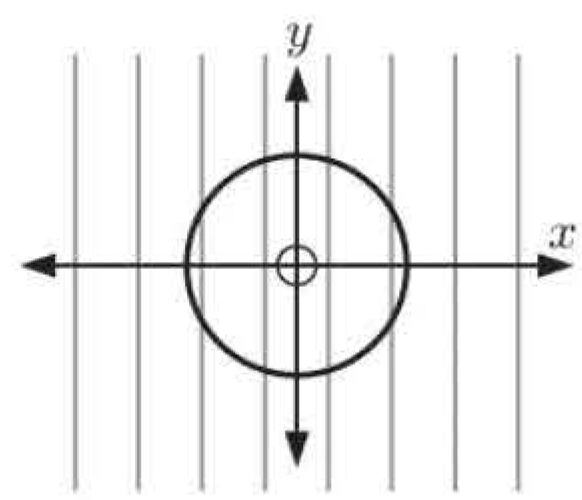
Each line cuts the graph no more than once, so it is a function.

c



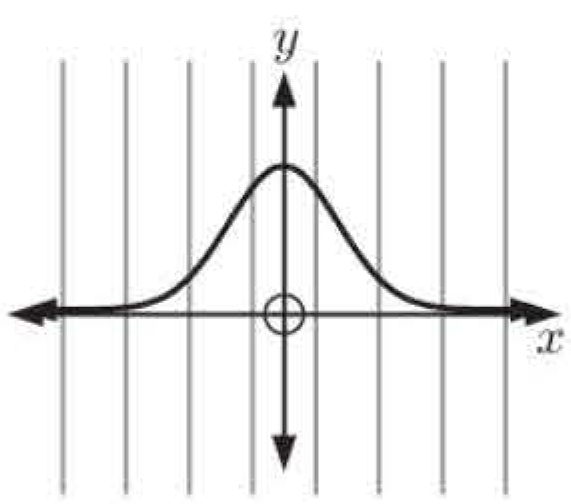
Each line cuts the graph no more than once, so it is a function.

d



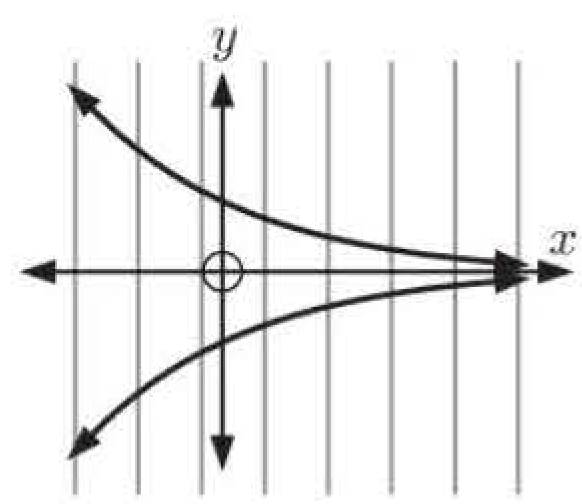
Some lines cut the graph more than once, so it is not a function.

e



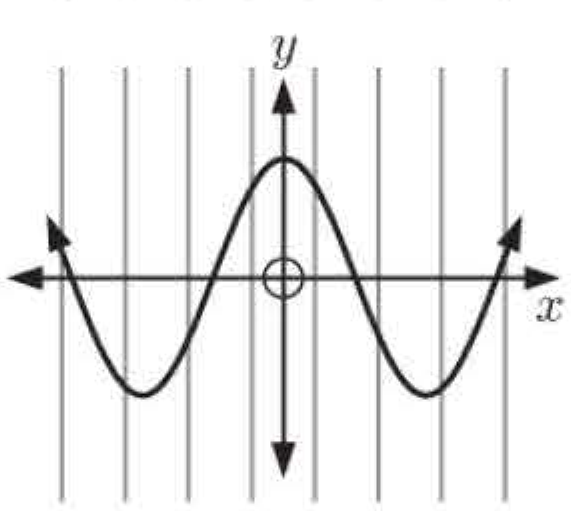
Each line cuts the graph no more than once, so it is a function.

f



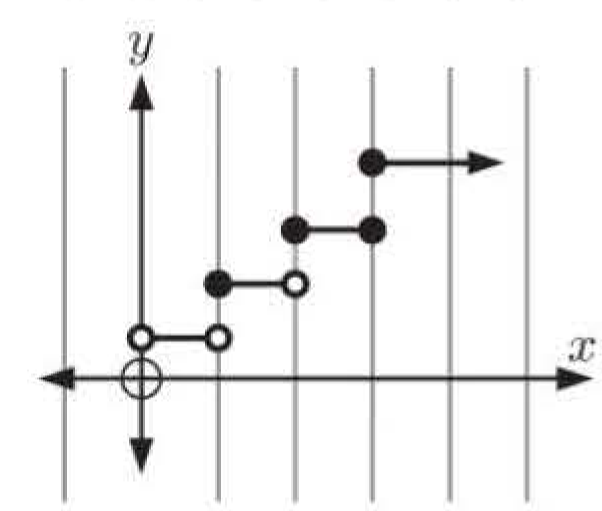
The lines cut the graph more than once, so it is not a function.

g



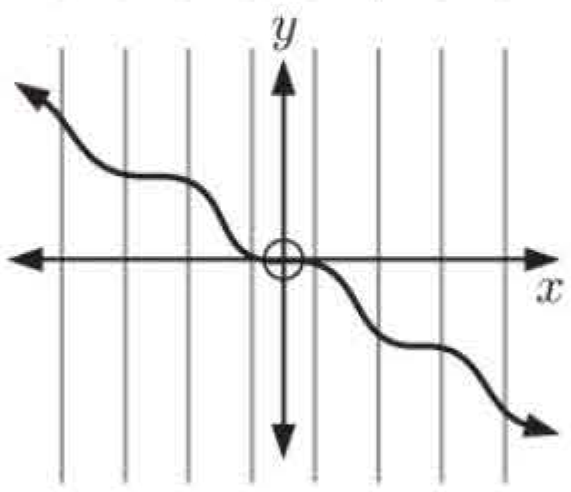
Each line cuts the graph no more than once, so it is a function.

h



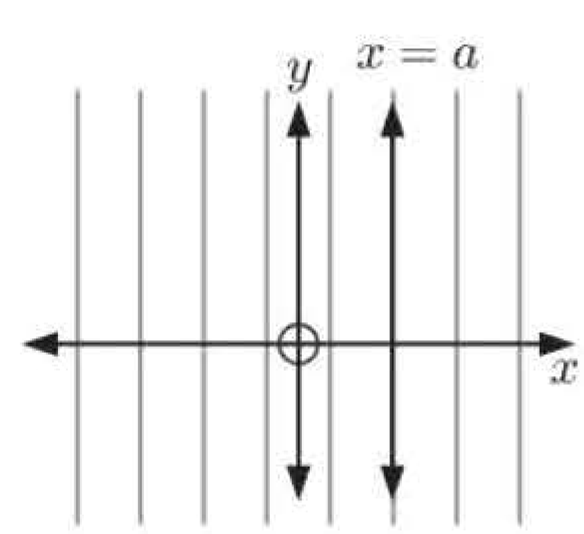
One line cuts the graph more than once, so it is not a function.

i



Each line cuts the graph no more than once, so it is a function.

3 The graph of a straight line is not a function if the graph is a vertical line. So, it is not a function if it has the form $x = a$ for some constant a.
The vertical line through $x = a$ cuts the graph at every point, so it is not a function.



4 $x^2 + y^2 = 9$ is the equation of a circle, centre $(0, 0)$ and radius 3.

Now $x^2 + y^2 = 9$

$\therefore\ y^2 = 9 - x^2$

$\therefore\ y = \pm\sqrt{9 - x^2}$

For any value of x where $-3 < x < 3$, y has two real values. Hence $x^2 + y^2 = 9$ is not a function.

EXERCISE 2B

1 **a** $f(0) = 3(0) + 2 = 2$ **b** $f(2) = 3(2) + 2 = 8$ **c** $f(-1) = 3(-1) + 2 = -1$

d $f(-5) = 3(-5) + 2 = -13$ **e** $f(-\frac{1}{3}) = 3(-\frac{1}{3}) + 2 = 1$

2 **a** $f(0) = 3(0) - 0^2 + 2$
$= 2$

b $f(3) = 3(3) - 3^2 + 2$
$= 9 - 9 + 2$
$= 2$

c $f(-3) = 3(-3) - (-3)^2 + 2$
$= -9 - 9 + 2$
$= -16$

d $f(-7) = 3(-7) - (-7)^2 + 2$
$= -21 - 49 + 2$
$= -68$

e $f(\frac{3}{2}) = 3(\frac{3}{2}) - (\frac{3}{2})^2 + 2$
$= \frac{9}{2} - \frac{9}{4} + 2$
$= \frac{17}{4}$

3 **a** $g(1) = 1 - \frac{4}{1} = -3$ **b** $g(4) = 4 - \frac{4}{4} = 3$ **c** $g(-1) = -1 - \frac{4}{(-1)} = 3$

d $g(-4) = -4 - \frac{4}{(-4)} = -3$ **e** $g(-\frac{1}{2}) = -\frac{1}{2} - \dfrac{4}{(-\frac{1}{2})} = -\frac{1}{2} + 8 = \frac{15}{2}$

4 **a** $f(a) = 7 - 3a$

b $f(-a) = 7 - 3(-a)$
$= 7 + 3a$

c $f(a + 3) = 7 - 3(a + 3)$
$= 7 - 3a - 9$
$= -3a - 2$

d $f(b - 1) = 7 - 3(b - 1)$
$= 7 - 3b + 3$
$= 10 - 3b$

e $f(x + 2) = 7 - 3(x + 2)$
$= 7 - 3x - 6$
$= 1 - 3x$

f $f(x + h) = 7 - 3(x + h)$
$= 7 - 3x - 3h$

5 **a** $F(x + 4)$
$= 2(x + 4)^2 + 3(x + 4) - 1$
$= 2(x^2 + 8x + 16) + 3x + 12 - 1$
$= 2x^2 + 16x + 32 + 3x + 11$
$= 2x^2 + 19x + 43$

b $F(2 - x)$
$= 2(2 - x)^2 + 3(2 - x) - 1$
$= 2(4 - 4x + x^2) + 6 - 3x - 1$
$= 8 - 8x + 2x^2 + 5 - 3x$
$= 2x^2 - 11x + 13$

c $F(-x)$
$= 2(-x)^2 + 3(-x) - 1$
$= 2x^2 - 3x - 1$

d $F(x^2)$
$= 2(x^2)^2 + 3(x^2) - 1$
$= 2x^4 + 3x^2 - 1$

e $F(x^2 - 1)$
$= 2(x^2 - 1)^2 + 3(x^2 - 1) - 1$
$= 2(x^4 - 2x^2 + 1) + 3x^2 - 3 - 1$
$= 2x^4 - 4x^2 + 2 + 3x^2 - 4$
$= 2x^4 - x^2 - 2$

f $F(x + h)$
$= 2(x + h)^2 + 3(x + h) - 1$
$= 2(x^2 + 2xh + h^2) + 3x + 3h - 1$
$= 2x^2 + 4xh + 2h^2 + 3x + 3h - 1$
$= 2x^2 + (4h + 3)x + 2h^2 + 3h - 1$

6 **a** **i** $G(2) = \dfrac{2(2)+3}{2-4} = \frac{7}{-2} = -\frac{7}{2}$

ii $G(0) = \dfrac{2(0)+3}{0-4} = \frac{3}{-4} = -\frac{3}{4}$

iii $G(-\frac{1}{2}) = \dfrac{2(-\frac{1}{2})+3}{-\frac{1}{2}-4} = \dfrac{-1+3}{(-\frac{9}{2})} = \dfrac{2}{(-\frac{9}{2})} = -\frac{4}{9}$

b $G(x) = \dfrac{2x+3}{x-4}$ is undefined when $x - 4 = 0$

$\therefore \quad x = 4$

So, when $x = 4$, $G(x)$ does not exist.

c $G(x+2) = \dfrac{2(x+2)+3}{(x+2)-4} = \dfrac{2x+4+3}{x+2-4} = \dfrac{2x+7}{x-2}$

d $G(x) = -3$, so $\dfrac{2x+3}{x-4} = -3$ $\quad \therefore \quad 2x+3 = -3(x-4)$

$\therefore \quad 2x+3 = -3x+12$

$\therefore \quad 5x = 9$ and so $x = \frac{9}{5}$

7 f is the function which converts x into $f(x)$ whereas $f(x)$ is the value of the function at any value of x.

8 **a** $V(4) = 9650 - 860(4)$
$= 9650 - 3440$
$= 6210$

The value of the photocopier 4 years after purchase is 6210 euros.

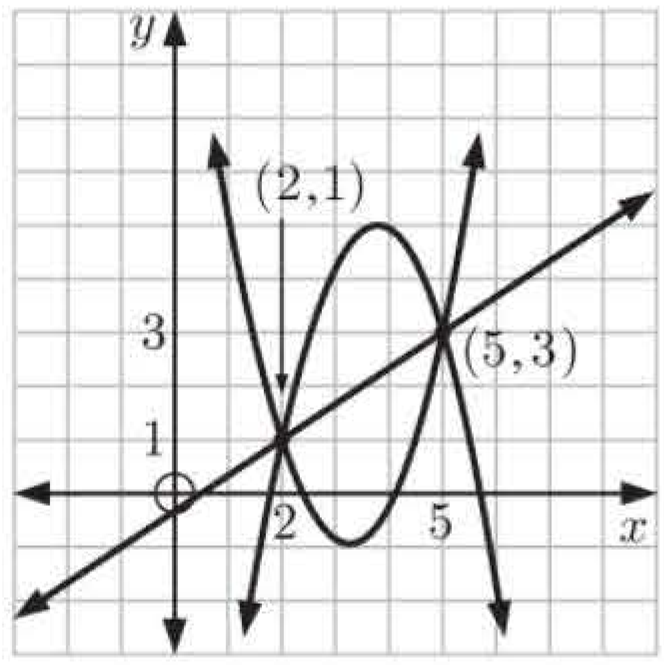
b If $V(t) = 5780$,
then $9650 - 860t = 5780$
$\therefore \quad 860t = 3870$
$\therefore \quad t = 4.5$

The value of the photocopier is 5780 euros after $4\frac{1}{2}$ years.

c Original purchase price is when $t = 0$,
$V(0) = 9650 - 860(0)$
$= 9650$

The original purchase price was 9650 euros.

9



First sketch the linear function which passes through the two points (2, 1) and (5, 3).
Then sketch two quadratic functions which also pass through the two points.

10 $f(x) = ax + b$ where $f(2) = 1$ and $f(-3) = 11$

So, $a(2) + b = 1$
$\therefore \quad 2a + b = 1$
$\therefore \quad b = 1 - 2a$ .... (1)

and $a(-3) + b = 11$
$\therefore \quad -3a + b = 11$
$\therefore \quad b = 11 + 3a$ .... (2)

Solving (1) and (2) simultaneously, $1 - 2a = 11 + 3a$
$\therefore \quad 5a = -10$
$\therefore \quad a = -2$

Substituting $a = -2$ into (1) gives $b = 1 - 2(-2) = 5$. So, $a = -2$, $b = 5$.

Hence $f(x) = -2x + 5$

11 $f(x) = ax + \dfrac{b}{x}$ where $f(1) = 1$ and $f(2) = 5$

So, $a(1) + \dfrac{b}{1} = 1$ and $a(2) + \dfrac{b}{2} = 5$

$\therefore\ a + b = 1$

$\therefore\ a = 1 - b$ .... (1)

$\therefore\ 2a + \dfrac{b}{2} = 5$ .... (2)

Substituting (1) into (2),
$$2(1-b) + \frac{b}{2} = 5$$
$$\therefore\ 2 - 2b + \frac{b}{2} = 5$$
$$\therefore\ -\frac{3b}{2} = 3$$
$$\therefore\ b = -2$$

Substituting $b = -2$ into (1) gives $a = 1 - (-2) = 3$.
So, $a = 3,\ b = -2$.

12 $T(x) = ax^2 + bx + c$ where $T(0) = -4$, $T(1) = -2$, and $T(2) = 6$

So, $a(0)^2 + b(0) + c = -4$

$\therefore\ c = -4$

Also, $a(1)^2 + b(1) + c = -2$ and $a(2)^2 + b(2) + c = 6$

$\therefore\ a + b + c = -2$ and $\therefore\ 4a + 2b + c = 6$

Substituting $c = -4$ into both equations gives

$a + b + (-4) = -2$ and $4a + 2b + (-4) = 6$

$\therefore\ a + b = 2$

$\therefore\ a = 2 - b$ .... (1)

$\therefore\ 4a + 2b = 10$ .... (2)

Substituting (1) into (2) gives $4(2-b) + 2b = 10$ $\therefore\ 8 - 4b + 2b = 10$

$\therefore\ -2b = 2$

$\therefore\ b = -1$

Substituting $b = -1$ into (1) gives $a = 2 - (-1) = 3$.

$\therefore\ a = 3,\ b = -1,$ and $c = -4$. So, $T(x) = 3x^2 - x - 4$.

EXERCISE 2C

1 **a** Domain is $\{x \mid x \geqslant -1\}$
Range is $\{y \mid y \leqslant 3\}$

b Domain is $\{x \mid -1 < x \leqslant 5\}$
Range is $\{y \mid 1 < y \leqslant 3\}$

c Domain is $\{x \mid x \neq 2\}$
Range is $\{y \mid y \neq -1\}$

d Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid 0 < y \leqslant 2\}$

e Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y \geqslant -1\}$

f Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y \leqslant 6\frac{1}{4}\}$ *or* $\{y \mid y \leqslant \frac{25}{4}\}$

g Domain is $\{x \mid x \geqslant -4\}$
Range is $\{y \mid y \geqslant -3\}$

h Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y > -2\}$

i Domain is $\{x \mid x \neq \pm 2\}$
Range is $\{y \mid y \leqslant -1 \text{ or } y > 0\}$

2 **a** $f(x)$ is defined when $x + 6 \geqslant 0$
$\therefore\ f(x)$ is defined for $x \geqslant -6$
$\therefore$ the domain is $\{x \mid x \geqslant -6\}$.

b $f(x)$ is defined when $x^2 \neq 0$
$\therefore\ f(x)$ is defined for $x \neq 0$
$\therefore$ the domain is $\{x \mid x \neq 0\}$.

c $f(x)$ is defined when $3 - 2x > 0$
$\therefore\ f(x)$ is defined for $x < \frac{3}{2}$
$\therefore$ the domain is $\{x \mid x < \frac{3}{2}\}$.

3 **a** $y = 2x - 1$ can take any x-value and any y-value.

$\therefore$ the domain is $\{x \mid x \in \mathbb{R}\}$ and the range is $\{y \mid y \in \mathbb{R}\}$.

b $y = 3$ can take any value of x, but the only permissible value for y is 3.

$\therefore$ the domain is $\{x \mid x \in \mathbb{R}\}$ and the range is $\{3\}$.

c $y = \sqrt{x}$ is defined when $x \geqslant 0$, and a square root cannot be negative.

$\therefore$ the domain is $\{x \mid x \geqslant 0\}$ and the range is $\{y \mid y \geqslant 0\}$.

d $y = \dfrac{1}{x+1}$ is defined when $x + 1 \neq 0$, or when $x \neq -1$.

$y = \dfrac{1}{x+1}$ cannot be 0 for any value of x.

$\therefore$ the domain is $\{x \mid x \neq -1\}$ and the range is $\{y \mid y \neq 0\}$.

e $y = -\dfrac{1}{\sqrt{x}}$ is defined when $x > 0$.

If x is always positive, then $y = -\dfrac{1}{\sqrt{x}}$ is always negative.

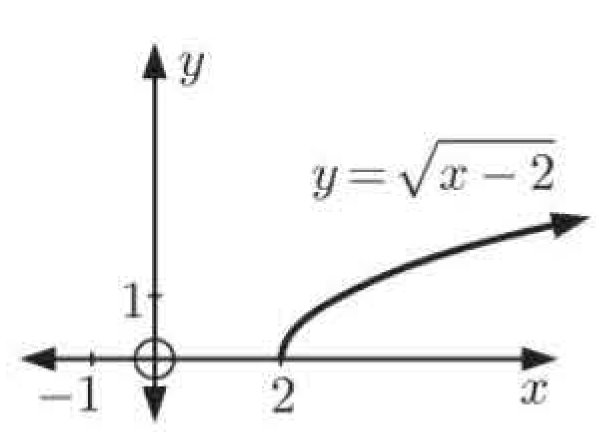
$\therefore$ the domain is $\{x \mid x > 0\}$ and the range is $\{y \mid y < 0\}$.

f $y = \dfrac{1}{3-x}$ is defined when $3 - x \neq 0$, or when $x \neq 3$.

$y = \dfrac{1}{3-x}$ cannot be 0 for any value of x.

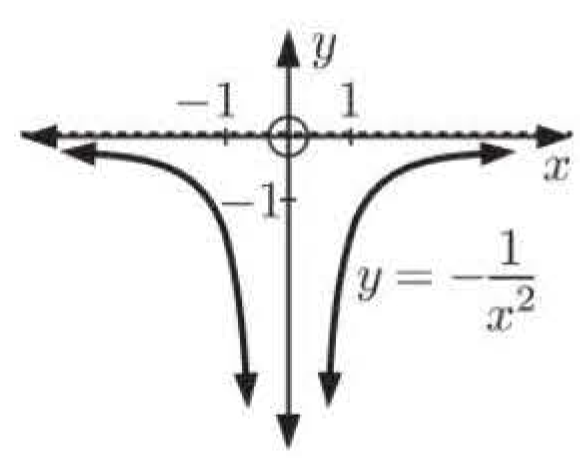
$\therefore$ the domain is $\{x \mid x \neq 3\}$ and the range is $\{y \mid y \neq 0\}$.

4 **a**



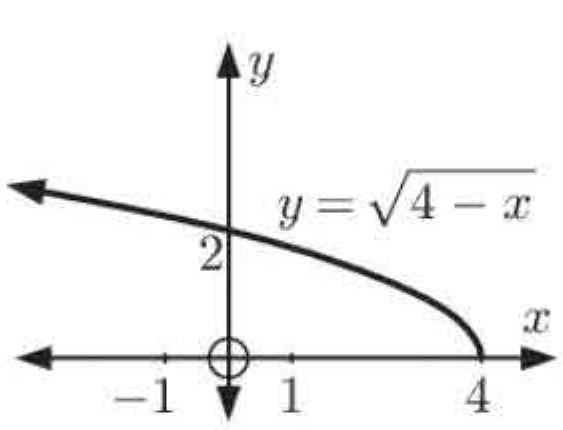
Domain is $\{x \mid x \geqslant 2\}$
Range is $\{y \mid y \geqslant 0\}$

b



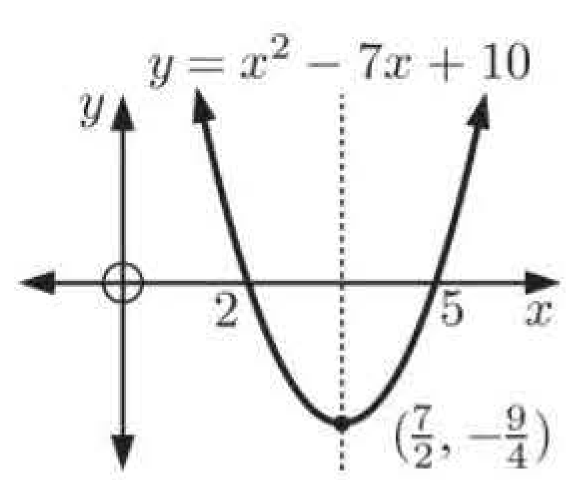
Domain is $\{x \mid x \neq 0\}$
Range is $\{y \mid y < 0\}$

c



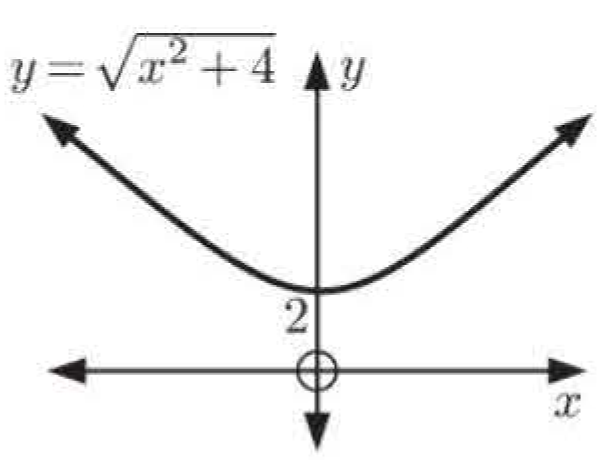
Domain is $\{x \mid x \leqslant 4\}$
Range is $\{y \mid y \geqslant 0\}$

d



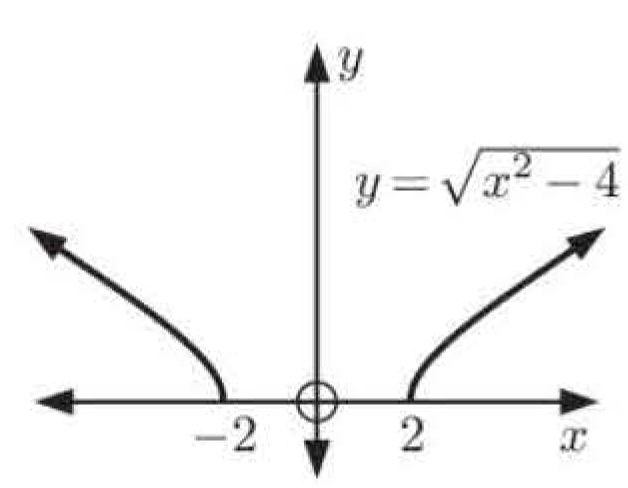
Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y \geqslant -2\frac{1}{4}\}$

e



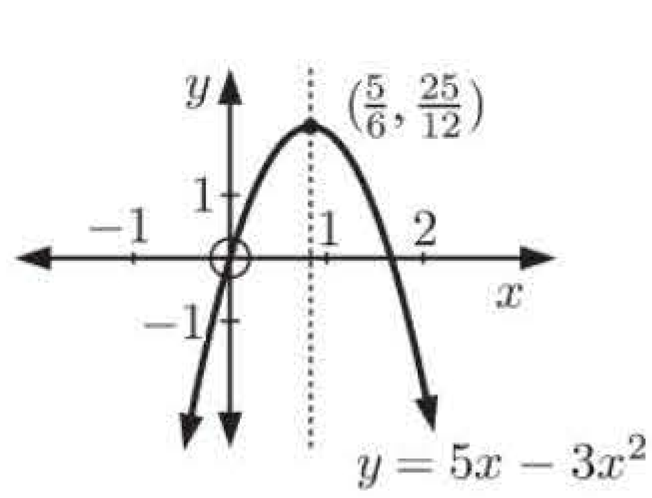
Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y \geqslant 2\}$

f



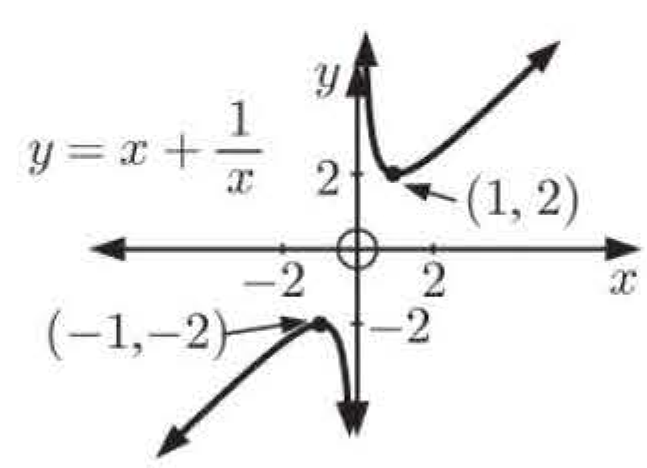
Domain is $\{x \mid x \leqslant -2$ or $x \geqslant 2\}$
Range is $\{y \mid y \geqslant 0\}$

g



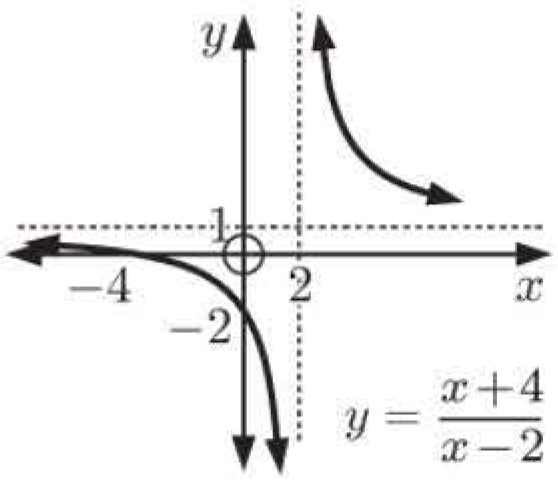
Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y \leqslant \frac{25}{12}\}$

h



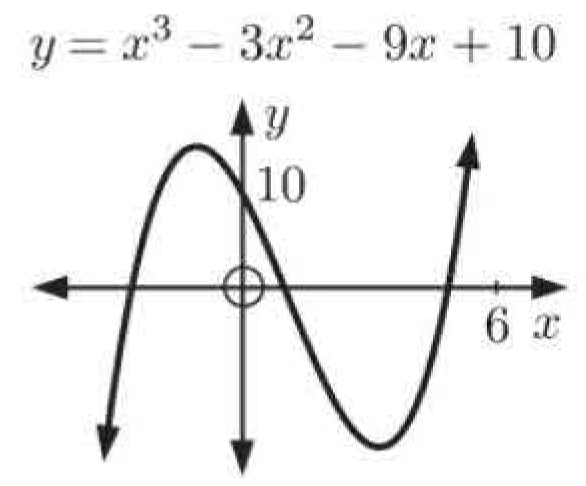
Domain is $\{x \mid x \neq 0\}$
Range is $\{y \mid y \leqslant -2$ or $y \geqslant 2\}$

i



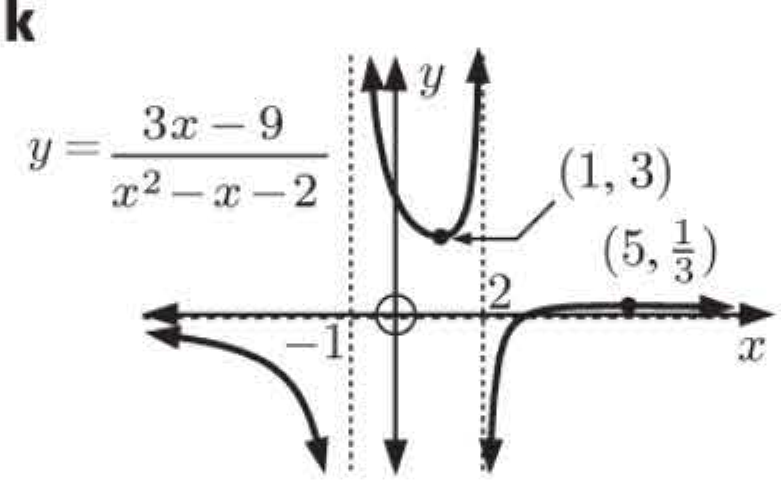
Domain is $\{x \mid x \neq 2\}$
Range is $\{y \mid y \neq 1\}$

j



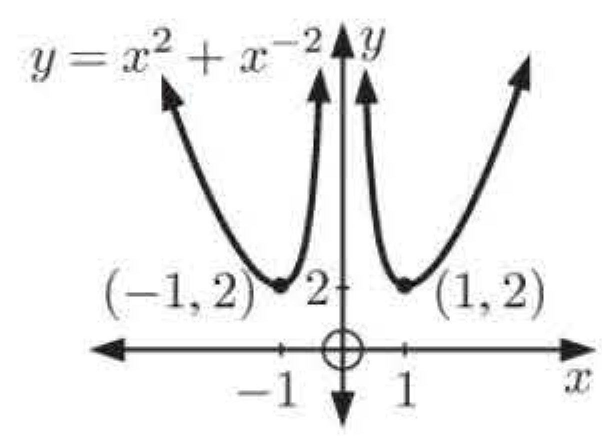
Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y \in \mathbb{R}\}$

k



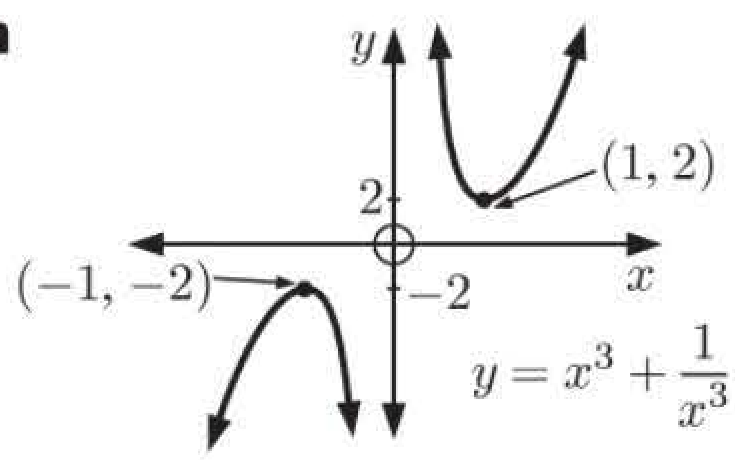
Domain is $\{x \mid x \neq -1$ and $x \neq 2\}$
Range is $\{y \mid y \leqslant \frac{1}{3}$ or $y \geqslant 3\}$

l



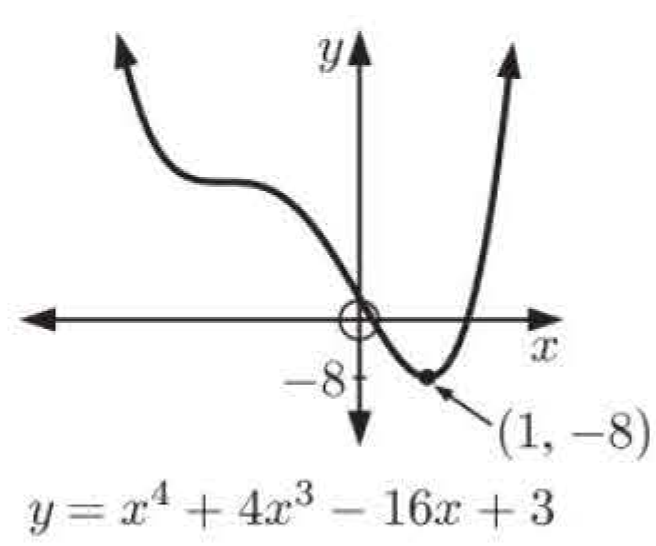
Domain is $\{x \mid x \neq 0\}$
Range is $\{y \mid y \geqslant 2\}$

m



Domain is $\{x \mid x \neq 0\}$
Range is $\{y \mid y \leqslant -2$ or $y \geqslant 2\}$

n



Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y \geqslant -8\}$

5 **a** The permissible values for x are 1, 2 and 3, so the domain is $\{1, 2, 3\}$.
The permissible values for y are 3, 5 and 7, so the range is $\{3, 5, 7\}$.

b The permissible values for x are -1, 0 and 2, so the domain is $\{-1, 0, 2\}$.
The permissible values for y are 3 and 5, so the range is $\{3, 5\}$.

c The permissible values for x are -3, -2, -1 and 3, so the domain is $\{-3, -2, -1, 3\}$.
The only permissible value for y is 1, so the range is $\{1\}$.

d The only solutions (x, y) to $x^2+y^2=4$, where $x \in \mathbb{Z}$ and $y \geqslant 0$, are $(-2, 0)$, $(-1, \sqrt{3})$, $(0, 2)$, $(1, \sqrt{3})$ and $(2, 0)$.
$\therefore$ the domain is $\{-2, -1, 0, 1, 2\}$ and the range is $\{0, \sqrt{3}, 2\}$.

EXERCISE 2D

1 **a**
$$\begin{aligned}&(f\circ g)(x)\\&=f(g(x))\\&=f(1-x)\\&=2(1-x)+3\\&=2-2x+3\\&=5-2x\end{aligned}$$

b
$$\begin{aligned}&(g\circ f)(x)\\&=g(f(x))\\&=g(2x+3)\\&=1-(2x+3)\\&=1-2x-3\\&=-2x-2\end{aligned}$$

c
$$\begin{aligned}&(f\circ g)(-3)\\&=5-2(-3) \quad \{\text{from } \textbf{a}\}\\&=11\end{aligned}$$

2 **a**
$$\begin{aligned}&(g\circ g)(x)\\&=g(g(x))\\&=g(5x-7)\\&=5(5x-7)-7\\&=25x-35-7\\&=25x-42\end{aligned}$$

b
$$\begin{aligned}(f\circ g)(1)&=f(g(1))\\ \text{Now}\quad g(1)&=5(1)-7\\&=-2\\ \therefore\quad (f\circ g)(1)&=f(-2)\\&=\sqrt{6-(-2)}\\&=\sqrt{8}\end{aligned}$$

c
$$\begin{aligned}(g\circ f)(6)&=g(f(6))\\ \text{Now}\quad f(6)&=\sqrt{6-6}\\&=0\\ \therefore\quad (g\circ f)(6)&=g(0)\\&=5(0)-7\\&=-7\end{aligned}$$

3
$$\begin{aligned}(f\circ g)(x)&=f(g(x))\\&=f(2-x)\\&=(2-x)^2\end{aligned}$$

Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y \geqslant 0\}$

$$\begin{aligned}(g\circ f)(x)&=g(f(x))\\&=g(x^2)\\&=2-x^2\end{aligned}$$

Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y \leqslant 2\}$

4 **a** **i** $(f\circ g)(x)$
$= f(g(x))$
$= f(3-x)$
$= (3-x)^2+1$
$= 9-6x+x^2+1$
$= x^2-6x+10$

ii $(g\circ f)(x)$
$= g(f(x))$
$= g(x^2+1)$
$= 3-(x^2+1)$
$= 3-x^2-1$
$= 2-x^2$

b $(g\circ f)(x) = f(x)$
$\therefore\ 2-x^2 = f(x)$ {from **a ii**}
$\therefore\ 2-x^2 = x^2+1$
$\therefore\ 2x^2 = 1$
$\therefore\ x^2 = \frac{1}{2}$
$\therefore\ x = \pm\frac{1}{\sqrt{2}}$

5 $(f\circ g)(0) = f(g(0)) = f(1) = 0$
$(f\circ g)(1) = f(g(1)) = f(2) = 1$
$(f\circ g)(2) = f(g(2)) = f(3) = 2$
$(f\circ g)(3) = f(g(3)) = f(0) = 3$
$\therefore\ f\circ g = \{(0, 0), (1, 1), (2, 2), (3, 3)\}$

6 **a** $(f\circ g)(2) = f(g(2)) = f(2) = 7$
$(f\circ g)(5) = f(g(5)) = f(0) = 2$
$(f\circ g)(7) = f(g(7)) = f(1) = 5$
$(f\circ g)(9) = f(g(9)) = f(3) = 9$
$\therefore\ f\circ g = \{(2, 7), (5, 2), (7, 5), (9, 9)\}$

b $(g\circ f)(0) = g(f(0)) = f(2) = 2$
$(g\circ f)(1) = g(f(1)) = f(5) = 0$
$(g\circ f)(2) = g(f(2)) = f(7) = 1$
$(g\circ f)(3) = g(f(3)) = f(9) = 3$
$\therefore\ g\circ f = \{(0, 2), (1, 0), (2, 1), (3, 3)\}$

7 **a** $(f\circ g)(x)$
$= f(g(x))$
$= f\left(\dfrac{x+1}{x-1}\right)$
$= \dfrac{\left(\frac{x+1}{x-1}\right)+3}{\left(\frac{x+1}{x-1}\right)+2}\times\dfrac{(x-1)}{(x-1)}$
$= \dfrac{x+1+3(x-1)}{x+1+2(x-1)}$
$= \dfrac{4x-2}{3x-1},\ x\neq 1$
$(f\circ g)(x)$ is undefined when $3x-1=0$, which is when $x=\frac{1}{3}$
$\therefore$ domain is $\{x \mid x\neq \frac{1}{3} \text{ or } 1\}$

b $(g\circ f)(x)$
$= g(f(x))$
$= g\left(\dfrac{x+3}{x+2}\right)$
$= \dfrac{\left(\frac{x+3}{x+2}\right)+1}{\left(\frac{x+3}{x+2}\right)-1}\times\dfrac{(x+2)}{(x+2)}$
$= \dfrac{x+3+(x+2)}{x+3-(x+2)}$
$= \dfrac{2x+5}{1}$
$= 2x+5,\ x\neq -2$
$\therefore$ domain is $\{x \mid x\neq -2\}$

c $(g\circ g)(x)$
$= g(g(x))$
$= g\left(\dfrac{x+1}{x-1}\right)$
$= \dfrac{\left(\frac{x+1}{x-1}\right)+1}{\left(\frac{x+1}{x-1}\right)-1}\times\dfrac{(x-1)}{(x-1)}$
$= \dfrac{x+1+(x-1)}{x+1-(x-1)}$
$= \dfrac{2x}{2}$
$= x,\ x\neq 1$
$\therefore$ domain is $\{x \mid x\neq 1\}$

8 **a** $(f\circ g)(x) = f(g(x))$
$= f(x^2)$
$= \sqrt{1-x^2}$

b $(f\circ g)(x) = \sqrt{1-x^2}$ is defined when $1-x^2\geqslant 0$
$\therefore\ x^2\leqslant 1$
$\therefore\ -1\leqslant x\leqslant 1$
$(f\circ g)(x) = \sqrt{1-x^2}$ is always positive, and always $\leqslant 1$.
$\therefore$ the domain is $\{x \mid -1\leqslant x\leqslant 1\}$
and the range is $\{y \mid 0\leqslant y\leqslant 1\}$

9 **a** $ax+b = cx+d$ is true for all x {given}
When $x=0$, $a(0)+b = c(0)+d$
$\therefore\ b = d$ .... $(*)$
When $x=1$, $a(1)+b = c(1)+d$
$\therefore\ a+b = c+d$
But from $(*)$, $b=d$, so $a+d = c+d$
$\therefore\ a = c$

b $(f \circ g)(x) = x$ for all x {given}

$\therefore\ f(g(x)) = x$

$\therefore\ f(ax+b) = x$

$\therefore\ 2(ax+b)+3 = x$

$\therefore\ 2ax+2b+3 = x$ for all x

$\therefore\ 2a = 1$ and $2b+3 = 0$ {using **a**}

$\therefore\ a = \frac{1}{2}$ and $2b = -3$

So, $a = \frac{1}{2}$ and $b = -\frac{3}{2}$ as required.

c If $(g \circ f)(x) = x$

then $g(f(x)) = x$

$\therefore\ g(2x+3) = x$

$\therefore\ a(2x+3)+b = x$

$\therefore\ 2ax+3a+b = x$

$\therefore\ 2a = 1$ and $3a+b = 0$ {using **a**}

$\therefore\ a = \frac{1}{2}$ and $b = -3a$

So, $a = \frac{1}{2}$ and $b = -\frac{3}{2}$

$\therefore$ the result in **b** is also true if $(g \circ f)(x) = x$ for all x.

EXERCISE 2E

1
$$\begin{aligned} f(x) &= \frac{1}{x^2}+2 \\ \therefore\ f(-x) &= \frac{1}{(-x)^2}+2 \\ &= \frac{1}{x^2}+2 \\ &= f(x) \end{aligned}$$

$\therefore\ f(x) = \dfrac{1}{x^2}+2$ is an even function.

2
$$\begin{aligned} f(x) &= x^3-3x \\ \therefore\ f(-x) &= (-x)^3-3(-x) \\ &= (-1)^3x^3+3x \\ &= -x^3+3x \\ &= -(x^3-3x) \\ &= -f(x) \end{aligned}$$

$\therefore\ f(x) = x^3-3x$ is an odd function.

3 **a**
$$\begin{aligned} f(x) &= 5x \\ \therefore\ f(-x) &= 5(-x) \\ &= -5x \\ &= -f(x) \end{aligned}$$

$\therefore\ f(x) = 5x$ is an odd function.

b
$$\begin{aligned} f(x) &= -4x+3 \\ \therefore\ f(-x) &= -4(-x)+3 \\ &= 4x+3 \end{aligned}$$

which is neither $f(x)$ or $-f(x)$.

$\therefore\ f(x) = -4x+3$ is neither even nor odd.

c
$$\begin{aligned} f(x) &= \frac{3}{x^2-4} \\ \therefore\ f(-x) &= \frac{3}{(-x)^2-4} \\ &= \frac{3}{x^2-4} \\ &= f(x) \end{aligned}$$

$\therefore\ f(x) = \dfrac{3}{x^2-4}$ is an even function.

d
$$\begin{aligned} f(x) &= 2x^3-\frac{5}{x} \\ \therefore\ f(-x) &= 2(-x)^3-\frac{5}{-x} \\ &= 2(-1)^3x^3+\frac{5}{x} \\ &= -2x^3+\frac{5}{x} \\ &= -\left(2x^3-\frac{5}{x}\right) \\ &= -f(x) \end{aligned}$$

$\therefore\ f(x) = 2x^3-\dfrac{5}{x}$ is an odd function.

e
$$\begin{aligned} f(x) &= x^2+\frac{7}{x^2}-3 \\ \therefore\ f(-x) &= (-x)^2+\frac{7}{(-x)^2}-3 \\ &= x^2+\frac{7}{x^2}-3 \\ &= f(x) \end{aligned}$$

$\therefore\ f(x) = x^2+\dfrac{7}{x^2}-3$ is an even function.

f
$$\begin{aligned} f(x) &= \sqrt{x} \\ \therefore\ f(-x) &= \sqrt{-x} \end{aligned}$$

which is neither $f(x)$ nor $-f(x) = -\sqrt{x}$.

$\therefore\ f(x) = \sqrt{x}$ is neither even nor odd.

4 $f(x) = (2x+3)(x+a)$ is an even function

$\therefore\ f(-x) = f(x)$

$\therefore\ (2(-x)+3)(-x+a) = (2x+3)(x+a)$

$\therefore\ (-2x+3)(-x+a) = (2x+3)(x+a)$

$\therefore\ \cancel{2x^2} - 2ax - 3x + \cancel{3a} = \cancel{2x^2} + 2ax + 3x + \cancel{3a}$

$\therefore\ -2a-3 = 2a+3$ {equating coefficients of x}

$\therefore\ 4a = -6$

$\therefore\ a = \dfrac{-6}{4} = -\frac{3}{2}$

5 $g(x) = (x+1)\left(\dfrac{1}{x}+b\right)$ is an odd function

$\therefore\ g(-x) = -g(x)$

$\therefore\ (-x+1)\left(\dfrac{1}{-x}+b\right) = -(x+1)\left(\dfrac{1}{x}+b\right)$

$\therefore\ \dfrac{x}{x} - bx - \dfrac{1}{x} + b = -\left(\dfrac{x}{x} + bx + \dfrac{1}{x} + b\right)$

$\therefore\ 1 \cancel{- bx} \cancel{- \dfrac{1}{x}} + b = -1 \cancel{- bx} \cancel{- \dfrac{1}{x}} - b$

$\therefore\ 1 + b = -1 - b$

$\therefore\ 2b = -2$

$\therefore\ b = -1$

6 **a** $f(x) = ax^2 + bx + c,\ a \neq 0$ is an even function

$\therefore\ f(-x) = f(x)$

$\therefore\ a(-x)^2 + b(-x) + c = ax^2 + bx + c$

$\therefore\ \cancel{ax^2} - bx + \cancel{c} = \cancel{ax^2} + bx + \cancel{c}$

$\therefore\ -b = b$ {equating coefficients of x}

$\therefore\ b = 0$

b If $g(x) = ax^3 + bx^2 + cx + d,\ a \neq 0$ is an odd function, then $g(-x) = -g(x)$

$\therefore\ a(-x)^3 + b(-x)^2 + c(-x) + d = -(ax^3 + bx^2 + cx + d)$

$\therefore\ a(-1)^3x^3 + bx^2 - cx + d = -ax^3 - bx^2 - cx - d$

$\therefore\ -ax^3 + bx^2 - cx + d = -ax^3 - bx^2 - cx - d$

Equating coefficients of x^3: $-a = -a$ $\therefore\ a \in \mathbb{R},\ a \neq 0$

Equating coefficients of x: $-c = -c$ $\therefore\ c \in \mathbb{R}$

Equating coefficients of x^2: $b = -b$ $\therefore\ b = 0$

Equating constant terms: $d = -d$ $\therefore\ d = 0$

So, the cubic function $g(x) = ax^3 + bx^2 + cx + d,\ a \neq 0$ is an odd function when $b = 0,\ d = 0$.

c If $h(x) = ax^4 + bx^3 + cx^2 + dx + e,\ a \neq 0$ is an even function, then $h(-x) = h(x)$

$\therefore\ a(-x)^4 + b(-x)^3 + c(-x)^2 + d(-x) + e = ax^4 + bx^3 + cx^2 + dx + e$

$\therefore\ ax^4 - bx^3 + cx^2 - dx + e = ax^4 + bx^3 + cx^2 + dx + e$

Equating coefficients of x^4: $a = a$ $\therefore\ a \in \mathbb{R},\ a \neq 0$

Equating coefficients of x: $-d = d$ $\therefore\ d = 0$

Equating coefficients of x^3: $-b = b$ $\therefore\ b = 0$

Equating constant terms: $e = e$ $\therefore\ e \in \mathbb{R}$

Equating coefficients of x^2: $c = c$ $\therefore\ c \in \mathbb{R}$

So, the quartic function $h(x) = ax^4 + bx^3 + cx^2 + dx + e,\ a \neq 0$ is an even function when $b = 0,\ d = 0$.

7 **a** Suppose $h(x) = f(x) + g(x)$ where $f(x)$ and $g(x)$ are even functions.

Now $h(-x) = f(-x) + g(-x)$
$= f(x) + g(x)$ $\{f(x)$ and $g(x)$ are even functions$\}$
$= h(x)$ for all x
$\therefore$ $h(x)$ is even.

Thus the sum of two even functions is an even function.

b Suppose $h(x) = f(x) - g(x)$ where $f(x)$ and $g(x)$ are odd functions.

Now $h(-x) = f(-x) - g(-x)$
$= -f(x) - -g(x)$ $\{f(x)$ and $g(x)$ are odd functions$\}$
$= -(f(x) - g(x))$
$= -h(x)$
$\therefore$ $h(x)$ is odd.

Thus the difference between two odd functions is an odd function.

c Suppose $h(x) = f(x)g(x)$ where $f(x)$ and $g(x)$ are odd functions.

Now $h(-x) = f(-x)g(-x)$
$= -f(x) \times -g(x)$ $\{f(x)$ and $g(x)$ are odd functions$\}$
$= f(x)g(x)$
$= h(x)$
$\therefore$ $h(x)$ is even.

Thus the product of two odd functions is an even function.

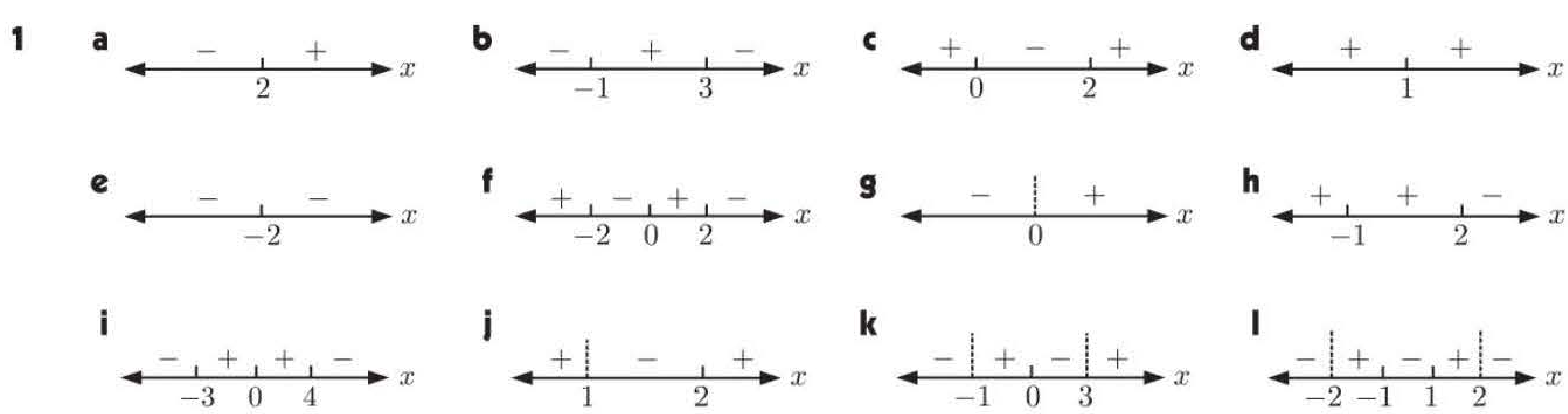
8 $f(x)$ is an even function and $g(x)$ is an odd function.

$(f \circ g)(-x) = f(g(-x))$
$= f(-g(x))$ $\{g(x)$ is an odd function$\}$
$= f(g(x))$ $\{f(x)$ is an even function$\}$
$= (f \circ g)(x)$

So $(f \circ g)(x)$ is an even function.

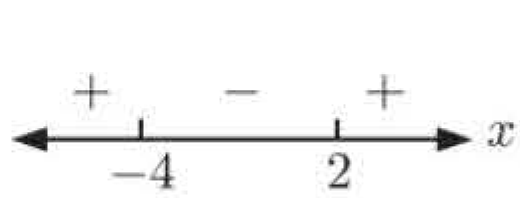
EXERCISE 2F

1



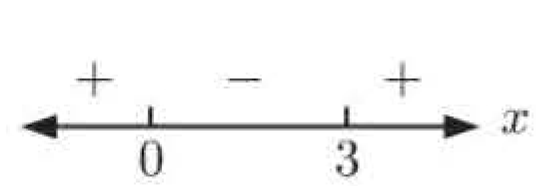
2 **a** $y = (x+4)(x-2)$ is zero when $x = -4$ or 2.
When $x = 0$, $y = (4)(-2) = -8 < 0$.
The factors are distinct and linear, so the signs alternate.

$\therefore$ sign diagram is:

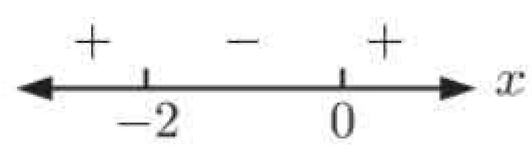


b $y = x(x-3)$ is zero when $x = 0$ or 3.
When $x = 10$, $y = 10(7) = 70 > 0$.
The factors are distinct and linear, so the signs alternate.

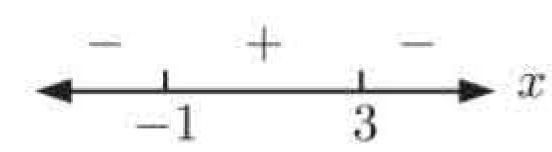
$\therefore$ sign diagram is:



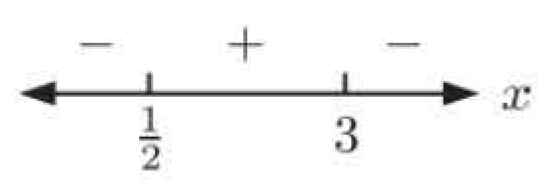
c $y = x(x+2)$ is zero when $x = -2$ or 0.
When $x = 10$, $y = 10(12) = 120 > 0$.
The factors are distinct and linear, so the signs alternate.

$\therefore$ sign diagram is:


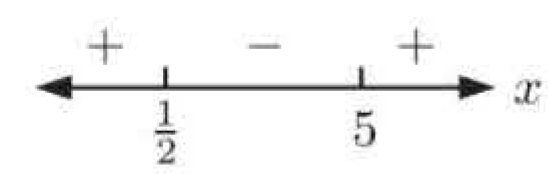
d $y = -(x+1)(x-3)$ is zero when $x = -1$ or 3.
When $x = 0$, $y = -(1)(-3) = 3 > 0$.
The factors are distinct and linear, so the signs alternate.

$\therefore$ sign diagram is:


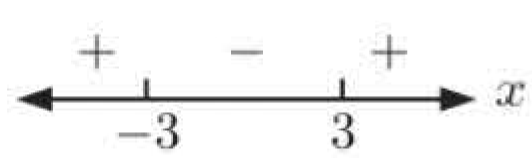
e $y = (2x-1)(3-x)$ is zero when $x = \frac{1}{2}$ or 3.
When $x = 0$, $y = (-1)(3) = -3 < 0$.
The factors are distinct and linear, so the signs alternate.

$\therefore$ sign diagram is:


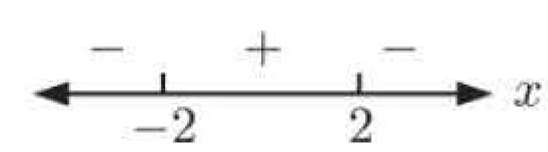
f $y = (5-x)(1-2x)$ is zero when $x = \frac{1}{2}$ or 5.
When $x = 0$, $y = (5)(1) = 5 > 0$.
The factors are distinct and linear, so the signs alternate.

$\therefore$ sign diagram is:


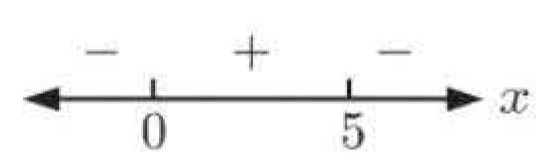
g $y = x^2 - 9 = (x+3)(x-3)$ is zero when $x = -3$ or 3.
When $x = 0$, $y = (3)(-3) = -9 < 0$.
The factors are distinct and linear, so the signs alternate.

$\therefore$ sign diagram is:


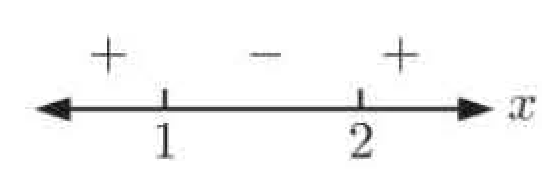
h $y = 4 - x^2 = (2+x)(2-x)$ is zero when $x = -2$ or 2.
When $x = 0$, $y = (2)(2) = 4 > 0$.
The factors are distinct and linear, so the signs alternate.

$\therefore$ sign diagram is:


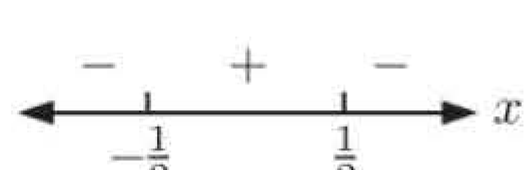
i $y = 5x - x^2 = x(5-x)$ is zero when $x = 0$ or 5.
When $x = 10$, $y = 10(-5) = -50 < 0$.
The factors are distinct and linear, so the signs alternate.

$\therefore$ sign diagram is:


j $y = x^2 - 3x + 2 = (x-1)(x-2)$ is zero when $x = 1$ or 2.
When $x = 0$, $y = (-1)(-2) = 2 > 0$.
The factors are distinct and linear, so the signs alternate.

$\therefore$ sign diagram is:


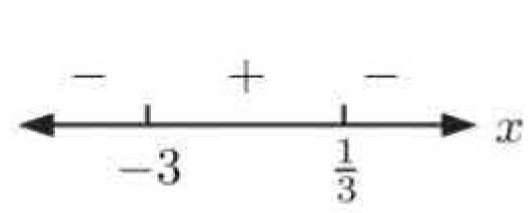
k $y = 2 - 8x^2 = 2(1+2x)(1-2x)$ is zero when $x = -\frac{1}{2}$ or $\frac{1}{2}$.
When $x = 0$, $y = 2(1)(1) = 2 > 0$.
The factors are distinct and linear, so the signs alternate.

$\therefore$ sign diagram is:


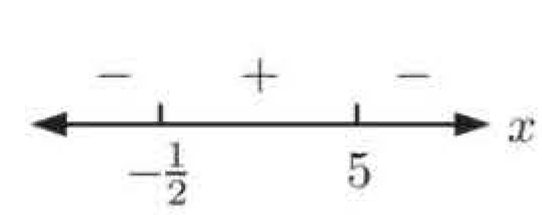
l $y = 6x^2 + x - 2 = (3x+2)(2x-1)$ is zero when $x = -\frac{2}{3}$ or $\frac{1}{2}$.
When $x = 0$, $y = (2)(-1) = -2 < 0$.
The factors are distinct and linear, so the signs alternate.

$\therefore$ sign diagram is:


m $y = 6 - 16x - 6x^2 = 2(3+x)(1-3x)$ is zero when $x = -3$ or $\frac{1}{3}$.
When $x = 0$, $y = 2(3)(1) = 6 > 0$.
The factors are distinct and linear, so the signs alternate.

$\therefore$ sign diagram is:


n $y = -2x^2 + 9x + 5 = (2x+1)(5-x)$ is zero when $x = -\frac{1}{2}$ or 5.
When $x = 0$, $y = (1)(5) = 5 > 0$.
The factors are distinct and linear, so the signs alternate.

$\therefore$ sign diagram is:


o $y = -15x^2 - x + 2 = (5x+2)(1-3x)$ is zero when $x = -\frac{2}{5}$ or $\frac{1}{3}$.
When $x = 0$, $y = (2)(1) = 2 > 0$.
The factors are distinct and linear, so the signs alternate. $\therefore$ sign diagram is:


3 **a** $y=(x+2)^2$ is zero when $x=-2$.
When $x=0$, $y=2^2=4>0$.
The linear factor is squared, so the sign does not change.
$\therefore$ sign diagram is: $\overset{+}{}\ \underset{-2}{|}\ \overset{+}{}\ \rightarrow x$

b $y=(x-3)^2$ is zero when $x=3$.
When $x=0$, $y=(-3)^2=9>0$.
The linear factor is squared, so the sign does not change.
$\therefore$ sign diagram is: $\overset{+}{}\ \underset{3}{|}\ \overset{+}{}\ \rightarrow x$

c $y=-(x+2)^2$ is zero when $x=-2$.
When $x=0$, $y=-(2^2)=-4<0$.
The linear factor is squared, so the sign does not change.
$\therefore$ sign diagram is: $\overset{-}{}\ \underset{-2}{|}\ \overset{-}{}\ \rightarrow x$

d $y=-(x-4)^2$ is zero when $x=4$.
When $x=0$, $y=-(-4)^2=-16<0$.
The linear factor is squared, so the sign does not change.
$\therefore$ sign diagram is: $\overset{-}{}\ \underset{4}{|}\ \overset{-}{}\ \rightarrow x$

e $y=x^2-2x+1=(x-1)^2$ is zero when $x=1$.
When $x=0$, $y=(-1)^2=1>0$.
The linear factor is squared, so the sign does not change.
$\therefore$ sign diagram is: $\overset{+}{}\ \underset{1}{|}\ \overset{+}{}\ \rightarrow x$

f $y=-x^2+4x-4=-(x-2)^2$ is zero when $x=2$.
When $x=0$, $y=-(-2)^2=-4<0$.
The linear factor is squared, so the sign does not change.
$\therefore$ sign diagram is: $\overset{-}{}\ \underset{2}{|}\ \overset{-}{}\ \rightarrow x$

g $y=4x^2-4x+1=(2x-1)^2$ is zero when $x=\frac{1}{2}$.
When $x=0$, $y=(-1)^2=1>0$.
The linear factor is squared, so the sign does not change.
$\therefore$ sign diagram is: $\overset{+}{}\ \underset{\frac{1}{2}}{|}\ \overset{+}{}\ \rightarrow x$

h $y=-x^2-6x-9=-(x+3)^2$ is zero when $x=-3$.
When $x=0$, $y=-(3^2)=-9<0$.
The linear factor is squared, so the sign does not change.
$\therefore$ sign diagram is: $\overset{-}{}\ \underset{-3}{|}\ \overset{-}{}\ \rightarrow x$

i $y=-4x^2+12x-9=-(2x-3)^2$ is zero when $x=\frac{3}{2}$.
When $x=0$, $y=-(-3)^2=-9<0$.
The linear factor is squared, so the sign does not change.
$\therefore$ sign diagram is: $\overset{-}{}\ \underset{\frac{3}{2}}{|}\ \overset{-}{}\ \rightarrow x$

4 **a** $y=\dfrac{x+2}{x-1}$ is zero when $x=-2$ and undefined when $x=1$.
When $x=0$, $y=\frac{2}{-1}=-2<0$.
Since the factors are distinct and linear, the signs alternate.
$\therefore$ sign diagram is: $\overset{+}{}\ \underset{-2}{|}\ \overset{-}{}\ \underset{1}{\vdots}\ \overset{+}{}\ \rightarrow x$

b $y=\dfrac{x}{x+3}$ is zero when $x=0$ and undefined when $x=-3$.
When $x=10$, $y=\frac{10}{13}>0$.
Since the factors are distinct and linear, the signs alternate.
$\therefore$ sign diagram is: $\overset{+}{}\ \underset{-3}{\vdots}\ \overset{-}{}\ \underset{0}{|}\ \overset{+}{}\ \rightarrow x$

c $y=\dfrac{2x+3}{4-x}$ is zero when $x=-\frac{3}{2}$ and undefined when $x=4$.
When $x=0$, $y=\frac{3}{4}>0$.
Since the factors are distinct and linear, the signs alternate.
$\therefore$ sign diagram is: $\overset{-}{}\ \underset{-\frac{3}{2}}{|}\ \overset{+}{}\ \underset{4}{\vdots}\ \overset{-}{}\ \rightarrow x$

d $y=\dfrac{4x-1}{2-x}$ is zero when $x=\frac{1}{4}$ and undefined when $x=2$.
When $x=0$, $y=\frac{-1}{2}=-\frac{1}{2}<0$.
Since the factors are distinct and linear, the signs alternate.
$\therefore$ sign diagram is: $\overset{-}{}\ \underset{\frac{1}{4}}{|}\ \overset{+}{}\ \underset{2}{\vdots}\ \overset{-}{}\ \rightarrow x$

e $y = \dfrac{3x}{x-2}$ is zero when $x = 0$ and undefined when $x = 2$.

When $x = 5$, $y = \frac{15}{3} = 5 > 0$.

Since the factors are distinct and linear, the signs alternate.

$\therefore$ sign diagram is: $+\ \underset{0}{|}\ -\ \underset{2}{\vdots}\ +$ $\ x$

f $y = \dfrac{-8x}{3-x}$ is zero when $x = 0$ and undefined when $x = 3$.

When $x = 5$, $y = \frac{-40}{-2} = 20 > 0$.

Since the factors are distinct and linear, the signs alternate.

$\therefore$ sign diagram is: $+\ \underset{0}{|}\ -\ \underset{3}{\vdots}\ +$ $\ x$

g $y = \dfrac{(x-1)^2}{x}$ is zero when $x = 1$ and undefined when $x = 0$.

When $x = 2$, $y = \frac{1^2}{2} = \frac{1}{2} > 0$.

Since the $(x-1)$ factor is squared, the sign does not change at $x = 1$.

$\therefore$ sign diagram is: $-\ \underset{0}{\vdots}\ +\ \underset{1}{|}\ +$ $\ x$

h $y = \dfrac{4x}{(x+1)^2}$ is zero when $x = 0$ and undefined when $x = -1$.

When $x = 1$, $y = \frac{4}{2^2} = 1 > 0$.

Since the $(x+1)$ factor is squared, the sign does not change at $x = -1$.

$\therefore$ sign diagram is: $-\ \underset{-1}{\vdots}\ -\ \underset{0}{|}\ +$ $\ x$

i $y = \dfrac{(x+2)(x-1)}{3-x}$ is zero when $x = -2$ or 1 and undefined when $x = 3$.

When $x = 0$, $y = \frac{(2)(-1)}{3} = -\frac{2}{3} < 0$.

Since the factors are distinct and linear, the signs alternate.

$\therefore$ sign diagram is: $+\ \underset{-2}{|}\ -\ \underset{1}{|}\ +\ \underset{3}{\vdots}\ -$ $\ x$

j $y = \dfrac{x(x-1)}{2-x}$ is zero when $x = 0$ or 1 and undefined when $x = 2$.

When $x = 3$, $y = \frac{3(2)}{-1} = -6 < 0$.

Since the factors are distinct and linear, the signs alternate.

$\therefore$ sign diagram is: $+\ \underset{0}{|}\ -\ \underset{1}{|}\ +\ \underset{2}{\vdots}\ -$ $\ x$

k $y = \dfrac{x^2-4}{-x} = \dfrac{(x-2)(x+2)}{-x}$ is zero when $x = \pm 2$ and undefined when $x = 0$.

When $x = 1$, $y = \dfrac{(-1)(3)}{-1} = 3 > 0$.

Since the factors are distinct and linear, the signs alternate.

$\therefore$ sign diagram is: $+\ \underset{-2}{|}\ -\ \underset{0}{\vdots}\ +\ \underset{2}{|}\ -$ $\ x$

l $y = \dfrac{3-x}{2x^2-x-6} = \dfrac{3-x}{(2x+3)(x-2)}$ is zero when $x = 3$ and undefined when $x = -\frac{3}{2}$ or 2.

When $x = 0$, $y = \frac{3}{-6} = -\frac{1}{2} < 0$.

Since the factors are distinct and linear, the signs alternate.

$\therefore$ sign diagram is: $+\ \underset{-\frac{3}{2}}{\vdots}\ -\ \underset{2}{\vdots}\ +\ \underset{3}{|}\ -$ $\ x$

m $y = \dfrac{x^2-3}{x+1} = \dfrac{(x+\sqrt{3})(x-\sqrt{3})}{x+1}$ is zero when $x = \pm\sqrt{3}$ and undefined when $x = -1$.

When $x = 0$, $y = \frac{-3}{1} = -3 < 0$.

Since the factors are distinct and linear, the signs alternate.

$\therefore$ sign diagram is: $-\ \underset{-\sqrt{3}}{|}\ +\ \underset{-1}{\vdots}\ -\ \underset{\sqrt{3}}{|}\ +$ $\ x$

n $y = \dfrac{x^2+1}{x}$ is never zero (since $x^2 + 1 > 0$ for all real x), and undefined when $x = 0$.

When $x = 1$, $y = \frac{2}{1} = 2 > 0$.

Since the factor is distinct and linear, the sign alternates.

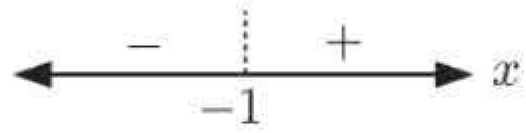
$\therefore$ sign diagram is: $-\ \underset{0}{\vdots}\ +$ $\ x$

o $y = \dfrac{x^2+2x+4}{x+1}$ is never zero (since $x^2+2x+4>0$ for all real x), and undefined when $x=-1$.

When $x=0$, $y=\frac{4}{1}=4>0$.

Since the factor is distinct and linear, the sign alternates.

$\therefore$ sign diagram is:

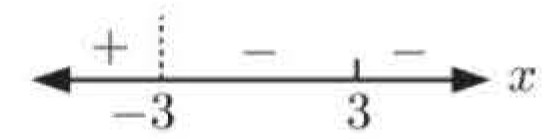


p $y = \dfrac{-(x-3)^2(x^2+2)}{x+3}$ is zero when $x=3$ and undefined when $x=-3$.

When $x=0$, $y=\dfrac{-(-3)^2(2)}{3}=-6<0$.

Since the $(x-3)$ factor is squared, the sign does not change at $x=3$.

$\therefore$ sign diagram is:

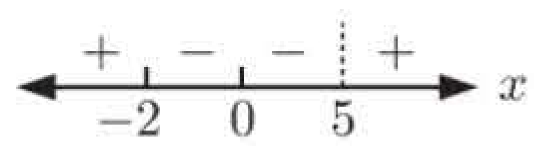


q $y = \dfrac{-x^2(x+2)}{5-x}$ is zero when $x=0$ or -2 and undefined when $x=5$.

When $x=1$, $y=\dfrac{-1^2(3)}{4}=-\frac{3}{4}<0$.

Since the x factor is squared, the sign does not change at $x=0$.

$\therefore$ sign diagram is:

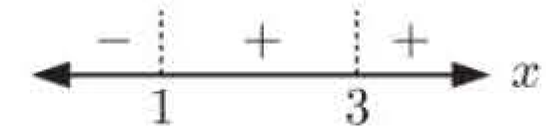


r $y = \dfrac{x^2+4}{(x-3)^2(x-1)}$ is never zero (since $x^2+4>0$ for all real x) and undefined when $x=3$ or 1.

When $x=0$, $y=\dfrac{4}{(-3)^2(-1)}=-\frac{4}{9}<0$.

Since the $(x-3)$ factor is squared, the sign does not change at $x=3$.

$\therefore$ sign diagram is:



s $y = \dfrac{x-5}{x+1}+3\dfrac{(x+1)}{(x+1)}$

$= \dfrac{x-5+3x+3}{x+1}$

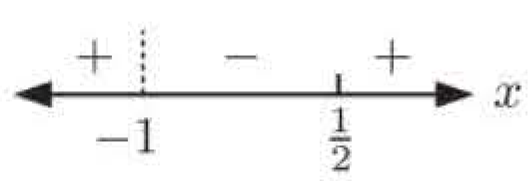
$= \dfrac{4x-2}{x+1}$

which is zero when $x=\frac{1}{2}$ and undefined when $x=-1$.

When $x=0$, $y=\frac{-2}{1}=-2<0$.

Since the factors are distinct and linear, the signs alternate.

$\therefore$ sign diagram is:



t $y = \dfrac{x-2}{x+3}-4\dfrac{(x+3)}{(x+3)}$

$= \dfrac{x-2-4x-12}{x+3}$

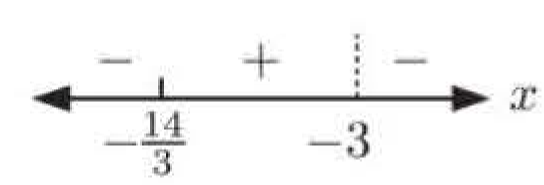
$= \dfrac{-3x-14}{x+3}$

which is zero when $x=-\frac{14}{3}$ and undefined when $x=-3$.

When $x=0$, $y=\frac{-14}{3}<0$.

Since the factors are distinct and linear, the signs alternate.

$\therefore$ sign diagram is:



u $y = \dfrac{3x+2}{x-2}-\dfrac{x-3}{x+3}$

$= \dfrac{(3x+2)(x+3)-(x-3)(x-2)}{(x-2)(x+3)}$

$= \dfrac{3x^2+11x+6-(x^2-5x+6)}{(x-2)(x+3)}$

$= \dfrac{2x^2+16x}{(x-2)(x+3)}$

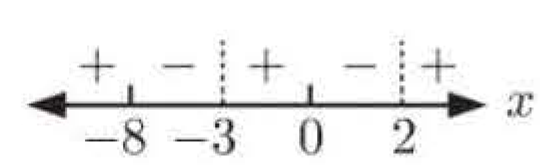
$= \dfrac{2x(x+8)}{(x-2)(x+3)}$

which is zero when $x=0$ or -8 and undefined when $x=2$ or -3.

When $x=1$, $y=\dfrac{2(1)(9)}{(-1)(4)}=-\frac{18}{4}<0$.

Since the factors are distinct and linear, the signs alternate.

$\therefore$ sign diagram is:



EXERCISE 2G

1 **a** Sign diagram of $(2-x)(x+3)$ is

$$\longleftarrow - \underset{-3}{|} + \underset{2}{|} - \longrightarrow x$$

$\therefore\ (2-x)(x+3) \geqslant 0$ when $x \in [-3, 2]$

b Sign diagram of $(x-1)^2$ is

$$\longleftarrow + \underset{1}{|} + \longrightarrow x$$

$\therefore\ (x-1)^2 < 0$ is never true

$\therefore$ no solutions

c Sign diagram of $(2x+1)(3-x)$ is

$$\longleftarrow - \underset{-\frac{1}{2}}{|} + \underset{3}{|} - \longrightarrow x$$

$\therefore\ (2x+1)(3-x) > 0$ when $x \in \left]-\frac{1}{2}, 3\right[$

d
$$x^2 \geqslant x$$
$$\therefore\ x^2 - x \geqslant 0$$
$$\therefore\ x(x-1) \geqslant 0$$

Sign diagram of LHS is

$$\longleftarrow + \underset{0}{|} - \underset{1}{|} + \longrightarrow x$$

$\therefore\ x \in \left]-\infty, 0\right]$ or $\left[1, \infty\right[$

e
$$x^2 \geqslant 3x$$
$$\therefore\ x^2 - 3x \geqslant 0$$
$$\therefore\ x(x-3) \geqslant 0$$

Sign diagram of LHS is

$$\longleftarrow + \underset{0}{|} - \underset{3}{|} + \longrightarrow x$$

$\therefore$ LHS $\geqslant 0$ when $x \in \left]-\infty, 0\right]$ or $\left[3, \infty\right[$

f
$$3x^2 + 2x < 0$$
$$\therefore\ x(3x+2) < 0$$

Sign diagram of LHS is

$$\longleftarrow + \underset{-\frac{2}{3}}{|} - \underset{0}{|} + \longrightarrow x$$

$\therefore$ LHS < 0 when $x \in \left]-\frac{2}{3}, 0\right[$

g
$$x^2 < 4$$
$$\therefore\ x^2 - 4 < 0$$
$$\therefore\ (x+2)(x-2) < 0$$

Sign diagram of LHS is

$$\longleftarrow + \underset{-2}{|} - \underset{2}{|} + \longrightarrow x$$

$\therefore$ LHS < 0 when $x \in \left]-2, 2\right[$

h
$$2x^2 \geqslant 4$$
$$\therefore\ 2x^2 - 4 \geqslant 0$$
$$\therefore\ 2(x+\sqrt{2})(x-\sqrt{2}) \geqslant 0$$

Sign diagram of LHS is

$$\longleftarrow + \underset{-\sqrt{2}}{|} - \underset{\sqrt{2}}{|} + \longrightarrow x$$

$\therefore$ LHS $\geqslant 0$ when $x \in \left]-\infty, -\sqrt{2}\right]$ or $\left[\sqrt{2}, \infty\right[$

i $x^2 + 4x + 4 > 0$

$\therefore\ (x+2)^2 > 0$

Sign diagram of LHS is

$$\longleftarrow + \underset{-2}{|} + \longrightarrow x$$

$\therefore$ LHS > 0 when $x \neq -2$

j
$$2x^2 \geqslant x + 3$$
$$\therefore\ 2x^2 - x - 3 \geqslant 0$$
$$\therefore\ (2x-3)(x+1) \geqslant 0$$

Sign diagram of LHS is

$$\longleftarrow + \underset{-1}{|} - \underset{\frac{3}{2}}{|} + \longrightarrow x$$

$\therefore$ LHS $\geqslant 0$ when $x \in \left]-\infty, -1\right]$ or $\left[\frac{3}{2}, \infty\right[$

k $4x^2 - 4x + 1 < 0$

$\therefore\ (2x-1)^2 < 0$

Sign diagram of LHS is

$$\longleftarrow + \underset{\frac{1}{2}}{|} + \longrightarrow x$$

$\therefore$ LHS < 0 is never true

$\therefore$ no solutions

l
$$6x^2 + 7x < 3$$
$$\therefore\ 6x^2 + 7x - 3 < 0$$
$$\therefore\ (3x-1)(2x+3) < 0$$

Sign diagram of LHS is

$$\longleftarrow + \underset{-\frac{3}{2}}{|} - \underset{\frac{1}{3}}{|} + \longrightarrow x$$

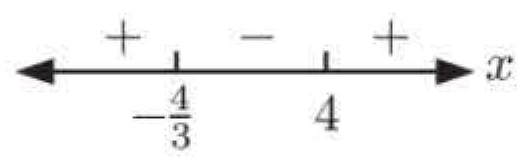
$\therefore$ LHS < 0 when $x \in \left]-\frac{3}{2}, \frac{1}{3}\right[$

m $3x^2 > 8(x+2)$

$\therefore \ 3x^2 - 8x - 16 > 0$

$\therefore \ (3x+4)(x-4) > 0$

Sign diagram of LHS is



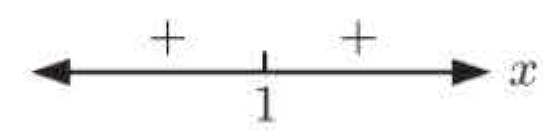
$\therefore$ LHS > 0 when $x \in \,]-\infty, -\frac{4}{3}[$
or $]4, \infty[$

n $2x^2 - 4x + 2 > 0$

$\therefore \ 2(x^2 - 2x + 1) > 0$

$\therefore \ 2(x-1)^2 > 0$

Sign diagram of LHS is



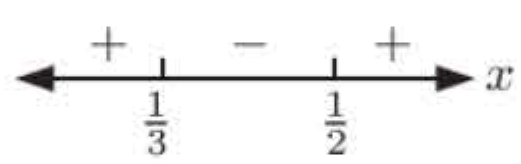
$\therefore$ LHS > 0 when $x \neq 1$

o $6x^2 + 1 \leqslant 5x$

$\therefore \ 6x^2 - 5x + 1 \leqslant 0$

$\therefore \ (3x-1)(2x-1) \leqslant 0$

Sign diagram of LHS is



$\therefore$ LHS $\leqslant 0$ when $x \in [\frac{1}{3}, \frac{1}{2}]$

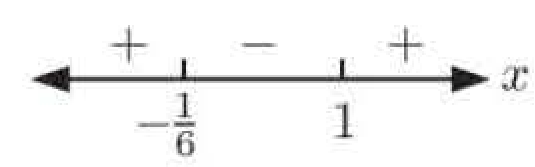
p $1 + 5x < 6x^2$

$\therefore \ -6x^2 + 5x + 1 < 0$

$\therefore \ 6x^2 - 5x - 1 > 0$

$\therefore \ (6x+1)(x-1) > 0$

Sign diagram of LHS is



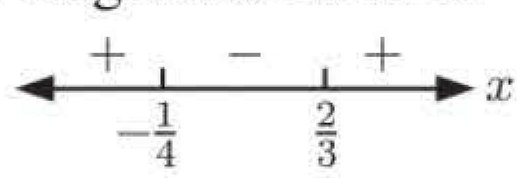
$\therefore$ LHS > 0 when $x \in \,]-\infty, -\frac{1}{6}[$
or $]1, \infty[$

q $12x^2 \geqslant 5x + 2$

$\therefore \ 12x^2 - 5x - 2 \geqslant 0$

$\therefore \ (3x-2)(4x+1) \geqslant 0$

Sign diagram of LHS is



$\therefore$ LHS $\geqslant 0$ when $x \in \,]-\infty, -\frac{1}{4}]$
or $[\frac{2}{3}, \infty[$

r $2x^2 + 9 > 9x$

$\therefore \ 2x^2 - 9x + 9 > 0$

$\therefore \ (2x-3)(x-3) > 0$

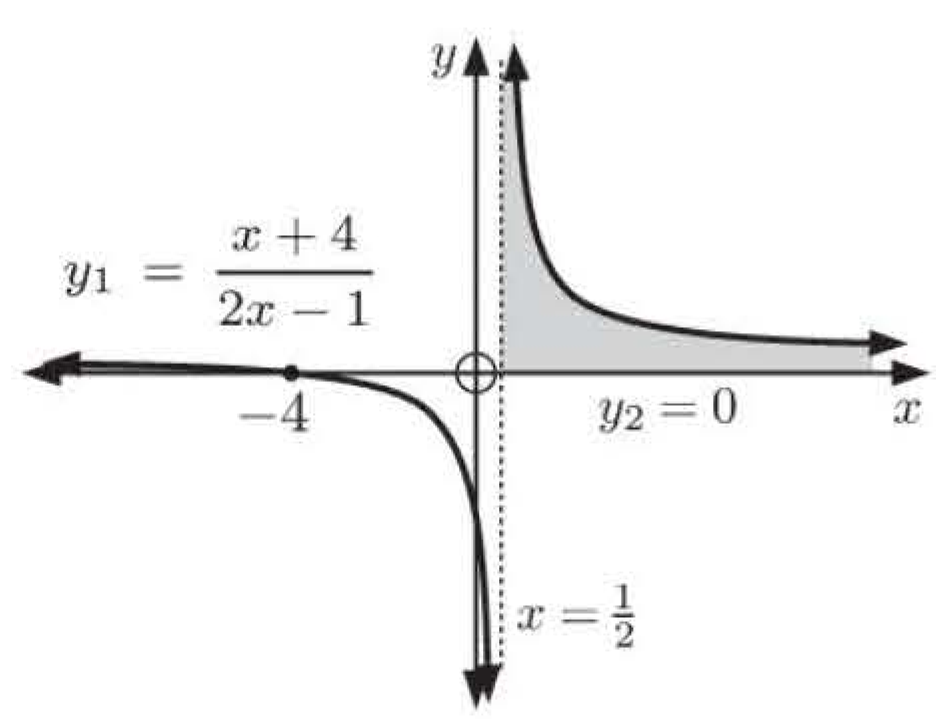
Sign diagram of LHS is

$\therefore$ LHS > 0 when $x \in \,]-\infty, \frac{3}{2}[$
or $]3, \infty[$

2 **a** The graphs of $y_1 = \dfrac{x+4}{2x-1}$ and $y_2 = 0$ intersect at $x = -4$.

$y_1 = \dfrac{x+4}{2x-1}$ has vertical asymptote $x = \frac{1}{2}$.

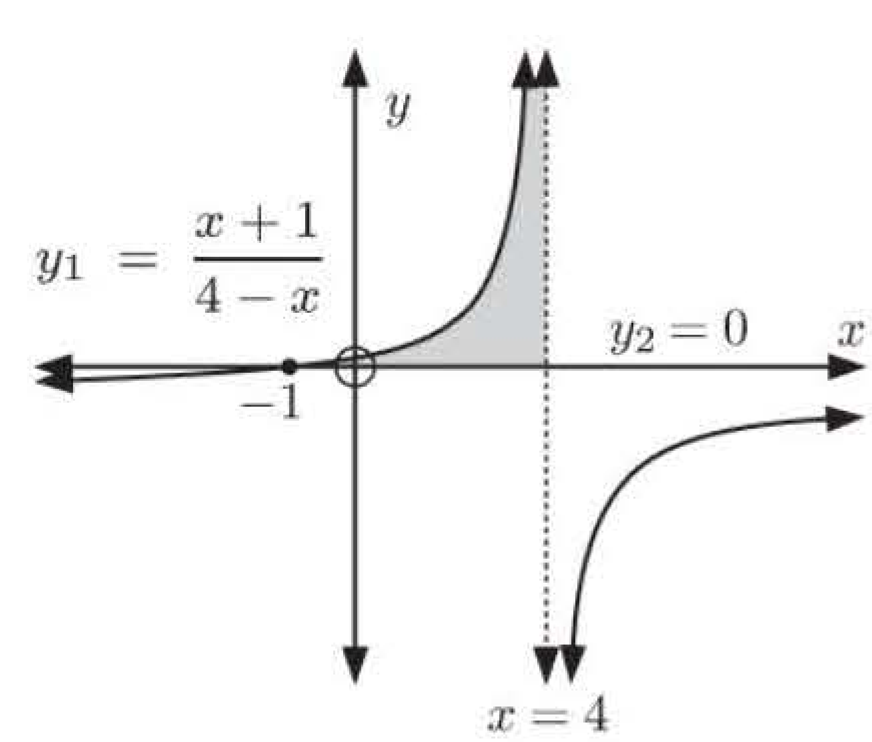
So, $\dfrac{x+4}{2x-1} > 0$ when $x < -4$ and $x > \frac{1}{2}$.



b The graphs of $y_1 = \dfrac{x+1}{4-x}$ and $y_2 = 0$ intersect at $x = -1$.

$y_1 = \dfrac{x+1}{4-x}$ has vertical asymptote $x = 4$.

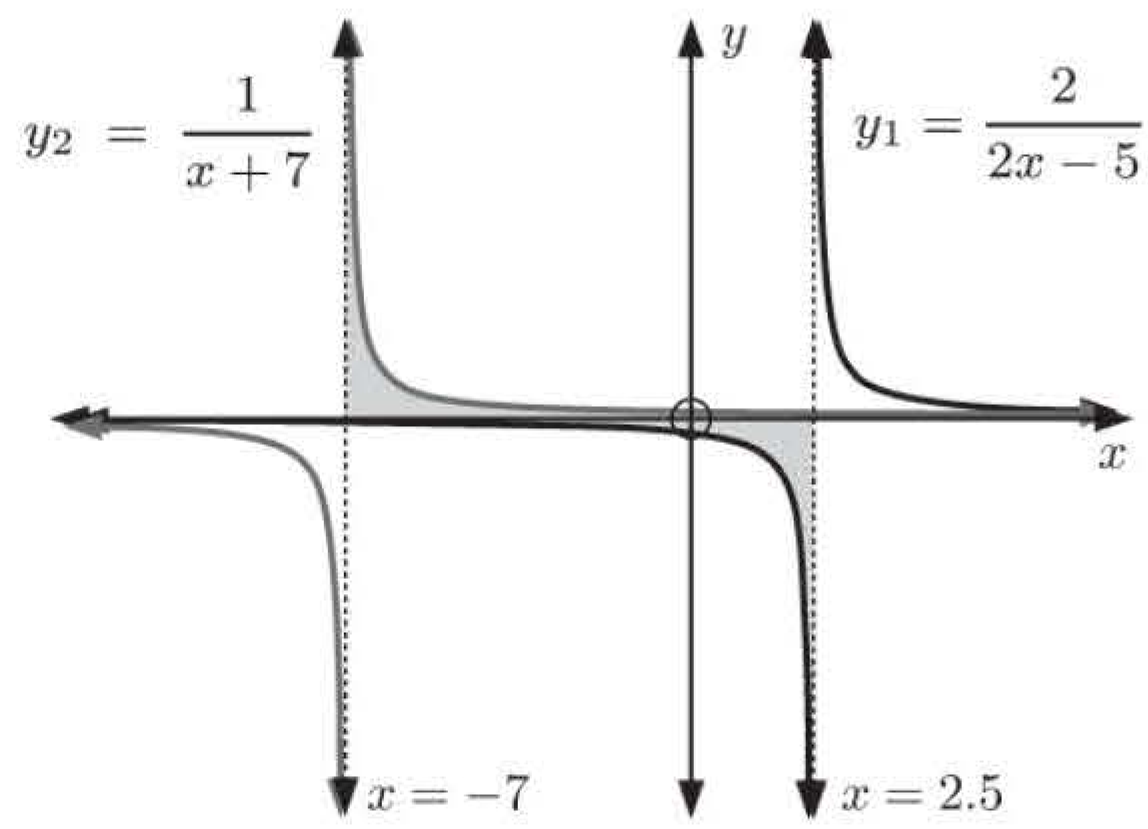
So, $\dfrac{x+1}{4-x} \geqslant 0$ when $-1 \leqslant x < 4$.



c The graphs of $y_1 = \dfrac{2}{2x-5}$ and $y_2 = \dfrac{1}{x+7}$ do not intersect.

$y_1 = \dfrac{2}{2x-5}$ has vertical asymptote $x = 2.5$ and $y_2 = \dfrac{1}{x+7}$ has vertical asymptote $x = -7$.

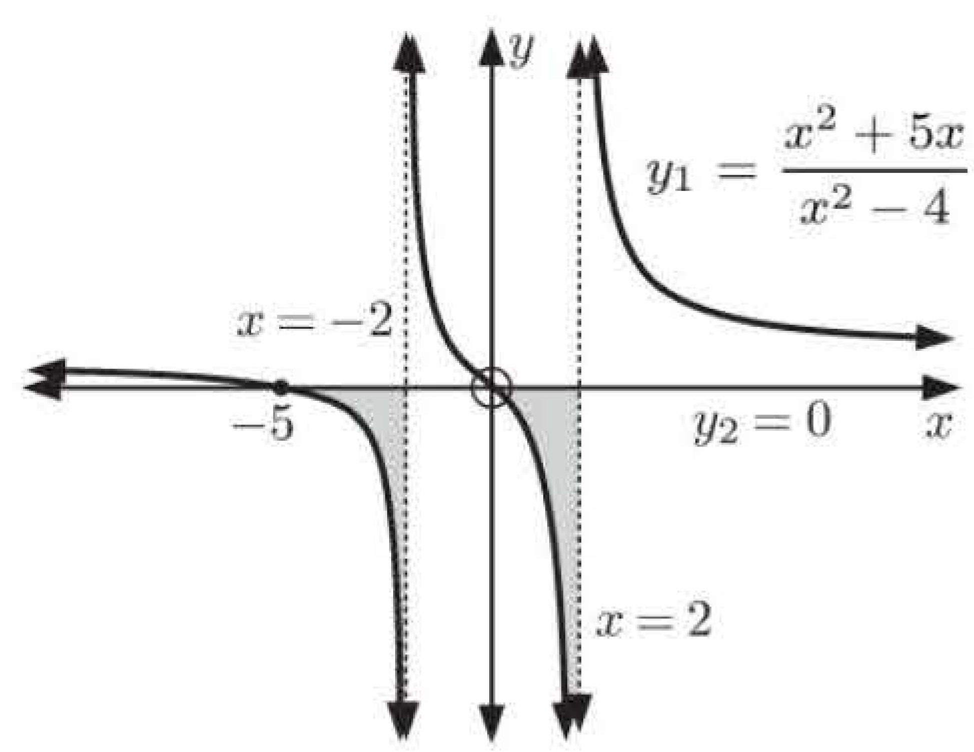
So, $\dfrac{2}{2x-5} < \dfrac{1}{x+7}$ when $-7 < x < 2.5$.



d The graphs of $y_1 = \dfrac{x^2+5x}{x^2-4}$ and $y_2 = 0$ intersect at $x = -5$ and $x = 0$.

$y_1 = \dfrac{x^2+5x}{x^2-4}$ has vertical asymptotes $x = -2$ and $x = 2$.

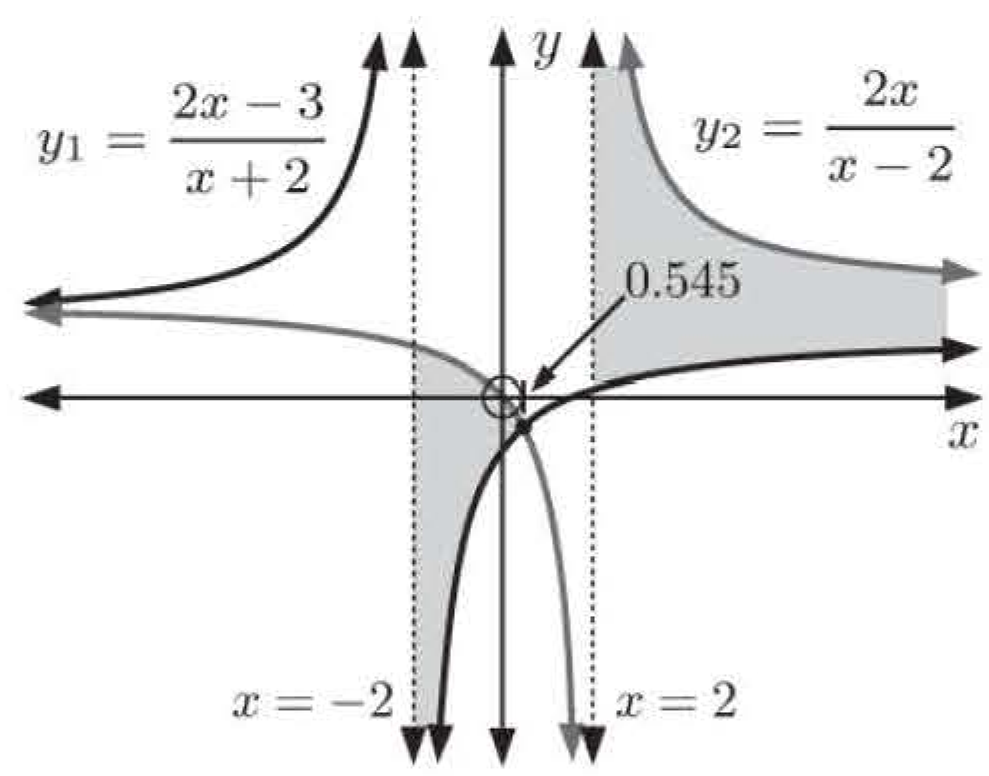
So, $\dfrac{x^2+5x}{x^2-4} \leqslant 0$ when $-5 \leqslant x < -2$ and $0 \leqslant x < 2$.



e The graphs of $y_1 = \dfrac{2x-3}{x+2}$ and $y_2 = \dfrac{2x}{x-2}$ intersect at $x \approx 0.545$.

$y_1 = \dfrac{2x-3}{x+2}$ has vertical asymptote $x = -2$ and $y_2 = \dfrac{2x}{x-2}$ has vertical asymptote $x = 2$.

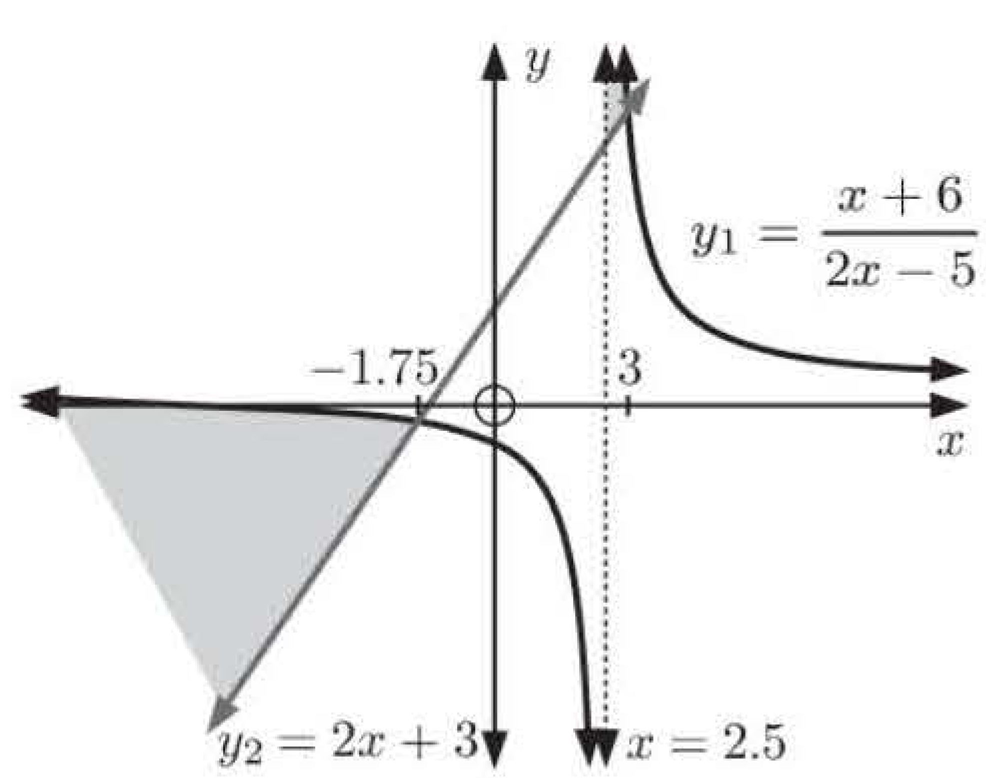
So, $\dfrac{2x-3}{x+2} < \dfrac{2x}{x-2}$ when $-2 < x < 0.545$ and $x > 2$.



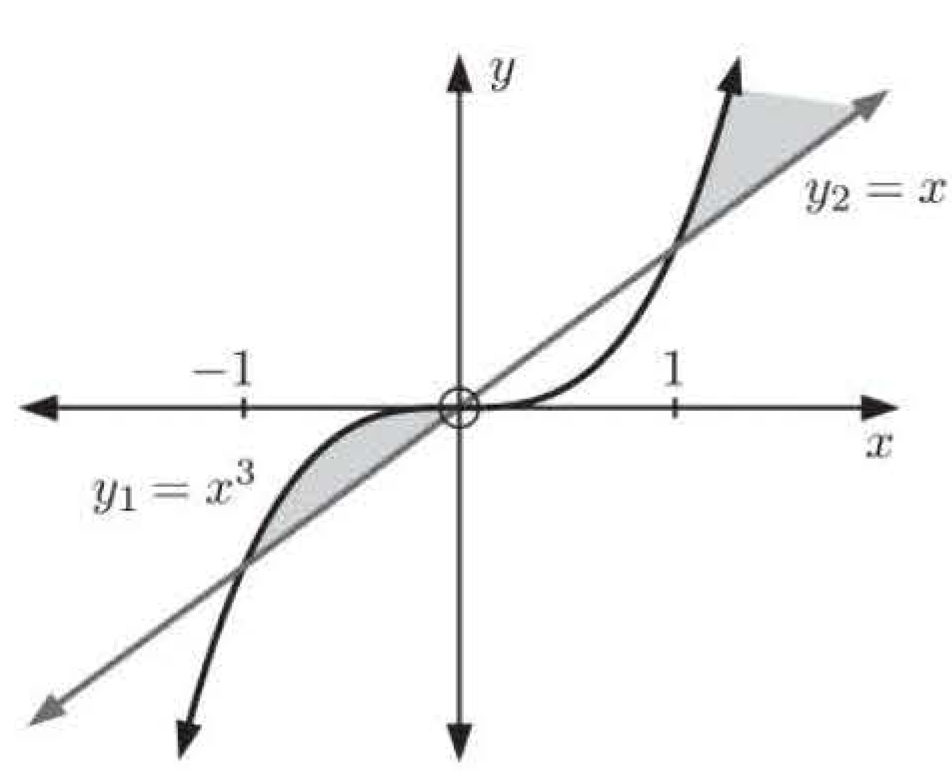
f The graphs of $y_1 = \dfrac{x+6}{2x-5}$ and $y_2 = 2x+3$ intersect at $x = -1.75$ and $x = 3$.

$y_1 = \dfrac{x+6}{2x-5}$ has vertical asymptote $x = 2.5$.

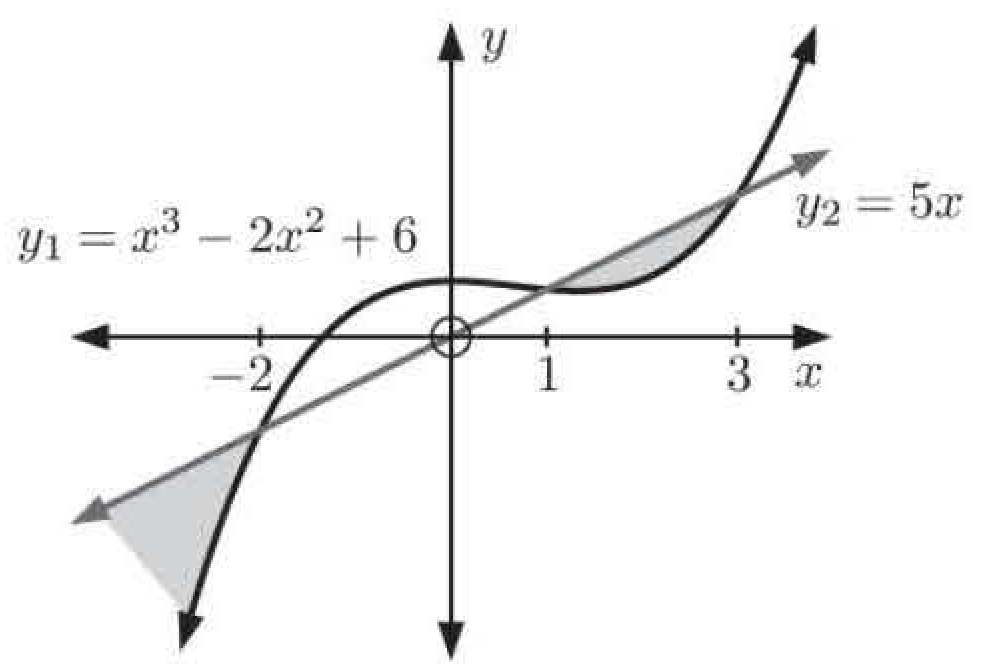
So, $\dfrac{x+6}{2x-5} > 2x+3$ when $x < -1.75$ and $2.5 < x < 3$.



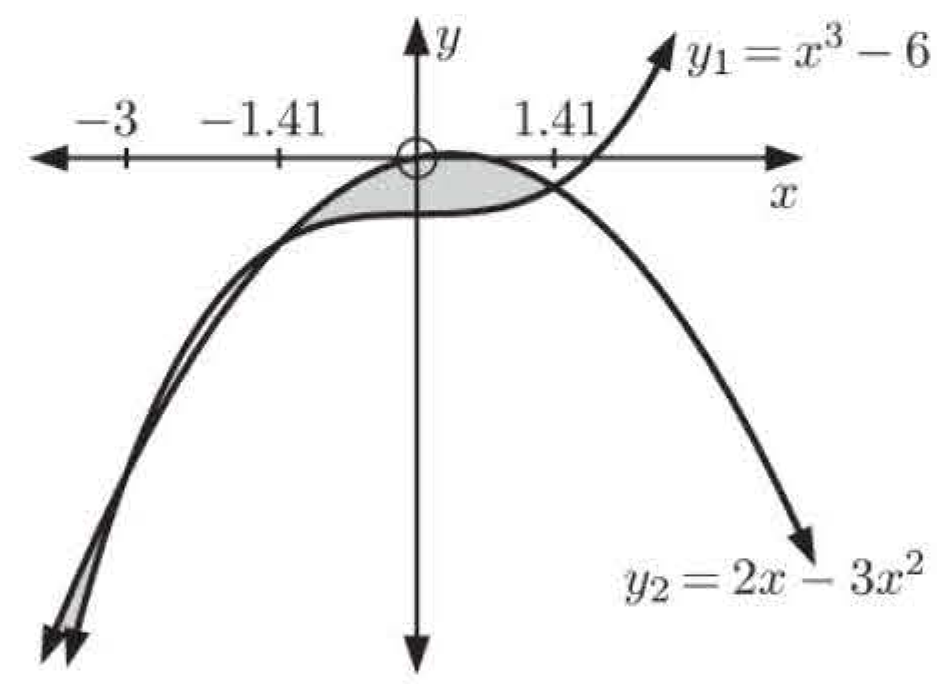
3 **a** The graphs of $y_1 = x^3$ and $y_2 = x$ intersect at $x = -1$, $x = 0$, and $x = 1$.
So, $x^3 \geqslant x$ when $-1 \leqslant x \leqslant 0$ and $x \geqslant 1$.



b The graphs of $y_1 = x^3 - 2x^2 + 6$ and $y_2 = 5x$ intersect at $x = -2$, $x = 1$, and $x = 3$.
So, $x^3 - 2x^2 + 6 < 5x$ when $x < -2$ and $1 < x < 3$.



c The graphs of $y_1 = x^3 - 6$ and $y_2 = 2x - 3x^2$ intersect at $x = -3$, $x \approx -1.41$, and $x \approx 1.41$.
So, $x^3 - 6 \leqslant 2x - 3x^2$ when $x \leqslant -3$ and $-1.41 \leqslant x \leqslant 1.41$.



EXERCISE 2H.1

1 **a** $|a| = |-2| = 2$

b $|b| = |3| = 3$

c $|a|\,|b| = |-2|\,|3| = 2 \times 3 = 6$

d $|ab| = |-2 \times 3| = |-6| = 6$

e $|a - b| = |-2 - 3| = |-5| = 5$

f $|a| - |b| = |-2| - |3| = 2 - 3 = -1$

g $|a + b| = |-2 + 3| = |1| = 1$

h $|a| + |b| = |-2| + |3| = 2 + 3 = 5$

i $|a|^2 = |-2|^2 = 2^2 = 4$

j $a^2 = (-2)^2 = 4$

k $\left|\frac{c}{a}\right| = \left|\frac{-4}{-2}\right| = |2| = 2$

l $\frac{|c|}{|a|} = \frac{|-4|}{|-2|} = \frac{4}{2} = 2$

2 **a** $|5 - x| = |5 - (-3)| = |8| = 8$

b $|5| - |x| = |5| - |-3| = 5 - 3 = 2$

c $\left|\frac{2x+1}{1-x}\right| = \left|\frac{2(-3)+1}{1-(-3)}\right| = \left|\frac{-5}{4}\right| = \frac{5}{4}$

d $\left|3 - 2x - x^2\right| = \left|3 - 2(-3) - (-3)^2\right| = |0| = 0$

3 **a**

a	b	$\lvert a\rvert + \lvert b\rvert$	$\lvert a\rvert - \lvert b\rvert$	$\lvert a+b\rvert$	$\lvert a-b\rvert$	$\lvert b-a\rvert$
6	2	8	4	8	4	4
6	−2	8	4	4	8	8
−6	2	8	4	4	8	8
−6	−2	8	4	8	4	4

b **i** False, $|a+b| \neq |a|+|b|$.
For example, $|6+(-2)| = 4$ but $|6|+|-2| = 8$.

ii False, $|a-b| \neq |a|-|b|$.
For example, $|6-(-2)| = 8$ but $|6|-|-2| = 4$.

c
$$\begin{aligned}|a-b| &= \sqrt{(a-b)^2}\\ &= \sqrt{(b-a)^2}\\ &= |b-a|\end{aligned}$$

4 **a**

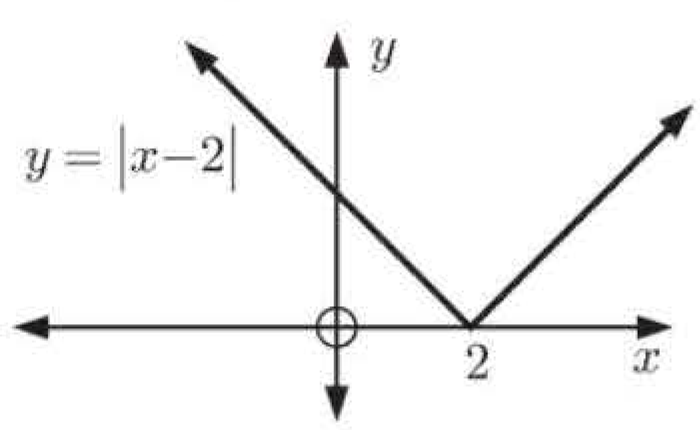
a	b	$\lvert ab\rvert$	$\lvert a\rvert\,\lvert b\rvert$	$\left\lvert\dfrac{a}{b}\right\rvert$	$\dfrac{\lvert a\rvert}{\lvert b\rvert}$
6	2	12	12	3	3
6	-2	12	12	3	3
-6	2	12	12	3	3
-6	-2	12	12	3	3

b **i**
$$\begin{aligned}|ab| &= \sqrt{(ab)^2}\\ &= \sqrt{a^2b^2}\\ &= \sqrt{a^2}\sqrt{b^2}\\ &= |a|\,|b|\end{aligned}$$

ii
$$\begin{aligned}\left|\frac{a}{b}\right| &= \sqrt{\left(\frac{a}{b}\right)^2}\\ &= \sqrt{\frac{a^2}{b^2}}\\ &= \frac{\sqrt{a^2}}{\sqrt{b^2}}\\ &= \frac{|a|}{|b|}\end{aligned}$$

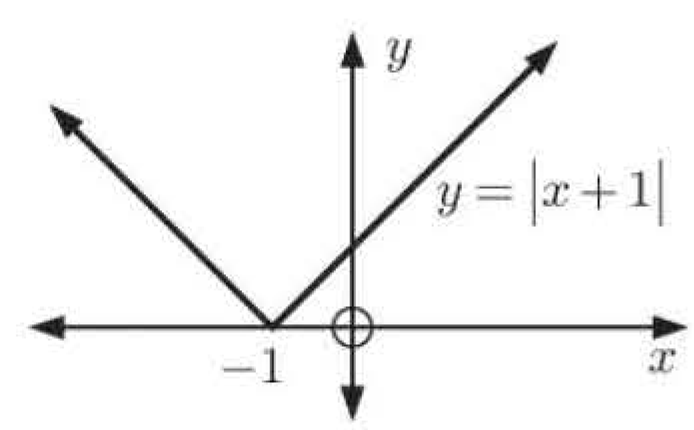
5 **a**
$$y = |x-2|$$
When $x \geqslant 2$, $x-2 \geqslant 0$,
so $y = x-2$
When $x < 2$, $x-2 < 0$,
so $y = -(x-2)$
$= 2-x$

$$\therefore \quad y = \begin{cases} x-2 & \text{when } x \geqslant 2\\ 2-x & \text{when } x < 2\end{cases}$$



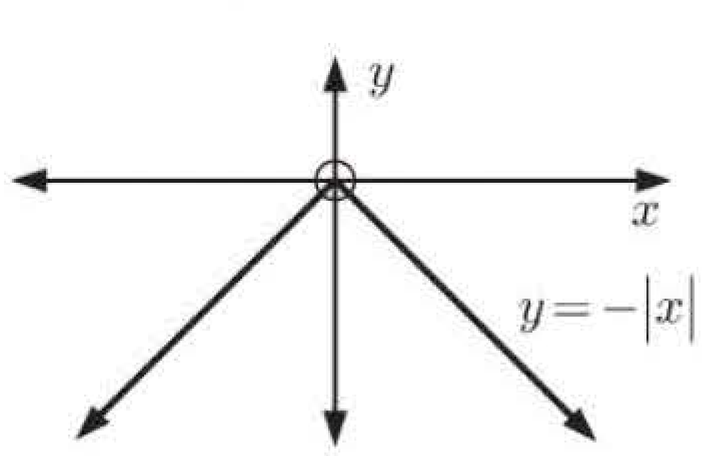
b
$$y = |x+1|$$
When $x \geqslant -1$, $x+1 \geqslant 0$,
so $y = x+1$
When $x < -1$, $x+1 < 0$,
so $y = -(x+1)$
$= -x-1$

$$\therefore \quad y = \begin{cases} x+1 & \text{when } x \geqslant -1\\ -x-1 & \text{when } x < -1\end{cases}$$



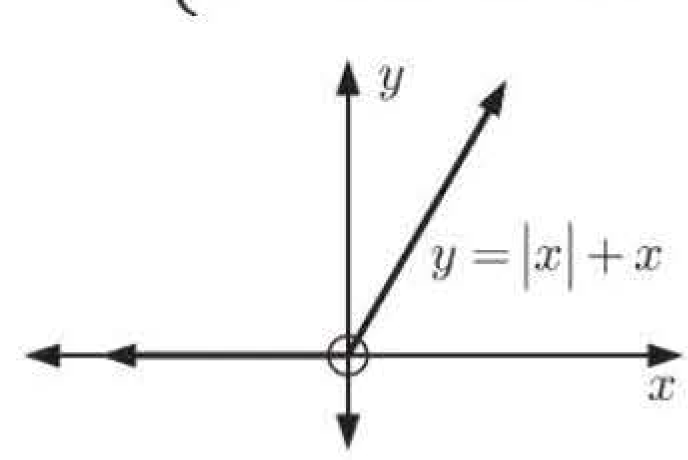
c
$$y = -|x|$$
When $x \geqslant 0$, $y = -x$
When $x < 0$, $y = -(-x)$
$= x$

$$\therefore \quad y = \begin{cases} -x & \text{when } x \geqslant 0\\ x & \text{when } x < 0\end{cases}$$



d
$$y = |x|+x$$
When $x \geqslant 0$, $y = x+x$
$= 2x$
When $x < 0$, $y = -x+x$
$= 0$

$$\therefore \quad y = \begin{cases} 2x & \text{when } x \geqslant 0\\ 0 & \text{when } x < 0\end{cases}$$



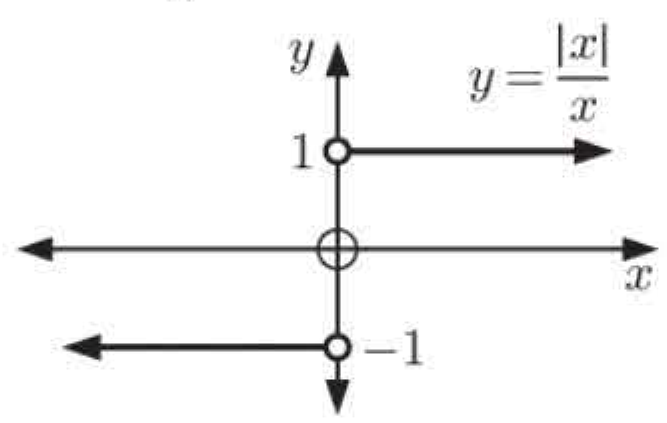
e
$$y = \frac{|x|}{x}$$

When $x > 0$, $y = \frac{x}{x} = 1$

When $x < 0$, $y = \frac{-x}{x} = -1$

When $x = 0$, y is undefined.

$$\therefore \quad y = \begin{cases} 1 & \text{when } x > 0 \\ \text{undefined} & \text{when } x = 0 \\ -1 & \text{when } x < 0 \end{cases}$$

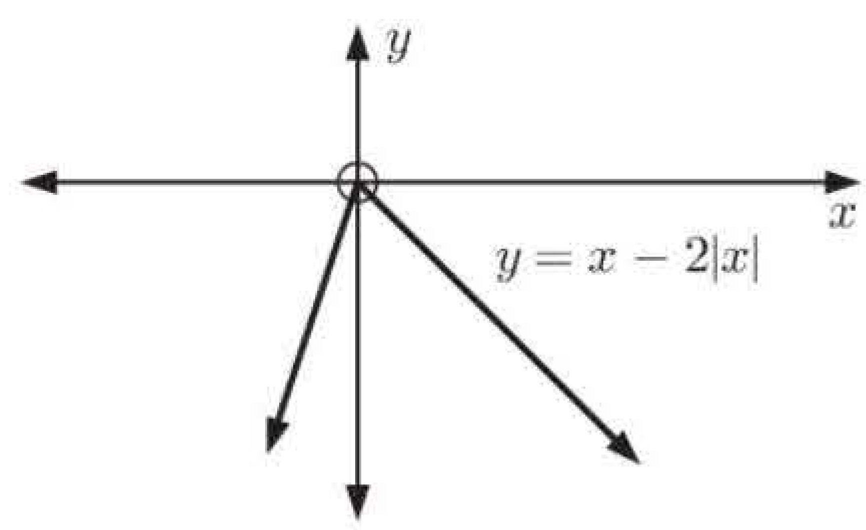


f
$$y = x - 2|x|$$

When $x \geqslant 0$, $\begin{aligned} y &= x - 2x \\ &= -x \end{aligned}$

When $x < 0$, $\begin{aligned} y &= x - 2(-x) \\ &= 3x \end{aligned}$

$$\therefore \quad y = \begin{cases} -x & \text{when } x \geqslant 0 \\ 3x & \text{when } x < 0 \end{cases}$$



g
$$y = |x| + |x-2|$$

When $x \geqslant 2$, $x - 2 \geqslant 0$ and $x \geqslant 0$

$\therefore \quad \begin{aligned} y &= x + (x-2) \\ &= 2x - 2 \end{aligned}$

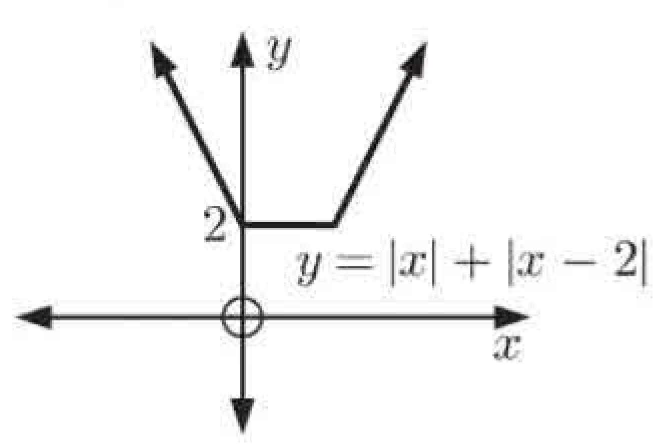
When $0 \leqslant x < 2$, $x \geqslant 0$ and $x - 2 < 0$

$\therefore \quad \begin{aligned} y &= x - (x-2) \\ &= 2 \end{aligned}$

When $x < 0$, $x - 2 < 0$

$\therefore \quad \begin{aligned} y &= -x - (x-2) \\ &= 2 - 2x \end{aligned}$

$$\therefore \quad y = \begin{cases} 2x-2 & \text{when } x \geqslant 2 \\ 2 & \text{when } 0 \leqslant x < 2 \\ 2-2x & \text{when } x < 0 \end{cases}$$



h
$$y = |x| - |x-1|$$

When $x \geqslant 1$, $x - 1 \geqslant 0$ and $x \geqslant 0$

$\therefore \quad \begin{aligned} y &= x - (x-1) \\ &= 1 \end{aligned}$

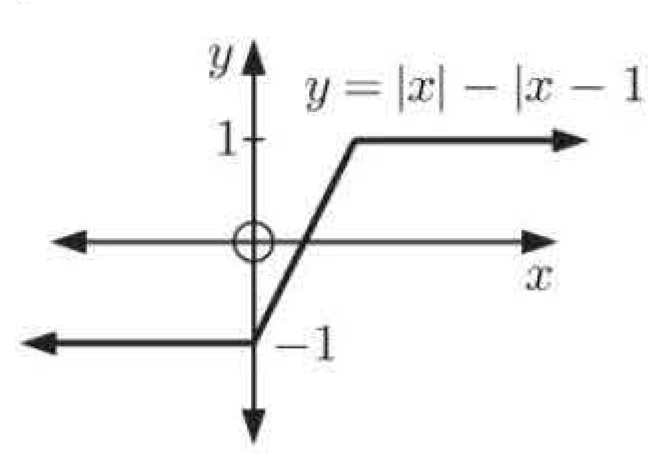
When $0 \leqslant x < 1$, $x \geqslant 0$ and $x - 1 < 0$

$\therefore \quad \begin{aligned} y &= x + (x-1) \\ &= 2x - 1 \end{aligned}$

When $x < 0$, $x - 1 < 0$

$\therefore \quad \begin{aligned} y &= -x + (x-1) \\ &= -1 \end{aligned}$

$$\therefore \quad y = \begin{cases} 1 & \text{when } x \geqslant 1 \\ 2x-1 & \text{when } 0 \leqslant x < 1 \\ -1 & \text{when } x < 0 \end{cases}$$



EXERCISE 2H.2

1 **a** $|x| = 3$

$\therefore \quad x = \pm 3$

b $|x| = -5$ has no solution as $|x|$ is never negative.

c $|x| = 0$

$\therefore \quad x = 0$

d $|x - 1| = 3$

$\therefore \quad x - 1 = \pm 3$

$\therefore \quad x = 1 + 3$ or $1 - 3$

$\therefore \quad x = 4$ or -2

e $|3 - x| = 4$

$\therefore \quad 3 - x = \pm 4$

$\therefore \quad x - 3 = \pm 4$

$\therefore \quad x = 3 + 4$ or $3 - 4$

$\therefore \quad x = 7$ or -1

f $|x + 5| = -1$ has no solution as $|x + 5|$ is never negative.

g $|3x - 2| = 1$

$\therefore \quad 3x - 2 = \pm 1$

$\therefore \quad 3x = 2 + 1$ or $2 - 1$

$\therefore \quad 3x = 3$ or 1

$\therefore \quad x = 1$ or $\frac{1}{3}$

h $|3 - 2x| = 3$

$\therefore \quad 3 - 2x = \pm 3$

$\therefore \quad -2x = -3 + 3$ or $-3 - 3$

$\therefore \quad -2x = 0$ or -6

$\therefore \quad x = 0$ or 3

i $|2 - 5x| = 12$

$\therefore \quad 2 - 5x = \pm 12$

$\therefore \quad -5x = -2 + 12$ or $-2 - 12$

$\therefore \quad -5x = 10$ or -14

$\therefore \quad x = -2$ or $\frac{14}{5}$

2 **a** $\left|\frac{x}{x-1}\right| = 3$

$\therefore \frac{x}{x-1} = \pm 3$

If $\frac{x}{x-1} = 3$

then $x = 3x - 3$

$\therefore -2x = -3$

$\therefore x = \frac{3}{2}$

If $\frac{x}{x-1} = -3$

then $x = -3x + 3$

$\therefore 4x = 3$

$\therefore x = \frac{3}{4}$

So, $x = \frac{3}{2}$ or $\frac{3}{4}$

b $\left|\frac{2x-1}{x+1}\right| = 5$

$\therefore \frac{2x-1}{x+1} = \pm 5$

If $\frac{2x-1}{x+1} = 5$

then $2x - 1 = 5x + 5$

$\therefore -3x = 6$

$\therefore x = -2$

If $\frac{2x-1}{x+1} = -5$

then $2x - 1 = -5x - 5$

$\therefore 7x = -4$

$\therefore x = -\frac{4}{7}$

So, $x = -2$ or $-\frac{4}{7}$

c $\left|\frac{x+3}{1-3x}\right| = \frac{1}{2}$

$\therefore \frac{x+3}{1-3x} = \pm\frac{1}{2}$

If $\frac{x+3}{1-3x} = \frac{1}{2}$

then $2(x+3) = 1 - 3x$

$\therefore 5x = -5$

$\therefore x = -1$

If $\frac{x+3}{1-3x} = -\frac{1}{2}$

then $2(x+3) = -(1-3x)$

$\therefore -x = -7$

$\therefore x = 7$

So, $x = -1$ or 7

3 **a** $|3x-1| = |x+2|$

$\therefore 3x - 1 = \pm(x+2)$

If $3x - 1 = x + 2$

then $2x = 3$

$\therefore x = \frac{3}{2}$

If $3x - 1 = -x - 2$

$\therefore 4x = -1$

$\therefore x = -\frac{1}{4}$

So, $x = \frac{3}{2}$ or $-\frac{1}{4}$

b $|2x+5| = |1-x|$

$\therefore 2x + 5 = \pm(1-x)$

If $2x + 5 = 1 - x$

then $3x = -4$

$\therefore x = -\frac{4}{3}$

If $2x + 5 = -(1-x)$

then $2x + 5 = -1 + x$

$\therefore x = -6$

So, $x = -\frac{4}{3}$ or -6

c $|x+1| = |2-x|$

$\therefore x + 1 = \pm(2-x)$

If $x + 1 = 2 - x$

then $2x = 1$

$\therefore x = \frac{1}{2}$

If $x + 1 = -(2-x)$

then $x + 1 = -2 + x$

$\therefore 1 = -2$

which is false

So, $x = \frac{1}{2}$ is the only solution.

d $|x| = |5-x|$

$\therefore x = \pm(5-x)$

If $x = 5 - x$

then $2x = 5$

$\therefore x = \frac{5}{2}$

If $x = -(5-x)$

then $x = -5 + x$

$\therefore 0 = -5$

which is false

So, $x = \frac{5}{2}$ is the only solution.

e $|1-4x| = 2|x-1|$

$\therefore 1 - 4x = \pm 2(x-1)$

If $1 - 4x = 2(x-1)$

then $1 - 4x = 2x - 2$

$\therefore -6x = -3$

$\therefore x = \frac{1}{2}$

If $1 - 4x = -2(x-1)$

then $1 - 4x = -2x + 2$

$\therefore -2x = 1$

$\therefore x = -\frac{1}{2}$

So, $x = \pm\frac{1}{2}$

f $|3x+2| = 2|2-x|$

$\therefore 3x + 2 = \pm 2(2-x)$

If $3x + 2 = 2(2-x)$

then $3x + 2 = 4 - 2x$

$\therefore 5x = 2$

$\therefore x = \frac{2}{5}$

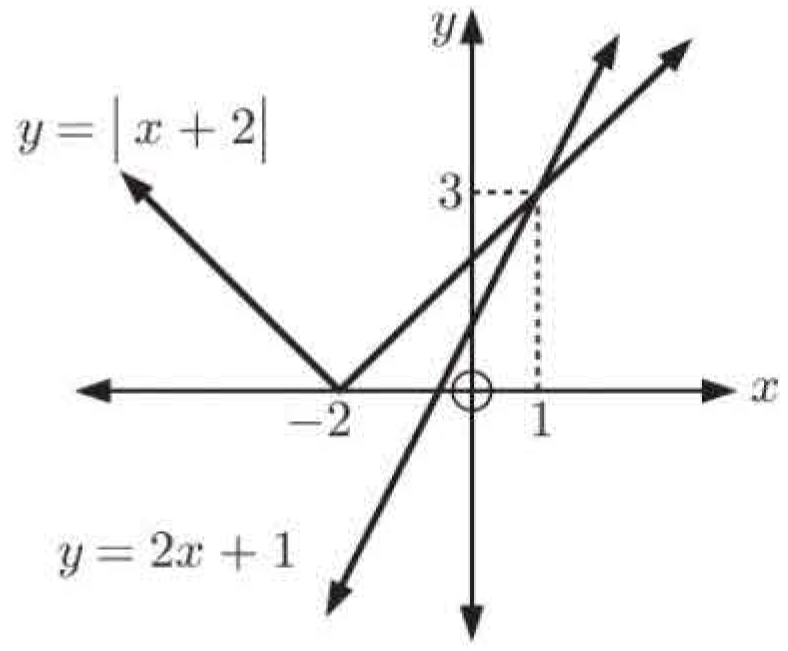
If $3x + 2 = -2(2-x)$

then $3x + 2 = -4 + 2x$

$\therefore x = -6$

So, $x = \frac{2}{5}$ or -6

4 **a** **i**



The lines $y = |x+2|$ and $y = 2x + 1$ intersect at $(1, 3)$.

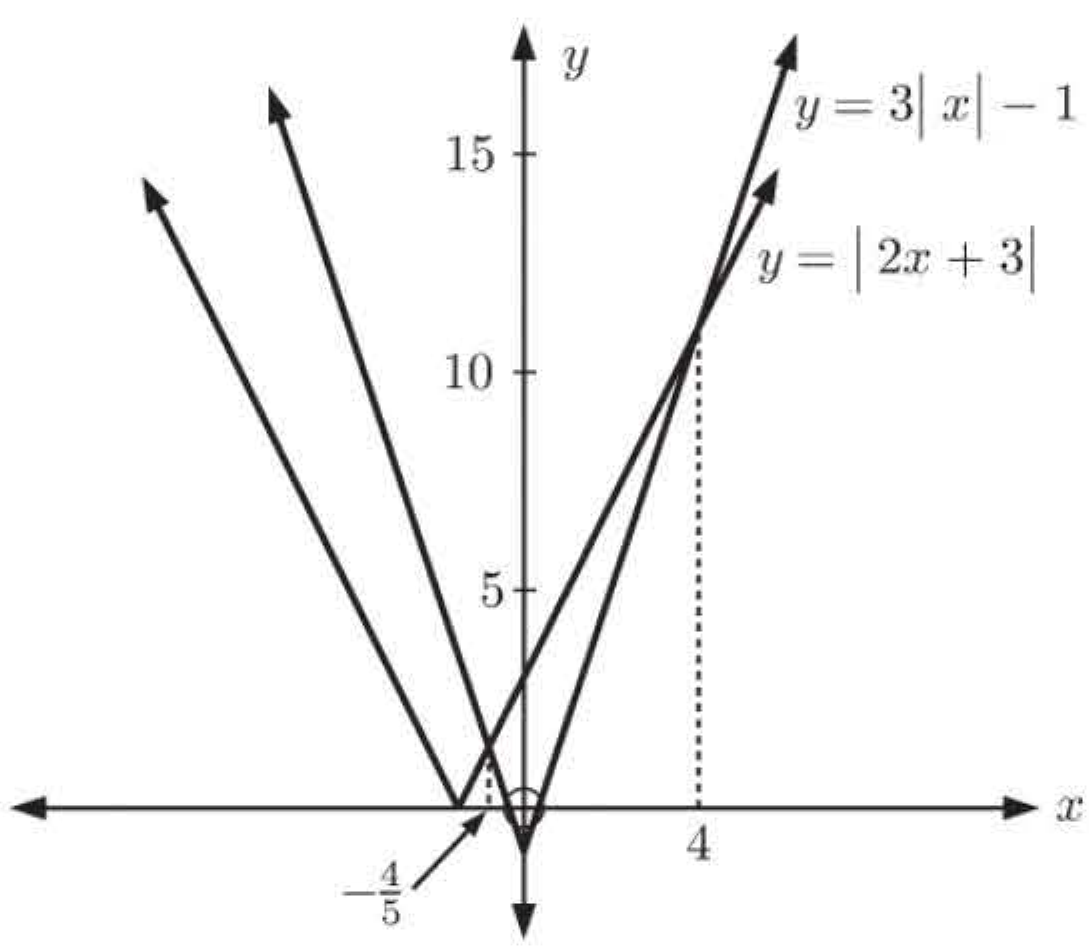
$\therefore$ the solution is $x = 1$.

ii $|x+2| = 2x+1$
$\therefore\ x+2 = \pm(2x+1)$
If $x+2 = 2x+1$
then $-x = -1$
$\therefore\ x = 1$
If $x+2 = -(2x+1)$
then $x+2 = -2x-1$
$\therefore\ 3x = -3$
$\therefore\ x = -1$

However, $x = -1$ is not a valid solution, because when $x=-1$, $2x+1<0$, and $|x+2|$ is never negative.

$\therefore\ x = 1$ is the only solution.

b **i**



The lines $y = |2x+3|$ and $y = 3|x|-1$ intersect at $(-\frac{4}{5}, \frac{7}{5})$ and $(4, 11)$.

$\therefore$ the solution is $x = -\frac{4}{5}$ or 4.

ii Let $y_1 = |2x+3|$, $y_2 = 3|x|-1$

When $x < -\frac{3}{2}$, $y_1 = -(2x+3)$
and $y_2 = 3(-x)-1$
$\therefore\ -(2x+3) = 3(-x)-1$
$\therefore\ -2x-3 = -3x-1$
$\therefore\ x = 2$

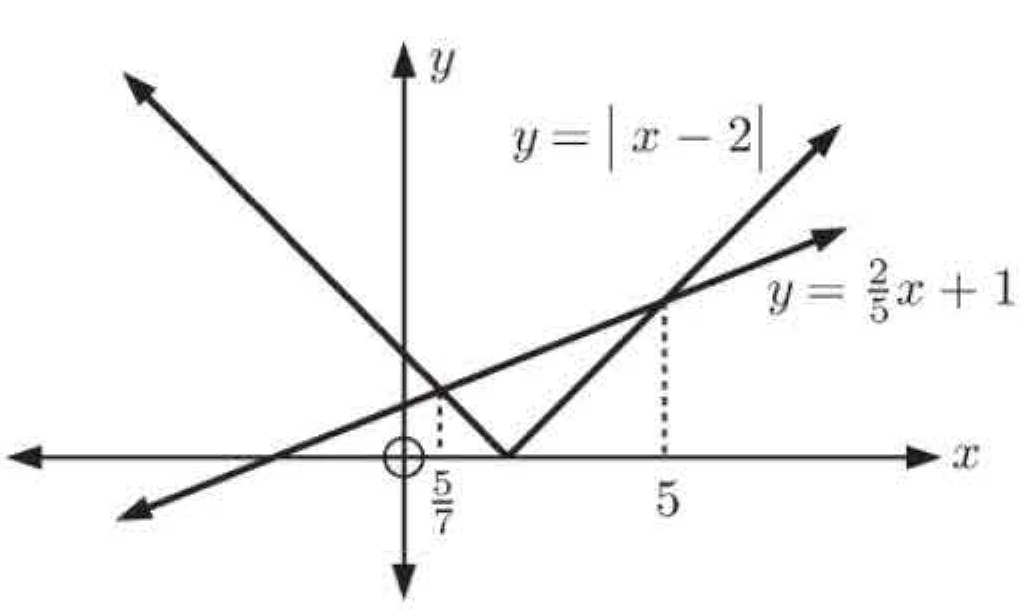
This is not in the domain $x < -\frac{3}{2}$, so is not a valid solution.

When $-\frac{3}{2} \leqslant x < 0$, $y_1 = 2x+3$
and $y_2 = 3(-x)-1$
$\therefore\ 2x+3 = 3(-x)-1$
$\therefore\ 2x+3 = -3x-1$
$\therefore\ 5x = -4$
$\therefore\ x = -\frac{4}{5}$

When $x \geqslant 0$, $y_1 = 2x+3$
and $y_2 = 3x-1$
$\therefore\ 2x+3 = 3x-1$
$\therefore\ -x = -4$
$\therefore\ x = 4$

So, the solution is $x = -\frac{4}{5}$ or 4.

c **i**



The lines $y = |x-2|$ and $y = \frac{2}{5}x+1$ intersect at $(\frac{5}{7}, \frac{9}{7})$ and $(5, 3)$.

$\therefore$ the solution is $x = \frac{5}{7}$ or 5.

ii $|x-2| = \frac{2}{5}x+1$
$\therefore\ x-2 = \pm(\frac{2}{5}x+1)$
If $x-2 = \frac{2}{5}x+1$
then $\frac{3}{5}x = 3$
$\therefore\ x = 5$
If $x-2 = -(\frac{2}{5}x+1)$
then $x-2 = -\frac{2}{5}x-1$
$\therefore\ \frac{7}{5}x = 1$
$\therefore\ x = \frac{5}{7}$

So, the solution is $x = \frac{5}{7}$ or 5.

5 **a** $|x| < 4$
$\therefore\ -4 < x < 4$
$\therefore\ x \in\ ]-4, 4[$

b $|x| \geqslant 3$
$\therefore\ x \leqslant -3$ or $x \geqslant 3$
$\therefore\ x \in\ ]-\infty, -3]$ or $[3, \infty[$

c $|x+3| \leqslant 1$

$\therefore \ -1 \leqslant x+3 \leqslant 1$

$\therefore \ -4 \leqslant x \leqslant -2$

$\therefore \ x \in [-4, -2]$

d $|2x-1| < 3$

$\therefore \ -3 < 2x-1 < 3$

$\therefore \ -2 < 2x < 4$

$\therefore \ -1 < x < 2$

$\therefore \ x \in \,]-1, 2[$

e $|3-4x| > 2$

$\therefore \ |4x-3| > 2$

$\therefore \ 4x-3 < -2$ or $4x-3 > 2$

$\therefore \ 4x < 1$ $\qquad \therefore \ 4x > 5$

$\therefore \ x < \frac{1}{4}$ $\qquad \therefore \ x > \frac{5}{4}$

$\therefore \ x \in \,]-\infty, \frac{1}{4}[$ or $]\frac{5}{4}, \infty[$

f $|x| \geqslant |2-x|$

$\therefore \ x^2 \geqslant (2-x)^2$

$\therefore \ x^2 - (2-x)^2 \geqslant 0$

$\therefore \ [x+(2-x)][x-(2-x)] \geqslant 0$

$\therefore \ 2(2x-2) \geqslant 0$

$\therefore \ x \geqslant 1$

So, $x \in [1, \infty[$

g $3\,|x| \leqslant |1-2x|$

$\therefore \ 9x^2 \leqslant (1-2x)^2$

$\therefore \ 9x^2 - (1-2x)^2 \leqslant 0$

$\therefore \ [3x+(1-2x)][3x-(1-2x)] \leqslant 0$

$\therefore \ (x+1)(5x-1) \leqslant 0$

$\therefore \ -1 \leqslant x \leqslant \frac{1}{5}$

So, $x \in [-1, \frac{1}{5}]$

h $\left|\dfrac{x}{x-2}\right| \geqslant 3$

$\therefore \ \left(\dfrac{x}{x-2}\right)^2 \geqslant 3^2$

$\therefore \ \left(\dfrac{x}{x-2}\right)^2 - 3^2 \geqslant 0$

$\therefore \ \left(\dfrac{x}{x-2}+3\right)\left(\dfrac{x}{x-2}-3\right) \geqslant 0$

$\therefore \ \left(\dfrac{x+3x-6}{x-2}\right)\left(\dfrac{x-3x+6}{x-2}\right) \geqslant 0$

$\therefore \ \dfrac{(4x-6)(-2x+6)}{(x-2)^2} \geqslant 0$

$\therefore \ x \in [\frac{3}{2}, 3]$ but $x \neq 2$

$\therefore \ x \in [\frac{3}{2}, 2[$ or $]2, 3]$

i $\left|\dfrac{2x+3}{x-1}\right| \geqslant 2$

$\therefore \ \left(\dfrac{2x+3}{x-1}\right)^2 \geqslant 2^2$

$\therefore \ \left(\dfrac{2x+3}{x-1}\right)^2 - 2^2 \geqslant 0$

$\therefore \ \left(\dfrac{2x+3}{x-1}+2\right)\left(\dfrac{2x+3}{x-1}-2\right) \geqslant 0$

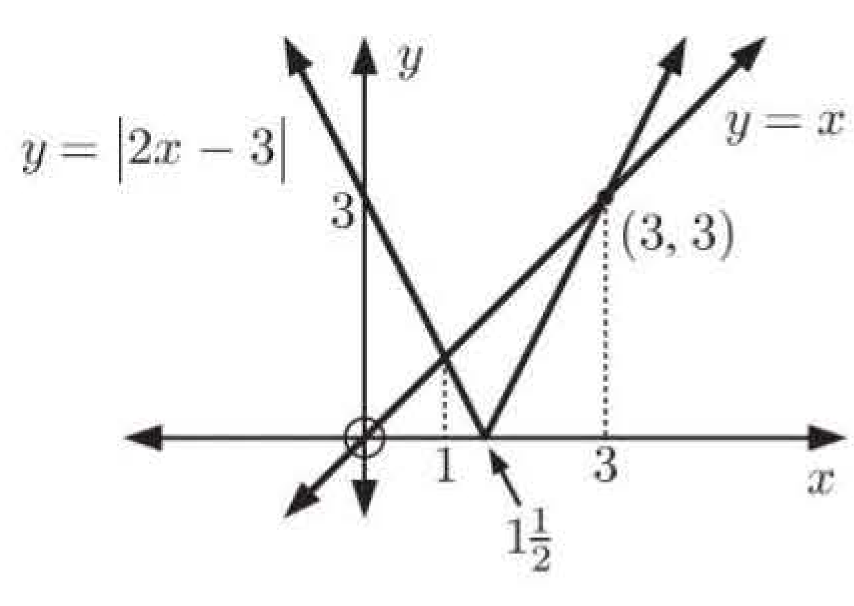
$\therefore \ \left(\dfrac{2x+3+2x-2}{x-1}\right)\left(\dfrac{2x+3-2x+2}{x-1}\right) \geqslant 0$

$\therefore \ \dfrac{(4x+1)(5)}{(x-1)^2} \geqslant 0$

$\therefore \ x \in [-\frac{1}{4}, \infty[$ but $x \neq 1$

$\therefore \ x \in [-\frac{1}{4}, 1[$ or $]1, \infty[$

j

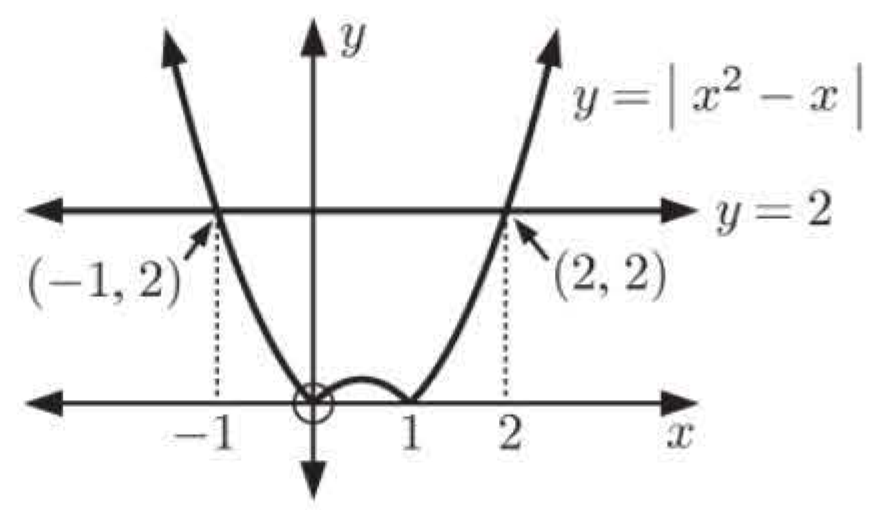


$|2x-3| < x$ when the modulus graph is below the line.

$\therefore \ 1 < x < 3$

$\therefore \ x \in \,]1, 3[$

k

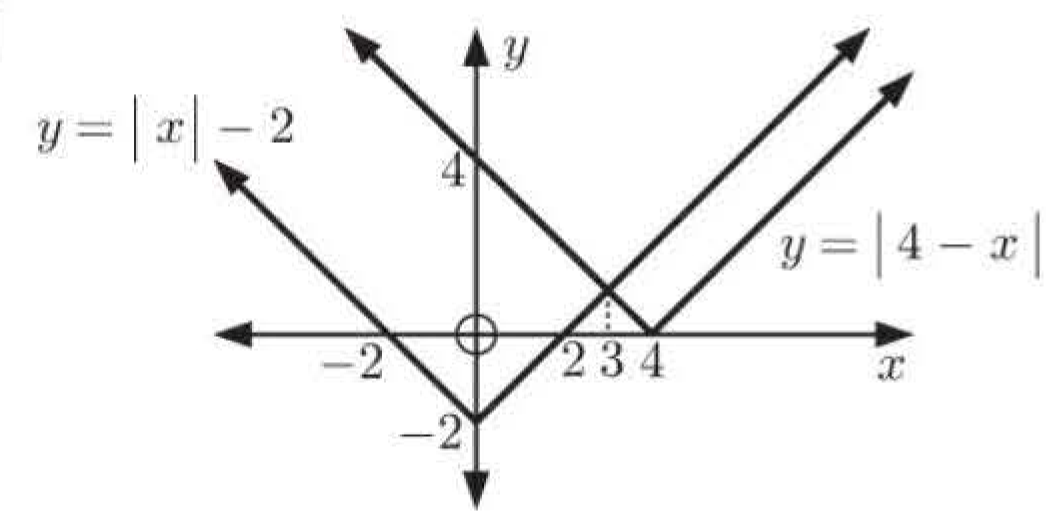


$\left|x^2 - x\right| > 2$ when the modulus graph is above the line.

$\therefore\ x < -1$ or $x > 2$

$\therefore\ x \in\ ]-\infty, -1[$ or $]2, \infty[$

l

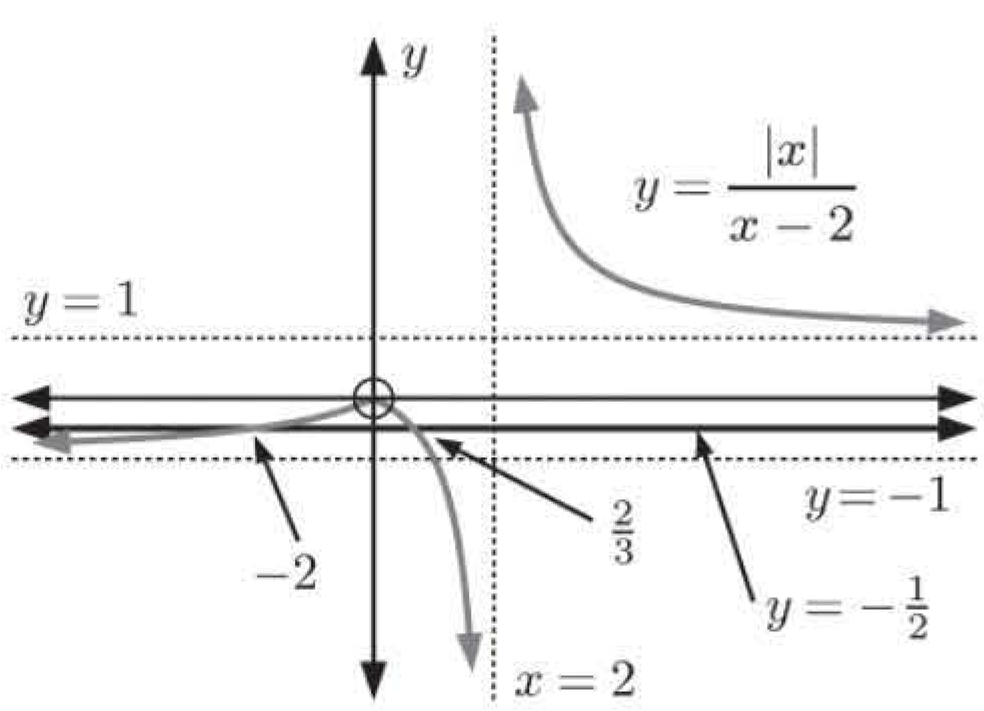


$|x| - 2 \geqslant |4 - x|$ when the graph of $y = |x| - 2$ is above or on the graph of $y = |4 - x|$.

$\therefore\ x \geqslant 3$

$\therefore\ x \in [3, \infty[$

6



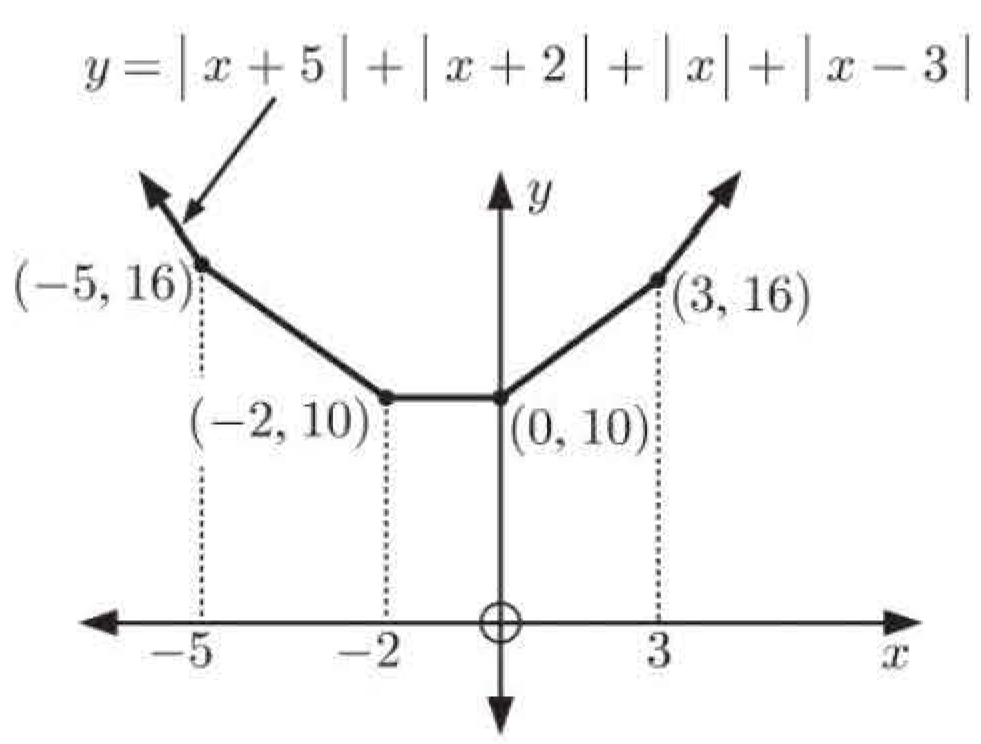
If $\dfrac{|x|}{x-2} \geqslant -\frac{1}{2}$ then the graph of $y = f(x)$ is above or on $y = -\frac{1}{2}$.

They intersect at -2 and $\frac{2}{3}$

$\therefore\ -2 \leqslant x \leqslant \frac{2}{3}$ or $x > 2$

$\therefore\ x \in [-2, \frac{2}{3}]$ or $]2, \infty[$

7 **a**



b **i** If x is a position along (AB) then:

$$\text{XP} = |x - (-5)| = |x + 5|$$
$$\text{XQ} = |x - (-2)| = |x + 2|$$
$$\text{XO} = |x - 0| = |x|$$
$$\text{XR} = |x - 3|$$

The total length is

$$|x + 5| + |x + 2| + |x| + |x - 3|$$

ii The minimum length is 10 km when $-2 \leqslant x \leqslant 0$, so x can be anywhere between O and Q.

iii We need to graph

$$y = |x + 5| + |x + 2| + |x| + |x - 3| + |x - 7|$$

From technology, the minimum cable length is 17 km when $x = 0$, so x is at O.

8 **a**

$$|x + y|^2 = (x + y)^2$$
$$= x^2 + 2xy + y^2$$

and $(|x| + |y|)^2 = |x|^2 + 2\,|x|\,|y| + |y|^2$
$$= x^2 + 2\,|x|\,|y| + y^2$$

Now $xy \leqslant |x|\,|y|$

$\therefore\ x^2 + 2xy + y^2 \leqslant x^2 + 2\,|x|\,|y| + y^2$

$\therefore\ |x + y|^2 \leqslant (|x| + |y|)^2$

$\therefore\ |x + y| \leqslant |x| + |y|$ {both sides $\geqslant 0$}

$\therefore$ the statement is true for all x, y.

b

$$|x - y|^2 = (x - y)^2$$
$$= x^2 - 2xy + y^2$$

and $(|x| - |y|)^2 = \left|x^2\right| - 2\,|x|\,|y| + |y|^2$
$$= x^2 - 2\,|x|\,|y| + y^2$$

Now $xy \leqslant |x|\,|y|$

$\therefore\ -2xy \geqslant -2\,|x|\,|y|$

$\therefore\ x^2 - 2xy + y^2 \geqslant x^2 - 2\,|x|\,|y| + y^2$

$\therefore\ |x - y|^2 \geqslant (|x| - |y|)^2$

$\therefore\ |x - y| \geqslant |x| - |y|$ {both sides $\geqslant 0$}

$\therefore$ the statement is true for all x, y.

EXERCISE 2I

1 **a** **i** $f : x \mapsto \dfrac{3}{x-2}$ is undefined when $x = 2$, so $x = 2$ is a vertical asymptote.

As $|x| \to \infty$, $f(x) \to 0$, so $y = 0$ is a horizontal asymptote.

ii Domain is $\{x \mid x \neq 2\}$, Range is $\{y \mid y \neq 0\}$

iii $f(0) = \dfrac{3}{0-2} = -\frac{3}{2}$

So, the y-intercept is $-\frac{3}{2}$.

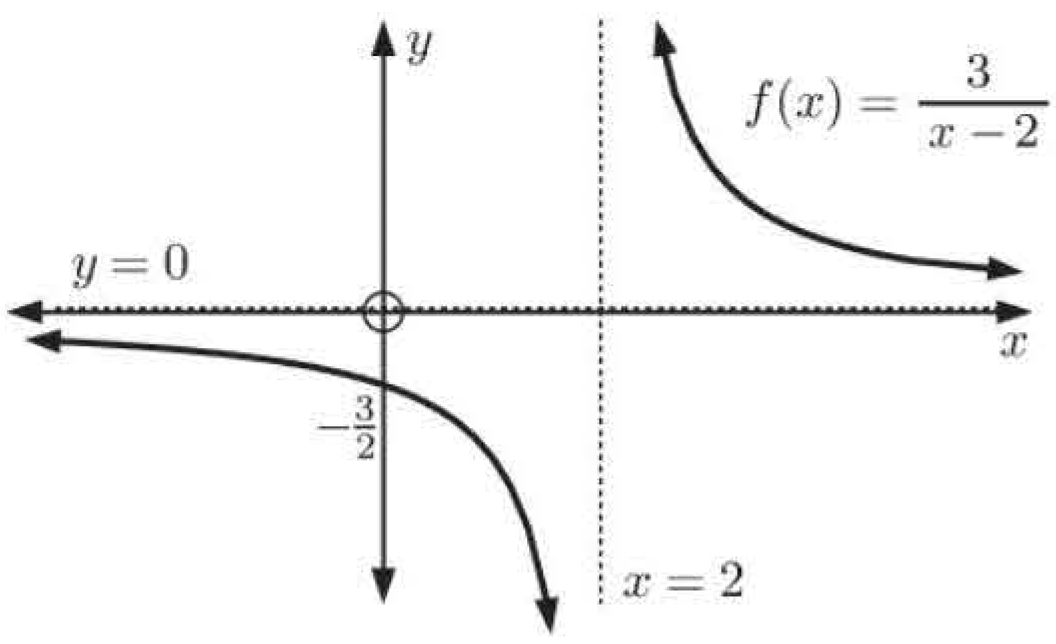
$f(x) = 0$ when $\dfrac{3}{x-2} = 0$,

which has no solutions.

$\therefore$ there is no x-intercept.

iv As $x \to 2^-$, $y \to -\infty$.
As $x \to 2^+$, $y \to \infty$.
As $x \to \infty$, $y \to 0^+$.
As $x \to -\infty$, $y \to 0^-$.

v



b **i** $f(x) = 2 - \dfrac{3}{x+1}$ is undefined when $x = -1$, so $x = -1$ is a vertical asymptote.

As $|x| \to \infty$, $\dfrac{3}{x+1} \to 0$, so $f(x) \to 2$ $\quad\therefore$ $y = 2$ is a horizontal asymptote.

ii Domain is $\{x \mid x \neq -1\}$, Range is $\{y \mid y \neq 2\}$

iii $f(0) = 2 - \dfrac{3}{0+1} = -1$

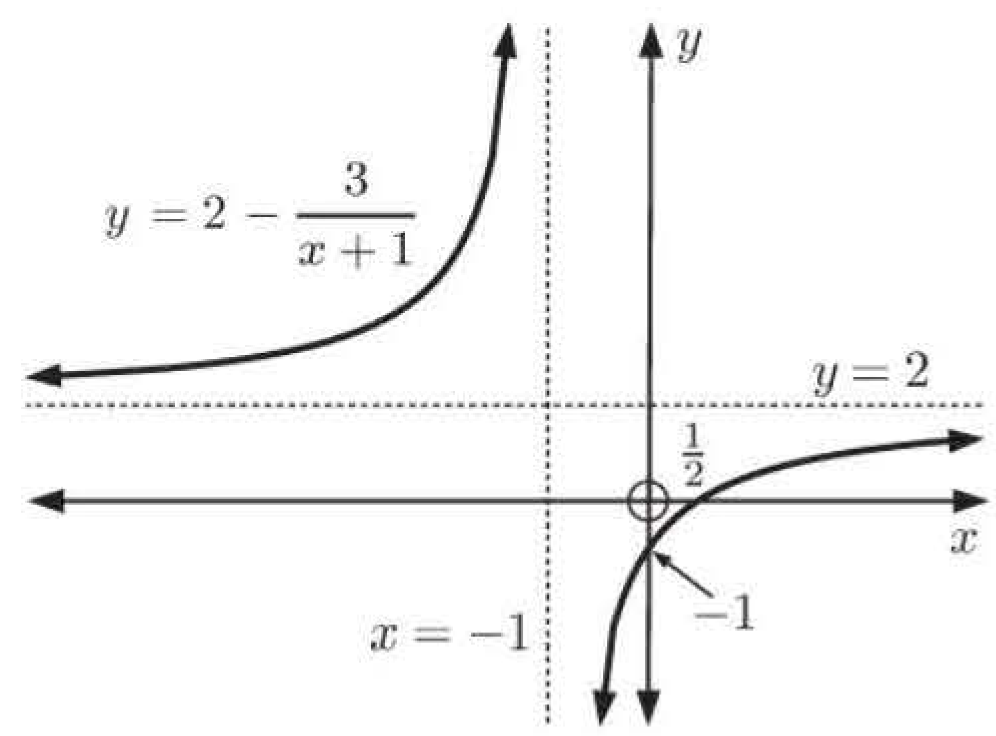
So, the y-intercept is -1.

$$f(x) = 0 \text{ when } 2 - \frac{3}{x+1} = 0$$
$$\therefore \frac{3}{x+1} = 2$$
$$\therefore x + 1 = \tfrac{3}{2}$$
$$\therefore x = \tfrac{1}{2}$$

So, the x-intercept is $\frac{1}{2}$.

iv As $x \to -1^-$, $y \to \infty$.
As $x \to -1^+$, $y \to -\infty$.
As $x \to \infty$, $y \to 2^-$.
As $x \to -\infty$, $y \to 2^+$.

v



c **i** $f : x \mapsto \dfrac{x+3}{x-2}$ is undefined when $x = 2$, so $x = 2$ is a vertical asymptote.

Now $f(x) = \dfrac{x+3}{x-2} = \dfrac{1 + \frac{3}{x}}{1 - \frac{2}{x}}$

$\therefore$ as $|x| \to \infty$, $f(x) \to \frac{1}{1} = 1$, and so $y = 1$ is a horizontal asymptote.

ii Domain is $\{x \mid x \neq 2\}$, Range is $\{y \mid y \neq 1\}$

iii $f(0) = \dfrac{0+3}{0-2} = -\frac{3}{2}$

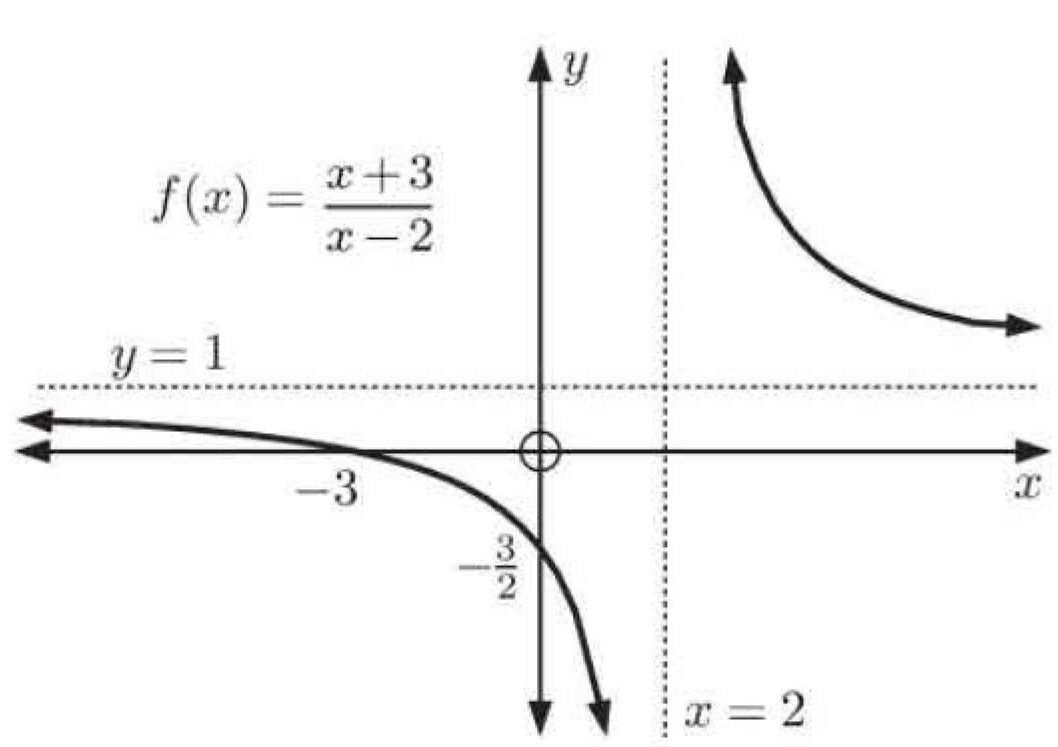
So, the y-intercept is $-\frac{3}{2}$.

$$f(x) = 0 \text{ when } \frac{x+3}{x-2} = 0$$
$$\therefore x + 3 = 0$$
$$\therefore x = -3$$

So, the x-intercept is -3.

iv As $x \to 2^-$, $y \to -\infty$.
As $x \to 2^+$, $y \to \infty$.
As $x \to \infty$, $y \to 1^+$.
As $x \to -\infty$, $y \to 1^-$.

v



d **i** $f(x) = \dfrac{3x-1}{x+2}$ is undefined when $x=-2$, so $x=-2$ is a vertical asymptote.

$f(x) = \dfrac{3x-1}{x+2} = \dfrac{3-\frac{1}{x}}{1+\frac{2}{x}}$

As $|x| \to \infty$, $f(x) \to \frac{3}{1} = 3$ and so $y = 3$ is a horizontal asymptote.

ii Domain is $\{x \mid x \neq -2\}$, Range is $\{y \mid y \neq 3\}$

iii $f(0) = \dfrac{3(0)-1}{0+2} = -\frac{1}{2}$

So, the y-intercept is $-\frac{1}{2}$.

$f(x) = 0$ when $\dfrac{3x-1}{x+2} = 0$

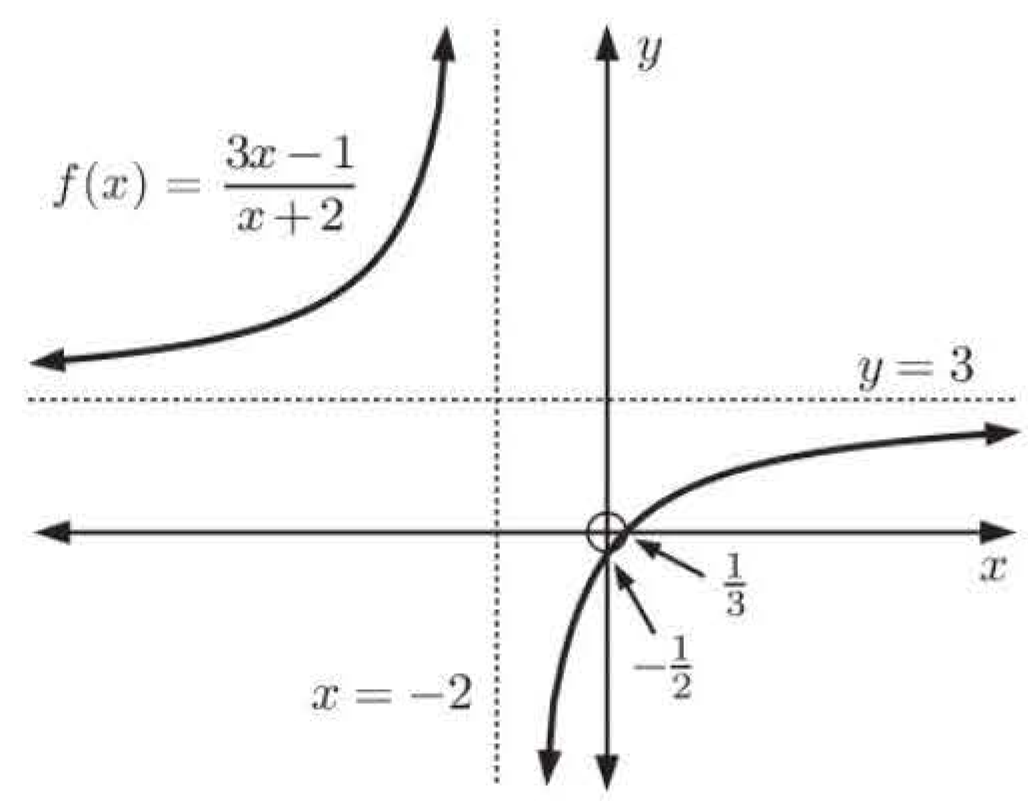
$\therefore \quad 3x - 1 = 0$

$\therefore \quad x = \frac{1}{3}$

So, the x-intercept is $\frac{1}{3}$.

iv As $x \to -2^-$, $y \to \infty$.
As $x \to -2^+$, $y \to -\infty$.
As $x \to \infty$, $y \to 3^-$.
As $x \to -\infty$, $y \to 3^+$.

v



2 **a** The function is defined when $cx + d \neq 0$, or when $x \neq -\dfrac{d}{c}$.

So, the domain is $\{x \mid x \neq -\dfrac{d}{c}\}$.

b The equation of the vertical asymptote is $x = -\dfrac{d}{c}$.

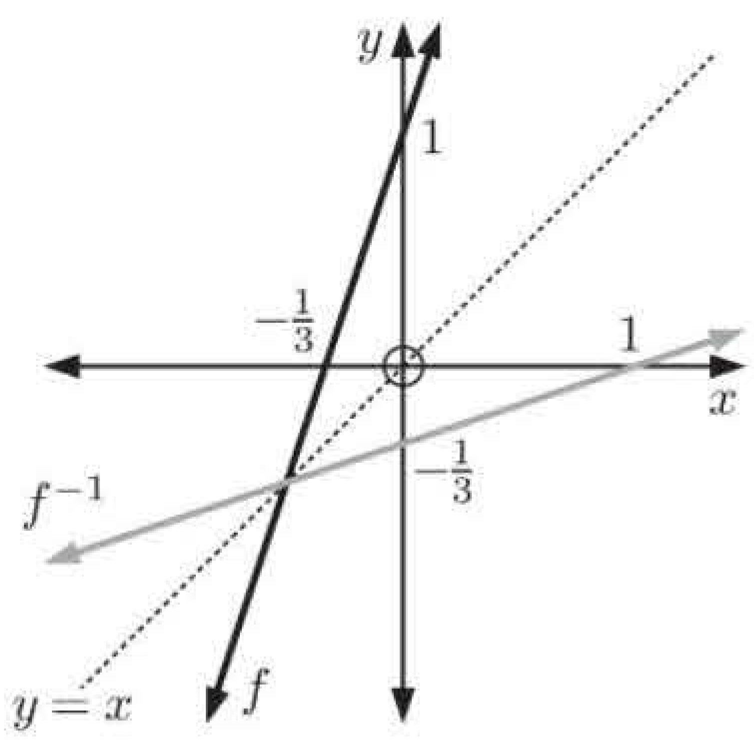
c To find the horizontal asymptote of $y = \dfrac{ax+b}{cx+d}$, we consider the function's behavior as $|x| \to \infty$.

Now for $c \neq 0$, $$\frac{ax+b}{cx+d} = \frac{acx+bc}{c(cx+d)} = \frac{a(cx+d)+bc-ad}{c(cx+d)} = \frac{a}{c} + \frac{bc-ad}{c(cx+d)} = \frac{a}{c} + \frac{b-\frac{ad}{c}}{cx+d}$$

$\therefore$ as $|x| \to \infty$, $y \to \dfrac{a}{c}$, and so $y = \dfrac{a}{c}$ is a horizontal asymptote.

EXERCISE 2J

1 **a** **i**



ii $f(x)$ passes through $(0, 1)$ and $(-\frac{1}{3}, 0)$

$\therefore$ $f^{-1}(x)$ passes through $(1, 0)$ and $(0, -\frac{1}{3})$

$f^{-1}(x)$ has gradient $\dfrac{-\frac{1}{3}-0}{0-1} = \dfrac{-\frac{1}{3}}{-1} = \frac{1}{3}$

So, its equation is $\dfrac{y-0}{x-1} = \frac{1}{3}$

which is $y = \dfrac{x-1}{3}$

So, $f^{-1}(x) = \dfrac{x-1}{3}$

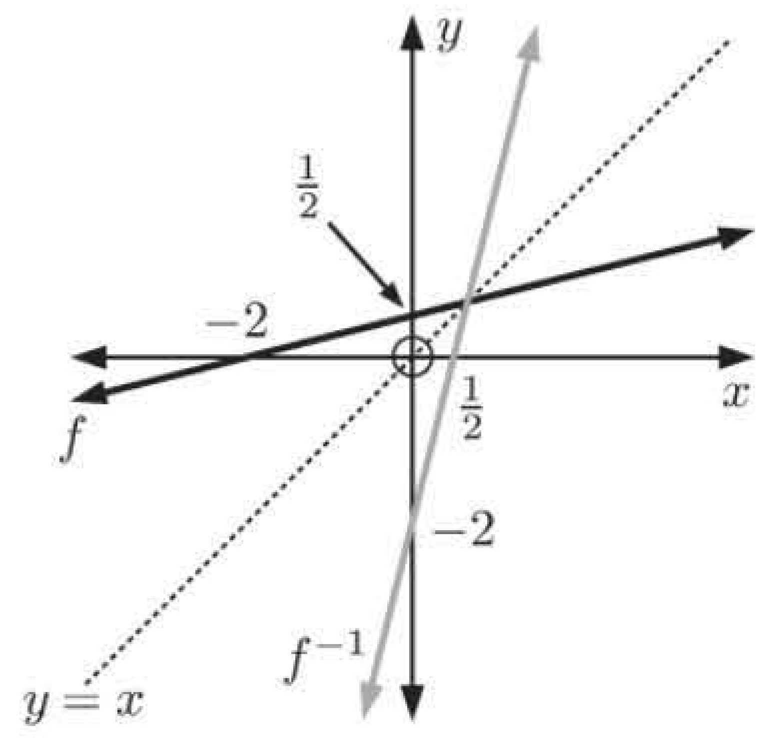
iii f is $y = 3x+1$

so f^{-1} is $x = 3y + 1$

$\therefore \quad x - 1 = 3y$

$\therefore \quad y = \dfrac{x-1}{3}$. So, $f^{-1}(x) = \dfrac{x-1}{3}$

b **i**



ii $f(x)$ passes through $(0,\ \frac{1}{2})$ and $(-2,\ 0)$

$\therefore$ $f^{-1}(x)$ passes through $(\frac{1}{2},\ 0)$ and $(0,\ -2)$

$f^{-1}(x)$ has gradient $\dfrac{-2-0}{0-\frac{1}{2}} = \dfrac{-2}{-\frac{1}{2}} = 4$

So, its equation is $\dfrac{y-0}{x-\frac{1}{2}} = 4$

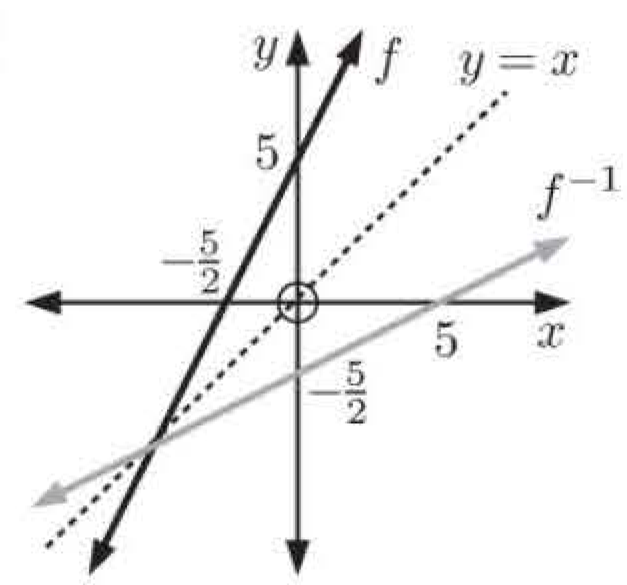
which is $y = 4x - 2$

So, $f^{-1}(x) = 4x - 2$

iii
$$f \text{ is } y = \frac{x+2}{4}$$
$$\text{so } f^{-1} \text{ is } x = \frac{y+2}{4}$$
$$\therefore\ 4x = y + 2$$
$$\therefore\ y = 4x - 2. \qquad \text{So, } f^{-1}(x) = 4x - 2$$

2 **a** **i**
$$f \text{ is } y = 2x + 5$$
$$\text{so } f^{-1} \text{ is } x = 2y + 5$$
$$\therefore\ x - 5 = 2y$$
$$\therefore\ y = \frac{x-5}{2}$$
$$\text{So, } f^{-1}(x) = \frac{x-5}{2}$$

ii



$f(x)$ passes through $(0,\ 5)$ and $(-\frac{5}{2},\ 0)$

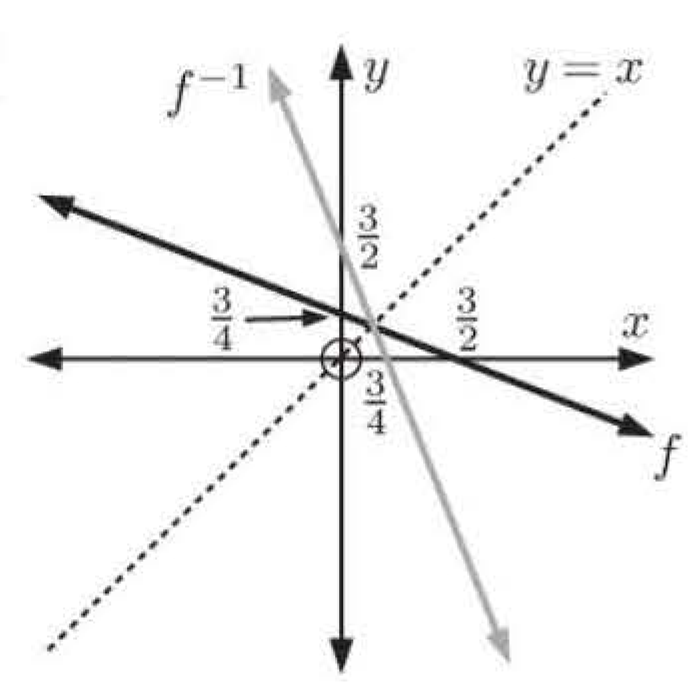
$\therefore$ $f^{-1}(x)$ passes through $(5,\ 0)$ and $(0,\ -\frac{5}{2})$.

iii
$$(f^{-1} \circ f)(x) = f^{-1}(2x+5) = \frac{2x+5-5}{2} = \frac{2x}{2} = x$$

and
$$(f \circ f^{-1})(x) = f(f^{-1}(x)) = f\left(\frac{x-5}{2}\right) = 2\left(\frac{x-5}{2}\right) + 5 = x - 5 + 5 = x$$

b **i**
$$f \text{ is } y = \frac{3-2x}{4}$$
$$\text{so } f^{-1} \text{ is } x = \frac{3-2y}{4}$$
$$\therefore\ 4x = 3 - 2y$$
$$\therefore\ 4x - 3 = -2y$$
$$\therefore\ y = -2x + \tfrac{3}{2}$$
$$\text{So, } f^{-1}(x) = -2x + \tfrac{3}{2}$$

ii



$f(x)$ passes through $(0,\ \frac{3}{4})$ and $(\frac{3}{2},\ 0)$

$\therefore$ $f^{-1}(x)$ passes through $(\frac{3}{4},\ 0)$ and $(0,\ \frac{3}{2})$.

iii
$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}\left(\frac{3-2x}{4}\right) = -2\left(\frac{3-2x}{4}\right) + \tfrac{3}{2} = \frac{3-2x}{-2} + \tfrac{3}{2} = -\tfrac{3}{2} + x + \tfrac{3}{2} = x$$

and
$$(f \circ f^{-1})(x) = f(f^{-1}(x)) = f\left(-2x + \tfrac{3}{2}\right) = \frac{3 - 2(-2x + \frac{3}{2})}{4} = \frac{3 + 4x - 3}{4} = \frac{4x}{4} = x$$

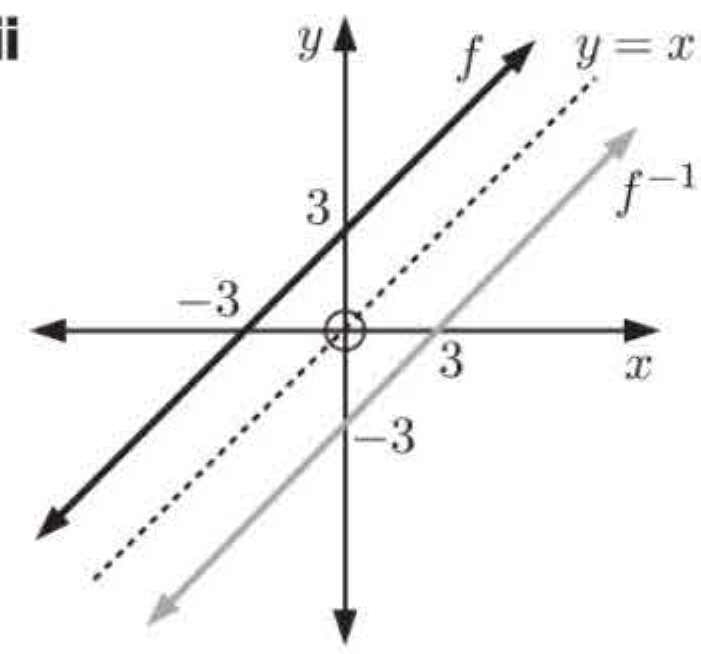
c **i** f is $y = x + 3$

so f^{-1} is $x = y + 3$

$\therefore \quad y = x - 3$

So, $f^{-1}(x) = x - 3$

ii



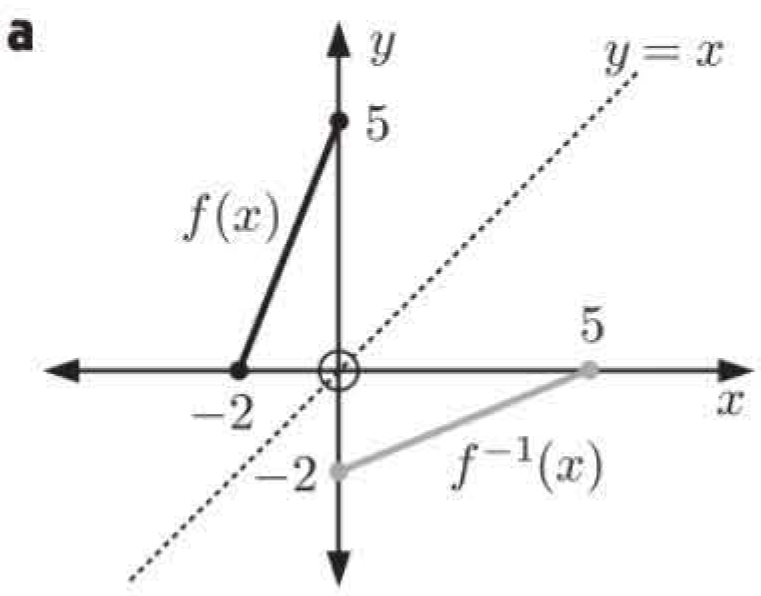
$f(x)$ passes through $(0, 3)$ and $(-3, 0)$

$\therefore$ $f^{-1}(x)$ passes through $(3, 0)$ and $(0, -3)$.

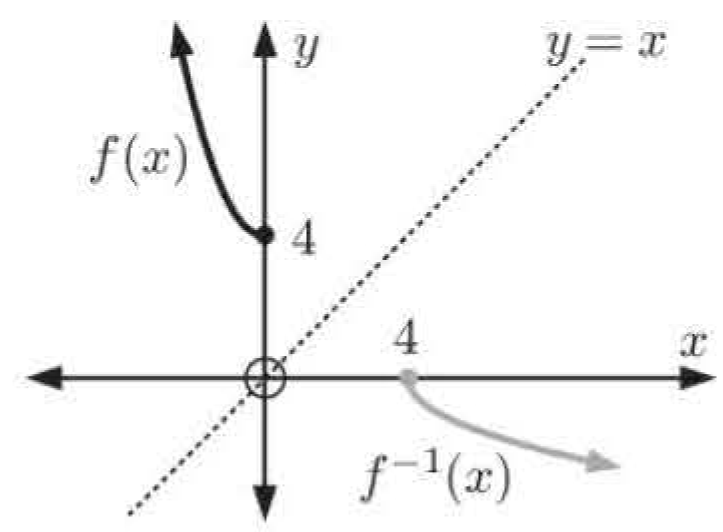
iii $(f^{-1} \circ f)(x) = f^{-1}(f(x))$
$= f^{-1}(x+3)$
$= (x+3) - 3$
$= x$

and $(f \circ f^{-1})(x) = f(f^{-1}(x))$
$= f(x-3)$
$= (x-3) + 3$
$= x$

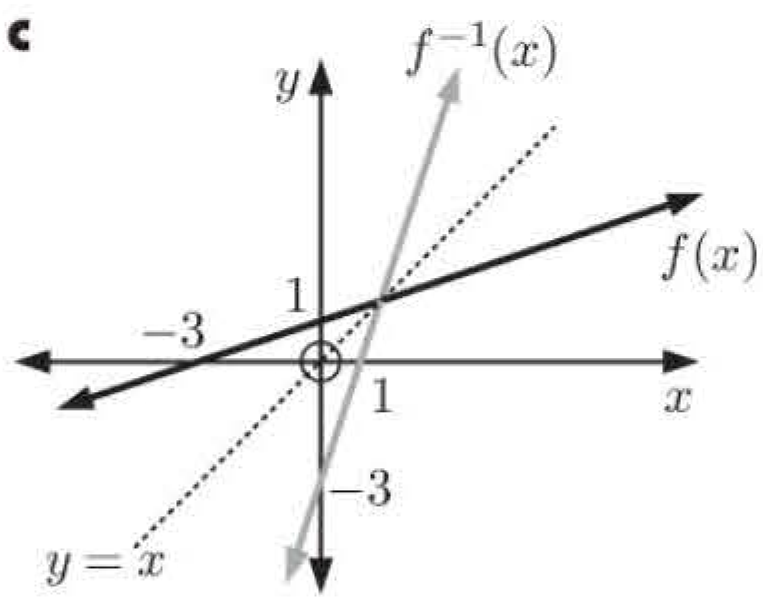
3 **a**



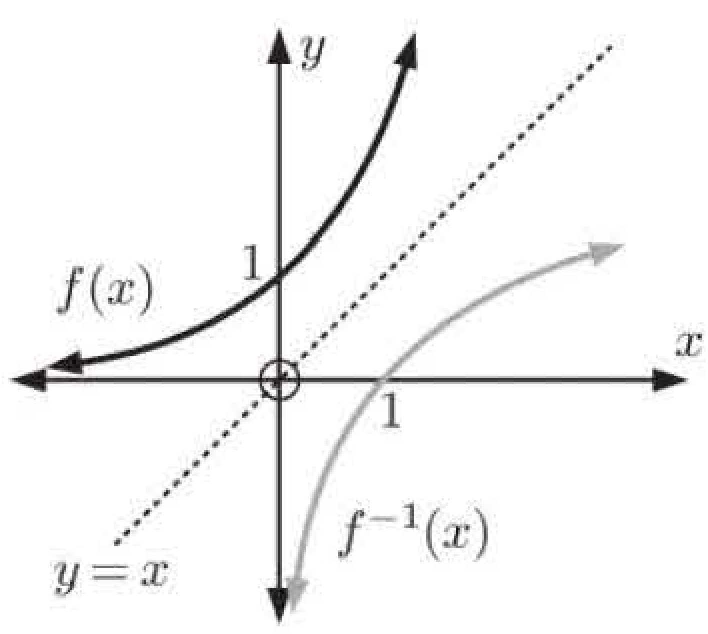
b



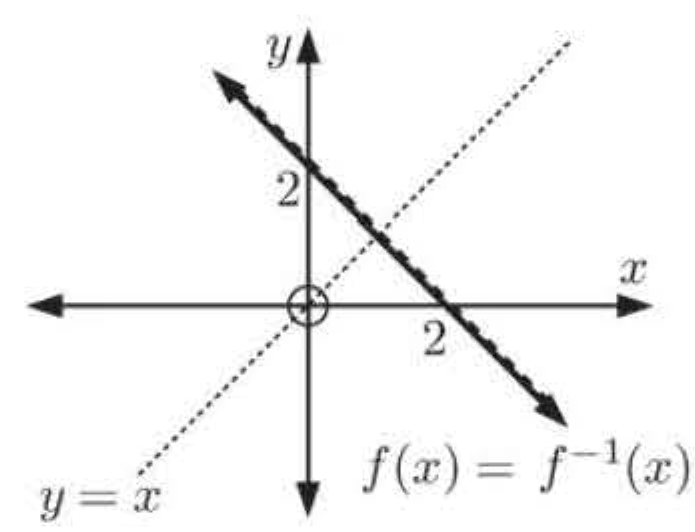
c



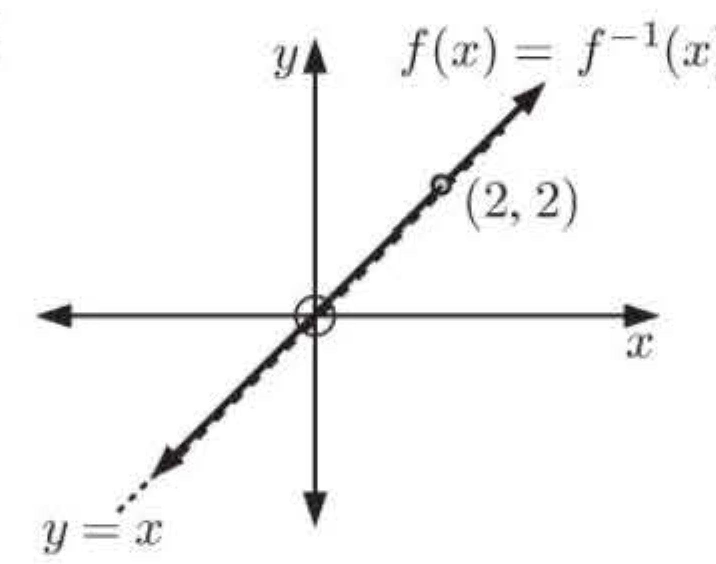
d



e



f

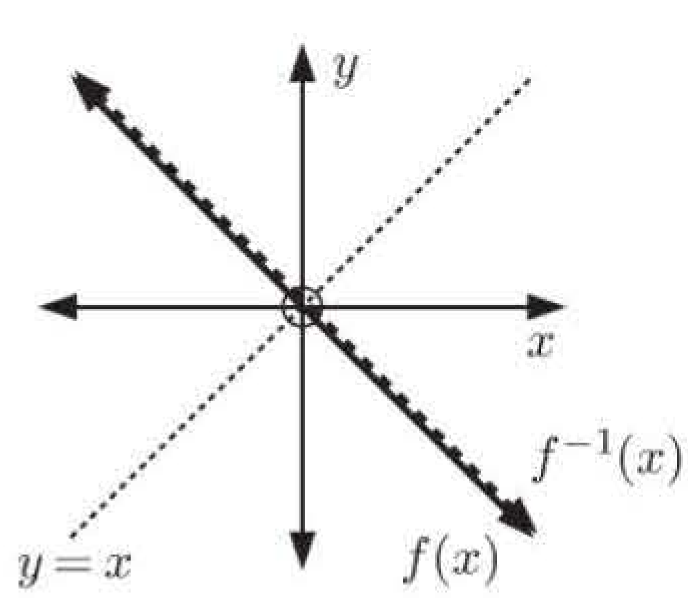


4 **a** Domain of $f(x)$ is $\{x \mid -2 \leqslant x \leqslant 0\}$ **b** Range of $f(x)$ is $\{y \mid 0 \leqslant y \leqslant 5\}$

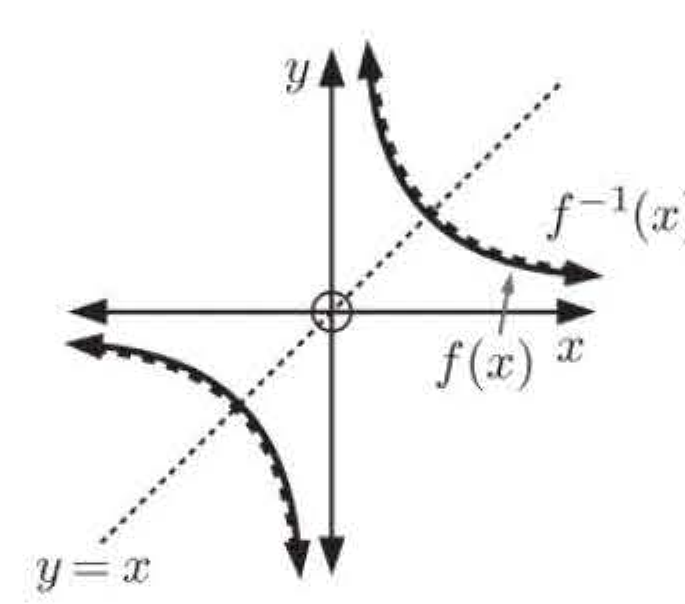
c Domain of $f^{-1}(x)$ is $\{x \mid 0 \leqslant x \leqslant 5\}$ **d** Range of $f^{-1}(x)$ is $\{y \mid -2 \leqslant y \leqslant 0\}$

5 **a** The functions in **3 e** and **3 f** are self-inverse functions.

b Any linear function of the form $y = a - x$ will be a self-inverse function, for example $y = -x$ (where $a = 0$):



c Any rational function of the form $y = \dfrac{a}{x}$ will be a self-inverse function, for example $y = \dfrac{2}{x}$ (where $a = 2$):



6 f is $y = 2x - 5$

$\therefore$ the inverse function is $x = 2y - 5$

$\therefore\ 2y = x + 5$

$\therefore\ y = \dfrac{x+5}{2}$

$\therefore\ f^{-1}(x) = \dfrac{x+5}{2}$

To find $(f^{-1})^{-1}(x)$, we need to find the inverse function for $y = \dfrac{x+5}{2}$

This is $x = \dfrac{y+5}{2}$

$\therefore\ 2x = y + 5$

$\therefore\ y = 2x - 5$

This is the original function $f(x)$.

So, $(f^{-1})^{-1}(x) = f(x)$.

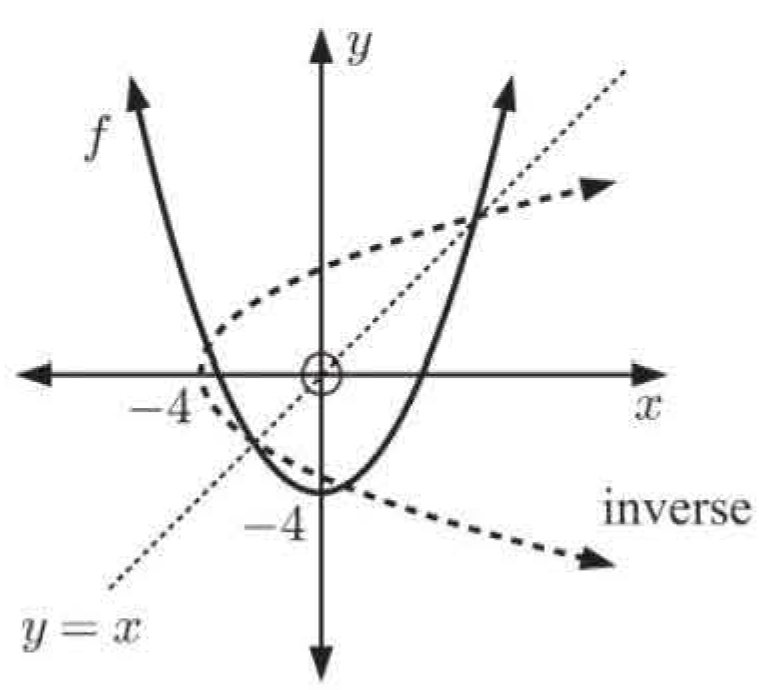
7 **a** For $\{(1, 2), (2, 4), (3, 5)\}$, there is at most one x-value corresponding to each y-value, so the function has an inverse. The inverse function is $\{(2, 1), (4, 2), (5, 3)\}$.

b For $\{(-1, 3), (0, 2), (1, 3)\}$, there are two x-values corresponding to the y-value of 3. So, the function does not have an inverse.

c For $\{(2, 1), (-1, 0), (0, 2), (1, 3)\}$, there is at most one x-value corresponding to each y-value, so the function has an inverse. The inverse function is $\{(0, -1), (1, 2), (2, 0), (3, 1)\}$.

d For $\{(-1, -1), (0, 0), (1, 1)\}$, there is at most one x-value corresponding to each y-value, so the function has an inverse. The inverse function is $\{(-1, -1), (0, 0), (1, 1)\}$.

8 **a**

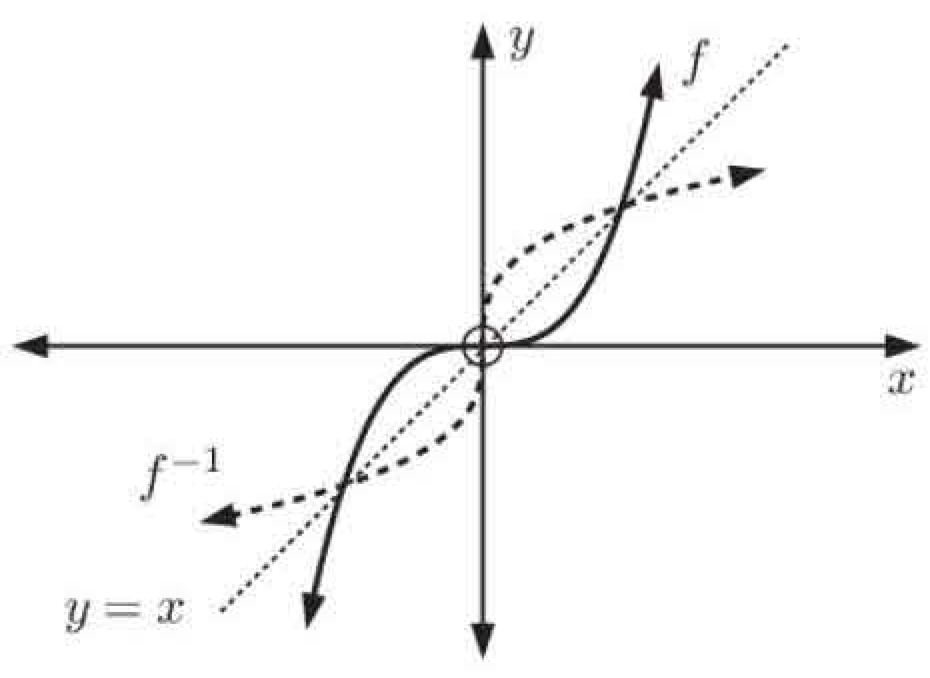


b Using the 'horizontal line test', f does not have an inverse function as a horizontal line through $y = x^2 - 4$ cuts it more than once.

c For $x \geqslant 0$, any horizontal line cuts it only once.

$\therefore$ f does have an inverse function for $x \geqslant 0$; $f^{-1}(x) = \sqrt{x+4}$.

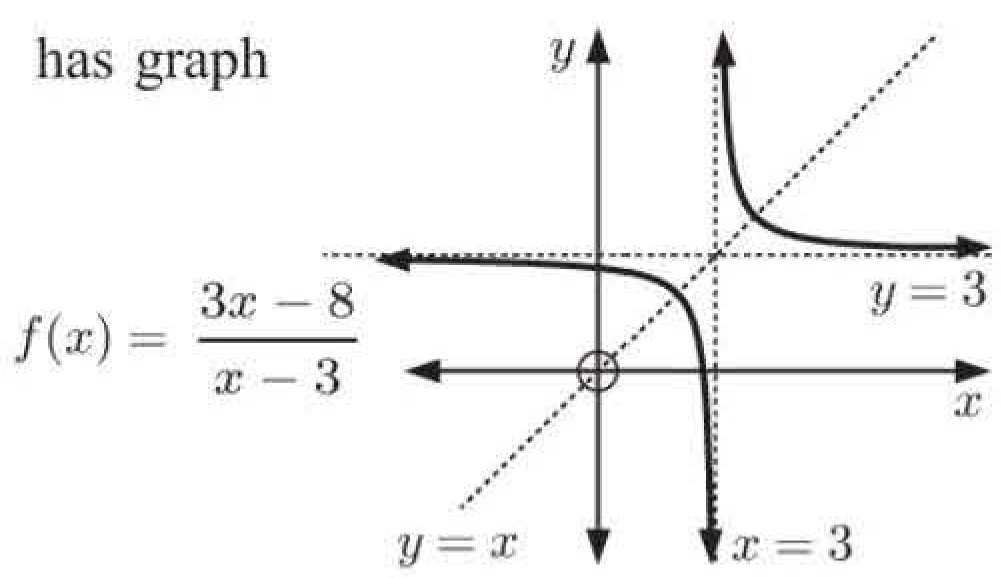
9



10 $f(x) = \dfrac{1}{x}$ has inverse function $x = \dfrac{1}{y}$ or $y = \dfrac{1}{x}$

So, $f^{-1}(x) = \dfrac{1}{x}$, which means $f(x)$ is a self-inverse function.

11 **a** $f(x) = \dfrac{3x-8}{x-3}$ has graph



The vertical line test shows it to be a function.

Symmetry about $y = x$ shows it is a self-inverse function.

b $f(x) = \dfrac{3x-8}{x-3}$ has inverse function $x = \dfrac{3y-8}{y-3}$

$$\therefore \quad x(y-3) = 3y-8$$
$$\therefore \quad xy - 3x = 3y - 8$$
$$\therefore \quad xy - 3y = 3x - 8$$
$$\therefore \quad y(x-3) = 3x-8$$
$$\therefore \quad y = \frac{3x-8}{x-3}$$
$$\therefore \quad f^{-1}(x) = \frac{3x-8}{x-3}$$

So, $f(x) = f^{-1}(x)$, which means $f(x)$ is a self-inverse function.

12 **a** If $y = f(x)$ has an inverse function, then the inverse function must also be a function. Thus, it must satisfy the 'vertical line test', i.e., no vertical line can cut it more than once. This condition for the inverse function cannot be satisfied if the original function does not satisfy the 'horizontal line test'. Thus, the 'horizontal line test' is a valid test for the existence of an inverse function.

b **i** This graph satisfies the 'horizontal line test' and therefore has an inverse function.

ii, iii These graphs both fail the 'horizontal line test' so neither of these have inverse functions.

c **ii** Domain $\{x \mid x \geqslant 1\}$ or $\{x \mid x \leqslant 1\}$ **iii** Domain $\{x \mid x \geqslant 1\}$ or $\{x \mid x \leqslant -2\}$

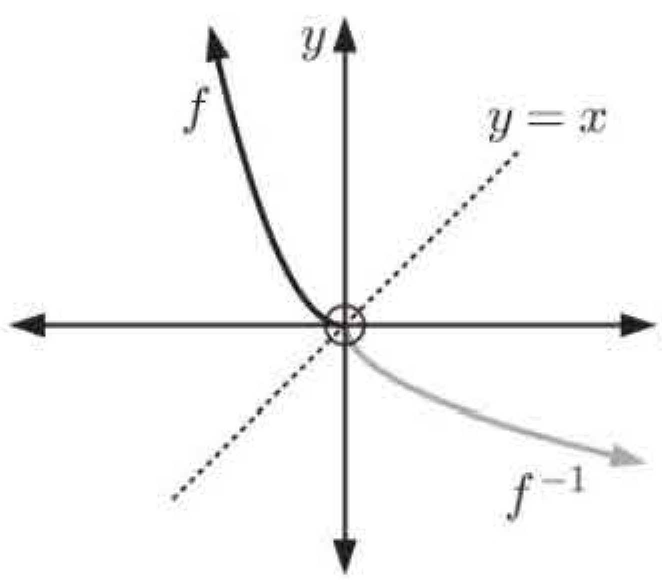
13 **a** f is $y = x^2, \; x \leqslant 0$

so f^{-1} is $x = y^2, \; y \leqslant 0$

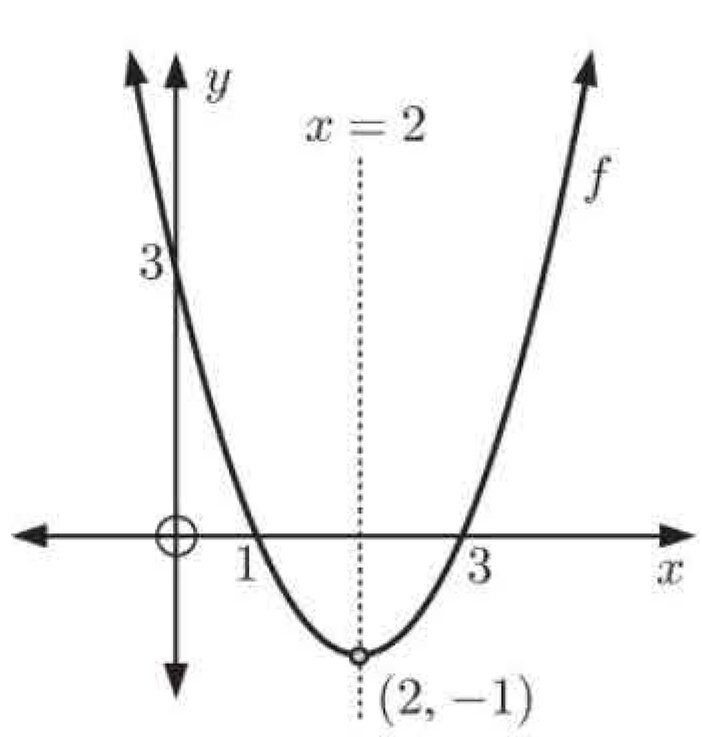
$\therefore \quad y = -\sqrt{x}$

So, $f^{-1}(x) = -\sqrt{x}$

b



14 **a**



$f : x \mapsto x^2 - 4x + 3$ satisfies the 'vertical line test' so is therefore a function. It does not however satisfy the horizontal line test as any horizontal line above the vertex cuts the graph twice. Therefore it does not have an inverse function.

b **i** For $g(x) = x^2 - 4x + 3$ where $x \geqslant 2$, all horizontal lines cut the graph no more than once. Therefore $g(x)$ has an inverse function for $x \geqslant 2$.

ii g is $y = x^2 - 4x + 3, \quad x \geqslant 2$

so g^{-1} is $x = y^2 - 4y + 3, \quad y \geqslant 2$

$\therefore \quad x = (y-2)^2 - 4 + 3$

$\therefore \quad x = (y-2)^2 - 1$

$\therefore \quad x + 1 = (y-2)^2$

$\therefore \quad y - 2 = \sqrt{x+1}, \quad y \geqslant 2, \; x \geqslant -1$

$\therefore \quad y = 2 + \sqrt{1+x}, \quad y \geqslant 2, \; x \geqslant -1$

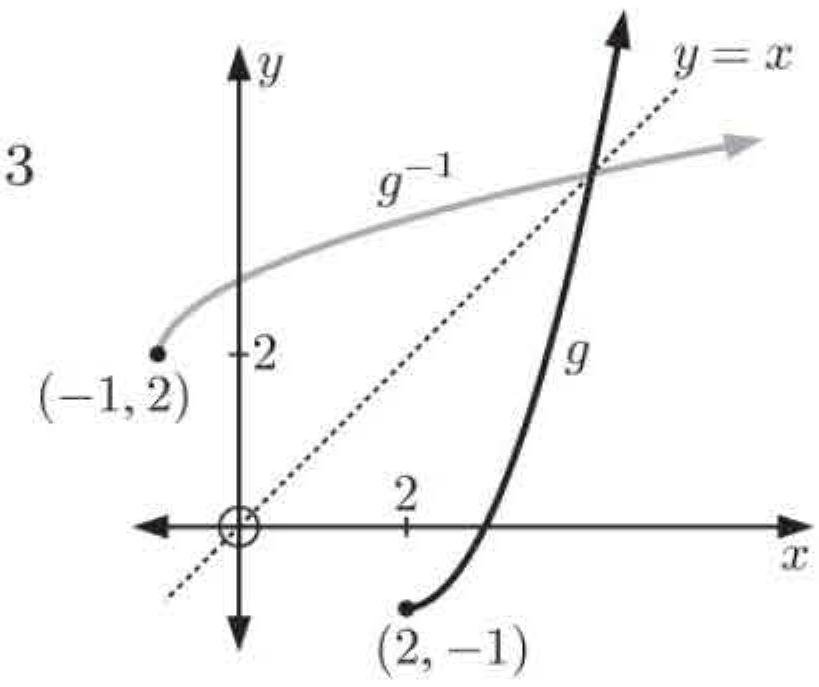
So, $g^{-1}(x) = 2 + \sqrt{1+x}$ as required.

iii **A** Domain of g is $\{x \mid x \geqslant 2\}$.
Range is $\{y \mid y \geqslant -1\}$

B Domain of g^{-1} is $\{x \mid x \geqslant -1\}$.
Range is $\{y \mid y \geqslant 2\}$

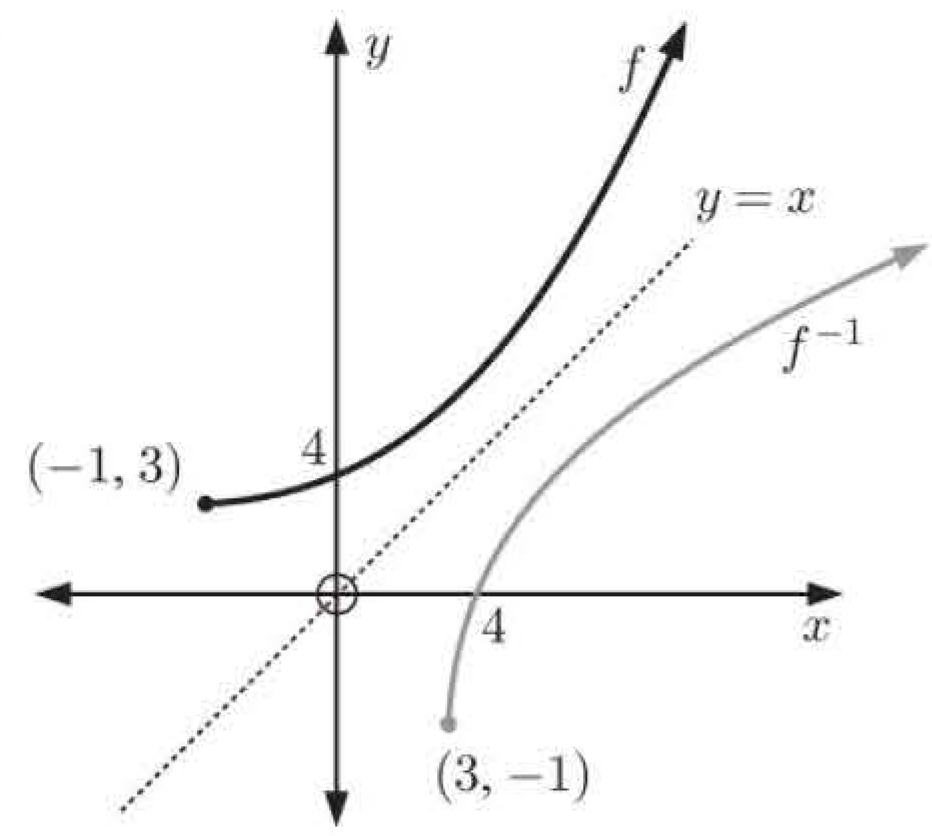
iv $(g \circ g^{-1})(x) = g(g^{-1}(x))$
$= \left(2+\sqrt{1+x}\right)^2 - 4\left(2+\sqrt{1+x}\right)+3$
$= 4+4\sqrt{1+x}+1+x-8-4\sqrt{1+x}+3$
$= x$

$(g^{-1} \circ g)(x) = g^{-1}(g(x))$
$= 2+\sqrt{1+x^2-4x+3}$
$= 2+\sqrt{(x-2)^2}$
$= 2+x-2$
$= x$



15 **a** f is $y=(x+1)^2+3, \quad x \geqslant -1$
so f^{-1} is $x=(y+1)^2+3, \quad y \geqslant -1$
$\therefore \quad x-3=(y+1)^2$
$\therefore \quad y+1=\sqrt{x-3}, \quad y \geqslant -1, \ x \geqslant 3$
$\therefore \quad y=\sqrt{x-3}-1, \quad y \geqslant -1, \ x \geqslant 3$

b



c **i** Domain $\{x \mid x \geqslant -1\}$. Range $\{y \mid y \geqslant 3\}$.
ii Domain $\{x \mid x \geqslant 3\}$. Range $\{y \mid y \geqslant -1\}$.

16 **a** g is $y=\dfrac{8-x}{2}$
so g^{-1} is $x=\dfrac{8-y}{2}$
$\therefore \quad 2x=8-y$
$\therefore \quad y=8-2x$
So, $g^{-1}(x)=8-2x$
$\therefore \quad g^{-1}(-1)=8-2(-1)=10$

b f is $y=2x+5$
so f^{-1} is $x=2y+5$
$\therefore \quad 2y=x-5$
$\therefore \quad y=\dfrac{x-5}{2}$
So, $f^{-1}(x)=\dfrac{x-5}{2}$
$\therefore \quad f^{-1}(-3)=\dfrac{-3-5}{2} = \dfrac{-8}{2} = -4$ and $g^{-1}(6)=8-2\times 6 = 8-12 = -4$
$\therefore \quad f^{-1}(-3)-g^{-1}(6) = -4-(-4)$
$= -4+4$
$= 0$ as required

c $(f \circ g^{-1})(x)=9$
$\therefore \quad f(g^{-1}(x))=9$
$\therefore \quad f(8-2x)=9$
$\therefore \quad 2(8-2x)+5=9$
$\therefore \quad 16-4x+5=9$
$\therefore \quad -4x=-12$
$\therefore \quad x=3$

17 **a** **i** f is $y=5^x$
so $f(2)=5^2$
$=25$

ii g is $y=\sqrt{x}$ where $y \geqslant 0$
so g^{-1} is $x=\sqrt{y}$ where $x \geqslant 0$
$\therefore \quad y=x^2$
$\therefore \quad g^{-1}(x)=x^2, \ x \geqslant 0$
$\therefore \quad g^{-1}(4)=4^2$
$=16$

b $(g^{-1} \circ f)(x)=25$
$\therefore \quad g^{-1}(f(x))=25$
$\therefore \quad g^{-1}(5^x)=25$
$\therefore \quad (5^x)^2=25 \quad \{$as $g^{-1}(x)=x^2, \ x \geqslant 0\}$
$\therefore \quad 5^{2x}=5^2$
$\therefore \quad 2x=2$
$\therefore \quad x=1$

18 f is $y = 2x$
so f^{-1} is $x = 2y$
$\therefore\ y = \dfrac{x}{2}$
$\therefore\ f^{-1}(x) = \dfrac{x}{2}$

g is $y = 4x - 3$
so g^{-1} is $x = 4y - 3$
$\therefore\ 4y = x + 3$
$\therefore\ y = \dfrac{x+3}{4}$
$\therefore\ g^{-1}(x) = \dfrac{x+3}{4}$

$$\begin{aligned}(g \circ f)(x) &= g(f(x))\\ &= g(2x)\\ &= 4(2x) - 3\\ \therefore\ (g \circ f)(x) &= 8x - 3\end{aligned}$$
$\therefore\ g \circ f$ is $y = 8x - 3$
so $(g \circ f)^{-1}$ is $x = 8y - 3$
$\therefore\ y = \dfrac{x+3}{8}$
So, $(g \circ f)^{-1}(x) = \dfrac{x+3}{8}$

$$\begin{aligned}\text{Now}\quad (f^{-1} \circ g^{-1})(x) &= f^{-1}(g^{-1}(x))\\ &= f^{-1}\left(\frac{x+3}{4}\right)\\ &= \frac{\left(\frac{x+3}{4}\right)}{2}\\ \therefore\ (f^{-1} \circ g^{-1})(x) &= \frac{x+3}{8} = (g \circ f)^{-1}(x) \quad \text{as required}\end{aligned}$$

19 **a** f is $y = 2x$
so f^{-1} is $x = 2y$
$\therefore\ y = \dfrac{x}{2}$
so $f^{-1}(x) = \dfrac{x}{2} \neq 2x$
So, $f^{-1}(x) \neq f(x)$

b f is $y = x$
so f^{-1} is $x = y$
$\therefore\ y = x$
so $f^{-1}(x) = x$
So, $f^{-1}(x) = f(x)$

c f is $y = -x$
so f^{-1} is $x = -y$
$\therefore\ y = -x$
so $f^{-1}(x) = -x$
So, $f^{-1}(x) = f(x)$

d f is $y = \dfrac{2}{x}$
so f^{-1} is $x = \dfrac{2}{y}$
$\therefore\ y = \dfrac{2}{x}$
so $f^{-1}(x) = \dfrac{2}{x}$
So, $f^{-1}(x) = f(x)$

e f is $y = -\dfrac{6}{x}$
so f^{-1} is $x = -\dfrac{6}{y}$
$\therefore\ y = -\dfrac{6}{x}$
so $f^{-1}(x) = -\dfrac{6}{x}$
So, $f^{-1}(x) = f(x)$

So, $f^{-1}(x) = f(x)$ is true for parts **b**, **c**, **d**, and **e**.

20 **a** $f(x)$ passes through $\text{A}(x, f(x))$, so $f^{-1}(x)$ passes through $\text{B}(f(x), x)$.

b Substituting the coordinates of $\text{B}(f(x), x)$ into $y = f^{-1}(x)$ gives $f^{-1}(f(x)) = x$.

c B has coordinates $(x, f^{-1}(x))$ since it lies on $y = f^{-1}(x)$,
so A has coordinates $(f^{-1}(x), x)$ as $f(x)$ is the inverse of $f^{-1}(x)$.
Substituting the coordinates of $\text{A}(f^{-1}(x), x)$ into $y = f(x)$ gives $x = f(f^{-1}(x))$.
$\therefore\ f(f^{-1}(x)) = x$ as required.

EXERCISE 2K

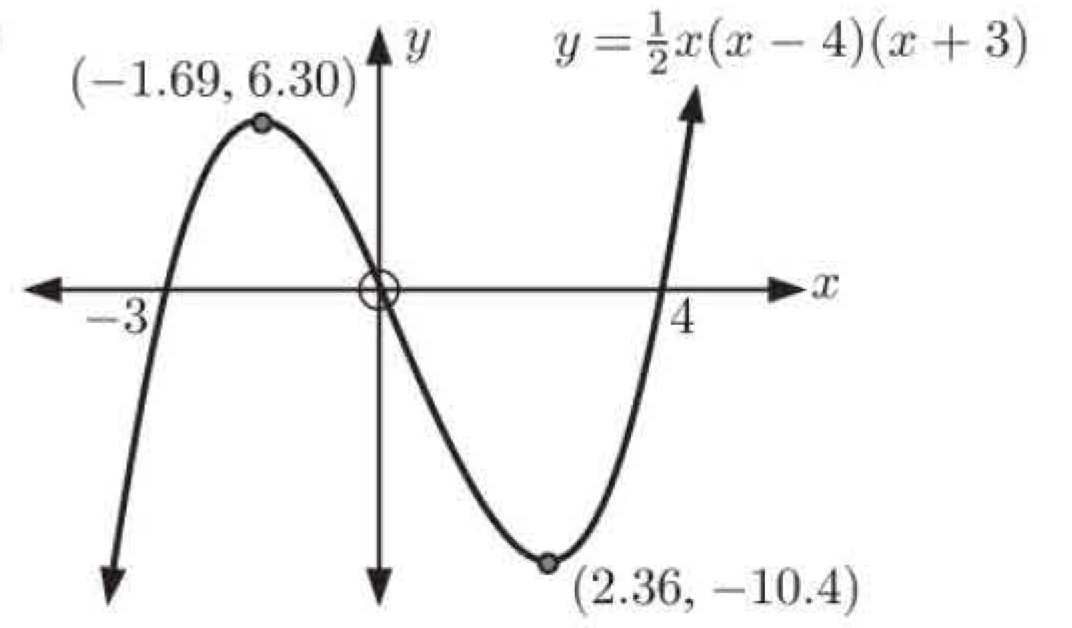
1 **a** **i** x-intercepts are -3, 0, and 4, y-intercept is 0 {using technology}

ii $(-1.69, 6.30)$ is a local maximum, $(2.36, -10.4)$ is a local minimum {using technology}

iii $y = \frac{1}{2}x(x-4)(x+3)$ is defined for all $x \in \mathbb{R}$.
$\therefore$ there are no vertical asymptotes.
As $x \to \infty$, $y \to \infty$
As $x \to -\infty$, $y \to -\infty$
$\therefore$ there are no horizontal asymptotes.

iv Domain is $\{x \mid x \in \mathbb{R}\}$.
Range is $\{y \mid y \in \mathbb{R}\}$.

v



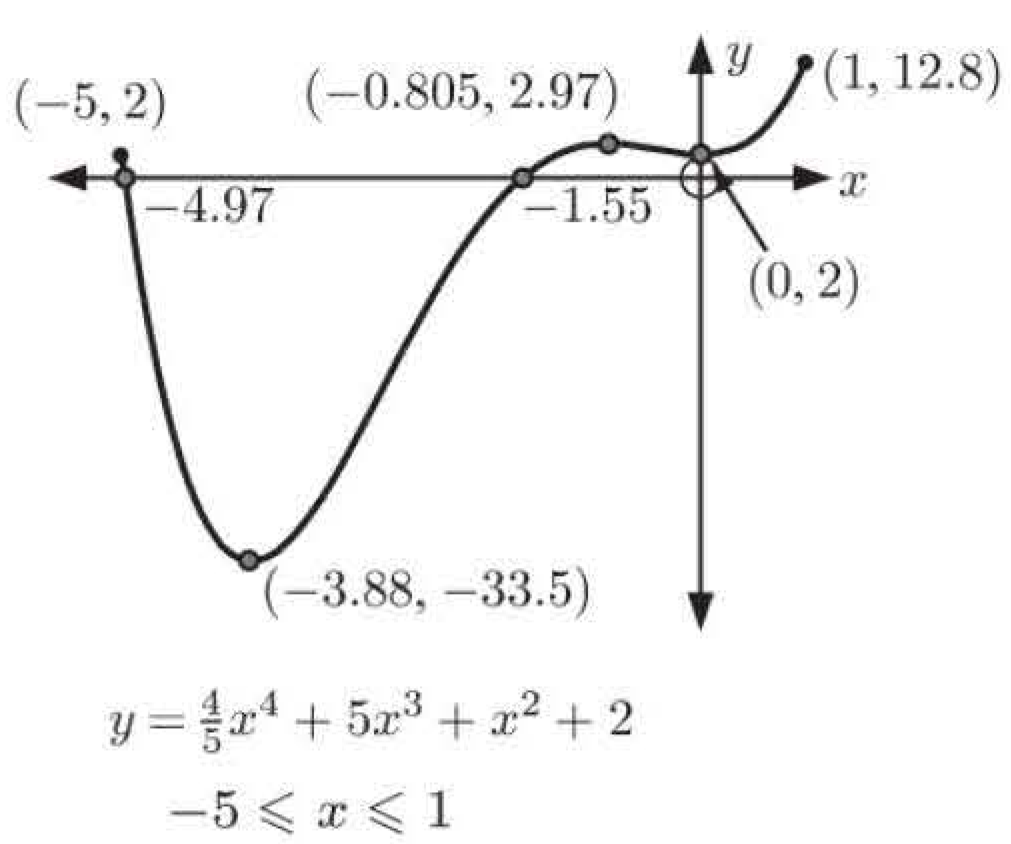
b **i** x-intercepts are ≈ -4.97 and ≈ -1.55, y-intercept is 2 {using technology}

ii $(-3.88, -33.5)$ and $(0, 2)$ are local minima,
$(-0.805, 2.97)$ is a local maximum {using technology}

iii $y = \frac{4}{5}x^4 + 5x^3 + 5x^2 + 2$ is defined for $-5 \leqslant x \leqslant 1$.
$\therefore$ there are no vertical asymptotes.
The function has no horizontal asymptotes.

iv We need to check the endpoints of the function.
When $x = -5$, $y = 2$.
When $x = 1$, $y = 12.8$.
So, the domain is $\{x \mid -5 \leqslant x \leqslant 1\}$,
and the range is $\{y \mid -33.5 \leqslant y \leqslant 12.8\}$.

v



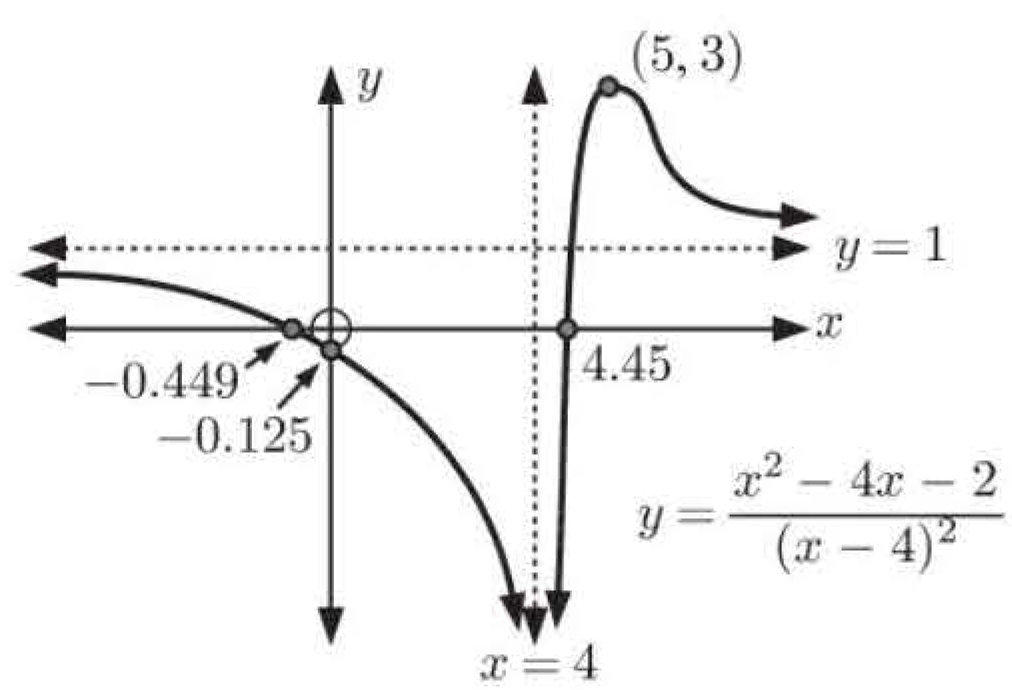
c **i** x-intercepts are ≈ -0.449 and ≈ 4.45, y-intercept is -0.125 {using technology}

ii $(5, 3)$ is a local maximum {using technology}

iii As $x \to 4^-$, $y \to -\infty$
As $x \to 4^+$, $y \to -\infty$
$\therefore$ the vertical asymptote is $x = 4$.
As $x \to \infty$, $y \to 1^+$
As $x \to -\infty$, $y \to 1^-$
$\therefore$ the horizontal asymptote is $y = 1$.

iv Domain is $\{x \mid x \neq 4\}$.
Range is $\{y \mid y \leqslant 3\}$.

v



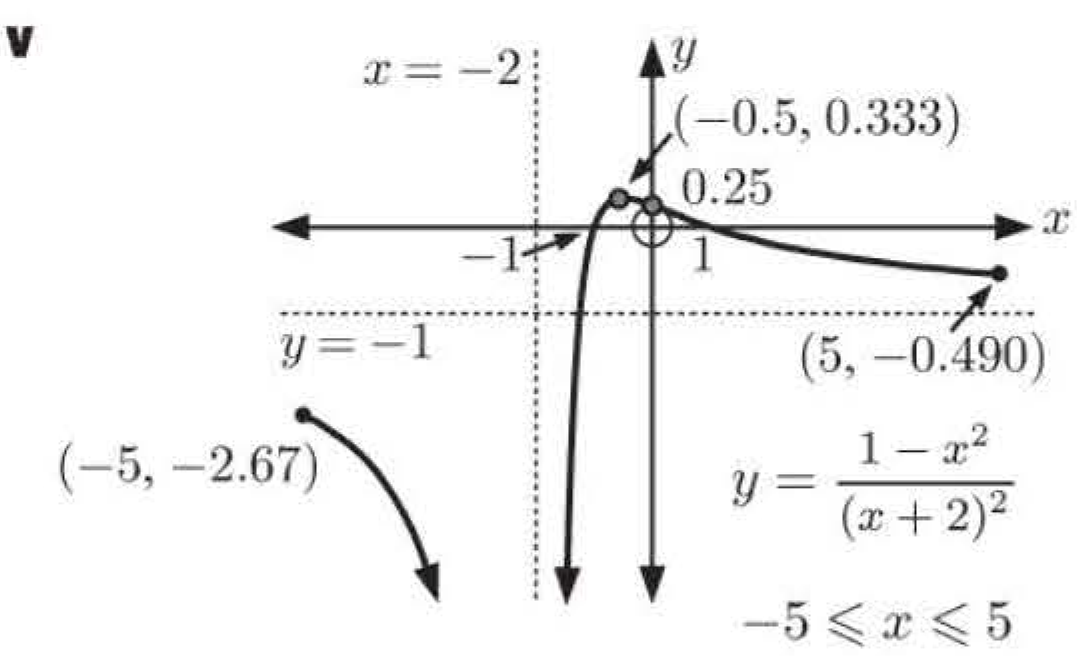
d **i** x-intercepts are -1 and 1, y-intercept is 0.25 {using technology}

ii $(-0.5, 0.333)$ is a local maximum {using technology}

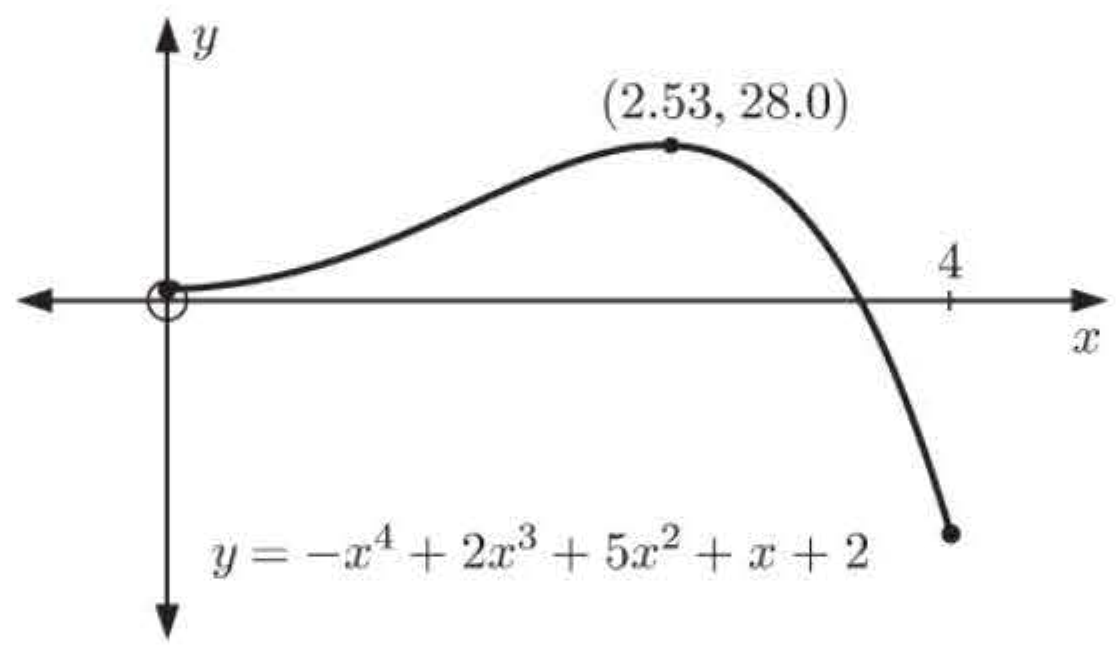
iii As $x \to -2^-$, $y \to -\infty$
As $x \to -2^+$, $y \to -\infty$
$\therefore$ the vertical asymptote is $x = -2$.
On $x \in \mathbb{R}$, $y = \dfrac{1-x^2}{(x+2)^2}$ has horizontal asymptote $y = -1$, but we are only considering the function on the domain $-5 \leqslant x \leqslant 5$.
Strictly speaking there is no horizontal asymptote on this domain.

iv Domain is $\{x \mid -5 \leqslant x \leqslant 5, \ x \neq 2\}$.
Range is $\{y \mid y \leqslant 0.333\}$.

v

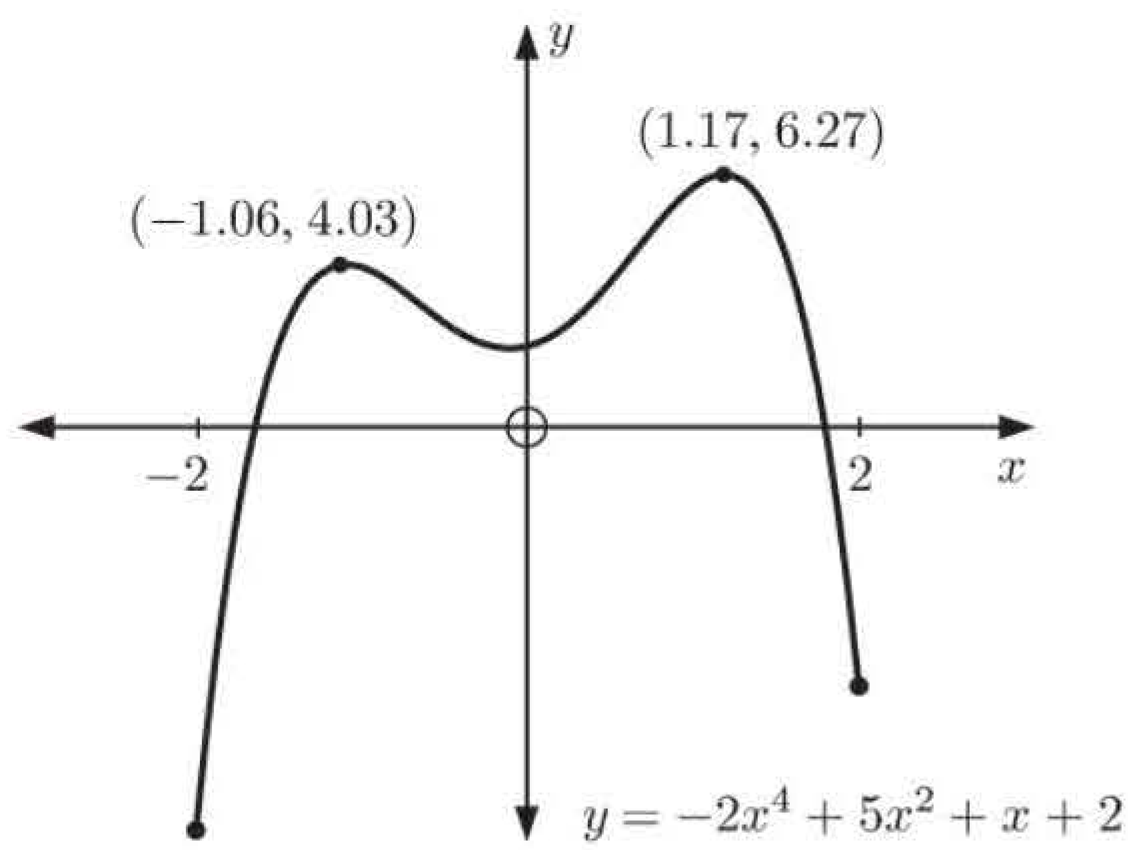


2 **a** Graphing $y = -x^4 + 2x^3 + 5x^2 + x + 2$ on $0 \leqslant x \leqslant 4$ using technology:



Clearly the maximum value occurs at the local maximum, which is $(2.53, 28.0)$, so the maximum value is 28.0.

b Graphing $y = -2x^4 + 5x^2 + x + 2$ on $-2 \leqslant x \leqslant 2$ using technology:



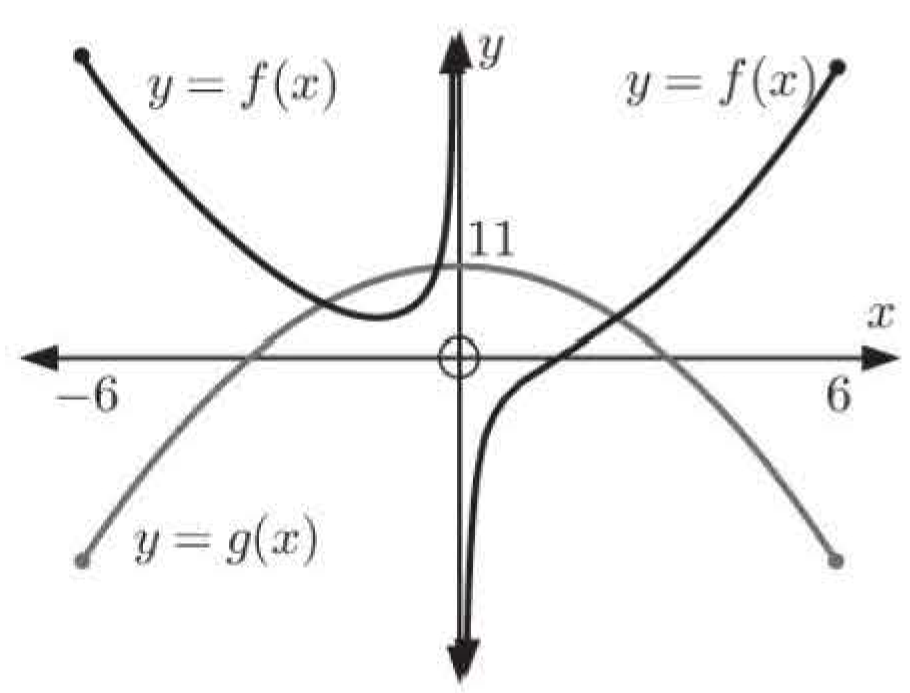
i The higher of the two maxima $(1.17, 6.27)$ is the global maximum on $-2 \leqslant x \leqslant 2$, so the maximum value is 6.27.

ii The global maximum on $-2 \leqslant x \leqslant 0$ occurs at the lower maximum $(-1.06, 4.03)$, so the maximum value on this interval is 4.03.

iii The higher maximum $(1.17, 6.27)$ is in $0 \leqslant x \leqslant 2$, so the maximum value on this interval is 6.27.

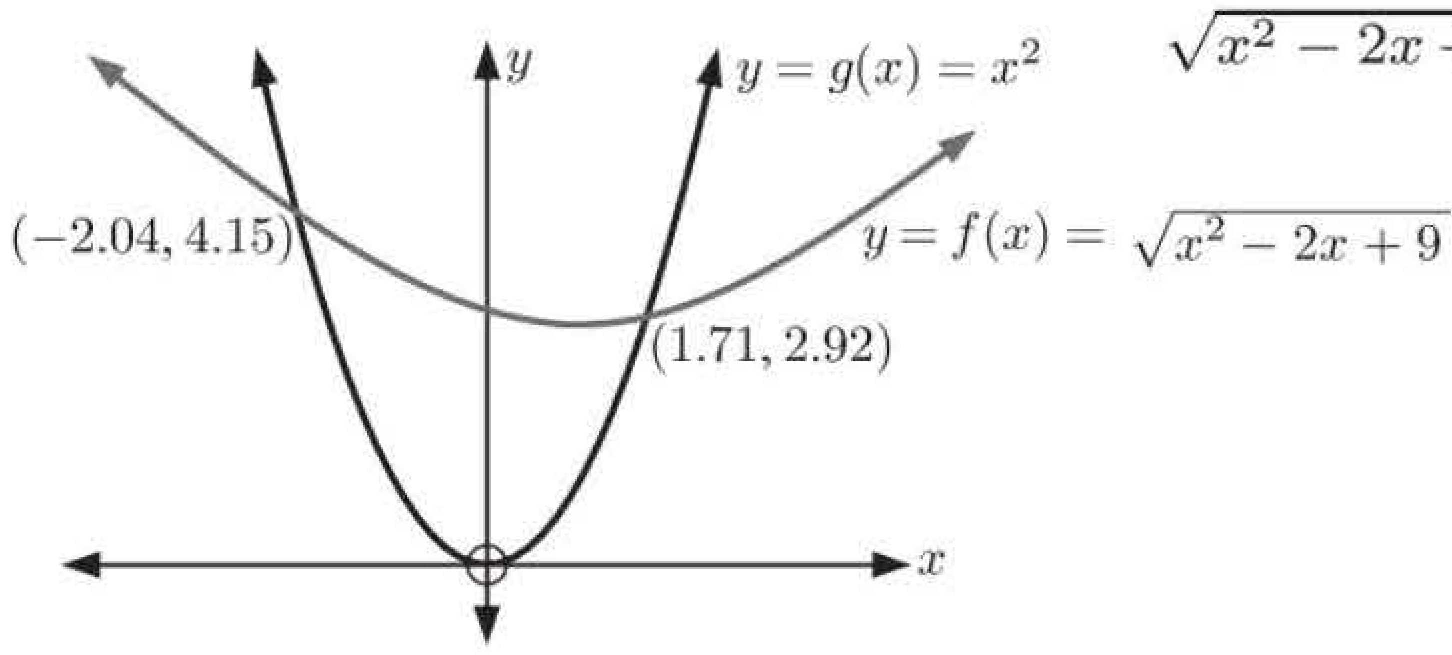
EXERCISE 2L

1 **a**



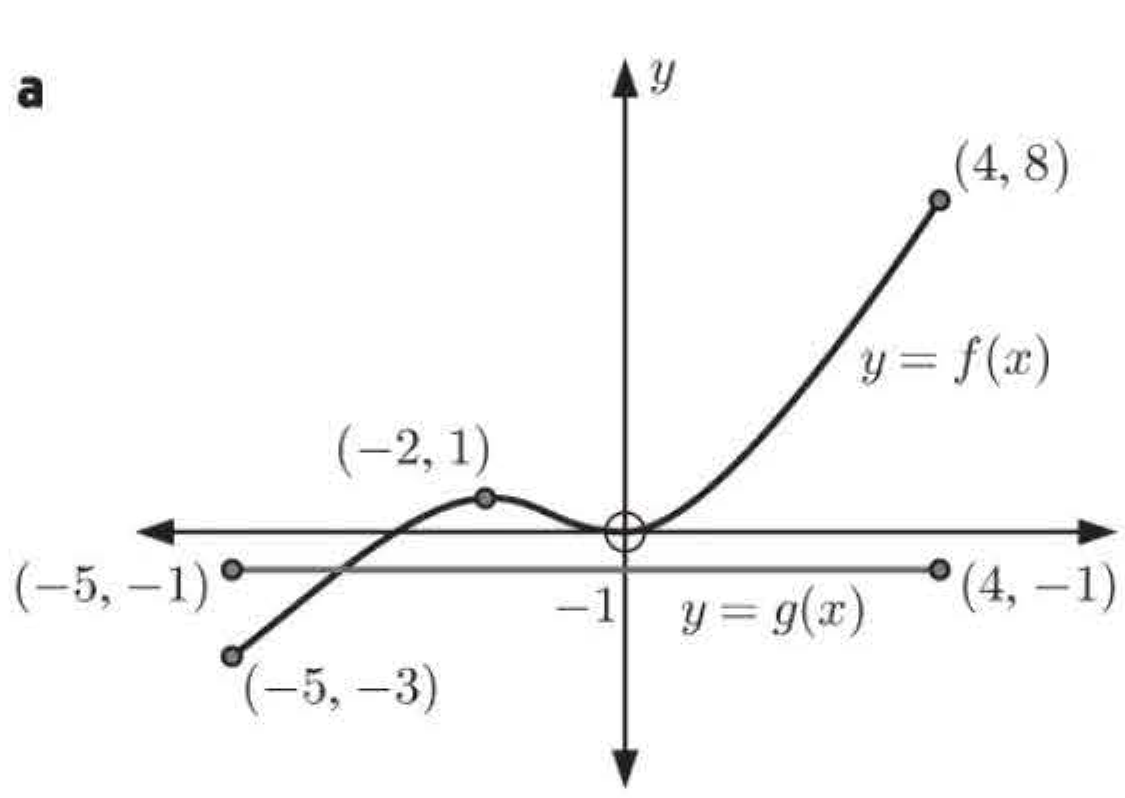
b Using technology, the other solutions are $x \approx -2.14$ and $x \approx -0.373$.

2



$\sqrt{x^2 - 2x + 9} = x^2$ when $x \approx -2.04$ or 1.71

3 **a**



b $f(x)$ and $g(x)$ intersect at one point, so $f(x) = g(x)$ has one solution on the domain $-5 \leqslant x \leqslant 4$.

c **i** $f(x) = h(x)$ on the domain $-5 \leqslant x \leqslant 4$ has three solutions when $0 < k < 1$.

ii $f(x) = h(x)$ on the domain $-5 \leqslant x \leqslant 4$ has two solutions when $k = 0$ or $k = 1$.

iii $f(x) = h(x)$ on the domain $-5 \leqslant x \leqslant 4$ has one solution when $-3 \leqslant k < 0$ or $1 < k \leqslant 8$.

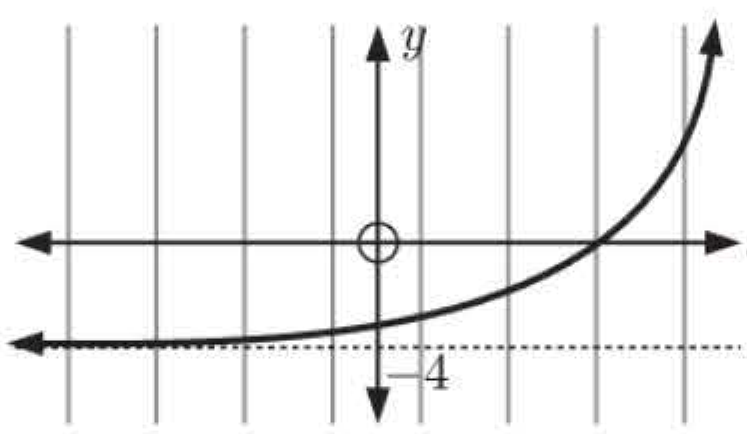
iv $f(x) = h(x)$ on the domain $-5 \leqslant x \leqslant 4$ has no solutions when $k < -3$ or $k > 8$.

REVIEW SET 2A

1 **a** **i** Domain is $\{x \mid x \in \mathbb{R}\}$.

ii Range is $\{y \mid y > -4\}$.

iii

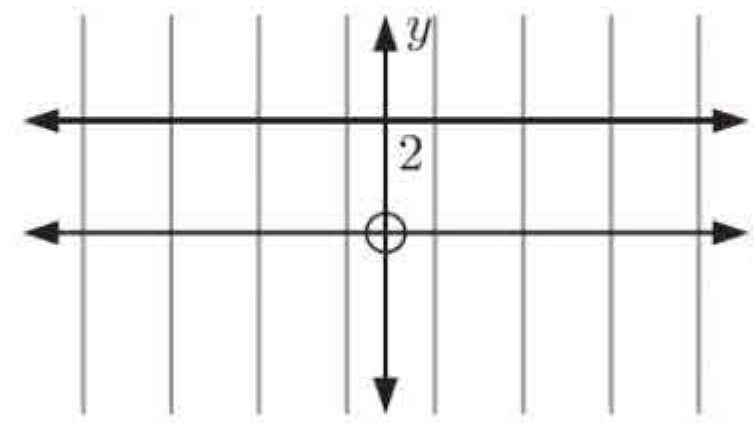


Each line cuts the graph no more than once, so the graph shows a function.

b **i** Domain is $\{x \mid x \in \mathbb{R}\}$.

ii Range is $\{2\}$.

iii

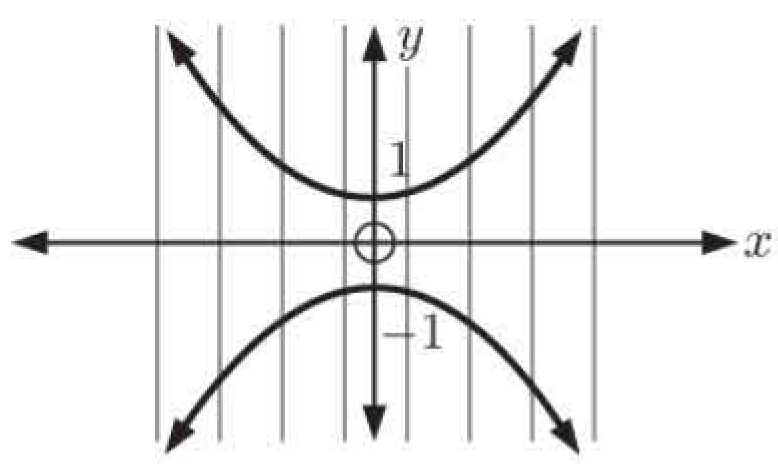


Each line cuts the graph no more than once, so the graph shows a function.

c **i** Domain is $\{x \mid x \in \mathbb{R}\}$.

ii Range is $\{y \mid y \leqslant -1 \text{ or } y \geqslant 1\}$.

iii

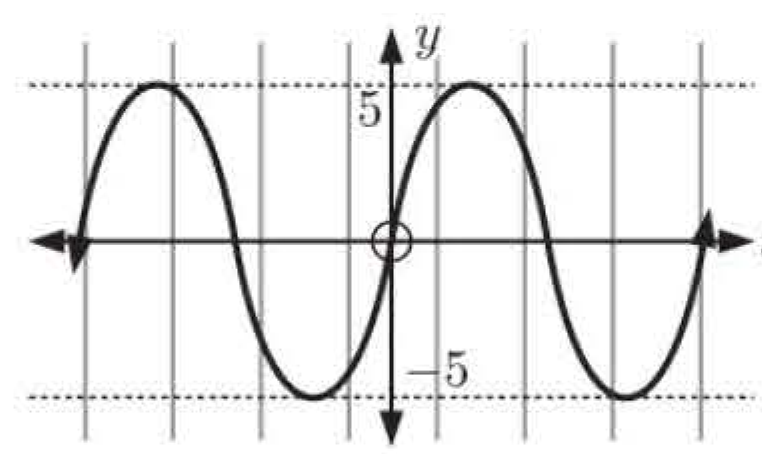


The lines cut the graph more than once, so the graph does not show a function.

d **i** Domain is $\{x \mid x \in \mathbb{R}\}$.

ii Range is $\{y \mid -5 \leqslant y \leqslant 5\}$.

iii



Each line cuts the graph no more than once, so the graph shows a function.

2 **a** $f(x) = 2x - x^2$

$f(2) = 2(2) - 2^2$
$= 0$

b $f(-3) = 2(-3) - (-3)^2$
$= -6 - 9$
$= -15$

c $f(-\frac{1}{2}) = 2(-\frac{1}{2}) - (-\frac{1}{2})^2$
$= -1 - \frac{1}{4}$
$= -\frac{5}{4}$

3 $f(x) = ax + b$, where $f(1) = 7$ and $f(3) = -5$

When $f(1) = 7$,
$7 = a(1) + b$
$\therefore\ 7 = a + b$
$\therefore\ a = 7 - b$ .... (1)

When $f(3) = -5$,
$-5 = a(3) + b$
$\therefore\ -5 = 3a + b$
$\therefore\ -5 = 3(7 - b) + b$ {using (1)}
$\therefore\ -5 = 21 - 3b + b$
$\therefore\ 2b = 26$ and so $b = 13$

Substituting $b = 13$ into (1), $a = 7 - 13 = -6$

$\therefore\ a = -6$ and $b = 13$

4 **a** $f(g(x)) = \sqrt{1 - x^2}$
$= f(1 - x^2)$

So, $f(x) = \sqrt{x}$
$g(x) = 1 - x^2$

Note: There are other possible functions f and g.

b $g(f(x)) = \left(\dfrac{x-2}{x+1}\right)^2$
$= g\left(\dfrac{x-2}{x+1}\right)$

So, $g(x) = x^2$
$f(x) = \dfrac{x-2}{x+1}$

5 **a** $|4x - 2| = |x + 7|$
$\therefore\ 4x - 2 = \pm(x + 7)$

If $4x - 2 = x + 7$
then $3x = 9$
$\therefore\ x = 3$

If $4x - 2 = -(x + 7)$
then $4x - 2 = -x - 7$
$\therefore\ 5x = -5$
$\therefore\ x = -1$

So, $x = -1$ or 3

b $x^2 + 6x > 16$

$\therefore\ x^2 + 6x - 16 > 0$

$\therefore\ (x+8)(x-2) > 0$

Sign diagram of LHS is: 

$\therefore\ x^2 + 6x - 16 > 0$ when $x \in]-\infty, -8[$ or $]2, \infty[$

$x^2 + 6x > 16$ when $x \in]-\infty, -8[$ or $]2, \infty[$

6 $g(x) = x^2 - 3x$

a $$\begin{aligned} g(x+1) &= (x+1)^2 - 3(x+1) \\ &= x^2 + 2x + 1 - 3x - 3 \\ &= x^2 - x - 2 \end{aligned}$$

b $$\begin{aligned} g(x^2-2) &= (x^2-2)^2 - 3(x^2-2) \\ &= x^4 - 4x^2 + 4 - 3x^2 + 6 \\ &= x^4 - 7x^2 + 10 \end{aligned}$$

7 **a** **i** Domain is $\{x \mid x \in \mathbb{R}\}$. Range is $\{y \mid y \geqslant -5\}$.

ii x-intercepts are -1 and 5, y-intercept is $-\frac{25}{9}$

iii The graph passes the 'vertical line test' so is therefore a function.

iv The graph does not pass the 'horizontal line test' so it does not have an inverse function.

b **i** Domain is $\{x \mid x \in \mathbb{R}\}$. Range is $\{y \mid y = 1 \text{ or } -3\}$.

ii There are no x-intercepts, y-intercept is 1.

iii The graph passes the 'vertical line test' so is therefore a function.

iv The graph does not pass the 'horizontal line test' so it does not have an inverse function.

8 **a** $$f(x) = -\frac{4}{x}$$

$$\begin{aligned} \therefore\ f(-x) &= -\frac{4}{-x} \\ &= \frac{4}{x} \\ &= -f(x) \end{aligned}$$

$\therefore\ f(x) = -\dfrac{4}{x}$ is an odd function.

b $$f(x) = \frac{2x-3}{x+1}$$

$$\begin{aligned} \therefore\ f(-x) &= \frac{2(-x)-3}{-x+1} \\ &= \frac{-2x-3}{-x+1} \end{aligned}$$

which is neither $f(x)$ or $-f(x)$.

$\therefore\ f(x) = \dfrac{2x-3}{x+1}$ is neither even nor odd.

c $$f(x) = \sqrt{x^2-5}$$

$$\begin{aligned} \therefore\ f(-x) &= \sqrt{(-x)^2-5} \\ &= \sqrt{x^2-5} \\ &= f(x) \end{aligned}$$

$\therefore\ f(x) = \sqrt{x^2-5}$ is an even function.

9 **a** f is $y = 4x + 2$

$\therefore\ f^{-1}(x)$ is $x = 4y + 2$

$\therefore\ y = \dfrac{x-2}{4}$

$\therefore\ f^{-1}(x) = \dfrac{x-2}{4}$

b f is $y = \dfrac{3-5x}{4}$

so $f^{-1}(x)$ is $x = \dfrac{3-5y}{4}$

$\therefore\ 4x = 3 - 5y$

$\therefore\ y = \dfrac{3-4x}{5}$

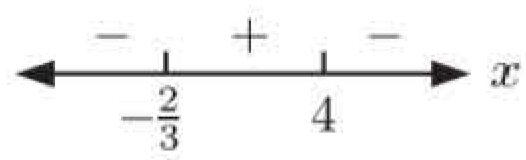
$\therefore\ f^{-1}(x) = \dfrac{3-4x}{5}$

10 **a** $y = (3x+2)(4-x)$ is zero when $x = -\frac{2}{3}$ or 4.

When $x = 0$, $y = (2)(4) = 8 > 0$.

Since the factors are distinct and linear, the signs alternate.

$\therefore$ sign diagram is



b $y = \dfrac{x-3}{x^2+4x+4} = \dfrac{x-3}{(x+2)^2}$ is zero when $x = 3$ and undefined when $x = -2$.

When $x = 0$, $y = \dfrac{-3}{2^2} = -\frac{3}{4} < 0$.

Since the $(x+2)$ factor is squared, the sign does not change at $x = -2$

$\therefore$ sign diagram is

11 $f(x) = ax + b$

Now $f(2) = 1$, so $a(2) + b = 1$

$\therefore$ $b = 1 - 2a$ .... $(*)$

Now $f^{-1}(3) = 4$, so

$$\begin{aligned} f(4) &= 3 \\ \therefore\ a(4) + b &= 3 \\ \therefore\ 4a + (1-2a) &= 3 \qquad \{\text{from } (*)\} \\ \therefore\ 2a &= 2 \\ \therefore\ a &= 1 \end{aligned}$$

Substituting $a = 1$ into $(*)$, $b = 1 - 2(1) = -1$

So, $a = 1$ and $b = -1$.

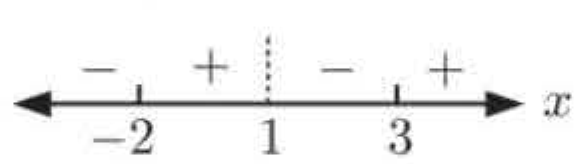
12 **a** $y = \dfrac{(x+2)(x-3)}{x-1}$ is zero when

$x = -2$ or 3, and undefined when $x = 1$.

When $x = 0$, $y = \frac{(2)(-3)}{-1} = 6 > 0$.

Since the factors are distinct and linear, the signs alternate.

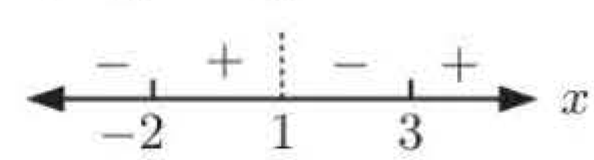
$\therefore$ the sign diagram is



b $$\frac{x^2-x-6}{x-1} < 0$$

$$\therefore\ \frac{(x+2)(x-3)}{x-1} < 0$$

From **a**, sign diagram of LHS is



$\therefore$ LHS < 0 when $x \in \,]-\infty, -2[$ or $]1, 3[$

13 **a**

$$\begin{aligned} f(x) &= x^2 \\ \therefore\ f(-3) &= (-3)^2 = 9 \\ g(x) &= 1 - 6x \\ \therefore\ g(-\tfrac{4}{3}) &= 1 - 6(-\tfrac{4}{3}) \\ &= 1 + 8 = 9 \\ \therefore\ f(-3) &= g(-\tfrac{4}{3}) \end{aligned}$$

b $(f \circ g)(-2) = f(g(-2))$

Now, $g(-2) = 1 - 6(-2)$
$= 13$

$$\begin{aligned} \therefore\ (f \circ g)(-2) &= f(13) \\ &= 13^2 \\ &= 169 \end{aligned}$$

c $f(5) = 5^2 = 25$

So, we need to find x such that $1 - 6x = 25$

$$\begin{aligned} \therefore\ -6x &= 24 \\ \therefore\ x &= -4 \end{aligned}$$

14

$$\begin{aligned} f \text{ is } y &= 3x + 6 \\ \text{so } f^{-1}(x) \text{ is } x &= 3y + 6 \\ \therefore\ y &= \frac{x-6}{3} \\ \therefore\ f^{-1}(x) &= \frac{x-6}{3} \end{aligned}$$

$$\begin{aligned} h \text{ is } y &= \frac{x}{3} \\ \text{so } h^{-1}(x) \text{ is } x &= \frac{y}{3} \\ \therefore\ y &= 3x \\ \therefore\ h^{-1}(x) &= 3x \end{aligned}$$

Now $(f^{-1} \circ h^{-1})(x) = f^{-1}(h^{-1}(x))$
$= f^{-1}(3x)$
$= \dfrac{3x-6}{3}$
$= x - 2$

$(h \circ f)(x) = h(f(x))$
$= h(3x+6)$
$= \dfrac{3x+6}{3}$
$\therefore\ h \circ f$ is $y = x + 2$
so $(h \circ f)^{-1}(x)$ is $x = y + 2$
$\therefore\ y = x - 2$
$\therefore\ (h \circ f)^{-1}(x) = x - 2$

$\therefore\ (f^{-1} \circ h^{-1})(x) = (h \circ f)^{-1}(x)$ as required.

15 **a** $h(x) = (x-4)^2 + 3, \quad x \geqslant 4$
$\therefore\ y = (x-4)^2 + 3, \quad x \geqslant 4$
$\therefore\ h^{-1}(x)$ is $x = (y-4)^2 + 3, \quad y \geqslant 4$
$\therefore\ x - 3 = (y-4)^2$
$\therefore\ y - 4 = \pm\sqrt{x-3}$
$\therefore\ y = 4 \pm \sqrt{x-3}$
But $y \geqslant 4$, so $y = 4 + \sqrt{x-3}$
$\therefore\ h^{-1}(x) = 4 + \sqrt{x-3}, \quad x \geqslant 3$

b $(h \circ h^{-1})(x)$
$= h\left(h^{-1}(x)\right)$
$= h(4 + \sqrt{x-3})$
$= (4 + \sqrt{x-3} - 4)^2 + 3$
$= \left(\sqrt{x-3}\right)^2 + 3$
$= x - 3 + 3$
$= x$

$(h^{-1} \circ h)(x)$
$= h^{-1}(h(x))$
$= h^{-1}\left((x-4)^2 + 3\right)$
$= 4 + \sqrt{(x-4)^2 + 3 - 3}$
$= 4 + \sqrt{(x-4)^2}$
$= 4 + x - 4 \qquad \{\text{as } x \geqslant 4\}$
$= x$

$\therefore\ (h \circ h^{-1})(x) = (h^{-1} \circ h)(x) = x$

REVIEW SET 2B

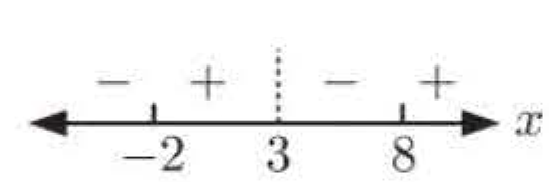
1 **a** $y = (x-1)(x-5)$
$\therefore$ the x-intercepts are $x = 1$ and 5
The vertex is at $x = 3$, with $y = (3-1)(3-5) = 2 \times (-2) = -4$
$\therefore$ the vertex is at $(3, -4)$
The domain is $\{x \mid x \in \mathbb{R}\}$. The range is $\{y \mid y \geqslant -4\}$.

b From the graph, the domain is $\{x \mid x \neq 0,\ x \neq 2\}$ and the range is $\{y \mid y \leqslant -1 \text{ or } y > 0\}$.

2 **a** $(f \circ g)(x) = f(g(x))$
$= f(x^2 + 2)$
$= 2(x^2 + 2) - 3$
$= 2x^2 + 4 - 3$
$= 2x^2 + 1$

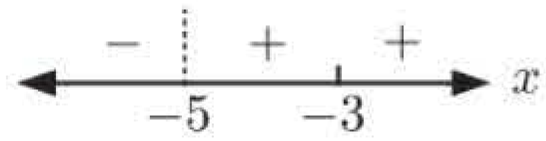
b $(g \circ f)(x) = g(f(x))$
$= g(2x - 3)$
$= (2x-3)^2 + 2$
$= 4x^2 - 12x + 9 + 2$
$= 4x^2 - 12x + 11$

3 **a** $y = \dfrac{x^2 - 6x - 16}{x-3} = \dfrac{(x+2)(x-8)}{x-3}$ is zero when $x = -2$ or 8 and undefined when $x = 3$.
When $x = 0$, $y = \frac{-16}{-3} > 0$.
Since the factors are single, the signs alternate. So, the sign diagram is:

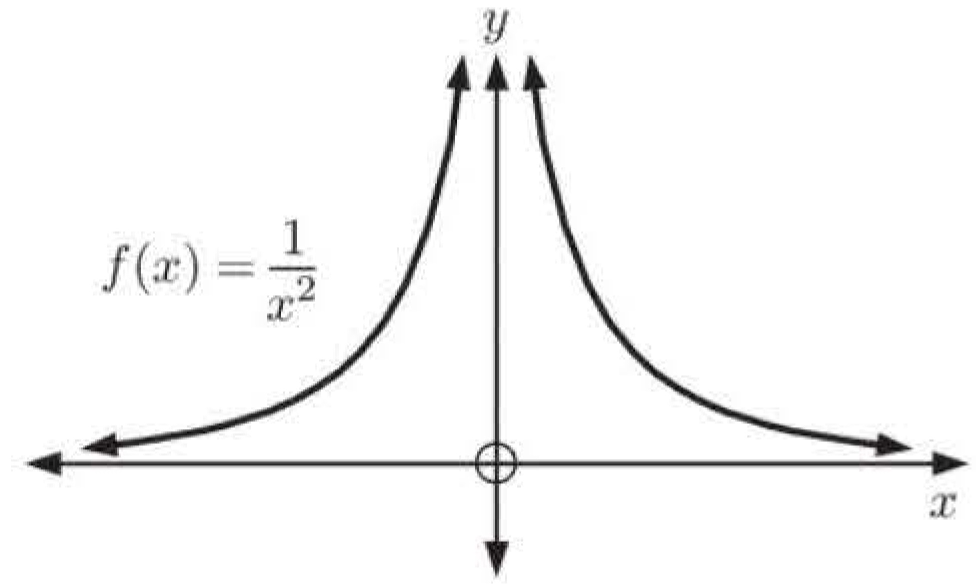


b $y = \dfrac{x+9}{x+5} + x\left(\dfrac{x+5}{x+5}\right) = \dfrac{x^2+6x+9}{x+5} = \dfrac{(x+3)^2}{x+5}$ is zero when $x = -3$ and undefined when $x = -5$.

When $x = 0$, $y = \dfrac{3^2}{5} > 0$. The $(x+3)$ factor is squared, so the sign does not change at $x = -3$.

So, the sign diagram is: 

4 **a** $f(x) = \dfrac{1}{x^2}$ is meaningless when $x = 0$.

b 

c Domain of $f(x)$ is $\{x \mid x \neq 0\}$.
Range of $f(x)$ is $\{y \mid y > 0\}$.

5 **a** $f(x) = \dfrac{ax+3}{x-b}$ has asymptotes $x = -1$, $y = 2$.

$f(x)$ is undefined when $x - b = 0$

$\therefore\ x = b$ is the vertical asymptote.

But $x = -1$ is the vertical asymptote, so $b = -1$.

So, $f(x) = \dfrac{ax+3}{x-(-1)} = \dfrac{ax+3}{x+1} = \dfrac{a+\frac{3}{x}}{1+\frac{1}{x}}$

As $|x| \to \infty$, $f(x) \to \dfrac{a}{1} = a$ so the horizontal asymptote is $y = a$.

But $y = 2$ is the horizontal asymptote, so $a = 2$.

$\therefore\ a = 2$ and $b = -1$.

b Domain of f is $\{x \mid x \neq -1\}$ and range of f is $\{y \mid y \neq 2\}$.

$\therefore$ domain of f^{-1} is $\{x \mid x \neq 2\}$ and range of f^{-1} is $\{y \mid y \neq -1\}$.

6 **a** Domain is $\{x \mid x > -3\}$.
Range is $\{y \mid -3 < y < 5\}$.

b Domain is $\{x \mid x \neq 1\}$.
Range is $\{y \mid y \leqslant -3 \text{ or } y \geqslant 5\}$.

7 **a** $\left|\dfrac{2x+1}{x-2}\right| = 3$

$\therefore\ \dfrac{2x+1}{x-2} = \pm 3$

If $\dfrac{2x+1}{x-2} = 3$

then $2x + 1 = 3x - 6$

$\therefore\ -x = -7$

$\therefore\ x = 7$

If $\dfrac{2x+1}{x-2} = -3$

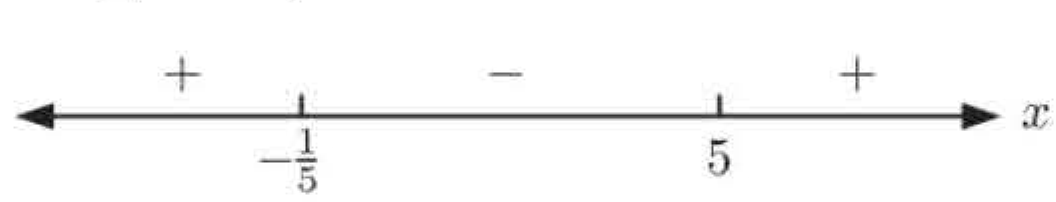
then $2x + 1 = -3x + 6$

$\therefore\ 5x = 5$

$\therefore\ x = 1$

So, $x = 1$ or 7

b
$$\begin{aligned}
|3x-2| &\geqslant |2x+3| \\
\therefore\ |3x-2|^2 &\geqslant |2x+3|^2 \\
\therefore\ (3x-2)^2 - (2x+3)^2 &\geqslant 0 \\
\therefore\ 9x^2 - 12x + 4 - (4x^2+12x+9) &\geqslant 0 \\
\therefore\ 5x^2 - 24x - 5 &\geqslant 0 \\
\therefore\ (5x+1)(x-5) &\geqslant 0
\end{aligned}$$

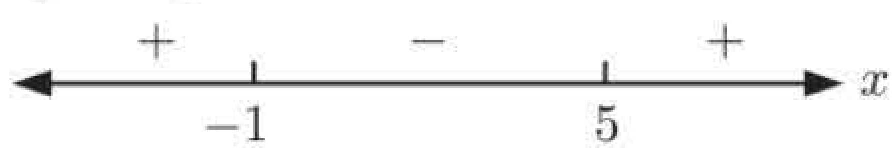
Sign diagram of LHS is 

$\therefore$ LHS $\geqslant 0$ when $x \in\ ]-\infty, -\frac{1}{5}]$ or $[5, \infty[$

8 **a**

$$x^2 - 5 \leqslant 4x$$
$$\therefore\ x^2 - 4x - 5 \leqslant 0$$
$$\therefore\ (x+1)(x-5) \leqslant 0$$

Sign diagram of LHS is



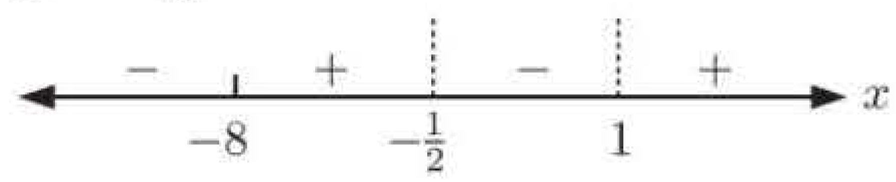
$\therefore\ x^2 - 4x - 5 \leqslant 0$ when $x \in [-1, 5]$

$\therefore\ x^2 - 5 \leqslant 4x$ when $x \in [-1, 5]$.

b

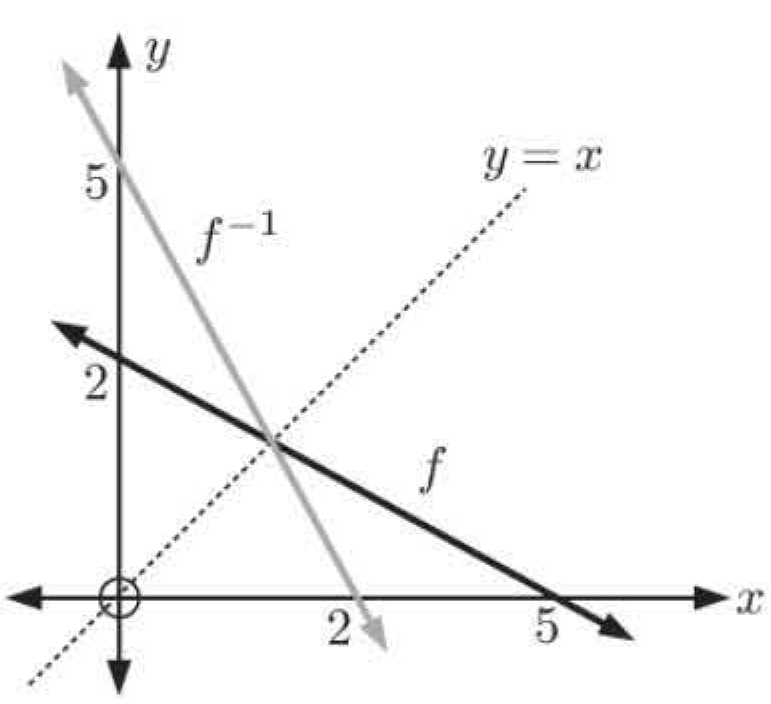
$$\frac{3}{x-1} > \frac{5}{2x+1}$$
$$\therefore\ \frac{3}{x-1} - \frac{5}{2x+1} > 0$$
$$\therefore\ \frac{3(2x+1) - 5(x-1)}{(x-1)(2x+1)} > 0$$
$$\therefore\ \frac{x+8}{(x-1)(2x+1)} > 0$$

Sign diagram of LHS is

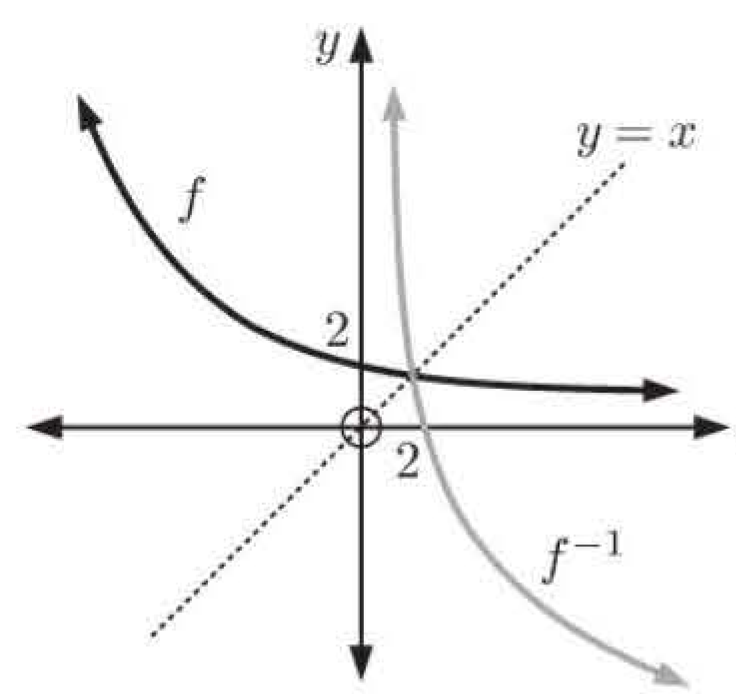


$\therefore$ LHS > 0 when $x \in\ ]-8, -\frac{1}{2}[$ or $]1, \infty[$

9 **a**



b



10 **a** $f : x \mapsto \dfrac{4x+1}{2-x}$ is undefined when $x = 2$

$\therefore\ x = 2$ is a vertical asymptote.

Now $f(x) = \dfrac{4x+1}{2-x} = \dfrac{4 + \frac{1}{x}}{-1 + \frac{2}{x}}$

$\therefore$ as $|x| \to \infty,\ f(x) \to \frac{4}{-1} = -4$,

and so $y = -4$ is a horizontal asymptote.

b The domain is $\{x \mid x \neq 2\}$.

The range is $\{y \mid y \neq -4\}$.

c As $x \to 2^-,\ \ y \to \infty$.

As $x \to 2^+,\ \ y \to -\infty$.

As $x \to \infty,\ \ y \to -4^-$.

As $x \to -\infty,\ \ y \to -4^+$.

d $f(0) = \dfrac{4(0)+1}{2-0} = \frac{1}{2}$

So, the y-intercept is $\frac{1}{2}$.

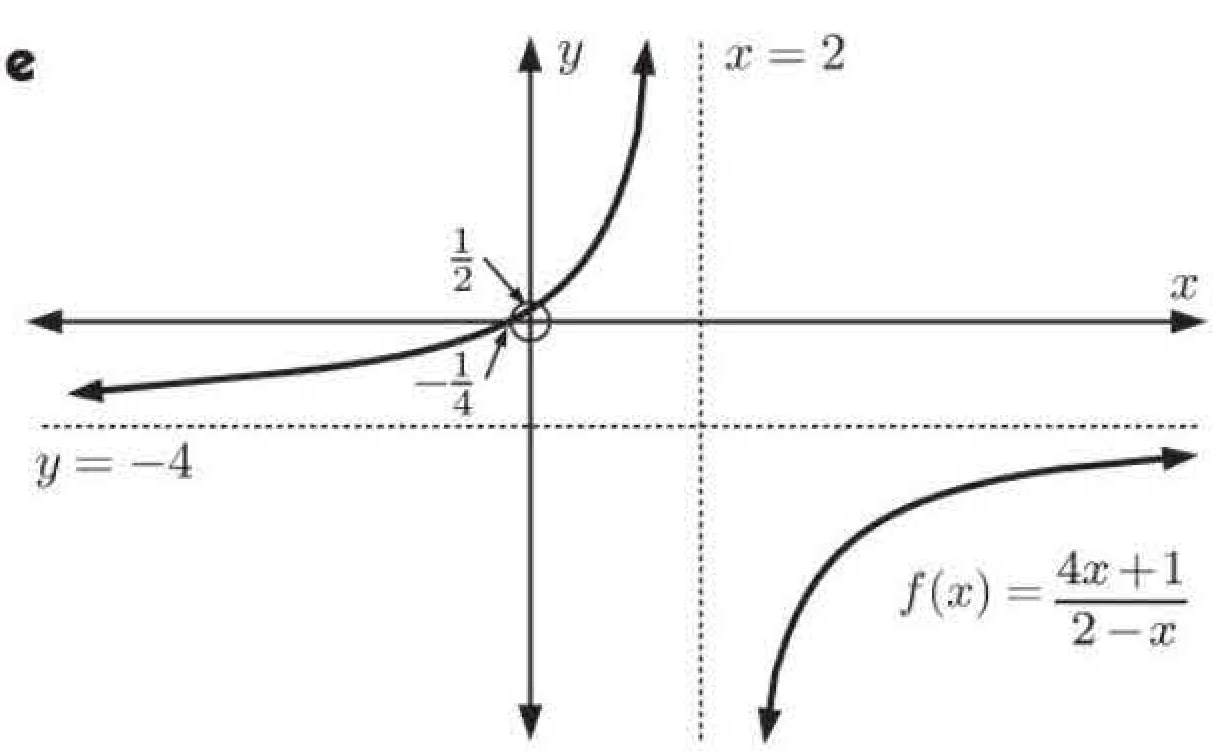
$f(x) = 0$ when $\dfrac{4x+1}{2-x} = 0$

$\therefore\ 4x + 1 = 0$

$\therefore\ x = -\frac{1}{4}$

So, the x-intercept is $-\frac{1}{4}$.

e



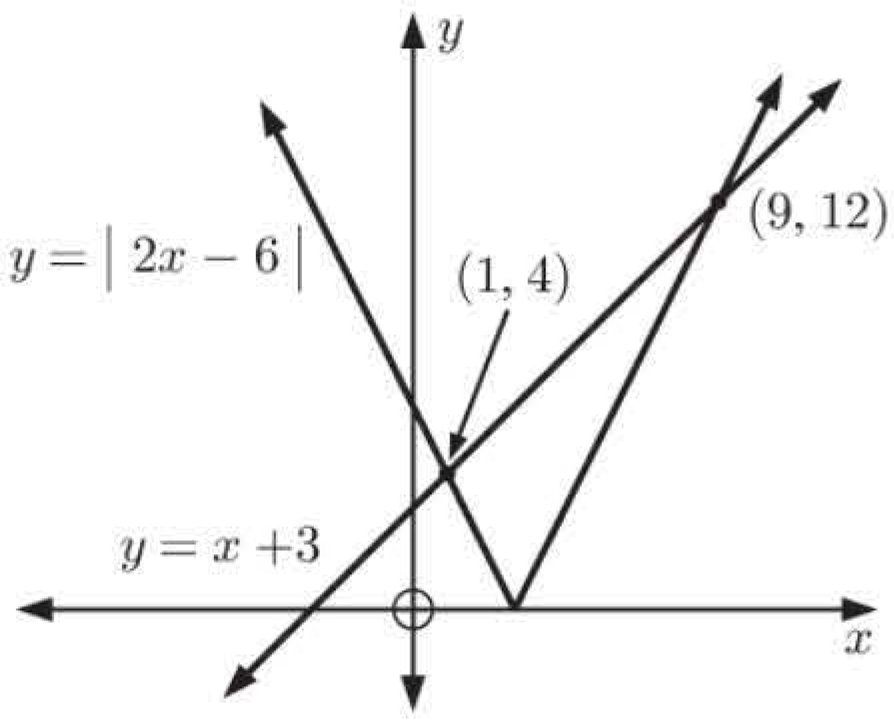
11

$$f(x) = (x-3)^2 + ax \quad \text{is an even function}$$
$$\therefore\ f(-x) = f(x)$$
$$\therefore\ (-x-3)^2 + a(-x) = (x-3)^2 + ax$$
$$\therefore\ \cancel{x^2} + 6x + \cancel{9} - ax = \cancel{x^2} - 6x + \cancel{9} + ax$$

Equating coefficients of x:

$$6 - a = -6 + a$$
$$\therefore\ 2a = 12$$
$$\therefore\ a = 6$$

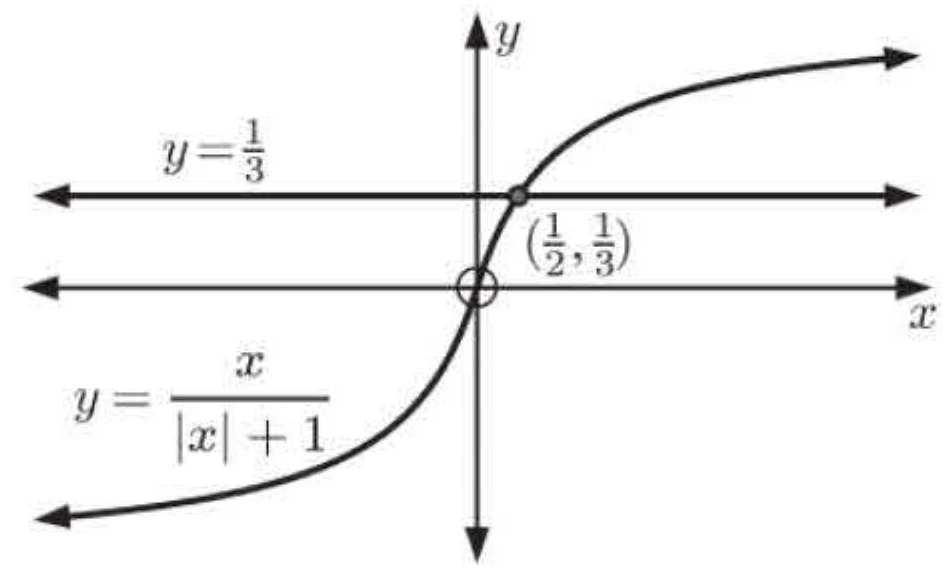
12 **a**



$y = |2x + 6|$ and $y = x + 3$ intersect at $x = 1$ and $x = 9$.

$\therefore$ from the graph, $|2x + 6| > x + 3$ when $x \in \,]-\infty, 1[$ or $]9, \infty[$

b



$\therefore \quad \dfrac{x}{|x|+1} \geqslant \frac{1}{3}$ for $x \in [\frac{1}{2}, \infty[$

13 **a**

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(3x+1) \\ &= \frac{2}{3x+1}\end{aligned}$$

b

$$\begin{aligned}(g \circ f)(x) &= -4 \\ \therefore \quad \frac{2}{3x+1} &= -4 \\ \therefore \quad -4(3x+1) &= 2 \\ \therefore \quad -12x - 4 &= 2 \\ \therefore \quad 12x &= -6 \\ \therefore \quad x &= -\tfrac{1}{2}\end{aligned}$$

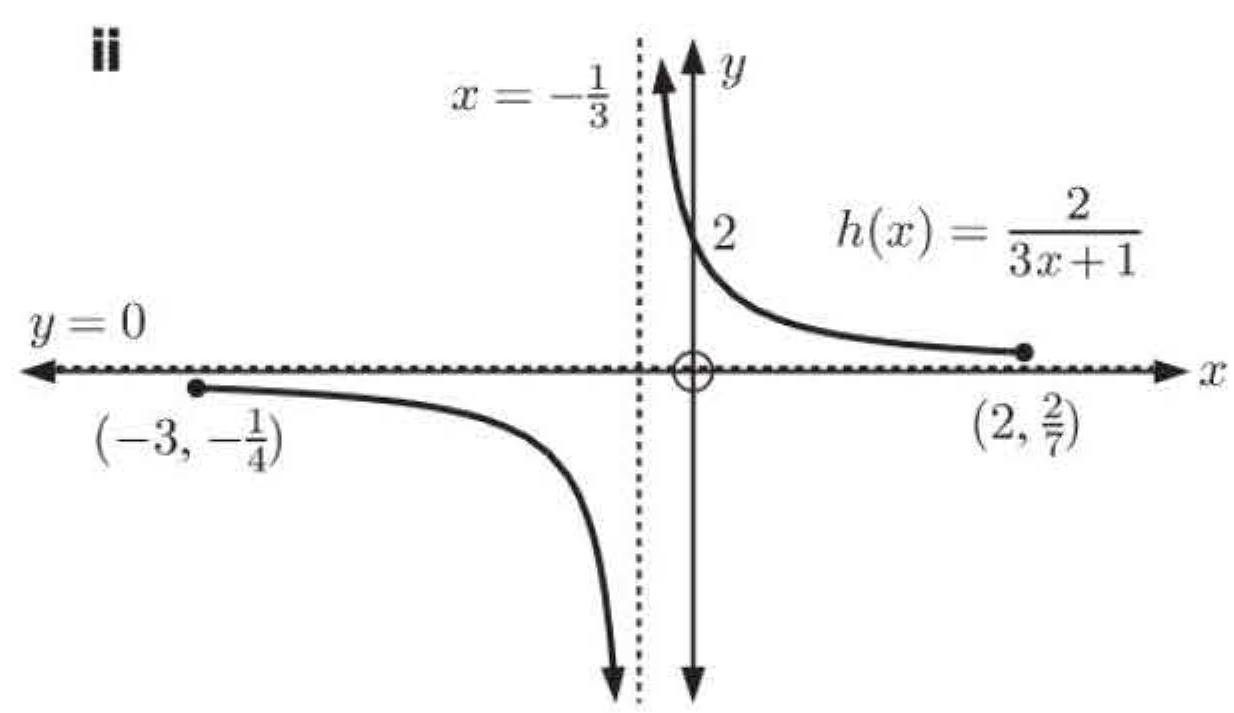
c **i** $h(x) = \dfrac{2}{3x+1}$ is undefined when $3x + 1 = 0$ or $x = -\frac{1}{3}$.

So, $x = -\frac{1}{3}$ is a vertical asymptote.

As $|x| \to \infty$, $h(x) \to 0$

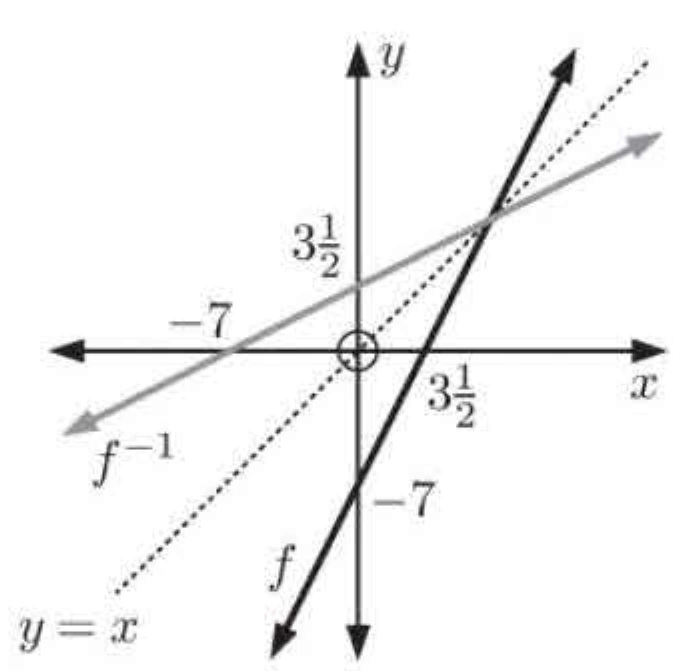
$\therefore$ $y = 0$ is a horizontal asymptote.

ii



iii Range of h is $\{y \mid y \leqslant -\frac{1}{4}$ or $y \geqslant \frac{2}{7}\}$.

14 **a**



b The function f is $y = 2x - 7$

so f^{-1} is $x = 2y - 7$

$\therefore \quad y = \dfrac{x+7}{2}$

So, $f^{-1}(x) = \dfrac{x+7}{2}$

c

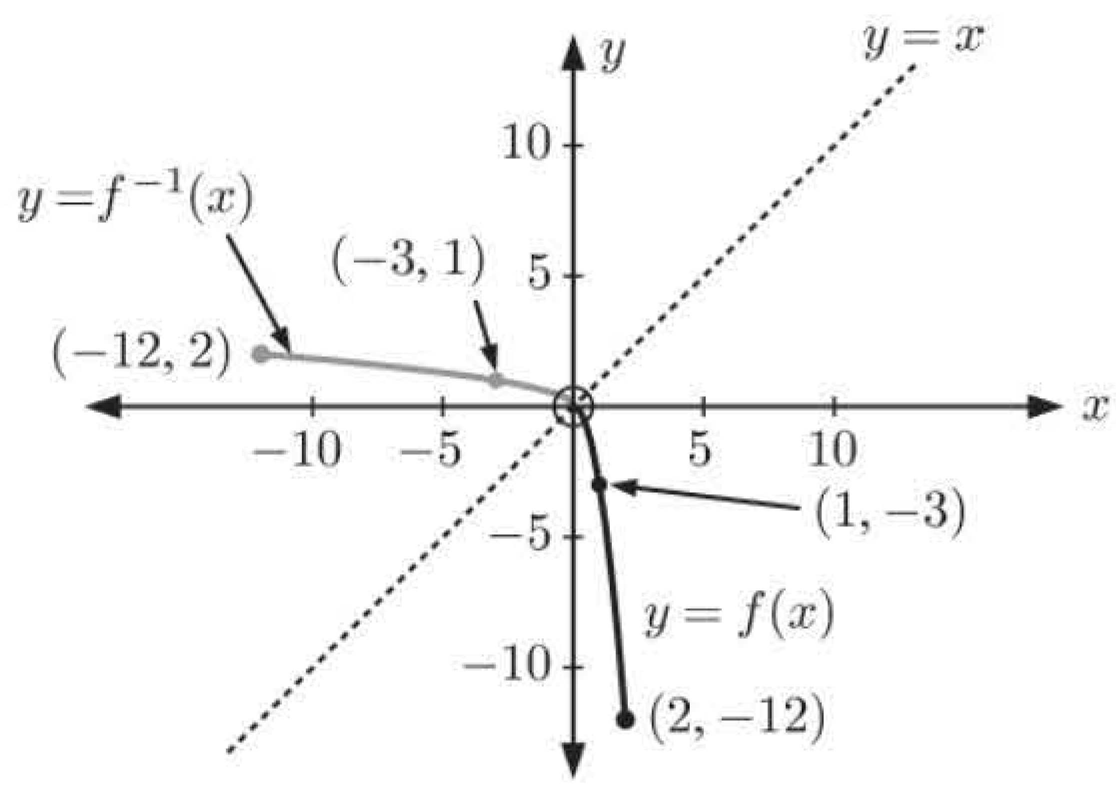
$(f \circ f^{-1})(x)$ and $(f^{-1} \circ f)(x)$

$$\begin{aligned}(f \circ f^{-1})(x) &= f\left(f^{-1}(x)\right) \\ &= f\left(\frac{x+7}{2}\right) \\ &= 2\left(\frac{x+7}{2}\right) - 7 \\ &= x + 7 - 7 \\ &= x\end{aligned}$$

$$\begin{aligned}(f^{-1} \circ f)(x) &= f^{-1}\left(f(x)\right) \\ &= f^{-1}(2x - 7) \\ &= \frac{2x - 7 + 7}{2} \\ &= \frac{2x}{2} \\ &= x\end{aligned}$$

So, $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$.

15 **a**



b Range of f^{-1} is $\{y \mid 0 \leqslant y \leqslant 2\}$.

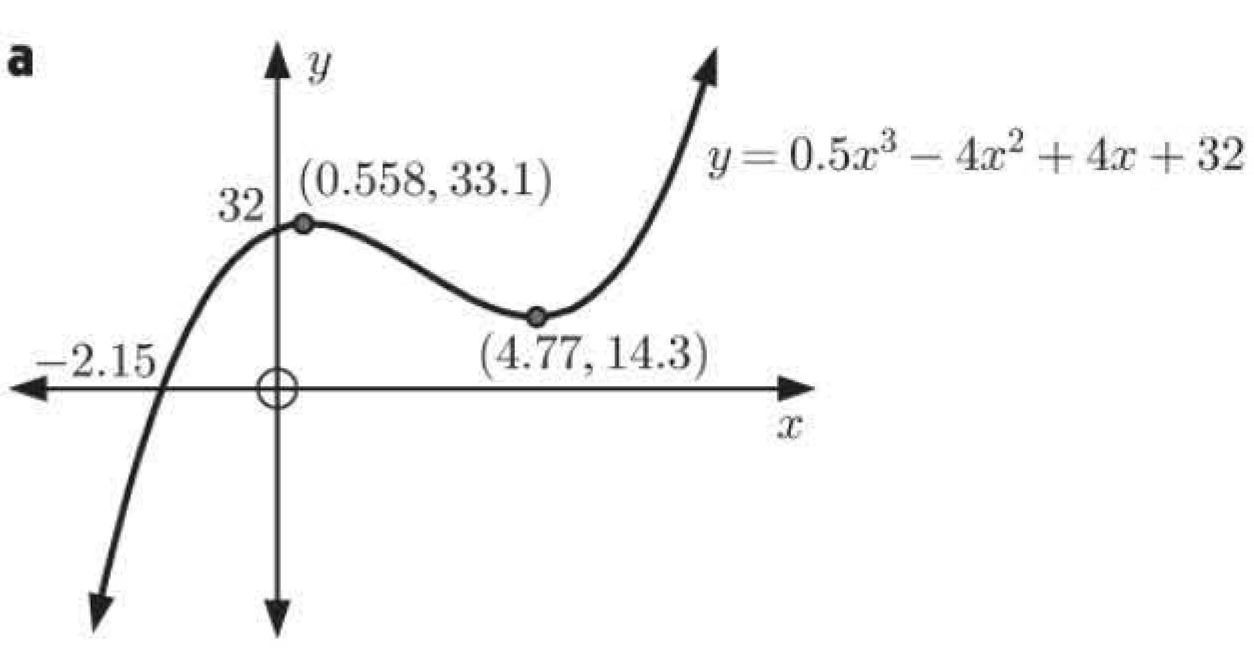
c **i**
$$f(x) = -10$$
$$\therefore\ -3x^2 = -10, \quad 0 \leqslant x \leqslant 2$$
$$\therefore\ x^2 = \tfrac{10}{3}$$
$$\therefore\ x = \sqrt{\tfrac{10}{3}} \quad \{0 \leqslant x \leqslant 2\}$$
$$\therefore\ x \approx 1.83$$

ii $f^{-1}(x) = 1$ so $f(1) = x$

The graph shows that $f(1) = -3$

$\therefore\ x = -3$

16 **a**



b $(0.558, 33.1)$ is a local maximum,

$(4.77, 14.3)$ is a local minimum

{using technology}

c Check endpoints:
$$f(0) = 32$$
$$f(6) = 0.5(6)^3 - 4(6)^2 + 4(6) + 32$$
$$= 20$$

$\therefore$ for $f(x)$ on the interval $0 \leqslant x \leqslant 6$, the maximum value is 33.1 and the minimum value is 14.3.

REVIEW SET 2C

1 **a** Domain is $\{x \mid x \geqslant -2\}$. Range is $\{y \mid 1 \leqslant y < 3\}$.

b Domain is $\{x \mid x \in \mathbb{R}\}$. Range is $\{y \mid y = -1, 1, \text{ or } 2\}$.

2 **a**
$$f(x) = x^2 + 3$$
$$\therefore\ f(-3) = (-3)^2 + 3$$
$$= 9 + 3$$
$$= 12$$

b
$$x^2 + 3 = 4$$
$$\therefore\ x^2 = 1$$
$$\therefore\ x = \pm 1$$

3 **a** $f(x) = 10 + \dfrac{3}{2x-1}$

is undefined when $2x - 1 = 0$

$\therefore\ x = \frac{1}{2}$

b $f(x) = \sqrt{x+7}$

is undefined when $x + 7 < 0$

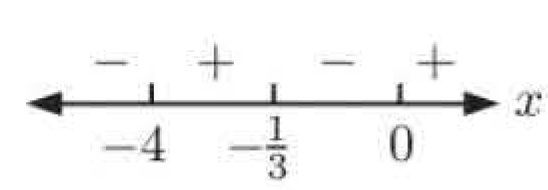
$\therefore\ x < -7$

4 **a** $f(x) = x(x+4)(3x+1)$ is zero when $x = 0$, -4, and $-\frac{1}{3}$.

When $x = 10$, $y = 10(14)(31) = 4340 > 0$.

The factors are single, so the signs alternate.

$\therefore$ sign diagram is:

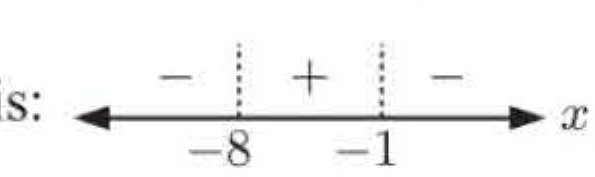


b $f(x) = \dfrac{-11}{(x+1)(x+8)}$ is undefined when $x = -1$ or -8.

When $x = 0$, $y = \frac{-11}{(1)(8)} = -\frac{11}{8} < 0$.

The factors are single, so the signs alternate.

$\therefore$ sign diagram is:



5 **a**
$$h(x) = 7 - 3x$$
$$h(2x-1) = 7 - 3(2x-1)$$
$$= 7 - 6x + 3$$
$$= 10 - 6x$$

b
$$h(2x-1) = -2$$
$$\therefore\ 10 - 6x = -2 \quad \{\text{using } \mathbf{a}\}$$
$$\therefore\ -6x = -12$$
$$\therefore\ x = 2$$

6 **a** **i** $(f \circ g)(x) = f(g(x))$
$= f(\sqrt{x})$
$= 1 - 2\sqrt{x}$

ii $(g \circ f)(x) = g(f(x))$
$= g(1 - 2x)$
$= \sqrt{1 - 2x}$

b **i** Domain is $\{x \mid x \geqslant 0\}$.
Range is $\{y \mid y \leqslant 1\}$.

ii Domain is $\{x \mid x \leqslant 0.5\}$.
Range is $\{y \mid y \geqslant 0\}$.

7 $f(x) = ax^2 + bx + c$, where $f(0) = 5$, $f(-2) = 21$, and $f(3) = -4$

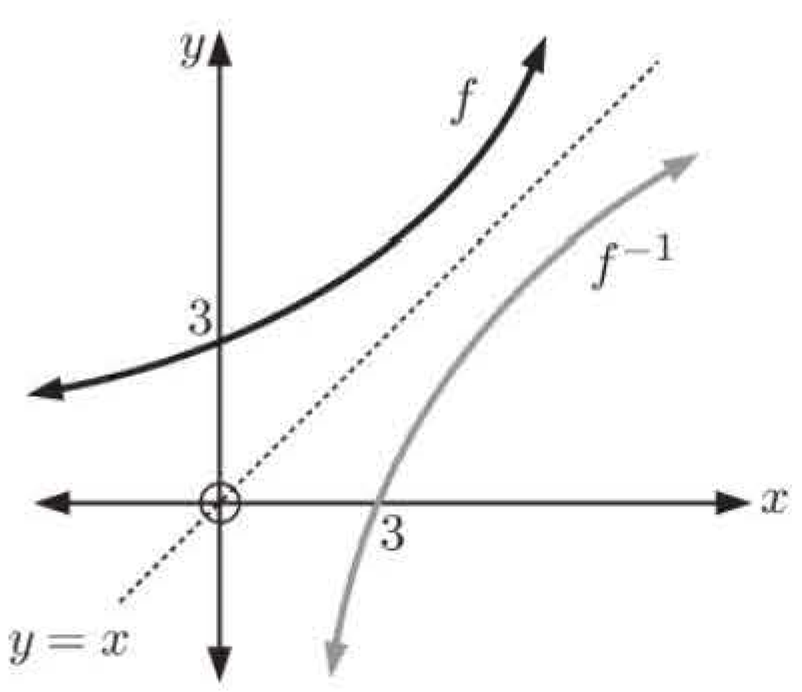
When $f(0) = 5$,
$5 = a(0)^2 + b(0) + c$
$\therefore\ 5 = c$
$\therefore\ c = 5$ .... (1)

When $f(-2) = 21$,
$21 = a(-2)^2 + b(-2) + c$
$= 4a - 2b + c$
$= 4a - 2b + 5$ {using (1)}
$\therefore\ 4a - 2b = 16$
$\therefore\ 2a - b = 8$ and so $b = 2a - 8$ .... (2)

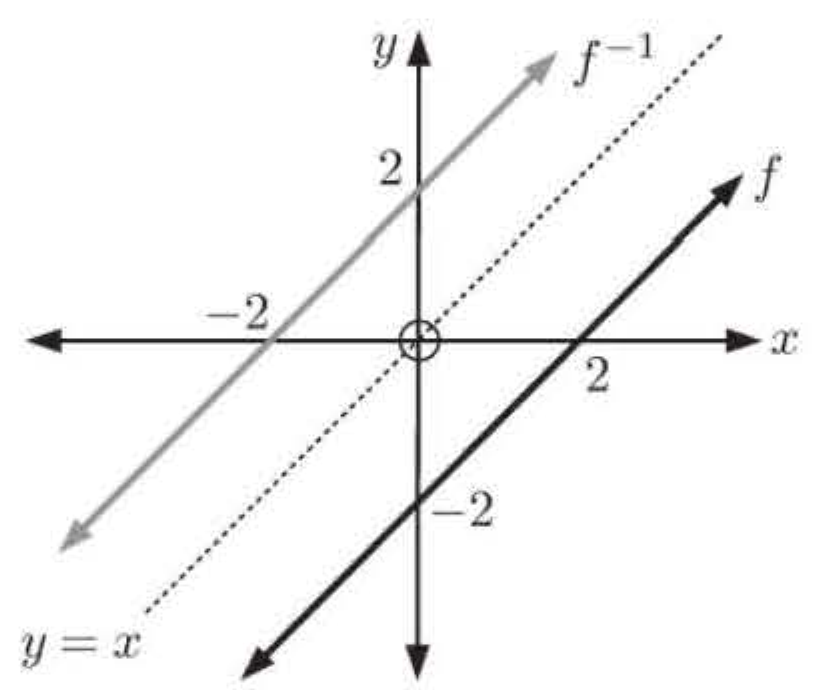
When $f(3) = -4$, $-4 = a(3)^2 + b(3) + c$
$\therefore\ -4 = 9a + 3b + c$
$\therefore\ -4 = 9a + 3b + 5$ {using (1)}
$\therefore\ -4 = 9a + 3(2a - 8) + 5$ {using (2)}
$\therefore\ -4 = 9a + 6a - 24 + 5$
$\therefore\ 15 = 15a$ and so $a = 1$

Now, substituting $a = 1$ into (2) gives $b = 2(1) - 8 = -6$
So, $a = 1$, $b = -6$, $c = 5$.

8 **a**



b



9 $g(x) = |f(x)|$
$$= \begin{cases} f(x) & \text{if } f(x) \geqslant 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$$

Now $g(-x) = |f(-x)|$
$= |-f(x)|$ $\{f(x)$ is an odd function$\}$
$$= \begin{cases} f(x) & \text{if } f(x) \geqslant 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$$
$= g(x)$

$\therefore\ g(x)$ is an even function.

10 **a** f is $y = 7 - 4x$
$\therefore\ f^{-1}$ is $x = 7 - 4y$
$\therefore\ y = \dfrac{7 - x}{4}$
So, $f^{-1}(x) = \dfrac{7 - x}{4}$

b f is $y = \dfrac{3 + 2x}{5}$
$\therefore\ f^{-1}$ is $x = \dfrac{3 + 2y}{5}$
$\therefore\ 5x = 3 + 2y$
$\therefore\ y = \dfrac{5x - 3}{2}$
So, $f^{-1}(x) = \dfrac{5x - 3}{2}$

11 f is $y = 5x - 2$

$\therefore$ f^{-1} is $x = 5y - 2$

$\therefore$ $y = \dfrac{x+2}{5}$

$\therefore$ $f^{-1}(x) = \dfrac{x+2}{5}$

h is $y = \dfrac{3x}{4}$

$\therefore$ h^{-1} is $x = \dfrac{3y}{4}$

$\therefore$ $y = \dfrac{4x}{3}$

$\therefore$ $h^{-1}(x) = \dfrac{4x}{3}$

Now
$$\begin{aligned}(f^{-1} \circ h^{-1})(x) &= f^{-1}(h^{-1}(x))\\ &= f^{-1}\left(\frac{4x}{3}\right)\\ &= \frac{\frac{4x}{3}+2}{5}\\ &= \frac{4x+6}{15}\end{aligned}$$

and
$$\begin{aligned}(h \circ f)(x) &= h(f(x))\\ &= h(5x-2)\\ &= \frac{3(5x-2)}{4}\end{aligned}$$

So, $y = \dfrac{15x-6}{4}$

$\therefore$ $(h \circ f)^{-1}(x)$ is $x = \dfrac{15y-6}{4}$

$\therefore$ $4x = 15y - 6$

$\therefore$ $y = \dfrac{4x+6}{15}$

$\therefore$ $(h \circ f)^{-1}(x) = \dfrac{4x+6}{15}$

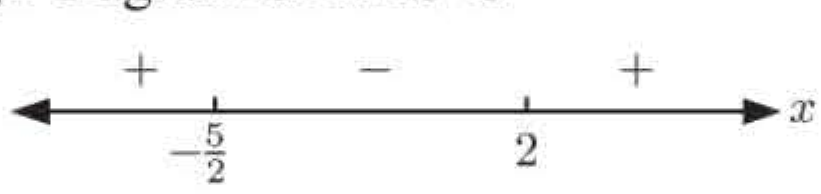
Hence, $(f^{-1} \circ h^{-1})(x) = (h \circ f)^{-1}(x)$ as required.

12 **a** $2x^2 + x \leqslant 10$

$\therefore$ $2x^2 + x - 10 \leqslant 0$

$\therefore$ $(2x+5)(x-2) \leqslant 0$

Sign diagram of LHS is



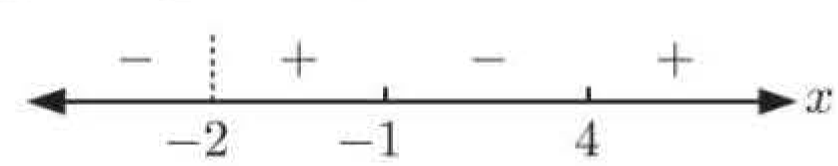
$\therefore$ LHS $\leqslant 0$ when $x \in [-\frac{5}{2}, 2]$

$\therefore$ $2x^2 + x \leqslant 10$ when $x \in [-\frac{5}{2}, 2]$

b $\dfrac{x^2 - 3x - 4}{x+2} > 0$

$\therefore$ $\dfrac{(x+1)(x-4)}{x+2} > 0$

Sign diagram of LHS is



$\therefore$ LHS > 0 when $x \in \,]-2, -1[$ or $]4, \infty[$

13 f is $y = 2x + 11$

so $f^{-1}(x)$ is $x = 2y + 11$

$\therefore$ $y = \dfrac{x-11}{2}$

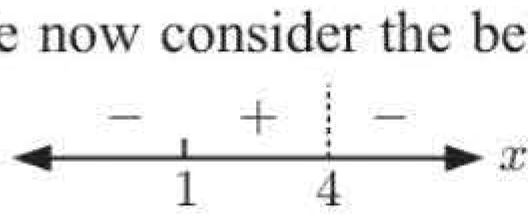
$\therefore$ $f^{-1}(x) = \dfrac{x-11}{2}$

$g(x) = x^2$

$$\begin{aligned}(g \circ f^{-1})(x) &= g(f^{-1}(x))\\ &= g\left(\frac{x-11}{2}\right)\\ &= \left(\frac{x-11}{2}\right)^2\end{aligned}$$

$$\therefore\quad (g \circ f^{-1})(3) = \left(\frac{3-11}{2}\right)^2 = (-4)^2 = 16$$

14 The domain is $\{x \mid x \neq 4\}$, so $x = 4$ is a vertical asymptote.
The range is $\{y \mid y \neq -1\}$, so $y = -1$ is a horizontal asymptote.
We now consider the behaviour of the function near the asymptotes, using the sign diagram



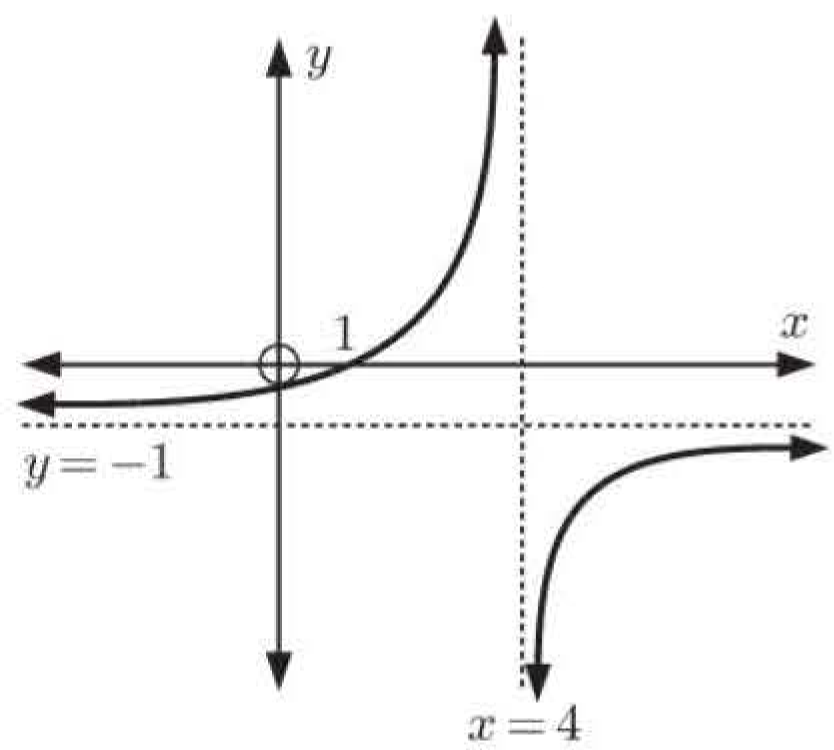
to help us.

As $x \to \infty$, $y \to -1$
As $x \to -\infty$, $y \to -1$

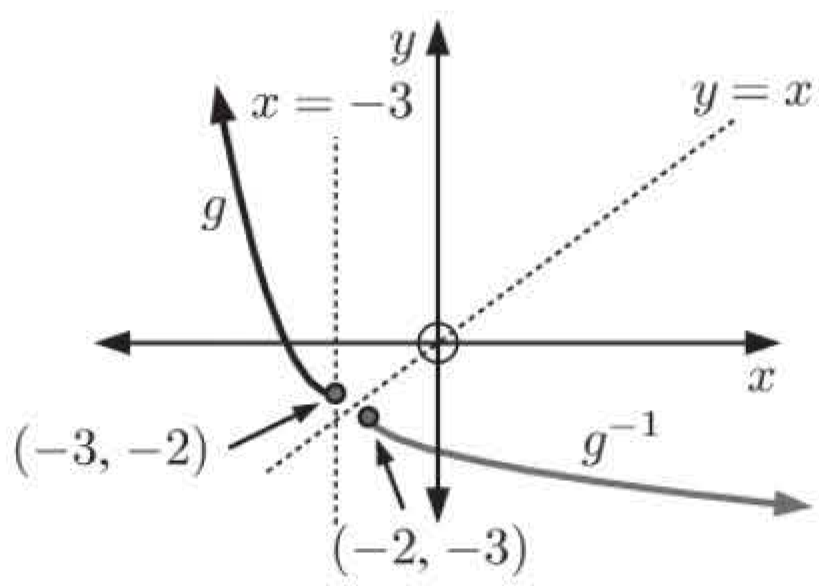
Note that we cannot tell whether the function tends to -1 from above or below.

As $x \to 4^-$, $y \to \infty$
As $x \to 4^+$, $y \to -\infty$

So, the function could be:
(**Note:** There may be other answers.)



15 **a**, **d**



b Any horizontal line through the graph cuts it no more than once. Therefore it has an inverse function.

c
$$\begin{aligned}
g(x) &= x^2 + 6x + 7, \quad x \leqslant -3 \\
\therefore \quad y &= x^2 + 6x + 7, \quad x \leqslant -3 \\
\therefore \quad g^{-1}(x) \text{ is } \quad x &= y^2 + 6y + 7, \quad y \leqslant -3 \\
&= (y+3)^2 - 9 + 7 \\
\therefore \quad x + 2 &= (y+3)^2 \\
\therefore \quad y + 3 &= \pm\sqrt{x+2} \\
\therefore \quad y &= -3 \pm \sqrt{x+2}
\end{aligned}$$

but $y \leqslant -3$, so $y = -3 - \sqrt{x+2}$

So, $g^{-1}(x) = -3 - \sqrt{x+2}$, $x \geqslant -2$

d The range of g is $\{y \mid y \geqslant -2\}$.

e The domain of g^{-1} is $\{x \mid x \geqslant -2\}$, and the range of g^{-1} is $\{y \mid y \leqslant -3\}$.

16 **a** x-intercept is 2.61, y-intercept is 4.29 {using technology}

b $(-0.973, 4.47)$ is a local maximum

c As $x \to 4^-$, $y \to -\infty$
As $x \to 4^+$, $y \to \infty$
$\therefore$ the vertical asymptote is $x = 4$.
As $x \to \infty$, $y \to 7^+$
As $x \to -\infty$, $y \to 3^+$
$\therefore$ the horizontal asymptotes are $y = 3$ and $y = 7$.

d Domain is $\{x \mid x \neq 4\}$.
Range is $\{y \mid y \leqslant 4.47,\ y > 7\}$.

e