Chapter 19

PROPERTIES OF CURVES

EXERCISE 19A

1 **a** We seek the tangent to $y = x - 2x^2 + 3$ at $x = 2$.

When $x = 2$, $y = 2 - 2(2)^2 + 3 = -3$

$\therefore$ the point of contact is $(2, -3)$.

Now $\dfrac{dy}{dx} = 1 - 4x$, so at $x = 2$,

$$\frac{dy}{dx} = 1 - 8 = -7$$

$\therefore$ the tangent has equation

$$\frac{y-(-3)}{x-2} = -7$$

$$\therefore\ y + 3 = -7(x-2)$$

$$\therefore\ y = -7x + 14 - 3$$

$$\therefore\ y = -7x + 11$$

b We seek the tangent to

$y = \sqrt{x} + 1 = x^{\frac{1}{2}} + 1$ at $x = 4$.

When $x = 4$, $y = \sqrt{4} + 1 = 3$

$\therefore$ the point of contact is $(4, 3)$.

Now $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}}$, so at $x = 4$,

$$\frac{dy}{dx} = \frac{1}{2\sqrt{4}} = \tfrac{1}{4}$$

$\therefore$ the tangent has equation

$$\frac{y-3}{x-4} = \tfrac{1}{4}$$

$$\therefore\ 4y - 12 = x - 4$$

$$\therefore\ 4y = x + 8$$

c We seek the tangent to $y = x^3 - 5x$ at $x = 1$.

When $x = 1$, $y = 1^3 - 5(1) = -4$

$\therefore$ the point of contact is $(1, -4)$.

Now $\dfrac{dy}{dx} = 3x^2 - 5$, so at $x = 1$,

$$\frac{dy}{dx} = 3 - 5 = -2$$

$\therefore$ the tangent has equation

$$\frac{y-(-4)}{x-1} = -2$$

$$\therefore\ y + 4 = -2x + 2$$

$$\therefore\ y = -2x - 2$$

d We seek the tangent to $y = \dfrac{4}{\sqrt{x}}$ at $(1, 4)$.

Now $y = \dfrac{4}{\sqrt{x}} = 4x^{-\frac{1}{2}}$

$\therefore$ $\dfrac{dy}{dx} = -2x^{-\frac{3}{2}}$ so at $x = 1$,

$$\frac{dy}{dx} = -2\left(1^{-\frac{3}{2}}\right) = -2$$

$\therefore$ the tangent has equation

$$\frac{y-4}{x-1} = -2$$

$$\therefore\ y - 4 = -2x + 2$$

$$\therefore\ y = -2x + 6$$

e We seek the tangent to

$y = \dfrac{3}{x} - \dfrac{1}{x^2} = 3x^{-1} - x^{-2}$ at $(-1, -4)$.

Now $\dfrac{dy}{dx} = -3x^{-2} + 2x^{-3}$

$= -\dfrac{3}{x^2} + \dfrac{2}{x^3}$ so at $x = -1$,

$$\frac{dy}{dx} = -\frac{3}{(-1)^2} + \frac{2}{(-1)^3}$$

$$= -3 - 2$$

$$= -5$$

$\therefore$ the tangent has equation

$$\frac{y-(-4)}{x-(-1)} = -5$$

$$\therefore\ y + 4 = -5x - 5$$

$$\therefore\ y = -5x - 9$$

f We seek the tangent to

$y = 3x^2 - \dfrac{1}{x} = 3x^2 - x^{-1}$ at $x = -1$.

When $x = -1$, $y = 3(-1)^2 - \frac{1}{(-1)} = 4$

$\therefore$ the point of contact is $(-1, 4)$.

Now $\dfrac{dy}{dx} = 6x + x^{-2}$

$= 6x + \dfrac{1}{x^2}$ so at $x = -1$,

$$\frac{dy}{dx} = 6(-1) + \frac{1}{(-1)^2} = -5$$

$\therefore$ the tangent has equation

$$\frac{y-4}{x-(-1)} = -5$$

$$\therefore\ y - 4 = -5x - 5$$

$$\therefore\ y = -5x - 1$$

2 **a** We seek the normal to $y = x^2$ at $(3, 9)$.

Now $\dfrac{dy}{dx} = 2x$ so at $x = 3$,

$\dfrac{dy}{dx} = 2(3) = 6$

$\therefore$ the normal at $(3, 9)$ has gradient $-\frac{1}{6}$, so the equation of the normal is

$\dfrac{y-9}{x-3} = -\frac{1}{6}$

$\therefore\ 6y - 54 = -x + 3$

$\therefore\ 6y = -x + 57$

b We seek the normal to $y = x^3 - 5x + 2$ at $x = -2$.

When $x = -2$, $y = (-2)^3 - 5(-2) + 2$
$= 4$

and so the point of contact is $(-2, 4)$.

Now $\dfrac{dy}{dx} = 3x^2 - 5$ so at $x = -2$,

$\dfrac{dy}{dx} = 3(-2)^2 - 5 = 7$

$\therefore$ the normal at $(-2, 4)$ has gradient $-\frac{1}{7}$, so the equation of the normal is

$\dfrac{y-4}{x-(-2)} = -\frac{1}{7}$

$\therefore\ 7y - 28 = -(x+2)$

$\therefore\ 7y = -x + 26$

c We seek the normal to $y = \dfrac{5}{\sqrt{x}} - \sqrt{x}$ at $(1, 4)$.

Now $y = 5x^{-\frac{1}{2}} - x^{\frac{1}{2}}$

$\therefore\ \dfrac{dy}{dx} = -\frac{5}{2}x^{-\frac{3}{2}} - \frac{1}{2}x^{-\frac{1}{2}}$ so at $x = 1$,

$\dfrac{dy}{dx} = -\frac{5}{2}\left(1^{-\frac{3}{2}}\right) - \frac{1}{2}\left(1^{-\frac{1}{2}}\right)$

$= -\frac{5}{2} - \frac{1}{2} = -3$

$\therefore$ the normal at $(1, 4)$ has gradient $\frac{1}{3}$, so the equation of the normal is

$\dfrac{y-4}{x-1} = \frac{1}{3}$

$\therefore\ 3y - 12 = x - 1$

$\therefore\ 3y = x + 11$

d We seek the normal to $y = 8\sqrt{x} - \dfrac{1}{x^2}$ at $x = 1$.

When $x = 1$, $y = 8\sqrt{1} - \dfrac{1}{1^2} = 7$

$\therefore$ the point of contact is $(1, 7)$.

Now $y = 8\sqrt{x} - \dfrac{1}{x^2} = 8x^{\frac{1}{2}} - x^{-2}$

$\therefore\ \dfrac{dy}{dx} = 4x^{-\frac{1}{2}} + 2x^{-3}$ so at $x = 1$,

$\dfrac{dy}{dx} = 4 + 2 = 6$

$\therefore$ the normal at $(1, 7)$ has gradient $-\frac{1}{6}$, so the equation of the normal is

$\dfrac{y-7}{x-1} = -\frac{1}{6}$

$\therefore\ 6y - 42 = -x + 1$

$\therefore\ 6y = -x + 43$

3 **a** $y = 2x^3 + 3x^2 - 12x + 1$

$\therefore\ \dfrac{dy}{dx} = 6x^2 + 6x - 12$

Horizontal tangents have gradient $= 0$

so $6x^2 + 6x - 12 = 0$

$\therefore\ x^2 + x - 2 = 0$

$\therefore\ (x+2)(x-1) = 0$

$\therefore\ x = -2$ or $x = 1$

Now at $x = -2$,

$y = 2(-2)^3 + 3(-2)^2 - 12(-2) + 1$
$= 21$

and at $x = 1$,

$y = 2(1)^3 + 3(1)^2 - 12(1) + 1$
$= -6$

$\therefore$ the points of contact are $(-2, 21)$ and $(1, -6)$

$\therefore$ the tangents are $y = 21$ and $y = -6$.

b Now $y = 2\sqrt{x} + \dfrac{1}{\sqrt{x}} = 2x^{\frac{1}{2}} + x^{-\frac{1}{2}}$

$\therefore\ \dfrac{dy}{dx} = x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}} = \dfrac{1}{\sqrt{x}} - \dfrac{1}{2x\sqrt{x}}$

Horizontal tangents have gradient $= 0$

$\therefore\ \dfrac{1}{\sqrt{x}} - \dfrac{1}{2x\sqrt{x}} = 0$

$\therefore\ \dfrac{2x-1}{2x\sqrt{x}} = 0$

$\therefore\ 2x - 1 = 0$

$\therefore\ x = \frac{1}{2}$

Now at $x = \frac{1}{2}$,

$y = 2\sqrt{\frac{1}{2}} + \dfrac{1}{\sqrt{\frac{1}{2}}} = \dfrac{2\left(\frac{1}{2}\right) + 1}{\sqrt{\frac{1}{2}}} = \dfrac{2}{\sqrt{\frac{1}{2}}}$

$= 2\sqrt{2}$

$\therefore$ the only horizontal tangent touches at the curve at $(\frac{1}{2}, 2\sqrt{2})$.

c Now $y = 2x^3 + kx^2 - 3$

$\therefore \dfrac{dy}{dx} = 6x^2 + 2kx$

When $x = 2$, $\dfrac{dy}{dx} = 4$

$\therefore 6(2)^2 + 2k(2) = 4$

$\therefore 24 + 4k = 4$

$\therefore 4k = -20$

$\therefore k = -5$

d Now $y = 1 - 3x + 12x^2 - 8x^3$

$\therefore \dfrac{dy}{dx} = -3 + 24x - 24x^2$

When $x = 1$, $\dfrac{dy}{dx} = -3 + 24 - 24 = -3$

$\therefore$ the tangent at $(1, 2)$ has gradient -3

The tangents to the curve have gradient -3 when $-3 + 24x - 24x^2 = -3$

$\therefore 24x^2 - 24x = 0$

$\therefore 24x(x - 1) = 0$

$\therefore$ when $x = 0$ or $x = 1$

So the other x-value for which the tangent to the curve has gradient -3 is $x = 0$,

and when $x = 0$, $y = 1 - 0 + 0 - 0 = 1$

$\therefore$ the tangent to the curve at $(0, 1)$ is parallel to the tangent at $(1, 2)$.

This tangent has equation $\dfrac{y-1}{x-0} = -3$

or $y = -3x + 1$.

4 **a** Now $y = x^2 + ax + b$

$\therefore \dfrac{dy}{dx} = 2x + a$

At $x = 1$, $\dfrac{dy}{dx} = 2 + a$

$\therefore$ the gradient of the tangent to the curve at $x = 1$ will be $2 + a$.

However the equation of the tangent is $2x + y = 6$ or $y = -2x + 6$

and so the gradient of the tangent is -2.

$\therefore 2 + a = -2$

$\therefore a = -4$

So, the curve is $y = x^2 - 4x + b$.

We also know that the tangent contacts the curve when $x = 1$.

$\therefore 1^2 - 4(1) + b = -2(1) + 6$

$\therefore 1 - 4 + b = 4$

$\therefore b = 7$

$\therefore a = -4, \quad b = 7$

b Now $y = a\sqrt{x} + \dfrac{b}{\sqrt{x}} = ax^{\frac{1}{2}} + bx^{-\frac{1}{2}}$

$\therefore \dfrac{dy}{dx} = \dfrac{a}{2}x^{-\frac{1}{2}} - \dfrac{b}{2}x^{-\frac{3}{2}}$

$\therefore$ at $x = 4$, $\dfrac{dy}{dx} = \dfrac{a}{2}\left(4^{-\frac{1}{2}}\right) - \dfrac{b}{2}\left(4^{-\frac{3}{2}}\right)$

$= \dfrac{a}{2}\left(\dfrac{1}{2}\right) - \dfrac{b}{2}\left(\dfrac{1}{8}\right)$

$= \dfrac{a}{4} - \dfrac{b}{16}$

$\therefore$ the gradient of the tangent to the curve at $x = 4$ will be $\dfrac{a}{4} - \dfrac{b}{16} = \dfrac{4a - b}{16}$

However the equation of the *normal* is $4x + y = 22$ or $y = -4x + 22$.

$\therefore$ the normal has gradient -4.

$\therefore$ the tangent has gradient $\frac{1}{4}$, and so

$\dfrac{4a - b}{16} = \frac{1}{4}$

$\therefore 4a - b = 4$

$\therefore b = 4a - 4 \quad .... (1)$

Also, at $x = 4$ the normal line intersects the curve.

$\therefore a\sqrt{4} + \dfrac{b}{\sqrt{4}} = -4(4) + 22$

$\therefore 2a + \dfrac{b}{2} = 6$

Consequently, $2a + \dfrac{4a - 4}{2} = 6$ {using (1)}

$\therefore 2a + 2a - 2 = 6$

$\therefore 4a = 8$

$\therefore a = 2$

and so $b = 4(2) - 4 = 4$ {from (1)}

$\therefore a = 2, \ b = 4$

c $y = 2x^2 - 1$

$\therefore \dfrac{dy}{dx} = 4x$

$\therefore$ at the point where $x = a$, $\dfrac{dy}{dx} = 4a$

$\therefore$ the gradient of the tangent at the point where $x = a$ is $4a$.

Also, at $x = a$, $y = 2a^2 - 1$.

$\therefore$ the tangent has equation

$\dfrac{y - (2a^2 - 1)}{x - a} = 4a$

$\therefore y - 2a^2 + 1 = 4a(x - a)$

$\therefore y - 2a^2 + 1 = 4ax - 4a^2$

$\therefore 4ax - y = 2a^2 + 1$

5 **a** $y = \sqrt{2x+1} = (2x+1)^{\frac{1}{2}}$

When $x = 4$, $y = \sqrt{2(4)+1} = 3$,

so the point of contact is $(4, 3)$.

Now $\dfrac{dy}{dx} = \frac{1}{2}(2x+1)^{-\frac{1}{2}}(2)$ {chain rule}

$= \dfrac{1}{\sqrt{2x+1}}$

$\therefore$ at $x = 4$, $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{2(4)+1}} = \frac{1}{3}$

$\therefore$ the tangent has equation $\dfrac{y-3}{x-4} = \frac{1}{3}$

$\therefore 3y - 9 = x - 4$

$\therefore 3y = x + 5$

b $y = \dfrac{1}{2-x} = (2-x)^{-1}$

$\therefore$ at $x = -1$, $y = \dfrac{1}{2-(-1)} = \frac{1}{3}$

So the point of contact is $(-1, \frac{1}{3})$.

Now $\dfrac{dy}{dx} = -1(2-x)^{-2}(-1)$ {chain rule}

$= \dfrac{1}{(2-x)^2}$

$\therefore$ at $x = -1$, $\dfrac{dy}{dx} = \dfrac{1}{(2-(-1))^2} = \frac{1}{9}$

$\therefore$ tangent has equation $\dfrac{y-\frac{1}{3}}{x-(-1)} = \frac{1}{9}$

$\therefore 9y - 3 = x + 1$

$\therefore 9y = x + 4$

c We seek the tangent to $f(x) = \dfrac{x}{1-3x}$ at $(-1, -\frac{1}{4})$.

$f(x)$ is a quotient where

$u = x$ and $v = 1 - 3x$

$\therefore u' = 1$ and $v' = -3$

Now $f'(x) = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$\therefore f'(x) = \dfrac{1(1-3x) - x(-3)}{(1-3x)^2}$

$= \dfrac{1}{(1-3x)^2}$

$\therefore f'(-1) = \dfrac{1}{(1-3(-1))^2} = \frac{1}{16}$

$\therefore$ tangent has equation $\dfrac{y-\left(-\frac{1}{4}\right)}{x-(-1)} = \frac{1}{16}$

$\therefore 16y + 4 = x + 1$

$\therefore 16y = x - 3$

d We seek the tangent to $f(x) = \dfrac{x^2}{1-x}$ at $(2, -4)$.

$f(x)$ is a quotient where

$u = x^2$ and $v = 1 - x$

$\therefore u' = 2x$ and $v' = -1$

Now $f'(x) = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$\therefore f'(x) = \dfrac{2x(1-x) - x^2(-1)}{(1-x)^2}$

$= \dfrac{2x - 2x^2 + x^2}{(1-x)^2} = \dfrac{2x - x^2}{(1-x)^2}$

$\therefore f'(2) = \dfrac{2(2) - 2^2}{(1-2)^2} = \dfrac{4-4}{1} = 0$

As the tangent has gradient 0, it is horizontal.

$\therefore$ its equation is $y = c$

Since the contact point is $(2, -4)$, the tangent has equation $y = -4$.

6 **a** We seek the normal to $y = \dfrac{1}{(x^2+1)^2}$ at $(1, \frac{1}{4})$.

As $y = (x^2+1)^{-2}$,

$\dfrac{dy}{dx} = -2(x^2+1)^{-3}(2x) = \dfrac{-4x}{(x^2+1)^3}$

$\therefore$ at $x = 1$, $\dfrac{dy}{dx} = \dfrac{-4}{(1+1)^3} = \frac{-4}{8} = -\frac{1}{2}$

$\therefore$ the normal at $(1, \frac{1}{4})$ has gradient 2.

So the equation of the normal is

$\dfrac{y-\frac{1}{4}}{x-1} = 2$

$\therefore y - \frac{1}{4} = 2x - 2$

$\therefore y = 2x - \frac{7}{4}$

b $y = \dfrac{1}{\sqrt{3-2x}}$

$\therefore$ at $x = -3$, $y = \dfrac{1}{\sqrt{3-2(-3)}} = \frac{1}{3}$

$\therefore$ the point of contact is $(-3, \frac{1}{3})$

Now $y = (3-2x)^{-\frac{1}{2}}$

$\therefore \dfrac{dy}{dx} = -\frac{1}{2}(3-2x)^{-\frac{3}{2}}(-2) = (3-2x)^{-\frac{3}{2}}$

$\therefore$ at $x = -3$, $\dfrac{dy}{dx} = (3-2(-3))^{-\frac{3}{2}}$

$= 9^{-\frac{3}{2}} = 3^{-3} = \frac{1}{27}$

$\therefore$ the normal at $(-3, \frac{1}{3})$ has gradient -27.

So the equation of the normal is

$\dfrac{y-\frac{1}{3}}{x-(-3)} = -27$

$\therefore y - \frac{1}{3} = -27(x+3)$

$\therefore y = -27x - \frac{242}{3}$

c $f(x) = \sqrt{x}(1-x)^2$

Since $f(4) = \sqrt{4}(1-4)^2 = 18$,
the point of contact is $(4, 18)$.
Now $f(x)$ is a product where

$u = x^{\frac{1}{2}}$ and $v = (1-x)^2$

$\therefore\ u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = 2(1-x)(-1)$
$= -2(1-x)$

Now $f'(x) = u'v + uv'$ {product rule}

$\therefore\ f'(x) = \frac{1}{2}x^{-\frac{1}{2}}(1-x)^2 - x^{\frac{1}{2}}2(1-x)$

$\therefore\ f'(4) = \frac{1}{2\sqrt{4}}(1-4)^2 - \sqrt{4}(2)(1-4)$
$= \frac{1}{4}(9) - 2(2)(-3) = \frac{57}{4}$

$\therefore$ the normal at $(4, 18)$ has gradient $-\frac{4}{57}$.
So, the equation of the normal is

$$\frac{y-18}{x-4} = -\frac{4}{57}$$

$\therefore\ 57(y-18) = -4(x-4)$

$\therefore\ 57y = -4x + 1042$

d $f(x) = \dfrac{x^2-1}{2x+3}$

Since $f(-1) = \dfrac{(-1)^2-1}{2(-1)+3} = \dfrac{0}{1} = 0$,
the point of contact is $(-1, 0)$.
Now $f(x)$ is a quotient where

$u = x^2 - 1$ and $v = 2x + 3$

$\therefore\ u' = 2x$ and $v' = 2$

Now $f'(x) = \dfrac{u'v - uv'}{v^2}$

$$= \frac{2x(2x+3) - (x^2-1)(2)}{(2x+3)^2}$$

$\therefore\ f'(-1)$

$$= \frac{2(-1)(-2+3) - ((-1)^2-1)(2)}{(2(-1)+3)^2}$$

$$= \frac{-2(1) - (0)(2)}{(1)^2} = -2$$

$\therefore$ the normal at $(-1, 0)$ has gradient $\frac{1}{2}$.
So, the equation of the normal is

$$\frac{y-0}{x-(-1)} = \frac{1}{2}$$

or $2y = x + 1$

7 The tangent has equation $3x + y = 5$ or $y = -3x + 5$

$\therefore$ the tangent has gradient -3 .... (1)

Also, at $x = -1$, $y = -3(-1) + 5 = 8$

$\therefore$ the tangent contacts the curve at $(-1, 8)$ .... (2)

Now $y = a(1-bx)^{\frac{1}{2}}$, so $\dfrac{dy}{dx} = \frac{1}{2}a(1-bx)^{-\frac{1}{2}}(-b)$

$\therefore\ -3 = \frac{1}{2}a(1+b)^{-\frac{1}{2}}(-b)$ {using (1)}

$\therefore\ 6 = \dfrac{ab}{\sqrt{1+b}}$ .... (3)

Using (2), $(-1, 8)$ must lie on the curve $y = a\sqrt{1-bx}$

$\therefore\ 8 = a\sqrt{1+b}$ .... (4)

$\therefore\ \dfrac{6\sqrt{1+b}}{b} = \dfrac{8}{\sqrt{1+b}}$ {equating as in (3) and (4)}

$\therefore\ 6(1+b) = 8b$

$\therefore\ 6 + 6b = 8b$

$\therefore\ 6 = 2b$

$\therefore\ b = 3$ and $a = \frac{8}{\sqrt{4}} = 4$

8 $f : x \mapsto \dfrac{x}{\sqrt{2-x}}$

a f is defined when $2 - x > 0$

$\therefore\ x < 2$

$\therefore$ domain of f is $\{x \mid x < 2\}$

b $f(x) = \dfrac{x}{\sqrt{2-x}}$

Now $f(x)$ is a quotient where

$u = x$ and $v = (2-x)^{\frac{1}{2}}$

$\therefore\ u' = 1$ and $v' = \frac{1}{2}(2-x)^{-\frac{1}{2}}(-1)$

Now $f'(x) = \dfrac{u'v - uv'}{v^2}$

$$= \frac{1(2-x)^{\frac{1}{2}} - x\frac{1}{2}(2-x)^{-\frac{1}{2}}(-1)}{(2-x)^{\frac{2}{2}}}$$

$$= \frac{\dfrac{2(2-x)}{2(2-x)^{\frac{1}{2}}} + \dfrac{x}{2(2-x)^{\frac{1}{2}}}}{(2-x)^{\frac{2}{2}}}$$

$$= \frac{2(2-x)+x}{2(2-x)^{\frac{3}{2}}}$$

$$= \frac{4-2x+x}{2(2-x)^{\frac{3}{2}}}$$

$$= \frac{4-x}{2(2-x)^{\frac{3}{2}}}$$

c $f(x) = \dfrac{x}{\sqrt{2-x}} = -1$

$\therefore\ x = -\sqrt{2-x}$

$\therefore\ x^2 = 2-x$

$\therefore\ x^2 + x - 2 = 0$

$\therefore\ (x+2)(x-1) = 0$

$\therefore\ x = -2$ or 1

but $x < 2$ $\therefore\ x = -2$

$\therefore$ the point of contact is $(-2, -1)$.

Now $f'(-2) = \dfrac{4-(-2)}{2(2-(-2))^{\frac{3}{2}}}$

$$= \frac{4+2}{2(4)^{\frac{3}{2}}}$$

$$= \frac{6}{2(2)^3}$$

$$= \tfrac{3}{8}$$

$\therefore$ the normal at $(-2, -1)$ has gradient $-\frac{8}{3}$.

So, the equation of the normal is

$$\frac{y-(-1)}{x-(-2)} = -\tfrac{8}{3}$$

$\therefore\ 3(y+1) = -8(x+2)$

$\therefore\ 3y + 3 = -8x - 16$

$\therefore\ 8x + 3y = -19$

9 **a** $f(x) = e^{-x}$

$\therefore\ f(1) = e^{-1}$

$\therefore$ the point of contact is $\left(1, \dfrac{1}{e}\right)$.

Now $f'(x) = -e^{-x}$

$\therefore\ f'(1) = -e^{-1} = -\dfrac{1}{e}$

So, the gradient of the tangent is $-\dfrac{1}{e}$

$\therefore$ the tangent has equation $\dfrac{y - \frac{1}{e}}{x-1} = -\dfrac{1}{e}$

$\therefore\ e\left(y - \dfrac{1}{e}\right) = -(x-1)$

$\therefore\ ey - 1 = -x + 1$

$\therefore\ x + ey = 2$

or $y = -\dfrac{1}{e}x + \dfrac{2}{e}$

b $y = \ln(2-x)$

so when $x = -1$, $y = \ln 3$

$\therefore$ the point of contact is $(-1, \ln 3)$.

Now $\dfrac{dy}{dx} = \dfrac{-1}{2-x}$

$\therefore$ when $x = -1$, $\dfrac{dy}{dx} = -\dfrac{1}{2+1} = -\frac{1}{3}$

So, the gradient of the tangent is $-\frac{1}{3}$.

$\therefore$ the tangent has equation $\dfrac{y - \ln 3}{x+1} = -\frac{1}{3}$

$\therefore\ 3(y - \ln 3) = -(x+1)$

$\therefore\ 3y - 3\ln 3 = -x - 1$

$\therefore\ x + 3y = 3\ln 3 - 1$

c $y = \ln\sqrt{x}$

$= \ln x^{\frac{1}{2}}$

$= \frac{1}{2}\ln x$

$\therefore$ when $y = -1$, $-1 = \frac{1}{2}\ln x$

$\therefore\ \ln x = -2$

$\therefore\ x = e^{-2}$

$\therefore\ x = \dfrac{1}{e^2}$

$\therefore$ the point of contact is $\left(\dfrac{1}{e^2}, -1\right)$

Now $\dfrac{dy}{dx} = \frac{1}{2}\left(\dfrac{1}{x}\right) = \dfrac{1}{2x}$, so at the point of contact, $\dfrac{dy}{dx} = \dfrac{1}{2e^{-2}} = \dfrac{e^2}{2}$

$\therefore$ the tangent has gradient $\dfrac{e^2}{2}$ and the normal has gradient $-\dfrac{2}{e^2}$

$\therefore$ the normal has equation $\dfrac{y+1}{x-\dfrac{1}{e^2}} = -\dfrac{2}{e^2}$

$$\therefore\ e^2(y+1) = -2\left(x - \frac{1}{e^2}\right)$$

$$\therefore\ e^2y + e^2 = -2x + \frac{2}{e^2}$$

$$\therefore\ 2x + e^2y = \frac{2}{e^2} - e^2 \quad \text{or} \quad y = -\frac{2}{e^2}x + \frac{2}{e^4} - 1$$

10 $y = \dfrac{\cos x}{1+\sin x}$

Now since y is a quotient where $u = \cos x$ and $v = 1 + \sin x$

$\therefore\ u' = -\sin x$ and $v' = \cos x$

So
$$\begin{aligned}\frac{dy}{dx} &= \frac{u'v - uv'}{v^2}\\ &= \frac{(-\sin x)(1+\sin x) - \cos x(\cos x)}{(1+\sin x)^2}\\ &= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2}\\ &= \frac{-1-\sin x}{(1+\sin x)^2} \qquad \{\sin^2 x + \cos^2 x = 1\}\\ &= -\frac{(1+\sin x)}{(1+\sin x)^2}\\ &= \frac{-1}{1+\sin x}\end{aligned}$$

Since $\dfrac{-1}{1+\sin x}$ never equals 0, there are no horizontal tangents.

11 **a** $y = \sin x \quad \therefore\ \dfrac{dy}{dx} = \cos x$

When $x = 0$, $\dfrac{dy}{dx} = \cos 0 = 1$

$\therefore$ the tangent has equation $\dfrac{y-0}{x-0} = 1$

or $y = x$

b $y = \tan x \quad \therefore\ \dfrac{dy}{dx} = \dfrac{1}{\cos^2 x}$

When $x = 0$, $\dfrac{dy}{dx} = \dfrac{1}{\cos^2 0} = 1$

$\therefore$ the tangent has equation $\dfrac{y-0}{x-0} = 1$

or $y = x$

c $y = \cos x \quad \therefore\ \dfrac{dy}{dx} = -\sin x$

When $x = \frac{\pi}{6}$, $y = \frac{\sqrt{3}}{2}$

and $\dfrac{dy}{dx} = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$

So, the normal has gradient 2,

and its equation is $\dfrac{y - \frac{\sqrt{3}}{2}}{x - \frac{\pi}{6}} = 2$

$$\therefore\ y - \frac{\sqrt{3}}{2} = 2x - \frac{\pi}{3}$$

$$\therefore\ 2x - y = \frac{\pi}{3} - \frac{\sqrt{3}}{2}$$

d $y = \dfrac{1}{\sin(2x)} = (\sin(2x))^{-1}$

$$\begin{aligned}\therefore\ \frac{dy}{dx} &= -1(\sin(2x))^{-2} \times 2\cos(2x)\\ &= -\frac{2\cos(2x)}{(\sin(2x))^2}\end{aligned}$$

When $x = \frac{\pi}{4}$, $y = 1$

and $\dfrac{dy}{dx} = -\dfrac{2\cos(\frac{\pi}{2})}{\left(\sin(\frac{\pi}{2})\right)^2} = 0$

$\therefore$ the gradient of the normal is undefined, so the normal is $x = \frac{\pi}{4}$.

12 **a** $y = \sec x$

$\therefore \dfrac{dy}{dx} = \sec x \tan x$

When $x = \frac{\pi}{4}$, $y = \sec\left(\frac{\pi}{4}\right) = \sqrt{2}$

and $\dfrac{dy}{dx} = \sec\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{4}\right) = \sqrt{2}$

$\therefore$ the gradient of the tangent at $(\frac{\pi}{4}, \sqrt{2})$ is $\sqrt{2}$ and the equation of the tangent is:

$$\frac{y - \sqrt{2}}{x - \frac{\pi}{4}} = \sqrt{2}$$

$$\therefore\ y - \sqrt{2} = \sqrt{2}x - \tfrac{\sqrt{2}\pi}{4}$$

$$\therefore\ \sqrt{2}x - y = \tfrac{\sqrt{2}\pi}{4} - \sqrt{2}$$

$$= \sqrt{2}(\tfrac{\pi}{4} - 1)$$

b $y = \cot(\frac{x}{2})$

$\therefore \dfrac{dy}{dx} = -\frac{1}{2}\csc^2(\frac{x}{2})$

When $x = \frac{\pi}{3}$, $y = \cot\left(\frac{\pi}{6}\right) = \sqrt{3}$

and $\dfrac{dy}{dx} = -\frac{1}{2}\csc^2\left(\frac{\pi}{6}\right) = -2$

$\therefore$ the gradient of the tangent at $(\frac{\pi}{3}, \sqrt{3})$ is -2 and the equation of the tangent is:

$$\frac{y - \sqrt{3}}{x - \frac{\pi}{3}} = -2$$

$$\therefore\ y - \sqrt{3} = -2x + \tfrac{2\pi}{3}$$

$$\therefore\ 2x + y = \tfrac{2\pi}{3} + \sqrt{3}$$

13 **a** $y = \csc x$

$\therefore \dfrac{dy}{dx} = -\csc x \cot x$

When $x = \frac{\pi}{6}$, $y = \csc\left(\frac{\pi}{6}\right)$

$$= \frac{1}{\sin\left(\frac{\pi}{6}\right)}$$

$$= \frac{1}{\frac{1}{2}} = 2$$

and $\dfrac{dy}{dx} = -\csc\left(\frac{\pi}{6}\right)\cot\left(\frac{\pi}{6}\right)$

$$= -2\,\frac{\cos\left(\frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{6}\right)}$$

$$= -2\left(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right) = -2\sqrt{3}$$

$\therefore$ the gradient of the normal at $(\frac{\pi}{6}, 2)$ is $\frac{1}{2\sqrt{3}}$

$$\therefore\ \frac{y - 2}{x - \frac{\pi}{6}} = \frac{1}{2\sqrt{3}}$$

$$\therefore\ 2\sqrt{3}(y - 2) = x - \tfrac{\pi}{6}$$

$$\therefore\ 2\sqrt{3}y - 4\sqrt{3} = x - \tfrac{\pi}{6}$$

$$\therefore\ x - 2\sqrt{3}y = \tfrac{\pi}{6} - 4\sqrt{3}$$

b $y = \sqrt{\sec(\frac{x}{3})} = \left(\sec(\frac{x}{3})\right)^{\frac{1}{2}}$

$$\frac{dy}{dx} = \tfrac{1}{2}\left(\sec(\tfrac{x}{3})\right)^{-\frac{1}{2}}\,\frac{\sec(\frac{x}{3})\tan(\frac{x}{3})}{3}$$

$$= \tfrac{1}{6}\sec^{\frac{1}{2}}(\tfrac{x}{3})\tan(\tfrac{x}{3})$$

When $x = \pi$, $y = \sqrt{\sec(\frac{\pi}{3})}$

$$= \sqrt{\frac{1}{\cos(\frac{\pi}{3})}}$$

$$= \sqrt{\frac{1}{\frac{1}{2}}} = \sqrt{2}$$

and $\dfrac{dy}{dx} = \frac{1}{6}\sec^{\frac{1}{2}}(\frac{\pi}{3})\tan(\frac{\pi}{3})$

$$= \tfrac{1}{6}\sqrt{2}\,\frac{\sin(\frac{\pi}{3})}{\cos(\frac{\pi}{3})}$$

$$= \frac{\sqrt{2}}{6}\,\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}$$

$$= \tfrac{\sqrt{2}}{6}\sqrt{3} = \tfrac{1}{\sqrt{6}}$$

$\therefore$ the gradient of the normal at $(\pi, \sqrt{2})$ is $-\sqrt{6}$ and the equation of the normal is:

$$\frac{y - \sqrt{2}}{x - \pi} = -\sqrt{6}$$

$$\therefore\ y - \sqrt{2} = -\sqrt{6}(x - \pi)$$

$$\therefore\ y - \sqrt{2} = -\sqrt{6}x + \pi\sqrt{6}$$

$$\therefore\ \sqrt{6}x + y = \pi\sqrt{6} + \sqrt{2}$$

14 **a** Consider the tangent to $y = x^3$ at $x = 2$.
When $x = 2$, $y = 2^3 = 8$ so the point of contact is $(2, 8)$

Now $\dfrac{dy}{dx} = 3x^2$ and so at $x = 2$,

$$\frac{dy}{dx} = 3(2)^2 = 12$$

$\therefore$ the tangent at $(2, 8)$ has gradient 12 and its equation is $\dfrac{y-8}{x-2} = 12$

$$\therefore\ y - 8 = 12x - 24$$
$$\therefore\ y = 12x - 16$$

$\therefore$ the tangent meets the curve where

$$12x - 16 = x^3$$

$\therefore\ x^3 - 12x + 16 = 0$

Because the tangent touches the curve at $x = 2$, there must be a repeated solution at this point.

$\therefore\ (x-2)^2$ must be a factor of this cubic

$\therefore\ (x-2)^2(x+4) = 0$

$\therefore$ the tangent meets the curve again when $x = -4$.

When $x = -4$, $y = (-4)^3 = -64$

$\therefore$ the tangent meets the curve again at $(-4, -64)$.

b Consider the tangent to $y = -x^3 + 2x^2 + 1$ at $x = -1$.
When $x = -1$, $y = -(-1)^3 + 2(-1)^2 + 1$
$= 4$

and so the point of contact is $(-1, 4)$.

Now $\dfrac{dy}{dx} = -3x^2 + 4x$ and so at $x = -1$,

$$\frac{dy}{dx} = -3(-1)^2 + 4(-1) = -7$$

$\therefore$ the tangent at $(-1, 4)$ has gradient -7 and its equation is

$$\frac{y-4}{x-(-1)} = -7$$
$$\therefore\ y - 4 = -7(x+1)$$
$$\therefore\ y = -7x - 3$$

$\therefore$ the tangent meets the curve where

$$-7x - 3 = -x^3 + 2x^2 + 1$$

$\therefore\ x^3 - 2x^2 - 7x - 4 = 0$

Because the tangent touches the curve at $x = -1$, there must be a repeated solution at this point.

$\therefore\ (x+1)^2$ must be a factor of this cubic

$\therefore\ (x+1)^2(x-4) = 0$

$\therefore$ the tangent meets the curve again when $x = 4$.

When $x = 4$, $y = -(4)^3 + 2(4)^2 + 1$
$= -64 + 32 + 1 = -31$

$\therefore$ the tangent meets the curve again at $(4, -31)$.

15 **a** $f(x) = x^2 + \dfrac{4}{x^2}$

$$\therefore\ f'(x) = 2x - 2 \times \frac{4}{x^3}$$
$$\therefore\ f'(x) = 2x - \frac{8}{x^3}$$

b Horizontal tangents have gradient 0, so

$$2x - \frac{8}{x^3} = 0$$
$$\therefore\ 2x^4 = 8$$
$$\therefore\ x^4 = 4$$
$$\therefore\ x = \pm\sqrt{2}$$

c When $x = -\sqrt{2}$,

$$f(-\sqrt{2}) = \left(-\sqrt{2}\right)^2 + \frac{4}{\left(-\sqrt{2}\right)^2} = 2 + \frac{4}{2} = 4$$

$\therefore$ the horizontal tangent at $(-\sqrt{2}, 4)$ is $y = 4$.

When $x = \sqrt{2}$,

$$f(\sqrt{2}) = \left(\sqrt{2}\right)^2 + \frac{4}{\left(\sqrt{2}\right)^2} = 2 + \frac{4}{2} = 4$$

$\therefore$ the horizontal tangent at $(\sqrt{2}, 4)$ is $y = 4$.

$\therefore$ the tangents are the same line because they have the same equation.

16 $y = x^2e^x$ so when $x = 1$, $y = e$

$\therefore$ the point of contact is $(1, e)$.

Now $\dfrac{dy}{dx} = 2xe^x + x^2e^x$

$\therefore$ when $x = 1$, $\dfrac{dy}{dx} = 2e + e = 3e$

$\therefore$ the tangent has equation $\dfrac{y-e}{x-1} = 3e$

$$\therefore\ y - e = 3ex - 3e$$
$$\therefore\ y - 3ex = -2e$$
$$\therefore\ 3ex - y = 2e$$

The tangent cuts the x-axis when

$$y = 0$$
$$\therefore\ 3ex = 2e$$
$$\therefore\ x = \tfrac{2}{3}$$

and the y-axis when

$$x = 0$$
$$\therefore\ -y = 2e$$
$$\therefore\ y = -2e$$

So, A is $(\frac{2}{3}, 0)$ and B is $(0, -2e)$.

17 **a** Consider the tangent to $y = x^2 - x + 9$ at $x = a$.

When $x = a$, $y = a^2 - a + 9$, so the point of contact is $(a, a^2 - a + 9)$.

Now $\dfrac{dy}{dx} = 2x - 1$ and so at $x = a$, $\dfrac{dy}{dx} = 2a - 1$

$\therefore$ the gradient of the tangent at $(a, a^2 - a + 9)$ is $2a - 1$

$\therefore$ the equation of the tangent is $\dfrac{y - (a^2 - a + 9)}{x - a} = 2a - 1$

$$\therefore\ y - (a^2 - a + 9) = (2a - 1)(x - a)$$
$$\therefore\ y = (2a - 1)x - 2a^2 + \cancel{a} + a^2 - \cancel{a} + 9$$
$$\therefore\ y = (2a - 1)x - a^2 + 9 \quad .... (1)$$

But this tangent passes through $(0, 0)$, so $0 = a^2 - 9$

$$\therefore\ (a + 3)(a - 3) = 0$$
$$\therefore\ a = \pm 3$$

$\therefore$ the tangents are: At $a = 3$, $y = (2(3) - 1)x - 3^2 + 9$ {from (1)}

$\therefore\ y = 5x$, with contact at $(3, 15)$.

At $a = -3$, $y = (2(-3) - 1)x - (-3)^2 + 9$ {from (1)}

$\therefore\ y = -7x$, with contact at $(-3, 21)$.

b Let (a, a^3) lie on $y = x^3$.

Now $\dfrac{dy}{dx} = 3x^2$, so at $x = a$, $\dfrac{dy}{dx} = 3a^2$

$\therefore$ the gradient of the tangent at (a, a^3) is $3a^2$

$\therefore$ the equation of the tangent is $\dfrac{y - a^3}{x - a} = 3a^2$ or $y - a^3 = (3a^2)(x - a)$

But this tangent passes through $(-2, 0)$, so $0 - a^3 = 3a^2(-2 - a)$

$$\therefore\ -a^3 = -6a^2 - 3a^3$$
$$\therefore\ 2a^3 + 6a^2 = 0$$
$$\therefore\ 2a^2(a + 3) = 0$$
$$\therefore\ a = 0 \text{ or } -3$$

If $a = 0$, the tangent equation is $y = 0$, with contact point $(0, 0)$.

If $a = -3$, the tangent equation is $y - (-27) = 27(x + 3)$

$\therefore\ y = 27x + 54$, with contact point $(-3, -27)$.

c Let $(a, \sqrt{a})$ lie on $y = \sqrt{x}$.

Now $\dfrac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x}}$, so at $x = a$, $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{a}}$

$\therefore$ the gradient of the tangent at $(a, \sqrt{a})$ is $\dfrac{1}{2\sqrt{a}}$

and the gradient of the normal at this point is $-2\sqrt{a}$.

$\therefore$ the normal has equation $\dfrac{y - \sqrt{a}}{x - a} = -2\sqrt{a}$

or $y - \sqrt{a} = -2\sqrt{a}(x - a)$

But this normal passes through $(4, 0)$, so $0 - \sqrt{a} = -2\sqrt{a}(4 - a)$

$$\therefore\ 2\sqrt{a}(4 - a) - \sqrt{a} = 0$$
$$\therefore\ \sqrt{a}(8 - 2a - 1) = 0$$
$$\therefore\ \sqrt{a}(7 - 2a) = 0$$
$$\therefore\ a = 0 \text{ or } \tfrac{7}{2}$$

But $a = 0$ is the endpoint of the function, so there is no normal here.

When $a = \frac{7}{2}$, $\quad y - \sqrt{\frac{7}{2}} = -2\sqrt{\frac{7}{2}}\left(x - \frac{7}{2}\right)$

$\therefore \quad \sqrt{2}y - \sqrt{7} = -2\sqrt{7}\left(x - \frac{7}{2}\right)$

$\therefore \quad \sqrt{2}y + 2\sqrt{7}x = 7\sqrt{7} + \sqrt{7}$

$\therefore \quad \sqrt{2}y + 2\sqrt{7}x = 8\sqrt{7}$

$\therefore \quad y = -\sqrt{14}x + 4\sqrt{14}$ with contact point $\left(\frac{7}{2}, \sqrt{\frac{7}{2}}\right)$.

18 $y = e^x$ so when $x = a$, $y = e^a$

$\therefore$ the point of contact is (a, e^a).

Now $\dfrac{dy}{dx} = e^x$

$\therefore$ at the point (a, e^a), $\dfrac{dy}{dx} = e^a$

$\therefore$ the tangent has equation $\dfrac{y - e^a}{x - a} = e^a$

or $y - e^a = e^a(x - a)$ .... $(*)$

$\therefore \quad y = e^a x + e^a - ae^a$

Since the tangent passes through the origin, $(0, 0)$ must satisfy $(*)$

$\therefore \quad 0 - e^a = e^a(0 - a)$

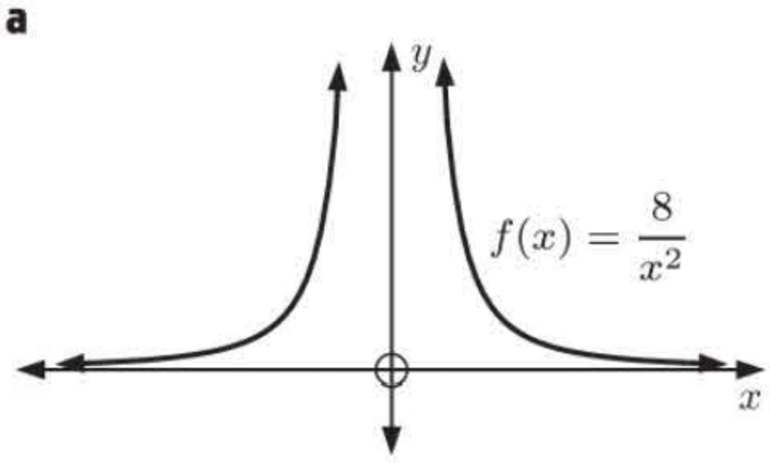
$\therefore \quad -e^a = -ae^a$

$\therefore \quad e^a(a - 1) = 0$

$\therefore \quad a = 1 \qquad \{\text{as } e^a > 0\}$

So the equation of the tangent is $y - e = ex - e$ or $y = ex$.

19 **a**



b Let $\left(a, \dfrac{8}{a^2}\right)$ lie on $f(x) = \dfrac{8}{x^2} = 8x^{-2}$

Now $f'(x) = -16x^{-3} = -\dfrac{16}{x^3}$

$\therefore \quad f'(a) = -\dfrac{16}{a^3}$

$\therefore$ the gradient of the tangent at $\left(a, \dfrac{8}{a^2}\right)$ is $-\dfrac{16}{a^3}$

$\therefore$ the equation of the tangent is $\dfrac{y - \frac{8}{a^2}}{x - a} = -\dfrac{16}{a^3}$

$\therefore \quad a^3y - 8a = -16x + 16a$

$\therefore \quad 16x + a^3y = 24a$

c The tangent cuts the x-axis when $y = 0$

$\therefore \quad 16x = 24a$

$\therefore \quad x = \frac{3}{2}a$

$\therefore$ A is $(\frac{3}{2}a, 0)$.

The tangent cuts the y-axis when $x = 0$

$\therefore \quad a^3y = 24a$

$\therefore \quad y = \dfrac{24}{a^2}$

$\therefore$ B is $\left(0, \dfrac{24}{a^2}\right)$.

d Area of triangle OAB

$= \left|\frac{1}{2} \times \left(\frac{3}{2}a\right) \times \left(\dfrac{24}{a^2}\right)\right|$

$= \dfrac{18}{|a|}$ units^2

As $a \to \infty$, $\dfrac{18}{a} \to 0$

$\therefore$ area $\to 0$

20 $y_1 = \sqrt{x + a}$ and $y_2 = \sqrt{2x - x^2}$

$\therefore \quad \dfrac{dy_1}{dx} = \frac{1}{2}(x + a)^{-\frac{1}{2}}$

$= \dfrac{1}{2\sqrt{x + a}}$

$\therefore \quad \dfrac{dy_2}{dx} = \frac{1}{2}(2x - x^2)^{-\frac{1}{2}}(2 - 2x)$

$= \dfrac{1 - x}{\sqrt{2x - x^2}}$

y_1 intersects y_2 when $y_1 = y_2$

$\therefore \quad \sqrt{x + a} = \sqrt{2x - x^2}$

$\therefore \quad x + a = 2x - x^2$ .... (1)

But $\dfrac{dy_1}{dx} = \dfrac{dy_2}{dx}$ {the gradients are equal when y_1 and y_2 intersect}

$\therefore \dfrac{1}{2\sqrt{x+a}} = \dfrac{1-x}{\sqrt{2x-x^2}}$

$\therefore \dfrac{1}{2\cancel{\sqrt{2x-x^2}}} = \dfrac{1-x}{\cancel{\sqrt{2x-x^2}}}$ {using (1)}

$\therefore \frac{1}{2} = 1 - x$

$\therefore x = 1 - \frac{1}{2} = \frac{1}{2}$

$\therefore \frac{1}{2} + a = 2(\frac{1}{2}) - (\frac{1}{2})^2$ {using (1)}

$\therefore a = 1 - \frac{1}{4} - \frac{1}{2}$

$= \frac{1}{4}$

$y = \sqrt{x+a}$

$= \sqrt{\frac{1}{2} + \frac{1}{4}}$

$= \sqrt{\frac{3}{4}}$

$= \frac{\sqrt{3}}{2}$

$\therefore a = \frac{1}{4}$ and the point of intersection is $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

21 P$(-2, 3)$, Q$(6, -3)$

$\therefore$ the gradient of line (PQ) is $\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{-3-3}{6-(-2)} = -\frac{6}{8} = -\frac{3}{4}$

and the equation of line (PQ) is $\dfrac{y-3}{x-(-2)} = -\frac{3}{4}$

$\therefore 4y - 12 = -3x - 6$

$\therefore y = \dfrac{-3x+6}{4}$

Line (PQ) intersects $y = \dfrac{b}{(x+1)^2}$ when $\dfrac{-3x+6}{4} = \dfrac{b}{(x+1)^2}$

$\therefore b = \dfrac{3(2-x)(x+1)^2}{4}$ .... (1)

Now $y = \dfrac{b}{(x+1)^2} = b(x+1)^{-2}$

$\therefore \dfrac{dy}{dx} = b(-2)(x+1)^{-3} = \dfrac{-2b}{(x+1)^3}$

The line (PQ) is a tangent to $y = \dfrac{b}{(x+1)^2}$ so their gradients are the same at the point of contact.

$\therefore \dfrac{-2b}{(x+1)^3} = -\frac{3}{4}$

$\therefore b = \dfrac{3(x+1)^3}{8}$ .... (2)

$\therefore \dfrac{\cancel{3}(x+1)^{\cancel{3}}}{8} = \dfrac{\cancel{3}(2-x)\cancel{(x+1)^2}}{4}$ {equating (1) and (2)}

$\therefore (x+1) = 2(2-x)$

$\therefore x + 1 = 4 - 2x$

$\therefore 3x = 3$

$\therefore x = 1$

$\therefore b = \dfrac{3(1+1)^3}{8}$

$= \dfrac{3 \times 2^3}{8}$

$= 3$

22 $y = 3e^{-x}$ and $y = 2 + e^x$ meet when $3e^{-x} = 2 + e^x$

$\therefore 3 = 2e^x + e^{2x}$ $\{\times e^x\}$

$\therefore e^{2x} + 2e^x - 3 = 0$

$\therefore (e^x + 3)(e^x - 1) = 0$

$\therefore e^x = -3$ or 1

$\therefore e^x = 1$ and so $x = 0$ {as $e^x > 0$}

Now when $x = 0$, $y = 3e^0 = 3$, so the graphs meet at $(0, 3)$.

For $y = 2 + e^x$, $\dfrac{dy}{dx} = e^x$,

so at the point $(0, 3)$, $\dfrac{dy}{dx} = e^0 = 1$

$\therefore$ the gradient of the tangent at this point is 1

$\therefore$ the tangent has direction vector $\binom{1}{1}$

For $y = 3e^{-x}$, $\dfrac{dy}{dx} = -3e^{-x}$,

so at the point $(0, 3)$, $\dfrac{dy}{dx} = -3$

$\therefore$ the gradient of the tangent at this point is -3

$\therefore$ the tangent has direction vector $\binom{1}{-3}$

If θ is the acute angle between the tangents, then

$$\cos\theta = \frac{|1(1) + 1(-3)|}{\sqrt{1^2 + 1^2}\sqrt{1^2 + (-3)^2}} = \frac{|-2|}{\sqrt{2}\sqrt{10}} = \frac{2}{\sqrt{20}} = \frac{1}{\sqrt{5}}$$

$$\therefore \quad \theta \approx 63.43^\circ$$

23 **a** $y = ax^2$, $a > 0$ touches $y = \ln x$ when $ax^2 = \ln x$

If the curves touch when $x = b$ then $ab^2 = \ln b$ .... (1)

Now for $y = ax^2$, $\dfrac{dy}{dx} = 2ax$

$\therefore$ when $x = b$, $\dfrac{dy}{dx} = 2ab$

and for $y = \ln x$, $\dfrac{dy}{dx} = \dfrac{1}{x}$

$\therefore$ when $x = b$, $\dfrac{dy}{dx} = \dfrac{1}{b}$

Since the curves touch each other, they share a common tangent. $\therefore \quad 2ab = \dfrac{1}{b}$ .... (2)

b Now $ab^2 = \frac{1}{2}$ {from (2)}

and $ab^2 = \ln b$ {from (1)}

$\therefore \quad \ln b = \frac{1}{2}$

$\therefore \quad b = e^{\frac{1}{2}} = \sqrt{e}$

When $x = b = \sqrt{e}$, $y = \ln x = \ln e^{\frac{1}{2}} = \frac{1}{2}$

$\therefore$ the point of contact is $(\sqrt{e}, \frac{1}{2})$.

c $a = \dfrac{1}{2b^2}$ {from (2)}

$\therefore \quad a = \dfrac{1}{2(\sqrt{e})^2} = \dfrac{1}{2e}$

d The tangent has gradient $\dfrac{1}{b} = \dfrac{1}{\sqrt{e}}$ and passes through $(\sqrt{e}, \frac{1}{2})$

$\therefore$ the tangent is $\dfrac{y - \frac{1}{2}}{x - \sqrt{e}} = \dfrac{1}{\sqrt{e}}$ $\quad \therefore \quad y - \frac{1}{2} = \dfrac{1}{\sqrt{e}}\left(x - \sqrt{e}\right)$

$$\therefore \quad y - \tfrac{1}{2} = \frac{1}{\sqrt{e}}x - 1$$

$$\therefore \quad y = e^{-\frac{1}{2}}x - \tfrac{1}{2}$$

24 **a** $x^2 + y^2 = 1$

$\therefore \quad y^2 = 1 - x^2$

$\therefore \quad y = \pm\sqrt{1 - x^2}$

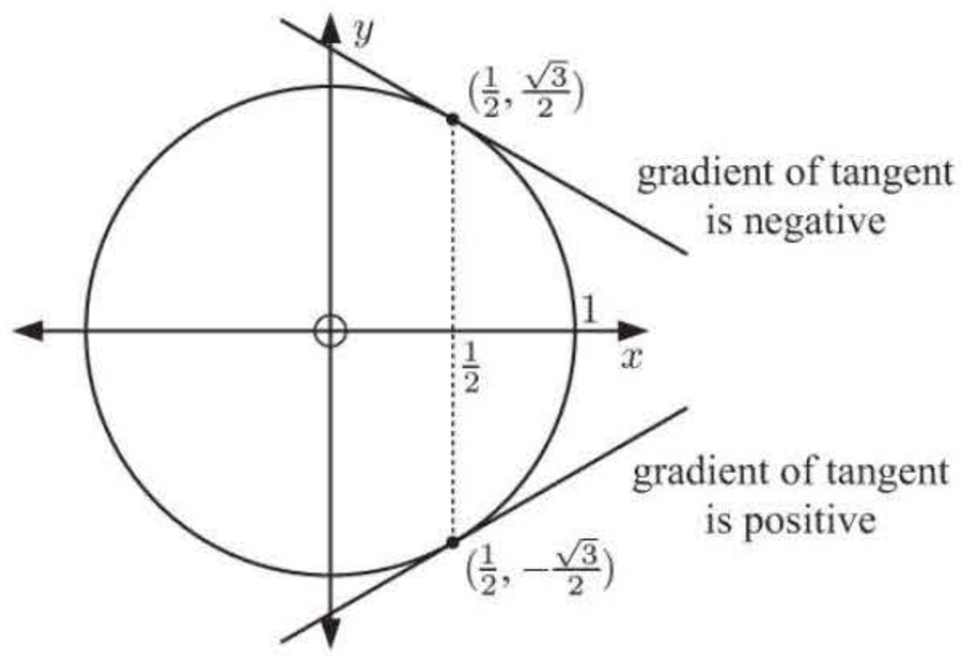
$\quad\quad = \pm(1 - x^2)^{\frac{1}{2}}$

When $x = \frac{1}{2}$, then $y = \pm\sqrt{1 - (\frac{1}{2})^2}$

$$= \pm\sqrt{1 - \tfrac{1}{4}}$$

$$= \pm\sqrt{\tfrac{3}{4}}$$

$$= \frac{\pm\sqrt{3}}{2}$$



So we need to find the tangents at $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

Now $\dfrac{dy}{dx} = \pm\frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)$

$= \dfrac{-x}{\pm\sqrt{1-x^2}}$

$= \mp\dfrac{x}{\sqrt{1-x^2}}$

When $x = \frac{1}{2}$, then $\dfrac{dy}{dx} = \mp\dfrac{\frac{1}{2}}{\sqrt{1-(\frac{1}{2})^2}} = \mp\dfrac{\frac{1}{2}}{\sqrt{\frac{3}{4}}} = \mp\frac{1}{2}\times\frac{2}{\sqrt{3}} = \mp\frac{1}{\sqrt{3}}$

$\therefore$ the gradient of the tangent at $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ is $-\frac{1}{\sqrt{3}}$

and the gradient of the tangent at $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ is $\frac{1}{\sqrt{3}}$

$\therefore$ the equation of the tangent at $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ is $\dfrac{y-\frac{\sqrt{3}}{2}}{x-\frac{1}{2}} = -\frac{1}{\sqrt{3}}$

$\therefore\ \sqrt{3}(y-\frac{\sqrt{3}}{2}) = -(x-\frac{1}{2})$

$\therefore\ \sqrt{3}y - \frac{3}{2} = -x + \frac{1}{2}$

$\therefore\ x + \sqrt{3}y = 2$

and the equation of the tangent at $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ is $\dfrac{y-(-\frac{\sqrt{3}}{2})}{x-\frac{1}{2}} = \frac{1}{\sqrt{3}}$

$\therefore\ \sqrt{3}(y+\frac{\sqrt{3}}{2}) = x - \frac{1}{2}$

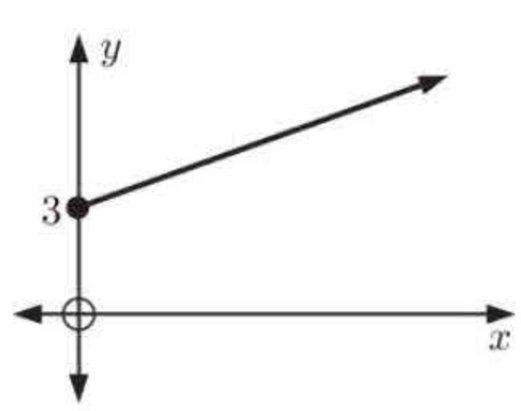
$\therefore\ \sqrt{3}y + \frac{3}{2} = x - \frac{1}{2}$

$\therefore\ x - \sqrt{3}y = 2$

b On the x-axis $y = 0$, $\therefore\ x = 2$.
So, the tangents intersect on the x-axis at $(2, 0)$.

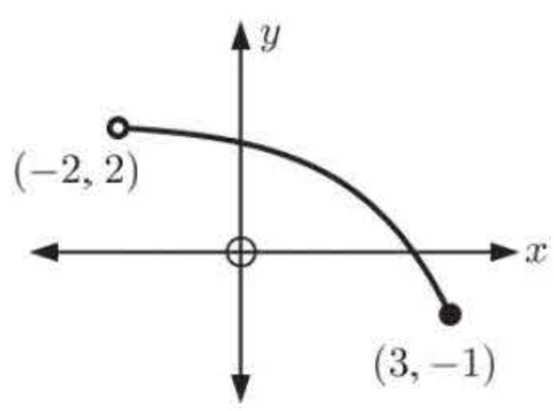
EXERCISE 19B

1 **a**



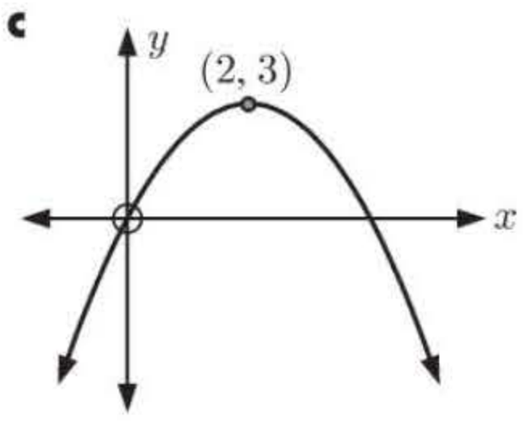
i $x \geqslant 0$ **ii** never

b



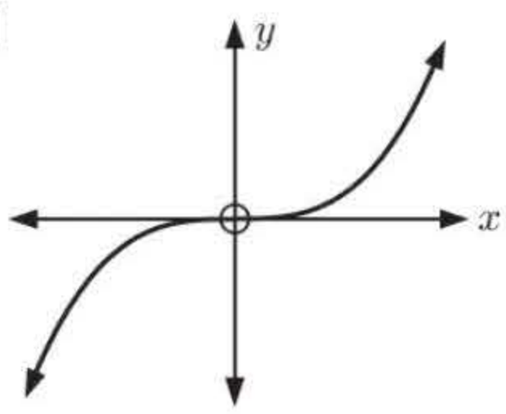
i never **ii** $-2 < x \leqslant 3$

c



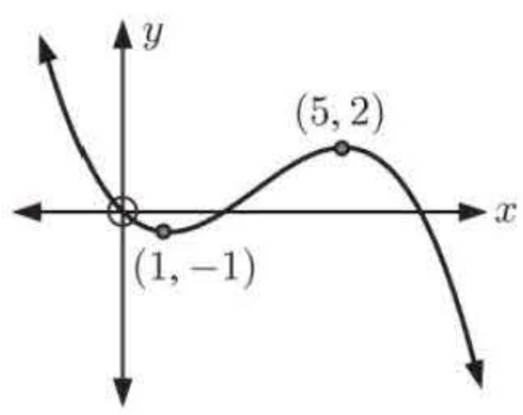
i $x \leqslant 2$ **ii** $x \geqslant 2$

d



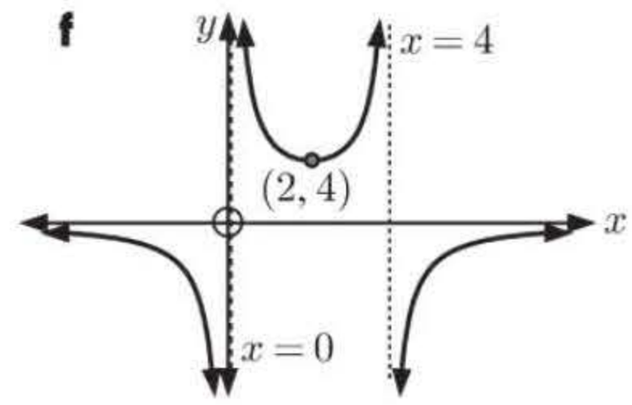
i all real x **ii** never

e



i $1 \leqslant x \leqslant 5$
ii $x \leqslant 1,\ x \geqslant 5$

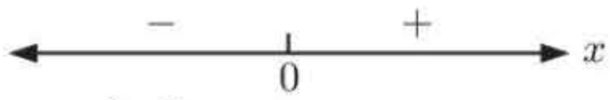
f



i $2 \leqslant x < 4,\quad x > 4$
ii $x < 0,\quad 0 < x \leqslant 2$

2 **a** $f(x) = x^2,\quad f'(x) = 2x$

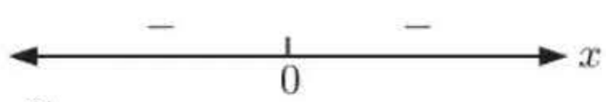
Sign diagram of $f'(x)$:



increasing when $x \geqslant 0$,
decreasing when $x \leqslant 0$

b $f(x) = -x^3,\quad f'(x) = -3x^2$

Sign diagram of $f'(x)$:

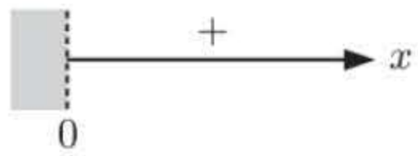


decreasing for all x

c $f(x) = \sqrt{x} = x^{\frac{1}{2}}$,

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x}}$

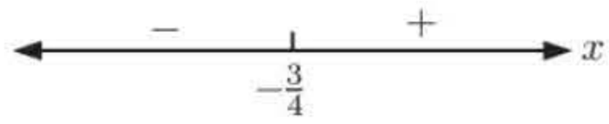
Sign diagram of $f'(x)$:



$f(x)$ is only defined when $x \geqslant 0$
increasing when $x \geqslant 0$, never decreasing

d $f(x) = 2x^2 + 3x - 4$, $f'(x) = 4x + 3$

Sign diagram of $f'(x)$:



increasing when $x \geqslant -\frac{3}{4}$,
decreasing when $x \leqslant -\frac{3}{4}$

e $f(x) = \dfrac{2}{\sqrt{x}} = 2x^{-\frac{1}{2}}$

$f'(x) = -x^{-\frac{3}{2}} = \dfrac{-1}{x\sqrt{x}}$

Sign diagram of $f'(x)$:

$f(x)$ is only defined for $x > 0$
never increasing, decreasing when $x > 0$

f $f(x) = x^3 - 6x^2$, $f'(x) = 3x^2 - 12x$
$= 3x(x-4)$

Sign diagram of $f'(x)$:

increasing when $x \leqslant 0$ or $x \geqslant 4$,
decreasing when $0 \leqslant x \leqslant 4$

g $f(x) = e^x$, $f'(x) = e^x$

Sign diagram of $f'(x)$:

$f(x)$ is increasing for all x

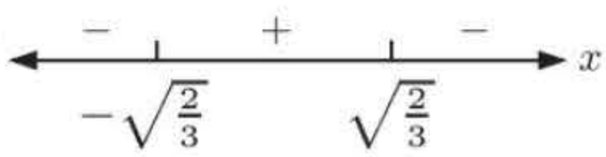
h $f(x) = \ln x$, $f'(x) = \dfrac{1}{x}$

Sign diagram of $f'(x)$:

$f(x)$ is only defined when $x > 0$
increasing when $x > 0$, never decreasing

i $f(x) = -2x^3 + 4x$
$f'(x) = -6x^2 + 4$
$= -2(3x^2 - 2)$

Sign diagram of $f'(x)$:



increasing for $-\sqrt{\frac{2}{3}} \leqslant x \leqslant \sqrt{\frac{2}{3}}$,
decreasing for $x \leqslant -\sqrt{\frac{2}{3}}$ or $x \geqslant \sqrt{\frac{2}{3}}$

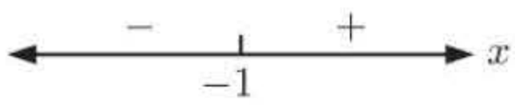
j $f(x) = 3 + e^{-x}$, $f'(x) = -e^{-x}$

Sign diagram of $f'(x)$:

$f(x)$ is decreasing for all x.

k $f(x) = xe^x$, $f'(x) = e^x + xe^x$
$= e^x(1+x)$

Sign diagram of $f'(x)$:

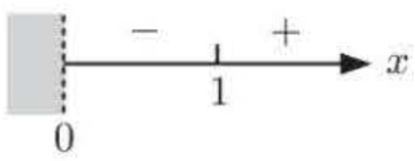


increasing when $x \geqslant -1$
decreasing when $x \leqslant -1$.

l $f(x) = x - 2\sqrt{x} = x - 2x^{\frac{1}{2}}$

$f'(x) = 1 - x^{-\frac{1}{2}} = 1 - \dfrac{1}{\sqrt{x}} = \dfrac{\sqrt{x} - 1}{\sqrt{x}}$

Sign diagram of $f'(x)$:



increasing when $x \geqslant 1$,
decreasing when $0 \leqslant x \leqslant 1$

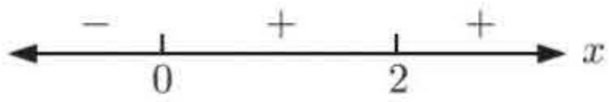
3 **a** $f(x) = -4x^3 + 15x^2 + 18x + 3$
$f'(x) = -12x^2 + 30x + 18$
$= -6(2x^2 - 5x - 3)$
$= -6(2x+1)(x-3)$

Sign diagram of $f'(x)$:

increasing when $-\frac{1}{2} \leqslant x \leqslant 3$,
decreasing when $x \leqslant -\frac{1}{2}$ or $x \geqslant 3$

b $f(x) = 3x^4 - 16x^3 + 24x^2 - 2$,
$f'(x) = 12x^3 - 48x^2 + 48x$
$= 12x(x^2 - 4x + 4)$
$= 12x(x-2)^2$

Sign diagram of $f'(x)$:



increasing when $x \geqslant 0$,
decreasing when $x \leqslant 0$

c $f(x) = 2x^3 + 9x^2 + 6x - 7,$

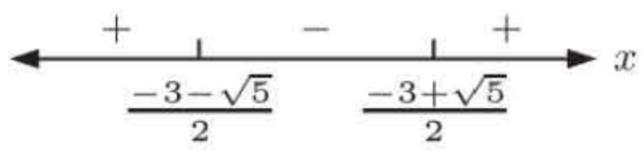
$f'(x) = 6x^2 + 18x + 6$

$= 6(x^2 + 3x + 1)$

$f'(x) = 0$ when $x = \dfrac{-3 \pm \sqrt{9-4}}{2}$

$= \dfrac{-3 \pm \sqrt{5}}{2}$

Sign diagram of $f'(x)$:



increasing for $x \leqslant \dfrac{-3-\sqrt{5}}{2}$ or $x \geqslant \dfrac{-3+\sqrt{5}}{2}$,

decreasing for $\dfrac{-3-\sqrt{5}}{2} \leqslant x \leqslant \dfrac{-3+\sqrt{5}}{2}$

d $f(x) = x^3 - 6x^2 + 3x - 1,$

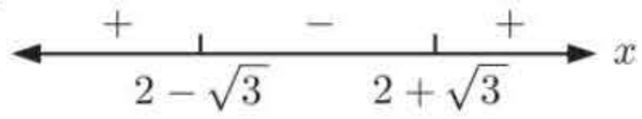
$f'(x) = 3x^2 - 12x + 3$

$= 3(x^2 - 4x + 1)$

$f'(x) = 0$ when $x = \dfrac{4 \pm \sqrt{16-4}}{2}$

$= 2 \pm \sqrt{3}$

Sign diagram of $f'(x)$:



increasing when $x \leqslant 2 - \sqrt{3}$ or $x \geqslant 2 + \sqrt{3}$,

decreasing when $2 - \sqrt{3} \leqslant x \leqslant 2 + \sqrt{3}$

4 **a** $f(x) = \dfrac{4x}{x^2+1}$ is a quotient with

$u = 4x$ and $v = x^2 + 1$

$\therefore\ u' = 4$ and $v' = 2x$

$\therefore\ f'(x) = \dfrac{4(x^2+1) - 4x \times 2x}{(x^2+1)^2}$

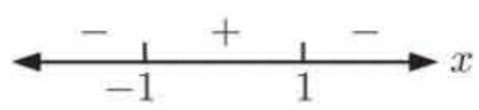
$= \dfrac{4x^2 + 4 - 8x^2}{(x^2+1)^2}$

$= \dfrac{4 - 4x^2}{(x^2+1)^2}$

$= \dfrac{-4(x^2-1)}{(x^2+1)^2}$

$= \dfrac{-4(x+1)(x-1)}{(x^2+1)^2}$

Sign diagram of $f'(x)$:



b $f(x)$ is increasing for $-1 \leqslant x \leqslant 1$, decreasing for $x \leqslant -1$ and $x \geqslant 1$

5 **a** $f(x) = \dfrac{4x}{(x-1)^2}$ is a quotient with

$u = 4x$ and $v = (x-1)^2$

$\therefore\ u' = 4$ and $v' = 2(x-1)$

$\therefore\ f'(x) = \dfrac{4(x-1)^2 - 8x(x-1)}{(x-1)^4}$

$= \dfrac{4(x-1)[(x-1) - 2x]}{(x-1)^4}$

$= \dfrac{4(-1-x)}{(x-1)^3}$

$= \dfrac{-4(x+1)}{(x-1)^3}$

Sign diagram of $f'(x)$:

b $f(x)$ is increasing for $-1 \leqslant x < 1$, decreasing for $x \leqslant -1$ and $x > 1$

6 **a** $f(x) = \dfrac{-x^2 + 4x - 7}{x - 1}$ is a quotient with $u = -x^2 + 4x - 7$ and $v = x - 1$

$\therefore\ u' = -2x + 4$ and $v' = 1$

$\therefore\ f'(x) = \dfrac{(-2x+4)(x-1) - (-x^2+4x-7)(1)}{(x-1)^2}$

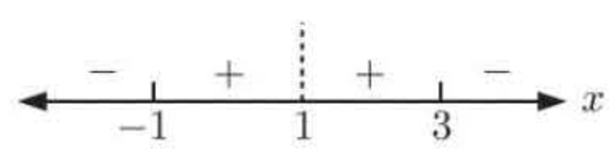
$= \dfrac{-2x^2 + 6x - 4 + x^2 - 4x + 7}{(x-1)^2}$

$= \dfrac{-x^2 + 2x + 3}{(x-1)^2}$

$= \dfrac{-(x^2 - 2x - 3)}{(x-1)^2}$

$= \dfrac{-(x+1)(x-3)}{(x-1)^2}$

Sign diagram of $f'(x)$:



b $f(x)$ is increasing for $-1 \leqslant x < 1$ and $1 < x \leqslant 3$, and decreasing for $x \leqslant -1$ and $x \geqslant 3$.

7 **a** $y = \arccos x, \quad \dfrac{dy}{dx} = \dfrac{-1}{\sqrt{1-x^2}},$

$x \in \,]-1, 1[$

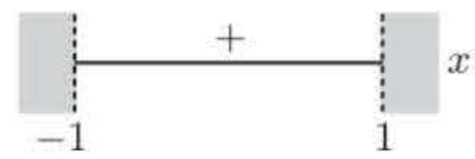
Sign diagram of y:

The function is decreasing for $-1 < x < 1$, never increasing.

b $y = \arcsin x, \quad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}},$

$x \in \,]-1, 1[$

Sign diagram of y:



The function is increasing for $-1 < x < 1$, never decreasing.

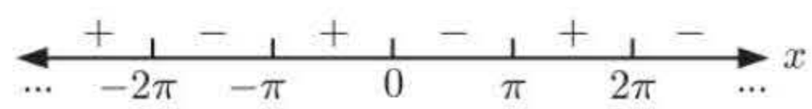
c $y = \arctan x, \quad \dfrac{dy}{dx} = \dfrac{1}{1+x^2}, \quad x \in \mathbb{R}$

Sign diagram of y:

The function is increasing for all x, never decreasing.

d $y = \cos x, \quad \dfrac{dy}{dx} = -\sin x, \quad x \in \mathbb{R}$

Sign diagram of y:



The function is increasing for the interval $-\pi \leqslant x \leqslant 0$ but this is repeated periodically every 2π for the whole real number line.

$\therefore \quad -\pi + 2k\pi \leqslant x \leqslant 0 + 2k\pi, \quad k \in \mathbb{Z}$

$\therefore \quad \pi(2k-1) \leqslant x \leqslant 2k\pi, \quad k \in \mathbb{Z}$

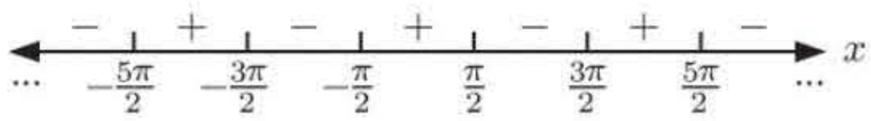
The function is decreasing for the interval $0 \leqslant x \leqslant \pi$ but this is repeated periodically every 2π for the whole real number line.

$\therefore \quad 0 + 2k\pi \leqslant x \leqslant \pi + 2k\pi, \quad k \in \mathbb{Z}$

$\therefore \quad 2k\pi \leqslant x \leqslant \pi(1+2k), \quad k \in \mathbb{Z}$

e $y = \sin x, \quad \dfrac{dy}{dx} = \cos x, \quad x \in \mathbb{R}$

Sign diagram of y:



The function is increasing for the interval $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$ but this is repeated periodically every 2π for the whole real number line.

$\therefore \quad -\frac{\pi}{2} + 2k\pi \leqslant x \leqslant \frac{\pi}{2} + 2k\pi, \quad k \in \mathbb{Z}$

$\therefore \quad \pi(2k - \frac{1}{2}) \leqslant x \leqslant \pi(2k + \frac{1}{2}), \quad k \in \mathbb{Z}$

The function is decreasing for the interval $\frac{\pi}{2} \leqslant x \leqslant \frac{3\pi}{2}$ but this is repeated periodically every 2π for the whole real number line.

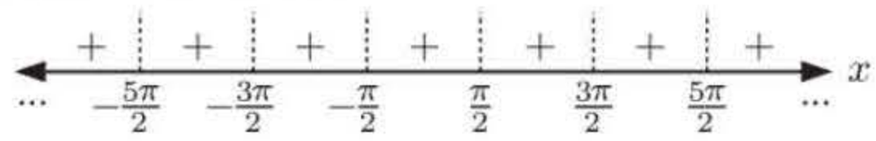
$\therefore \quad \frac{\pi}{2} + 2k\pi \leqslant x \leqslant \frac{3\pi}{2} + 2k\pi, \quad k \in \mathbb{Z}$

$\therefore \quad \pi(\frac{1}{2} + 2k) \leqslant x \leqslant \pi(\frac{3}{2} + 2k), \quad k \in \mathbb{Z}$

f $y = \tan x, \quad \dfrac{dy}{dx} = \sec^2 x, \quad x \in \mathbb{R},$

$x \neq \frac{\pi}{2} \pm k\pi, \quad k \in \mathbb{Z}$

Sign diagram of y:



The function is increasing for the interval $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$ but this is repeated periodically every π for the whole real number line.

$\therefore \quad -\frac{\pi}{2} + k\pi \leqslant x \leqslant \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$

$\therefore \quad \pi(k - \frac{1}{2}) \leqslant x \leqslant \pi(\frac{1}{2} + k), \quad k \in \mathbb{Z}$

It is never decreasing.

8 **a** $f(x) = \dfrac{x^3}{x^2-1}$ is a quotient with $u = x^3$ and $v = x^2 - 1$

$\therefore \quad u' = 3x^2$ and $v' = 2x$

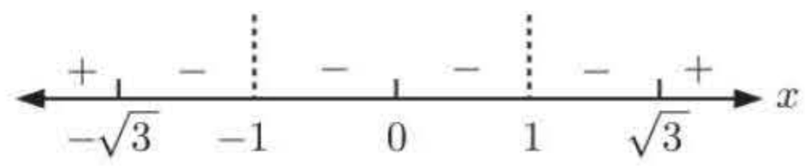
$$\therefore \quad f'(x) = \frac{3x^2(x^2-1) - x^3 \times 2x}{(x^2-1)^2}$$

$$= \frac{3x^4 - 3x^2 - 2x^4}{(x^2-1)^2}$$

$\therefore\ f'(x) = \dfrac{x^2(x^2-3)}{(x^2-1)^2}$

$= \dfrac{x^2(x+\sqrt{3})(x-\sqrt{3})}{(x-1)^2(x+1)^2}$

Sign diagram of $f'(x)$:

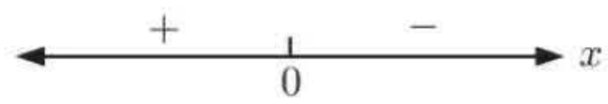


$\therefore\ f(x)$ is increasing for $x \leqslant -\sqrt{3}$ and $x \geqslant \sqrt{3}$, and decreasing for $-\sqrt{3} \leqslant x < -1$, $-1 < x < 1$, and $1 < x \leqslant \sqrt{3}$.

b $f(x) = e^{-x^2}$

$\therefore\ f'(x) = -2xe^{-x^2}$

Sign diagram of $f'(x)$:



$\therefore\ f(x)$ is increasing for $x \leqslant 0$ and decreasing for $x \geqslant 0$.

c $f(x) = x^2 + \dfrac{4}{x-1} = x^2 + 4(x-1)^{-1}$

$\therefore\ f'(x) = 2x - 4(x-1)^{-2} \times 1$

$= 2x - \dfrac{4}{(x-1)^2}$

$= \dfrac{2x(x-1)^2 - 4}{(x-1)^2}$

$= \dfrac{2x(x^2-2x+1) - 4}{(x-1)^2}$

$= \dfrac{2x^3 - 4x^2 + 2x - 4}{(x-1)^2}$

$= \dfrac{(x-2)(2x^2+2)}{(x-1)^2}$

Sign diagram of $f'(x)$:

$\therefore\ f(x)$ is increasing for $x \geqslant 2$, and decreasing for $x < 1$ and $1 < x \leqslant 2$.

d $f(x) = \dfrac{e^{-x}}{x}$ is a quotient with $u = e^{-x}$ and $v = x$

$\therefore\ u' = -e^{-x}$ and $v' = 1$

$\therefore\ f'(x) = \dfrac{-e^{-x}x - e^{-x} \times 1}{x^2}$

$= \dfrac{-e^{-x}(x+1)}{x^2}$

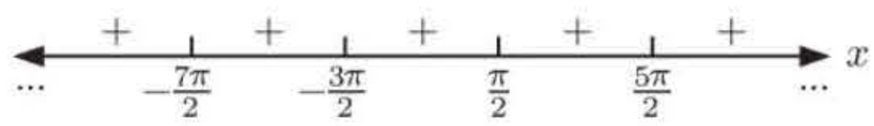
Sign diagram of $f'(x)$:

$\therefore\ f(x)$ is increasing for $x \leqslant -1$, and decreasing for $-1 \leqslant x < 0$ and $x > 0$.

e $f(x) = x + \cos x$

$\therefore\ f'(x) = 1 - \sin x$

Sign diagram of $f'(x)$:

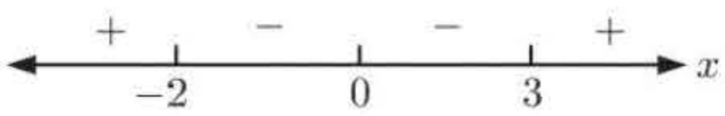


$\therefore\ f(x)$ is increasing for all x, and never decreasing.

EXERCISE 19C

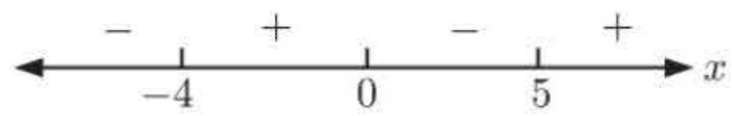
1 **a** A is a *local maximum*, B is a *stationary inflection*, C is a *local minimum*.

b $f'(x)$ has sign diagram:



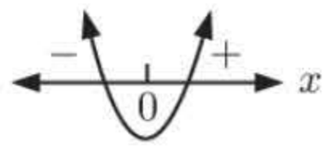
c **i** $f(x)$ is increasing for $x \leqslant -2$ and $x \geqslant 3$ **ii** $f(x)$ is decreasing for $-2 \leqslant x \leqslant 3$

d $f(x)$ has sign diagram:

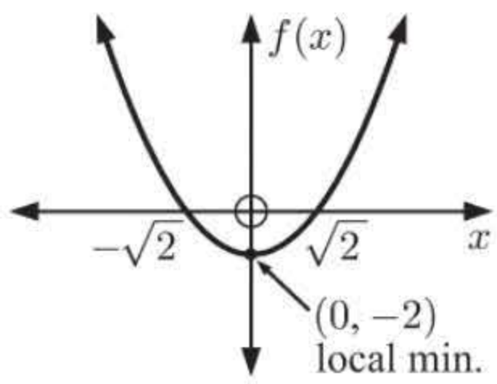


2 **a** $f(x) = x^2 - 2 \quad \therefore \quad f'(x) = 2x$

with sign diagram:

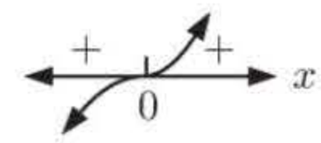


Now $f(0) = -2$,
so there is a local minimum at $(0, -2)$.

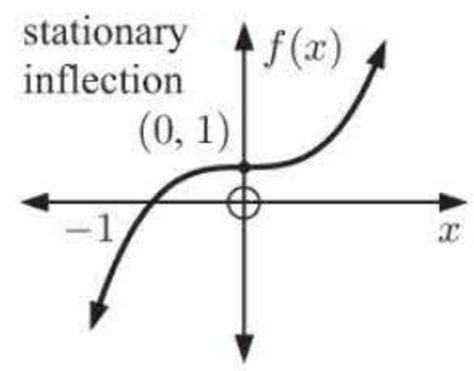


b $f(x) = x^3 + 1 \quad \therefore \quad f'(x) = 3x^2$

with sign diagram:



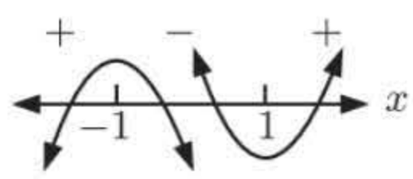
Now $f(0) = 1$,
so there is a stationary inflection at $(0, 1)$.



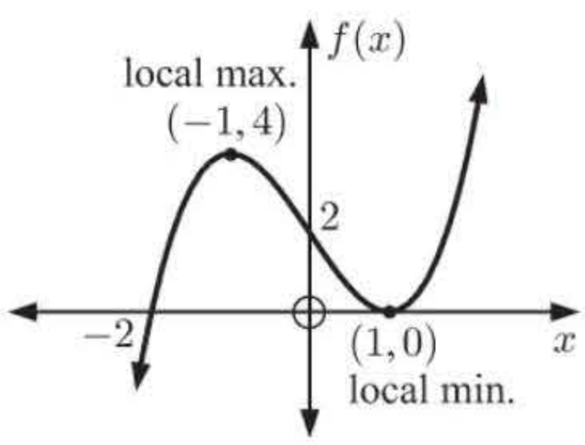
c $f(x) = x^3 - 3x + 2$

$$\begin{aligned} \therefore \quad f'(x) &= 3x^2 - 3 \\ &= 3(x^2 - 1) \\ &= 3(x+1)(x-1) \end{aligned}$$

with sign diagram:



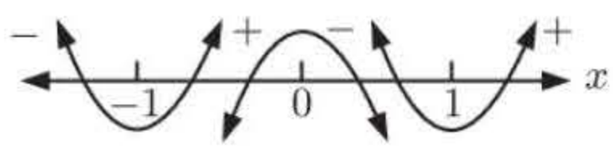
Now $f(-1) = 4$, $f(1) = 0$,
so there is a local maximum at $(-1, 4)$,
and a local minimum at $(1, 0)$.



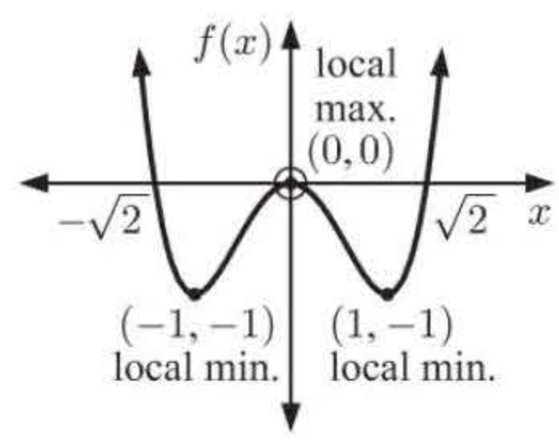
d $f(x) = x^4 - 2x^2$

$$\begin{aligned} \therefore \quad f'(x) &= 4x^3 - 4x \\ &= 4x(x^2 - 1) \\ &= 4x(x+1)(x-1) \end{aligned}$$

with sign diagram:



Now $f(-1) = -1$, $f(1) = -1$, $f(0) = 0$,
so there are local minima at $(-1, -1)$ and $(1, -1)$, and a local maximum at $(0, 0)$.

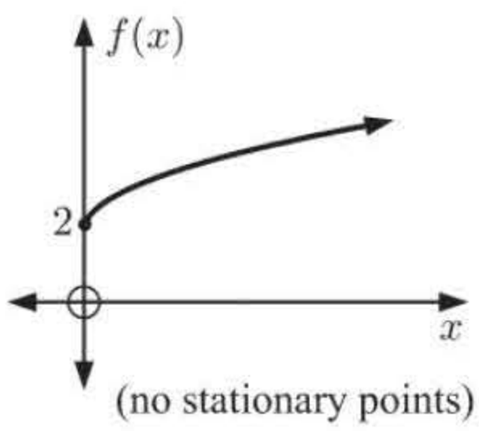


e $f(x) = \sqrt{x} + 2$

$$\begin{aligned} \therefore \quad f'(x) &= \tfrac{1}{2}x^{-\frac{1}{2}} \\ &= \frac{1}{2\sqrt{x}} \neq 0 \end{aligned}$$

with sign diagram:

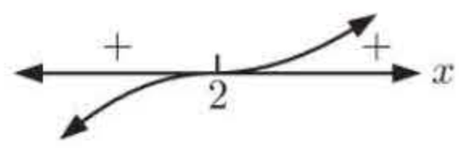
$\therefore$ no stationary points.



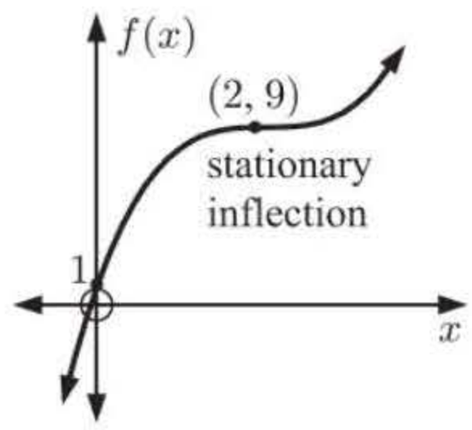
f $f(x) = x^3 - 6x^2 + 12x + 1$

$$\begin{aligned} \therefore \quad f'(x) &= 3x^2 - 12x + 12 \\ &= 3(x^2 - 4x + 4) \\ &= 3(x-2)^2 \end{aligned}$$

with sign diagram:



Now $f(2) = 9$, so there is a stationary inflection at $(2, 9)$.

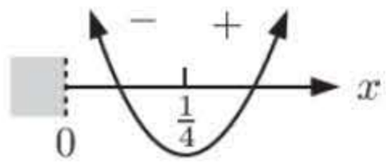


g $f(x) = x - \sqrt{x} = x - x^{\frac{1}{2}}$

$\therefore\ f'(x) = 1 - \frac{1}{2}x^{-\frac{1}{2}}$

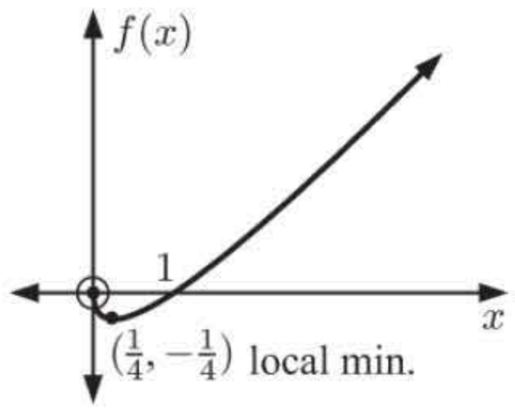
$= 1 - \dfrac{1}{2\sqrt{x}} = \dfrac{2\sqrt{x} - 1}{2\sqrt{x}}$

with sign diagram:



$f(x)$ is defined for all $x \geqslant 0$

Now $f(\frac{1}{4}) = -\frac{1}{4}$, so there is a local minimum at $(\frac{1}{4}, -\frac{1}{4})$.



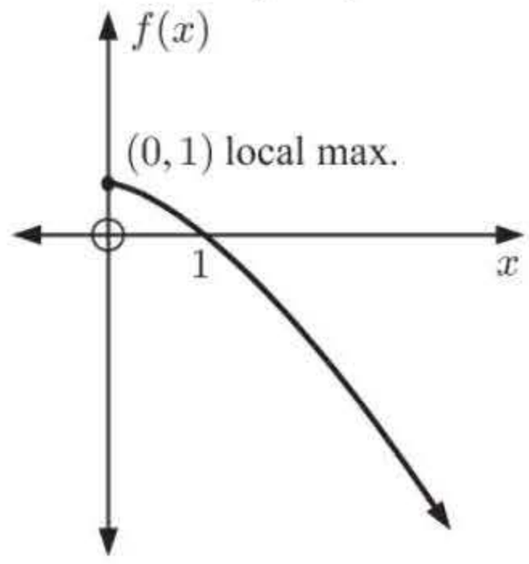
h $f(x) = 1 - x\sqrt{x} = 1 - x^{\frac{3}{2}}$

$\therefore\ f'(x) = -\frac{3}{2}x^{\frac{1}{2}} = \dfrac{-3\sqrt{x}}{2}$

with sign diagram:

$f(x)$ is only defined when $x \geqslant 0$

Now $f(0) = 1$, so there is a local maximum at $(0, 1)$.



i $f(x) = x^4 - 6x^2 + 8x - 3$

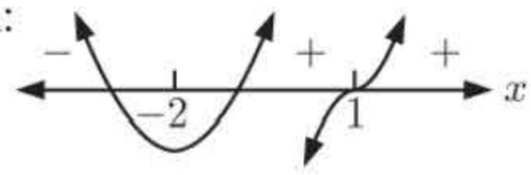
$\therefore\ f'(x) = 4x^3 - 12x + 8$

$= 4(x^3 - 3x + 2)$

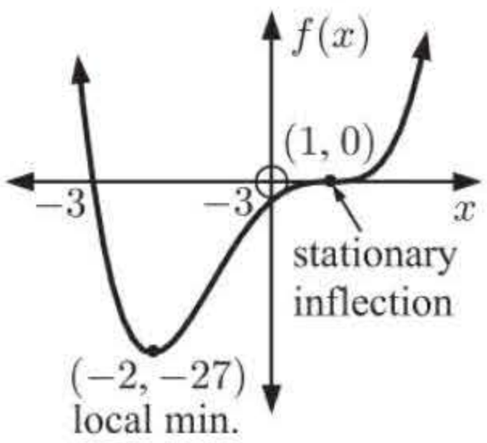
$= 4(x - 1)(x^2 + x - 2)$

$= 4(x - 1)(x + 2)(x - 1)$

with sign diagram:



Now $f(-2) = -27$, $f(1) = 0$, so there is a local minimum at $(-2, -27)$, and a stationary inflection at $(1, 0)$.

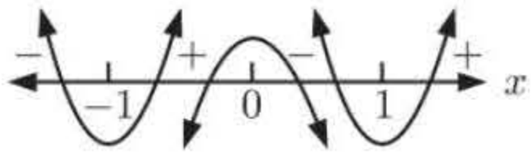


j $f(x) = x^4 - 2x^2 - 8$

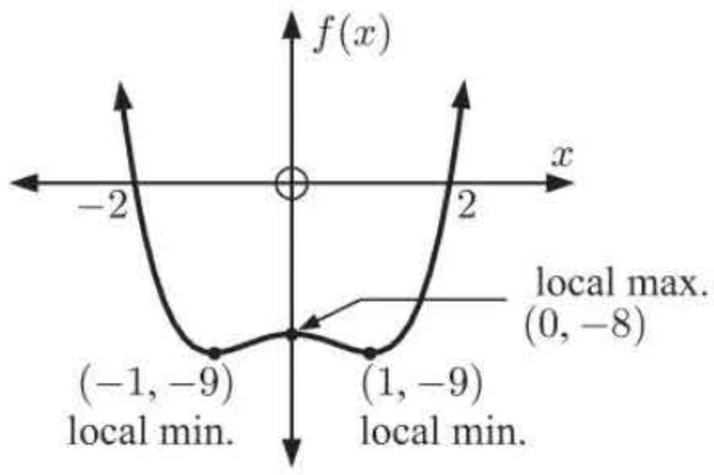
$\therefore\ f'(x) = 4x^3 - 4x$

$= 4x(x^2 - 1) = 4x(x + 1)(x - 1)$

with sign diagram:



Now $f(-1) = -9$, $f(1) = -9$, $f(0) = -8$, so there are local minima at $(-1, -9)$ and $(1, -9)$, and a local maximum at $(0, -8)$.



3 $f(x) = ax^2 + bx + c, \quad a \neq 0$

$\therefore\ f'(x) = 2ax + b$

$f(x)$ has a stationary point when $f'(x) = 0$

$\therefore\ x = -\dfrac{b}{2a}$

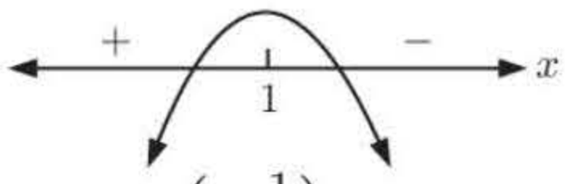
There is a local maximum when $a < 0$ and there is a local minimum when $a > 0$

4 **a** $y = xe^{-x}$

$\therefore\ \dfrac{dy}{dx} = 1e^{-x} - xe^{-x}$ {product rule}

$= e^{-x}(1 - x)$

$= \dfrac{1 - x}{e^x}$ which has sign diagram:

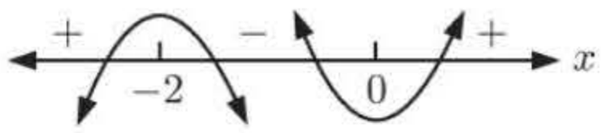


When $x = 1$, $y = 1e^{-1} = \dfrac{1}{e}$, so we have a local maximum at $\left(1, \dfrac{1}{e}\right)$.

b $y = x^2e^x$

$\therefore \dfrac{dy}{dx} = 2xe^x + x^2e^x$ {product rule}

$= xe^x(2+x)$ which has sign diagram:



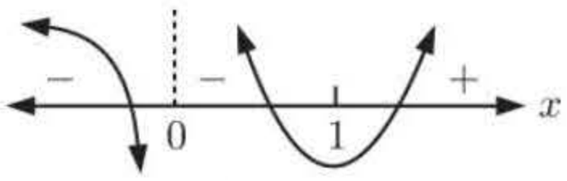
When $x = -2$, $y = 4e^{-2}$, and when $x = 0$, $y = 0$.

So, we have a local maximum at $\left(-2, \dfrac{4}{e^2}\right)$, and a local minimum at $(0, 0)$.

c $y = \dfrac{e^x}{x}$

$\therefore \dfrac{dy}{dx} = \dfrac{e^x x - e^x(1)}{x^2}$ {quotient rule}

$= \dfrac{e^x(x-1)}{x^2}$ which has sign diagram:



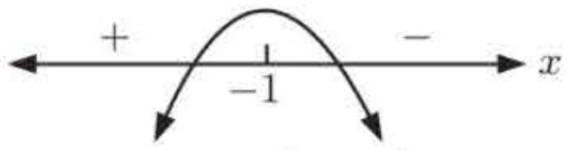
When $x = 1$, $y = \dfrac{e^1}{1} = e$, so we have a local minimum at $(1, e)$.

d $y = e^{-x}(x+2)$

$\therefore \dfrac{dy}{dx} = -e^{-x}(x+2) + e^{-x}$ {product rule}

$= e^{-x}(-x-2+1)$

$= e^{-x}(-x-1)$ which has sign diagram:



When $x = -1$, $y = e(-1+2) = e$, so we have a local maximum at $(-1, e)$.

5 $f(x) = 2x^3 + ax^2 - 24x + 1$

$\therefore f'(x) = 6x^2 + 2ax - 24$

But $f'(-4) = 0$, so $6(-4)^2 + 2a(-4) - 24 = 0$

$\therefore 96 - 8a - 24 = 0$

$\therefore 72 = 8a$

$\therefore a = 9$

6 **a** $f(x) = x^3 + ax + b$

$\therefore f'(x) = 3x^2 + a$

But $f'(-2) = 0$

$\therefore 3(-2)^2 + a = 0$

$\therefore 12 + a = 0$

$\therefore a = -12$

Also, $f(-2) = 3$

$\therefore (-2)^3 - 12(-2) + b = 3$

$\therefore -8 + 24 + b = 3$

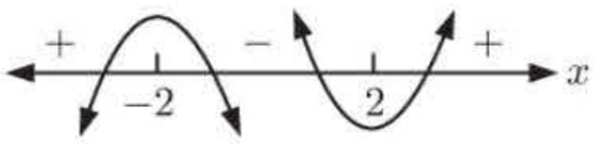
$\therefore b = -13$

b Now $f(x) = x^3 - 12x - 13$

$\therefore f'(x) = 3x^2 - 12$

$= 3(x^2 - 4)$

$= 3(x+2)(x-2)$ with sign diagram:



Now $f(2) = -29$, so there is a local minimum at $(2, -29)$ and a local maximum at $(-2, 3)$.

7 **a** $f(x)$ is defined when $\ln x$ is defined $\therefore$ $f(x)$ is defined for $x > 0$

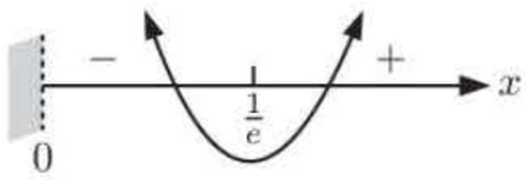
b $f'(x) = \ln x + \dfrac{x}{x}$ {product rule}

$= \ln x + 1$

which is 0 when $\ln x = -1$

$\therefore x = e^{-1} = \frac{1}{e}$

Sign diagram of $f'(x)$ is:



So, there is a local minimum at $\left(\frac{1}{e}, \frac{1}{e}\ln\frac{1}{e}\right)$

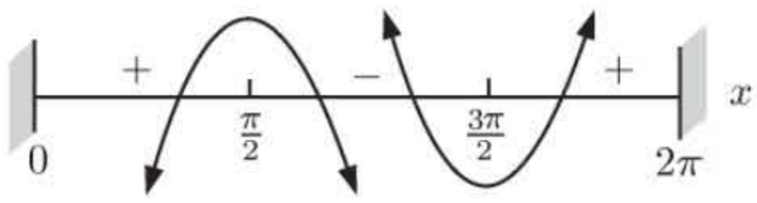
$\therefore$ the global minimum value of $f(x)$ is $\frac{1}{e}\ln e^{-1} = -\frac{1}{e}$

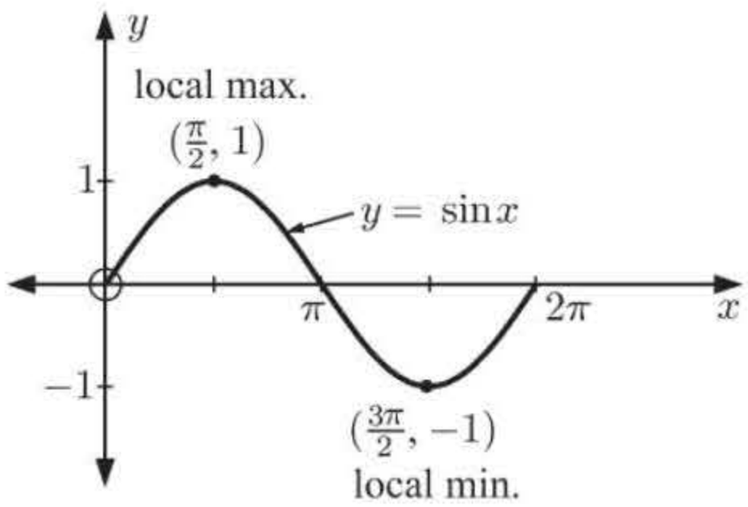
8 **a** If $f(x) = \sin x$ then $f'(x) = \cos x$

Stationary points occur when $f'(x) = 0$,

which is when $x = \frac{\pi}{2}, \frac{3\pi}{2}$

Sign diagram for $f'(x)$ is:





There is a local maximum at $(\frac{\pi}{2}, 1)$

and a local minimum at $(\frac{3\pi}{2}, -1)$.

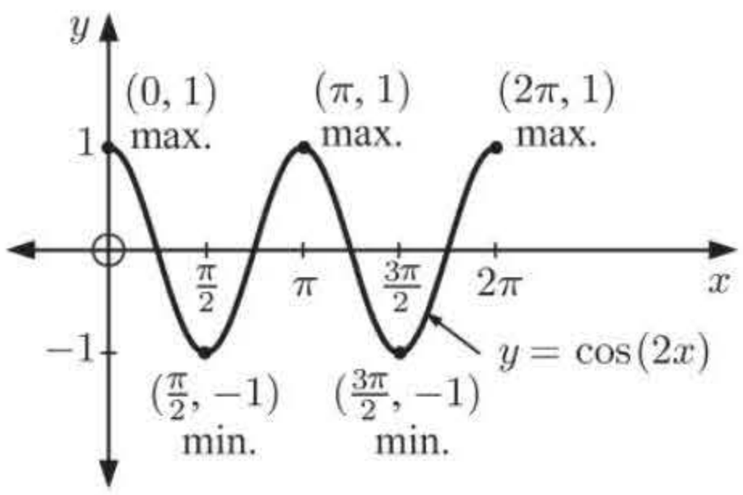
b If $f(x) = \cos(2x)$ then $f'(x) = -2\sin(2x)$

$\therefore\ f'(x) = 0$ when $-2\sin(2x) = 0$

$\therefore\ \sin(2x) = 0$

$\therefore\ 2x = k\pi$ for any integer k

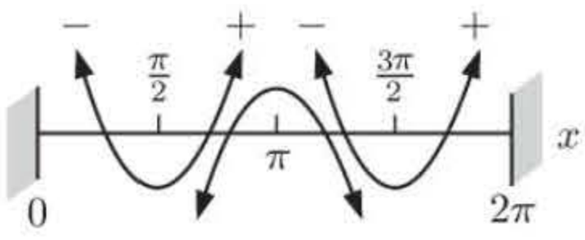
$\therefore\ x = \dfrac{k\pi}{2}$



On the domain $0 \leqslant x \leqslant 2\pi$, $f'(x) = 0$

when $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and 2π.

Sign diagram for $f'(x)$ is:



There are local maxima at $(0, 1)$, $(\pi, 1)$, $(2\pi, 1)$ and local minima at $(\frac{\pi}{2}, -1)$, $(\frac{3\pi}{2}, -1)$.

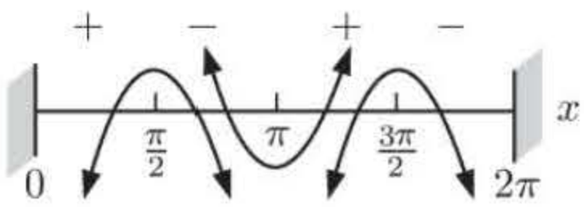
c If $f(x) = \sin^2 x$ then $f'(x) = 2\sin x\cos x = \sin(2x)$

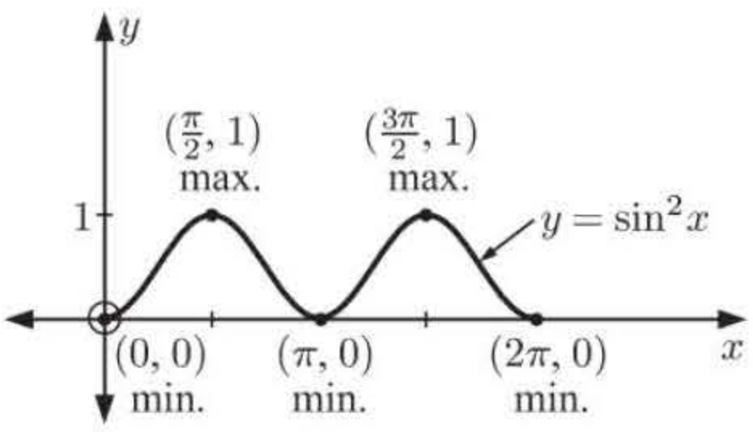
$\therefore\ f'(x) = 0$ when $\sin(2x) = 0$

Using **b**, we know that on the domain $0 \leqslant x \leqslant 2\pi$

$f'(x) = 0$ when $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and 2π.

Sign diagram for $f'(x)$ is:





There are local mimima at $(0, 0)$, $(\pi, 0)$, $(2\pi, 0)$ and local maxima at $(\frac{\pi}{2}, 1)$, $(\frac{3\pi}{2}, 1)$.

d If $f(x) = e^{\sin x}$ then $f'(x) = e^{\sin x} \times \cos x$

$\therefore\ f'(x) = 0$ when $\cos x\, e^{\sin x} = 0$

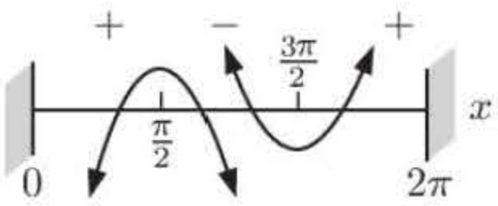
$\therefore\ \cos x = 0$ $\{e^{\sin x} > 0$ for all $x\}$

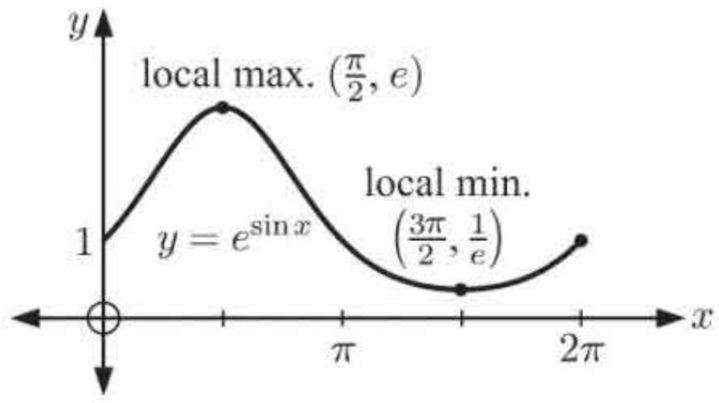
$\therefore\ x = \frac{\pi}{2} + k\pi$, k an integer

On the domain $0 \leqslant x \leqslant 2\pi$, $f'(x) = 0$

when $x = \frac{\pi}{2}, \frac{3\pi}{2}$.

Sign diagram for $f'(x)$ is:





There is a local maximum at $(\frac{\pi}{2}, e)$

and a local minimum at $(\frac{3\pi}{2}, \frac{1}{e})$.

e If $f(x) = \sin(2x) + 2\cos x$ then $f'(x) = 2\cos(2x) - 2\sin x$

$\therefore\ f'(x) = 0$ when $2\cos(2x) - 2\sin x = 0$

$\therefore\ 2(1 - 2\sin^2 x) - 2\sin x = 0$

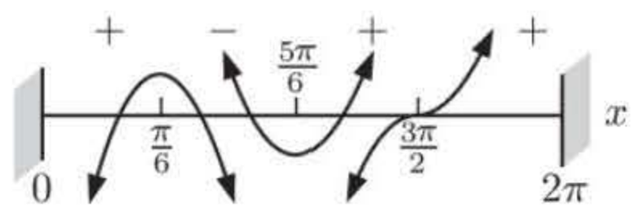
$\therefore\ -2(2\sin^2 x + \sin x - 1) = 0$

$\therefore\ -2(2\sin x - 1)(\sin x + 1) = 0$

$\therefore$ when $\sin x = \frac{1}{2}$ or $\sin x = -1$

On the domain $0 \leqslant x \leqslant 2\pi$, when $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$.

Sign diagram of $f'(x)$:



$f(\frac{\pi}{6}) = \sin(\frac{2\pi}{6}) + 2\cos(\frac{\pi}{6})$

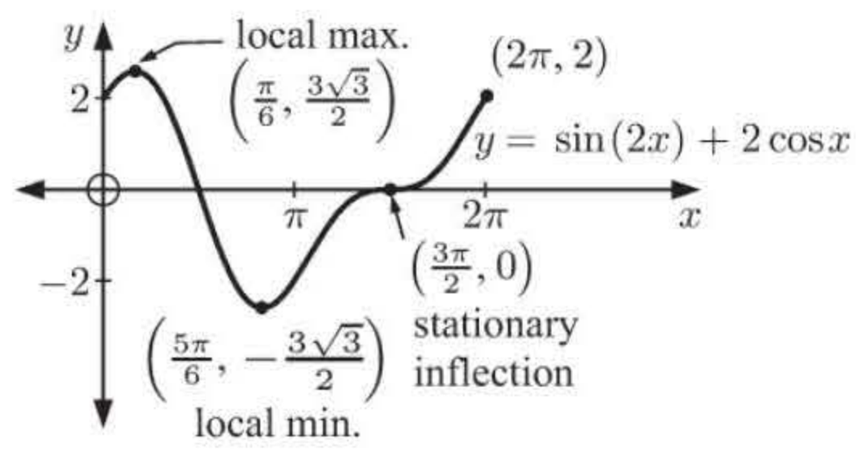
$= \frac{\sqrt{3}}{2} + 2 \times \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$

$f(\frac{5\pi}{6}) = \sin(\frac{10\pi}{6}) + 2\cos(\frac{5\pi}{6})$

$= -\frac{\sqrt{3}}{2} + 2(-\frac{\sqrt{3}}{2}) = -\frac{3\sqrt{3}}{2}$

$f(\frac{3\pi}{2}) = \sin(3\pi) + 2\cos(\frac{3\pi}{2})$

$= 0 + 2 \times 0 = 0$



$\therefore$ there is a local maximum at $(\frac{\pi}{6}, \frac{3\sqrt{3}}{2})$,

a local minimum at $(\frac{5\pi}{6}, -\frac{3\sqrt{3}}{2})$

and a stationary point of inflection at $(\frac{3\pi}{2}, 0)$.

9 Let the cubic polynomial be

$P(x) = ax^3 + bx^2 + cx + d$

$\therefore\ P'(x) = 3ax^2 + 2bx + c$ .... (1)

Now $(0, 2)$ lies on $P(x)$, so $P(0) = 2$

$\therefore\ a(0) + b(0) + c(0) + d = 2$

$\therefore\ d = 2$

The tangent at $(0, 2)$ is $y = 9x + 2$, so

$P'(0) = 9$

$\therefore\ 3a(0) + 2b(0) + c = 9$

$\therefore\ c = 9$ .... (2)

There is a stationary point at $(-1, -7)$, so

$P'(-1) = 0$

$\therefore\ 3a(-1)^2 + 2b(-1) + c = 0$ {using (1)}

$\therefore\ 3a - 2b + c = 0$

So, using (2), $3a - 2b = -9$ .... (3)

Finally, $(-1, -7)$ lies on $P(x)$

$\therefore\ a(-1)^3 + b(-1)^2 + c(-1) + d = -7$

$\therefore\ -a + b - 9 + 2 = -7$

$\therefore\ b - a = 0$

$\therefore\ a = b$

So, using (3), $3a - 2a = -9$

$\therefore\ a = -9$

$\therefore\ a = b = -9$

$\therefore\ P(x) = -9x^3 - 9x^2 + 9x + 2$

10 **a** $f(x) = x^3 - 12x - 2$, for $-3 \leqslant x \leqslant 5$

$\therefore\ f'(x) = 3x^2 - 12$

$= 3(x + 2)(x - 2)$

which is 0 when $x = -2$ or 2

x	-3	-2	2	5
$f(x)$	7	14	-18	63

$\therefore$ the greatest value is 63 when $x = 5$, and the least value is -18 when $x = 2$.

b $f(x) = 4 - 3x^2 + x^3$, for $-2 \leqslant x \leqslant 3$

$\therefore\ f'(x) = -6x + 3x^2$

$= 3x(x - 2)$

which is 0 when $x = 0$ or 2

x	-2	0	2	3
$f(x)$	-16	4	0	4

$\therefore$ greatest value is 4 when $x = 0$ or $x = 3$, least value is -16 when $x = -2$.

11 $y = 4e^{-x}\sin x$

$\therefore \quad \dfrac{dy}{dx} = -4e^{-x}\sin x + 4e^{-x}\cos x$

$\therefore$ stationary points occur when $-4e^{-x}\sin x + 4e^{-x}\cos x = 0$

$\therefore \quad 4e^{-x}(\cos x - \sin x) = 0$

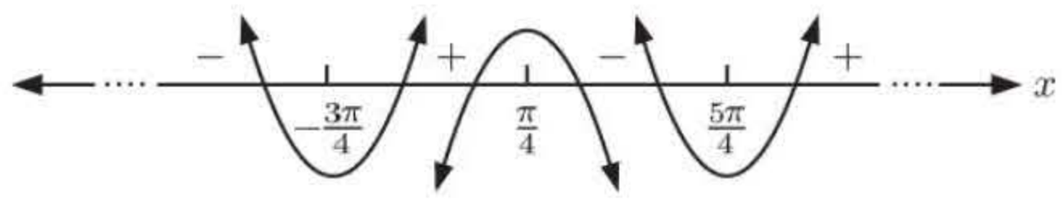
$\therefore \quad \cos x - \sin x = 0 \quad \{e^{-x} > 0 \text{ for all } x\}$

$\therefore \quad \sin x = \cos x$

$\therefore \quad \tan x = 1$

$\therefore \quad x = \frac{\pi}{4} + k\pi, \quad k$ an integer

Sign diagram of $\dfrac{dy}{dx}$ is:



$\therefore \quad y = 4e^{-x}\sin x$ has a local maximum when $x = \frac{\pi}{4}$.

12 $f(t) = ate^{bt^2}$

$\therefore \quad f'(t) = ae^{bt^2} + at \times 2bte^{bt^2}$

$= ae^{bt^2}(1 + 2bt^2)$

There is a maximum of 1 when $t = 2$

$\therefore \quad f'(2) = 0$ and $ae^{b2^2}(1 + 2b2^2) = 0$

$\therefore \quad ae^{4b}(1 + 8b) = 0 \quad \ldots. (1)$

Also the point $(2, 1)$ lies on $f(t)$

$\therefore \quad f(2) = a2e^{b2^2} = 1$

$\therefore \quad ae^{4b} = \frac{1}{2} \quad \ldots. (2)$

Substituting this into (1) $\quad \therefore \quad \frac{1}{2}(1 + 8b) = 0$

$\therefore \quad 1 + 8b = 0$

$\therefore \quad b = -\frac{1}{8}$

Substituting this into (2) $\quad \therefore \quad ae^{4(-\frac{1}{8})} = \frac{1}{2}$

$\therefore \quad ae^{-\frac{1}{2}} = \frac{1}{2}$

$\therefore \quad a = \frac{1}{2}e^{\frac{1}{2}} = \dfrac{\sqrt{e}}{2}$

13 Consider $f(x) = \dfrac{\ln x}{x}$

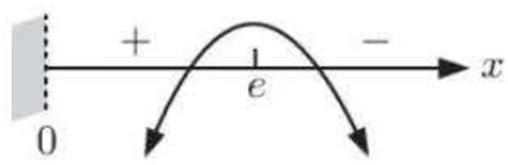
$\therefore \quad f'(x) = \dfrac{\left(\frac{1}{x}\right)x - \ln x(1)}{x^2} = \dfrac{1 - \ln x}{x^2}$

$\therefore \quad f'(x) = 0$ when $1 - \ln x = 0$

$\therefore \quad \ln x = 1$

$\therefore \quad x = e$

Sign diagram of $f'(x)$ is:



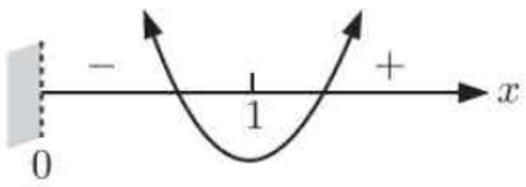
Now $f(e) = \dfrac{\ln e}{e} = \dfrac{1}{e}$

$\therefore$ there is a local maximum at $\left(e, \dfrac{1}{e}\right)$

$\therefore \quad f(x) \leqslant \dfrac{1}{e}$ for all x, and so $\dfrac{\ln x}{x} \leqslant \dfrac{1}{e}$ for all $x > 0$

14 **a** $f(x) = x - \ln x$

$\therefore\ f'(x) = 1 - \dfrac{1}{x} = \dfrac{x-1}{x}$ and the sign diagram of $f'(x)$ is:



$\therefore\ f(x)$ has a local minimum at $(1,\ 1 - \ln 1)$ or $(1,\ 1)$. This is the only turning point.

b $f(x) \geqslant 1$ for all $x > 0$

$\therefore\ x - \ln x \geqslant 1$

$\therefore\ \ln x \leqslant x - 1$ for all $x > 0$

15 **a** $f(x) = \sec x = \dfrac{1}{\cos x}$

$\therefore\ f(x)$ is undefined when $\cos x = 0$.

But $\cos x = 0$ when $x = \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$

$\therefore$ for $0 \leqslant x \leqslant 2\pi$, $f(x)$ is undefined when $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$.

b $f'(x) = \sec x \tan x$

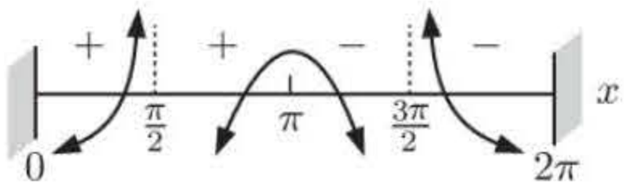
$= \dfrac{1}{\cos x} \times \dfrac{\sin x}{\cos x}$

$f'(x) = 0$ when $\sin x = 0$

$\therefore\ f'(x) = 0$ when $x = 0 + k\pi,\ k \in \mathbb{Z}$

$\therefore$ for $0 \leqslant x \leqslant 2\pi$, $x = 0,\ \pi,\ 2\pi$

$\therefore$ sign diagram of $f'(x)$ is:



$f(0) = \sec 0 = 1$

$f(\pi) = \sec \pi = -1$

$f(2\pi) = \sec 2\pi = 1$

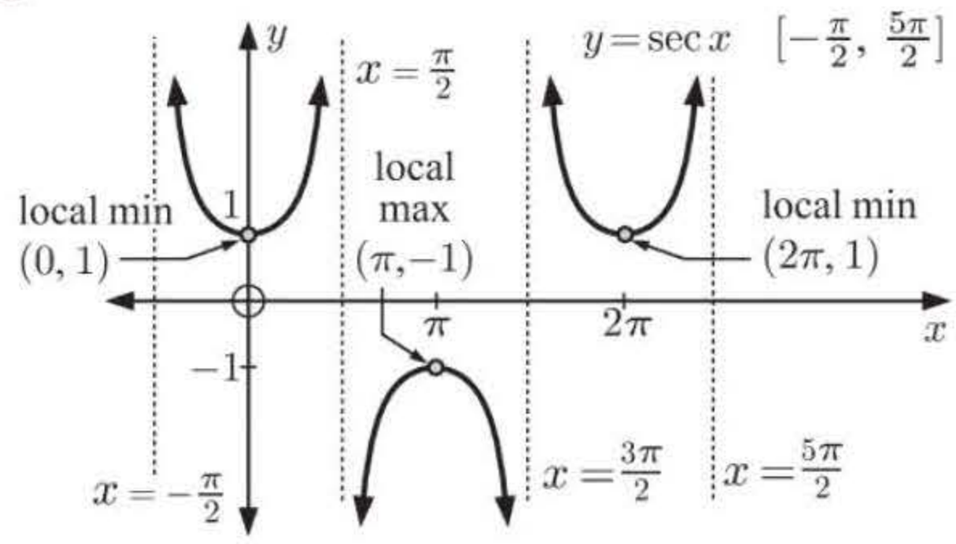
$\therefore$ there is a local maximum at $(\pi,\ -1)$ and local minima at $(0,\ 1)$ and $(2\pi,\ 1)$.

c $f(x) = \sec x$

$$\begin{aligned}\therefore\ f(x+2\pi) &= \sec(x+2\pi)\\ &= \frac{1}{\cos(x+2\pi)}\\ &= \frac{1}{\cos x\cos 2\pi - \sin x \sin 2\pi}\\ &= \frac{1}{\cos x \times 1 - \sin x \times 0}\\ &= \frac{1}{\cos x}\\ &= \sec x\\ &= f(x),\ \ x \in \mathbb{R}\end{aligned}$$

$\therefore$ the graph has a period of 2π.

d



EXERCISE 19D.1

1 **a**

Point	$f(x)$	$f'(x)$	$f''(x)$
A	+	−	+
B	−	0	+
C	+	+	0
D	+	0	−
E	0	−	−

b The turning points of $y = f(x)$ are point B, a local minimum, and point D, a local maximum.

c The inflection point of $y = f(x)$ is point C, a non-stationary point of inflection.

2 **a** $f(x) = x^2 + 3$

$\therefore\ f'(x) = 2x$

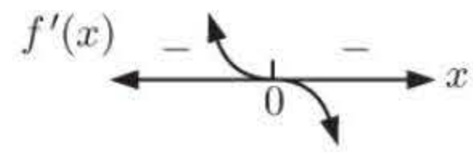
$\therefore\ f''(x) = 2$

Since $f''(x) \neq 0$,

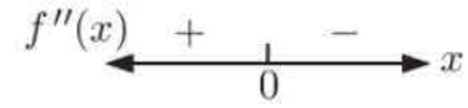
no points of inflection exist.

b $f(x) = 2 - x^3$

$\therefore\ f'(x) = -3x^2$



$\therefore\ f''(x) = -6x$



Now $f''(x) = 0$ when $x = 0$,

and $f'(0) = 0$

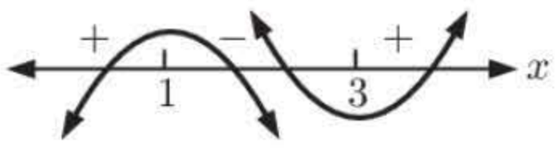
$\therefore$ there is a stationary inflection at $(0, 2)$.

c $f(x) = x^3 - 6x^2 + 9x + 1$

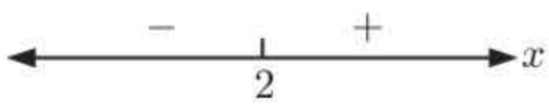
$\therefore\ f'(x) = 3x^2 - 12x + 9$

$= 3(x^2 - 4x + 3)$

$= 3(x - 3)(x - 1)$



and $f''(x) = 6x - 12 = 6(x - 2)$



Now $f''(x) = 0$ when $x = 2$

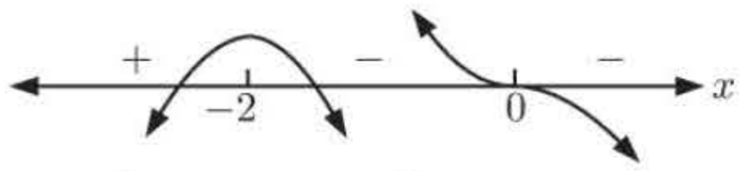
and $f'(2) \neq 0$

$\therefore$ there is a non-stationary inflection at $(2, f(2))$ which is $(2, 3)$.

d $f(x) = -3x^4 - 8x^3 + 2$

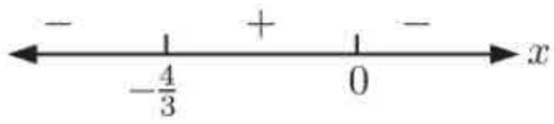
$\therefore\ f'(x) = -12x^3 - 24x^2$

$= -12x^2(x + 2)$



and $f''(x) = -36x^2 - 48x$

$= -12x(3x + 4)$



$\therefore$ there is a stationary inflection at $(0, 2)$, and a non-stationary inflection at $(-\frac{4}{3}, f(-\frac{4}{3}))$, which is $\left(-\frac{4}{3}, \frac{310}{27}\right)$.

e $f(x) = 3 - \dfrac{1}{\sqrt{x}} = 3 - x^{-\frac{1}{2}}$

$\therefore\ f'(x) = \frac{1}{2}x^{-\frac{3}{2}}$

and $f''(x) = -\frac{3}{4}x^{-\frac{5}{2}} = \dfrac{-3}{4x^2\sqrt{x}}$

Now $f''(x) \neq 0$ for all x

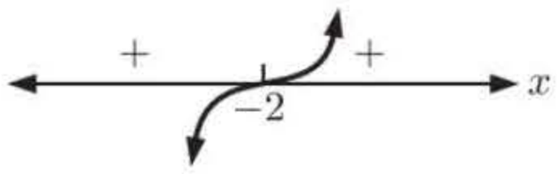
$\therefore$ there are no points of inflection.

f $f(x) = x^3 + 6x^2 + 12x + 5$

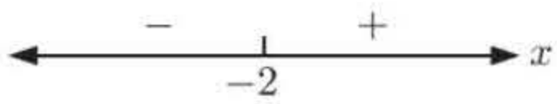
$\therefore\ f'(x) = 3x^2 + 12x + 12$

$= 3(x^2 + 4x + 4)$

$= 3(x + 2)^2$



and $f''(x) = 6x + 12 = 6(x + 2)$



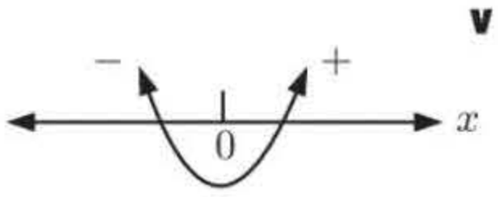
Now $f''(x) = 0$ when $x = -2$

and $f'(-2) = 0$

$\therefore$ there is a stationary inflection at $(-2, f(-2))$ which is $(-2, -3)$.

3 **a** $f(x) = x^2$

$\therefore\ f'(x) = 2x$ which has sign diagram:



and $f''(x) = 2$

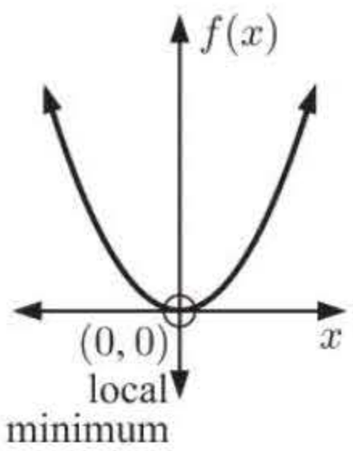
i There is a local minimum at $(0, 0)$.

ii There are no points of inflection as $f''(x) \neq 0$.

iii $f(x)$ is increasing when $x \geqslant 0$, and decreasing when $x \leqslant 0$.

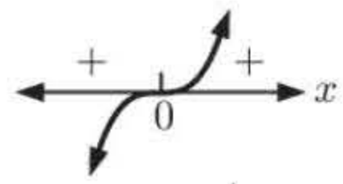
iv $f(x)$ is concave up for all x as $f''(x) > 0$ for all x.

v

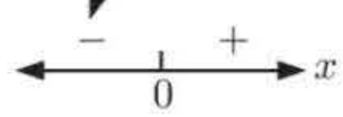


b $f(x) = x^3$

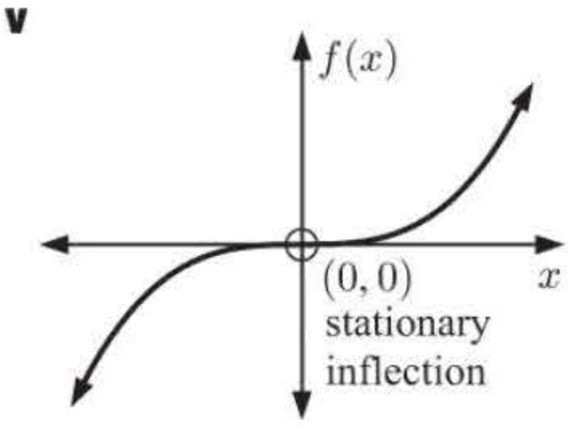
$\therefore\ f'(x) = 3x^2$ which has sign diagram:



and $f''(x) = 6x$ which has sign diagram:



v



i A stationary inflection at $(0, 0)$.

ii A stationary inflection at $(0, 0)$.

iii $f(x)$ is increasing for all real x.

iv $f(x)$ is concave up when $x \geqslant 0$, and concave down when $x \leqslant 0$.

c $f(x) = \sqrt{x}$

$\therefore\ f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x}}$ which has sign diagram:

and $f''(x) = -\frac{1}{4}x^{-\frac{3}{2}} = \dfrac{-1}{4x\sqrt{x}}$ which has sign diagram:

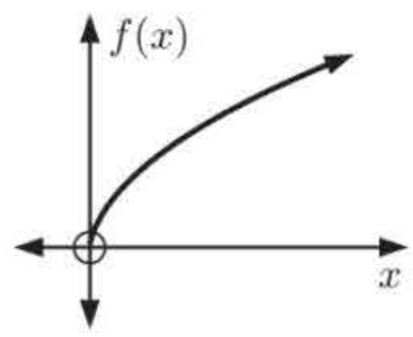
i There are no stationary points as $f'(x) \neq 0$.

ii There are no points of inflection as $f''(x) \neq 0$.

iii $f(x)$ is increasing for all $x \geqslant 0$, never decreasing.

iv $f(x)$ is concave down for all $x \geqslant 0$ as $f''(x) < 0$ for all $x > 0$, never concave up.

v

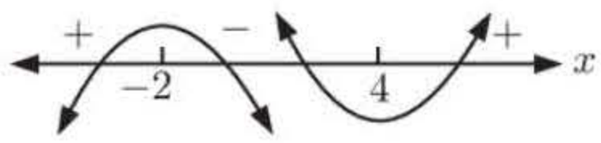


d $f(x) = x^3 - 3x^2 - 24x + 1$

$\therefore\ f'(x) = 3x^2 - 6x - 24$

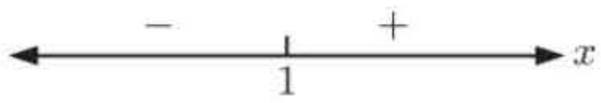
$= 3(x^2 - 2x - 8)$

$= 3(x - 4)(x + 2)$ which has sign diagram:



and $f''(x) = 6x - 6$

$= 6(x - 1)$ which has sign diagram:



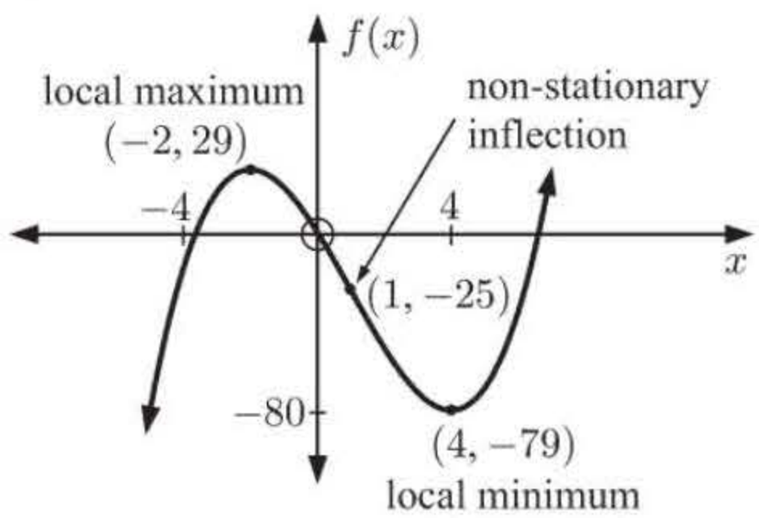
i $f(-2) = 29$, $f(4) = -79$, so there is a local maximum at $(-2, 29)$, and a local minimum at $(4, -79)$.

ii $f(1) = -25$, so there is a non-stationary inflection at $(1, -25)$.

iii $f(x)$ is increasing for $x \leqslant -2$ and $x \geqslant 4$, and decreasing for $-2 \leqslant x \leqslant 4$.

iv $f(x)$ is concave down for $x \leqslant 1$, and concave up for $x \geqslant 1$.

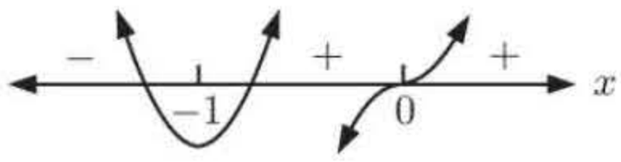
v



e $f(x) = 3x^4 + 4x^3 - 2$

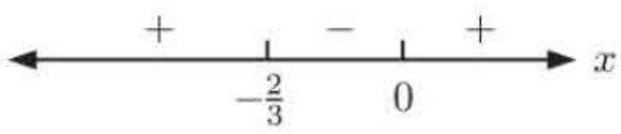
$\therefore\ f'(x) = 12x^3 + 12x^2$

$= 12x^2(x + 1)$ which has sign diagram:



and $f''(x) = 36x^2 + 24x$

$= 12x(3x + 2)$ which has sign diagram:



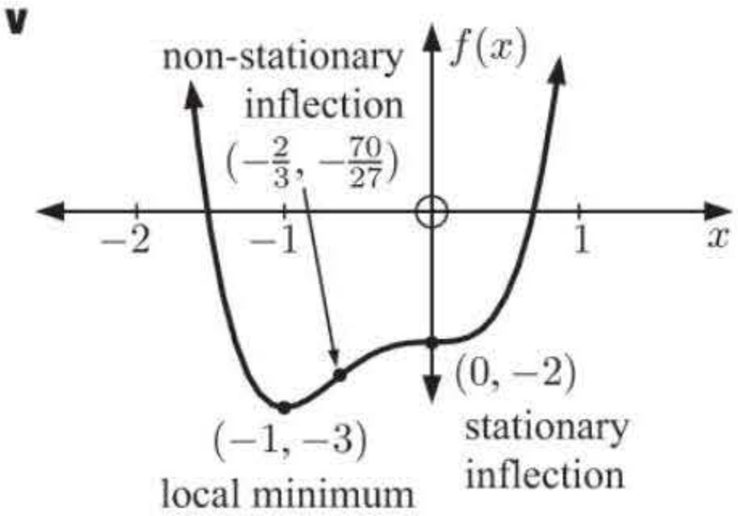
i There is a local minimum at $(-1, f(-1))$ which is $(-1, -3)$, and a stationary inflection at $(0, -2)$.

ii There is a non-stationary inflection at $(-\frac{2}{3}, f(-\frac{2}{3}))$ which is $(-\frac{2}{3}, -\frac{70}{27})$, and a stationary inflection at $(0, -2)$.

iii $f(x)$ is increasing for $x \geqslant -1$, and decreasing for $x \leqslant -1$.

iv $f(x)$ is concave down for $-\frac{2}{3} \leqslant x \leqslant 0$, and concave up for $x \leqslant -\frac{2}{3}$ and $x \geqslant 0$.

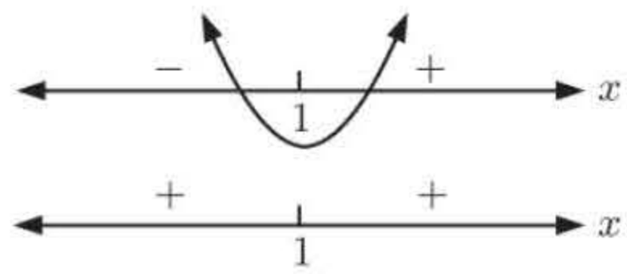
v



f $f(x) = (x-1)^4$

$\therefore\ f'(x) = 4(x-1)^3$ which has sign diagram:

and $f''(x) = 12(x-1)^2$ which has sign diagram:



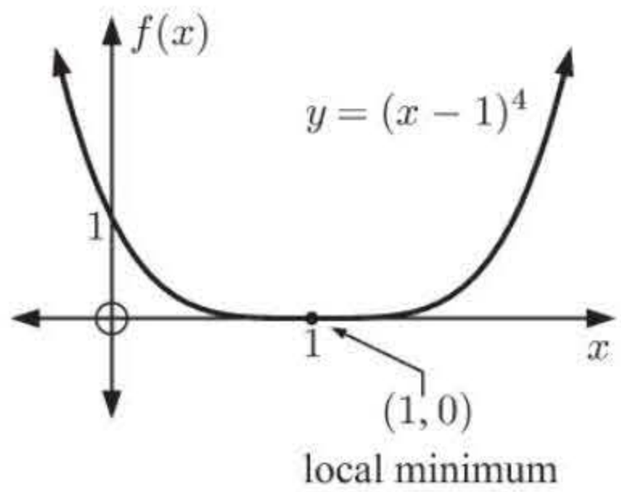
i There is a local minimum at $(1, 0)$.

ii Since there is no sign change in $f''(x)$ at $x = 1$, there are no points of inflection.

iii $f(x)$ is increasing for $x \geqslant 1$, and decreasing for $x \leqslant 1$.

iv $f(x)$ is concave up for all x.

v



g $f(x) = x^4 - 4x^2 + 3$

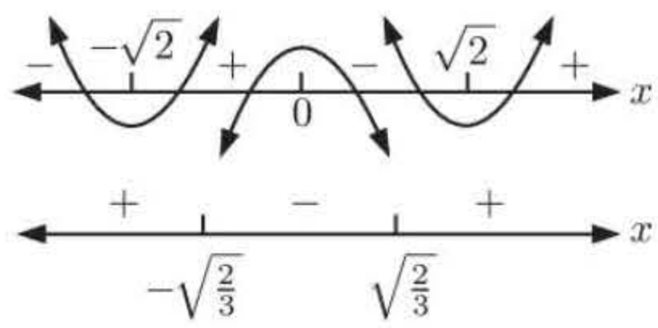
$f'(x) = 4x^3 - 8x$

$= 4x(x^2 - 2)$

$= 4x(x + \sqrt{2})(x - \sqrt{2})$ which has sign diagram:

$f''(x) = 12x^2 - 8 = 4(3x^2 - 2)$

$= 4(\sqrt{3}x + \sqrt{2})(\sqrt{3}x - \sqrt{2})$ which has sign diagram:



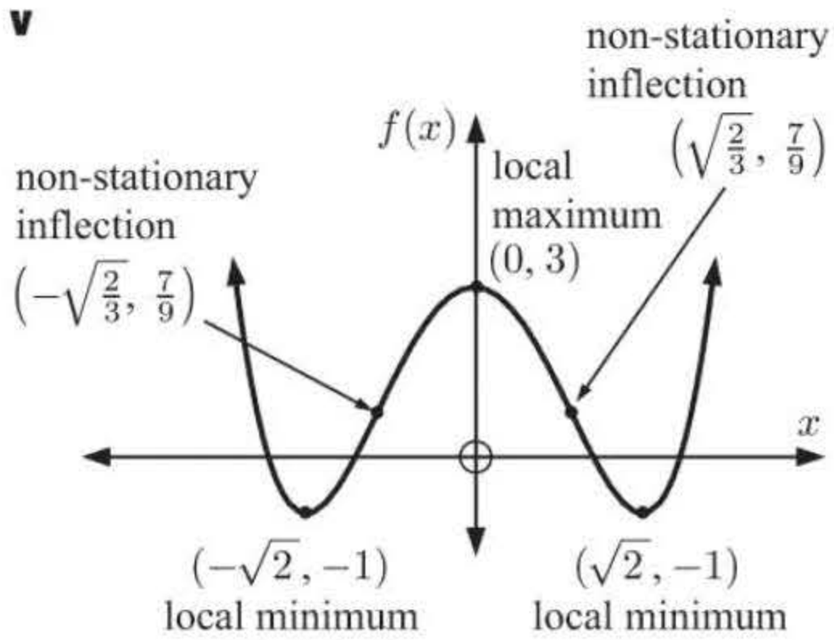
i There is a local maximum at $(0, 3)$, and $f(-\sqrt{2}) = f(\sqrt{2}) = -1$, so there are local minima at $(\sqrt{2}, -1)$ and $(-\sqrt{2}, -1)$.

ii $f\left(\sqrt{\frac{2}{3}}\right) = f\left(-\sqrt{\frac{2}{3}}\right) = \frac{7}{9}$, so there are non-stationary inflections at $(\sqrt{\frac{2}{3}}, \frac{7}{9})$ and $(-\sqrt{\frac{2}{3}}, \frac{7}{9})$.

iii $f(x)$ is increasing for $-\sqrt{2} \leqslant x \leqslant 0$ and $x \geqslant \sqrt{2}$, and decreasing for $x \leqslant -\sqrt{2}$ and $0 \leqslant x \leqslant \sqrt{2}$.

iv $f(x)$ is concave down for $-\sqrt{\frac{2}{3}} \leqslant x \leqslant \sqrt{\frac{2}{3}}$, and concave up for $x \leqslant -\sqrt{\frac{2}{3}}$ and $x \geqslant \sqrt{\frac{2}{3}}$.

v



h $f(x) = 3 - \dfrac{4}{\sqrt{x}} = 3 - 4x^{-\frac{1}{2}}, \quad x > 0$

$\therefore \; f'(x) = 2x^{-\frac{3}{2}} = \dfrac{2}{x\sqrt{x}}$ with sign diagram:

and $f''(x) = -3x^{-\frac{5}{2}} = -\dfrac{3}{x^2\sqrt{x}}$ with sign diagram:

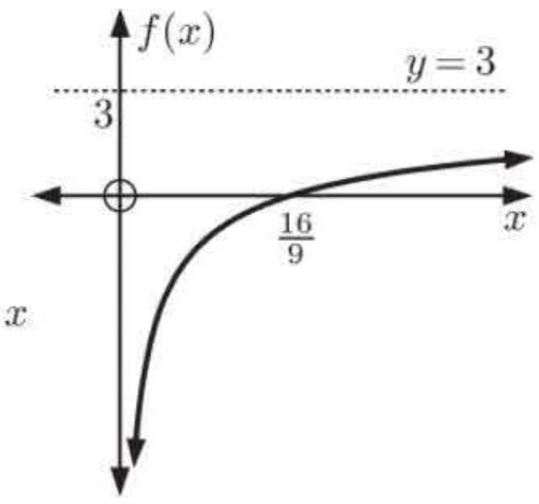
i There are no stationary points as $f'(x) \neq 0$.

ii There are no points of inflection as $f''(x) \neq 0$.

iii $f(x)$ is increasing for all $x > 0$ and never decreasing.

iv $f(x)$ is concave down for all $x > 0$ and never concave up.

v



4 **a** Consider $f(x) = e^{2x} - 3$

$f(x)$ cuts the x-axis at A when $f(x) = 0$

$\therefore \; e^{2x} - 3 = 0$

$\therefore \; e^{2x} = 3$

$\therefore \; 2x = \ln 3$

$\therefore \; x = \dfrac{\ln 3}{2}$

$\therefore$ A is $\left(\frac{\ln 3}{2}, 0\right)$

$f(x)$ cuts the y-axis at B when $x = 0$

$\therefore \; f(0) = e^{2\times 0} - 3$

$= e^0 - 3$

$= -2$

$\therefore$ B is $(0, -2)$

b $f(x) = e^{2x} - 3$

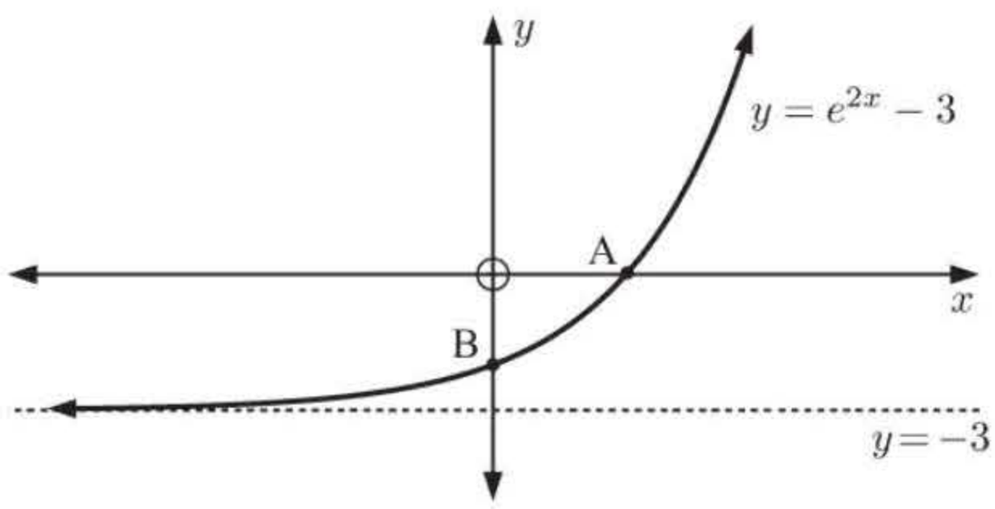
$\therefore \; f'(x) = 2e^{2x}$

Since $e^{2x} > 0$ for all x, $f'(x) > 0$ for all x, and hence $f(x)$ is increasing for all x.

c $f''(x) = 4e^{2x}$, which is always > 0.

$\therefore \; f(x)$ is concave up for all x.

d



e As $x \to -\infty$, $e^{2x} \to 0$, so $e^{2x} - 3 \to -3^+$

$\therefore \; y = -3$ is a horizontal asymptote.

5 **a** The x-intercepts occur when $y = 0$

For $f(x) = e^x - 3$, $e^x - 3 = 0$

$\therefore \; e^x = 3$

$\therefore \; x = \ln 3$

and for $g(x) = 3 - \dfrac{5}{e^x}$, $3 - \dfrac{5}{e^x} = 0$

$\therefore \; \dfrac{3e^x - 5}{e^x} = 0$

$\therefore \; 3e^x - 5 = 0$

$\therefore \; e^x = \frac{5}{3}$

$\therefore \; x = \ln(\frac{5}{3})$

$\therefore \; f(x)$ has x-intercept $\ln 3$

and $g(x)$ has x-intercept $\ln(\frac{5}{3})$.

The y-intercepts occur when $x = 0$

Now $f(0) = e^0 - 3 = -2$ and $g(0) = 3 - \dfrac{5}{e^0} = 3 - 5 = -2$

$\therefore$ both $f(x)$ and $g(x)$ have y-intercept -2.

b As $x \to \infty$, $f(x) \to \infty$ As $x \to \infty$, $g(x) \to 3^-$
$x \to -\infty$, $f(x) \to -3^+$ $x \to -\infty$, $g(x) \to -\infty$

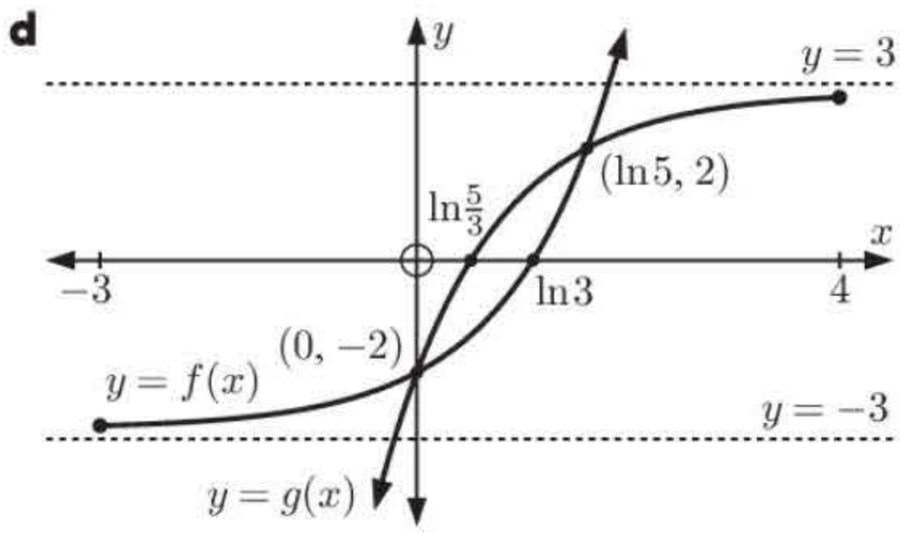
c $f(x)$ and $g(x)$ meet when

$$e^x - 3 = 3 - 5e^{-x}$$
$$\therefore\ e^{2x} - 3e^x = 3e^x - 5 \quad \{\times e^x\}$$
$$\therefore\ e^{2x} - 6e^x + 5 = 0$$
$$\therefore\ (e^x - 5)(e^x - 1) = 0$$
$$\therefore\ e^x = 5 \text{ or } 1$$
$$\therefore\ x = \ln 5 \text{ or } 0$$

Now $f(\ln 5) = e^{\ln 5} - 3 = 5 - 3 = 2$
and $f(0) = -2$

$\therefore$ $f(x)$ and $g(x)$ meet at $(\ln 5, 2)$ and $(0, -2)$.

d



6 **a** Consider $y = e^x - 3e^{-x}$

It cuts the x-axis at P when $y = 0$

$$\therefore\ e^x - 3e^{-x} = 0$$
$$\therefore\ e^{2x} - 3 = 0 \quad \{\times e^x\}$$
$$\therefore\ e^{2x} = 3$$
$$\therefore\ 2x = \ln 3$$
$$\therefore\ x = \tfrac{1}{2}\ln 3$$

It cuts the y-axis at Q when $x = 0$

$$\therefore\ y = e^0 - 3e^0 = 1 - 3 = -2$$

$\therefore$ P is $(\frac{1}{2}\ln 3, 0)$ and Q is $(0, -2)$.

b
$$\frac{dy}{dx} = e^x + 3e^{-x} = e^x + \frac{3}{e^x}$$

Since $e^x > 0$ for all x,

$$\frac{dy}{dx} > 0 \text{ for all } x$$

$\therefore$ the function is increasing for all x

c
$$\frac{dy}{dx} = e^x + 3e^{-x}$$
$$\therefore\ \frac{d^2y}{dx^2} = e^x - 3e^{-x} = y$$

Above the x-axis $y > 0$ $\therefore$ $\dfrac{d^2y}{dx^2} > 0$

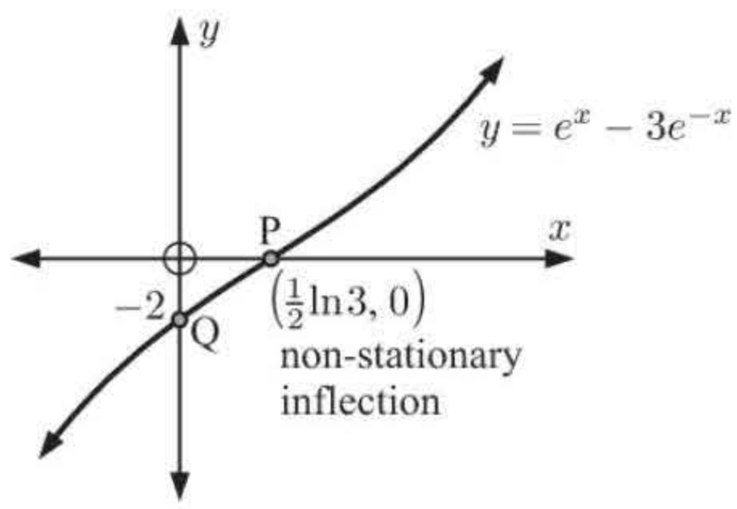
$\therefore$ the function is concave up

Below the x-axis $y < 0$ $\therefore$ $\dfrac{d^2y}{dx^2} < 0$

$\therefore$ the function is concave down

$\therefore$ a non-stationary inflection occurs when $y = 0$

d



7 $f(x) = \ln(2x - 1) - 3$

a $f(x) = 0$ when $\ln(2x - 1) = 3$

$$\therefore\ 2x - 1 = e^3$$
$$\therefore\ 2x = e^3 + 1$$
$$\therefore\ x = \frac{e^3 + 1}{2} \approx 10.5$$

$\therefore$ the x-intercept is $\dfrac{e^3 + 1}{2}$

b $f(0)$ cannot be found as $\ln(-1)$ is not defined. $\therefore$ there is no y-intercept.

c $f'(x) = \dfrac{2}{2x - 1}$ $\therefore$ $f'(1) = \dfrac{2}{2 - 1} = 2$ $\therefore$ gradient of tangent $= 2$

d $\ln(2x - 1)$ has meaning provided $2x - 1 > 0$ $\therefore$ $2x > 1$ and so $x > \frac{1}{2}$

$\therefore$ the domain of f is $\{x \mid x > \frac{1}{2}\}$

e $f'(x) = 2(2x-1)^{-1}$

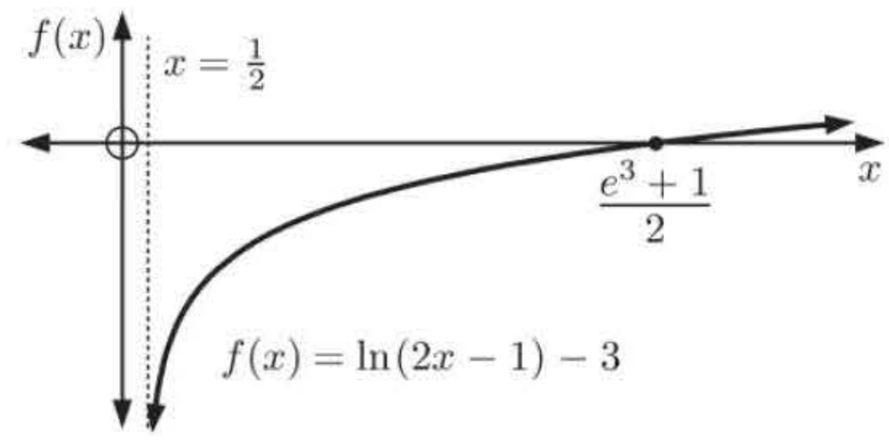
$\therefore\ f''(x) = -2(2x-1)^{-2}(2)$

$= \dfrac{-4}{(2x-1)^2}, \quad x > \frac{1}{2}$

$\therefore$ provided $x > \frac{1}{2}$, $f''(x) < 0$

$\therefore$ $f(x)$ is concave down for all x in the domain of f.

f



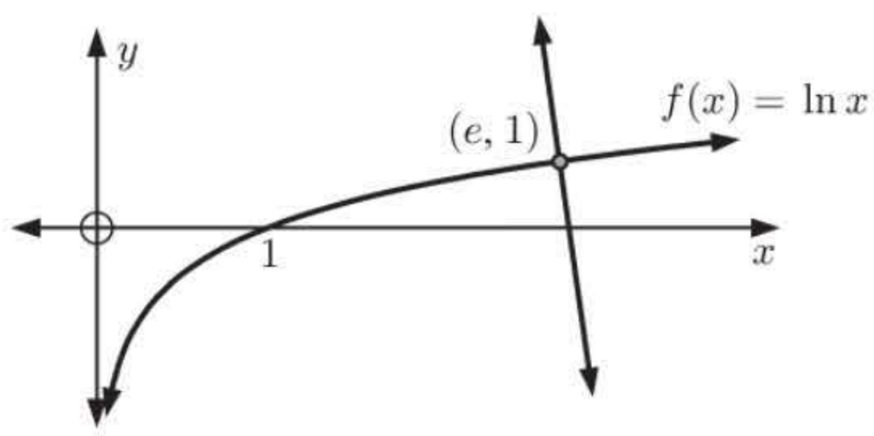
8 **a** $f(x) = \ln x$ is defined for all $x > 0$.

b $f'(x) = \dfrac{1}{x}$ which is > 0 for all $x > 0$

$\therefore$ $f(x)$ is increasing on $x > 0$; its gradient is always positive.

$f''(x) = -x^{-2} = \dfrac{-1}{x^2}$ which is < 0 for all $x > 0$ $\quad\therefore$ $f(x)$ is concave down on $x > 0$.

c



At $y = 1$, $1 = \ln x$

$\therefore\ x = e^1 = e$

$\therefore$ the point of contact is $(e, 1)$

Now $\dfrac{dy}{dx} = \dfrac{1}{x}$

$\therefore$ at $(e, 1)$, $\dfrac{dy}{dx} = \dfrac{1}{e}$

$\therefore$ the gradient of the tangent is $\dfrac{1}{e}$, and the gradient of the normal is $-e$

$\therefore$ the equation of the normal is $\dfrac{y-1}{x-e} = -e$ $\quad\therefore\ y - 1 = -e(x-e)$

$\therefore\ y - 1 = -ex + e^2$

$\therefore\ y = -ex + 1 + e^2$

9 Consider $f(x) = \dfrac{e^x}{x}$.

a $e^x \neq 0$ for all x, so $f(x) \neq 0$ and there is no x-intercept.

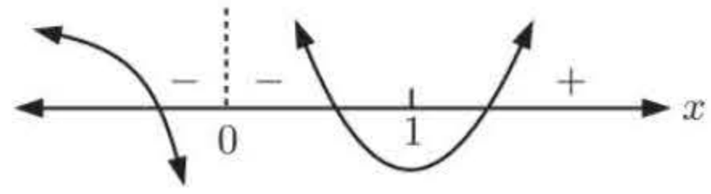
$f(0) = \dfrac{e^0}{0}$ is undefined, so there is also no y-intercept.

b As $x \to +\infty$ $f(x) \to \infty$, and as $x \to -\infty$, $f(x) \to 0^-$

$\left(\begin{array}{l}\text{As } x \to 0^+,\ y \to +\infty, \text{ and as } x \to 0^-,\ y \to -\infty \\ \therefore\ x = 0 \text{ is a vertical asymptote.}\end{array}\right)$

c Using the quotient rule, $f'(x) = \dfrac{e^x x - e^x(1)}{x^2} = \dfrac{e^x(x-1)}{x^2}$

with sign diagram:



$f(1) = \dfrac{e^1}{1} = e$, so there is a local minimum at $(1, e)$.

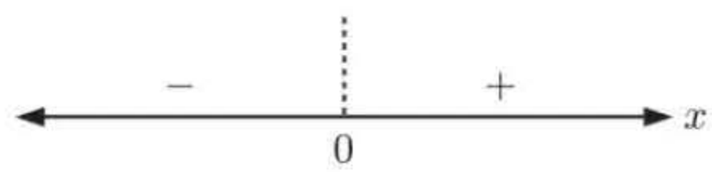
d Using the product and quotient rules,

$$f''(x) = \frac{[e^x(x-1) + e^x]x^2 - e^x(x-1)2x}{x^4}$$

$$= \frac{x^3e^x - \cancel{x^2e^x} + \cancel{x^2e^x} - 2x^2e^x + 2xe^x}{x^4}$$

$$= \frac{e^x(x^2 - 2x + 2)}{x^3}$$

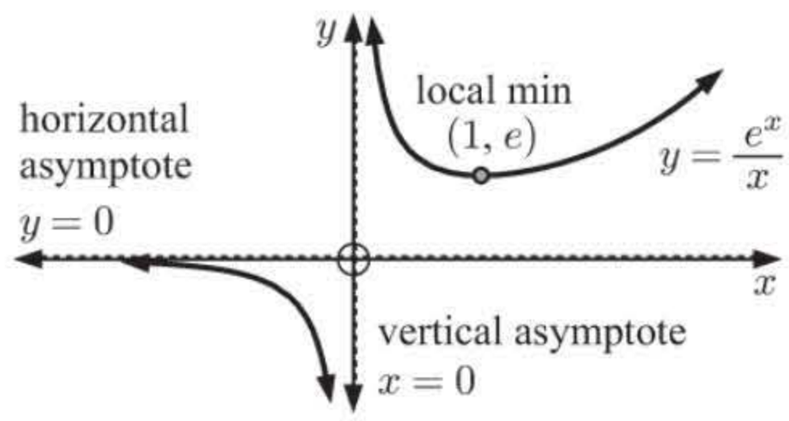
with sign diagram:



i $f(x)$ is concave up for $x > 0$.

ii $f(x)$ is concave down for $x < 0$.

e



f Now $f'(x) = \dfrac{e^x(x-1)}{x^2}$

$\therefore\ f'(-1) = \dfrac{e^{-1}(-1-1)}{(-1)^2} = -\dfrac{2}{e}$

$\therefore$ the gradient of the tangent is $= -\dfrac{2}{e}$

When $x = -1$, $y = \dfrac{e^{-1}}{-1} = -\dfrac{1}{e}$

$\therefore$ the equation of the tangent is

$$\frac{y - \left(-\frac{1}{e}\right)}{x - (-1)} = -\frac{2}{e}$$

$$\therefore\ \frac{y + \frac{1}{e}}{x+1} = -\frac{2}{e}$$

$$\therefore\ e\left(y + \frac{1}{e}\right) = -2(x+1)$$

$$ey + 1 = -2x - 2$$

$$\therefore\ ey = -2x - 3$$

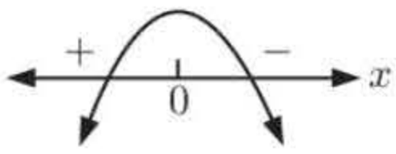
10 **a** $f(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}$

$\therefore\ f'(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}(-x)$

$= \dfrac{-x}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}$

$\therefore\ f'(x) = 0$ when $x = 0$

$f'(x)$ has sign diagram:



Now $f(0) = \frac{1}{\sqrt{2\pi}}$

so there is a local maximum at $\left(0, \frac{1}{\sqrt{2\pi}}\right)$.

The function is increasing for $x \leqslant 0$ and decreasing for $x \geqslant 0$

b $f'(x) = \dfrac{-x}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2} = \frac{1}{\sqrt{2\pi}}\left(-xe^{-\frac{1}{2}x^2}\right)$

$\therefore\ f''(x) = \frac{1}{\sqrt{2\pi}}\left((-1)\, e^{-\frac{1}{2}x^2} + (-x)\, e^{-\frac{1}{2}x^2}\,(-x)\right)$ {product rule}

$= \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}\left(x^2 - 1\right)$

$= \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}\,(x+1)\,(x-1)$ which has sign diagram:

Now $f(1) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}} = \frac{1}{\sqrt{2e\pi}}$ and $f(-1) = \frac{1}{\sqrt{2e\pi}}$

$\therefore$ there are non-stationary points of inflection at $\left(1, \frac{1}{\sqrt{2e\pi}}\right)$ and $\left(-1, \frac{1}{\sqrt{2e\pi}}\right)$.

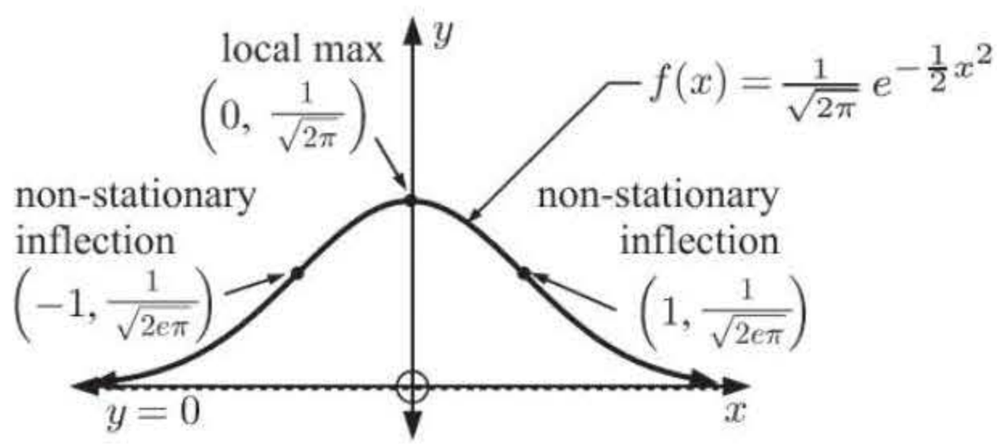
c As $x \to \infty$, $e^{-\frac{1}{2}x^2} \to 0^+$

$\therefore\ f(x) \to 0^+$

As $x \to -\infty$, $e^{-\frac{1}{2}x^2} \to 0^+$

$\therefore\ f(x) \to 0^+$

d



11 $y = 4^x - 2^x$

a When $x = 0$, $y = 4^0 - 2^0 = 1 - 1 = 0$ $\quad\therefore$ the y-intercept is 0.

When $y = 0$, $4^x - 2^x = 0$

$\therefore\ 4^x = 2^x$

$\therefore\ \dfrac{4^x}{2^x} = 1$

$\therefore\ \left(\frac{4}{2}\right)^x = 1$

$\therefore\ 2^x = 1$

$\therefore\ x = 0$ $\quad\therefore$ the x-intercept is 0.

b As $x \to \infty$, $y \to \infty$
As $x \to -\infty$, $y \to 0^-$

c $\dfrac{dy}{dx} = 4^x \ln 4 - 2^x \ln 2$ $\quad \{\dfrac{d}{dx}(a^x) = a^x \ln a\}$

When $\dfrac{dy}{dx} = 0$, $4^x \ln 4 - 2^x \ln 2 = 0$

$\therefore\ 2^x \times 2^x \times 2\ln 2 - 2^x \ln 2 = 0$ $\quad \{\ln a^x = x \ln a\}$

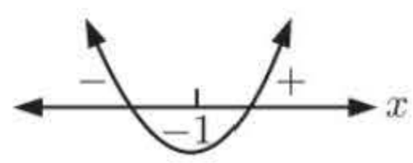
$\therefore\ 2^x \ln 2(2^{x+1} - 1) = 0$

$\therefore\ 2^{x+1} = 1$ $\quad \{\text{since } 2^x \ln 2 \neq 0\}$

$\therefore\ x + 1 = 0$ $\quad \{\text{since } 2^0 = 1\}$

$\therefore\ x = -1$

$\therefore$ sign diagram of $\dfrac{dy}{dx}$ is:



When $x = -1$, $y = 4^{-1} - 2^{-1}$
$= \frac{1}{4} - \frac{1}{2}$
$= -\frac{1}{4}$

$\therefore$ there is a local minimum at $(-1, -\frac{1}{4})$.

$\dfrac{d^2y}{dx^2} = 4^x(\ln 4)^2 - 2^x(\ln 2)^2$ $\quad \{\dfrac{d}{dx}(a^x) = a^x \ln a\}$

$= 2^x \times 2^x(2\ln 2)^2 - 2^x(\ln 2)^2$ $\quad \{\ln a^x = x \ln a\}$

$= 2^x(\ln 2)^2(2^x \times 2^2 - 1)$

$= 2^x(\ln 2)^2(2^{x+2} - 1)$

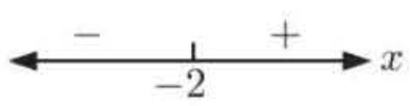
When $\dfrac{d^2y}{dx^2} = 0$, $2^x(\ln 2)^2(2^{x+2} - 1) = 0$

$\therefore\ 2^{x+2} = 1$ $\quad \{\text{since } 2^x(\ln 2)^2 \neq 0\}$

$\therefore\ x + 2 = 0$ $\quad \{\text{since } 2^0 = 1\}$

$\therefore\ x = -2$

$\therefore$ the sign diagram of $\dfrac{d^2y}{dx^2}$ is:

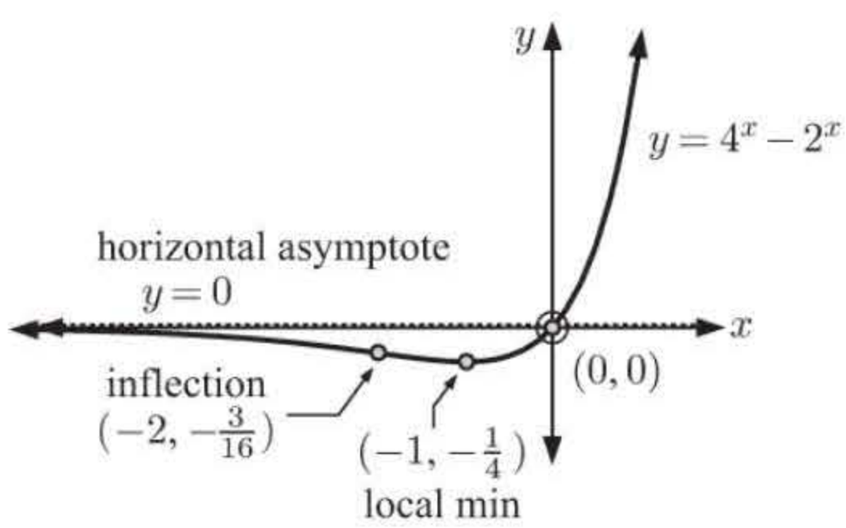


When $x = -2$, $y = 4^{-2} - 2^{-2}$
$= \frac{1}{16} - \frac{1}{4}$
$= -\frac{3}{16}$

$\therefore$ at $(-2, -\frac{3}{16})$ there is a non-stationary point of inflection.

d concave down for $x \leqslant -2$, concave up for $x \geqslant -2$

e



12 $f(t) = Ate^{-bt}, \quad t \geqslant 0, \quad A, b > 0$

a **i** $f'(t) = Ae^{-bt} - Abte^{-bt}$ {product rule}

$= Ae^{-bt}(1 - bt)$

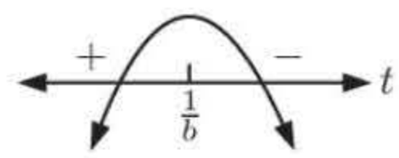
When $f'(t) = 0$ then $Ae^{-bt}(1 - bt) = 0$ but $A > 0$ and $e^{-bt} > 0$

$\therefore \; 1 - bt = 0$

$\therefore \; -bt = -1$

$\therefore \; t = \dfrac{1}{b}$

Sign diagram of $f'(t)$ is:



$\therefore \; t = \dfrac{1}{b}$ is a local maximum.

ii $f''(t) = -Abe^{-bt} - (Abe^{-bt} - Ab^2te^{-bt})$ {product rule}

$= -2Abe^{-bt} + Ab^2te^{-bt}$

$= Abe^{-bt}(bt - 2)$

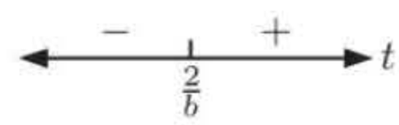
When $f''(t) = 0$ then $Abe^{-bt}(bt - 2) = 0$ but $A, b > 0$ and $e^{-bt} > 0$

$\therefore \; bt - 2 = 0$

$\therefore \; bt = 2$

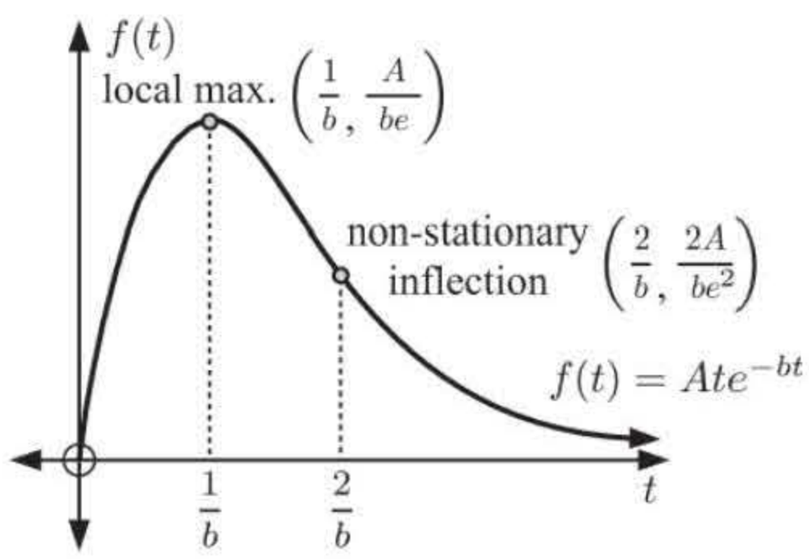
$\therefore \; t = \dfrac{2}{b}$

Sign diagram of $f''(x)$ is:



$\therefore \; t = \dfrac{2}{b}$ is a non-stationary point of inflection.

b



13 $f(t) = \dfrac{C}{1 + Ae^{-bt}}, \quad t \geqslant 0, \quad A, b, C > 0$

a When $t = 0$, then $f(0) = \dfrac{C}{1 + Ae^{-b0}}$

$= \dfrac{C}{1 + A}$

b **i** As $t \to \infty$, $Ae^{-bt} \to 0^+$ $\quad \therefore \; f(t) \to \dfrac{C}{1 + 0} = C^-$

$\therefore \; y = C$ is a horizontal asymptote.

ii $f(t) = C(1 + Ae^{-bt})^{-1}$

$f'(t) = C(-1)(1 + Ae^{-bt})^{-2}(-bAe^{-bt})$

$= AbCe^{-bt}(1 + Ae^{-bt})^{-2}$

$f''(t) = (-b)AbCe^{-bt}(1 + Ae^{-bt})^{-2} + AbCe^{-bt}(-2)(1 + Ae^{-bt})^{-3}(-bAe^{-bt})$

{product rule}

$= -\dfrac{Ab^2C}{e^{bt}(1 + Ae^{-bt})^2} + \dfrac{2A^2b^2C}{e^{2bt}(1 + Ae^{-bt})^3}$

When $f''(t) = 0$,

$$\frac{2A^{\cancel{2}}\cancel{b^2C}}{e^{\cancel{2}bt}(1+Ae^{-bt})^{\cancel{3}}} = \frac{\cancel{Ab^2C}}{\cancel{e^{bt}(1+Ae^{-bt})^2}}$$

$$\therefore \quad \frac{2A}{e^{bt}(1+Ae^{-bt})} = 1$$

$$\therefore \quad 2A = e^{bt} + A\cancel{e^{-bt}}\cancel{e^{bt}}$$

$$\therefore \quad 2A = e^{bt} + A$$

$$\therefore \quad A = e^{bt}$$

$$\therefore \quad \ln A = bt$$

$$\therefore \quad t = \frac{\ln A}{b}, \quad \text{since } t \geqslant 0, \text{ then } \frac{\ln A}{b} \geqslant 0 \quad \therefore \quad b > 0, \ A > 1$$

$$f\left(\frac{\ln A}{b}\right) = \frac{C}{1+Ae^{-b\left(\frac{\ln A}{b}\right)}}$$

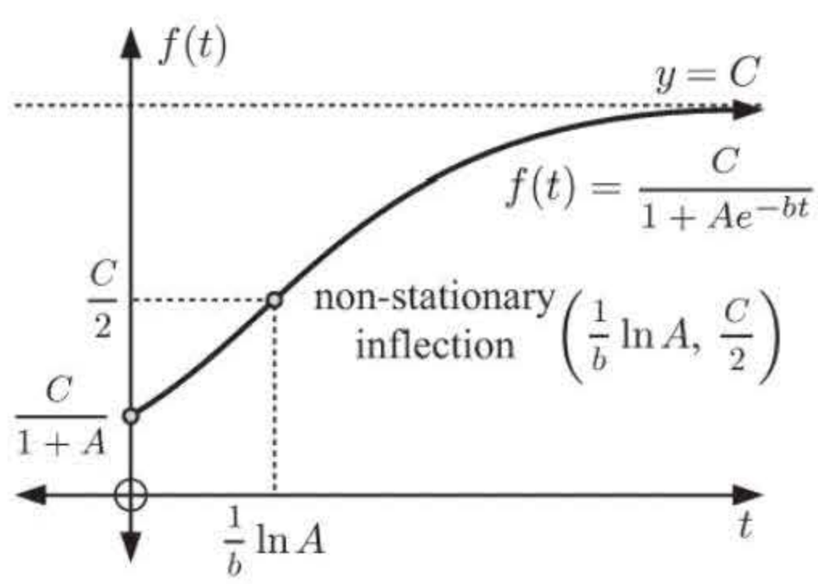
$$\therefore \quad = \frac{C}{1+Ae^{-\ln A}}$$

$$= \frac{C}{1+Ae^{\ln A^{-1}}}$$

$$= \frac{C}{1+AA^{-1}}$$

$$= \frac{C}{1+1} = \frac{C}{2}$$

c



EXERCISE 19D.2

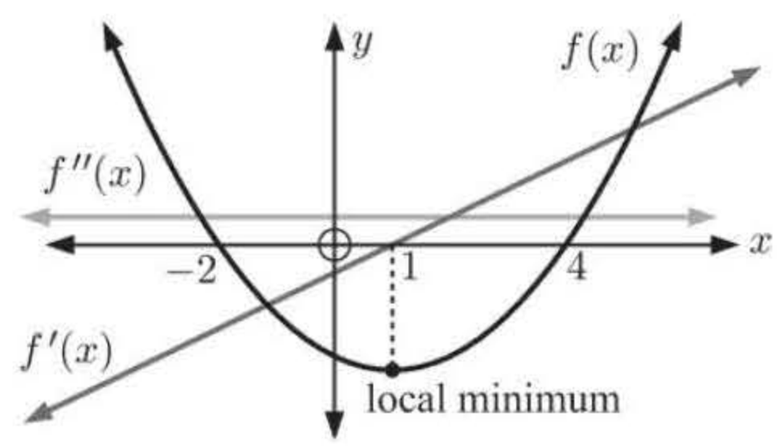
1 **a** $f(x)$ is quadratic, so $f'(x)$ will be linear and $f''(x)$ will be constant.

$f(x)$ is decreasing for $x \leqslant 1$ and increasing for $x \geqslant 1$

$\therefore \quad f'(x) \leqslant 0$ for $x \leqslant 1$ and $f'(x) \geqslant 0$ for $x \geqslant 1$

$\therefore \quad f'(x)$ is an increasing linear function which cuts the x-axis at 1.

As $f'(x)$ is increasing, $f''(x) > 0$.



b $f(x)$ is cubic, so $f'(x)$ will be quadratic and $f''(x)$ will be linear.

$f(x)$ has turning points at $x \approx \pm 1$

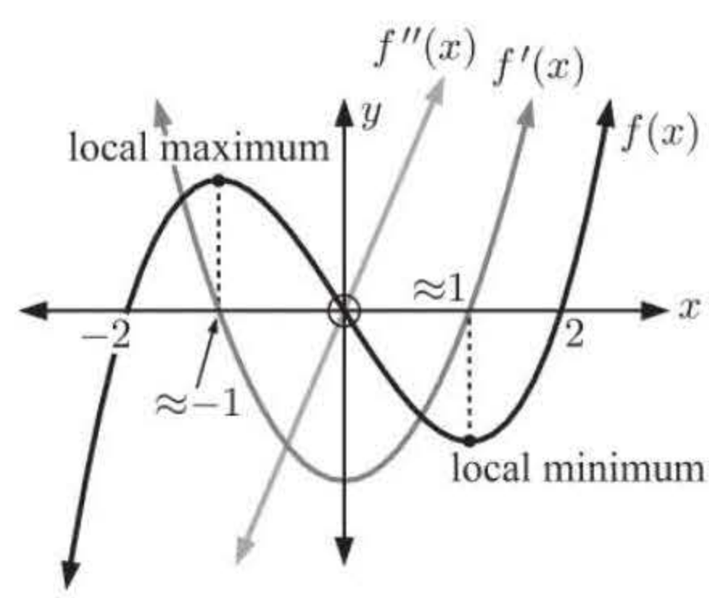
$\therefore \quad f'(x)$ cuts the x-axis at these points.

$f(x)$ has a non-stationary inflection point at $x = 0$

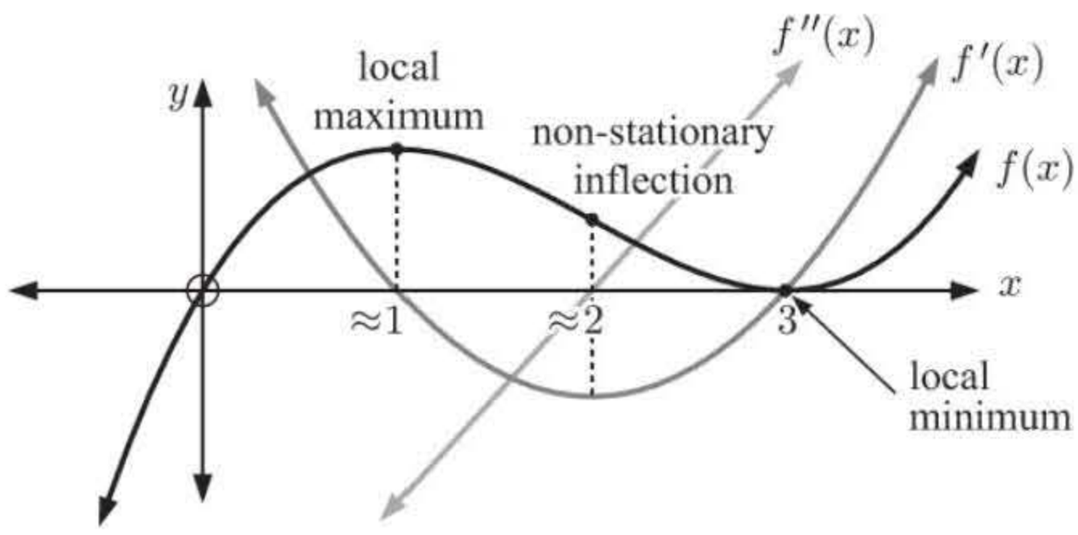
$\therefore \quad f'(x)$ has a turning point at $x = 0$, and $f''(0) = 0$.

$f(x)$ is concave down for $x \leqslant 0$ and concave up for $x \geqslant 0$

$\therefore \quad f'(x)$ is decreasing for $x \leqslant 0$ and increasing for $x \geqslant 0$ and $f''(x) \leqslant 0$ for $x \leqslant 0$ and $\geqslant 0$ for $x \geqslant 0$.

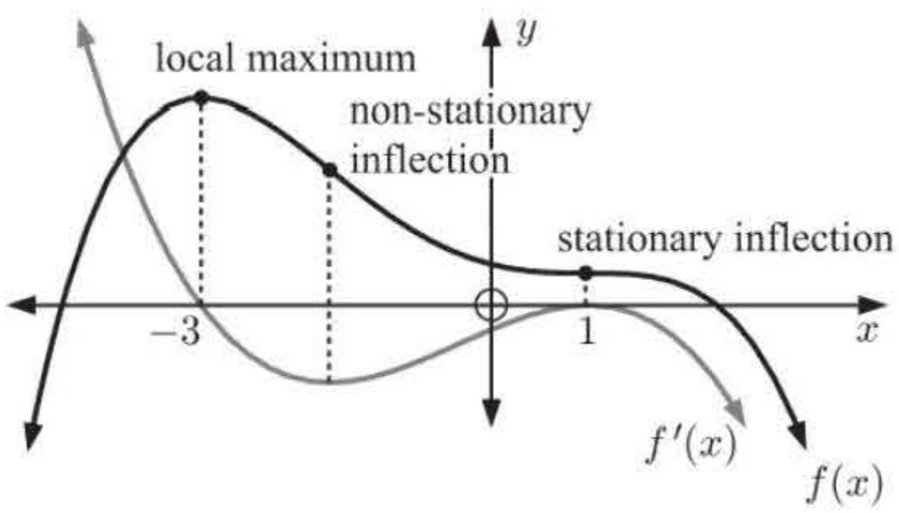


c $f(x)$ is cubic, so $f'(x)$ will be quadratic and $f''(x)$ will be linear.
$f(x)$ has turning points at $x \approx 1$ and $x = 3$
$\therefore$ $f'(x)$ cuts the x-axis at these points.
$f(x)$ has a non-stationary inflection point at $x \approx 2$
$\therefore$ $f'(x)$ has a turning point at $x \approx 2$, and $f''(2) = 0$
$f(x)$ is concave down for $x \leqslant 2$ and concave up for $x \geqslant 2$
$\therefore$ $f'(x)$ is decreasing for $x \leqslant 2$ and increasing for $x \geqslant 2$
and $f''(x) \leqslant 0$ for $x \leqslant 2$ and $\geqslant 0$ for $x \geqslant 2$.

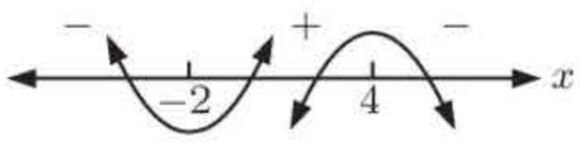


2 **a** $f'(x)$ has sign diagram:

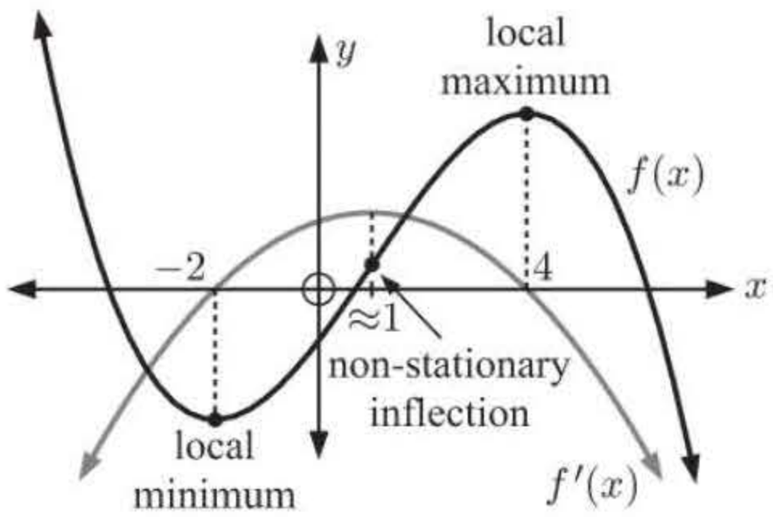
$\therefore$ $f(x)$ is increasing for $x \leqslant -3$, and decreasing for $x \geqslant -3$
$\therefore$ $f(x)$ has a local maximum at $x = -3$
$f'(x)$ has a turning point at $x \approx -1.7$.
At this point, $f''(x) = 0$, but $f'(x) \neq 0$
$\therefore$ $f(x)$ has a non-stationary inflection point here.
$f'(x)$ has another turning point at $x = 1$.
At this point, $f''(x) = 0$ and $f'(x) = 0$
$\therefore$ $f(x)$ has a stationary inflection point at $x = 1$.
A possible graph of $f(x)$ is shown alongside:



b $f'(x)$ has sign diagram:



$\therefore$ $f(x)$ has a local minimum at $x = -2$ and a local maximum at $x = 4$
$f'(x)$ has a turning point at $x \approx 1$.
At this point, $f''(x) = 0$, but $f'(x) \neq 0$
$\therefore$ $f(x)$ has a non-stationary inflection point at $x \approx 1$.
A possible graph of $f(x)$ is shown alongside:



REVIEW SET 19A

1 Consider $y = -2x^2$.
When $x = -1$, $y = -2(-1)^2 = -2$, so the point of contact is $(-1, -2)$.

Now $\dfrac{dy}{dx} = -4x$

$\therefore$ at $x = -1$, $\dfrac{dy}{dx} = -4(-1) = 4$

$\therefore$ the tangent has equation $\dfrac{y-(-2)}{x-(-1)} = 4$

or $y = 4x + 2$

2 Consider $y = \dfrac{1-2x}{x^2}$.

When $x = 1$, $y = \dfrac{1-2(1)}{1^2} = -1$, so the point of contact is $(1, -1)$.

Since $y = \dfrac{1}{x^2} - \dfrac{2}{x}$, $\dfrac{dy}{dx} = -2x^{-3} + 2x^{-2} = -\dfrac{2}{x^3} + \dfrac{2}{x^2}$

$\therefore$ at $x = 1$, $\dfrac{dy}{dx} = -2 + 2 = 0$

So, the tangent is a horizontal line, and the normal must be a vertical line of the form $x = k$.
As the normal passes through $(1, -1)$, its equation must be $x = 1$.

3 **a** The vertical asymptote is
$x + 3 = 0$ or $x = -3$.

b When $y = 0$, $\dfrac{3x-2}{x+3} = 0$

$\therefore$ $3x - 2 = 0$

$\therefore$ $x = \frac{2}{3}$

$\therefore$ the x-intercept is $\frac{2}{3}$.

When $x = 0$, $f(0) = \frac{-2}{3}$

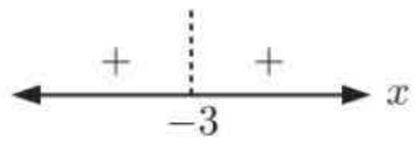
$\therefore$ the y-intercept is $-\frac{2}{3}$.

c $f'(x) = \dfrac{3(x+3) - (3x-2)(1)}{(x+3)^2}$ {quotient rule}

$= \dfrac{3x + 9 - 3x + 2}{(x+3)^2}$

$= \dfrac{11}{(x+3)^2}$

$f'(x)$ has sign diagram:



d $f'(x) \neq 0$ for any x, so $f(x)$ has no stationary points.

4 $y = e^{-x^2}$ so when $x = 1$, $y = e^{-1} = \dfrac{1}{e}$

$\therefore$ the point of contact is $\left(1, \dfrac{1}{e}\right)$

Now $\dfrac{dy}{dx} = -2xe^{-x^2}$

$\therefore$ when $x = 1$, $\dfrac{dy}{dx} = -2e^{-1}$

$\therefore$ the gradient of the tangent is $-\dfrac{2}{e}$

and the gradient of the normal is $\dfrac{e}{2}$

$\therefore$ the equation of the normal is $\dfrac{y - \frac{1}{e}}{x - 1} = \dfrac{e}{2}$

$\therefore$ $2\left(y - \dfrac{1}{e}\right) = e(x-1)$

$\therefore$ $2y - \dfrac{2}{e} = ex - e$

$\therefore$ $2y = ex + \dfrac{2}{e} - e$

$\therefore$ $y = \dfrac{e}{2}x + \dfrac{1}{e} - \dfrac{e}{2}$

5 $y = x \tan x$

$\therefore$ $\dfrac{dy}{dx} = 1 \times \tan x + x \times \left(\dfrac{1}{\cos^2 x}\right)$

$= \tan x + \dfrac{x}{\cos^2 x}$

Now $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and $\tan\frac{\pi}{4} = 1$

$\therefore$ at $x = \frac{\pi}{4}$, $y = \frac{\pi}{4}$,

and $\dfrac{dy}{dx} = 1 + \dfrac{\frac{\pi}{4}}{\left(\frac{1}{\sqrt{2}}\right)^2} = 1 + \frac{\pi}{2}$

$\therefore$ the equation of the tangent is

$\dfrac{y - \frac{\pi}{4}}{x - \frac{\pi}{4}} = 1 + \frac{\pi}{2}$

$\therefore$ $y - \frac{\pi}{4} = (1 + \frac{\pi}{2})(x - \frac{\pi}{4})$

$= x - \frac{\pi}{4} + \frac{\pi}{2}x - \frac{\pi^2}{8}$

$\therefore$ $y = (1 + \frac{\pi}{2})x - \frac{\pi^2}{8}$

$\therefore$ $2y = (2 + \pi)x - \frac{\pi^2}{4}$

$\therefore$ $(2 + \pi)x - 2y = \frac{\pi^2}{4}$ as required

6 $y = \dfrac{ax+b}{\sqrt{x}} = a\sqrt{x} + \dfrac{b}{\sqrt{x}} = ax^{\frac{1}{2}} + bx^{-\frac{1}{2}}$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{a}{2}x^{-\frac{1}{2}} - \dfrac{b}{2}x^{-\frac{3}{2}} = \dfrac{a}{2\sqrt{x}} - \dfrac{b}{2x\sqrt{x}}$

The equation of the tangent at $x = 1$

is $2x - y = 1$

or $y = 2x - 1$

so the gradient of the tangent is 2

$\therefore$ at $x = 1$, $\dfrac{dy}{dx} = \dfrac{a}{2} - \dfrac{b}{2} = 2$

$\therefore \quad a - b = 4$

$\therefore \quad a = b + 4 \quad .... (1)$

Also at $x = 1$, the tangent touches the curve

$\therefore \quad \dfrac{a(1)+b}{\sqrt{1}} = 2(1) - 1$

$\therefore \quad a + b = 1$

$\therefore \quad b + 4 + b = 1 \quad \{\text{using (1)}\}$

$\therefore \quad 2b = -3$

$\therefore \quad b = -\frac{3}{2}$ and $a = 4 - \frac{3}{2} = \frac{5}{2}$

7 $f(x) = 4\ln(2x), \qquad P(1, 4\ln 2)$

$\therefore \quad f'(x) = 4 \times \dfrac{2}{2x}$

$= \dfrac{4}{x}$

$\therefore$ at $x = 1$, $f'(1) = \dfrac{4}{1} = 4$

$\therefore$ the tangent has equation

$\dfrac{y - 4\ln 2}{x - 1} = 4$

$\therefore \quad y - 4\ln 2 = 4x - 4$

$\therefore \quad y = 4x + 4\ln 2 - 4$

8 **a** $f(x) = \dfrac{e^x}{x-1}$

Now $f(0) = \dfrac{e^0}{-1} = -1$ so the y-intercept is -1.

b $f(x)$ is defined for all $x \neq 1$.

c $f'(x) = \dfrac{e^x(x-1) - e^x(1)}{(x-1)^2}$ {quotient rule}

$= \dfrac{e^x(x-2)}{(x-1)^2}$ and has sign diagram:

$\therefore \quad f'(x) \leqslant 0$ for $x < 1$ and $1 < x \leqslant 2$ and $f'(x) \geqslant 0$ for $x \geqslant 2$

$\therefore \quad f(x)$ is decreasing for $x < 1$ and $1 < x \leqslant 2$, and increasing for $x \geqslant 2$.

$f''(x) = \dfrac{[e^x(x-2) + e^x(1)](x-1)^2 - e^x(x-2)[2(x-1)^1(1)]}{(x-1)^4}$ {product and quotient rules}

$= \dfrac{[e^x(x-2+1)(x-1)^2] \; - \; 2e^x(x-2)(x-1)}{(x-1)^4}$

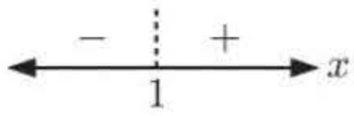
$= \dfrac{e^x(x-1)(x-1)^2 \; - \; 2e^x(x-2)(x-1)}{(x-1)^4}$

$= \dfrac{e^x(x-1)\left[(x-1)^2 \; - \; 2(x-2)\right]}{(x-1)^4}$

$= \dfrac{e^x(x-1)\left[x^2 - 2x + 1 - 2x + 4\right]}{(x-1)^4}$

$= \dfrac{e^x(x^2 - 4x + 5)}{(x-1)^3}$ where the quadratic term has $\Delta < 0$

The sign diagram of $f''(x)$ is:



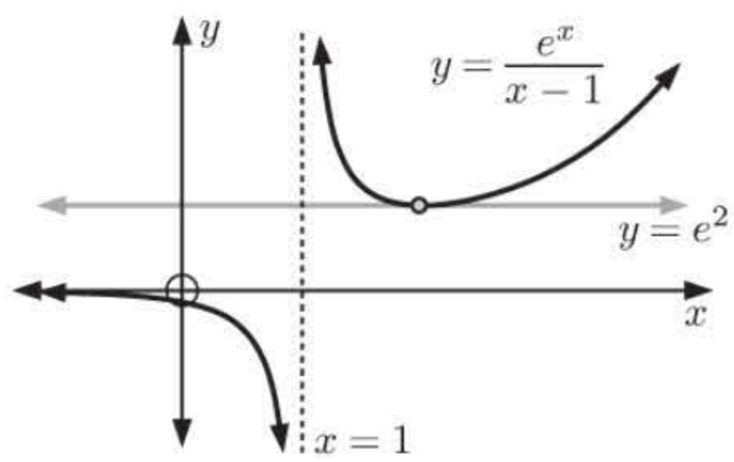
$\therefore \quad f''(x) > 0$ for $x > 1$

and $f''(x) < 0$ for $x < 1$.

$\therefore \quad f(x)$ is concave down for all $x < 1$

and concave up for all $x > 1$.

d



e Now $f(2) = \dfrac{e^2}{2-1} = e^2$

Using **c**, we have a local minimum at $(2, e^2)$

$\therefore$ the tangent at $x = 2$ is horizontal and is $y = e^2$.

9 $y = \dfrac{a}{(x+2)^2} = a(x+2)^{-2}$

The gradient of the line (AB) is

$\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{8-4}{0-2} = \dfrac{4}{-2} = -2$

$\therefore$ the equation of the tangent is

$\dfrac{y-8}{x-0} = -2$ or $y = -2x + 8$

Now $\dfrac{dy}{dx} = -2a(x+2)^{-3}$, so for the given tangent, $-2a(x+2)^{-3} = -2$

$\therefore \dfrac{a}{(x+2)^3} = 1$

$\therefore a = (x+2)^3$ .... (1)

The line (AB) meets the curve where

$-2x + 8 = \dfrac{a}{(x+2)^2}$

$\therefore -2x + 8 = \dfrac{(x+2)^3}{(x+2)^2}$ {using (1)}

$\therefore -2x + 8 = x + 2$

$\therefore -3x = -6$

$\therefore x = 2$

and so $a = (2+2)^3 = 64$

10 $y = \dfrac{5}{\sqrt{x}} = 5x^{-\frac{1}{2}}$

$\therefore \dfrac{dy}{dx} = -\frac{5}{2}x^{-\frac{3}{2}}$

$\therefore$ the gradient of the tangent at the point $(1, 5)$ is $-\frac{5}{2}(1)^{-\frac{3}{2}} = -\frac{5}{2}$

$\therefore$ the equation of the tangent is

$\dfrac{y-5}{x-1} = -\dfrac{5}{2}$

$\therefore y - 5 = -\frac{5}{2}x + \frac{5}{2}$

$\therefore y = -\frac{5}{2}x + \frac{15}{2}$

Now, P and Q are the y- and x-intercepts, so:

P: $y = -\frac{5}{2}(0) + \frac{15}{2}$

$= \frac{15}{2}$

Q: $0 = -\frac{5}{2}x + \frac{15}{2}$

$\therefore \frac{5}{2}x = \frac{15}{2}$

$\therefore x = 3$

So P is $(0, 7.5)$ and Q is $(3, 0)$.

11 At $x = A$, $f'(x) = 0$ and $f''(x) = 0$

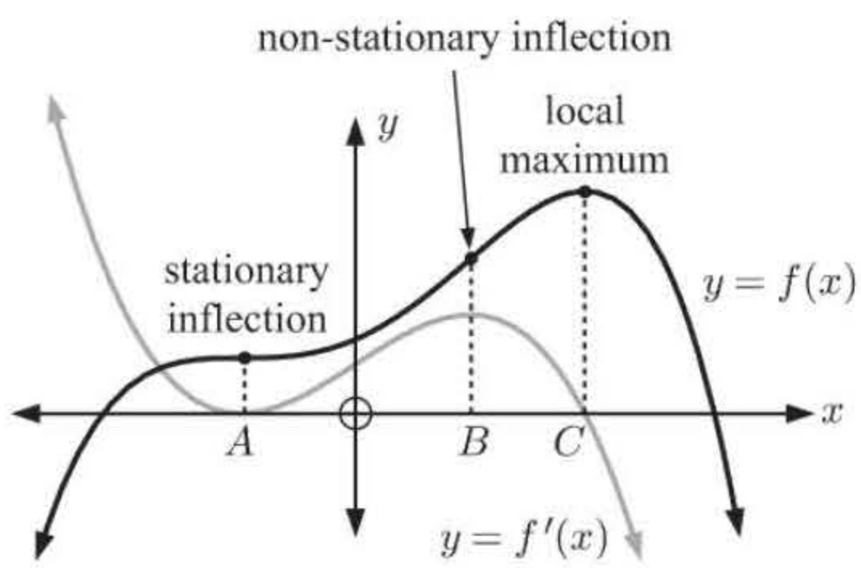
$\therefore f(x)$ has a stationary inflection point at $x = A$.

At $x = B$, $f''(x) = 0$ but $f'(x) \neq 0$

$\therefore f(x)$ has a non-stationary inflection point at $x = B$.

$f'(x)$ is above the x-axis for $x \leqslant C$, and below the x-axis for $x \geqslant C$

$\therefore f(x)$ is increasing for $x \leqslant C$ and decreasing for $x \geqslant C$, so $f(x)$ has a local maximum at $x = C$.



12 $y = \ln(x^2 + 3)$ $\quad \therefore \dfrac{dy}{dx} = \dfrac{2x}{x^2+3}$

When $x = 0$, $\dfrac{dy}{dx} = 0$ so the gradient of the tangent at this point is 0.

But when $x = 0$, $y = \ln(0+3) = \ln 3$

$\therefore$ the tangent is $y = \ln 3$ which does not cut the x-axis.

13 **a** $y = \sec x = \dfrac{1}{\cos x}$

When $x = \frac{\pi}{3}$, then $y = \dfrac{1}{\cos\frac{\pi}{3}} = \dfrac{1}{\frac{1}{2}} = 2$

$\dfrac{dy}{dx} = \sec x \tan x = \dfrac{1}{\cos x}\,\dfrac{\sin x}{\cos x}$

When $x = \frac{\pi}{3}$, then $\dfrac{dy}{dx} = \dfrac{\sin\frac{\pi}{3}}{(\cos\frac{\pi}{3})^2}$

$= \dfrac{\frac{\sqrt{3}}{2}}{(\frac{1}{2})^2}$

$= \frac{\sqrt{3}}{2} \times \frac{4}{1}$

$= 2\sqrt{3}$

$\therefore$ the equation of the tangent at $(\frac{\pi}{3}, 2)$ is:

$$\frac{y-2}{x-\frac{\pi}{3}} = 2\sqrt{3}$$

$$\therefore\ y - 2 = 2\sqrt{3}x - \frac{2\sqrt{3}\pi}{3}$$

$$\therefore\ 2\sqrt{3}x - y = \frac{2\sqrt{3}\pi}{3} - 2$$

b $y = \arctan x$

When $x = \sqrt{3}$, then $y = \arctan\sqrt{3} = \frac{\pi}{3}$

$\dfrac{dy}{dx} = \dfrac{1}{1+x^2}$, $x \in \mathbb{R}$

When $x = \sqrt{3}$, then $\dfrac{dy}{dx} = \dfrac{1}{1+\sqrt{3}^2}$

$= \frac{1}{1+3}$

$= \frac{1}{4}$

$\therefore$ the gradient of the normal is -4

$\therefore$ the equation of the normal at $(\sqrt{3}, \frac{\pi}{3})$ is:

$$\frac{y-\frac{\pi}{3}}{x-\sqrt{3}} = -4$$

$$\therefore\ y - \tfrac{\pi}{3} = -4x + 4\sqrt{3}$$

$$\therefore\ 4x + y = 4\sqrt{3} + \tfrac{\pi}{3}$$

14 $x^2 + y^2 = 1$

$\therefore\ y^2 = 1 - x^2$

$\therefore\ y = (1-x^2)^{\frac{1}{2}} = \pm\sqrt{1-x^2}, \quad x \in [-1, 1]$

For any point on the circle let $x = a$, then $y = \pm\sqrt{1-a^2}$, $a \in [-1, 1]$

If $y > 0$, $\dfrac{dy}{dx} = \frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x) = \dfrac{-x}{\sqrt{1-x^2}}$, $x \in [-1, 1]$

$\therefore$ the gradient of the normal would be $\dfrac{\sqrt{1-x^2}}{x}$

$\therefore$ for the points $(a, \sqrt{1-a^2})$ the gradient is $\dfrac{\sqrt{1-a^2}}{a}$

$\therefore$ the equation of the normal at $(a, \sqrt{1-a^2})$ is:

$$\frac{y-\sqrt{1-a^2}}{x-a} = \frac{\sqrt{1-a^2}}{a}$$

$$\therefore\ ay - \cancel{a\sqrt{1-a^2}} = x\sqrt{1-a^2} - \cancel{a\sqrt{1-a^2}}$$

$$\therefore\ y = \frac{\sqrt{1-a^2}}{a}\,x$$

which for any $a \in [-1, 1]$ will pass through the origin.

If $y < 0$, $\dfrac{dy}{dx} = \dfrac{x}{\sqrt{1-x^2}}$, $x \in [-1, 1]$

$\therefore$ the gradient of the normal is $-\dfrac{\sqrt{1-x^2}}{x}$

$\therefore$ for the points $(a, -\sqrt{1-a^2})$ the gradient is $-\dfrac{\sqrt{1-a^2}}{a}$

$\therefore$ the equation of the normal at $(a, -\sqrt{1-a^2})$ is:

$$\frac{y+\sqrt{1-a^2}}{x-a} = -\frac{\sqrt{1-a^2}}{a}$$

$\therefore\ y = -\dfrac{\sqrt{1-a^2}}{a}\,x$ which will pass through the origin for any $a \in [-1, 1]$.

REVIEW SET 19B

1 $y = x^3 - 3x^2 - 9x + 2 \quad \therefore \quad \dfrac{dy}{dx} = 3x^2 - 6x - 9$

Horizontal tangents occur when $\dfrac{dy}{dx} = 0$

$$\therefore \ 3x^2 - 6x - 9 = 0$$
$$\therefore \ x^2 - 2x - 3 = 0$$
$$\therefore \ (x-3)(x+1) = 0$$
$$\therefore \ x = 3 \text{ or } x = -1$$

When $x = 3$, the horizontal tangent has equation $y = -25$.
When $x = -1$, the horizontal tangent has equation $y = 7$.

2 Consider the tangent to $y = x^2\sqrt{1-x}$ at $x = -3$.

When $x = -3$, $y = (-3)^2\sqrt{1-(-3)} = 9\sqrt{4} = 18 \quad \{y \geqslant 0\}$

$\therefore$ the point of contact is $(-3, 18)$.

Also, $y = x^2\sqrt{1-x}$ is a product with $u = x^2$ and $v = (1-x)^{\frac{1}{2}}$

$$\therefore \ u' = 2x \quad \text{and} \quad v' = \tfrac{1}{2}(1-x)^{-\frac{1}{2}}(-1)$$

$$\therefore \ \frac{dy}{dx} = 2x(1-x)^{\frac{1}{2}} - x^2(\tfrac{1}{2})(1-x)^{-\frac{1}{2}}$$

$$\therefore \text{ at } x = -3, \ \frac{dy}{dx} = 2(-3)(1-(-3))^{\frac{1}{2}} - (-3)^2(\tfrac{1}{2})(1-(-3))^{-\frac{1}{2}}$$
$$= -6(2) - 9(\tfrac{1}{2})(\tfrac{1}{2})$$
$$= -\tfrac{57}{4}$$

$\therefore$ the tangent at $(-3, 18)$ has equation $\dfrac{y-18}{x-(-3)} = -\frac{57}{4}$

$$\therefore \ 4y - 72 = -57x - 171$$
$$\therefore \ 4y = -57x - 99$$

Now when $x = 0$, $y = -\frac{99}{4}$ and when $y = 0$, $x = -\frac{99}{57}$

$\therefore$ the area of $\triangle$OAB $= \frac{1}{2}\left(\frac{99}{4}\right)\left(\frac{99}{57}\right) = \frac{3267}{152} \approx 21.5$ units^2

3 **a** $f(x) = x^3 + ax, \quad a < 0$

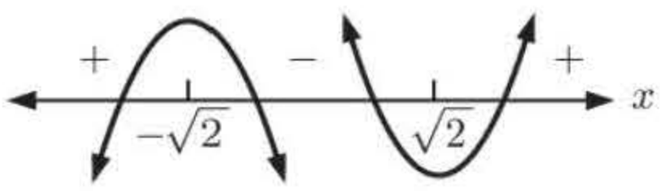
$\therefore \ f'(x) = 3x^2 + a$

$f(x)$ has a turning point at $x = \sqrt{2}$, so $f'(\sqrt{2}) = 0$

$$\therefore \ 3(\sqrt{2})^2 + a = 0$$
$$\therefore \ a = -6$$

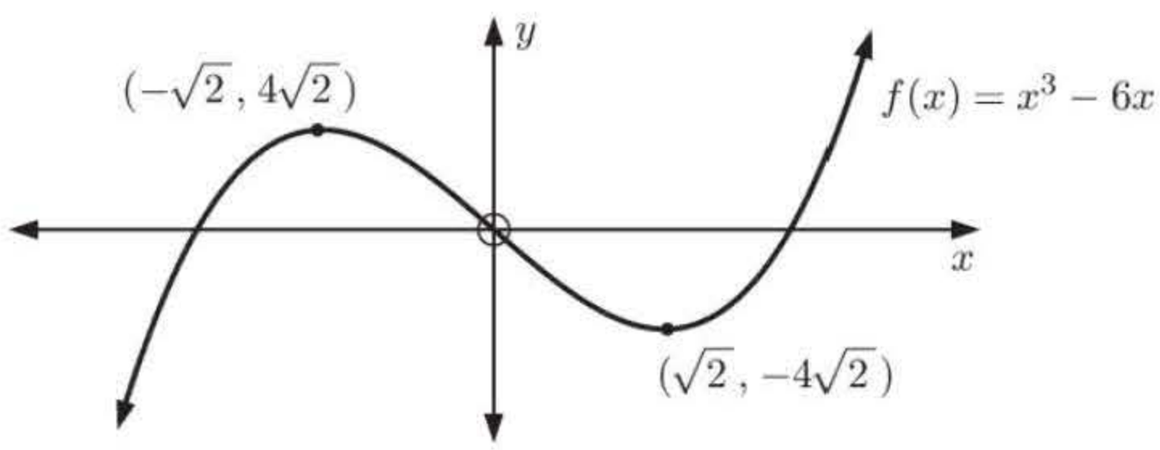
b $f'(x) = 3x^2 - 6 = 3(x^2 - 2) = 3(x + \sqrt{2})(x - \sqrt{2})$

$\therefore \ f'(x)$ has sign diagram:



$\therefore \ f(x)$ has a local maximum at $\left(-\sqrt{2}, (-\sqrt{2})^3 - 6(-\sqrt{2})\right)$ or $\left(-\sqrt{2}, 4\sqrt{2}\right)$,
and a local minimum at $\left(\sqrt{2}, (\sqrt{2})^3 - 6\sqrt{2}\right)$ or $\left(\sqrt{2}, -4\sqrt{2}\right)$.

c



4 $f(x) = e^{4x} + px + q$

$\therefore\ f'(x) = 4e^{4x} + p$

At the point where $x = 0$, the tangent to $f(x)$ has equation $y = 5x - 7$, so $f'(0) = 5$

$\therefore\ 4e^0 + p = 5$

$\therefore\ p = 1$

The tangent meets $f(x)$ when $x = 0$ and $y = 5(0) - 7 = -7$, so $(0, -7)$ must lie on $f(x)$ too.

$\therefore\ e^{4(0)} + p(0) + q = -7$

$\therefore\ 1 + q = -7$

$\therefore\ q = -8$

5 Consider the tangent to $y = 2x^3 + 4x - 1$ at $(1, 5)$.

$\dfrac{dy}{dx} = 6x^2 + 4$ $\quad\therefore$ at $x = 1$, $\dfrac{dy}{dx} = 6(1)^2 + 4 = 10$

$\therefore$ the tangent has equation $\dfrac{y-5}{x-1} = 10$ or $y = 10x - 5$

Now the tangent meets the curve again where $10x - 5 = 2x^3 + 4x - 1$

$\therefore\ 2x^3 - 6x + 4 = 0$

$\therefore\ x^3 - 3x + 2 = 0$

We know that $(x-1)^2$ is a factor since the line is tangent to the curve at $x = 1$.

Consequently, $x^3 - 3x + 2 = (x-1)^2(x+2) = 0$ {since the constant term is 2}

Thus $x = -2$ is the other solution and when $x = -2$, $y = 2(-2)^3 + 4(-2) - 1 = -25$

$\therefore$ the tangent meets the curve again at $(-2, -25)$.

6 Consider $y = 4(ax+1)^{-2}$.

When $x = 0$, $y = 4(0+1)^{-2} = 4$, so the point of contact is $(0, 4)$.

Now $\dfrac{dy}{dx} = -8(ax+1)^{-3}(a) = \dfrac{-8a}{(ax+1)^3}$ $\quad\therefore$ at $x = 0$, $\dfrac{dy}{dx} = -8a$

$\therefore$ the tangent has equation $\dfrac{y-4}{x-0} = -8a$ or $y - 4 = -8ax$

This tangent passes through $(1, 0)$, so $0 - 4 = -8a(1)$ $\quad\therefore\ a = \frac{1}{2}$

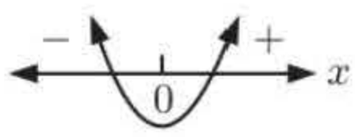
7 $f(x) = e^x - x$

a $f'(x) = e^x - 1$

so $f'(x) = 0$ when $e^x = 1$

$\therefore\ x = 0$

Sign diagram of $f'(x)$ is:



Now $f(0) = e^0 - 0 = 1$

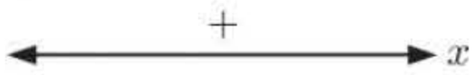
$\therefore$ there is a local minimum at $(0, 1)$.

b As $x \to \infty$, $e^x \to \infty$ faster than x

$\therefore\ f(x) \to \infty$

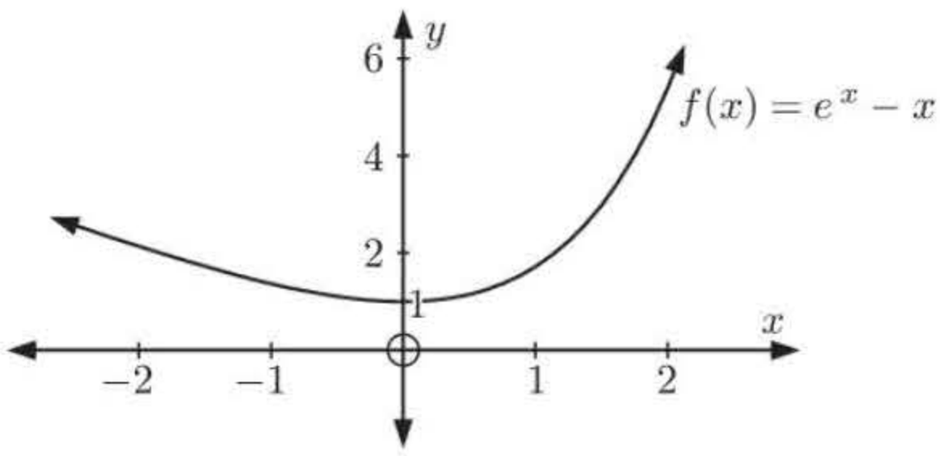
c $f''(x) = e^x$

$\therefore\ f''(x) > 0$ for all x



$\therefore\ f(x)$ is concave up for all x

d



e Since a local minimum exists at $(0, 1)$,

$f(x) \geqslant 1$ for all x

$\therefore\ e^x - x \geqslant 1$

$\therefore\ e^x \geqslant x + 1$ for all x

8 Consider $y = \dfrac{x+1}{x^2-2}$.

When $x = 1$, $y = \dfrac{1+1}{1^2-2} = -2$

$\therefore$ the point of contact is $(1, -2)$.

$y = \dfrac{x+1}{x^2-2}$ is a quotient with $u = x+1$ and $v = x^2-2$

$\therefore\ u' = 1$ and $v' = 2x$

$\dfrac{dy}{dx} = \dfrac{1(x^2-2)-(x+1)(2x)}{(x^2-2)^2}$ {quotient rule}

$\therefore$ at $x = 1$, $\dfrac{dy}{dx} = \dfrac{(1-2)-2(1+1)}{(1-2)^2}$

$= \dfrac{-1-4}{1} = -5$

$\therefore$ the normal at $(1, -2)$ has gradient $\frac{1}{5}$.

So the normal has equation $\dfrac{y-(-2)}{x-1} = \frac{1}{5}$

$\therefore\ 5y + 10 = x - 1$

$\therefore\ y = \frac{1}{5}x - \frac{11}{5}$ (or $x - 5y = 11$)

9 $y = 2 - \dfrac{7}{1+2x} = 2 - 7(1+2x)^{-1}$

$\therefore\ \dfrac{dy}{dx} = 7(1+2x)^{-2} \times 2$ {chain rule}

$= \dfrac{14}{(1+2x)^2}$

The tangent is horizontal when the gradient $\dfrac{dy}{dx} = 0$.

But $\dfrac{14}{(1+2x)^2}$ is never 0, so

$y = 2 - \dfrac{7}{1+2x}$ has no horizontal tangents.

10 Let $g(x) = ax^2 + bx + c$.

$g(x)$ has y-intercept $(0, 3)$, so $g(0) = a(0)^2 + b(0) + c = 3$

$\therefore\ c = 3$

$\therefore\ g(x) = ax^2 + bx + 3$

The point $(2, 7)$ lies on $g(x)$, so $g(2) = a(2)^2 + b(2) + 3 = 7$

$\therefore\ 4a + 2b = 4$ .... (1)

Also, $g'(x) = 2ax + b$

$\therefore\ g'(2) = 2a(2) + b = 4a + b$

$\therefore$ the gradient of the tangent to $g(x)$ at $(2, 7)$ is $4a + b$.

But, the tangent at $(2, 7)$ passes through $(0, 11)$, so the gradient $= \dfrac{7-11}{2-0} = -2$

$\therefore\ 4a + b = -2$ .... (2)

Solving (1) and (2) simultaneously,

$$\begin{aligned} 4a + 2b &= 4 \\ 4a + b &= -2 \\ \hline \text{subtracting:} \quad b &= 6 \end{aligned}$$

Using (2), $4a + 6 = -2$, so $a = -2$

So, $g(x) = -2x^2 + 6x + 3$

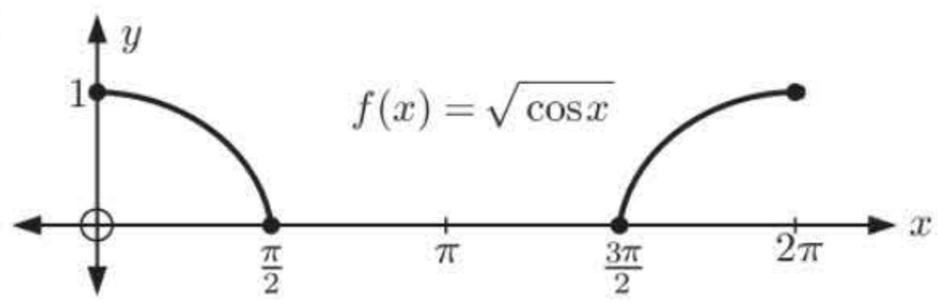
11 **a** $f(x) = \sqrt{\cos x}, \quad 0 \leqslant x \leqslant 2\pi$

$f(x)$ is defined when $\cos x \geqslant 0$,

which is when $0 \leqslant x \leqslant \frac{\pi}{2}$

and $\frac{3\pi}{2} \leqslant x \leqslant 2\pi$

c



b $f(x) = (\cos x)^{\frac{1}{2}}$

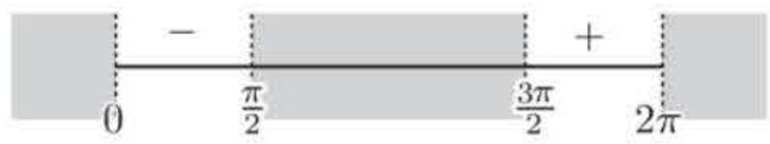
$\therefore \quad f'(x) = \frac{1}{2}(\cos x)^{-\frac{1}{2}}(-\sin x)$

$= \dfrac{-\sin x}{2\sqrt{\cos x}}$

$\therefore \quad f'(x) = 0$ when $-\sin x = 0$

For $0 \leqslant x \leqslant 2\pi$, this is when $x = 0, \pi, 2\pi$.

Sign diagram for $f'(x)$ is:



$f(x)$ is increasing for $\frac{3\pi}{2} \leqslant x \leqslant 2\pi$

and decreasing for $0 \leqslant x \leqslant \frac{\pi}{2}$.

12 $f(x)$ has a turning point at $x = 0$

$\therefore \quad f'(0) = 0$

$f(x)$ is increasing for $x \geqslant 0$,

except at the asymptote,

so $f'(x)$ is positive for $x \geqslant 0$.

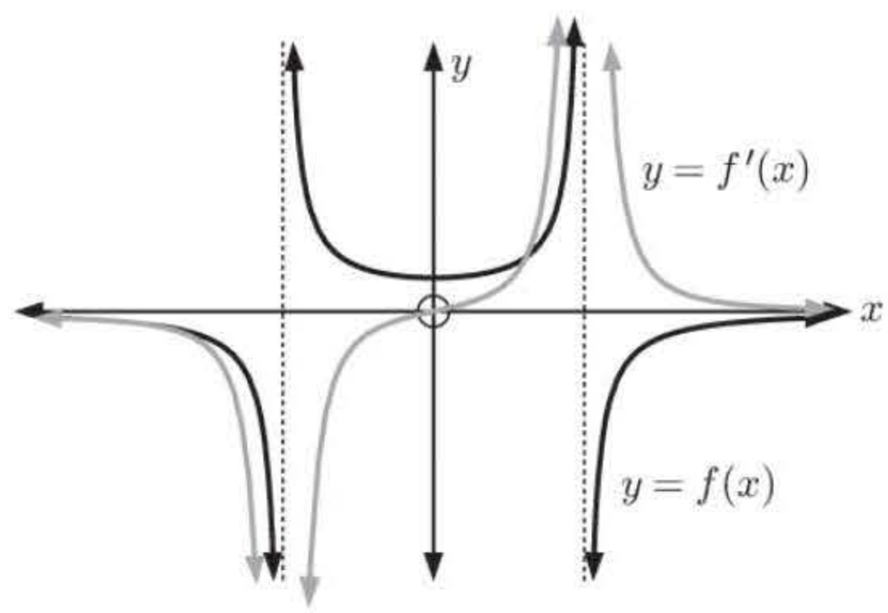
$f(x)$ is decreasing for $x \leqslant 0$,

except at the asymptote,

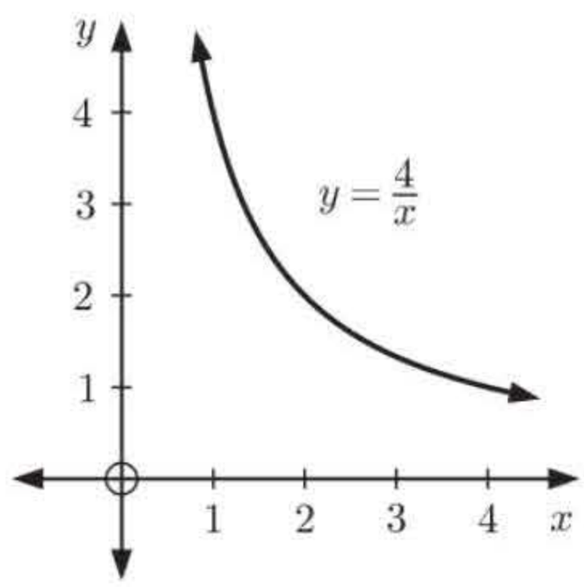
so $f'(x)$ is negative for $x \leqslant 0$.

As $x \to \pm\infty$, $f(x)$ becomes

closer to horizontal so $f'(x) \to 0$.



13 **a**



b For $f(x) = \dfrac{4}{x} = 4x^{-1}$,

$f'(x) = -4x^{-2} = -\dfrac{4}{x^2}$ and $f'(k) = -\dfrac{4}{k^2}, \quad k > 0$

$\therefore$ the gradient of the tangent to $f(x)$ at $\left(k, \dfrac{4}{k}\right)$ is $-\dfrac{4}{k^2}$

$\therefore$ the equation of the tangent is $\dfrac{y - \frac{4}{k}}{x - k} = -\dfrac{4}{k^2}$

$\therefore \quad yk^2 - 4k = -4x + 4k$

$\therefore \quad k^2 y = -4x + 8k$

$\therefore \quad y = -\dfrac{4}{k^2}x + \dfrac{8}{k}$

c $y = -\dfrac{4}{k^2}x + \dfrac{8}{k}$ cuts the x-axis when $y = 0$

$\therefore \quad -\dfrac{4}{k^2}x + \dfrac{8}{k} = 0$

$\therefore \quad \dfrac{4}{k^2}x = \dfrac{8}{k}$

$\therefore \quad x = 2k \qquad \therefore$ A is at $(2k, 0)$

$y = -\dfrac{4}{k^2}x + \dfrac{8}{k}$ cuts the y-axis when $x = 0$

$\therefore \quad y = \dfrac{8}{k} \qquad \therefore$ B is at $\left(0, \dfrac{8}{k}\right)$

d Area of triangle OAB $= \frac{1}{2}(2k)\left(\dfrac{8}{k}\right) = 8$ units^2

e The gradient of the tangent to $f(x)$ at $\left(k, \frac{4}{k}\right)$ is $-\frac{4}{k^2}$

$\therefore$ the gradient of the normal to $f(x)$ at $\left(k, \frac{4}{k}\right)$ is $\frac{k^2}{4}$

$\therefore$ the equation of the normal is $\dfrac{y - \frac{4}{k}}{x - k} = \dfrac{k^2}{4}$

$$\therefore \quad 4y - \frac{16}{k} = k^2x - k^3$$

$$\therefore \quad 4ky - k^3x = 16 - k^4$$

This normal passes through $(1, 1)$, so $4k - k^3 = 16 - k^4$

$$\therefore \quad k^4 - k^3 + 4k - 16 = 0$$

$$\therefore \quad (k-2)(k+2)(k^2 - k + 4) = 0 \quad \{\text{using technology}\}$$

$$\therefore \quad k = \pm 2$$

But $k > 0$, so $k = 2$

14 $y = \dfrac{x}{\sqrt{1-x}} = x(1-x)^{-\frac{1}{2}}$

When $x = -3$, $y = \dfrac{-3}{\sqrt{1-(-3)}} = \frac{-3}{\sqrt{4}} = -\frac{3}{2}$

$$\frac{dy}{dx} = x(-\tfrac{1}{2})(1-x)^{-\frac{3}{2}}(-1) + (1-x)^{-\frac{1}{2}}$$

$$= \frac{x}{2(1-x)^{\frac{3}{2}}} + \frac{1}{(1-x)^{\frac{1}{2}}}$$

When $x = -3$, $\dfrac{dy}{dx} = \dfrac{-3}{2(1-(-3))^{\frac{3}{2}}} + \dfrac{1}{(1-(-3))^{\frac{1}{2}}}$

$$= \frac{-3}{2 \times 4^{\frac{3}{2}}} + \frac{1}{4^{\frac{1}{2}}}$$

$$= \tfrac{-3}{16} + \tfrac{8}{16}$$

$$= \tfrac{5}{16}$$

$\therefore$ the equation of the tangent at $(-3, -\frac{3}{2})$ is: $\quad \therefore \dfrac{y - (-\frac{3}{2})}{x - (-3)} = \frac{5}{16}$

$$\therefore \quad 16(y + \tfrac{3}{2}) = 5(x+3)$$

$$\therefore \quad 16y + 24 = 5x + 15$$

$$\therefore \quad 5x - 16y = 9$$

Given $5x + by = a \quad \therefore \quad a = 9$ and $b = -16$

REVIEW SET 19C

1 Consider the normal to the curve $y = \dfrac{1}{\sqrt{x}}$ at $x = 4$.

When $x = 4$, $y = \frac{1}{\sqrt{4}} = \frac{1}{2}$, so the point of contact is $(4, \frac{1}{2})$.

Now $\dfrac{dy}{dx} = -\frac{1}{2}x^{-\frac{3}{2}} \quad \therefore$ at $x = 4$, $\dfrac{dy}{dx} = -\frac{1}{2}\left(4^{-\frac{3}{2}}\right) = -\frac{1}{2}\left(\frac{1}{8}\right) = -\frac{1}{16}$

$\therefore$ the normal at $(4, \frac{1}{2})$ has gradient 16.

So the equation is $\dfrac{y - \frac{1}{2}}{x - 4} = 16$

$$\therefore \quad y - \tfrac{1}{2} = 16x - 64$$

$$\therefore \quad y = 16x - \tfrac{127}{2}$$

2 **a** The tangent shown on the graph passes through (0, 5) and (5, 0).

$\therefore$ the gradient of the tangent is

$$\frac{0-5}{5-0} = -1, \quad \text{so} \quad f'(3) = -1.$$

Also, since the tangent passes through (0, 5),

it has equation $\dfrac{y-5}{x-0} = -1$

$$\therefore \quad y - 5 = -x$$
$$\therefore \quad y = -x + 5$$

So when $x = 3$, $y = -3 + 5 = 2$

$\therefore$ the point of contact is (3, 2), and hence $f(3) = 2$.

b $f(x)$ has the form $f(x) = ax^2 + bx + c$

The y-intercept is 14 $\therefore$ $f(0) = 14$.

$$\therefore \quad a(0)^2 + b(0) + c = 14$$
$$\therefore \quad c = 14$$

$f(3) = 2$

$$\therefore \quad a(3)^2 + b(3) + 14 = 2$$
$$\therefore \quad 9a + 3b = -12 \quad .... (1)$$

$f'(3) = -1$

and $f'(x) = 2ax + b$

$$\therefore \quad 2a(3) + b = -1$$
$$\therefore \quad 6a + b = -1$$
$$\therefore \quad b = -6a - 1 \quad .... (2)$$

Substituting (2) into (1),

$$9a + 3(-6a - 1) = -12$$
$$\therefore \quad 9a - 18a - 3 = -12$$
$$\therefore \quad -9a = -9$$
$$\therefore \quad a = 1$$

Using (2), $b = -6(1) - 1$

$$\therefore \quad b = -7$$

So, $f(x) = x^2 - 7x + 14$

3 $y = x^3 + ax + b$ $\quad \therefore \quad \dfrac{dy}{dx} = 3x^2 + a$

$$\therefore \quad \text{at} \quad x = 1, \quad \frac{dy}{dx} = 3 + a$$

The equation of the tangent at $x = 1$ is $y = 2x$, so the gradient is 2.

$\therefore$ $3 + a = 2$ and so $a = -1$

Also at $x = 1$, the tangent touches the curve.

$$\therefore \quad x^3 + ax + b = 2x \quad \text{when} \quad x = 1$$
$$\therefore \quad (1)^3 + (-1)(1) + b = 2(1)$$
$$\therefore \quad 1 - 1 + b = 2$$
$$\therefore \quad b = 2$$

4 **a** $y = x^3 + ax^2 - 4x + 3$ $\quad \therefore \quad \dfrac{dy}{dx} = 3x^2 + 2ax - 4$

The tangent at $x = 1$ is parallel to $y = 3x$, so when $x = 1$, $\dfrac{dy}{dx} = 3$

$$\therefore \quad 3 = 3(1)^2 + 2a(1) - 4$$
$$\therefore \quad 2a = 4$$
$$\therefore \quad a = 2$$

When $x = 1$, $y = 1^3 + 2(1)^2 - 4(1) + 3 = 2$

The contact point is (1, 2) and since the gradient is 3, the tangent at (1, 2) has equation

$$\frac{y-2}{x-1} = 3 \qquad \therefore \quad y - 2 = 3x - 3$$
$$\therefore \quad y = 3x - 1$$

b The tangent meets the curve where $x^3 + 2x^2 - 4x + 3 = 3x - 1$

$$\therefore \quad x^3 + 2x^2 - 7x + 4 = 0$$

Since the line touches the curve at $x = 1$, $(x-1)^2$ must be a factor.

Consequently, $x^3 + 2x^2 - 7x + 4 = (x-1)^2(x+4) = 0$ $\quad$ {since the constant term is 4}

$\therefore$ the curve cuts the tangent when $x = -4$.

When $x = -4$, $y = (-4)^3 + 2(-4)^2 - 4(-4) + 3 = -13$

$\therefore$ the curve cuts the tangent at $(-4, -13)$.

5 $y = \ln(x^4 + 3)$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{4x^3}{x^4+3}$

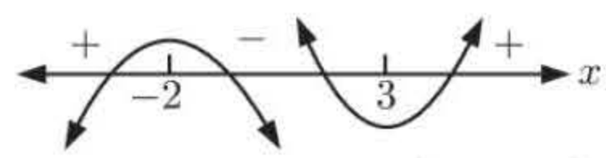
$\therefore$ when $x = 1$, $\dfrac{dy}{dx} = \dfrac{4(1)^3}{1^4+3} = 1$ and $y = \ln(1^4+3) = \ln 4$

$\therefore$ the tangent has equation $\dfrac{y - \ln 4}{x - 1} = 1$ or $y = x - 1 + \ln 4$

Now when $x = 0$, $y = \ln 4 - 1$, so the tangent cuts the y-axis at $(0, \ln 4 - 1)$.

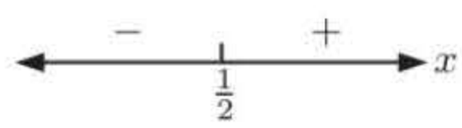
6 **a** $f(x) = 2x^3 - 3x^2 - 36x + 7$

$\therefore \quad f'(x) = 6x^2 - 6x - 36$
$= 6(x^2 - x - 6)$
$= 6(x-3)(x+2)$ with sign diagram:



Now $f(-2) = 51$, $f(3) = -74$, so there is a local maximum at $(-2, 51)$, and a local minimum at $(3, -74)$.

$f''(x) = 12x - 6$
$= 6(2x - 1)$ with sign diagram:

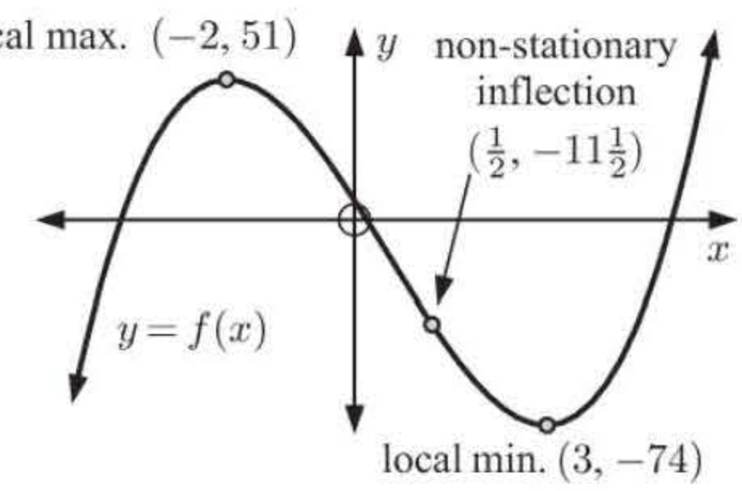


Now $f(\frac{1}{2}) = -\frac{23}{2}$, so there is a non-stationary inflection at $(\frac{1}{2}, -\frac{23}{2})$.

b $f(x)$ is increasing when $x \leqslant -2$ or $x \geqslant 3$, and decreasing when $-2 \leqslant x \leqslant 3$.

c $f(x)$ is concave up when $x \geqslant \frac{1}{2}$, and concave down when $x \leqslant \frac{1}{2}$.

d



7 Consider the normal to $f(x) = \dfrac{3x}{1+x}$ at $(2, 2)$.

$f(x)$ is a quotient with $u = 3x$ and $v = 1 + x$
$\therefore \quad u' = 3$ and $v' = 1$

$\therefore \quad f'(x) = \dfrac{3(1+x) - 1(3x)}{(1+x)^2} = \dfrac{3}{(1+x)^2}$ {quotient rule}

$\therefore \quad f'(2) = \frac{3}{9} = \frac{1}{3}$

$\therefore$ the normal at $(2, 2)$ has gradient -3

So, the equation of the normal is $\dfrac{y-2}{x-2} = -3$

$\therefore \quad y - 2 = -3(x - 2)$
$\therefore \quad y = -3x + 8$

When $x = 0$, $y = 8$ and when $y = 0$, $x = \frac{8}{3}$

$\therefore$ B and C are at $(0, 8)$ and $(\frac{8}{3}, 0)$,

and the distance BC $= \sqrt{\left(0 - \frac{8}{3}\right)^2 + (8-0)^2} = \sqrt{\frac{64}{9} + 64} = \sqrt{\frac{640}{9}} = \frac{8\sqrt{10}}{3}$ units

8 $f(x) = x^3 - 4x^2 + 4x$
$= x(x^2 - 4x + 4)$
$= x(x-2)^2$

a $f(0) = 0$, so the y-intercept is 0.

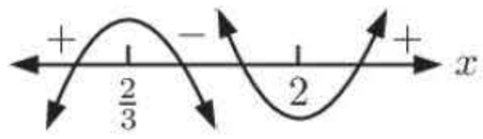
$f(x)$ cuts the x-axis when $y = 0$ $\quad \therefore \quad x(x-2)^2 = 0$
$\therefore \quad x = 0$ or 2

$\therefore$ the x-intercepts are 0 and 2.

b $f'(x) = 3x^2 - 8x + 4$
$= (3x-2)(x-2)$

which is 0 when $x = \frac{2}{3}$ or 2

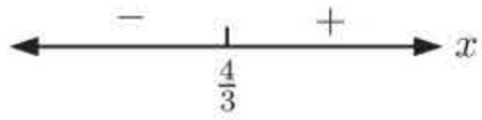
Sign diagram of $f'(x)$ is:



Now $f(\frac{2}{3}) = \frac{32}{27}$, so there is a local maximum at $(\frac{2}{3}, \frac{32}{27})$, and a local minimum at $(2, 0)$.

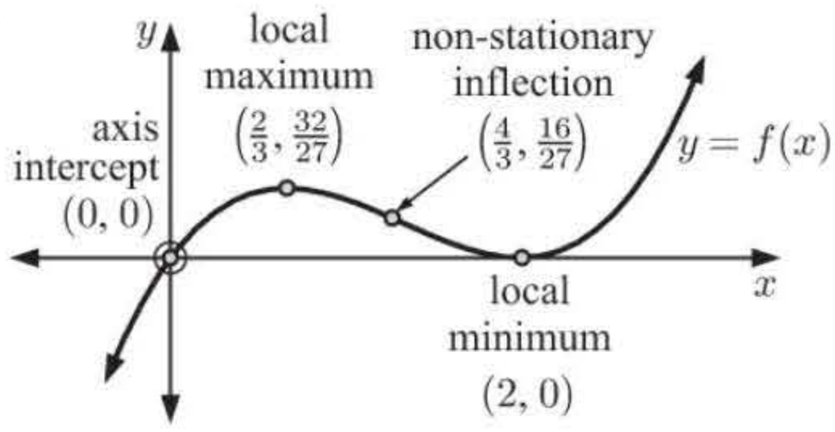
$f''(x) = 6x - 8 = 2(3x-4)$

Sign diagram of $f''(x)$ is:



Now $f(\frac{4}{3}) = \frac{16}{27}$, so there is a non-stationary inflection at $(\frac{4}{3}, \frac{16}{27})$.

c



9 **a** $y = \dfrac{1}{\sin x} = (\sin x)^{-1}$

$\therefore \dfrac{dy}{dx} = -(\sin x)^{-2}(\cos x)$
$= -\dfrac{\cos x}{\sin^2 x}$

When $x = \frac{\pi}{3}$, $y = \dfrac{1}{\sin(\frac{\pi}{3})} = \frac{2}{\sqrt{3}}$

and $\dfrac{dy}{dx} = -\dfrac{\cos(\frac{\pi}{3})}{\sin^2(\frac{\pi}{3})} = -\dfrac{\frac{1}{2}}{(\frac{\sqrt{3}}{2})^2} = -\frac{2}{3}$

$\therefore$ the tangent has equation $\dfrac{y - \frac{2}{\sqrt{3}}}{x - \frac{\pi}{3}} = -\frac{2}{3}$ which is $3y - 2\sqrt{3} = -2x + \frac{2\pi}{3}$

or $2x + 3y = 2\sqrt{3} + \frac{2\pi}{3}$

b $y = \cos(\frac{x}{2})$

$\therefore \dfrac{dy}{dx} = -\frac{1}{2}\sin(\frac{x}{2})$

When $x = \frac{\pi}{2}$, $y = \cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

and $\dfrac{dy}{dx} = -\frac{1}{2}\sin(\frac{\pi}{4}) = -\frac{1}{2\sqrt{2}}$

$\therefore$ the normal has gradient $2\sqrt{2}$, and its equation is $\dfrac{y - \frac{1}{\sqrt{2}}}{x - \frac{\pi}{2}} = 2\sqrt{2}$

$\therefore y - \frac{1}{\sqrt{2}} = 2\sqrt{2}x - \pi\sqrt{2}$

$\therefore y - 2\sqrt{2}x = \frac{1}{\sqrt{2}} - \pi\sqrt{2}$

or $\sqrt{2}y - 4x = 1 - 2\pi$

10 $f(x) = 3x^3 + ax^2 + b$

$\therefore f'(x) = 9x^2 + 2ax$

Since the tangent at $(-2, 14)$ has gradient 0, $f'(-2) = 0$

$\therefore 36 - 4a = 0$

$\therefore a = 9$

As the point $(-2, 14)$ lies on the curve, $14 = 3(-2)^3 + 9(-2)^2 + b$

$\therefore b = 14 + 24 - 36$

$\therefore b = 2$

$\therefore f'(x) = 9x^2 + 18x$

$\therefore f''(x) = 18x + 18$ and so $f''(-2) = -36 + 18 = -18$

11 The curves $y = \sqrt{3x+1}$ and $y = \sqrt{5x - x^2}$ meet when $\sqrt{3x+1} = \sqrt{5x - x^2}$

Squaring both sides, $3x + 1 = 5x - x^2$

$\therefore x^2 - 2x + 1 = 0$

$\therefore (x-1)^2 = 0$

$\therefore x = 1$

When $x = 1$, $y = \sqrt{3+1} = 2$, so the curves meet at $(1, 2)$.

Now for $y = \sqrt{3x+1} = (3x+1)^{\frac{1}{2}}$

$\dfrac{dy}{dx} = \frac{1}{2}(3x+1)^{-\frac{1}{2}}(3)$

$\therefore$ at $(1, 2)$, $\dfrac{dy}{dx} = \dfrac{3}{2(3+1)^{\frac{1}{2}}} = \frac{3}{4}$

Check: $y = \sqrt{5x - x^2} = (5x - x^2)^{\frac{1}{2}}$

$\dfrac{dy}{dx} = \frac{1}{2}(5x - x^2)^{-\frac{1}{2}}(5 - 2x) = \dfrac{5-2x}{2\sqrt{5x - x^2}}$

$\therefore$ at $(1, 2)$, $\dfrac{dy}{dx} = \dfrac{5-2}{2\sqrt{5-1}} = \frac{3}{4}$ ✓

$\therefore$ the curves have a common tangent at their point of intersection.

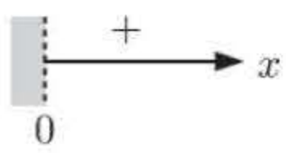
The equation of the common tangent at $(1, 2)$ is $\dfrac{y-2}{x-1} = \frac{3}{4}$

$\therefore\ 4(y-2) = 3(x-1)$

$\therefore\ 4y = 3x + 5$

12 **a** $f(x) = x + \ln x$ is defined when $x > 0$

b $f'(x) = 1 + \dfrac{1}{x} = \dfrac{x+1}{x}$ which has sign diagram:

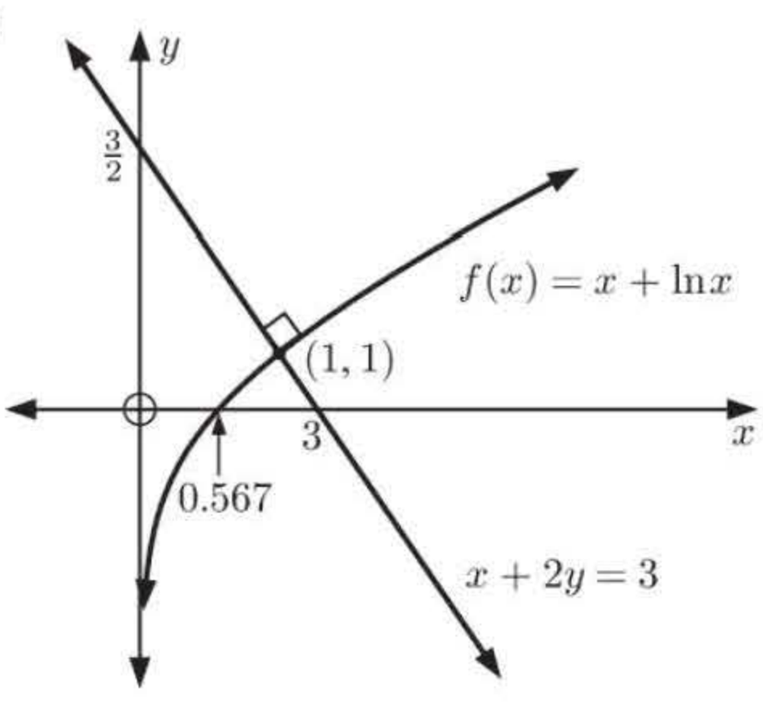


$\therefore$ $f(x)$ is increasing for all $x > 0$.

$f''(x) = -\dfrac{1}{x^2}$ which has sign diagram:

$\therefore$ $f(x)$ is concave down for all $x > 0$.

c



d $f(1) = 1 + \ln(1) = 1$

$\therefore$ $(1, 1)$ is the point of contact.

$f'(1) = \dfrac{1+1}{1} = 2$

$\therefore$ the tangent at $x = 1$ has gradient 2,

so the normal has gradient $-\frac{1}{2}$

$\therefore$ the normal has equation $\dfrac{y-1}{x-1} = -\frac{1}{2}$

$\therefore\ 2y - 2 = -x + 1$

$\therefore\ x + 2y = 3$

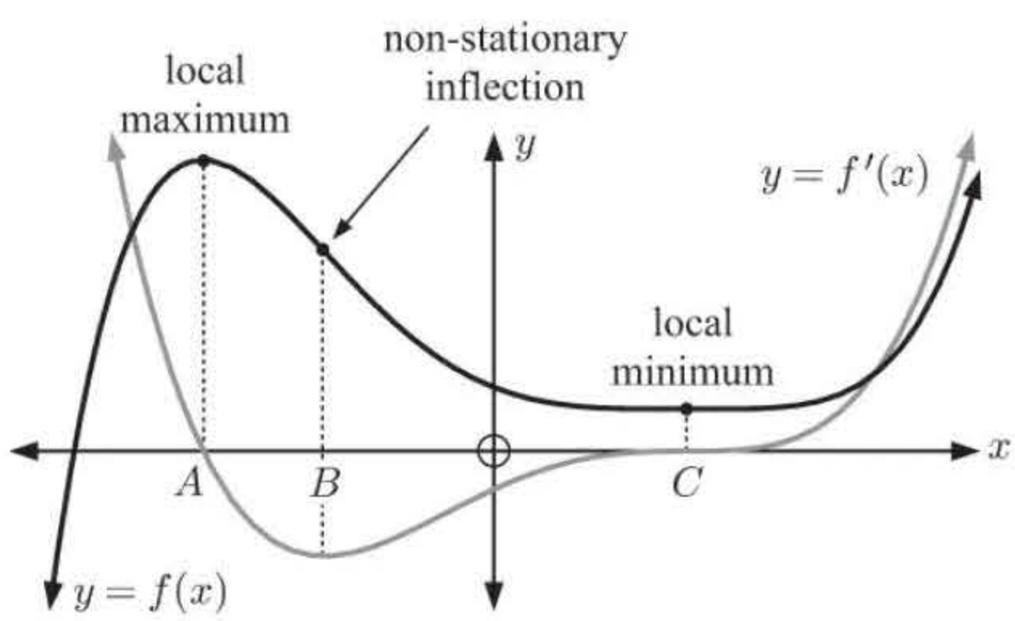
13 At $x = B$, $f''(x) = 0$ but $f'(x) \neq 0$

$\therefore$ $f(x)$ has a non-stationary inflection point at $x = B$.

$f'(x)$ is above the x-axis for $x \leqslant A$ and $x \geqslant C$, and below the x-axis for $A \leqslant x \leqslant C$

$\therefore$ $f(x)$ is increasing for $x \leqslant A$, decreasing for $A \leqslant x \leqslant C$, then increasing for $x \geqslant C$

$\therefore$ $f(x)$ has a local maximum at $x = A$ and a local minimum at $x = C$.



14 $f(x) = \arcsin x + \arccos x$

$$f'(x) = \frac{1}{\sqrt{1-x^2}} + \left(\frac{-1}{\sqrt{1-x^2}}\right), \quad x \in]-1, 1[$$
$$= 0$$

$\therefore$ the gradient of the function is zero.

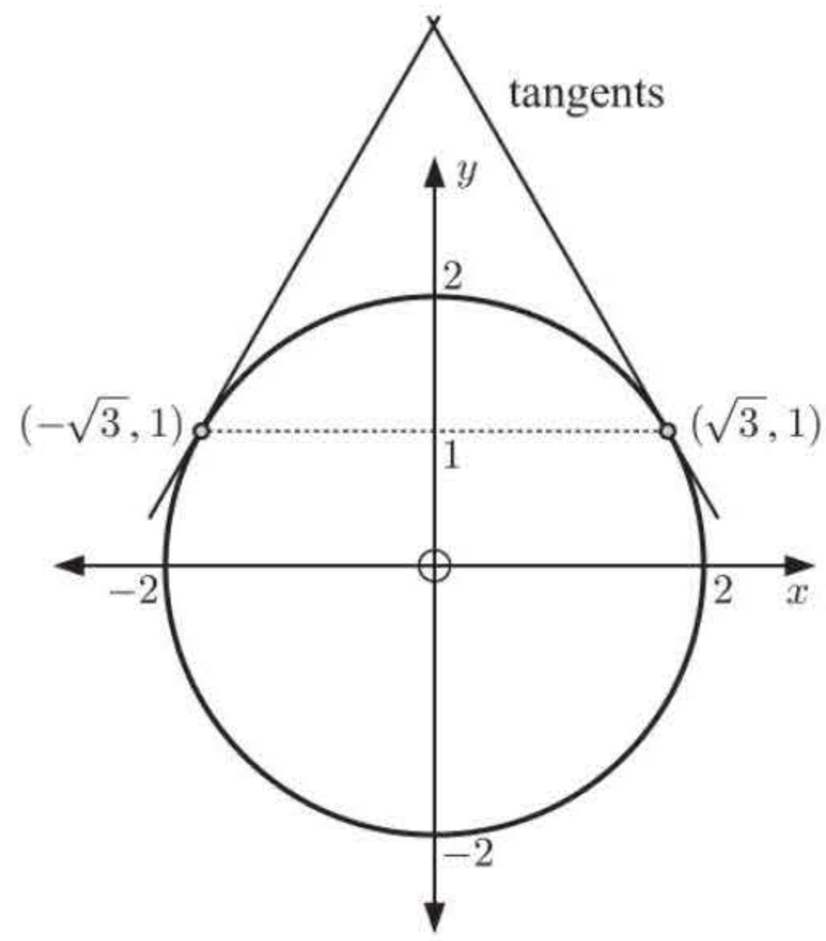
$$f(0) = \arcsin(0) + \arccos(0)$$
$$= 0 + \tfrac{\pi}{2}$$

$\therefore$ $f(x) = \frac{\pi}{2}$, a constant.

15 **a** $x^2 + y^2 = 4$

$$\therefore \quad y = \pm\sqrt{4-x^2} = \pm(4-x^2)^{\frac{1}{2}}$$

$$\therefore \quad \frac{dy}{dx} = \pm\tfrac{1}{2}(4-x^2)^{-\frac{1}{2}}(-2x)$$
$$= \frac{\mp x}{\sqrt{4-x^2}}$$



$\therefore$ when $y = (4-x^2)^{\frac{1}{2}}$, $\quad \dfrac{dy}{dx} = \dfrac{-x}{\sqrt{4-x^2}}$

and when $y = -(4-x^2)^{\frac{1}{2}}$, $\quad \dfrac{dy}{dx} = \dfrac{x}{\sqrt{4-x^2}}$

When $y = 1$, then $x^2 + 1^2 = 4$

$$\therefore \quad x^2 = 3$$
$$\therefore \quad x = \pm\sqrt{3}$$

At the point $(\sqrt{3}, 1)$, $\quad \dfrac{dy}{dx} = \dfrac{-\sqrt{3}}{\sqrt{4-\sqrt{3}^2}} = \frac{-\sqrt{3}}{1} = -\sqrt{3}$

At the point $(-\sqrt{3}, 1)$, $\quad \dfrac{dy}{dx} = \dfrac{-(-\sqrt{3})}{\sqrt{4-\left(-\sqrt{3}\right)^2}} = \frac{\sqrt{3}}{1} = \sqrt{3}$

$\therefore$ the equation of the tangent at $(\sqrt{3}, 1)$, is:

$$\frac{y-1}{x-\sqrt{3}} = -\sqrt{3}$$
$$\therefore \quad y - 1 = -\sqrt{3}x + 3$$
$$\therefore \quad \sqrt{3}x + y = 4$$

$\therefore$ the equation of the tangent at $(-\sqrt{3}, 1)$ is:

$$\frac{y-1}{x-(-\sqrt{3})} = \sqrt{3}$$
$$\therefore \quad y - 1 = \sqrt{3}(x+\sqrt{3})$$
$$\therefore \quad y - 1 = \sqrt{3}x + 3$$
$$\therefore \quad \sqrt{3}x - y = -4$$

b The tangents intersect when $\sqrt{3}x = 4 - y$, $\sqrt{3}x = -4 + y$

$$\therefore \quad 4 - y = -4 + y$$
$$\therefore \quad 8 = 2y$$
$$\therefore \quad y = 4$$

When $y = 4$, then $\sqrt{3}x = 4 - 4$

$$\therefore \quad \sqrt{3}x = 0$$
$$\therefore \quad x = 0$$

$\therefore$ the tangents intersect at $(0, 4)$.