Chapter 1
QUADRATICS

EXERCISE 1A.1

1 **a**
$$4x^2 + 7x = 0$$
$$\therefore\ x(4x+7) = 0$$
$$\therefore\ x = 0 \text{ or } 4x+7 = 0 \quad \{\text{Null Factor law}\}$$
$$\therefore\ x = 0 \text{ or } -\tfrac{7}{4}$$

b
$$3x^2 - 7x = 0$$
$$\therefore\ x(3x-7) = 0$$
$$\therefore\ x = 0 \text{ or } 3x-7 = 0 \quad \{\text{Null Factor law}\}$$
$$\therefore\ x = 0 \text{ or } \tfrac{7}{3}$$

c
$$2x^2 - 11x = 0$$
$$\therefore\ x(2x-11) = 0$$
$$\therefore\ x = 0 \text{ or } 2x-11 = 0 \quad \{\text{Null Factor law}\}$$
$$\therefore\ x = 0 \text{ or } \tfrac{11}{2}$$

d
$$9x = 6x^2$$
$$\therefore\ 6x^2 - 9x = 0$$
$$\therefore\ 3x(2x-3) = 0$$
$$\therefore\ x = 0 \text{ or } 2x-3 = 0 \quad \{\text{Null Factor law}\}$$
$$\therefore\ x = 0 \text{ or } \tfrac{3}{2}$$

e
$$x^2 - 5x + 6 = 0$$
$$\therefore\ (x-2)(x-3) = 0$$
$$\therefore\ x-2 = 0 \text{ or } x-3 = 0 \quad \{\text{Null Factor law}\}$$
$$\therefore\ x = 2 \text{ or } 3$$

f
$$x^2 + 21 = 10x$$
$$\therefore\ x^2 - 10x + 21 = 0$$
$$\therefore\ (x-3)(x-7) = 0$$
$$\therefore\ x-3 = 0 \text{ or } x-7 = 0 \quad \{\text{Null Factor law}\}$$
$$\therefore\ x = 3 \text{ or } 7$$

g
$$9 + x^2 = 6x$$
$$\therefore\ x^2 - 6x + 9 = 0$$
$$\therefore\ (x-3)^2 = 0$$
$$\therefore\ x-3 = 0$$
$$\therefore\ x = 3$$

h
$$x^2 + x = 12$$
$$\therefore\ x^2 + x - 12 = 0$$
$$\therefore\ (x+4)(x-3) = 0$$
$$\therefore\ x+4 = 0 \text{ or } x-3 = 0 \quad \{\text{Null Factor law}\}$$
$$\therefore\ x = -4 \text{ or } 3$$

i
$$x^2 + 8x = 33$$
$$\therefore\ x^2 + 8x - 33 = 0$$
$$\therefore\ (x+11)(x-3) = 0$$
$$\therefore\ x+11 = 0 \text{ or } x-3 = 0 \quad \{\text{Null Factor law}\}$$
$$\therefore\ x = -11 \text{ or } 3$$

2 **a**
$$9x^2 - 12x + 4 = 0$$
$$\therefore\ (3x-2)^2 = 0$$
$$\therefore\ x = \tfrac{2}{3}$$

b
$$2x^2 - 13x - 7 = 0$$
$$\therefore\ (2x+1)(x-7) = 0$$
$$\therefore\ x = -\tfrac{1}{2} \text{ or } 7$$

c
$$3x^2 = 16x + 12$$
$$\therefore\ 3x^2 - 16x - 12 = 0$$
$$\therefore\ (3x+2)(x-6) = 0$$
$$\therefore\ x = -\tfrac{2}{3} \text{ or } 6$$

d
$$3x^2 + 5x = 2$$
$$\therefore\ 3x^2 + 5x - 2 = 0$$
$$\therefore\ (3x-1)(x+2) = 0$$
$$\therefore\ x = \tfrac{1}{3} \text{ or } -2$$

e
$$2x^2 + 3 = 5x$$
$$\therefore\ 2x^2 - 5x + 3 = 0$$
$$\therefore\ (2x-3)(x-1) = 0$$
$$\therefore\ x = \tfrac{3}{2} \text{ or } 1$$

f
$$3x^2 + 8x + 4 = 0$$
$$\therefore\ (3x+2)(x+2) = 0$$
$$\therefore\ x = -\tfrac{2}{3} \text{ or } -2$$

g
$$3x^2 = 10x + 8$$
$$\therefore\ 3x^2 - 10x - 8 = 0$$
$$\therefore\ (3x+2)(x-4) = 0$$
$$\therefore\ x = -\tfrac{2}{3} \text{ or } 4$$

h
$$4x^2 + 4x = 3$$
$$\therefore\ 4x^2 + 4x - 3 = 0$$
$$\therefore\ (2x+3)(2x-1) = 0$$
$$\therefore\ x = -\tfrac{3}{2} \text{ or } \tfrac{1}{2}$$

i
$$4x^2 = 11x + 3$$
$$\therefore\ 4x^2 - 11x - 3 = 0$$
$$\therefore\ (4x+1)(x-3) = 0$$
$$\therefore\ x = -\tfrac{1}{4} \text{ or } 3$$

3 **a**
$$(x+1)^2 = 2x^2 - 5x + 11$$
$$\therefore\ x^2 + 2x + 1 = 2x^2 - 5x + 11$$
$$\therefore\ x^2 - 7x + 10 = 0$$
$$\therefore\ (x-2)(x-5) = 0$$
$$\therefore\ x = 2 \text{ or } 5$$

b
$$5 - 4x^2 = 3(2x+1) + 2$$
$$\therefore\ 5 - 4x^2 = 6x + 3 + 2$$
$$\therefore\ 4x^2 + 6x = 0$$
$$\therefore\ 2x(2x+3) = 0$$
$$\therefore\ x = 0 \text{ or } -\tfrac{3}{2}$$

c
$$2x - \frac{1}{x} = -1$$
$$\therefore\ 2x^2 - 1 = -x$$
$$\therefore\ 2x^2 + x - 1 = 0$$
$$\therefore\ (2x-1)(x+1) = 0$$
$$\therefore\ x = \tfrac{1}{2} \text{ or } -1$$

d
$$\frac{x+3}{1-x} = -\frac{9}{x}$$
$$\therefore\ x(x+3) = -9(1-x)$$
$$\therefore\ x^2 + 3x = -9 + 9x$$
$$\therefore\ x^2 - 6x + 9 = 0$$
$$\therefore\ (x-3)^2 = 0$$
$$\therefore\ x = 3$$

EXERCISE 1A.2

1 **a** $(x+5)^2 = 2$
$\therefore\ x+5 = \pm\sqrt{2}$
$\therefore\ x = -5 \pm \sqrt{2}$

b $(x+6)^2 = -11$
has no real solutions as $(x+6)^2$ cannot be negative

c $(x-4)^2 = 8$
$\therefore\ x-4 = \pm\sqrt{8}$
$\therefore\ x = 4 \pm 2\sqrt{2}$

d $3(x-2)^2 = 18$
$\therefore\ (x-2)^2 = 6$
$\therefore\ x-2 = \pm\sqrt{6}$
$\therefore\ x = 2 \pm \sqrt{6}$

e $(2x+1)^2 = 3$
$\therefore\ 2x+1 = \pm\sqrt{3}$
$\therefore\ 2x = -1 \pm \sqrt{3}$
$\therefore\ x = -\frac{1}{2} \pm \frac{1}{2}\sqrt{3}$

f $(1-3x)^2 - 7 = 0$
$\therefore\ (1-3x)^2 = 7$
$\therefore\ 1-3x = \pm\sqrt{7}$
$\therefore\ 3x = 1 \pm \sqrt{7}$
$\therefore\ x = \frac{1}{3} \pm \frac{\sqrt{7}}{3}$

2 **a**
$$x^2 - 4x + 1 = 0$$
$$\therefore\ x^2 - 4x = -1$$
$$\therefore\ x^2 - 4x + (-2)^2 = -1 + (-2)^2$$
$$\therefore\ (x-2)^2 = 3$$
$$\therefore\ x-2 = \pm\sqrt{3}$$
$$\therefore\ x = 2 \pm \sqrt{3}$$

b
$$x^2 + 6x + 2 = 0$$
$$\therefore\ x^2 + 6x = -2$$
$$\therefore\ x^2 + 6x + 3^2 = -2 + 3^2$$
$$\therefore\ (x+3)^2 = 7$$
$$\therefore\ x+3 = \pm\sqrt{7}$$
$$\therefore\ x = -3 \pm \sqrt{7}$$

c
$$x^2 - 14x + 46 = 0$$
$$\therefore\ x^2 - 14x = -46$$
$$\therefore\ x^2 - 14x + (-7)^2 = -46 + (-7)^2$$
$$\therefore\ (x-7)^2 = 3$$
$$\therefore\ x-7 = \pm\sqrt{3}$$
$$\therefore\ x = 7 \pm \sqrt{3}$$

d
$$x^2 = 4x + 3$$
$$\therefore\ x^2 - 4x = 3$$
$$\therefore\ x^2 - 4x + (-2)^2 = 3 + (-2)^2$$
$$\therefore\ (x-2)^2 = 7$$
$$\therefore\ x-2 = \pm\sqrt{7}$$
$$\therefore\ x = 2 \pm \sqrt{7}$$

e
$$x^2 + 6x + 7 = 0$$
$$\therefore\ x^2 + 6x = -7$$
$$\therefore\ x^2 + 6x + 3^2 = -7 + 3^2$$
$$\therefore\ (x+3)^2 = 2$$
$$\therefore\ x+3 = \pm\sqrt{2}$$
$$\therefore\ x = -3 \pm \sqrt{2}$$

f
$$x^2 = 2x + 6$$
$$\therefore\ x^2 - 2x = 6$$
$$\therefore\ x^2 - 2x + (-1)^2 = 6 + (-1)^2$$
$$\therefore\ (x-1)^2 = 7$$
$$\therefore\ x-1 = \pm\sqrt{7}$$
$$\therefore\ x = 1 \pm \sqrt{7}$$

g
$$x^2 + 6x = 2$$
$$\therefore\ x^2 + 6x + 3^2 = 2 + 3^2$$
$$\therefore\ (x+3)^2 = 11$$
$$\therefore\ x+3 = \pm\sqrt{11}$$
$$\therefore\ x = -3 \pm \sqrt{11}$$

h
$$x^2 + 10 = 8x$$
$$\therefore\ x^2 - 8x = -10$$
$$\therefore\ x^2 - 8x + (-4)^2 = -10 + (-4)^2$$
$$\therefore\ (x-4)^2 = 6$$
$$\therefore\ x-4 = \pm\sqrt{6}$$
$$\therefore\ x = 4 \pm \sqrt{6}$$

i
$$x^2 + 6x = -11$$
$$\therefore\ x^2 + 6x + 3^2 = -11 + 3^2$$
$$\therefore\ (x+3)^2 = -2$$

$\therefore$ x has no real solutions, since the perfect square cannot be negative.

3 **a**
$$2x^2 + 4x + 1 = 0$$
$$\therefore\ x^2 + 2x + \tfrac{1}{2} = 0$$
$$\therefore\ x^2 + 2x = -\tfrac{1}{2}$$
$$\therefore\ x^2 + 2x + 1^2 = -\tfrac{1}{2} + 1^2$$
$$\therefore\ (x+1)^2 = \tfrac{1}{2}$$
$$\therefore\ x+1 = \pm\tfrac{1}{\sqrt{2}}$$
$$\therefore\ x = -1 \pm \tfrac{1}{\sqrt{2}}$$

b
$$2x^2 - 10x + 3 = 0$$
$$\therefore\ x^2 - 5x + \tfrac{3}{2} = 0$$
$$\therefore\ x^2 - 5x = -\tfrac{3}{2}$$
$$\therefore\ x^2 - 5x + (-\tfrac{5}{2})^2 = -\tfrac{3}{2} + (-\tfrac{5}{2})^2$$
$$\therefore\ (x - \tfrac{5}{2})^2 = -\tfrac{3}{2} + \tfrac{25}{4}$$
$$\therefore\ (x - \tfrac{5}{2})^2 = \tfrac{19}{4}$$
$$\therefore\ x - \tfrac{5}{2} = \pm\tfrac{\sqrt{19}}{2}$$
$$\therefore\ x = \tfrac{5}{2} \pm \tfrac{\sqrt{19}}{2}$$

c
$$\begin{aligned} 3x^2+12x+5&=0\\ \therefore\ x^2+4x+\tfrac{5}{3}&=0\\ \therefore\ x^2+4x&=-\tfrac{5}{3}\\ \therefore\ x^2+4x+2^2&=-\tfrac{5}{3}+2^2\\ \therefore\ (x+2)^2&=\tfrac{7}{3}\\ \therefore\ x+2&=\pm\sqrt{\tfrac{7}{3}}\\ \therefore\ x&=-2\pm\sqrt{\tfrac{7}{3}} \end{aligned}$$

d
$$\begin{aligned} 3x^2&=6x+4\\ \therefore\ x^2&=2x+\tfrac{4}{3}\\ \therefore\ x^2-2x&=\tfrac{4}{3}\\ \therefore\ x^2-2x+(-1)^2&=\tfrac{4}{3}+(-1)^2\\ \therefore\ (x-1)^2&=\tfrac{7}{3}\\ \therefore\ x-1&=\pm\sqrt{\tfrac{7}{3}}\\ \therefore\ x&=1\pm\sqrt{\tfrac{7}{3}} \end{aligned}$$

e
$$\begin{aligned} 5x^2-15x+2&=0\\ \therefore\ x^2-3x+\tfrac{2}{5}&=0\\ \therefore\ x^2-3x&=-\tfrac{2}{5}\\ \therefore\ x^2-3x+(-\tfrac{3}{2})^2&=-\tfrac{2}{5}+(-\tfrac{3}{2})^2\\ \therefore\ (x-\tfrac{3}{2})^2&=-\tfrac{2}{5}+\tfrac{9}{4}=\tfrac{37}{20}\\ \therefore\ x-\tfrac{3}{2}&=\pm\sqrt{\tfrac{37}{20}}\\ \therefore\ x&=\tfrac{3}{2}\pm\sqrt{\tfrac{37}{20}} \end{aligned}$$

f
$$\begin{aligned} 4x^2+4x&=5\\ \therefore\ x^2+x&=\tfrac{5}{4}\\ \therefore\ x^2+x+(\tfrac{1}{2})^2&=\tfrac{5}{4}+(\tfrac{1}{2})^2\\ \therefore\ (x+\tfrac{1}{2})^2&=\tfrac{6}{4}\\ \therefore\ x+\tfrac{1}{2}&=\pm\tfrac{\sqrt{6}}{2}\\ \therefore\ x&=-\tfrac{1}{2}\pm\tfrac{\sqrt{6}}{2} \end{aligned}$$

EXERCISE 1A.3

1 **a** $x^2-4x-3=0$
has $a=1,\ b=-4,\ c=-3$
$$\begin{aligned} \therefore\ x&=\frac{-(-4)\pm\sqrt{(-4)^2-4(1)(-3)}}{2(1)}\\ &=\frac{4\pm\sqrt{28}}{2}\\ &=\frac{4\pm2\sqrt{7}}{2}\\ &=2\pm\sqrt{7} \end{aligned}$$

b $x^2+6x+7=0$
has $a=1,\ b=6,\ c=7$
$$\begin{aligned} \therefore\ x&=\frac{-6\pm\sqrt{6^2-4(1)(7)}}{2(1)}\\ &=\frac{-6\pm\sqrt{8}}{2}\\ &=\frac{-6\pm2\sqrt{2}}{2}\\ &=-3\pm\sqrt{2} \end{aligned}$$

c
$$\begin{aligned} x^2+1&=4x\\ \therefore\ x^2-4x+1&=0 \end{aligned}$$
which has $a=1,\ b=-4,\ c=1$
$$\begin{aligned} \therefore\ x&=\frac{-(-4)\pm\sqrt{(-4)^2-4(1)(1)}}{2(1)}\\ &=\frac{4\pm\sqrt{12}}{2}\\ &=\frac{4\pm2\sqrt{3}}{2}\\ &=2\pm\sqrt{3} \end{aligned}$$

d
$$\begin{aligned} x^2+4x&=1\\ \therefore\ x^2+4x-1&=0 \end{aligned}$$
which has $a=1,\ b=4,\ c=-1$
$$\begin{aligned} \therefore\ x&=\frac{-4\pm\sqrt{4^2-4(1)(-1)}}{2(1)}\\ &=\frac{-4\pm\sqrt{20}}{2}\\ &=\frac{-4\pm2\sqrt{5}}{2}\\ &=-2\pm\sqrt{5} \end{aligned}$$

e $x^2 - 4x + 2 = 0$
has $a = 1, \quad b = -4, \quad c = 2$

$$\therefore \quad x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(2)}}{2(1)}$$
$$= \frac{4 \pm \sqrt{8}}{2}$$
$$= \frac{4 \pm 2\sqrt{2}}{2}$$
$$= 2 \pm \sqrt{2}$$

f $2x^2 - 2x - 3 = 0$
has $a = 2, \quad b = -2, \quad c = -3$

$$\therefore \quad x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(2)(-3)}}{2(2)}$$
$$= \frac{2 \pm \sqrt{28}}{4}$$
$$= \frac{2 \pm 2\sqrt{7}}{4}$$
$$= \tfrac{1}{2} \pm \tfrac{\sqrt{7}}{2}$$

g $(3x + 1)^2 = -2x$
$\therefore \quad 9x^2 + 6x + 1 = -2x$
$\therefore \quad 9x^2 + 8x + 1 = 0$
which has $a = 9, \quad b = 8, \quad c = 1$

$$\therefore \quad x = \frac{-8 \pm \sqrt{8^2 - 4(9)(1)}}{2(9)}$$
$$= \frac{-8 \pm \sqrt{28}}{18}$$
$$= \frac{-8 \pm 2\sqrt{7}}{18} \quad \text{or} \quad -\tfrac{4}{9} \pm \tfrac{\sqrt{7}}{9}$$

h $(x + 3)(2x + 1) = 9$
$\therefore \quad 2x^2 + x + 6x + 3 = 9$
$\therefore \quad 2x^2 + 7x - 6 = 0$
which has $a = 2, \quad b = 7, \quad c = -6$

$$\therefore \quad x = \frac{-7 \pm \sqrt{7^2 - 4(2)(-6)}}{2(2)}$$
$$= \frac{-7 \pm \sqrt{49 + 48}}{4}$$
$$= -\tfrac{7}{4} \pm \tfrac{\sqrt{97}}{4}$$

i $x^2 - 2\sqrt{2}x + 2 = 0$
has $a = 1, \quad b = -2\sqrt{2}, \quad c = 2$

$$\therefore \quad x = \frac{-(-2\sqrt{2}) \pm \sqrt{(-2\sqrt{2})^2 - 4(1)(2)}}{2(1)}$$
$$= \frac{2\sqrt{2} \pm \sqrt{8 - 8}}{2}$$
$$= \frac{2\sqrt{2} \pm 0}{2}$$
$$= \sqrt{2}$$

2 **a** $(x + 2)(x - 1) = 2 - 3x$
$\therefore \quad x^2 - x + 2x - 2 = 2 - 3x$
$\therefore \quad x^2 + 4x - 4 = 0$
which has $a = 1, \quad b = 4, \quad c = -4$

$$\therefore \quad x = \frac{-4 \pm \sqrt{4^2 - 4(1)(-4)}}{2(1)}$$
$$= \frac{-4 \pm \sqrt{32}}{2}$$
$$= \frac{-4 \pm 4\sqrt{2}}{2}$$
$$= -2 \pm 2\sqrt{2}$$

b $(2x + 1)^2 = 3 - x$
$\therefore \quad 4x^2 + 4x + 1 = 3 - x$
$\therefore \quad 4x^2 + 5x - 2 = 0$
which has $a = 4, \quad b = 5, \quad c = -2$

$$\therefore \quad x = \frac{-5 \pm \sqrt{5^2 - 4(4)(-2)}}{2(4)}$$
$$= \frac{-5 \pm \sqrt{25 + 32}}{8}$$
$$= -\tfrac{5}{8} \pm \tfrac{\sqrt{57}}{8}$$

c
$$(x-2)^2 = 1 + x$$
$$\therefore \quad x^2 - 4x + 4 = 1 + x$$
$$\therefore \quad x^2 - 5x + 3 = 0$$
which has $a = 1, \quad b = -5, \quad c = 3$
$$\therefore \quad x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(3)}}{2(1)}$$
$$= \frac{5 \pm \sqrt{25-12}}{2}$$
$$= \tfrac{5}{2} \pm \tfrac{\sqrt{13}}{2}$$

d
$$\frac{x-1}{2-x} = 2x + 1$$
$$\therefore \quad x - 1 = (2x+1)(2-x)$$
$$\therefore \quad x - 1 = 4x - 2x^2 + 2 - x$$
$$\therefore \quad 2x^2 - 2x - 3 = 0$$
which has $a = 2, \quad b = -2, \quad c = -3$
$$\therefore \quad x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(2)(-3)}}{2(2)}$$
$$= \frac{2 \pm \sqrt{28}}{4}$$
$$= \frac{2 \pm 2\sqrt{7}}{4} \text{ or } \tfrac{1}{2} \pm \tfrac{\sqrt{7}}{2}$$

e
$$x - \frac{1}{x} = 1$$
$$\therefore \quad x^2 - 1 = x$$
$$\therefore \quad x^2 - x - 1 = 0$$
which has $a = 1, \quad b = -1, \quad c = -1$
$$\therefore \quad x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$
$$= \frac{1 \pm \sqrt{1+4}}{2}$$
$$= \tfrac{1}{2} \pm \tfrac{\sqrt{5}}{2}$$

f
$$2x - \frac{1}{x} = 3$$
$$\therefore \quad 2x^2 - 1 = 3x$$
$$\therefore \quad 2x^2 - 3x - 1 = 0$$
which has $a = 2, \quad b = -3, \quad c = -1$
$$\therefore \quad x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-1)}}{2(2)}$$
$$= \frac{3 \pm \sqrt{9+8}}{4}$$
$$= \tfrac{3}{4} \pm \tfrac{\sqrt{17}}{4}$$

EXERCISE 1B

1 **a** $x^2 + 7x - 3 = 0$
has $a = 1, \quad b = 7, \quad c = -3$
$$\therefore \quad \Delta = b^2 - 4ac$$
$$= 7^2 - 4(1)(-3)$$
$$= 61$$
Since $\Delta > 0$, there are two distinct real solutions.

b $x^2 - 3x + 2 = 0$
has $a = 1, \quad b = -3, \quad c = 2$
$$\therefore \quad \Delta = b^2 - 4ac$$
$$= (-3)^2 - 4(1)(2)$$
$$= 1$$
Since $\Delta > 0$ and Δ is a square, there are two distinct rational solutions.

c $3x^2 + 2x - 1 = 0$
has $a = 3, \quad b = 2, \quad c = -1$
$$\therefore \quad \Delta = b^2 - 4ac$$
$$= 2^2 - 4(3)(-1)$$
$$= 16$$
Since $\Delta > 0$ and Δ is a square, there are two distinct rational solutions.

d $5x^2 + 4x - 3 = 0$
has $a = 5, \quad b = 4, \quad c = -3$
$$\therefore \quad \Delta = b^2 - 4ac$$
$$= 4^2 - 4(5)(-3)$$
$$= 76$$
Since $\Delta > 0$, there are two distinct real solutions.

e $x^2 + x + 5 = 0$
has $a = 1, \quad b = 1, \quad c = 5$
$$\therefore \quad \Delta = b^2 - 4ac$$
$$= 1^2 - 4(1)(5)$$
$$= -19$$
Since $\Delta < 0$, there are no real solutions.

f $16x^2 - 8x + 1 = 0$
has $a = 16, \quad b = -8, \quad c = 1$
$$\therefore \quad \Delta = b^2 - 4ac$$
$$= (-8)^2 - 4(16)(1)$$
$$= 0$$
$\therefore$ there is one repeated real solution.

2 **a** $6x^2 - 5x - 6 = 0$
has $a = 6,\ \ b = -5,\ \ c = -6$
$\therefore\ \Delta = b^2 - 4ac$
$= (-5)^2 - 4(6)(-6)$
$= 169$
$\therefore\ \sqrt{\Delta} = 13,$ so the equation has rational roots.

b $2x^2 - 7x - 5 = 0$
has $a = 2,\ \ b = -7,\ \ c = -5$
$\therefore\ \Delta = b^2 - 4ac$
$= (-7)^2 - 4(2)(-5)$
$= 89$
$\therefore\ \sqrt{\Delta} = \sqrt{89},$ so the equation does not have rational roots.

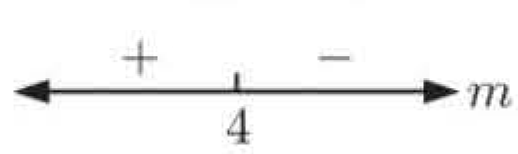
c $3x^2 + 4x + 1 = 0$
has $a = 3,\ \ b = 4,\ \ c = 1$
$\therefore\ \Delta = b^2 - 4ac$
$= 4^2 - 4(3)(1)$
$= 4$
$\therefore\ \sqrt{\Delta} = 2,$ so the equation has rational roots.

d $6x^2 - 47x - 8 = 0$
has $a = 6,\ \ b = -47,\ \ c = -8$
$\therefore\ \Delta = b^2 - 4ac$
$= (-47)^2 - 4(6)(-8)$
$= 2401$
$\therefore\ \sqrt{\Delta} = 49,$ so the equation has rational roots.

e $4x^2 - 3x + 2 = 0$
has $a = 4,\ \ b = -3,\ \ c = 2$
$\therefore\ \Delta = b^2 - 4ac$
$= (-3)^2 - 4(4)2$
$= -23$
Since $\Delta < 0,$ the equation does not have rational roots.

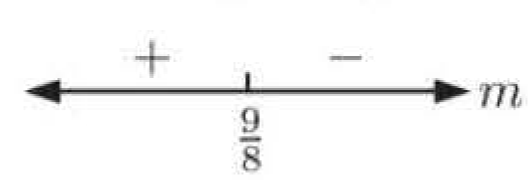
f $8x^2 + 2x - 3 = 0$
has $a = 8,\ \ b = 2,\ \ c = -3$
$\therefore\ \Delta = b^2 - 4ac$
$= 2^2 - 4(8)(-3)$
$= 100$
$\therefore\ \sqrt{\Delta} = 10,$ so the equation has rational roots.

3 **a** For $x^2 + 4x + m = 0,$
$a = 1,\ \ b = 4,\ \ c = m$
So, $\Delta = b^2 - 4ac$
$= 4^2 - 4(1)(m)$
$= 16 - 4m$
which has sign diagram



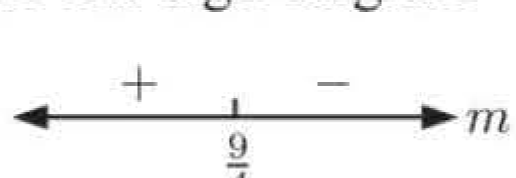
i For a repeated root, $\Delta = 0$
$\therefore\ m = 4$
ii For two distinct real roots, $\Delta > 0$
$\therefore\ m < 4$
iii For no real roots, $\Delta < 0$
$\therefore\ m > 4$

b For $mx^2 + 3x + 2 = 0,$
$a = m,\ \ b = 3,\ \ c = 2$
So, $\Delta = b^2 - 4ac$
$= 3^2 - 4(m)(2)$
$= 9 - 8m$
which has sign diagram



i For a repeated root, $\Delta = 0$
$\therefore\ m = \frac{9}{8}$
ii For two distinct real roots, $\Delta > 0$
$\therefore\ m < \frac{9}{8}$
iii For no real roots, $\Delta < 0$
$\therefore\ m > \frac{9}{8}$

c For $mx^2 - 3x + 1 = 0,$
$a = m,\ \ b = -3,\ \ c = 1$
So, $\Delta = b^2 - 4ac$
$= (-3)^2 - 4(m)(1)$
$= 9 - 4m$
which has sign diagram

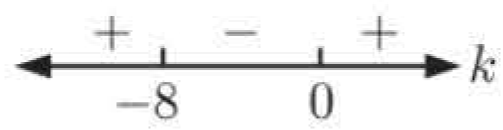


i For a repeated root, $\Delta = 0$ $\quad\therefore\ m = \frac{9}{4}$
ii For two distinct real roots, $\Delta > 0$ $\quad\therefore\ m < \frac{9}{4}$
iii For no real roots, $\Delta < 0$ $\quad\therefore\ m > \frac{9}{4}$

4 **a** For $2x^2 + kx - k = 0$,
$a = 2, \quad b = k, \quad c = -k$

So, $\Delta = b^2 - 4ac$
$= k^2 - 4(2)(-k)$
$= k^2 + 8k$
$= k(k+8)$

which has sign diagram



i For two distinct real roots, $\Delta > 0$
$\therefore \; k < -8$ or $k > 0$

ii For two real roots, $\Delta \geqslant 0$
$\therefore \; k \leqslant -8$ or $k \geqslant 0$

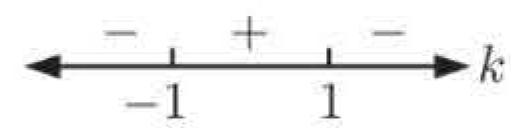
iii For a repeated root, $\Delta = 0$
$\therefore \; k = -8$ or 0

iv For no real roots, $\Delta < 0$
$\therefore \; -8 < k < 0$

b For $kx^2 - 2x + k = 0$,
$a = k, \quad b = -2, \quad c = k$

So, $\Delta = b^2 - 4ac$
$= (-2)^2 - 4(k)(k)$
$= 4 - 4k^2$
$= 4(1+k)(1-k)$

which has sign diagram



i For two distinct real roots, $\Delta > 0$
$\therefore \; -1 < k < 1$

ii For two real roots, $\Delta \geqslant 0$
$\therefore \; -1 \leqslant k \leqslant 1$

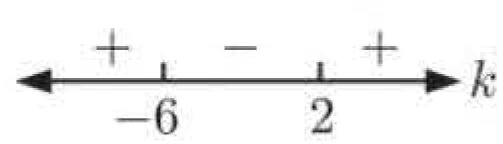
iii For a repeated root, $\Delta = 0$
$\therefore \; k = -1$ or 1

iv For no real roots, $\Delta < 0$
$\therefore \; k < -1$ or $k > 1$

c For $x^2 + (k+2)x + 4 = 0$,
$a = 1, \quad b = k+2, \quad c = 4$

So, $\Delta = b^2 - 4ac$
$= (k+2)^2 - 4(1)(4)$
$= k^2 + 4k + 4 - 16$
$= k^2 + 4k - 12$
$= (k+6)(k-2)$

which has sign diagram



i For two distinct real roots, $\Delta > 0$
$\therefore \; k < -6$ or $k > 2$

ii For two real roots, $\Delta \geqslant 0$
$\therefore \; k \leqslant -6$ or $k \geqslant 2$

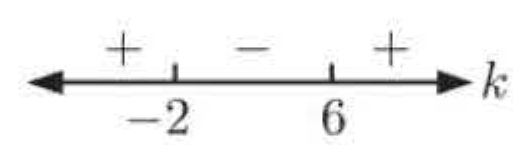
iii For a repeated root, $\Delta = 0$
$\therefore \; k = -6$ or 2

iv For no real roots, $\Delta < 0$
$\therefore \; -6 < k < 2$

d For $2x^2 + (k-2)x + 2 = 0$,
$a = 2, \quad b = k-2, \quad c = 2$

So, $\Delta = b^2 - 4ac$
$= (k-2)^2 - 4(2)(2)$
$= k^2 - 4k + 4 - 16$
$= k^2 - 4k - 12$
$= (k-6)(k+2)$

which has sign diagram



i For two distinct real roots, $\Delta > 0$
$\therefore \; k < -2$ or $k > 6$

ii For two real roots, $\Delta \geqslant 0$
$\therefore \; k \leqslant -2$ or $k \geqslant 6$

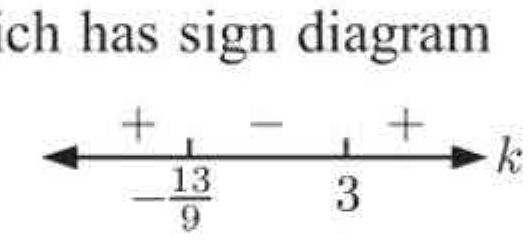
iii For a repeated root, $\Delta = 0$
$\therefore \; k = -2$ or 6

iv For no real roots, $\Delta < 0$
$\therefore \; -2 < k < 6$

e For $x^2 + (3k-1)x + (2k+10) = 0$,
$a = 1, \quad b = 3k-1, \quad c = 2k+10$

So, $\Delta = b^2 - 4ac$
$= (3k-1)^2 - 4(1)(2k+10)$
$= 9k^2 - 6k + 1 - 8k - 40$
$= 9k^2 - 14k - 39$
$= (9k+13)(k-3)$

which has sign diagram



i For two distinct real roots, $\Delta > 0$
$\therefore \; k < -\frac{13}{9}$ or $k > 3$

ii For two real roots, $\Delta \geqslant 0$
$\therefore \; k \leqslant -\frac{13}{9}$ or $k \geqslant 3$

iii For a repeated root, $\Delta = 0$
$\therefore \; k = -\frac{13}{9}$ or 3

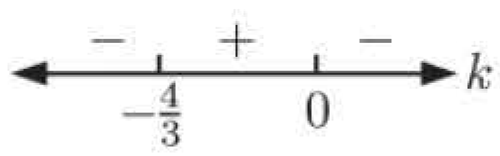
iv For no real roots, $\Delta < 0$
$\therefore \; -\frac{13}{9} < k < 3$

f For $(k+1)x^2 + kx + k = 0$,

$a = k+1, \quad b = k, \quad c = k$

So, $\Delta = b^2 - 4ac$
$= k^2 - 4(k+1)(k)$
$= k^2 - 4k^2 - 4k$
$= -3k^2 - 4k$
$= -k(3k+4)$

which has sign diagram



i For two distinct real roots, $\Delta > 0$
$\therefore \; -\frac{4}{3} < k < 0$

ii For two real roots, $\Delta \geqslant 0$
$\therefore \; -\frac{4}{3} \leqslant k \leqslant 0$

iii For a repeated root, $\Delta = 0$
$\therefore \; k = -\frac{4}{3}$ or 0

iv For no real roots, $\Delta < 0$
$\therefore \; k < -\frac{4}{3}$ or $k > 0$

EXERCISE 1C

1 **a** For $3x^2 - 2x + 7 = 0$:

sum of roots $= -\dfrac{b}{a} = -\dfrac{(-2)}{3} = \frac{2}{3}$

product of roots $= \dfrac{c}{a} = \frac{7}{3}$

b For $x^2 + 11x - 13 = 0$:

sum of roots $= -\dfrac{b}{a} = -\frac{11}{1} = -11$

product of roots $= \dfrac{c}{a} = \frac{-13}{1} = -13$

c For $5x^2 - 6x - 14 = 0$:

sum of roots $= -\dfrac{b}{a} = -\dfrac{(-6)}{5} = \frac{6}{5}$

product of roots $= \dfrac{c}{a} = -\frac{14}{5}$

2 For $kx^2 - (1+k)x + (3k+2) = 0, \qquad a = k, \quad b = -(1+k), \quad c = 3k+2$

$\therefore$ sum of roots $= -\dfrac{b}{a} = -\dfrac{-(1+k)}{k} = \dfrac{k+1}{k}$, product of roots $= \dfrac{c}{a} = \dfrac{3k+2}{k}$

Now, sum of the roots is twice their product

$\therefore \; \dfrac{k+1}{k} = 2\left(\dfrac{3k+2}{k}\right)$

$\therefore \; k+1 = 2(3k+2)$

$\therefore \; k+1 = 6k+4$

$\therefore \; -3 = 5k$

$\therefore \; k = -\frac{3}{5}$

Substituting $k = -\frac{3}{5}$ into the equation gives

$-\frac{3}{5}x^2 - (1 - \frac{3}{5})x + (-\frac{9}{5} + 2) = 0$

$\therefore \; -\frac{3}{5}x^2 - \frac{2}{5}x + \frac{1}{5} = 0$

$\therefore \; -\frac{1}{5}(3x^2 + 2x - 1) = 0$

$\therefore \; -\frac{1}{5}(3x-1)(x+1) = 0$

$\therefore \; x = \frac{1}{3}$ or -1

$\therefore$ the roots of the equation are $\frac{1}{3}$ and -1

3 For $ax^2 - 6x + a - 2 = 0, \qquad$ "a" $= a, \quad b = -6, \quad c = a - 2$

a sum of roots $= -\dfrac{b}{a}$

$\therefore \; \alpha + 2\alpha = -\dfrac{(-6)}{a}$

$\therefore \; 3\alpha = \dfrac{6}{a} \quad \{\text{or } \alpha = \dfrac{2}{a} \quad .... \text{ (1)}\}$

product of roots $= \dfrac{c}{a}$

$\therefore \; \alpha \times 2\alpha = \dfrac{a-2}{a}$

$\therefore \; 2\alpha^2 = \dfrac{a-2}{a} \quad .... \text{ (2)}$

b Substituting (1) into (2) gives

$2\left(\dfrac{2}{a}\right)^2 = \dfrac{a-2}{a}$

$\therefore \; 2\left(\dfrac{4}{a^2}\right) = \dfrac{a-2}{a}$

$\therefore \; \dfrac{8}{a} = a - 2$

$\therefore \; 8 = a^2 - 2a$

$\therefore \; a^2 - 2a - 8 = 0$

$\therefore \; (a-4)(a+2) = 0$

$\therefore \; a = 4$ or -2

If $a = 4$, then $\alpha = \frac{2}{4}$ {using (1)}
$= \frac{1}{2}$
and $2\alpha = 1$

If $a = -2$, then $\alpha = \frac{2}{-2}$ {using (1)}
$= -1$
and $2\alpha = -2$

$\therefore$ $a = 4$ and the roots are $\frac{1}{2}$ and 1 or $a = -2$ and the roots are -1 and -2.

4 For $kx^2 + (k-8)x + (1-k) = 0$, $a = k$, $b = k-8$, $c = 1-k$

Let the roots of the equation be m and $m+2$.

$$\text{sum of roots} = -\frac{b}{a}$$
$$\therefore\ m + (m+2) = -\frac{k-8}{k}$$
$$\therefore\ 2m + 2 = \frac{8-k}{k}$$
$$\therefore\ 2m = \frac{8-3k}{k}$$
$$\therefore\ m = \frac{8-3k}{2k} \quad \text{.... (1)}$$

$$\text{and product of roots} = \frac{c}{a}$$
$$\therefore\ m(m+2) = \frac{1-k}{k} \quad \text{.... (2)}$$

Substituting (1) into (2) gives

$$\left(\frac{8-3k}{2k}\right)\left(\frac{8-3k}{2k} + 2\right) = \frac{1-k}{k}$$
$$\therefore\ \left(\frac{8-3k}{2k}\right)\left(\frac{8+k}{2k}\right) = \frac{1-k}{k}$$
$$\therefore\ \frac{64-16k-3k^2}{4k^2} = \frac{1-k}{k}$$
$$\therefore\ \frac{64-16k-3k^2}{4k} = 1-k$$
$$\therefore\ 64 - 16k - 3k^2 = 4k - 4k^2$$
$$\therefore\ k^2 - 20k + 64 = 0$$
$$\therefore\ (k-4)(k-16) = 0$$
$$\therefore\ k = 4 \text{ or } 16$$

Using (1), if $k = 4$ then $m = \dfrac{8-3(4)}{2(4)} = \dfrac{-4}{8} = -\frac{1}{2}$, and $m + 2 = \frac{3}{2}$

and if $k = 16$ then $m = \dfrac{8-3(16)}{2(16)} = \dfrac{-40}{32} = -\frac{5}{4}$, and $m + 2 = \frac{3}{4}$

$\therefore$ $k = 4$ and the roots are $-\frac{1}{2}$ and $\frac{3}{2}$ or $k = 16$ and the roots are $-\frac{5}{4}$ and $\frac{3}{4}$.

5 The roots of $x^2 - 6x + 7 = 0$ are α and β.

sum of roots $= -\frac{-6}{1}$ and product of roots $= \frac{7}{1}$

$\therefore$ $\alpha + \beta = 6$ $\therefore$ $\alpha\beta = 7$

Now consider a quadratic equation with roots $\alpha + \dfrac{1}{\beta}$ and $\beta + \dfrac{1}{\alpha}$

$$\begin{aligned}\text{sum of roots} &= \alpha + \frac{1}{\beta} + \beta + \frac{1}{\alpha}\\ &= (\alpha+\beta) + \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)\\ &= (\alpha+\beta) + \frac{\alpha+\beta}{\alpha\beta}\\ &= 6 + \tfrac{6}{7}\\ &= \tfrac{48}{7}\end{aligned}$$

$$\begin{aligned}\text{product of roots} &= \left(\alpha + \frac{1}{\beta}\right)\left(\beta + \frac{1}{\alpha}\right)\\ &= \alpha\beta + 1 + 1 + \frac{1}{\alpha\beta}\\ &= 7 + 2 + \tfrac{1}{7}\\ &= 9 + \tfrac{1}{7}\\ &= \tfrac{64}{7}\end{aligned}$$

$\therefore$ a quadratic equation $ax^2 + bx + c = 0$ with roots $\alpha + \dfrac{1}{\beta}$ and $\beta + \dfrac{1}{\alpha}$ has $-\dfrac{b}{a} = \frac{48}{7}$ and $\dfrac{c}{a} = \frac{64}{7}$.

The simplest solution to this is $a = 7$, $b = -48$, $c = 64$.

$\therefore$ the simplest quadratic equation with roots $\alpha + \dfrac{1}{\beta}$ and $\beta + \dfrac{1}{\alpha}$ is $7x^2 - 48x + 64 = 0$.

6 The roots of $2x^2 - 3x - 5 = 0$ are p and q.

$\therefore$ sum of roots $= -\frac{-3}{2}$ and product of roots $= \frac{-5}{2}$

$\therefore\ p + q = \frac{3}{2}$ $\qquad\therefore\ pq = -\frac{5}{2}$

Now consider a quadratic equation with roots $p^2 + q$ and $q^2 + p$.

$$\begin{aligned}\text{sum of roots} &= p^2 + q + q^2 + p\\ &= (p^2 + q^2) + (p + q)\\ &= [(p+q)^2 - 2pq] + (p+q)\\ &= (\tfrac{3}{2})^2 - 2(-\tfrac{5}{2}) + \tfrac{3}{2}\\ &= \tfrac{9}{4} + 5 + \tfrac{3}{2}\\ &= \tfrac{35}{4}\end{aligned}$$

$$\begin{aligned}\text{product of roots} &= (p^2 + q)(q^2 + p)\\ &= p^2q^2 + p^3 + q^3 + qp\\ &= (pq)^2 + [(p+q)^3 - 3p^2q - 3pq^2] + pq\\ &= (pq)^2 + [(p+q)^3 - 3pq(p+q)] + pq\\ &= (-\tfrac{5}{2})^2 + (\tfrac{3}{2})^3 - 3(-\tfrac{5}{2})(\tfrac{3}{2}) + (-\tfrac{5}{2})\\ &= \tfrac{25}{4} + \tfrac{27}{8} + \tfrac{45}{4} - \tfrac{5}{2}\\ &= \tfrac{147}{8}\end{aligned}$$

$\therefore$ a quadratic equation $ax^2 + bx + c = 0$ with roots $p^2 + q$ and $q^2 + p$

has $-\dfrac{b}{a} = \frac{35}{4}$ and $\dfrac{c}{a} = \frac{147}{8}$.

$\therefore\ b = -\frac{35}{4}a$ and $c = \frac{147}{8}a$

$\therefore$ the quadratic equation is $ax^2 - \frac{35}{4}ax + \frac{147}{8}a = 0,\quad a \neq 0$

$\therefore\ a(x^2 - \frac{35}{4}x + \frac{147}{8}) = 0,\quad a \neq 0$

$\therefore\ a(8x^2 - 70x + 147) = 0,\quad a \neq 0$

7 $kx^2 + (k+2)x - 3 = 0$ will have roots which are real and positive if:

(1) $\Delta \geqslant 0$ (2) sum of roots > 0 (3) product of roots > 0

(1) $\Delta \geqslant 0$

$\therefore\ (k+2)^2 - 4(k)(-3) \geqslant 0$

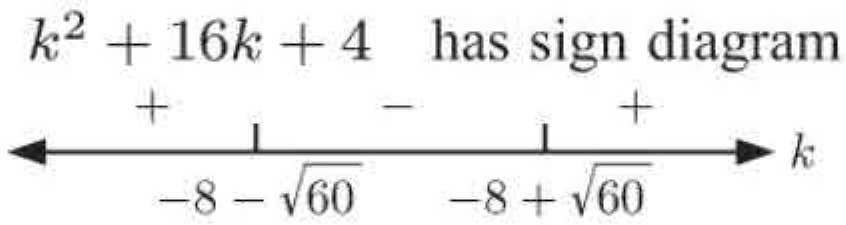
$\therefore\ k^2 + 4k + 4 + 12k \geqslant 0$

$\therefore\ k^2 + 16k + 4 \geqslant 0$

Now $k^2 + 16k + 4 = 0$ when

$$\begin{aligned}k &= \frac{-16 \pm \sqrt{16^2 - 4(1)(4)}}{2(1)}\\ &= \frac{-16 \pm \sqrt{240}}{2}\\ &= \frac{-16 \pm 2\sqrt{60}}{2}\\ &= -8 \pm \sqrt{60}\end{aligned}$$

$\therefore\ k^2 + 16k + 4$ has sign diagram



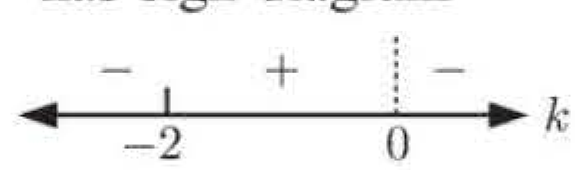
$\therefore\ \Delta \geqslant 0$ when

$k \leqslant -8 - \sqrt{60}$ or $k \geqslant -8 + \sqrt{60}$

(2) sum of roots > 0

$\therefore\ -\dfrac{k+2}{k} > 0$

$-\dfrac{k+2}{k}$ has sign diagram



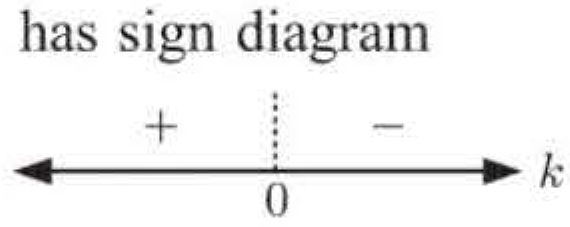
$\therefore$ sum of roots > 0 when

$-2 < k < 0$

(3) product of roots > 0

$\therefore\ \dfrac{-3}{k} > 0$

$-\dfrac{3}{k}$ has sign diagram



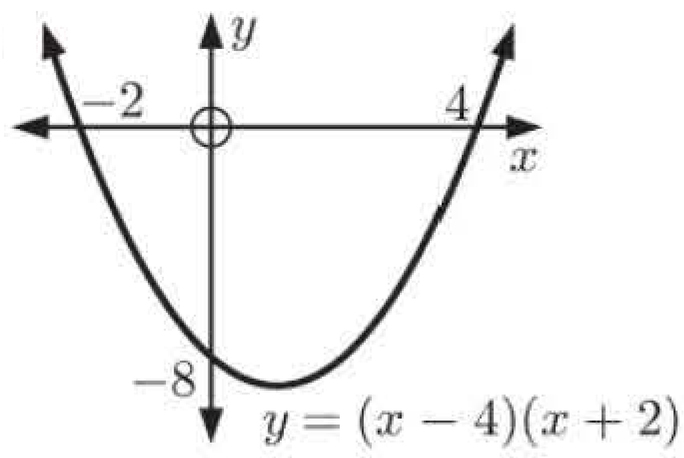
$\therefore$ product of roots > 0 when

$k < 0$

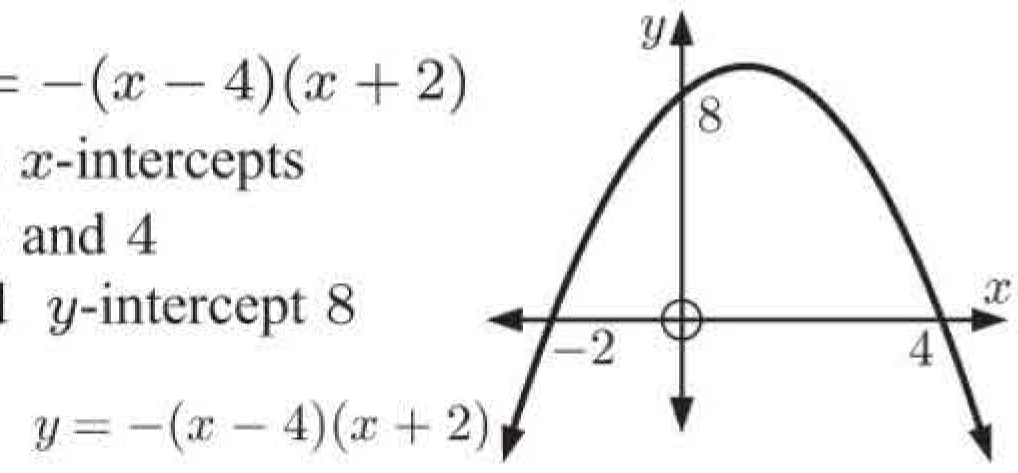
$\therefore$ the only values of k which satisfy all three conditions are $-8 + \sqrt{60} \leqslant k < 0$.

EXERCISE 1D.1

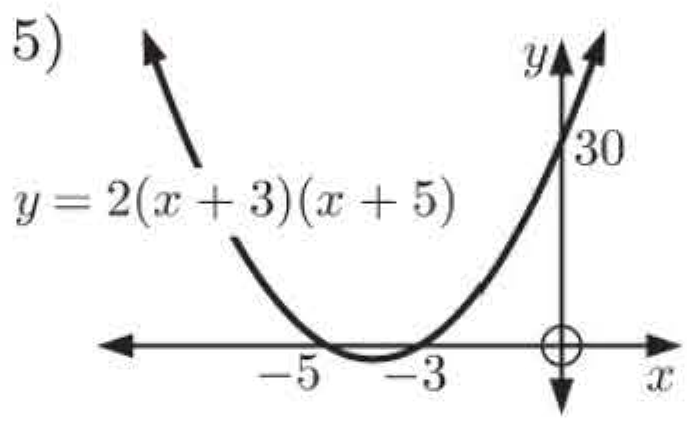
1 **a** $y=(x-4)(x+2)$ has x-intercepts -2 and 4 and y-intercept -8



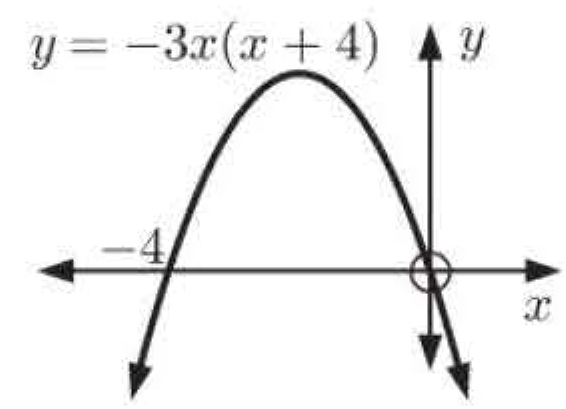
b $y=-(x-4)(x+2)$ has x-intercepts -2 and 4 and y-intercept 8



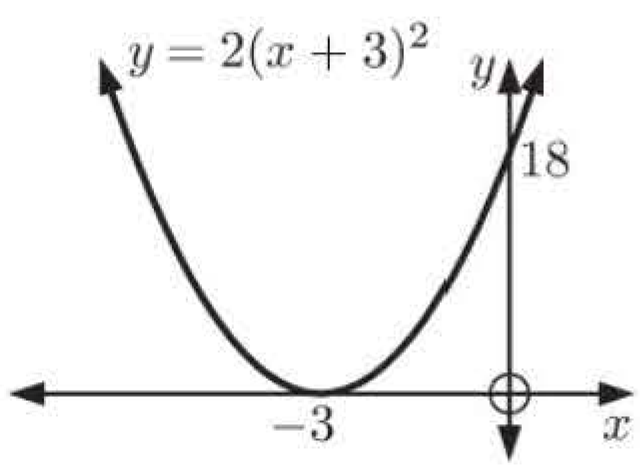
c $y=2(x+3)(x+5)$ has x-intercepts -5 and -3 and y-intercept 30



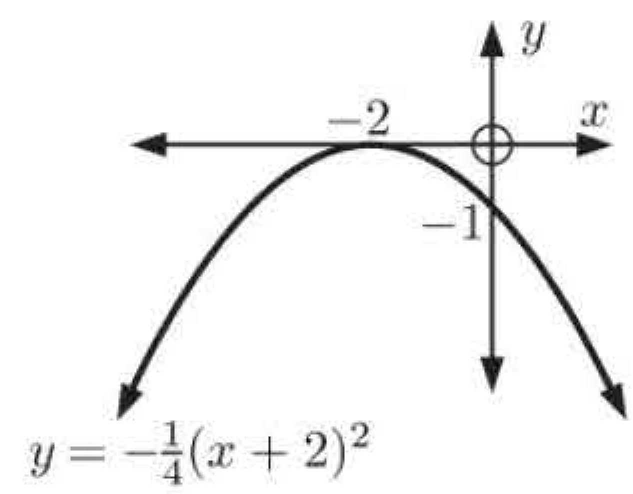
d $y=-3x(x+4)$ has x-intercepts 0 and -4 and y-intercept 0



e $y=2(x+3)^2$ has x-intercept -3 and y-intercept 18



f $y=-\frac{1}{4}(x+2)^2$ has x-intercept -2 and y-intercept -1



2 **a** The average of the x-intercepts is 1, so the axis of symmetry is $x=1$.

b The average of the x-intercepts is 1, so the axis of symmetry is $x=1$.

c The average of the x-intercepts is -4, so the axis of symmetry is $x=-4$.

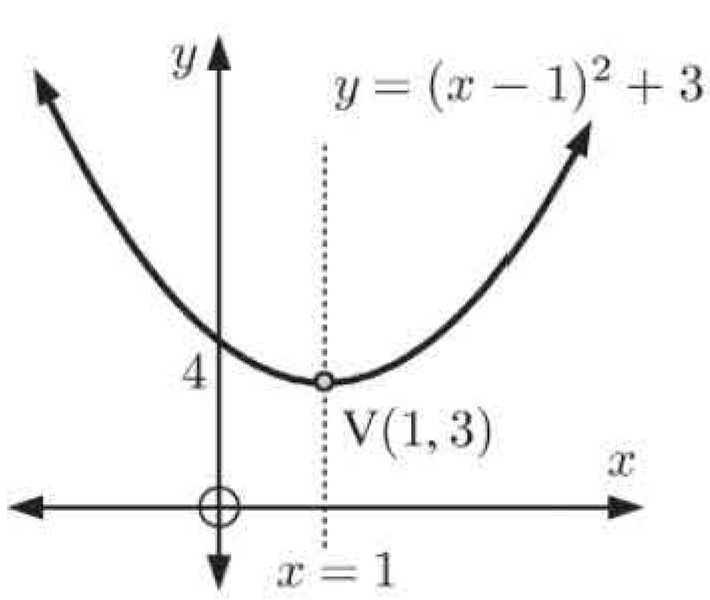
d The average of the x-intercepts is -2, so the axis of symmetry is $x=-2$.

e The only x-intercept is -3, so the axis of symmetry is $x=-3$.

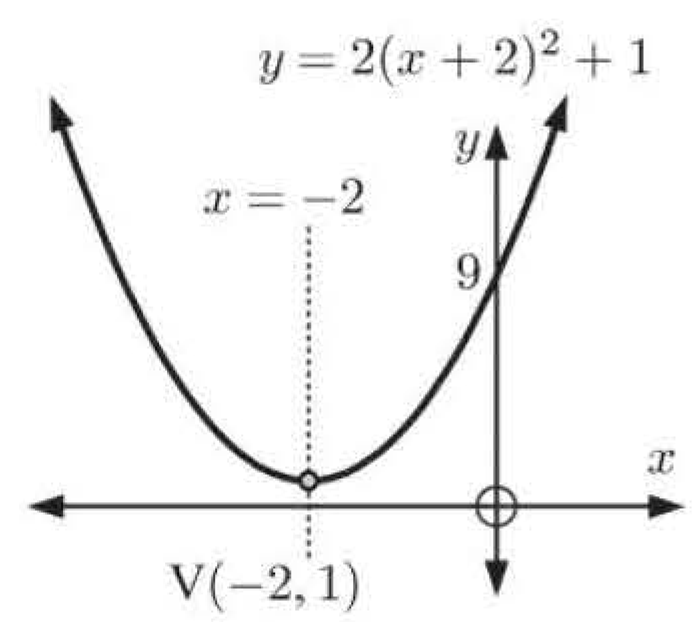
f The only x-intercept is -2, so the axis of symmetry is $x=-2$.

3 **a** C **b** E **c** B **d** F **e** G **f** H **g** A **h** D

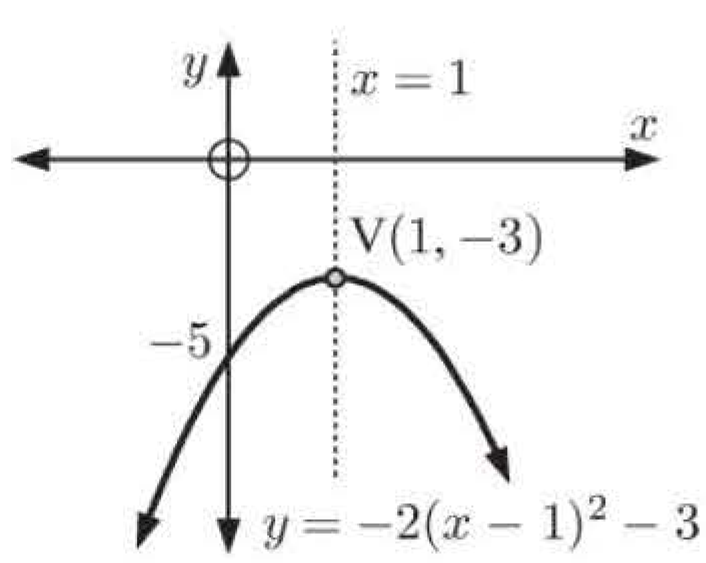
4 **a** The vertex is $(1,\ 3)$. The axis of symmetry is $x=1$. The y-intercept is 4.



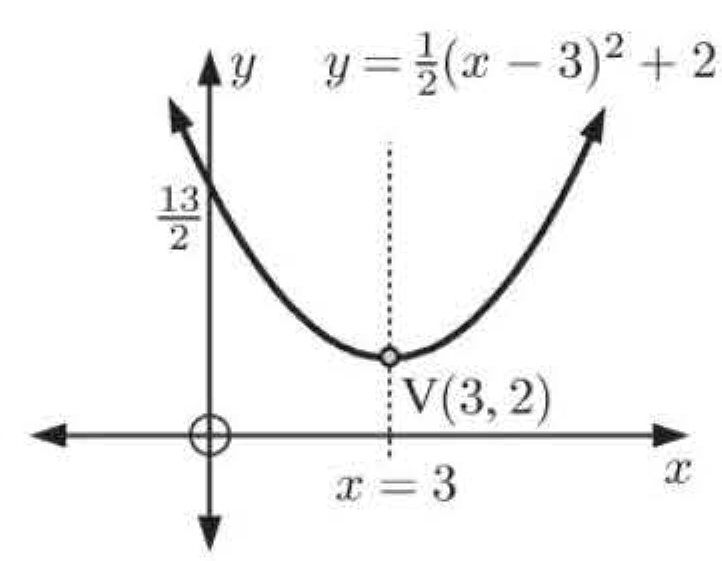
b The vertex is $(-2,\ 1)$. The axis of symmetry is $x=-2$. The y-intercept is 9.



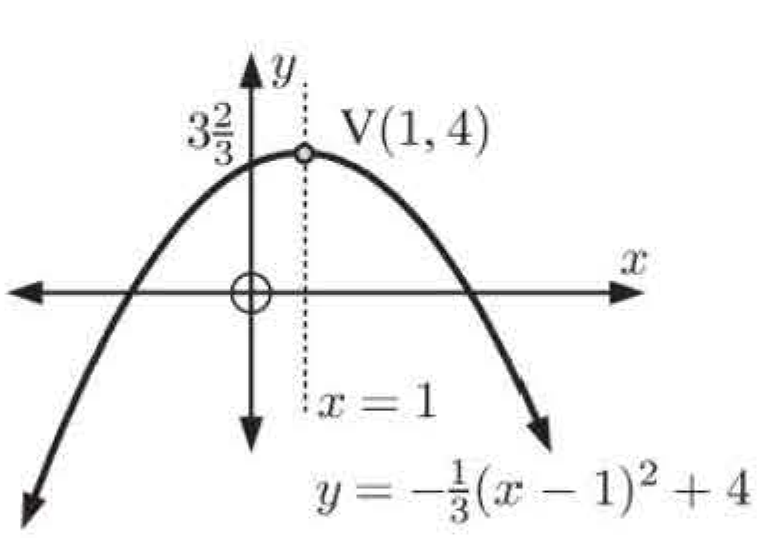
c The vertex is $(1,\ -3)$. The axis of symmetry is $x=1$. The y-intercept is -5.



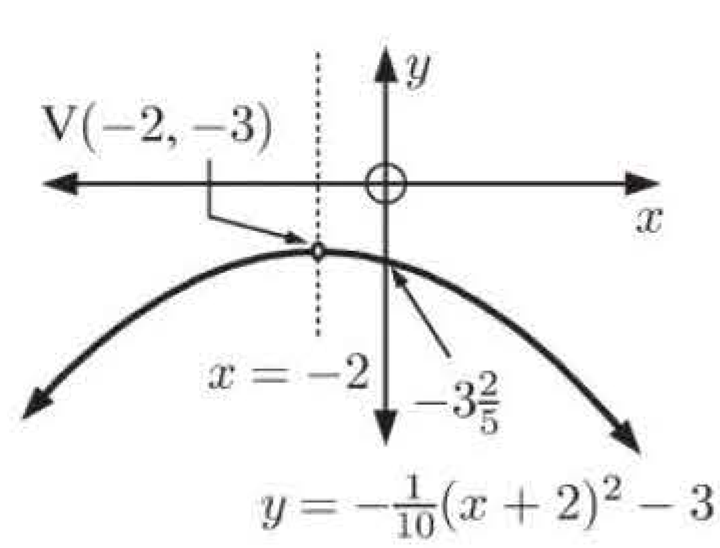
d The vertex is $(3,\ 2)$. The axis of symmetry is $x=3$. The y-intercept is $6\frac{1}{2}$.



e The vertex is $(1,\ 4)$. The axis of symmetry is $x=1$. The y-intercept is $3\frac{2}{3}$.



f The vertex is $(-2,\ -3)$. The axis of symmetry is $x=-2$. The y-intercept is $-3\frac{2}{5}$.



5 **a** G **b** A **c** E **d** B **e** I **f** C **g** D **h** F **i** H

6 **a** $y = x^2 - 4x + 2$
has $a = 1$, $b = -4$, $c = 2$
$\therefore \ -\dfrac{b}{2a} = -\dfrac{(-4)}{2(1)} = 2$
$\therefore$ the axis of symmetry is $x = 2$.
When $x = 2$,
$y = 2^2 - 4 \times 2 + 2 = -2$
$\therefore$ the vertex is at $(2, -2)$.

b $y = 2x^2 + 4$
has $a = 2$, $b = 0$, $c = 4$
$\therefore \ -\dfrac{b}{2a} = -\dfrac{0}{2(2)} = 0$
$\therefore$ the axis of symmetry is $x = 0$.
When $x = 0$, $y = 4$
$\therefore$ the vertex is at $(0, 4)$.

c $y = -3x^2 + 1$
has $a = -3$, $b = 0$, $c = 1$
$\therefore \ -\dfrac{b}{2a} = -\dfrac{0}{2(-3)} = 0$
$\therefore$ the axis of symmetry is $x = 0$.
When $x = 0$, $y = 1$
$\therefore$ the vertex is at $(0, 1)$.

d $y = -x^2 - 4x - 9$
has $a = -1$, $b = -4$, $c = -9$
$\therefore \ -\dfrac{b}{2a} = -\dfrac{(-4)}{2(-1)} = -2$
$\therefore$ the axis of symmetry is $x = -2$.
When $x = -2$, $y = -(-2)^2 - 4(-2) - 9$
$= -4 + 8 - 9$
$= -5$
$\therefore$ the vertex is at $(-2, -5)$.

e $y = 2x^2 + 6x - 1$
has $a = 2$, $b = 6$, $c = -1$
$\therefore \ -\dfrac{b}{2a} = -\dfrac{6}{2(2)} = -\dfrac{3}{2}$
$\therefore$ the axis of symmetry is $x = -\frac{3}{2}$.
When $x = -\frac{3}{2}$, $y = 2(-\frac{3}{2})^2 + 6(-\frac{3}{2}) - 1$
$= \frac{9}{2} - 9 - 1$
$= -\frac{11}{2}$
$\therefore$ the vertex is at $(-\frac{3}{2}, -\frac{11}{2})$.

f $y = -\frac{1}{2}x^2 + x - 5$
has $a = -\frac{1}{2}$, $b = 1$, $c = -5$
$\therefore \ -\dfrac{b}{2a} = -\dfrac{1}{2(-\frac{1}{2})} = 1$
$\therefore$ the axis of symmetry is $x = 1$.
When $x = 1$,
$y = -\frac{1}{2}(1)^2 + 1 - 5 = -\frac{9}{2}$
$\therefore$ the vertex is at $(1, -\frac{9}{2})$.

7 **a** When $y = 0$, $x^2 - 9 = 0$
$\therefore \ (x+3)(x-3) = 0$
$\therefore \ x = \pm 3$
$\therefore$ the x-intercepts are ± 3

b When $y = 0$, $2x^2 - 6 = 0$
$\therefore \ x^2 - 3 = 0$
$\therefore \ (x + \sqrt{3})(x - \sqrt{3}) = 0$
$\therefore \ x = \pm\sqrt{3}$
$\therefore$ the x-intercepts are $\pm\sqrt{3}$

c When $y = 0$, $x^2 + x - 12 = 0$
$\therefore \ (x+4)(x-3) = 0$
$\therefore \ x = -4$ or 3
$\therefore$ the x-intercepts are -4 and 3

d When $y = 0$, $4x - x^2 = 0$
$\therefore \ x(4 - x) = 0$
$\therefore \ x = 0$ or 4
$\therefore$ the x-intercepts are 0 and 4

e When $y = 0$, $-x^2 - 6x - 8 = 0$
$\therefore \ x^2 + 6x + 8 = 0$
$\therefore \ (x+4)(x+2) = 0$
$\therefore \ x = -4$ or -2
$\therefore$ the x-intercepts are -4 and -2

f When $y = 0$, $-2x^2 - 4x - 2 = 0$
$\therefore \ x^2 + 2x + 1 = 0$
$\therefore \ (x+1)^2 = 0$
$\therefore \ x = -1$
$\therefore$ the x-intercept is -1 (touching)

g When $y = 0$, $4x^2 - 24x + 36 = 0$
$\therefore \ x^2 - 6x + 9 = 0$
$\therefore \ (x-3)^2 = 0$
$\therefore \ x = 3$
$\therefore$ the x-intercept is 3 (touching)

h When $y=0$, $x^2-4x+1=0$
$a=1$, $b=-4$, and $c=1$

$$\therefore \quad x=\frac{-(-4)\pm\sqrt{(-4)^2-4(1)(1)}}{2(1)} = \frac{4\pm\sqrt{12}}{2} = \frac{4\pm2\sqrt{3}}{2} = 2\pm\sqrt{3}$$

$\therefore$ the x-intercepts are $2\pm\sqrt{3}$

i When $y=0$, $x^2+8x+11=0$
$a=1$, $b=8$, and $c=11$

$$\therefore \quad x=\frac{-8\pm\sqrt{8^2-4(1)(11)}}{2(1)} = \frac{-8\pm\sqrt{20}}{2} = \frac{-8\pm2\sqrt{5}}{2} = -4\pm\sqrt{5}$$

$\therefore$ the x-intercepts are $-4\pm\sqrt{5}$

8 **a** **i** $y=x^2-2x+5$
has $a=1$, $b=-2$, $c=5$

$$\therefore \quad -\frac{b}{2a}=-\frac{(-2)}{2(1)}=1$$

$\therefore$ the axis of symmetry is $x=1$

ii When $x=1$,

$$y=1^2-2(1)+5 = 1-2+5 = 4$$

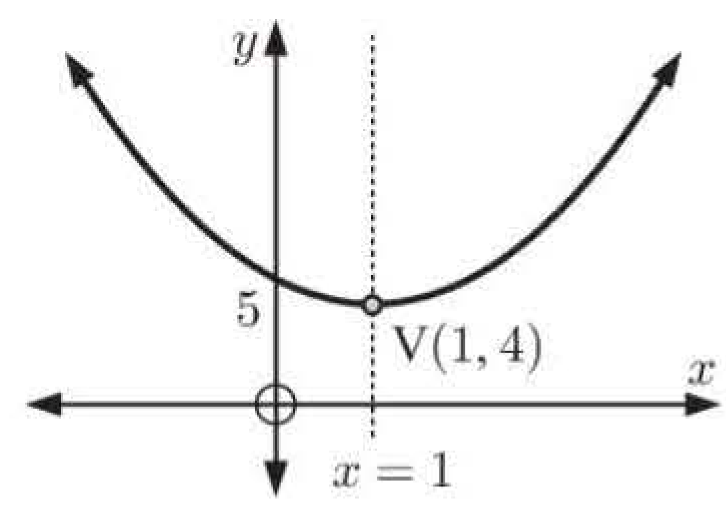
$\therefore$ the vertex is at $(1, 4)$

iii When $x=0$, $y=5$,
so the y-intercept is 5
When $y=0$, $x^2-2x+5=0$

$$\therefore \quad x=\frac{-(-2)\pm\sqrt{(-2)^2-4(1)(5)}}{2(1)} = \frac{2\pm\sqrt{4-20}}{2}$$

This has no real solutions,
so there are no x-intercepts.

iv



b **i** $y=2x^2-5x+2$
has $a=2$, $b=-5$, $c=2$

$$\therefore \quad -\frac{b}{2a}=-\frac{(-5)}{2(2)}=\frac{5}{4}$$

$\therefore$ the axis of symmetry is $x=\frac{5}{4}$

ii When $x=\frac{5}{4}$,

$$y=2(\tfrac{5}{4})^2-5(\tfrac{5}{4})+2 = \tfrac{25}{8}-\tfrac{25}{4}+2 = -\tfrac{9}{8}$$

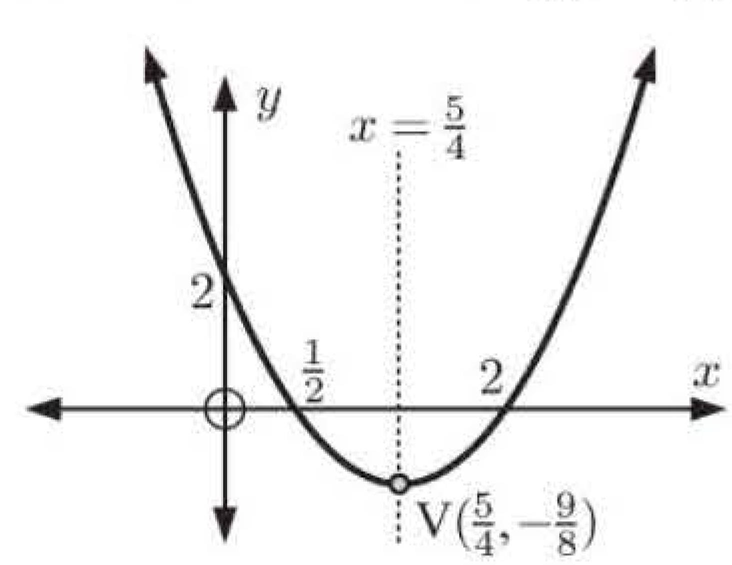
$\therefore$ the vertex is at $(\frac{5}{4}, -\frac{9}{8})$

iii When $x=0$, $y=2$,
so the y-intercept is 2.
When $y=0$, $2x^2-5x+2=0$

$$\therefore \quad (2x-1)(x-2)=0$$
$$\therefore \quad x=\tfrac{1}{2} \text{ or } 2$$

$\therefore$ the x-intercepts are $\frac{1}{2}$ and 2

iv



c **i** $y=-x^2+3x-2$
has $a=-1$, $b=3$, $c=-2$

$$\therefore \quad -\frac{b}{2a}=-\frac{3}{2(-1)}=\frac{3}{2}$$

$\therefore$ the axis of symmetry is $x=\frac{3}{2}$

ii When $x=\frac{3}{2}$, $y=-(\frac{3}{2})^2+3(\frac{3}{2})-2$

$$= -\tfrac{9}{4}+\tfrac{9}{2}-2 = \tfrac{1}{4}$$

$\therefore$ the vertex is at $(\frac{3}{2}, \frac{1}{4})$

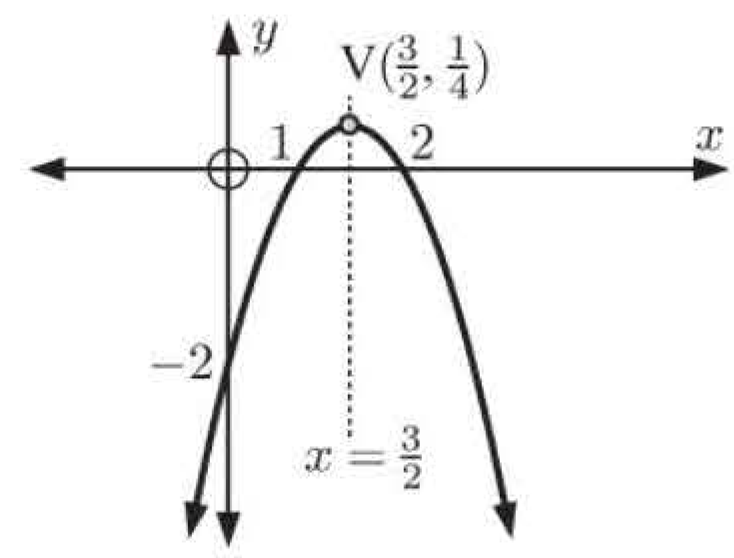
iii When $x=0$, $y=-2$,
so the y-intercept is -2.

When $y=0$, $-x^2+3x-2=0$

$$\therefore\ x^2-3x+2=0$$
$$\therefore\ (x-1)(x-2)=0$$
$$\therefore\ x=1 \text{ or } 2$$

$\therefore$ the x-intercepts are 1 and 2

iv



d **i** $y=-2x^2+x+1$
has $a=-2$, $b=1$, $c=1$

$$\therefore\ -\frac{b}{2a}=-\frac{1}{2(-2)}=\frac{1}{4}$$

$\therefore$ the axis of symmetry is $x=\frac{1}{4}$

ii When $x=\frac{1}{4}$, $y=-2(\frac{1}{4})^2+\frac{1}{4}+1$

$$=-\tfrac{1}{8}+\tfrac{1}{4}+1$$
$$=\tfrac{9}{8}$$

$\therefore$ the vertex is at $(\frac{1}{4}, \frac{9}{8})$

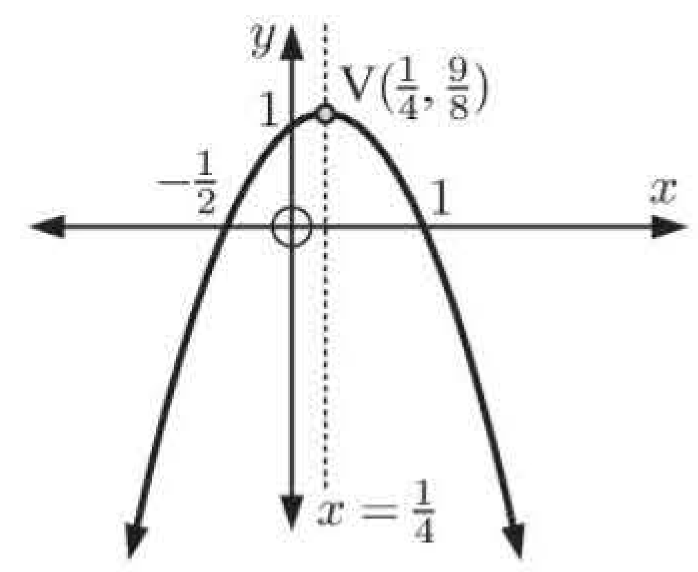
iii When $x=0$, $y=1$,
so the y-intercept is 1.

When $y=0$, $-2x^2+x+1=0$

$$\therefore\ 2x^2-x-1=0$$
$$\therefore\ (2x+1)(x-1)=0$$
$$\therefore\ x=-\tfrac{1}{2} \text{ or } 1$$

$\therefore$ the x-intercepts are $-\frac{1}{2}$ and 1

iv



e **i** $y=6x-x^2$
has $a=-1$, $b=6$, $c=0$

$$\therefore\ -\frac{b}{2a}=-\frac{6}{2(-1)}=3$$

$\therefore$ the axis of symmetry is $x=3$

ii When $x=3$, $y=6\times 3-3^2$

$$=9$$

$\therefore$ the vertex is at $(3, 9)$

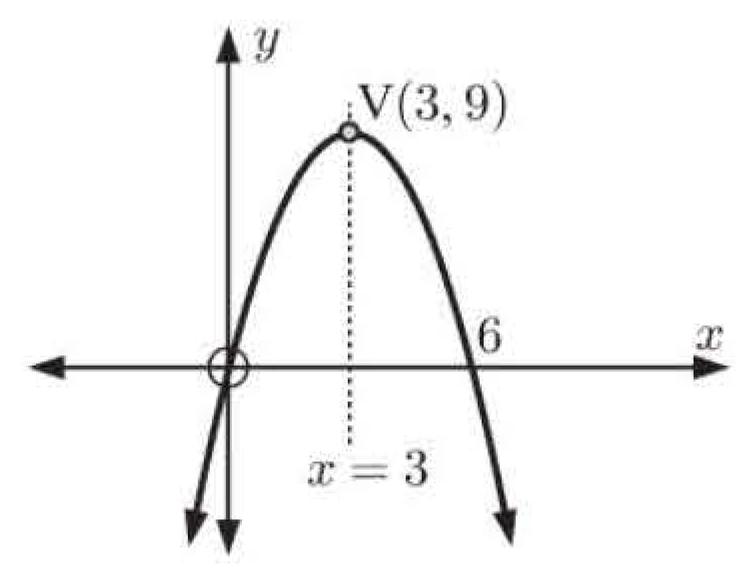
iii When $x=0$, $y=0$,
so the y-intercept is 0.

When $y=0$, $6x-x^2=0$

$$\therefore\ x(6-x)=0$$
$$\therefore\ x=0 \text{ or } 6$$

$\therefore$ the x-intercepts are 0 and 6

iv



f **i** $y=-\frac{1}{4}x^2+2x+1$
has $a=-\frac{1}{4}$, $b=2$, $c=1$

$$\therefore\ -\frac{b}{2a}=-\frac{2}{2(-\frac{1}{4})}=4$$

$\therefore$ the axis of symmetry is $x=4$

ii When $x=4$, $y=-\frac{1}{4}(4)^2+2(4)+1$

$$=-4+8+1$$
$$=5$$

$\therefore$ the vertex is at $(4, 5)$

iii When $x=0$, $y=1$,
so the y-intercept is 1.

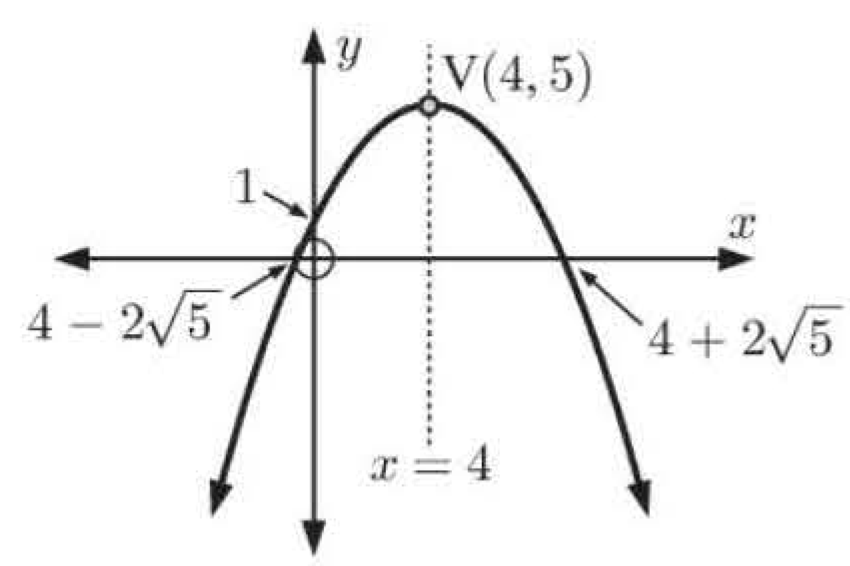
When $y=0$, $-\frac{1}{4}x^2+2x+1=0$

$$\therefore\ x^2-8x-4=0$$

$$\therefore\ x=\frac{-(-8)\pm\sqrt{(-8)^2-4(1)(-4)}}{2(1)}$$
$$=\frac{8\pm\sqrt{80}}{2}$$
$$=\frac{8\pm 4\sqrt{5}}{2}$$
$$=4\pm 2\sqrt{5}$$

$\therefore$ the x-intercepts are $4\pm 2\sqrt{5}$.

iv



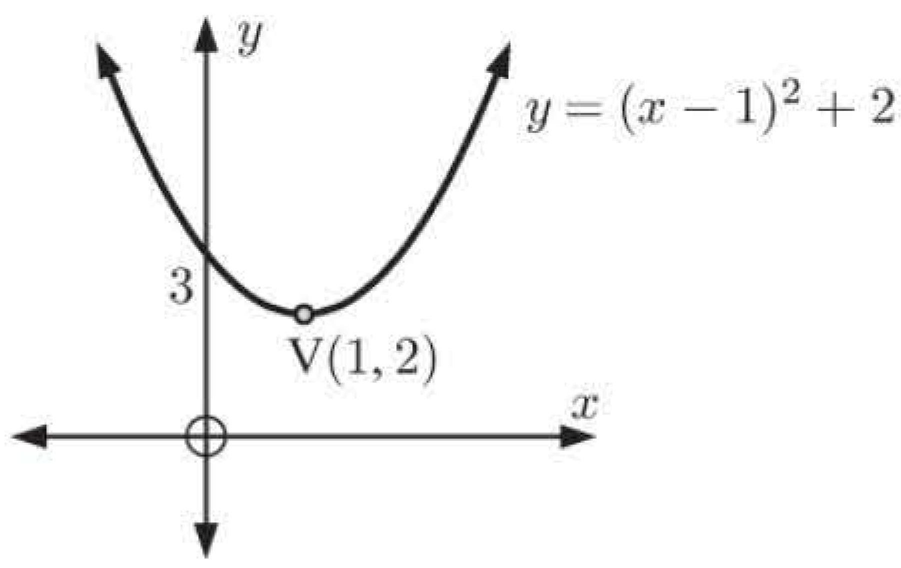
EXERCISE 1D.2

1 **a** $y = x^2 - 2x + 3$

$\therefore\ y = x^2 - 2x + 1^2 + 3 - 1^2$

$\therefore\ y = (x-1)^2 + 2$

$\therefore$ vertex is $(1, 2)$, y-intercept is 3

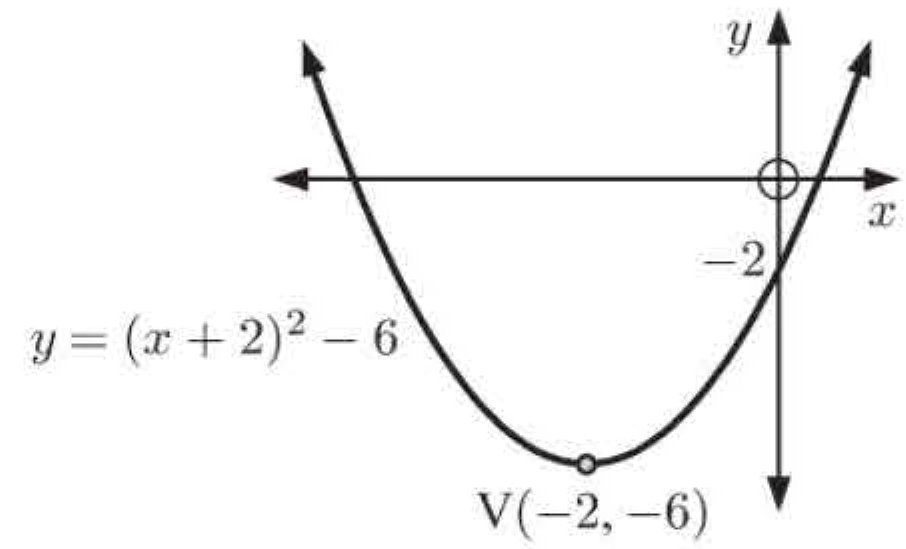


b $y = x^2 + 4x - 2$

$\therefore\ y = x^2 + 4x + 2^2 - 2 - 2^2$

$\therefore\ y = (x+2)^2 - 6$

$\therefore$ vertex is $(-2, -6)$, y-intercept is -2

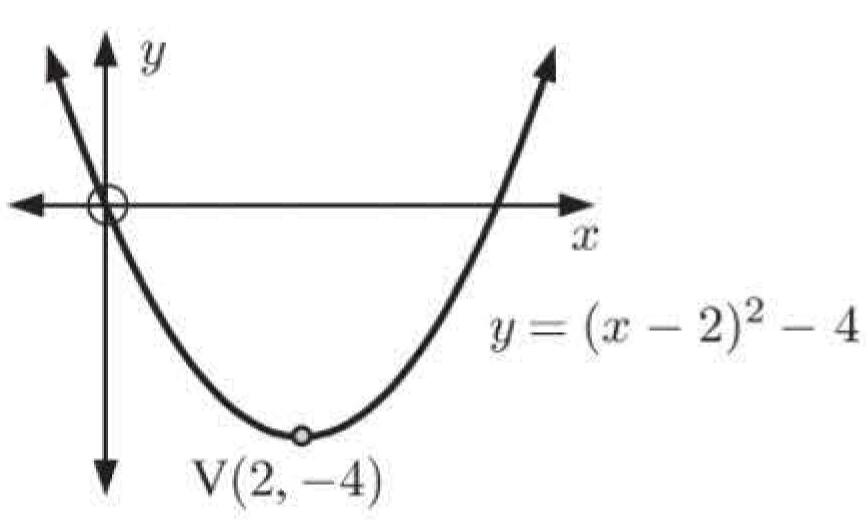


c $y = x^2 - 4x$

$\therefore\ y = x^2 - 4x + 2^2 - 2^2$

$\therefore\ y = (x-2)^2 - 4$

$\therefore$ vertex is $(2, -4)$, y-intercept is 0

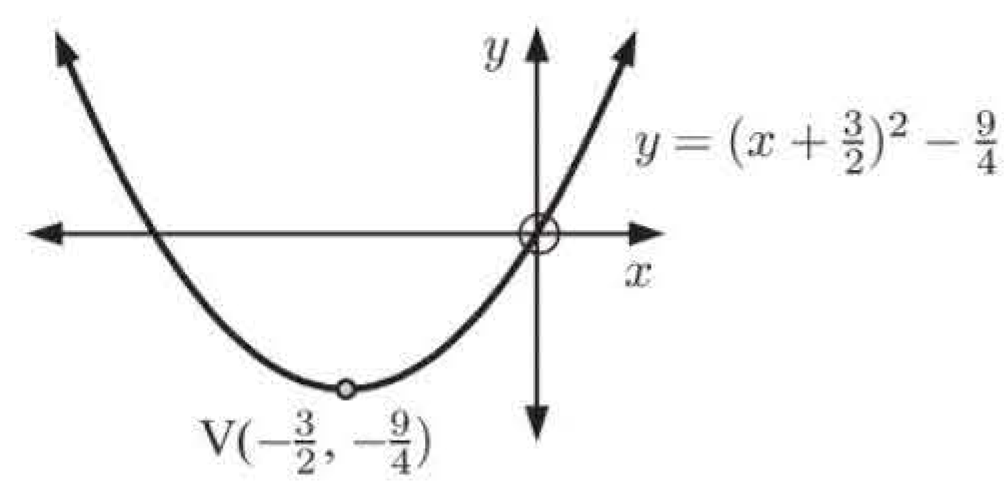


d $y = x^2 + 3x$

$\therefore\ y = x^2 + 3x + (\frac{3}{2})^2 - (\frac{3}{2})^2$

$\therefore\ y = (x + \frac{3}{2})^2 - \frac{9}{4}$

$\therefore$ vertex is $(-\frac{3}{2}, -\frac{9}{4})$, y-intercept is 0

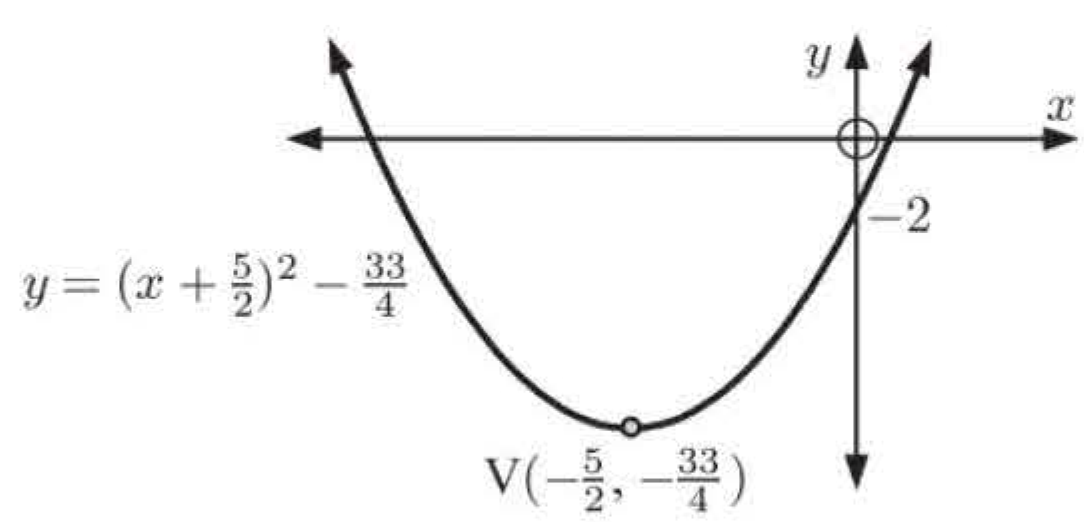


e $y = x^2 + 5x - 2$

$\therefore\ y = x^2 + 5x + (\frac{5}{2})^2 - 2 - (\frac{5}{2})^2$

$\therefore\ y = (x + \frac{5}{2})^2 - \frac{33}{4}$

$\therefore$ vertex is $(-\frac{5}{2}, -\frac{33}{4})$, y-intercept is -2

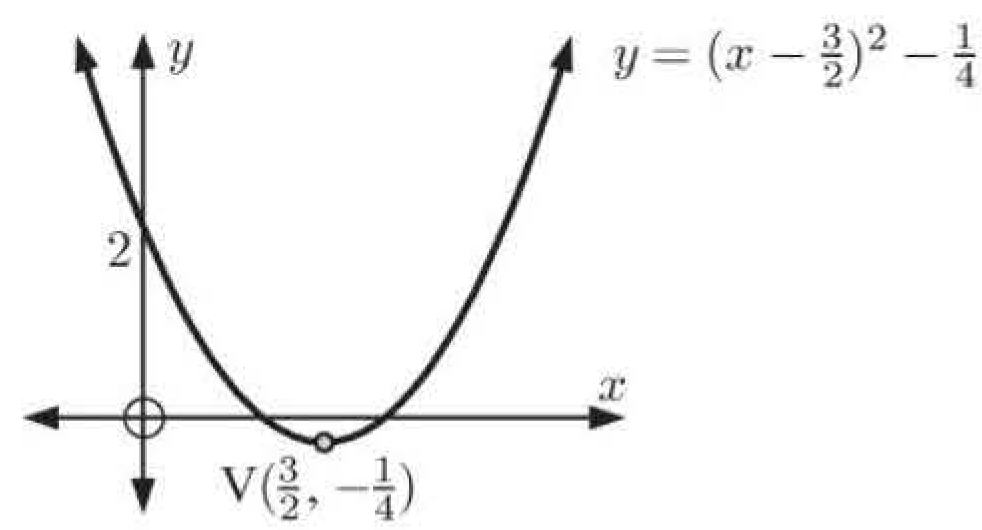


f $y = x^2 - 3x + 2$

$\therefore\ y = x^2 - 3x + (\frac{3}{2})^2 + 2 - (\frac{3}{2})^2$

$\therefore\ y = (x - \frac{3}{2})^2 - \frac{1}{4}$

$\therefore$ vertex is $(\frac{3}{2}, -\frac{1}{4})$, y-intercept is 2

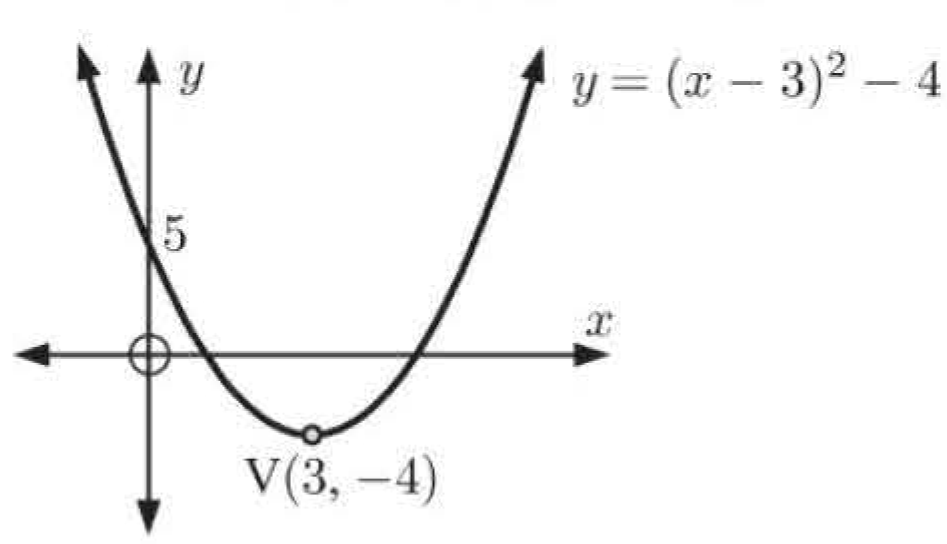


g $y = x^2 - 6x + 5$

$\therefore\ y = x^2 - 6x + 3^2 + 5 - 3^2$

$\therefore\ y = (x-3)^2 - 4$

$\therefore$ vertex is $(3, -4)$, y-intercept is 5

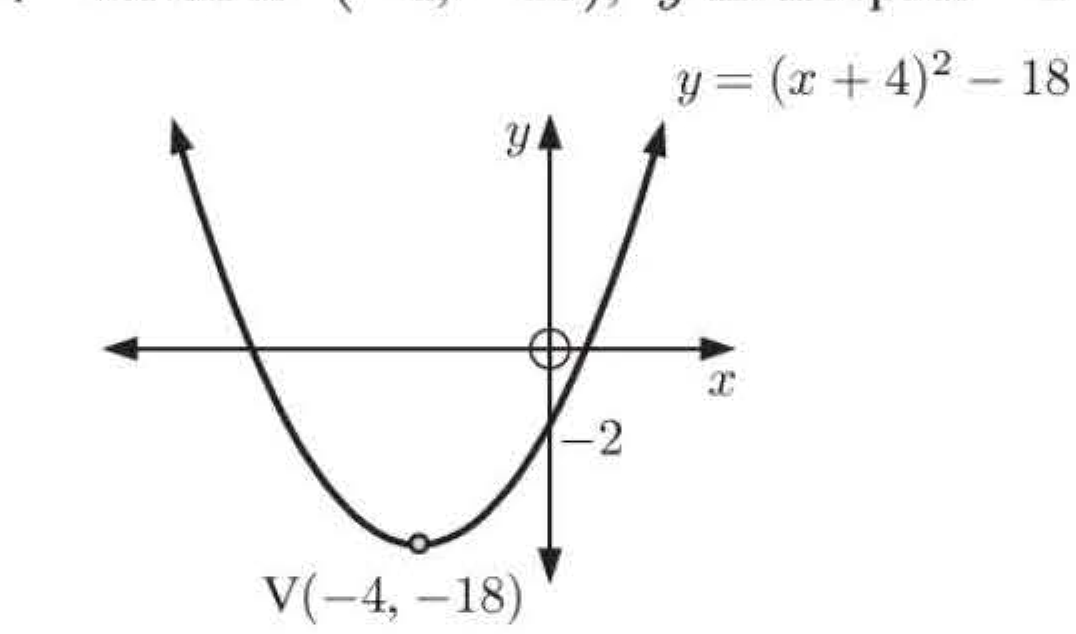


h $y = x^2 + 8x - 2$

$\therefore\ y = x^2 + 8x + 4^2 - 2 - 4^2$

$\therefore\ y = (x+4)^2 - 18$

$\therefore$ vertex is $(-4, -18)$, y-intercept is -2

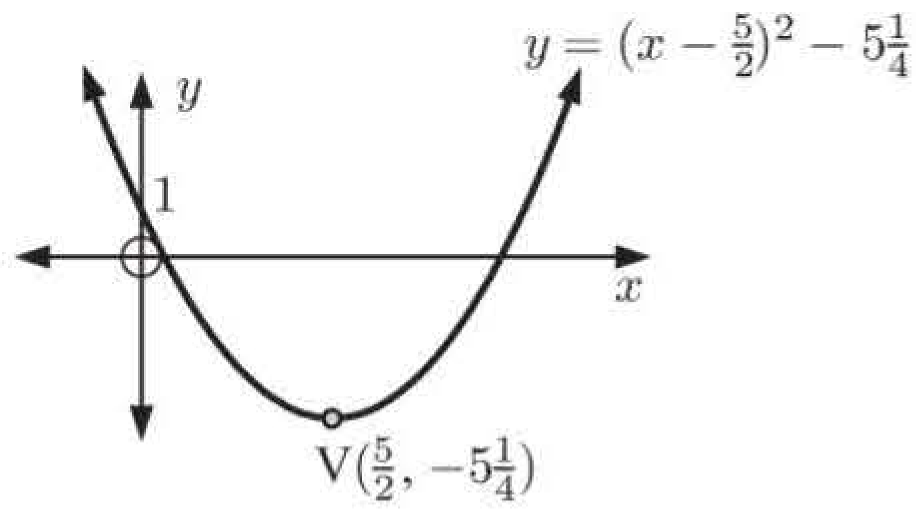


i $y = x^2 - 5x + 1$

$\therefore\ y = x^2 - 5x + (\frac{5}{2})^2 + 1 - (\frac{5}{2})^2$

$\therefore\ y = (x - \frac{5}{2})^2 - \frac{21}{4}$

$\therefore$ vertex is $(\frac{5}{2}, -5\frac{1}{4})$, y-intercept is 1



2 **a** **i** $y = 2x^2 + 4x + 5$

$= 2[x^2 + 2x + \frac{5}{2}]$

$= 2[x^2 + 2x + 1^2 - 1^2 + \frac{5}{2}]$

$= 2[(x+1)^2 + \frac{3}{2}]$

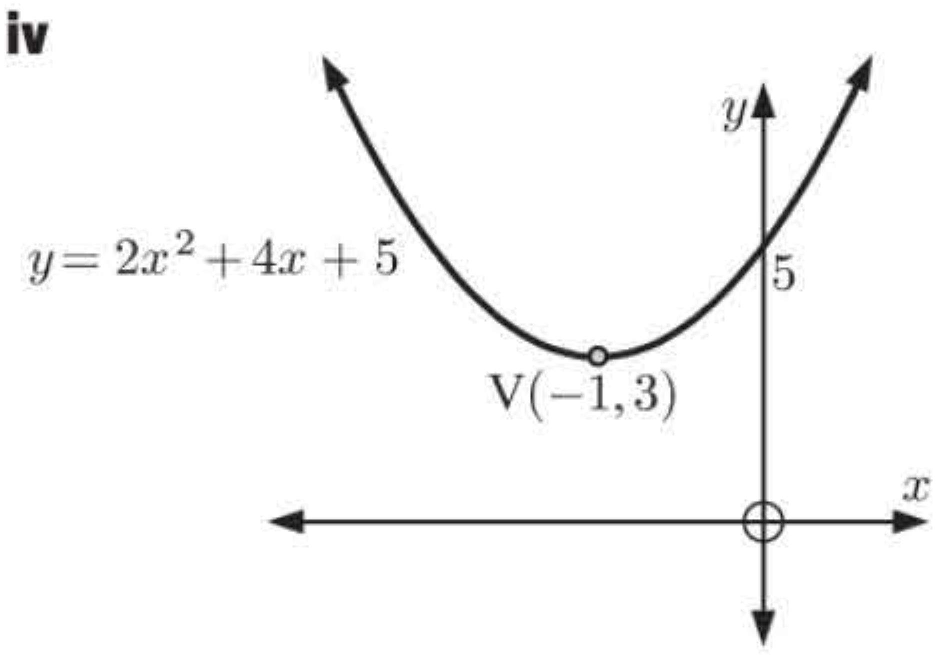
$= 2(x+1)^2 + 3$

ii The vertex is $(-1, 3)$.

iii When $x = 0$, $y = 5$

$\therefore$ the y-intercept is 5

iv



b **i** $y = 2x^2 - 8x + 3$

$= 2[x^2 - 4x + \frac{3}{2}]$

$= 2[x^2 - 4x + 2^2 - 2^2 + \frac{3}{2}]$

$= 2[(x-2)^2 - \frac{5}{2}]$

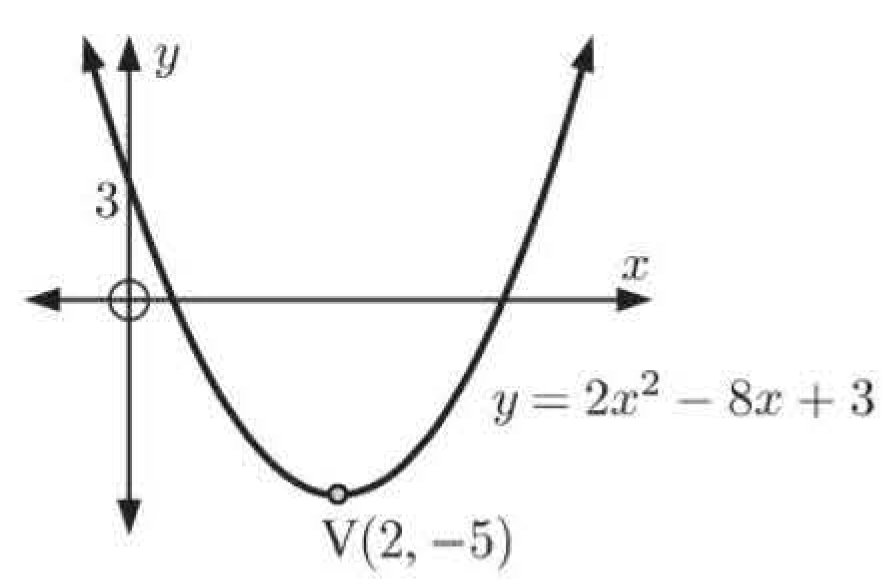
$= 2(x-2)^2 - 5$

ii The vertex is $(2, -5)$.

iii When $x = 0$, $y = 3$

$\therefore$ the y-intercept is 3

iv



c **i** $y = 2x^2 - 6x + 1$

$= 2[x^2 - 3x + \frac{1}{2}]$

$= 2[x^2 - 3x + (\frac{3}{2})^2 - (\frac{3}{2})^2 + \frac{1}{2}]$

$= 2[(x - \frac{3}{2})^2 - \frac{7}{4}]$

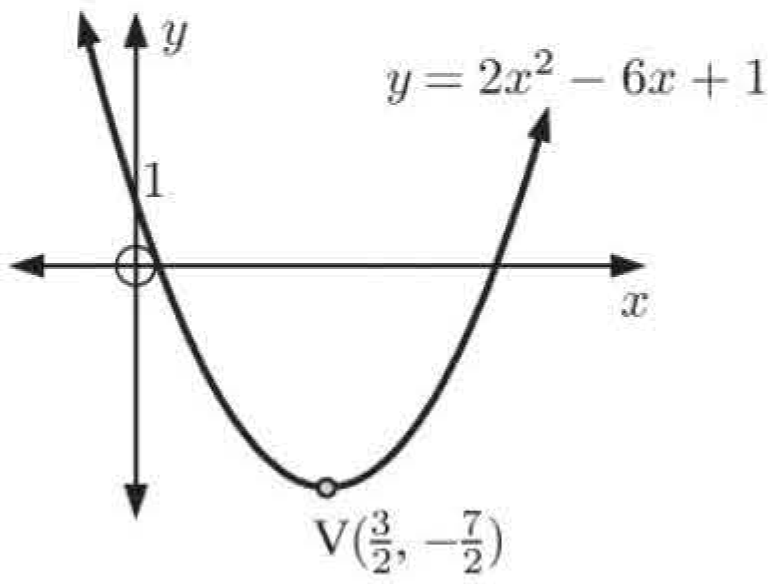
$= 2(x - \frac{3}{2})^2 - \frac{7}{2}$

ii The vertex is $(\frac{3}{2}, -\frac{7}{2})$.

iii When $x = 0$, $y = 1$

$\therefore$ the y-intercept is 1

iv



d **i** $y = 3x^2 - 6x + 5$

$= 3[x^2 - 2x + \frac{5}{3}]$

$= 3[x^2 - 2x + 1^2 - 1^2 + \frac{5}{3}]$

$= 3[(x-1)^2 + \frac{2}{3}]$

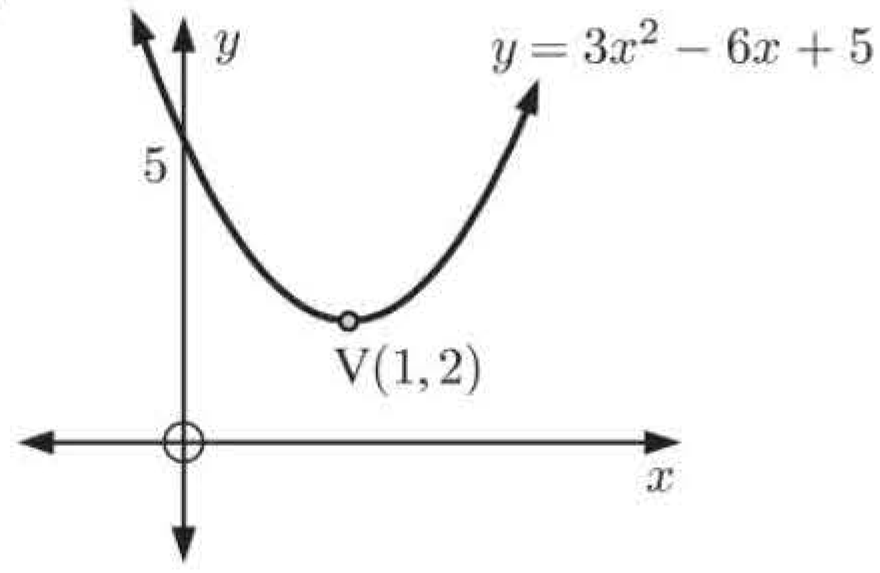
$= 3(x-1)^2 + 2$

ii The vertex is $(1, 2)$.

iii When $x = 0$, $y = 5$

$\therefore$ the y-intercept is 5

iv



e **i** $y = -x^2 + 4x + 2$

$= -[x^2 - 4x - 2]$

$= -[x^2 - 4x + 2^2 - 2^2 - 2]$

$= -[(x-2)^2 - 6]$

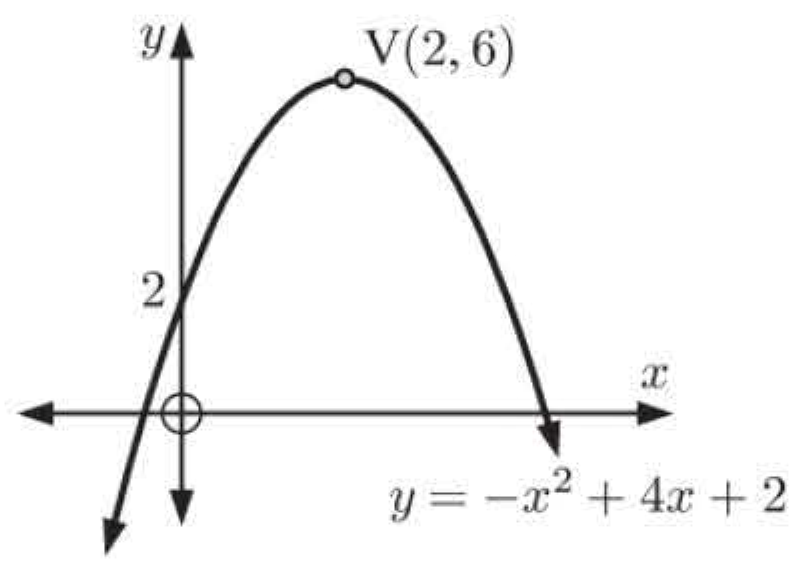
$= -(x-2)^2 + 6$

ii The vertex is $(2, 6)$.

iii When $x = 0$, $y = 2$

$\therefore$ the y-intercept is 2

iv

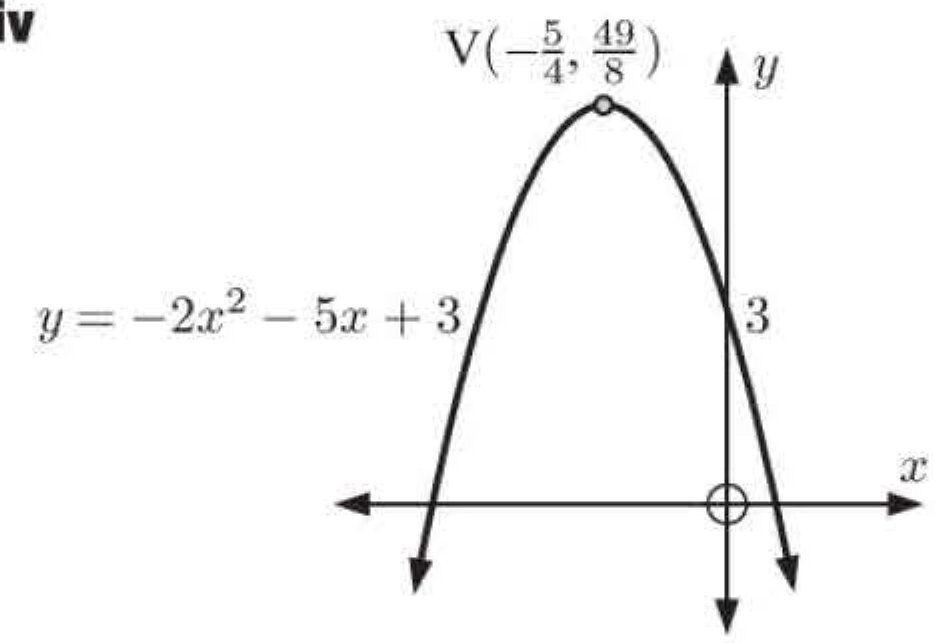


f **i** $y = -2x^2 - 5x + 3$
$= -2[x^2 + \frac{5}{2}x - \frac{3}{2}]$
$= -2[x^2 + \frac{5}{2}x + (\frac{5}{4})^2 - (\frac{5}{4})^2 - \frac{3}{2}]$
$= -2[(x + \frac{5}{4})^2 - \frac{25}{16} - \frac{24}{16}]$
$= -2[(x + \frac{5}{4})^2 - \frac{49}{16}]$
$= -2(x + \frac{5}{4})^2 + \frac{49}{8}$

ii The vertex is $(-\frac{5}{4}, \frac{49}{8})$.

iii When $x = 0$, $y = 3$
$\therefore$ the y-intercept is 3

iv



EXERCISE 1D.3

1 **a** $y = x^2 + x - 2$
has $a = 1$, $b = 1$, $c = -2$
$\therefore$ $\Delta = b^2 - 4ac$
$= 1^2 - 4(1)(-2)$
$= 9 > 0$
$\therefore$ the graph cuts the x-axis twice, and since $a > 0$, the graph is concave up.

b $y = x^2 + 7x - 2$
has $a = 1$, $b = 7$, $c = -2$
$\therefore$ $\Delta = b^2 - 4ac$
$= 7^2 - 4(1)(-2)$
$= 57 > 0$
$\therefore$ the graph cuts the x-axis twice, and since $a > 0$, the graph is concave up.

c $y = x^2 + 8x + 16$
has $a = 1$, $b = 8$, $c = 16$
$\therefore$ $\Delta = b^2 - 4ac$
$= 8^2 - 4(1)(16)$
$= 0$
$\therefore$ the graph touches the x-axis, and since $a > 0$, the graph is concave up.

d $y = x^2 + 4\sqrt{2}x + 8$
has $a = 1$, $b = 4\sqrt{2}$, $c = 8$
$\therefore$ $\Delta = b^2 - 4ac$
$= (4\sqrt{2})^2 - 4(1)(8)$
$= 0$
$\therefore$ the graph touches the x-axis, and since $a > 0$, the graph is concave up.

e $y = -x^2 + x + 6$
has $a = -1$, $b = 1$, $c = 6$
$\therefore$ $\Delta = b^2 - 4ac$
$= 1^2 - 4(-1)(6)$
$= 25 > 0$
$\therefore$ the graph cuts the x-axis twice, and since $a < 0$, the graph is concave down.

f $y = 9x^2 + 6x + 1$
has $a = 9$, $b = 6$, $c = 1$
$\therefore$ $\Delta = b^2 - 4ac$
$= 6^2 - 4(9)(1)$
$= 0$
$\therefore$ the graph touches the x-axis, and since $a > 0$, the graph is concave up.

2 **a** $x^2 - 3x + 6$
has $a = 1$, $b = -3$, $c = 6$
$\therefore$ $\Delta = b^2 - 4ac$
$= (-3)^2 - 4(1)(6)$
$= -15$
Since $a > 0$ and $\Delta < 0$,
$x^2 - 3x + 6$ is positive definite.
$\therefore$ $x^2 - 3x + 6 > 0$ for all x.

b $4x - x^2 - 6$
has $a = -1$, $b = 4$, $c = -6$
$\therefore$ $\Delta = b^2 - 4ac$
$= 4^2 - 4(-1)(-6)$
$= -8$
Since $a < 0$ and $\Delta < 0$,
$4x - x^2 - 6$ is negative definite.
$\therefore$ $4x - x^2 - 6 < 0$ for all x.

c $2x^2 - 4x + 7$
has $a = 2$, $b = -4$, $c = 7$
$\therefore \Delta = b^2 - 4ac$
$= (-4)^2 - 4(2)(7)$
$= -40$
Since $a > 0$ and $\Delta < 0$,
$2x^2 - 4x + 7$ is positive definite.

d $-2x^2 + 3x - 4$
has $a = -2$, $b = 3$, $c = -4$
$\therefore \Delta = b^2 - 4ac$
$= 3^2 - 4(-2)(-4)$
$= -23$
Since $a < 0$ and $\Delta < 0$,
$-2x^2 + 3x - 4$ is negative definite.

3 $3x^2 + kx - 1$
has $a = 3$, $b = k$, $c = -1$
$\therefore \Delta = b^2 - 4ac$
$= k^2 - 4(3)(-1)$
$= k^2 + 12$
$\therefore \Delta > 0$ for all k
{as $k^2 \geqslant 0$ for all k}
$\therefore$ $3x^2 + kx - 1$ has two real distinct roots for all k.
$\therefore$ it can never be positive definite.

4 $2x^2 + kx + 2$
has $a = 2$, $b = k$, $c = 2$
$\therefore \Delta = b^2 - 4ac$
$= k^2 - 4(2)(2)$
$= k^2 - 16$
Now $2x^2 + kx + 2$ has $a > 0$.
$\therefore$ it is positive definite provided $k^2 - 16 < 0$
$\therefore k^2 < 16$
$\therefore -4 < k < 4$

EXERCISE 1E

1 **a** The x-intercepts are 1 and 2.
$\therefore y = a(x-1)(x-2)$ for some $a \neq 0$.
But the y-intercept is 4.
$\therefore a(-1)(-2) = 4$
$\therefore 2a = 4$
$\therefore a = 2$
$\therefore y = 2(x-1)(x-2)$

b The graph touches the x-axis when $x = 2$.
$\therefore y = a(x-2)^2$ for some $a \neq 0$.
But the y-intercept is 8.
$\therefore a(-2)^2 = 8$
$\therefore 4a = 8$
$\therefore a = 2$
$\therefore y = 2(x-2)^2$

c The x-intercepts are 1 and 3.
$\therefore y = a(x-1)(x-3)$ for some $a \neq 0$.
But the y-intercept is 3.
$\therefore a(-1)(-3) = 3$
$\therefore 3a = 3$
$\therefore a = 1$
$\therefore y = (x-1)(x-3)$

d The x-intercepts are -1 and 3.
$\therefore y = a(x+1)(x-3)$ for some $a \neq 0$.
But the y-intercept is 3.
$\therefore a(1)(-3) = 3$
$\therefore -3a = 3$
$\therefore a = -1$
$\therefore y = -(x+1)(x-3)$

e The graph touches the x-axis when $x = 1$.
$\therefore y = a(x-1)^2$ for some $a \neq 0$.
But the y-intercept is -3.
$\therefore a(-1)^2 = -3$
$\therefore a = -3$
$\therefore y = -3(x-1)^2$

f The x-intercepts are -2 and 3.
$\therefore y = a(x+2)(x-3)$ for some $a \neq 0$.
But the y-intercept is 12.
$\therefore a(2)(-3) = 12$
$\therefore -6a = 12$
$\therefore a = -2$
$\therefore y = -2(x+2)(x-3)$

2 **a** As the axis of symmetry is $x = 3$, the other x-intercept is 4.
$\therefore y = a(x-2)(x-4)$ for some $a \neq 0$.
But the y-intercept $= 12$
$\therefore a(-2)(-4) = 12$
$\therefore 8a = 12$
$\therefore a = \frac{12}{8} = \frac{3}{2}$
$\therefore y = \frac{3}{2}(x-2)(x-4)$

b As the axis of symmetry is $x = -1$, the other x-intercept is 2.
$\therefore y = a(x+4)(x-2)$ for some $a \neq 0$.
But the y-intercept $= 4$
$\therefore a(4)(-2) = 4$
$\therefore -8a = 4$
$\therefore a = -\frac{1}{2}$
$\therefore y = -\frac{1}{2}(x+4)(x-2)$

c The graph touches the x-axis at $x = -3$,

$\therefore\ y = a(x+3)^2$ for some $a \neq 0$.

But the y-intercept is -12, so
$$\begin{aligned} a(3)^2 &= -12 \\ \therefore\ 9a &= -12 \\ \therefore\ a &= -\tfrac{12}{9} = -\tfrac{4}{3} \\ \therefore\ y &= -\tfrac{4}{3}(x+3)^2 \end{aligned}$$

3 **a** Since the x-intercepts are 5 and 1, the equation is $y = a(x-5)(x-1)$ for some $a \neq 0$.

But when $x = 2$, $y = -9$
$$\begin{aligned} \therefore\ -9 &= a(2-5)(2-1) \\ \therefore\ -9 &= a(-3)(1) \\ \therefore\ -3a &= -9 \\ \therefore\ a &= 3 \end{aligned}$$
$\therefore$ the equation is
$$\begin{aligned} y &= 3(x-5)(x-1) \\ \therefore\ y &= 3(x^2-6x+5) \\ \therefore\ y &= 3x^2-18x+15 \end{aligned}$$

b Since the x-intercepts are 2 and $-\frac{1}{2}$, the equation is $y = a(x-2)(x+\frac{1}{2})$ for some $a \neq 0$.

But when $x = 3$, $y = -14$
$$\begin{aligned} \therefore\ -14 &= a(3-2)(3+\tfrac{1}{2}) \\ \therefore\ -14 &= a(1)(\tfrac{7}{2}) \\ \therefore\ \tfrac{7}{2}a &= -14 \\ \therefore\ a &= -4 \end{aligned}$$
$\therefore$ the equation is
$$\begin{aligned} y &= -4(x-2)(x+\tfrac{1}{2}) \\ \therefore\ y &= -4(x^2-\tfrac{3}{2}x-1) \\ \therefore\ y &= -4x^2+6x+4 \end{aligned}$$

c Since the graph touches the x-axis at 3, its equation is $y = a(x-3)^2$, for some $a \neq 0$.

But when $x = -2$, $y = -25$
$$\begin{aligned} \therefore\ -25 &= a(-2-3)^2 \\ \therefore\ -25 &= 25a \\ \therefore\ a &= -1 \end{aligned}$$
$\therefore$ the equation is
$$\begin{aligned} y &= -(x-3)^2 \\ \therefore\ y &= -(x^2-6x+9) \\ \therefore\ y &= -x^2+6x-9 \end{aligned}$$

d Since the graph touches the x-axis at -2, its equation is $y = a(x+2)^2$, for some $a \neq 0$.

But when $x = -1$, $y = 4$
$$\begin{aligned} \therefore\ 4 &= a(-1+2)^2 \\ \therefore\ 4 &= a \end{aligned}$$
$\therefore$ the equation is
$$\begin{aligned} y &= 4(x+2)^2 \\ \therefore\ y &= 4(x^2+4x+4) \\ \therefore\ y &= 4x^2+16x+16 \end{aligned}$$

e Since the graph cuts the x-axis at 3 and has axis of symmetry $x = 2$, it must also cut the x-axis at 1.

$\therefore$ the x-intercepts are 3 and 1, and the equation is $y = a(x-3)(x-1)$ for some $a \neq 0$.

But when $x = 5$, $y = 12$
$$\begin{aligned} \therefore\ 12 &= a(5-3)(5-1) \\ \therefore\ 12 &= a(2)(4) \\ \therefore\ 8a &= 12 \\ \therefore\ a &= \tfrac{12}{8} = \tfrac{3}{2} \end{aligned}$$
$\therefore$ the equation is
$$\begin{aligned} y &= \tfrac{3}{2}(x-3)(x-1) \\ \therefore\ y &= \tfrac{3}{2}(x^2-4x+3) \\ \therefore\ y &= \tfrac{3}{2}x^2-6x+\tfrac{9}{2} \end{aligned}$$

f Since the graph cuts the x-axis at 5 and has axis of symmetry $x = 1$, it must also cut the x-axis at -3.

$\therefore$ the x-intercepts are 5 and -3, and the equation is $y = a(x-5)(x+3)$ for some $a \neq 0$.

But when $x = 2$, $y = 5$
$$\begin{aligned} \therefore\ 5 &= a(2-5)(2+3) \\ \therefore\ 5 &= a(-3)(5) \\ \therefore\ -15a &= 5 \\ \therefore\ a &= -\tfrac{5}{15} = -\tfrac{1}{3} \end{aligned}$$
$\therefore$ the equation is
$$\begin{aligned} y &= -\tfrac{1}{3}(x-5)(x+3) \\ \therefore\ y &= -\tfrac{1}{3}(x^2-2x-15) \\ \therefore\ y &= -\tfrac{1}{3}x^2+\tfrac{2}{3}x+5 \end{aligned}$$

4 **a** The vertex is $(2, 4)$,
so the quadratic has equation
$y = a(x-2)^2 + 4$ for some $a \neq 0$.
But the graph passes through the origin
$\therefore \quad 0 = a(0-2)^2 + 4$
$\therefore \quad 4a + 4 = 0$
$\therefore \quad a = -1$
$\therefore$ the equation is $y = -(x-2)^2 + 4$

b The vertex is $(2, -1)$,
so the quadratic has equation
$y = a(x-2)^2 - 1$ for some $a \neq 0$.
But the graph passes through $(0, 7)$
$\therefore \quad 7 = a(0-2)^2 - 1$
$\therefore \quad 7 = 4a - 1$
$\therefore \quad a = 2$
$\therefore$ the equation is $y = 2(x-2)^2 - 1$

c The vertex is $(3, 8)$,
so the quadratic has equation
$y = a(x-3)^2 + 8$ for some $a \neq 0$.
But the graph passes through $(1, 0)$
$\therefore \quad 0 = a(1-3)^2 + 8$
$\therefore \quad 0 = 4a + 8$
$\therefore \quad a = -2$
$\therefore$ the equation is $y = -2(x-3)^2 + 8$

d The vertex is $(4, -6)$,
so the quadratic has equation
$y = a(x-4)^2 - 6$ for some $a \neq 0$.
But the graph passes through $(7, 0)$
$\therefore \quad 0 = a(7-4)^2 - 6$
$\therefore \quad 9a - 6 = 0$
$\therefore \quad a = \frac{2}{3}$
$\therefore$ the equation is $y = \frac{2}{3}(x-4)^2 - 6$

e The vertex is $(2, 3)$,
so the quadratic has equation
$y = a(x-2)^2 + 3$ for some $a \neq 0$.
But the graph passes through $(3, 1)$
$\therefore \quad 1 = a(3-2)^2 + 3$
$\therefore \quad 1 = a + 3$
$\therefore \quad a = -2$
$\therefore$ the equation is $y = -2(x-2)^2 + 3$

f The vertex is $(\frac{1}{2}, -\frac{3}{2})$,
so the quadratic has equation
$y = a(x-\frac{1}{2})^2 - \frac{3}{2}$ for some $a \neq 0$.
But the graph passes through $(\frac{3}{2}, \frac{1}{2})$
$\therefore \quad \frac{1}{2} = a(\frac{3}{2}-\frac{1}{2})^2 - \frac{3}{2}$
$\therefore \quad \frac{1}{2} = a - \frac{3}{2}$
$\therefore \quad a = 2$
$\therefore$ the equation is $y = 2(x-\frac{1}{2})^2 - \frac{3}{2}$

EXERCISE 1F

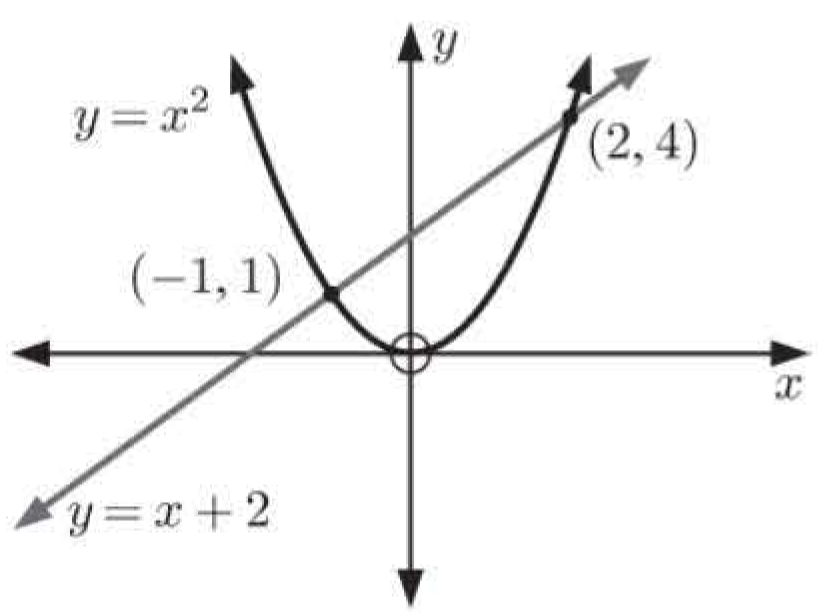
1 **a** $y = x^2 - 2x + 8$ meets $y = x + 6$
when $x^2 - 2x + 8 = x + 6$
$\therefore \quad x^2 - 3x + 2 = 0$
$\therefore \quad (x-1)(x-2) = 0$
$\therefore \quad x = 1$ or 2
Substituting into $y = x + 6$,
when $x = 1$, $y = 7$
and when $x = 2$, $y = 8$
$\therefore$ the graphs intersect at $(1, 7)$ and $(2, 8)$.

b $y = -x^2 + 3x + 9$ meets $y = 2x - 3$
when $-x^2 + 3x + 9 = 2x - 3$
$\therefore \quad x^2 - x - 12 = 0$
$\therefore \quad (x-4)(x+3) = 0$
$\therefore \quad x = 4$ or -3
Substituting into $y = 2x - 3$,
when $x = -3$, $y = 2(-3) - 3 = -9$
and when $x = 4$, $y = 2(4) - 3 = 5$
$\therefore$ the graphs intersect at $(-3, -9)$ and $(4, 5)$.

c $y = x^2 - 4x + 3$ meets $y = 2x - 6$
when $x^2 - 4x + 3 = 2x - 6$
$\therefore \quad x^2 - 6x + 9 = 0$
$\therefore \quad (x-3)^2 = 0$
$\therefore \quad x = 3$
Substituting into $y = 2x - 6$,
when $x = 3$, $y = 0$
$\therefore$ the graphs touch at $(3, 0)$.

d $y = -x^2 + 4x - 7$ meets $y = 5x - 4$
when $-x^2 + 4x - 7 = 5x - 4$
$\therefore \quad x^2 + x + 3 = 0$
which has $a = 1$, $b = 1$, $c = 3$
$$\therefore \quad x = \frac{-1 \pm \sqrt{1^2 - 4(1)(3)}}{2(1)}$$
$$\therefore \quad x = \frac{-1 \pm \sqrt{-11}}{2}$$
$\therefore$ there are no real solutions
$\therefore$ the graphs do not intersect.

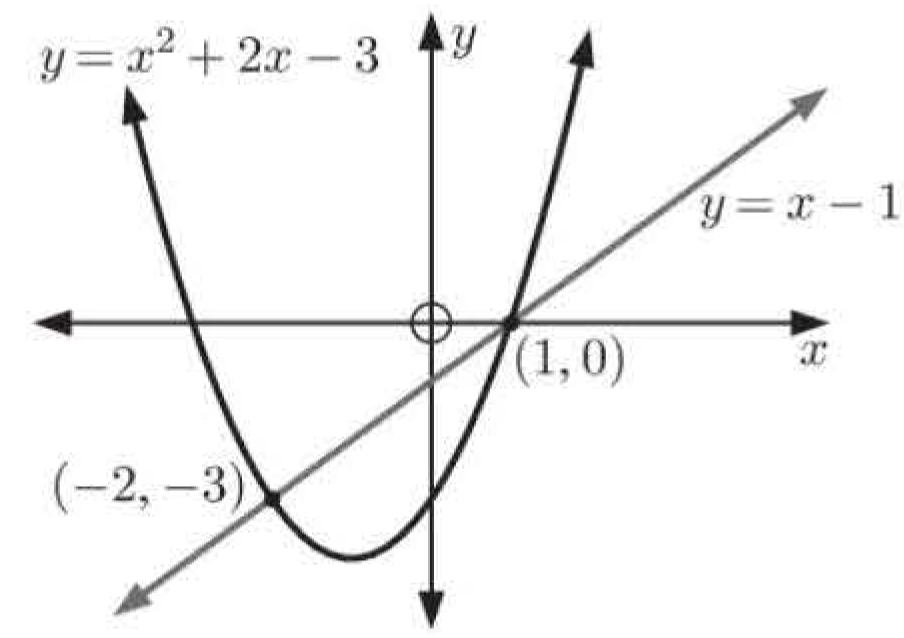
2 **a** **i**



$\therefore\ y = x^2$ meets $y = x + 2$ at the points $(-1, 1)$ and $(2, 4)$.

ii Using the graph in **a i**, $x^2 > x + 2$ when $x < -1$ or $x > 2$.

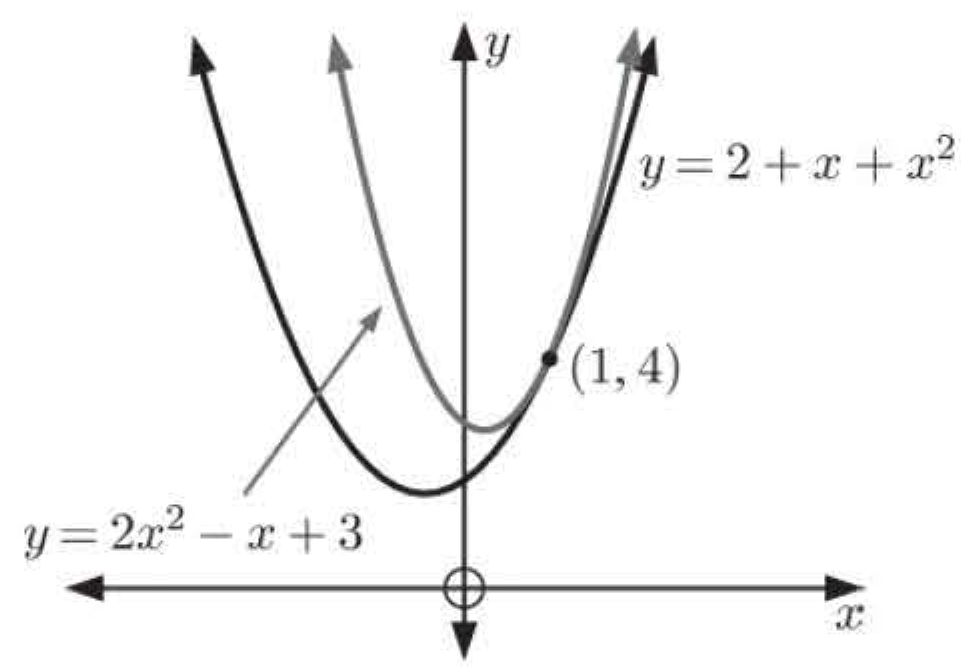
b **i**



$\therefore\ y = x^2 + 2x - 3$ meets $y = x - 1$ at the points $(-2, -3)$ and $(1, 0)$.

ii Using the graph in **b i**, $x^2 + 2x - 3 > x - 1$ when $x < -2$ or $x > 1$.

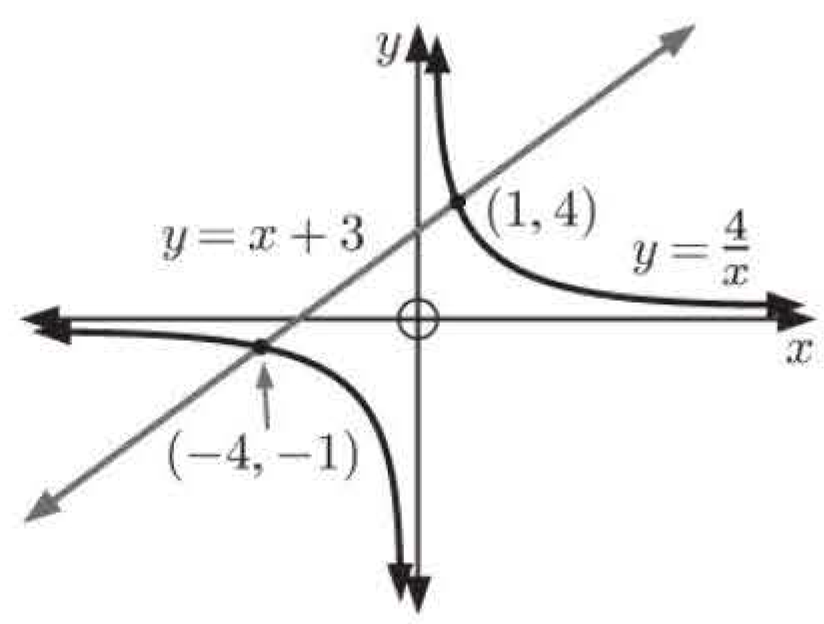
c **i**



$\therefore\ y = 2x^2 - x + 3$ touches $y = 2 + x + x^2$ at the point $(1, 4)$.

ii Using the graph in **c i**, $2x^2 - x + 3 > 2 + x + x^2$ when $x \neq 1$.

d **i**



$\therefore\ y = \dfrac{4}{x}$ meets $y = x + 3$ at the points $(-4, -1)$ and $(1, 4)$.

ii Using the graph in **d i**, $\dfrac{4}{x} > x + 3$ when $x < -4$ or $0 < x < 1$.

3 $y = 3x + c$ is a tangent to $y = x^2 - 5x + 7$ if they meet at exactly one point (touch).
$y = x^2 - 5x + 7$ meets $y = 3x + c$ when $x^2 - 5x + 7 = 3x + c$

$$\therefore\ x^2 - 8x + 7 - c = 0$$

The graphs meet exactly once when this equation has a repeated root $\therefore\ \Delta = 0$

$$\begin{aligned}\therefore\ (-8)^2 - 4(1)(7-c) &= 0\\ \therefore\ 64 - 28 + 4c &= 0\\ \therefore\ 4c &= -36\\ \therefore\ c &= -9\end{aligned}$$

4 $y = mx - 2$ is a tangent to $y = x^2 - 4x + 2$ if they meet at exactly one point (touch).
$y = x^2 - 4x + 2$ meets $y = mx - 2$ when $x^2 - 4x + 2 = mx - 2$

$$\therefore\ x^2 - (m+4)x + 4 = 0$$

The graphs meet exactly once when this equation has a repeated root $\therefore\ \Delta = 0$

$$\begin{aligned}\therefore\ (-(m+4))^2 - 4(1)(4) &= 0\\ \therefore\ m^2 + 8m + 16 - 16 &= 0\\ \therefore\ m(m+8) &= 0\\ \therefore\ m &= 0 \text{ or } -8\end{aligned}$$

5 Lines with y-intercept 1 have the form $y = mx + 1$.
$y = mx + 1$ is a tangent to $y = 3x^2 + 5x + 4$ if they meet at exactly one point (touch).
$y = 3x^2 + 5x + 4$ meets $y = mx + 1$ when $3x^2 + 5x + 4 = mx + 1$

$$\therefore\ 3x^2 + (5-m)x + 3 = 0$$

The graphs meet exactly once when this equation has a repeated root $\therefore\ \Delta = 0$

$$\therefore\ (5-m)^2 - 4(3)(3) = 0$$
$$\therefore\ 25 - 10m + m^2 - 36 = 0$$
$$\therefore\ m^2 - 10m - 11 = 0$$
$$\therefore\ (m+1)(m-11) = 0$$
$$\therefore\ m = -1 \text{ or } 11$$

$\therefore$ the required lines have gradient -1 or 11.

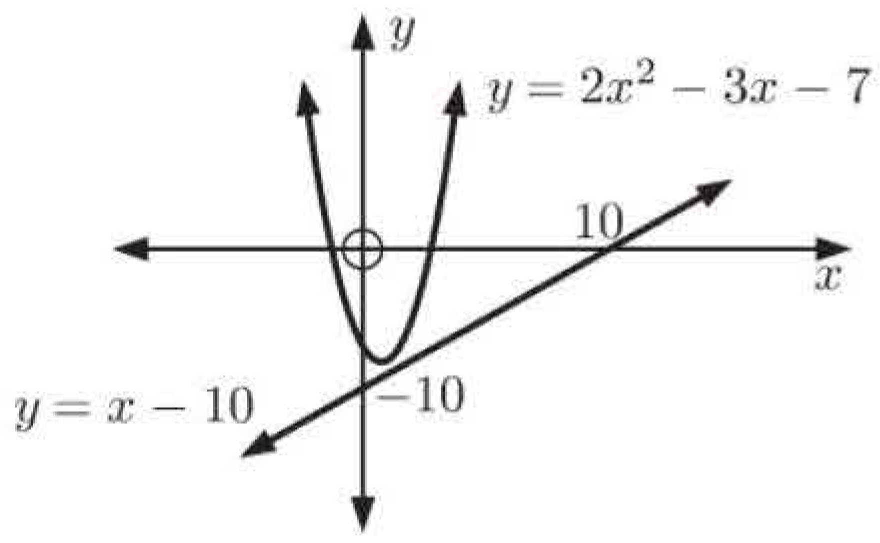
6 **a** $y = x + c$ meets $y = 2x^2 - 3x - 7$ when $2x^2 - 3x - 7 = x + c$

$\therefore\ 2x^2 - 4x - 7 - c = 0$

The graphs will never meet if this equation has no real roots $\therefore\ \Delta < 0$

$$\therefore\ (-4)^2 - 4(2)(-7-c) < 0$$
$$\therefore\ 16 + 56 + 8c < 0$$
$$\therefore\ 8c < -72$$
$$\therefore\ c < -9$$

b Choose c such that $c < -9$, for example $c = -10$:



EXERCISE 1G

1 Let the smaller of the integers be x.

The other integer is $(x+12)$.

$\therefore$ the sum of their squares is

$$x^2 + (x+12)^2 = 74$$
$$\therefore\ x^2 + x^2 + 24x + 144 = 74$$
$$\therefore\ 2x^2 + 24x + 70 = 0$$
$$\therefore\ x^2 + 12x + 35 = 0$$
$$\therefore\ (x+7)(x+5) = 0$$
$$\therefore\ x = -7 \text{ or } -5$$

So, the integers are -7 and 5, or -5 and 7.

2 Let the number be x, so its reciprocal is $\dfrac{1}{x}$.

They have sum $x + \dfrac{1}{x} = 5\frac{1}{5}$

$$\therefore\ x^2 + 1 = \tfrac{26}{5}x$$
$$\therefore\ x^2 - \tfrac{26}{5}x + 1 = 0$$
$$\therefore\ 5x^2 - 26x + 5 = 0$$
$$\therefore\ (5x-1)(x-5) = 0$$
$$\therefore\ x = \tfrac{1}{5} \text{ or } 5$$

So, the number is either $\frac{1}{5}$ or 5.

3 Let the number be x so its square is x^2.

$\therefore$ the sum is $x + x^2 = 210$

$$\therefore\ x^2 + x - 210 = 0$$
$$\therefore\ (x+15)(x-14) = 0$$
$$\therefore\ x = -15 \text{ or } 14$$

But x is a natural number, so $x > 0$,

$\therefore$ the number is 14.

4 Suppose the numbers are x and $(x+2)$.

Then $x(x+2) = 255$

$$\therefore\ x^2 + 2x - 255 = 0$$
$$\therefore\ (x+17)(x-15) = 0$$
$$\therefore\ x = -17 \text{ or } 15$$

$\therefore$ the numbers are -17 and -15, or 15 and 17.

5 If the width of the rectangle is w cm, then its length is $(w+4)$ cm.

$\therefore$ the area is $w(w+4) = 26$

$$\therefore\ w^2 + 4w - 26 = 0$$

which has $a = 1$, $b = 4$, $c = -26$

$$\therefore\ w = \frac{-4 \pm \sqrt{4^2 - 4(1)(-26)}}{2(1)}$$

$$\therefore\ w = \frac{-4 \pm \sqrt{120}}{2} = -2 \pm \sqrt{30}$$

But $w > 0$, so $w = -2 + \sqrt{30}$
≈ 3.477

So, the width is approximately 3.48 cm.

6 **a** The base has sides of length x cm, so the areas of the top and bottom surfaces are both x^2 cm^2.
The box has height $(x+1)$ cm, so the area of each of the side faces is $x(x+1)$ cm^2.
$\therefore$ the total surface area is

$$\begin{aligned} A &= 2x^2 + 4x(x+1) \\ &= 2x^2 + 4x^2 + 4x \\ &= (6x^2 + 4x) \text{ cm}^2 \end{aligned}$$

b

$$\begin{aligned} 6x^2 + 4x &= 240 \\ \therefore\ 3x^2 + 2x - 120 &= 0 \\ \therefore\ (3x+20)(x-6) &= 0 \\ \therefore\ x &= -\tfrac{20}{3} \text{ or } 6 \end{aligned}$$

but $x > 0$, so $x = 6$
$\therefore$ the box is $6 \text{ cm} \times 6 \text{ cm} \times 7 \text{ cm}$.

7 Suppose the tinplate was $x \text{ cm} \times x \text{ cm}$.
When $3 \text{ cm} \times 3 \text{ cm}$ squares are cut from the corners, the base of the open box formed is $(x-6) \text{ cm} \times (x-6) \text{ cm}$.
The open box has height 3 cm, so its volume is $3 \times (x-6) \times (x-6) = 80$

$$\begin{aligned} \therefore\ 3(x^2 - 12x + 36) &= 80 \\ 3x^2 - 36x + 108 &= 80 \\ \therefore\ 3x^2 - 36x + 28 &= 0 \end{aligned}$$

which has $a = 3$, $b = -36$, $c = 28$

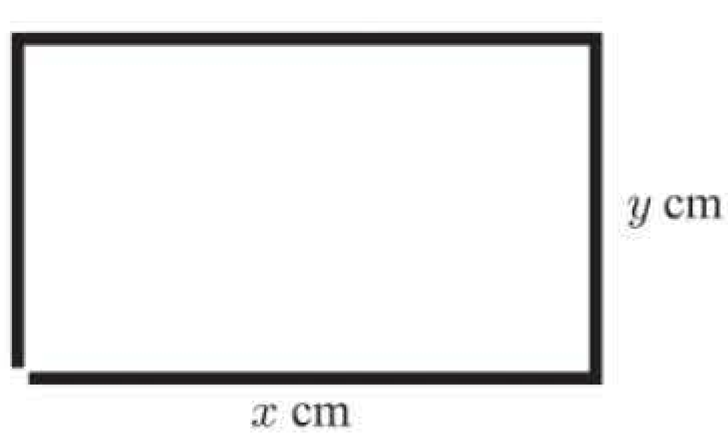
$$\therefore\ x = \frac{-(-36) \pm \sqrt{(-36)^2 - 4(3)(28)}}{2(3)}$$

$$= \frac{36 \pm \sqrt{960}}{6} \quad \text{and since } x > 6,$$

$$x = 6 + \frac{\sqrt{960}}{6} \approx 11.16$$

$\therefore$ the original piece of tinplate was about 11.2 cm square.

8



Suppose one side of the rectangle has length x cm and the other has length y cm.
The perimeter is $(2x + 2y)$ cm,
so $2x + 2y = 20$

$$\begin{aligned} \therefore\ 2y &= 20 - 2x \\ \therefore\ y &= 10 - x \end{aligned}$$

The area of the rectangle is therefore $x(10-x)$ cm^2.

If the area is 30 cm^2, then

$$\begin{aligned} x(10-x) &= 30 \\ \therefore\ 10x - x^2 &= 30 \\ \therefore\ x^2 - 10x + 30 &= 0 \end{aligned}$$

which has $a = 1$, $b = -10$, $c = 30$

$$\therefore\ x = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(30)}}{2(1)}$$

$$= \frac{10 \pm \sqrt{100 - 120}}{2}$$

$$= \frac{10 \pm \sqrt{-20}}{2}$$

$\therefore$ x has no real solutions, so it is not possible.

9 The smaller rectangle is similar to the original rectangle.

$$\therefore\ \frac{\text{AB}}{\text{AD}} = \frac{\text{BC}}{\text{BY}}$$

Suppose $\text{AB} = x$ units, and $\text{AD} = \text{BC} = 1$ unit

$$\therefore\ \frac{x}{1} = \frac{1}{x-1}$$

$$\begin{aligned} \therefore\ x(x-1) &= 1 \\ \therefore\ x^2 - x - 1 &= 0 \end{aligned}$$

which has $a = 1$, $b = -1$, $c = -1$

$$\therefore\ x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{1 \pm \sqrt{1+4}}{2}$$

$$= \frac{1 \pm \sqrt{5}}{2}$$

$$\therefore\ x = \frac{1+\sqrt{5}}{2}, \quad \text{since } x > 0$$

But $x = \dfrac{\text{AB}}{\text{AD}}$, which is the golden ratio

$\therefore$ the golden ratio is $\dfrac{1+\sqrt{5}}{2}$.

10 Suppose the express train travels at x km h^{-1}. We know $\text{speed} = \dfrac{\text{distance}}{\text{time}}$, so $\text{time} = \dfrac{\text{distance}}{\text{speed}}$.

$\therefore$ it takes the express train $\dfrac{160}{x}$ hours and the normal train $\dfrac{160}{x-10}$ hours.

$$\therefore \quad \frac{160}{x} + \tfrac{1}{2} = \frac{160}{x-10}$$

$$\therefore \quad 160(x-10) + \tfrac{1}{2}x(x-10) = 160x$$

$$\therefore \quad 160x - 1600 + \tfrac{1}{2}x^2 - 5x = 160x$$

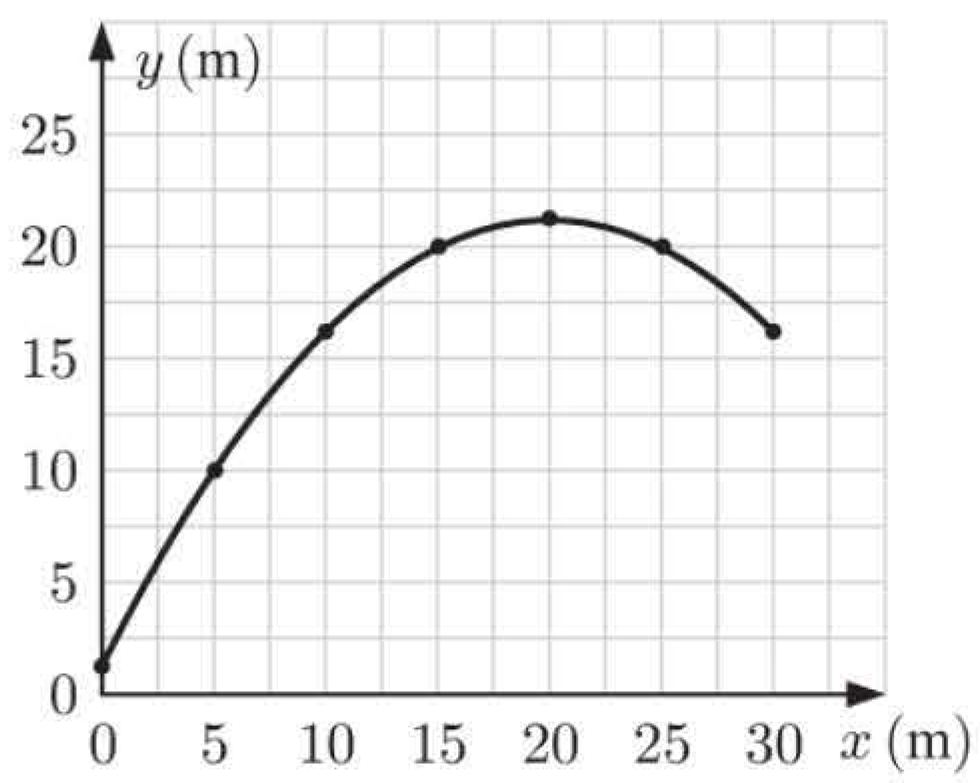
$\therefore \quad x^2 - 10x - 3200 = 0$ which has $a = 1, \quad b = -10, \quad c = -3200$

$$\therefore \quad x = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(-3200)}}{2(1)} = \frac{10 \pm \sqrt{12\,900}}{2}$$

But $x > 0$, so $x = \dfrac{10 + \sqrt{12\,900}}{2} \approx 61.8 \text{ km h}^{-1}$

$\therefore$ the express train travels on average at about 61.8 km h^{-1}.

11 **a**



b The graph is a parabola.

c The maximum height reached by the ball is 21.25 m.

d Let $f(x) = ax^2 + bx + c$ be the form of the formula for height given horizontal distance x.

$f(0) = 1.25$

$\therefore \quad a(0)^2 + b(0) + c = 1.25$

$\therefore \quad c = 1.25$

Also $f(5) = 10$

$\therefore \quad a(5)^2 + b(5) + 1.25 = 10$

$\therefore \quad 25a + 5b = 8.75 \quad (1)$

And $f(15) = 20$

$\therefore \quad a(15)^2 + b(15) + 1.25 = 20$

$\therefore \quad 225a + 15b = 18.75$

$\therefore \quad 75a + 5b = 6.25 \quad \{\div 3\} \quad (2)$

Solving simultaneously (or using technology):

$$\begin{aligned} 75a + 5b &= 6.25 \quad (2) \\ -25a - 5b &= -8.75 \quad -(1) \\ \hline 50a \quad &= -2.5 \end{aligned}$$

$\therefore \quad a = -0.05$

$\therefore \quad -1.25 + 5b = 8.75 \quad$ {substitute in (1)}

$\therefore \quad 5b = 10$

$\therefore \quad b = 2$

So, $f(x) = -0.05x^2 + 2x + 1.25$

e The ball bounces when it hits the ground

$\therefore \quad f(x) = 0$

$\therefore \quad -0.05x^2 + 2x + 1.25 = 0$

Using technology, $x = -0.616$ or $x = 40.6$

But $x \geqslant 0 \quad \therefore$ ball bounces at 40.6 m.

But Badrani is 40 m from Abiola, so the ball will reach Badrani before it bounces.

12 **a** The parabola has vertex $(0, 8)$, so it has equation

$y = a(x-0)^2 + 8$

$\therefore \quad y = ax^2 + 8$

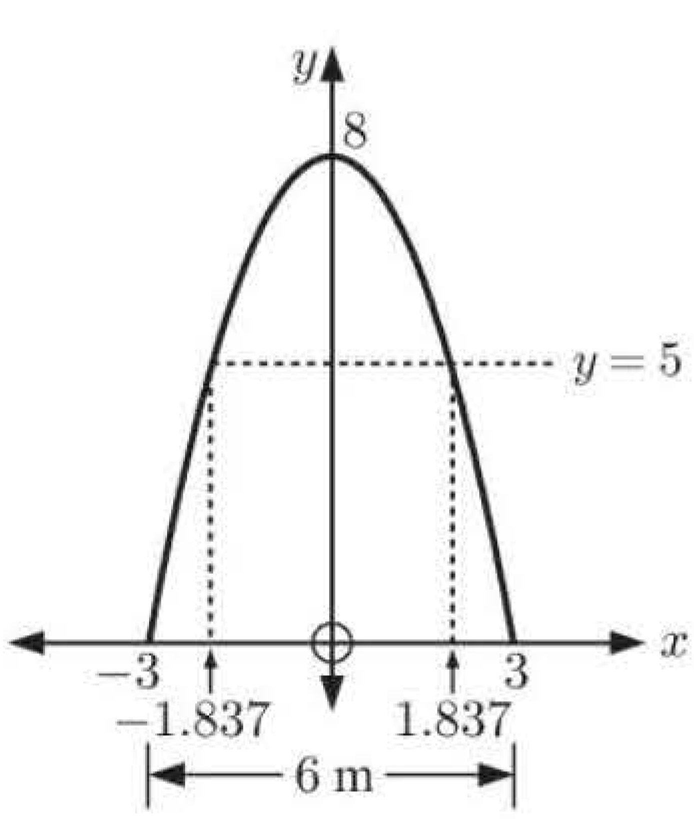
When $x = 3$, $y = 0$, so

$0 = a(3^2) + 8$

$\therefore \quad 9a = -8$

$\therefore \quad a = -\frac{8}{9}$

$\therefore$ the equation of the parabola is $y = -\frac{8}{9}x^2 + 8$.



b The truck is 4 m wide, so we use the equation in **a** to find the width of the tunnel 5 m above ground level.

When $y = 5$, $\quad -\frac{8}{9}x^2 + 8 = 5$

$\therefore \; -\frac{8}{9}x^2 = -3$

$\therefore \; x^2 = \frac{27}{8}$

$\therefore \; x = \pm\sqrt{\frac{27}{8}}$

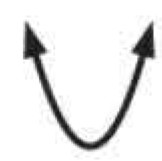
$\therefore \; x \approx \pm 1.837$

So, the tunnel is $2 \times 1.837 \approx 3.67$ m wide, 5 m above ground level. But the truck is 4 m wide.

$\therefore$ the truck will not fit through the tunnel.

EXERCISE 1H

1 **a** For $y = x^2 - 2x$,
$a = 1, \quad b = -2, \quad c = 0.$

As $a > 0$, the shape is 

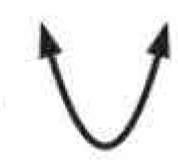
$\therefore$ the minimum value occurs when

$$x = \frac{-b}{2a} = \frac{2}{2} = 1$$

and $y = 1^2 - 2(1) = -1$

$\therefore$ the minimum value of $y = x^2 - 2x$ is -1, occurring when $x = 1$.

b For $y = 4x^2 - x + 5$,
$a = 4, \quad b = -1, \quad c = 5.$

As $a > 0$, the shape is 

$\therefore$ the minimum value occurs when

$$x = \frac{-b}{2a} = \frac{1}{8}$$

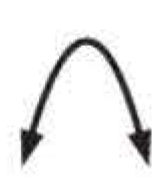
and $y = 4(\frac{1}{8})^2 - \frac{1}{8} + 5$

$= \frac{1}{16} - \frac{1}{8} + 5$

$= 4\frac{15}{16}$

$\therefore$ the minimum value of $y = 4x^2 - x + 5$ is $4\frac{15}{16}$, occuring when $x = \frac{1}{8}$.

c For $y = 7x - 2x^2$,
$a = -2, \quad b = 7, \quad c = 0.$

As $a < 0$, the shape is 

$\therefore$ the maximum value occurs when

$$x = \frac{-b}{2a} = \frac{-7}{-4} = \frac{7}{4}$$

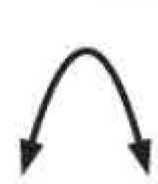
and $y = 7(\frac{7}{4}) - 2(\frac{7}{4})^2$

$= \frac{49}{4} - \frac{49}{8}$

$= \frac{49}{8}$ or $6\frac{1}{8}$

$\therefore$ the maximum value of $y = 7x - 2x^2$ is $6\frac{1}{8}$, occuring when $x = \frac{7}{4}$.

2 **a** For $P = -3x^2 + 240x - 800$,
$a = -3, \quad b = 240, \quad c = -800.$

As $a < 0$, the shape is 

$\therefore$ the maximum profit occurs when

$$x = \frac{-b}{2a} = \frac{-240}{-6} = 40$$

$\therefore$ 40 refrigerators should be made each day to maximise the total profit.

b $P = -3(40)^2 + 240(40) - 800$

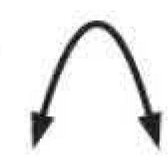
$= 4000$

$\therefore$ the maximum profit is \$4000.

3 **a** Let the other side be y m long.
The perimeter is 200 m.

$\therefore \quad 2x + 2y = 200$
$\therefore \quad x + y = 100$
$\therefore \quad y = 100 - x$
$\therefore$ the area $A = xy$
$\therefore \quad A = x(100 - x)$
$\therefore \quad A = 100x - x^2$

b $A = 100x - x^2$ is a quadratic function with $a = -1, \quad b = 100, \quad c = 0.$

As $a < 0$, the shape is 

$\therefore$ the area is maximised when

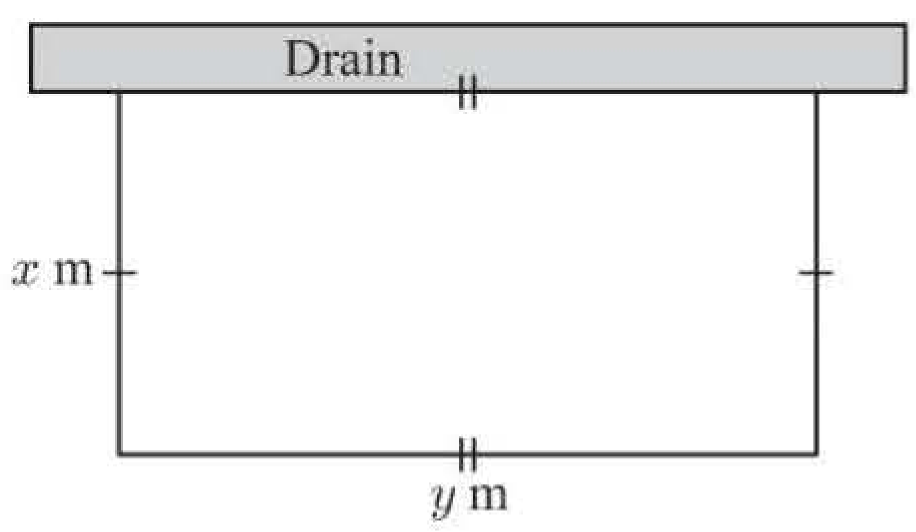
$$x = \frac{-b}{2a} = \frac{-100}{-2} = 50$$

and $y = 100 - 50 = 50$

$\therefore$ the area of the rectangle is maximised when $x = y = 50$, which is when the rectangle is a square.

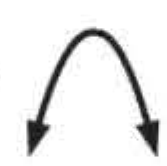
4 Let the dimensions of the paddock be x m $\times$ y m.
If 1000 m of fence is available, then

$2x + y = 1000 \qquad$ {perimeter}
$\therefore \quad y = 1000 - 2x \quad .... (1)$



The area of the enclosure $A = xy$

Since $y = 1000 - 2x, \quad A = x(1000 - 2x)$
$= 1000x - 2x^2$
$\therefore \quad A = -2x^2 + 1000x$

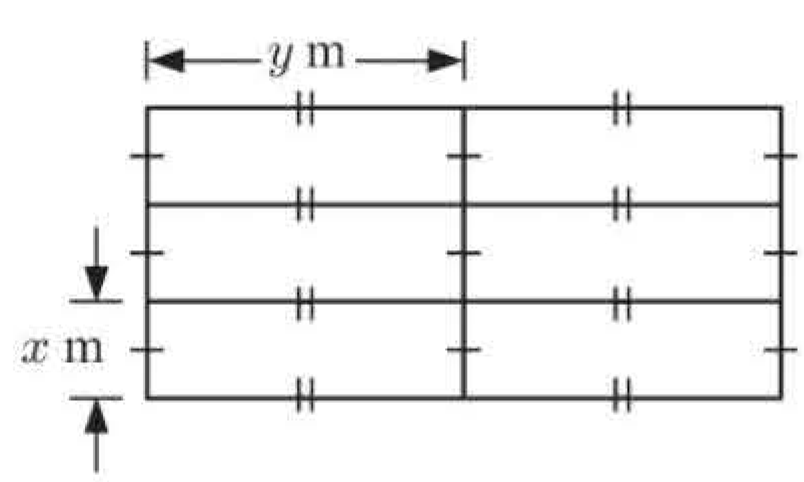
A is a quadratic and $a < 0$, so its shape is 

So, area is maximised when $x = \dfrac{-b}{2a} = \dfrac{-1000}{2 \times (-2)} = 250$

and when $x = 250, \; y = 1000 - 2(250) = 500$

$\therefore$ the paddock has a maximum area when the dimensions are 250 m $\times$ 500 m.

5 **a**



The length of fence required for this enclosure is $9x + 8y$. If 1800 m is available for this enclosure, then $9x + 8y = 1800$.

b If $9x + 8y = 1800$, then $y = \dfrac{1800 - 9x}{8}$.

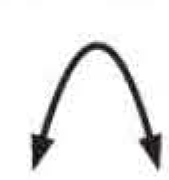
The area of each pen is $A = xy$.

Substituting $y = \dfrac{1800 - 9x}{8}$ into A we get

$$A = x\left(\frac{1800 - 9x}{8}\right)$$

$$\therefore \quad A = \frac{1800x}{8} - \frac{9x^2}{8}$$

$$\therefore \quad A = -\tfrac{9}{8}x^2 + 225x \;\; \text{m}^2$$

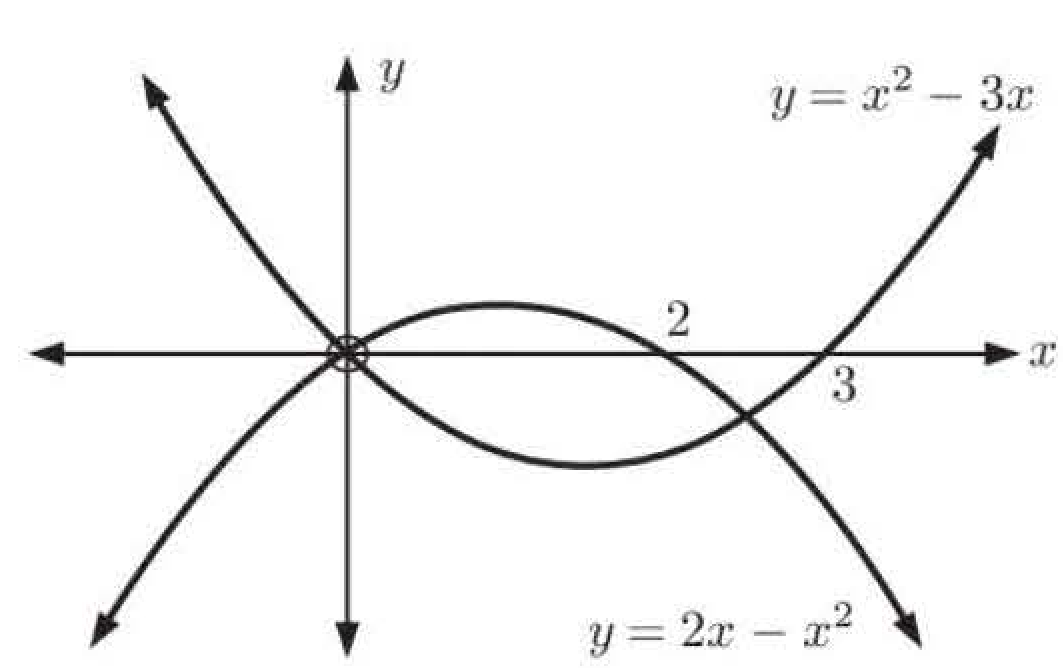
c The area is a quadratic function with $a < 0$, so its shape is 

So, at $x = \dfrac{-b}{2a}$ we have a maximum

$\therefore \quad x = \dfrac{-225}{2 \times (-\frac{9}{8})} = 100$, and when $x = 100, \quad y = \dfrac{1800 - 9(100)}{8} = 112.5$

Hence, the area is maximised when the dimensions are 100 m $\times$ 112.5 m.

6 **a** The graphs of $y = x^2 - 3x$ and $y = 2x - x^2$
meet where $x^2 - 3x = 2x - x^2$
$\therefore \quad 2x^2 - 5x = 0$
$\therefore \quad x(2x - 5) = 0$
$\therefore \quad x = 0 \text{ or } 2\frac{1}{2}$

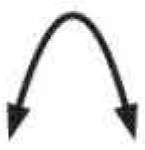


b The vertical separation between the curves is given by

$S = (2x - x^2) - (x^2 - 3x)$ $\{y = 2x - x^2$ is above $y = x^2 - 3x$ for $0 \leqslant x \leqslant 2\frac{1}{2}\}$

$\therefore$ $S = 2x - x^2 - x^2 + 3x$

$\therefore$ $S = -2x^2 + 5x$

Thus S is a quadratic function with $a < 0$ so the shape is 

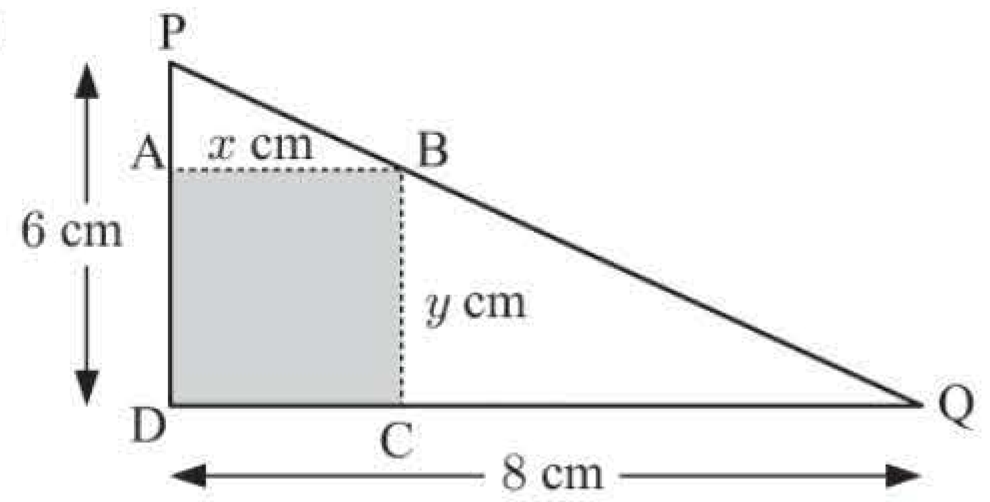
$\therefore$ the maximum separation occurs when $x = \dfrac{-b}{2a} = \dfrac{-5}{-4} = \frac{5}{4}$

and $S = -2(\frac{5}{4})^2 + 5(\frac{5}{4})$

$= -\frac{25}{8} + \frac{25}{4} = \frac{25}{8}$ or $3\frac{1}{8}$

$\therefore$ the maximum vertical separation between the curves for $0 \leqslant x \leqslant 2\frac{1}{2}$ is $3\frac{1}{8}$ units.

7 **a**



$\triangle$s PAB and PDQ are similar

$\{\widehat{APB}$ is common, $\widehat{ABP} = \widehat{DQP}$ as $[AB] \parallel [DQ]\}$

$\therefore$ $\dfrac{PA}{PD} = \dfrac{AB}{DQ}$

$\therefore$ $\dfrac{6-y}{6} = \dfrac{x}{8}$

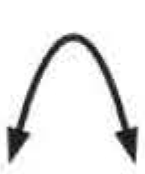
$\therefore$ $6 - y = \frac{3}{4}x$

$\therefore$ $y = 6 - \frac{3}{4}x$

b Rectangle ABCD has area $A = xy$

$= x(6 - \frac{3}{4}x)$

$= -\frac{3}{4}x^2 + 6x$

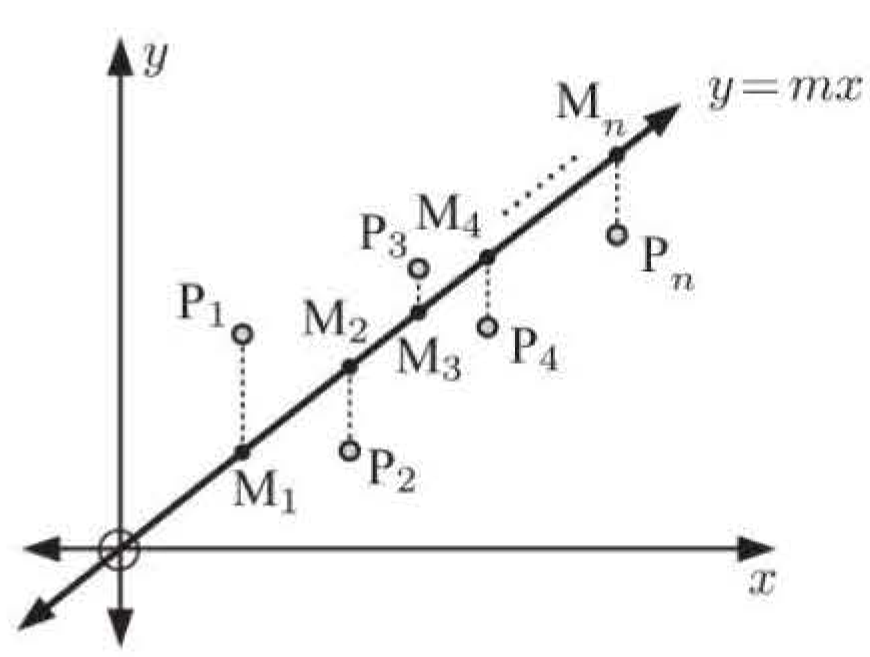
which is a quadratic with $a < 0$ $\therefore$ the shape is 

$\therefore$ the area is maximised when $x = \dfrac{-b}{2a} = \dfrac{-6}{-\frac{3}{2}} = 4$

and when $x = 4$, $y = 6 - \frac{3}{4}(4) = 3$

$\therefore$ the dimensions of rectangle ABCD of maximum area are 4 cm $\times$ 3 cm.

8 Let the 'line of best fit' through (0, 0) have slope m.



$\therefore$ the line has equation $y = mx$.

$\therefore$ for $P_1(a_1, b_1)$, the coordinates of M_1 are (a_1, ma_1).

$\therefore$ the distance between P_1 and M_1 is $b_1 - ma_1$.

In general, $P_iM_i = |b_i - ma_i|$, $i = 1, 2, ..., n$.

$\therefore$ $(P_1M_1)^2 + (P_2M_2)^2 + ... + (P_nM_n)^2$

$= |b_1 - ma_1|^2 + |b_2 - ma_2|^2 + ... + |b_n - ma_n|^2$

$= (b_1 - ma_1)^2 + (b_2 - ma_2)^2 + ... + (b_n - ma_n)^2$ $\{|z|^2 = z^2\}$

$= b_1^2 - 2b_1ma_1 + m^2a_1^2 + b_2^2 - 2b_2ma_2 + m^2a_2^2 + ... + b_n^2 - 2b_nma_n^2 + m^2a_n^2$

$= m^2(a_1^2 + a_2^2 + ... + a_n^2) - m(2a_1b_1 + 2a_2b_2 + ... + 2a_nb_n) + (b_1^2 + b_2^2 + ... + b_n^2)$

which is a quadratic in m, with $a = a_1^2 + a_2^2 + ... + a_n^2$, $b = -(2a_1b_1 + 2a_2b_2 + ... + 2a_nb_n)$, $c = b_1^2 + b_2^2 + ... + b_n^2$.

$a = a_1^2 + a_2^2 + ... + a_n^2 > 0$, so the quadratic has shape 

$\therefore$ the sum $(P_1M_1)^2 + (P_2M_2)^2 + ... + (P_nM_n)^2$ is minimised when

$$m = \frac{-b}{2a} = \frac{2a_1b_1 + 2a_2b_2 + ... + 2a_nb_n}{2a_1^2 + 2a_2^2 + ... + 2a_n^2}$$

$$\therefore\ m = \frac{a_1b_1 + a_2b_2 + ... + a_nb_n}{a_1^2 + a_2^2 + ... + a_n^2}$$

9 $y = (x-a-b)(x-a+b)(x+a-b)(x+a+b)$

$$\begin{aligned}
&= [x-(a+b)][x+(a+b)][x-(a-b)][x+(a-b)] \qquad \{\text{rearranging}\}\\
&= [x^2-(a+b)^2][x^2-(a-b)^2]\\
&= x^4 - x^2(a-b)^2 - x^2(a+b)^2 + (a+b)^2(a-b)^2\\
&= x^4 - x^2[(a-b)^2+(a+b)^2] + [(a+b)(a-b)]^2\\
&= x^4 - x^2(a^2 \cancel{-2ab} + b^2 + a^2 \cancel{+2ab} + b^2) + (a^2-b^2)^2\\
&= x^4 - 2(a^2+b^2)x^2 + (a^2-b^2)^2
\end{aligned}$$

which is a quadratic in x^2 with "a" $= 1$, "b" $= -2(a^2+b^2)$, "c" $= (a^2-b^2)^2$

"a" > 0, so the quadratic has shape 

$\therefore$ y is minimised when $x^2 = \dfrac{2(a^2+b^2)}{2} = a^2+b^2$

When $x^2 = a^2+b^2$,

$$\begin{aligned}
y &= (a^2+b^2)^2 - 2(a^2+b^2)(a^2+b^2) + (a^2-b^2)^2\\
&= (a^2+b^2)^2 - 2(a^2+b^2)^2 + (a^2-b^2)^2\\
&= (a^2-b^2)^2 - (a^2+b^2)^2\\
&= a^4 - 2a^2b^2 + b^4 - a^4 - 2a^2b^2 - b^4\\
&= -4a^2b^2
\end{aligned}$$

$\therefore$ the least value of y is $-4a^2b^2$.

10 $y = (a_1x-b_1)^2 + (a_2x-b_2)^2$

$$\begin{aligned}
&= a_1^2x^2 - 2a_1b_1x + b_1^2 + a_2^2x^2 - 2a_2b_2x + b_2^2\\
&= (a_1^2+a_2^2)x^2 - 2(a_1b_1+a_2b_2)x + (b_1^2+b_2^2)
\end{aligned}$$

which is a quadratic in x with $a = a_1^2 + a_2^2 > 0$, so it has shape 

$\therefore$ y is minimised when $x = \dfrac{-b}{2a} = \dfrac{2(a_1b_1+a_2b_2)}{2(a_1^2+a_2^2)} = \dfrac{a_1b_1+a_2b_2}{a_1^2+a_2^2}$

When $x = \dfrac{a_1b_1+a_2b_2}{a_1^2+a_2^2}$,

$$\begin{aligned}
y &= \left(a_1^2+a_2^2\right)\left(\frac{a_1b_1+a_2b_2}{a_1^2+a_2^2}\right)^2 - 2(a_1b_1+a_2b_2)\left(\frac{a_1b_1+a_2b_2}{a_1^2+a_2^2}\right) + b_1^2 + b_2^2\\
&= \frac{(a_1b_1+a_2b_2)^2}{a_1^2+a_2^2} - \frac{2(a_1b_1+a_2b_2)^2}{a_1^2+a_2^2} + b_1^2 + b_2^2\\
&= b_1^2 + b_2^2 - \frac{(a_1b_1+a_2b_2)^2}{a_1^2+a_2^2}
\end{aligned}$$

But since $y = (a_1x-b_1)^2 + (a_2x-b_2)^2$, $y \geqslant 0$ for all x {sum of 2 squared terms}

$$\begin{aligned}
\therefore\quad b_1^2 + b_2^2 - \frac{(a_1b_1+a_2b_2)^2}{a_1^2+a_2^2} &\geqslant 0\\
\therefore\quad b_1^2 + b_2^2 &\geqslant \frac{(a_1b_1+a_2b_2)^2}{a_1^2+a_2^2}\\
\therefore\quad (a_1^2+a_2^2)(b_1^2+b_2^2) &\geqslant (a_1b_1+a_2b_2)^2 \quad \{a_1^2+a_2^2 \geqslant 0\}\\
\therefore\quad \sqrt{a_1^2+a_2^2}\sqrt{b_1^2+b_2^2} &\geqslant \sqrt{(a_1b_1+a_2b_2)^2}\\
\therefore\quad |a_1b_1+a_2b_2| &\leqslant \sqrt{a_1^2+a_2^2}\sqrt{b_1^2+b_2^2}
\end{aligned}$$

11 Suppose one of the equations, say $x^2+b_1x+c_1=0$ does not have two real roots.

$$\therefore \text{ its discriminant } \Delta<0$$
$$\therefore b_1^{\,2}-4(1)(c_1)<0$$
$$\therefore b_1^{\,2}<4c_1 \quad .... (1)$$
$$\therefore \left(\frac{2(c_1+c_2)}{b_2}\right)^2<4c_1 \qquad \{b_1b_2=2(c_1+c_2)\}$$
$$\therefore \frac{\cancel{4}(c_1+c_2)^2}{b_2^{\,2}}<\cancel{4}c_1$$
$$\therefore c_1^{\,2}+2c_1c_2+c_2^{\,2}<c_1b_2^{\,2} \qquad \{b_2^{\,2}>0\}$$
$$\therefore c_1^{\,2}-2c_1c_2+c_2^{\,2}<c_1b_2^{\,2}-4c_1c_2 \qquad \{\text{subtract } 4c_1c_2 \text{ from both sides}\}$$
$$\therefore (c_1-c_2)^2<c_1(b_2^{\,2}-4c_2)$$
$$\therefore c_1(b_2^{\,2}-4c_2)>(c_1-c_2)^2$$

Now $(c_1-c_2)^2\geqslant 0 \quad \therefore c_1(b_2^{\,2}-4c_2)>0$

We know $c_1>0 \qquad \{4c_1>b_1^{\,2}>0 \text{ using (1)}\}$

$$\therefore b_2^{\,2}-4c_2>0$$

$\therefore$ $x^2+b_2x+c_2$ has two real roots $\{b_2^{\,2}-4c_2$ is the discriminant$\}$

$\therefore$ if one of the equations does not have two real roots, the other equation does have two real roots.

$\therefore$ at least one of the equations has two real roots.

REVIEW SET 1A

1 **a** The x-intercepts are -2 and 1.

b The axis of symmetry lies midway between the x-intercepts, so its equation is $x=-\frac{1}{2}$.

c When $x=0$, $y=-2(2)(-1)=4$

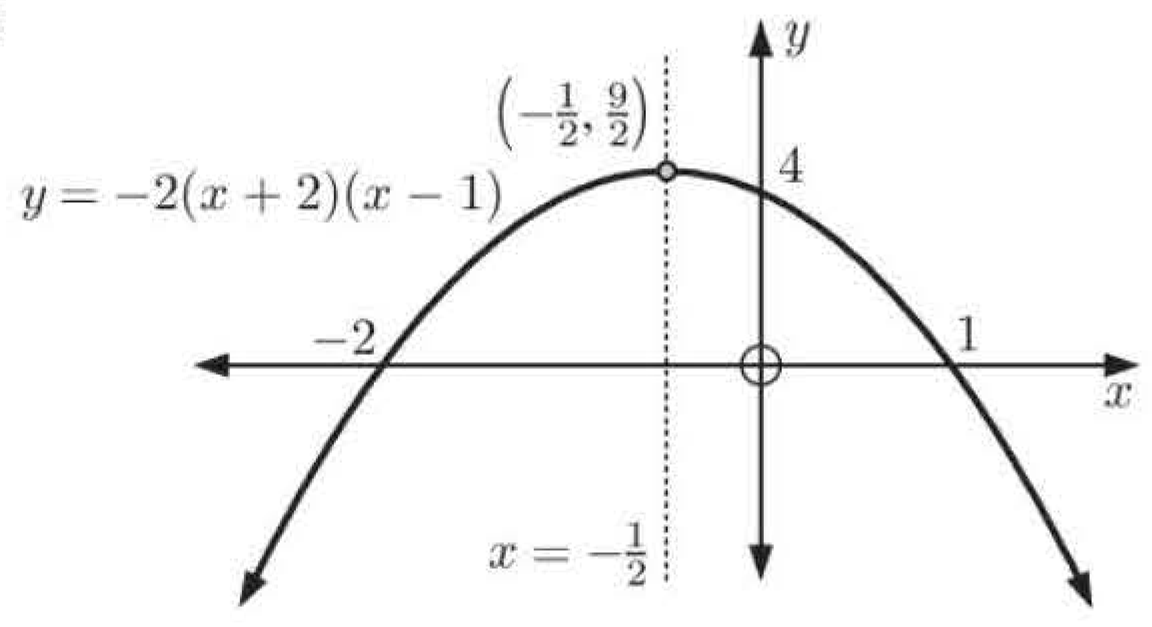
$\therefore$ the y-intercept is 4

d When $x=-\frac{1}{2}$, $y=-2(-\frac{1}{2}+2)(-\frac{1}{2}-1)$

$=-2(\frac{3}{2})(-\frac{3}{2})=\frac{9}{2}$

$\therefore$ the vertex is $(-\frac{1}{2},\frac{9}{2})$.

e



2 **a** $3x^2-12x=0$

$\therefore 3x(x-4)=0$

$\therefore x=0$ or 4

b $3x^2-x-10=0$

$\therefore (3x+5)(x-2)=0$

$\therefore x=-\frac{5}{3}$ or 2

c $x^2-11x=60$

$\therefore x^2-11x-60=0$

$\therefore (x+4)(x-15)=0$

$\therefore x=-4$ or 15

3 **a** $x^2+5x+3=0$

has $a=1$, $b=5$, $c=3$

$$\therefore x=\frac{-5\pm\sqrt{5^2-4(1)(3)}}{2(1)}$$

$$\therefore x=-\tfrac{5}{2}\pm\tfrac{\sqrt{13}}{2}$$

b $3x^2+11x-2=0$

has $a=3$, $b=11$, $c=-2$

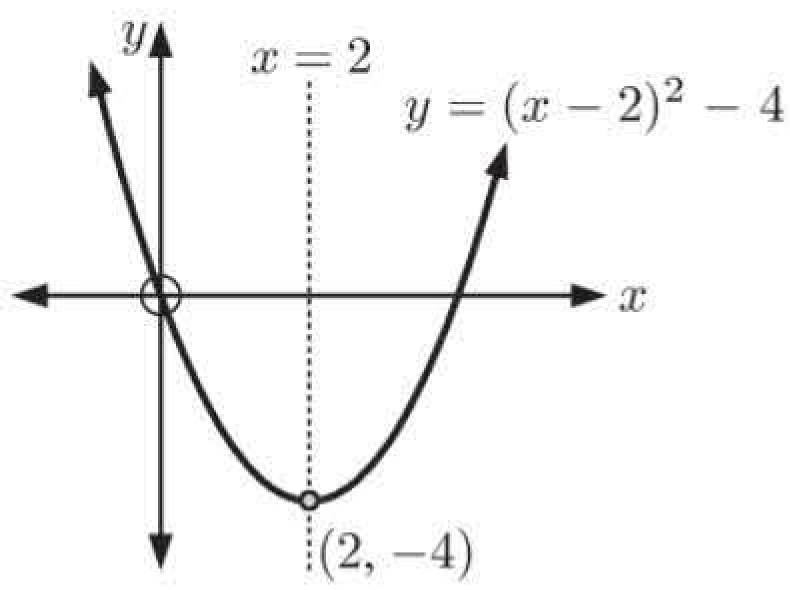
$$\therefore x=\frac{-11\pm\sqrt{11^2-4(3)(-2)}}{2(3)}$$

$$\therefore x=-\tfrac{11}{6}\pm\tfrac{\sqrt{145}}{6}$$

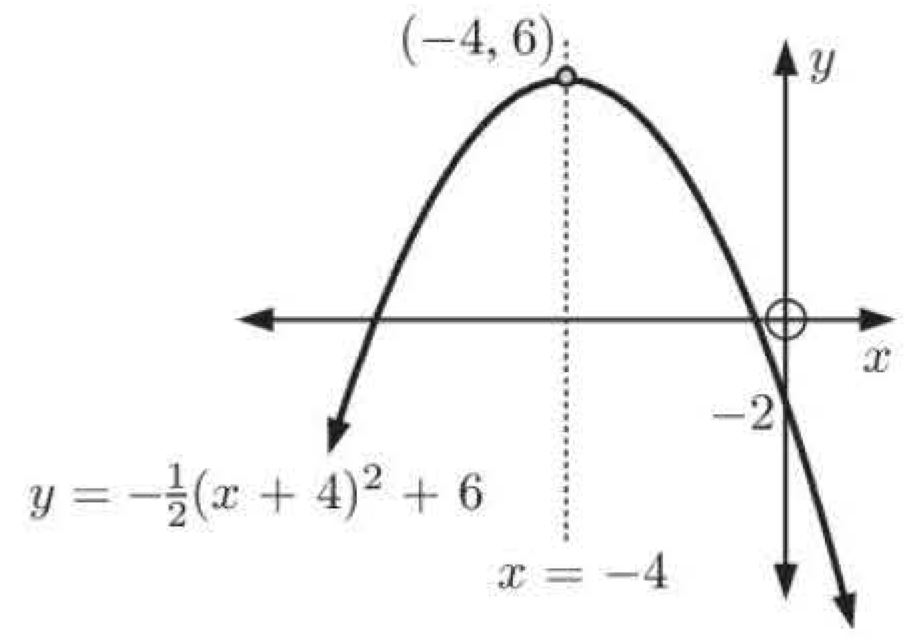
4

$$x^2+7x-4=0$$
$$\therefore x^2+7x+(\tfrac{7}{2})^2-(\tfrac{7}{2})^2-4=0$$
$$\therefore (x+\tfrac{7}{2})^2-\tfrac{49}{4}-4=0$$
$$\therefore (x+\tfrac{7}{2})^2=\tfrac{65}{4}$$
$$\therefore x+\tfrac{7}{2}=\pm\tfrac{\sqrt{65}}{2}$$
$$\therefore x=-\tfrac{7}{2}\pm\tfrac{\sqrt{65}}{2}$$

5 **a** $y = (x-2)^2 - 4$ has vertex $(2, -4)$
and axis of symmetry $x = 2$.
When $x = 0$, $y = (-2)^2 - 4 = 0$
so the y-intercept is 0.



b $y = -\frac{1}{2}(x+4)^2 + 6$ has vertex $(-4, 6)$
and axis of symmetry $x = -4$.
When $x = 0$, $y = -\frac{1}{2}(4)^2 + 6 = -2$
so the y-intercept is -2.



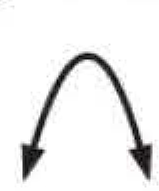
6 **a** The graph touches the x-axis at 4, so its vertex is $(4, 0)$.
$\therefore$ its equation is $y = a(x-4)^2$ for some $a \neq 0$.
The graph also passes through $(2, 12)$ $\therefore$ $a(2-4)^2 = 12$
$\therefore$ $4a = 12$
$\therefore$ $a = 3$

$\therefore$ the equation is $y = 3(x-4)^2$ which is $y = 3(x^2 - 8x + 16)$
or $y = 3x^2 - 24x + 48$

b The quadratic has vertex $(-4, 1)$, so its equation is $y = a(x+4)^2 + 1$ for some $a \neq 0$.
The graph also passes through $(1, 11)$ $\therefore$ $11 = a(1+4)^2 + 1$
$\therefore$ $25a = 10$
$\therefore$ $a = \frac{2}{5}$

$\therefore$ the equation is $y = \frac{2}{5}(x+4)^2 + 1$ which is $y = \frac{2}{5}(x^2 + 8x + 16) + 1$
or $y = \frac{2}{5}x^2 + \frac{16}{5}x + \frac{37}{5}$

7 $y = -2x^2 + 4x + 3$ has $a = -2$, $b = 4$, $c = 3$

Since $a < 0$, the graph has shape  and will have a maximum.

The axis of symmetry is $x = -\dfrac{b}{2a} = -\dfrac{4}{2(-2)} = 1$

When $x = 1$, $y = -2(1)^2 + 4(1) + 3$
$= 5$

$\therefore$ the maximum is 5, and this occurs when $x = 1$.

8 The roots of $2x^2 - 3x = 4$ or $2x^2 - 3x - 4 = 0$ are α and β.

sum of roots $= -\dfrac{b}{a}$ and product of roots $= \dfrac{c}{a}$

$\therefore$ $\alpha + \beta = -\dfrac{-3}{2}$ $\therefore$ $\alpha\beta = \dfrac{-4}{2}$

$\therefore$ $\alpha + \beta = \frac{3}{2}$ $\therefore$ $\alpha\beta = -2$

Now consider a quadratic equation with roots $\dfrac{1}{\alpha}$ and $\dfrac{1}{\beta}$.

$$\text{sum of roots} = \frac{1}{\alpha}+\frac{1}{\beta} = \frac{\beta}{\alpha\beta}+\frac{\alpha}{\alpha\beta} = \frac{\alpha+\beta}{\alpha\beta} = \frac{\frac{3}{2}}{-2} = -\tfrac{3}{4}$$

and

$$\text{product of roots} = \frac{1}{\alpha}\times\frac{1}{\beta} = \frac{1}{\alpha\beta} = \frac{1}{-2} = -\tfrac{1}{2}$$

$\therefore$ a quadratic equation $ax^2+bx+c=0$ with roots $\dfrac{1}{\alpha}$ and $\dfrac{1}{\beta}$ has $-\dfrac{b}{a} = -\frac{3}{4}$ and $\dfrac{c}{a} = -\frac{1}{2}$

$\therefore$ $b = \frac{3}{4}a$ and $c = -\frac{1}{2}a$

The simplest solution to this is $a = 4$, $\therefore$ $b = 3$ and $c = -2$.

$\therefore$ the simplest quadratic equation with roots $\dfrac{1}{\alpha}$ and $\dfrac{1}{\beta}$ is $4x^2+3x-2=0$.

9 **a**
$$\begin{aligned} x^2+10 &= 7x \\ \therefore\ x^2-7x+10 &= 0 \\ \therefore\ (x-2)(x-5) &= 0 \\ \therefore\ x &= 2 \text{ or } 5 \end{aligned}$$

b
$$\begin{aligned} x+\frac{12}{x} &= 7 \\ \therefore\ x^2+12 &= 7x \\ \therefore\ x^2-7x+12 &= 0 \\ \therefore\ (x-3)(x-4) &= 0 \\ \therefore\ x &= 3 \text{ or } 4 \end{aligned}$$

c
$$\begin{aligned} 2x^2-7x+3 &= 0 \\ \therefore\ (2x-1)(x-3) &= 0 \\ \therefore\ x &= \tfrac{1}{2} \text{ or } 3 \end{aligned}$$

10 $y = x^2-3x$ meets $y = 3x^2-5x-24$

$$\begin{aligned} \text{when } x^2-3x &= 3x^2-5x-24 \\ \therefore\ 2x^2-2x-24 &= 0 \\ \therefore\ x^2-x-12 &= 0 \\ \therefore\ (x-4)(x+3) &= 0 \\ \therefore\ x &= 4 \text{ or } -3 \end{aligned}$$

Substituting into $y = x^2-3x$,
when $x = 4$, $y = 4^2-3\times4 = 4$
and when $x=-3$, $y = (-3)^2-3(-3) = 9+9 = 18$

$\therefore$ the graphs meet at $(4, 4)$ and $(-3, 18)$.

11 $y = -2x^2+5x+k$
has $a=-2$, $b=5$, $c=k$.

$$\begin{aligned} \therefore\ \Delta &= b^2-4ac \\ &= 5^2-4(-2)k \\ &= 25+8k \end{aligned}$$

The graph does not cut the x-axis if $\Delta < 0$

$$\begin{aligned} \therefore\ 25+8k &< 0 \\ \therefore\ 8k &< -25 \\ \therefore\ k &< -\tfrac{25}{8} \end{aligned}$$

So, $k < -3\frac{1}{8}$

12 $2x^2-3x+m=0$
has $a=2$, $b=-3$, $c=m$

$$\begin{aligned} \therefore\ \Delta &= b^2-4ac \\ &= (-3)^2-4(2)m \\ &= 9-8m \end{aligned}$$

a There is a repeated root if $\Delta = 0$
$$\begin{aligned} \therefore\ 9-8m &= 0 \\ \therefore\ m &= \tfrac{9}{8} \end{aligned}$$

b There are two distinct real roots if $\Delta > 0$
$$\begin{aligned} \therefore\ 9-8m &> 0 \\ \therefore\ 8m &< 9 \\ \therefore\ m &< \tfrac{9}{8} \end{aligned}$$

c There are no real roots if $\Delta < 0$
$$\begin{aligned} \therefore\ 9-8m &< 0 \\ \therefore\ 8m &> 9 \\ \therefore\ m &> \tfrac{9}{8} \end{aligned}$$

13 Let the number be x, so its reciprocal is $\dfrac{1}{x}$.

$$\begin{aligned} \therefore\ x+\frac{1}{x} &= 2\tfrac{1}{30} = \tfrac{61}{30} \\ \therefore\ x^2+1 &= \tfrac{61}{30}x \\ \therefore\ 30x^2+30 &= 61x \\ \therefore\ 30x^2-61x+30 &= 0 \\ \therefore\ (6x-5)(5x-6) &= 0 \\ \therefore\ x &= \tfrac{5}{6} \text{ or } \tfrac{6}{5} \end{aligned}$$

$\therefore$ the number is $\frac{5}{6}$ or $\frac{6}{5}$

14 Let the line with y-intercept $(0, 10)$ have equation $y = mx + 10$.

$y = 3x^2 + 7x - 2$ meets this line when $3x^2 + 7x - 2 = mx + 10$

$\therefore\ 3x^2 + (7 - m)x - 12 = 0$

For $y = mx + 10$ to be tangential to $y = 3x^2 + 7x - 2$, this equation must have exactly one solution, so there is a repeated root.

$$\therefore\ \Delta = 0$$
$$\therefore\ (7-m)^2 - 4(3)(-12) = 0$$
$$\therefore\ 49 - 14m + m^2 + 144 = 0$$
$$\therefore\ m^2 - 14m + 193 = 0$$
$$\therefore\ m = \frac{14 \pm \sqrt{(-14)^2 - 4(1)(193)}}{2}$$
$$\therefore\ m = \frac{14 \pm \sqrt{-576}}{2} \quad \text{which has no real solutions}$$

$\therefore$ no line with y-intercept $(0, 10)$ can be tangential to $y = 3x^2 + 7x - 2$.

15 $kx^2 + (1 - 3k)x + (k - 6) = 0$ has $a = k$, $b = 1 - 3k$ and $c = k - 6$.

Let the roots be α and $-\dfrac{1}{\alpha}$.

$\therefore$ product of roots $= -1$

$$\therefore\ \frac{c}{a} = \frac{k-6}{k} = -1$$
$$\therefore\ k - 6 = -k$$
$$\therefore\ 2k = 6$$
$$\therefore\ k = 3$$

For $k = 3$ the equation is

$$3x^2 + (1 - 3(3))x + (3 - 6) = 0$$
$$\therefore\ 3x^2 - 8x - 3 = 0$$
$$\therefore\ (3x+1)(x-3) = 0$$
$$\therefore\ x = -\tfrac{1}{3} \text{ or } 3$$

$\therefore\ k = 3$, and the two roots of the equation are $-\frac{1}{3}$ and 3.

REVIEW SET 1B

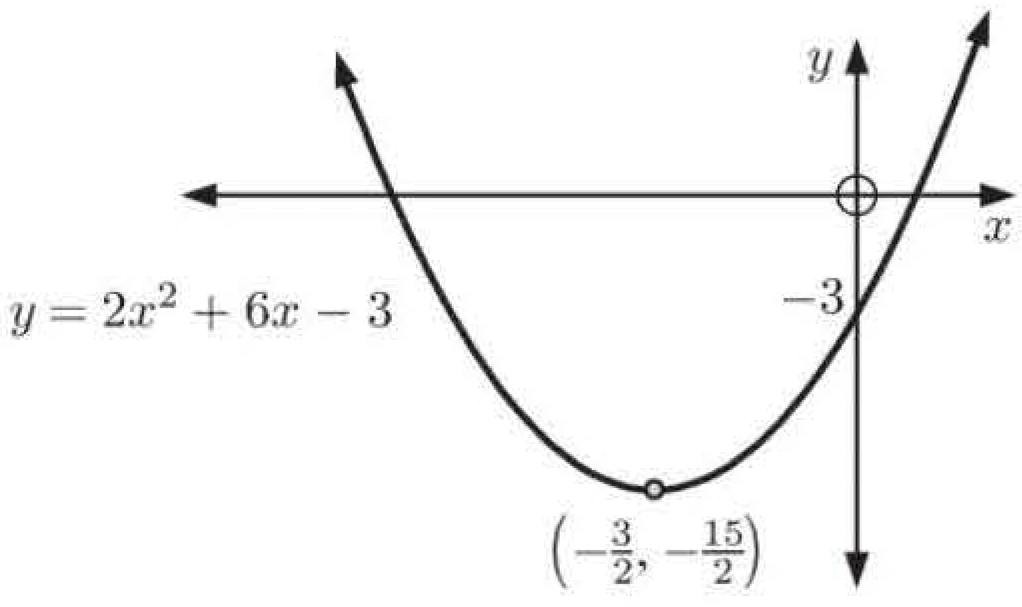
1 **a**
$$\begin{aligned} y &= 2x^2 + 6x - 3 \\ &= 2[x^2 + 3x - \tfrac{3}{2}] \\ &= 2[x^2 + 3x + (\tfrac{3}{2})^2 - (\tfrac{3}{2})^2 - \tfrac{3}{2}] \\ &= 2[(x + \tfrac{3}{2})^2 - \tfrac{9}{4} - \tfrac{3}{2}] \\ &= 2[(x + \tfrac{3}{2})^2 - \tfrac{15}{4}] \\ &= 2(x + \tfrac{3}{2})^2 - \tfrac{15}{2} \end{aligned}$$

b The vertex is $(-\frac{3}{2}, -\frac{15}{2})$.

c When $x = 0$, $y = -3$

$\therefore$ the y-intercept is -3.

d



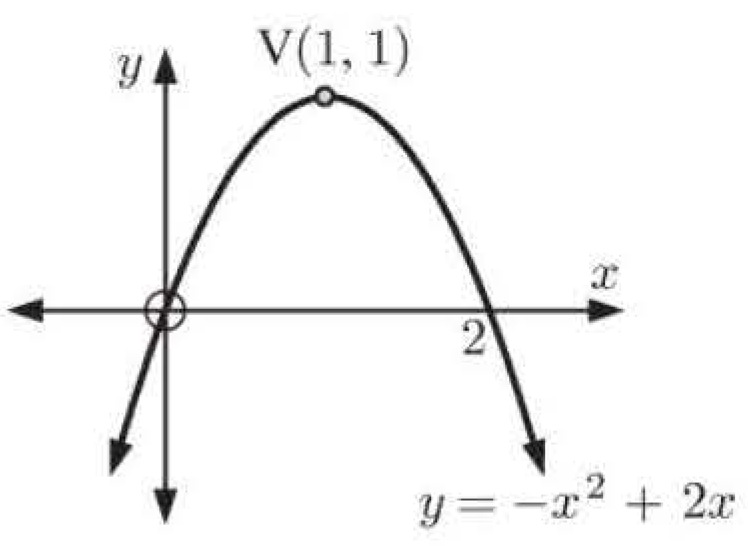
2 **a** Using technology, $x \approx 0.586$ or 3.414 **b** Using technology, $x \approx -0.186$ or 2.686

3 $y = -x^2 + 2x = x(2 - x)$

$\therefore$ the graph has x-intercepts 0 and 2, and y-intercept 0

Its axis of symmetry is midway between the x-intercepts, at $x = 1$, and when $x = 1$, $y = -1^2 + 2 = 1$

$\therefore$ the vertex is $(1, 1)$.



4 $y = -3x^2 + 8x + 7$ has $a = -3,\ b = 8,\ c = 7$

The axis of symmetry is $x = -\dfrac{b}{2a} = -\dfrac{8}{2(-3)} = \frac{4}{3}$

When $x = \frac{4}{3}$, $y = -3(\frac{4}{3})^2 + 8(\frac{4}{3}) + 7$

$= -\frac{16}{3} + \frac{32}{3} + 7 = \frac{37}{3}$

$\therefore$ the axis of symmetry is $x = \frac{4}{3}$ and the vertex is $(\frac{4}{3}, \frac{37}{3})$ or $(\frac{4}{3}, 12\frac{1}{3})$.

5 **a** $2x^2 - 5x - 7 = 0$

has $a = 2,\ b = -5,\ c = -7$

$\therefore\ \Delta = b^2 - 4ac$
$= (-5)^2 - 4(2)(-7)$
$= 25 + 56 = 81$

$\therefore\ \Delta > 0$ and $\sqrt{\Delta} = 9$

$\therefore$ there are two distinct real rational roots

b $3x^2 - 24x + 48 = 0$

has $a = 3,\ b = -24,\ c = 48$

$\therefore\ \Delta = b^2 - 4ac$
$= (-24)^2 - 4(3)(48)$
$= 576 - 576$
$= 0$

$\therefore$ there is a repeated real root

6 **a** $y = 3x + c$ intersects the parabola $y = x^2 + x - 5$ when $x^2 + x - 5 = 3x + c$

$\therefore\ x^2 - 2x - 5 - c = 0$

The graphs meet in two distinct points when this equation has two distinct real roots.

$\therefore\ \Delta > 0$

$\therefore\ (-2)^2 - 4(1)(-5 - c) > 0$

$\therefore\ 4 + 20 + 4c > 0$

$\therefore\ 4c > -24$

$\therefore\ c > -6$

b Choose c such that $c > -6$, for example, $c = -2$.

The graphs meet where $x^2 + x - 5 = 3x - 2$

$\therefore\ x^2 - 2x - 3 = 0$

$\therefore\ (x+1)(x-3) = 0$

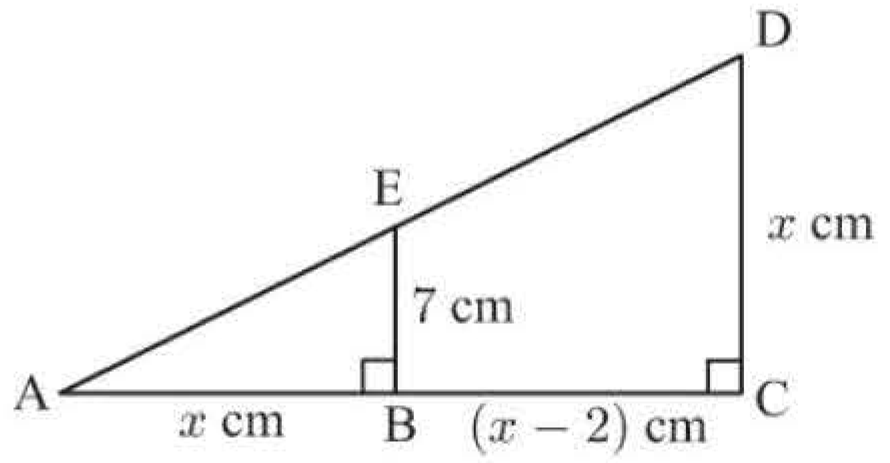
$\therefore\ x = -1$ or 3

Using the line $y = 3x - 2$, when $x = -1$, $y = 3(-1) - 2 = -5$

and when $x = 3$, $y = 3(3) - 2 = 7$

$\therefore$ the points of intersection are $(-1, -5)$ and $(3, 7)$.

7 Suppose [AB] is x cm in length. Then, using the information given, we can label the diagram:



Now by similar triangles, $\dfrac{\text{BE}}{\text{AB}} = \dfrac{\text{CD}}{\text{AC}}$

$\therefore\ \dfrac{7}{x} = \dfrac{x}{x + (x-2)}$

$\therefore\ \dfrac{7}{x} = \dfrac{x}{2x - 2}$

$\therefore\ 7(2x - 2) = x^2$

$\therefore\ 14x - 14 = x^2$

$\therefore\ x^2 - 14x + 14 = 0$

which has $a = 1,\ b = -14$ and $c = 14$

$$\therefore\ x = \frac{-(-14) \pm \sqrt{(-14)^2 - 4(1)(14)}}{2(1)} = \frac{14 \pm \sqrt{140}}{2}$$

Now $x - 2 > 0$, so $x = \dfrac{14 + \sqrt{140}}{2} \approx 12.92$ cm

$\therefore$ [AB] is approximately 12.9 cm long.

8 **a** The total length of wire for the fence is 60 m.

$\therefore\ AB + BC + CD = 60$

Since the enclosure is rectangular,

$$CD = AB$$
$$\therefore\ 2AB + x = 60$$
$$\therefore\ 2AB = 60 - x$$
$$\therefore\ AB = 30 - \tfrac{1}{2}x$$

$\therefore$ the area of the rectangle is

$$A = x\left(30 - \tfrac{1}{2}x\right) = \left(30x - \tfrac{1}{2}x^2\right)\ \text{m}^2$$

b $A = 30x - \frac{1}{2}x^2$ has $a = -\frac{1}{2}$ and $b = 30$.

Since $a < 0$, A has a maximum at the axis of symmetry, and this is at

$$x = -\frac{b}{2a} = -\frac{30}{2(-\frac{1}{2})} = 30$$

When $x = 30$, $AB = 30 - \frac{1}{2} \times 30 = 15$ m

$\therefore$ the enclosure is 15 m by 30 m.

9 **a** $y = 2x^2 + 4x - 1$ has $a = 2$, $b = 4$, $c = -1$

The axis of symmetry is $x = -\dfrac{b}{2a}$

$$\therefore\ x = -\frac{4}{2 \times 2}$$
$$\therefore\ x = -1$$

b When $x = -1$, $y = 2(-1)^2 + 4(-1) - 1 = 2 - 4 - 1 = -3$

$\therefore$ the vertex is $(-1, -3)$

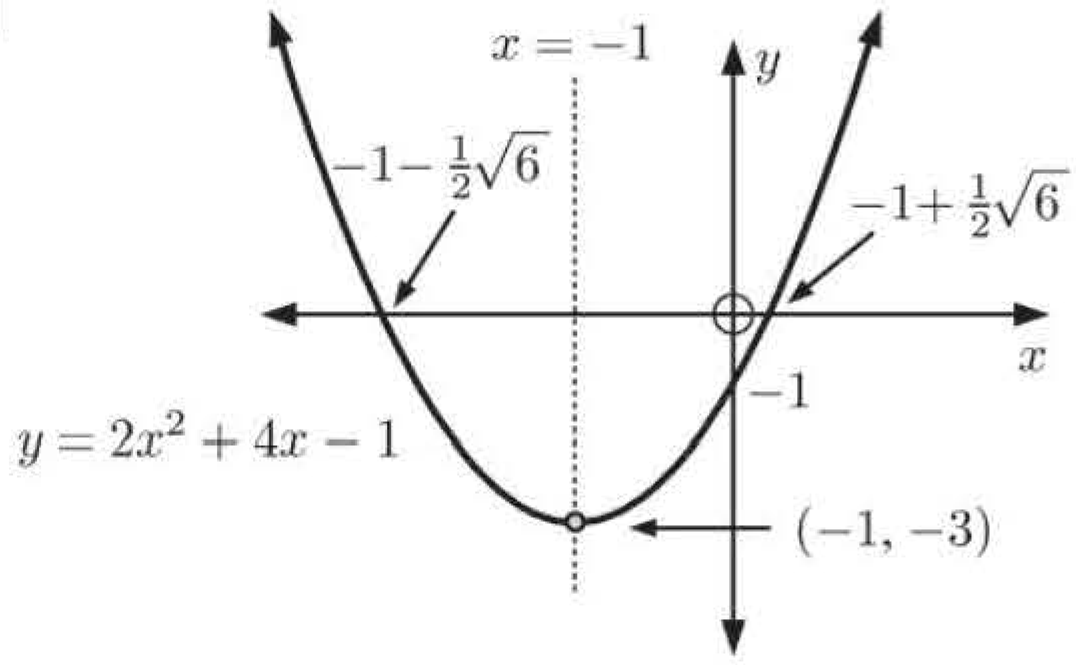
c When $x = 0$, $y = -1$, so the y-intercept is -1.

When $y = 0$, $2x^2 + 4x - 1 = 0$

$$\therefore\ x = \frac{-4 \pm \sqrt{4^2 - 4(2)(-1)}}{2(2)} = \frac{-4 \pm \sqrt{24}}{4} = \frac{-4 \pm 2\sqrt{6}}{4} = -1 \pm \tfrac{1}{2}\sqrt{6}$$

$\therefore$ the x-intercepts are $-1 \pm \frac{1}{2}\sqrt{6}$.

d



10 Since the container has a square base, the original tinplate must have been square.

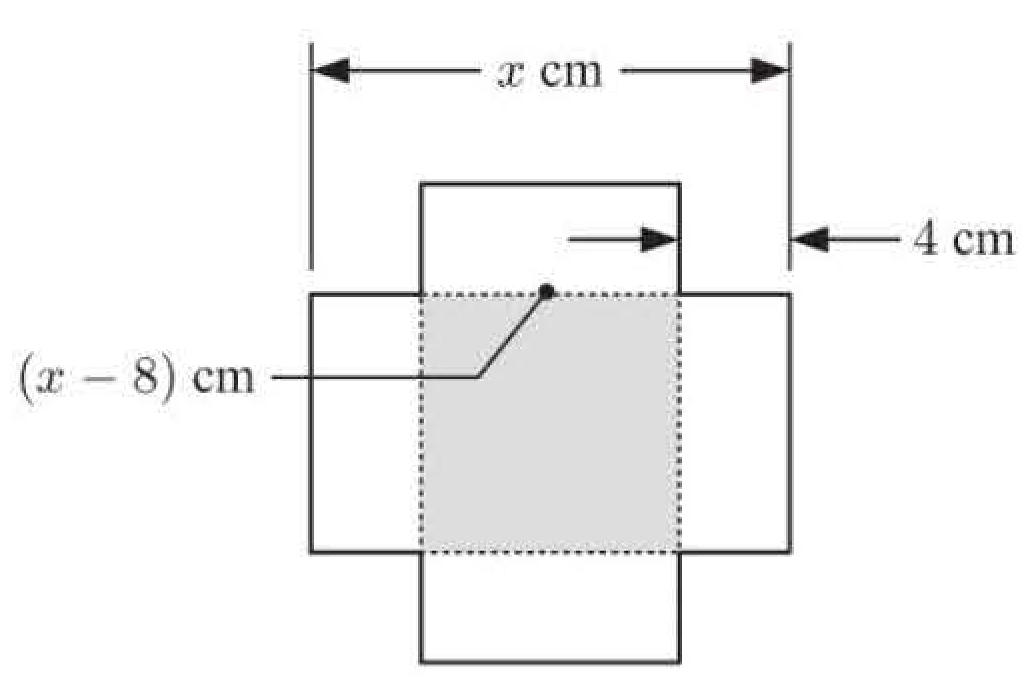
Suppose its side was x cm long, so the base of the container is $(x - 8)$ cm by $(x - 8)$ cm.

The height of the container is 4 cm, so its capacity is $4(x - 8)(x - 8)$ cm^3.

$$\therefore\ 4(x - 8)^2 = 120$$
$$\therefore\ (x - 8)^2 = 30$$
$$\therefore\ x - 8 = \pm\sqrt{30}$$
$$\therefore\ x = 8 \pm \sqrt{30}$$

Clearly, $x > 8$, so $x = 8 + \sqrt{30} \approx 13.48$

$\therefore$ the tinplate was about 13.5 cm by 13.5 cm.



11 **a**

$$-x^2 - 5x + 3 = x^2 + 3x + 11$$
$$\therefore\ 2x^2 + 8x + 8 = 0$$
$$\therefore\ x^2 + 4x + 4 = 0$$
$$\therefore\ (x + 2)^2 = 0$$
$$\therefore\ x = -2$$

b From **a**, $x^2 + 3x + 11 = -x^2 - 5x + 3$ has only one solution.

$\therefore$ the lines touch but do not cross.

So $y = x^2 + 3x + 11$ is either above or below $y = -x^2 - 5x + 3$ for all $x \neq -2$.

If $x = 1$, $(1)^2 + 3(1) + 11 = 15$ and $-(1)^2 - 5(1) + 3 = -3$

$\therefore$ $x^2 + 3x + 11 > -x^2 - 5x + 3$ for $x \neq -2$.

12 **a** $y = 3x^2 + 4x + 7$

has $a = 3$, $b = 4$, and $c = 7$

Since $a > 0$,

the graph has shape

and so has a minimum.

This occurs on the axis of symmetry

$$x = -\frac{b}{2a}$$

$$\therefore\ x = -\frac{4}{2(3)} = -\tfrac{2}{3}$$

When $x = -\frac{2}{3}$,

$$\begin{aligned} y &= 3(-\tfrac{2}{3})^2 + 4(-\tfrac{2}{3}) + 7 \\ &= \tfrac{4}{3} - \tfrac{8}{3} + 7 \\ &= \tfrac{17}{3} = 5\tfrac{2}{3} \end{aligned}$$

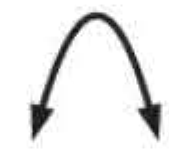
$\therefore$ the minimum is $5\frac{2}{3}$ when $x = -\frac{2}{3}$

b $y = -2x^2 - 5x + 2$

has $a = -2$, $b = -5$, and $c = 2$

Since $a < 0$,

the graph has shape



and so has a maximum.

This occurs on the axis of symmetry

$$x = -\frac{b}{2a}$$

$$\therefore\ x = -\frac{-5}{2(-2)} = -\tfrac{5}{4}$$

When $x = -\frac{5}{4}$,

$$\begin{aligned} y &= -2(-\tfrac{5}{4})^2 - 5(-\tfrac{5}{4}) + 2 \\ &= -\tfrac{25}{8} + \tfrac{25}{4} + 2 \\ &= \frac{-25 + 50 + 16}{8} \\ &= \tfrac{41}{8} = 5\tfrac{1}{8} \end{aligned}$$

$\therefore$ the maximum is $5\frac{1}{8}$ when $x = -\frac{5}{4}$

13 **a** The total length of fencing is $(8x + 9y)$ m

$$\begin{aligned} \therefore\ 8x + 9y &= 600 \\ \therefore\ 9y &= 600 - 8x \\ \therefore\ y &= \frac{600 - 8x}{9} \end{aligned}$$

The area of each pen is

$$\begin{aligned} A &= xy \\ &= x\left(\frac{600 - 8x}{9}\right)\ \text{m}^2 \end{aligned}$$

b $$\begin{aligned} A &= x\left(\frac{600 - 8x}{9}\right) \\ &= \tfrac{600}{9}x - \tfrac{8}{9}x^2 \end{aligned}$$

which has $a = -\frac{8}{9}$, $b = \frac{600}{9}$

Since $a < 0$, A is maximised at the axis of symmetry, which is $x = -\dfrac{b}{2a}$

$$\therefore\ x = -\frac{\frac{600}{9}}{2(-\frac{8}{9})} = \frac{600}{16}$$

$$\therefore\ x = \tfrac{75}{2}$$

When $x = \frac{75}{2}$, $y = \dfrac{600 - 8(\frac{75}{2})}{9} = 33\frac{1}{3}$

$\therefore$ for maximum area, each pen should be $37\frac{1}{2}$ m $\times$ $33\frac{1}{3}$ m.

c The maximum area of each pen is

$$\begin{aligned} &37\tfrac{1}{2} \times 33\tfrac{1}{3} \\ &= \tfrac{75}{2} \times \tfrac{100}{3} \\ &= 1250\ \text{m}^2 \end{aligned}$$

14 **a** $9x^2 - kx + 4$ touches the x-axis if

$$\begin{aligned} \Delta &= 0 \\ \therefore\ (-k)^2 - 4(9)(4) &= 0 \\ \therefore\ k^2 - 144 &= 0 \\ \therefore\ k &= \pm 12 \end{aligned}$$

b The functions intersect when

$$\begin{aligned} 9x^2 - 12x + 4 &= 9x^2 + 12x + 4 \\ \therefore\ 24x &= 0 \\ \therefore\ x &= 0 \end{aligned}$$

$f(0) = 9(0)^2 - 12(0) + 4 = 4$

The two functions intersect at $(0, 4)$.

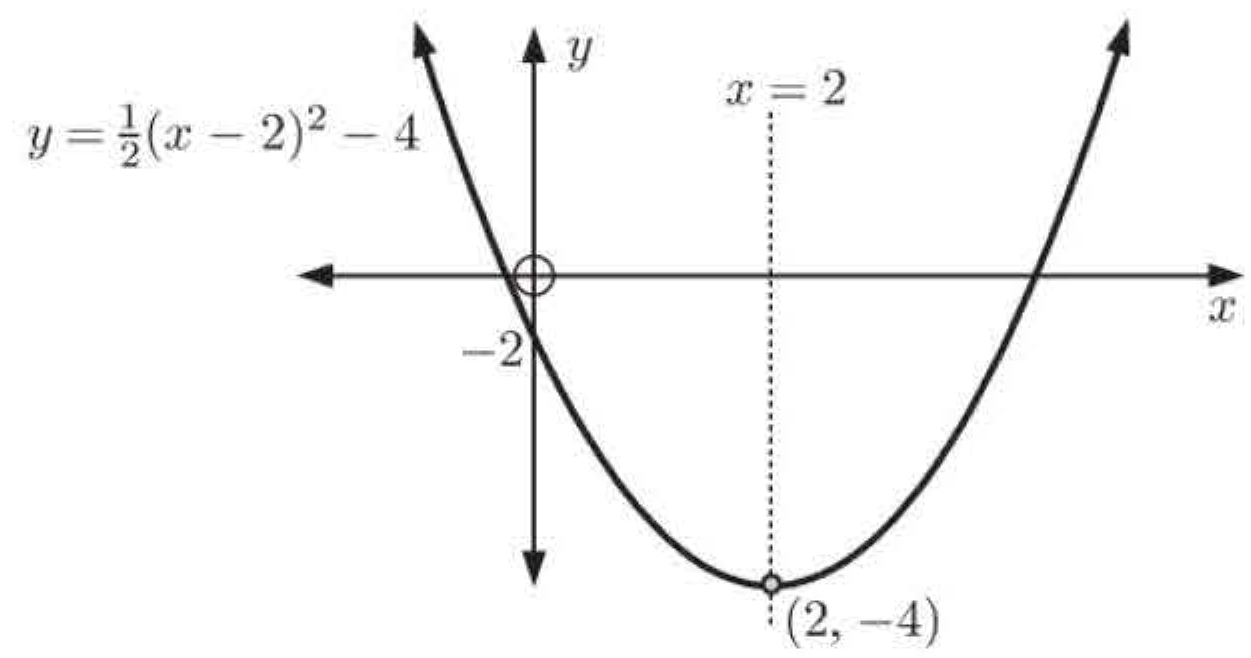
REVIEW SET 1C

1 **a** The axis of symmetry is $x = 2$.

b When $x = 2$, $y = \frac{1}{2}(2-2)^2 - 4$
$= -4$
$\therefore$ the vertex is $(2, -4)$

c When $x = 0$, $y = \frac{1}{2}(-2)^2 - 4$
$= -2$
$\therefore$ the y-intercept is -2

d



2 **a** $x^2 - 5x - 3 = 0$
has $a = 1$, $b = -5$, $c = -3$

$$\therefore \quad x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(-3)}}{2(1)}$$
$$= \tfrac{5}{2} \pm \tfrac{\sqrt{37}}{2}$$

b $2x^2 - 7x - 3 = 0$
has $a = 2$, $b = -7$, $c = -3$

$$\therefore \quad x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(2)(-3)}}{2(2)}$$
$$= \tfrac{7}{4} \pm \tfrac{\sqrt{73}}{4}$$

3 **a** $x^2 - 7x + 3 = 0$
has $a = 1$, $b = -7$, $c = 3$

$$\therefore \quad x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(3)}}{2(1)}$$
$$= \frac{7 \pm \sqrt{49-12}}{2}$$
$$= \frac{7 \pm \sqrt{37}}{2}$$
$$\therefore \quad x = \tfrac{7}{2} \pm \tfrac{\sqrt{37}}{2}$$

b $2x^2 - 5x + 4 = 0$
has $a = 2$, $b = -5$, $c = 4$

$$\therefore \quad x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(2)(4)}}{2(2)}$$
$$= \frac{5 \pm \sqrt{25-32}}{4}$$
$$\therefore \quad x = \frac{5 \pm \sqrt{-7}}{4}$$

$\therefore$ x has no real solutions.

4 **a** The graph has vertex $(2, -20)$, so its equation is
$y = a(x-2)^2 - 20$ for some $a \neq 0$.
Now an x-intercept is 5
$\therefore$ $a(5-2)^2 - 20 = 0$
$\therefore$ $9a = 20$
and so $a = \frac{20}{9}$

So, the equation is $y = \frac{20}{9}(x-2)^2 - 20$.

b Since one x-intercept is 7 and the axis of symmetry is $x = 4$, the other x-intercept is $x = 1$.
$\therefore$ the graph has equation
$y = a(x-7)(x-1)$ for some $a \neq 0$.
The y-intercept is -2
$\therefore$ $a(-7)(-1) = -2$
$\therefore$ $a = -\frac{2}{7}$
$\therefore$ the equation is $y = -\frac{2}{7}(x-7)(x-1)$.

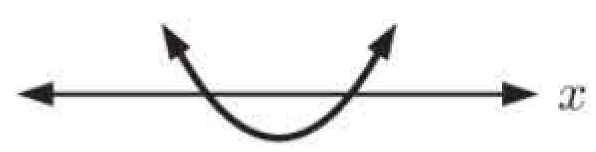
c The graph has vertex $(-3, 0)$, so its equation is $y = a(x+3)^2$ for some $a \neq 0$.
The y-intercept is 2
$\therefore$ $a(3)^2 = 2$
$\therefore$ $9a = 2$
and so $a = \frac{2}{9}$

So, the equation is $y = \frac{2}{9}(x+3)^2$.

5 **a** $y = 2x^2 + 3x - 7$
has $a = 2$, $b = 3$, $c = -7$

$\therefore \Delta = b^2 - 4ac$
$= 3^2 - 4(2)(-7)$
$= 65$

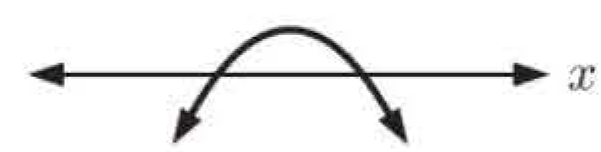
Since $\Delta > 0$, the graph cuts the x-axis twice.
Note that since $a > 0$, the graph is



b $y = -3x^2 - 7x + 4$
has $a = -3$, $b = -7$, $c = 4$

$\therefore \Delta = b^2 - 4ac$
$= (-7)^2 - 4(-3)4$
$= 97$

Since $\Delta > 0$, the graph cuts the x-axis twice.
Note that since $a < 0$, the graph is



6 **a** $y = -2x^2 + 3x + 2$
has $a = -2$, $b = 3$, $c = 2$

$\therefore \Delta = b^2 - 4ac$
$= 3^2 - 4(-2)(2)$
$= 25$

Since $\Delta > 0$, the function is neither positive definite nor negative definite.

b $y = 3x^2 + x + 11$
has $a = 3$, $b = 1$, $c = 11$

$\therefore \Delta = b^2 - 4ac$
$= 1^2 - 4(3)(11)$
$= -131$

$\therefore \Delta < 0$, and since $a > 0$, the function is positive definite.

7 The quadratic has vertex $(2, 25)$.
$\therefore$ its equation is $y = a(x-2)^2 + 25$
The y-intercept is 1, so $a(-2)^2 + 25 = 1$
$\therefore 4a = -24$
$\therefore a = -6$
$\therefore$ the equation is $y = -6(x-2)^2 + 25$

8 Let the line with gradient -3 and y-intercept c have equation $y = -3x + c$.
$y = -3x + c$ is tangential to $y = 2x^2 - 5x + 1$ if they meet at exactly one point.
$y = 2x^2 - 5x + 1$ meets $y = -3x + c$ when $2x^2 - 5x + 1 = -3x + c$
$\therefore 2x^2 - 2x + 1 - c = 0$

The graphs meet exactly once when this equation has a repeated root $\therefore \Delta = 0$
$\therefore (-2)^2 - 4(2)(1-c) = 0$
$\therefore 4 - 8 + 8c = 0$
$\therefore 8c = 4$
$\therefore c = \frac{1}{2}$

$\therefore$ the y-intercept of the line is $\frac{1}{2}$.

9 $y = x^2 - 2x + k$
has $a = 1$, $b = -2$, $c = k$
$\therefore \Delta = b^2 - 4ac$
$= (-2)^2 - 4(1)k$
$= 4 - 4k$

The graph cuts the x-axis twice if $\Delta > 0$
$\therefore 4 - 4k > 0$
$\therefore 4k < 4$
$\therefore k < 1$

10 The x-intercepts are 3 and -2, so the equation is $y = a(x-3)(x+2)$ for some $a \neq 0$.
But the y-intercept is 24 $\therefore a(-3)(2) = 24$
$\therefore -6a = 24$
$\therefore a = -4$

$\therefore$ the equation is $y = -4(x-3)(x+2)$
$\therefore y = -4(x^2 - x - 6)$
$\therefore y = -4x^2 + 4x + 24$

11 $y = mx - 10$ is a tangent to $y = 3x^2 + 7x + 2$ if they meet at exactly one point (touch).

$y = 3x^2 + 7x + 2$ meets $y = mx - 10$ when $3x^2 + 7x + 2 = mx - 10$

$\therefore\ 3x^2 + (7 - m)x + 12 = 0$

The graphs meet exactly once when this equation has a repeated root $\therefore\ \Delta = 0$

$$\therefore\ (7-m)^2 - 4(3)(12) = 0$$
$$\therefore\ 49 - 14m + m^2 - 144 = 0$$
$$\therefore\ m^2 - 14m - 95 = 0$$
$$\therefore\ (m+5)(m-19) = 0$$
$$\therefore\ m = -5 \text{ or } 19$$

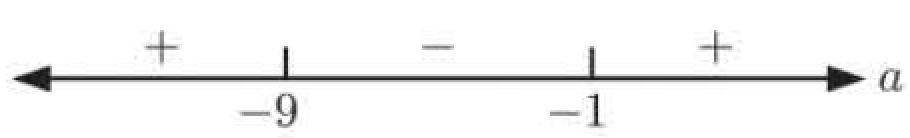
12 $ax^2 + [3 - a]x - 4 = 0$ will have roots which are real and positive if:

(1) $\Delta \geqslant 0$ (2) sum of roots > 0 (3) product of roots > 0

(1) $\Delta \geqslant 0$

$$\therefore\ (3-a)^2 - 4(a)(-4) \geqslant 0$$
$$\therefore\ 9 - 6a + a^2 + 16a \geqslant 0$$
$$\therefore\ a^2 + 10a + 9 \geqslant 0$$
$$\therefore\ (a+9)(a+1) \geqslant 0$$

$(a+9)(a+1)$ has sign diagram

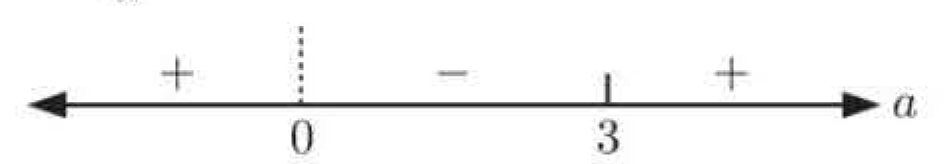


$\therefore\ \Delta \geqslant 0$ when $a \leqslant -9$ or $a \geqslant -1$

(2) sum of roots > 0

$$\therefore\ -\frac{3-a}{a} > 0$$

$-\dfrac{3-a}{a}$ has sign diagram

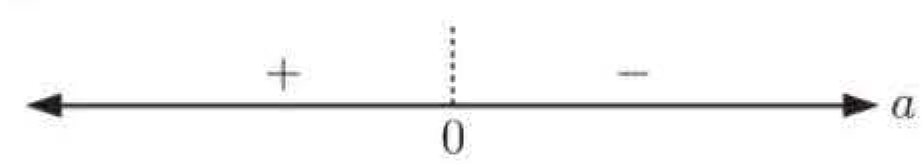


$\therefore$ sum of roots > 0 when $a < 0$ or $a > 3$

(3) product of roots > 0

$$\therefore\ \frac{-4}{a} > 0$$

$-\dfrac{4}{a}$ has sign diagram



$\therefore$ product of roots > 0 when $a < 0$

$\therefore$ the only values of a which satisfy all three conditions are $a \leqslant -9$ and $-1 \leqslant a < 0$.

13 **a** **i** The quadratic has x-intercepts ± 3, so its equation is $y = a(x+3)(x-3)$ for some $a \neq 0$.

Its y-intercept is -27, so

$$a(3)(-3) = -27$$
$$\therefore\ -9a = -27$$
$$\therefore\ a = 3$$

$\therefore$ the equation is $y = 3(x+3)(x-3)$

or $y = 3x^2 - 27$

ii The straight line has gradient $\dfrac{0-(-27)}{3-0} = 9$

and its y-intercept is -27.

So, its equations is $y = 9x - 27$.

b From the graph, the straight line is above the curve when $0 < x < 3$.

14 $y = -5x + k$ meets $y = x^2 - 3x + c$ when $x^2 - 3x + c = -5x + k$

$\therefore\ x^2 + 2x + c - k = 0$

For $y = -5x + k$ to be a tangent to $y = x^2 - 3x + c$, this equation must have exactly one solution, so there is a repeated root.

$$\therefore\ \Delta = 0$$
$$\therefore\ 2^2 - 4(1)(c-k) = 0$$
$$\therefore\ 4 = 4(c-k)$$
$$\therefore\ c - k = 1$$

15 The roots of $4x^2 - 3x - 3 = 0$ are p and q.

$\therefore$ sum of roots $= -\dfrac{-3}{4}$ and $\therefore$ product of roots $= \dfrac{-3}{4}$

$\therefore\ p + q = \frac{3}{4}$ $\qquad\therefore\ pq = -\frac{3}{4}$

Now consider a quadratic equation with roots p^3 and q^3.

$$\begin{aligned}\text{sum of roots} &= p^3 + q^3\\ &= (p+q)^3 - 3p^2q - 3pq^2\\ &= (p+q)^3 - 3pq(p+q)\\ &= (\tfrac{3}{4})^3 - 3(-\tfrac{3}{4})(\tfrac{3}{4})\\ &= \tfrac{27}{64} + \tfrac{27}{16}\\ &= \tfrac{135}{64}\end{aligned}$$

$$\begin{aligned}\text{product of roots} &= p^3 \times q^3\\ &= (pq)^3\\ &= (-\tfrac{3}{4})^3\\ &= -\tfrac{27}{64}\end{aligned}$$

$\therefore$ a quadratic equation $ax^2 + bx + c = 0$ with roots p^3 and q^3 has $-\dfrac{b}{a} = \frac{135}{64}$ and $\dfrac{c}{a} = -\frac{27}{64}$

$\therefore\ b = -\frac{135}{64}a$ and $c = -\frac{27}{64}a$

$\therefore$ the quadratic equation is $ax^2 - \frac{135}{64}ax - \frac{27}{64}a = 0,\quad a \neq 0$

$\therefore\ a(x^2 - \frac{135}{64}x - \frac{27}{64}) = 0,\quad a \neq 0$

$\therefore\ a(64x^2 - 135x - 27) = 0,\quad a \neq 0$