Chapter 25

DISCRETE RANDOM VARIABLES

EXERCISE 25A

1 **a** The quantity of fat in a sausage is a continuous random variable.

b The mark out of 50 for a geography test is a discrete random variable.

c The weight of a seventeen year old student is a continuous random variable.

d The volume of water in a cup of coffee is a continuous random variable.

e The number of trout in a lake is a discrete random variable.

f The number of the hairs on a cat is a discrete random variable.

g The length of hairs on a horse is a continuous random variable.

h The height of a sky-scraper is a continuous random variable.

2 **a** **i** The random variable X is the height of water in the rain gauge.

ii $0 \leqslant X \leqslant 400$ mm **iii** The variable is a continuous random variable.

b **i** The random variable X is the stopping distance.

ii $0 \leqslant X \leqslant 50$ m **iii** The variable is a continuous random variable.

c **i** The random variable X is the number of times that the switch is turned on or off before it fails.

ii X can be any integer $\geqslant 1$ **iii** The variable is a discrete random variable.

3 **a** Since X is the number of weighing devices that are accurate, $X = 0, 1, 2, 3$, or 4.

b

$(X=4)$	$(X=3)$	$(X=2)$	$(X=1)$	$(X=0)$
YYYY	YYYN	YYNN	NNNY	NNNN
	YYNY	YNYN	NNYN	
	YNYY	YNNY	NYNN	
	NYYY	NYYN	YNNN	
		NYNY		
		NNYY		

c **i** If two are accurate then $X = 2$.

ii If at least two are accurate then 2, 3, or 4 are accurate $\therefore\ X = 2, 3$, or 4.

4 **a** If 3 coins are tossed then the number of heads X can be 0, 1, 2, or 3.

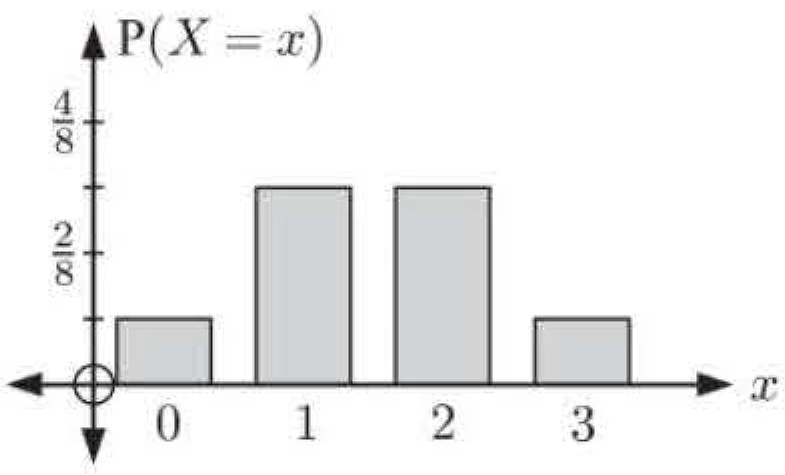
b Suppose H represents heads, T represents tails.

$(X=3)$	$(X=2)$	$(X=1)$	$(X=0)$
HHH	HHT	TTH	TTT
	HTH	THT	
	THH	HTT	

c $P(X=0) = \frac{1}{8}$ $\quad P(X=1) = \frac{3}{8}$

$P(X=2) = \frac{3}{8}$ $\quad P(X=3) = \frac{1}{8}$

d



EXERCISE 25B

1 **a**
$$\sum_{x=0}^{2} P(X=x) = 1$$
$$\therefore\ 0.3 + k + 0.5 = 1$$
$$\therefore\ k = 0.2$$

b
$$\sum_{x=0}^{3} P(X=x) = 1$$
$$\therefore\ k + 2k + 3k + k = 1$$
$$\therefore\ 7k = 1$$
$$\therefore\ k = \tfrac{1}{7}$$

2 **a** $P(2) = 0.1088$ (from table)

b Since this is a probability distribution, $\sum P(x_i) = 1$

$\therefore\ a + 0.3333 + 0.1088 + 0.0084 + 0.0007 + 0.0000 = 1$

$\therefore\ a + 0.4512 = 1$

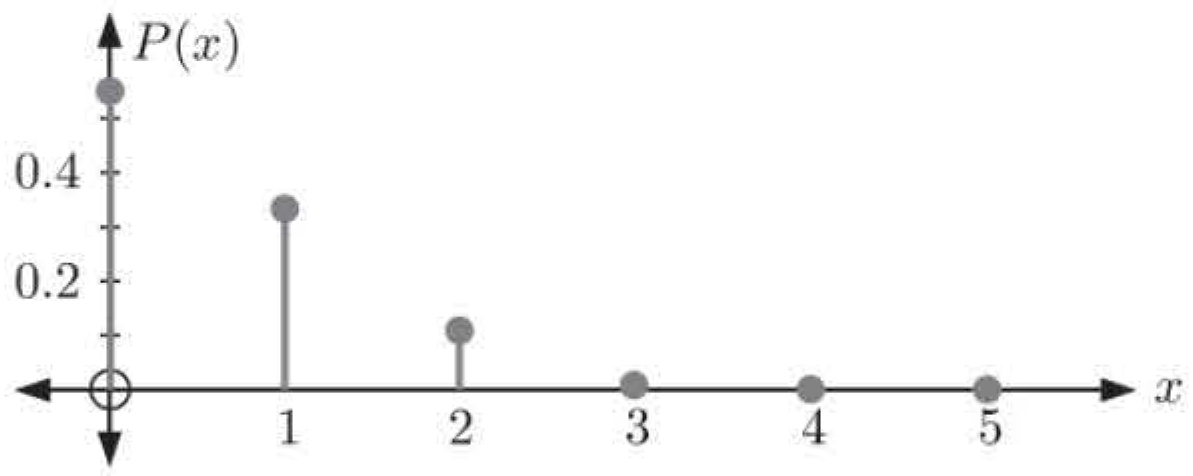
$\therefore\ a = 0.5488$

This is the probability that Jason does not hit a home run in a game.

c $P(1) + P(2) + P(3) + P(4) + P(5) = 0.3333 + 0.1088 + 0.0084 + 0.0007 + 0.0000$
$= 0.4512$

This represents the probability that Jason hits at least one home run in a game.

d



e Jason is most likely to score 0 home runs, so this is the mode of the distribution.
$p_0 = 0.5488$ Since $p_0 \geqslant 0.5$, the median is 0 home runs.

3 **a** Sum of probabilities $\sum P(x_i) = 0.2 + 0.3 + 0.4 + 0.2 = 1.1$
Since this sum $\neq 1$, this is not a valid probability distribution.

b $P(5) = -0.2$, so not all of the probabilities lie in $0 \leqslant P(x_i) \leqslant 1$.
$\therefore$ this is not a valid probability distribution.

4 **a** The random variable represents the number of hits that Sally has in each game.
$X = 0, 1, 2, 3, 4$, or 5.

b **i** $0.07 + 0.14 + k + 0.46 + 0.08 + 0.02 = 1$ $\{\text{since } \sum_{x=0}^{5} \mathrm{P}(X = x) = 1\}$

$\therefore\ k + 0.77 = 1$

$\therefore\ k = 0.23$

ii $\mathrm{P}(X \geqslant 2)$
$= \mathrm{P}(X = 2 \text{ or } X = 3 \text{ or } X = 4 \text{ or } X = 5)$
$= P(2) + P(3) + P(4) + P(5)$
$= 0.23 + 0.46 + 0.08 + 0.02$
$= 0.79$

iii $\mathrm{P}(1 \leqslant X \leqslant 3)$
$= P(1) + P(2) + P(3)$
$= 0.14 + 0.23 + 0.46$
$= 0.83$

c Sally is most likely to have 3 hits, so this is the mode of the distribution.

$p = 0.07$
$p_0 + p_1 = 0.07 + 0.14 = 0.21$
$p_0 + p_1 + p_2 = 0.21 + 0.23 = 0.44$
$p_0 + p_1 + p_2 + p_3 = 0.44 + 0.46 = 0.90$

Since $p_0 + p_1 + p_2 + p_3 \geqslant 0.5$, the median is 3 hits.

5 **a** When rolling a die twice, the sample space is:

roll 1 \ roll 2	1	2	3	4	5	6
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)

b **i** $P(0) = 0$ $\quad P(1) = 0$

$P(2) = \frac{1}{36}$ $\quad P(3) = \frac{2}{36}$

$P(4) = \frac{3}{36}$ $\quad P(5) = \frac{4}{36}$

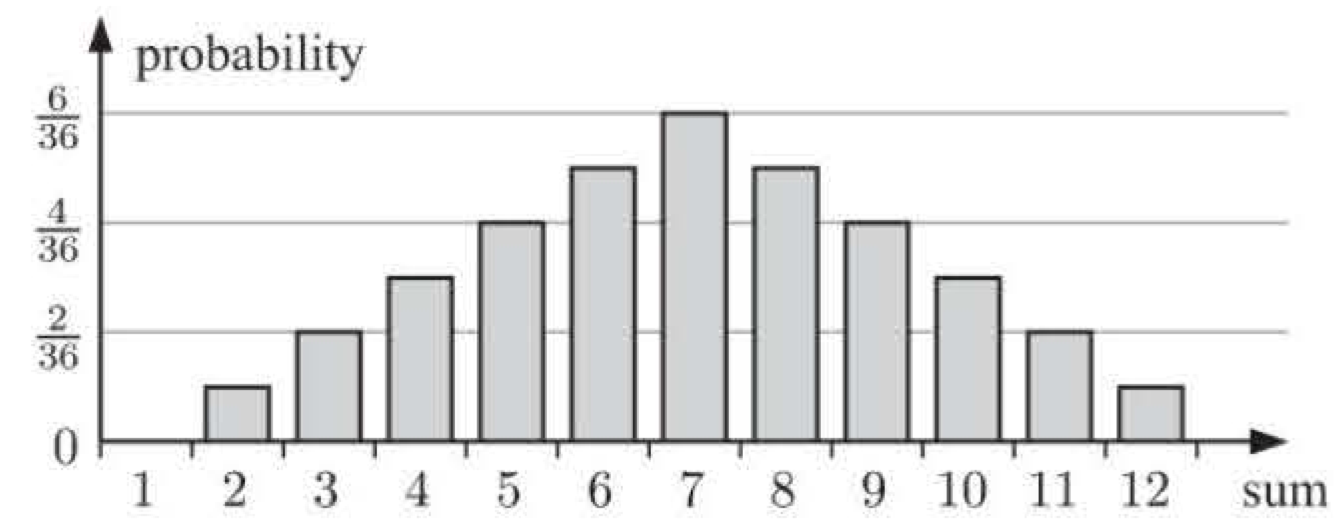
$P(6) = \frac{5}{36}$ $\quad P(7) = \frac{6}{36}$

$P(8) = \frac{5}{36}$ $\quad P(9) = \frac{4}{36}$

$P(10) = \frac{3}{36}$ $\quad P(11) = \frac{2}{36}$

$P(12) = \frac{1}{36}$

ii



iii The sum of the results for the two rolls is most likely to be 7, so this is the mode.

$$p_2 = \tfrac{1}{36}$$
$$p_2 + p_3 = \tfrac{1}{36} + \tfrac{2}{36} = \tfrac{3}{36}$$
$$p_2 + p_3 + p_4 = \tfrac{3}{36} + \tfrac{3}{36} = \tfrac{6}{36}$$
$$p_2 + p_3 + p_4 + p_5 = \tfrac{6}{36} + \tfrac{4}{36} = \tfrac{10}{36}$$
$$p_2 + p_3 + p_4 + p_5 + p_6 = \tfrac{10}{36} + \tfrac{5}{36} = \tfrac{15}{36}$$
$$p_2 + p_3 + p_4 + p_5 + p_6 + p_7 = \tfrac{15}{36} + \tfrac{6}{36} = \tfrac{21}{36} \approx 0.583$$

Since $p_2 + p_3 + p_4 + p_5 + p_6 + p_7 \approx 0.583 \geqslant 0.5$, the median is 7.

6 **a** $P(x) = k(x+2), \quad x = 1, 2, 3$

$\therefore \; P(1) = 3k, \quad P(2) = 4k, \quad P(3) = 5k$

Since this is a probability distribution, $3k + 4k + 5k = 1$

$\therefore \; 12k = 1$

$\therefore \; k = \frac{1}{12}$

b $P(x) = \dfrac{k}{x+1}, \quad x = 0, 1, 2, 3$

$\therefore \; P(0) = k, \quad P(1) = \dfrac{k}{2},$

$P(2) = \dfrac{k}{3}, \quad P(3) = \dfrac{k}{4}$

Since $\sum P(x_i) = 1, \quad k + \dfrac{k}{2} + \dfrac{k}{3} + \dfrac{k}{4} = 1$

$\therefore \; \dfrac{12k + 6k + 4k + 3k}{12} = 1$

$\therefore \; \dfrac{25k}{12} = 1$

$\therefore \; k = \frac{12}{25}$

7 **a** $\mathrm{P}(X = x) = k\left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{4-x}, \qquad x = 0, 1, 2, 3, 4$

$\mathrm{P}(X = 0) = k\left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^4 = \dfrac{16k}{81} \approx 0.1975k$ $\qquad \mathrm{P}(X = 1) = k\left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^3 = \dfrac{8k}{81} \approx 0.0988k$

$\mathrm{P}(X = 2) = k\left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^2 = \dfrac{4k}{81} \approx 0.0494k$ $\qquad \mathrm{P}(X = 3) = k\left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^1 = \dfrac{2k}{81} \approx 0.0247k$

$\mathrm{P}(X = 4) = k\left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^0 = \dfrac{k}{81} \approx 0.0123k$

b Since $\sum \mathrm{P}(X = i) = 1,$

$\therefore \; \dfrac{16k}{81} + \dfrac{8k}{81} + \dfrac{4k}{81} + \dfrac{2k}{81} + \dfrac{k}{81} = 1$

$\therefore \; \dfrac{31k}{81} = 1$

$\therefore \; k = \frac{81}{31}$

$\therefore \; k \approx 2.61$

$\therefore \; \mathrm{P}(X \geqslant 2) = P(2) + P(3) + P(4)$

$= \dfrac{4k}{81} + \dfrac{2k}{81} + \dfrac{k}{81}$

$= \dfrac{7k}{81}$

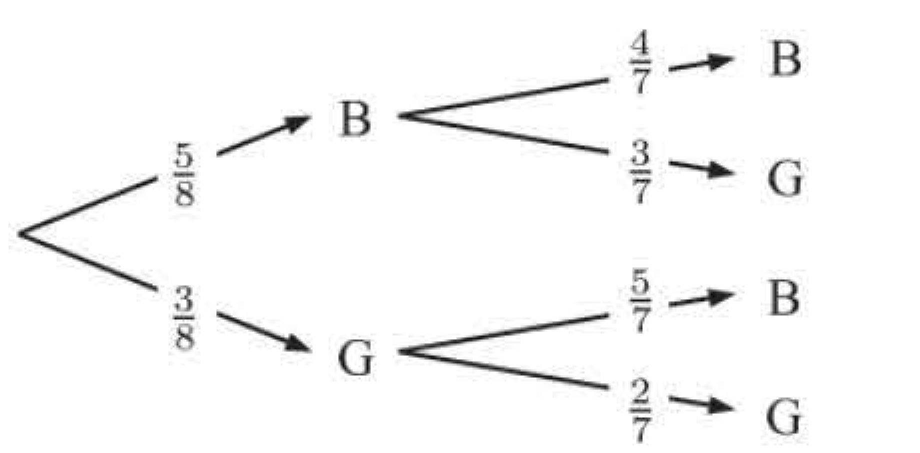
$= \frac{7}{81} \times \frac{81}{31}$

$= \frac{7}{31} \quad (\approx 0.226)$

8 **a** P(no faulty component)
$= \mathrm{P}(X = 0)$
$= \binom{10}{0}(0.04)^0(0.96)^{10-0}$
$= (0.96)^{10}$
≈ 0.665

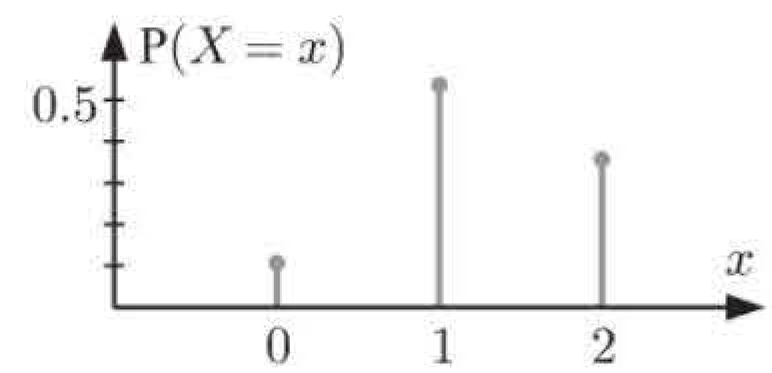
b P(at least one faulty component)
$= \mathrm{P}(X \geqslant 1)$
$= 1 - \mathrm{P}(\text{none are faulty})$
$\approx 1 - (0.96)^{10}$
≈ 0.335

9 **a**

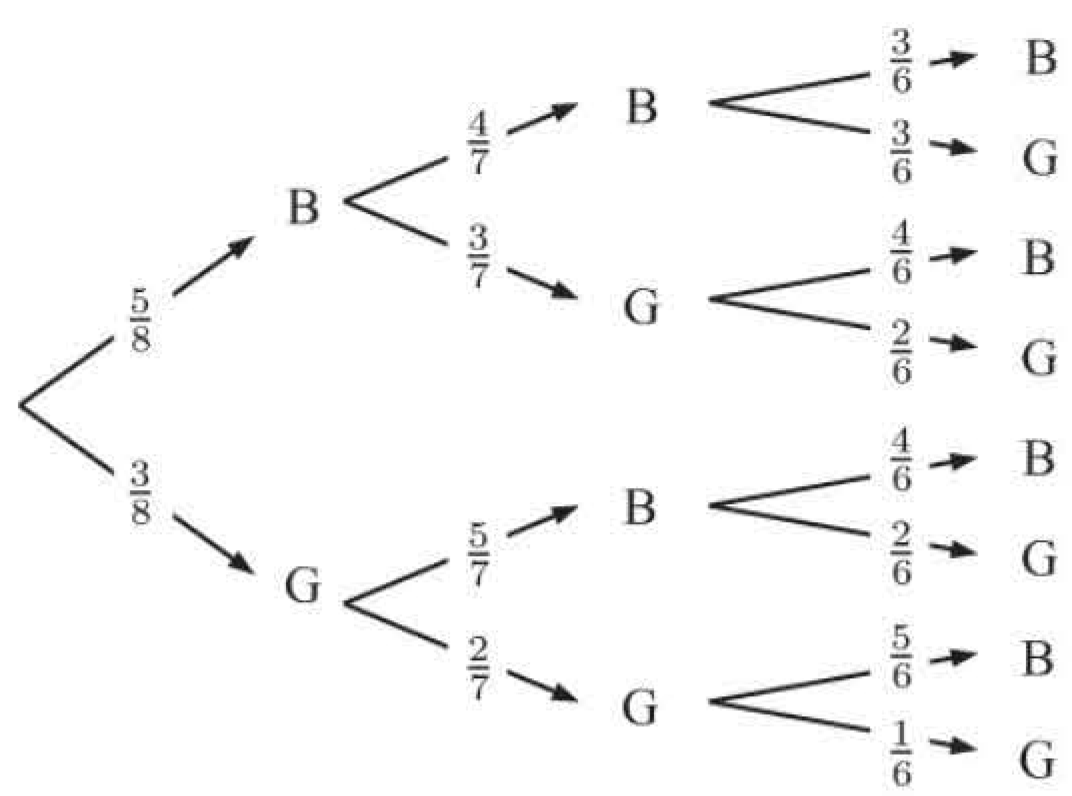


Event	X	Probability
BB	2	$\frac{5}{8} \times \frac{4}{7} = \frac{20}{56}$
BG	1	$\frac{5}{8} \times \frac{3}{7} = \frac{15}{56}$
GB	1	$\frac{3}{8} \times \frac{5}{7} = \frac{15}{56}$
GG	0	$\frac{3}{8} \times \frac{2}{7} = \frac{6}{56}$

x	0	1	2
$\mathrm{P}(X = x)$	$\frac{3}{28}$	$\frac{15}{28}$	$\frac{10}{28}$

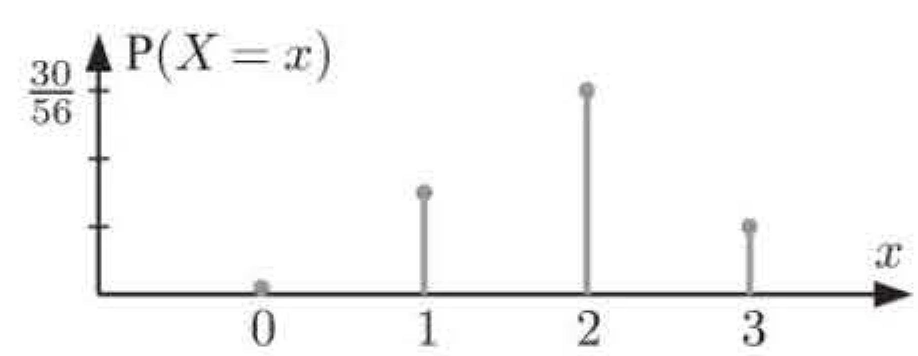


b



Event	X	Probability
BBB	3	$\frac{5}{8} \times \frac{4}{7} \times \frac{3}{6} = \frac{10}{56}$
BBG	2	$\frac{5}{8} \times \frac{4}{7} \times \frac{3}{6} = \frac{10}{56}$
BGB	2	$\frac{5}{8} \times \frac{3}{7} \times \frac{4}{6} = \frac{10}{56}$
BGG	1	$\frac{5}{8} \times \frac{3}{7} \times \frac{2}{6} = \frac{5}{56}$
GBB	2	$\frac{3}{8} \times \frac{5}{7} \times \frac{4}{6} = \frac{10}{56}$
GBG	1	$\frac{3}{8} \times \frac{5}{7} \times \frac{2}{6} = \frac{5}{56}$
GGB	1	$\frac{3}{8} \times \frac{2}{7} \times \frac{5}{6} = \frac{5}{56}$
GGG	0	$\frac{3}{8} \times \frac{2}{7} \times \frac{1}{6} = \frac{1}{56}$

x	0	1	2	3
$\mathrm{P}(X = x)$	$\frac{1}{56}$	$\frac{15}{56}$	$\frac{30}{56}$	$\frac{10}{56}$



10 **a**

Die 2

		1	2	3	4	5	6
	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
Die 1	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

36 possible results

b $\mathrm{P}(D = 7) = \frac{6}{36} = \frac{1}{6}$

c

d	2	3	4	5	6	7	8	9	10	11	12
$\mathrm{P}(D = d)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

d $\mathrm{P}(D \geqslant 8 \mid D \geqslant 6) = \dfrac{\mathrm{P}(D \geqslant 8 \cap D \geqslant 6)}{\mathrm{P}(D \geqslant 6)}$
$= \dfrac{\mathrm{P}(D \geqslant 8)}{\mathrm{P}(D \geqslant 6)}$
$= \frac{15}{36} \div \frac{26}{36}$
$= \frac{15}{26}$

11 **a**

Die 2

		1	2	3	4	5	6
	1	0	1	2	3	4	5
	2	1	0	1	2	3	4
Die 1	3	2	1	0	1	2	3
	4	3	2	1	0	1	2
	5	4	3	2	1	0	1
	6	5	4	3	2	1	0

b

N	0	1	2	3	4	5
$\mathrm{P}(N = n)$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{8}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{2}{36}$

c **i** $\text{P}(N = 3) = \frac{6}{36}$
$= \frac{1}{6}$

ii $\text{P}(N \geqslant 3 \mid N \geqslant 1) = \dfrac{\text{P}(N \geqslant 3 \cap N \geqslant 1)}{\text{P}(N \geqslant 1)}$
$= \dfrac{\text{P}(N \geqslant 3)}{\text{P}(N \geqslant 1)}$
$= \frac{12}{36} \div \frac{30}{36}$
$= \frac{2}{5}$

12 **a** $e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!} = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots.$

$$\sum_{n=0}^{\infty} \frac{(0.2)^n e^{-0.2}}{n!} = e^{-0.2} \sum_{n=0}^{\infty} \frac{(0.2)^n}{n!}$$
$= e^{-0.2} \times e^{0.2}$ {definition of e^x}
$= e^0$
$= 1$

b **i** $0 \leqslant \dfrac{0.2^x e^{-0.2}}{x!} \leqslant 1$ for all $x = 0, 1, 2, \ldots.$

$\therefore$ $0 \leqslant p_x \leqslant 1$ for all $x = 0, 1, 2, \ldots.$

$\sum_{x=0}^{\infty} p_x = \sum_{x=0}^{\infty} \dfrac{0.2^x e^{-0.2}}{x!}$
$= 1$ {from **a**}

ii $\text{P}(X = 0) = \dfrac{(0.2)^0 e^{-0.2}}{0!} = e^{-0.2} \approx 0.819$

$\text{P}(X = 1) = \dfrac{(0.2)^1 e^{-0.2}}{1!} = 0.2e^{-0.2} \approx 0.164$

$\text{P}(X = 2) = \dfrac{(0.2)^2 e^{-0.2}}{2!} = 0.02e^{-0.2} \approx 0.0164$

iii P(at least 3 cars will pass) $= \text{P}(X \geqslant 3)$
$= 1 - \text{P}(X \leqslant 2)$
$= 1 - [\text{P}(X = 0) + \text{P}(X = 1) + \text{P}(X = 2)]$
$= 1 - (e^{-0.2} + 0.2e^{-0.2} + 0.02e^{-0.2})$
$\approx 0.001\,15$

EXERCISE 25C

1 P(rain) $= 0.28$ $\therefore$ we would expect rain on $0.28 \times 365.25 \approx 102$ days a year.

2 **a** $\text{P(HHH)} = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$
$= \frac{1}{8}$

b For 200 tosses, we expect $200 \times \frac{1}{8} = 25$ to be ‘3 heads’.

3 P(double) $=$ P(1, 1 or 2, 2 or 3, 3 or 4, 4 or 5, 5 or 6, 6)
$= \frac{6}{36}$ {6 of the possible 36 outcomes}
$= \frac{1}{6}$

$\therefore$ when rolling the dice 180 times, we expect $180 \times \frac{1}{6} = 30$ doubles.

4

result	*win*
H	\$2
T	−\$1

For playing *once*,
we would expect to win $\frac{1}{2} \times \$2 + \frac{1}{2} \times (-\$1) = \$0.50$
$\therefore$ for 3 games we would expect to win \$1.50.

5 Udo could expect to see snow falling on $\frac{3}{7} \times 5 \times 7 = 15$ days.

6 The goalkeeper would expect to save $\frac{3}{10} \times 90 = 27$ goals.

7 **a** $165 + 87 + 48 = 300$

i $P(A) \approx \frac{165}{300} = 0.55$ **ii** $P(B) \approx \frac{87}{300} = 0.29$ **iii** $P(C) \approx \frac{48}{300} = 0.16$

b **i** We expect $7500 \times 0.55 = 4125$ to vote for A.
ii We expect $7500 \times 0.29 = 2175$ to vote for B.
iii We expect $7500 \times 0.16 = 1200$ to vote for C.

8 **a** **i** P(wins \$10) $=$ P(rolls a 6) $= \frac{1}{6}$

ii P(wins \$4) $=$ P(rolls 4 or 5) $= \frac{2}{6}$ (or $\frac{1}{3}$)

iii P(wins \$1) $=$ P(rolls 1, 2, or 3) $= \frac{3}{6}$ (or $\frac{1}{2}$)

b **i** Expectation $= \frac{2}{6} \times \$4 \approx \$1.33$

ii Expectation $= \frac{3}{6} \times \$1 = \$0.50$

iii Expectation $= \frac{1}{6} \times \$10 + \frac{2}{6} \times \$4 + \frac{3}{6} \times \$1 = \frac{1}{6}(\$21) = \$3.50$

c It costs \$4 to play and the expected return is \$3.50.
$\therefore$ you expect to lose \$0.50 per game. (50 cents)

d Over 100 games you expect to lose $100 \times \$0.50 = \$50$.

9 **a** Expect to win $\frac{1}{6} \times €1 + \frac{1}{6} \times €2 + \frac{1}{6} \times €3 + \frac{1}{6} \times €4 + \frac{1}{6} \times €5 + \frac{1}{6} \times €6$
$= \frac{1}{6} \times €21 = €3.50$

b The expected gain is $€3.50 - €4 = -€0.50$
$\therefore$ the player should not play several games, as on each occasion he would expect to lose an average of €0.50.

c **i** The game is fair when the expected gain is 0.
$\therefore\ 3.50 - k = 0$, so $k = 3.50$.

ii $k > 3.50$

10

result	*win*
HH	£10
HT or TH	£3
TT	$-$£5

a Expectation $= \frac{1}{4} \times £10 + \frac{2}{4} \times £3 + \frac{1}{4} \times (-£5) = £2.75$

b Expected win per game (payout) $= £2.75$
$\therefore$ the organiser would charge $£2.75 + £1.00 = £3.75$ to play each game.

11 **a**

Die 2

Die 1	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

36 possible results

x	2	3	4	5	6	7	8	9	10	11	12
$P(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

So, $P(X \leqslant 3) = \frac{1}{36} + \frac{2}{36} = \frac{1}{12}$
$P(4 \leqslant X \leqslant 6) = \frac{3}{36} + \frac{4}{36} + \frac{5}{36} = \frac{1}{3}$
$P(7 \leqslant X \leqslant 9) = \frac{6}{36} + \frac{5}{36} + \frac{4}{36} = \frac{5}{12}$
$P(X \geqslant 10) = \frac{3}{36} + \frac{2}{36} + \frac{1}{36} = \frac{1}{6}$

b The expected gain is
$\left(\frac{1}{12} + \frac{5}{12}\right)\left(-\frac{a}{3}\right) + \frac{1}{3}(7) + \frac{1}{6}(21) - a$
$= -\frac{a}{6} + \frac{7}{3} + \frac{21}{6} - a$
$= -\frac{7a}{6} + \frac{35}{6}$
$= \frac{1}{6}(35 - 7a)$ dollars, as required.

c The game is fair when the expected gain is 0.
$\therefore\ \frac{1}{6}(35 - 7a) = 0$
$\therefore\ 35 - 7a = 0$
$\therefore\ a = 5$

d If $a = 4$, expected gain $= \frac{1}{6}(35 - 7(4))$

$= \frac{7}{6}$ dollars

So, the people playing would expect to win about \$1.17 per game, which means the organisers expect to lose \$1.17 per game.

e If $a = 6$, expected gain $= \frac{1}{6}(35 - 7(6))$

$= -\frac{7}{6}$ dollars

Expectation from 2406 games is $-\frac{7}{6} \times 2406$

$= -2807$

$\therefore$ the organisers would expect to gain \$2807.

EXERCISE 25D

1

x	0	1	2	3	4	5	> 5
$\mathrm{P}(X = x)$	0.54	0.26	0.15	k	0.01	0.01	0.00

a $0.54 + 0.26 + 0.15 + k + 0.01 + 0.01 = 1$

$\therefore \quad k + 0.97 = 1$

$\therefore \quad k = 0.03$

b $\mu = \sum x_i p_i$

$= 0 \times 0.54 + 1 \times 0.26 + 2 \times 0.15 + 3 \times 0.03 + 4 \times 0.01 + 5 \times 0.01$

$= 0.26 + 0.30 + 0.09 + 0.04 + 0.05$

$= 0.74$ So, over a long period the mean number of deaths per dozen crayfish is 0.74.

c $\sigma = \sqrt{\sum (x_i - \mu)^2 p_i}$

$= \sqrt{(0-0.74)^2 \times 0.54 + (1-0.74)^2 \times 0.26 + (2-0.74)^2 \times 0.15 + ... + (5-0.74)^2 \times 0.01}$

≈ 0.996

2 $\mathrm{P}(X = x) = \dfrac{x^2 + x}{20}$ for $x = 1, 2, 3$

x	1	2	3
$\mathrm{P}(X = x)$	$\frac{2}{20} = 0.1$	$\frac{6}{20} = 0.3$	$\frac{12}{20} = 0.6$

a The most likely value of X is 3, so this is the mode of the distribution.

b $p_1 = 0.1$

$p_1 + p_2 = 0.1 + 0.3 = 0.4$

$p_1 + p_2 + p_3 = 0.4 + 0.6 = 1.0$

Since $p_1 + p_2 + p_3 \geqslant 0.5$, the median is 3.

c $\mu = \sum x_i p_i$

$= 1 \times 0.1 + 2 \times 0.3 + 3 \times 0.6$

$= 2.5$

d $\sigma = \sqrt{\sum (x_i - \mu)^2 p_i}$

$= \sqrt{(1 - 2.5)^2 \times 0.1 + (2 - 2.5)^2 \times 0.3 + (3 - 2.5)^2 \times 0.6}$

≈ 0.671

3 **a** $P(x) = \binom{3}{x}(0.4)^x(0.6)^{3-x}$ for $x = 0, 1, 2, 3$

$\therefore \quad P(0) = \binom{3}{0}(0.4)^0(0.6)^3 = (0.6)^3 = 0.216$

$P(1) = \binom{3}{1}(0.4)^1(0.6)^2 = 3(0.4)(0.6)^2 = 0.432$

$P(2) = \binom{3}{2}(0.4)^2(0.6)^1 = 3(0.16)(0.6) = 0.288$

$P(3) = \binom{3}{3}(0.4)^3(0.6)^0 = 1(0.4)^3 = 0.064$

x_i	0	1	2	3
$P(x_i)$	0.216	0.432	0.288	0.064

b $\mu = \sum x_i p_i = 0(0.216) + 1(0.432) + 2(0.288) + 3(0.064) = 1.2$

$\sigma = \sqrt{\sum (x_i - \mu)^2 p_i}$

$= \sqrt{(0 - 1.2)^2(0.216) + (1 - 1.2)^2(0.432) + (2 - 1.2)^2 \times 0.288 + (3 - 1.2)^2 \times 0.064}$

≈ 0.849

4 $$\sigma = \sqrt{\sum (x_i - \mu)^2 p_i}$$
$$\begin{aligned}
\therefore\quad \sigma^2 &= \sum (x_i - \mu)^2 p_i \\
&= (x_1 - \mu)^2 p_1 + (x_2 - \mu)^2 p_2 + \dots\dots + (x_n - \mu)^2 p_n \\
&= (x_1{}^2 - 2x_1\mu + \mu^2)p_1 + (x_2{}^2 - 2x_2\mu + \mu^2)p_2 + \dots\dots + (x_n{}^2 - 2x_n\mu + \mu^2)p_n \\
&= (x_1{}^2 p_1 + x_2{}^2 p_2 + x_3{}^2 p_3 + \dots\dots + x_n{}^2 p_n) - 2\mu(x_1p_1 + x_2p_2 + \dots\dots + x_np_n) \\
&\quad + \mu^2(p_1 + p_2 + p_3 + \dots\dots + p_n)
\end{aligned}$$

Now $p_1 + p_2 + \dots\dots + p_n = 1$

$$\begin{aligned}
\therefore\quad \sigma^2 &= \sum x_i{}^2 p_i - 2\mu(\sum x_ip_i) + \mu^2(1) \\
&= \sum x_i{}^2 p_i - 2\mu(\mu) + \mu^2 \qquad \{\text{since } \sum x_ip_i = \mu\} \\
&= \sum x_i{}^2 p_i - \mu^2
\end{aligned}$$

5 **a**

x_i	1	2	3	4	5
$P(x_i)$	0.1	0.2	0.4	0.2	0.1

b $$\begin{aligned}
\mu &= \sum x_ip_i \\
&= 1(0.1) + 2(0.2) + \dots\dots + 5(0.1) \\
&= 0.1 + 0.4 + 1.2 + 0.8 + 0.5 \\
&= 3
\end{aligned}$$

$$\begin{aligned}
\sigma &= \sqrt{\sum (x_i - \mu)^2 p_i} \\
&= \sqrt{\sum x_i^2 p_i - \mu^2} \\
&= \sqrt{1^2(0.1) + 2^2(0.2) + \dots\dots + 5^2(0.1) - (3.0)^2} \\
&= \sqrt{0.1 + 0.8 + 3.6 + 3.2 + 2.5 - 9} \\
&= \sqrt{1.2} \\
&\approx 1.10
\end{aligned}$$

c **i** $$\begin{aligned}
&\mathrm{P}(\mu - \sigma < X < \mu + \sigma) \\
&= \mathrm{P}(3 - 1.095 < X < 3 + 1.095) \\
&= \mathrm{P}(1.905 < X < 4.095) \\
&= \mathrm{P}(X = 2, 3, 4) \\
&= 0.2 + 0.4 + 0.2 \\
&= 0.8
\end{aligned}$$

ii $$\begin{aligned}
&\mathrm{P}(\mu - 2\sigma < X < \mu + 2\sigma) \\
&= \mathrm{P}(3 - 2.19 < X < 3 + 2.19) \\
&= \mathrm{P}(0.81 < X < 5.19) \\
&= \mathrm{P}(X = 1, 2, 3, 4 \text{ or } 5) \\
&= 0.1 + 0.2 + 0.4 + 0.2 + 0.1 \\
&= 1
\end{aligned}$$

6 Let X be the payout, so $x = \$20\,000$, $\$8000$, or $\$0$.

$\therefore$ the probability distribution is

x_i	20 000	8000	0
$P(x_i) = p_i$	0.0025	0.03	0.9675

The expectation is $\mu = \sum x_ip_i = 20\,000(0.0025) + 8000(0.03) + 0(0.9675)$
$= \$290$

The company expects to pay out \$290 on average in the long run.

$\therefore$ the company should charge $\$290 + \$100 = \$390$.

7

Die 2

Die 1		1	2	3	4	5	6
	1	1	2	3	4	5	6
	2	2	2	3	4	5	6
	3	3	3	3	4	5	6
	4	4	4	4	4	5	6
	5	5	5	5	5	5	6
	6	6	6	6	6	6	6

a

m_i	1	2	3	4	5	6
$P(m_i)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

b **i** The most likely result is 6, so this is the mode.

ii $p_1 = \frac{1}{36}$

$p_1 + p_2 = \frac{1}{36} + \frac{3}{36} = \frac{4}{36}$

$p_1 + p_2 + p_3 = \frac{4}{36} + \frac{5}{36} = \frac{9}{36}$

$p_1 + p_2 + p_3 + p_4 = \frac{9}{36} + \frac{7}{36} = \frac{16}{36}$

$p_1 + p_2 + p_3 + p_4 + p_5 = \frac{16}{36} + \frac{9}{36} = \frac{25}{36} \approx 0.694$

Since $p_1 + p_2 + p_3 + p_4 + p_5 \approx 0.694 \geqslant 0.5$, the median is 5.

iii $\mu = \sum m_i p_i$

$= 1\left(\frac{1}{36}\right) + 2\left(\frac{3}{36}\right) + 3\left(\frac{5}{36}\right) + \ldots. + 6\left(\frac{11}{36}\right)$

$= \frac{1}{36} + \frac{6}{36} + \frac{15}{36} + \frac{28}{36} + \frac{45}{36} + \frac{66}{36}$

$= \frac{161}{36}$

≈ 4.47

iv $\sigma = \sqrt{\sum m_i^{\,2} p_i - \mu^2}$

$= \sqrt{1^2\left(\frac{1}{36}\right) + 2^2\left(\frac{3}{36}\right) + \ldots. + 6^2\left(\frac{11}{36}\right) - \left(\frac{161}{36}\right)^2}$

$\approx \sqrt{1.971\,45}$

≈ 1.40

8 Examples are:

(1) Tossing one coin, where X is the number of 'heads' resulting. $x = 0$ or 1

x	0	1
$P(x)$	$\frac{1}{2}$	$\frac{1}{2}$

(2) Rolling one die, where X is the number on the uppermost face. $x = 1, 2, 3, 4, 5,$ or 6

x	1	2	3	4	5	6
$P(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

EXERCISE 25E

1 **a** mean of $X = \mathrm{E}(X)$

$= \sum x_i p_i$

$= 2(0.3) + 3(0.3) + 4(0.2) + 5(0.1) + 6(0.1)$

$= 3.4$

b $\mathrm{E}(X^2) = \sum x_i^{\,2} p_i = 4(0.3) + 9(0.3) + 16(0.2) + 25(0.1) + 36(0.1) = 13.2$

Now $\mathrm{Var}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2$

$= 13.2 - (3.4)^2$

$= 1.64$

c $\sigma = \sqrt{\mathrm{Var}(X)} \approx 1.28$

2 **a** $\sum p_i = 1$

$\therefore \quad 0.2 + k + 0.4 + 0.1 = 1$

$\therefore \quad k = 0.3$

b $\mathrm{E}(X) = \sum x_i p_i$

$= 5(0.2) + 6(0.3) + 7(0.4) + 8(0.1)$

$= 6.4$

c $\mathrm{Var}(X) = \sum x_i^{\,2} p_i - (\mathrm{E}(X))^2$

$= 25(0.2) + 36(0.3) + 49(0.4) + 64(0.1) - 6.4^2$

$= 0.84$

3 **a** $\mathrm{E}(X)$

$= \sum x_i p_i$

$= 1(0.4) + 2(0.3) + 3(0.2) + 4(0.1)$

$= 2$

b $\mathrm{E}(X^2)$

$= \sum x_i^{\,2} p_i$

$= 1(0.4) + 4(0.3) + 9(0.2) + 16(0.1)$

$= 5$

c $\text{Var}(X)$
$= \text{E}(X^2) - (\text{E}(X))^2$
$= 5 - 2^2$
$= 1$

d $\sigma = \sqrt{\text{Var}(X)}$
$= \sqrt{1}$
$= 1$

e $\text{E}(X+1)$
$= \text{E}(X) + \text{E}(1)$
$= 2 + 1$
$= 3$

f $\text{Var}(X+1)$
$= \text{E}\left((X+1)^2\right) - (\text{E}(X+1))^2$
$= \text{E}\left(X^2 + 2X + 1\right) - 3^2$
$= \text{E}(X^2) + 2\,\text{E}(X) + \text{E}(1) - 9$
$= 5 + 2(2) + 1 - 9$
$= 1$

g $\text{E}(2X^2 + 3X - 7)$
$= 2\,\text{E}(X^2) + 3\,\text{E}(X) - \text{E}(7)$
$= 2(5) + 3(2) - 7$
$= 9$

4 **a** $\text{E}(X) = 2.8$
$\therefore\ 1(0.2) + 2a + 3(0.3) + 4b = 2.8$
$\therefore\ 0.2 + 2a + 0.9 + 4b = 2.8$
$\therefore\ 2a + 4b = 1.7$ .... (1)
Also, $0.2 + a + 0.3 + b = 1$
$\therefore\ b = 0.5 - a$ .... (2)

Substituting (2) into (1) gives
$2a + 4(0.5 - a) = 1.7$
$\therefore\ 2a + 2 - 4a = 1.7$
$\therefore\ -2a = -0.3$
$\therefore\ a = 0.15$
and $b = 0.5 - 0.15$
$= 0.35$

b $\text{E}(X^2) = \sum {x_i}^2 p_i$
$= 1(0.2) + 4(0.15) + 9(0.3) + 16(0.35)$
$= 9.1$

$\therefore\ \text{Var}(X) = \text{E}(X)^2 - (\text{E}(X))^2$
$= 9.1 - 2.8^2$
$= 1.26$

5 **a** $\text{P}(X=0) = a(0) = 0$
$\text{P}(X=1) = a(-7) = -7a$
$\text{P}(X=2) = a(-12) = -12a$
$\text{P}(X=3) = a(-15) = -15a$
$\text{P}(X=4) = a(-16) = -16a$
$\text{P}(X=5) = a(-15) = -15a$
$\text{P}(X=6) = a(-12) = -12a$
$\text{P}(X=7) = a(-7) = -7a$
$\text{P}(X=8) = a(0) = 0$

$\therefore\ 2(-7a - 12a - 15a) - 16a = 1$
$\therefore\ a(-84) = 1$
$\therefore\ a = -\frac{1}{84}$

b $\text{E}(X) = \sum x_i p_i$
$= 1\left(\frac{7}{84}\right) + 2\left(\frac{12}{84}\right) + 3\left(\frac{15}{84}\right) + 4\left(\frac{16}{84}\right) + 5\left(\frac{15}{84}\right) + 6\left(\frac{12}{84}\right) + 7\left(\frac{7}{84}\right)$
$= \frac{336}{84} = 4$

c $\text{E}(X^2) = \sum {x_i}^2 p_i$
$= 1\left(\frac{7}{84}\right) + 4\left(\frac{12}{84}\right) + 9\left(\frac{15}{84}\right) + 16\left(\frac{16}{84}\right) + 25\left(\frac{15}{84}\right) + 36\left(\frac{12}{84}\right) + 49\left(\frac{7}{84}\right)$
$= \frac{1596}{84} = 19$

$\therefore\ \text{Var}(X) = \text{E}(X^2) - (\text{E}(X))^2$
$= 19 - 4^2$
$= 3$

$\therefore\ \sigma = \sqrt{\text{Var}(X)} = \sqrt{3}$

6 **a** $\left(\frac{1}{2}+\frac{1}{2}\right)^4 = \left(\frac{1}{2}\right)^4 + 4\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^1 + 6\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4$

$= \frac{1}{16} + \frac{4}{16} + \frac{6}{16} + \frac{4}{16} + \frac{1}{16}$

So, the probability distribution for X, the number of heads occurring, is:

x	0	1	2	3	4
$P(x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

b **i** $\mathrm{E}(X) = \sum x_i p_i = 0\left(\frac{1}{16}\right) + 1\left(\frac{4}{16}\right) + 2\left(\frac{6}{16}\right) + 3\left(\frac{4}{16}\right) + 4\left(\frac{1}{16}\right) = 2$

$\therefore$ mean $= 2$

ii $\mathrm{E}(X^2) = \sum x_i^2 p_i = 0\left(\frac{1}{16}\right) + 1\left(\frac{4}{16}\right) + 4\left(\frac{6}{16}\right) + 9\left(\frac{4}{16}\right) + 16\left(\frac{1}{16}\right) = 5$

$\therefore \quad \sigma = \sqrt{\mathrm{Var}\,(X)} = \sqrt{\mathrm{E}(X^2) - (\mathrm{E}(X))^2} = \sqrt{5-2^2} = 1$

7 **a** P(0 bitter, 3 not bitter) $= \dfrac{\binom{2}{0}\binom{8}{3}}{\binom{10}{3}} = \frac{42}{90} = \frac{7}{15}$

P(1 bitter, 2 not bitter) $= \dfrac{\binom{2}{1}\binom{8}{2}}{\binom{10}{3}} = \frac{7}{15}$

P(2 bitter, 1 not bitter) $= \dfrac{\binom{2}{2}\binom{8}{1}}{\binom{10}{3}} = \frac{6}{90} = \frac{1}{15}$

x	0	1	2
$P(x)$	$\frac{7}{15}$	$\frac{7}{15}$	$\frac{1}{15}$

b **i** $\mathrm{E}(X)$

$= 0\left(\frac{7}{15}\right) + 1\left(\frac{7}{15}\right) + 2\left(\frac{1}{15}\right)$

$= \frac{9}{15}$

$= 0.6$

$\therefore$ mean $= 0.6$ bitter almonds

ii $\mathrm{E}(X^2) = 0^2\left(\frac{7}{15}\right) + 1^2\left(\frac{7}{15}\right) + 2^2\left(\frac{1}{15}\right) = \frac{11}{15}$

$\therefore \quad \mathrm{Var}\,(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2$

$= \frac{11}{15} - \left(\frac{3}{5}\right)^2$

$= \frac{11}{15} - \frac{9}{25}$

≈ 0.3733 and so $\sigma \approx 0.611$

8 **a**

$$\begin{aligned} \mathrm{E}(Y) &= 0.9 \\ \therefore\quad -1(0.1) + 0(a) + 1(0.3) + 2b &= 0.9 \\ \therefore\quad 0.2 + 2b &= 0.9 \\ \therefore\quad 2b &= 0.7 \\ \therefore\quad b &= 0.35 \end{aligned}$$

Also, $0.1 + a + 0.3 + b = 1$

$\therefore \quad a = 1 - 0.1 - 0.3 - 0.35$

$\therefore \quad a = 0.25$

b $\mathrm{E}(Y^2) = (-1)^2(0.1) + 0^2(0.25) + 1^2(0.3) + 2^2(0.35) = 1.8$

$\therefore \quad \mathrm{Var}(Y) = \mathrm{E}(Y^2) - (\mathrm{E}(Y))^2 = 1.8 - 0.9^2 = 0.99$

9 **a** $\frac{1}{6} + \frac{1}{3} + \frac{1}{12} + a + \frac{1}{6} = 1$

$\therefore \quad a = \frac{1}{4}$

b **i** $\mathrm{E}(X) = 1\left(\frac{1}{6}\right) + 2\left(\frac{1}{3}\right) + 3\left(\frac{1}{12}\right) + 4\left(\frac{1}{4}\right) + 5\left(\frac{1}{6}\right) = \frac{35}{12} = 2\frac{11}{12}$

$\mathrm{E}(X^2) = 1^2\left(\frac{1}{6}\right) + 2^2\left(\frac{1}{3}\right) + 3^2\left(\frac{1}{12}\right) + 4^2\left(\frac{1}{4}\right) + 5^2\left(\frac{1}{6}\right) = \frac{125}{12}$

$\therefore \quad \mathrm{Var}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2$

$= \frac{125}{12} - \left(\frac{35}{12}\right)^2$

$= \frac{1500}{144} - \frac{1225}{144}$

$= \frac{275}{144} \approx 1.91$

ii $\begin{aligned}&\mathrm{E}(2X)\\&= 2\,\mathrm{E}(X)\\&= 2\times\tfrac{35}{12}\\&= \tfrac{35}{6}\\&= 5\tfrac{5}{6}\end{aligned}$ $\quad$ $\begin{aligned}&\mathrm{Var}(2X)\\&= 2^2\,\mathrm{Var}(X)\\&= 4\times\tfrac{275}{144}\\&= \tfrac{275}{36}\\&= 7\tfrac{23}{36}\\&\approx 7.64\end{aligned}$

iii $\begin{aligned}&\mathrm{E}(X-1)\\&= \mathrm{E}(X)-\mathrm{E}(1)\\&= \tfrac{35}{12}-1\\&= \tfrac{23}{12}\\&= 1\tfrac{11}{12}\end{aligned}$ $\quad$ $\begin{aligned}&\mathrm{Var}(X-1)\\&= \mathrm{Var}(X)\\&= \tfrac{275}{144}\\&\approx 1.91\end{aligned}$

10 X has mean 6 and standard deviation 2.

$$\begin{aligned}\mathrm{E}(Y) &= \mathrm{E}(2X+5)\\&= 2\mathrm{E}(X)+\mathrm{E}(5)\\&= 2\times 6+5\\&= 17\end{aligned}$$

$\therefore$ mean of Y distribution is 17

$$\begin{aligned}\mathrm{Var}(Y) &= \mathrm{Var}(2X+5)\\&= 2^2\mathrm{Var}(X)\\&= 4\times 2^2\\&= 16\end{aligned}$$

$\therefore$ standard deviation of Y distribution is $\sqrt{16}=4$

11 **a** $$\begin{aligned}\mathrm{E}(aX+b) &= \mathrm{E}(aX)+\mathrm{E}(b) && \{\text{using } \mathrm{E}(A+B)=\mathrm{E}(A)+\mathrm{E}(B)\}\\&= a\,\mathrm{E}(X)+\mathrm{E}(b) && \{\text{using } \mathrm{E}(kX)=k\,\mathrm{E}(X)\}\\&= a\,\mathrm{E}(X)+b && \{\text{using } \mathrm{E}(k)=k,\ \ k \text{ a constant}\}\end{aligned}$$

b **i** $$\begin{aligned}\mathrm{E}(Y) &= \mathrm{E}(3X+4)\\&= 3\,\mathrm{E}(X)+4\\&= 3(3)+4\\&= 13\end{aligned}$$

ii $$\begin{aligned}\mathrm{E}(Y) &= \mathrm{E}(-2X+1)\\&= -2\,\mathrm{E}(X)+1\\&= -2(3)+1\\&= -5\end{aligned}$$

iii $$\begin{aligned}\mathrm{E}(Y) &= \mathrm{E}\left(\frac{4X-2}{3}\right)\\&= \mathrm{E}\left(\tfrac{4}{3}X-\tfrac{2}{3}\right)\\&= \tfrac{4}{3}\,\mathrm{E}(X)-\tfrac{2}{3}\\&= \tfrac{4}{3}(3)-\tfrac{2}{3}=3\tfrac{1}{3}\end{aligned}$$

12 X has mean 5 and standard deviation 2.

a $\mathrm{E}(Y)=\mathrm{E}(2X+3)=2\,\mathrm{E}(X)+3=2\times 5+3=13$

$\mathrm{Var}(Y)=\mathrm{Var}(2X+3)=2^2\ \mathrm{Var}(X)=4\times 2^2=16$

b $\mathrm{E}(Y)=\mathrm{E}(-2X+3)=-2\,\mathrm{E}(X)+3=-2\times 5+3=-7$

$\mathrm{Var}(Y)=\mathrm{Var}(-2X+3)=(-2)^2\ \mathrm{Var}(X)=4\times 2^2=16$

c $Y=\dfrac{X-5}{2}=\tfrac{1}{2}X-\tfrac{5}{2}$

$\mathrm{E}(Y)=\mathrm{E}\left(\tfrac{1}{2}X-\tfrac{5}{2}\right)=\tfrac{1}{2}\mathrm{E}(X)-\tfrac{5}{2}=\tfrac{1}{2}\times 5-\tfrac{5}{2}=0$

$\mathrm{Var}(Y)=\mathrm{Var}\left(\tfrac{1}{2}X-\tfrac{5}{2}\right)=\left(\tfrac{1}{2}\right)^2\ \mathrm{Var}(X)=\tfrac{1}{4}\times 2^2=1$

13 $Y=2X+3$

a $$\begin{aligned}\mathrm{E}(Y) &= \mathrm{E}(2X+3)\\&= 2\,\mathrm{E}(X)+3\end{aligned}$$

b $$\begin{aligned}\mathrm{E}(Y^2) &= \mathrm{E}(4X^2+12X+9)\\&= 4\,\mathrm{E}(X^2)+12\,\mathrm{E}(X)+9\end{aligned}$$

c $$\begin{aligned}\mathrm{Var}(Y) &= \mathrm{E}(Y^2)-(\mathrm{E}(Y))^2\\&= [4\mathrm{E}(X^2)+12\mathrm{E}(X)+9]-[2\mathrm{E}(X)+3]^2\\&= 4\mathrm{E}(X^2)+12\mathrm{E}(X)+9-[4(\mathrm{E}(X))^2+12\mathrm{E}(X)+9]\\&= 4\mathrm{E}(X^2)-4(\mathrm{E}(X))^2\end{aligned}$$

14 $$\begin{aligned}\mathrm{Var}(aX+b) &= \mathrm{E}\left((aX+b)^2\right)-(\mathrm{E}(aX+b))^2\\&= \mathrm{E}\left(a^2X^2+2abX+b^2\right)-(a\,\mathrm{E}(X)+b)^2\\&= a^2\,\mathrm{E}(X^2)+2ab\,\mathrm{E}(X)+b^2-\left[a^2(\mathrm{E}(X))^2+2ab\,\mathrm{E}(X)+b^2\right]\\&= a^2\,\mathrm{E}(X^2)+\cancel{2ab\mathrm{E}(X)}+\cancel{b^2}-a^2(\mathrm{E}(X))^2-\cancel{2ab\mathrm{E}(X)}-\cancel{b^2}\\&= a^2\left(\mathrm{E}(X^2)-(\mathrm{E}(X))^2\right)\\&= a^2\ \mathrm{Var}(X)\end{aligned}$$

EXERCISE 25F.1

1 **a** $(p+q)^4$
$= p^4 + 4p^3q + 6p^2q^2 + 4pq^3 + q^4$

b $\text{P(3 heads)} = 4p^3q$
$= 4\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)$ $\{\text{as } p = q = \frac{1}{2}\}$
$= \frac{1}{4}$

2 **a** $(p+q)^5 = p^5 + 5p^4q + 10p^3q^2 + 10p^2q^3 + 5pq^4 + q^5$

b **i** P(4H and 1T)
$= 5p^4q$
$= 5\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)$
$= \frac{5}{32}$

ii P(2H and 3T)
$= 10p^2q^3$
$= 10\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3$
$= \frac{10}{32}$
$= \frac{5}{16}$

iii P(HHHHT)
$= \left(\frac{1}{2}\right)^4 \times \frac{1}{2}$
$= \frac{1}{32}$

3 **a** $\left(\frac{2}{3}+\frac{1}{3}\right)^4 = \left(\frac{2}{3}\right)^4 + 4\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right) + 6\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2 + 4\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^4$

b $\text{P(S)} = \frac{2}{3}$, $\text{P(S}') = \frac{1}{3}$ $\quad$ S$'$ represents an almond centre

i P(all S)
$= \left(\frac{2}{3}\right)^4$
$= \frac{16}{81}$

ii P(two of each)
$= 6\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2$
$= \frac{8}{27}$

iii P(at least 2 strawberry creams)
$= \text{P(all S or 3S, 1S}' \text{ or 2S, 2S}')$
$= \left(\frac{2}{3}\right)^4 + 4\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right) + 6\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2$
$= \frac{16}{81} + \frac{32}{81} + \frac{24}{81}$
$= \frac{72}{81}$
$= \frac{8}{9}$

4 **a** $\left(\frac{3}{4}+\frac{1}{4}\right)^5 = \left(\frac{3}{4}\right)^5 + 5\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right) + 10\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2 + 10\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^3 + 5\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^4 + \left(\frac{1}{4}\right)^5$

b P('normal' kiwi) $= \frac{3}{4}$, P('flat back') $= \frac{1}{4}$

i P(2 'flat backs')
$= \text{P(3F}', \text{2F)}$
$= 10 \times \left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2$
$= \frac{135}{512}$

ii P(at least 3 'flat backs')
$= \text{P(2F}', \text{3F or 1F}', \text{4F or 5F)}$
$= 10\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^3 + 5\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^4 + \left(\frac{1}{4}\right)^5$
$= \frac{53}{512}$ on simplifying

iii P(at most 3 'normal' kiwis) $= 1 - $ P(4 or 5 normal kiwis)
$= 1 - \text{P(4F}', \text{1F or 5F}')$
$= 1 - \left(5\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right) + \left(\frac{3}{4}\right)^5\right)$
$= \frac{47}{128}$

5 Let X be the number of Huy's hits.

a Using the binomial expansion,
$P(X=2) = 6\left(\frac{4}{5}\right)^2\left(\frac{1}{5}\right)^2 \approx 0.154$

b $P(X \geqslant 2)$
$= 1 - P(X \leqslant 1)$
$\approx 1 - \left(4\left(\frac{4}{5}\right)\left(\frac{1}{5}\right)^3 + \left(\frac{1}{5}\right)^4\right)$
≈ 0.973

EXERCISE 25F.2

1 **a** The binomial distribution applies, as tossing a coin has two possible outcomes (a head or a tail) and each toss is independent of every other toss.

b The binomial distribution applies, as this is equivalent to tossing one coin 100 times.

c The binomial distribution applies as we can draw out a red or a blue marble with the same chances each time.

d The binomial distribution does not apply as the result of each draw is dependent upon the results of previous draws.

e The binomial distribution does not apply, assuming that ten bolts are drawn without replacement, as we do not have a repetition of independent trials.

2 Let X be the number of defective light bulbs.

a $P(X = 2) \approx 0.0305$ {using technology}

b $P(X \geqslant 1) \approx 0.265$

3 If X is the number of questions Raj answers correctly, then X is binomial. There are $n = 10$ independent trials with probability $p = \frac{1}{5}$ of a corrrect answer for each.

$P(\text{Raj passes}) = P(X \geqslant 7)$
$\approx 0.000\,864$ {or about 9 in 10 000}

4 X is the random variable for the number working night-shift.
$\therefore$ $X = 0, 1, 2, 3, 4, 5, 6, 7$ and $X \sim B(7, 0.35)$.

a $P(X = 3)$
$= \binom{7}{3}(0.35)^3(0.65)^4$
≈ 0.268

b $P(X < 4)$
$= P(X \leqslant 3)$
≈ 0.800

c P(at least 4 work night-shift)
$= P(X \geqslant 4)$
≈ 0.200

5 X is the number of faulty items.
$\therefore$ $X = 0, 1, 2, 3, ..., 12$ and $X \sim B(12, 0.06)$.

a $P(X = 0) = \binom{12}{0}(0.06)^0(0.94)^{12}$
≈ 0.476

b P(at most one is faulty) $= P(X \leqslant 1)$
≈ 0.840

c P(at least two are faulty) $= P(X \geqslant 2)$
≈ 0.160

d P(less than four are faulty) $= P(X < 4)$
$= P(X \leqslant 3)$
≈ 0.996

6 X is the random variable for the number of apples with a blemish.
$\therefore$ $X = 0, 1, 2, 3, ..., 25$ and $X \sim B(25, 0.05)$.

a $P(X = 2)$
$= \binom{25}{2}(0.05)^2(0.95)^{23}$
≈ 0.231

b $P(X \geqslant 1)$
≈ 0.723

c $E(X) = np$
$= 25 \times 0.05$
$= 1.25$ apples

7 X is the random variable for the number of times in a week that the bus is on time.
Since it is late 2 in every 5 days, and on time 3 in every 5 days,
$X = 0, 1, 2, 3, 4, 5, 6,$ or 7 and $X \sim B(7, 0.6)$.

a $P(X = 7) = \binom{7}{7}(0.6)^7(0.4)^0$
≈ 0.0280

b P(on time only on Monday) $= 0.6 \times (0.4)^6$
$\approx 0.002\,46$

c $P(X = 6) = \binom{7}{6}(0.6)^6(0.4)$
≈ 0.131

d $P(X \geqslant 4) \approx 0.710$

8 X is the random variable for the number of students with the flu.
$\therefore$ $X = 0, 1, 2, 3, ..., 25$ and $X \sim B(25, 0.3)$.

a **i** $P(X \geqslant 2) \approx 0.998$

ii P(test cancelled) $= P(X \geqslant 6)$ {20% of 25 = 5}
≈ 0.807

b Expected absentees from 350 students $= 0.3 \times 350$
$= 105$ students

9 X is the random variable for the number of successful shots from the free throw line.

$\therefore \quad X = 0, 1, 2, 3, ..., 20 \quad \text{and} \quad X \sim \text{B}(20, 0.94).$

a **i** $\text{P}(X = 20) = \binom{20}{20}(0.94)^{20}(0.06)^0$
≈ 0.290

ii $\text{P}(X \geqslant 18) \approx 0.885$

b $\text{E}(X) = np = 20 \times 0.94$
$= 18.8$ successful throws

10 $\text{P(M wins a game against J)} = \frac{2}{3} \quad \therefore \quad \text{P(M wins)} = \frac{2}{3} \quad \text{P(J wins)} = \frac{1}{3}$

$$\text{P(J wins a set 6 games to 4)} = \text{P}(\underbrace{\text{J wins 5 of the first 9 games}}_{\text{this is binomial with } n=9 \text{ trials of probability } p=\frac{1}{3}} \textbf{ and } \text{J wins the 10th game})$$

$\approx 0.1024 \times \frac{1}{3}$
≈ 0.0341

11 If there are n dice thrown, $\text{P(no sixes)} = \left(\frac{5}{6}\right)^n$

$\therefore \quad \text{P(at least 1 six)} = 1 - \left(\frac{5}{6}\right)^n$

$\therefore$ need to find the smallest integer n such that $1 - \left(\frac{5}{6}\right)^n > 0.5$

$\therefore \quad \left(\frac{5}{6}\right)^n < 0.5$

$\therefore \quad n\log\left(\frac{5}{6}\right) < \log(0.5)$

$\therefore \quad n > \dfrac{\log(0.5)}{\log\left(\frac{5}{6}\right)} \qquad \{\log\left(\frac{5}{6}\right) < 0\}$

$\therefore \quad n > 3.80$

$\therefore$ at least 4 dice are needed.

12 If a fair coin is tossed 200 times, then $n = 200$ and $p = \frac{1}{2}$.

a $\text{P}(90 \leqslant X \leqslant 110)$
≈ 0.863

b $\text{P}(95 < X < 105)$
$= \text{P}(96 \leqslant X \leqslant 104)$
≈ 0.475

13 $n = 38, \quad p = 0.75$

$\text{P}(24 \leqslant X \leqslant 31) \approx 0.837$

14 **a**

$$\text{P}(x) = \binom{n}{x}p^{n-x}(1-p)^x = \left(\frac{n!}{x!(n-x)!}\right)p^{n-x}(1-p)^x$$

$$\therefore \quad \text{P}(x+1) = \binom{n}{x+1}p^{n-(x+1)}(1-p)^{x+1}$$

$$= \frac{n!}{(x+1)![n-(x+1)]!}\,p^{n-x-1}(1-p)^{x+1}$$

$$= \frac{n!}{(x+1)x!(n-x-1)!}\left(\frac{p^{n-x}}{p}\right)(1-p)^x \times (1-p)$$

$$= \frac{n!(n-x)}{(x+1)x!(n-x)!}\,p^{n-x}(1-p)^x\left(\frac{1-p}{p}\right)$$

$$= \left(\frac{n-x}{x+1}\right)\left(\frac{1-p}{p}\right)\left(\frac{n!}{x!(n-x)!}\right)p^{n-x}(1-p)^x$$

$$= \left(\frac{n-x}{x+1}\right)\left(\frac{1-p}{p}\right)\text{P}(x), \quad \text{where} \quad \text{P}(0) = \binom{n}{0}p^{n-0}(1-p)^0 = p^n$$

b $\mathrm{P}(0) = p^n = \left(\frac{1}{2}\right)^5 = \frac{1}{32}$

$$\mathrm{P}(1) = \left(\frac{n-x}{x+1}\right)\left(\frac{1-p}{p}\right)\mathrm{P}(0)$$

$$= \left(\frac{5-0}{0+1}\right)(1)\left(\frac{1}{32}\right) \qquad \{p = \tfrac{1}{2} \text{ and } 1-p = \tfrac{1}{2} \quad \therefore \quad \frac{1-p}{p} = 1\}$$

$$= \frac{5}{32}$$

$$\mathrm{P}(2) = \left(\frac{5-1}{1+1}\right)(1)\left(\frac{5}{32}\right) = \frac{10}{32}$$

$$\mathrm{P}(3) = \left(\frac{5-2}{2+1}\right)(1)\left(\frac{10}{32}\right) = \frac{10}{32}$$

$$\mathrm{P}(4) = \left(\frac{5-3}{3+1}\right)(1)\left(\frac{10}{32}\right) = \frac{5}{32}$$

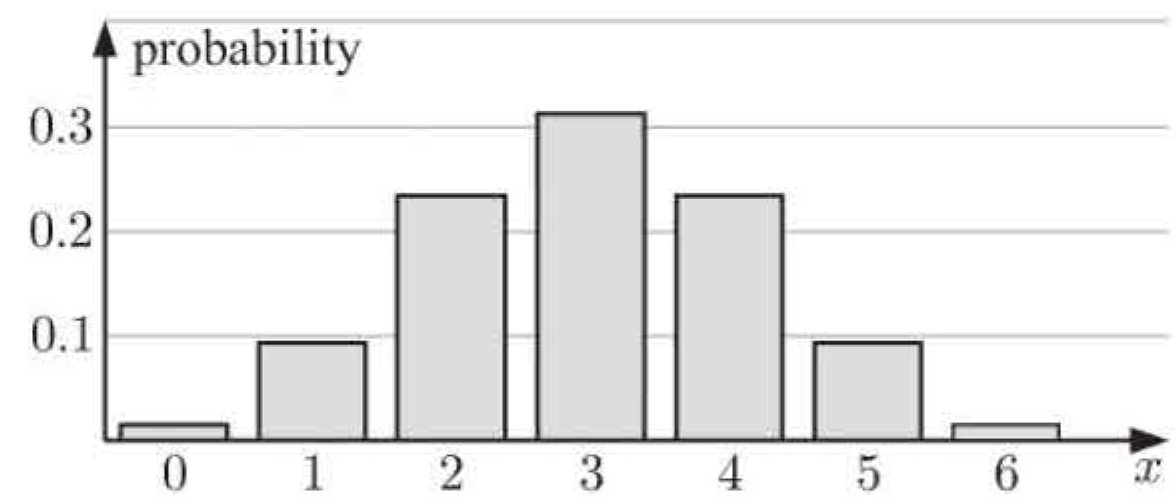
$$\mathrm{P}(5) = \left(\frac{5-4}{4+1}\right)(1)\left(\frac{5}{32}\right) = \frac{1}{32}$$

EXERCISE 25F.3

1 **a** **i** $\mu = np = 6 \times 0.5 = 3$ $\qquad \sigma = \sqrt{np(1-p)} = \sqrt{6 \times 0.5 \times 0.5} \approx 1.22$

ii

x	0	1	2	3	4	5	6
$P(x)$	0.0156	0.0938	0.2344	0.3125	0.2344	0.0938	0.0156

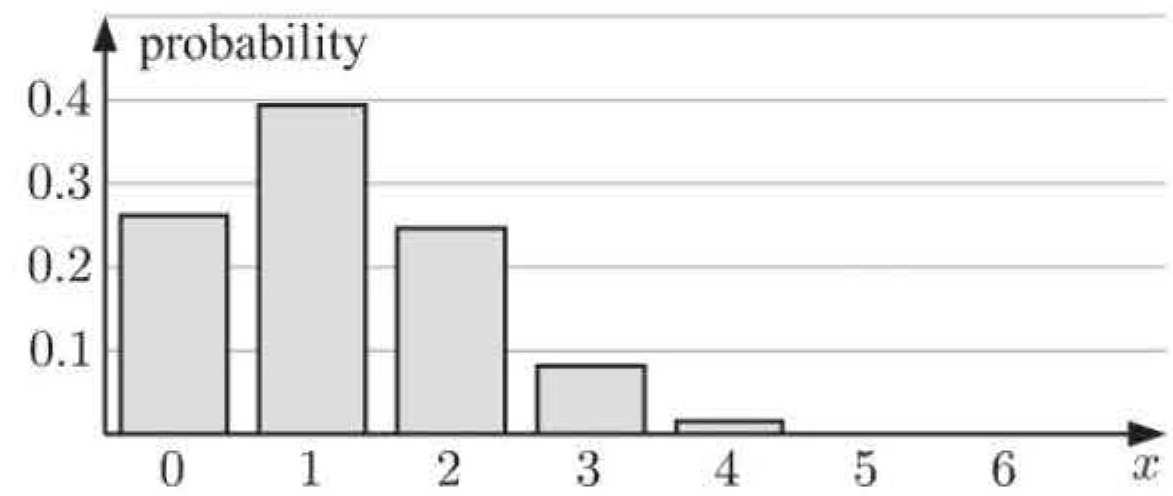


iii The distribution is bell-shaped.

b **i** $\mu = np = 6 \times 0.2 = 1.2$ $\qquad \sigma = \sqrt{np(1-p)} = \sqrt{6 \times 0.2 \times 0.8} \approx 0.980$

ii

x	0	1	2	3	4	5	6
$P(x)$	0.2621	0.3932	0.2458	0.0819	0.0154	0.0015	0.0001

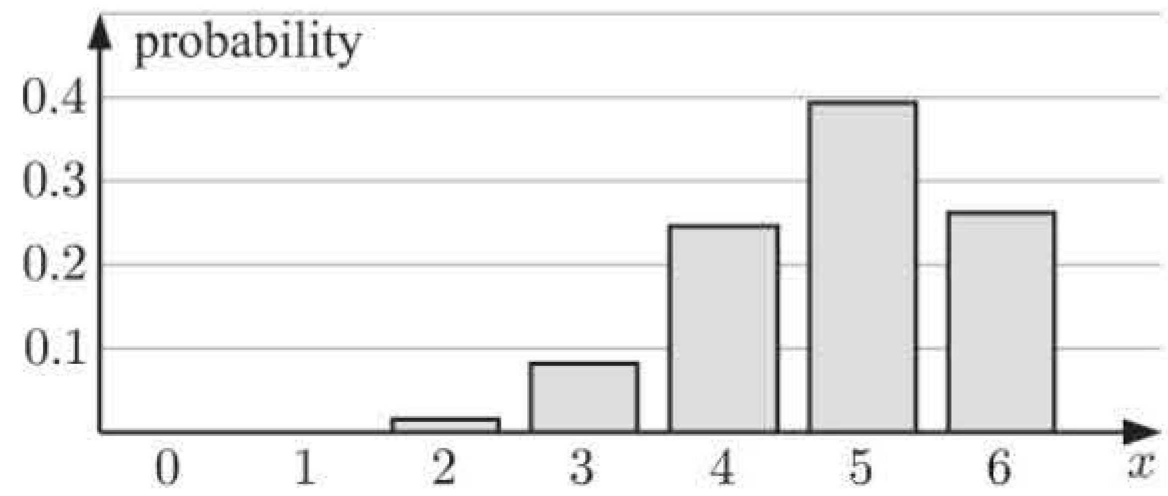


iii The distribution is positively skewed.

c **i** $\mu = np = 6 \times 0.8 = 4.8$ $\qquad \sigma = \sqrt{np(1-p)} = \sqrt{6 \times 0.8 \times 0.2} \approx 0.980$

ii

x	0	1	2	3	4	5	6
$P(x)$	0.0001	0.0015	0.0154	0.0819	0.2458	0.3932	0.2621



iii The distribution is negatively skewed, and is the exact reflection of the distribution in **b**.

2 $n = 10$, $p = \frac{1}{2}$, mean $\mu = np = 10 \times \frac{1}{2} = 5$ and variance $\sigma^2 = np(1-p) = 10 \times \frac{1}{2} \times \frac{1}{2} = 2.5$

3 **a** $X \sim \mathrm{B}(3, p)$

$\mathrm{P}(X=0) = \binom{3}{0}p^0(1-p)^3 = (1-p)^3$

$\mathrm{P}(X=1) = \binom{3}{1}p^1(1-p)^2 = 3p(1-p)^2$

$\mathrm{P}(X=2) = \binom{3}{2}p^2(1-p)^1 = 3p^2(1-p)$

$\mathrm{P}(X=3) = \binom{3}{3}p^3(1-p)^0 = p^3$

x_i	0	1	2	3
$P(x_i)$	$(1-p)^3$	$3p(1-p)^2$	$3p^2(1-p)$	p^3

b
$$\begin{aligned}\mu &= \sum x_i p_i \\ &= 0(1-p)^3 + 1 \times 3p(1-p)^2 + 2 \times 3p^2(1-p) + 3p^3 \\ &= 3p(1-p)^2 + 6p^2(1-p) + 3p^3 \\ &= 3p(1-2p+p^2) + 6p^2 - 6p^3 + 3p^3 \\ &= 3p - 6p^2 + 3p^3 + 6p^2 - 6p^3 + 3p^3 \\ &= 3p \quad \text{as required}\end{aligned}$$

c
$$\begin{aligned}\sigma^2 &= \sum x_i^2 p_i - \mu^2 \\ &= 0^2 \times (1-p)^3 + 1^2 \times 3p(1-p)^2 + 2^2 \times 3p^2(1-p) + 3^2p^3 - (3p)^2 \\ &= 3p(1-p)^2 + 12p^2(1-p) + 9p^2(p-1) \\ &= (1-p)\left[3p(1-p) + 12p^2 - 9p^2\right] \\ &= (1-p)\left[3p - 3p^2 + 3p^2\right] \\ &= 3p(1-p)\end{aligned}$$

$\therefore\ \sigma = \sqrt{3p(1-p)}$ as required

4 **a** $n = 30$, $p = 0.04$

$$\begin{aligned}\mu &= np = 30 \times 0.04 = 1.2 \\ \sigma &= \sqrt{np(1-p)} = \sqrt{30 \times 0.04 \times 0.96} \approx 1.07\end{aligned}$$

b $n = 30$, $p = 0.96$

$$\begin{aligned}\mu &= np = 30 \times 0.96 = 28.8 \\ \sigma &= \sqrt{np(1-p)} = \sqrt{30 \times 0.96 \times 0.04} \approx 1.07\end{aligned}$$

5 $n = 30$, $p = 0.13$

$\therefore$ mean $\mu = np = 30 \times 0.13 = 3.9$ and standard deviation $\sigma = \sqrt{np(1-p)} = \sqrt{30 \times 0.13 \times 0.87} \approx 1.84$

EXERCISE 25G

1 **a** mean $= \dfrac{\sum f_i x_i}{\sum f_i} = \dfrac{0+18+24+18+12+0+6}{52} = \dfrac{78}{52} = 1.5$

b Using $m = 1.5$, we find $p_x = \dfrac{(1.5)^x e^{-1.5}}{x!}$, $x = 0, 1, 2, 3, ...$

So, we can obtain:

$$p_0 = \frac{(1.5)^0 e^{-1.5}}{0!} \approx 0.2231 \quad \therefore \quad 52p_0 \approx 11.6$$

$$p_1 = \frac{(1.5)^1 e^{-1.5}}{1!} \approx 0.3347 \quad \therefore \quad 52p_1 \approx 17.4$$

$$p_2 = \frac{(1.5)^2 e^{-1.5}}{2!} \approx 0.2510 \quad \therefore \quad 52p_2 \approx 13.1$$

$$p_3 = \frac{(1.5)^3 e^{-1.5}}{3!} \approx 0.1255 \quad \therefore \quad 52p_3 \approx 6.5$$

$$p_4 = \frac{(1.5)^4 e^{-1.5}}{4!} \approx 0.0471 \quad \therefore \quad 52p_4 \approx 2.4$$

$$p_5 = \frac{(1.5)^5 e^{-1.5}}{5!} \approx 0.0141 \quad \therefore \quad 52p_5 \approx 0.7$$

$$p_6 = \frac{(1.5)^6 e^{-1.5}}{6!} \approx 0.0035 \quad \therefore \quad 52p_6 \approx 0.2$$

Comparison:

x	0	1	2	3	4	5	6
f	12	18	12	6	3	0	1
$52p_x$	11.6	17.4	13.1	6.5	2.4	0.7	0.2

The fit is excellent.

2 Standard deviation $= 2.67$

a mean $= \sigma^2$
$= 2.67^2$
$= 7.1289$
≈ 7.13

b $m = 7.1289$
$\therefore \; p_x \approx \dfrac{(7.1289)^x e^{-7.1289}}{x!}$ where $x = 0, 1, 2, 3, 4, 5, ...$

c **i** $\text{P}(X = 2)$
$= \dfrac{(7.1289)^2 e^{-7.1289}}{2!}$
≈ 0.0204

ii $\text{P}(X \leqslant 3)$
≈ 0.0753

iii $\text{P}(X \geqslant 5)$
$= 1 - \text{P}(X \leqslant 4)$
$\approx 1 - 0.162$
≈ 0.838

iv $\text{P}(X \geqslant 3 \mid X \geqslant 1)$
$= \dfrac{\text{P}(X \geqslant 3 \cap X \geqslant 1)}{\text{P}(X \geqslant 1)}$
$= \dfrac{\text{P}(X \geqslant 3)}{\text{P}(X \geqslant 1)}$
$= \dfrac{1 - \text{P}(X \leqslant 2)}{1 - \text{P}(X = 0)}$
$\approx \dfrac{1 - 0.0269}{1 - 0.0008}$
≈ 0.974

3 **a** $\mu = \dfrac{1 \times 156 + 2 \times 132 + 3 \times 75 + 4 \times 33 + 5 \times 9 + 6 \times 3 + 7 \times 1}{91 + 156 + 132 + 75 + 33 + 9 + 3 + 1}$
$= \dfrac{847}{500}$
$= 1.694$

b Using $m = 1.694$, we find $p_x = \dfrac{(1.694)^x e^{-1.694}}{x!}$ where $x = 0, 1, 2, 3, 4, ...$

So, we can obtain:

$$500p_0 = 500 \times 1.694^0 \times e^{-1.694} \times \tfrac{1}{0!} \approx 91.9$$
$$500p_1 = 500 \times 1.694^1 \times e^{-1.694} \times \tfrac{1}{1!} \approx 155.7$$
$$500p_2 = 500 \times 1.694^2 \times e^{-1.694} \times \tfrac{1}{2!} \approx 131.8$$
$$500p_3 = 500 \times 1.694^3 \times e^{-1.694} \times \tfrac{1}{3!} \approx 74.4$$
$$500p_4 = 500 \times 1.694^4 \times e^{-1.694} \times \tfrac{1}{4!} \approx 31.5$$
$$500p_5 = 500 \times 1.694^5 \times e^{-1.694} \times \tfrac{1}{5!} \approx 10.7$$
$$500p_6 = 500 \times 1.694^6 \times e^{-1.694} \times \tfrac{1}{6!} \approx 3.0$$
$$500p_7 = 500 \times 1.694^7 \times e^{-1.694} \times \tfrac{1}{7!} \approx 0.7$$

Comparison:

x	0	1	2	3	4	5	6	7
f	91	156	132	75	33	9	3	1
$500p_x$	92	156	132	74	32	11	3	1

The fit is excellent.

c $\text{Var}(X)$
$= \text{E}(X^2) - (\text{E}(X))^2$
$= \sum {x_i}^2 p_i - (1.694)^2$
$= 1 \times \frac{156}{500} + 4 \times \frac{132}{500} + 9 \times \frac{75}{500} + 16 \times \frac{33}{500} + 25 \times \frac{9}{500} + 36 \times \frac{3}{500} + 49 \times \frac{1}{500} - (1.694)^2$
≈ 1.6683

$\therefore$ $\sigma \approx 1.29$ and $\sqrt{m} = \sqrt{1.694} \approx 1.30$
$\therefore$ σ is very close to the square root of the mean.

4 $p_x = \dfrac{3^x e^{-3}}{x!}$ where $x = 0, 1, 2, 3, 4, 5, ...$

a $\text{P}(X = 0)$
$= \dfrac{3^0 e^{-3}}{0!}$
≈ 0.0498

b $\text{P}(X \geqslant 3)$
$= 1 - \text{P}(X \leqslant 2)$
$\approx 1 - 0.423$
≈ 0.577

c P(some requests are refused)
$= \text{P}(X \geqslant 5)$
$= 1 - \text{P}(X \leqslant 4)$
$\approx 1 - 0.815$
≈ 0.185

d $\text{P}(X \geqslant 4 \mid X \geqslant 2)$
$= \dfrac{\text{P}(X \geqslant 4 \cap X \geqslant 2)}{\text{P}(X \geqslant 2)}$
$= \dfrac{\text{P}(X \geqslant 4)}{\text{P}(X \geqslant 2)}$
$= \dfrac{1 - \text{P}(X \leqslant 3)}{1 - \text{P}(X \leqslant 1)}$
$\approx \dfrac{1 - 0.647\,23}{1 - 0.199\,14}$
≈ 0.440

5 $\mathrm{P}(X=x)=\dfrac{m^x e^{-m}}{x!}$ where $x=0, 1, 2, 3, 4, \ldots$

a If $\mathrm{P}(X=1)+\mathrm{P}(X=2)=\mathrm{P}(X=3)$,

then $\dfrac{me^{-m}}{1!}+\dfrac{m^2e^{-m}}{2!}=\dfrac{m^3e^{-m}}{3!}$

$$\therefore \quad m+\frac{m^2}{2}=\frac{m^3}{6} \quad \{\div e^{-m}\}$$

$$\therefore \quad 6m+3m^2=m^3$$

$$\therefore \quad m(m^2-3m-6)=0 \quad \text{where } m\neq 0$$

$$\therefore \quad m^2-3m-6=0$$

$$\therefore \quad m=\frac{3\pm\sqrt{9-4(1)(-6)}}{2}=\frac{3\pm\sqrt{33}}{2}$$

But $m>0$, so $m=\dfrac{3+\sqrt{33}}{2}$

b **i**
$$\begin{aligned}&\mathrm{P}(X\geqslant 3)\\&=1-\mathrm{P}(X\leqslant 2)\\&\approx 1-0.494\\&\approx 0.506\end{aligned}$$

ii
$$\begin{aligned}&\mathrm{P}(X\leqslant 4 \mid X\geqslant 2)\\&=\frac{\mathrm{P}(X\leqslant 4\cap X\geqslant 2)}{\mathrm{P}(X\geqslant 2)}\\&=\frac{\mathrm{P}(X=2,\ 3 \text{ or } 4)}{\mathrm{P}(X\geqslant 2)}\\&=\frac{\mathrm{P}(X\leqslant 4)-\mathrm{P}(X\leqslant 1)}{1-\mathrm{P}(X\leqslant 1)}\\&\approx\frac{0.8629-0.248\,66}{1-0.248\,66}\\&\approx 0.818\end{aligned}$$

6 Let X be the number of aerofoils which disintegrate from a sample of 100.
Each aerofoil has a 2% chance of disintegrating, so $m=\mathrm{E}(X)=0.02\times 100=2$

$\therefore$ $\mathrm{P}(X=x)=\dfrac{2^x e^{-2}}{x!}$ where $x=0, 1, 2, 3, \ldots$

a $\mathrm{P}(X=1)=\dfrac{2^1e^{-2}}{1!}=\dfrac{2}{e^2}\approx 0.271$

b $\mathrm{P}(X=2)=\dfrac{2^2e^{-2}}{2!}=\dfrac{4}{2e^2}\approx 0.271$

c
$$\begin{aligned}\mathrm{P}(X\leqslant 2)&=\mathrm{P}(X=0)+\mathrm{P}(X=1)+\mathrm{P}(X=2)\\&=\frac{2^0e^{-2}}{0!}+\frac{2^1e^{-2}}{1!}+\frac{2^2e^{-2}}{2!}\\&=\frac{1}{e^2}+\frac{2}{e^2}+\frac{2}{e^2}\\&=\frac{5}{e^2}\approx 0.677\end{aligned}$$

7 **a** A person who drives 10 times per week will drive $10\times 52=520$ times in one year.
Let X be the number of fatalities from driving 520 times.

$$m=\mathrm{E}(X)=0.0002\times 520=0.104$$

$$\therefore \quad \mathrm{P}(X=x)=\frac{0.104^x e^{-0.104}}{x!}$$

$\therefore$ $\mathrm{P}(X=0)=\dfrac{0.104^0e^{-0.104}}{0!}\approx 0.901$ $\quad\therefore$ the probability of surviving is 0.901.

b P(driving for n years and surviving) $= (0.901)^n$

$\therefore$ we need to find n such that $(0.901)^n = 0.5$

$$\therefore \quad n \log 0.901 = \log 0.5$$

$$\therefore \quad n = \frac{\log 0.5}{\log 0.901} \approx 6.66$$

$\therefore$ you can drive for 6 years and still have a better than even chance of surviving.

8 Let X be the number of flaws in 1 metre of material.

$m = 1.7 \quad \therefore \ \mathrm{P}(X = x) = \dfrac{1.7^x e^{-1.7}}{x!}, \quad x = 0, 1, 2, 3, ...$

a $\mathrm{P}(X = 3) = \dfrac{1.7^3 e^{-1.7}}{3!} \approx 0.150$

b P(at least one flaw in 2 metres)
$= 1 - \mathrm{P}(\text{no flaws in 2 metres})$
$= 1 - (\mathrm{P}(X = 0))^2$
$\approx 1 - (0.1827)^2$
≈ 0.967

c $\mathrm{P}(X = 0) \approx 0.183$
$\mathrm{P}(X = 1) \approx 0.311$
$\mathrm{P}(X = 2) \approx 0.264$
$\mathrm{P}(X = 3) \approx 0.150$
$\mathrm{P}(X = 4) \approx 0.064$
Finding the highest of the probabilities, the mode is 1 flaw per metre.

9 **a** $\mathrm{P}(Y = y) = \dfrac{m^y e^{-m}}{y!}, \ y = 0, 1, 2, 3, ...$

$\mathrm{P}(Y = 3) = \mathrm{P}(Y = 1) + 2\mathrm{P}(Y = 2)$

$$\therefore \quad \frac{m^3 e^{-m}}{3!} = \frac{m^1 e^{-m}}{1!} + 2\frac{m^2 e^{-m}}{2!}$$

$$\therefore \quad \frac{m^3}{6} = m + m^2 \qquad \{\times\, e^m\}$$

$$\therefore \quad m^3 = 6m + 6m^2$$

$$\therefore \quad m(m^2 - 6m - 6) = 0 \ \text{ where } \ m \neq 0$$

$$\therefore \quad m^2 - 6m - 6 = 0$$

$$\therefore \quad m = \frac{6 \pm \sqrt{36 - 4(1)(-6)}}{2} = 3 \pm \sqrt{15}$$

But $m > 0$, so $m = 3 + \sqrt{15} \approx 6.8730$

b **i** $\mathrm{P}(1 < Y < 5)$
$= \mathrm{P}(Y \leqslant 4) - \mathrm{P}(Y \leqslant 1)$
≈ 0.177

ii $\mathrm{P}(2 \leqslant Y \leqslant 6 \mid Y \geqslant 4)$

$$= \frac{\mathrm{P}(2 \leqslant Y \leqslant 6 \,\cap\, Y \geqslant 4)}{\mathrm{P}(Y \geqslant 4)}$$

$$= \frac{\mathrm{P}(4 \leqslant Y \leqslant 6)}{\mathrm{P}(Y \geqslant 4)}$$

$$= \frac{\mathrm{P}(Y \leqslant 6) - \mathrm{P}(Y \leqslant 3)}{1 - \mathrm{P}(Y \leqslant 3)}$$

≈ 0.417

10 $\mathrm{P}(U = u) = \dfrac{x^u e^{-x}}{u!}$ where $u = 0, 1, 2, 3, ...$

a $y = \mathrm{P}(U = 0, 1 \text{ or } 2)$

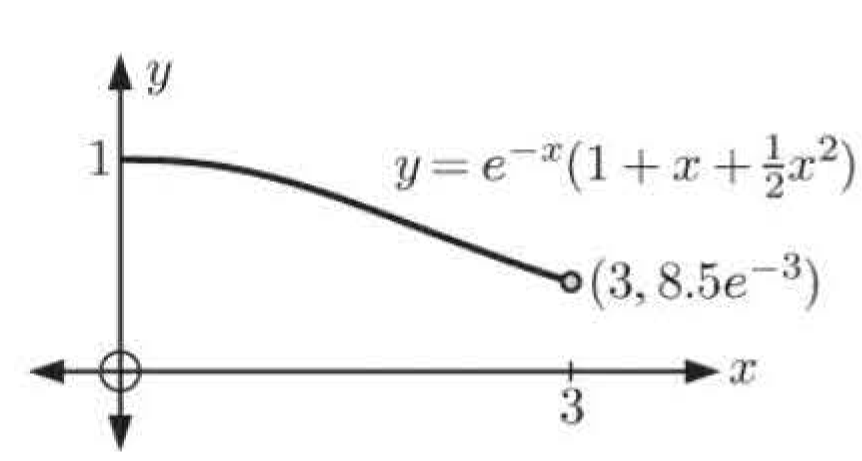
$= \mathrm{P}(U = 0) + \mathrm{P}(U = 1) + \mathrm{P}(U = 2)$

$$= \frac{x^0 e^{-x}}{0!} + \frac{x^1 e^{-x}}{1!} + \frac{x^2 e^{-x}}{2!}$$

$= e^{-x} + xe^{-x} + \frac{1}{2}x^2 e^{-x}$

$\therefore \quad y = e^{-x}(1 + x + \frac{1}{2}x^2)$

b



c $$y = e^{-x}(1 + x + \tfrac{1}{2}x^2)$$
$$\therefore\ \frac{dy}{dx} = -e^{-x}(1 + x + \tfrac{1}{2}x^2) + e^{-x}(1 + x)$$
$$= -e^{-x} - xe^{-x} - \tfrac{1}{2}x^2e^{-x} + e^{-x} + xe^{-x}$$
$$= -\tfrac{1}{2}x^2e^{-x}$$

Since $x^2e^{-x} > 0$, $\dfrac{dy}{dx} < 0$ for all $x > 0$.

$\therefore$ as the mean x increases, $y = \text{P}(U \leqslant 2)$ decreases.

REVIEW SET 25A

1 **a** $\text{P}(X = x) = \dfrac{a}{x^2 + 1}$ for $x = 0, 1, 2, 3$

x	0	1	2	3
$\text{P}(X = x)$	a	$\frac{a}{2}$	$\frac{a}{5}$	$\frac{a}{10}$

Now $a + \dfrac{a}{2} + \dfrac{a}{5} + \dfrac{a}{10} = 1$ $\{$as $\sum_{x=0}^{3} \text{P}(X = x) = 1\}$

$\therefore\ 10a + 5a + 2a + a = 10$

$\therefore\ 18a = 10$

$\therefore\ a = \frac{5}{9}$

b $\text{P}(X \geqslant 1) = \text{P}(X = 1, 2, \text{ or } 3)$
$= \text{P}(X = 1) + \text{P}(X = 2) + \text{P}(X = 3)$
$= \frac{5}{18} + \frac{1}{9} + \frac{5}{90}$
$= \frac{4}{9}$

or $\text{P}(X \geqslant 1) = 1 - \text{P}(X < 1)$
$= 1 - \text{P}(X = 0)$
$= 1 - \frac{5}{9}$
$= \frac{4}{9}$

2 Let X be the number of defective toothbrushes.

$\therefore\ X \sim \text{B}(120, 0.04)$

$\mu = np$
$= 120 \times 0.04$
$= 4.8$ defectives

$\sigma = \sqrt{np(1-p)}$
$= \sqrt{120 \times 0.04 \times 0.96}$
≈ 2.15

3

x	0	1	2	3	4
$\text{P}(X = x)$	0.10	0.30	0.45	0.10	k

a If this is a probability distribution then $\sum P(x_i) = 1$

$\therefore\ 0.1 + 0.3 + 0.45 + 0.1 + k = 1$

$\therefore\ 0.95 + k = 1$

$\therefore\ k = 0.05$

b $\text{P}(X \geqslant 3)$
$= \text{P}(X = 3) + \text{P}(X = 4)$
$= 0.10 + 0.05$
$= 0.15$

c $\text{E}(X) = \sum x_i p_i$
$= 0(0.10) + 1(0.30) + 2(0.45) + 3(0.10) + 4(0.05)$
$= 0 + 0.3 + 0.9 + 0.3 + 0.2$
$= 1.7$

d $\text{E}(X^2) = \sum {x_i}^2 p_i$
$= 0(0.10) + 1(0.30) + 4(0.45) + 9(0.10) + 16(0.05)$
$= 3.8$

$\sigma = \sqrt{\text{Var}(X)} = \sqrt{\text{E}(X^2) - (\text{E}(X))^2}$
$= \sqrt{3.8 - 1.7^2}$
≈ 0.954

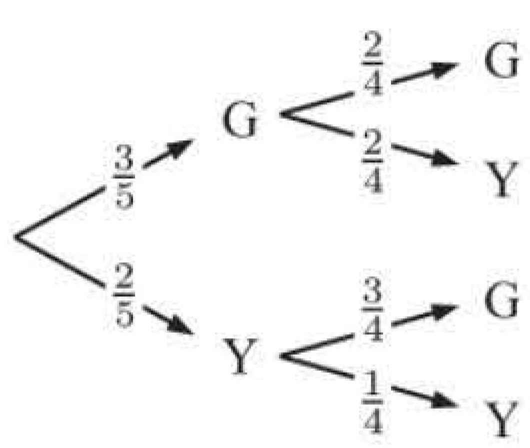
4 **a** $\left(\frac{3}{5}+\frac{2}{5}\right)^4 = \underbrace{\left(\frac{3}{5}\right)^4}_{4\text{B}} + \underbrace{4\left(\frac{3}{5}\right)^3\left(\frac{2}{5}\right)}_{3\text{B}\ 1\text{B}'} + \underbrace{6\left(\frac{3}{5}\right)^2\left(\frac{2}{5}\right)^2}_{2\text{B}\ 2\text{B}'} + \underbrace{4\left(\frac{3}{5}\right)\left(\frac{2}{5}\right)^3}_{1\text{B}\ 3\text{B}'} + \underbrace{\left(\frac{2}{5}\right)^4}_{4\text{B}'}$

$\text{P(B)} = \frac{12}{20}$
$= \frac{3}{5}$
$\therefore\ \text{P(B}') = \frac{2}{5}$

b **i** P(2 Blue inks)
$= \text{P(2B and 2B}')$
$= 6\left(\frac{3}{5}\right)^2\left(\frac{2}{5}\right)^2$
$= \dfrac{6\times 9\times 4}{5^4}$
$= \frac{216}{625}$

ii P(at most 2 Blue inks)
$= \text{P(2B and 2B}'\ \text{ or }\ 1\text{B and 3B}'\ \text{ or }\ 4\text{B}')$
$= 6\left(\frac{3}{5}\right)^2\left(\frac{2}{5}\right)^2 + 4\left(\frac{3}{5}\right)\left(\frac{2}{5}\right)^3 + \left(\frac{2}{5}\right)^4$
$= \dfrac{6\times 9\times 4 + 4\times 3\times 8 + 16}{625}$
$= \frac{328}{625}$

5 1st draw 2nd draw



Event	X	Probability
GG	2	$\frac{3}{5}\times\frac{2}{4}=\frac{3}{10}$
GY	1	$\frac{3}{5}\times\frac{2}{4}=\frac{3}{10}$
YG	1	$\frac{2}{5}\times\frac{3}{4}=\frac{3}{10}$
YY	0	$\frac{2}{5}\times\frac{1}{4}=\frac{1}{10}$

	a	**b**	**c**
x	0	1	2
$\text{P}(X=x)$	$\frac{1}{10}$	$\frac{3}{5}$	$\frac{3}{10}$

d $\text{E}(X) = 0\times\frac{1}{10} + 1\times\frac{3}{5} + 2\times\frac{3}{10} = \frac{6}{5}\quad (=1\frac{1}{5})$

6 **a**

Result	*Pays*
1	£2
2	£4
3	£6
4	£8
5	£10
6	£12

Expectation
$= \frac{1}{6}\times £2 + \frac{1}{6}\times £4 + \frac{1}{6}\times £6 + \frac{1}{6}\times £8 + \frac{1}{6}\times £10 + \frac{1}{6}\times £12$
$= \frac{1}{6}\times £42$
$= £7$

b Expected gain is $£7 - £8 = -£1$.
$\therefore$ advise Lakshmi against playing several games, as £1 is expected to be lost per game in the long run.

7 **a** $n=7$ and $r = \{0, 1, 2, 3, ..., 7\}$
$= x$

$\therefore\ k = \binom{7}{x}$

b $n=7,\ \ p=\frac{1}{3}$

$\therefore\ \mu = np$
$= \frac{7}{3}\quad (\approx 2.33)$

and $\sigma^2 = np(1-p)$
$= 7\times\frac{1}{3}\times\frac{2}{3}$
$= \frac{14}{9}\quad (\approx 1.56)$

8 **a** $\left(\frac{4}{5}+\frac{1}{5}\right)^5 = \left(\frac{4}{5}\right)^5 + 5\left(\frac{4}{5}\right)^4\left(\frac{1}{5}\right)^1 + 10\left(\frac{4}{5}\right)^3\left(\frac{1}{5}\right)^2 + 10\left(\frac{4}{5}\right)^2\left(\frac{1}{5}\right)^3 + 5\left(\frac{4}{5}\right)^1\left(\frac{1}{5}\right)^4 + \left(\frac{1}{5}\right)^5$

b Let X = the number of goals scored

i P(3 goals **then** 2 misses)
$= \text{P(GGGG}'\text{G}')$
$= \left(\frac{4}{5}\right)^3\times\left(\frac{1}{5}\right)^2$
$= \frac{64}{3125}$
≈ 0.0205

ii P(3 goals and 2 misses)
$= \text{P}(X=3)$
$= 10\left(\frac{4}{5}\right)^3\left(\frac{1}{5}\right)^2$
$= \frac{128}{625}$
≈ 0.205

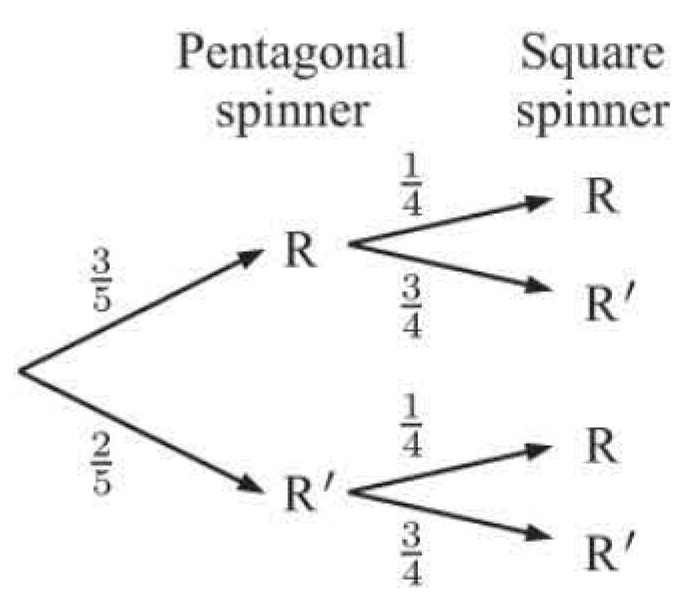
9

Result	*Pays*
1, 3, 5	\$2
2	\$3
4	\$6
6	\$9

a $\text{Expected return} = \frac{3}{6} \times \$2 + \frac{1}{6} \times \$3 + \frac{1}{6} \times \$6 + \frac{1}{6} \times \$9$
$= \frac{1}{6}(\$24)$
$= \$4$

b For a \$5 amount to play the game, the club expects a \$1 return per game
$\therefore$ for 75 people, the return expected is \$75.

10 **a**



b $\text{P(exactly one red)} = \text{P}(RR') + \text{P}(R'R)$
$= \frac{3}{5} \times \frac{3}{4} + \frac{2}{5} \times \frac{1}{4}$
$= \frac{9}{20} + \frac{1}{10}$
$= \frac{11}{20}$

c **i** $n = 10, \quad p = \frac{11}{20}$
$\text{P}(X = 1) = \binom{10}{1}\left(\frac{11}{20}\right)^1\left(\frac{9}{20}\right)^9$
$\text{P}(X = 9) = \binom{10}{9}\left(\frac{11}{20}\right)^9\left(\frac{9}{20}\right)^1$

ii $\binom{10}{1} = \binom{10}{9} = 10$ so we need only consider the parts $\left(\frac{11}{20}\right)^1\left(\frac{9}{20}\right)^9$ and $\left(\frac{11}{20}\right)^9\left(\frac{9}{20}\right)^1$.
Now, $11 \times 9^9 < 11^9 \times 9$, and the denominators are the same in each case.
$\therefore$ it is more likely that exactly one red will occur 9 times.

11 **a** $\frac{1}{3} + \frac{1}{6} + \frac{1}{4} + y = 1$
$\therefore \quad y = \frac{1}{4}$
$\therefore$ the probability of obtaining the number 24 is $y = \frac{1}{4}$.

b $\text{E}(X) = 6\left(\frac{1}{3}\right) + 12\left(\frac{1}{6}\right) + x\left(\frac{1}{4}\right) + 24\left(\frac{1}{4}\right) = 14$
$\therefore \quad 2 + 2 + \dfrac{x}{4} + 6 = 14$
$\therefore \quad \dfrac{x}{4} = 4$
$\therefore \quad x = 16$
So, the fourth number is 16.

c $p_1 = \frac{1}{3}$
$p_1 + p_2 = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$
Since $p_1 + p_2 = 0.5$, the median is $\dfrac{12 + 16}{2} = 14$.
The most likely result is 6, so this is the mode.

12 X has mean μ and standard deviation σ.
$\therefore \quad \text{E}(X) = \mu$ and $\sigma^2 = \text{Var}(X) = \text{E}(X^2) - (\text{E}(X))^2$
Now $Y = aX + b \quad \therefore$ mean of $Y = \text{E}(Y)$
$= \text{E}(aX + b)$
$= \text{E}(aX) + \text{E}(b)$
$= a\,\text{E}(X) + b$
$= a\mu + b$

Also, $\text{Var}(aX + b) = \text{E}\left((aX + b)^2\right) - (\text{E}(aX + b))^2$
$= \text{E}\left(a^2X^2 + 2abX + b^2\right) - (a\,\text{E}(X) + b)^2$
$= a^2\,\text{E}(X^2) + 2ab\,\text{E}(X) + b^2 - \left[a^2(\text{E}(X))^2 + 2ab\,\text{E}(X) + b^2\right]$
$= a^2\,\text{E}(X^2) + \cancel{2ab\,\text{E}(X)} + \cancel{b^2} - a^2(\text{E}(X))^2 - \cancel{2ab\,\text{E}(X)} - \cancel{b^2}$
$= a^2\left(\text{E}(X^2) - (\text{E}(X))^2\right)$
$= a^2\sigma^2$

$\therefore$ standard deviation of $Y = \sqrt{a^2\sigma^2}$

$= \sqrt{a^2}\,\sigma$ $\quad$ {since $\sigma > 0$}

$= |a|\sigma$

REVIEW SET 25B

1 **a** $P(x) = k\left(\frac{3}{4}\right)^x\left(\frac{1}{4}\right)^{3-x}$ for $x = 0, 1, 2, 3$

$P(0) = k\left(\frac{3}{4}\right)^0\left(\frac{1}{4}\right)^3 = \dfrac{k}{64}$

$P(1) = k\left(\frac{3}{4}\right)^1\left(\frac{1}{4}\right)^2 = \dfrac{3k}{64}$

$P(2) = k\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^1 = \dfrac{9k}{64}$

$P(3) = k\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^0 = \dfrac{27k}{64}$

x	0	1	2	3
$P(x)$	$\frac{k}{64}$	$\frac{3k}{64}$	$\frac{9k}{64}$	$\frac{27k}{64}$

Now $\dfrac{k}{64} + \dfrac{3k}{64} + \dfrac{9k}{64} + \dfrac{27k}{64} = 1$ {as $\sum P(x_i) = 1$}

$\therefore \dfrac{40k}{64} = 1$

$\therefore k = \frac{8}{5}$ $(= 1.6)$

b $\text{P}(X \geqslant 1) = 1 - \text{P}(X = 0)$

$= 1 - \dfrac{k}{64}$

$= 1 - \dfrac{1.6}{64}$

$= 0.975$

c $\text{E}(X) = \sum x_i p_i$

$= 0 \times \frac{1.6}{64} + 1 \times \frac{3\times1.6}{64} + 2 \times \frac{9\times1.6}{64} + 3 \times \frac{27\times1.6}{64}$

$= 2.55$

d $\sigma = \sqrt{\sum(x_i - \mu)^2 p_i}$

$= \sqrt{(0-2.55)^2 \times \frac{1.6}{64} + (1-2.55)^2 \times \frac{3\times1.6}{64} + (2-2.55)^2 \times \frac{9\times1.6}{64} + (3-2.55)^2 \times \frac{27\times1.6}{64}}$

≈ 0.740

2 X is the number of defectives. Then $X \sim \text{B}(10, 0.18)$. $X = 0, 1, 2, 3, ..., 10$.

a $\text{P}(X = 1)$

$= \binom{10}{1}(0.18)^1(0.82)^9$

≈ 0.302

b $\text{P}(X = 2)$

$= \binom{10}{2}(0.18)^2(0.82)^8$

≈ 0.298

c $\text{P}(X \geqslant 2)$

≈ 0.561

3 Expected number of major knee surgeries $= np$

$= 487 \times 0.0132$

≈ 6.43

4 X is the number of visitors who make a voluntary donation upon entry.
Then $X = 0, 1, 2, 3, ..., 175$ and $X \sim \text{B}(175, 0.24)$.

a $\text{E}(X) = np$

$= 175 \times 0.24$

$= 42$

b $\text{P}(X < 40) = \text{P}(X \leqslant 39)$

≈ 0.334

5 If X is the number of X-rays which show the fracture, then $X = 0, 1, 2, 3, 4$ and $X \sim \text{B}(4, 0.96)$.

a $\text{P}(X = 4) = \binom{4}{4}(0.96)^4(0.04)^0$

≈ 0.849

b $\text{P}(X = 0) = \binom{4}{0}(0.96)^0(0.04)^4$

$\approx 2.56 \times 10^{-6}$

c $\text{P}(X \geqslant 3) \approx 0.991$

d $\text{P}(X = 1) = \binom{4}{1}(0.96)^1(0.04)^3$

$\approx 0.000\,246$

6 X is the number of players who turn up to a game.
Then $X = 0, 1, 2, 3, ..., 8$ and $X \sim B(8, 0.75)$.

a **i** $P(X = 8) = \binom{8}{8}(0.75)^8(0.25)^0$
≈ 0.100

ii $P(\text{team has to forfeit}) = P(X \leqslant 4)$
≈ 0.114

b Expected number of games forfeited in $30 = np$
$\approx 30 \times 0.1138$ {from **a ii**}
≈ 3.41

7 **a** If the mean is 30 then $np = 30$ .... (1)
If the variance is 22.5 then $np(1-p) = 22.5$ .... (2)
Substituting (1) into (2), we get $30(1-p) = 22.5$

$$\therefore \quad 1 - p = \frac{22.5}{30}$$
$$\therefore \quad p = 1 - \frac{22.5}{30}$$
$$\therefore \quad p = 0.25$$
and so $n \times 0.25 = 30$
$$\therefore \quad n = 120$$

So, $n = 120$ and $p = 0.25$ (or $\frac{1}{4}$).

b **i** $P(X = 25)$
≈ 0.0501

ii $P(X \geqslant 25)$
≈ 0.878

iii $P(15 \leqslant X \leqslant 25)$
≈ 0.172

8 $P(X = 0) + P(X = 1) + P(X = 2) + ... = 1$

$$\therefore \quad a\left(\tfrac{5}{6}\right)^0 + a\left(\tfrac{5}{6}\right)^1 + a\left(\tfrac{5}{6}\right)^2 + ... = 1$$
$$\therefore \quad a\underbrace{\left(1 + \tfrac{5}{6} + \left(\tfrac{5}{6}\right)^2 + ...\right)}_{\text{infinite geometric series with } u_1 = 1,\ r = \frac{5}{6}} = 1$$
$$\therefore \quad a\left(\frac{1}{1 - \frac{5}{6}}\right) = 1$$
$$\therefore \quad a(6) = 1$$
$$\therefore \quad a = \tfrac{1}{6}$$

9 **a** $P(\text{X wins}) = \frac{3}{5} = 0.6$ $\quad P(\text{Y wins}) = \frac{2}{5} = 0.4$
Probability generator is $(0.6 + 0.4)^6$

$$= \underbrace{(0.6)^6}_{\substack{\text{X wins 6}}} + \underbrace{6(0.6)^5(0.4)}_{\substack{\text{X wins 5} \\ \text{Y wins 1}}} + \underbrace{15(0.6)^4(0.4)^2}_{\substack{\text{X wins 4} \\ \text{Y wins 2}}} + \underbrace{20(0.6)^3(0.4)^3}_{\substack{\text{X wins 3} \\ \text{Y wins 3}}} + \underbrace{15(0.6)^2(0.4)^4}_{\substack{\text{X wins 2} \\ \text{Y wins 4}}} + \underbrace{6(0.6)(0.4)^5}_{\substack{\text{X wins 1} \\ \text{Y wins 5}}} + \underbrace{(0.4)^6}_{\substack{\text{X wins 0} \\ \text{Y wins 6}}}$$

b **i** $P(\text{Y wins 3})$
$= 20(0.6)^3(0.4)^3$
≈ 0.276

ii $P(\text{Y wins at least 5})$
$= 6(0.6)^1(0.4)^5 + (0.4)^6$
≈ 0.0410

10 Let X be the number of objects broken by the first glass blower.
$\therefore$ X has mean $m = 20 \times \frac{1}{200} = 0.1$
$\therefore$ $X \sim Po(0.1)$
Let Y be the number of objects broken by the second glass blower.
$\therefore$ Y has mean $m = 40 \times \frac{3}{200} = 0.6$
$\therefore$ $Y \sim Po(0.6)$

$\therefore$ P(glass blowers break 2 or more objects between them)
$= 1 -$ P(glass blowers break 0 or 1 objects between them)
$= 1 - \mathrm{P}(X = 0, Y = 0 \quad \text{or} \quad X = 1, Y = 0 \quad \text{or} \quad X = 0, Y = 1)$

$$= 1 - \left[\frac{0.1^0 e^{-0.1}}{0!} \times \frac{0.6^0 e^{-0.6}}{0!} + \frac{0.1^1 e^{-0.1}}{1!} \times \frac{0.6^0 e^{-0.6}}{0!} + \frac{0.1^0 e^{-0.1}}{0!} \times \frac{0.6^1 e^{-0.6}}{1!}\right]$$

≈ 0.156

11 **a** $\mathrm{P}(X = 3) = 0.226\,89$

$\therefore \binom{7}{3} p^3 (1-p)^{7-3} = 0.226\,89$

$\therefore 35p^3(1-p)^4 = 0.226\,89$

$\therefore p \approx 0.300$ or 0.564 {using technology}

$\therefore p \approx 0.300$ {smallest p}

b $\mathrm{P}(X \leqslant 4) \approx 0.850$

12 $\mathrm{P}(X = x) = k\left(x + \dfrac{1}{x}\right)$

a $\mathrm{P}(X = 1) = k(1 + \frac{1}{1}) = 2k$

$\mathrm{P}(X = 2) = k(2 + \frac{1}{2}) = \frac{5}{2}k$

$\mathrm{P}(X = 3) = k(3 + \frac{1}{3}) = \frac{10}{3}k$

$\mathrm{P}(X = 4) = k(4 + \frac{1}{4}) = \frac{17}{4}k$

Now $\sum P(x_i) = 1$

$\therefore 2k + \frac{5}{2}k + \frac{10}{3}k + \frac{17}{4}k = 1$

$\therefore \frac{145}{12}k = 1$

$\therefore k = \frac{12}{145}$

b Using $k = \frac{12}{145}$ we obtain:

x	1	2	3	4
$P(x)$	$\frac{24}{145}$	$\frac{30}{145}$	$\frac{40}{145}$	$\frac{51}{145}$

$\therefore \mathrm{E}(X) = 1\left(\frac{24}{145}\right) + 2\left(\frac{30}{145}\right) + 3\left(\frac{40}{145}\right) + 4\left(\frac{51}{145}\right) = \frac{408}{145} \approx 2.81$

$\mathrm{E}(X^2) = 1^2\left(\frac{24}{145}\right) + 2^2\left(\frac{30}{145}\right) + 3^2\left(\frac{40}{145}\right) + 4^2\left(\frac{51}{145}\right) = \frac{264}{29}$

$\therefore \mathrm{Var}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2$

$= \frac{264}{29} - \left(\frac{408}{145}\right)^2$

≈ 1.19

c $p_1 = \frac{24}{145}$

$p_1 + p_2 = \frac{24}{145} + \frac{30}{145} = \frac{54}{145}$

$p_1 + p_2 + p_3 = \frac{54}{145} + \frac{40}{145} = \frac{94}{145} \approx 0.648$

Since $p_1 + p_2 + p_3 \geqslant 0.5$, the median is 3.

The most likely value of X is 4, so this is the mode.

13 $\mathrm{P}(Y > 3) \approx 0.033\,768\,97$

$\therefore \mathrm{P}(Y \leqslant 3) \approx 1 - 0.033\,768\,97$

$\therefore \mathrm{P}(Y = 0, 1, 2, \text{ or } 3) \approx 0.966\,231\,03$

$$\therefore \frac{m^0 e^{-m}}{0!} + \frac{m^1 e^{-m}}{1!} + \frac{m^2 e^{-m}}{2!} + \frac{m^3 e^{-m}}{3!} \approx 0.966\,231\,03$$

$$\therefore e^{-m}\left(1 + m + \frac{m^2}{2} + \frac{m^3}{6}\right) \approx 0.966\,231\,03$$

Using technology on the domain $m > 0$, we find $m = 1.2$

$\therefore \mathrm{P}(Y < 3) = \mathrm{P}(Y \leqslant 2)$

≈ 0.879

REVIEW SET 25C

1 **a** $\sum P(X = x_i) = 1$

$\therefore \dfrac{k}{2\times 1} + \dfrac{k}{2\times 2} + \dfrac{k}{2\times 3} = 1$

$\therefore \dfrac{k}{2} + \dfrac{k}{4} + \dfrac{k}{6} = 1$

$\therefore 6k + 3k + 2k = 12$

$\therefore 11k = 12$

$\therefore k = \frac{12}{11}$

b $\sum P(x_i) = 1$

$\therefore \dfrac{k}{2} + 0.2 + k^2 + 0.3 = 1$

$\therefore 2k^2 + k + 1 = 2$

$\therefore 2k^2 + k - 1 = 0$

$\therefore (2k-1)(k+1) = 0$

$\therefore k = -1, \frac{1}{2}$

If $k = -1$, then $P(0) = \frac{-1}{2} < 0$, so $P(x)$ would not be a valid probability distribution function.

$\therefore k = \frac{1}{2}$

2 **a** $P(X = x) = \binom{4}{x}\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{4-x}$

$\therefore P(X = 0) = \binom{4}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^4 = 0.0625$

$P(X = 1) = \binom{4}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^3 = 0.25$

$P(X = 2) = \binom{4}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 = 0.375$

$P(X = 3) = \binom{4}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^1 = 0.25$

$P(X = 4) = \binom{4}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^0 = 0.0625$

x	0	1	2	3	4
$P(X = x)$	0.0625	0.25	0.375	0.25	0.0625

b $\mu = \sum x_i P(X = x_i)$
$= 0\times 0.0625 + 1\times 0.25 + 2\times 0.375 + 3\times 0.25 + 4\times 0.0625$
$= 2$

c $n = 4,\ p = \frac{1}{2}$ $\quad \therefore \sigma = \sqrt{np(1-p)} = \sqrt{4\times\frac{1}{2}\times\frac{1}{2}}$
$= \sqrt{1}$
$= 1$

3 $X \sim B(1200, 0.4)$

So, mean of $X = np$
$= 1200 \times 0.4$
$\therefore \mu = 480$

and standard deviation of $X = \sqrt{np(1-p)}$
$= \sqrt{1200\times 0.4\times 0.6}$
$\therefore \sigma \approx 17.0$

4 X is the number of trees that survive the first year.

$\therefore X = 0, 1, 2, 3, 4, 5$ and $X \sim B(5, 0.4)$

a $P(X = 1) = \binom{5}{1}(0.4)^1(0.6)^4$
≈ 0.259

b $P(X \leqslant 1) \approx 0.337$

c $P(X \geqslant 1) \approx 0.922$

5 **a** **i** In the numbers 1 to 20, there are 10 even numbers.
However, '4' and '16' are square numbers, so 8 of the numbers in the bag win \$3.
$\therefore$ P(player wins \$3) $= \frac{8}{20} = \frac{2}{5}$

ii 1, 4, 9, and 16 are the only square numbers in the bag.
But '4' and '16' are even, so 2 of the numbers in the bag win \$6.
$\therefore$ P(player wins \$6) $= \frac{2}{20} = \frac{1}{10}$

iii 2 numbers are both even and square (4 and 16).
$\therefore$ P(player wins \$9) $= \frac{2}{20} = \frac{1}{10}$

b Expected winnings $= \frac{2}{5}\times \$3 + \frac{1}{10}\times \$6 + \frac{1}{10}\times \$9$
$= \$2.70$

$\therefore$ for the game to be fair, players should be charged \$2.70 per game.

6 Suppose X is the event that a 6 is rolled. $\therefore\ n = 360,\quad p = \frac{1}{6}$

a $\text{P}(X < 50) = \text{P}(X \leqslant 49)$
≈ 0.0660

b $\text{P}(55 \leqslant X \leqslant 65) \approx 0.563$

7 With n tosses, P(getting at least 2 heads) $= 1 -$ P(getting at most 1 head)

We need to find n such that P(getting at least 2 heads) > 0.99

So, $1 - \left(\binom{n}{0}\left(\frac{1}{2}\right)^n + \binom{n}{1}\left(\frac{1}{2}\right)^n\right) > 0.99$

Using technology, $n \geqslant 11 \quad \{n \in \mathbb{Z}\}$

$\therefore\ n = 11$ is the smallest value of n.

8 **a** P(hot water unit fails within one year) $=$ P(all 20 components fail)

$$= (0.85)^{20} \approx 0.0388$$

b P(hot water unit with n components fails within one year) $= 0.85^n$

$\therefore$ P(hot water unit with n components is operating after one year) $= 1 - 0.85^n$

$\therefore$ we need to find the smallest integer n such that $1 - 0.85^n \geqslant 0.98$

$$\therefore\ 0.85^n \leqslant 0.02$$
$$\therefore\ n \log 0.85 \leqslant \log 0.02$$
$$\therefore\ n \geqslant \frac{\log 0.02}{\log 0.85} \quad \{\log 0.85 < 0\}$$
$$\therefore\ n \geqslant 24.1$$

$\therefore$ at least 25 solar components are needed.

9 **a** $\text{P(hit)} = \frac{1}{3},\quad \text{P(miss)} = \frac{2}{3}$

Probability generator is

$$\left(\tfrac{1}{3}+\tfrac{2}{3}\right)^5 = \underbrace{\left(\tfrac{1}{3}\right)^5}_{X=5} + \underbrace{5\left(\tfrac{1}{3}\right)^4\left(\tfrac{2}{3}\right)}_{X=4} + \underbrace{10\left(\tfrac{1}{3}\right)^3\left(\tfrac{2}{3}\right)^2}_{X=3} + \underbrace{10\left(\tfrac{1}{3}\right)^2\left(\tfrac{2}{3}\right)^3}_{X=2} + \underbrace{5\left(\tfrac{1}{3}\right)\left(\tfrac{2}{3}\right)^4}_{X=1} + \underbrace{\left(\tfrac{2}{3}\right)^5}_{X=0}$$

b
$$\begin{aligned}\text{P}(X \text{ odd} \mid X \geqslant 2) &= \frac{\text{P}(X \text{ odd} \cap X \geqslant 2)}{\text{P}(X \geqslant 2)} \\ &= \frac{\text{P}(X = 3 \text{ or } 5)}{1 - \text{P}(X \leqslant 1)} \\ &= \frac{10\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^5}{1 - \left(5\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^4 + \left(\frac{2}{3}\right)^5\right)} \\ &\approx 0.313\end{aligned}$$

10 **a**
$$\text{P}(X = x) = \frac{m^x e^{-m}}{x!},\quad x = 0, 1, 2, \ldots..$$

Now $\text{P}(X = 1) = \text{P}(2 \leqslant x \leqslant 4)$

$$\therefore\ \text{P}(X = 1) = \text{P}(X = 2) + \text{P}(X = 3) + \text{P}(X = 4)$$
$$\therefore\ \frac{me^{-m}}{1!} = \frac{m^2e^{-m}}{2!} + \frac{m^3e^{-m}}{3!} + \frac{m^4e^{-m}}{4!}$$
$$\therefore\ m = \frac{m^2}{2} + \frac{m^3}{6} + \frac{m^4}{24} \qquad \{\times\ e^m\}$$
$$\therefore\ 24m = 12m^2 + 4m^3 + m^4$$
$$\therefore\ m(m^3 + 4m^2 + 12m - 24) = 0 \quad \text{where} \quad m \neq 0$$
$$\therefore\ m \approx 1.28 \quad \{\text{using technology}\}$$

i mean of $X = m \approx 1.28$

standard deviation $= \sqrt{m} \approx 1.13$

ii $Y = \dfrac{X+1}{2} = \frac{1}{2}X + \frac{1}{2}$

$$\therefore \text{ mean of } Y = \mathrm{E}(Y) = \mathrm{E}\left(\tfrac{1}{2}X + \tfrac{1}{2}\right) = \tfrac{1}{2}\mathrm{E}(X) + \mathrm{E}(\tfrac{1}{2}) \approx 1.14$$

Now, $\mathrm{Var}(X) \approx 1.28$

and $\mathrm{Var}(Y) = \mathrm{Var}(\frac{1}{2}X) \approx (\frac{1}{2})^2 \times 1.28 \approx 0.320$

$$\therefore \sigma_Y \approx \sqrt{0.320} \approx 0.566$$

b $\mathrm{P}(X \geqslant 2) = 1 - \mathrm{P}(X \leqslant 1) \approx 1 - 0.634 \approx 0.366$

11 **a** For a Poisson random variable X,

$\mathrm{E}(X) = \mathrm{Var}(X) = m$

$$\therefore 5m = 2m^2 - 12$$
$$\therefore 2m^2 - 5m - 12 = 0$$
$$\therefore (2m+3)(m-4) = 0$$
$$\therefore m = 4 \quad \{\text{as } m > 0\}$$

$\therefore$ the mean of X is 4.

b $$\mathrm{P}(X = x) = \frac{4^x e^{-4}}{x!}, \quad x = 0, 1, 2, 3, \ldots.$$

$$\therefore \mathrm{P}(X < 3) = \frac{4^0 e^{-4}}{0!} + \frac{4^1 e^{-4}}{1!} + \frac{4^2 e^{-4}}{2!} = e^{-4}(1+4+8) = \frac{13}{e^4} \approx 0.238$$

12 $$\mathrm{P}(X > 2) \approx 0.070\,198$$
$$\therefore \mathrm{P}(X \leqslant 2) \approx 1 - 0.070\,198$$
$$\therefore \mathrm{P}(X = 0, 1 \text{ or } 2) \approx 0.929\,802$$
$$\therefore \binom{10}{0} p^0(1-p)^{10} + \binom{10}{1} p^1(1-p)^9 + \binom{10}{2} p^2(1-p)^8 \approx 0.929\,802$$
$$\therefore (1-p)^{10} + 10p(1-p)^9 + 45p^2(1-p)^8 \approx 0.929\,802$$

Using technology and the domain $0 \leqslant p \leqslant 1$, we find $p \approx 0.100$

$$\therefore \mathrm{P}(X < 2) = \mathrm{P}(X \leqslant 1) \approx 0.736$$

13 **a** Let X be the number of customers arriving at the shop in a 15 minute period.

$X \sim \mathrm{Po}(20) \quad \therefore \mathrm{P}(X = x) = \dfrac{20^x e^{-20}}{x!}$, where $x = 0, 1, 2, 3, \ldots.$

$$\mathrm{P}(X = 15) = \frac{20^{15} e^{-20}}{15!} \approx 0.0516$$

b Let Y be the number of customers arriving at the shop in a 10 minute period.

$\therefore$ Y has mean $m = \frac{10}{15} \times 20 = \frac{40}{3}$

$\therefore$ $Y \sim \mathrm{Po}\left(\frac{40}{3}\right)$

$$\mathrm{P}(Y > 10) = 1 - \mathrm{P}(Y \leqslant 10) \approx 1 - 0.224 \approx 0.776 < 0.8$$

$\therefore$ the probability that more than 10 customers will arrive at the shop in a 10 minute period is *not* greater than 80%.

$\therefore$ the manager will not hire an extra shop assistant.