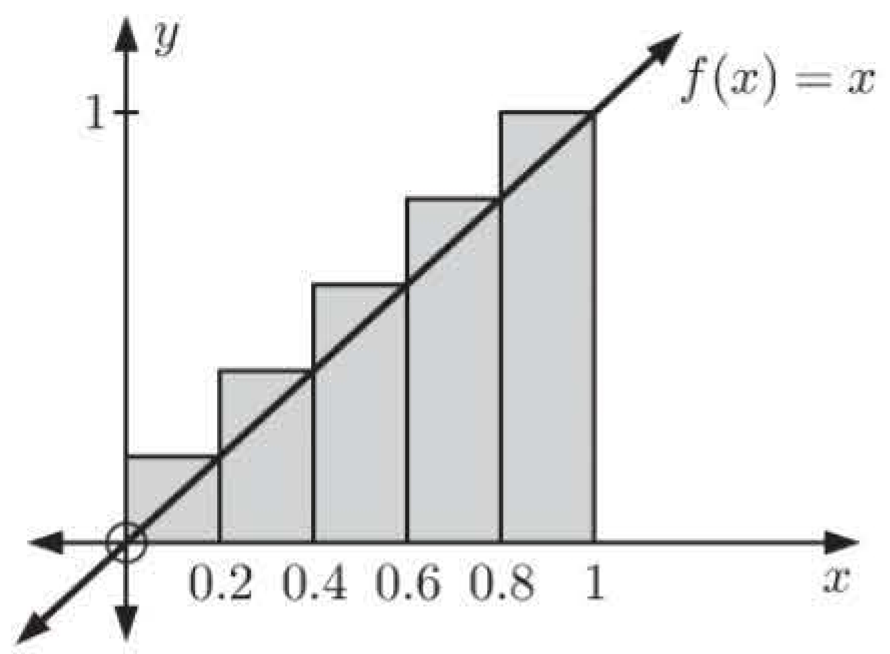
Chapter 21
INTEGRATION

EXERCISE 21A.1

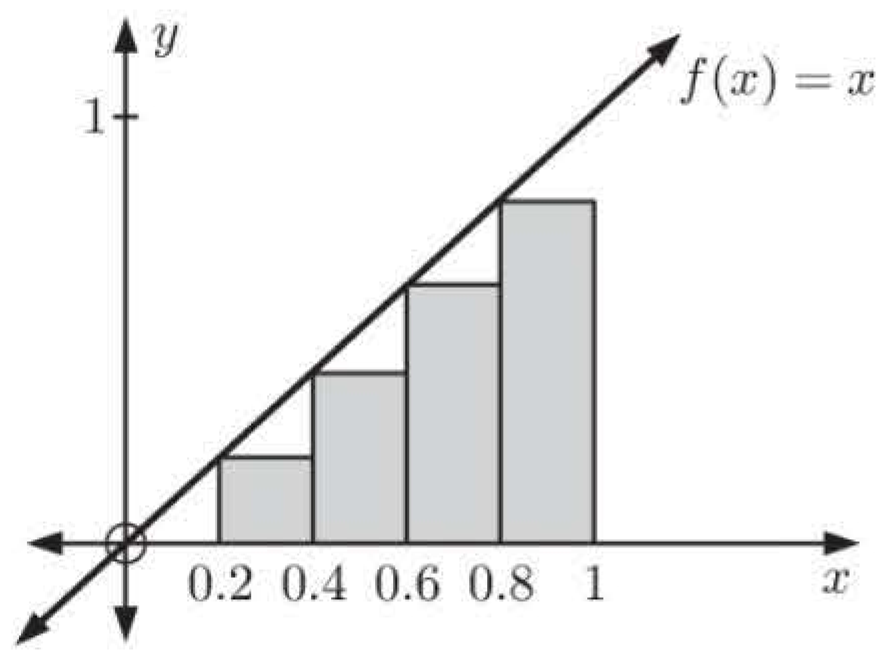
1 **a** **i**



The rectangles are $\frac{1}{5} = 0.2$ units wide.

$$\begin{aligned} A_U &= 0.2 \times f(0.2) + 0.2 \times f(0.4) + 0.2 \times f(0.6) \\ &\quad + 0.2 \times f(0.8) + 0.2 \times f(1) \\ &= 0.2 \times 0.2 + 0.2 \times 0.4 + 0.2 \times 0.6 \\ &\quad + 0.2 \times 0.8 + 0.2 \times 1 \\ &= 0.6 \text{ units}^2 \end{aligned}$$

ii



$$\begin{aligned} A_L &= 0.2 \times f(0) + 0.2 \times f(0.2) + 0.2 \times f(0.4) \\ &\quad + 0.2 \times f(0.6) + 0.2 \times f(0.8) \\ &= 0.2 \times 0 + 0.2 \times 0.2 + 0.2 \times 0.4 \\ &\quad + 0.2 \times 0.6 + 0.2 \times 0.8 \\ &= 0.4 \text{ units}^2 \end{aligned}$$

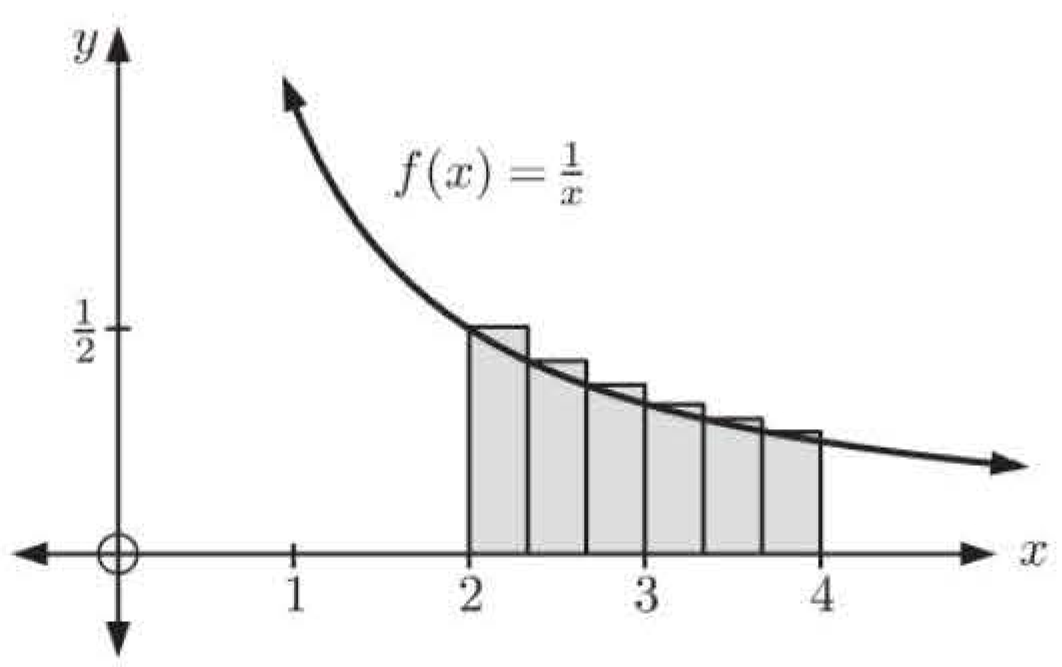
b The area between $y = x$ and the x-axis from $x = 0$ to $x = 1$ is a triangle.

$$\begin{aligned} \therefore\ \text{area} &= \tfrac{1}{2} \times \text{base} \times \text{height} \\ &= \tfrac{1}{2} \times 1 \times 1 \\ &= \tfrac{1}{2} \text{ unit}^2 \end{aligned}$$

$\therefore\ A_L < \text{area} < A_U$, and both A_L and A_U are within 0.1 unit^2, or 20%, of the actual area.

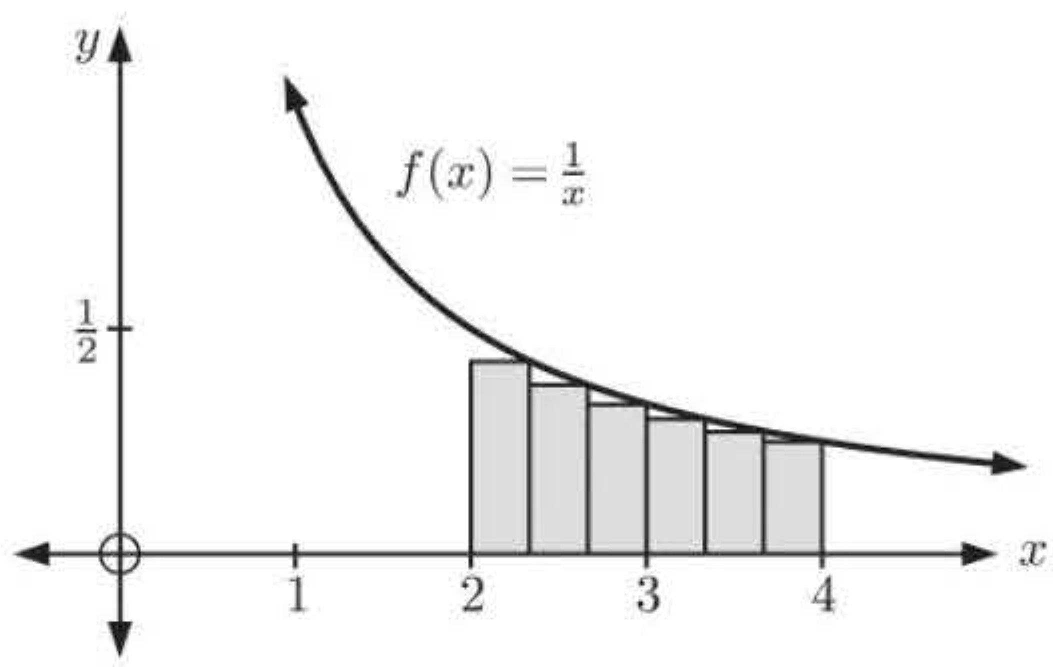
2 The rectangles are $\frac{2}{6} = \frac{1}{3}$ units wide.

a



$$\begin{aligned} A_U &= \tfrac{1}{3}f(2) + \tfrac{1}{3}f(\tfrac{7}{3}) + \tfrac{1}{3}f(\tfrac{8}{3}) + \tfrac{1}{3}f(3) \\ &\quad + \tfrac{1}{3}f(\tfrac{10}{3}) + \tfrac{1}{3}f(\tfrac{11}{3}) \\ &= \tfrac{1}{3} \times \tfrac{1}{2} + \tfrac{1}{3} \times \tfrac{3}{7} + \tfrac{1}{3} \times \tfrac{3}{8} + \tfrac{1}{3} \times \tfrac{1}{3} \\ &\quad + \tfrac{1}{3} \times \tfrac{3}{10} + \tfrac{1}{3} \times \tfrac{3}{11} \\ &\approx 0.737 \text{ units}^2 \end{aligned}$$

b



$$\begin{aligned} A_L &= \tfrac{1}{3}f(\tfrac{7}{3}) + \tfrac{1}{3}f(\tfrac{8}{3}) + \tfrac{1}{3}f(3) + \tfrac{1}{3}f(\tfrac{10}{3}) \\ &\quad + \tfrac{1}{3}f(\tfrac{11}{3}) + \tfrac{1}{3}f(4) \\ &= \tfrac{1}{3} \times \tfrac{3}{7} + \tfrac{1}{3} \times \tfrac{3}{8} + \tfrac{1}{3} \times \tfrac{1}{3} + \tfrac{1}{3} \times \tfrac{3}{10} \\ &\quad + \tfrac{1}{3} \times \tfrac{3}{11} + \tfrac{1}{3} \times \tfrac{1}{4} \\ &\approx 0.653 \text{ units}^2 \end{aligned}$$

3 Using provided software,

n	A_L	A_U
10	2.1850	2.4850
25	2.2736	2.3936
50	2.3034	2.3634
100	2.3184	2.3484
500	2.3303	2.3363

A_L and A_U converge to $\frac{7}{3}$

4 **a** **i**

n	A_L	A_U
5	0.160 00	0.360 00
10	0.202 50	0.302 50
50	0.240 10	0.260 10
100	0.245 03	0.255 03
500	0.249 00	0.251 00
1000	0.249 50	0.250 50
10 000	0.249 95	0.250 05

ii

n	A_L	A_U
5	0.400 00	0.600 00
10	0.450 00	0.550 00
50	0.490 00	0.510 00
100	0.495 00	0.505 00
500	0.499 00	0.501 00
1000	0.499 50	0.500 50
10 000	0.499 95	0.500 05

iii

n	A_L	A_U
5	0.549 74	0.749 74
10	0.610 51	0.710 51
50	0.656 10	0.676 10
100	0.661 46	0.671 46
500	0.665 65	0.667 65
1000	0.666 16	0.667 16
10 000	0.666 62	0.666 72

iv

n	A_L	A_U
5	0.618 67	0.818 67
10	0.687 40	0.787 40
50	0.738 51	0.758 51
100	0.744 41	0.754 41
500	0.748 93	0.750 93
1000	0.749 47	0.750 47
10 000	0.749 95	0.750 05

b **i** A_L and A_U converge to $0.25 = \frac{1}{4} = \frac{1}{3+1}$

ii A_L and A_U converge to $0.5 = \frac{1}{2} = \frac{1}{1+1}$

iii A_L and A_U converge to $0.6\overline{6} = \frac{2}{3} = \frac{1}{\frac{1}{2}+1}$

iv A_L and A_U converge to $0.75 = \frac{3}{4} = \frac{1}{\frac{1}{3}+1}$

c From **b**, it appears that the area between the graph of $y = x^a$ and the x-axis for $0 \leqslant x \leqslant 1$ is $\dfrac{1}{a+1}$.

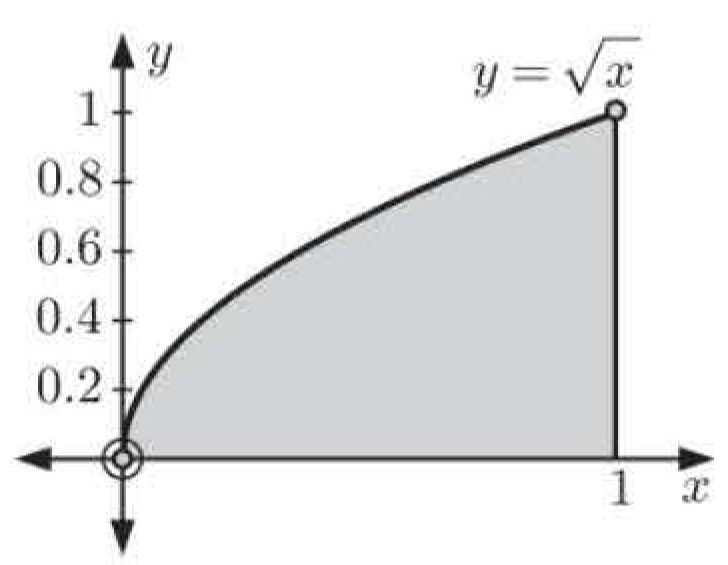
5 **a**

n	*Rational bounds for* π
10	$2.9045 < \pi < 3.3045$
50	$3.0983 < \pi < 3.1783$
100	$3.1204 < \pi < 3.1604$
200	$3.1312 < \pi < 3.1512$
1000	$3.1396 < \pi < 3.1436$
10 000	$3.1414 < \pi < 3.1418$

b $3\frac{10}{71} < \pi < 3\frac{1}{7}$ is approximately
$3.1408 < \pi < 3.1429$
From **a**, this is a better approximation than our estimate using $n = 10, 50, 100, 200, 1000$.
Only $n = 10\,000$ gives us a better estimate.

EXERCISE 21A.2

1 **a**

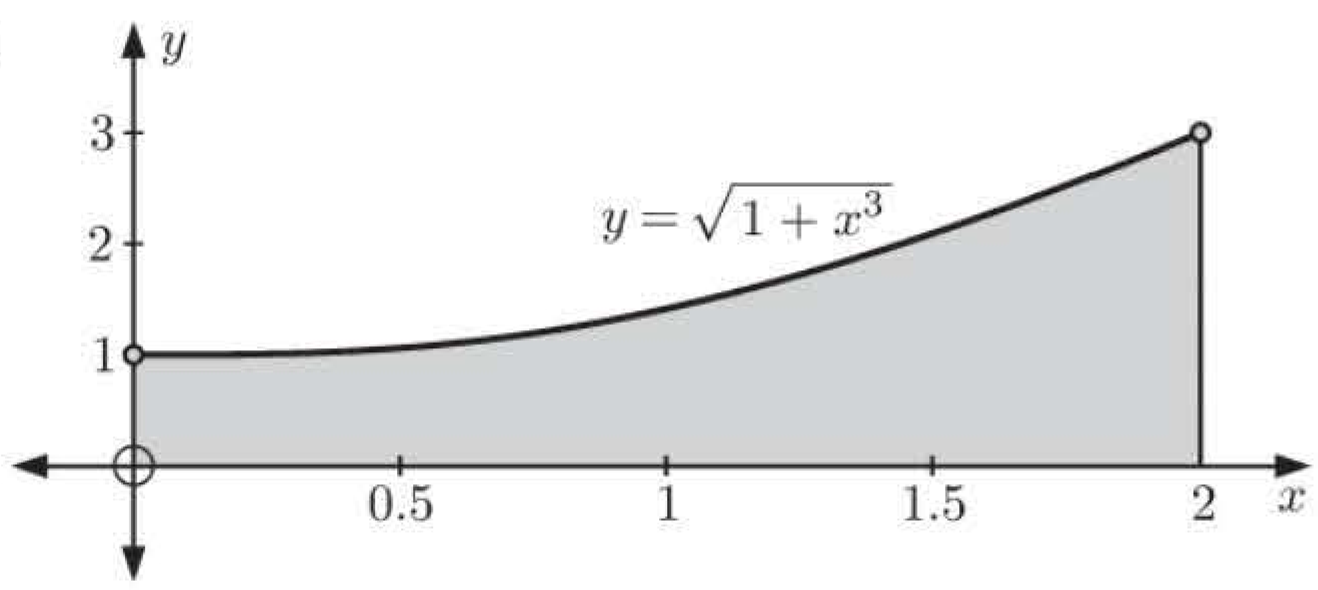


b

n	A_L	A_U
5	0.5497	0.7497
10	0.6105	0.7105
50	0.6561	0.6761
100	0.6615	0.6715
500	0.6656	0.6676

c $\int_0^1 \sqrt{x}\, dx \approx 0.67$

2 **a**



b The rectangles will have width $\dfrac{2-0}{n} = \dfrac{2}{n}$.

The lower rectangle sum will be

$$A_L = \frac{2}{n} \times \sqrt{1+x_0{}^3} + \frac{2}{n} \times \sqrt{1+x_1{}^3} + ... + \frac{2}{n} \times \sqrt{1+x_{n-1}^{\ 3}}$$
$$= \frac{2}{n} \sum_{i=0}^{n-1} \sqrt{1+x_i{}^3}$$

The upper rectangle sum will be

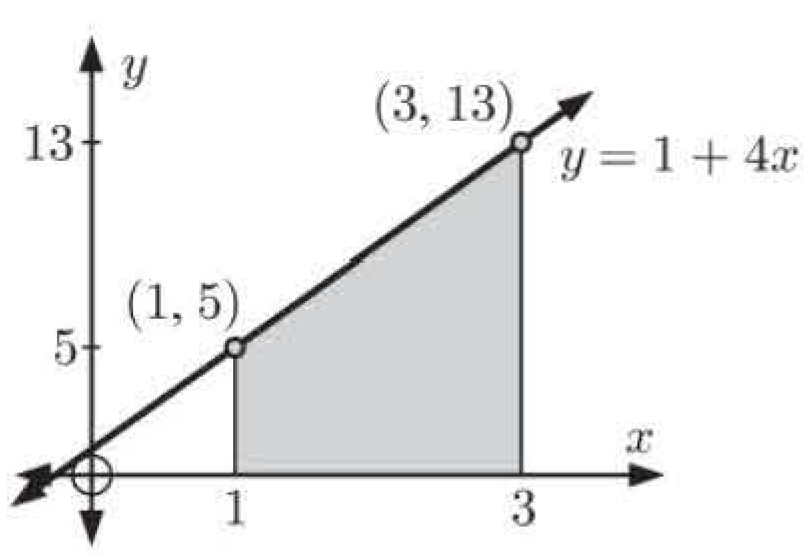
$$A_U = \frac{2}{n}\sqrt{1+x_1{}^3} + \frac{2}{n}\sqrt{1+x_2{}^3} + ... + \frac{2}{n}\sqrt{1+x_n^{\ 3}}$$
$$= \frac{2}{n} \sum_{i=1}^{n} \sqrt{1+x_i{}^3}$$

c

n	A_L	A_U
50	3.2016	3.2816
100	3.2214	3.2614
500	3.2373	3.2453

d $\int_0^2 \sqrt{1+x^3}\, dx \approx 3.24$

3 **a**



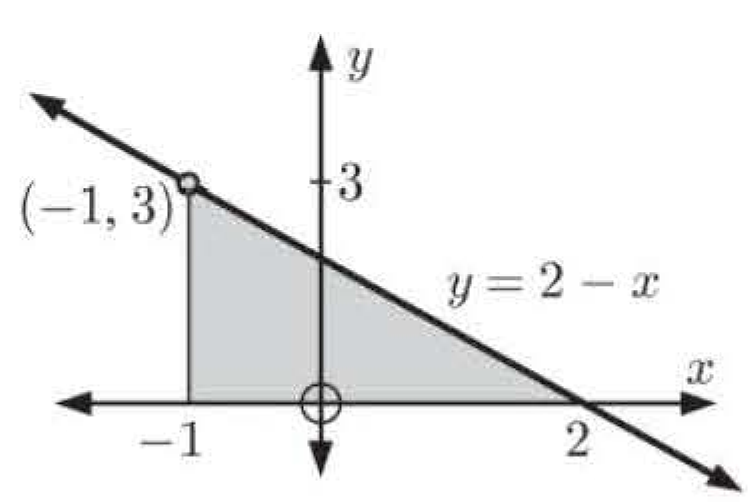
$\int_1^3 (1+4x)\ dx$

$=$ area of the shaded trapezium

$= \left(\dfrac{5+13}{2}\right) \times 2$

$= 18$

b



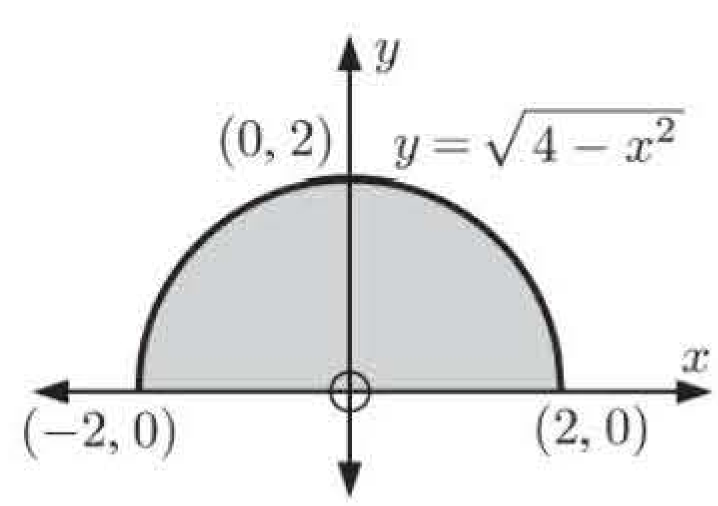
$\int_{-1}^2 (2-x)\ dx$

$=$ area of shaded triangle

$= \frac{1}{2}(3 \times 3)$

$= 4.5$

c



$\int_{-2}^2 \sqrt{4-x^2}\, dx$

$=$ area of semi-circle, radius 2

$= \frac{1}{2}\left(\pi \times 2^2\right)$

$= 2\pi$

EXERCISE 21B

1 **a** **i** $\dfrac{d}{dx}(x^2) = 2x$

$\therefore\ \dfrac{d}{dx}\left(\frac{1}{2}x^2\right) = x$

$\therefore$ the antiderivative of x is $\frac{1}{2}x^2$

ii $\dfrac{d}{dx}(x^3) = 3x^2$

$\therefore\ \dfrac{d}{dx}\left(\frac{1}{3}x^3\right) = x^2$

$\therefore$ the antiderivative of x^2 is $\frac{1}{3}x^3$

iii $\dfrac{d}{dx}(x^6) = 6x^5$

$\therefore\ \dfrac{d}{dx}\left(\frac{1}{6}x^6\right) = x^5$

$\therefore$ the antiderivative of x^5 is $\frac{1}{6}x^6$

iv $\dfrac{d}{dx}(x^{-1}) = -x^{-2}$

$\therefore\ \dfrac{d}{dx}(-x^{-1}) = x^{-2}$

$\therefore$ the antiderivative of x^{-2} is $-x^{-1}$ or $-\dfrac{1}{x}$

v $\dfrac{d}{dx}(x^{-3}) = -3x^{-4}$

$\therefore\ \dfrac{d}{dx}\left(-\frac{1}{3}x^{-3}\right) = x^{-4}$

$\therefore$ the antiderivative of x^{-4} is $-\frac{1}{3}x^{-3} = -\dfrac{1}{3x^3}$

vi $\dfrac{d}{dx}\left(x^{\frac{4}{3}}\right) = \frac{4}{3}x^{\frac{1}{3}}$

$\therefore\ \dfrac{d}{dx}\left(\frac{3}{4}x^{\frac{4}{3}}\right) = x^{\frac{1}{3}}$

$\therefore$ the antiderivative of $x^{\frac{1}{3}}$ is $\frac{3}{4}x^{\frac{4}{3}}$

vii $\dfrac{d}{dx}(x^{\frac{1}{2}}) = \frac{1}{2}x^{-\frac{1}{2}}$

$\therefore\ \dfrac{d}{dx}(2x^{\frac{1}{2}}) = x^{-\frac{1}{2}}$

$\therefore$ the antiderivative of $x^{-\frac{1}{2}}$ is $2x^{\frac{1}{2}} = 2\sqrt{x}$

b The antiderivative of x^n is $\dfrac{x^{n+1}}{n+1}$ $(n \neq -1)$.

2 **a** **i** $\dfrac{d}{dx}(e^{2x}) = 2e^{2x}$

$\therefore\ \dfrac{d}{dx}\left(\frac{1}{2}e^{2x}\right) = e^{2x}$

$\therefore$ the antiderivative of e^{2x} is $\frac{1}{2}e^{2x}$

ii $\dfrac{d}{dx}(e^{5x}) = 5e^{5x}$

$\therefore\ \dfrac{d}{dx}\left(\frac{1}{5}e^{5x}\right) = e^{5x}$

$\therefore$ the antiderivative of e^{5x} is $\frac{1}{5}e^{5x}$

iii $\dfrac{d}{dx}(e^{\frac{1}{2}x}) = \frac{1}{2}e^{\frac{1}{2}x}$

$\therefore\ \dfrac{d}{dx}\left(2e^{\frac{1}{2}x}\right) = e^{\frac{1}{2}x}$

$\therefore$ the antiderivative of $e^{\frac{1}{2}x}$ is $2e^{\frac{1}{2}x}$

iv $\dfrac{d}{dx}(e^{0.01x}) = 0.01e^{0.01x}$

$\therefore\ \dfrac{d}{dx}\left(100e^{0.01x}\right) = e^{0.01x}$

$\therefore$ the antiderivative of $e^{0.01x}$ is $100e^{0.01x}$

v $\dfrac{d}{dx}(e^{\pi x}) = \pi e^{\pi x}$

$\therefore\ \dfrac{d}{dx}\left(\frac{1}{\pi}e^{\pi x}\right) = e^{\pi x}$

$\therefore$ the antiderivative of $e^{\pi x}$ is $\frac{1}{\pi}e^{\pi x}$

vi $\dfrac{d}{dx}\left(e^{\frac{x}{3}}\right) = \frac{1}{3}e^{\frac{x}{3}}$

$\therefore\ \dfrac{d}{dx}\left(3e^{\frac{x}{3}}\right) = e^{\frac{x}{3}}$

$\therefore$ the antiderivative of $e^{\frac{x}{3}}$ is $3e^{\frac{x}{3}}$

b The antiderivative of e^{kx} is $\dfrac{1}{k}e^{kx}$.

3 **a** $\dfrac{d}{dx}(x^3 + x^2) = 3x^2 + 2x$

$\therefore\ \dfrac{d}{dx}\left(2x^3 + 2x^2\right) = 6x^2 + 4x$

$\therefore$ the antiderivative of $6x^2 + 4x$ is $2x^3 + 2x^2$

b $\dfrac{d}{dx}(e^{3x+1}) = 3e^{3x+1}$

$\therefore\ \dfrac{d}{dx}\left(\frac{1}{3}e^{3x+1}\right) = e^{3x+1}$

$\therefore$ the antiderivative of e^{3x+1} is $\frac{1}{3}e^{3x+1}$

c $\dfrac{d}{dx}(x\sqrt{x}) = \dfrac{d}{dx}(x^{\frac{3}{2}}) = \frac{3}{2}x^{\frac{1}{2}}$
$= \frac{3}{2}\sqrt{x}$

$\therefore\ \dfrac{d}{dx}\left(\frac{2}{3}x\sqrt{x}\right) = \sqrt{x}$

$\therefore$ the antiderivative of $\sqrt{x}$ is $\frac{2}{3}x\sqrt{x}$

d $\dfrac{d}{dx}((2x+1)^4) = 4(2x+1)^3 \times 2$
$= 8(2x+1)^3$

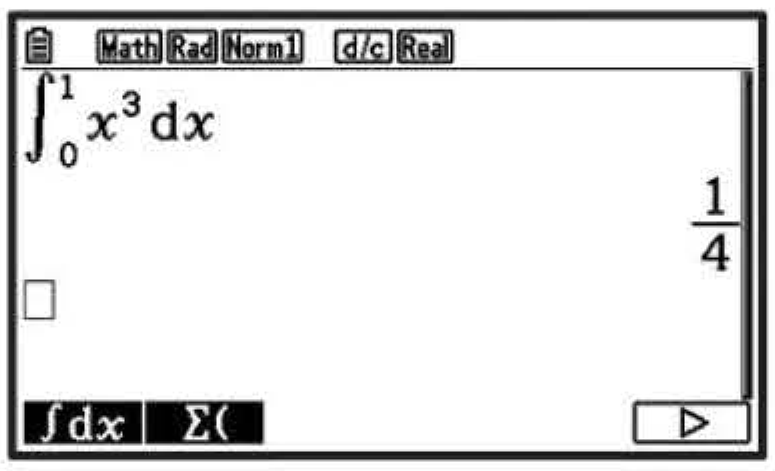
$\therefore\ \dfrac{d}{dx}\left(\frac{1}{8}(2x+1)^4\right) = (2x+1)^3$

$\therefore$ the antiderivative of $(2x+1)^3$ is $\frac{1}{8}(2x+1)^4$

EXERCISE 21C

1 **a** $f(x) = x^3$ has antiderivative $F(x) = \dfrac{x^4}{4}$

$\therefore$ area $= \int_0^1 x^3\,dx$
$= F(1) - F(0)$
$= \frac{1}{4} - 0 = \frac{1}{4}$ units^2

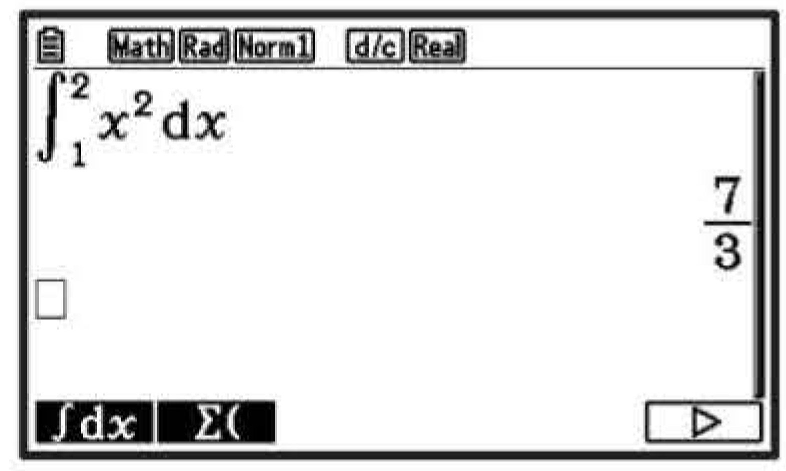


b $f(x)=x^2$ has antiderivative $F(x)=\dfrac{x^3}{3}$

$\therefore$ area $=\int_1^2 x^2\,dx$

$=F(2)-F(1)$

$=\frac{8}{3}-\frac{1}{3}=2\frac{1}{3}$ units^2



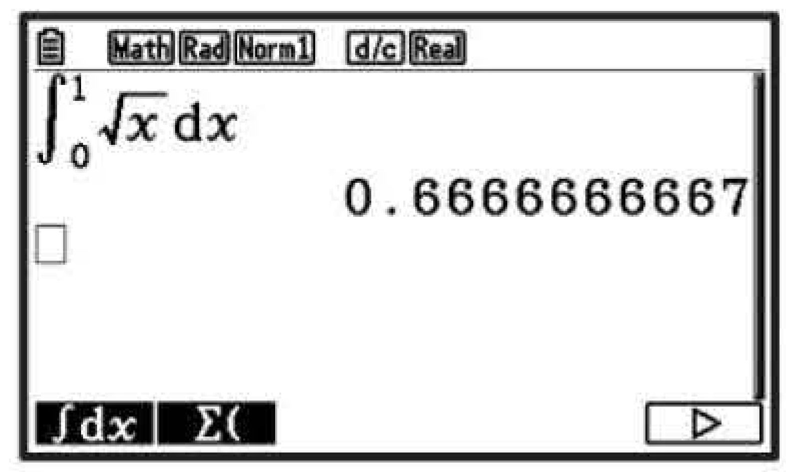
c $f(x)=\sqrt{x}=x^{\frac{1}{2}}$ has antiderivative

$F(x)=\dfrac{x^{\frac{3}{2}}}{\frac{3}{2}}=\frac{2}{3}x\sqrt{x}$

$\therefore$ area $=\int_0^1 \sqrt{x}\,dx$

$=F(1)-F(0)$

$=\frac{2}{3}\times 1\sqrt{1}-0=\frac{2}{3}$ units^2



2 **a** $\int_a^a f(x)\,dx=F(a)-F(a)=0$

$\int_a^a f(x)\,dx=$ area of the strip between $x=a$ and $x=a$.

This strip has 0 width, so its area $=0$.

b The antiderivative of c is cx.

$\therefore\ \int_a^b c\,dx=F(b)-F(a)$

$=cb-ca$

$=c(b-a)$

c $\int_b^a f(x)\,dx=F(a)-F(b)$

$=-[F(b)-F(a)]$

$=-\int_a^b f(x)\,dx$

d If $\dfrac{d}{dx}F(x)=f(x)$ then

$\dfrac{d}{dx}c\,F(x)=c\,f(x)$

$\therefore\ \int_a^b c\,f(x)\,dx=c\,F(b)-c\,F(a)$

$=c[F(b)-F(a)]$

$=c\int_a^b f(x)\,dx$

e $\int_a^b (f(x)+g(x))\,dx$

$=[F(b)+G(b)]-[F(a)+G(a)]$

$=[F(b)-F(a)]+[G(b)-G(a)]$

$=\int_a^b f(x)\,dx+\int_a^b g(x)\,dx$

3 **a** $f(x)=x^3$ has antiderivative $F(x)=\dfrac{x^4}{4}$

$\therefore$ area $=\int_1^2 x^3\,dx$

$=F(2)-F(1)$

$=\frac{16}{4}-\frac{1}{4}$

$=3\frac{3}{4}$ units^2

b $f(x)=x^2+3x+2$ has antiderivative

$F(x)=\dfrac{x^3}{3}+\dfrac{3x^2}{2}+2x$

$\therefore$ area $=\int_1^3 (x^2+3x+2)\,dx$

$=F(3)-F(1)$

$=(\frac{27}{3}+\frac{27}{2}+6)-(\frac{1}{3}+\frac{3}{2}+2)$

$=24\frac{2}{3}$ units^2

c $f(x)=\sqrt{x}=x^{\frac{1}{2}}$ has antiderivative

$F(x)=\dfrac{x^{\frac{3}{2}}}{\frac{3}{2}}=\frac{2}{3}x\sqrt{x}$

$\therefore$ area $=\int_1^2 \sqrt{x}\,dx$

$=F(2)-F(1)$

$=\frac{2}{3}2\sqrt{2}-\frac{2}{3}1\sqrt{1}$

$=\frac{4\sqrt{2}}{3}-\frac{2}{3}$

$=\frac{-2+4\sqrt{2}}{3}$ units^2

d $f(x)=e^x$ has antiderivative $F(x)=e^x$

$\therefore$ area $=\int_0^{1.5} e^x\,dx$

$=F(1.5)-F(0)$

$=e^{1.5}-e^0$

$=e^{1.5}-1$

≈ 3.48 units^2

e $f(x) = \dfrac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$ has antiderivative

$F(x) = \dfrac{x^{\frac{1}{2}}}{\frac{1}{2}} = 2\sqrt{x}$

$\therefore$ area $= \int_1^4 \dfrac{1}{\sqrt{x}}\,dx$

$= F(4) - F(1)$

$= 2\sqrt{4} - 2\sqrt{1} = 2$ units^2

f $f(x) = x^3 + 2x^2 + 7x + 4$ has antiderivative

$F(x) = \dfrac{x^4}{4} + \dfrac{2x^3}{3} + \dfrac{7x^2}{2} + 4x$

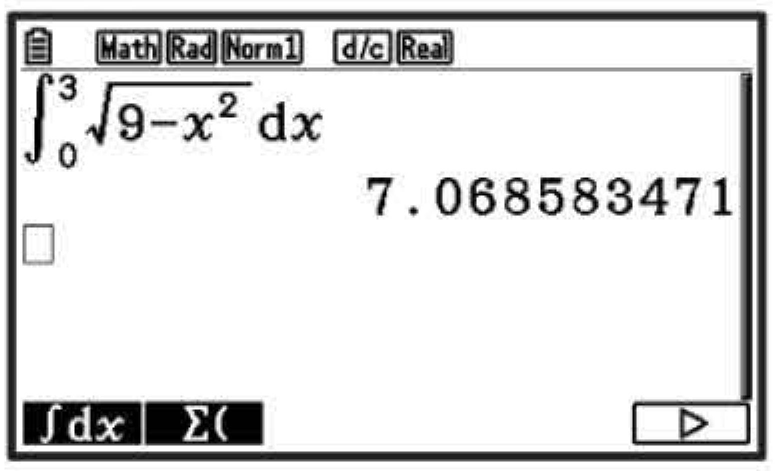
$\therefore$ area $= \int_1^{1.25} \left(x^3 + 2x^2 + 7x + 4\right)\,dx$

$= F(1.25) - F(1)$

$= [12.381\,18 - 8.416\,67]$

≈ 3.96 units^2

4



$\therefore$ using technology, area $= \int_0^3 \sqrt{9-x^2}\,dx \approx 7.07$ units^2

Check: The area is a quarter circle with radius 3 units.

$\therefore$ area $= \frac{1}{4}\pi r^2$

$= \frac{1}{4} \times \pi \times 3^2 = \frac{9}{4}\pi \approx 7.07$ units^2 ✓

5 **a** If $\dfrac{d}{dx}F(x) = f(x)$ then $\dfrac{d}{dx}(-F(x)) = -f(x)$

$\therefore \int_a^b (-f(x))\,dx = -F(b) - (-F(a))$

$= -(F(b) - F(a))$

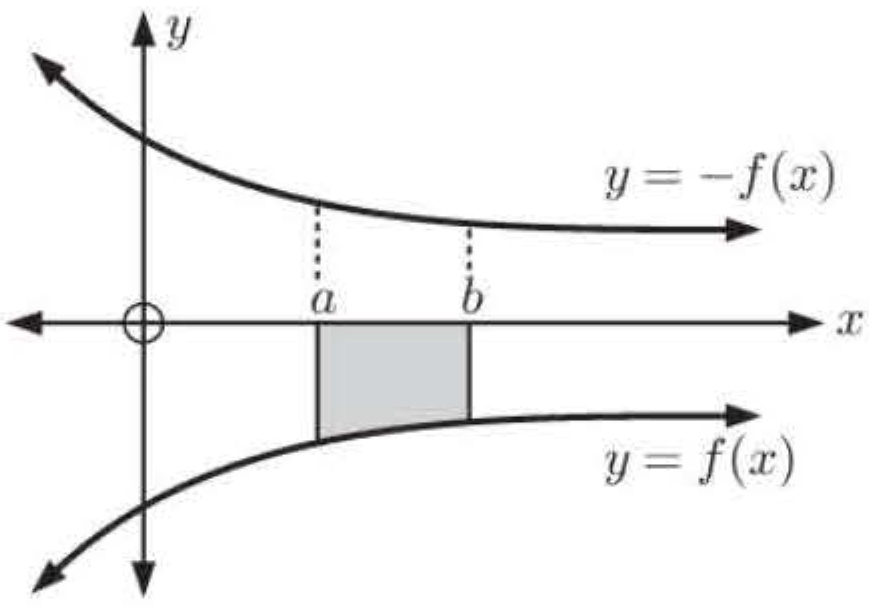
$= -\int_a^b f(x)\,dx$

b Since $y = -f(x)$ is a reflection of $y = f(x)$ in the x-axis, then

shaded area = area between the x-axis and $y = -f(x)$ from $x = a$ to $x = b$

$= \int_a^b (-f(x))\,dx$

$= -\int_a^b f(x)\,dx$ {using **a**}



c **i** $\int_0^1 (-x^2)\,dx = -\int_0^1 x^2\,dx$

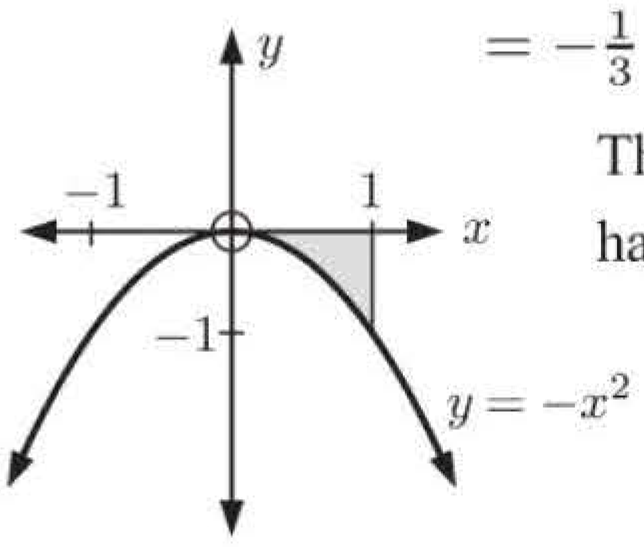
Now $f(x) = x^2$ has antiderivative

$F(x) = \frac{1}{3}x^3$

$\therefore \int_0^1 (-x^2)\,dx = -(F(1) - F(0))$

$= -\left(\frac{1}{3} - 0\right)$

$= -\frac{1}{3}$



The shaded region has area $\frac{1}{3}$ units^2.

ii $\int_0^1 (x^2 - x)\,dx = -\int_0^1 (x - x^2)\,dx$

$\{x^2 - x \leqslant 0$ for all $x \in [0, 1]\}$

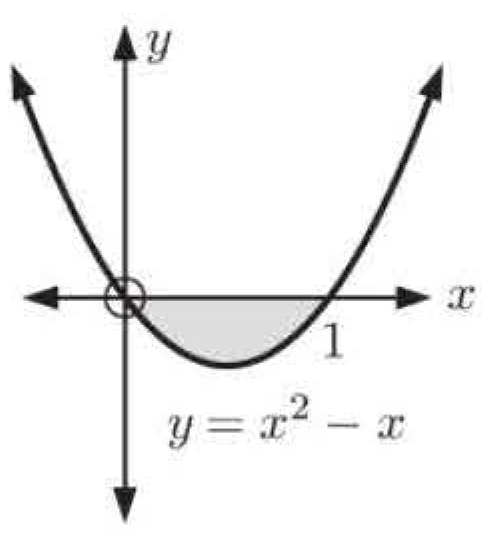
Now $f(x) = x - x^2$ has antiderivative

$F(x) = \frac{1}{2}x^2 - \frac{1}{3}x^3$

$\therefore \int_0^1 (x^2 - x)\,dx = -(F(1) - F(0))$

$= -\left(\frac{1}{2} - \frac{1}{3} - (0 - 0)\right)$

$= -\frac{1}{6}$

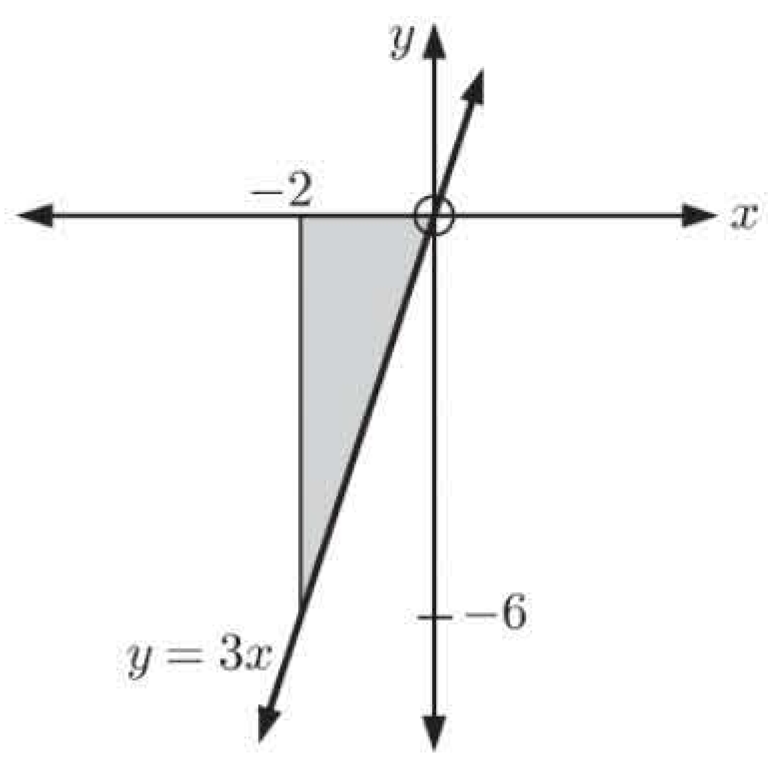


The shaded region has area $\frac{1}{6}$ units^2.

iii $\int_{-2}^{0} 3x\,dx = -\int_{-2}^{0} -3x\,dx$

Now $f(x) = -3x$ has antiderivative $F(x) = -\frac{3}{2}x^2$

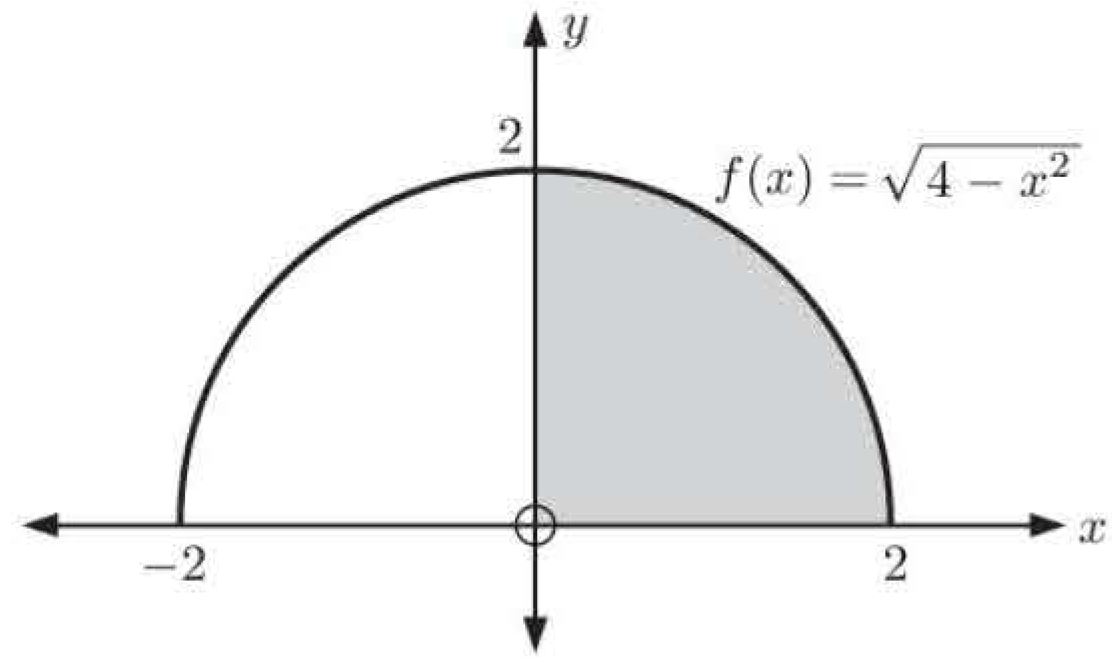
$\therefore \int_{-2}^{0} 3x\,dx = -(F(0) - F(-2))$
$= -(0 - (-6))$
$= -6$



The shaded region has area 6 units^2.

d $\int_0^2 \left(-\sqrt{4-x^2}\right) dx = -\int_0^2 \sqrt{4-x^2}\,dx$

Now $f(x) = \sqrt{4-x^2}$ is the top half of a circle with radius 2 units and centre (0, 0).



$\therefore \int_0^2 \left(-\sqrt{4-x^2}\right) dx = -\int_0^2 \sqrt{4-x^2}\,dx$
$= -$(shaded area)
$= -\frac{1}{4} \times \pi \times 2^2$
$= -\pi$

EXERCISE 21D

1 If $y = x^7$ then $\frac{dy}{dx} = 7x^6$

$\therefore \int 7x^6\,dx = x^7 + c$
$\therefore 7\int x^6\,dx = x^7 + c$
$\therefore \int x^6\,dx = \frac{1}{7}x^7 + c$

2 If $y = x^3 + x^2$ then $\frac{dy}{dx} = 3x^2 + 2x$

$\therefore \int (3x^2 + 2x)\,dx = x^3 + x^2 + c$

3 If $y = e^{2x+1}$ then $\frac{dy}{dx} = 2e^{2x+1}$

$\therefore \int 2e^{2x+1}\,dx = e^{2x+1} + c$
$\therefore 2\int e^{2x+1}\,dx = e^{2x+1} + c$
$\therefore \int e^{2x+1}\,dx = \frac{1}{2}e^{2x+1} + c$

4 If $y = (2x+1)^4$

then $\frac{dy}{dx} = 4(2x+1)^3 \times 2 = 8(2x+1)^3$ {chain rule}

$\therefore \int 8(2x+1)^3\,dx = (2x+1)^4 + c$
$\therefore 8\int (2x+1)^3\,dx = (2x+1)^4 + c$
$\therefore \int (2x+1)^3\,dx = \frac{1}{8}(2x+1)^4 + c$

5 If $y = x\sqrt{x} = x^{\frac{3}{2}}$

then $\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}$

$\therefore \int \frac{3}{2}\sqrt{x}\,dx = x\sqrt{x} + c$
$\therefore \frac{3}{2}\int \sqrt{x}\,dx = x\sqrt{x} + c$
$\therefore \int \sqrt{x}\,dx = \frac{2}{3}x\sqrt{x} + c$

6 If $y = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$

then $\frac{dy}{dx} = -\frac{1}{2}x^{-\frac{3}{2}} = -\frac{1}{2x\sqrt{x}}$

$\therefore \int -\frac{1}{2}\left(\frac{1}{x\sqrt{x}}\right) dx = \frac{1}{\sqrt{x}} + c$
$\therefore -\frac{1}{2}\int \frac{1}{x\sqrt{x}}\,dx = \frac{1}{\sqrt{x}} + c$
$\therefore \int \frac{1}{x\sqrt{x}}\,dx = -\frac{2}{\sqrt{x}} + c$

7

If $y = \cos 2x$

then $\dfrac{dy}{dx} = -2\sin 2x$

$\therefore \int -2\sin 2x\,dx = \cos 2x + c$

$\therefore -2\int \sin 2x\,dx = \cos 2x + c$

$\therefore \int \sin 2x\,dx = -\frac{1}{2}\cos 2x + c$

8

If $y = \sin(1-5x)$

then $\dfrac{dy}{dx} = -5\cos(1-5x)$

$\therefore \int -5\cos(1-5x)\,dx = \sin(1-5x) + c$

$\therefore -5\int \cos(1-5x)\,dx = \sin(1-5x) + c$

$\therefore \int \cos(1-5x)\,dx = -\frac{1}{5}\sin(1-5x) + c$

9

$\dfrac{d}{dx}\left[(x^2-x)^3\right] = 3(x^2-x)^2(2x-1)$ {chain rule}

$\therefore \int 3(2x-1)(x^2-x)^2\,dx = (x^2-x)^3 + c$

$\therefore 3\int (2x-1)(x^2-x)^2\,dx = (x^2-x)^3 + c$

$\therefore \int (2x-1)(x^2-x)^2\,dx = \frac{1}{3}(x^2-x)^3 + c$

10 Suppose $F(x)$ is the antiderivative of $f(x)$ and $G(x)$ is the antiderivative of $g(x)$.

$\therefore \dfrac{d}{dx}(F(x)+G(x)) = f(x)+g(x)$

$\therefore \int [f(x)+g(x)]\,dx$

$= F(x)+G(x)+c$

$= (F(x)+c_1)+(G(x)+c_2)$

$= \int f(x)\,dx + \int g(x)\,dx$

11

$y = \sqrt{1-4x} = (1-4x)^{\frac{1}{2}}$

$\therefore \dfrac{dy}{dx} = \frac{1}{2}(1-4x)^{-\frac{1}{2}}(-4)$ {chain rule}

$= \dfrac{-2}{\sqrt{1-4x}}$

$\therefore \displaystyle\int \frac{-2}{\sqrt{1-4x}}\,dx = \sqrt{1-4x} + c$

$\therefore \displaystyle -2\int \frac{1}{\sqrt{1-4x}}\,dx = \sqrt{1-4x} + c$

$\therefore \displaystyle\int \frac{1}{\sqrt{1-4x}}\,dx = -\frac{1}{2}\sqrt{1-4x} + c$

12

$\dfrac{d}{dx}(\ln(5-3x+x^2)) = \dfrac{2x-3}{5-3x+x^2}$

Now $5-3x+x^2 > 0$ for all x,

as $a > 0$ and $\Delta = -11 < 0$

$\therefore \displaystyle\int \frac{2x-3}{5-3x+x^2}\,dx = \ln(5-3x+x^2) + c$

$\therefore \displaystyle\int \frac{4x-6}{5-3x+x^2}\,dx = 2\ln(5-3x+x^2) + c$

13

$\dfrac{d}{dx}(\csc x) = -\csc x\cot x$

$= -\dfrac{1}{\sin x} \times \dfrac{\cos x}{\sin x}$

$= -\dfrac{\cos x}{\sin^2 x}$

$\therefore \displaystyle\int -\frac{\cos x}{\sin^2 x}\,dx = \csc x + c$

$\therefore \displaystyle -\int \frac{\cos x}{\sin^2 x}\,dx = \csc x + c$

$\therefore \displaystyle\int \frac{\cos x}{\sin^2 x}\,dx = -\csc x + c$

14

$\dfrac{d}{dx}(\arctan x) = \dfrac{1}{x^2+1}, \quad x \in \mathbb{R}$

$\therefore \displaystyle\int \frac{1}{x^2+1}\,dx = \arctan x + c$

$\therefore \displaystyle -3\int \frac{1}{x^2+1}\,dx = -3\arctan x + c$

$\therefore \displaystyle\int \frac{-3}{x^2+1}\,dx = -3\arctan x + c$

15
$$\frac{d}{dx}(2^x) = \frac{d}{dx}(e^{\ln 2})^x$$
$$= e^{(\ln 2)x} \times \ln 2$$
$$= 2^x \ln 2$$
$$\therefore \int 2^x \ln 2 \, dx = 2^x + c$$
$$\therefore \ln 2 \int 2^x \, dx = 2^x + c$$
$$\therefore \int 2^x \, dx = \frac{2^x}{\ln 2} + c$$

16
$$\frac{d}{dx}(x \ln x) = 1 \times \ln x + x \times \frac{1}{x}$$
{chain rule}
$$= \ln x + 1$$
$$\therefore \int (\ln x + 1) \, dx = x \ln x + c$$
$$\therefore \int \ln x \, dx + \int 1 \, dx = x \ln x + c$$
$$\therefore \int \ln x \, dx + x = x \ln x + c$$
$$\therefore \int \ln x \, dx = x \ln x - x + c$$

EXERCISE 21E.1

1 **a**
$$\int (x^4 - x^2 - x + 2) \, dx$$
$$= \frac{x^5}{5} - \frac{x^3}{3} - \frac{x^2}{2} + 2x + c$$

b
$$\int (\sqrt{x} + e^x) \, dx$$
$$= \int (x^{\frac{1}{2}} + e^x) \, dx$$
$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + e^x + c$$
$$= \tfrac{2}{3}x^{\frac{3}{2}} + e^x + c$$

c
$$\int \left(3e^x - \frac{1}{x}\right) dx$$
$$= 3e^x - \ln|x| + c$$

d
$$\int \left(x\sqrt{x} - \frac{2}{x}\right) dx$$
$$= \int \left(x^{\frac{3}{2}} - \frac{2}{x}\right) dx$$
$$= \frac{x^{\frac{5}{2}}}{\frac{5}{2}} - 2\ln|x| + c$$
$$= \tfrac{2}{5}x^{\frac{5}{2}} - 2\ln|x| + c$$

e
$$\int \left(\frac{1}{x\sqrt{x}} + \frac{4}{x}\right) dx$$
$$= \int \left(x^{-\frac{3}{2}} + \frac{4}{x}\right) dx$$
$$= \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + 4\ln|x| + c$$
$$= -2x^{-\frac{1}{2}} + 4\ln|x| + c$$

f
$$\int (\tfrac{1}{2}x^3 - x^4 + x^{\frac{1}{3}}) \, dx$$
$$= \tfrac{1}{2}\frac{x^4}{4} - \frac{x^5}{5} + \frac{x^{\frac{4}{3}}}{\frac{4}{3}} + c$$
$$= \tfrac{1}{8}x^4 - \tfrac{1}{5}x^5 + \tfrac{3}{4}x^{\frac{4}{3}} + c$$

g
$$\int \left(x^2 + \frac{3}{x}\right) dx$$
$$= \tfrac{1}{3}x^3 + 3\ln|x| + c$$

h
$$\int \left(\frac{1}{2x} + x^2 - e^x\right) dx$$
$$= \tfrac{1}{2}\ln|x| + \tfrac{1}{3}x^3 - e^x + c$$

i
$$\int \left(5e^x + \tfrac{1}{3}x^3 - \frac{4}{x}\right) dx$$
$$= 5e^x + \tfrac{1}{3}\frac{x^4}{4} - 4\ln|x| + c$$
$$= 5e^x + \tfrac{1}{12}x^4 - 4\ln|x| + c$$

2 **a**
$$\int (3\sin x - 2) \, dx$$
$$= -3\cos x - 2x + c$$

b
$$\int (4x - 2\cos x) \, dx$$
$$= 2x^2 - 2\sin x + c$$

c
$$\int (\sin x - 2\cos x + e^x) \, dx$$
$$= -\cos x - 2\sin x + e^x + c$$

d
$$\int (x^2\sqrt{x} - 10\sin x) \, dx$$
$$= \int (x^{\frac{5}{2}} - 10\sin x) \, dx$$
$$= \tfrac{2}{7}x^{\frac{7}{2}} + 10\cos x + c$$
$$= \tfrac{2}{7}x^3\sqrt{x} + 10\cos x + c$$

e
$$\int \left(\frac{x(x-1)}{3} + \cos x\right) dx$$
$$= \int \left(\tfrac{1}{3}x^2 - \tfrac{1}{3}x + \cos x\right) dx$$
$$= \tfrac{1}{9}x^3 - \tfrac{1}{6}x^2 + \sin x + c$$

f
$$\int (-\sin x + 2\sqrt{x}) \, dx$$
$$= \int (-\sin x + 2x^{\frac{1}{2}}) \, dx$$
$$= \cos x + \tfrac{4}{3}x^{\frac{3}{2}} + c$$
$$= \cos x + \tfrac{4}{3}x\sqrt{x} + c$$

3 **a**
$$\int (x^2 + 3x - 2) \, dx$$
$$= \tfrac{1}{3}x^3 + \tfrac{3}{2}x^2 - 2x + c$$

b
$$\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) dx$$
$$= \int \left(x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right) dx$$
$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$$
$$= \tfrac{2}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + c$$

c
$$\int \left(2e^x - \frac{1}{x^2}\right) dx$$
$$= \int \left(2e^x - x^{-2}\right) dx$$
$$= 2e^x - \frac{x^{-1}}{-1} + c$$
$$= 2e^x + \frac{1}{x} + c$$

d $\int \left(\frac{1-4x}{x\sqrt{x}}\right) dx$
$= \int \left(\frac{1}{x\sqrt{x}} - \frac{4}{\sqrt{x}}\right) dx$
$= \int (x^{-\frac{3}{2}} - 4x^{-\frac{1}{2}})\, dx$
$= \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} - \frac{4x^{\frac{1}{2}}}{\frac{1}{2}} + c$
$= -2x^{-\frac{1}{2}} - 8x^{\frac{1}{2}} + c$

e $\int (2x+1)^2\, dx$
$= \int (4x^2+4x+1)\, dx$
$= \frac{4x^3}{3} + \frac{4x^2}{2} + x + c$
$= \frac{4}{3}x^3 + 2x^2 + x + c$

f $\int \frac{x^2+x-3}{x}\, dx$
$= \int \left(x+1-\frac{3}{x}\right) dx$
$= \frac{1}{2}x^2 + x - 3\ln|x| + c$

g $\int \frac{2x-1}{\sqrt{x}}\, dx$
$= \int \left(2x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right) dx$
$= \frac{2x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$
$= \frac{4}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + c$

h $\int \frac{x^2-4x+10}{x^2\sqrt{x}}\, dx$
$= \int \left(\frac{x^2}{x^2\sqrt{x}} - \frac{4x}{x^2\sqrt{x}} + \frac{10}{x^2\sqrt{x}}\right) dx$
$= \int \left(x^{-\frac{1}{2}} - 4x^{-\frac{3}{2}} + 10x^{-\frac{5}{2}}\right) dx$
$= \frac{x^{\frac{1}{2}}}{\frac{1}{2}} - \frac{4x^{-\frac{1}{2}}}{-\frac{1}{2}} + \frac{10x^{-\frac{3}{2}}}{-\frac{3}{2}} + c$
$= 2x^{\frac{1}{2}} + 8x^{-\frac{1}{2}} - \frac{20}{3}x^{-\frac{3}{2}} + c$

i $\int (x+1)^3\, dx$
$= \int (x^3+3x^2+3x+1)\, dx$
$= \frac{1}{4}x^4 + x^3 + \frac{3}{2}x^2 + x + c$

4 **a** $\int (\sqrt{x} + \frac{1}{2}\cos x)\, dx$
$= \int (x^{\frac{1}{2}} + \frac{1}{2}\cos x)\, dx$
$= \frac{2}{3}x^{\frac{3}{2}} + \frac{1}{2}\sin x + c$

b $\int (2e^t - 4\sin t)\, dt$
$= 2e^t + 4\cos t + c$

c $\int \left(3\cos t - \frac{1}{t}\right) dt$
$= 3\sin t - \ln|t| + c$

d $\int (\sec^2 x + 2\sin x)\, dx$
$= \tan x - 2\cos x + c$

e $\int (\theta - \sin\theta)\, d\theta$
$= \frac{1}{2}\theta^2 - (-\cos\theta) + c$
$= \frac{1}{2}\theta^2 + \cos\theta + c$

f $\int \left(\frac{2}{\theta} - \sec^2\theta\right) d\theta$
$= 2\ln|\theta| - \tan\theta + c$

5 **a** $\frac{dy}{dx} = 6$
$\therefore\ y = \int 6\, dx$
$\therefore\ y = 6x + c$

b $\frac{dy}{dx} = 4x^2$
$\therefore\ y = \int 4x^2\, dx$
$\therefore\ y = \frac{4}{3}x^3 + c$

c $\frac{dy}{dx} = 5\sqrt{x} - x^2 = 5x^{\frac{1}{2}} - x^2$
$\therefore\ y = \int (5x^{\frac{1}{2}} - x^2)\, dx$
$\therefore\ y = \frac{10}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3 + c$
$\therefore\ y = \frac{10}{3}x\sqrt{x} - \frac{1}{3}x^3 + c$

d $\frac{dy}{dx} = \frac{1}{x^2} = x^{-2}$
$\therefore\ y = \int x^{-2}\, dx$
$\therefore\ y = \frac{x^{-1}}{-1} + c$
$\therefore\ y = -\frac{1}{x} + c$

e $\frac{dy}{dx} = 2e^x - 5$
$\therefore\ y = \int (2e^x - 5)\, dx$
$\therefore\ y = 2e^x - 5x + c$

f $\frac{dy}{dx} = 4x^3 + 3x^2$
$\therefore\ y = \int (4x^3 + 3x^2)\, dx$
$= \frac{4x^4}{4} + \frac{3x^3}{3} + c$
$\therefore\ y = x^4 + x^3 + c$

6 **a** $\dfrac{dy}{dx} = (1-2x)^2$

$\therefore\ y = \int (1-2x)^2\, dx$

$= \int (1 - 4x + 4x^2)\, dx$

$= x - \dfrac{4x^2}{2} + \dfrac{4x^3}{3} + c$

$= x - 2x^2 + \frac{4}{3}x^3 + c$

b $\dfrac{dy}{dx} = \sqrt{x} - \dfrac{2}{\sqrt{x}}$

$= x^{\frac{1}{2}} - 2x^{-\frac{1}{2}}$

$\therefore\ y = \int (x^{\frac{1}{2}} - 2x^{-\frac{1}{2}})\, dx$

$= \dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} - \dfrac{2x^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= \frac{2}{3}x^{\frac{3}{2}} - 4\sqrt{x} + c$

c $\dfrac{dy}{dx} = \dfrac{x^2+2x-5}{x^2}$

$= 1 + 2x^{-1} - 5x^{-2}$

$\therefore\ y = \int (1 + 2x^{-1} - 5x^{-2})\, dx$

$= x + 2\ln|x| - \dfrac{5x^{-1}}{-1} + c$

$= x + 2\ln|x| + \dfrac{5}{x} + c$

7 **a** $f'(x) = x^3 - 5\sqrt{x} + 3$

$= x^3 - 5x^{\frac{1}{2}} + 3$

$\therefore\ f(x) = \int (x^3 - 5x^{\frac{1}{2}} + 3)\, dx$

$= \frac{1}{4}x^4 - \frac{10}{3}x^{\frac{3}{2}} + 3x + c$

$= \frac{1}{4}x^4 - \frac{10}{3}x\sqrt{x} + 3x + c$

b $f'(x) = 2\sqrt{x}(1-3x)$

$= 2x^{\frac{1}{2}} - 6x^{\frac{3}{2}}$

$\therefore\ f(x) = \int (2x^{\frac{1}{2}} - 6x^{\frac{3}{2}})\, dx$

$= \dfrac{2x^{\frac{3}{2}}}{\frac{3}{2}} - \dfrac{6x^{\frac{5}{2}}}{\frac{5}{2}} + c$

$= \frac{4}{3}x^{\frac{3}{2}} - \frac{12}{5}x^{\frac{5}{2}} + c$

c $f'(x) = 3e^x - \dfrac{4}{x}$

$\therefore\ f(x) = \displaystyle\int \left(3e^x - \frac{4}{x}\right) dx$

$= 3e^x - 4\ln|x| + c$

8 $\dfrac{d}{dx}(e^x \sin x) = e^x \sin x + e^x \cos x$

$\therefore\ \int e^x(\sin x + \cos x)\, dx$

$= \int (e^x \sin x + e^x \cos x)\, dx$

$= e^x \sin x + c$

9 $\dfrac{d}{dx}(e^{-x}\sin x) = -e^{-x}\sin x + e^{-x}\cos x$

$= \dfrac{\cos x - \sin x}{e^x}$

$\therefore\ \displaystyle\int \frac{\cos x - \sin x}{e^x}\, dx = e^{-x}\sin x + c$

10 $\dfrac{d}{dx}(x\cos x) = \cos x + x(-\sin x)$

$= \cos x - x\sin x$

$\therefore\ \int (\cos x - x\sin x)\, dx = x\cos x + c$

$\therefore\ \int \cos x\, dx - \int x\sin x\, dx = x\cos x + c$

$\therefore\ \sin x - \int x\sin x\, dx = x\cos x + c$

$\therefore\ \int x\sin x\, dx = \sin x - x\cos x + c$

11 $\dfrac{d}{dx}(\sec x) = \sec x \tan x$

$\therefore\ \int \tan x \sec x\, dx = \sec x + c$

12 $\dfrac{d}{dx}(x - 3\arctan x) = 1 - 3\left(\dfrac{1}{x^2+1}\right), \quad x \in \mathbb{R}$

$= 1 - \dfrac{3}{x^2+1}$

$= \dfrac{1 + x^2 - 3}{x^2+1}$

$= \dfrac{x^2-2}{x^2+1}$

$\therefore\ \displaystyle\int \frac{x^2-2}{x^2+1}\, dx = x - 3\arctan x + c$

13 **a** $\dfrac{d}{dx}(\arccos x) = -\dfrac{1}{\sqrt{1-x^2}}, \quad x \in]-1, 1[$

$\dfrac{d}{dx}(\arcsin x) = \dfrac{1}{\sqrt{1-x^2}}, \quad x \in]-1, 1[$

b $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + c$

Also, $\displaystyle\int -\frac{1}{\sqrt{1-x^2}}\,dx = \arccos x + c$

$\therefore \ \displaystyle -\int \frac{1}{\sqrt{1-x^2}}\,dx = \arccos x + c$

$\therefore \ \displaystyle \int \frac{1}{\sqrt{1-x^2}}\,dx = -\arccos x + c$

c $\dfrac{d}{dx}(\arcsin x) = -\dfrac{d}{dx}(\arccos x)$

So, the solution can be written in terms of either.

EXERCISE 21E.2

1 **a**
$$\begin{aligned} f'(x) &= 2x - 1 \\ \therefore\ f(x) &= \int (2x-1)\,dx \\ &= \frac{2x^2}{2} - x + c \\ &= x^2 - x + c \end{aligned}$$

But $f(0) = 3$, so $0 - 0 + c = 3$

$\therefore\ c = 3$

$\therefore\ f(x) = x^2 - x + 3$

b
$$\begin{aligned} f'(x) &= 3x^2 + 2x \\ \therefore\ f(x) &= \int (3x^2+2x)\,dx \\ &= \frac{3x^3}{3} + \frac{2x^2}{2} + c \\ &= x^3 + x^2 + c \end{aligned}$$

But $f(2) = 5$, so $8 + 4 + c = 5$

$\therefore\ c = -7$

$\therefore\ f(x) = x^3 + x^2 - 7$

c
$$\begin{aligned} f'(x) &= e^x + \frac{1}{\sqrt{x}} = e^x + x^{-\frac{1}{2}} \\ \therefore\ f(x) &= \int (e^x + x^{-\frac{1}{2}})\,dx \\ &= e^x + 2x^{\frac{1}{2}} + c \end{aligned}$$

But $f(1) = 1$, so $e^1 + 2 + c = 1$

$\therefore\ c = -1 - e$

$\therefore\ f(x) = e^x + 2\sqrt{x} - 1 - e$

d
$$\begin{aligned} f'(x) &= x - \frac{2}{\sqrt{x}} = x - 2x^{-\frac{1}{2}} \\ \therefore\ f(x) &= \int (x - 2x^{-\frac{1}{2}})\,dx \\ &= \frac{x^2}{2} - \frac{2x^{\frac{1}{2}}}{\frac{1}{2}} + c \\ &= \tfrac{1}{2}x^2 - 4\sqrt{x} + c \end{aligned}$$

But $f(1) = 2$, so $\frac{1}{2} - 4 + c = 2$

$\therefore\ c = \frac{11}{2}$

$\therefore\ f(x) = \frac{1}{2}x^2 - 4\sqrt{x} + \frac{11}{2}$

2 **a**
$$\begin{aligned} f'(x) &= x^2 - 4\cos x \\ \therefore\ f(x) &= \int (x^2 - 4\cos x)\,dx \\ &= \frac{x^3}{3} - 4\sin x + c \end{aligned}$$

But $f(0) = 3 \quad \therefore\ 0 - 4\sin(0) + c = 3$

$\therefore\ c = 3$

$\therefore\ f(x) = \dfrac{x^3}{3} - 4\sin x + 3$

b
$$\begin{aligned} f'(x) &= 2\cos x - 3\sin x \\ \therefore\ f(x) &= \int (2\cos x - 3\sin x)\,dx \\ &= 2\sin x + 3\cos x + c \end{aligned}$$

But $f(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

$\therefore\ 2\sin\frac{\pi}{4} + 3\cos\frac{\pi}{4} + c = \frac{1}{\sqrt{2}}$

$\therefore\ 2(\frac{1}{\sqrt{2}}) + 3(\frac{1}{\sqrt{2}}) + c = \frac{1}{\sqrt{2}}$

$\therefore\ c = -\frac{4}{\sqrt{2}}$

$\therefore\ c = -2\sqrt{2}$

$\therefore\ f(x) = 2\sin x + 3\cos x - 2\sqrt{2}$

c
$$\begin{aligned} f'(x) &= \sqrt{x} - 2\sec^2 x \\ \therefore\ f(x) &= \int (x^{\frac{1}{2}} - 2\sec^2 x)\,dx \\ &= \tfrac{2}{3}x^{\frac{3}{2}} - 2\tan x + c \end{aligned}$$

But $f(\pi) = 0 \quad \therefore\ \frac{2}{3}\pi^{\frac{3}{2}} - 2\tan\pi + c = 0$

$\therefore\ c = -\frac{2}{3}\pi^{\frac{3}{2}}$

$\therefore\ f(x) = \frac{2}{3}x^{\frac{3}{2}} - 2\tan x - \frac{2}{3}\pi^{\frac{3}{2}}$

3 **a** Given:

$f''(x) = 2x + 1, \quad f'(1) = 3, \quad f(2) = 7$

$\therefore \; f'(x) = \int (2x + 1)\, dx$

$= \dfrac{2x^2}{2} + x + c$

$= x^2 + x + c$

But $f'(1) = 3$ so $1 + 1 + c = 3$

$\therefore \; c = 1$

$\therefore \; f'(x) = x^2 + x + 1$

Then $f(x) = \int (x^2 + x + 1)\, dx$

$= \dfrac{x^3}{3} + \dfrac{x^2}{2} + x + k$

But $f(2) = 7$, so $\frac{8}{3} + 2 + 2 + k = 7$

$\therefore \; k = 7 - 4 - \frac{8}{3}$

$\therefore \; k = \frac{1}{3}$

$\therefore \; f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + \frac{1}{3}$

b Given: $f''(x) = 15\sqrt{x} + \dfrac{3}{\sqrt{x}}$,

$f'(1) = 12, \quad f(0) = 5$

Now $f''(x) = 15x^{\frac{1}{2}} + 3x^{-\frac{1}{2}}$

$\therefore \; f'(x) = \dfrac{15x^{\frac{3}{2}}}{\frac{3}{2}} + \dfrac{3x^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= 10x^{\frac{3}{2}} + 6x^{\frac{1}{2}} + c$

But $f'(1) = 12$ so $10 + 6 + c = 12$

$\therefore \; c = -4$

$\therefore \; f'(x) = 10x^{\frac{3}{2}} + 6x^{\frac{1}{2}} - 4$

$\therefore \; f(x) = \int (10x^{\frac{3}{2}} + 6x^{\frac{1}{2}} - 4)\, dx$

$= \dfrac{10x^{\frac{5}{2}}}{\frac{5}{2}} + \dfrac{6x^{\frac{3}{2}}}{\frac{3}{2}} - 4x + k$

$= 4x^{\frac{5}{2}} + 4x^{\frac{3}{2}} - 4x + k$

But $f(0) = 5$, so $k = 5$

$\therefore \; f(x) = 4x^{\frac{5}{2}} + 4x^{\frac{3}{2}} - 4x + 5$

c Given: $f''(x) = \cos x$, $f'(\frac{\pi}{2}) = 0$ and $f(0) = 3$

Now $f'(x) = \int \cos x\, dx = \sin x + c$

But $f'(\frac{\pi}{2}) = 0$ so $\sin(\frac{\pi}{2}) + c = 0$

$\therefore \; c = -1$

$\therefore \; f'(x) = \sin x - 1$

So, $f(x) = \int (\sin x - 1)\, dx$

$= -\cos x - x + k$

But $f(0) = 3$ so $-\cos 0 - 0 + k = 3$

$\therefore \; -1 + k = 3$

$\therefore \; k = 4$

So, $f(x) = -\cos x - x + 4$

d Given: $f''(x) = 2x$ and that $(1, 0)$ and $(0, 5)$ lie on the curve

Now $f'(x) = \int 2x\, dx = \dfrac{2x^2}{2} + c = x^2 + c$

$\therefore \; f(x) = \int (x^2 + c)\, dx = \dfrac{x^3}{3} + cx + k$

But $f(0) = 5$ so $0 + 0 + k = 5$ and so $k = 5$

and $f(1) = 0$ so $\frac{1}{3} + c + 5 = 0$ and so $c = -5\frac{1}{3}$

$\therefore \; f(x) = \frac{1}{3}x^3 - \frac{16}{3}x + 5$

EXERCISE 21F

1 **a** $\int (2x + 5)^3\, dx$

$= \frac{1}{2} \times \dfrac{(2x + 5)^4}{4} + c$

$= \frac{1}{8}(2x + 5)^4 + c$

b $\displaystyle\int \frac{1}{(3 - 2x)^2}\, dx$

$= \int (3 - 2x)^{-2}\, dx$

$= \frac{1}{-2} \times \dfrac{(3 - 2x)^{-1}}{-1} + c$

$= \dfrac{1}{2(3 - 2x)} + c$

c $\displaystyle\int \frac{4}{(2x - 1)^4}\, dx$

$= \int 4\,(2x - 1)^{-4}\, dx$

$= 4(\frac{1}{2}) \times \dfrac{(2x - 1)^{-3}}{-3} + c$

$= \dfrac{-2}{3(2x - 1)^3} + c$

d $\int (4x-3)^7\,dx$
$= \frac{1}{4} \times \frac{(4x-3)^8}{8} + c$
$= \frac{1}{32}(4x-3)^8 + c$

e $\int \sqrt{3x-4}\,dx$
$= \int (3x-4)^{\frac{1}{2}}\,dx$
$= \frac{1}{3} \times \frac{(3x-4)^{\frac{3}{2}}}{\frac{3}{2}} + c$
$= \frac{2}{9}(3x-4)^{\frac{3}{2}} + c$

f $\int \frac{10}{\sqrt{1-5x}}\,dx$
$= \int 10(1-5x)^{-\frac{1}{2}}\,dx$
$= 10(\frac{1}{-5}) \times \frac{(1-5x)^{\frac{1}{2}}}{\frac{1}{2}} + c$
$= -4\sqrt{1-5x} + c$

g $\int 3(1-x)^4\,dx$
$= 3\int (1-x)^4\,dx$
$= 3(\frac{1}{-1}) \times \frac{(1-x)^5}{5} + c$
$= -\frac{3}{5}(1-x)^5 + c$

h $\int \frac{4}{\sqrt{3-4x}}\,dx$
$= \int 4(3-4x)^{-\frac{1}{2}}\,dx$
$= 4(\frac{1}{-4}) \times \frac{(3-4x)^{\frac{1}{2}}}{\frac{1}{2}} + c$
$= -2\sqrt{3-4x} + c$

2 **a** $\int \sin(3x)\,dx$
$= -\frac{1}{3}\cos(3x) + c$

b $\int (2\cos(-4x)+1)\,dx$
$= 2 \times (\frac{1}{-4})\sin(-4x) + x + c$
$= -\frac{1}{2}\sin(-4x) + x + c$

c $\int \sec^2(2x)\,dx$
$= \frac{1}{2}\tan(2x) + c$

d $\int 3\cos\left(\frac{x}{2}\right)\,dx$
$= 6\sin\left(\frac{x}{2}\right) + c$

e $\int (3\sin(2x) - e^{-x})\,dx$
$= -\frac{3}{2}\cos(2x) + e^{-x} + c$

f $\int \left[e^{2x} - 2\sec^2\left(\frac{x}{2}\right)\right]\,dx$
$= \frac{1}{2}e^{2x} - 2 \times 2\tan\left(\frac{x}{2}\right) + c$
$= \frac{1}{2}e^{2x} - 4\tan\left(\frac{x}{2}\right) + c$

g $\int 2\sin(2x + \frac{\pi}{6})\,dx$
$= -\frac{2}{2}\cos(2x + \frac{\pi}{6}) + c$
$= -\cos(2x + \frac{\pi}{6}) + c$

h $\int -3\cos(\frac{\pi}{4} - x)\,dx$
$= -3 \times (-1)\sin(\frac{\pi}{4} - x) + c$
$= 3\sin(\frac{\pi}{4} - x) + c$

i $\int 4\sec^2\left(\frac{\pi}{3} - 2x\right)\,dx$
$= 4 \times (-\frac{1}{2})\tan\left(\frac{\pi}{3} - 2x\right) + c$
$= -2\tan\left(\frac{\pi}{3} - 2x\right) + c$

j $\int (\cos(2x) + \sin(2x))\,dx$
$= \frac{1}{2}\sin(2x) - \frac{1}{2}\cos(2x) + c$

k $\int (2\sin(3x) + 5\cos(4x))\,dx$
$= -\frac{2}{3}\cos(3x) + \frac{5}{4}\sin(4x) + c$

l $\int \left(\frac{1}{2}\cos(8x) - 3\sin x\right)\,dx$
$= \frac{1}{2}(\frac{1}{8})\sin(8x) + 3\cos x + c$
$= \frac{1}{16}\sin(8x) + 3\cos x + c$

3 **a** $\frac{dy}{dx} = \sqrt{2x-7} = (2x-7)^{\frac{1}{2}}$
$\therefore\ y = \frac{1}{2} \times \frac{(2x-7)^{\frac{3}{2}}}{\frac{3}{2}} + c$
$= \frac{1}{3}(2x-7)^{\frac{3}{2}} + c$

But $y = 11$ when $x = 8$
$\therefore\ \frac{1}{3}(16-7)^{\frac{3}{2}} + c = 11$
$\therefore\ \frac{1}{3}(27) + c = 11$
$\therefore\ 9 + c = 11$ and so $c = 2$
$\therefore\ y = \frac{1}{3}(2x-7)^{\frac{3}{2}} + 2$

b $f(x)$ has gradient function $f'(x) = \dfrac{4}{\sqrt{1-x}} = 4(1-x)^{-\frac{1}{2}}$

$\therefore \quad f(x) = 4(\frac{1}{-1}) \times \dfrac{(1-x)^{\frac{1}{2}}}{\frac{1}{2}} + c$
$= -8\sqrt{1-x} + c$

But $y = -11$ when $x = -3$

$\therefore \quad -8\sqrt{1-(-3)} + c = -11$
$\therefore \quad -8\sqrt{4} + c = -11$
$\therefore \quad -16 + c = -11$ and so $c = 5$
$\therefore \quad f(x) = 5 - 8\sqrt{1-x}$

Now $f(-8) = 5 - 8\sqrt{1-(-8)} = 5 - 8(3) = -19$, so the point is $(-8, -19)$.

4 **a** $\int \cos^2 x \, dx$
$= \int (\frac{1}{2} + \frac{1}{2}\cos(2x)) \, dx$
$= \frac{1}{2}x + \frac{1}{4}\sin(2x) + c$

b $\int \sin^2 x \, dx$
$= \int (\frac{1}{2} - \frac{1}{2}\cos(2x)) \, dx$
$= \frac{1}{2}x - \frac{1}{4}\sin(2x) + c$

c $\int (1 + \cos^2(2x)) \, dx$
$= \int (1 + \frac{1}{2} + \frac{1}{2}\cos(4x)) \, dx$
$= \int (\frac{3}{2} + \frac{1}{2}\cos(4x)) \, dx$
$= \frac{3}{2}x + \frac{1}{8}\sin(4x) + c$

d $\int (3 - \sin^2(3x)) \, dx$
$= \int (3 - (\frac{1}{2} - \frac{1}{2}\cos(6x))) \, dx$
$= \int (\frac{5}{2} + \frac{1}{2}\cos(6x)) \, dx$
$= \frac{5}{2}x + \frac{1}{12}\sin(6x) + c$

e $\int \frac{1}{2}\cos^2(4x) \, dx$
$= \int \frac{1}{2}(\frac{1}{2} + \frac{1}{2}\cos(8x)) \, dx$
$= \int (\frac{1}{4} + \frac{1}{4}\cos(8x)) \, dx$
$= \frac{1}{4}x + \frac{1}{32}\sin(8x) + c$

f $\int (1 + \cos x)^2 \, dx$
$= \int (1 + 2\cos x + \cos^2 x) \, dx$
$= \int (1 + 2\cos x + \frac{1}{2} + \frac{1}{2}\cos(2x)) \, dx$
$= \int \left(\frac{3}{2} + 2\cos x + \frac{1}{2}\cos(2x)\right) dx$
$= \frac{3}{2}x + 2\sin x + \frac{1}{4}\sin(2x) + c$

5 **a** $\int 3(2x-1)^2 \, dx$
$= 3\int (2x-1)^2 \, dx$
$= 3(\frac{1}{2}) \dfrac{(2x-1)^3}{3} + c$
$= \frac{1}{2}(2x-1)^3 + c$

b $\int (x^2 - x)^2 \, dx$
$= \int \left(x^4 - 2x^3 + x^2\right) dx$
$= \dfrac{x^5}{5} - \dfrac{2x^4}{4} + \dfrac{x^3}{3} + c$
$= \frac{1}{5}x^5 - \frac{1}{2}x^4 + \frac{1}{3}x^3 + c$

c $\int (1-3x)^3 \, dx$
$= (\frac{1}{-3}) \dfrac{(1-3x)^4}{4} + c$
$= -\frac{1}{12}(1-3x)^4 + c$

d $\int (1-x^2)^2 \, dx$
$= \int (1 - 2x^2 + x^4) \, dx$
$= x - \frac{2}{3}x^3 + \frac{1}{5}x^5 + c$

e $\int 4\sqrt{5-x} \, dx$
$= 4\int (5-x)^{\frac{1}{2}} \, dx$
$= 4(\frac{1}{-1}) \dfrac{(5-x)^{\frac{3}{2}}}{\frac{3}{2}} + c$
$= -\frac{8}{3}(5-x)^{\frac{3}{2}} + c$

f $\int (x^2+1)^3 \, dx$
$= \int (x^6 + 3x^4 + 3x^2 + 1) \, dx$
$= \dfrac{x^7}{7} + \dfrac{3x^5}{5} + \dfrac{3x^3}{3} + x + c$
$= \frac{1}{7}x^7 + \frac{3}{5}x^5 + x^3 + x + c$

6 **a** $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$

$\therefore \quad \cos^4 x = (\frac{1}{2} + \frac{1}{2}\cos(2x))^2$
$= \frac{1}{4} + \frac{1}{2}\cos(2x) + \frac{1}{4}\cos^2(2x)$
$= \frac{1}{4} + \frac{1}{2}\cos(2x) + \frac{1}{4}(\frac{1}{2} + \frac{1}{2}\cos(4x))$
$= \frac{1}{4} + \frac{1}{2}\cos(2x) + \frac{1}{8} + \frac{1}{8}\cos(4x)$
$= \frac{1}{8}\cos(4x) + \frac{1}{2}\cos(2x) + \frac{3}{8}$ as required

b $\int \cos^4 x \, dx = \int (\frac{1}{8}\cos(4x) + \frac{1}{2}\cos(2x) + \frac{3}{8}) \, dx$
$= \frac{1}{32}\sin(4x) + \frac{1}{4}\sin(2x) + \frac{3}{8}x + c$

7 **a** $\int \left(2e^x + 5e^{2x}\right) dx$
$= 2e^x + 5(\frac{1}{2})e^{2x} + c$
$= 2e^x + \frac{5}{2}e^{2x} + c$

b $\int (3e^{5x-2})\, dx$
$= 3(\frac{1}{5})e^{5x-2} + c$
$= \frac{3}{5}e^{5x-2} + c$

c $\int (e^{7-3x})\, dx$
$= \frac{1}{-3}e^{7-3x} + c$
$= -\frac{1}{3}e^{7-3x} + c$

d $\displaystyle\int \frac{1}{2x-1}\, dx$
$= \frac{1}{2}\ln|2x-1| + c$

e $\displaystyle\int \frac{5}{1-3x}\, dx$
$= 5\displaystyle\int \frac{1}{1-3x}\, dx$
$= 5(\frac{1}{-3})\ln|1-3x| + c$
$= -\frac{5}{3}\ln|1-3x| + c$

f $\displaystyle\int \left(e^{-x} - \frac{4}{2x+1}\right) dx$
$= \frac{1}{-1}e^{-x} - 4(\frac{1}{2})\ln|2x+1| + c$
$= -e^{-x} - 2\ln|2x+1| + c$

g $\int (e^x + e^{-x})^2\, dx$
$= \int \left(e^{2x} + 2 + e^{-2x}\right) dx$
$= \frac{1}{2}e^{2x} + 2x + (\frac{1}{-2})e^{-2x} + c$
$= \frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x} + c$

h $\int (e^{-x} + 2)^2\, dx$
$= \int \left(e^{-2x} + 4e^{-x} + 4\right) dx$
$= \frac{1}{-2}e^{-2x} + 4(\frac{1}{-1})e^{-x} + 4x + c$
$= -\frac{1}{2}e^{-2x} - 4e^{-x} + 4x + c$

i $\displaystyle\int \left(x - \frac{5}{1-x}\right) dx$
$= \dfrac{x^2}{2} - 5(\frac{1}{-1})\ln|1-x| + c$
$= \frac{1}{2}x^2 + 5\ln|1-x| + c$

8 **a** $\dfrac{dy}{dx} = (1-e^x)^2$
$= 1 - 2e^x + e^{2x}$
$\therefore\ y = x - 2e^x + \frac{1}{2}e^{2x} + c$

b $\dfrac{dy}{dx} = 1 - 2x + \dfrac{3}{x+2}$
$\therefore\ y = x - \dfrac{2x^2}{2} + 3\ln|x+2| + c$
$= x - x^2 + 3\ln|x+2| + c$

c $\dfrac{dy}{dx} = e^{-2x} + \dfrac{4}{2x-1}$
$\therefore\ y = \frac{1}{-2}e^{-2x} + 4(\frac{1}{2})\ln|2x-1| + c$
$= -\frac{1}{2}e^{-2x} + 2\ln|2x-1| + c$

9 Differentiating Tracy's answer gives
$\dfrac{d}{dx}\left(\frac{1}{4}\ln(4x) + c\right) = \frac{1}{4}\left(\dfrac{1}{4x}\right) \times 4 + 0,\ \ x > 0$
$= \dfrac{1}{4x},\ \ x > 0$

Differentiating Nadine's answer gives
$\dfrac{d}{dx}\left(\frac{1}{4}\ln(x) + c\right) = \frac{1}{4}\left(\dfrac{1}{x}\right) + 0,\ \ x > 0$
$= \dfrac{1}{4x},\ \ x > 0$

Both answers give the correct derivative and both are correct. This result occurs because $\ln(4x) = \ln 4 + \ln x$. Their answers differ by a constant which is accounted for by c.

10 Given: $f'(x) = p\sin(\frac{1}{2}x)$, $f(0) = 1$ and $f(2\pi) = 0$
$\therefore\ f(x) = -2p\cos(\frac{1}{2}x) + c$

But $f(0) = 1$,
so $-2p\cos(0) + c = 1$
$\therefore\ -2p + c = 1$
$\therefore\ c = 1 + 2p$ .... (1)

Also, $f(2\pi) = 0$, so $-2p\cos(\pi) + c = 0$
$\therefore\ 2p + c = 0$
$\therefore\ 2p + 1 + 2p = 0$ {using (1)}
$\therefore\ p = -\frac{1}{4}$
$\therefore\ c = \frac{1}{2}$ {from (1)}

$\therefore\ f(x) = \frac{1}{2}\cos(\frac{1}{2}x) + \frac{1}{2}$

11 $g''(x) = -\sin 2x$

Integrating both sides with respect to x, we get $g'(x) = \frac{1}{2}\cos 2x + c$, c some constant.

So, $g'(\pi) = \frac{1}{2}\cos(2\pi) + c = \frac{1}{2} + c$ and $g'(-\pi) = \frac{1}{2}\cos(-2\pi) + c = \frac{1}{2} + c = g'(\pi)$

$\therefore$ the gradients of the tangents to $y = g(x)$ at $x = \pi$ and $x = -\pi$ are equal.

12 **a**

$$\begin{aligned} f'(x) &= 2e^{-2x} \\ \therefore\ f(x) &= 2(\tfrac{1}{-2})e^{-2x} + c \\ &= -e^{-2x} + c \end{aligned}$$

But $f(0) = 3$ so $-e^0 + c = 3$
$\therefore\ c = 4$

$\therefore\ f(x) = -e^{-2x} + 4$

b

$$\begin{aligned} f'(x) &= 2x - \frac{2}{1-x} \\ \therefore\ f(x) &= \frac{2x^2}{2} - \frac{2}{-1}\ln|1-x| + c \\ &= x^2 + 2\ln|1-x| + c \end{aligned}$$

But $f(-1) = 3$ so $1 + 2\ln|2| + c = 3$
$\therefore\ c = 2 - 2\ln 2$

$\therefore\ f(x) = x^2 + 2\ln|1-x| + 2 - 2\ln 2$

c

$$\begin{aligned} f'(x) &= \sqrt{x} + \tfrac{1}{2}e^{-4x} \\ &= x^{\frac{1}{2}} + \tfrac{1}{2}e^{-4x} \\ \therefore\ f(x) &= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \tfrac{1}{2}(\tfrac{1}{-4})e^{-4x} + c \\ &= \tfrac{2}{3}x^{\frac{3}{2}} - \tfrac{1}{8}e^{-4x} + c \end{aligned}$$

But $f(1) = 0$

$\therefore\ \frac{2}{3} - \frac{1}{8}e^{-4} + c = 0$

$\therefore\ c = \frac{1}{8}e^{-4} - \frac{2}{3}$

$\therefore\ f(x) = \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{8}e^{-4x} + \frac{1}{8}e^{-4} - \frac{2}{3}$

13

$$\begin{aligned} (\sin x + \cos x)^2 &= \sin^2 x + 2\sin x\cos x + \cos^2 x \\ &= 1 + \sin 2x \\ \therefore\ \int (\sin x + \cos x)^2\, dx &= \int (1 + \sin 2x)\, dx \\ &= x - \tfrac{1}{2}\cos(2x) + c \end{aligned}$$

14

$$\begin{aligned} (\cos x + 1)^2 &= \cos^2 x + 2\cos x + 1 \\ &= (\tfrac{1}{2} + \tfrac{1}{2}\cos 2x) + 2\cos x + 1 \\ &= \tfrac{1}{2}\cos 2x + 2\cos x + \tfrac{3}{2} \\ \therefore\ \int (\cos x + 1)^2\, dx &= \int (\tfrac{1}{2}\cos 2x + 2\cos x + \tfrac{3}{2})\, dx \\ &= \tfrac{1}{4}\sin 2x + 2\sin x + \tfrac{3}{2}x + c \end{aligned}$$

15

$$\begin{aligned} \frac{3}{x+2} - \frac{1}{x-2} &= \frac{3(x-2) - 1(x+2)}{(x+2)(x-2)} \\ &= \frac{3x - 6 - x - 2}{x^2 - 4} \\ &= \frac{2x - 8}{x^2 - 4} \\ \therefore\ \int \frac{2x-8}{x^2-4}\, dx &= \int \left(\frac{3}{x+2} - \frac{1}{x-2}\right) dx \\ &= 3\ln|x+2| - \ln|x-2| + c \end{aligned}$$

16

$$\begin{aligned} \frac{1}{2x-1} - \frac{1}{2x+1} &= \frac{1(2x+1) - 1(2x-1)}{(2x-1)(2x+1)} \\ &= \frac{2x + 1 - 2x + 1}{(2x-1)(2x+1)} \\ &= \frac{2}{4x^2 - 1} \end{aligned}$$

$$\begin{aligned} \therefore\ &\int \frac{2}{4x^2 - 1}\, dx \\ &= \int \left(\frac{1}{2x-1} - \frac{1}{2x+1}\right) dx \\ &= \tfrac{1}{2}\ln|2x-1| - \tfrac{1}{2}\ln|2x+1| + c \end{aligned}$$

EXERCISE 21G.1

1 **a** $u = x^3+1, \quad \dfrac{du}{dx} = 3x^2$

$$\begin{aligned}\therefore \int 3x^2(x^3+1)^4\,dx &= \int u^4 \frac{du}{dx}\,dx\\ &= \int u^4\,du\\ &= \tfrac{1}{5}u^5 + c\\ &= \tfrac{1}{5}(x^3+1)^5 + c\end{aligned}$$

b $u = x^3+1, \quad \dfrac{du}{dx} = 3x^2$

$$\begin{aligned}\therefore \int x^2 e^{x^3+1}\,dx &= \tfrac{1}{3}\int (3x^2)e^{x^3+1}\,dx\\ &= \tfrac{1}{3}\int e^u \frac{du}{dx}\,dx\\ &= \tfrac{1}{3}\int e^u\,du\\ &= \tfrac{1}{3}e^u + c\\ &= \tfrac{1}{3}e^{x^3+1} + c\end{aligned}$$

c $u = \sin x, \quad \dfrac{du}{dx} = \cos x$

$$\begin{aligned}\therefore \int \sin^4 x\cos x\,dx &= \int u^4\frac{du}{dx}\,dx\\ &= \int u^4\,du\\ &= \frac{u^5}{5} + c\\ &= \tfrac{1}{5}\sin^5 x + c\end{aligned}$$

d $u = \dfrac{x-1}{x} = 1 - x^{-1}, \quad \dfrac{du}{dx} = \dfrac{1}{x^2}$

$$\begin{aligned}\therefore \int \frac{e^{\frac{x-1}{x}}}{x^2}\,dx &= \int e^u\frac{du}{dx}\,dx\\ &= \int e^u\,du\\ &= e^u + c\\ &= e^{\frac{x-1}{x}} + c\end{aligned}$$

2 **a**

$$\begin{aligned}&\int 4x^3(2+x^4)^3\,dx\\ &= \int u^3\frac{du}{dx}\,dx \quad \left\{u = 2+x^4, \ \frac{du}{dx} = 4x^3\right\}\\ &= \int u^3\,du\\ &= \frac{u^4}{4} + c\\ &= \tfrac{1}{4}(2+x^4)^4 + c\end{aligned}$$

b

$$\begin{aligned}&\int \frac{2x}{\sqrt{x^2+3}}\,dx\\ &= \int ((x^2+3)^{-\frac{1}{2}} \times 2x)\,dx\\ &= \int u^{-\frac{1}{2}}\frac{du}{dx}\,dx \quad \left\{u = x^2+3, \ \frac{du}{dx} = 2x\right\}\\ &= \int u^{-\frac{1}{2}}\,du\\ &= \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c\\ &= 2\sqrt{u} + c\\ &= 2\sqrt{x^2+3} + c\end{aligned}$$

c

$$\begin{aligned}&\int \frac{x}{(1-x^2)^5}\,dx\\ &= -\tfrac{1}{2}\int (1-x^2)^{-5} \times (-2x)\,dx\\ &= -\tfrac{1}{2}\int u^{-5}\frac{du}{dx}\,dx \quad \{u = 1-x^2,\\ &= -\tfrac{1}{2}\int u^{-5}\,du \qquad \frac{du}{dx} = -2x\}\\ &= -\tfrac{1}{2}\,\frac{u^{-4}}{-4} + c\\ &= \frac{1}{8(1-x^2)^4} + c\end{aligned}$$

d

$$\begin{aligned}&\int \sqrt{x^3+x}\,(3x^2+1)\,dx\\ &= \int \sqrt{u}\,\frac{du}{dx}\,dx \quad \{u = x^3+x,\\ &= \int u^{\frac{1}{2}}\,du \qquad \frac{du}{dx} = 3x^2+1\}\\ &= \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c\\ &= \tfrac{2}{3}u^{\frac{3}{2}} + c\\ &= \tfrac{2}{3}(x^3+x)^{\frac{3}{2}} + c\end{aligned}$$

e

$$\begin{aligned}&\int (x^3+2x+1)^4(3x^2+2)\,dx\\ &= \int u^4\frac{du}{dx}\,dx \quad \{u = x^3+2x+1,\\ &= \int u^4\,du \qquad \frac{du}{dx} = 3x^2+2\}\\ &= \frac{u^5}{5} + c\\ &= \tfrac{1}{5}(x^3+2x+1)^5 + c\end{aligned}$$

f

$$\int \frac{x^2}{(3x^3-1)^4}\,dx$$
$$= \int (3x^3-1)^{-4} \times x^2\,dx$$
$$= \tfrac{1}{9}\int (3x^3-1)^{-4} \times 9x^2\,dx$$
$$= \tfrac{1}{9}\int u^{-4}\,\frac{du}{dx}\,dx$$
$$\left\{u = 3x^3-1,\quad \frac{du}{dx} = 9x^2\right\}$$
$$= \tfrac{1}{9}\int u^{-4}\,du$$
$$= \tfrac{1}{9}\,\frac{u^{-3}}{-3} + c$$
$$= -\frac{1}{27(3x^3-1)^3} + c$$

g

$$\int \frac{x+2}{(x^2+4x-3)^2}\,dx$$
$$= \tfrac{1}{2}\int (x^2+4x-3)^{-2}(2x+4)\,dx$$
$$= \tfrac{1}{2}\int u^{-2}\,\frac{du}{dx}\,dx \quad \left\{u = x^2+4x-3,\ \frac{du}{dx} = 2x+4\right\}$$
$$= \tfrac{1}{2}\int u^{-2}\,du$$
$$= \tfrac{1}{2}\,\frac{u^{-1}}{-1} + c$$
$$= \frac{-1}{2(x^2+4x-3)} + c$$

h

$$\int x^4(x+1)^4(2x+1)\,dx$$
$$= \int (x^2+x)^4(2x+1)\,dx$$
$$= \int u^4\,\frac{du}{dx}\,dx \quad \left\{u = x^2+x,\quad \frac{du}{dx} = 2x+1\right\}$$
$$= \int u^4\,du$$
$$= \tfrac{1}{5}u^5 + c$$
$$= \tfrac{1}{5}(x^2+x)^5 + c$$

3 **a**

$$\int -2e^{1-2x}\,dx$$
$$= \int e^u\,\frac{du}{dx}\,dx \quad \left\{u = 1-2x,\quad \frac{du}{dx} = -2\right\}$$
$$= \int e^u\,du$$
$$= e^u + c$$
$$= e^{1-2x} + c$$

b

$$\int 2xe^{x^2}\,dx$$
$$= \int e^u\,\frac{du}{dx}\,dx \quad \left\{u = x^2,\quad \frac{du}{dx} = 2x\right\}$$
$$= \int e^u\,du$$
$$= e^u + c$$
$$= e^{x^2} + c$$

c

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int \frac{e^{\sqrt{x}}}{2\sqrt{x}}\,dx$$
$$= 2\int e^u\,\frac{du}{dx}\,dx \quad \left\{u = \sqrt{x},\ \frac{du}{dx} = \frac{1}{2\sqrt{x}}\right\}$$
$$= 2\int e^u\,du$$
$$= 2e^u + c$$
$$= 2e^{\sqrt{x}} + c$$

d

$$\int (2x-1)e^{x-x^2}\,dx$$
$$= -\int (1-2x)e^{x-x^2}\,dx$$
$$= -\int e^u\,\frac{du}{dx}\,dx \quad \left\{u = x-x^2,\ \frac{du}{dx} = 1-2x\right\}$$
$$= -\int e^u\,du$$
$$= -e^u + c$$
$$= -e^{x-x^2} + c$$

4 **a** Let $u = x^2+1$, $\dfrac{du}{dx} = 2x$

$$\therefore\ \int \frac{2x}{x^2+1}\,dx = \int \frac{1}{x^2+1}\,(2x)\,dx$$
$$= \int \frac{1}{u}\,\frac{du}{dx}\,dx$$
$$= \int \frac{1}{u}\,du$$
$$= \ln|u| + c$$
$$= \ln\left|x^2+1\right| + c$$

b Let $u = 2-x^2$, $\dfrac{du}{dx} = -2x$

$$\therefore\ \int \frac{x}{2-x^2}\,dx = -\tfrac{1}{2}\int \frac{1}{2-x^2}\,(-2x)\,dx$$
$$= -\tfrac{1}{2}\int \frac{1}{u}\,\frac{du}{dx}\,dx$$
$$= -\tfrac{1}{2}\int \frac{1}{u}\,du$$
$$= -\tfrac{1}{2}\ln|u| + c$$
$$= -\tfrac{1}{2}\ln\left|2-x^2\right| + c$$

c Let $u = x^2 - 3x$, $\dfrac{du}{dx} = 2x - 3$

$$\therefore \int \frac{2x-3}{x^2-3x}\,dx = \int \frac{1}{x^2-3x}(2x-3)\,dx = \int \frac{1}{u}\frac{du}{dx}\,dx = \int \frac{1}{u}\,du = \ln|u| + c = \ln\left|x^2-3x\right| + c$$

d Let $u = x^3 - x$, $\dfrac{du}{dx} = 3x^2 - 1$

$$\therefore \int \frac{6x^2-2}{x^3-x}\,dx = 2\int \frac{1}{x^3-3x}(3x^2-1)\,dx = 2\int \frac{1}{u}\frac{du}{dx}\,dx = 2\int \frac{1}{u}\,du = 2\ln|u| + c = 2\ln\left|x^3-x\right| + c$$

e Let $u = 5x - x^2$, $\dfrac{du}{dx} = 5 - 2x$

$$\therefore \int \frac{4x-10}{5x-x^2}\,dx = -2\int \frac{1}{5x-x^2}(5-2x)\,dx = -2\int \frac{1}{u}\frac{du}{dx}\,dx = -2\int \frac{1}{u}\,du = -2\ln|u| + c = -2\ln\left|5x-x^2\right| + c$$

f Let $u = x^3 - 3x$, $\dfrac{du}{dx} = 3x^2 - 3$

$$\therefore \int \frac{1-x^2}{x^3-3x}\,dx = -\tfrac{1}{3}\int \frac{1}{x^3-3x}(3x^2-3)\,dx = -\tfrac{1}{3}\int \frac{1}{u}\frac{du}{dx}\,dx = -\tfrac{1}{3}\int \frac{1}{u}\,du = -\tfrac{1}{3}\ln|u| + c = -\tfrac{1}{3}\ln\left|x^3-3x\right| + c$$

5 **a** Let $u = 3 - x^3$, $\dfrac{du}{dx} = -3x^2$

$$\therefore f(x) = \int x^2(3-x^3)^2\,dx = -\tfrac{1}{3}\int (-3x^2)(3-x^3)^2\,dx = -\tfrac{1}{3}\int u^2\frac{du}{dx}\,dx = -\tfrac{1}{3}\int u^2\,du = -\tfrac{1}{3}\times\frac{u^3}{3} + c = -\tfrac{1}{9}(3-x^3)^3 + c$$

b Let $u = \ln x$, $\dfrac{du}{dx} = \dfrac{1}{x}$

$$\therefore f(x) = \int \frac{4}{x\ln x}\,dx = 4\int \frac{1}{\ln x}\times\frac{1}{x}\,dx = 4\int u^{-1}\frac{du}{dx}\,dx = 4\int \frac{1}{u}\,du = 4\ln|u| + c = 4\ln|\ln x| + c$$

c Let $u = 1 - x^2$, $\dfrac{du}{dx} = -2x$

$$\therefore f(x) = \int x\sqrt{1-x^2}\,dx = -\tfrac{1}{2}\int (-2x)\sqrt{1-x^2}\,dx = -\tfrac{1}{2}\int \sqrt{u}\frac{du}{dx}\,dx = -\tfrac{1}{2}\int u^{\frac{1}{2}}\,du = -\tfrac{1}{2}\times\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c = -\tfrac{1}{3}u^{\frac{3}{2}} + c = -\tfrac{1}{3}(1-x^2)^{\frac{3}{2}} + c$$

d Let $u = 1 - x^2$, $\dfrac{du}{dx} = -2x$

$$\therefore f(x) = \int xe^{1-x^2}\,dx = -\tfrac{1}{2}\int (-2x)e^{1-x^2}\,dx = -\tfrac{1}{2}\int e^u\frac{du}{dx}\,dx = -\tfrac{1}{2}\int e^u\,du = -\tfrac{1}{2}e^u + c = -\tfrac{1}{2}e^{1-x^2} + c$$

e Let $u = x^3 - x$, $\dfrac{du}{dx} = 3x^2 - 1$

$$\begin{aligned}\therefore\ f(x) &= \int \frac{1-3x^2}{x^3-x}\,dx \\ &= -\int \frac{3x^2-1}{x^3-x}\,dx \\ &= -\int \frac{1}{u}\frac{du}{dx}\,dx \\ &= -\int \frac{1}{u}\,du \\ &= -\ln|u| + c \\ &= -\ln\left|x^3-x\right| + c\end{aligned}$$

f Let $u = \ln x$, $\dfrac{du}{dx} = \dfrac{1}{x}$

$$\begin{aligned}\therefore\ f(x) &= \int \frac{(\ln x)^3}{x}\,dx \\ &= \int u^3 \frac{du}{dx}\,dx \\ &= \int u^3\,du \\ &= \frac{u^4}{4} + c \\ &= \tfrac{1}{4}(\ln x)^4 + c\end{aligned}$$

6 **a** Let $u = \sin x$, $\dfrac{du}{dx} = \cos x$

$$\begin{aligned}\therefore\ \int \sin^4 x \cos x\,dx &= \int u^4 \frac{du}{dx}\,dx \\ &= \int u^4\,du \\ &= \frac{u^5}{5} + c \\ &= \tfrac{1}{5}\sin^5 x + c\end{aligned}$$

b Let $u = \cos x$, $\dfrac{du}{dx} = -\sin x$

$$\begin{aligned}\therefore\ \int \frac{\sin x}{\sqrt{\cos x}}\,dx &= -\int \frac{-\sin x}{\sqrt{\cos x}}\,dx \\ &= -\int u^{-\frac{1}{2}} \frac{du}{dx}\,dx \\ &= -\int u^{-\frac{1}{2}}\,du \\ &= -\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c \\ &= -2\sqrt{\cos x} + c\end{aligned}$$

c Let $u = \cos x$, $\dfrac{du}{dx} = -\sin x$

$$\begin{aligned}\therefore\ \int \tan x\,dx &= \int \frac{\sin x}{\cos x}\,dx \\ &= -\int \frac{-\sin x}{\cos x}\,dx \\ &= -\int \frac{1}{u}\frac{du}{dx}\,dx \\ &= -\int \frac{1}{u}\,du \\ &= -\ln|u| + c \\ &= -\ln|\cos x| + c\end{aligned}$$

d Let $u = \sin x$, $\dfrac{du}{dx} = \cos x$

$$\begin{aligned}\therefore\ \int \sqrt{\sin x}\cos x\,dx &= \int u^{\frac{1}{2}} \frac{du}{dx}\,dx \\ &= \int u^{\frac{1}{2}}\,du \\ &= \tfrac{2}{3}u^{\frac{3}{2}} + c \\ &= \tfrac{2}{3}(\sin x)^{\frac{3}{2}} + c\end{aligned}$$

e Let $u = 2 + \sin x$, $\dfrac{du}{dx} = \cos x$

$$\begin{aligned}\therefore\ \int \frac{\cos x}{(2+\sin x)^2}\,dx &= \int u^{-2} \frac{du}{dx}\,dx \\ &= \int u^{-2}\,du \\ &= -u^{-1} + c \\ &= \frac{-1}{2+\sin x} + c\end{aligned}$$

f Let $u = \cos x$, $\dfrac{du}{dx} = -\sin x$

$$\begin{aligned}\therefore\ \int \frac{\sin x}{\cos^3 x}\,dx &= -\int \frac{-\sin x}{\cos^3 x}\,dx \\ &= -\int u^{-3} \frac{du}{dx}\,dx \\ &= -\int u^{-3}\,du \\ &= \frac{-u^{-2}}{-2} + c \\ &= \tfrac{1}{2}u^{-2} + c \\ &= \frac{1}{2\cos^2 x} + c\end{aligned}$$

g Let $u = 1 - \cos x$, $\dfrac{du}{dx} = \sin x$

$$\therefore \int \frac{\sin x}{1-\cos x}\,dx = \int \frac{1}{u}\frac{du}{dx}\,dx = \int \frac{1}{u}\,du = \ln|u| + c = \ln|1-\cos x| + c$$

h Let $u = \sin(2x) - 3$, $\dfrac{du}{dx} = 2\cos(2x)$

$$\therefore \int \frac{\cos(2x)}{\sin(2x)-3}\,dx = \tfrac{1}{2}\int \frac{2\cos(2x)}{\sin(2x)-3}\,dx = \tfrac{1}{2}\int \frac{1}{u}\frac{du}{dx}\,dx = \tfrac{1}{2}\int \frac{1}{u}\,du = \tfrac{1}{2}\ln|u| + c = \tfrac{1}{2}\ln|\sin(2x)-3| + c$$

i Let $u = x^2$, $\dfrac{du}{dx} = 2x$

$$\therefore \int x\sin(x^2)\,dx = \tfrac{1}{2}\int (2x)\sin(x^2)\,dx = \tfrac{1}{2}\int \sin u\,\frac{du}{dx}\,dx = \tfrac{1}{2}\int \sin u\,du = \tfrac{1}{2}(-\cos u) + c = -\tfrac{1}{2}\cos(x^2) + c$$

j Now $\displaystyle\int \frac{\sin^3 x}{\cos^5 x}\,dx = \int \tan^3 x\sec^2 x\,dx$

Let $u = \tan x$, $\dfrac{du}{dx} = \sec^2 x$

$$\therefore \int \frac{\sin^3 x}{\cos^5 x}\,dx = \int u^3\frac{du}{dx}\,dx = \int u^3\,du = \frac{u^4}{4} + c = \frac{\tan^4 x}{4} + c$$

k Let $u = \csc(2x)$,

$\dfrac{du}{dx} = -\csc(2x)\cot(2x) \times 2$

Now $\int \csc^3(2x)\cot(2x)\,dx$

$$= \int \csc^2(2x)\csc(2x)\cot(2x)\,dx = \int u^2\left(-\tfrac{1}{2}\frac{du}{dx}\right)dx = -\tfrac{1}{2}\int u^2\frac{du}{dx}\,dx = -\tfrac{1}{2}\int u^2\,du = -\tfrac{1}{2}\left(\frac{u^3}{3}\right) + c = -\tfrac{1}{6}\csc^3(2x) + c$$

l $$\int \cos^3 x\,dx = \int \cos^2 x\cos x\,dx = \int (1-\sin^2 x)\cos x\,dx$$

Let $u = \sin x$, $\dfrac{du}{dx} = \cos x$

$$\therefore \int \cos^3 x\,dx = \int (1-u^2)\frac{du}{dx}\,dx = \int (1-u^2)\,du = u - \frac{u^3}{3} + c = \sin x - \frac{\sin^3 x}{3} + c = \sin x - \tfrac{1}{3}\sin^3 x + c$$

7 **a** $$\int \sin^5 x\,dx = \int \sin^4 x\sin x\,dx = \int (1-\cos^2 x)^2\sin x\,dx = \int (1 - 2\cos^2 x + \cos^4 x)\sin x\,dx$$

Let $u = \cos x$, $\dfrac{du}{dx} = -\sin x$

$$\therefore \int \sin^5 x\,dx = -\int (1-2u^2+u^4)\frac{du}{dx}\,dx = -\int (1-2u^2+u^4)\,du = -u + \tfrac{2}{3}u^3 - \tfrac{1}{5}u^5 + c = -\cos x + \tfrac{2}{3}\cos^3 x - \tfrac{1}{5}\cos^5 x + c$$

b
$$\begin{aligned}&\int \sin^4 x\cos^3 x\,dx\\&= \int \sin^4 x\cos^2 x\cos x\,dx\\&= \int \sin^4 x(1-\sin^2 x)\cos x\,dx\\&= \int (\sin^4 x-\sin^6 x)\cos x\,dx\end{aligned}$$

Let $u=\sin x$, $\dfrac{du}{dx}=\cos x$

$$\begin{aligned}\therefore\ \int \sin^4 x\cos^3 x\,dx &= \int (u^4-u^6)\,\frac{du}{dx}\,dx\\&= \int (u^4-u^6)\,du\\&= \tfrac{1}{5}u^5-\tfrac{1}{7}u^7+c\\&= \tfrac{1}{5}\sin^5 x-\tfrac{1}{7}\sin^7 x+c\end{aligned}$$

c
$$\begin{aligned}&\int \sin^3(2x)\cos(2x)\,dx\\&= \tfrac{1}{2}\int \sin^3(2x)(2\cos(2x))\,dx\end{aligned}$$

Let $u=\sin(2x)$, $\dfrac{du}{dx}=2\cos(2x)$

$$\begin{aligned}\therefore\ &\int \sin^3(2x)\cos(2x)\,dx\\&= \tfrac{1}{2}\int u^3\,\frac{du}{dx}\,dx\\&= \tfrac{1}{2}\int u^3\,du\\&= \tfrac{1}{2}\times\frac{u^4}{4}+c\\&= \tfrac{1}{8}\sin^4(2x)+c\end{aligned}$$

8 **a** Let $u=\cos x$, $\dfrac{du}{dx}=-\sin x$

$$\begin{aligned}\therefore\ f(x)&=\int \sin x\, e^{\cos x}\,dx\\&= -\int e^{\cos x}(-\sin x)\,dx\\&= -\int e^u\,\frac{du}{dx}\,dx\\&= -\int e^u\,du\\&= -e^u+c\\&= -e^{\cos x}+c\end{aligned}$$

b Let $u=\sin x-\cos x$,

$$\frac{du}{dx}=\cos x+\sin x$$

$$\begin{aligned}\therefore\ f(x)&=\int \frac{\sin x+\cos x}{\sin x-\cos x}\,dx\\&= \int \frac{1}{u}\,\frac{du}{dx}\,dx\\&= \int \frac{1}{u}\,du\\&= \ln|u|+c\\&= \ln|\sin x-\cos x|+c\end{aligned}$$

c Let $u=\tan x$,

$$\frac{du}{dx}=\sec^2 x=\frac{1}{\cos^2 x}$$

$$\begin{aligned}\therefore\ \int \frac{e^{\tan x}}{\cos^2 x}\,dx &= \int e^u\,\frac{du}{dx}\,dx\\&= \int e^u\,du\\&= e^u+c\\&= e^{\tan x}+c\end{aligned}$$

9 **a**
$$\int \cot x\,dx=\int \frac{\cos x}{\sin x}\,dx$$

Let $u=\sin x$, $\dfrac{du}{dx}=\cos x$

$$\begin{aligned}\therefore\ \int \cot x\,dx &= \int \frac{1}{u}\,\frac{du}{dx}\,dx\\&= \int \frac{1}{u}\,du\\&= \ln|u|+c\\&= \ln|\sin x|+c\end{aligned}$$

b
$$\int \cot(3x)\,dx=\int \frac{\cos(3x)}{\sin(3x)}\,dx$$

Let $u=\sin(3x)$, $\dfrac{du}{dx}=3\cos(3x)$

$$\begin{aligned}\therefore\ \int \cot(3x)\,dx &= \tfrac{1}{3}\int \frac{3\cos(3x)}{\sin(3x)}\,dx\\&= \tfrac{1}{3}\int \frac{1}{u}\,\frac{du}{dx}\,dx\\&= \tfrac{1}{3}\int \frac{1}{u}\,du\\&= \tfrac{1}{3}\ln|u|+c\\&= \tfrac{1}{3}\ln|\sin(3x)|+c\end{aligned}$$

c Let $u = \cot x = \dfrac{\cos x}{\sin x}$,

$$\frac{du}{dx} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x}$$
$$= -\csc^2 x$$

$$\therefore \int \csc^2 x\, dx = -\int -\csc^2 x\, dx$$
$$= -\int 1 \frac{du}{dx}\, dx$$
$$= -\int 1\, du$$
$$= -u + c$$
$$= -\cot x + c$$

d $\displaystyle\int \sec x \tan x\, dx = \int \frac{1}{\cos x}\frac{\sin x}{\cos x}\, dx$

Let $u = \cos x$, $\dfrac{du}{dx} = -\sin x$

$$\therefore \int \sec x \tan x\, dx = -\int \frac{1}{u^2}\frac{du}{dx}\, dx$$
$$= -\int u^{-2}\, du$$
$$= -\frac{u^{-1}}{(-1)} + c$$
$$= \frac{1}{\cos x} + c = \sec x + c$$

e $\displaystyle\int \csc x \cot x\, dx = \int \frac{1}{\sin x}\frac{\cos x}{\sin x}\, dx$

Let $u = \sin x$, $\dfrac{du}{dx} = \cos x$

$$\therefore \int \csc x \cot x\, dx = \int \frac{1}{u^2}\frac{du}{dx}\, dx$$
$$= \int u^{-2}\, du$$
$$= \frac{u^{-1}}{-1} + c$$
$$= -\frac{1}{\sin x} + c$$
$$= -\csc x + c$$

f $\displaystyle\int \tan(3x)\sec(3x)\, dx = \int \frac{\sin(3x)}{\cos(3x)}\frac{1}{\cos(3x)}\, dx$

Let $u = \cos(3x)$, $\dfrac{du}{dx} = -3\sin(3x)$

$$\therefore \int \tan(3x)\sec(3x)\, dx = -\tfrac{1}{3}\int \frac{1}{u^2}\frac{du}{dx}\, dx$$
$$= -\tfrac{1}{3}\int u^{-2}\, du$$
$$= -\tfrac{1}{3} \times \frac{u^{-1}}{-1} + c$$
$$= \tfrac{1}{3} \times \frac{1}{\cos(3x)} + c$$
$$= \tfrac{1}{3}\sec(3x) + c$$

g $\displaystyle\int \csc\left(\tfrac{x}{2}\right)\cot\left(\tfrac{x}{2}\right)\, dx = \int \frac{1}{\sin\left(\frac{x}{2}\right)}\frac{\cos\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)}\, dx$

Let $u = \sin\left(\frac{x}{2}\right)$, $\dfrac{du}{dx} = \tfrac{1}{2}\cos\left(\tfrac{x}{2}\right)$

$$\therefore \int \csc\left(\tfrac{x}{2}\right)\cot\left(\tfrac{x}{2}\right)\, dx = 2\int \frac{1}{u^2}\frac{du}{dx}\, dx$$
$$= 2\int u^{-2}\, du$$
$$= 2 \times \frac{u^{-1}}{-1} + c$$
$$= \frac{-2}{\sin\left(\frac{x}{2}\right)} + c$$
$$= -2\csc\left(\tfrac{x}{2}\right) + c$$

h $\displaystyle\int \sec^3 x \sin x\, dx = \int (\cos x)^{-3}\sin x\, dx$

Let $u = \cos x$, $\dfrac{du}{dx} = -\sin x$

$$\therefore \int \sec^3 x \sin x\, dx = -\int \sec^3 x(-\sin x)\, dx$$
$$= -\int u^{-3}\frac{du}{dx}\, dx$$
$$= -\int u^{-3}\, du$$
$$= \tfrac{1}{2}u^{-2} + c$$
$$= \tfrac{1}{2}\cos^{-2} x + c$$
$$= \tfrac{1}{2}\sec^2 x + c$$

i Let $u = \cot x$, $\dfrac{du}{dx} = -\csc^2 x$ {see **c**}

$$\therefore \int \frac{\csc^2 x}{\sqrt{\cot x}}\, dx = -\int \frac{-\csc^2 x}{\sqrt{\cot x}}\, dx$$
$$= -\int \frac{1}{\sqrt{u}}\frac{du}{dx}\, dx$$
$$= -\int u^{-\frac{1}{2}}\, du$$
$$= -2u^{\frac{1}{2}} + c$$
$$= -2\sqrt{\cot x} + c$$

10 **a**
$$u = \ln(x^2+7)$$
$$\therefore\ \frac{du}{dx} = \frac{2x}{x^2+7}$$
$$\therefore\ \int \frac{x\ln(x^2+7)}{x^2+7}\,dx$$
$$= \tfrac{1}{2}\int \frac{2x}{x^2+7}\ \ln(x^2+7)\,dx$$
$$= \tfrac{1}{2}\int u\,\frac{du}{dx}\,dx$$
$$= \tfrac{1}{2}\int u\,du$$
$$= \tfrac{1}{2}\times\frac{u^2}{2}+c$$
$$= \tfrac{1}{4}\left[\ln(x^2+7)\right]^2+c$$

b
$$u = x-16$$
$$\therefore\ \frac{du}{dx} = 1$$
Also, $x = u+16$
$$\therefore\ \int x^2\sqrt{x-16}\,dx$$
$$= \int (u+16)^2\sqrt{u}\,\frac{du}{dx}\,dx$$
$$= \int (u^2+32u+256)u^{\frac{1}{2}}\,du$$
$$= \int u^{\frac{5}{2}}+32u^{\frac{3}{2}}+256u^{\frac{1}{2}}\,du$$
$$= \frac{u^{\frac{7}{2}}}{\frac{7}{2}}+\frac{32u^{\frac{5}{2}}}{\frac{5}{2}}+\frac{256u^{\frac{3}{2}}}{\frac{3}{2}}+c$$
$$= \tfrac{2}{7}(x-16)^{\frac{7}{2}}+\tfrac{64}{5}(x-16)^{\frac{5}{2}}+\tfrac{512}{3}(x-16)^{\frac{3}{2}}+c$$

c
$$x = 3\tan\theta$$
$$\therefore\ \theta = \arctan\left(\tfrac{x}{3}\right)$$
Also, $\displaystyle\frac{d\theta}{dx} = \tfrac{1}{3}\left(\frac{1}{1+\left(\frac{x}{3}\right)^2}\right)$
$$\therefore\ 3\,\frac{d\theta}{dx} = \frac{1}{1+\frac{x^2}{9}}$$
$$\therefore\ \int \frac{1}{36+4x^2}\,dx = \tfrac{1}{36}\int \frac{1}{1+\frac{x^2}{9}}\,dx$$
$$= \tfrac{1}{36}\int 3\,\frac{d\theta}{dx}\,dx$$
$$= \tfrac{3}{36}\int 1\,d\theta$$
$$= \tfrac{1}{12}\theta+c$$
$$= \tfrac{1}{12}\arctan\left(\tfrac{x}{3}\right)+c$$

d Let $u = \sqrt{x-1}$
$$\therefore\ \frac{du}{dx} = \tfrac{1}{2}(x-1)^{-\frac{1}{2}}(1) = \frac{1}{2\sqrt{x-1}}$$
Also, $x = u^2+1$
$$\therefore\ \int \frac{\sqrt{x-1}}{x}\,dx = 2\int \frac{x-1}{x}\ \frac{1}{2\sqrt{x-1}}\,dx$$
$$= 2\int \frac{u^2}{u^2+1}\ \frac{du}{dx}\,dx$$
$$= 2\int \frac{u^2}{u^2+1}\,du$$
$$= 2\int \left(1-\frac{1}{u^2+1}\right)\,du$$
$$= 2(u-\arctan u)+c$$
$$= 2\sqrt{x-1}-2\arctan\sqrt{x-1}+c$$

e
$$x = \tfrac{1}{2}\sec\theta$$
$$\therefore\ x^2 = \tfrac{1}{4}\sec^2\theta$$
$$\therefore\ x^2 = \tfrac{1}{4}(1+\tan^2\theta)$$
$$\therefore\ 4x^2-1 = \tan^2\theta$$
$$\therefore\ \sqrt{4x^2-1} = \tan\theta$$
Also, $\displaystyle\frac{dx}{d\theta} = \tfrac{1}{2}\sec\theta\tan\theta$

Also, $2x = \sec\theta$
$$\therefore\ \cos\theta = \frac{1}{2x}$$
$$\therefore\ \theta = \arccos\left(\tfrac{1}{2x}\right)$$

$$\therefore\ \int \frac{\sqrt{4x^2-1}}{5x}\,dx = \tfrac{1}{5}\int \frac{\tan\theta}{\frac{1}{2}\sec\theta}\times\frac{dx}{d\theta}\times d\theta$$
$$= \tfrac{1}{5}\int \frac{\tan\theta}{\cancel{\frac{1}{2}\sec\theta}}\times\cancel{\tfrac{1}{2}\sec\theta}\tan\theta\,d\theta$$
$$= \tfrac{1}{5}\int \tan^2\theta\,d\theta$$
$$= \tfrac{1}{5}\int \sec^2\theta-1\,d\theta$$
$$= \tfrac{1}{5}(\tan\theta-\theta)+c$$
$$= \tfrac{1}{5}\sqrt{4x^2-1}-\tfrac{1}{5}\arccos\left(\tfrac{1}{2x}\right)+c$$

11 **a** $\dfrac{d}{dx}(\arcsin x) = \dfrac{1}{\sqrt{1-x^2}}, \quad x \in \left]-1, 1\right[$

$\therefore \displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + c$

b Let $\theta = \dfrac{x}{a}$

$\therefore \; x = a\theta$

$\therefore \; \dfrac{dx}{d\theta} = a$

$$\begin{aligned}\therefore \int \frac{1}{\sqrt{a^2-x^2}}\,dx &= \int \frac{1}{\sqrt{a^2-(a\theta)^2}}\,\frac{dx}{d\theta} \times d\theta \\ &= \int \frac{1}{\sqrt{a^2(1-\theta^2)}} \times a\,d\theta \\ &= \int \frac{1}{\sqrt{1-\theta^2}}\,d\theta \\ &= \arcsin\theta + c \\ &= \arcsin\left(\tfrac{x}{a}\right) + c\end{aligned}$$

c Domain $= \{x \mid -a \leqslant x \leqslant a, \quad a \neq 0\}$

d **i**
$$\begin{aligned}&\int \frac{4}{\sqrt{1-x^2}}\,dx \\ &= 4\int \frac{1}{\sqrt{1-x^2}}\,dx \\ &= 4\arcsin\left(\tfrac{x}{1}\right) + c \\ &= 4\arcsin x + c\end{aligned}$$

ii
$$\begin{aligned}&\int \frac{3}{\sqrt{4-x^2}}\,dx \\ &= 3\int \frac{1}{\sqrt{2^2-x^2}}\,dx \\ &= 3\arcsin\left(\tfrac{x}{2}\right) + c\end{aligned}$$

iii
$$\begin{aligned}&\int \frac{1}{\sqrt{1-4x^2}}\,dx \\ &= \tfrac{1}{2}\int \frac{1}{\sqrt{\frac{1}{4}-x^2}}\,dx \\ &= \tfrac{1}{2}\arcsin\left(\frac{x}{\frac{1}{2}}\right) + c \\ &= \tfrac{1}{2}\arcsin(2x) + c\end{aligned}$$

iv
$$\begin{aligned}&\int \frac{2}{\sqrt{4-9x^2}}\,dx \\ &= 2\int \frac{1}{\sqrt{4-9x^2}}\,dx \\ &= \tfrac{2}{3}\int \frac{1}{\sqrt{\frac{4}{9}-x^2}}\,dx \\ &= \tfrac{2}{3}\arcsin\left(\tfrac{3x}{2}\right) + c\end{aligned}$$

12 **a** $\dfrac{d}{dx}(\arctan x) = \dfrac{1}{x^2+1}, \quad x \in \mathbb{R}$

$\therefore \displaystyle\int \frac{1}{x^2+1}\,dx = \arctan x$

b Let $\theta = \dfrac{x}{a}$

$\therefore \; x = a\theta$

$\therefore \; \dfrac{dx}{d\theta} = a$

$\therefore \; dx = a\,d\theta$

$$\begin{aligned}\therefore \int \frac{1}{x^2+a^2}\,dx &= \int \frac{1}{(a\theta)^2+a^2} \times a\,d\theta \\ &= \int \frac{a}{a^2(\theta^2+1)}\,d\theta \\ &= \frac{1}{a}\int \frac{1}{\theta^2+1}\,d\theta \\ &= \frac{1}{a}\arctan\theta + c \\ &= \frac{1}{a}\arctan\left(\tfrac{x}{a}\right) + c, \quad a \neq 0\end{aligned}$$

c Domain $= \{a, x \mid a, x \in \mathbb{R}, \quad a \neq 0\}$

d **i** $\displaystyle\int \frac{1}{x^2+16}\,dx$

$\displaystyle= \int \frac{1}{x^2+4^2}\,dx$

$= \frac{1}{4}\arctan\left(\frac{x}{4}\right) + c$

ii $\displaystyle\int \frac{1}{4x^2+1}\,dx$

$\displaystyle= \frac{1}{4}\int \frac{1}{x^2+(\frac{1}{2})^2}\,dx$

$= \frac{1}{4}\left(\frac{1}{\frac{1}{2}}\right)\arctan\left(\frac{x}{\frac{1}{2}}\right) + c$

$= \frac{1}{2}\arctan(2x) + c$

iii $\displaystyle\int \frac{1}{4+2x^2}\,dx$

$\displaystyle= \frac{1}{2}\int \frac{1}{2+x^2}\,dx$

$= \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)\arctan\left(\frac{x}{\sqrt{2}}\right) + c$

$= \frac{1}{2\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right) + c$

iv $\displaystyle\int \frac{5}{9+4x^2}\,dx$

$\displaystyle= \frac{5}{4}\int \frac{1}{\frac{9}{4}+x^2}\,dx$

$= \frac{5}{4}\left(\frac{1}{\frac{3}{2}}\right)\arctan\left(\frac{x}{\frac{3}{2}}\right) + c$

$= \frac{5}{6}\arctan\left(\frac{2x}{3}\right) + c$

13 **a** Let $u = \sqrt{\dfrac{x-\xi}{x+\xi}}$

$$\begin{aligned}
\therefore\ u^2 &= \frac{x-\xi}{x+\xi} \\
&= \frac{x+\xi-2\xi}{x+\xi} \\
&= 1 - \frac{2\xi}{x+\xi} \\
&= 1 - 2\xi(x+\xi)^{-1}
\end{aligned}$$

$$\therefore\ 2u\,\frac{du}{dx} = \frac{2\xi}{(x+\xi)^2}$$

$$\therefore\ \frac{u}{\xi}\,du = \frac{1}{(x+\xi)^2}\,dx$$

$$\begin{aligned}
\therefore\ \int \frac{x-\xi}{(x+\xi)^3}\,dx &= \int \frac{x-\xi}{x+\xi} \times \frac{1}{(x+\xi)^2}\,dx \\
&= \int u^2 \times \frac{u}{\xi}\,du \\
&= \frac{1}{\xi}\int u^3\,du \\
&= \frac{1}{\xi}\left(\frac{u^4}{4} + c\right) \\
&= \frac{1}{4\xi}\left(\frac{x-\xi}{x+\xi}\right)^2 + c \\
&= \frac{x^2-2\xi x+\xi^2}{4\xi(x+\xi)^2} + c \\
&= \frac{x^2-4\xi x+2\xi x+\xi^2}{4\xi(x+\xi)^2} + c \\
&= \frac{(x+\xi)^2-4\xi x}{4\xi(x+\xi)^2} + c \\
&= \frac{1}{4\xi} - \frac{x}{(x+\xi)^2} + c
\end{aligned}$$

b $\displaystyle\int (x-\xi)^{\frac{1}{2}}(x+\xi)^{-\frac{5}{2}}\,dx = \int \sqrt{\frac{x-\xi}{x+\xi}} \times \frac{1}{(x+\xi)^2}\,dx$

$\displaystyle = \int u \times \frac{u}{\xi}\,du \quad \{\text{from part } \mathbf{a},\ \ \frac{u}{\xi}\,du = \frac{1}{(x+\xi)^2}\,dx\}$

$\displaystyle = \frac{1}{\xi}\int u^2\,du$

$\displaystyle = \frac{1}{\xi}\left(\frac{u^3}{3}+c\right)$

$\displaystyle = \frac{1}{3\xi}\left(\frac{x-\xi}{x+\xi}\right)^{\frac{3}{2}} + c$

14 Let $u = x^{\frac{1}{6}}$

$\therefore\ x = u^6$

$\therefore\ \dfrac{dx}{du} = 6u^5$

$$\therefore\ \int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx = \int \frac{1}{\sqrt{u^6}+\sqrt[3]{u^6}}\,\frac{dx}{du}\,du$$

$$= \int \frac{1}{u^3+u^2} \times 6u^5\,du$$

$$= 6\int \frac{u^5}{u^3+u^2}\,du$$

$$= 6\int \frac{u^3}{u+1}\,du$$

Let $v = u+1 \quad \therefore\ u = v-1$ and $\dfrac{dv}{du} = 1$

$$\therefore\ \int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx = 6\int \frac{(v-1)^3}{v-1+1}\,\frac{dv}{du}\,du$$

$$= 6\int \frac{v^3-3v^2+3v-1}{v}\,dv$$

$$= 6\int v^2-3v+3-\frac{1}{v}\,dv$$

$$= 6\left(\frac{v^3}{3}-\frac{3v^2}{2}+3v-\ln|v|+c\right)$$

$$= 2v^3-9v^2+18v-6\ln|v|+c$$

$$= 2(u+1)^3-9(u+1)^2+18(u+1)-6\ln|u+1|+c \quad \{v=u+1\}$$

$$= 2(u^3+3u^2+3u+1)-9(u^2+2u+1)+18u+18-6\ln|u+1|+c$$

$$= 2u^3+6u^2+6u+2-9u^2-\cancel{18u}-9+\cancel{18u}+18-6\ln|u+1|+c$$

$$= 2u^3-3u^2+6u-6\ln|u+1|+c$$

$$= 2(x^{\frac{1}{6}})^3-3(x^{\frac{1}{6}})^2+6(x^{\frac{1}{6}})-6\ln(x^{\frac{1}{6}}+1)+c \quad \{u=x^{\frac{1}{6}}\}$$

$$= 2x^{\frac{1}{2}}-3x^{\frac{1}{3}}+6x^{\frac{1}{6}}-6\ln(x^{\frac{1}{6}}+1)+c$$

EXERCISE 21G.2

1 **a** Let $u = x - 3$, $\dfrac{du}{dx} = 1$

$\therefore\ x = u + 3$

$$\begin{aligned}\therefore\ &\int x\sqrt{x-3}\,dx\\
&= \int (u+3)\sqrt{u}\,\frac{du}{dx}\,dx\\
&= \int \left(u^{\frac{3}{2}} + 3u^{\frac{1}{2}}\right) du\\
&= \frac{u^{\frac{5}{2}}}{\frac{5}{2}} + \frac{3u^{\frac{3}{2}}}{\frac{3}{2}} + c\\
&= \tfrac{2}{5}u^{\frac{5}{2}} + 2u^{\frac{3}{2}} + c\\
&= \tfrac{2}{5}(x-3)^{\frac{5}{2}} + 2(x-3)^{\frac{3}{2}} + c\end{aligned}$$

b Let $u = x + 1$, $\dfrac{du}{dx} = 1$

$\therefore\ x = u - 1$

$$\begin{aligned}\therefore\ &\int x^2\sqrt{x+1}\,dx\\
&= \int (u-1)^2\sqrt{u}\,\frac{du}{dx}\,dx\\
&= \int (u^2 - 2u + 1)u^{\frac{1}{2}}\,du\\
&= \int \left(u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du\\
&= \frac{u^{\frac{7}{2}}}{\frac{7}{2}} - \frac{2u^{\frac{5}{2}}}{\frac{5}{2}} + \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c\\
&= \tfrac{2}{7}u^{\frac{7}{2}} - \tfrac{4}{5}u^{\frac{5}{2}} + \tfrac{2}{3}u^{\frac{3}{2}} + c\\
&= \tfrac{2}{7}(x+1)^{\frac{7}{2}} - \tfrac{4}{5}(x+1)^{\frac{5}{2}} + \tfrac{2}{3}(x+1)^{\frac{3}{2}} + c\end{aligned}$$

c Let $u = 3 - x^2$, $\dfrac{du}{dx} = -2x$

$$\begin{aligned}\therefore\ &\int x^3\sqrt{3-x^2}\,dx\\
&= -\tfrac{1}{2}\int x^2\sqrt{3-x^2}(-2x)\,dx\\
&= -\tfrac{1}{2}\int x^2\sqrt{3-x^2}\,\frac{du}{dx}\,dx\\
&= -\tfrac{1}{2}\int (3-u)\sqrt{u}\,du\\
&= -\tfrac{1}{2}\int \left(3u^{\frac{1}{2}} - u^{\frac{3}{2}}\right) du\\
&= -\tfrac{1}{2}\left[\frac{3u^{\frac{3}{2}}}{\frac{3}{2}} - \frac{u^{\frac{5}{2}}}{\frac{5}{2}}\right] + c\\
&= -\tfrac{1}{2}\left[2u^{\frac{3}{2}} - \tfrac{2}{5}u^{\frac{5}{2}}\right] + c\\
&= -u^{\frac{3}{2}} + \tfrac{1}{5}u^{\frac{5}{2}} + c\\
&= -(3-x^2)^{\frac{3}{2}} + \tfrac{1}{5}(3-x^2)^{\frac{5}{2}} + c\end{aligned}$$

d Let $u = t^2 + 2$, $\dfrac{du}{dt} = 2t$

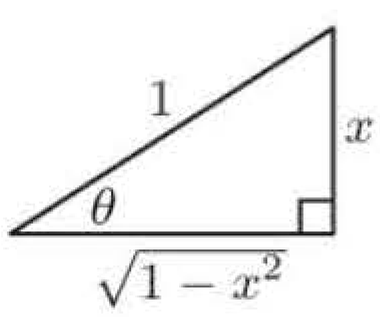
$$\begin{aligned}\therefore\ &\int t^3\sqrt{t^2+2}\,dt\\
&= \tfrac{1}{2}\int t^2\sqrt{t^2+2}(2t)\,dt\\
&= \tfrac{1}{2}\int t^2\sqrt{t^2+2}\,\frac{du}{dt}\,dt\\
&= \tfrac{1}{2}\int (u-2)\sqrt{u}\,du\\
&= \tfrac{1}{2}\int \left(u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right) du\\
&= \tfrac{1}{2}\left[\frac{u^{\frac{5}{2}}}{\frac{5}{2}} - \frac{2u^{\frac{3}{2}}}{\frac{3}{2}}\right] + c\\
&= \tfrac{1}{5}u^{\frac{5}{2}} - \tfrac{2}{3}u^{\frac{3}{2}} + c\\
&= \tfrac{1}{5}(t^2+2)^{\frac{5}{2}} - \tfrac{2}{3}(t^2+2)^{\frac{3}{2}} + c\end{aligned}$$

2 **a** Let $x = 3\tan\theta$, $\dfrac{dx}{d\theta} = 3\sec^2\theta$

$$\begin{aligned}\therefore\ &\int \frac{x^2}{9+x^2}\,dx\\
&= \int \frac{9\tan^2\theta}{9+9\tan^2\theta}\,3\sec^2\theta\,d\theta\\
&= 3\int \frac{\tan^2\theta}{1+\tan^2\theta}\,\sec^2\theta\,d\theta\\
&= 3\int \tan^2\theta\,d\theta \qquad \{\sec^2\theta = 1+\tan^2\theta\}\\
&= 3\int (\sec^2\theta - 1)\,d\theta\\
&= 3\tan\theta - 3\theta + c\\
&= x - 3\arctan\left(\tfrac{x}{3}\right) + c\end{aligned}$$

b Let $x = \sin\theta$, $\dfrac{dx}{d\theta} = \cos\theta$

$$\begin{aligned}\therefore\ &\int \frac{x^2}{\sqrt{1-x^2}}\,dx\\
&= \int \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}}\,\cos\theta\,d\theta\\
&= \int \frac{\sin^2\theta}{\cos\theta}\,\cos\theta\,d\theta\\
&= \int \sin^2\theta\,d\theta\\
&= \int \left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta\right) d\theta\\
&= \tfrac{1}{2}\theta - \tfrac{1}{2}\left(\tfrac{1}{2}\right)\sin 2\theta + c\\
&= \tfrac{1}{2}\arcsin x - \tfrac{1}{2}\sin\theta\cos\theta + c\\
&= \tfrac{1}{2}\arcsin x - \tfrac{1}{2}x\sqrt{1-x^2} + c\end{aligned}$$



$\{\text{since} \quad \cos\theta = \sqrt{1-\sin^2\theta}\}$

c $\displaystyle\int \frac{2x}{x^2+9}\,dx$ has the form $\displaystyle\int \frac{f'(x)}{f(x)}\,dx$

$\therefore \displaystyle\int \frac{2x}{x^2+9}\,dx$

$= \ln\left|x^2+9\right| + c$

$= \ln(x^2+9) + c \quad \{x^2+9>0 \text{ for all } x\}$

d Let $u = \ln x, \quad \dfrac{du}{dx} = \dfrac{1}{x}$

$\therefore \displaystyle\int \frac{4\ln x}{x\,(1+[\ln x]^2)}\,dx$

$= \displaystyle\int \frac{4u}{1+u^2}\,\frac{du}{dx}\,dx$

$= 2\displaystyle\int \frac{2u}{1+u^2}\,du$

which has the form $\displaystyle\int \frac{f'(x)}{f(x)}\,dx.$

$\therefore$ the integral

$= 2\ln\left|1+u^2\right| + c$

$= 2\ln(1+u^2) + c \quad \{1+u^2>0\}$

$= 2\ln\left(1+[\ln x]^2\right) + c$

e Let $x = 2\sec\theta, \quad \dfrac{dx}{d\theta} = 2\sec\theta\tan\theta$

$\therefore \displaystyle\int \frac{\sqrt{x^2-4}}{x}\,dx$

$= \displaystyle\int \frac{\sqrt{4\sec^2\theta-4}}{2\sec\theta}\,2\sec\theta\tan\theta\,d\theta$

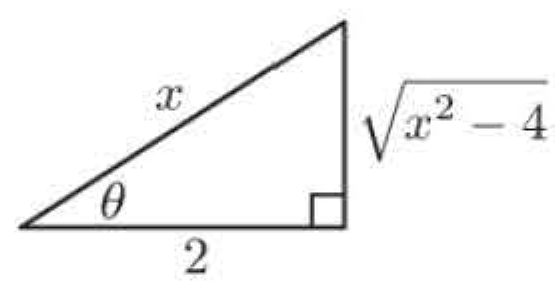
$= \frac{2}{2}\displaystyle\int \frac{\sqrt{\sec^2\theta-1}}{\cancel{\sec\theta}} \times 2\cancel{\sec\theta}\tan\theta\,d\theta$

$= 2\int \sqrt{\sec^2\theta-1}\,\tan\theta\,d\theta$

$= 2\int \tan\theta\tan\theta\,d\theta \quad \{\sec^2\theta - 1 = \tan^2\theta\}$

$= 2\int \tan^2\theta\,d\theta$

$= 2\int (\sec^2\theta - 1)\,d\theta$



$= 2\tan\theta - 2\theta + c$

$= 2\,\dfrac{\sqrt{x^2-4}}{2} - 2\arccos\left(\frac{2}{x}\right) + c$

$= \sqrt{x^2-4} - 2\arccos\left(\frac{2}{x}\right) + c$

f $\int \sin x\cos 2x\,dx$

$= \int \sin x(2\cos^2 x - 1)\,dx$

$= 2\int \cos^2 x\sin x\,dx - \int \sin x\,dx$

$= -2\int [\cos x]^2(-\sin x)\,dx - \int \sin x\,dx$

$= -2\dfrac{[\cos x]^3}{3} - (-\cos x) + c$

$= -\frac{2}{3}\cos^3 x + \cos x + c$

$= \cos x - \frac{2}{3}\cos^3 x + c$

g $\displaystyle\int \frac{1}{\sqrt{9-4x^2}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{\frac{9}{4}-x^2}}\,dx$

Let $x = \frac{3}{2}\sin\theta$, so $\dfrac{dx}{d\theta} = \frac{3}{2}\cos\theta$

$\therefore$ the integral

$= \frac{1}{2}\displaystyle\int \frac{1}{\sqrt{\frac{9}{4}-\frac{9}{4}\sin^2\theta}}\,\frac{3}{2}\cos\theta\,d\theta$

$= \frac{1}{2}\displaystyle\int \frac{1}{\frac{3}{2}\sqrt{1-\sin^2\theta}} \times \frac{3}{2}\cos\theta\,d\theta$

$= \frac{1}{2}\displaystyle\int \frac{\cos\theta}{\cos\theta}\,d\theta$

$= \frac{1}{2}\int 1\,d\theta$

$= \frac{1}{2}\theta + c$

$= \frac{1}{2}\arcsin\left(\frac{2x}{3}\right) + c \quad \{\text{since } \sin\theta = \frac{2x}{3}\}$

h $\displaystyle\int \frac{x^3}{1+x^2}\,dx$

$= \displaystyle\int \frac{x(1+x^2)-x}{1+x^2}\,dx$

$= \displaystyle\int \left(x - \frac{x}{1+x^2}\right)dx$

$= \displaystyle\int \left(x - \frac{1}{2}\left(\frac{2x}{1+x^2}\right)\right)dx$

$= \dfrac{x^2}{2} - \frac{1}{2}\ln\left|1+x^2\right| + c$

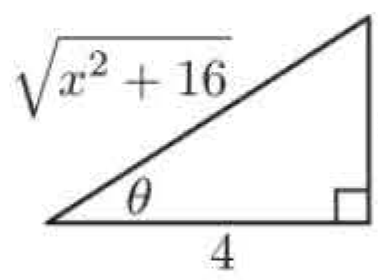
$= \dfrac{x^2}{2} - \frac{1}{2}\ln(1+x^2) + c \quad \{\text{as } 1+x^2>0\}$

i Let $u = \ln x, \quad \dfrac{du}{dx} = \dfrac{1}{x}$

$$\therefore \quad \int \frac{1}{x\,(9+4[\ln x]^2)}\,dx$$
$$= \int \frac{1}{9+4u^2}\,\frac{du}{dx}\,dx$$
$$= \int \frac{1}{9+4u^2}\,du$$
$$= \tfrac{1}{4}\int \frac{1}{u^2+\frac{9}{4}}\,du$$
$$= \tfrac{1}{4}\left(\frac{1}{\frac{3}{2}}\right)\arctan\left(\frac{u}{\frac{3}{2}}\right) + c$$
$$= \tfrac{1}{6}\arctan\left(\frac{2u}{3}\right) + c$$
$$= \tfrac{1}{6}\arctan\left(\frac{2\ln x}{3}\right) + c$$

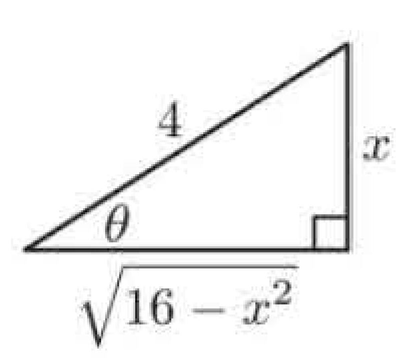
j Let $x = 4\tan\theta, \quad \dfrac{dx}{d\theta} = 4\sec^2\theta$

$$\therefore \quad \int \frac{1}{x(x^2+16)}\,dx$$
$$= \int \frac{1}{4\tan\theta(16\tan^2\theta+16)} \times 4\sec^2\theta\,d\theta$$
$$= \int \frac{1}{4\tan\theta \times 16\sec^2\theta} \times 4\sec^2\theta\,d\theta$$
$$= \tfrac{1}{16}\int \frac{1}{\tan\theta}\,d\theta$$
$$= \tfrac{1}{16}\int \frac{\cos\theta}{\sin\theta}\,d\theta$$
$$= \tfrac{1}{16}\ln|\sin\theta| + c \qquad \left\{\text{form } \frac{f'(\theta)}{f(\theta)}\right\}$$
$$= \tfrac{1}{16}\ln\left|\frac{x}{\sqrt{x^2+16}}\right| + c$$
$$= \tfrac{1}{16}\ln\left(\frac{|x|}{\sqrt{x^2+16}}\right) + c$$



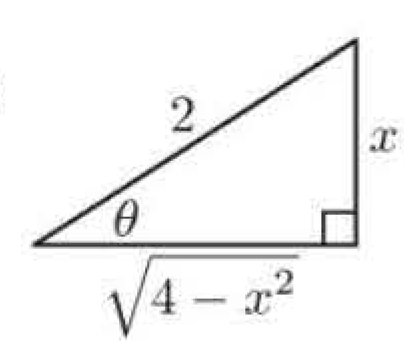
k Let $x = 4\sin\theta, \quad \dfrac{dx}{d\theta} = 4\cos\theta$

$$\therefore \quad \int \frac{1}{x^2\sqrt{16-x^2}}\,dx$$
$$= \int \frac{1}{16\sin^2\theta\sqrt{16-16\sin^2\theta}}\,4\cos\theta\,d\theta$$
$$= \int \frac{1}{16\sin^2\theta \times 4\cos\theta}\,4\cos\theta\,d\theta$$
$$= \tfrac{1}{16}\int \frac{1}{\sin^2\theta}\,d\theta$$
$$= \tfrac{1}{16}\int \csc^2\theta\,d\theta$$
$$= \tfrac{1}{16}(-\cot\theta) + c$$
$$= -\tfrac{1}{16}\cot\theta + c$$
$$= -\tfrac{1}{16}\,\frac{\sqrt{16-x^2}}{x} + c$$
$$= -\frac{\sqrt{16-x^2}}{16x} + c$$



l Let $x = 2\sin\theta, \quad \dfrac{dx}{d\theta} = 2\cos\theta$

$$\therefore \quad \int x^2\sqrt{4-x^2}\,dx$$
$$= \int 4\sin^2\theta\sqrt{4-4\sin^2\theta}\,2\cos\theta\,d\theta$$
$$= \int 4\sin^2\theta\,2\cos\theta\,2\cos\theta\,d\theta$$
$$= 4\int 4\sin^2\theta\cos^2\theta\,d\theta$$
$$= 4\int \sin^2(2\theta)\,d\theta$$
$$= 4\int \left(\tfrac{1}{2} - \tfrac{1}{2}\cos(4\theta)\right)\,d\theta$$
$$= 2\theta - 2(\tfrac{1}{4})\sin(4\theta) + c$$
$$= 2\theta - \tfrac{1}{2}\sin(4\theta) + c$$



Now $\sin\theta = \dfrac{x}{2}, \quad \cos\theta = \dfrac{\sqrt{4-x^2}}{2}$

$$\therefore \quad \sin 2\theta = 2\left(\frac{x}{2}\right)\frac{\sqrt{4-x^2}}{2} = \frac{x\sqrt{4-x^2}}{2}$$

and $\cos 2\theta = \cos^2\theta - \sin^2\theta$
$$= \frac{4-x^2}{4} - \frac{x^2}{4}$$
$$= \frac{4-2x^2}{4}$$

$$\therefore \quad \sin 4\theta = 2\left(\frac{x\sqrt{4-x^2}}{2}\right)\left(\frac{4-2x^2}{4}\right)$$
$$= \frac{x\sqrt{4-x^2}(2-x^2)}{2}$$

$$\therefore \quad \int x^2\sqrt{4-x^2}\,dx$$
$$= 2\arcsin\left(\tfrac{x}{2}\right) - \tfrac{1}{4}x\sqrt{4-x^2}(2-x^2) + c$$

EXERCISE 21H

1 **a** We integrate by parts with

$$u = x \qquad v' = e^x$$
$$u' = 1 \qquad v = e^x$$

$$\therefore \int xe^x\,dx = xe^x - \int 1e^x\,dx$$
$$= xe^x - e^x + c$$

b We integrate by parts with

$$u = x \qquad v' = \sin x$$
$$u' = 1 \qquad v = -\cos x$$

$$\therefore \int x\sin x\,dx$$
$$= x(-\cos x) - \int 1(-\cos x)\,dx$$
$$= -x\cos x + \int \cos x\,dx$$
$$= -x\cos x + \sin x + c$$

c We integrate by parts with

$$u = \ln x \qquad v' = x^2$$
$$u' = \frac{1}{x} \qquad v = \frac{x^3}{3}$$

$$\therefore \int x^2 \ln x\,dx$$
$$= \ln x\left(\frac{x^3}{3}\right) - \int \frac{1}{x}\,\frac{x^3}{3}\,dx$$
$$= \frac{x^3\ln x}{3} - \tfrac{1}{3}\int x^2\,dx$$
$$= \frac{x^3\ln x}{3} - \tfrac{1}{3}\,\frac{x^3}{3} + c$$
$$= \tfrac{1}{3}x^3\ln x - \tfrac{1}{9}x^3 + c$$

d We integrate by parts with

$$u = x \qquad v' = \sin 3x$$
$$u' = 1 \qquad v = -\tfrac{1}{3}\cos 3x$$

$$\therefore \int x\sin 3x\,dx$$
$$= x(-\tfrac{1}{3}\cos 3x) - \int \left(-\tfrac{1}{3}\cos 3x\right)dx$$
$$= -\tfrac{1}{3}x\cos 3x + \tfrac{1}{3}(\tfrac{1}{3})\sin 3x + c$$
$$= -\tfrac{1}{3}x\cos 3x + \tfrac{1}{9}\sin 3x + c$$

e We integrate by parts with

$$u = x \qquad v' = \cos 2x$$
$$u' = 1 \qquad v = \tfrac{1}{2}\sin 2x$$

$$\therefore \int x\cos 2x\,dx$$
$$= x(\tfrac{1}{2}\sin 2x) - \int \tfrac{1}{2}\sin 2x\,dx$$
$$= \tfrac{1}{2}x\sin 2x - \tfrac{1}{2}(-\tfrac{1}{2})\cos 2x + c$$
$$= \tfrac{1}{2}x\sin 2x + \tfrac{1}{4}\cos 2x + c$$

f We integrate by parts with

$$u = x \qquad v' = \sec^2 x$$
$$u' = 1 \qquad v = \tan x$$

$$\therefore \int x\sec^2 x\,dx$$
$$= x\tan x - \int \tan x\,dx$$
$$= x\tan x + \int \frac{-\sin x}{\cos x}\,dx$$
$$= x\tan x + \ln|\cos x| + c$$

2 **a** $\int \ln x\,dx = \int 1 \times \ln x\,dx$

So, we integate by parts with

$$u = \ln x \qquad v' = 1$$
$$u' = \frac{1}{x} \qquad v = x$$

$$\therefore \int \ln x\,dx = x\ln x - \int \left(\frac{1}{x}\right)x\,dx$$
$$= x\ln x - \int 1\,dx$$
$$= x\ln x - x + c$$

b $\int (\ln x)^2\,dx = \int (\ln x)(\ln x)\,dx$

So, we integrate by parts with

$$u = \ln x \qquad v' = \ln x$$
$$u' = \frac{1}{x} \qquad v = x\ln x - x \quad \{\text{using } \mathbf{g}\}$$

$$\therefore \int (\ln x)^2\,dx$$
$$= \ln x(x\ln x - x) - \int \frac{1}{x}(x\ln x - x)\,dx$$
$$= x(\ln x)^2 - x\ln x - \int (\ln x - 1)\,dx$$
$$= x(\ln x)^2 - x\ln x - [x\ln x - x] + x + c$$
$$= x(\ln x)^2 - 2x\ln x + 2x + c$$

3 $\int \arctan x\,dx = \int 1\arctan x\,dx$

So, we integrate by parts with

$$u = \arctan x \qquad v' = 1$$
$$u' = \frac{1}{x^2+1} \qquad v = x$$

$$\therefore \int \arctan x\,dx$$
$$= x\arctan x - \int \frac{x}{x^2+1}\,dx$$
$$= x\arctan x - \tfrac{1}{2}\int \frac{2x}{x^2+1}\,dx$$
$$= x\arctan x - \tfrac{1}{2}\ln\left|x^2+1\right| + c$$
$$= x\arctan x - \tfrac{1}{2}\ln(x^2+1) + c \quad \{\text{as } x^2+1 > 0\}$$

4 **a** We integrate by parts with $u = x^2 \quad v' = e^{-x}$
$u' = 2x \quad v = -e^{-x}$

$$\therefore \int x^2 e^{-x}\,dx = -x^2 e^{-x} - \int 2x(-e^{-x})\,dx$$
$$= -x^2 e^{-x} + 2\int x e^{-x}\,dx$$

We integrate by parts again, this time with $u = x \quad v' = e^{-x}$
$u' = 1 \quad v = -e^{-x}$

$$\therefore \int x^2 e^{-x}\,dx = -x^2 e^{-x} + 2\left[x(-e^{-x}) - \int -e^{-x}\,dx\right]$$
$$= -x^2 e^{-x} - 2xe^{-x} + 2\int e^{-x}\,dx$$
$$= -x^2 e^{-x} - 2xe^{-x} - 2e^{-x} + c$$

b We integrate by parts with $u = e^x \quad v' = \cos x$
$u' = e^x \quad v = \sin x$

$$\therefore \int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx$$

We again integrate by parts with $u = e^x \quad v' = \sin x$
$u' = e^x \quad v = -\cos x$

$$\therefore \int e^x \cos x\,dx = e^x \sin x - \left[-e^x \cos x - \int e^x(-\cos x)\,dx\right] + c$$
$$= e^x \sin x + e^x \cos x - \int e^x \cos x\,dx + c$$
$$\therefore 2\int e^x \cos x\,dx = e^x(\sin x + \cos x) + c$$
$$\therefore \int e^x \cos x\,dx = \tfrac{1}{2}e^x(\sin x + \cos x) + c$$

c We integrate by parts with $u = e^{-x} \quad v' = \sin x$
$u' = -e^{-x} \quad v = -\cos x$

$$\therefore \int e^{-x} \sin x\,dx = -e^{-x}\cos x - \int -e^{-x}(-\cos x)\,dx$$
$$= -e^{-x}\cos x - \int e^{-x}\cos x\,dx$$

We integrate by parts again, this time with $u = e^{-x} \quad v' = \cos x$
$u' = -e^{-x} \quad v = \sin x$

$$\therefore \int e^{-x} \sin x\,dx = -e^{-x}\cos x - \left[e^{-x}\sin x - \int -e^{-x}\sin x\,dx\right] + c$$
$$= -e^{-x}\cos x - e^{-x}\sin x - \int e^{-x}\sin x\,dx + c$$
$$\therefore 2\int e^{-x}\sin x\,dx = -e^{-x}(\sin x + \cos x) + c$$
$$\therefore \int e^{-x}\sin x\,dx = -\tfrac{1}{2}e^{-x}(\sin x + \cos x) + c$$

d We integrate by parts with $u = x^2 \quad v' = \sin x$
$u' = 2x \quad v = -\cos x$

$$\therefore \int x^2 \sin x\,dx = -x^2\cos x - \int -2x\cos x\,dx$$
$$= -x^2\cos x + \int 2x\cos x\,dx$$

We integrate by parts again, this time with $u = 2x \quad v' = \cos x$
$u' = 2 \quad v = \sin x$

$$\therefore \int x^2 \sin x\,dx = -x^2\cos x + \left[2x\sin x - \int 2\sin x\,dx\right]$$
$$= -x^2\cos x + 2x\sin x - 2\int \sin x\,dx$$
$$= -x^2\cos x + 2x\sin x - 2(-\cos x) + c$$
$$= -x^2\cos x + 2x\sin x + 2\cos x + c$$

5 **a** We integrate by parts with $a = u^2 \quad b' = e^u$
$a' = 2u \quad b = e^u$

$$\therefore \int u^2 e^u\,du = u^2 e^u - \int 2ue^u\,du$$
$$= u^2 e^u - 2\int ue^u\,du$$

We integrate by parts again, this time with $\begin{array}{ll} a = u & b' = e^u \\ a' = 1 & b = e^u \end{array}$

$$\therefore \int u^2 e^u\,du = u^2 e^u - 2\left[ue^u - \int e^u\,du\right]$$
$$= u^2 e^u - 2ue^u + 2e^u + c$$

b Let $u = \ln x$, $\dfrac{du}{dx} = \dfrac{1}{x} = \dfrac{1}{e^u}$ $\quad\therefore \int (\ln x)^2\,dx = \int u^2 e^u\,du$

$$= u^2e^u - 2ue^u + 2e^u + c \quad \{\text{using } \mathbf{a}\}$$
$$= (\ln x)^2 e^{\ln x} - 2\ln x e^{\ln x} + 2e^{\ln x} + c$$
$$= x(\ln x)^2 - 2x\ln x + 2x + c$$

6 **a** We integrate by parts with $\begin{array}{ll} a = u & b' = \sin u \\ a' = 1 & b = -\cos u \end{array}$

$$\therefore \int u\sin u\,du = -u\cos u - \int -\cos u\,du$$
$$= -u\cos u + \sin u + c$$

b Let $u^2 = 2x$, $2u\dfrac{du}{dx} = 2$

$$\therefore \frac{du}{dx} = \frac{1}{u}$$

$$\therefore \int \sin\sqrt{2x}\,dx = \int \sin u\,(u\,du)$$
$$= \int u\sin u\,du$$
$$= -u\cos u + \sin u + c$$
$$= -\sqrt{2x}\cos\sqrt{2x} + \sin\sqrt{2x} + c$$

7 Let $u^2 = 3x$, $2u\dfrac{du}{dx} = 3$

$$\therefore \frac{du}{dx} = \frac{3}{2u}$$

$$\therefore \int \cos\sqrt{3x}\,dx = \int \cos u\left(\frac{2u}{3}\right)du$$
$$= \tfrac{2}{3}\int u\cos u\,du$$

We integrate by parts with $\begin{array}{ll} a = u & b' = \cos u \\ a' = 1 & b = \sin u \end{array}$

$$\therefore \int \cos\sqrt{3x}\,dx = \tfrac{2}{3}\left[u\sin u - \int \sin u\,du\right]$$
$$= \tfrac{2}{3}u\sin u - \tfrac{2}{3}(-\cos u) + c$$
$$= \tfrac{2}{3}\sqrt{3x}\,\sin\sqrt{3x} + \tfrac{2}{3}\cos\sqrt{3x} + c$$

EXERCISE 21I

1 **a** $\displaystyle\int \frac{e^x + e^{-x}}{e^x - e^{-x}}\,dx$ has the form $\displaystyle\int \frac{f'(x)}{f(x)}\,dx$

$$\therefore \int \frac{e^x + e^{-x}}{e^x - e^{-x}}\,dx = \ln\left|e^x - e^{-x}\right| + c$$

b $$\int 7^x\,dx = \frac{1}{\ln 7}\int 7^x \ln 7\,dx$$
$$= \frac{7^x}{\ln 7} + c$$

c $$\int (3x+5)^5\,dx = \tfrac{1}{3}\,\frac{(3x+5)^6}{6} + c$$
$$= \frac{(3x+5)^6}{18} + c$$

d $\displaystyle\int \frac{\sin x}{2 - \cos x}\,dx$ has the form $\displaystyle\int \frac{f'(x)}{f(x)}\,dx$

$$\therefore \int \frac{\sin x}{2-\cos x}\,dx = \ln|2 - \cos x| + c$$
$$= \ln(2 - \cos x) + c$$
$$\{\text{since } 2 - \cos x > 0\}$$

e We integrate by parts with

$$\begin{array}{ll} u = x & v' = \sec^2 x \\ u' = 1 & v = \tan x \end{array}$$

$$\therefore \int x\sec^2 x\,dx = x\tan x - \int \tan x\,dx$$
$$= x\tan x + \int \frac{-\sin x}{\cos x}\,dx$$
$$\left\{\text{which has form } \int \frac{f'(x)}{f(x)}\,dx\right\}$$
$$= x\tan x + \ln|\cos x| + c$$

f $$\int \cot 2x\,dx$$
$$= \int \frac{\cos 2x}{\sin 2x}\,dx$$
$$= \tfrac{1}{2}\int \frac{2\cos 2x}{\sin 2x}\,dx$$
$$\left\{\text{which has form } \int \frac{f'(x)}{f(x)}\,dx\right\}$$
$$= \tfrac{1}{2}\ln|\sin 2x| + c$$

g Let $u = x + 3$, $\dfrac{du}{dx} = 1$

$\therefore \int x(x+3)^3\,dx$
$= \int (u-3)u^3\,du$
$= \int (u^4 - 3u^3)\,du$
$= \dfrac{u^5}{5} - \dfrac{3u^4}{4} + c$
$= \frac{1}{5}(x+3)^5 - \frac{3}{4}(x+3)^4 + c$

h $\displaystyle\int \frac{(x+1)^3}{x}\,dx$
$\displaystyle= \int \frac{x^3 + 3x^2 + 3x + 1}{x}\,dx$
$\displaystyle= \int \left(x^2 + 3x + 3 + \frac{1}{x}\right)\,dx$
$= \frac{1}{3}x^3 + \frac{3}{2}x^2 + 3x + \ln|x| + c$

2 **a** We integrate by parts with $u = x^2$ $\quad v' = e^{-x}$
$u' = 2x \quad v = -e^{-x}$

$\therefore \int x^2 e^{-x} dx = x^2(-e^{-x}) - \int 2x(-e^{-x})\,dx$
$= -x^2e^{-x} + 2\int xe^{-x}\,dx$

We integrate by parts again, this time with $u = x \quad v' = e^{-x}$
$u' = 1 \quad v = -e^{-x}$

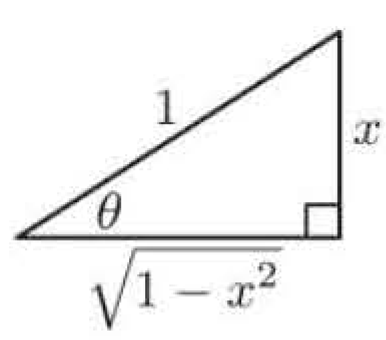
$\therefore \int x^2e^{-x}dx = -x^2e^{-x} + 2\left[x(-e^{-x}) - \int -e^{-x}\,dx\right]$
$= -x^2e^{-x} - 2xe^{-x} + 2\int e^{-x}\,dx$
$= -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + c$

b Let $u = 1 - x$, $\dfrac{du}{dx} = -1$

$\therefore \int x\sqrt{1-x}\,dx$
$= \int (1-u)\sqrt{u}\,(-du)$
$= \int (u-1)u^{\frac{1}{2}}\,du$
$= \int (u^{\frac{3}{2}} - u^{\frac{1}{2}})\,du$
$= \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + c$
$= \frac{2}{5}(1-x)^{\frac{5}{2}} - \frac{2}{3}(1-x)^{\frac{3}{2}} + c$

c Let $x = \sin\theta$, $\dfrac{dx}{d\theta} = \cos\theta$

$\therefore \int x^2\sqrt{1-x^2}\,dx$
$= \int \sin^2\theta\sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta$
$= \frac{1}{4}\int 4\sin^2\theta\cos^2\theta\,d\theta$
$= \frac{1}{4}\int \sin^2(2\theta)\,d\theta$
$= \frac{1}{4}\int \left(\frac{1}{2} - \frac{1}{2}\cos(4\theta)\right)\,d\theta$
$= \frac{1}{4}\left(\frac{1}{2}\theta - \frac{1}{8}\sin 4\theta\right) + c$
$= \frac{1}{8}\theta - \frac{1}{32}\sin 4\theta + c$



Now $\sin\theta = x$, $\quad \cos\theta = \sqrt{1-x^2}$

$\therefore \sin 2\theta = 2x\sqrt{1-x^2}$

and $\cos 2\theta = 1 - 2\sin^2\theta = 1 - 2x^2$

$\therefore \sin 4\theta = 2\left(2x\sqrt{1-x^2}\right)\left(1-2x^2\right)$
$= 4x\sqrt{1-x^2}(1-2x^2)$

$\therefore \int x^2\sqrt{1-x^2}\,dx$
$= \frac{1}{8}\arcsin x - \frac{1}{32}(4x\sqrt{1-x^2}(1-2x^2)) + c$
$= \frac{1}{8}\arcsin x - \frac{1}{8}x\sqrt{1-x^2}(1-2x^2) + c$
$= \frac{1}{8}\arcsin x - \frac{1}{8}x\sqrt{1-x^2} + \frac{1}{4}x^3\sqrt{1-x^2} + c$

d Let $x = 2\sec\theta$, $\dfrac{dx}{d\theta} = 2\sec\theta\tan\theta$

$\displaystyle\therefore \int \frac{3}{x\sqrt{x^2-4}}\,dx$
$\displaystyle= \int \frac{3}{2\cancel{\sec\theta}\sqrt{4\sec^2\theta - 4}}\,2\cancel{\sec\theta}\tan\theta\,d\theta$
$\displaystyle= \int \frac{3\tan\theta}{2\sqrt{\sec^2\theta - 1}}\,d\theta$
$= \int \frac{3}{2}\,d\theta \qquad \{\sqrt{\sec^2\theta - 1} = \tan\theta\}$
$= \frac{3}{2}\theta + c$

Now $x = \dfrac{2}{\cos\theta}$ so $\cos\theta = \dfrac{2}{x}$

$\displaystyle\therefore \int \frac{3}{x\sqrt{x^2-4}}\,dx = \frac{3}{2}\arccos\left(\frac{2}{x}\right) + c$

e Let $u = x - 3$, $\dfrac{du}{dx} = 1$

$\therefore \int x^2\sqrt{x-3}\,dx$
$= \int (u+3)^2\sqrt{u}\,du$
$= \int (u^2+6u+9)\,u^{\frac{1}{2}}\,du$
$= \int (u^{\frac{5}{2}} + 6u^{\frac{3}{2}} + 9u^{\frac{1}{2}})\,du$
$= \frac{2}{7}u^{\frac{7}{2}} + \frac{12}{5}u^{\frac{5}{2}} + 6u^{\frac{3}{2}} + c$
$= \frac{2}{7}(x-3)^{\frac{7}{2}} + \frac{12}{5}(x-3)^{\frac{5}{2}} + 6(x-3)^{\frac{3}{2}} + c$

f Let $u = \cos x$, $\dfrac{du}{dx} = -\sin x$

$\therefore \int \tan^3 x\,dx$
$= \displaystyle\int \frac{\sin^3 x}{\cos^3 x}\,dx$
$= \displaystyle\int \frac{\sin x(1-\cos^2 x)}{\cos^3 x}\,dx$
$= \displaystyle\int \left(\frac{1}{\cos x} - \frac{1}{\cos^3 x}\right)(-\sin x)\,dx$
$= \int (u^{-1} - u^{-3})\,du$
$= \ln|u| + \dfrac{1}{2u^2} + c$
$= \ln|\cos x| + \dfrac{1}{2\cos^2 x} + c$

g Let $u = x + 2$, $\dfrac{du}{dx} = 1$

$\therefore \displaystyle\int \frac{\ln(x+2)}{(x+2)^2}\,dx = \int u^{-2}\ln u\,du$

We integrate by parts with

$a = \ln u \qquad b' = u^{-2}$
$a' = \dfrac{1}{u} \qquad b = -\dfrac{1}{u}$

$$\therefore \int \frac{\ln(x+2)}{(x+2)^2}\,dx = -\frac{\ln u}{u} - \int -u^{-2}\,du$$
$$= -\frac{\ln u}{u} - \frac{1}{u} + c$$
$$= -\frac{\ln u + 1}{u} + c$$
$$= -\frac{\ln(x+2)+1}{x+2} + c$$

h $\dfrac{1}{x^2+2x+3} = \dfrac{1}{(x+1)^2+2}$
$= \dfrac{1}{(x+1)^2+(\sqrt{2})^2}$

We let $x + 1 = \sqrt{2}\tan\theta$, $\dfrac{dx}{d\theta} = \sqrt{2}\sec^2\theta$

$\therefore \displaystyle\int \frac{1}{x^2+2x+3}\,dx$
$= \displaystyle\int \frac{1}{(x+1)^2+(\sqrt{2})^2}\,dx$
$= \displaystyle\int \frac{1}{2\tan^2\theta+2}(\sqrt{2}\sec^2\theta\,d\theta)$
$= \displaystyle\int \frac{\sqrt{2}\sec^2\theta}{2\sec^2\theta}\,d\theta = \int \tfrac{1}{\sqrt{2}}\,d\theta$
$= \frac{1}{\sqrt{2}}\theta + c$

Now $\tan\theta = \dfrac{x+1}{\sqrt{2}}$, so $\theta = \arctan\left(\dfrac{x+1}{\sqrt{2}}\right)$

$\therefore \displaystyle\int \frac{1}{x^2+2x+3}\,dx = \tfrac{1}{\sqrt{2}}\arctan\left(\frac{x+1}{\sqrt{2}}\right) + c$

3 **a** Let $x = 3\tan\theta$ so $\dfrac{dx}{d\theta} = 3\sec^2\theta$

$\therefore \displaystyle\int \frac{1}{x^2+9}\,dx$
$= \displaystyle\int \frac{1}{9\tan^2\theta+9} \times 3\sec^2\theta\,d\theta$
$= \displaystyle\int \frac{3\sec^2\theta}{9(\tan^2\theta+1)}\,d\theta$
$= \int \frac{1}{3}\,d\theta \quad \{\tan^2\theta + 1 = \sec^2\theta\}$
$= \frac{1}{3}\theta + c$
$= \frac{1}{3}\arctan\left(\frac{x}{3}\right) + c$

b Let $x = \sin^2\theta$, $\dfrac{dx}{d\theta} = 2\sin\theta\cos\theta$

$\therefore \displaystyle\int \frac{4}{\sqrt{x}\sqrt{1-x}}\,dx$
$= \displaystyle\int \frac{4}{\sqrt{\sin^2\theta}\sqrt{1-\sin^2\theta}} \times 2\sin\theta\cos\theta\,d\theta$
$= \displaystyle\int \frac{8\sin\theta\cos\theta}{\sin\theta\cos\theta}\,d\theta$
$= \int 8\,d\theta$
$= 8\theta + c$
$= 8\arcsin(\sqrt{x}) + c$

c Let $u = 2x$, $\dfrac{du}{dx} = 2$ $\quad\therefore\quad \int \ln(2x)\,dx = \frac{1}{2}\int \ln(2x) \times 2\,dx$

$$\begin{aligned}
&= \tfrac{1}{2}\int \ln u\,du \\
&= \tfrac{1}{2}(u\ln u - u) + c \quad \{\textbf{Ex. 21H Q 2 a}\} \\
&= \tfrac{1}{2}(2x)\ln(2x) - \tfrac{1}{2}(2x) + c \\
&= x\ln(2x) - x + c
\end{aligned}$$

d We integrate by parts with $u = e^{-x}$ $\quad v' = \cos x$

$u' = -e^{-x}$ $\quad v = \sin x$

$\therefore\ \int e^{-x}\cos x\,dx = e^{-x}\sin x + \int e^{-x}\sin x\,dx$

We integrate by parts again, this time with $u = e^{-x}$ $\quad v' = \sin x$

$u' = -e^{-x}$ $\quad v = -\cos x$

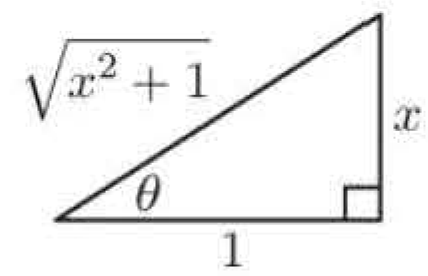
$\therefore\ \int e^{-x}\cos x\,dx = e^{-x}\sin x - e^{-x}\cos x - \int e^{-x}\cos x\,dx + c$

$\therefore\ 2\int e^{-x}\cos x\,dx = e^{-x}(\sin x - \cos x) + c$

$\therefore\ \int e^{-x}\cos x\,dx = \frac{1}{2}e^{-x}(\sin x - \cos x) + c$

e Let $x = \tan\theta$, $\dfrac{dx}{d\theta} = \sec^2\theta$

$$\begin{aligned}
\therefore\ &\int \frac{1}{x(1+x^2)}\,dx \\
&= \int \frac{1}{\tan\theta(1+\tan^2\theta)} \times \sec^2\theta\,d\theta \\
&= \int \frac{1}{\tan\theta}\,d\theta \quad \{1+\tan^2\theta = \sec^2\theta\} \\
&= \int \frac{\cos\theta}{\sin\theta}\,d\theta \\
&= \ln|\sin\theta| + c \\
&= \ln\left|\frac{x}{\sqrt{x^2+1}}\right| + c
\end{aligned}$$



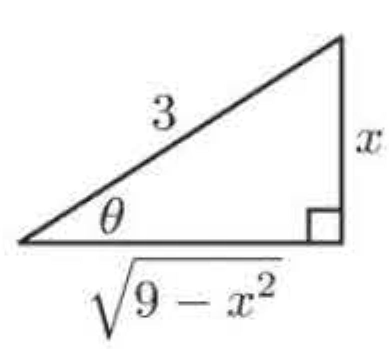
f Let $x = \tan\theta$, $\dfrac{dx}{d\theta} = \sec^2\theta$

$$\begin{aligned}
\therefore\ &\int \frac{\arctan x}{1+x^2}\,dx \\
&= \int \frac{\arctan(\tan\theta)}{1+\tan^2\theta}\sec^2\theta\,d\theta \\
&= \int \frac{\theta \times \sec^2\theta}{\sec^2\theta}\,d\theta \\
&= \int \theta\,d\theta \\
&= \tfrac{1}{2}\theta^2 + c \\
&= \tfrac{1}{2}\arctan^2 x + c
\end{aligned}$$

g Let $x = 3\sin\theta$, $\dfrac{dx}{d\theta} = 3\cos\theta$

$$\begin{aligned}
\therefore\ \int \sqrt{9-x^2}\,dx &= \int \sqrt{9-9\sin^2\theta} \times 3\cos\theta\,d\theta \\
&= 9\int \cos^2\theta\,d\theta \\
&= 9\int \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta\right)d\theta \\
&= 9\left(\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right) + c \\
&= \tfrac{9}{2}\theta + \tfrac{9}{4}\sin 2\theta + c
\end{aligned}$$

Now $\sin\theta = \dfrac{x}{3}$, so $\cos\theta = \dfrac{\sqrt{9-x^2}}{3}$



$\therefore\ \sin 2\theta = 2\left(\dfrac{x}{3}\right)\dfrac{\sqrt{9-x^2}}{3} = \dfrac{2x\sqrt{9-x^2}}{9}$

$$\begin{aligned}
\therefore\ \int \sqrt{9-x^2}\,dx &= \tfrac{9}{2}\arcsin\left(\tfrac{x}{3}\right) + \tfrac{9}{4}\left(\frac{2x\sqrt{9-x^2}}{9}\right) + c \\
&= \tfrac{9}{2}\arcsin\left(\tfrac{x}{3}\right) + \frac{x\sqrt{9-x^2}}{2} + c
\end{aligned}$$

h $\displaystyle\int \frac{(\ln x)^2}{x^2}\,dx = \int x^{-2}(\ln x)^2\,dx$

We integrate by parts with $u = (\ln x)^2 \quad v' = x^{-2}$

$u' = \dfrac{2\ln x}{x} \quad v = -\dfrac{1}{x}$

$$\therefore \int \frac{(\ln x)^2}{x^2}\,dx = -\frac{(\ln x)^2}{x} - \int \left(\frac{2\ln x}{x}\right)\left(-\frac{1}{x}\right)dx$$

$$= -\frac{(\ln x)^2}{x} + 2\int x^{-2}\ln x\,dx$$

We integrate by parts again, this time with $u = \ln x \quad v' = x^{-2}$

$u' = \dfrac{1}{x} \quad v = -\dfrac{1}{x}$

$$\therefore \int \frac{(\ln x)^2}{x^2}\,dx = \frac{-(\ln x)^2}{x} + 2\left(-\frac{\ln x}{x} - \int -x^{-2}\,dx\right)$$

$$= \frac{-(\ln x)^2}{x} - \frac{2\ln x}{x} - 2\left(\frac{1}{x}\right) + c$$

$$= -\frac{(\ln x)^2 + 2\ln x + 2}{x} + c$$

i Let $u = x - 3$, $\dfrac{du}{dx} = 1$ $\quad\therefore \displaystyle\int \frac{x}{\sqrt{x-3}}\,dx = \int \frac{u+3}{\sqrt{u}}\,du$

$$= \int (u^{\frac{1}{2}} + 3u^{-\frac{1}{2}})\,du$$

$$= \tfrac{2}{3}u^{\frac{3}{2}} + 6u^{\frac{1}{2}} + c$$

$$= \tfrac{2}{3}(x-3)^{\frac{3}{2}} + 6\sqrt{x-3} + c$$

j We integrate by parts with $u = \sin 4x \quad v' = \cos x$

$u' = 4\cos 4x \quad v = \sin x$

$$\therefore \int \sin 4x\cos x\,dx = \sin 4x\sin x - 4\int \cos 4x\sin x\,dx$$

We integrate by parts again, this time with $u = \cos 4x \quad v' = \sin x$

$u' = -4\sin 4x \quad v = -\cos x$

$$\therefore \int \sin 4x\cos x\,dx = \sin 4x\sin x - 4\left[-\cos 4x\cos x - \int 4\sin 4x\cos x\,dx\right] + c$$

$$= \sin 4x\sin x + 4\cos 4x\cos x + 16\int \sin 4x\cos x\,dx + c$$

$$\therefore -15\int \sin 4x\cos x\,dx = \sin 4x\sin x + 4\cos 4x\cos x + c$$

$$\therefore \int \sin 4x\cos x\,dx = -\tfrac{1}{15}(\sin 4x\sin x + 4\cos 4x\cos x) + c$$

k
$$\frac{2x+3}{x^2-2x+5} = \frac{2x-2}{x^2-2x+5} + \frac{5}{x^2-2x+5}$$

$$\therefore \int \frac{2x+3}{x^2-2x+5}\,dx = \int \frac{2x-2}{x^2-2x+5}\,dx + \int \frac{5}{x^2-2x+5}\,dx$$

Now $\displaystyle\int \frac{2x-2}{x^2-2x+5}\,dx$ has the form $\displaystyle\int \frac{f'(x)}{f(x)}\,dx$,

so $\displaystyle\int \frac{2x-2}{x^2-2x+5}\,dx = \ln\left|x^2-2x+5\right| + c$

$= \ln(x^2-2x+5) + c$ since $x^2 - 2x + 5 > 0$

For $\displaystyle\int \frac{5}{x^2-2x+5}\,dx$ we let $x - 1 = 2\tan\theta$, $\dfrac{dx}{d\theta} = 2\sec^2\theta$

$$\therefore \int \frac{5}{x^2-2x+5}\,dx = \int \frac{5}{(x-1)^2+4}\,dx$$
$$= \int \frac{5}{(2\tan\theta)^2+4} \times 2\sec^2\theta\,d\theta$$
$$= \int \frac{10\sec^2\theta}{4(\tan^2\theta+1)}\,d\theta$$
$$= \int \tfrac{5}{2}\,d\theta \qquad \{\tan^2\theta+1=\sec^2\theta\}$$
$$= \tfrac{5}{2}\theta + c$$
$$= \tfrac{5}{2}\arctan\left(\tfrac{x-1}{2}\right) + c$$

So, $$\int \frac{2x+3}{x^2-2x+5}\,dx = \ln(x^2-2x+5) + \tfrac{5}{2}\arctan\left(\tfrac{x-1}{2}\right) + c, \quad \text{as} \quad x^2-2x+5>0$$

l Let $u = \sin x$, $\dfrac{du}{dx} = \cos x$ $\quad\therefore$ $\int \cos^3 x\,dx = \int \cos^2 x \cos x\,dx$
$$= \int (1-\sin^2 x)\cos x\,dx$$
$$= \int (1-u^2)\,du$$
$$= u - \frac{u^3}{3} + c$$
$$= \sin x - \tfrac{1}{3}\sin^3 x + c$$

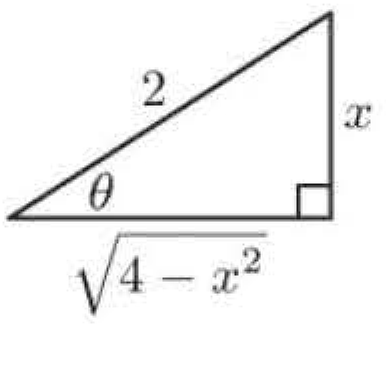
m $$\int \frac{x+4}{x^2+4}\,dx = \tfrac{1}{2}\int \frac{2x}{x^2+4}\,dx + \int \frac{4}{x^2+4}\,dx$$
$$= \tfrac{1}{2}\ln|x^2+4| + \int \frac{4}{x^2+2^2}\,dx \quad \text{where} \quad x^2+4>0$$

Let $x = 2\tan\theta$, $\dfrac{dx}{d\theta} = 2\sec^2\theta$

$$\therefore \int \frac{x+4}{x^2+4}\,dx = \tfrac{1}{2}\ln(x^2+4) + \int \frac{4}{4\tan^2\theta+4} \times 2\sec^2\theta\,d\theta$$
$$= \tfrac{1}{2}\ln(x^2+4) + \int \frac{8\sec^2\theta}{4(\tan^2\theta+1)}\,d\theta$$
$$= \tfrac{1}{2}\ln(x^2+4) + \int 2\,d\theta$$
$$= \tfrac{1}{2}\ln(x^2+4) + 2\theta + c$$
$$= \tfrac{1}{2}\ln(x^2+4) + 2\arctan\left(\tfrac{x}{2}\right) + c$$

n Let $x = 2\sin\theta$, $\dfrac{dx}{d\theta} = 2\cos\theta$

$$\therefore \int \frac{1-2x}{\sqrt{4-x^2}}\,dx = \int \frac{1-4\sin\theta}{\sqrt{4-4\sin^2\theta}}(2\cos\theta)\,d\theta$$
$$= \int \frac{1-4\sin\theta}{2\sqrt{1-\sin^2\theta}}(2\cos\theta)\,d\theta$$
$$= \int \frac{1-4\sin\theta}{2\cos\theta}(2\cos\theta)\,d\theta$$
$$= \int (1-4\sin\theta)\,d\theta$$
$$= \theta + 4\cos\theta + c$$
$$= \arcsin\left(\tfrac{x}{2}\right) + 4\left(\frac{\sqrt{4-x^2}}{2}\right) + c$$
$$= \arcsin\left(\tfrac{x}{2}\right) + 2\sqrt{4-x^2} + c$$



o Let $u = 2 - x$, $\dfrac{du}{dx} = -1$

$$\therefore \int \frac{x^3}{(2-x)^3}\,dx$$
$$= \int \frac{(2-u)^3}{u^3}\,(-du)$$
$$= \int \frac{(u-2)^3}{u^3}\,du$$
$$= \int \frac{u^3 - 6u^2 + 12u - 8}{u^3}\,du$$
$$= \int \left(1 - \frac{6}{u} + 12u^{-2} - 8u^{-3}\right) du$$
$$= u - 6\ln|u| - 12u^{-1} + 4u^{-2} + c$$
$$= (2-x) - 6\ln|2-x| - \frac{12}{2-x} + \frac{4}{(2-x)^2} + c$$

p Let $u = \sin x$, $\dfrac{du}{dx} = \cos x$

$$\therefore \int \sin^5 x \cos^5 x\,dx$$
$$= \int \sin^5 x\,(1 - \sin^2 x)(1 - \sin^2 x)\cos x\,dx$$
$$= \int u^5(1-u^2)(1-u^2)\,du$$
$$= \int u^5(1 - 2u^2 + u^4)\,du$$
$$= \int (u^5 - 2u^7 + u^9)\,du$$
$$= \frac{u^6}{6} - \frac{u^8}{4} + \frac{u^{10}}{10} + c$$
$$= \frac{\sin^6 x}{6} - \frac{\sin^8 x}{4} + \frac{\sin^{10} x}{10} + c$$

EXERCISE 21J.1

1 **a** $\int_0^1 x^3\,dx = \left[\dfrac{x^4}{4}\right]_0^1$
$= \frac{1}{4} - 0$
$= \frac{1}{4}$

b $\int_0^2 (x^2 - x)\,dx = \left[\dfrac{x^3}{3} - \dfrac{x^2}{2}\right]_0^2$
$= (\frac{8}{3} - 2) - (0 - 0)$
$= \frac{2}{3}$

c $\int_0^1 e^x\,dx = [e^x]_0^1$
$= e^1 - e^0$
$= e - 1$
≈ 1.72

d $$\int_1^4 \left(x - \frac{3}{\sqrt{x}}\right) dx$$
$$= \int_1^4 (x - 3x^{-\frac{1}{2}})\,dx$$
$$= \left[\frac{x^2}{2} - \frac{3x^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^4$$
$$= \left[\frac{x^2}{2} - 6\sqrt{x}\right]_1^4$$
$$= \left[\tfrac{16}{2} - 12\right] - (\tfrac{1}{2} - 6)$$
$$= 1\tfrac{1}{2}$$

e $$\int_4^9 \frac{x-3}{\sqrt{x}}\,dx$$
$$= \int_4^9 (x^{\frac{1}{2}} - 3x^{-\frac{1}{2}})\,dx$$
$$= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{3x^{\frac{1}{2}}}{\frac{1}{2}}\right]_4^9$$
$$= \left[\tfrac{2}{3}x^{\frac{3}{2}} - 6x^{\frac{1}{2}}\right]_4^9$$
$$= \left[\tfrac{2}{3}(27) - 6(3)\right] - \left[\tfrac{2}{3}(8) - 6(2)\right]$$
$$= (18 - 18) - (\tfrac{16}{3} - 12)$$
$$= 6\tfrac{2}{3}$$

f $$\int_1^3 \frac{1}{x}\,dx = [\ln|x|]_1^3$$
$= \ln 3 - \ln 1$
$= \ln 3 - 0$
$= \ln 3$
≈ 1.10

g $$\int_1^2 (e^{-x} + 1)^2\,dx$$
$$= \int_1^2 (e^{-2x} + 2e^{-x} + 1)\,dx$$
$$= \left[(\tfrac{1}{-2})e^{-2x} + 2(\tfrac{1}{-1})e^{-x} + x\right]_1^2$$
$$= \left[-\frac{e^{-2x}}{2} - 2e^{-x} + x\right]_1^2$$
$$= \left(-\frac{e^{-4}}{2} - 2e^{-2} + 2\right) - \left(-\frac{e^{-2}}{2} - 2e^{-1} + 1\right)$$
≈ 1.52

h $$\int_2^6 \frac{1}{\sqrt{2x-3}}\,dx = \int_2^6 (2x-3)^{-\frac{1}{2}}\,dx$$
$$= \left[\tfrac{1}{2}\,\frac{(2x-3)^{\frac{1}{2}}}{\frac{1}{2}}\right]_2^6$$
$$= \left[\sqrt{2x-3}\right]_2^6$$
$$= \sqrt{9} - \sqrt{1}$$
$$= 2$$

i $\int_0^1 e^{1-x}\,dx = \left[\left(\frac{1}{-1}\right)e^{1-x}\right]_0^1$

$= \left(\dfrac{e^0}{-1}\right) - \left(\dfrac{e^1}{-1}\right)$

$= -1 + e$

≈ 1.72

2 **a** $\int_0^{\frac{\pi}{6}} \cos x\,dx$

$= \left[\sin x\right]_0^{\frac{\pi}{6}}$

$= \sin\frac{\pi}{6} - \sin 0$

$= \frac{1}{2}$

b $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x\,dx$

$= \left[-\cos x\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$

$= -\cos\frac{\pi}{2} + \cos\frac{\pi}{3}$

$= \frac{1}{2}$

c $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^2 x\,dx$

$= \left[\tan x\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$

$= \tan\frac{\pi}{3} - \tan\frac{\pi}{4}$

$= \sqrt{3} - 1$

d $\int_0^{\frac{\pi}{6}} \sin(3x)\;dx$

$= \left[-\frac{1}{3}\cos(3x)\right]_0^{\frac{\pi}{6}}$

$= -\frac{1}{3}[\cos\frac{\pi}{2} - \cos 0]$

$= -\frac{1}{3}[0 - 1]$

$= \frac{1}{3}$

e $\int_0^{\frac{\pi}{4}} \cos^2 x\,dx$

$= \int_0^{\frac{\pi}{4}} \left(\frac{1}{2} + \frac{1}{2}\cos(2x)\right)dx$

$= \left[\dfrac{x}{2} + \frac{1}{4}\sin(2x)\right]_0^{\frac{\pi}{4}}$

$= [\frac{\pi}{8} + \frac{1}{4}\sin\frac{\pi}{2}] - 0$

$= \frac{\pi}{8} + \frac{1}{4}$

f $\int_0^{\frac{\pi}{2}} \sin^2 x\,dx$

$= \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right)dx$

$= \left[\dfrac{x}{2} - \frac{1}{4}\sin(2x)\right]_0^{\frac{\pi}{2}}$

$= [\frac{\pi}{4} - \frac{1}{4}\sin\pi] - 0$

$= \frac{\pi}{4}$

3 $\dfrac{4x+1}{x-1} = \dfrac{4x-4+1+4}{x-1}$

$= \dfrac{4(x-1)+5}{x-1}$

$= \dfrac{\cancel{4(x-1)}}{\cancel{x-1}} + \dfrac{5}{x-1}$

$= 4 + \dfrac{5}{x-1}$ as required

$\therefore \displaystyle\int_3^5 \frac{4x+1}{x-1}\,dx = \int_3^5 4 + \frac{5}{x-1}\,dx$

$= [4x + 5\ln|x-1|]_3^5$

$= 4(5) + 5\ln|5-1| - (4(3) + 5\ln|3-1|)$

$= 20 + 5\ln 4 - 12 - 5\ln 2$

$= 20 + 5\ln 2^2 - 12 - 5\ln 2$

$= 8 + 10\ln 2 - 5\ln 2$

$= 8 + 5\ln 2$

4 **a** $\displaystyle\int_m^{-2} \frac{1}{4-x}\,dx = \ln\tfrac{3}{2}$

$\therefore\ [-\ln|4-x|]_m^{-2} = \ln\frac{3}{2}$

$\therefore\ -\ln|4--2| + \ln|4-m| = \ln\frac{3}{2}$

$\therefore\ \ln|4-m| - \ln 6 = \ln\frac{3}{2}$

$\therefore\ \ln\left|\dfrac{4-m}{6}\right| = \ln\frac{3}{2}$

$\therefore\ \left|\dfrac{4-m}{6}\right| = \frac{3}{2}$

$\therefore\ \dfrac{4-m}{6} = \pm\frac{3}{2}$

$\therefore\ 4 - m = \pm 9$

$\therefore\ m = 4 \pm 9$

$\therefore\ m = -5$ or 13

However, the solution $m = 13$ is invalid, since the vertical asymptote $x = 4$ lies between -2 and 13.

$\therefore\ m = -5$ is the only valid answer.

b $\displaystyle\int_m^{2m} (2x-1)\,dx = 4$

$\therefore\ \left[x^2 - x\right]_m^{2m} = 4$

$\therefore\ (2m)^2 - 2m - (m^2 - m) = 4$

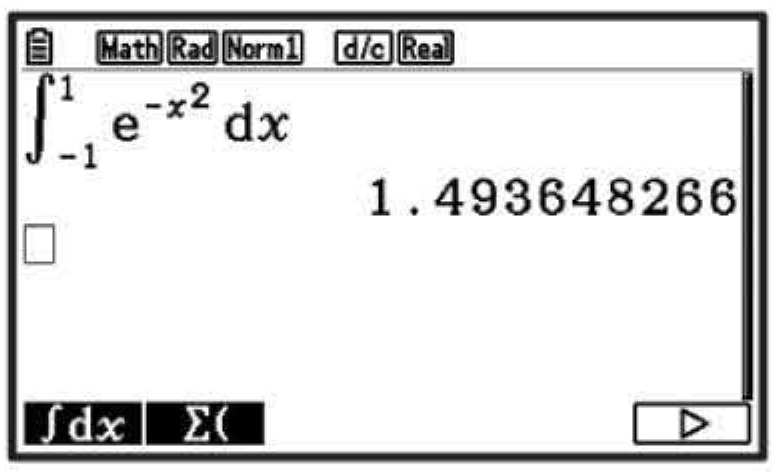
$\therefore\ 4m^2 - 2m - m^2 + m = 4$

$\therefore\ 3m^2 - m - 4 = 0$

$\therefore\ (3m-4)(m+1) = 0$

$\therefore\ m = \frac{4}{3}$ or -1

5



$\therefore \int_{-1}^{1} e^{-x^2}\, dx \approx 1.49$

6 **a** $\int_0^1 -xe^{-x}\, dx$

Let $u = -x, \quad v' = e^{-x}$

$\therefore u' = -1, \quad v = -e^{-x}$

Using $\int uv'\, dx = uv - \int u'v\, dx$

$$\begin{aligned}\therefore \int -xe^{-x}\, dx &= -x(-e^{-x}) - \int(-1)(-e^{-x})\, dx \\ &= xe^{-x} - \int e^{-x}\, dx \\ &= xe^{-x} + e^{-x} + c \\ &= e^{-x}(x+1) + c\end{aligned}$$

$$\begin{aligned}\therefore \int_0^1 -xe^{-x}\, dx &= \left[e^{-x}(x+1)\right]_0^1 \\ &= e^{-1}(1+1) - e^0(0+1) \\ &= 2e^{-1} - 1 \\ &\approx -0.264\,241\,1177 \quad \{\text{using technology}\} \\ &\approx -0.264\end{aligned}$$

b $\int_0^{\frac{\pi}{2}} x \sin x\, dx$

Let $u = x, \quad v' = \sin x$

$\therefore u' = 1, \quad v = -\cos x$

Using $\int uv'\, dx = uv - \int u'v\, dx$

$$\begin{aligned}\therefore \int x\sin x\, dx &= x(-\cos x) - \int 1(-\cos x)\, dx \\ &= -x\cos x + \int \cos x\, dx \\ &= -x\cos x + \sin x + c\end{aligned}$$

$$\begin{aligned}\therefore \int_0^{\frac{\pi}{2}} x\sin x\, dx &= \Big[\sin x - x\cos x\Big]_0^{\frac{\pi}{2}} \\ &= (\sin(\tfrac{\pi}{2}) - \tfrac{\pi}{2}\cos(\tfrac{\pi}{2})) - (\sin 0 - 0 \times \cos 0) \\ &= (1-0) - (0-0) \\ &= 1\end{aligned}$$

c $\int_1^3 \ln x\, dx$

Let $u = \ln x, \quad v' = 1$

$\therefore u' = \dfrac{1}{x}, \quad v = x$

Using $\int uv'\, dx = uv - \int u'v\, dx$

$$\begin{aligned}\therefore \int \ln x\, dx &= \ln x \times x - \int \frac{1}{x} \times x\, dx \\ &= x\ln x - \int 1\, dx \\ &= x\ln x - x + c\end{aligned}$$

$$\begin{aligned}\therefore \int_1^3 \ln x\, dx &= [x\ln x - x]_1^3 \\ &= (3\ln 3 - 3) - (1\ln 1 - 1) \\ &= 3\ln 3 - \ln 1 - 3 + 1 \\ &= 3\ln 3 - 2 \\ &\approx 1.295\,836\,866 \quad \{\text{using technology}\} \\ &\approx 1.30\end{aligned}$$

EXERCISE 21J.2

1 **a** In $\displaystyle\int_1^2 \frac{x}{(x^2+2)^2}\,dx$

we let $u = x^2+2, \quad \dfrac{du}{dx} = 2x$

when $x = 1, \quad u = 3$

when $x = 2, \quad u = 6$

$$\therefore \int_1^2 \frac{x}{(x^2+2)^2}\,dx = \int_1^2 u^{-2}\left(\tfrac{1}{2}\frac{du}{dx}\right)dx$$
$$= \tfrac{1}{2}\int_3^6 u^{-2}\,du$$
$$= \tfrac{1}{2}\left[\frac{u^{-1}}{-1}\right]_3^6 = \tfrac{1}{2}\left[-\tfrac{1}{6} - (-\tfrac{1}{3})\right]$$
$$= \tfrac{1}{12}$$

b In $\int_0^1 x^2 e^{x^3+1}\,dx$

we let $u = x^3+1, \quad \dfrac{du}{dx} = 3x^2$

when $x = 0, \quad u = 1$

when $x = 1, \quad u = 2$

$$\therefore \int_0^1 x^2 e^{x^3+1}\,dx = \int_0^1 e^u\left(\tfrac{1}{3}\frac{du}{dx}\right)dx$$
$$= \tfrac{1}{3}\int_1^2 e^u\,du$$
$$= \tfrac{1}{3}\left[e^u\right]_1^2$$
$$= \tfrac{1}{3}(e^2 - e)$$
$$\approx 1.56$$

c In $\int_0^3 x\sqrt{x^2+16}\,dx$

we let $u = x^2+16, \quad \dfrac{du}{dx} = 2x$

when $x = 0, \quad u = 16$

when $x = 3, \quad u = 25$

$$\therefore \int_0^3 x\sqrt{x^2+16}\,dx$$
$$= \tfrac{1}{2}\int_0^3 2x\sqrt{x^2+16}\,dx$$
$$= \tfrac{1}{2}\int_0^3 u^{\frac{1}{2}}\frac{du}{dx}\,dx$$
$$= \tfrac{1}{2}\int_{16}^{25} u^{\frac{1}{2}}\,du$$
$$= \tfrac{1}{2}\left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_{16}^{25}$$
$$= \tfrac{1}{2}\times\tfrac{2}{3}\left[u^{\frac{3}{2}}\right]_{16}^{25}$$
$$= \tfrac{1}{3}(125 - 64)$$
$$= 20\tfrac{1}{3}$$

d In $\int_1^2 xe^{-2x^2}\,dx$

we let $u = -2x^2, \quad \dfrac{du}{dx} = -4x$

when $x = 1, \quad u = -2$

when $x = 2, \quad u = -8$

$$\therefore \int_1^2 xe^{-2x^2}\,dx$$
$$= -\tfrac{1}{4}\int_1^2 -4xe^{-2x^2}\,dx$$
$$= -\tfrac{1}{4}\int_1^2 e^u\frac{du}{dx}\,dx$$
$$= -\tfrac{1}{4}\int_{-2}^{-8} e^u\,du$$
$$= -\tfrac{1}{4}\left[e^u\right]_{-2}^{-8}$$
$$= -\tfrac{1}{4}(e^{-8} - e^{-2})$$
$$\approx 0.0337$$

e In $\displaystyle\int_2^3 \frac{x}{2-x^2}\,dx$

we let $u = 2-x^2, \quad \dfrac{du}{dx} = -2x$

when $x = 2, \quad u = -2$

when $x = 3, \quad u = -7$

$$\therefore \int_2^3 \frac{x}{2-x^2}\,dx = -\tfrac{1}{2}\int_2^3 \frac{1}{u}\frac{du}{dx}\,dx$$
$$= -\tfrac{1}{2}\int_{-2}^{-7}\frac{1}{u}\,du$$
$$= -\tfrac{1}{2}\left[\ln|u|\right]_{-2}^{-7}$$
$$= -\tfrac{1}{2}(\ln 7 - \ln 2)$$
$$= -\tfrac{1}{2}\ln(\tfrac{7}{2})$$
$$\approx -0.626$$

f In $\displaystyle\int_1^2 \frac{\ln x}{x}\,dx$

we let $u = \ln x$, $\dfrac{du}{dx} = \dfrac{1}{x}$

when $x = 1$, $u = 0$

when $x = 2$, $u = \ln 2$

$$\begin{aligned}\therefore \int_1^2 \frac{\ln x}{x}\,dx &= \int_1^2 u\,\frac{du}{dx}\,dx\\ &= \int_0^{\ln 2} u\,du\\ &= \left[\frac{u^2}{2}\right]_0^{\ln 2}\\ &= \frac{(\ln 2)^2}{2} - 0\\ &\approx 0.240\end{aligned}$$

g In $\displaystyle\int_0^1 \frac{1-3x^2}{1-x^3+x}\,dx$

we let $u = 1 - x^3 + x$,

$\dfrac{du}{dx} = -3x^2 + 1$

when $x = 0$, $u = 1$

when $x = 1$, $u = 1$

$$\begin{aligned}\therefore \int_0^1 \frac{1-3x^2}{1-x^3+x}\,dx &= \int_0^1 \frac{1}{u}\,\frac{du}{dx}\,dx\\ &= \int_1^1 \frac{1}{u}\,du\\ &= 0\end{aligned}$$

h In $\displaystyle\int_2^4 \frac{6x^2-4x+4}{x^3-x^2+2x}\,dx$

we let $u = x^3 - x^2 + 2x$, $\dfrac{du}{dx} = 3x^2 - 2x + 2$

when $x = 2$, $u = 8$

when $x = 4$, $u = 56$

$$\begin{aligned}\therefore \int_2^4 \frac{6x^2-4x+4}{x^3-x^2+2x}\,dx &= 2\int_2^4 \frac{3x^2-2x+2}{x^3-x^2+2x}\,dx\\ &= 2\int_2^4 \frac{1}{u}\,\frac{du}{dx}\,dx\\ &= 2\int_8^{56} \frac{1}{u}\,du\\ &= 2\left[\ln|u|\right]_8^{56}\\ &= 2(\ln 56 - \ln 8)\\ &= 2\ln 7\\ &\approx 3.89\end{aligned}$$

2 **a** Let $u = \cos x$, $\dfrac{du}{dx} = -\sin x$

when $x = 0$, $u = \cos 0 = 1$

when $x = \frac{\pi}{3}$, $u = \cos\frac{\pi}{3} = \frac{1}{2}$

$$\begin{aligned}\therefore \int_0^{\frac{\pi}{3}} \frac{\sin x}{\sqrt{\cos x}}\,dx &= -\int_0^{\frac{\pi}{3}} \frac{-\sin x}{\sqrt{\cos x}}\,dx\\ &= -\int_0^{\frac{\pi}{3}} u^{-\frac{1}{2}}\,\frac{du}{dx}\,dx\\ &= \int_{\frac{1}{2}}^1 u^{-\frac{1}{2}}\,du = \left[2u^{\frac{1}{2}}\right]_{\frac{1}{2}}^1\\ &= 2\sqrt{1} - 2\sqrt{\tfrac{1}{2}}\\ &= 2 - \sqrt{2}\end{aligned}$$

b Let $u = \sin x$, $\dfrac{du}{dx} = \cos x$

when $x = 0$, $u = \sin 0 = 0$

when $x = \frac{\pi}{6}$, $u = \sin\frac{\pi}{6} = \frac{1}{2}$

$$\begin{aligned}\therefore \int_0^{\frac{\pi}{6}} \sin^2 x\cos x\,dx &= \int_0^{\frac{\pi}{6}} u^2\,\frac{du}{dx}\,dx\\ &= \int_0^{\frac{1}{2}} u^2\,du\\ &= \left[\frac{u^3}{3}\right]_0^{\frac{1}{2}}\\ &= \tfrac{1}{3}\left(\tfrac{1}{2}\right)^3\\ &= \tfrac{1}{24}\end{aligned}$$

c Let $u = \cos x$, $\dfrac{du}{dx} = -\sin x$

when $x = 0$, $u = \cos 0 = 1$

when $x = \frac{\pi}{4}$, $u = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$$\begin{aligned}\therefore\ \int_0^{\frac{\pi}{4}} \tan x\,dx &= \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x}\,dx\\ &= -\int_0^{\frac{\pi}{4}} \frac{1}{u}\frac{du}{dx}\,dx\\ &= -\int_1^{\frac{1}{\sqrt{2}}} \frac{1}{u}\,du\\ &= \int_{\frac{1}{\sqrt{2}}}^1 \frac{1}{u}\,du\\ &= \Big[\ln|u|\Big]_{\frac{1}{\sqrt{2}}}^1\\ &= \ln 1 - \ln\tfrac{1}{\sqrt{2}}\\ &= \ln\sqrt{2} = \tfrac{1}{2}\ln 2\end{aligned}$$

d Let $u = \sin x$, $\dfrac{du}{dx} = \cos x$

when $x = \frac{\pi}{6}$, $u = \sin\frac{\pi}{6} = \frac{1}{2}$

when $x = \frac{\pi}{2}$, $u = \sin\frac{\pi}{2} = 1$

$$\begin{aligned}\therefore\ \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cot x\,dx &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x}{\sin x}\,dx\\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{u}\frac{du}{dx}\,dx\\ &= \int_{\frac{1}{2}}^1 \frac{1}{u}\,du\\ &= \Big[\ln|u|\Big]_{\frac{1}{2}}^1\\ &= \ln 1 - \ln\tfrac{1}{2}\\ &= \ln 2\end{aligned}$$

e Let $u = 1 - \sin x$, $\dfrac{du}{dx} = -\cos x$

when $x = 0$, $u = 1 - \sin 0 = 1$

when $x = \frac{\pi}{6}$, $u = 1 - \sin\frac{\pi}{6} = \frac{1}{2}$

$$\begin{aligned}\therefore\ \int_0^{\frac{\pi}{6}} \frac{\cos x}{1-\sin x}\,dx &= -\int_0^{\frac{\pi}{6}} \frac{-\cos x}{1-\sin x}\,dx\\ &= -\int_0^{\frac{\pi}{6}} \frac{1}{u}\frac{du}{dx}\,dx\\ &= -\int_1^{\frac{1}{2}} \frac{1}{u}\,du\\ &= \int_{\frac{1}{2}}^1 \frac{1}{u}\,du\\ &= [\ln|u|]_{\frac{1}{2}}^1\\ &= \ln 1 - \ln\tfrac{1}{2}\\ &= \ln 2\end{aligned}$$

f Let $u = \tan x$, $\dfrac{du}{dx} = \sec^2 x$

when $x = 0$, $u = \tan 0 = 0$

when $x = \frac{\pi}{4}$, $u = \tan\frac{\pi}{4} = 1$

$$\begin{aligned}\therefore\ \int_0^{\frac{\pi}{4}} \sec^2 x\tan^3 x\,dx &= \int_0^{\frac{\pi}{4}} u^3\frac{du}{dx}\,dx\\ &= \int_0^1 u^3\,du\\ &= \left[\frac{u^4}{4}\right]_0^1\\ &= \tfrac{1}{4}\end{aligned}$$

3 In $\int_0^1 (x^2+2x)^n(x+1)\,dx$

we let $u = x^2 + 2x$, $\dfrac{du}{dx} = 2x + 2$

when $x = 0$, $u = 0$

when $x = 1$, $u = 3$

$$\begin{aligned}\therefore\ \int_0^1 (x^2+2x)^n(x+1)\,dx &= \tfrac{1}{2}\int_0^1 (x^2+2x)^n(2x+2)\,dx\\ &= \tfrac{1}{2}\int_0^1 u^n\frac{du}{dx}\,dx\\ &= \tfrac{1}{2}\int_0^3 u^n\,du\end{aligned}$$

If $n \neq -1$, the integral $= \frac{1}{2}\left[\dfrac{u^{n+1}}{n+1}\right]_0^3 = \frac{1}{2}\left(\dfrac{3^{n+1}}{n+1}\right)$

If $n = -1$, the integral $= \frac{1}{2}\displaystyle\int_0^3 \frac{1}{u}\,du = \frac{1}{2}[\ln|u|]_0^3$ which is undefined as $\ln 0$ is not defined.

4 **a** Let $u = x - 1,\quad \dfrac{du}{dx} = 1$

when $x = 4,\quad u = 3$

when $x = 3,\quad u = 2$

$$\therefore\quad \int_3^4 x\sqrt{x-1}\,dx$$
$$= \int_2^3 (u+1)\sqrt{u}\,du$$
$$= \int_2^3 (u^{\frac{3}{2}} + u^{\frac{1}{2}})\,du$$
$$= \left[\frac{u^{\frac{5}{2}}}{\frac{5}{2}} + \frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_2^3$$
$$= \left(\tfrac{2}{5}(3)^{\frac{5}{2}} + \tfrac{2}{3}(3)^{\frac{3}{2}}\right) - \left(\tfrac{2}{5}(2)^{\frac{5}{2}} + \tfrac{2}{3}(2)^{\frac{3}{2}}\right)$$
$$= \tfrac{2}{5}\left(9\sqrt{3}\right) + \tfrac{2}{3}\left(3\sqrt{3}\right) - \tfrac{2}{5}\left(4\sqrt{2}\right) - \tfrac{2}{3}\left(2\sqrt{2}\right)$$
$$= \left(\tfrac{18}{5} + 2\right)\sqrt{3} - \left(\tfrac{8}{5} + \tfrac{4}{3}\right)\sqrt{2}$$
$$= \tfrac{28\sqrt{3}}{5} - \tfrac{44\sqrt{2}}{15}$$

b Let $u = x + 6,\quad \dfrac{du}{dx} = 1$

when $x = 3,\quad u = 9$

when $x = 0,\quad u = 6$

$$\therefore\quad \int_0^3 x\sqrt{x+6}\,dx$$
$$= \int_6^9 (u-6)\sqrt{u}\,du$$
$$= \int_6^9 (u^{\frac{3}{2}} - 6u^{\frac{1}{2}})\,du$$
$$= \left[\frac{u^{\frac{5}{2}}}{\frac{5}{2}} - \frac{6u^{\frac{3}{2}}}{\frac{3}{2}}\right]_6^9$$
$$= \left(\tfrac{2}{5}(9)^{\frac{5}{2}} - 4(9)^{\frac{3}{2}}\right) - \left(\tfrac{2}{5}(6)^{\frac{5}{2}} - 4(6)^{\frac{3}{2}}\right)$$
$$= \tfrac{2}{5}\left(3^5\right) - 4\left(3^3\right) - \tfrac{2}{5} \times 36\sqrt{6} + 4 \times 6\sqrt{6}$$
$$= \tfrac{486}{5} - 108 - \tfrac{72}{5}\sqrt{6} + 24\sqrt{6}$$
$$= -\tfrac{54}{5} + \tfrac{48}{5}\sqrt{6}$$
$$= \tfrac{1}{5}(48\sqrt{6} - 54)$$

c Let $u = x - 2,\quad \dfrac{du}{dx} = 1$

when $x = 5,\quad u = 3$

when $x = 2,\quad u = 0$

$$\therefore\quad \int_2^5 x^2\sqrt{x-2}\,dx = \int_0^3 (u+2)^2\sqrt{u}\,du$$
$$= \int_0^3 \left(u^2 + 4u + 4\right)\sqrt{u}\,du$$
$$= \int_0^3 (u^{\frac{5}{2}} + 4u^{\frac{3}{2}} + 4u^{\frac{1}{2}})\,du$$
$$= \left[\frac{u^{\frac{7}{2}}}{\frac{7}{2}} + \frac{4u^{\frac{5}{2}}}{\frac{5}{2}} + \frac{4u^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^3$$
$$= \tfrac{2}{7}(3)^{\frac{7}{2}} + \tfrac{8}{5}(3)^{\frac{5}{2}} + \tfrac{8}{3}(3)^{\frac{3}{2}} - 0$$
$$= \tfrac{2}{7}\left(27\sqrt{3}\right) + \tfrac{8}{5}\left(9\sqrt{3}\right) + \tfrac{8}{3}\left(3\sqrt{3}\right)$$
$$= \tfrac{1054}{35}\sqrt{3}$$

EXERCISE 21J.3

1 **a** $\int_1^4 \sqrt{x}\,dx = \int_1^4 x^{\frac{1}{2}}\,dx$
$$= \left[\tfrac{2}{3}x^{\frac{3}{2}}\right]_1^4$$
$$= \tfrac{2}{3}(8) - \tfrac{2}{3}(1)$$
$$= \tfrac{14}{3} \approx 4.67$$

$\int_1^4 (-\sqrt{x})\,dx = \int_1^4 \left(-x^{\frac{1}{2}}\right)\,dx$
$$= \left[-\tfrac{2}{3}x^{\frac{3}{2}}\right]_1^4$$
$$= -\tfrac{2}{3}(8) - (-\tfrac{2}{3}(1))$$
$$= -\tfrac{14}{3} \approx -4.67$$

b $\int_0^1 x^7\,dx = \left[\tfrac{1}{8}x^8\right]_0^1$
$$= \tfrac{1}{8} - 0 = \tfrac{1}{8}$$

$\int_0^1 (-x^7)\,dx = \left[-\tfrac{1}{8}x^8\right]_0^1$
$$= -\tfrac{1}{8} - 0 = -\tfrac{1}{8}$$

Property: $\int_a^b [-f(x)]\,dx = -\int_a^b f(x)\,dx$

2 **a** $\int_0^1 x^2\,dx$
$= \left[\frac{1}{3}x^3\right]_0^1$
$= \frac{1}{3} - 0$
$= \frac{1}{3}$

b $\int_1^2 x^2\,dx$
$= \left[\frac{1}{3}x^3\right]_1^2$
$= \frac{1}{3}(8) - \frac{1}{3}(1)$
$= \frac{7}{3}$

c $\int_0^2 x^2\,dx$
$= \left[\frac{1}{3}x^3\right]_0^2$
$= \frac{1}{3}(8) - 0$
$= \frac{8}{3}$

d $\int_0^1 3x^2\,dx$
$= \left[x^3\right]_0^1$
$= 1 - 0$
$= 1$

Properties: $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
$\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$, c a constant

3 **a** $\int_0^2 (x^3 - 4x)\,dx$
$= \left[\frac{1}{4}x^4 - 2x^2\right]_0^2$
$= \left[\frac{1}{4}(16) - 2(4)\right] - [0 - 0]$
$= -4$

b $\int_2^3 (x^3 - 4x)\,dx$
$= \left[\frac{1}{4}x^4 - 2x^2\right]_2^3$
$= \left[\frac{1}{4}(81) - 2(9)\right] - \left[\frac{1}{4}(16) - 2(4)\right]$
$= \frac{25}{4} = 6.25$

c $\int_0^3 (x^3 - 4x)\,dx = \left[\frac{1}{4}x^4 - 2x^2\right]_0^3$
$= \left[\frac{1}{4}(81) - 2(9)\right] - [0 - 0]$
$= \frac{9}{4} = 2.25$

Property: $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

4 **a** $\int_0^1 x^2\,dx = \left[\frac{1}{3}x^3\right]_0^1$
$= \frac{1}{3}(1) - 0$
$= \frac{1}{3}$

b $\int_0^1 \sqrt{x}\,dx = \int_0^1 x^{\frac{1}{2}}\,dx$
$= \left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^1$
$= \frac{2}{3}(1) - 0$
$= \frac{2}{3}$

c $\int_0^1 (x^2 + \sqrt{x})\,dx$
$= \int_0^1 (x^2 + x^{\frac{1}{2}})\,dx$
$= \left[\frac{1}{3}x^3 + \frac{2}{3}x^{\frac{3}{2}}\right]_0^1$
$= \left[\frac{1}{3}(1) + \frac{2}{3}(1)\right] - [0 + 0]$
$= 1$

Property: $\int_a^b f(x)\,dx + \int_a^b g(x)\,dx = \int_a^b [f(x) + g(x)]\,dx$

5 **a** $\int_0^3 f(x)\,dx =$ area between $f(x)$ and the x-axis from $x = 0$ to $x = 3$
$= 2 + 3 + 1.5 = 6.5$

b $\int_3^7 f(x)\,dx = -$(area between $f(x)$ and the x-axis from $x = 3$ to $x = 7$)
$= -\left(\frac{3}{2} + 3 + \frac{5}{2} + 2\right) = -9$

c $\int_2^4 f(x)\,dx =$ (area between $f(x)$ and the x-axis from $x = 2$ to $x = 3$)
$-$(area between $f(x)$ and the x-axis from $x = 3$ to $x = 4$)
$= 1.5 - 1.5 = 0$

d $\int_0^7 f(x)\,dx =$ (area between $f(x)$ and the x-axis from $x = 0$ to $x = 3$)
$-$(area between $f(x)$ and the x-axis from $x = 3$ to $x = 7$)
$= 6.5 - 9 = -2.5$

6 **a** $\int_0^4 f(x)\,dx$
$=$ area of semi-circle with radius 2
$= \frac{1}{2}\pi(2)^2 = 2\pi$

b $\int_4^6 f(x)\,dx$
$= -$(area of 2 by 2 rectangle)
$= -(2 \times 2) = -4$

c $\int_6^8 f(x)\,dx$
$=$ area of semi-circle with radius 1
$= \frac{1}{2}\pi(1)^2 = \frac{\pi}{2}$

d $\int_0^8 f(x)\,dx$
$= \int_0^4 f(x)\,dx + \int_4^6 f(x)\,dx + \int_6^8 f(x)\,dx$
$= 2\pi + (-4) + \frac{\pi}{2} = \frac{5\pi}{2} - 4$

7 **a** $\int_2^4 f(x)\,dx + \int_4^7 f(x)\,dx$
$= \int_2^7 f(x)\,dx$

b $\int_1^3 g(x)\,dx + \int_3^8 g(x)\,dx + \int_8^9 g(x)\,dx$
$= \int_1^9 g(x)\,dx$

8 **a** $\int_1^3 f(x)\,dx + \int_3^6 f(x)\,dx = \int_1^6 f(x)\,dx$

$$\begin{aligned} \therefore\ \int_3^6 f(x)\,dx &= \int_1^6 f(x)\,dx - \int_1^3 f(x)\,dx \\ &= (-3) - 2 \\ &= -5 \end{aligned}$$

b $\int_0^2 f(x)\,dx + \int_2^4 f(x)\,dx + \int_4^6 f(x)\,dx = \int_0^6 f(x)\,dx$

$$\begin{aligned} \therefore\ \int_2^4 f(x)\,dx &= \int_0^6 f(x)\,dx - \int_4^6 f(x)\,dx - \int_0^2 f(x)\,dx \\ &= (7) - (-2) - (5) \\ &= 4 \end{aligned}$$

9 **a**
$$\begin{aligned} \int_1^{-1} f(x)\,dx &= -\int_{-1}^1 f(x)\,dx \\ &= -(-4) \\ &= 4 \end{aligned}$$

b
$$\begin{aligned} \int_{-1}^1 (2 + f(x))\,dx &= \int_{-1}^1 2\,dx + \int_{-1}^1 f(x)\,dx \\ &= [2x]_{-1}^1 + (-4) \\ &= (2 - (-2)) - 4 \\ &= 0 \end{aligned}$$

c
$$\begin{aligned} \int_{-1}^1 2\,f(x)\,dx &= 2\int_{-1}^1 f(x)\,dx \\ &= 2(-4) \\ &= -8 \end{aligned}$$

d
$$\begin{aligned} \int_{-1}^1 k\,f(x)\,dx &= 7 \\ \therefore\ k\int_{-1}^1 f(x)\,dx &= 7 \\ \therefore\ k(-4) &= 7 \\ \therefore\ k &= -\tfrac{7}{4} \end{aligned}$$

10
$$\begin{aligned} \int_2^3 (g'(x) - 1)\,dx &= \int_2^3 g'(x)\,dx + \int_2^3 -1\,dx \\ &= [g(x)]_2^3 + [-x]_2^3 \\ &= (g(3) - g(2)) + (-3 - (-2)) \\ &= 5 - 4 - 1 \\ &= 0 \end{aligned}$$

REVIEW SET 21A

1 **a**
$$\begin{aligned} \int_0^4 f(x)\,dx &= \text{area of semi-circle with radius 2} \\ &= \tfrac{1}{2} \times \pi \times 2^2 \\ &= 2\pi \text{ units}^2 \end{aligned}$$

b
$$\begin{aligned} \int_4^6 f(x)\,dx &= \text{area of square} \\ &= 2 \times 2 \\ &= 4 \text{ units}^2 \end{aligned}$$

2 **a**
$$\begin{aligned} \int \frac{4}{\sqrt{x}}\,dx &= 4\int x^{-\frac{1}{2}}\,dx \\ &= 4\,\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c = 8\sqrt{x} + c \end{aligned}$$

b
$$\begin{aligned} \int \frac{3}{1-2x}\,dx &= 3\int \frac{1}{1-2x}\,dx \\ &= 3(\tfrac{1}{-2})\ln|1-2x| + c \\ &= -\tfrac{3}{2}\ln|1-2x| + c \end{aligned}$$

c $\int \sin(4x - 5)\,dx = -\frac{1}{4}\cos(4x - 5) + c$

d
$$\begin{aligned} \int e^{4-3x}\,dx &= \tfrac{1}{-3}e^{4-3x} + c \\ &= -\tfrac{1}{3}e^{4-3x} + c \end{aligned}$$

3 **a** $\int_{-5}^{-1} \sqrt{1-3x}\,dx = \int_{-5}^{-1} (1-3x)^{\frac{1}{2}}\,dx$

$$= \left[\tfrac{1}{-3} \times \frac{(1-3x)^{\frac{3}{2}}}{\frac{3}{2}}\right]_{-5}^{-1}$$

$$= -\tfrac{2}{9}\left[(1-3x)^{\frac{3}{2}}\right]_{-5}^{-1}$$

$$= -\tfrac{2}{9}\left(4^{\frac{3}{2}} - 16^{\frac{3}{2}}\right)$$

$$= -\tfrac{2}{9}(8-64) = 12\tfrac{4}{9}$$

b $\int_0^{\frac{\pi}{2}} \cos\left(\frac{x}{2}\right)\,dx = \left[2\sin\left(\frac{x}{2}\right)\right]_0^{\frac{\pi}{2}}$

$$= 2\sin(\tfrac{\pi}{4}) - 2\sin(0)$$

$$= 2\left(\tfrac{1}{\sqrt{2}}\right) - 2(0)$$

$$= \sqrt{2}$$

c In $\displaystyle\int_0^1 \frac{4x^2}{(x^3+2)^3}\,dx$

we let $u = x^3+2$, $\dfrac{du}{dx} = 3x^2$

when $x=0$, $u=2$

when $x=1$, $u=3$

$$\therefore \int_0^1 \frac{4x^2}{(x^3+2)^3}\,dx = \tfrac{4}{3}\int_0^1 \frac{3x^2}{(x^2+2)^3}\,dx$$

$$= \tfrac{4}{3}\int_0^1 \frac{1}{u^3}\frac{du}{dx}\,dx$$

$$= \tfrac{4}{3}\int_2^3 u^{-3}\,du$$

$$= \tfrac{4}{3}\left[\frac{u^{-2}}{-2}\right]_2^3$$

$$= -\tfrac{2}{3}\left[\frac{1}{u^2}\right]_2^3$$

$$= -\tfrac{2}{3}\left[\tfrac{1}{9} - \tfrac{1}{4}\right]$$

$$= \tfrac{5}{54}$$

4

$$y = \sqrt{x^2-4} = (x^2-4)^{\frac{1}{2}}$$

$$\therefore \frac{dy}{dx} = \tfrac{1}{2}\left(x^2-4\right)^{-\frac{1}{2}} \times 2x$$

$$= \frac{x}{(x^2-4)^{\frac{1}{2}}}$$

$$= \frac{x}{\sqrt{x^2-4}}$$

$$\therefore \int \frac{x}{\sqrt{x^2-4}}\,dx = \sqrt{x^2-4} + c$$

5

$$\int_0^b \cos x\,dx = \tfrac{1}{\sqrt{2}}, \quad 0 < b < \pi$$

$$\therefore [\sin x]_0^b = \tfrac{1}{\sqrt{2}}$$

$$\therefore \sin b - \sin 0 = \tfrac{1}{\sqrt{2}}$$

$$\therefore \sin b = \tfrac{1}{\sqrt{2}}$$

$$\therefore b = \tfrac{\pi}{4}, \tfrac{3\pi}{4} \quad \{0 < b < \pi\}$$

6 **a**

$$\int (2-\cos x)^2\,dx$$

$$= \int (4 - 4\cos x + \cos^2 x)\,dx$$

$$= \int (4 - 4\cos x + \tfrac{1}{2} + \tfrac{1}{2}\cos 2x)\,dx$$

$$= \tfrac{9}{2}x - 4\sin x + \tfrac{1}{4}\sin 2x + c$$

b

$$y = \sin(x^2)$$

$$\therefore \frac{dy}{dx} = \cos(x^2) \times 2x$$

$$\therefore \int 2x\cos(x^2)\,dx = \sin(x^2) + c$$

$$\therefore 2\int x\cos(x^2)\,dx = \sin(x^2) + c$$

$$\therefore \int x\cos(x^2)\,dx = \tfrac{1}{2}\sin(x^2) + c$$

7

$$\frac{d}{dx}(3x^2+x)^3 = 3(3x^2+x)^2(6x+1)$$

$$\therefore \int 3(3x^2+x)^2(6x+1)\,dx = (3x^2+x)^3 + c$$

$$\therefore 3\int (3x^2+x)^2(6x+1)\,dx = (3x^2+x)^3 + c$$

$$\therefore \int (3x^2+x)^2(6x+1)\,dx = \tfrac{1}{3}(3x^2+x)^3 + c$$

8 **a** $\int_1^4 (f(x)+1)\ dx$
$= \int_1^4 f(x)\ dx + \int_1^4 1\ dx$
$= 3 + [x]_1^4$
$= 3 + (4-1) = 6$

b $\int_1^2 f(x)\ dx - \int_4^2 f(x)\ dx$
$= \int_1^2 f(x)\ dx + \int_2^4 f(x)\ dx$
$= \int_1^4 f(x)\ dx$
$= 3$

9 **a** Let $u = \sin x$, $\dfrac{du}{dx} = \cos x$

$$\therefore \int \sin^7 x \cos x\, dx = \int u^7 \frac{du}{dx}\, dx = \int u^7\, du = \frac{u^8}{8} + c = \frac{\sin^8 x}{8} + c$$

b Let $u = \cos 2x$, $\dfrac{du}{dx} = -2\sin 2x$

$$\therefore \int \tan 2x\ dx = \int \frac{\sin 2x}{\cos 2x}\, dx = -\tfrac{1}{2}\int \frac{-2\sin 2x}{\cos 2x}\, dx = -\tfrac{1}{2}\int \frac{1}{u}\frac{du}{dx}\, dx = -\tfrac{1}{2}\int \frac{1}{u}\, du = -\tfrac{1}{2}\ln|u| + c = -\tfrac{1}{2}\ln|\cos(2x)| + c$$

c Let $u = \sin x$, $\dfrac{du}{dx} = \cos x$

$$\therefore \int e^{\sin x} \cos x\ dx = \int e^u \frac{du}{dx}\, dx = \int e^u\, du = e^u + c = e^{\sin x} + c$$

10 Given: $f''(x) = 2\sin(2x)$, $f'(\frac{\pi}{2}) = 0$, and $f(0) = 3$

Now $f'(x) = \int 2\sin(2x)\ dx$
$= -\cos(2x) + c$

But $f'(\frac{\pi}{2}) = 0$, so $-\cos(\pi) + c = 0$
$\therefore -(-1) + c = 0$
$\therefore c = -1$

$\therefore f'(x) = -\cos(2x) - 1$

$\therefore f(x) = \int(-\cos(2x) - 1)\, dx$
$= -\frac{1}{2}\sin(2x) - x + k$

But $f(0) = 3$, so $-\frac{1}{2}\sin(0) - 0 + k = 3$
$\therefore k = 3$

so $f(x) = -\frac{1}{2}\sin(2x) - x + 3$
$\therefore f(\frac{\pi}{2}) = -\frac{1}{2}\sin(\pi) - \frac{\pi}{2} + 3$
$= 3 - \frac{\pi}{2}$

11 **a** $\int_0^{\frac{\pi}{6}} 4\sin^2\left(\frac{x}{2}\right)\ dx$
$= 4\int_0^{\frac{\pi}{6}} \left(\frac{1}{2} - \frac{1}{2}\cos x\right)\ dx$
$= 4\left[\frac{1}{2}x - \frac{1}{2}\sin x\right]_0^{\frac{\pi}{6}}$
$= 4\left[\frac{\pi}{12} - \frac{1}{2}(\frac{1}{2}) - 0 + 0\right]$
$= 4(\frac{\pi}{12} - \frac{1}{4})$
$= \frac{\pi}{3} - 1$

b Let $u = \sin\theta$, $\dfrac{du}{d\theta} = \cos\theta$

when $\theta = \frac{\pi}{6}$, $\sin\theta = \sin\frac{\pi}{6} = \frac{1}{2}$
when $\theta = \frac{\pi}{2}$, $\sin\theta = \sin\frac{\pi}{2} = 1$

$$\therefore \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cot\theta\ d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos\theta}{\sin\theta}\, d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{u}\frac{du}{d\theta}\, d\theta = \int_{\frac{1}{2}}^{1} \frac{1}{u}\, du = \Big[\ln|u|\Big]_{\frac{1}{2}}^{1} = \ln 1 - \ln\tfrac{1}{2} = \ln 2$$

c Let $u = \tan x$, $\dfrac{du}{dx} = \sec^2 x$

when $x = \frac{\pi}{4}$, $u = \tan\frac{\pi}{4} = 1$

when $x = \frac{\pi}{3}$, $u = \tan\frac{\pi}{3} = \sqrt{3}$

$$\therefore \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2 x}{\tan x}\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{u}\frac{du}{dx}\,dx$$
$$= \int_1^{\sqrt{3}} \frac{1}{u}\,du$$
$$= \Big[\ln|u|\Big]_1^{\sqrt{3}}$$
$$= \ln\sqrt{3} - \ln 1$$
$$= \tfrac{1}{2}\ln 3$$

12 Let $u = 4 - x$, $\dfrac{du}{dx} = -1$

$$\therefore \int x^2\sqrt{4-x}\,dx$$
$$= \int (4-u)^2\sqrt{u}\,(-du)$$
$$= -\int (16 - 8u + u^2)u^{\frac{1}{2}}\,du$$
$$= -\int (16u^{\frac{1}{2}} - 8u^{\frac{3}{2}} + u^{\frac{5}{2}})\,du$$
$$= -\left(\frac{16u^{\frac{3}{2}}}{\frac{3}{2}} - \frac{8u^{\frac{5}{2}}}{\frac{5}{2}} + \frac{u^{\frac{7}{2}}}{\frac{7}{2}}\right) + c$$
$$= -\tfrac{32}{3}u^{\frac{3}{2}} + \tfrac{16}{5}u^{\frac{5}{2}} - \tfrac{2}{7}u^{\frac{7}{2}} + c$$
$$= -\tfrac{32}{3}(4-x)^{\frac{3}{2}} + \tfrac{16}{5}(4-x)^{\frac{5}{2}} - \tfrac{2}{7}(4-x)^{\frac{7}{2}} + c$$

13 $\int \arctan x\,dx = \int 1\arctan x\,dx$

so we integrate by parts with $u = \arctan x$ $\quad v' = 1$

$u' = \dfrac{1}{x^2+1}$ $\quad v = x$

$$\therefore \int \arctan x\,dx$$
$$= x\arctan x - \int \frac{1}{x^2+1}(x)\,dx \quad \left\{\int uv'\,dx = uv - \int u'v\,dx\right\}$$
$$= x\arctan x - \tfrac{1}{2}\int \frac{2x}{x^2+1}\,dx$$
$$= x\arctan x - \tfrac{1}{2}\ln\left|x^2+1\right| + c$$
$$= x\arctan x - \tfrac{1}{2}\ln(x^2+1) + c \quad \{x^2+1>0\}$$

Check: $\dfrac{d}{dx}\left(x\arctan x - \tfrac{1}{2}\ln(x^2+1) + c\right)$

$$= \arctan x + \frac{x}{x^2+1} - \tfrac{1}{2}\,\frac{2x}{x^2+1}$$
$$= \arctan x \quad \checkmark$$

14 **a** If $u(x) = x^2 + 1$, then $\dfrac{du}{dx} = 2x$

$$\therefore \int 2x(x^2+1)^3\,dx = \int \frac{du}{dx}\,u^3\,dx$$
$$= \int u^3\,du$$
$$= \frac{u^4}{4} + c$$
$$= \tfrac{1}{4}(x^2+1)^4 + c$$

b **i** $\int_0^1 2x(x^2+1)^3\,dx$

$$= \left[\frac{(x^2+1)^4}{4}\right]_0^1 \quad \{\text{using } \mathbf{a}\}$$
$$= \frac{(1^2+1)^4}{4} - \frac{(0+1)^4}{4}$$
$$= \tfrac{16}{4} - \tfrac{1}{4}$$
$$= \tfrac{15}{4}$$

ii $\int_{-1}^2 -x(1+x^2)^3\,dx$

$$= \int_{-1}^2 -\tfrac{1}{2} \times 2x(1+x^2)^3\,dx$$
$$= -\tfrac{1}{2}\int_{-1}^2 2x(x^2+1)^3\,dx$$
$$= -\tfrac{1}{2}\left[\frac{(x^2+1)^4}{4}\right]_{-1}^2$$
$$= -\tfrac{1}{2}\left[\frac{(4+1)^4}{4} - \frac{(1+1)^4}{4}\right]$$
$$= -\tfrac{1}{2}\left(\tfrac{625}{4} - \tfrac{16}{4}\right)$$
$$= -\tfrac{609}{8}$$

15 **a** $\dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{x-1}$

$$= \frac{A(x^2-1) + Bx(x-1) + Cx(x+1)}{x(x+1)(x-1)}$$
$$= \frac{Ax^2 - A + Bx^2 - Bx + Cx^2 + Cx}{x(x^2-1)}$$
$$= \frac{x^2(A+B+C) + x(C-B) - A}{x(x^2-1)}$$
$$= \frac{x^2(-A-B-C) + x(B-C) + A}{x(1-x^2)}$$

So, if $\dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{x-1} = \dfrac{4}{x(1-x^2)}$, then

$A = 4$, $B - C = 0$ and $-A - B - C = 0$

$\therefore$ $B = C$

$\therefore$ $-4 - 2B = 0$

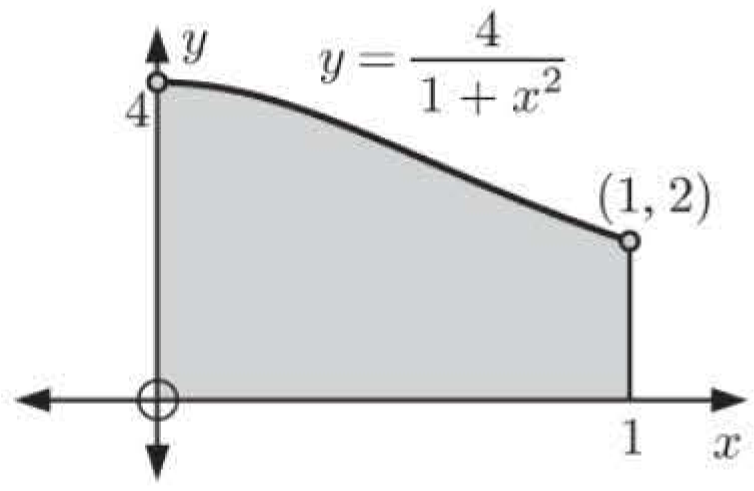
So, $A = 4$, $B = -2$, $C = -2$

b $$\int \frac{4}{x(1-x^2)}\,dx = \int \left(\frac{4}{x} - \frac{2}{x+1} - \frac{2}{x-1}\right) dx$$
$$= 4\ln|x| - 2\ln|x+1| - 2\ln|x-1| + c$$

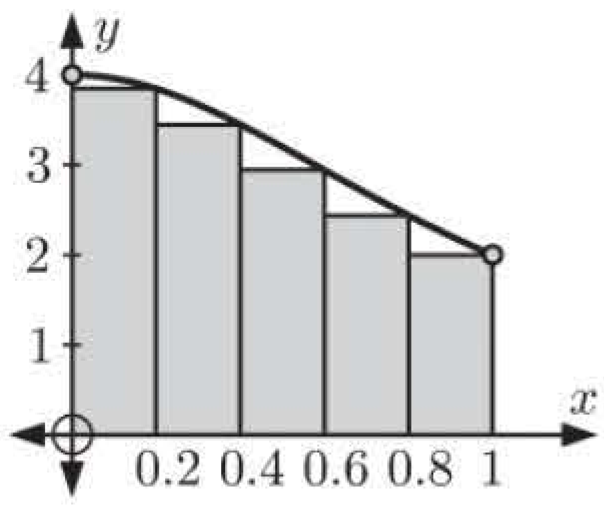
c $$\int_2^4 \frac{4}{x(1-x^2)}\,dx = [4\ln|x| - 2\ln|x+1| - 2\ln|x-1|]_2^4$$
$$= 4\ln 4 - 2\ln 5 - \cancel{2\ln 3} - 4\ln 2 + \cancel{2\ln 3} + 2\ln 1$$
$$= \ln\left(\frac{4^4}{5^2 \times 2^4}\right) = \ln\left(\tfrac{16}{25}\right)$$

REVIEW SET 21B

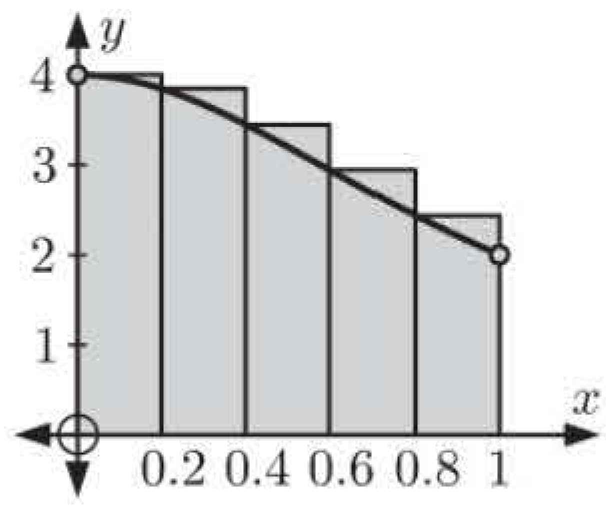
1 **a**



lower rectangles



upper rectangles



b

n	A_L	A_U
5	2.9349	3.3349
50	3.1215	3.1615
100	3.1316	3.1516
500	3.1396	3.1436

c $\displaystyle\int_0^1 \frac{4}{1+x^2}\,dx \approx 3.1416$

(the average of A_L and A_U for $n = 500$). This value agrees with π to 4 decimal places.

2 **a** $\dfrac{dy}{dx} = (x^2-1)^2$

$$\begin{aligned} \therefore\ y &= \int (x^2-1)^2\,dx \\ &= \int (x^4 - 2x^2 + 1)\,dx \\ &= \tfrac{1}{5}x^5 - \tfrac{2}{3}x^3 + x + c \end{aligned}$$

b $\dfrac{dy}{dx} = 400 - 20e^{-\frac{x}{2}}$

$$\begin{aligned} \therefore\ y &= \int (400 - 20e^{-\frac{x}{2}})\,dx \\ &= 400x - \frac{20e^{-\frac{x}{2}}}{-\frac{1}{2}} + c \\ &= 400x + 40e^{-\frac{x}{2}} + c \end{aligned}$$

c
$$\begin{aligned} \frac{dy}{dx} &= x(x^2-1)^2 \\ &= x^5 - 2x^3 + x \\ \therefore\ y &= \int (x^5 - 2x^3 + x)\,dx \\ &= \tfrac{1}{6}x^6 - \tfrac{2}{4}x^4 + \tfrac{1}{2}x^2 + c \\ &= \tfrac{1}{6}x^6 - \tfrac{1}{2}x^4 + \tfrac{1}{2}x^2 + c \end{aligned}$$

3 Using technology: **a** $\int_{-2}^{0} 4e^{-x^2}\,dx \approx 3.528$ **b** $\displaystyle\int_0^1 \frac{10x}{\sqrt{3x+1}}\,dx \approx 2.963$

4
$$\begin{aligned} \frac{d}{dx}(\ln x)^2 &= 2(\ln x)^1\left(\frac{1}{x}\right) \\ &= \frac{2\ln x}{x} \\ \therefore\ \int \frac{2\ln x}{x}\,dx &= (\ln x)^2 + c \\ \therefore\ \int \frac{\ln x}{x}\,dx &= \tfrac{1}{2}(\ln x)^2 + c \end{aligned}$$

5 Given: $f''(x) = 18x + 10$, $f(0) = -1$, $f(1) = 13$

$$\begin{aligned} f'(x) &= \int (18x+10)\,dx \\ &= 9x^2 + 10x + c \\ \therefore\ f(x) &= 3x^3 + 5x^2 + cx + d \end{aligned}$$

But $f(0) = -1$ so $d = -1$

$\therefore\ f(x) = 3x^3 + 5x^2 + cx - 1$

And $f(1) = 13$ so $3 + 5 + c - 1 = 13$

$\therefore\ c + 7 = 13$

$\therefore\ c = 6$

$\therefore\ f(x) = 3x^3 + 5x^2 + 6x - 1$

6
$$\int_0^a e^{1-2x}\,dx = \frac{e}{4}$$
$$\therefore\quad \left[\tfrac{1}{-2}e^{1-2x}\right]_0^a = \frac{e}{4}$$
$$\therefore\quad (-\tfrac{1}{2}e^{1-2a}) - (-\tfrac{1}{2}e^1) = \frac{e}{4}$$
$$\therefore\quad -\tfrac{1}{2}e^{1-2a} + \frac{e}{2} = \frac{e}{4}$$
$$\therefore\quad \tfrac{1}{2}e^{1-2a} = \frac{e}{4}$$
$$\therefore\quad e^{1-2a} = \frac{e}{2}$$
$$\therefore\quad 1-2a = \ln\left(\frac{e}{2}\right) = \ln e - \ln 2$$
$$\therefore\quad 1-2a = 1 - \ln 2$$
$$\therefore\quad 2a = \ln 2$$
$$\therefore\quad a = \tfrac{1}{2}\ln 2$$
$$\therefore\quad a = \ln 2^{\frac{1}{2}}$$
$$\therefore\quad a = \ln\sqrt{2}$$

7 **a** Let $y = \sqrt{2x+1} = (2x+1)^{\frac{1}{2}}$
$$\therefore\quad \frac{dy}{dx} = \cancel{\tfrac{1}{2}}(2x+1)^{-\frac{1}{2}} \times \cancel{2} = \frac{1}{\sqrt{2x+1}}$$
$$\therefore\quad \int \frac{1}{\sqrt{2x+1}}\,dx = \sqrt{2x+1} + c$$
$$\therefore\quad \int_3^4 \frac{1}{\sqrt{2x+1}}\,dx = \left[\sqrt{2x+1}\right]_3^4$$
$$= \sqrt{9} - \sqrt{7}$$
$$= 3 - \sqrt{7}$$
$$\approx 0.354\,248\,688\,9$$

b Let $u = x^2$, $v' = e^{x+1}$
$\therefore\ u' = 2x \qquad v = e^{x+1}$
$$\therefore\quad \int x^2e^{x+1}\,dx = x^2e^{x+1} - \int 2xe^{x+1}\,dx \quad \{\text{using } \int uv'\,dx = uv - \int u'v\,dx\}$$
$$= x^2e^{x+1} - 2\int xe^{x+1}\,dx$$
Let $u = x$, $v' = e^{x+1}$
$\therefore\ u' = 1 \qquad v = e^{x+1}$
$$\therefore\quad \int xe^{x+1}\,dx = xe^{x+1} - \int 1 \times e^{x+1}\,dx$$
$$= xe^{x+1} - e^{x+1} + c$$
$$\therefore\quad \int x^2e^{x+1}\,dx = x^2e^{x+1} - 2(xe^{x+1} - e^{x+1} + c)$$
$$= x^2e^{x+1} - 2xe^{x+1} + 2e^{x+1} + c$$
$$\therefore\quad \int_0^1 x^2e^{x+1}\,dx = \left[x^2e^{x+1} - 2xe^{x+1} + 2e^{x+1}\right]_0^1$$
$$= (e^2 - \cancel{2e^2} + \cancel{2e^2}) - (0e^1 - 0e^1 + 2e^1)$$
$$= e^2 - 2e^1$$
$$\approx 1.952\,492\,442$$

8 **a**
$$f''(x) = 3x^2 + 2x$$
$$\therefore\ f'(x) = \frac{3x^3}{3} + \frac{2x^2}{2} + c$$
$$= x^3 + x^2 + c$$
$$\therefore\ f(x) = \frac{x^4}{4} + \frac{x^3}{3} + cx + d$$
But $f(0) = 3$ so $d = 3$
$$\therefore\ f(x) = \frac{x^4}{4} + \frac{x^3}{3} + cx + 3$$
Also, $f(2) = 3$ so $4 + \frac{8}{3} + 2c + 3 = 3$
$$\therefore\ \tfrac{20}{3} = -2c$$
$$\therefore\ c = -\tfrac{10}{3}$$
$$\therefore\ f(x) = \tfrac{1}{4}x^4 + \tfrac{1}{3}x^3 - \tfrac{10}{3}x + 3$$

b Now $f'(2) = 2^3 + 2^2 - \frac{10}{3}$
$$= 12 - \tfrac{10}{3}$$
$$= \tfrac{26}{3}$$
$\therefore$ the normal has gradient $-\frac{3}{26}$

$\therefore$ equation is $\dfrac{y-3}{x-2} = -\frac{3}{26}$
$$\therefore\ y - 3 = -\tfrac{3}{26}(x-2)$$
$$\therefore\ y = -\tfrac{3}{26}x + \tfrac{6}{26} + 3$$
or $3x + 26y = 84$

9 **a**
$$(e^x + 2)^3$$
$$= (e^x)^3 + 3(e^x)^2(2) + 3(e^x)(2)^2 + (2)^3$$
$$= e^{3x} + 6e^{2x} + 12e^x + 8$$

b
$$\int_0^1 (e^x + 2)^3\, dx$$
$$= \left[\tfrac{1}{3}e^{3x} + 3e^{2x} + 12e^x + 8x\right]_0^1$$
$$= \left(\tfrac{1}{3}e^3 + 3e^2 + 12e + 8\right) - \left(\tfrac{1}{3} + 3 + 12\right)$$
$$= \tfrac{1}{3}e^3 + 3e^2 + 12e - 7\tfrac{1}{3}$$
$$\approx 54.148\,395\,88$$

10
$$\int \sin^5 x \cos x\, dx$$
$$= \int (\sin x)^5 \cos x\, dx$$
$$= \int u^5 \frac{du}{dx}\, dx \qquad \left\{u = \sin x,\ \frac{du}{dx} = \cos x\right\}$$
$$= \int u^5\, du$$
$$= \frac{u^6}{6} + c$$
$$= \frac{\sin^6 x}{6} + c$$
$$\therefore\ \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin^5 x \cos x\, dx$$
$$= \left[\frac{\sin^6 x}{6}\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$
$$= \tfrac{1}{6}\left(\left(\tfrac{\sqrt{3}}{2}\right)^6 - \left(\tfrac{1}{\sqrt{2}}\right)^6\right)$$
$$= \tfrac{19}{384}$$
$$\approx 0.049\,479\,166\,67$$

11
$$f''(x) = 4x^2 - 3$$
$$\therefore\ f'(x) = \frac{4x^3}{3} - 3x + c$$
But $f'(0) = 6$ so $c = 6$
$$\therefore\ f'(x) = \frac{4x^3}{3} - 3x + 6$$
$$\therefore\ f(x) = \tfrac{4}{3}\,\frac{x^4}{4} - \frac{3x^2}{2} + 6x + d$$
$$= \tfrac{1}{3}x^4 - \tfrac{3}{2}x^2 + 6x + d$$
But $f(2) = 3$, so $\frac{16}{3} - 6 + 12 + d = 3$
$$\therefore\ d = -3 - \tfrac{16}{3} = -\tfrac{25}{3}$$
$$\therefore\ f(x) = \tfrac{1}{3}x^4 - \tfrac{3}{2}x^2 + 6x - \tfrac{25}{3}$$
and $f(3) = 27 - \frac{27}{2} + 18 - \frac{25}{3}$
$$= 23\tfrac{1}{6}$$

12 If $y = x\tan x$ then $\dfrac{dy}{dx} = \tan x + x\sec^2 x$
$$\therefore\ \int (\tan x + x\sec^2 x)\, dx = x\tan x + c$$
$$\therefore\ \int \tan x\, dx + \int x\sec^2 x\, dx = x\tan x + c$$
$$\therefore\ -\ln|\cos x| + \int x\sec^2 x\, dx = x\tan x + c$$ {see **Ex 21G.1**, Q **6 c**}
$$\therefore\ \int x\sec^2 x\, dx = x\tan x + \ln|\cos x| + c$$

13 **a** $\dfrac{4x-3}{2x+1}=\dfrac{2(2x+1)-5}{2x+1}$

$=2+\dfrac{-5}{2x+1}$

$\therefore\ A=2,\ \ B=-5$

b $\displaystyle\int_0^2 \frac{4x-3}{2x+1}\,dx=\int_0^2\left(2-5\left(\frac{1}{2x+1}\right)\right)dx$

$=\left[2x-5\left(\frac{1}{2}\right)\ln|2x+1|\right]_0^2$

$=[4-\frac{5}{2}\ln 5]-[0-\frac{5}{2}\ln 1]$

$=4-\frac{5}{2}\ln 5$

≈ -0.0236

14 **a** We integrate by parts with $u=e^{-x}$ $\quad v'=\cos x$
$u'=-e^{-x}$ $\quad v=\sin x$

$\therefore\ \int e^{-x}\cos x\,dx=e^{-x}\sin x-\int -e^{-x}\sin x\,dx$
$=e^{-x}\sin x+\int e^{-x}\sin x\,dx$

We integrate by parts again, this time with $u=e^{-x}$ $\quad v'=\sin x$
$u'=-e^{-x}$ $\quad v=-\cos x$

$\therefore\ \int e^{-x}\cos x\,dx=e^{-x}\sin x+e^{-x}(-\cos x)-\int(-e^{-x})(-\cos x)\,dx+c$
$=e^{-x}\sin x-e^{-x}\cos x-\int e^{-x}\cos x\,dx+c$

$\therefore\ 2\int e^{-x}\cos x\,dx=e^{-x}(\sin x-\cos x)+c$

$\therefore\ \int e^{-x}\cos x\,dx=\frac{1}{2}e^{-x}(\sin x-\cos x)+c$

b We integrate by parts with $u=x^2$ $\quad v'=e^x$
$u'=2x$ $\quad v=e^x$

$\therefore\ \int x^2e^x\,dx=x^2e^x-\int 2xe^x\,dx$

We integrate by parts again, this time with $u=2x$ $\quad v'=e^x$
$u'=2$ $\quad v=e^x$

$\therefore\ \int x^2e^x\,dx=x^2e^x-\left[2xe^x-\int 2e^x\,dx\right]$
$=x^2e^x-2xe^x+2\int e^x\,dx$
$=x^2e^x-2xe^x+2e^x+c$
$=e^x(x^2-2x+2)+c$

c Let $u=9-x^2,\ \ \dfrac{du}{dx}=-2x$

$$\therefore\ \int\frac{x^3}{\sqrt{9-x^2}}\,dx=-\frac{1}{2}\int\frac{x^2}{\sqrt{9-x^2}}(-2x)\,dx$$
$$=-\frac{1}{2}\int\frac{9-u}{\sqrt{u}}\,\frac{du}{dx}\,dx$$
$$=-\frac{1}{2}\int\frac{9-u}{\sqrt{u}}\,du$$
$$=-\frac{1}{2}\int\left(9u^{-\frac{1}{2}}-u^{\frac{1}{2}}\right)du$$
$$=-\frac{1}{2}\left[\frac{9u^{\frac{1}{2}}}{\frac{1}{2}}-\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]+c$$
$$=-9u^{\frac{1}{2}}+\frac{1}{3}u^{\frac{3}{2}}+c$$
$$=-9\sqrt{9-x^2}+\frac{1}{3}\left(9-x^2\right)^{\frac{3}{2}}+c$$

15 **a** $\displaystyle\int \frac{1}{x+2}\,dx - \int \frac{2}{x-1}\,dx = \ln|x+2| - 2\ln|x-1| + c = \ln\left(\frac{|x+2|}{(x-1)^2}\right) + c$

b $$\frac{1}{x+2} - \frac{2}{x-1} = \frac{(x-1)-2(x+2)}{(x+2)(x-1)} = \frac{-x-5}{(x+2)(x-1)} = -\frac{x+5}{(x+2)(x-1)}$$

$$\therefore \int \frac{x+5}{(x+2)(x-1)}\,dx = -\left[\int \frac{1}{x+2}\,dx - \int \frac{2}{x-1}\,dx\right] = -\ln\left(\frac{|x+2|}{(x-1)^2}\right) + c = \ln\left(\frac{(x-1)^2}{|x+2|}\right) + c$$

REVIEW SET 21C

1 **a** $$\int \left(2e^{-x} - \frac{1}{x} + 3\right) dx = -2e^{-x} - \ln|x| + 3x + c$$

b $$\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx = \int \left(x - 2 + \frac{1}{x}\right) dx = \tfrac{1}{2}x^2 - 2x + \ln|x| + c$$

c $$\int (3 + e^{2x-1})^2\,dx = \int (9 + 6e^{2x-1} + e^{4x-2})\,dx = 9x + 3e^{2x-1} + \tfrac{1}{4}e^{4x-2} + c$$

2 $$f'(x) = x^2 - 3x + 2$$
$$\therefore \quad f(x) = \frac{x^3}{3} - \frac{3x^2}{2} + 2x + c$$

But $f(1) = 3$ so $\frac{1}{3} - \frac{3}{2} + 2 + c = 3$
$$\therefore \quad c = 1 - \tfrac{1}{3} + 1\tfrac{1}{2}$$
$$\therefore \quad c = 2\tfrac{1}{6}$$
$$\therefore \quad f(x) = \tfrac{1}{3}x^3 - \tfrac{3}{2}x^2 + 2x + 2\tfrac{1}{6}$$

3 **a** $$\int_2^3 \frac{1}{\sqrt{3x-4}}\,dx = \int_2^3 (3x-4)^{-\frac{1}{2}}\,dx = \left[\tfrac{1}{3}\frac{(3x-4)^{\frac{1}{2}}}{\frac{1}{2}}\right]_2^3 = \left[\tfrac{2}{3}\sqrt{3x-4}\right]_2^3 = \tfrac{2}{3}\sqrt{5} - \tfrac{2}{3}\sqrt{2} = \tfrac{2}{3}(\sqrt{5} - \sqrt{2})$$

b $$\int_0^{\frac{\pi}{3}} \cos^2\left(\tfrac{x}{2}\right) dx = \int_0^{\frac{\pi}{3}} \left(\tfrac{1}{2} + \tfrac{1}{2}\cos x\right) dx = \left[\tfrac{1}{2}x + \tfrac{1}{2}\sin x\right]_0^{\frac{\pi}{3}} = \tfrac{\pi}{6} + \tfrac{1}{2}\left(\tfrac{\sqrt{3}}{2}\right) - 0 - 0 = \tfrac{\pi}{6} + \tfrac{\sqrt{3}}{4}$$

c Let $u = \cos x$, $\dfrac{du}{dx} = -\sin x$

when $x = 0$, $u = \cos 0 = 1$

when $x = \frac{\pi}{4}$, $u = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$$\therefore \int_0^{\frac{\pi}{4}} \tan x\,dx = \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x}\,dx = -\int_0^{\frac{\pi}{4}} \frac{-\sin x}{\cos x}\,dx = -\int_0^{\frac{\pi}{4}} \frac{1}{u}\frac{du}{dx}\,dx = -\int_1^{\frac{1}{\sqrt{2}}} \frac{1}{u}\,du = \int_{\frac{1}{\sqrt{2}}}^1 \frac{1}{u}\,du = [\ln|u|]_{\frac{1}{\sqrt{2}}}^1 = \ln 1 - \ln\tfrac{1}{\sqrt{2}} = \tfrac{1}{2}\ln 2$$

4 $\dfrac{d}{dx}(e^{-2x}\sin x) = -2e^{-2x}\sin x + e^{-2x}\cos x$ {product rule}

$= e^{-2x}(\cos x - 2\sin x)$

$\therefore \int_0^{\frac{\pi}{2}} e^{-2x}(\cos x - 2\sin x)\,dx = \left[e^{-2x}\sin x\right]_0^{\frac{\pi}{2}}$

$= e^{-\pi}(1) - e^0(0) = e^{-\pi}$

5 If $n \neq -1$, $\int (2x+3)^n\,dx = \frac{1}{2}\dfrac{(2x+3)^{n+1}}{n+1} + c = \dfrac{1}{2(n+1)}(2x+3)^{n+1} + c$

If $n = -1$, $\int (2x+3)^{-1}\,dx = \displaystyle\int \frac{1}{2x+3}\,dx = \frac{1}{2}\ln|2x+3| + c$

So, $$\int (2x+3)^n\,dx = \begin{cases} \dfrac{1}{2(n+1)}(2x+3)^{n+1} + c & \text{if } n \neq -1 \\ \frac{1}{2}\ln|2x+3| + c & \text{if } n = -1 \end{cases}$$

6 $f'(x) = 2\sqrt{x} + \dfrac{a}{\sqrt{x}}$

$= 2x^{\frac{1}{2}} + ax^{-\frac{1}{2}}$

$\therefore f(x) = \frac{4}{3}x^{\frac{3}{2}} + 2ax^{\frac{1}{2}} + c$

$= \dfrac{4x\sqrt{x}}{3} + 2a\sqrt{x} + c$

Now $f(0) = 2$ so $c = 2$

$\therefore f(x) = \dfrac{4x\sqrt{x}}{3} + 2a\sqrt{x} + 2$

Also, $f(1) = 4$ so $\frac{4}{3} + 2a + 2 = 4$

$\therefore 2a = \frac{2}{3}$

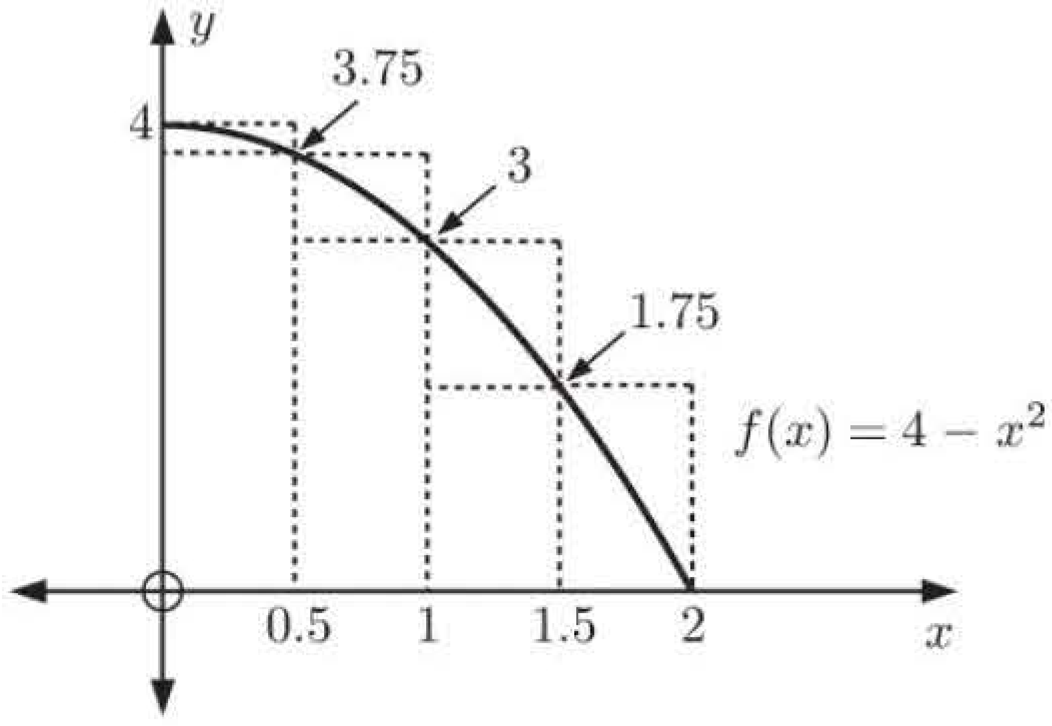
$\therefore a = \frac{1}{3}$

$\therefore f'(x) = 2\sqrt{x} + \dfrac{1}{3\sqrt{x}} = \dfrac{6x+1}{3\sqrt{x}}$

Now $f(x)$ is only defined for $x > 0$,
so $f'(x) > 0$ for all x in the domain.
$\therefore$ the function has no stationary points.

7
$$\int_a^{2a} (x^2 + ax + 2)\,dx = \frac{73a}{2}$$
$$\therefore \left[\frac{x^3}{3} + \frac{ax^2}{2} + 2x\right]_a^{2a} = \frac{73a}{2}$$
$$\therefore \left(\frac{8a^3}{3} + \frac{a}{2}(4a^2) + 4a\right) - \left(\frac{a^3}{3} + \frac{a^3}{2} + 2a\right) = \frac{73a}{2}$$
$$\frac{8a^3}{3} + 2a^3 + 4a - \frac{a^3}{3} - \frac{a^3}{2} - 2a = \frac{73a}{2}$$
$$\therefore 16a^3 + 12a^3 + 24a - 2a^3 - 3a^3 - 12a = 219a$$
$$\therefore 23a^3 - 207a = 0$$
$$\therefore 23a(a^2 - 9) = 0$$
$$\therefore 23a(a+3)(a-3) = 0$$
$$\therefore a = 0 \text{ or } a = \pm 3$$

8 **a**



$A_U = 0.5\,[f(0) + f(0.5) + f(1) + f(1.5)]$
$= 0.5(4 + 3.75 + 3 + 1.75)$
$= 6.25$

$A_L = 0.5\,[f(0.5) + f(1) + f(1.5) + f(2)]$
$= 0.5(3.75 + 3 + 1.75 + 0)$
$= 4.25$

$\therefore 4.25 < \int_0^2 (4 - x^2)\,dx < 6.25$

$\therefore A = 4.25 = \frac{17}{4}$, $B = 6.25 = \frac{25}{4}$

b An estimate of $\int_0^2 (4-x^2)\,dx \approx \dfrac{A+B}{2} \approx \frac{42}{8} \approx \frac{21}{4}$.

9 **a**
$$\begin{aligned}
&\int \frac{2x}{\sqrt{x^2-5}}\,dx\\
&= \int 2x(x^2-5)^{-\frac{1}{2}}\,dx\\
&= \int u^{-\frac{1}{2}}\frac{du}{dx}\,dx \quad \left\{u = x^2-5,\ \frac{du}{dx} = 2x\right\}\\
&= \int u^{-\frac{1}{2}}\,du\\
&= \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c\\
&= 2\sqrt{u} + c\\
&= 2\sqrt{x^2-5} + c
\end{aligned}$$

b Let $u = \cos x$, $\dfrac{du}{dx} = -\sin x$

$$\begin{aligned}
\therefore \int \frac{\sin x}{\cos^4 x}\,dx &= -\int \frac{-\sin x}{\cos^4 x}\,dx\\
&= -\int u^{-4}\frac{du}{dx}\,dx\\
&= -\int u^{-4}\,du\\
&= \frac{u^{-3}}{3} + c = \frac{1}{3\cos^3 x} + c
\end{aligned}$$

c
$$\begin{aligned}
&\int 3x^2\sqrt{x^3-1}\,dx\\
&= \int 3x^2(x^3-1)^{\frac{1}{2}}\,dx\\
&= \int u^{\frac{1}{2}}\frac{du}{dx}\,dx \quad \left\{u = x^3-1,\ \frac{du}{dx} = 3x^2\right\}\\
&= \int u^{\frac{1}{2}}\,du\\
&= \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c\\
&= \tfrac{2}{3}u\sqrt{u} + c\\
&= \tfrac{2}{3}(x^3-1)\sqrt{x^3-1} + c
\end{aligned}$$

$$\begin{aligned}
\therefore\ &\int_1^2 3x^2\sqrt{x^3-1}\,dx\\
&= \tfrac{2}{3}\left[(x^3-1)\sqrt{x^3-1}\right]_1^2\\
&= \tfrac{2}{3}\left((8-1)\sqrt{8-1} - 0\right)\\
&= \tfrac{2}{3} \times 7\sqrt{7}\\
&= \tfrac{14\sqrt{7}}{3}
\end{aligned}$$

10
$$\begin{aligned}
&y = \ln(\sec x), \quad \sec x > 0\\
\therefore\ &\frac{dy}{dx} = \frac{\sec x \tan x}{\sec x} = \tan x\\
\therefore\ &\int \tan x\,dx = \ln(\sec x) + c, \quad \sec x > 0
\end{aligned}$$

11 **a**
$$\begin{aligned}
&\int \frac{5}{\sqrt{9-x^2}}\,dx\\
&= 5\int \frac{1}{\sqrt{3^2-x^2}}\,dx\\
&= 5\arcsin\left(\tfrac{x}{3}\right) + c
\end{aligned}$$

{using $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\tfrac{x}{a}\right) + c,\ a \neq 0$ from **Ex 21G.1**, Q **11 b**}

b
$$\begin{aligned}
&\int \frac{1}{9+4x^2}\,dx\\
&= \tfrac{1}{4}\int \frac{1}{\frac{9}{4}+x^2}\,dx\\
&= \tfrac{1}{4}\left(\frac{1}{\frac{3}{2}}\right)\arctan\left(\frac{x}{\frac{3}{2}}\right) + c
\end{aligned}$$

{using $\displaystyle\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\tfrac{x}{a}\right) + c,\ a \neq 0$ from **Ex 21G.1**, Q **12 b**}

$$= \tfrac{1}{6}\arctan\left(\tfrac{2x}{3}\right) + c$$

c Let $\quad u = x - 5, \quad \dfrac{du}{dx} = 1$

when $\quad x = 10, \quad u = 5, \quad$ and when $\quad x = 7, \quad u = 2$

$$\begin{aligned}
\therefore \quad \int_7^{10} x\sqrt{x-5}\, dx &= \int_2^5 (u+5)\sqrt{u}\, du \\
&= \int_2^5 (u^{\frac{3}{2}} + 5u^{\frac{1}{2}})\, du \\
&= \left[\frac{u^{\frac{5}{2}}}{\frac{5}{2}} + \frac{5u^{\frac{3}{2}}}{\frac{3}{2}} \right]_2^5 \\
&= \tfrac{2}{5}\left(5^{\frac{5}{2}}\right) + \tfrac{10}{3}\left(5^{\frac{3}{2}}\right) - \left[\tfrac{2}{5}\left(2^{\frac{5}{2}}\right) + \tfrac{10}{3}\left(2^{\frac{3}{2}}\right)\right] \\
&= \tfrac{2}{5}\left(25\sqrt{5}\right) + \tfrac{10}{3}\left(5\sqrt{5}\right) - \tfrac{2}{5}\left(4\sqrt{2}\right) - \tfrac{10}{3}\left(2\sqrt{2}\right) \\
&= 10\sqrt{5} + \tfrac{50}{3}\sqrt{5} - \tfrac{8}{5}\sqrt{2} - \tfrac{20}{3}\sqrt{2} \\
&= \tfrac{80}{3}\sqrt{5} - \tfrac{124}{15}\sqrt{2}
\end{aligned}$$

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$$\begin{array}{r|rrr|r}
2 & 1 & 0 & -3 & 2 \\
 & 0 & 2 & 4 & 2 \\
\hline
 & 1 & 2 & 1 & 4
\end{array}$$

$$\therefore \quad \frac{x^3 - 3x + 2}{x - 2} = x^2 + 2x + 1 + \frac{4}{x-2}$$

$$\therefore \quad A = 1, \ B = 2, \ C = 1, \ D = 4$$

$$\begin{aligned}
\therefore \quad & \int \frac{x^3 - 3x + 2}{x - 2}\, dx \\
&= \int \left(x^2 + 2x + 1 + \frac{4}{x-2} \right) dx \\
&= \frac{x^3}{3} + \frac{2x^2}{2} + x + 4\ln|x-2| + c \\
&= \tfrac{1}{3}x^3 + x^2 + x + 4\ln|x-2| + c
\end{aligned}$$

or

$$\begin{aligned}
& \int \frac{x^3 - 3x + 2}{x - 2}\, dx \\
&= \int \left((x+1)^2 + \frac{4}{x-2} \right) dx \\
&= \frac{(x+1)^3}{3} + 4\ln|x-2| + c
\end{aligned}$$

13 **a** We integrate by parts with

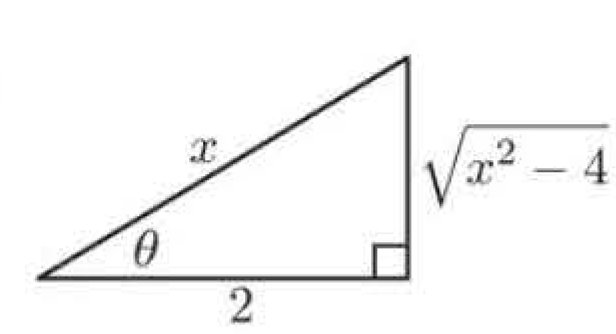
$$\begin{aligned}
u &= x & v' &= \cos x \\
u' &= 1 & v &= \sin x
\end{aligned}$$

Using $\int uv'\, dx = uv - \int u'v\, dx$,

$$\begin{aligned}
\therefore \quad & \int x \cos x\, dx \\
&= x\sin x - \int \sin x\, dx \\
&= x\sin x - (-\cos x) + c \\
&= x\sin x + \cos x + c
\end{aligned}$$

b Let $\quad x = 2\sec\theta, \quad \dfrac{dx}{d\theta} = 2\sec\theta\tan\theta$

$$\begin{aligned}
\therefore \quad & \int \frac{\sqrt{x^2 - 4}}{x}\, dx \\
&= \int \frac{\sqrt{4\sec^2\theta - 4}}{2\sec\theta} \times 2\sec\theta\tan\theta\, d\theta \\
&= \int \sqrt{4(\sec^2\theta - 1)}\ \tan\theta\, d\theta \\
&= \int 2\tan\theta\tan\theta\, d\theta \\
&= 2\int \tan^2\theta\, d\theta \\
&= 2\int (\sec^2\theta - 1)\, d\theta \\
&= 2[\tan\theta - \theta] + c \\
&= 2\tan\theta - 2\theta + c \\
&= 2\left(\frac{\sqrt{x^2-4}}{2}\right) - 2\arccos\left(\tfrac{2}{x}\right) + c \\
&= \sqrt{x^2 - 4} - 2\arccos\left(\tfrac{2}{x}\right) + c
\end{aligned}$$



14
$$\frac{d}{dx}\left(\frac{e^{1-x}}{x^2}\right) = \frac{-e^{1-x}x^2 - e^{1-x}(2x)}{x^4} \quad \{\text{quotient rule}\}$$
$$= -\frac{e^{1-x}(x+2)}{x^3}$$
$$\therefore \quad \int_1^2 \frac{e^{1-x}(x+2)}{x^3}\,dx = -\int_1^2 -\frac{e^{1-x}(x+2)}{x^3}\,dx$$
$$= -\left[\frac{e^{1-x}}{x^2}\right]_1^2$$
$$= -\left(\frac{e^{-1}}{4} - \frac{e^0}{1}\right)$$
$$= 1 - \frac{1}{4e}$$

15 Let $u = \cos x$, $\dfrac{du}{dx} = -\sin x$

$$\therefore \quad \int \frac{\sin x}{\sqrt{\cos^n x}}\,dx = -\int \frac{-\sin x}{\sqrt{\cos^n x}}\,dx$$
$$= -\int \frac{1}{\sqrt{u^n}}\frac{du}{dx}\,dx$$
$$= -\int u^{-\frac{n}{2}}\,du$$
$$= -\frac{u^{1-\frac{n}{2}}}{1-\frac{n}{2}} + c \quad \text{provided } n \neq 2$$
$$= \frac{\cos^{1-\frac{n}{2}} x}{\frac{n}{2}-1} + c, \quad \text{for } n \neq 2$$

If $n = 2$,
$$\int \frac{\sin x}{\sqrt{\cos^2 x}}\,dx = \int \frac{\sin x}{\cos x}\,dx$$
$$= \int \tan x\,dx$$
$$= -\ln|\cos x| + c, \quad \text{for } n = 2 \qquad \{\text{see } \textbf{Ex 21G.1} \text{ Q } \textbf{6 c}\}$$

So, the integral is defined for all n.