Chapter 7
SEQUENCES AND SERIES

EXERCISE 7A

1 **a** 4, 13, 22, 31 **b** 45, 39, 33, 27 **c** 2, 6, 18, 54 **d** 96, 48, 24, 12

2 **a** The sequence starts at 8 and each term is 8 more than the previous term. The next two terms are 40 and 48.

b The sequence starts at 2 and each term is 3 more than the previous term. The next two terms are 14 and 17.

c The sequence starts at 36 and each term is 5 less than the previous term. The next two terms are 16 and 11.

d The sequence starts at 96 and each term is 7 less than the previous term. The next two terms are 68 and 61.

e The sequence starts at 1 and each term is 4 times the previous term. The next two terms are 256 and 1024.

f The sequence starts at 2 and each term is 3 times the previous term. The next two terms are 162 and 486.

g The sequence starts at 480 and each term is half the previous term. The next two terms are 30 and 15.

h The sequence starts at 243 and each term is one third of the previous term. The next two terms are 3 and 1.

i The sequence starts at 50 000 and each term is one fifth of the previous term. The next two terms are 80 and 16.

3 **a** Each term is the square of the number of the term. The next three terms are 25, 36, and 49.

b Each term is the cube of the number of the term. The next three terms are 125, 216, and 343.

c Each term is $n \times (n+1)$ where n is the number of the term. The next three terms are 30, 42, and 56.

4 **a** 79, 75 (subtracting 4 each time) **b** 1280, 5120 (multiplying by 4 each time)

c 625, 1296 $(1^4, 2^4, 3^4, 4^4, \ldots.)$ **d** 13, 17 (prime numbers)

e 16, 22 (the difference between terms increases by 1) **f** 6, 12 $(-1, +2, -3, +4, \ldots.)$

EXERCISE 7B.1

1 **a** $u_1 = 3(1) - 2$
$= 3 - 2$
$= 1$

b $u_5 = 3(5) - 2$
$= 15 - 2$
$= 13$

c $u_{27} = 3(27) - 2$
$= 81 - 2$
$= 79$

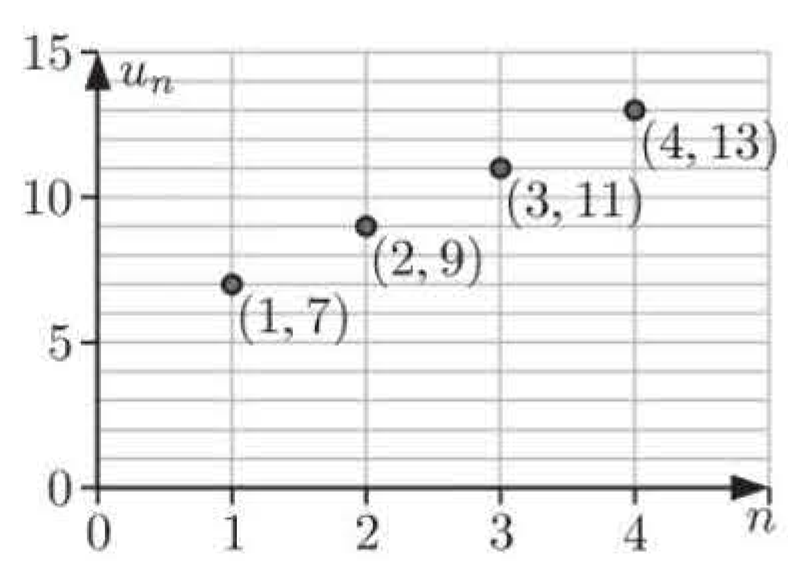
2 **a** $u_1 = 2(1) + 5$
$= 2 + 5$
$= 7$

$u_2 = 2(2) + 5$
$= 4 + 5$
$= 9$

$u_3 = 2(3) + 5$
$= 6 + 5$
$= 11$

$u_4 = 2(4) + 5$
$= 8 + 5$
$= 13$

b



3 **a** The sequence $\{2n\}$ begins 2, 4, 6, 8, 10 (letting $n = 1, 2, 3, 4, 5, \ldots.$).

b The sequence $\{2n + 2\}$ begins 4, 6, 8, 10, 12 (letting $n = 1, 2, 3, 4, 5, \ldots.$).

c The sequence $\{2n - 1)$ begins 1, 3, 5, 7, 9 (letting $n = 1, 2, 3, 4, 5, \ldots.$).

d The sequence $\{2n - 3\}$ begins -1, 1, 3, 5, 7 (letting $n = 1, 2, 3, 4, 5, \ldots.$).

e The sequence $\{2n + 3\}$ begins 5, 7, 9, 11, 13 (letting $n = 1, 2, 3, 4, 5, \ldots.$).

f The sequence $\{2n + 11\}$ begins 13, 15, 17, 19, 21 (letting $n = 1, 2, 3, 4, 5, \ldots.$).

g The sequence $\{3n+1\}$ begins $4, 7, 10, 13, 16$ (letting $n = 1, 2, 3, 4, 5, ...$).

h The sequence $\{4n-3\}$ begins $1, 5, 9, 13, 17$ (letting $n = 1, 2, 3, 4, 5, ...$).

4 **a** The sequence $\{2^n\}$ begins $2, 4, 8, 16, 32$ (letting $n = 1, 2, 3, 4, 5, ...$).

b The sequence $\{3 \times 2^n\}$ begins $6, 12, 24, 48, 96$ (letting $n = 1, 2, 3, 4, 5, ...$).

c The sequence $\{6 \times (\frac{1}{2})^n\}$ begins $3, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \frac{3}{16}$ (letting $n = 1, 2, 3, 4, 5, ...$).

d The sequence $\{(-2)^n\}$ begins $-2, 4, -8, 16, -32$ (letting $n = 1, 2, 3, 4, 5, ...$).

5 $\{15-(-2)^n\}$ generates the sequence with first five terms:
$t_1 = 15-(-2)^1 = 17, \quad t_2 = 15-(-2)^2 = 11, \quad t_3 = 15-(-2)^3 = 23,$
$t_4 = 15-(-2)^4 = -1, \quad t_5 = 15-(-2)^5 = 47$

EXERCISE 7B.2

1 $u_1 = 3, \ u_n = u_{n-1} - 4, \ n > 1$

a $u_2 = u_1 - 4 = 3 - 4 = -1$

b $u_3 = u_2 - 4 = -1 - 4 = -5$

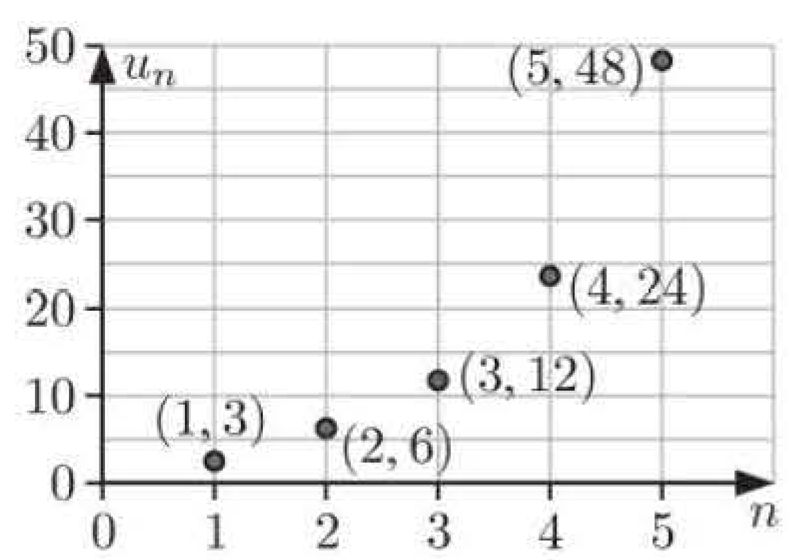
c $u_4 = u_3 - 4 = -5 - 4 = -9$

d $u_5 = u_4 - 4 = -9 - 4 = -13$

2 **a** The first term of the sequence is 3. So, $u_1 = 3$.
Each subsequent term is double the previous one. So, $u_n = 2u_{n-1}$.
$\therefore$ the sequence is defined by $u_1 = 3, \ u_n = 2u_{n-1}, \ n > 1$.

b $u_2 = 2u_1 = 2 \times 3 = 6$ $\quad u_3 = 2u_2 = 2 \times 6 = 12$

$u_4 = 2u_3 = 2 \times 12 = 24$ $\quad u_5 = 2u_4 = 2 \times 24 = 48$



3 $u_1 = 1, \ u_n = \dfrac{1}{1+u_{n-1}}, \ n > 1$.

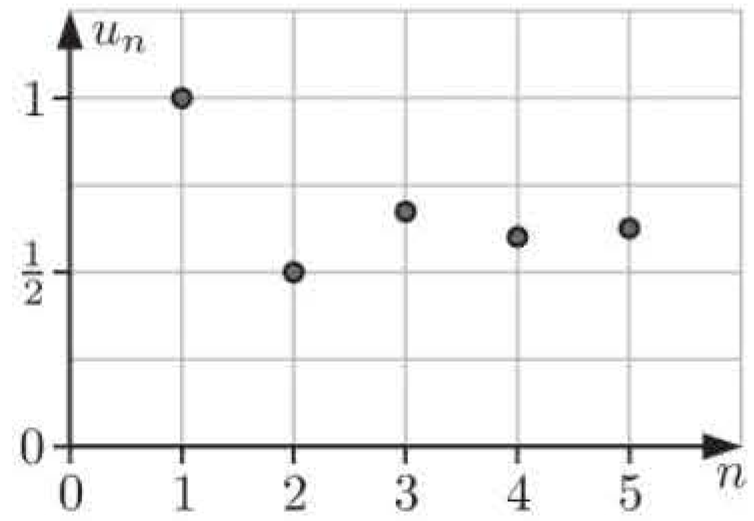
a $u_2 = \dfrac{1}{1+u_1} = \dfrac{1}{1+1} = \frac{1}{2}$

$u_3 = \dfrac{1}{1+u_2} = \dfrac{1}{1+\frac{1}{2}} = \dfrac{1}{\frac{3}{2}} = \frac{2}{3}$

$u_4 = \dfrac{1}{1+u_3} = \dfrac{1}{1+\frac{2}{3}} = \dfrac{1}{\frac{5}{3}} = \frac{3}{5}$

$u_5 = \dfrac{1}{1+u_4} = \dfrac{1}{1+\frac{3}{5}} = \dfrac{1}{\frac{8}{5}} = \frac{5}{8}$

b



c **i** As $n \to \infty$, $u_{n-1} \to u$ and $u_n \to u$

$\therefore$ as $n \to \infty$, $u_n = \dfrac{1}{1+u_{n-1}} \to \dfrac{1}{1+u}$

$\therefore$ as $n \to \infty$, $u_n \to \dfrac{1}{1+u}$ and also $u_n \to u$

$$\therefore \quad u = \frac{1}{1+u}$$
$$\therefore \quad u(1+u) = 1$$
$$\therefore \quad u + u^2 = 1$$
$$\therefore \quad u^2 + u - 1 = 0$$

ii $u^2 + u - 1 = 0$

$$\therefore \quad u = \frac{-1 \pm \sqrt{1-4(1)(-1)}}{2(1)}$$
$$= \frac{-1 \pm \sqrt{5}}{2}$$

But $u_n > 0$ for all n, so $u = \dfrac{-1+\sqrt{5}}{2}$

Now, $$u_n = \frac{1}{1+u_{n-1}}$$
$$= \frac{1}{1+\frac{1}{1+u_{n-2}}}$$
$$= \frac{1}{1+\frac{1}{1+\frac{1}{1+u_{n-3}}}}$$
$$= \ldots.$$

$$\therefore \quad u = \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\ldots}}}} = \frac{-1+\sqrt{5}}{2}$$

4 $u_1 = 1, \; u_n = \sqrt{\dfrac{1}{1+u_{n-1}}}, \quad n > 1$

a $$u_2 = \sqrt{\frac{1}{1+u_1}} = \sqrt{\frac{1}{1+1}} = \sqrt{\tfrac{1}{2}} \approx 0.707\,11$$

$$u_3 = \sqrt{\frac{1}{1+u_2}} = \sqrt{\frac{1}{1+\frac{1}{\sqrt{2}}}} \approx 0.765\,37$$

$$u_4 = \sqrt{\frac{1}{1+u_3}} \approx 0.752\,63$$

$$u_5 = \sqrt{\frac{1}{1+u_4}} \approx 0.755\,36$$

b As $n \to \infty, \; u_{n-1} \to u$ and $u_n \to u$

$\therefore$ as $n \to \infty, \; u_n = \sqrt{\dfrac{1}{1+u_{n-1}}} \to \sqrt{\dfrac{1}{1+u}}$

So, as $n \to \infty, \quad u_n \to \sqrt{\dfrac{1}{1+u}}$ and $u_n \to u$

$$\therefore \quad u = \sqrt{\frac{1}{1+u}}$$
$$\therefore \quad u^2 = \frac{1}{1+u}$$
$$\therefore \quad u^2(1+u) = 1$$
$$\therefore \quad u^2 + u^3 = 1$$
$$\therefore \quad u^3 + u^2 - 1 = 0$$

c $u^3 + u^2 - 1 = 0$

$\therefore \quad u \approx 0.754\,88 \qquad$ {using technology}

and $\quad u = \sqrt{\dfrac{1}{1+\sqrt{\frac{1}{1+\sqrt{\frac{1}{1+\sqrt{....}}}}}}} \approx 0.754\,88$

5 Suppose a sequence is defined by $u_1 = 1,\ u_n = \dfrac{1}{1+u_{n-1}^2},\ n > 0.$

Assume that the sequence terms tend to the constant value u, so that as $n \to \infty,\ u_{n-1} \to u$ and $u_n \to u$.

$\therefore \quad u_n = \dfrac{1}{1+u_{n-1}^2} \to \dfrac{1}{1+u^2} \qquad$ {since as $n \to \infty,\ u_{n-1} \to u$}

So, as $n \to \infty, \quad u_n \to \dfrac{1}{1+u^2}$ and $u_n \to u$

$$\therefore \quad u = \frac{1}{1+u^2}$$

$$\therefore \quad u(1+u^2) = 1$$

$$\therefore \quad u + u^3 = 1$$

$$\therefore \quad u^3 + u - 1 = 0$$

$\therefore \quad u \approx 0.682\,33 \qquad$ {using technology}

and $\quad u = \dfrac{1}{1+\left(\frac{1}{1+\left(\frac{1}{1+\left(\frac{1}{1+(....)}\right)^2}\right)^2}\right)^2} \approx 0.682\,33$

EXERCISE 7C

1 **a** The sequence begins with 19 and the common difference is $25 - 19 = 6$.

$$\begin{aligned} u_n &= u_1 + (n-1)d \\ \therefore \quad u_n &= 19 + 6(n-1) \\ \text{So,} \quad u_{10} &= 19 + 6(10-1) \\ &= 19 + 6 \times 9 \\ &= 73 \end{aligned}$$

b The sequence begins with 101 and the common difference is $97 - 101 = -4$.

$$\begin{aligned} u_n &= u_1 + (n-1)d \\ \therefore \quad u_n &= 101 + (-4)(n-1) \\ \text{So,} \quad u_{10} &= 101 + (-4)(10-1) \\ &= 101 - 4 \times 9 \\ &= 65 \end{aligned}$$

c The sequence begins with 8 and the common difference is $9\frac{1}{2} - 8 = \frac{3}{2}$.

$$\begin{aligned} u_n &= u_1 + (n-1)d \\ \therefore \quad u_n &= 8 + \tfrac{3}{2}(n-1) \\ \text{So,} \quad u_{10} &= 8 + \tfrac{3}{2}(10-1) \\ &= 8 + \tfrac{3}{2} \times 9 \\ &= 21\tfrac{1}{2} \end{aligned}$$

2 **a** The first term of the arithmetic sequence is 31 and the common difference is $36 - 31 = 5$.

$$\begin{aligned} u_n &= u_1 + (n-1)d \\ \therefore\ u_n &= 31 + 5(n-1) \\ \text{So, } u_{15} &= 31 + 5(15-1) \\ &= 31 + 5 \times 14 \\ &= 101 \end{aligned}$$

b The first term of the arithmetic sequence is 5 and the common difference is $-3 - 5 = -8$.

$$\begin{aligned} u_n &= u_1 + (n-1)d \\ \therefore\ u_n &= 5 + (-8)(n-1) \\ \text{So, } u_{15} &= 5 + (-8)(15-1) \\ &= 5 - 8 \times 14 \\ &= -107 \end{aligned}$$

c The first term of the arithmetic sequence is a and the common difference is $a + d - a = d$.

$$\begin{aligned} u_n &= u_1 + (n-1)d \\ \therefore\ u_n &= a + (n-1)d \\ \text{So, } u_{15} &= a + d(15-1) \\ &= a + 14d \end{aligned}$$

3 **a** $17 - 6 = 11$
$28 - 17 = 11$
$39 - 28 = 11$
$50 - 39 = 11$

Assuming that the pattern continues, consecutive terms differ by 11.
$\therefore$ the sequence is arithmetic with $u_1 = 6,\ \ d = 11$.

b $$\begin{aligned} u_n &= u_1 + (n-1)d \\ &= 6 + (n-1)11 \\ &= 11n - 5 \end{aligned}$$

c $$\begin{aligned} u_{50} &= 11(50) - 5 \\ &= 545 \end{aligned}$$

d Let $u_n = 325 = 11n - 5$
$\therefore\ 330 = 11n$
$\therefore\ n = 30$
So, 325 is the 30th member.

e Let $u_n = 761 = 11n - 5$
$\therefore\ 766 = 11n$
$\therefore\ n = 69\frac{7}{11}$, but n must be an integer, so 761 is not a member of the sequence.

4 **a** $83 - 87 = -4$
$79 - 83 = -4$
$75 - 79 = -4$

Assuming that the pattern continues, consecutive terms differ by -4.
$\therefore$ the sequence is arithmetic with $u_1 = 87,\ \ d = -4$.

b $$\begin{aligned} u_n &= u_1 + (n-1)d \\ &= 87 + (n-1)(-4) \\ &= 87 - 4n + 4 \\ &= 91 - 4n \end{aligned}$$

c $$\begin{aligned} u_{40} &= 91 - 4(40) \\ &= 91 - 160 \\ &= -69 \end{aligned}$$

d Let $u_n = -297 = 91 - 4n$
$\therefore\ 4n = 388$
$\therefore\ n = 97$
So, -297 is the 97th term of the sequence.

5 **a** $u_n = 3n - 2,\ \ u_{n+1} = 3(n+1) - 2 = 3n + 1$

$$\begin{aligned} u_{n+1} - u_n &= (3n+1) - (3n-2) \\ &= 3, \quad \text{a constant} \end{aligned}$$

Consecutive terms differ by 3.
$\therefore$ the sequence is arithmetic.

b $u_1 = 3(1) - 2 = 1,\ \ d = 3$

c $u_{57} = 3(57) - 2 = 169$

d Let $u_n = 450 = 3n - 2$, so $3n = 452$ and hence $n = 150\frac{2}{3}$.
We try the two values on either side of $n = 150\frac{2}{3}$, which are $n = 150$ and $n = 151$:
$u_{150} = 3(150) - 2 = 448$ and $u_{151} = 3(151) - 2 = 451$
So, $u_{150} = 448$ is the largest term which is smaller than 450.

6 **a** $$u_n = \frac{71 - 7n}{2} = 35\tfrac{1}{2} - \tfrac{7}{2}n$$

$$u_{n+1} = \frac{71 - 7(n+1)}{2} = \frac{71 - 7n - 7}{2} = \frac{64 - 7n}{2} = 32 - \tfrac{7}{2}n$$

$$u_{n+1} - u_n = (32 - \tfrac{7}{2}n) - (35\tfrac{1}{2} - \tfrac{7}{2}n) = -\tfrac{7}{2}, \quad \text{a constant}$$

So, consecutive terms differ by $-\frac{7}{2}$. $\therefore$ the sequence is arithmetic.

b $$u_1 = \frac{71 - 7(1)}{2} = 32, \quad d = -\tfrac{7}{2}$$

c $$u_{75} = \frac{71 - 7(75)}{2} = -227$$

d Let $u_n = -200 = \dfrac{71-7n}{2}$ so $-400 = 71 - 7n$ $\therefore\ 7n = 471$

$\therefore\ n = 67\frac{2}{7}$

We try the two values on either side of $n = 67\frac{2}{7}$, which are $n = 67$ and $n = 68$:

$u_{67} = \dfrac{71-7(67)}{2} = -199$ and $u_{68} = \dfrac{71-7(68)}{2} = -202\frac{1}{2}$

So, the terms of the sequence are less than -200 for $n \geqslant 68$.

7 **a** The terms are consecutive, so we equate common differences:

$k - 32 = 3 - k$

$\therefore\ 2k = 35$

$\therefore\ k = 17\frac{1}{2}$

b The terms are consecutive, so we equate common differences:

$7 - k = 10 - 7$

$\therefore\ 7 - k = 3$

$\therefore\ k = 4$

c The terms are consecutive, so we equate common differences:

$(2k+1) - (k+1) = 13 - (2k+1)$

$\therefore\ k = 12 - 2k$

$\therefore\ 3k = 12$

$\therefore\ k = 4$

d The terms are consecutive, so we equate common differences:

$(2k+3) - (k-1) = (7-k) - (2k+3)$

$\therefore\ k + 4 = 4 - 3k$

$\therefore\ 4k = 0$

$\therefore\ k = 0$

e The terms are consecutive, so we equate common differences:

$k^2 - k = (k^2+6) - k^2$

$\therefore\ k^2 - k - 6 = 0$

$\therefore\ (k+2)(k-3) = 0$

$\therefore\ k = -2$ or 3

f The terms are consecutive, so we equate common differences:

$k - 5 = k^2 - 8 - k$

$\therefore\ k^2 - 2k - 3 = 0$

$\therefore\ (k-3)(k+1) = 0$

$\therefore\ k = -1$ or 3

8 **a** $u_7 = 41$ $\therefore\ u_1 + 6d = 41$ .... (1)

$u_{13} = 77$ $\therefore\ u_1 + 12d = 77$ .... (2)

Solving simultaneously,

$-u_1 - 6d = -41$

$u_1 + 12d = 77$

$\therefore\ 6d = 36$ {adding the equations}

$\therefore\ d = 6$

So in (1), $u_1 + 6(6) = 41$

$\therefore\ u_1 + 36 = 41$ $\therefore\ u_1 = 5$

Now $u_n = u_1 + (n-1)d$

$\therefore\ u_n = 5 + (n-1)6$

$\therefore\ u_n = 6n - 1$

b $u_5 = -2$ $\therefore\ u_1 + 4d = -2$ .... (1)

$u_{12} = -12\frac{1}{2}$ $\therefore\ u_1 + 11d = -12\frac{1}{2}$ .... (2)

Solving simultaneously,

$-u_1 - 4d = 2$

$u_1 + 11d = -12\frac{1}{2}$

$\therefore\ 7d = -10\frac{1}{2}$ {adding the equations}

$\therefore\ d = -\frac{3}{2}$

So in (1), $u_1 + 4(-\frac{3}{2}) = -2$ $\therefore\ u_1 = 4$

Now $u_n = u_1 + (n-1)d$

$\therefore\ u_n = 4 + (n-1)(-\frac{3}{2})$

$\therefore\ u_n = -\frac{3}{2}n + \frac{11}{2}$

c $u_7 = 1$ $\therefore\ u_1 + 6d = 1$ .... (1)

$u_{15} = -39$ $\therefore\ u_1 + 14d = -39$ .... (2)

Solving simultaneously,

$-u_1 - 6d = -1$

$u_1 + 14d = -39$

$\therefore\ 8d = -40$ {adding the equations}

$\therefore\ d = -5$

So in (1), $u_1 + 6(-5) = 1$

$\therefore\ u_1 - 30 = 1$ $\therefore\ u_1 = 31$

Now $u_n = u_1 + (n-1)d$

$\therefore\ u_n = 31 + (n-1)(-5)$

$\therefore\ u_n = 31 - 5n + 5$

$\therefore\ u_n = -5n + 36$

d $u_{11} = -16$ $\therefore\ u_1 + 10d = -16$ .... (1)

$u_8 = -11\frac{1}{2}$ $\therefore\ u_1 + 7d = -11\frac{1}{2}$ .... (2)

Solving simultaneously,

$-u_1 - 10d = 16$

$u_1 + 7d = -11\frac{1}{2}$

$\therefore\ -3d = 4\frac{1}{2}$ {adding the equations}

$\therefore\ d = -\frac{3}{2}$

So in (1), $u_1 + 10(-\frac{3}{2}) = -16$

$\therefore\ u_1 - 15 = -16$ $\therefore\ u_1 = -1$

Now $u_n = u_1 + (n-1)d$

$\therefore\ u_n = -1 + (n-1)(-\frac{3}{2})$

$\therefore\ u_n = -\frac{3}{2}n + \frac{1}{2}$

9 **a** Let the numbers be
$5,\ 5+d,\ 5+2d,\ 5+3d,\ 10.$

Then $5+4d=10$
$\therefore\ 4d=5$
$\therefore\ d=\frac{5}{4}=1\frac{1}{4}$

So, the numbers are
$5,\ 6\frac{1}{4},\ 7\frac{1}{2},\ 8\frac{3}{4},\ 10.$

b Let the numbers be $-1,\ -1+d,\ -1+2d,$
$-1+3d,\ -1+4d,\ -1+5d,\ -1+6d,\ 32.$

Then $-1+7d=32$
$\therefore\ 7d=33$
$\therefore\ d=\frac{33}{7}=4\frac{5}{7}$

So, the numbers are
$-1,\ 3\frac{5}{7},\ 8\frac{3}{7},\ 13\frac{1}{7},\ 17\frac{6}{7},\ 22\frac{4}{7},\ 27\frac{2}{7},\ 32.$

10 **a** $u_1=36,\ \ 35\frac{1}{3}-36=-\frac{2}{3},$
$34\frac{2}{3}-35\frac{1}{3}=-\frac{2}{3}$
So, $d=-\frac{2}{3}.$

b $u_n=u_1+(n-1)d$
$\therefore\ -30=36+(n-1)(-\frac{2}{3})$ {letting $u_n=-30$}
$\therefore\ -66=-\frac{2}{3}n+\frac{2}{3}$
$\therefore\ \frac{2}{3}n=66\frac{2}{3}$
$\therefore\ n=100$ So, -30 is the 100th term of the sequence.

11 $u_1=23,\ \ 36-23=13$
$49-36=13$
$62-49=13$
so $d=13$

$u_n=u_1+(n-1)d$
$\therefore\ u_n=23+(n-1)13$
$=23+13n-13$
$\therefore\ u_n=13n+10$

Let $u_n=100\,000$
$=13n+10$
$\therefore\ 99\,990=13n$
$\therefore\ n=7691\frac{7}{13}$

We try the two values on either side of $n=7691\frac{7}{13}$, which are $n=7691$ and $n=7692$:
$u_{7691}=13(7691)+10=99\,993$ and $u_{7692}=13(7692)+10=100\,006$
So, the first term to exceed 100 000 is $u_{7692}=100\,006$.

12 **a** *Month 1*: 5 cars *Month 2*: $5+13=18$ cars *Month 3*: $18+13=31$ cars
Month 4: $31+13=44$ cars *Month 5*: $44+13=57$ cars *Month 6*: $57+13=70$ cars

b Every month after the first, the factory assembles 13 cars, so the difference between successive months is always 13. Thus we have an arithmetic sequence with $u_1=5$ and $d=13$.

c $u_n=u_1+(n-1)d$
$=5+(n-1)\times 13$
$=13n-8$
$\therefore\ u_{12}=13\times 12-8$ {12 months = 1 year}
$=148$

So, 148 cars are made in the first year.

d $u_n=250=13n-8$
$\therefore\ 258=13n$
$\therefore\ n=\dfrac{258}{13}\approx 19.85$
So, the 250th car is made in the 20th month.

13 **a** $41-34=48-41=55-48=7$
Assuming the pattern continues, there is a common difference of 7, and so the number of Valéria's friends forms an arithmetic sequence with $u_1=34$ and $d=7$.

b $u_n=u_1+(n-1)d$
$=34+(n-1)\times 7$
$=27+7n$
$\therefore\ u_{12}=27+7\times 12=111$

So, after 12 weeks Valéria will have 111 online friends.

c $u_n=150=27+7n$
$\therefore\ 123=7n$
$\therefore\ n=\frac{123}{7}\approx 17.57$

So, Valéria will have 150 online friends after the 18th week.

14 **a** *July 1st*: $100-2.7\times 1=97.3$ tonnes of hay
July 2nd: $100-2.7\times 2=94.6$ tonnes of hay
July 3rd: $100-2.7\times 3=91.9$ tonnes of hay

b $94.6-97.3=91.9-94.6=-2.7$
So $d=-2.7$, which means the cows eat 2.7 tonnes of hay per day.

c $u_{25}=100-2.7\times 25=32.5$ tonnes
So, at the end of July 25th there are 32.5 tonnes of hay remaining in the barn.

d July has 31 days, so the end of day 31 is the start of August.
$u_{31} = 100 - 2.7 \times 31 = 16.3$ tonnes.
Hence there are 16.3 tonnes of hay in the barn at the beginning of August.

EXERCISE 7D.1

1 **a** $\frac{6}{2} = 3 \quad \therefore \quad r = 3, \quad u_1 = 2 \qquad \therefore \quad b = 6 \times 3 = 18$ and $c = 18 \times 3 = 54$

b $\frac{5}{10} = \frac{1}{2} \quad \therefore \quad r = \frac{1}{2}, \quad u_1 = 10 \qquad \therefore \quad b = 5 \times \frac{1}{2} = 2\frac{1}{2}$ and $c = 2\frac{1}{2} \times \frac{1}{2} = 1\frac{1}{4}$

c $\frac{-6}{12} = -\frac{1}{2} \quad \therefore \quad r = -\frac{1}{2}, \quad u_1 = 12 \qquad \therefore \quad b = -6 \times -\frac{1}{2} = 3$ and $c = 3 \times -\frac{1}{2} = -1\frac{1}{2}$

2 **a** $\frac{6}{3} = 2 \quad \therefore \quad r = 2, \quad u_1 = 3 \quad \therefore \quad u_6 = 3 \times 2^{6-1} = 96$

b $\frac{10}{2} = 5 \quad \therefore \quad r = 5, \quad u_1 = 2 \quad \therefore \quad u_6 = 2 \times 5^{6-1} = 6250$

c $\frac{256}{512} = \frac{1}{2} \quad \therefore \quad r = \frac{1}{2}, \quad u_1 = 512 \quad \therefore \quad u_6 = 512 \times (\frac{1}{2})^{6-1} = 16$

3 **a** $\frac{3}{1} = 3 \quad \therefore \quad r = 3, \quad u_1 = 1 \quad \therefore \quad u_9 = 1 \times 3^{9-1} = 6561$

b $\frac{18}{12} = \frac{3}{2} \quad \therefore \quad r = \frac{3}{2}, \quad u_1 = 12 \quad \therefore \quad u_9 = 12 \times (\frac{3}{2})^{9-1} = 307\frac{35}{64}$ or $\frac{19\,683}{64}$

c $\dfrac{-\frac{1}{8}}{\frac{1}{16}} = -2 \quad \therefore \quad r = -2, \quad u_1 = \frac{1}{16} \quad \therefore \quad u_9 = \frac{1}{16} \times (-2)^{9-1} = 16$

d $\dfrac{ar}{a} = r \quad \therefore \quad r = r, \quad u_1 = a \quad \therefore \quad u_9 = a \times r^{9-1} = ar^8$

4 **a** $\frac{10}{5} = \frac{20}{10} = \frac{40}{20} = 2$

Assuming the pattern continues, consecutive terms have a common ratio of 2.
$\therefore$ the sequence is geometric with $u_1 = 5$ and $r = 2$.

b
$$u_n = u_1 r^{n-1}$$
$$\therefore \quad u_n = 5 \times 2^{n-1}$$
$$\text{so} \quad u_{15} = 5 \times 2^{14} = 81\,920$$

5 **a**
$$\frac{-6}{12} = -\frac{1}{2}$$
$$\frac{3}{-6} = -\frac{1}{2}$$
$$\frac{(-\frac{3}{2})}{3} = -\frac{1}{2}$$

Assuming the pattern continues, consecutive terms have a common ratio of $-\frac{1}{2}$.
$\therefore$ the sequence is geometric with $u_1 = 12$ and $r = -\frac{1}{2}$.

b
$$u_n = u_1 r^{n-1}$$
$$\therefore \quad u_n = 12 \times \left(-\tfrac{1}{2}\right)^{n-1}$$

$$\begin{aligned} \text{so} \quad u_{13} &= 12 \times (-\tfrac{1}{2})^{13-1} \\ &= 12 \times (-\tfrac{1}{2})^{12} \\ &= 12 \times \tfrac{1}{4096} \\ &= 3 \times \tfrac{1}{1024} = \tfrac{3}{1024} \end{aligned}$$

6
$$\frac{-6}{8} = -\frac{3}{4}$$
$$\frac{4.5}{-6} = -\frac{(\frac{9}{2})}{6} = -\frac{3}{4}$$
$$\frac{-3.375}{4.5} = \frac{(-\frac{27}{8})}{(\frac{9}{2})} = -\frac{3}{4}$$

Assuming the pattern continues, consecutive terms have a common ratio of $-\frac{3}{4}$.
$\therefore$ the sequence is geometric with $u_1 = 8$ and $r = -\frac{3}{4}$.

$u_n = u_1 r^{n-1} = 8 \times (-\frac{3}{4})^{n-1}$ So, $u_{10} = 8 \times (-\frac{3}{4})^9 \approx -0.600\,677\,490\,2$

7 $\dfrac{4\sqrt{2}}{8} = \dfrac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

$\dfrac{4}{4\sqrt{2}} = \frac{1}{\sqrt{2}}$

$\dfrac{2\sqrt{2}}{4} = \dfrac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

Assuming the pattern continues, consecutive terms have a common ratio of $\frac{1}{\sqrt{2}}$.

$\therefore$ the sequence is geometric with $u_1 = 8$ and $r = \frac{1}{\sqrt{2}}$.

$$u_n = u_1 r^{n-1} = 8\left(\tfrac{1}{\sqrt{2}}\right)^{n-1} = 2^3 \times \left(2^{-\frac{1}{2}}\right)^{n-1} = 2^3 \times 2^{-\frac{1}{2}n+\frac{1}{2}}$$

So, $u_n = 2^{\frac{7}{2}-\frac{1}{2}n}$

8 **a** Since the terms are geometric,

$\dfrac{k}{7} = \dfrac{28}{k}$

$\therefore\ k^2 = 196$

$\therefore\ k = \pm 14$

b Since the terms are geometric,

$\dfrac{3k}{k} = \dfrac{20-k}{3k} = 3$

$\therefore\ 20 - k = 9k$

$\therefore\ 20 = 10k$

$\therefore\ k = 2$

c Since the terms are geometric,

$\dfrac{k+8}{k} = \dfrac{9k}{k+8}$

$\therefore\ (k+8)^2 = 9k^2$

$\therefore\ k^2 + 16k + 64 = 9k^2$

$\therefore\ 8k^2 - 16k - 64 = 0$

$\therefore\ 8(k^2 - 2k - 8) = 0$

$\therefore\ 8(k+2)(k-4) = 0$ and so $k = -2$ or 4

9 **a** $u_4 = 24 \quad \therefore\ u_1 \times r^3 = 24 \quad .... (1)$

$u_7 = 192 \quad \therefore\ u_1 \times r^6 = 192 \quad .... (2)$

So, $\dfrac{u_1 r^6}{u_1 r^3} = \dfrac{192}{24} \quad \{(2) \div (1)\}$

$\therefore\ r^3 = 8 \qquad \therefore\ r = 2$

So in (1), $u_1 \times 2^3 = 24$

$\therefore\ u_1 \times 8 = 24$

$\therefore\ u_1 = 3$

$\therefore\ u_n = 3 \times 2^{n-1}$

b $u_3 = 8 \quad \therefore\ u_1 \times r^2 = 8 \quad .... (1)$

$u_6 = -1 \quad \therefore\ u_1 \times r^5 = -1 \quad .... (2)$

So, $\dfrac{u_1 r^5}{u_1 r^2} = -\dfrac{1}{8} \quad \{(2) \div (1)\}$

$\therefore\ r^3 = -\frac{1}{8} \qquad \therefore\ r = -\frac{1}{2}$

So in (1), $u_1 \times (-\frac{1}{2})^2 = 8$

$\therefore\ u_1 \times \frac{1}{4} = 8$

$\therefore\ u_1 = 32$

$\therefore\ u_n = 32 \times (-\frac{1}{2})^{n-1}$

c $u_7 = 24 \quad \therefore\ u_1 \times r^6 = 24 \quad .... (1)$

$u_{15} = 384 \quad \therefore\ u_1 \times r^{14} = 384 \quad .... (2)$

So, $\dfrac{u_1 r^{14}}{u_1 r^6} = \dfrac{384}{24} \quad \{(2) \div (1)\}$

$\therefore\ r^8 = 16 \qquad \therefore\ r = \pm\sqrt{2}$

So in (1), $u_1 \times (\pm\sqrt{2})^6 = 24$

$\therefore\ u_1 \times 8 = 24$

$\therefore\ u_1 = 3$

Now $u_n = u_1 r^{n-1}$

$\therefore\ u_n = 3 \times (\sqrt{2})^{n-1}$

or $u_n = 3 \times (-\sqrt{2})^{n-1}$

d $u_3 = 5 \quad \therefore\ u_1 \times r^2 = 5 \quad .... (1)$

$u_7 = \frac{5}{4} \quad \therefore\ u_1 \times r^6 = \frac{5}{4} \quad .... (2)$

So, $\dfrac{u_1 r^6}{u_1 r^2} = \dfrac{(\frac{5}{4})}{5} \quad \{(2) \div (1)\}$

$\therefore\ r^4 = \frac{1}{4} \qquad \therefore\ r = \pm\frac{1}{\sqrt{2}}$

So in (1), $u_1 \times (\pm\frac{1}{\sqrt{2}})^2 = 5$

$\therefore\ u_1 \times \frac{1}{2} = 5$

$\therefore\ u_1 = 10$

Now $u_n = u_1 r^{n-1}$

$\therefore\ u_n = 10 \times (\frac{1}{\sqrt{2}})^{n-1}$

$= 10 \times (\sqrt{2})^{1-n}$

or $u_n = 10 \times (-\frac{1}{\sqrt{2}})^{n-1}$

$= 10 \times (-\sqrt{2})^{1-n}$

10 **a** 2, 6, 18, 54, ... has $u_1 = 2$ and $r = 3$

$u_n = u_1 r^{n-1}$ $\therefore$ $u_n = 2 \times 3^{n-1}$

Let $u_n = 10\,000 = 2 \times 3^{n-1}$, so $5000 = 3^{n-1}$

$\therefore$ $n \approx 8.7527$ {using technology}

We try the two values on either side of $n = 8.7527$, which are $n = 8$ and $n = 9$:

$u_8 = 2 \times 3^7 = 4374$ and $u_9 = 2 \times 3^8 = 13\,122$

So, the first term to exceed 10 000 is $u_9 = 13\,122$.

b $4, 4\sqrt{3}, 12, 12\sqrt{3}, ...$ has $u_1 = 4$ and $r = \sqrt{3}$

$u_n = u_1 r^{n-1}$ $\therefore$ $u_n = 4 \times \left(\sqrt{3}\right)^{n-1}$

Let $u_n = 4800 = 4 \times \left(\sqrt{3}\right)^{n-1}$, so $1200 = \left(\sqrt{3}\right)^{n-1}$

$\therefore$ $n \approx 13.91$ {using technology}

We try the two values on either side of $n \approx 13.91$, which are $n = 13$ and $n = 14$:

$u_{13} = 4 \times (\sqrt{3})^{12} = 2916$ and $u_{14} = 4 \times (\sqrt{3})^{13} = 2916\sqrt{3} \approx 5050.7$

So, the first term to exceed 4800 is $u_{14} = 2916\sqrt{3} \approx 5050.7$.

c 12, 6, 3, 1.5, ... has $u_1 = 12$ and $r = \frac{1}{2}$

$u_n = u_1 r^{n-1}$ $\therefore$ $u_n = 12 \times (\frac{1}{2})^{n-1}$ Let $0.0001 = u_n = 12 \times \left(\frac{1}{2}\right)^{n-1}$

$\therefore$ $0.000\,008\,\overline{3} = \left(\frac{1}{2}\right)^{n-1}$

$\therefore$ $n \approx 17.87$ {using technology}

We try the two values on either side of $n \approx 17.87$, which are $n = 17$ and $n = 18$:

$u_{17} = 12 \times \left(\frac{1}{2}\right)^{16} \approx 0.000\,183\,1$ and $u_{18} = 12 \times \left(\frac{1}{2}\right)^{17} \approx 0.000\,091\,55$

So, the first term of the sequence which is less than 0.0001 is $u_{18} \approx 0.000\,091\,55$.

EXERCISE 7D.2

1 There is a fixed percentage increase each week, so the population forms a geometric sequence.

$u_{n+1} = u_1 \times r^n$, where $u_1 = 500$, $r = 1.12$

a **i** $u_{11} = 500 \times (1.12)^{10}$
≈ 1552.92

There will be approximately 1550 ants.

ii $u_{21} = 500 \times (1.12)^{20}$
≈ 4823.15

There will be approximately 4820 ants.

b For the population to reach 2000, $u_{n+1} = 500 \times (1.12)^n = 2000$

$\therefore$ $(1.12)^n = 4$

$\therefore$ $n \approx 12.23$ {using technology}

It will take approximately 12.2 weeks.

2 $u_{n+1} = u_1 \times r^n$, where $u_1 = 555$, $r = 0.955$

a $u_{16} = 555 \times (0.955)^{15}$
≈ 278.19 The population is approximately 278 animals in the year 2010.

b For the population to have declined to 50,

$u_{n+1} = 555 \times (0.955)^n = 50$

$\therefore$ $(0.955)^n = 0.0\overline{900}$

$\therefore$ $n \approx 52.3$ {using technology}

So, in the 53rd year the population is 50. This is the year 2047.

3 $u_{n+1} = u_1 \times r^n$, where $u_1 = 32$, $r = 1.18$

a **i** $u_6 = 32 \times (1.18)^5$
≈ 73.21

There will be approximately 73 deer.

ii $u_{11} = 32 \times (1.18)^{10}$
≈ 167.48

There will be approximately 167 deer.

b For the population to reach 5000, $u_{n+1} = 32 \times (1.18)^n = 5000$

$\therefore \; n \approx 30.52$ {using technology}

So, it will take approximately 30.5 years.

4 $u_{n+1} = u_1 \times r^n$, where $u_1 = 178$, $r = 1.32$

a **i** $u_{11} = 178 \times (1.32)^{10}$
≈ 2858.6

There will be approximately 2860 marsupials.

ii $u_{26} = 178 \times (1.32)^{25}$
$\approx 183\,979.0$

There will be approximately 184 000 marsupials.

b For the population to reach 10 000, $u_{n+1} = 178 \times (1.32)^n = 10\,000$

$\therefore \; n \approx 14.5$ {using technology}

So, it will take approximately 14.5 years.

EXERCISE 7D.3

1 **a** $u_{n+1} = u_1 \times r^n$

where $u_1 = 3000$, $r = 1.1$, $n = 3$

$\therefore \; u_4 = 3000 \times (1.1)^3$
$= 3993$

The investment will amount to \$3993.

b Interest = amount after 3 years − initial amount
= \$3993 − \$3000
= \$993

2 $u_{n+1} = u_1 \times r^n$ where $u_1 = 20\,000$, $r = 1.12$, $n = 4$

$\therefore \; u_5 = 20\,000 \times (1.12)^4$
$\approx 31\,470.39$

Interest = €31 470.39 − €20 000
= €11 470.39

3 **a** $u_{n+1} = u_1 \times r^n$

where $u_1 = 30\,000$, $r = 1.1$, $n = 4$

$\therefore \; u_5 = 30\,000 \times (1.1)^4$
$= 43\,923$

The investment amounts to ¥43 923.

b Interest = amount after 4 years − initial amount
= ¥43 923 − ¥30 000
= ¥13 923

4 $u_{n+1} = u_1 \times r^n$ where $u_1 = 80\,000$, $r = 1.09$, $n = 3$

$\therefore \; u_4 = 80\,000 \times (1.09)^3$
$= 103\,602.32$

Interest = amount after 3 years − initial amount
= \$103 602.32 − \$80 000
= \$23 602.32

5 $u_{n+1} = u_1 \times r^n$ where $u_1 = 100\,000$, $r = 1 + \dfrac{0.08}{2} = 1.04$, $n = 10$

$\therefore \; u_{11} = 100\,000 \times (1.04)^{10}$
$\approx 148\,024.43$ It amounts to ¥148 024.43.

6 $u_{n+1} = u_1 \times r^n$ where $u_1 = 45\,000$, $r = 1 + \dfrac{0.075}{4} = 1.018\,75$, $n = 7$ {21 months = 7 'quarters'}

$\therefore \; u_{10} = 45\,000 \times (1.018\,75)^7$
$\approx 51\,249.06$ It amounts to £51 249.06.

7 The initial investment u_1 is unknown.

The interest rate is compounded annually, so the multiplier $r = 1 + 0.075 = 1.075$.

There are 4 compounding periods, so $n = 4$

$\therefore \; u_{n+1} = u_5$

Now $u_5 = u_1 \times r^4$ {using $u_{n+1} = u_1 \times r^n$}

$\therefore\ 20\,000 = u_1 \times (1.075)^4$

$\therefore\ u_1 = \dfrac{20\,000}{(1.075)^4}$

$\therefore\ u_1 \approx 14\,976.01$ So, $14 976.01 should be invested now.

8 The initial investment is unknown.

The interest rate is compounded annually, so the multiplier $r = 1 + 0.055 = 1.055$.

There are $\frac{60}{12} = 5$ compounding periods, so $n = 5$

$\therefore\ u_{n+1} = u_6$

Now $u_6 = u_1 \times r^5$ {using $u_{n+1} = u_1 \times r^n$}

$\therefore\ 15\,000 = u_1 \times (1.055)^5$

$\therefore\ u_1 = \dfrac{15\,000}{(1.055)^5}$

$\therefore\ u_1 \approx 11\,477.02$ The initial investment required is £11 477.02.

9 The initial investment is unknown.

The interest rate is compounded quarterly, so the multiplier $r = 1 + \dfrac{0.08}{4} = 1.02$.

There are $3 \times 4 = 12$ compounding periods, so $n = 12$

$\therefore\ u_{n+1} = u_{13}$

Now $u_{13} = u_1 \times r^{12}$ {using $u_{n+1} = u_1 \times r^n$}

$\therefore\ 25\,000 = u_1 \times (1.02)^{12}$

$\therefore\ u_1 = \dfrac{25\,000}{(1.02)^{12}}$

$\therefore\ u_1 \approx 19\,712.33$ I should invest €19 712.33 now.

10 The initial investment is unknown.

The interest rate is compounded monthly, so the multiplier $r = 1 + \dfrac{0.09}{12} = 1.0075$.

There are $8 \times 12 = 96$ compounding periods, so $n = 96$

$\therefore\ u_{n+1} = u_{97}$

Now $u_{97} = u_1 \times r^{96}$ {using $u_{n+1} = u_1 \times r^n$}

$\therefore\ 40\,000 = u_1 \times (1.0075)^{96}$

$\therefore\ u_1 = \dfrac{40\,000}{(1.0075)^{96}}$

$\therefore\ u_1 \approx 19\,522.47$ The initial investment should be ¥19 522.47.

EXERCISE 7E

1 **a** **i** 3, 11, 19, 27, ... is arithmetic with $u_1 = 3,\ d = 8,$ so $u_n = 3 + (n-1)8 = 8n - 5$

$S_n = \sum_{k=1}^{n} (8k - 5)$

ii $S_5 = 3 + 11 + 19 + 27 + 35 = 95$

b **i** 42, 37, 32, 27, ... is arithmetic with $u_1 = 42,\ d = -5,$

so $u_n = 42 + (n-1)(-5) = 47 - 5n$

$\therefore\ S_n = \sum_{k=1}^{n} (47 - 5k)$

ii $S_5 = 42 + 37 + 32 + 27 + 22 = 160$

c **i** 12, 6, 3, $1\frac{1}{2}$, ... is geometric with $u_1 = 12$, $r = \frac{1}{2}$, so $u_n = 12 \times (\frac{1}{2})^{n-1}$

$S_n = \sum_{k=1}^{n} 12(\frac{1}{2})^{k-1}$

ii $S_5 = 12 + 6 + 3 + 1\frac{1}{2} + \frac{3}{4} = 23\frac{1}{4}$

d **i** 2, 3, $4\frac{1}{2}$, $6\frac{3}{4}$, ... is geometric with $u_1 = 2$, $r = \frac{3}{2}$, so $u_n = 2 \times (\frac{3}{2})^{n-1}$

$S_n = \sum_{k=1}^{n} 2(\frac{3}{2})^{k-1}$

ii $S_5 = 2 + 3 + 4\frac{1}{2} + 6\frac{3}{4} + 10\frac{1}{8} = 26\frac{3}{8}$

e **i** 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, ... is geometric with $u_1 = 1$, $r = \frac{1}{2}$, so $u_n = 1 \times (\frac{1}{2})^{n-1} = \dfrac{1}{2^{n-1}}$

$S_n = \sum_{k=1}^{n} \dfrac{1}{2^{k-1}}$

ii $S_5 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = 1\frac{15}{16}$

f **i** 1, 8, 27, 64, ...

$S_n = \sum_{k=1}^{n} k^3$ {since $1 = 1^3$, $8 = 2^3$, $27 = 3^3$, $64 = 4^3$}

ii $S_5 = 1 + 8 + 27 + 64 + 125 = 225$

2 **a** $\sum_{k=1}^{3} 4k = 4 + 8 + 12 = 24$

b $\sum_{k=1}^{6} (k+1) = 2 + 3 + 4 + 5 + 6 + 7 = 27$

c $\sum_{k=1}^{4} (3k - 5) = -2 + 1 + 4 + 7 = 10$

d $\sum_{k=1}^{5} (11 - 2k) = 9 + 7 + 5 + 3 + 1 = 25$

e $\sum_{k=1}^{7} k(k+1) = 2 + 6 + 12 + 20 + 30 + 42 + 56 = 168$

f $\sum_{k=1}^{5} 10 \times 2^{k-1} = 10 + 20 + 40 + 80 + 160 = 310$

3 $u_n = 3n - 1$

$$\begin{aligned}
\therefore \quad u_1 + u_2 + u_3 + ... + u_{20} &= \sum_{n=1}^{20} (3n - 1) \\
&= 2 + 5 + 8 + 11 + 14 + 17 + 20 + 23 + 26 + 29 + 32 + 35 \\
&\quad + 38 + 41 + 44 + 47 + 50 + 53 + 56 + 59 \\
&= 610
\end{aligned}$$

4 **a** $\sum_{k=1}^{n} c = \underbrace{c + c + c + ... + c}_{n \text{ times}} = cn$

b $$\begin{aligned}
\sum_{k=1}^{n} ca_k &= ca_1 + ca_2 + ... + ca_n \\
&= c(a_1 + a_2 + ... + a_n) \\
&= c\sum_{k=1}^{n} a_k
\end{aligned}$$

c $$\begin{aligned}
\sum_{k=1}^{n} (a_k + b_k) &= (a_1 + b_1) + (a_2 + b_2) + ... + (a_n + b_n) \\
&= (a_1 + a_2 + ... + a_n) + (b_1 + b_2 + ... + b_n) \\
&= \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k
\end{aligned}$$

5 **a** $\sum_{k=1}^{5} k(k+1)(k+2) = 6 + 24 + 60 + 120 + 210 = 420$

b $\displaystyle\sum_{k=6}^{12}\left(100(1.2)^{k-3}\right) = 172.8 + 207.36 + 248.832 + 298.5984 + 358.318\,08 + 429.981\,696 + 515.978\,0352 \approx 2232$

6 **a** $\displaystyle\sum_{k=1}^{n} k = 1+2+3+\ldots.+(n-2)+(n-1)+n$

b
$$\begin{array}{rcccccccccccc} & 1 & + & 2 & + & 3 & + \ldots. + & (n-2) & + & (n-1) & + & n \\ & n & + & (n-1) & + & (n-2) & + \ldots. + & 3 & + & 2 & + & 1 \\ \hline & (n+1) & + & (n+1) & + & (n+1) & + \ldots. + & (n+1) & + & (n+1) & + & (n+1) & = n(n+1) \end{array}$$

c $S_n = \displaystyle\sum_{k=1}^{n} k = 1+2+3+\ldots.+(n-2)+(n-1)+n$ {from **a**}

and $2[1+2+3+\ldots.+(n-2)+(n-1)+n] = n(n+1)$ {from **b**}

$\therefore\ 2S_n = n(n+1)$

$\therefore\ S_n = \dfrac{n(n+1)}{2}$

d $\displaystyle\sum_{k=1}^{n}(ak+b) = \sum_{k=1}^{n} ak + \sum_{k=1}^{n} b$

$= a\displaystyle\sum_{k=1}^{n} k + nb$

$= \dfrac{an(n+1)}{2} + nb$ {using **c**}

But $\displaystyle\sum_{k=1}^{n}(ak+b) = 8n^2+11n$

$\therefore\ \dfrac{an(n+1)}{2} + nb = 8n^2+11n$

$\therefore\ \dfrac{an^2+an}{2} + nb = 8n^2+11n$

$\therefore\ \dfrac{a}{2}n^2 + \dfrac{a}{2}n + nb = 8n^2+11n$

$\therefore\ \dfrac{a}{2}n^2 + \left(\dfrac{a}{2}+b\right)n = 8n^2+11n$

Comparing coefficients, we get $\dfrac{a}{2} = 8$ and $\dfrac{a}{2}+b = 11$

$\therefore\ a = 16$ $\quad\therefore\ 8+b = 11$

$\therefore\ b = 3$

7 **a** $\displaystyle\sum_{k=1}^{n}(3k^2+4k-3) = \sum_{k=1}^{n} 3k^2 + \sum_{k=1}^{n} 4k - \sum_{k=1}^{n} 3$ {using **4 c**}

$= 3\displaystyle\sum_{k=1}^{n} k^2 + 4\sum_{k=1}^{n} k - 3n$ {using **4 a**, **b**}

b $\displaystyle\sum_{k=1}^{n}(k+1)(k+2) = \sum_{k=1}^{n}(k^2+3k+2)$

$= \displaystyle\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + 2n$

$= \dfrac{n(n+1)(2n+1)}{6} + 3\dfrac{n(n+1)}{2} + 2n$

$= \dfrac{n(n+1)(2n+1)+9n(n+1)+12n}{6}$

$= \dfrac{2n^3+n^2+2n^2+n+9n^2+9n+12n}{6}$

$= \dfrac{2n^3+12n^2+22n}{6}$

$= \dfrac{n(n^2+6n+11)}{3}$

When $n = 10$,

$$\text{LHS} = \sum_{k=1}^{10} (k+1)(k+2)$$
$$= 2\times 3 + 3\times 4 + 4\times 5 + 5\times 6 + 6\times 7 + 7\times 8 + 8\times 9 + 9\times 10 + 10\times 11 + 11\times 12$$
$$= 6 + 12 + 20 + 30 + 42 + 56 + 72 + 90 + 110 + 132$$
$$= 570$$

$$\text{RHS} = \frac{10(10^2 + 6(10) + 11)}{3} = \frac{10\times 171}{3} = 570 \quad \checkmark$$

EXERCISE 7F

1 **a** The series is arithmetic with $u_1 = 3,\quad d = 4,\quad n = 20$

$$S_n = \frac{n}{2}(2u_1 + (n-1)d)$$

So, $S_{20} = \frac{20}{2}(2\times 3 + 19\times 4)$
$= 10(6 + 76)$
$= 820$

b The series is arithmetic with $u_1 = \frac{1}{2},\quad d = \frac{5}{2},\quad n = 50$

$$S_n = \frac{n}{2}(2u_1 + (n-1)d)$$

So, $S_{50} = \frac{50}{2}\left(2\times\frac{1}{2} + 49\times\frac{5}{2}\right)$
$= 25(1 + 122\frac{1}{2})$
$= 3087\frac{1}{2}$

c The series is arithmetic with $u_1 = 100,\quad d = -7,\quad n = 40$

$$S_n = \frac{n}{2}(2u_1 + (n-1)d)$$

So, $S_{40} = \frac{40}{2}(2\times 100 + 39\times(-7))$
$= 20(200 - 273)$
$= -1460$

d The series is arithmetic with $u_1 = 50,\quad d = -\frac{3}{2},\quad n = 80$

$$S_n = \frac{n}{2}(2u_1 + (n-1)d)$$

So, $S_{80} = \frac{80}{2}\left(2\times 50 + 79\times(-\frac{3}{2})\right)$
$= 40(100 - \frac{237}{2})$
$= -740$

2 **a** The series is arithmetic with $u_1 = 5,\quad d = 3,\quad u_n = 101$

Since $u_n = 101$,
then $u_1 + (n-1)d = 101$
$\therefore\ 5 + 3(n-1) = 101$
$\therefore\ 5 + 3n - 3 = 101$
$\therefore\ 3n = 99$
$\therefore\ n = 33$

So, $S_n = \frac{n}{2}(u_1 + u_n)$
$= \frac{33}{2}(5 + 101)$
$= 1749$

b The series is arithmetic with $u_1 = 50,\quad d = -\frac{1}{2},\quad u_n = -20$

Since $u_n = -20$,
then $u_1 + (n-1)d = -20$
$\therefore\ 50 + (-\frac{1}{2})(n-1) = -20$
$\therefore\ -\frac{1}{2}n + \frac{1}{2} = -70$
$\therefore\ -\frac{1}{2}n = -\frac{141}{2}$
$\therefore\ n = 141$

So, $S_n = \frac{n}{2}(u_1 + u_n)$
$= \frac{141}{2}(50 + (-20))$
$= 2115$

c The series is arithmetic with $u_1 = 8,\quad d = \frac{5}{2},\quad u_n = 83$

Since $u_n = 83$,
then $u_1 + (n-1)d = 83$
$\therefore\ 8 + \frac{5}{2}(n-1) = 83$
$\therefore\ \frac{5}{2}n - \frac{5}{2} = 75$
$\therefore\ \frac{5}{2}n = \frac{155}{2}$
$\therefore\ n = 31$

So, $S_n = \frac{n}{2}(u_1 + u_n)$
$= \frac{31}{2}(8 + 83)$
$= 1410\frac{1}{2}$

3 **a** $\sum_{k=1}^{10}(2k+5) = 7+9+11+ .... +25$

This series is arithmetic with $u_1 = 7$, $d = 2$, and $n = 10$.

$\therefore$ sum $= \frac{n}{2}[2u_1 + (n-1)d] = \frac{10}{2}[14 + 9 \times 2] = 160$

b $\sum_{k=1}^{15}(k-50) = (-49) + (-48) + (-47) + ... + (-35)$

This series is arithmetic with $u_1 = -49$, $d = 1$, and $n = 15$.

$\therefore$ sum $= \frac{n}{2}[2u_1 + (n-1)d] = \frac{15}{2}[-98 + 14 \times 1] = -630$

c $\sum_{k=1}^{20}\left(\frac{k+3}{2}\right) = 2 + \frac{5}{2} + 3 + ... + \frac{23}{2}$

This series is arithmetic with $u_1 = 2$, $r = \frac{1}{2}$, and $n = 20$.

$\therefore$ sum $= \frac{n}{2}[2u_1 + (n-1)d] = \frac{20}{2}[4 + 19 \times \frac{1}{2}] = 135$

4 $u_1 = 5$, $n = 7$, $u_n = 53$

$$\begin{aligned} S_n &= \frac{n}{2}(u_1 + u_n) \\ &= \frac{7}{2}(5 + 53) \\ &= 203 \end{aligned}$$

5 $u_1 = 6$, $n = 11$, $u_n = -27$

$$\begin{aligned} S_n &= \frac{n}{2}(u_1 + u_n) \\ &= \frac{11}{2}(6 + (-27)) \\ &= -115\tfrac{1}{2} \end{aligned}$$

6 The total number of bricks can be expressed as an arithmetic series: $1 + 2 + 3 + 4 + ... + n$

We know that the total number of bricks is 171, so $S_n = 171$.

Also, $u_1 = 1$, $d = 1$ and we need to find n, the number of members (layers) of the series.

$$S_n = \frac{n}{2}(2u_1 + (n-1)d) = 171$$

$$\therefore \quad \frac{n}{2}(2 \times 1 + (n-1) \times 1) = 171$$

$$\therefore \quad n(2 + n - 1) = 342$$

$$\therefore \quad n(n+1) = 342$$

$$\therefore \quad n^2 + n - 342 = 0$$

$$\therefore \quad (n-18)(n+19) = 0$$

$$\therefore \quad n = -19 \text{ or } 18$$

But $n > 0$, so $n = 18$. So, there are 18 layers placed.

7 The total number of seats in n rows can be expressed as an arithmetic series:

$22 + 23 + 24 + ... + u_n$

Row 1 has 22 seats, so $u_1 = 22$. Row 2 has 23 seats, so $d = 1$.

$$\begin{aligned} S_n &= \frac{n}{2}(2u_1 + (n-1)d) \\ &= \frac{n}{2}(2 \times 22 + 1(n-1)) \\ &= \frac{n}{2}(44 + n - 1) \end{aligned}$$

$\therefore \quad S_n = \frac{n}{2}(n + 43)$ which is the total number of seats in n rows.

a Number of seats in row 44 of one section $=$ total no. of seats in every row $-$ no. of seats in the first 43 rows

$$\begin{aligned} &= S_{44} - S_{43} \\ &= \frac{44}{2}(44 + 43) - \frac{43}{2}(43 + 43) \\ &= 1914 - 1849 \\ &= 65 \end{aligned}$$

b Number of seats in a section $= S_{44} = 1914$ (from **a**)

c Number of seats in 25 sections $= S_{44} \times 25 = 1914 \times 25 = 47\,850$

8 **a** The first 50 multiples of 11 can be expressed as an arithmetic series:

$11 + 22 + 33 + ... + u_{50}$ where $u_1 = 11, \quad d = 11, \quad n = 50$

$S_n = \frac{n}{2}(2u_1 + (n-1)d) \qquad \therefore \quad S_{50} = \frac{50}{2}(2 \times 11 + 11(50-1))$

$$= 25(22 + 539)$$
$$= 14\,025$$

b The multiples of 7 between 0 and 1000 can be expressed as an arithmetic series:

$7 + 14 + 21 + 28 + ... + u_n$ where $u_1 = 7, \quad d = 7$

To find u_n, we need to find the largest multiple of 7 less than 1000.

Now $\frac{1000}{7} \approx 142.9$, so $u_n = 142 \times 7 = 994$

But $u_n = u_1 + (n-1)d$

$\therefore \quad 994 = 7 + 7(n-1)$

$\therefore \quad 987 = 7n - 7$

$\therefore \quad 7n = 994$

$\therefore \quad n = 142$

So, $S_{142} = \frac{142}{2}(7 + 994) = 71\,071$

c The integers between 1 and 100 which are not divisible by 3 can be expressed as:

1, 2, 4, 5, 7, 8, ..., 100 where $u_1 = 1, \quad u_n = 100$.

Alternatively, these integers can be expressed as two separate arithmetic series:

$S_A = 1 + 4 + 7 + ... + 97 + 100$ where $u_1 = 1, \quad d = 3, \quad u_n = 100$

and $S_B = 2 + 5 + 8 + ... + 95 + 98$ where $u_1 = 2, \quad d = 3, \quad u_n = 98$

Now for S_A, $u_n = u_1 + (n-1)d$

$\therefore \quad 100 = 1 + 3(n-1)$

$\therefore \quad 99 = 3n - 3$

$\therefore \quad 3n = 102$

$\therefore \quad n = 34$

and for S_B, $u_n = u_1 + (n-1)d$

$\therefore \quad 98 = 2 + 3(n-1)$

$\therefore \quad 96 = 3n - 3$

$\therefore \quad 3n = 99$

$\therefore \quad n = 33$

So, $S_A = \frac{34}{2}(1 + 100) = 1717$ and $S_B = \frac{33}{2}(2 + 98) = 1650$

The total sum $= S_A + S_B = 1717 + 1650 = 3367$

9 The series of odd numbers can be expressed as an arithmetic series:

$1 + 3 + 5 + 7 + ...$ where $u_1 = 1, \quad d = 2$

a Now $u_n = u_1 + (n-1)d = 1 + 2(n-1)$

$\therefore \quad u_n = 2n - 1$

b We need to show that S_n is n^2.

The sum of the first n odd numbers can be expressed as an arithmetic series:

$1 + 3 + 5 + 7 + ... + (2n-1)$ $\quad$ {using $u_n = 2n - 1$ from **a**}

So, $S_n = \frac{n}{2}(u_1 + u_n)$

$$= \frac{n}{2}(1 + (2n-1))$$
$$= \frac{n}{2}(2n)$$

Hence $S_n = n^2$ as required.

c $S_1 = 1 \qquad\qquad\quad = 1 = 1^2 = n^2$ for $n = 1$ ✓

$S_2 = 1 + 3 \qquad\quad = 4 = 2^2 = n^2$ for $n = 2$ ✓

$S_3 = 1 + 3 + 5 \quad\; = 9 = 3^2 = n^2$ for $n = 3$ ✓

$S_4 = 1 + 3 + 5 + 7 = 16 = 4^2 = n^2$ for $n = 4$ ✓

10 $u_6 = 21$, $S_{17} = 0$. We need to find u_1 and u_2.

$S_n = \frac{n}{2}(2u_1 + (n-1)d)$

$\therefore\ S_{17} = \frac{17}{2}(2u_1 + 16d) = 0$

$\therefore\ u_1 + 8d = 0$

$\therefore\ u_1 = -8d$ .... (1)

Also, $u_n = u_1 + (n-1)d$

$\therefore\ u_6 = u_1 + 5d$

$\therefore\ 21 = -8d + 5d$ {using (1)}

$\therefore\ 21 = -3d$

$\therefore\ d = -7$

So, $u_1 = -8(-7) = 56$ and $u_2 = 56 - 7 = 49$

The first two terms are 56 and 49.

11 Let the three consecutive terms be $x-d$, x, and $x+d$.

Now, sum of terms $= 12$

$\therefore\ (x-d) + x + (x+d) = 12$

$\therefore\ 3x = 12$

$\therefore\ x = 4$

So, the terms are $4-d$, 4, $4+d$

Also, product of terms $= -80$

$\therefore\ (4-d)4(4+d) = -80$

$\therefore\ 4(4^2 - d^2) = -80$

$\therefore\ 16 - d^2 = -20$

$\therefore\ d^2 = 36$ $\therefore\ d = \pm 6$

So, the three terms could be $4-6$, 4, $4+6$, which are -2, 4, 10

or $4-(-6)$, 4, $4+(-6)$, which are 10, 4, -2.

12 Let the five consecutive terms be $n-2d$, $n-d$, n, $n+d$, $n+2d$.

Now, sum of terms $= 40$ $\therefore\ (n-2d) + (n-d) + n + (n+d) + (n+2d) = 40$

$\therefore\ 5n = 40$

$\therefore\ n = 8$

So the terms are $8-2d$, $8-d$, 8, $8+d$, $8+2d$

Also, the product of the first, middle and last terms $= (8-2d) \times 8 \times (8+2d) = 224$

$\therefore\ 8(8^2 - 4d^2) = 224$

$\therefore\ 64 - 4d^2 = 28$

$\therefore\ 4d^2 = 36$

$\therefore\ d^2 = 9$

$\therefore\ d = \pm 3$

So, the five terms could be $8-2(3)$, $8-3$, 8, $8+3$, $8+2(3)$, which are 2, 5, 8, 11, 14

or $8-2(-3)$, $8-(-3)$, 8, $8+(-3)$, $8+2(-3)$, which are 14, 11, 8, 5, 2.

13 11, 14, 17, 20, ... is arithmetic with $u_1 = 11$, $d = 3$

$$\therefore\ S_n = \frac{n}{2}(2u_1 + (n-1)d) = \frac{n}{2}(22 + 3(n-1)) = \frac{n}{2}(3n+19)$$

Suppose $S_n = 2000$

$\therefore\ \frac{n}{2}(3n+19) = 2000$

$\therefore\ 3n^2 + 19n = 4000$

Using technology to list the terms of $3n^2 + 19n$:

$n = 33$ gives $3n^2 + 19n = 3894$ $\therefore\ S_{33} = 1947$

$n = 34$ gives $3n^2 + 19n = 4114$ $\therefore\ S_{34} = 2057$

$\therefore$ Henk would sell the 2000th TV set in week 34.

14 **a** $S_n = \dfrac{n(3n+11)}{2}$

$\therefore\ S_1 = u_1 = \dfrac{1(14)}{2} = 7$

and $S_2 = u_1 + u_2 = \dfrac{2(17)}{2} = 17$

$\therefore\ u_1 = 7$ and $u_2 = 10$

b $u_1 = 7$ and $d = 3$

$\therefore\ u_{20} = u_1 + 19d = 7 + 19 \times 3 = 64$

$\therefore$ the twentieth term is 64.

EXERCISE 7G.1

1 **a** The series is geometric with $u_1 = 12, \quad r = \frac{1}{2}, \quad n = 10$

$$\text{Now} \quad S_n = \frac{u_1(1-r^n)}{1-r}$$

$$\therefore \quad S_{10} = \frac{12\left(1-(\frac{1}{2})^{10}\right)}{1-\frac{1}{2}} \approx 23.9766 \approx 24.0$$

b The series is geometric with $u_1 = \sqrt{7}, \quad r = \sqrt{7}, \quad n = 12$

$$\text{Now} \quad S_n = \frac{u_1(r^n-1)}{r-1}$$

$$\therefore \quad S_{12} = \frac{\sqrt{7}\left((\sqrt{7})^{12}-1\right)}{\sqrt{7}-1} \approx 189\,134$$

c The series is geometric with $u_1 = 6, \quad r = -\frac{1}{2}, \quad n = 15$

$$\text{Now} \quad S_n = \frac{u_1(1-r^n)}{1-r}$$

$$\therefore \quad S_{15} = \frac{6\left(1-(-\frac{1}{2})^{15}\right)}{1-(-\frac{1}{2})} \approx 4.000$$

d The series is geometric with $u_1 = 1, \quad r = -\frac{1}{\sqrt{2}}, \quad n = 20$

$$\text{Now} \quad S_n = \frac{u_1(1-r^n)}{1-r}$$

$$\therefore \quad S_{20} = \frac{1\left(1-(-\frac{1}{\sqrt{2}})^{20}\right)}{1-(-\frac{1}{\sqrt{2}})} \approx 0.5852$$

2 **a** The series is geometric with $u_1 = \sqrt{3}, \quad r = \sqrt{3}$

$$S_n = \frac{u_1(r^n-1)}{r-1} = \frac{\sqrt{3}\left((\sqrt{3})^n-1\right)}{\sqrt{3}-1} \times \left(\frac{\sqrt{3}+1}{\sqrt{3}+1}\right) = \frac{\left(3+\sqrt{3}\right)\left((\sqrt{3})^n-1\right)}{3-1} = \frac{3+\sqrt{3}}{2}\left((\sqrt{3})^n-1\right)$$

b The series is geometric with $u_1 = 12, \quad r = \frac{1}{2}$

$$S_n = \frac{u_1(1-r^n)}{1-r} = \frac{12\left(1-(\frac{1}{2})^n\right)}{1-\frac{1}{2}} = 24\left(1-(\tfrac{1}{2})^n\right)$$

c The series is geometric with $u_1 = 0.9, \quad r = 0.1$

$$S_n = \frac{u_1(1-r^n)}{1-r} = \frac{0.9\,(1-(0.1)^n)}{1-0.1} = 1-(0.1)^n$$

d The series is geometric with $u_1 = 20, \quad r = -\frac{1}{2}$

$$S_n = \frac{u_1(1-r^n)}{1-r} = \frac{20\left(1-(-\frac{1}{2})^n\right)}{1-(-\frac{1}{2})} = \frac{20\left(1-(-\frac{1}{2})^n\right)}{(\frac{3}{2})} = \frac{40}{3}\left(1-(-\tfrac{1}{2})^n\right)$$

3 **a** $S_1 = u_1 \quad \therefore \quad u_1 = 3$

b $u_2 = S_2 - S_1 = 4 - 3 = 1$

So, $r = \frac{1}{3}$

c $u_1 = 3, \quad r = \frac{1}{3}$

so $u_n = 3 \times (\frac{1}{3})^{n-1}$

$\therefore \quad u_5 = 3 \times (\frac{1}{3})^4 = \frac{1}{27}$

4 **a** $\displaystyle\sum_{k=1}^{10} 3 \times 2^{k-1} = 3 + 6 + 12 + ... + 384 + 768 + 1536$

This series is geometric with $u_1 = 3, \quad r = 2, \quad$ and $\quad n = 10.$

$$\therefore \quad \text{sum} = \frac{u_1(r^n-1)}{r-1} = \frac{3(2^{10}-1)}{1} = 3069$$

b $\displaystyle\sum_{k=1}^{12} \left(\tfrac{1}{2}\right)^{k-2} = 2 + 1 + \tfrac{1}{2} + ... + \tfrac{1}{256} + \tfrac{1}{512} + \tfrac{1}{1024}$

This series is geometric with $u_1 = 2$, $r = \frac{1}{2}$, and $n = 12$.

$$\therefore \quad \text{sum} = \frac{u_1(1-r^n)}{1-r} = \frac{2\left(1-(\frac{1}{2})^{12}\right)}{\frac{1}{2}} = 4\left(1-\left(\tfrac{1}{2}\right)^{12}\right) = \frac{2^{12}-1}{2^{10}}$$

$$\therefore \quad \text{sum} = \frac{4095}{1024} \approx 4.00$$

c $\displaystyle\sum_{k=1}^{25} 6 \times (-2)^k = -12 + 24 + (-48) + ... + 100\,663\,296 + (-201\,326\,592)$

This series is geometric with $u_1 = -12$, $r = -2$, and $n = 25$.

$$\therefore \quad \text{sum} = \frac{u_1(1-r^n)}{1-r} = \frac{-12\left(1-(-2)^{25}\right)}{1+2} = -4\left(1-(-2)^{25}\right)$$

$$\therefore \quad \text{sum} = -134\,217\,732$$

5 **a**

$$\begin{aligned} A_2 &= A_1 \times 1.06 + 2000 \\ &= (A_0 \times 1.06 + 2000) \times 1.06 + 2000 \\ &= (2000 \times 1.06 + 2000) \times 1.06 + 2000 \end{aligned}$$

$\therefore \; A_2 = 2000 + 2000 \times 1.06 + 2000 \times (1.06)^2$ as required

b

$$\begin{aligned} A_3 &= A_2 \times 1.06 + 2000 \\ &= \left[2000 + 2000 \times 1.06 + 2000 \times (1.06)^2\right] \times 1.06 + 2000 \quad \{\text{from } \mathbf{a}\} \end{aligned}$$

$\therefore \; A_3 = 2000\left[1 + 1.06 + (1.06)^2 + (1.06)^3\right]$ as required

c

$$\begin{aligned} A_9 = 2000[&1 + 1.06 + (1.06)^2 + (1.06)^3 + (1.06)^4 + (1.06)^5 + (1.06)^6 + (1.06)^7 \\ &+ (1.06)^8 + (1.06)^9] \end{aligned}$$

$\therefore \; A_9 \approx 26\,361.59$

$\therefore$ the total bank balance after 10 years is \$26 361.59

6 **a** $S_1 = \frac{1}{2}$, $S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$, $S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$, $S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$,
$S_5 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = \frac{31}{32}$

b $S_n = \dfrac{2^n - 1}{2^n}$

c

$$\begin{aligned} S_n &= \frac{u_1(1-r^n)}{1-r}, \quad \text{where } u_1 = \tfrac{1}{2},\; r = \tfrac{1}{2} \\ &= \frac{\frac{1}{2}\left(1-(\frac{1}{2})^n\right)}{1-\frac{1}{2}} \end{aligned}$$

$$\therefore \; S_n = 1 - (\tfrac{1}{2})^n = 1 - \frac{1}{2^n} = \frac{2^n-1}{2^n}$$

d As $n \to \infty$, $\left(\frac{1}{2}\right)^n \to 0$,
and so $S_n \to 1$ (from below)

e The diagram represents one whole unit divided into smaller and smaller fractions.
As $n \to \infty$, the area which the fraction represents becomes smaller and smaller, and the total area approaches the area of a 1×1 unit square.

7 $u_2 = u_1 r = 6$ and so $u_1 = \dfrac{6}{r}$

$$\begin{aligned} S_3 = u_1 + u_1 r + u_1 r^2 &= -14 \\ \therefore \quad \frac{6}{r} + 6 + 6r &= -14 \\ \therefore \quad \frac{6}{r} + 20 + 6r &= 0 \\ \therefore \quad 6 + 20r + 6r^2 &= 0 \\ \therefore \quad 3r^2 + 10r + 3 &= 0 \\ \therefore \quad (3r+1)(r+3) &= 0 \\ \therefore \quad r &= -\tfrac{1}{3} \text{ or } -3 \end{aligned}$$

When $r = -\frac{1}{3}$, $u_1 = -18$

When $r = -3$, $u_1 = -2$

Since $u_4 = u_1 r^3$,

$$u_4 = -18 \times \left(-\tfrac{1}{3}\right)^3 \quad \text{or} \quad -2 \times (-3)^3$$

$$\therefore \quad u_4 = \tfrac{2}{3} \text{ or } 54$$

8 $u_1 = 1$

Now $u_2 = u_1 + d$ {for arithmetic}
and $u_2 = u_1 r$ {for geometric}
$\therefore\ u_1 + d = u_1 r$
$\therefore\ 1 + d = r$ .... (1)

Also, $u_{14} = u_1 + 13d$ {for arithmetic}
and $u_3 = u_1 r^2$ {for geometric}
$\therefore\ u_1 + 13d = 3u_1 r^2$
$\therefore\ 1 + 13d = 3r^2$ .... (2)

From (1), $d = r - 1$ $\therefore$ in (2)

$$\begin{aligned} 1 + 13(r-1) &= 3r^2 \\ \therefore\ 1 + 13r - 13 &= 3r^2 \\ \therefore\ 3r^2 - 13r + 12 &= 0 \\ \therefore\ (3r-4)(r-3) &= 0 \\ \therefore\ r &= \tfrac{4}{3} \text{ or } 3 \end{aligned}$$

When $r = \frac{4}{3}$, $d = \frac{1}{3}$: $u_{20} = u_1 + 19d = 1 + 19(\frac{1}{3}) = \frac{22}{3} = 7\frac{1}{3}$ {arithmetic}
$u_{20} = u_1 r^{19} = 1 \times \left(\frac{4}{3}\right)^{19} = \left(\frac{4}{3}\right)^{19}$ {geometric}

When $r = 3$, $d = 2$: $u_{20} = u_1 + 19d = 1 + 19(2) = 39$ {arithmetic}
$u_{20} = u_1 r^{19} = 1 \times 3^{19} = 3^{19}$ {geometric}

So, the 20th terms are $7\frac{1}{3}$ for arithmetic and $\left(\frac{4}{3}\right)^{19}$ for geometric or
39 for arithmetic and 3^{19} for geometric.

9 **a** The LHS is arithmetic with $u_1 = 5$, $d = 2$, 'n' $= n$.

Now $\frac{n}{2}(2u_1 + (n-1)d) = 1517$

$$\begin{aligned} \therefore\ \frac{n}{2}(10 + 2(n-1)) &= 1517 \\ \therefore\ \frac{n}{2}(2n+8) &= 1517 \\ \therefore\ n(n+4) - 1517 &= 0 \\ \therefore\ n^2 + 4n - 1517 &= 0 \quad \text{where } n > 1 \end{aligned}$$

and $n = 37$ {using technology}

b The LHS is geometric with $u_1 = 2$, $r = 3$, 'n' $= n$.

Now $u_1\left(\frac{r^n - 1}{r - 1}\right) = 177\,146$

$$\begin{aligned} \therefore\ 2\left(\frac{3^n - 1}{3 - 1}\right) &= 177\,146 \\ \therefore\ 3^n - 1 &= 177\,146 \\ \therefore\ 3^n &= 177\,147 \\ \therefore\ n &= 11 \quad \text{\{using technology\}} \end{aligned}$$

10 $u_1, u_2, ..., u_n$ is a geometric sequence with common ratio r.

$\therefore\ u_n = u_1 r^{n-1}$

Now

$$\begin{aligned} &(u_1+u_2)^2 + (u_2+u_3)^2 + (u_3+u_4)^2 + ... + (u_{n-1}+u_n)^2 \\ &= (u_1^2 + 2u_1u_2 + u_2^2) + (u_2^2 + 2u_2u_3 + u_3^2) + (u_3^2 + 2u_3u_4 + u_4^2) + ... \\ &\quad + (u_{n-1}^2 + 2u_{n-1}u_n + u_n^2) \\ &= (2u_1u_2 + 2u_2u_3 + 2u_3u_4 + ... + 2u_{n-1}u_n) + (u_1^2 + 2u_2^2 + 2u_3^2 + 2u_4^2 + ... + 2u_{n-1}^2 + u_n^2) \\ &= \underbrace{2(u_1u_2 + u_2u_3 + u_3u_4 + ... + u_{n-1}u_n)}_{(1)} + \underbrace{2(u_1^2 + u_2^2 + u_3^2 + ... + u_{n-1}^2 + u_n^2)}_{(2)} - (u_1^2 + u_n^2) \quad .... (*) \end{aligned}$$

Consider (1):

$$\begin{aligned} &2(u_1u_2 + u_2u_3 + u_3u_4 + ... + u_{n-1}u_n) \\ &= 2\left[u_1(u_1r) + (u_1r)(u_1r^2) + (u_1r^2(u_1r^3) + ... + (u_1r^{n-2})(u_1r^{n-1})\right] \end{aligned}$$

{using the sequence definition $u_n = u_1r^{n-1}$ to substitute values for $u_2, u_3, u_4, ..., u_{n-1}, u_n$}

$= 2u_1(u_1r + u_1r^3 + u_1r^5 + ... + u_1r^{2n-3})$ {taking out common factor u_1}

Consider (2):

$$\begin{aligned} &2(u_1^2 + u_2^2 + u_3^2 + ... + u_{n-1}^2 + u_n^2) \\ &= 2\left[u_1^2 + (u_1r)^2 + (u_1r^2)^2 + ... + (u_1r^{n-2})^2 + (u_1r^{n-1})^2\right] \end{aligned}$$

{using the sequence definition $u_n = u_1r^{n-1}$ to substitute values for $u_2, u_3, ..., u_{n-1}, u_n$}

$= 2(u_1^2 + u_1^2r^2 + u_1^2r^4 + ... + u_1^2r^{2n-4} + u_1^2r^{2n-2})$

$= 2u_1(u_1 + u_1r^2 + u_1r^4 + ... + u_1r^{2n-4} + u_1r^{2n-2})$ {taking out common factor u_1}

Now $(*)$:

$$2(u_1u_2+u_2u_3+u_3u_4+....+u_{n-1}u_n)+2(u_1^{\,2}+u_2^{\,2}+u_3^{\,2}+....+u_{n-1}^{\,2}+u_n^{\,2})-(u_1^{\,2}+u_n^{\,2})$$
$$=2u_1(u_1r+u_1r^3+u_1r^5+....+u_1r^{2n-3})+2u_1(u_1+u_1r^2+u_1r^4+....+u_1r^{2n-4}+u_1r^{2n-2})$$
$$-(u_1^{\,2}+u_n^{\,2})$$
$$=2u_1\underbrace{(u_1+u_1r+u_1r^2+u_1r^3+u_1r^4+u_1r^5+....+u_1r^{2n-4}+u_1r^{2n-3}+u_1r^{2n-2})}_{(3)}-(u_1^{\,2}+u_n^{\,2})$$

{Notice that (3) is the sum of the first $(2n-1)$ terms of a geometric sequence with first term u_1 and common ratio r.}

$$=2u_1\times\frac{u_1(r^{2n-1}-1)}{r-1}-(u_1^{\,2}+u_n^{\,2})$$
$$=\frac{2u_1^{\,2}(r^{2n-1}-1)}{r-1}-(u_1^{\,2}+u_n^{\,2})$$

11 **a** $A_3=A_2\times 1.03-R$
$$=(\$8000\times(1.03)^2-1.03R-R)\times 1.03-R$$
$$=\$8000\times(1.03)^3-(1.03)^2R-(1.03)R-R$$

b $A_8=\$8000\times(1.03)^8-(1.03)^7R-(1.03)^6R-(1.05)^5R-....-1.03R-R$

c $A_8=0$

$$\therefore\quad R\left(1+1.03+(1.03)^2+(1.03)^3+....+(1.03)^7\right)=\$8000\times(1.03)^8$$
$$\therefore\quad R\left(1\left[\frac{(1.03)^8-1}{1.03-1}\right]\right)=\$8000\times(1.03)^8$$
$$\therefore\quad R=\frac{\$8000\times(1.03)^8\times 0.03}{(1.03)^8-1}\approx\$1139.65$$

d Notice in **c** that $\$8000=P$ and $(1.03)^8=\left(1+\frac{3}{100}\right)^8=\left(1+\frac{r}{100}\right)^m$

$$0.03=\tfrac{3}{100}=\frac{r}{100}$$
$$\text{and}\quad (1.03)^8-1=\left(1+\frac{r}{100}\right)^m-1$$

So, in the general case $R=\dfrac{P\times\left(1+\frac{r}{100}\right)^m\times\frac{r}{100}}{\left(1+\frac{r}{100}\right)^m-1}$

EXERCISE 7G.2

1 **a** **i** $u_1=\frac{3}{10}$

ii $r=\dfrac{\left(\frac{3}{100}\right)}{\left(\frac{3}{10}\right)}=\frac{1}{10}$
$=0.1$

b We need to show that $0.\overline{3}=\frac{1}{3}$.

Now $0.\overline{3}=\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+....$

So, let $S_n=\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+....$

Since $n\to\infty$, then $S=\dfrac{u_1}{1-r}=\dfrac{\frac{3}{10}}{1-\left(\frac{1}{10}\right)}=\frac{1}{3}$

So, $0.\overline{3}=\frac{1}{3}$ as required.

2 **a** $0.\overline{4} = 0.444\,444\,....$
$= \frac{4}{10} + \frac{4}{100} + \frac{4}{1000} + ...$
which is a geometric series with
$u_1 = 0.4, \quad r = 0.1$
$\therefore \quad S = \dfrac{u_1}{1-r} = \dfrac{0.4}{1-0.1} = \dfrac{0.4}{0.9}$
$= \frac{4}{9}$
So, $0.\overline{4} = \frac{4}{9}$

b $0.\overline{16} = 0.161\,616\,....$
$= \dfrac{16}{10^2} + \dfrac{16}{10^4} + \dfrac{16}{10^6} + ...$
which is a geometric series with
$u_1 = 0.16, \; r = 0.01$
$\therefore \quad S = \dfrac{u_1}{1-r} = \dfrac{0.16}{0.99} = \dfrac{16}{99}$
So, $0.\overline{16} = \frac{16}{99}$

c $0.\overline{312} = 0.312\,312\,312\,....$
$= \dfrac{312}{10^3} + \dfrac{312}{10^6} + \dfrac{312}{10^9} + ...$
which is a geometric series with $u_1 = 0.312, \quad r = 0.001$
$\therefore \quad S = \dfrac{u_1}{1-r} = \dfrac{0.312}{0.999} = \dfrac{312}{999} = \dfrac{104}{333}$ So, $0.\overline{312} = \frac{104}{333}$

3 Checking **Exercise 7G.1 6b**: $S = \dfrac{u_1}{1-r} = \dfrac{\frac{1}{2}}{1-\frac{1}{2}} = 1$ ✓

4 **a** $18 + 12 + 8 + ...$ is an infinite geometric series with $u_1 = 18, \; r = \frac{2}{3}$.
$\therefore \quad S = \dfrac{u_1}{1-r} = \dfrac{18}{\frac{1}{3}} = 54$

b $18.9 - 6.3 + 2.1 - ....$ is an infinite geometric series with $u_1 = 18.9, \; r = -\frac{1}{3}$.
$\therefore \quad S = \dfrac{u_1}{1-r} = \dfrac{18.9}{\frac{4}{3}} = 14.175$

5 **a** $\displaystyle\sum_{k=1}^{\infty} \frac{3}{4^k} = \frac{3}{4} + \frac{3}{16} + \frac{3}{64} + ...$
is an infinite geometric series with
$u_1 = \frac{3}{4}, \; r = \frac{1}{4}$.
$\therefore \quad S = \dfrac{u_1}{1-r} = \dfrac{\frac{3}{4}}{\frac{3}{4}} = 1$

b $\displaystyle\sum_{k=0}^{\infty} 6(-\tfrac{2}{5})^k = 6 - 6 \times (\tfrac{2}{5}) + 6 \times (\tfrac{2}{5})^2 - ....$
is an infinite geometric series with
$u_1 = 6, \; r = -\frac{2}{5}$.
$\therefore \quad S = \dfrac{u_1}{1-r} = \dfrac{6}{\frac{7}{5}} = \frac{30}{7} = 4\frac{2}{7}$

6 Let the terms of the geometric series be $u_1, \; u_1 r, \; u_1 r^2, \; ....$

Then $u_1 + u_1 r + u_1 r^2 = 19$
$\therefore \quad u_1(1 + r + r^2) = 19$
$\therefore \quad u_1 = \dfrac{19}{1 + r + r^2}$ .... (1)

and $\dfrac{u_1}{1-r} = 27$
$\therefore \quad u_1 = 27(1-r)$ .... (2)

Equating (1) and (2), $\dfrac{19}{1+r+r^2} = 27(1-r)$
$\therefore \quad \frac{19}{27} = (1-r)(1+r+r^2)$
$\therefore \quad \frac{19}{27} = 1 + r + r^2 - r - r^2 - r^3$
$\therefore \quad \frac{19}{27} = 1 - r^3$
$\therefore \quad r^3 = \frac{8}{27}$
$\therefore \quad r = \frac{2}{3}$

Substituting $r = \frac{2}{3}$ into (2) gives $u_1 = 27(1 - \frac{2}{3}) = 9$
$\therefore$ the first term is 9 and the common ratio is $\frac{2}{3}$.

7 Let the terms of the geometric series be $u_1,\ u_1r,\ u_1r^2, ...$

Then $u_1 r = \frac{8}{5}$ and $\dfrac{u_1}{1-r} = 10$

$\therefore\ u_1 = \dfrac{8}{5r}$ .... (1) $\qquad \therefore\ u_1 = 10 - 10r$ .... (2)

Equating (1) and (2), $\dfrac{8}{5r} = 10 - 10r$

$\therefore\ 8 = 50r - 50r^2$

$\therefore\ 50r^2 - 50r + 8 = 0$

$\therefore\ 2(25r^2 - 25r + 4) = 0$

$\therefore\ 2(5r-1)(5r-4) = 0$

$\therefore\ r = \frac{1}{5}$ or $\frac{4}{5}$

Using (2), if $r = \frac{1}{5}$, $u_1 = 10 - 10(\frac{1}{5}) = 8$

if $r = \frac{4}{5}$, $u_1 = 10 - 10(\frac{4}{5}) = 2$

$\therefore$ either $u_1 = 8,\ r = \frac{1}{5}$ or $u_1 = 2,\ r = \frac{4}{5}$

8 **a** Total time of motion $= 1 + (90\% \times 1) + (90\% \times 1) + (90\% \times 90\% \times 1)$
$+ (90\% \times 90\% \times 1) + (90\% \times 90\% \times 90\% \times 1) + ...$
$= 1 + 0.9 + 0.9 + (0.9)^2 + (0.9)^2 + (0.9)^3 + ...$
$= 1 + 2(0.9) + 2(0.9)^2 + 2(0.9)^3 + ...$ as required

b The total time of motion can be written as $[2 + 2(0.9) + 2(0.9)^2 + 2(0.9)^3 + ...] - 1$

So, $S_n = \dfrac{u_1(1-r^n)}{1-r} - 1$, where $u_1 = 2,\ r = 0.9$

$\therefore\ S_n = \dfrac{2(1-0.9^n)}{1-0.9} - 1$

$\therefore\ S_n = \dfrac{2(1-0.9^n)}{0.1} - 1$

$\therefore\ S_n = 20(1-0.9^n) - 1$

$\therefore\ S_n = 20 - 20 \times 0.9^n - 1$

$\therefore\ S_n = 19 - 20 \times 0.9^n$

c For the ball to come to rest, n must approach infinity.
As $n \to \infty$, $0.9^n \to 0$ and so $20 \times 0.9^n \to 0$ also.
$\therefore\ S_n \to 19^-$.
So, it takes 19 seconds for the ball to come to rest.

9 **a** $18 - 9 + 4.5 - ....$ is an infinite geometric series with $u_1 = 18,\ r = -\frac{1}{2}$.
Since $|r| < 1$, the series converges.

$\therefore\ S = \dfrac{u_1}{1-r}$

$= \dfrac{18}{\frac{3}{2}}$

$= 12$

$\therefore$ the series is convergent, and its sum is 12.

b $1.2 + 1.8 + 2.7 + ...$ is an infinite geometric series with $u_1 = 1.2,\ r = 1.5$.
Since $|r| > 1$, the series is divergent.

$S_n > 100$ when $\dfrac{u_1(r^n - 1)}{r-1} > 100$

$\therefore\ \dfrac{1.2(1.5^n - 1)}{\frac{1}{2}} > 100$

$\therefore\ 2.4(1.5^n - 1) > 100$

$\therefore\ 1.5^n - 1 > \frac{125}{3}$

$\therefore\ 1.5^n > \frac{128}{3}$

$\therefore\ n > 9.26$

{using technology}

$\therefore\ n = 10$ is the smallest value of n such that $S_n > 100$.

10 Total distance travelled

$= h + 2\left(\frac{3}{4}\right)h + 2\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)h + ...$

$= h + 2\left(\frac{3}{4}\right)h\left[1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + ...\right]$

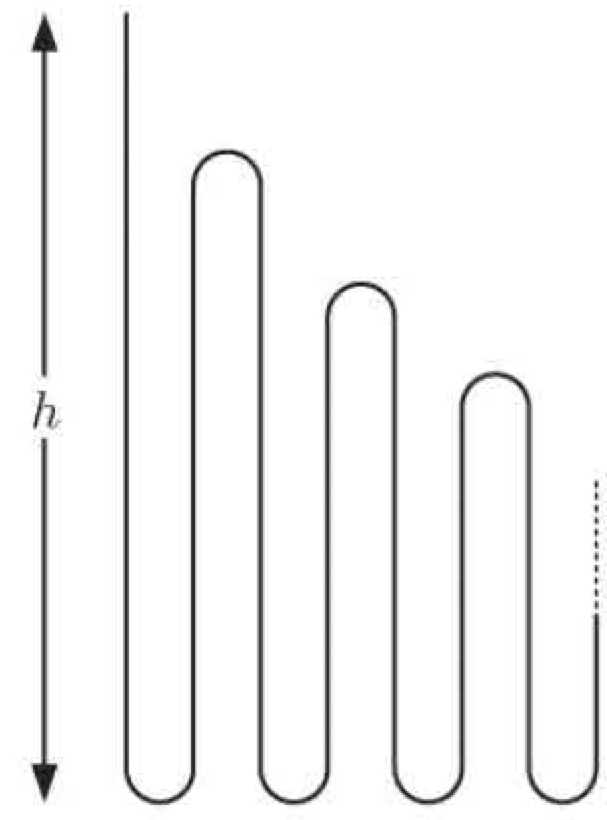
$= h + \frac{3}{2}h\left(\frac{1}{1-\frac{3}{4}}\right)$ $\left\{\text{as } |r| = \left|\frac{3}{4}\right| < 1 \text{ and } S = \frac{u_1}{1-r}\right\}$

$= h + \frac{3}{2}h(4)$

$= 7h$

But $7h = 490$, so $h = 70$.

The ball was dropped from a height of 70 cm.



11

$$\sum_{k=1}^{\infty}\left(\frac{3x}{2}\right)^{k-1} = \left(\frac{3x}{2}\right)^0 + \left(\frac{3x}{2}\right)^1 + \left(\frac{3x}{2}\right)^2 + \left(\frac{3x}{2}\right)^3 + ...$$

$$= 1 + \frac{3x}{2} + \left(\frac{3x}{2}\right)^2 + \left(\frac{3x}{2}\right)^3 + ...$$

$$= \frac{u_1}{1-r} \qquad \{\text{as it converges to 4 and is geometric}\}$$

$$= \frac{1}{1-\frac{3x}{2}} = \frac{2}{2-3x}$$

$\therefore \frac{2}{2-3x} = 4$ and so $2 - 3x = \frac{1}{2}$

$\therefore 3x = 1\frac{1}{2}$

$\therefore x = \frac{1}{2}$

REVIEW SET 7A

1 **a** arithmetic, $d = -8$ **b** geometric, $r = 1$ or arithmetic, $d = 0$

c geometric, $r = -\frac{1}{2}$ **d** neither

e 4, 8, 12, 16, ... arithmetic, $d = 4$

2 Since the terms are consecutive,

$(k-2) - 3k = k + 7 - (k-2)$ {equating common differences}

$\therefore k - 2 - 3k = k + 7 - k + 2$

$\therefore -2 - 2k = 9$

$\therefore 2k = -11$

$\therefore k = -\frac{11}{2}$

3 28, 23, 18, 13, ...

$23 - 28 = -5$

$18 - 23 = -5$

$13 - 18 = -5$

Assuming that the pattern continues, consecutive terms differ by -5.

$\therefore$ the sequence is arithmetic with $u_1 = 28$, $d = -5$.

$u_n = u_1 + (n-1)d$

$= 28 + (n-1)(-5)$

$= 28 - 5n + 5$

$= 33 - 5n$

$S_n = \frac{n}{2}(2u_1 + (n-1)d)$

$= \frac{n}{2}(2 \times 28 + (n-1)(-5))$

$= \frac{n}{2}(56 - 5n + 5)$

$= \frac{n}{2}(61 - 5n)$

4 The terms are geometric, so
$$\frac{k}{4} = \frac{k^2-1}{k}$$
$$\therefore\ k^2 = 4(k^2-1)$$
$$\therefore\ 3k^2 = 4$$
$$\therefore\ k^2 = \tfrac{4}{3}$$
$$\therefore\ k = \pm\tfrac{2}{\sqrt{3}} = \pm\tfrac{2\sqrt{3}}{3}$$

5 $u_6 = \frac{16}{3}$ $\quad\therefore\ u_1 \times r^5 = \frac{16}{3}$ .... (1)

$u_{10} = \frac{256}{3}$ $\quad\therefore\ u_1 \times r^9 = \frac{256}{3}$ .... (2)

So, $\dfrac{u_1 r^9}{u_1 r^5} = \dfrac{(\frac{256}{3})}{(\frac{16}{3})}$ $\quad\{(2) \div (1)\}$
$$\therefore\ r^4 = 16$$
$$\therefore\ r = \pm 2$$

Substituting $r = 2$ into (1) gives
$$u_1 \times 2^5 = \tfrac{16}{3}$$
$$\therefore\ u_1 \times 32 = \tfrac{16}{3}$$
$$\therefore\ u_1 = \tfrac{1}{6}$$

Substituting $r = -2$ into (1) gives
$$u_1 \times (-2)^5 = \tfrac{16}{3}$$
$$\therefore\ u_1 \times (-32) = \tfrac{16}{3}$$
$$\therefore\ u_1 = -\tfrac{1}{6}$$

Now $u_n = u_1 r^{n-1}$ $\quad\therefore\ u_n = \frac{1}{6} \times 2^{n-1}$ or $-\frac{1}{6} \times (-2)^{n-1}$

6 Let the numbers be $23,\ 23+d,\ 23+2d,\ 23+3d,\ 23+4d,\ 23+5d,\ 23+6d,\ 9$

Then $23 + 7d = 9$
$$\therefore\ 7d = -14$$
$$\therefore\ d = -2$$
So, the numbers are 23, 21, 19, 17, 15, 13, 11, 9.

7 **a** The sequence 86, 83, 80, 77, ... is arithmetic with $u_1 = 86,\ d = -3$.
$$u_n = u_1 + (n-1)d$$
$$\therefore\ u_n = 86 + (n-1)(-3) = 86 - 3n + 3$$
$$\therefore\ u_n = 89 - 3n$$

b $\frac{3}{4}, 1, \frac{7}{6}, \frac{9}{7}$, ... can also be written as $\frac{3}{4}, \frac{5}{5}, \frac{7}{6}, \frac{9}{7}$, ...

So, the numerator starts at 3 and increases by 2 each time, whilst the denominator starts at 4 and increases by 1 each time.

The nth term is $\dfrac{2n+1}{n+3}$, and so $u_n = \dfrac{2n+1}{n+3}$

c The sequence 100, 90, 81, 72.9, ... is geometric with $u_1 = 100,\ r = \frac{90}{100} = 0.9$
$$u_n = u_1 r^{n-1}$$
$$\therefore\ u_n = 100(0.9)^{n-1}$$

8 **a**
$$\sum_{k=1}^{7} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2$$
$$= 1 + 4 + 9 + 16 + 25 + 36 + 49$$
$$= 140$$

b
$$\sum_{k=1}^{4} \frac{k+3}{k+2} = \tfrac{4}{3} + \tfrac{5}{4} + \tfrac{6}{5} + \tfrac{7}{6}$$
$$= \tfrac{99}{20}$$

9 **a** $18 - 12 + 8 - ....$

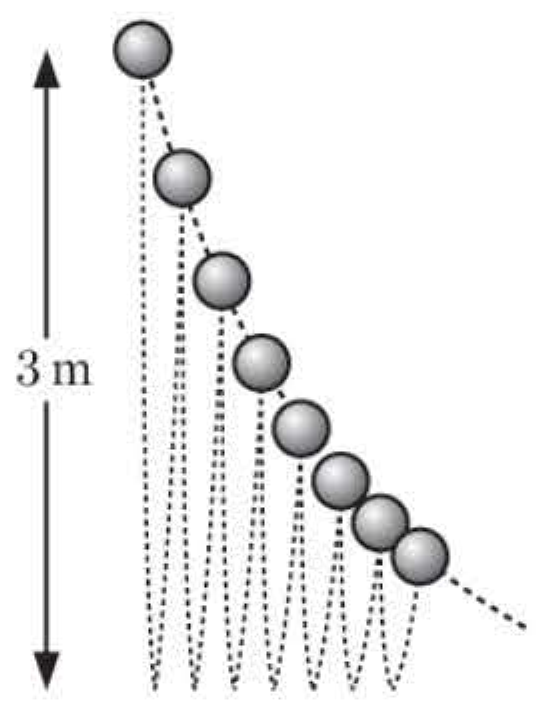
The series is geometric with $u_1 = 18,\ r = -\frac{2}{3}$
$$\therefore\ S = \frac{u_1}{1-r}$$
$$= \frac{18}{\frac{5}{3}}$$
$$= \tfrac{54}{5}$$
$$= 10\tfrac{4}{5}$$

b $8+4\sqrt{2}+4+....$

The series is geometric with $u_1=8,\ r=\frac{1}{\sqrt{2}}$

$$\therefore\ S=\frac{u_1}{1-r}$$
$$=\frac{8}{(1-\frac{1}{\sqrt{2}})}\times\frac{(1+\frac{1}{\sqrt{2}})}{(1+\frac{1}{\sqrt{2}})}$$
$$=\frac{8+\frac{8}{\sqrt{2}}}{1-\frac{1}{2}}$$
$$=\frac{8+4\sqrt{2}}{\frac{1}{2}}$$
$$=16+8\sqrt{2}$$

10



Total distance travelled

$$=3+3\times0.8\times2+3\times(0.8)^2\times2+3\times(0.8)^3\times2+....$$
$$=3+3\times0.8\times2\left[1+0.8+(0.8)^2+(0.8)^3+....\right]$$
$$=3+4.8\times\frac{1}{1-0.8}\quad\left\{\text{as } r=0.8,\quad |r|<1 \text{ so converges to } \frac{u_1}{1-r}\right\}$$
$$=3+\tfrac{4.8}{0.2}$$
$$=3+24=27 \text{ metres}$$

11 **a** $S_n=\dfrac{3n^2+5n}{2}$

$$\therefore\ u_n=S_n-S_{n-1}$$
$$=\frac{3n^2+5n}{2}-\frac{3(n-1)^2+5(n-1)}{2}$$
$$=\frac{3n^2+5n-3(n^2-2n+1)-5(n-1)}{2}$$
$$=\frac{3n^2+5n-3n^2+6n-3-5n+5}{2}$$
$$=\frac{6n+2}{2}$$
$$\therefore\ u_n=3n+1$$

b Using part **a**,

$$u_n-u_{n-1}=[3n+1]-[3(n-1)+1]$$
$$=3n+1-3n+3-1$$
$$=3$$

The difference between consecutive terms is constant for all n, so the sequence is arithmetic.

12 If a, b, and c are consecutive terms of a geometric sequence with constant ratio r, then $b=ar$ and $c=ar^2$ .... (1)

If a, b, and c are consecutive terms of an arithmetic sequence then

$$b-a=c-b$$
$$\therefore\ ar-a=ar^2-ar\quad\{\text{using (1)}\}$$
$$\therefore\ ar^2-2ar+a=0$$
$$\therefore\ a(r^2-2r+1)=0$$
$$\therefore\ a(r-1)^2=0$$
$$\therefore\ a=0 \text{ or } r=1$$

If $a=0$ then $b=0$ and $c=0$ {using (1)}

If $r=1$ then $b=a(1)=a$ and $c=a(1)^2=a$

In either case, $a=b=c$.

13 If x, y, and z are consecutive terms of a geometric sequence, then

$$\frac{y}{x} = \frac{z}{y} \quad \{\text{equating constant ratios}\}$$

$$\therefore \quad y^2 = xz \quad .... (1)$$

Now $x + y + z = \frac{7}{3}$ .... (2)

$$\therefore \quad (x+y+z)^2 = \tfrac{49}{9}$$

$$\therefore \quad x^2 + y^2 + z^2 + 2xy + 2xz + 2yz = \tfrac{49}{9} \qquad \{\text{expanding LHS}\}$$

$$\therefore \quad \tfrac{91}{9} + 2(xy + xz + yz) = \tfrac{49}{9} \qquad \{x^2 + y^2 + z^2 = \tfrac{91}{9}\}$$

$$\therefore \quad 2(xy + xz + yz) = -\tfrac{42}{9}$$

$$\therefore \quad xy + xz + yz = -\tfrac{7}{3}$$

$$\therefore \quad xy + y^2 + yz = -\tfrac{7}{3} \qquad \{\text{using (1)}\}$$

$$\therefore \quad y(x+y+z) = -(x+y+z) \qquad \{x + y + z = \tfrac{7}{3}\}$$

$$\therefore \quad y = -1$$

Substituting $y = -1$ into (1) and (2) gives $xz = 1$ and $x + z = \frac{10}{3}$ .... (4)

$$\therefore \quad z = \frac{1}{x} \quad .... (3)$$

Substituting (3) into (4) gives $x + \dfrac{1}{x} = \frac{10}{3}$

$$\therefore \quad 3x^2 - 10x + 3 = 0$$

$$\therefore \quad (3x-1)(x-3) = 0$$

$$\therefore \quad x = \tfrac{1}{3} \text{ or } 3$$

Using (3), if $x = \frac{1}{3}$, $z = 3$ and if $x = 3$, $z = \frac{1}{3}$

$$\therefore \quad x = \tfrac{1}{3},\ y = -1,\ z = 3 \quad \text{or} \quad x = 3,\ y = -1,\ z = \tfrac{1}{3}$$

14 The first two terms of a geometric series are $2x$ and $x - 2$, so $u_1 = 2x$ and $r = \dfrac{x-2}{2x}$

Now $S = \dfrac{u_1}{1-r} = \dfrac{2x}{1 - \left(\frac{x-2}{2x}\right)}$

$$= \frac{4x^2}{2x - (x-2)}$$

$$= \frac{4x^2}{x+2}$$

The sum of the series is $\frac{18}{7}$, so $\dfrac{4x^2}{x+2} = \frac{18}{7}$

$$\therefore \quad 28x^2 = 18x + 36$$

$$\therefore \quad 28x^2 - 18x - 36 = 0$$

$$\therefore \quad 2(14x^2 - 9x - 18) = 0$$

$$\therefore \quad 2(7x+6)(2x-3) = 0$$

$$\therefore \quad x = -\tfrac{6}{7} \text{ or } \tfrac{3}{2}$$

When $x = -\frac{6}{7}$, $r = \dfrac{-\frac{6}{7} - 2}{2(-\frac{6}{7})}$

$$= \frac{-\frac{20}{7}}{-\frac{12}{7}}$$

$$= \tfrac{5}{3}$$

When $x = \frac{3}{2}$, $r = \dfrac{\frac{3}{2} - 2}{2(\frac{3}{2})}$

$$= \frac{-\frac{1}{2}}{3}$$

$$= -\tfrac{1}{6}$$

$|r| < 1$ only when $x = \frac{3}{2}$, so $x = \frac{3}{2}$ is the only solution.

15 a, b, and c are arithmetic, so $a - b = b - c = d$ where d is a constant.

a $(c+a) - (b+c) = c + a - b - c$
$= a - b$
$= d$

$(a+b) - (c+a) = a + b - c - a$
$= b - c$
$= d$

$\therefore$ the differences between the terms are equal.

$\therefore$ $b + c$, $c + a$, and $a + b$ are also consecutive terms of an arithmetic sequence.

b $\dfrac{1}{\sqrt{b}+\sqrt{c}} = \dfrac{1}{(\sqrt{b}+\sqrt{c})}\dfrac{(\sqrt{b}-\sqrt{c})}{(\sqrt{b}-\sqrt{c})} = \dfrac{\sqrt{b}-\sqrt{c}}{b-c} = \dfrac{\sqrt{b}-\sqrt{c}}{d}$ .... (1)

$\dfrac{1}{\sqrt{c}+\sqrt{a}} = \dfrac{1}{(\sqrt{c}+\sqrt{a})}\dfrac{(\sqrt{c}-\sqrt{a})}{(\sqrt{c}-\sqrt{a})} = \dfrac{\sqrt{c}-\sqrt{a}}{c-a} = \dfrac{\sqrt{c}-\sqrt{a}}{-2d}$ .... (2)

$\dfrac{1}{\sqrt{a}+\sqrt{b}} = \dfrac{1}{(\sqrt{a}+\sqrt{b})}\dfrac{(\sqrt{a}-\sqrt{b})}{(\sqrt{a}-\sqrt{b})} = \dfrac{\sqrt{a}-\sqrt{b}}{a-b} = \dfrac{\sqrt{a}-\sqrt{b}}{d}$ .... (3)

Using (2) and (1):

$$\frac{1}{\sqrt{c}+\sqrt{a}} - \frac{1}{\sqrt{b}+\sqrt{c}} = \frac{\sqrt{c}-\sqrt{a}}{-2d} - \frac{\sqrt{b}-\sqrt{c}}{d} = \frac{\sqrt{a}-\sqrt{c}-2\sqrt{b}+2\sqrt{c}}{2d}$$
$$= \frac{\sqrt{a}-2\sqrt{b}+\sqrt{c}}{2d}$$

Using (3) and (2):

$$\frac{1}{\sqrt{a}+\sqrt{b}} - \frac{1}{\sqrt{c}+\sqrt{a}} = \frac{\sqrt{a}-\sqrt{b}}{d} - \frac{\sqrt{c}-\sqrt{a}}{-2d} = \frac{2\sqrt{a}-2\sqrt{b}+\sqrt{c}-\sqrt{a}}{2d}$$
$$= \frac{\sqrt{a}-2\sqrt{b}+\sqrt{c}}{2d}$$

$\therefore$ the differences between the terms are equal.

$\therefore$ $\dfrac{1}{\sqrt{b}+\sqrt{c}}$, $\dfrac{1}{\sqrt{c}+\sqrt{a}}$ and $\dfrac{1}{\sqrt{a}+\sqrt{b}}$ are also consecutive terms of an arithmetic sequence.

REVIEW SET 7B

1 **a** $u_n = 3^{n-2}$ $\quad\therefore\quad u_1 = 3^{-1} = \frac{1}{3},\quad u_2 = 3^0 = 1,\quad u_3 = 3^1 = 3,\quad u_4 = 3^2 = 9$

b $u_n = \dfrac{3n+2}{n+3}$ $\quad\therefore\quad u_1 = \frac{5}{4},\quad u_2 = \frac{8}{5},\quad u_3 = \frac{11}{6},\quad u_4 = \frac{14}{7} = 2$

c $u_n = 2^n - (-3)^n$

$\therefore\quad u_1 = 2-(-3) = 5,\quad u_2 = 4-9 = -5,\quad u_3 = 8-(-27) = 35,\quad u_4 = 16-81 = -65$

2 $u_n = 6\left(\frac{1}{2}\right)^{n-1}$

a $\dfrac{u_{n+1}}{u_n} = \dfrac{6\left(\frac{1}{2}\right)^{n+1-1}}{6\left(\frac{1}{2}\right)^{n-1}} = \frac{1}{2}$ for all n

$\therefore$ $\{u_n\}$ is a geometric sequence.

b $u_1 = 6,$
$r = \frac{1}{2}$

c $u_{16} = 6\left(\frac{1}{2}\right)^{15}$
$\approx 0.000\,183$

3 **a** Given $24, 23\frac{1}{4}, 22\frac{1}{2}, ..., -36$ we have $u_1 = 24,\quad u_n = -36,$ and we need to find n.

The sequence is arithmetic with $d = -\frac{3}{4}$.

Now $u_n = u_1 + (n-1)d$

$\therefore\quad -36 = 24 + (n-1)(-\frac{3}{4})$

$\therefore\quad -60 = -\frac{3}{4}n + \frac{3}{4}$

$\therefore\quad \frac{3}{4}n = \frac{243}{4}$

$\therefore\quad n = 81$ So, -36 is the 81st term in the sequence.

b $u_{35} = 24 + (35-1)(-\frac{3}{4})$
$= 24 - \frac{102}{4}$
$= -\frac{3}{2}$
$= -1\frac{1}{2}$

c $S_n = \dfrac{n}{2}(2u_1 + (n-1)d)$

$\therefore\quad S_{40} = \frac{40}{2}(2\times 24 + (40-1)(-\frac{3}{4}))$
$= 20(48 - \frac{117}{4})$
$= 375$

4 **a** $3+9+15+21+....$
The series is arithmetic with
$u_1 = 3, \quad d = 6, \quad n = 23$

Now $S_n = \frac{n}{2}(2u_1 + (n-1)d)$

$\therefore \quad S_{23} = \frac{23}{2}(2 \times 3 + 6(23-1))$

$\therefore \quad S_{23} = \frac{23}{2}(6+132)$
$= 1587$

b $24+12+6+3+....$
The series is geometric with
$u_1 = 24, \quad r = \frac{1}{2}, \quad n = 12$

$$S_n = \frac{u_1(1-r^n)}{1-r}$$

$$\therefore \quad S_{12} = \frac{24\left(1-(\frac{1}{2})^{12}\right)}{1-\frac{1}{2}}$$
$$= 48\left(1-(\tfrac{1}{2})^{12}\right)$$
$$= 47\tfrac{253}{256} \approx 48.0$$

5 5, 10, 20, 40, ... The sequence is geometric with $u_1 = 5, \quad r = 2$
$u_n = u_1 r^{n-1} = 5 \times 2^{n-1}$
Let $u_n = 10\,000 = 5 \times 2^{n-1}$
$\therefore \quad 2000 = 2^{n-1}$
$\therefore \quad n \approx 11.97$ {using technology}
We try the two values on either side of $n \approx 11.97$, which are $n = 11$ and $n = 12$:
$u_{11} = 5 \times 2^{10} = 5120$ and $u_{12} = 5 \times 2^{11} = 10\,240$
So, the first term to exceed 10 000 is $u_{12} = 10\,240$.

6 **a** $u_6 = u_1 \times r^5$ is the amount after 5 years, where $u_1 = 6000, r = 1.07$
$= 6000 \times (1.07)^5$
≈ 8415.31 So, the value of the investment will be €8415.31.

b If interest is compounded quarterly, then $r = 1 + \frac{0.07}{4} = 1.0175$
and $n = 5 \times 4 = 20$

$\therefore \quad u_{21} = u_1 \times r^{20}$
$= 6000 \times (1.0175)^{20}$
≈ 8488.67 So, the value of the investment will be €8488.67.

c If interest is compounded monthly, then $r = 1 + \frac{0.07}{12} = 1.005\,8\bar{3}$
and $n = 5 \times 12 = 60$

$\therefore \quad u_{61} = u_1 \times r^{60}$
$= 6000 \times (1.005\,8\bar{3})^{60}$
≈ 8505.75 So, the value of the investment will be €8505.75

7 **a** $u_n = 5n - 8$
$\therefore \quad u_{10} = 5 \times 10 - 8 = 42$

b $u_{n+1} - u_n = (5(n+1) - 8) - (5n - 8)$
$= 5n + 5 - 8 - 5n + 8$
$= 5$

c The difference between consecutive terms u_n and u_{n+1} is constant for all n, so the sequence is arithmetic.

d $S_n = \frac{n}{2}(2u_1 + (n-1)d)$ where $d = 5$ and $u_1 = 5 \times 1 - 8 = -3$
Now, $u_{15} + u_{16} + u_{17} + ... + u_{30} = S_{30} - S_{14}$
$= \frac{30}{2}(2(-3) + (30-1) \times 5) - \frac{14}{2}(2(-3) + (14-1) \times 5)$
$= 1672$

8 $u_6 = 24 \quad \therefore \quad u_1 \times r^5 = 24 \quad .... (1)$

$u_{11} = 768 \quad \therefore \quad u_1 \times r^{10} = 768 \quad .... (2)$

So $\dfrac{u_1 r^{10}}{u_1 r^5} = \dfrac{768}{24} \quad \{(2) \div (1)\}$

$\therefore \quad r^5 = 32$

$\therefore \quad r = 2$

Substituting $r = 2$ into (1) gives $u_1 \times 2^5 = 24$

$\therefore \quad u_1 = \frac{24}{32} = \frac{3}{4}$

$u_n = u_1 r^{n-1} = \left(\frac{3}{4}\right) 2^{n-1}$

a $u_{17} = \left(\frac{3}{4}\right) 2^{17-1}$

$= 49\,152$

b $S_n = \dfrac{u_1(r^n - 1)}{r - 1} = \dfrac{\frac{3}{4}(2^n - 1)}{2 - 1}$

$= \frac{3}{4}(2^n - 1)$

$\therefore \quad S_{15} = \frac{3}{4}(2^{15} - 1) = 24\,575.25$

9 $24, 8, \frac{8}{3}, \frac{8}{9}, ...$ is geometric with $u_1 = 24, \quad r = \frac{1}{3}$

$u_n = u_1 r^{n-1} = 24 \left(\frac{1}{3}\right)^{n-1}$

Given $u_n = 0.001$, we need to find n, so $u_n = 24 \left(\frac{1}{3}\right)^{n-1} = 0.001$

$\therefore \quad \left(\frac{1}{3}\right)^{n-1} = \dfrac{0.001}{24}$

$\therefore \quad n \approx 10.18 \quad$ {using technology}

We try the two values on either side of $n \approx 10.18$, which are $n = 10$ and $n = 11$:

$u_{10} = 24 \left(\frac{1}{3}\right)^9 = \dfrac{8}{6561} \approx 0.001\,22$ and $u_{11} = 24 \left(\frac{1}{3}\right)^{10} = \dfrac{8}{19\,683} \approx 0.000\,406$

$\therefore \quad u_{11} \approx 0.000\,406$ is the first term of the sequence which is less than 0.001.

10 **a** $128, 64, 32, 16, ..., \frac{1}{512}$

is geometric with:

$u_1 = 128, \quad r = \frac{1}{2}, \quad u_n = \frac{1}{512}$

$u_n = u_1 r^{n-1}$

$= 128 \left(\frac{1}{2}\right)^{n-1}$

$= 2^7 \times 2^{1-n}$

$\therefore \quad \frac{1}{512} = 2^7 \times 2^{1-n}$

$\therefore \quad 2^{-9} = 2^{8-n}$

$\therefore \quad -9 = 8 - n$

$\therefore \quad n = 17$

So, there are 17 terms in the sequence.

b $S_n = \dfrac{u_1(1 - r^n)}{1 - r}$

$\therefore \quad S_{17} = \dfrac{128\left(1 - \left(\frac{1}{2}\right)^{17}\right)}{1 - \frac{1}{2}}$

$= 255\frac{511}{512}$

≈ 255.998

≈ 256

11 **a** $1.21 - 1.1 + 1 - ....$ is an infinite geometric series with $u_1 = 1.21, \quad r = -\frac{10}{11}$.

$\therefore \quad S = \dfrac{u_1}{1 - r} = \dfrac{1.21}{\frac{21}{11}} = \frac{1331}{2100}$

$\therefore \quad S \approx 0.634$

b $\frac{14}{3} + \frac{4}{3} + \frac{8}{21} + ...$ is an infinite geometric series with $u_1 = \frac{14}{3}, \quad r = \frac{2}{7}$.

$\therefore \quad S = \dfrac{u_1}{1 - r} = \dfrac{\frac{14}{3}}{\frac{5}{7}}$

$\therefore \quad S = \frac{98}{15} = 6\frac{8}{15}$

12 $u_{n+1} = u_1 \times r^n$ where $u_{n+1} = 20\,000$, $r = 1 + \dfrac{0.09}{12} = 1.0075$, $n = 4 \times 12 = 48$

$\therefore\ 20\,000 = u_1 \times (1.0075)^{48}$

$\therefore\ u_1 = \dfrac{20\,000}{(1.0075)^{48}}$

$\therefore\ u_1 \approx 13\,972.28$ So, $13 972.28 should be invested.

13 **a** $u_{n+1} = u_1 \times r^n$ where $u_1 = 3000$, $r = 1.05$, $n = 3$

$\therefore\ u_{n+1} = 3000 \times (1.05)^3$

$= 3472.875$ There were approximately 3470 iguanas.

b $u_{n+1} = u_1 \times r^n$ where $u_1 = 3000$, $u_{n+1} = 5000$, $r = 1.05$

$\therefore\ 10\,000 = 3000 \times (1.05)^n$

$\therefore\ n \approx 24.68$ {using technology}

After 24.68 years the population will exceed 10 000. This is during the year 2029.

14 $u_1 = x + 3$, $u_2 = u_1 r = x - 2$

$\therefore\ r = \dfrac{u_2}{u_1} = \dfrac{x-2}{x+3}$

The series will converge if $|r| < 1$

$$\therefore\ \left|\frac{x-2}{x+3}\right| < 1$$

$$\therefore\ |x-2| < |x+3| \qquad \left\{\left|\frac{a}{b}\right| = \frac{|a|}{|b|}\right\}$$

$$\therefore\ (x-2)^2 < (x+3)^2$$

$$\therefore\ x^2 - 4x + 4 < x^2 + 6x + 9$$

$$\therefore\ -10x < 5$$

$$\therefore\ x > -\tfrac{1}{2}$$

REVIEW SET 7C

1 $u_n = 68 - 5n$

a $u_{n+1} - u_n = [68 - 5(n+1)] - [68 - 5n]$

$= 68 - 5n - 5 - 68 + 5n$

$= -5$ for all n

$\therefore$ the sequence is arithmetic with common difference $d = -5$.

b $u_1 = 68 - 5(1) = 63$, $d = -5$

c $u_{37} = 68 - 5(37) = -117$

d Let $u_n = -200$, and we need to find n.

$u_n = 68 - 5n = -200$

$\therefore\ 5n = 268$

$\therefore\ n = 53\frac{3}{5}$

We try the two values on either side of $n = 53\frac{3}{5}$, which are $n = 53$ and $n = 54$:

$u_{53} = 68 - 5(53) = -197$ and $u_{54} = 68 - 5(54) = -202$

So, the first term of the sequence less than -200 is $u_{54} = -202$.

2 **a** 3, 12, 48, 192, ...

$\frac{12}{3} = 4 \qquad \frac{48}{12} = 4 \qquad \frac{192}{48} = 4$

Assuming the pattern continues, consecutive terms have a common ratio of 4.

$\therefore$ the sequence is geometric with $u_1 = 3$ and $r = 4$.

b $u_n = u_1 r^{n-1}$

$\therefore \quad u_n = 3 \times 4^{n-1}$

$\therefore \quad u_9 = 3 \times 4^8 = 196\,608$

3 $u_7 = 31 \quad \therefore \quad u_1 + 6d = 31 \quad .... (1)$

$u_{15} = -17 \quad \therefore \quad u_1 + 14d = -17 \quad .... (2)$

So, $(u_1 + 14d) - (u_1 + 6d) = -17 - 31 \qquad \{(2) - (1)\}$

$\therefore \quad 8d = -48$

$\therefore \quad d = -6$

So in (1), $u_1 + 6(-6) = 31$

$\therefore \quad u_1 - 36 = 31$

$\therefore \quad u_1 = 67$

Now $u_n = u_1 + (n-1)d$

$\therefore \quad u_n = 67 + (n-1)(-6)$

$\therefore \quad u_n = 67 - 6n + 6$

$\therefore \quad u_n = 73 - 6n$

So, $u_{34} = 73 - 6(34) = -131$

4 **a** $4 + 11 + 18 + 25 + ...$

The series is arithmetic with $u_1 = 4, \ d = 7, \ u_k = u_1 + (k-1)d$

$= 4 + 7(k-1)$

$= 7k - 3$

So, the series is $\sum\limits_{k=1}^{n} (7k - 3)$.

b $\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + ...$

The series is geometric with $u_1 = \frac{1}{4}, \ r = \frac{1}{2}$,

$$u_k = u_1 r^{k-1} = \tfrac{1}{4} \times \left(\tfrac{1}{2}\right)^{k-1} = \left(\tfrac{1}{2}\right)^2 \left(\tfrac{1}{2}\right)^{k-1} = \left(\tfrac{1}{2}\right)^{k+1}$$

So, the series is $\sum\limits_{k=1}^{n} (\frac{1}{2})^{k+1}$.

5 **a** $\sum\limits_{k=1}^{8} \left(\dfrac{31 - 3k}{2}\right) = 14 + 12\frac{1}{2} + 11 + 9\frac{1}{2} + 8 + 6\frac{1}{2} + 5 + 3\frac{1}{2}$

This series is arithmetic with $u_1 = 14, \ n = 8,$ and $u_n = 3\frac{1}{2}$.

$\therefore$ the sum is $\frac{8}{2}(14 + 3\frac{1}{2}) = 70$

b $\sum\limits_{k=1}^{15} 50(0.8)^{k-1} \approx 50 + 40 + 32 + ... + 3.436 + 2.749 + 2.199$

This series is geometric with $u_1 = 50, \ r = 0.8,$ and $n = 15$.

$\therefore$ the sum is $\dfrac{50\left[1 - (0.8)^{15}\right]}{1 - 0.8} \approx 241$

c $\sum\limits_{k=7}^{\infty} 5(\frac{2}{5})^{k-1} = 5(\frac{2}{5})^6 + 5(\frac{2}{5})^7 + 5(\frac{2}{5})^8 + ...$

The series is an infinite geometric series with $u_1 = 5(\frac{2}{5})^6, \ r = \frac{2}{5}$

$$\therefore \ S = \frac{u_1}{1 - r} = \frac{5(\frac{2}{5})^6}{\frac{3}{5}}$$

$$= \frac{2^6}{3 \times 5^4}$$

$$= \tfrac{64}{1875}$$

6 $11+16+21+26+....$ is arithmetic with $u_1 = 11$, $d = 5$

$\therefore \quad S_n = \frac{n}{2}(2u_1 + (n-1)d)$

$= \frac{n}{2}(2 \times 11 + 5(n-1))$

$= \frac{n}{2}(22 + 5n - 5)$

$= \frac{n}{2}(5n+17)$

Given $S_n = 450$, we need to find n,

so $S_n = \frac{n}{2}(5n+17) = 450$

$\therefore \quad \frac{5}{2}n^2 + \frac{17}{2}n - 450 = 0$

$\therefore \quad 5n^2 + 17n - 900 = 0$

$\therefore \quad n \approx -15.2,\ 11.8$ {using technology}

But $n > 0$, so $n \approx 11.8$

We try the two values on either side of $n \approx 11.8$, which are $n = 11$ and $n = 12$:

$S_{11} = \frac{11}{2}(5(11)+17) = 396$ and $S_{12} = \frac{12}{2}(5(12)+17) = 462$

$\therefore$ 12 terms of the series are required to exceed a sum of 450.

7 **a** $u_{n+1} = u_1 \times r^n$ where $u_1 = 12\,500$, $r = 1 + \dfrac{0.0825}{2} = 1.041\,25$, $n = 5 \times 2 = 10$

So, $u_{n+1} = 12\,500 \times (1.041\,25)^{10}$

$\approx 18\,726.65$ The value of the investment is £18 726.65.

b $u_{n+1} = u_1 \times r^n$ where $u_1 = 12\,500$, $r = 1 + \dfrac{0.0825}{12} = 1.006\,875$, $n = 5 \times 12 = 60$

So, $u_{n+1} = 12\,500 \times (1.006\,875)^{60}$

$\approx 18\,855.74$ The value of the investment is £18 855.74.

8 **a** Let the terms of the geometric series be $u_1, u_1r, u_1r^2, ...$

Then $u_1 + u_1 r = 90$ and $u_1 r^2 = 24$

$\therefore \quad u_1(1+r) = 90$

$\therefore \quad u_1 = \dfrac{90}{1+r}$ .... (1)

$\therefore \quad u_1 = \dfrac{24}{r^2}$ .... (2)

Equating (1) and (2) gives $\dfrac{90}{1+r} = \dfrac{24}{r^2}$

$\therefore \quad 90r^2 = 24r + 24$

$\therefore \quad 90r^2 - 24r - 24 = 0$

$\therefore \quad 6(15r^2 - 4r - 4) = 0$

$\therefore \quad 6(5r+2)(3r-2) = 0$

$\therefore \quad r = -\frac{2}{5}$ or $\frac{2}{3}$

Using (2), if $r = -\frac{2}{5}$ then $u_1 = \dfrac{24}{(-\frac{2}{5})^2} = \dfrac{24}{\frac{4}{25}} = 150$

if $r = \frac{2}{3}$ then $u_1 = \dfrac{24}{(\frac{2}{3})^2} = \dfrac{24}{\frac{4}{9}} = 54$

$\therefore$ either $u_1 = 150$, $r = -\frac{2}{5}$ or $u_1 = 54$, $r = \frac{2}{3}$

b Since $|r| < 1$ in each case, both series converge.

When $u_1 = 150$, $r = -\frac{2}{5}$

$\therefore \quad S = \dfrac{u_1}{1-r}$

$= \dfrac{150}{\frac{7}{5}}$

$= \frac{750}{7} = 107\frac{1}{7}$

When $u_1 = 54$, $r = \frac{2}{3}$

$\therefore \quad S = \dfrac{u_1}{1-r}$

$= \dfrac{54}{\frac{1}{3}}$

$= 162$

9 Since Seve walks an additional 500 m $=$ 0.5 km each week, we have an arithmetic sequence with $u_1 = 10$ and constant difference $d = 0.5$.

$$u_n = u_1 + (n-1)d$$
$$\therefore \quad u_n = 10 + (n-1)0.5$$

a $u_{52} = 10 + (52-1)0.5 \qquad \{52 \text{ weeks in a year}\}$
$\quad = 35.5$

$\therefore$ Seve walks 35.5 km in the last week.

b In total, Seve walks $10 + 10.5 + 11 + ... + 35.5$, which is an arithmetic series.

$$S_n = \frac{n}{2}(u_1 + u_n)$$
$$\therefore \quad S_{52} = \tfrac{52}{2}(10 + 35.5) = 1183$$

$\therefore$ Seve walks 1183 km in total.

10 **a** $\sum_{k=1}^{\infty} 50\,(2x-1)^{k-1}$ is a geometric series with $r = 2x-1$ and converges if $-1 < r < 1$

$\therefore \quad -1 < 2x - 1 < 1$
$\therefore \quad 0 < 2x < 2$
$\therefore \quad 0 < x < 1$

b When $x = 0.3$, $2x - 1 = 0.6 - 1 = -0.4$

and $\sum_{k=1}^{\infty} 50\,(2x-1)^{k-1} = 50(-0.4)^0 + 50(-0.4)^1 + 50(-0.4)^2 + ...$

which is geometric with $u_1 = 50$, $r = -0.4$

Now as $0 < 0.3 < 1$, the series converges and $S = \dfrac{u_1}{1-r} = \dfrac{50}{1+0.4} = \dfrac{50}{\frac{7}{5}} = 35\frac{5}{7}$

11 Since $a, b, c, d,$ and e are consecutive terms of an arithmetic sequence,
$b - a = c - b = d - c = e - d$

Taking the first and last equalities, $\quad b - a = e - d$
$\therefore \quad b + d = a + e$

Taking the middle equalities, $\quad c - b = d - c$
$\therefore \quad 2c = b + d$

$\therefore \quad a + e = b + d = 2c$

12 Let the geometric sequence be $\quad 1, \underbrace{r, r^2, r^3, ..., r^{n-1}, r^n}_{n \text{ terms}}, 2$

$\therefore \quad r^{n+1} = 2$ and so $r = 2^{\frac{1}{n+1}}$

The required sum is $r + r^2 + r^3 + ... + r^{n-1} + r^n$,
which is geometric with $u_1 = r$, 'r' $= r$, and 'n' $= n$.

$$\therefore \quad S_n = \frac{u_1(r^n - 1)}{r-1} = \frac{r(r^n-1)}{r-1} = \frac{r^{n+1} - r}{r-1} = \frac{2 - 2^{\frac{1}{n+1}}}{2^{\frac{1}{n+1}} - 1}$$

13 Let the first three terms of the arithmetic sequence be $u_1,\ u_1+d,\ u_1+2d$, and the first three terms of the geometric sequence be $u_1,\ u_1r,\ u_1r^2$.

The second terms are equal

$\therefore\ u_1+d=u_1r$

$\therefore\ d=u_1(r-1)$

$\therefore\ u_1=\dfrac{d}{r-1}$ .... (1)

Now $$\frac{u_1r^2}{u_1+2d}=\frac{(\frac{d}{r-1})r^2}{(\frac{d}{r-1})+2d}\quad\{\text{using (1)}\}$$
$$=\frac{dr^2}{d+2d(r-1)}$$
$$=\frac{dr^2}{d(2r-1)}$$
$$=\frac{r^2}{2r-1}$$

$\therefore$ the third term of the geometric sequence is $\dfrac{r^2}{2r-1}$ times the third term of the arithmetic sequence.

14 The value of $\underbrace{111....11}_{2n\text{ lots of }1}$ is $1+10+10^2+....+10^{2n-1}$, which is a geometric series with $u_1=1,\ r=10,\ `n\text{'}=2n$.

$\therefore$ the value of this sum is $$S_1=\frac{u_1(r^n-1)}{r-1}=\frac{1(10^{2n}-1)}{10-1}$$
$$=\frac{10^{2n}-1}{9}$$

The value of $\underbrace{222....22}_{n\text{ lots of }2}$ is $2+2\times10+2\times10^2+....+2\times10^{n-1}$,

which is a geometric series with $u_1=2,\ r=10,\ `n\text{'}=n$.

$\therefore$ the value of this sum is $$S_2=\frac{u_1(r^n-1)}{r-1}=\frac{2(10^n-1)}{10-1}$$
$$=\frac{2\times10^n-2}{9}$$

$\therefore$ the value of $$\underbrace{111....11}_{2n\text{ lots of }1}-\underbrace{222....22}_{n\text{ lots of }2}=S_1-S_2=\frac{10^{2n}-1}{9}-\frac{2\times10^n-2}{9}$$
$$=\frac{10^{2n}-1-2\times10^n+2}{9}$$
$$=\frac{10^{2n}-2\times10^n+1}{9}$$
$$=\frac{(10^n-1)^2}{3^2}$$
$$=\left(\frac{10^n-1}{3}\right)^2$$

$10^n-1=\underbrace{99....9}_{n\text{ lots of }9}$ is divisible by 3, so $\dfrac{10^n-1}{3}$ is an integer

$\therefore\ \underbrace{111....11}_{2n\text{ lots of }1}-\underbrace{222....22}_{n\text{ lots of }2}$ is a perfect square.