Chapter 17

INTRODUCTION TO DIFFERENTIAL CALCULUS

EXERCISE 17A

1 **a** As $x \to 3$, $x + 4 \to 7$

$\therefore \lim_{x \to 3} (x + 4) = 7$

b As $x \to -1$, $5 - 2x \to 7$

$\therefore \lim_{x \to -1} (5 - 2x) = 7$

c As $x \to 4$, $3x - 1 \to 11$

$\therefore \lim_{x \to 4} (3x - 1) = 11$

d As $x \to 2$, $5x^2 - 3x + 2 \to 5(4) - 3(2) + 2 = 16$

$\therefore \lim_{x \to 2} (5x^2 - 3x + 2) = 16$

e As $h \to 0$, $h^2 \to 0$ and $1 - h \to 1$

$\therefore \lim_{h \to 0} h^2(1 - h) = 0 \times 1 = 0$

f As $x \to 0$, $x^2 + 5 \to 5$

$\therefore \lim_{x \to 0} (x^2 + 5) = 5$

2 **a** $\lim_{x \to 0} 5 = 5$

b $\lim_{h \to 2} 7 = 7$

c $\lim_{x \to 0} c = c$ (when c is a constant)

3 **a**
$$\begin{aligned}\lim_{x \to 1} \frac{x^2 - 3x}{x} &= \lim_{x \to 1} \frac{x(x-3)}{x}\\ &= \lim_{x \to 1} (x - 3) \quad \text{since } x \neq 0\\ &= -2\end{aligned}$$

b
$$\begin{aligned}\lim_{h \to 2} \frac{h^2 + 5h}{h} &= \lim_{h \to 2} \frac{h(h+5)}{h}\\ &= \lim_{h \to 2} (h + 5) \quad \text{since } h \neq 0\\ &= 7\end{aligned}$$

c $\dfrac{x-1}{x+1}$ can be made as close as we like to -1 by making x sufficiently close to 0.

$\therefore \lim_{x \to 0} \dfrac{x-1}{x+1} = -1$

d
$$\begin{aligned}\lim_{x \to 0} \frac{x}{x} &= \lim_{x \to 0} 1 \quad \text{since } x \neq 0\\ &= 1\end{aligned}$$

4 **a** $f(x) = \dfrac{1}{x}$ is not defined when $x = 0$

$\therefore f(x) = \dfrac{1}{x}$ is not continuous at $x = 0$.

b $f(x) = \dfrac{x^2 - x}{x}$ is not defined when $x = 0$

$\therefore f(x) = \dfrac{x^2 - x}{x}$ is not continuous at $x = 0$.

5 **a**
$$\begin{aligned}\lim_{x \to 0} \frac{x^2 - 3x}{x} &= \lim_{x \to 0} \frac{x(x-3)}{x}\\ &= \lim_{x \to 0} (x - 3) \quad \text{since } x \neq 0\\ &= -3\end{aligned}$$

b
$$\begin{aligned}\lim_{x \to 0} \frac{x^2 + 5x}{x} &= \lim_{x \to 0} \frac{x(x+5)}{x}\\ &= \lim_{x \to 0} (x + 5) \quad \text{since } x \neq 0\\ &= 5\end{aligned}$$

c
$$\begin{aligned}&\lim_{x \to 0} \frac{2x^2 - x}{x}\\ &= \lim_{x \to 0} \frac{x(2x-1)}{x}\\ &= \lim_{x \to 0} (2x - 1) \quad \text{since } x \neq 0\\ &= -1\end{aligned}$$

d
$$\begin{aligned}&\lim_{h \to 0} \frac{2h^2 + 6h}{h}\\ &= \lim_{h \to 0} \frac{2h(h+3)}{h}\\ &= \lim_{h \to 0} 2(h + 3) \quad \text{since } h \neq 0\\ &= 6\end{aligned}$$

e
$$\begin{aligned}&\lim_{h \to 0} \frac{3h^2 - 4h}{h}\\ &= \lim_{h \to 0} \frac{h(3h-4)}{h}\\ &= \lim_{h \to 0} (3h - 4) \quad \text{since } h \neq 0\\ &= -4\end{aligned}$$

f
$$\begin{aligned}&\lim_{h \to 0} \frac{h^3 - 8h}{h}\\ &= \lim_{h \to 0} \frac{h(h^2-8)}{h}\\ &= \lim_{h \to 0} (h^2 - 8) \quad \text{since } h \neq 0\\ &= -8\end{aligned}$$

g $\lim\limits_{x\to1}\dfrac{x^2-x}{x-1}=\lim\limits_{x\to1}\dfrac{x(x-1)}{x-1}$
$=\lim\limits_{x\to1}x \quad \text{since } x\neq1$
$=1$

h $\lim\limits_{x\to2}\dfrac{x^2-2x}{x-2}=\lim\limits_{x\to2}\dfrac{x(x-2)}{x-2}$
$=\lim\limits_{x\to2}x \quad \text{since } x\neq2$
$=2$

i $\lim\limits_{x\to3}\dfrac{x^2-x-6}{x-3}=\lim\limits_{x\to3}\dfrac{(x+2)(x-3)}{x-3}$
$=\lim\limits_{x\to3}(x+2) \quad \text{since } x\neq3$
$=5$

EXERCISE 17B.1

1 As x gets larger and positive, $\dfrac{1}{x^2}$ gets smaller and closer to 0.
$\therefore \lim\limits_{x\to\infty}\dfrac{1}{x^2}=0$

2 **a** $\lim\limits_{x\to\infty}\dfrac{3x-2}{x+1}$
$=\lim\limits_{x\to\infty}\dfrac{3-\frac{2}{x}}{1+\frac{1}{x}}$
$=\frac{3}{1}=3$

b $\lim\limits_{x\to\infty}\dfrac{1-2x}{3x+2}$
$=\lim\limits_{x\to\infty}\dfrac{\frac{1}{x}-2}{3+\frac{2}{x}}$
$=-\frac{2}{3}$

c $\lim\limits_{x\to\infty}\dfrac{x}{1-x}$
$=\lim\limits_{x\to\infty}\dfrac{1}{\frac{1}{x}-1}$
$=\frac{1}{-1}=-1$

d $\lim\limits_{x\to\infty}\dfrac{x^2+3}{x^2-1}$
$=\lim\limits_{x\to\infty}\dfrac{1+\frac{3}{x^2}}{1-\frac{1}{x^2}}$
$=\frac{1}{1}$
$=1$

e $\lim\limits_{x\to\infty}\dfrac{x^2-2x+4}{x^2+x-1}$
$=\lim\limits_{x\to\infty}\dfrac{1-\frac{2}{x}+\frac{4}{x^2}}{1+\frac{1}{x}-\frac{1}{x^2}}$
$=\frac{1}{1}$
$=1$

EXERCISE 17B.2

1 **a** **i** As $x\to0^-$, $f(x)\to-\infty$
As $x\to0^+$, $f(x)\to\infty$
As $x\to\infty$, $f(x)\to0^+$
As $x\to-\infty$, $f(x)\to0^-$
The vertical asymptote is $x=0$.
The horizontal asymptote is $y=0$.

ii $\lim\limits_{x\to-\infty}f(x)=0$, $\lim\limits_{x\to\infty}f(x)=0$

b **i** As $x\to-3^-$, $f(x)\to\infty$
As $x\to-3^+$, $f(x)\to-\infty$
As $x\to\infty$, $f(x)\to3^-$
As $x\to-\infty$, $f(x)\to3^+$
The vertical asymptote is $x=-3$.
The horizontal asymptote is $y=3$.

ii $\lim\limits_{x\to-\infty}f(x)=3$, $\lim\limits_{x\to\infty}f(x)=3$

c **i** As $x\to-\frac{2}{3}^-$, $f(x)\to-\infty$
As $x\to-\frac{2}{3}^+$, $f(x)\to\infty$
As $x\to\infty$, $f(x)\to-\frac{2}{3}^+$
As $x\to-\infty$, $f(x)\to-\frac{2}{3}^-$
The vertical asymptote is $x=-\frac{2}{3}$.
The horizontal asymptote is $y=-\frac{2}{3}$.

ii $\lim\limits_{x\to-\infty}f(x)=-\frac{2}{3}$,
$\lim\limits_{x\to\infty}f(x)=-\frac{2}{3}$

d **i** As $x\to1^-$, $f(x)\to\infty$
As $x\to1^+$, $f(x)\to-\infty$
As $x\to\infty$, $f(x)\to-1^-$
As $x\to-\infty$, $f(x)\to-1^+$
The vertical asymptote is $x=1$.
The horizontal asymptote is $y=-1$.

ii $\lim\limits_{x\to-\infty}f(x)=-1$, $\lim\limits_{x\to\infty}f(x)=-1$

e **i** Since there are no real values of x that make $x^2+1=0$, $f(x)$ is defined for all $x \in \mathbb{R}$.

$\therefore$ there are no vertical asymptotes.

As $x \to \infty$, $f(x) \to 1^-$

As $x \to -\infty$, $f(x) \to 1^-$

The horizontal asymptote is $y = 1$.

ii $\lim\limits_{x \to -\infty} f(x) = 1$, $\lim\limits_{x \to \infty} f(x) = 1$

f **i** Since there are no real values of x that make $x^2+1=0$, $f(x)$ is defined for all $x \in \mathbb{R}$.

$\therefore$ there are no vertical asymptotes.

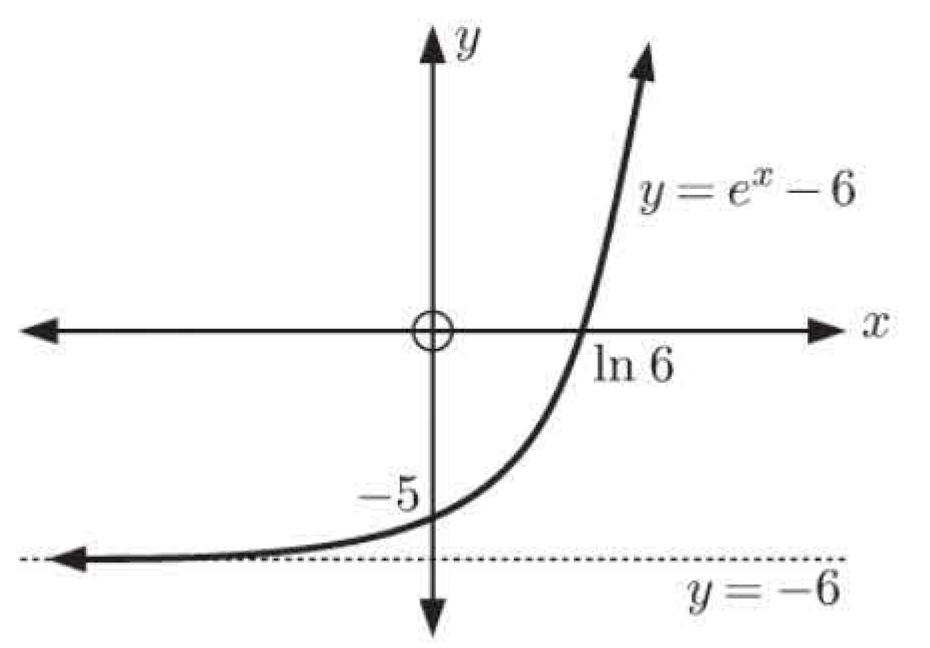
As $x \to \infty$, $f(x) \to 0^+$

As $x \to -\infty$, $f(x) \to 0^-$

The horizontal asymptote is $y = 0$.

ii $\lim\limits_{x \to -\infty} f(x) = 0$, $\lim\limits_{x \to \infty} f(x) = 0$

2 **a**



b **i** As $x \to -\infty$, $e^x - 6 \to -6^+$

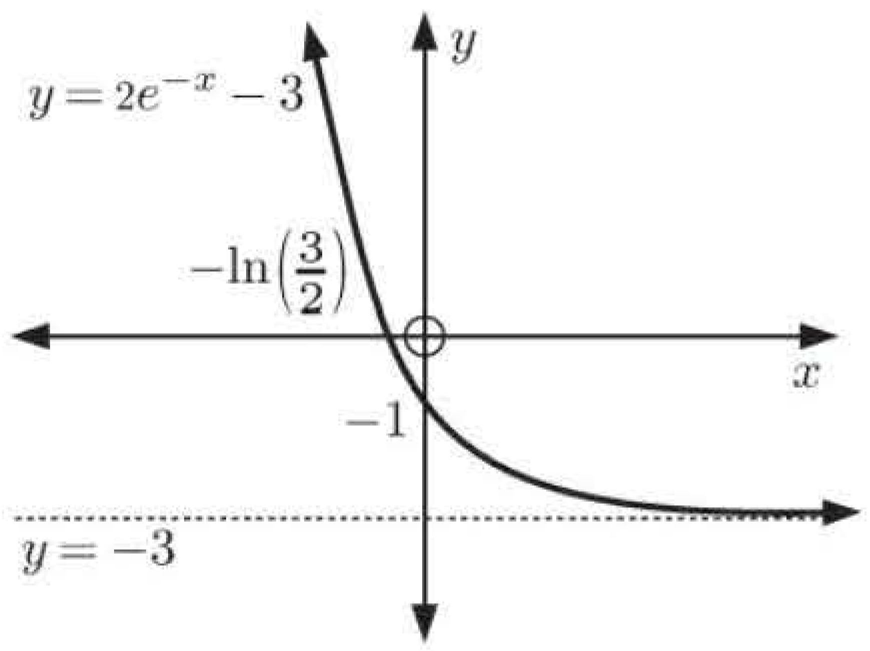
$\therefore$ $\lim\limits_{x \to -\infty} (e^x - 6) = -6$

$\therefore$ the function has horizontal asymptote $y = -6$.

ii As $x \to \infty$, $e^x - 6 \to \infty$

$\therefore$ $\lim\limits_{x \to \infty} (e^x - 6)$ does not exist.

3 We sketch the graph of $y = 2e^{-x} - 3$:



As $x \to -\infty$, $2e^{-x} - 3 \to \infty$

$\therefore$ $\lim\limits_{x \to -\infty} (2e^{-x} - 3)$ does not exist.

As $x \to \infty$, $2e^{-x} - 3 \to -3^+$

$\therefore$ $\lim\limits_{x \to \infty} (2e^{-x} - 3) = -3$.

4 **a** $f(x) = \ln x$

$f(x)$ is undefined when $x \leqslant 0$

$\therefore$ $x = 0$ is a vertical asymptote.

As $x \to 0^+$, $y \to -\infty$

b $f(x) = e^{x - \frac{1}{x}}$

$f(x)$ is undefined when $x = 0$

$\therefore$ $x = 0$ is a vertical asymptote.

As $x \to -\infty$, $f(x) \to 0$

$\therefore$ $y = 0$ is a horizontal asymptote.

As $x \to 0^-$, $f(x) \to \infty$

As $x \to 0^+$, $f(x) \to 0^+$

As $x \to \infty$, $f(x) \to \infty$

As $x \to -\infty$, $f(x) \to 0^+$

5 **a** $f(x) = x + \ln x$

$f(x)$ is undefined for $x \leqslant 0$

$\therefore$ $x = 0$ is a vertical asymptote

As $x \to 0^+$, $f(x) \to -\infty$

b $f(x) = e^x - x$

$f(x)$ is defined for all $x \in \mathbb{R}$

$\therefore$ no vertical asymptotes exist.

As $x \to -\infty$, $f(x) \to -x$

$\therefore$ $y = -x$ is an oblique asymptote

As $x \to \infty$, $f(x) \to \infty$

As $x \to -\infty$, $f(x) \to (-x)^+$

c $f(x) = \dfrac{x^3-2}{x^2+1}$

$$\begin{array}{r|l} & \quad\quad\quad\quad x \\ \hline x^2+1 & x^3+0x^2+0x-2 \\ & x^3 \quad\quad + \; x \\ \hline & \quad\quad -\; x-2 \end{array}$$

$\therefore \quad f(x) = x - \dfrac{x+2}{x^2+1}$

x is defined for all $x \in \mathbb{R}$

$\therefore$ no vertical asymptotes exist.

As $|x| \to \infty$, $f(x) \to x$

$\therefore$ $y = x$ is an oblique asymptote

As $x \to \infty$, $f(x) \to x^-$

As $x \to -\infty$, $f(x) \to x^+$

d $f(x) = (x-2)e^{-x} = \dfrac{x-2}{e^x}$

$f(x)$ is defined for all $x \in \mathbb{R}$

$\therefore$ no vertical asymptotes exist.

As $x \to \infty$, $f(x) \to 0^+$

$\therefore$ $y = 0$ is a horizontal asymptote.

EXERCISE 17C

1 **a**
$$\begin{aligned} &\lim_{\theta\to 0} \frac{\sin 2\theta}{\theta} \\ &= \lim_{\theta\to 0} \frac{\sin 2\theta}{2\theta} \times 2 \\ &= 2 \times \lim_{2\theta\to 0} \frac{\sin 2\theta}{2\theta} \quad \{2\theta \to 0 \text{ as } \theta \to 0\} \\ &= 2\times 1 \\ &= 2 \end{aligned}$$

b
$$\begin{aligned} &\lim_{\theta\to 0} \frac{\theta}{\sin\theta} \\ &= \lim_{\theta\to 0} \frac{1}{\frac{\sin\theta}{\theta}} \\ &= \frac{1}{\lim_{\theta\to 0} \frac{\sin\theta}{\theta}} \\ &= \frac{1}{1} \\ &= 1 \end{aligned}$$

c
$$\begin{aligned} &\lim_{\theta\to 0} \frac{\tan\theta}{\theta} \\ &= \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \times \frac{1}{\cos\theta} \\ &= 1 \times \frac{1}{1} \\ &= 1 \end{aligned}$$

d
$$\begin{aligned} &\lim_{\theta\to 0} \frac{\sin\theta\sin 4\theta}{\theta^2} \\ &= \lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right) \lim_{\theta\to 0}\left(\frac{\sin 4\theta}{\theta}\right) \\ &= 1\times \lim_{\theta\to 0}\left(\frac{\sin 4\theta}{4\theta}\right)\times 4 \\ &= 4\times \lim_{4\theta\to 0}\left(\frac{\sin 4\theta}{4\theta}\right) \quad \{4\theta\to 0 \text{ as } \theta\to 0\} \\ &= 4\times 1 \\ &= 4 \end{aligned}$$

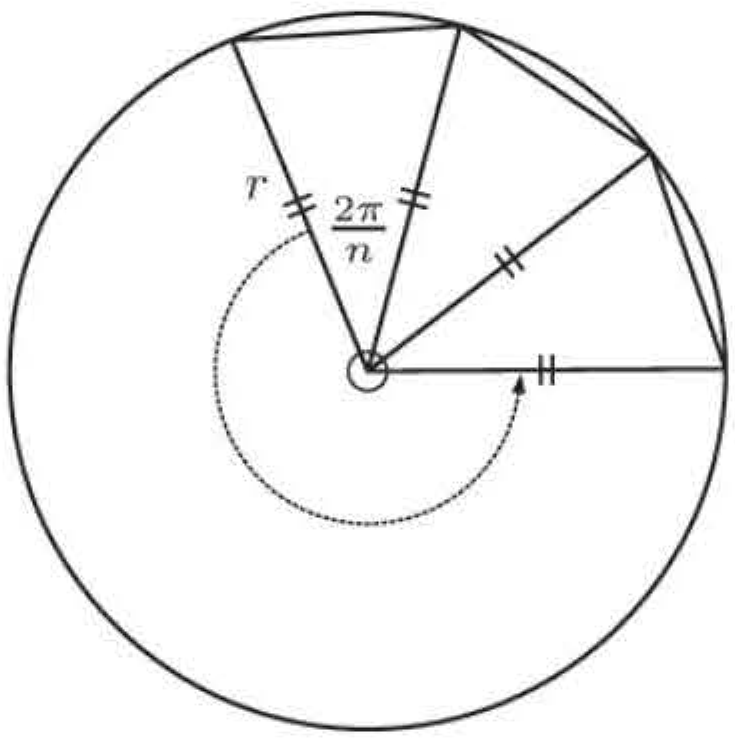
e
$$\begin{aligned} &\lim_{h\to 0} \frac{\sin\left(\frac{h}{2}\right)\cos h}{h} \\ &= \lim_{h\to 0}\cos h \lim_{h\to 0} \frac{\sin\left(\frac{h}{2}\right)}{h} \\ &= 1\times \lim_{h\to 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \times \tfrac{1}{2} \\ &= \tfrac{1}{2}\times \lim_{\frac{h}{2}\to 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \quad \{\tfrac{h}{2}\to 0 \text{ as } h\to 0\} \\ &= \tfrac{1}{2}\times 1 \\ &= \tfrac{1}{2} \end{aligned}$$

f
$$\begin{aligned} &\lim_{n\to\infty} n\sin\left(\tfrac{2\pi}{n}\right) \\ &= \lim_{\frac{1}{n}\to 0^+} \frac{\sin\left(\frac{2\pi}{n}\right)}{\frac{1}{n}} \quad \{\tfrac{1}{n}\to 0^+ \text{ as } n\to\infty\} \\ &= \lim_{\frac{1}{n}\to 0^+} \frac{\sin\left(\frac{2\pi}{n}\right)}{\frac{2\pi}{n}} \times 2\pi \\ &= 2\pi \times \lim_{\frac{2\pi}{n}\to 0^+} \frac{\sin\left(\frac{2\pi}{n}\right)}{\frac{2\pi}{n}} \quad \left\{\begin{array}{l} \frac{2\pi}{n}\to 0^+ \\ \text{as } \frac{1}{n}\to 0^+ \end{array}\right\} \\ &= 2\pi\times 1 \\ &= 2\pi \end{aligned}$$

2 **a** The angle at the apex of each triangle $= \frac{2\pi}{n}$ {angles at a point}

$\therefore$ area of each triangle $= \frac{1}{2}r^2 \sin\left(\frac{2\pi}{n}\right)$

$\therefore$ area of the n triangles $S_n = \frac{1}{2}nr^2 \sin\left(\frac{2\pi}{n}\right)$



b **i** As the number of triangles increases, the triangles cover more of the circle. As $n \to \infty$, the triangles get closer to covering the whole circle.

$\therefore \lim\limits_{n\to\infty} S_n =$ area of the circle

ii $$\begin{aligned}\lim_{n\to\infty} S_n &= \lim_{n\to\infty} \tfrac{1}{2}nr^2 \sin\left(\tfrac{2\pi}{n}\right)\\ &= \tfrac{1}{2}r^2 \lim_{n\to\infty} n\sin\left(\tfrac{2\pi}{n}\right)\\ &= \tfrac{1}{2}r^2(2\pi) \qquad \{\text{using } \mathbf{1\ f}\}\\ &= \pi r^2\end{aligned}$$

c Area of circle $= \pi r^2$

3 **a** $$\begin{aligned}\cos(A+B) - \cos(A-B) &= \cos A\cos B - \sin A\sin B - (\cos A\cos B + \sin A\sin B)\\ &= \cancel{\cos A\cos B} - \sin A\sin B - \cancel{\cos A\cos B} - \sin A\sin B\\ &= -2\sin A\sin B\end{aligned}$$

b $$\begin{aligned}\cos S - \cos D &= \cos(A+B) - \cos(A-B)\\ &= -2\sin A\sin B \qquad \{\text{using } \mathbf{a}\}\end{aligned}$$

Now $S + D = A + B + A - B = 2A$ and $S - D = A + B - (A - B) = 2B$

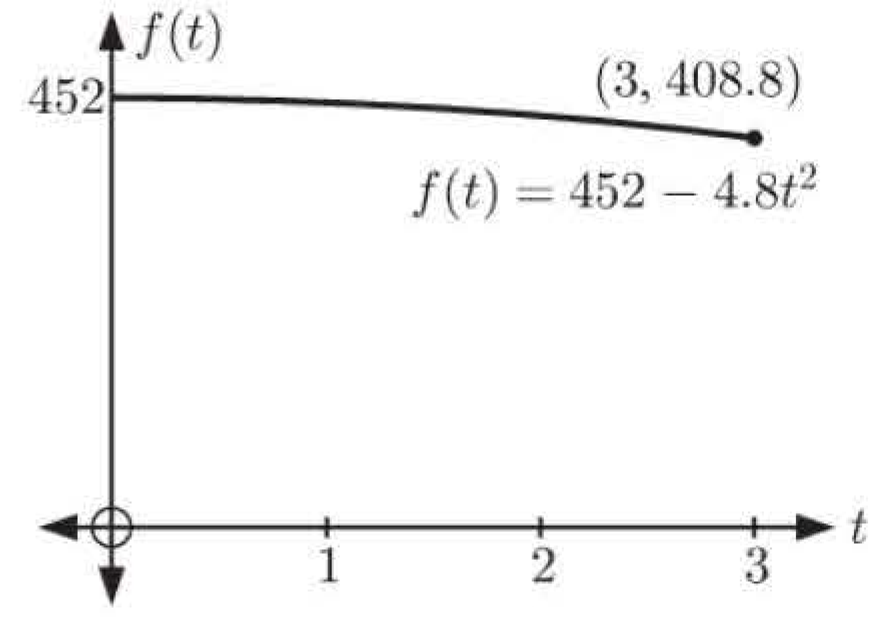
$\therefore A = \dfrac{S+D}{2}$ $\qquad \therefore B = \dfrac{S-D}{2}$

So, $\cos S - \cos D = -2\sin\left(\frac{S+D}{2}\right)\sin\left(\frac{S-D}{2}\right)$

c $$\begin{aligned}\lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} &= \lim_{h\to 0}\frac{-2\sin\left[\frac{(x+h)+x}{2}\right]\sin\left[\frac{(x+h)-x}{2}\right]}{h} \qquad \{x+h=S,\ x=D\}\\ &= \lim_{h\to 0}\frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}\\ &= -2\lim_{h\to 0}\frac{\sin\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}\\ &= -2\lim_{h\to 0}\sin\left(x+\tfrac{h}{2}\right)\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\times\tfrac{1}{2}\\ &= -1\lim_{h\to 0}\sin\left(x+\tfrac{h}{2}\right)\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \qquad \{\tfrac{h}{2}\to 0 \text{ as } h\to 0\}\\ &= -1\lim_{h\to 0}\sin\left(x+\tfrac{h}{2}\right)\\ &= -\sin x\end{aligned}$$

EXERCISE 17D

1 **a**

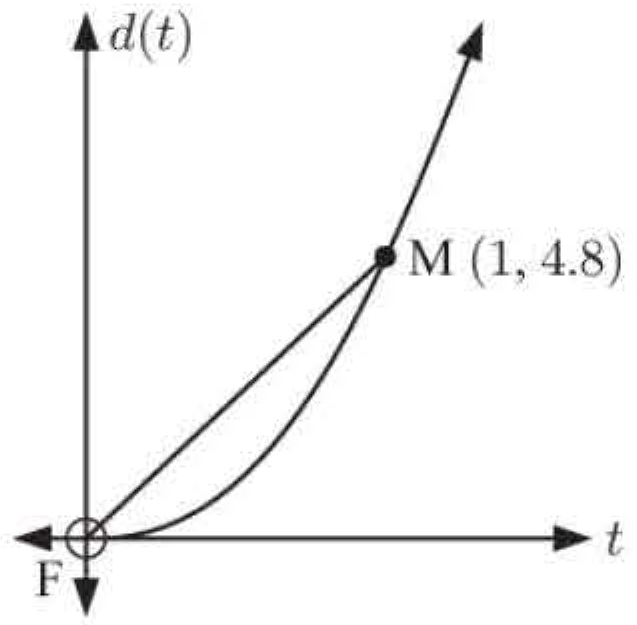


b The graph of $f(t)$ is not a straight line, so the jumper is not travelling with constant speed.

c If the *altitude* of the jumper is given by $f(t) = 452 - 4.8t^2$,
then the *distance* covered by the jumper $d(t) = 452 - f(t)$

$$\therefore\ d(t) = 452 - (452 - 4.8t^2)$$
$$\therefore\ d(t) = 452 - 452 + 4.8t^2$$
$$\therefore\ d(t) = 4.8t^2$$

i We choose a fixed point F on $d(t)$ when $t = 0$ seconds. This is the point $(0, 0)$.
We then choose another point M on the curve, for example the point $(1, 4.8)$.
The average speed in the interval $0 \leqslant t \leqslant 1$ is $\dfrac{4.8 - 0}{1 - 0} = 4.8\ \text{m s}^{-1}$.

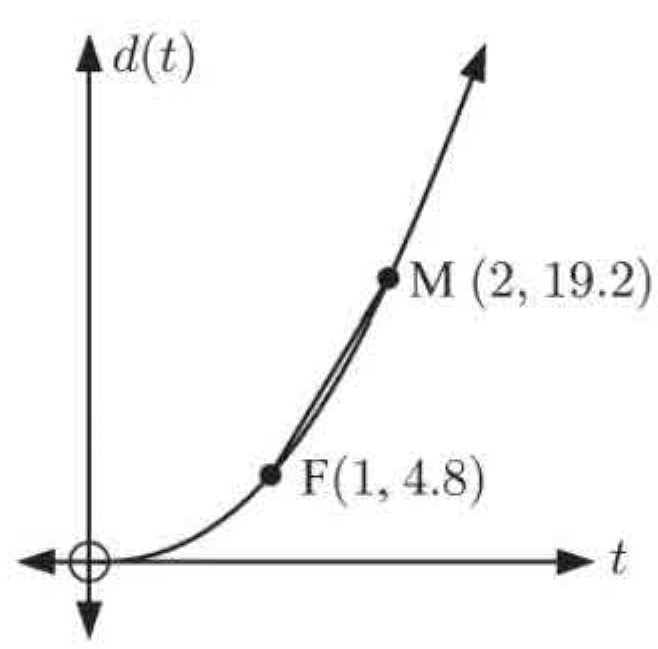


We repeat this process, moving M closer to F each time, and get the following results:

t	gradient of [FM]
1	4.8
0.5	2.4
0.1	0.48
0.01	0.048
0.001	0.0048

So, as M approaches F, the gradient of [FM] approaches 0.
$\therefore$ the speed of the jumper at $t = 0$ seconds is $0\ \text{m s}^{-1}$.

ii We now choose point F on $d(t)$ when $t = 1$ second. This is the point $(1, 4.8)$.
We then choose another point M on the curve, for example the point $(2, 19.2)$.
The average speed in the interval $1 \leqslant t \leqslant 2$ is $\dfrac{19.2 - 4.8}{2 - 1} = 14.4\ \text{m s}^{-1}$.



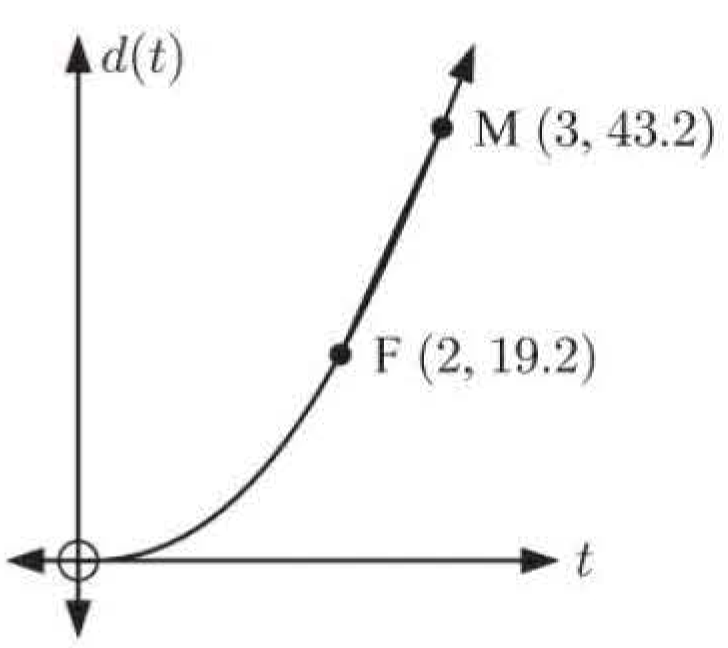
We repeat this process, moving M closer to F each time, and get the following results:

t	gradient of [FM]
2	14.4
1.5	12
1.1	10.08
1.01	9.648
1.001	9.6048
1.0001	9.600 48

t	gradient of [FM]
0	4.8
0.5	7.2
0.9	9.12
0.99	9.552
0.999	9.5952
0.9999	9.599 52

So, as M approaches F (from either direction), the gradient of [FM] approaches 9.6 .
$\therefore$ the speed of the jumper at $t = 1$ second is $9.6\ \text{m s}^{-1}$.

iii We now choose point F on $d(t)$ when $t = 2$ seconds. This is the point $(2, 19.2)$.
We then choose another point M on the curve, for example the point $(3, 43.2)$.
The average speed in the interval $2 \leqslant t \leqslant 3$ is $\dfrac{43.2 - 19.2}{3 - 2} = 24\ \text{m s}^{-1}$.



We repeat this process, moving M closer to F each time, and get the following results:

t	gradient of [FM]
3	24
2.5	21.6
2.1	19.68
2.01	19.248
2.001	19.2048
2.0001	19.200 48

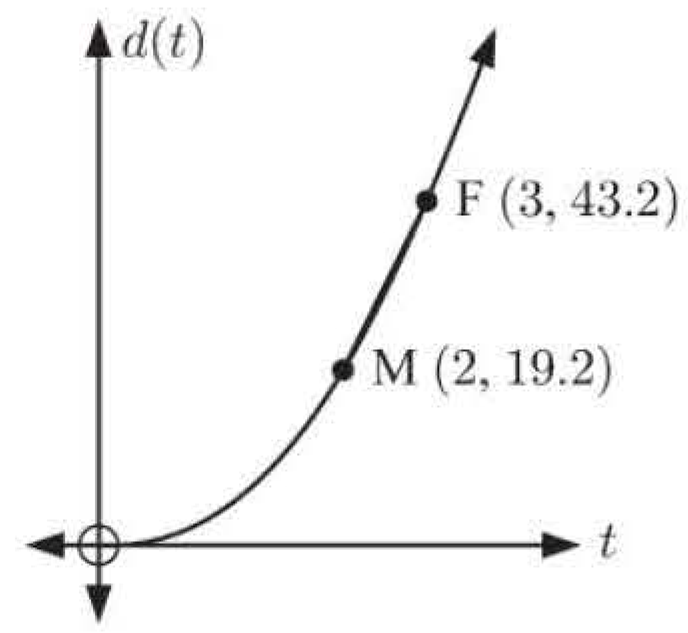
t	gradient of [FM]
1	14.4
1.5	16.8
1.9	18.72
1.99	19.152
1.999	19.1952
1.9999	19.199 52

So, as M approaches F (from either direction), the gradient of [FM] approaches 19.2.

$\therefore$ the speed of the jumper at $t = 2$ seconds is 19.2 m s^{-1}.

iv We choose a fixed point F on $d(t)$ when $t = 3$ seconds. This is the point (3, 43.2).
We then choose another point M on the curve, for example the point (2, 19.2).
The gradient of [MF] is

$$\frac{43.2 - 19.2}{3 - 2} = 24 \text{ m s}^{-1}.$$



We repeat this process, moving M closer to F each time, and get the following results:

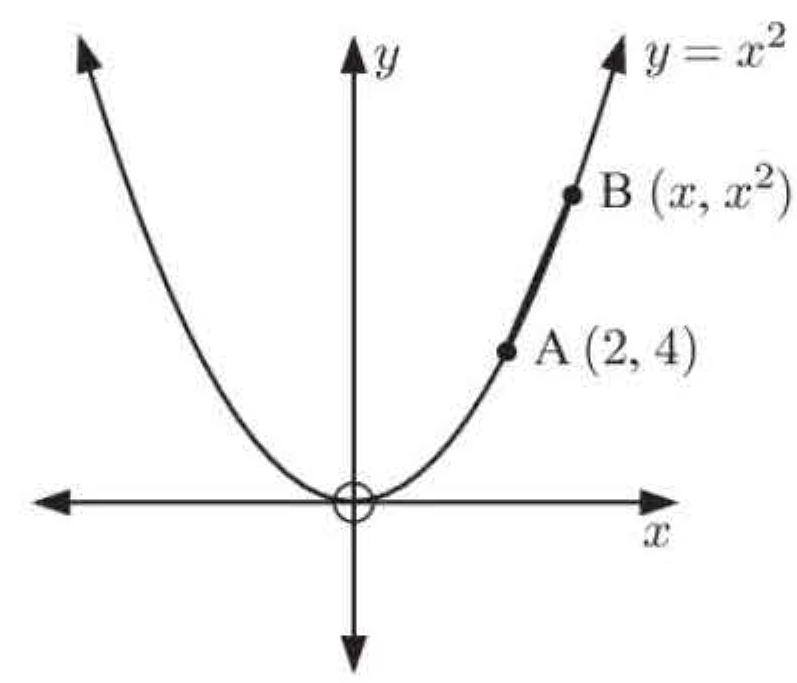
t	gradient of [MF]
2	24
2.5	26.4
2.9	28.32
2.99	28.752
2.999	28.7952
2.9999	28.799 52

So, as M approaches F, the gradient of [MF] approaches 28.8.

$\therefore$ the speed of the jumper at $t = 3$ seconds is 28.8 m s^{-1}.

2 **a** Suppose A is the point (2, 4) and B is a point on $y = x^2$ with coordinates (x, x^2).
The chord [AB] has gradient

$$\frac{x^2 - 4}{x - 2} \quad \left(\text{or } \frac{4 - x^2}{2 - x}\right).$$



As B moves closer to A (from either side), we get the following results:

x	Point B	Gradient of [AB]
0	(0, 0)	2
1	(1, 1)	3
1.5	(1.5, 2.25)	3.5
1.9	(1.9, 3.61)	3.9
1.99	(1.99, 3.9601)	3.99
1.999	(1.999, 3.996 001)	3.999

x	Point B	Gradient of [AB]
5	(5, 25)	7
3	(3, 9)	5
2.5	(2.5, 6.25)	4.5
2.1	(2.1, 4.41)	4.1
2.01	(2.01, 4.0401)	4.01
2.001	(2.001, 4.004 001)	4.001

So, as B approaches A, the gradient of [AB] approaches 4.

$\therefore$ the gradient of the tangent to $y = x^2$ at the point (2, 4) is 4.

b
$$\begin{aligned}\lim_{x\to 2} \frac{x^2 - 4}{x - 2} &= \lim_{x \to 2} \frac{(x+2)(x-2)}{x-2} \\ &= \lim_{x\to 2} (x + 2) \quad \text{since } x \neq 2 \\ &= 4\end{aligned}$$

This is the gradient of the tangent to $y = x^2$ at the point where $x = 2$.

EXERCISE 17E

1 **a** $f(2) = 3$

b $f'(2)$ is the gradient of the tangent to $f(x)$ at the point where $x = 2$.
Since $f(x)$ is a straight line, this is the same as the gradient of $f(x)$ itself.
$f(x)$ is a horizontal line, and hence has gradient 0.
$\therefore\ f'(2) = 0$

2 **a** $f(0) = 4$

b $f'(0)$ is the gradient of the tangent to $f(x)$ at the point where $x = 0$.
Since $f(x)$ is a straight line, this is the same as the gradient of $f(x)$ itself.
$f(x)$ passes through (0, 4) and (4, 0), so it has gradient $= \dfrac{0-4}{4-0} = -1$
$\therefore\ f'(0) = -1$

3 The graph shows the tangent to the curve $y = f(x)$ at the point where $x = 2$.
The tangent passes through (0, 1) and (4, 5), so its gradient is $f'(2) = \dfrac{5-1}{4-0} = 1$.

The equation of the tangent is $\dfrac{y-1}{x-0} = 1$
$\therefore\ y = x + 1$

When $x = 2$, $y = 3$, so the point of contact is (2, 3).
$\therefore\ f(2) = 3$ and $f'(2) = 1$.

EXERCISE 17F

1 **a** **i** $f(x) = x$

$$\begin{aligned}\therefore\ f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0} \frac{(x+h)-x}{h}\\ &= \lim_{h\to 0} \frac{h}{h}\\ &= \lim_{h\to 0} 1 \qquad \{\text{as } h \neq 0\}\\ &= 1\end{aligned}$$

ii $f(x) = 5$

$$\begin{aligned}\therefore\ f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0} \frac{5-5}{h}\\ &= \lim_{h\to 0} \frac{0}{h}\\ &= \lim_{h\to 0} 0 \qquad \{\text{as } h \neq 0\}\\ &= 0\end{aligned}$$

iii $f(x) = x^3$

$$\begin{aligned}&\therefore\ f'(x)\\ &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0} \frac{(x+h)^3-x^3}{h}\\ &= \lim_{h\to 0} \frac{\cancel{x^3}+3x^2h+3xh^2+h^3-\cancel{x^3}}{h}\\ &= \lim_{h\to 0} \frac{3x^2h+3xh^2+h^3}{h}\\ &= \lim_{h\to 0} 3x^2+3xh+h^2 \qquad \{\text{as } h \neq 0\}\\ &= 3x^2\end{aligned}$$

iv $f(x) = x^4$

$$\begin{aligned}&\therefore\ f'(x)\\ &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0} \frac{(x+h)^4-x^4}{h}\\ &= \lim_{h\to 0} \frac{\cancel{x^4}+4x^3h+6x^2h^2+4xh^3+h^4-\cancel{x^4}}{h}\\ &= \lim_{h\to 0} \frac{4x^3h+6x^2h^2+4xh^3+h^4}{h}\\ &= \lim_{h\to 0} 4x^3+6x^2h+4xh^2+h^3 \qquad \{\text{as } h \neq 0\}\\ &= 4x^3\end{aligned}$$

b From **a**, we predict that if $f(x) = x^n$, $f'(x) = nx^{n-1}$, $n \in \mathbb{N}$.

2 **a** $f(x) = 2x + 5$

$\therefore\ f'(x)$

$= \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim\limits_{h \to 0} \dfrac{(2(x+h)+5) - (2x+5)}{h}$

$= \lim\limits_{h \to 0} \dfrac{\cancel{2x} + 2h + \cancel{5} - \cancel{2x} - \cancel{5}}{h}$

$= \lim\limits_{h \to 0} \dfrac{2h}{h}$

$= \lim\limits_{h \to 0} 2 \qquad \{\text{as } h \neq 0\}$

$= 2$

b $f(x) = x^2 - 3x$

$\therefore\ f'(x)$

$= \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim\limits_{h \to 0} \dfrac{[(x+h)^2 - 3(x+h)] - [x^2 - 3x]}{h}$

$= \lim\limits_{h \to 0} \dfrac{\cancel{x^2} + 2xh + h^2 - \cancel{3x} - 3h - \cancel{x^2} + \cancel{3x}}{h}$

$= \lim\limits_{h \to 0} \dfrac{2xh + h^2 - 3h}{h}$

$= \lim\limits_{h \to 0} 2x + h - 3 \qquad \{\text{as } h \neq 0\}$

$= 2x - 3$

c $f(x) = -x^2 + 5x - 3$

$\therefore\ f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim\limits_{h \to 0} \dfrac{[-(x+h)^2 + 5(x+h) - 3] - [-x^2 + 5x - 3]}{h}$

$= \lim\limits_{h \to 0} \dfrac{-\cancel{x^2} - 2xh - h^2 + \cancel{5x} + 5h - \cancel{3} + \cancel{x^2} - \cancel{5x} + \cancel{3}}{h}$

$= \lim\limits_{h \to 0} \dfrac{-2xh - h^2 + 5h}{h}$

$= \lim\limits_{h \to 0} -2x + 5 - h \qquad \{\text{as } h \neq 0\}$

$= -2x + 5$

3 **a** $y = f(x) = 4 - x$

$\therefore\ \dfrac{dy}{dx}$

$= \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim\limits_{h \to 0} \dfrac{[4 - (x+h)] - [4 - x]}{h}$

$= \lim\limits_{h \to 0} \dfrac{\cancel{4} - \cancel{x} - h - \cancel{4} + \cancel{x}}{h}$

$= \lim\limits_{h \to 0} \dfrac{-h}{h}$

$= \lim\limits_{h \to 0} -1 \qquad \{\text{as } h \neq 0\}$

$= -1$

b $y = f(x) = 2x^2 + x - 1$

$\therefore\ \dfrac{dy}{dx}$

$= \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim\limits_{h \to 0} \dfrac{[2(x+h)^2 + (x+h) - 1] - [2x^2 + x - 1]}{h}$

$= \lim\limits_{h \to 0} \dfrac{\cancel{2x^2} + 4xh + 2h^2 + \cancel{x} + h - \cancel{1} - \cancel{2x^2} - \cancel{x} + \cancel{1}}{h}$

$= \lim\limits_{h \to 0} \dfrac{4xh + 2h^2 + h}{h}$

$= \lim\limits_{h \to 0} 4x + 1 + 2h \qquad \{\text{as } h \neq 0\}$

$= 4x + 1$

c $y = f(x) = x^3 - 2x^2 + 3$

$\therefore\ \dfrac{dy}{dx} = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim\limits_{h \to 0} \dfrac{[(x+h)^3 - 2(x+h)^2 + 3] - [x^3 - 2x^2 + 3]}{h}$

$= \lim\limits_{h \to 0} \dfrac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{2x^2} - 4xh - 2h^2 + \cancel{3} - \cancel{x^3} + \cancel{2x^2} - \cancel{3}}{h}$

$= \lim\limits_{h \to 0} \dfrac{3x^2h + 3xh^2 + h^3 - 4xh - 2h^2}{h}$

$= \lim\limits_{h \to 0} 3x^2 + 3xh + h^2 - 4x - 2h \qquad \{\text{as } h \neq 0\}$

$= 3x^2 - 4x$

4 **a** $f(x) = x^3$

$\therefore\ f'(2)$

$= \lim\limits_{h \to 0} \dfrac{f(2+h) - f(2)}{h}$ where $f(2) = 2^3 = 8$

$= \lim\limits_{h \to 0} \dfrac{(2+h)^3 - 8}{h}$

$= \lim\limits_{h \to 0} \dfrac{\cancel{8} + 12h + 6h^2 + h^3 - \cancel{8}}{h}$

$= \lim\limits_{h \to 0} \dfrac{12h + 6h^2 + h^3}{h}$

$= \lim\limits_{h \to 0} 12 + 6h + h^2 \quad \{\text{as } h \neq 0\}$

$= 12$

b $f(x) = x^4$

$\therefore\ f'(3)$

$= \lim\limits_{h \to 0} \dfrac{f(3+h) - f(3)}{h}$ where $f(3) = 3^4 = 81$

$= \lim\limits_{h \to 0} \dfrac{(3+h)^4 - 81}{h}$

$= \lim\limits_{h \to 0} \dfrac{\cancel{81} + 108h + 54h^2 + 12h^3 + h^4 - \cancel{81}}{h}$

$= \lim\limits_{h \to 0} \dfrac{108h + 54h^2 + 12h^3 + h^4}{h}$

$= \lim\limits_{h \to 0} 108 + 54h + 12h^2 + h^3 \quad \{\text{as } h \neq 0\}$

$= 108$

5 **a** $f(x) = 3x + 5$

We need to find $f'(-2)$.

$f'(-2) = \lim\limits_{h \to 0} \dfrac{f(-2+h) - f(-2)}{h}$

where $f(-2) = 3(-2) + 5 = -1$

$= \lim\limits_{h \to 0} \dfrac{[3(-2+h)+5] - [-1]}{h}$

$= \lim\limits_{h \to 0} \dfrac{-\cancel{6} + 3h + \cancel{5} + \cancel{1}}{h}$

$= \lim\limits_{h \to 0} \dfrac{3h}{h}$

$= \lim\limits_{h \to 0} 3 \quad \{\text{as } h \neq 0\}$

$= 3$

$\therefore$ the gradient of the tangent to $f(x) = 3x + 5$ at $x = -2$ is 3.

b $f(x) = 5 - 2x^2$

We need to find $f'(3)$.

$f'(3) = \lim\limits_{h \to 0} \dfrac{f(3+h) - f(3)}{h}$

where $f(3) = 5 - 2(3)^2 = -13$

$= \lim\limits_{h \to 0} \dfrac{[5 - 2(3+h)^2] - [-13]}{h}$

$= \lim\limits_{h \to 0} \dfrac{5 - 2(9 + 6h + h^2) + 13}{h}$

$= \lim\limits_{h \to 0} \dfrac{\cancel{5} - \cancel{18} - 12h - 2h^2 + \cancel{13}}{h}$

$= \lim\limits_{h \to 0} \dfrac{-12h - 2h^2}{h}$

$= \lim\limits_{h \to 0} \dfrac{-2h(6+h)}{h}$

$= \lim\limits_{h \to 0} -2(6+h) \quad \{\text{as } h \neq 0\}$

$= -12$

c $f(x) = x^2 + 3x - 4$

We need to find $f'(3)$.

$f'(3) = \lim\limits_{h \to 0} \dfrac{f(3+h) - f(3)}{h}$ where $f(3) = 3^2 + 3(3) - 4 = 14$

$= \lim\limits_{h \to 0} \dfrac{[(3+h)^2 + 3(3+h) - 4] - 14}{h}$

$= \lim\limits_{h \to 0} \dfrac{\cancel{9} + 6h + h^2 + \cancel{9} + 3h - \cancel{4} - \cancel{14}}{h}$

$= \lim\limits_{h \to 0} \dfrac{9h + h^2}{h}$

$= \lim\limits_{h \to 0} \dfrac{h(9+h)}{h}$

$= \lim\limits_{h \to 0} (9+h) \quad \{\text{as } h \neq 0\}$

$= 9$

d $f(x) = 5 - 2x - 3x^2$

We need to find $f'(-2)$.

$$\begin{aligned} f'(-2) &= \lim_{h \to 0} \frac{f(-2+h) - f(-2)}{h} \quad \text{where } f(-2) = 5 - 2(-2) - 3(-2)^2 = -3 \\ &= \lim_{h \to 0} \frac{[5 - 2(-2+h) - 3(-2+h)^2] - [-3]}{h} \\ &= \lim_{h \to 0} \frac{5 + 4 - 2h - 3(4 - 4h + h^2) + 3}{h} \\ &= \lim_{h \to 0} \frac{\cancel{5} + \cancel{4} - 2h - \cancel{12} + 12h - 3h^2 + \cancel{3}}{h} \\ &= \lim_{h \to 0} \frac{10h - 3h^2}{h} \\ &= \lim_{h \to 0} \frac{h(10 - 3h)}{h} \\ &= \lim_{h \to 0} (10 - 3h) \qquad \{\text{as } h \neq 0\} \\ &= 10 \end{aligned}$$

6 **a** $y = x^3 - 3x$

$$\begin{aligned} \therefore \ \frac{dy}{dx} & \\ &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{[(x+h)^3 - 3(x+h)] - [x^3 - 3x]}{h} \\ &= \lim_{h \to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{3x} - 3h - \cancel{x^3} + \cancel{3x}}{h} \\ &= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 - 3h}{h} \\ &= \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2 - 3)}{h} \\ &= \lim_{h \to 0} (3x^2 + 3xh + h^2 - 3) \qquad \{\text{as } h \neq 0\} \\ &= 3x^2 - 3 \end{aligned}$$

b The tangent has zero gradient when

$$\begin{aligned} f'(x) &= 0 \\ \therefore \ 3x^2 - 3 &= 0 \\ \therefore \ 3x^2 &= 3 \\ \therefore \ x^2 &= 1 \\ \therefore \ x &= \pm 1 \end{aligned}$$

When $x = -1$,
$y = (-1)^3 - 3(-1) = 2$

When $x = 1$,
$y = (1)^3 - 3(1) = -2$

So, the points on the graph at which the tangent has zero gradient are $(-1, 2)$ and $(1, -2)$.

7 **a** $y = f(x) = \dfrac{4}{x}$

$$\begin{aligned} \frac{dy}{dx} &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \left(\frac{\frac{4}{x+h} - \frac{4}{x}}{h} \right) \\ &= \lim_{h \to 0} \frac{\left(\frac{4x - 4(x+h)}{x(x+h)} \right)}{h} \\ &= \lim_{h \to 0} \frac{4x - 4x - 4h}{xh(x+h)} \\ &= \lim_{h \to 0} \frac{-4\cancel{h}}{x\cancel{h}(x+h)} \\ &= \lim_{h \to 0} \frac{-4}{x(x+h)} \\ &= \frac{-4}{x^2} \end{aligned}$$

b $y = f(x) = \dfrac{4x+1}{x-2}$

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\left(\frac{4(x+h)+1}{x+h-2} - \frac{4x+1}{x-2}\right)}{h} \\
&= \lim_{h\to 0} \frac{\left(\frac{4x+4h+1}{x+h-2} - \frac{4x+1}{x-2}\right)}{h} \\
&= \lim_{h\to 0} \frac{\left(\frac{(4x+4h+1)(x-2)-(4x+1)(x+h-2)}{(x+h-2)(x-2)}\right)}{h} \\
&= \lim_{h\to 0} \frac{4x^2-8x+4hx-8h+x-2-(4x^2+4hx-8x+x+h-2)}{h(x+h-2)(x-2)} \\
&= \lim_{h\to 0} \frac{\cancel{4x^2}-\cancel{7x}+\cancel{4hx}-8h-\cancel{2}-\cancel{4x^2}-\cancel{4hx}+\cancel{7x}-h+\cancel{2}}{h(x+h-2)(x-2)} \\
&= \lim_{h\to 0} \frac{-9\cancel{h}}{\cancel{h}(x+h-2)(x-2)} \\
&= \lim_{h\to 0} \frac{-9}{(x+h-2)(x-2)} \\
&= \frac{-9}{(x-2)^2}
\end{aligned}$$

8 **a** $f(x) = \dfrac{1}{x^2}$

$$\begin{aligned}
\therefore\ f(3) &= \frac{1}{3^2} = \frac{1}{9} \\
f'(3) &= \lim_{h\to 0} \frac{f(3+h)-f(3)}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{1}{(3+h)^2} - \dfrac{1}{9}}{h} \\
&= \lim_{h\to 0} \frac{9-(3+h)^2}{9h(3+h)^2} \\
&= \lim_{h\to 0} \frac{\cancel{9}-\cancel{9}-6h-h^2}{9h(3+h)^2} \\
&= \lim_{h\to 0} \frac{-\cancel{h}(6+h)}{9\cancel{h}(3+h)^2} \\
&= \lim_{h\to 0} \frac{-(6+h)}{9(3+h)^2} \quad \{\text{as } h \neq 0\} \\
&= \tfrac{-6}{81} \\
&= -\tfrac{2}{27}
\end{aligned}$$

b $f(x) = \dfrac{3x}{x^2+1}$

$$\begin{aligned}
\therefore\ f(-4) &= -\tfrac{12}{17} \\
f'(-4) &= \lim_{x\to -4} \frac{f(x)-f(-4)}{x-(-4)} \\
&= \lim_{x\to -4} \frac{\dfrac{3x}{x^2+1} - \left(-\dfrac{12}{17}\right)}{x+4} \\
&= \lim_{x\to -4} \frac{51x+12(x^2+1)}{17(x^2+1)(x+4)} \\
&= \lim_{x\to -4} \frac{12x^2+51x+12}{17(x^2+1)(x+4)} \\
&= \lim_{x\to -4} \frac{\cancel{(x+4)}(12x+3)}{17(x^2+1)\cancel{(x+4)}} \\
&= \lim_{x\to -4} \frac{12x+3}{17(x^2+1)} \quad \{x \neq -4\} \\
&= -\frac{45}{17\times 17} \\
&= -\tfrac{45}{289}
\end{aligned}$$

c $f(x) = \sqrt{x}$ and $f(4) = \sqrt{4} = 2$

$$\begin{aligned} f'(4) &= \lim_{x \to 4} \frac{f(x) - f(4)}{x - 4} \\ &= \lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} \\ &= \lim_{x \to 4} \frac{\cancel{\sqrt{x} - 2}}{(\sqrt{x} + 2)\cancel{(\sqrt{x} - 2)}} \\ &= \lim_{x \to 4} \frac{1}{(\sqrt{x} + 2)} \quad \{\text{as } x \neq 4\} \\ &= \frac{1}{2 + 2} \\ &= \tfrac{1}{4} \end{aligned}$$

d $f(x) = \dfrac{1}{\sqrt{x}}$

$\therefore \ f(1) = \dfrac{1}{\sqrt{1}} = 1$

$$\begin{aligned} f'(1) &= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h \to 0} \frac{\frac{1}{\sqrt{1+h}} - \frac{1}{1}}{h} \\ &= \lim_{h \to 0} \frac{1 - \sqrt{1+h}}{h\sqrt{1+h}} \\ &= \lim_{h \to 0} \frac{(1 - \sqrt{1+h})}{h\sqrt{1+h}} \left(\frac{1 + \sqrt{1+h}}{1 + \sqrt{1+h}} \right) \\ &= \lim_{h \to 0} \frac{1 - (1+h)}{h(\sqrt{1+h})(1 + \sqrt{1+h})} \\ &= \lim_{h \to 0} \frac{{}^{-1}\cancel{-h}}{\cancel{h}\sqrt{1+h}(1 + \sqrt{1+h})} \\ &= \lim_{h \to 0} \frac{-1}{\sqrt{1+h}(1 + \sqrt{1+h})} \quad \{h \neq 0\} \\ &= \frac{-1}{1(1+1)} = -\tfrac{1}{2} \end{aligned}$$

REVIEW SET 17A

1 **a** We can make $6x - 7$ as close as we like to -1 by making x sufficiently close to 1.

$\therefore \ \lim\limits_{x \to 1} (6x - 7) = -1$

b
$$\begin{aligned} \lim_{h \to 0} \frac{2h^2 - h}{h} &= \lim_{h \to 0} \frac{h(2h - 1)}{h} \\ &= \lim_{h \to 0} (2h - 1) \quad \{\text{as } h \neq 0\} \\ &= -1 \end{aligned}$$

c
$$\begin{aligned} \lim_{x \to 4} \frac{x^2 - 16}{x - 4} &= \lim_{x \to 4} \frac{(x+4)(x-4)}{x - 4} \\ &= \lim_{x \to 4} (x + 4) \quad \{\text{as } x \neq 0\} \\ &= 8 \end{aligned}$$

2 **a** $f(x) = x^2 + 2x$

$\therefore \ f'(x)$
$$\begin{aligned} &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{[(x+h)^2 + 2(x+h)] - [x^2 + 2x]}{h} \\ &= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 + \cancel{2x} + 2h - \cancel{x^2} - \cancel{2x}}{h} \\ &= \lim_{h \to 0} \frac{2xh + h^2 + 2h}{h} \\ &= \lim_{h \to 0} \frac{h(2x + h + 2)}{h} \\ &= \lim_{h \to 0} (2x + h + 2) \quad \{\text{as } h \neq 0\} \\ &= 2x + 2 \end{aligned}$$

b $y = f(x) = 4 - 3x^2$

$\therefore \ \dfrac{dy}{dx}$
$$\begin{aligned} &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{[4 - 3(x+h)^2] - [4 - 3x^2]}{h} \\ &= \lim_{h \to 0} \frac{4 - 3(x^2 + 2xh + h^2) - 4 + 3x^2}{h} \\ &= \lim_{h \to 0} \frac{\cancel{4} - \cancel{3x^2} - 6xh - 3h^2 - \cancel{4} + \cancel{3x^2}}{h} \\ &= \lim_{h \to 0} \frac{-3h(2x + h)}{h} \\ &= \lim_{h \to 0} -3(2x + h) \quad \{\text{as } h \neq 0\} \\ &= -6x \end{aligned}$$

3 **a** $f(x) = e^{x-2} - 3$

$f(x)$ is defined for all $x \in \mathbb{R}$

$\therefore$ no vertical asymptotes exist

As $x \to -\infty$, $f(x) \to -3$

$\therefore$ $y = -3$ is a horizontal asymptote

As $x \to \infty$, $f(x) \to \infty$

As $x \to -\infty$, $f(x) \to -3^+$

b $f(x) = \ln(x^2 + 3)$ has no asymptotes

c $f(x) = \ln(-x) + 2$

$f(x)$ is undefined for $x \geqslant 0$

$\therefore$ $x = 0$ is a vertical asymptote

As $x \to 0^-$, $f(x) \to -\infty$

4 **a**

$$\begin{aligned}
&\lim_{\theta \to 0} \frac{\sin 4\theta}{\theta} \\
&= \lim_{\theta \to 0} \frac{\sin 4\theta}{4\theta} \times 4 \\
&= 4 \lim_{4\theta \to 0} \frac{\sin 4\theta}{4\theta} \qquad \{4\theta \to 0 \text{ as } \theta \to 0\} \\
&= 4 \times 1 \\
&= 4
\end{aligned}$$

b

$$\begin{aligned}
&\lim_{\theta \to 0} \frac{2\theta}{\sin 3\theta} \\
&= \lim_{\theta \to 0} \frac{3\theta}{\sin 3\theta} \times \tfrac{2}{3} \\
&= \tfrac{2}{3} \lim_{\theta \to 0} \frac{1}{\frac{\sin 3\theta}{3\theta}} \\
&= \tfrac{2}{3} \frac{1}{\lim\limits_{3\theta \to 0} \frac{\sin 3\theta}{3\theta}} \qquad \{3\theta \to 0 \text{ as } \theta \to 0\} \\
&= \tfrac{2}{3} \times \tfrac{1}{1} \\
&= \tfrac{2}{3}
\end{aligned}$$

c

$$\begin{aligned}
&\lim_{n \to \infty} n \sin\left(\tfrac{\pi}{n}\right) \\
&= \lim_{\frac{1}{n} \to 0} \frac{\sin\left(\frac{\pi}{n}\right)}{\frac{1}{n}} \qquad \{\tfrac{1}{n} \to 0 \text{ as } n \to \infty\} \\
&= \lim_{\frac{1}{n} \to 0} \frac{\sin\left(\frac{\pi}{n}\right)}{\frac{\pi}{n}} \times \pi \\
&= \pi \times \lim_{\frac{\pi}{n} \to 0} \frac{\sin\left(\frac{\pi}{n}\right)}{\frac{\pi}{n}} \qquad \{\tfrac{\pi}{n} \to 0 \text{ as } \tfrac{1}{n} \to 0\} \\
&= \pi \times 1 \\
&= \pi
\end{aligned}$$

5 $f(x) = 5x - x^2$

$$\begin{aligned}
f'(1) &= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \qquad \text{where } f(1) = 5(1) - (1)^2 = 4 \\
&= \lim_{h \to 0} \frac{[5(1+h) - (1+h)^2] - 4}{h} \\
&= \lim_{h \to 0} \frac{5 + 5h - (1 + 2h + h^2) - 4}{h} \\
&= \lim_{h \to 0} \frac{\cancel{5} + 5h - \cancel{1} - 2h - h^2 - \cancel{4}}{h} \\
&= \lim_{h \to 0} \frac{3h - h^2}{h} \\
&= \lim_{h \to 0} \frac{h(3-h)}{h} \\
&= \lim_{h \to 0} (3 - h) \qquad \{\text{as } h \neq 0\} \\
&= 3
\end{aligned}$$

6 **a** $f(t) = 452 - 4.8t^2$

$\therefore\ f'(t)$

$= \lim\limits_{h\to 0} \dfrac{f(t+h) - f(t)}{h}$

$= \lim\limits_{h\to 0} \dfrac{[452 - 4.8(t+h)^2] - [452 - 4.8t^2]}{h}$

$= \lim\limits_{h\to 0} \dfrac{\cancel{452} - 4.8(t^2 + 2th + h^2) - \cancel{452} + 4.8t^2}{h}$

$= \lim\limits_{h\to 0} \dfrac{-\cancel{4.8t^2} - 9.6th - 4.8h^2 + \cancel{4.8t^2}}{h}$

$= \lim\limits_{h\to 0} \dfrac{h(-9.6t - 4.8h)}{h}$

$= \lim\limits_{h\to 0} (-9.6t - 4.8h) \qquad \{\text{as } h \neq 0\}$

$= -9.6t \text{ m s}^{-1}$

b To find the speed of the jumper at $t = 2$ seconds, we need to find $f'(2)$.

Now $f'(t) = -9.6t$ {from **a**}

$\therefore\ f'(2) = -9.6 \times 2$

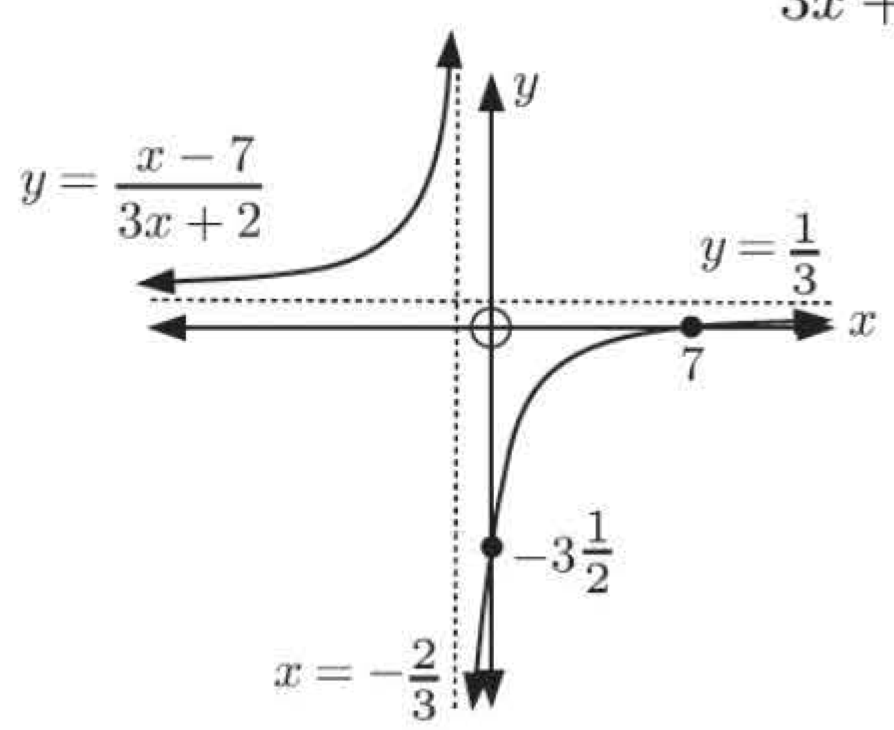
$= -19.2$

$\therefore$ the speed of the jumper at $t = 2$ seconds is 19.2 m s^{-1}.

(The $-$ sign indicates the jumper is moving downwards.)

REVIEW SET 17B

1 **a** We sketch the graph of $y = \dfrac{x-7}{3x+2}$:



As $x \to -\frac{2}{3}^-$, $y \to \infty$

As $x \to -\frac{2}{3}^+$, $y \to -\infty$

As $x \to \infty$, $y \to \frac{1}{3}^-$

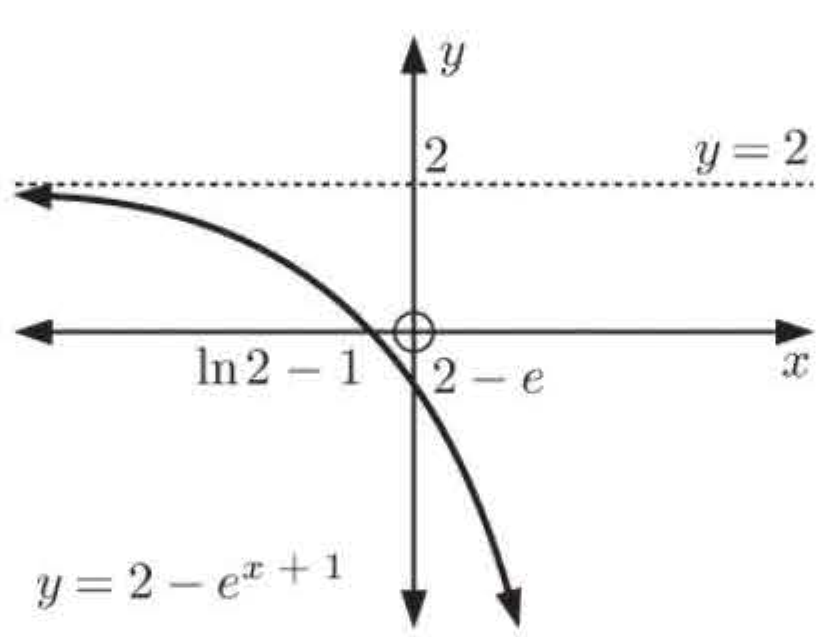
As $x \to -\infty$, $y \to \frac{1}{3}^+$

The vertical asymptote is $x = -\frac{2}{3}$.

The horizontal asymptote is $y = \frac{1}{3}$.

b $\lim\limits_{x\to -\infty} \left(\dfrac{x-7}{3x+2}\right) = \frac{1}{3}, \qquad \lim\limits_{x\to \infty} \left(\dfrac{x-7}{3x+2}\right) = \frac{1}{3}$

2 **a**



b $\lim\limits_{x\to -\infty} (2 - e^{x+1}) = 2,$

$\lim\limits_{x\to \infty} (2 - e^{x+1})$ does not exist

c The horizontal asymptote is $y = 2$.

3 **a** $f(x) = \ln(x^2)$ is not defined when $x = 0$

$\therefore\ f(x) = \ln(x^2)$ is not continuous at $x = 0$.

b $f(x) = \dfrac{x^2 - 1}{1 - x}$ is not defined when $x = 1$

$\therefore\ f(x) = \dfrac{x^2 - 1}{1 - x}$ is not continuous at $x = 1$.

4 $$\lim_{h\to 0} \frac{2\cos\left(x + \frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{h} = 2 \lim_{h\to 0} \cos\left(x + \tfrac{h}{2}\right) \lim_{h\to 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \times \tfrac{1}{2}$$

$$= 1 \times \lim_{h\to 0} \cos\left(x + \tfrac{h}{2}\right) \lim_{\frac{h}{2}\to 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \qquad \{\tfrac{h}{2} \to 0 \text{ as } h \to 0\}$$

$$= 1 \times \lim_{h\to 0} \cos\left(x + \tfrac{h}{2}\right)$$

$$= \cos x$$

It is assumed that x and h are in radians.

5 **a** $$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{2(x+h)^2-2x^2}{h}\\ &= \frac{2(x^2+2xh+h^2)-2x^2}{h}\\ &= \frac{\cancel{2x^2}+4xh+2h^2-\cancel{2x^2}}{h}\\ &= \frac{h(4x+2h)}{h}\\ &= 4x+2h \quad \text{provided } h\neq 0\end{aligned}$$

b If $x=3$ then $$\begin{aligned}\frac{f(3+h)-f(3)}{h} &= 4(3)+2h \quad \{\text{using } \mathbf{a}\}\\ &= 12+2h\end{aligned}$$

i When $h=0.1$,
$$\begin{aligned}\frac{f(3+h)-f(3)}{h} &= 12+2(0.1)\\ &= 12+0.2\\ &= 12.2\end{aligned}$$

ii When $h=0.01$,
$$\begin{aligned}\frac{f(3+h)-f(3)}{h} &= 12+2(0.01)\\ &= 12+0.02\\ &= 12.02\end{aligned}$$

iii $$\begin{aligned}\lim_{h\to 0}\frac{f(3+h)-f(3)}{h} &= \lim_{h\to 0} 12+2h\\ &= 12\end{aligned}$$

c The gradient of the tangent to $y=2x^2$ at the point (3, 18) is 12.

6 $\displaystyle\lim_{x\to -\infty}\left(\frac{2x+3}{4-x}\right)=-2, \qquad \lim_{x\to\infty}\left(\frac{2x+3}{4-x}\right)=-2$

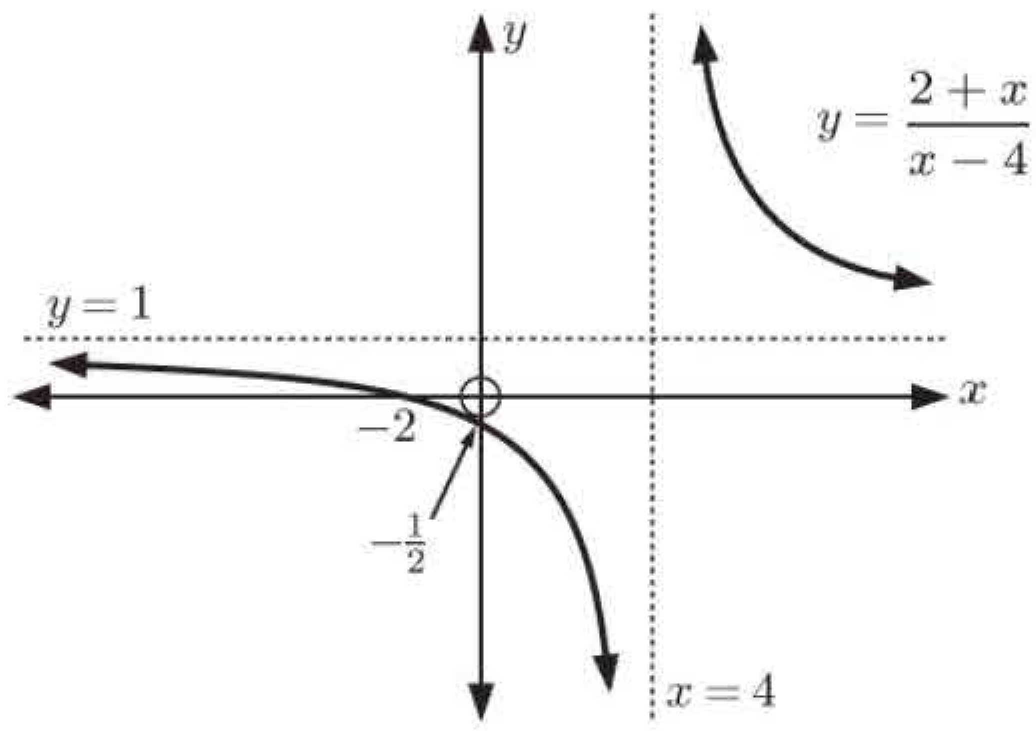
REVIEW SET 17C

1 **a** $$\begin{aligned}\lim_{h\to 0}\frac{h^3-3h}{h} &= \lim_{h\to 0}\frac{h(h^2-3)}{h}\\ &= \lim_{h\to 0} h^2-3 \quad \{\text{as } h\neq 0\}\\ &= -3\end{aligned}$$

b $$\begin{aligned}\lim_{x\to 1}\frac{3x^2-3x}{x-1} &= \lim_{x\to 1}\frac{3x(x-1)}{x-1}\\ &= \lim_{x\to 1} 3x \quad \{\text{as } x\neq 1\}\\ &= 3\end{aligned}$$

c $$\begin{aligned}\lim_{x\to 2}\frac{x^2-3x+2}{2-x} &= \lim_{x\to 2}\frac{(x-1)(x-2)}{-(x-2)}\\ &= \lim_{x\to 2} -(x-1) \quad \{\text{as } x\neq 2\}\\ &= -1\end{aligned}$$

2 **a**



b As $x\to 4^-$, $y\to -\infty$
As $x\to 4^+$, $y\to\infty$
As $x\to\infty$, $y\to 1^+$
As $x\to -\infty$, $y\to 1^-$
The vertical asymptote is $x=4$.
The horizontal asymptote is $y=1$.

c $\displaystyle\lim_{x\to -\infty}\frac{2+x}{x-4}=1, \quad \lim_{x\to\infty}\frac{2+x}{x-4}=1$

3 $f(x) = x^4 - 2x$

$$\begin{aligned}
f'(1) &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} \quad \text{where } f(1) = 1^4 - 2(1) = -1 \\
&= \lim_{h\to 0} \frac{[(1+h)^4 - 2(1+h)] - [-1]}{h} \\
&= \lim_{h\to 0} \frac{\cancel{1} + 4h + 6h^2 + 4h^3 + h^4 - \cancel{2} - 2h + \cancel{1}}{h} \\
&= \lim_{h\to 0} \frac{h^4 + 4h^3 + 6h^2 + 2h}{h} \\
&= \lim_{h\to 0} \frac{h(h^3 + 4h^2 + 6h + 2)}{h} \\
&= \lim_{h\to 0} (h^3 + 4h^2 + 6h + 2) \qquad \{\text{as } h \neq 0\} \\
&= 2
\end{aligned}$$

4 **a**
$$\begin{aligned}
\sin(A+B) - \sin(A-B) &= \sin A\cos B + \cos A\sin B - (\sin A\cos B - \cos A\sin B) \\
&= \cancel{\sin A\cos B} + \cos A\sin B - \cancel{\sin A\cos B} + \cos A\sin B \\
&= 2\cos A\sin B
\end{aligned}$$

b
$$\begin{aligned}
\sin S - \sin D &= \sin(A+B) - \sin(A-B) \\
&= 2\cos A\sin B \qquad \{\text{using } \mathbf{a}\}
\end{aligned}$$

Now $S + D = A + B + (A - B) = 2A$ and $S - D = A + B - (A - B) = 2B$

$\therefore\ A = \dfrac{S+D}{2}$ $\qquad$ $\therefore\ B = \dfrac{S-D}{2}$

$\therefore\ \sin S - \sin D = 2\cos\left(\frac{S+D}{2}\right)\sin\left(\frac{S-D}{2}\right)$

c
$$\begin{aligned}
\lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} &= \lim_{h\to 0} \frac{2\cos\left[\frac{(x+h)+x}{2}\right]\sin\left[\frac{(x+h)-x}{2}\right]}{h} \qquad \{x + h = S,\ \ x = D\} \\
&= \lim_{h\to 0} \frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h} \\
&= 2\lim_{h\to 0} \frac{\cos\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{h} \\
&= 2\lim_{h\to 0} \cos\left(x+\tfrac{h}{2}\right) \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \times \tfrac{1}{2} \\
&= 1\lim_{h\to 0} \cos\left(x+\tfrac{h}{2}\right) \lim_{\frac{h}{2}\to 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \qquad \{\tfrac{h}{2} \to 0 \text{ as } h \to 0\} \\
&= \lim_{h\to 0} \cos\left(x+\tfrac{h}{2}\right) \\
&= \cos x
\end{aligned}$$

d $\displaystyle\lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} = \cos x$ which is the gradient function of $f(x) = \sin x$.

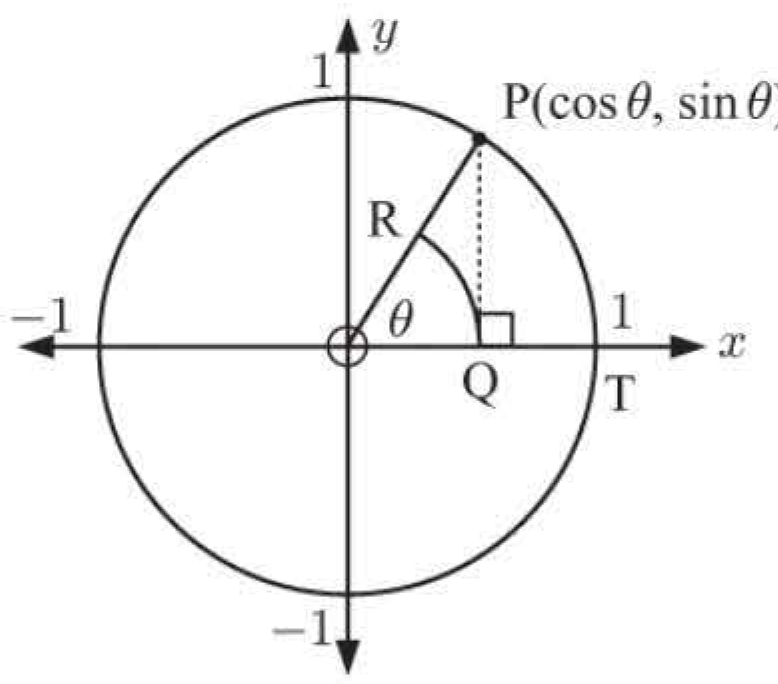
5 **a**

$$\begin{aligned} y &= 2x^2 - 1 \\ \therefore\ \frac{dy}{dx} &= \lim_{h \to 0} \frac{[2(x+h)^2 - 1] - [2x^2 - 1]}{h} \\ &= \lim_{h \to 0} \frac{2(x^2 + 2hx + h^2) - 1 - 2x^2 + 1}{h} \\ &= \lim_{h \to 0} \frac{\cancel{2x^2} + 4hx + 2h^2 - \cancel{1} - \cancel{2x^2} + \cancel{1}}{h} \\ &= \lim_{h \to 0} \frac{4hx + 2h^2}{h} \\ &= \lim_{h \to 0} \frac{h(4x + 2h)}{h} \\ &= \lim_{h \to 0} 4x + 2h \qquad \{\text{as } h \neq 0\} \\ &= 4x \end{aligned}$$

b The gradient of the tangent to $y = 2x^2 - 1$ at the point where $x = 4$ is $4 \times 4 = 16$.

c If the gradient of the tangent is equal to -12, then $4x = -12$
$\therefore\ x = -3$

6 **a** **i**



Suppose $P(\cos\theta,\ \sin\theta)$ lies on the unit circle in the first quadrant.
[PQ] is drawn perpendicular to the x-axis, and arc QR with centre O is drawn. Now,
area of sector OQR $\leqslant$ area $\triangle$OQP $\leqslant$ area sector OTP

$$\therefore\ \tfrac{1}{2}(\text{OQ})^2 \times \theta \leqslant \tfrac{1}{2}(\text{OQ})(\text{PQ}) \leqslant \tfrac{1}{2}(\text{OT})^2 \times \theta$$

$$\therefore\ \tfrac{1}{2}\theta\cos^2\theta \leqslant \tfrac{1}{2}\cos\theta\sin\theta \leqslant \tfrac{1}{2}\theta$$

Dividing throughout by $\frac{1}{2}\theta\cos\theta$, which is > 0, $\quad \cos\theta \leqslant \dfrac{\sin\theta}{\theta} \leqslant \dfrac{1}{\cos\theta}$

Now as $\theta \to 0$, both $\cos\theta \to 1$ and $\dfrac{1}{\cos\theta} \to 1$

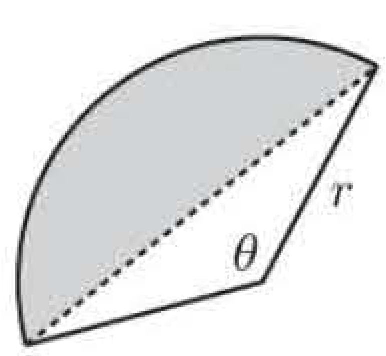
$\therefore$ as $\theta \to 0^+$, $\dfrac{\sin\theta}{\theta} \to 1$. So, if $\theta > 0$ then $\displaystyle\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$.

ii If $f(\theta) = \dfrac{\sin\theta}{\theta}$, $f(-\theta) = \dfrac{\sin(-\theta)}{-\theta} = \dfrac{-\sin\theta}{-\theta} = \dfrac{\sin\theta}{\theta} = f(\theta)$

$\therefore$ $\dfrac{\sin\theta}{\theta}$ is an even function, so as $\theta \to 0^-$, $\dfrac{\sin\theta}{\theta} \to 1$ also.

So, if $\theta < 0$, $\displaystyle\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

b



Area of shaded segment

$$\begin{aligned} &= (\text{area of sector}) - (\text{area of triangle}) \\ &= \tfrac{1}{2}r^2\theta - \tfrac{1}{2}r^2\sin\theta \\ &= \tfrac{1}{2}r^2(\theta - \sin\theta) \end{aligned}$$

c As $\theta \to 0$, area of shaded segment $\to 0$

$$\begin{aligned} &\therefore\ \tfrac{1}{2}r^2(\theta - \sin\theta) \to 0 \\ &\therefore\ \theta - \sin\theta \to 0 \\ &\therefore\ \theta \to \sin\theta \\ &\therefore\ \frac{\sin\theta}{\theta} \to 1 \\ &\therefore\ \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 \end{aligned}$$