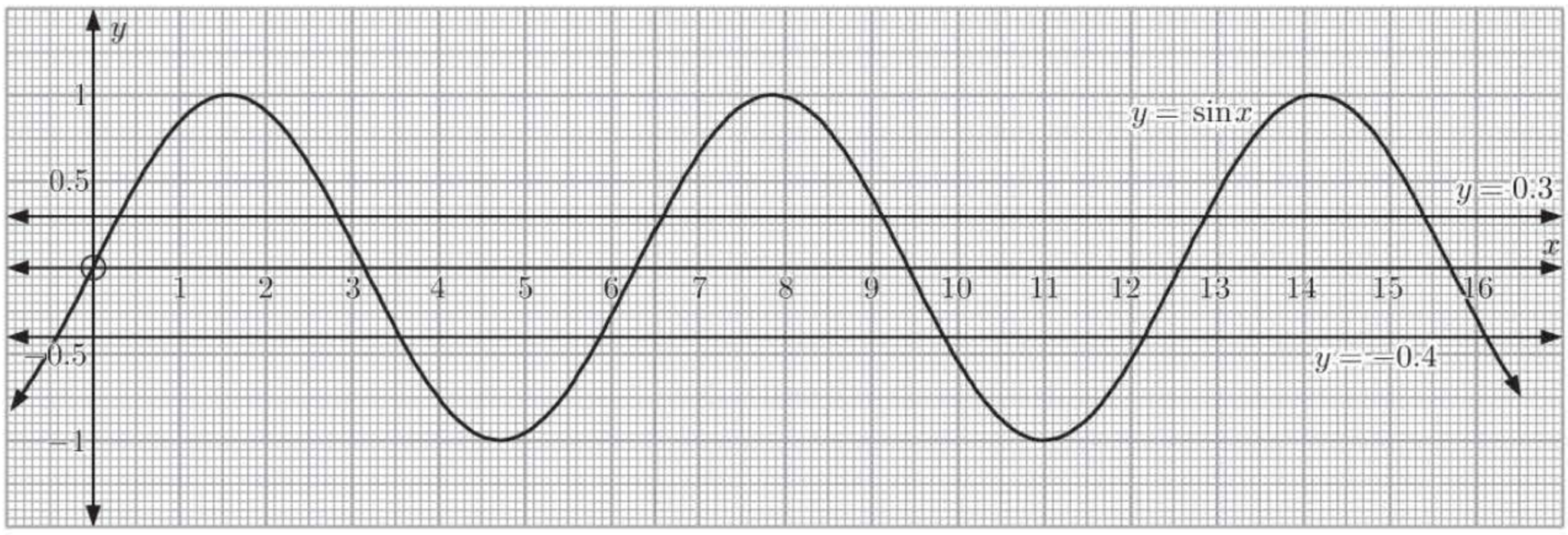
Chapter 13

TRIGONOMETRIC EQUATIONS AND IDENTITIES

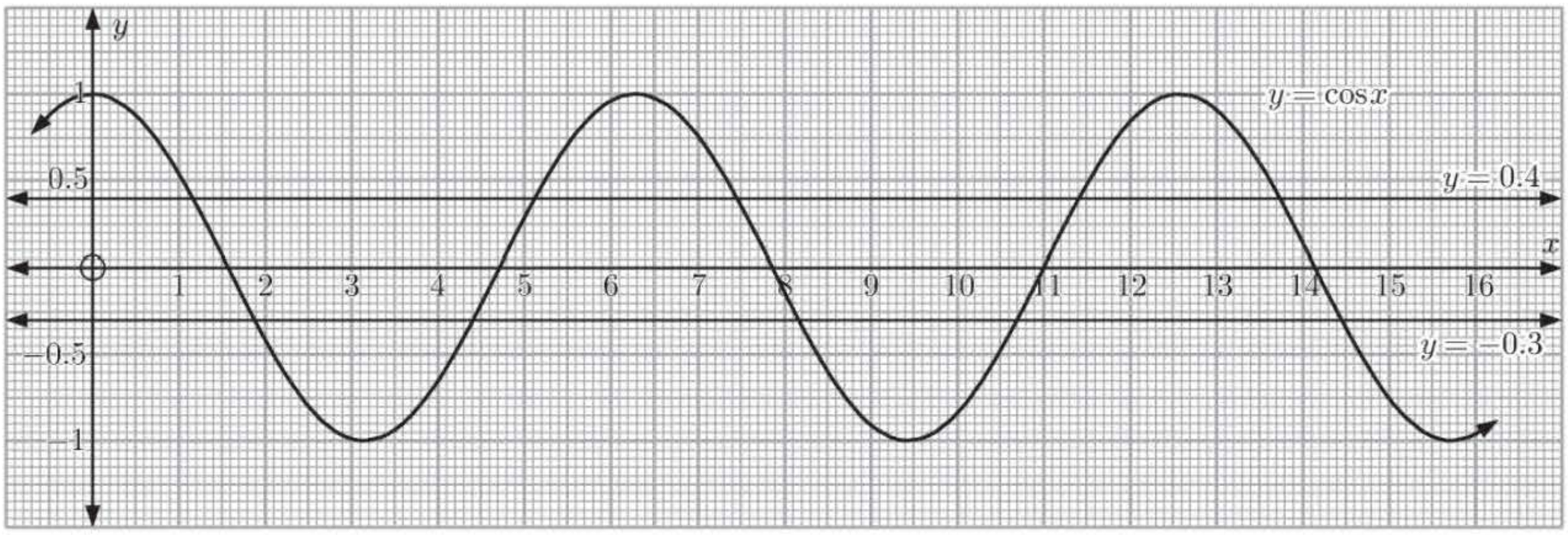
EXERCISE 13A.1

1



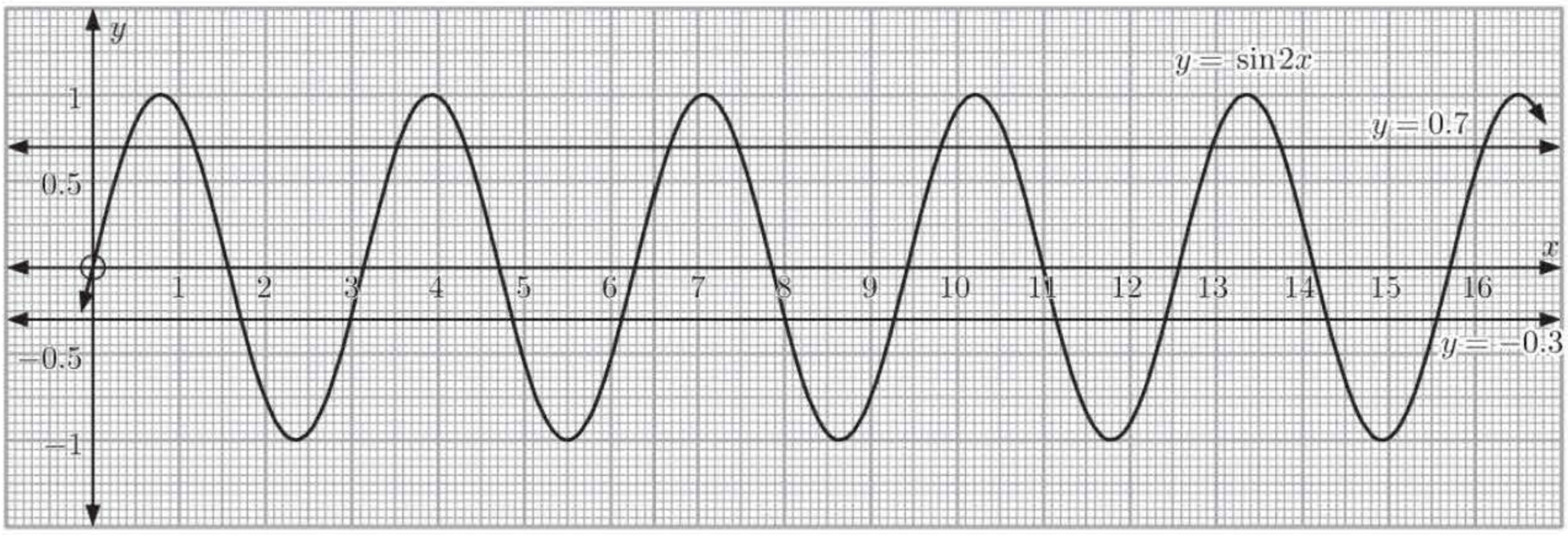
a When $\sin x = 0.3$, $x \approx 0.3, 2.8, 6.6, 9.1, 12.9$ **b** When $\sin x = -0.4$, $x \approx 5.9, 9.8, 12.2$

2



a When $\cos x = 0.4$, $x \approx 1.2, 5.1, 7.4$ **b** When $\cos x = -0.3$, $x \approx 4.4, 8.2, 10.7$

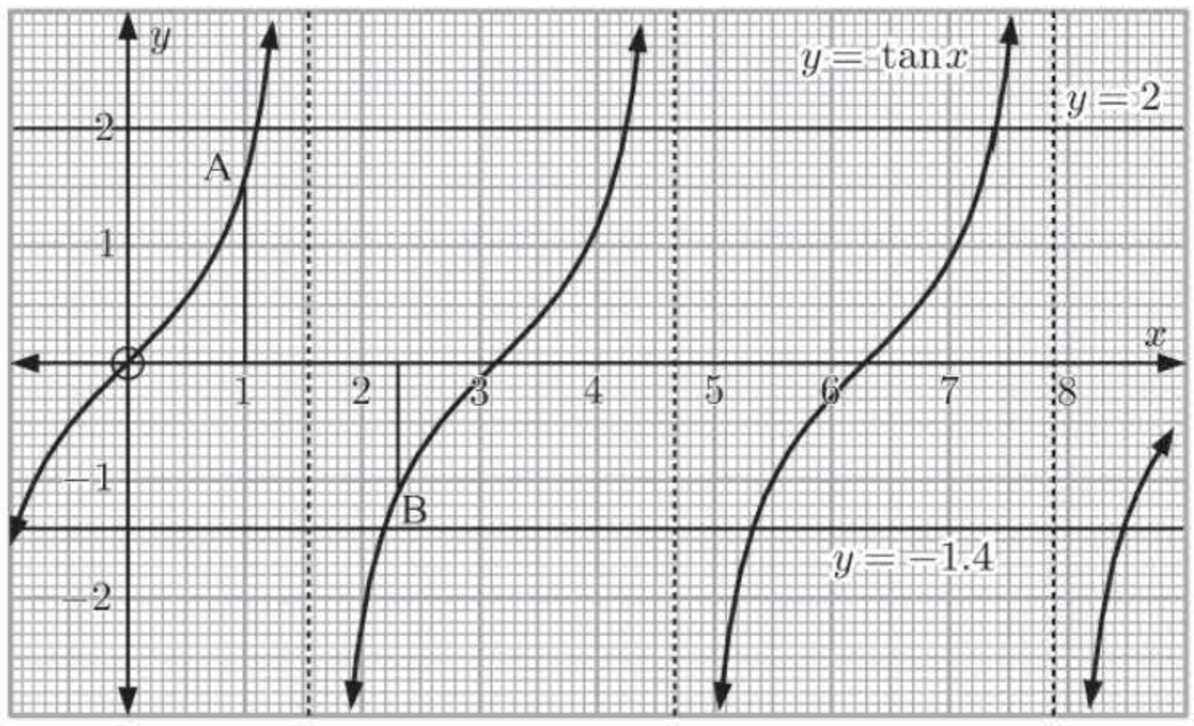
3



a When $\sin 2x = 0.7$, $x \approx 0.4, 1.2, 3.5, 4.3, 6.7, 7.5, 9.8, 10.6, 13.0, 13.7$

b When $\sin 2x = -0.3$, $x \approx 1.7, 3.0, 4.9, 6.1, 8.0, 9.3, 11.1, 12.4, 14.3, 15.6$

4



a **i** $\tan 1 \approx 1.6$ {point A}

ii $\tan 2.3 \approx -1.1$ {point B}

Using technology, we see that $\tan 1 \approx 1.557$ and $\tan 2.3 \approx -1.119$.

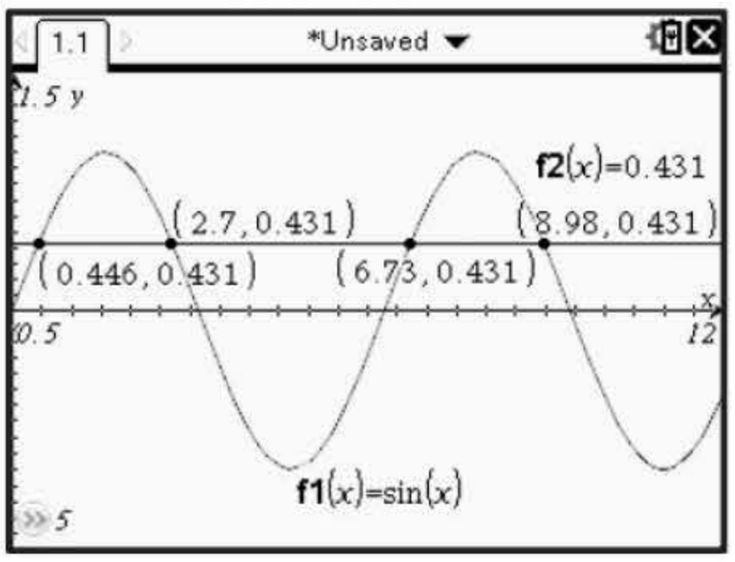
b **i** When $\tan x = 2$,
$x \approx 1.1, 4.2, 7.4$

ii When $\tan x = -1.4$,
$x \approx 2.2, 5.3$

EXERCISE 13A.2

1 **a** $\sin x = 0.431, \quad 0 < x < 12$

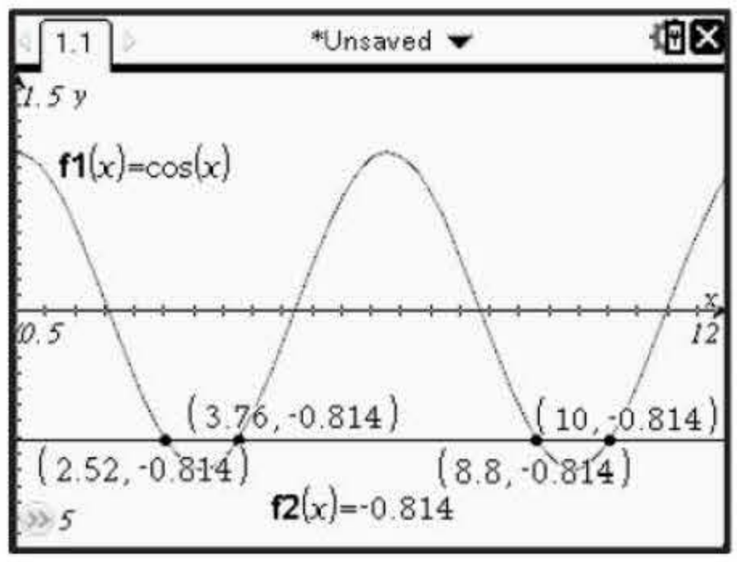
We graph $Y_1 = \sin x$ and $Y_2 = 0.431$ on the same set of axes.



$\therefore \quad x \approx 0.446, 2.70, 6.73, 8.98$

b $\cos x = -0.814, \quad 0 < x < 12$

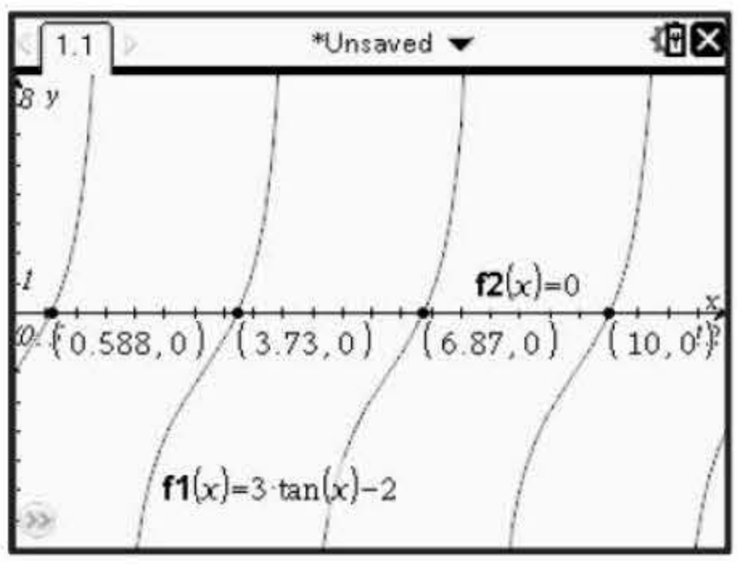
We graph $Y_1 = \cos x$ and $Y_2 = -0.814$ on the same set of axes.



$\therefore \quad x \approx 2.52, 3.76, 8.80, 10.0$

c $3\tan x - 2 = 0, \quad 0 < x < 12$

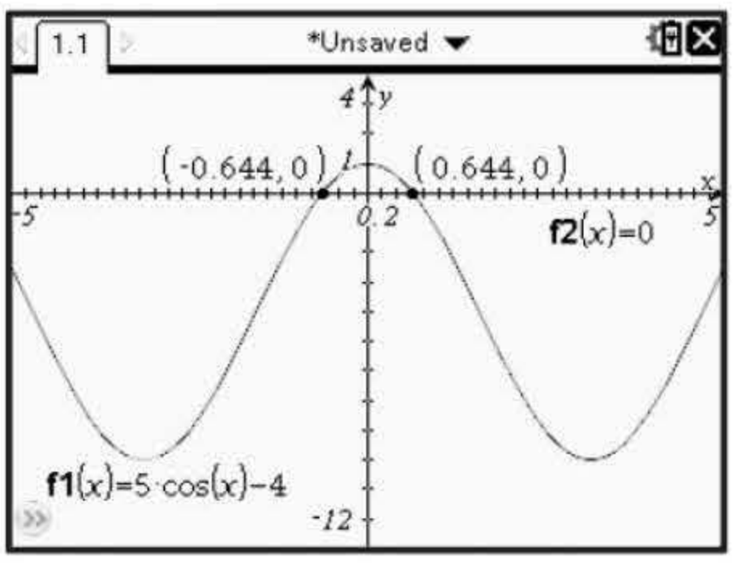
We graph $Y_1 = 3\tan x - 2$ and $Y_2 = 0$ on the same set of axes.



$\therefore \quad x \approx 0.588, 3.73, 6.87, 10.0$

2 **a** $5\cos x - 4 = 0, \quad -5 \leqslant x \leqslant 5$

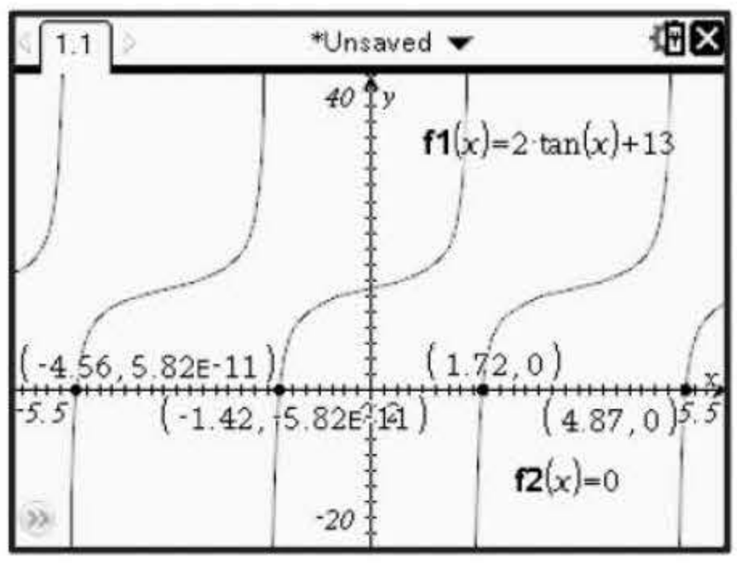
We graph $Y_1 = 5\cos x - 4$ and $Y_2 = 0$ on the same set of axes.



$\therefore \quad x \approx -0.644, 0.644$

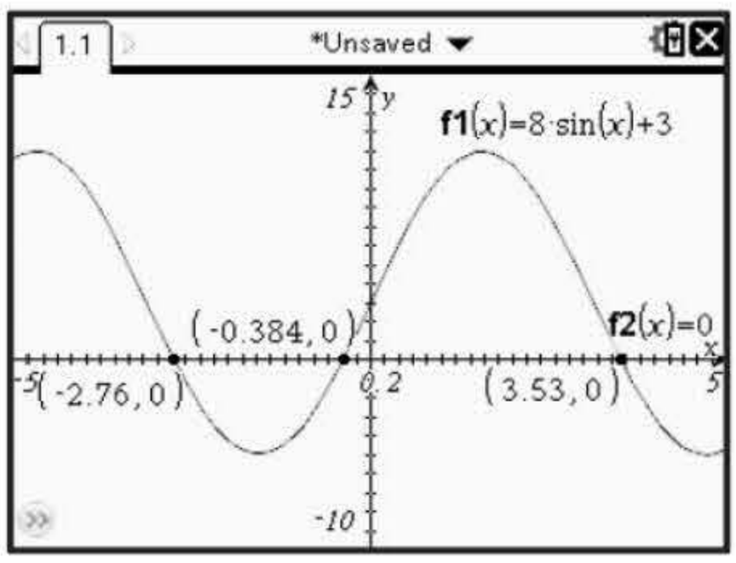
b $2\tan x + 13 = 0, \quad -5 \leqslant x \leqslant 5$

We graph $Y_1 = 2\tan x + 13$ and $Y_2 = 0$ on the same set of axes.



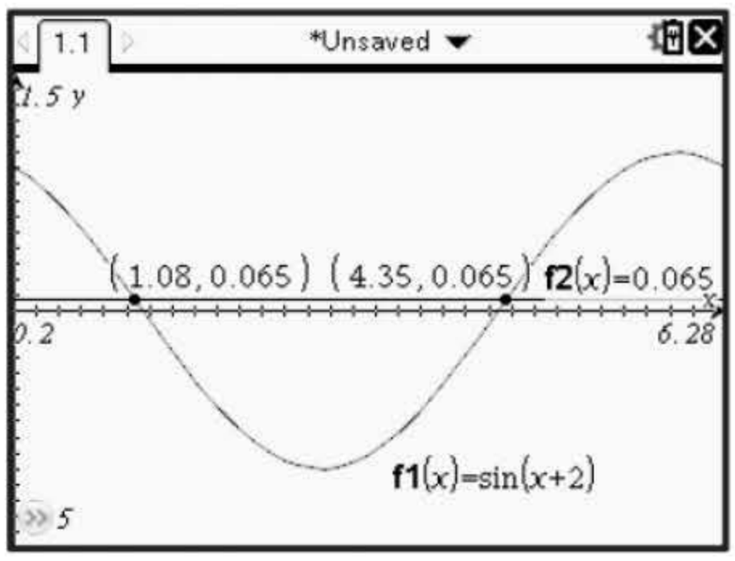
$\therefore \quad x \approx -4.56, -1.42, 1.72, 4.87$

c $8\sin x + 3 = 0, \quad -5 \leqslant x \leqslant 5$
We graph $Y_1 = 8\sin x + 3$ and $Y_2 = 0$ on the same set of axes.



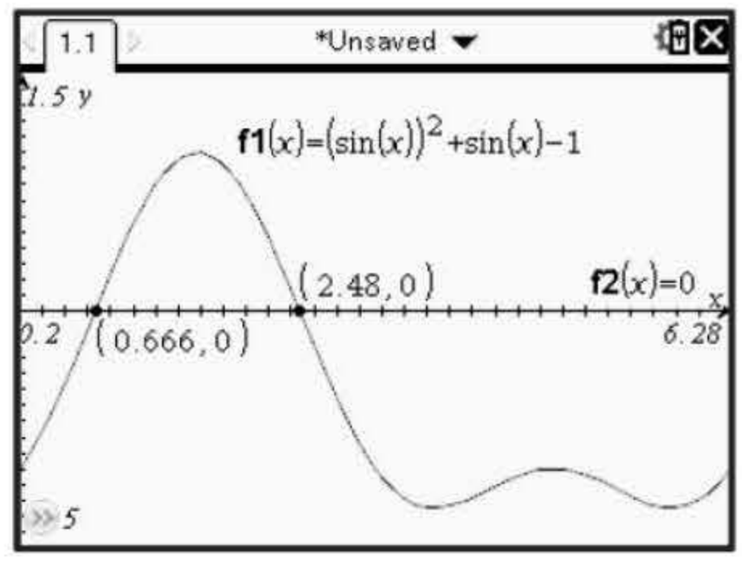
$\therefore \quad x \approx -2.76, -0.384, 3.53$

3 **a** $\sin(x+2) = 0.0652, \quad 0 \leqslant x \leqslant 2\pi$
We graph $Y_1 = \sin(x+2)$ and $Y_2 = 0.0652$ on the same set of axes.



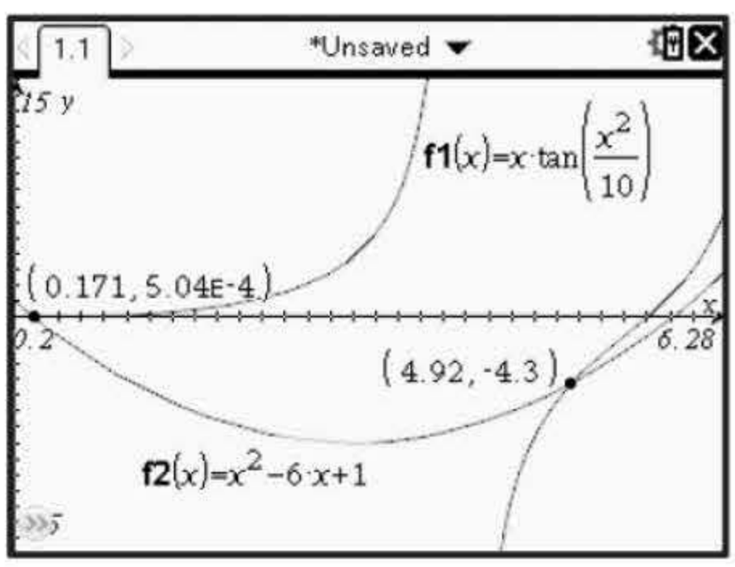
$\therefore \quad x \approx 1.08, 4.35$

b $\sin^2 x + \sin x - 1 = 0, \quad 0 \leqslant x \leqslant 2\pi$
We graph $Y_1 = \sin^2 x + \sin x - 1$ and $Y_2 = 0$ on the same set of axes.



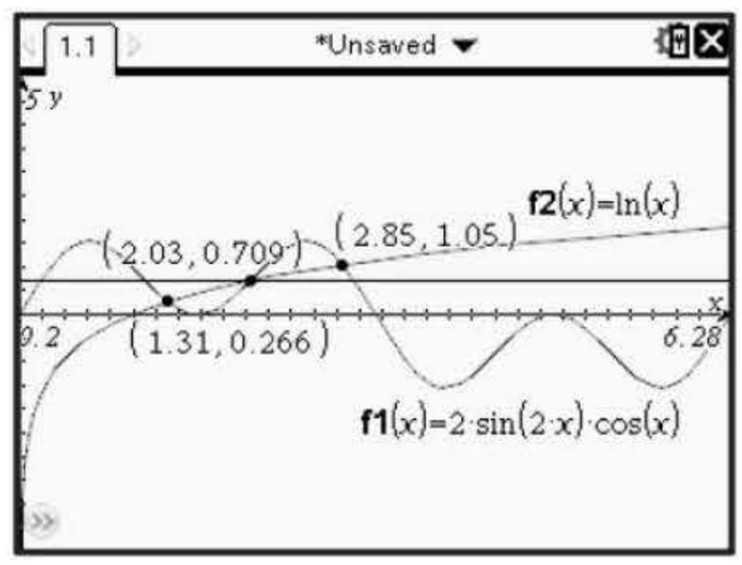
$\therefore \quad x \approx 0.666, 2.48$

c $x\tan\left(\frac{x^2}{10}\right) = x^2 - 6x + 1, \quad 0 \leqslant x \leqslant 2\pi$
We graph $Y_1 = x\tan\left(\frac{x^2}{10}\right)$ and $Y_2 = x^2 - 6x + 1$ on the same set of axes.



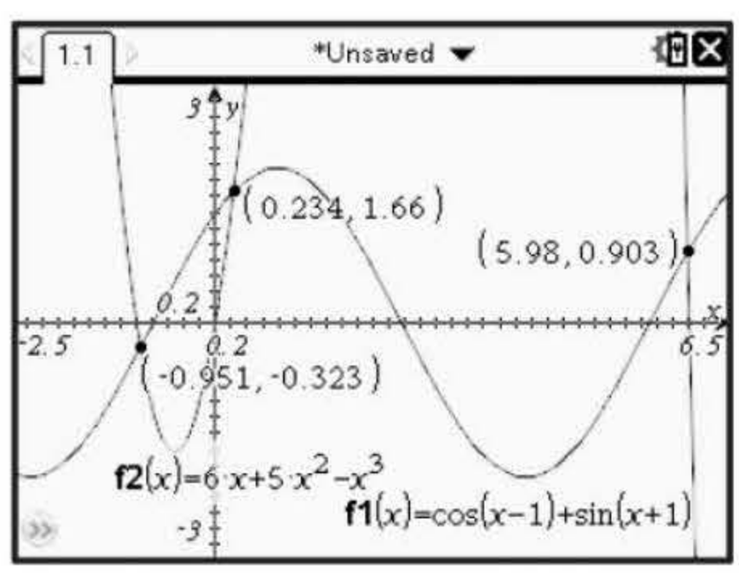
$\therefore \quad x \approx 0.171, 4.92$

d $2\sin(2x)\cos x = \ln x, \quad 0 \leqslant x \leqslant 2\pi$
We graph $Y_1 = 2\sin(2x)\cos x$ and $Y_2 = \ln x$ on the same set of axes.



$\therefore \quad x \approx 1.31, 2.03, 2.85$

4 $\cos(x-1) + \sin(x+1) = 6x + 5x^2 - x^3, \quad -2 \leqslant x \leqslant 6$
We graph $Y_1 = \cos(x-1) + \sin(x+1)$ and $Y_2 = 6x + 5x^2 - x^3$ on the same set of axes.



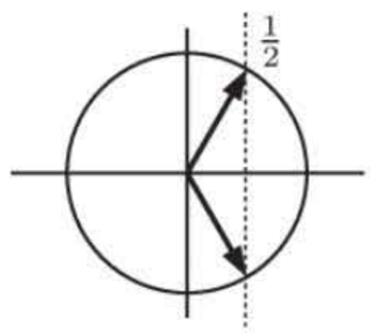
$\therefore \quad x \approx -0.951, 0.234, 5.98$

EXERCISE 13A.3

1 **a** $2\cos x - 1 = 0$

$\therefore \quad \cos x = \frac{1}{2}$

There are two points on the unit circle with cosine $\frac{1}{2}$.



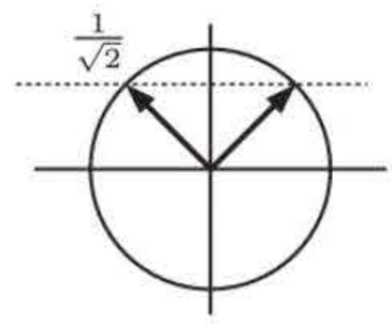
They correspond to angles $\frac{\pi}{3}$ and $\frac{5\pi}{3}$.

For the domain $0 \leqslant x \leqslant 4\pi$ we have 4 solutions: $x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$, or $\frac{11\pi}{3}$.

b $\sqrt{2}\sin x = 1$

$\therefore \quad \sin x = \frac{1}{\sqrt{2}}$

There are two points on the unit circle with sine $\frac{1}{\sqrt{2}}$.

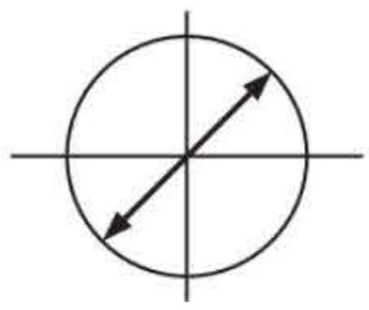


They correspond to angles $\frac{\pi}{4}$ and $\frac{3\pi}{4}$.

For the domain $0 \leqslant x \leqslant 4\pi$ we have 4 solutions: $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}$, or $\frac{11\pi}{4}$.

c $\tan x = 1$

There are two points on the unit circle with tangent 1.



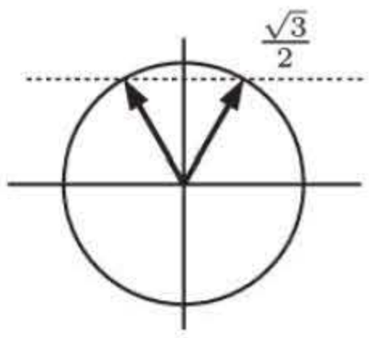
They correspond to angles $\frac{\pi}{4}$ and $\frac{5\pi}{4}$.

For the domain $0 \leqslant x \leqslant 4\pi$ we have 4 solutions: $x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}$, or $\frac{13\pi}{4}$.

2 **a** $2\sin x - \sqrt{3} = 0$

$\therefore \quad \sin x = \frac{\sqrt{3}}{2}$

There are two points on the unit circle with sine $\frac{\sqrt{3}}{2}$.



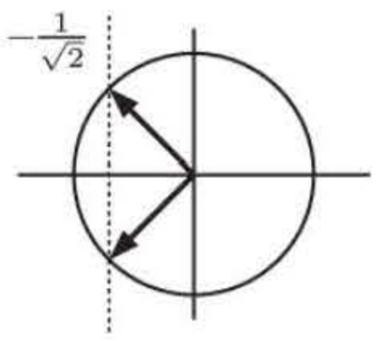
They correspond to angles $\frac{\pi}{3}$ and $\frac{2\pi}{3}$.

For the domain $-2\pi \leqslant x \leqslant 2\pi$ we have 4 solutions: $x = -\frac{5\pi}{3}, -\frac{4\pi}{3}, \frac{\pi}{3}$, or $\frac{2\pi}{3}$.

b $\sqrt{2}\cos x + 1 = 0$

$\therefore \quad \cos x = -\frac{1}{\sqrt{2}}$

There are two points on the unit circle with cosine $-\frac{1}{\sqrt{2}}$.

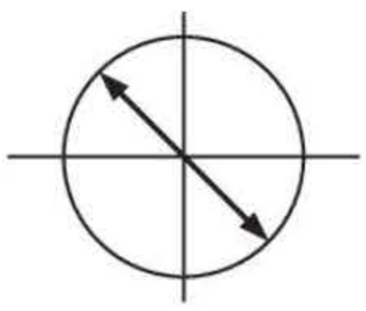


They correspond to angles $\frac{3\pi}{4}$ and $\frac{5\pi}{4}$.

For the domain $-2\pi \leqslant x \leqslant 2\pi$ we have 4 solutions: $x = -\frac{5\pi}{4}, -\frac{3\pi}{4}, \frac{3\pi}{4}$, or $\frac{5\pi}{4}$.

c $\tan x = -1$

There are two points on the unit circle with tangent -1.



They correspond to angles $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$.

For the domain $-2\pi \leqslant x \leqslant 2\pi$ we have 4 solutions: $x = -\frac{5\pi}{4}, -\frac{\pi}{4}, \frac{3\pi}{4}$, or $\frac{7\pi}{4}$.

3 **a** If $0 \leqslant x \leqslant 2\pi$
then $0 \leqslant 2x \leqslant 4\pi$

b If $0 \leqslant x \leqslant 2\pi$
then $0 \leqslant \dfrac{x}{3} \leqslant \frac{2\pi}{3}$

c If $0 \leqslant x \leqslant 2\pi$
then $\frac{\pi}{2} \leqslant x + \frac{\pi}{2} \leqslant \frac{5\pi}{2}$

d If $0 \leqslant x \leqslant 2\pi$
then $-\frac{\pi}{6} \leqslant x - \frac{\pi}{6} \leqslant \frac{11\pi}{6}$

e If $0 \leqslant x \leqslant 2\pi$
then $-\frac{\pi}{4} \leqslant (x - \frac{\pi}{4}) \leqslant \frac{7\pi}{4}$
and so $-\frac{\pi}{2} \leqslant 2(x - \frac{\pi}{4}) \leqslant \frac{7\pi}{2}$

f If $0 \leqslant x \leqslant 2\pi$
then $-2\pi \leqslant -x \leqslant 0$

4 **a** If $-\pi \leqslant x \leqslant \pi$
then $-3\pi \leqslant 3x \leqslant 3\pi$

b If $-\pi \leqslant x \leqslant \pi$
then $-\frac{\pi}{4} \leqslant \dfrac{x}{4} \leqslant \frac{\pi}{4}$

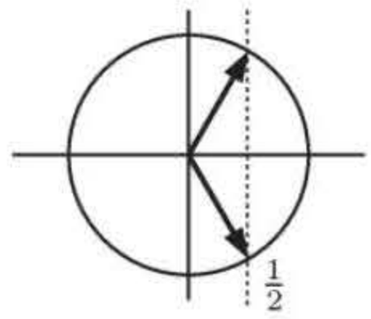
c If $-\pi \leqslant x \leqslant \pi$
then $-\frac{3\pi}{2} \leqslant x - \frac{\pi}{2} \leqslant \frac{\pi}{2}$

d If $-\pi \leqslant x \leqslant \pi$
then $-2\pi \leqslant 2x \leqslant 2\pi$
and so $-\frac{3\pi}{2} \leqslant 2x + \frac{\pi}{2} \leqslant \frac{5\pi}{2}$

e If $-\pi \leqslant x \leqslant \pi$
then $2\pi \geqslant -2x \geqslant -2\pi$
and so $-2\pi \leqslant -2x \leqslant 2\pi$

f If $-\pi \leqslant x \leqslant \pi$
then $\pi - -\pi \geqslant \pi - x \geqslant \pi - \pi$
and so $0 \leqslant \pi - x \leqslant 2\pi$

5 The three equations all have the form $\cos\theta = \frac{1}{2}$.
There are two points on the unit circle with cosine $\frac{1}{2}$.
They correspond to angles $\frac{\pi}{3}$ and $\frac{5\pi}{3}$.

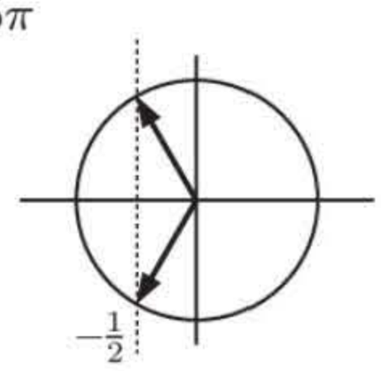


a In this case θ is simply x, so we have the domain $0 \leqslant x \leqslant 3\pi$.
The solutions for this domain are $x = \frac{\pi}{3}, \frac{5\pi}{3}$, or $\frac{7\pi}{3}$.

b In this case θ is $2x$.
If $0 \leqslant x \leqslant 3\pi$ then $0 \leqslant 2x \leqslant 6\pi$.
$\therefore\ 2x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \frac{13\pi}{3}$, or $\frac{17\pi}{3}$
$\therefore\ x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}$, or $\frac{17\pi}{6}$

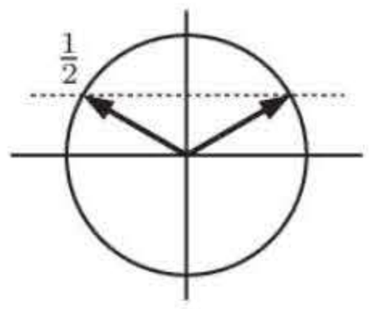
c In this case θ is $x + \frac{\pi}{3}$.
If $0 \leqslant x \leqslant 3\pi$ then $\frac{\pi}{3} \leqslant x + \frac{\pi}{3} \leqslant \frac{10\pi}{3}$.
$\therefore\ x + \frac{\pi}{3} = \frac{\pi}{3}, \frac{5\pi}{3}$, or $\frac{7\pi}{3}$
$\therefore\ x = 0, \frac{4\pi}{3}$, or 2π

6 **a** $\cos x = -\frac{1}{2}, \quad 0 \leqslant x \leqslant 5\pi$
There are two points on the unit circle with cosine $-\frac{1}{2}$.
They correspond to angles $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$.



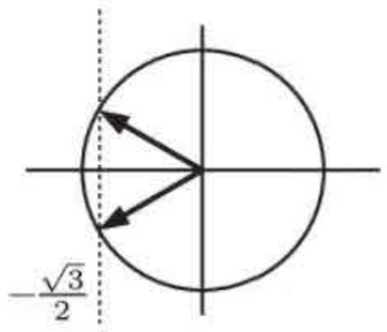
For the domain $0 \leqslant x \leqslant 5\pi$:
$x = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}, \frac{14\pi}{3}$

b $2\sin x - 1 = 0, \quad -360° \leqslant x \leqslant 360°$
$\therefore\ \sin x = \frac{1}{2}$
There are two points on the unit circle with sine $\frac{1}{2}$.
They correspond to angles $30°$ and $150°$.



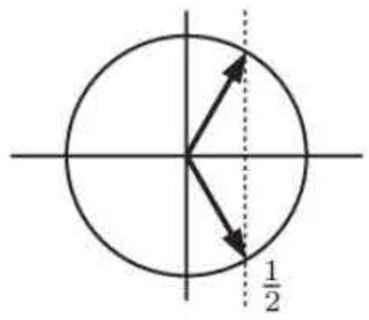
For the domain $-360° \leqslant x \leqslant 360°$:
$x = -330°, -210°, 30°, 150°$

c $2\cos x + \sqrt{3} = 0, \quad 0 \leqslant x \leqslant 3\pi$
$\therefore\ \cos x = -\frac{\sqrt{3}}{2}$
There are two points on the unit circle with cosine $-\frac{\sqrt{3}}{2}$.
They correspond to angles $\frac{5\pi}{6}$ and $\frac{7\pi}{6}$.



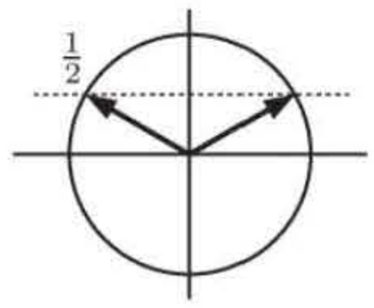
For the domain $0 \leqslant x \leqslant 3\pi$:
$x = \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{17\pi}{6}$

d $\cos(x - \frac{2\pi}{3}) = \frac{1}{2}, \quad -2\pi \leqslant x \leqslant 2\pi$
There are two points on the unit circle with cosine $\frac{1}{2}$.
They correspond to angles $\frac{\pi}{3}$ and $\frac{5\pi}{3}$.



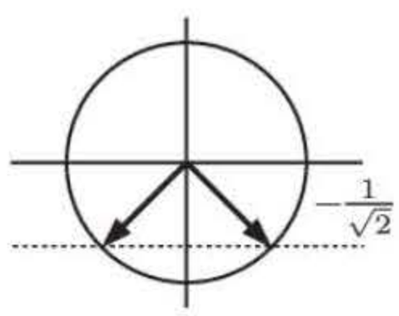
$-2\pi \leqslant x \leqslant 2\pi$
$\therefore\ -\frac{8\pi}{3} \leqslant x - \frac{2\pi}{3} \leqslant \frac{4\pi}{3}$
So, $x - \frac{2\pi}{3} = -\frac{7\pi}{3}, -\frac{5\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}$
$\therefore\ x = -\frac{5\pi}{3}, -\pi, \frac{\pi}{3}, \pi$

e $2\sin(x + \frac{\pi}{3}) = 1, \quad -3\pi \leqslant x \leqslant 3\pi$
$\therefore\ \sin(x + \frac{\pi}{3}) = \frac{1}{2}$
There are two points on the unit circle with sine $\frac{1}{2}$.
They correspond to angles $\frac{\pi}{6}$ and $\frac{5\pi}{6}$.



$-3\pi \leqslant x \leqslant 3\pi$
$\therefore\ -\frac{8\pi}{3} \leqslant x + \frac{\pi}{3} \leqslant \frac{10\pi}{3}$
So, $x + \frac{\pi}{3} = -\frac{11\pi}{6}, -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$
$\therefore\ x = -\frac{13\pi}{6}, -\frac{3\pi}{2}, -\frac{\pi}{6}, \frac{\pi}{2}, \frac{11\pi}{6}, \frac{5\pi}{2}$

f $\sqrt{2}\sin(x - \frac{\pi}{4}) + 1 = 0, \quad 0 \leqslant x \leqslant 3\pi$
$\therefore\ \sin(x - \frac{\pi}{4}) = -\frac{1}{\sqrt{2}}$
There are two points on the unit circle with sine $-\frac{1}{\sqrt{2}}$.
They correspond to angles $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$.

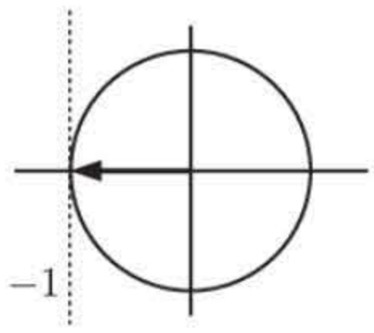


$0 \leqslant x \leqslant 3\pi$
$\therefore\ -\frac{\pi}{4} \leqslant x - \frac{\pi}{4} \leqslant \frac{11\pi}{4}$
So, $x - \frac{\pi}{4} = -\frac{\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$
$\therefore\ x = 0, \frac{3\pi}{2}, 2\pi$

g $3\cos 2x + 3 = 0, \quad 0 \leqslant x \leqslant 3\pi$

$\therefore \quad \cos 2x = -1$

The one point on the unit circle with cosine -1 corresponds to angle π.



$0 \leqslant x \leqslant 3\pi$

$\therefore \quad 0 \leqslant 2x \leqslant 6\pi$

So, $2x = \pi, 3\pi, 5\pi$

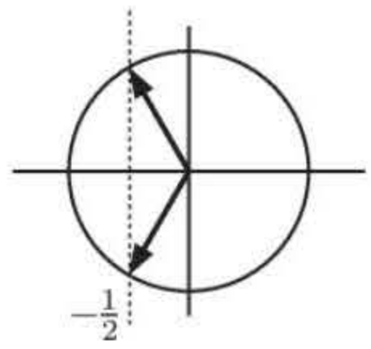
$\therefore \quad x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$

h $4\cos 3x + 2 = 0, \quad -\pi \leqslant x \leqslant \pi$

$\therefore \quad \cos 3x = -\frac{1}{2}$

There are two points on the unit circle with cosine $-\frac{1}{2}$.

They correspond to angles $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$.



$-\pi \leqslant x \leqslant \pi$

$\therefore \quad -3\pi \leqslant 3x \leqslant 3\pi$

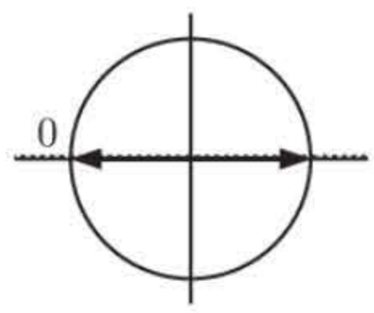
So, $3x = -\frac{8\pi}{3}, -\frac{4\pi}{3}, -\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}$

$\therefore \quad x = -\frac{8\pi}{9}, -\frac{4\pi}{9}, -\frac{2\pi}{9}, \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}$

i $\sin\left(4(x - \frac{\pi}{4})\right) = 0, \quad 0 \leqslant x \leqslant \pi$

There are two points on the unit circle with sine 0.

They correspond to angles 0 and π.



$0 \leqslant x \leqslant \pi$

$\therefore \quad -\frac{\pi}{4} \leqslant x - \frac{\pi}{4} \leqslant \frac{3\pi}{4}$

$\therefore \quad -\pi \leqslant 4(x - \frac{\pi}{4}) \leqslant 3\pi$

So, $4(x - \frac{\pi}{4}) = -\pi, 0, \pi, 2\pi, 3\pi$

$\therefore \quad x - \frac{\pi}{4} = -\frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$

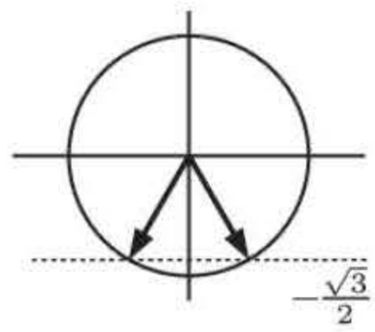
$\therefore \quad x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$

j $2\sin\left(2(x - \frac{\pi}{3})\right) = -\sqrt{3}, \quad 0 \leqslant x \leqslant 2\pi$

$\therefore \quad \sin\left(2(x - \frac{\pi}{3})\right) = -\frac{\sqrt{3}}{2}$

There are two points on the unit circle with sine $-\frac{\sqrt{3}}{2}$.

They correspond to angles $\frac{4\pi}{3}$ and $\frac{5\pi}{3}$.



$0 \leqslant x \leqslant 2\pi$

$\therefore \quad -\frac{\pi}{3} \leqslant x - \frac{\pi}{3} \leqslant \frac{5\pi}{3}$

$\therefore \quad -\frac{2\pi}{3} \leqslant 2(x - \frac{\pi}{3}) \leqslant \frac{10\pi}{3}$

So, $2(x - \frac{\pi}{3}) = -\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{10\pi}{3}$

$\therefore \quad x - \frac{\pi}{3} = -\frac{\pi}{3}, -\frac{\pi}{6}, \frac{2\pi}{3}, \frac{5\pi}{6}, \frac{5\pi}{3}$

$\therefore \quad x = 0, \frac{\pi}{6}, \pi, \frac{7\pi}{6}, 2\pi$

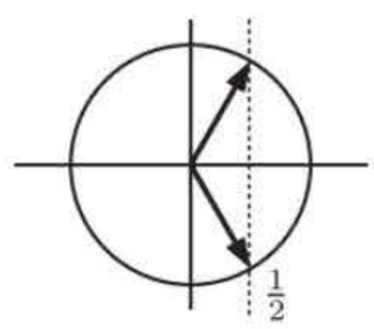
7 **a** $\sec x = 2, \quad x \in [0, 2\pi]$

$\therefore \quad \dfrac{1}{\cos x} = 2$

$\therefore \quad \cos x = \frac{1}{2}, \quad 0 \leqslant x \leqslant 2\pi$

There are two points on the unit circle with cosine $\frac{1}{2}$.

They correspond to angles $\frac{\pi}{3}$ and $\frac{5\pi}{3}$.



For the domain $0 \leqslant x \leqslant 2\pi$:

$x = \frac{\pi}{3}, \frac{5\pi}{3}$

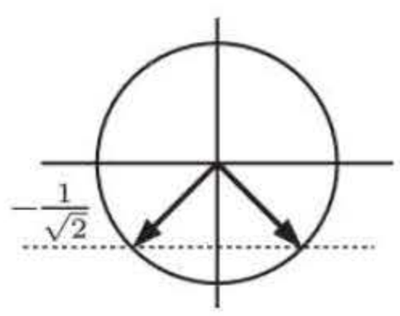
b $\csc x = -\sqrt{2}, \quad x \in [0, 2\pi]$

$\therefore \quad \dfrac{1}{\sin x} = -\sqrt{2}$

$\therefore \quad \sin x = -\frac{1}{\sqrt{2}}, \quad 0 \leqslant x \leqslant 2\pi$

There are two points on the unit circle with sine $-\frac{1}{\sqrt{2}}$.

They correspond to angles $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$.



For the domain $0 \leqslant x \leqslant 2\pi$:

$x = \frac{5\pi}{4}, \frac{7\pi}{4}$

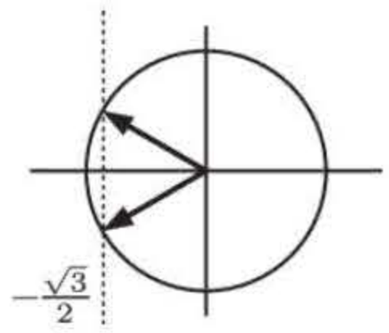
c $\sqrt{3}\sec 2x = -2, \quad x \in [0, 2\pi]$

$\therefore \quad \dfrac{\sqrt{3}}{\cos 2x} = -2$

$\therefore \quad \cos 2x = -\frac{\sqrt{3}}{2}, \quad 0 \leqslant x \leqslant 2\pi$

There are two points on the unit circle with cosine $-\frac{\sqrt{3}}{2}$.

They correspond to angles $\frac{5\pi}{6}$ and $\frac{7\pi}{6}$.



$0 \leqslant x \leqslant 2\pi$

$\therefore \quad 0 \leqslant 2x \leqslant 4\pi$

So, $2x = \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{17\pi}{6}, \frac{19\pi}{6}$

$\therefore \quad x = \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{17\pi}{12}, \frac{19\pi}{12}$

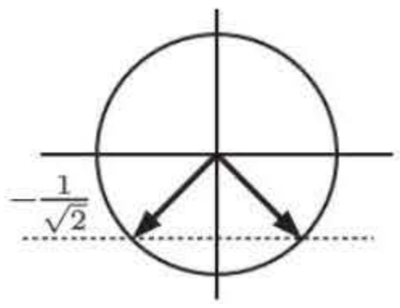
d $\csc(x+\frac{\pi}{6})+\sqrt{2}=0, \quad x\in[0,\,2\pi]$

$\therefore \quad \dfrac{1}{\sin(x+\frac{\pi}{6})}=-\sqrt{2}$

$\therefore \quad \sin(x+\frac{\pi}{6})=-\frac{1}{\sqrt{2}}, \quad 0\leqslant x\leqslant 2\pi$

There are two points on the unit circle with sine $-\frac{1}{\sqrt{2}}$.

They correspond to angles $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$.



$0\leqslant x\leqslant 2\pi$

$\therefore \quad \frac{\pi}{6}\leqslant x+\frac{\pi}{6}\leqslant\frac{13\pi}{6}$

So, $x+\frac{\pi}{6}=\frac{5\pi}{4},\ \frac{7\pi}{4}$

$\therefore \quad x=\frac{13\pi}{12},\ \frac{19\pi}{12}$

8 $\tan x=\sqrt{3}, \quad 0\leqslant x\leqslant 2\pi$

There are two points on the unit circle with tangent $\sqrt{3}$. They correspond to angles $\frac{\pi}{3}$ and $\frac{4\pi}{3}$.

a $0\leqslant x\leqslant 2\pi$

$\therefore \quad -\frac{\pi}{6}\leqslant x-\frac{\pi}{6}\leqslant\frac{11\pi}{6}$

$\therefore \quad x-\frac{\pi}{6}=\frac{\pi}{3},\ \frac{4\pi}{3}$

$\therefore \quad x=\frac{\pi}{2},\ \frac{3\pi}{2}$

b $0\leqslant x\leqslant 2\pi$

$\therefore \quad 0\leqslant 4x\leqslant 8\pi$

$\therefore \quad 4x=\frac{\pi}{3},\ \frac{4\pi}{3},\ \frac{7\pi}{3},\ \frac{10\pi}{3},\ \frac{13\pi}{3},\ \frac{16\pi}{3},\ \frac{19\pi}{3},\ \frac{22\pi}{3}$

$\therefore \quad x=\frac{\pi}{12},\ \frac{\pi}{3},\ \frac{7\pi}{12},\ \frac{5\pi}{6},\ \frac{13\pi}{12},\ \frac{4\pi}{3},\ \frac{19\pi}{12},\ \frac{11\pi}{6}$

c $\tan^2 x=3, \quad 0\leqslant x\leqslant 2\pi$

$\therefore \quad \tan x=\pm\sqrt{3}$

$\therefore \quad x=\frac{\pi}{3},\ \frac{2\pi}{3},\ \frac{4\pi}{3},\ \frac{5\pi}{3}$

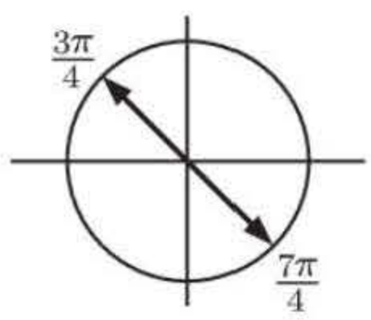
9 **a** $\cot x+1=0, \quad 0\leqslant x\leqslant 2\pi$

$\therefore \quad \dfrac{1}{\tan x}=-1$

$\therefore \quad \tan x=-1$

There are two points on the unit circle with tangent -1.

They correspond to angles $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$.



For the domain $0\leqslant x\leqslant 2\pi$:

$x=\frac{3\pi}{4},\ \frac{7\pi}{4}$

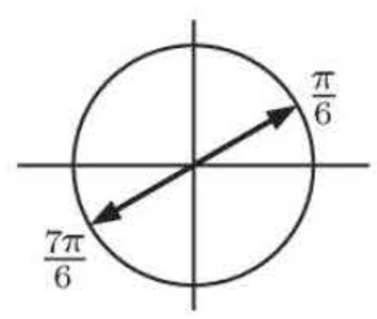
b $\cot(2x-\frac{\pi}{4})-\sqrt{3}=0, \quad 0\leqslant x\leqslant 2\pi$

$\therefore \quad \dfrac{1}{\tan(2x-\frac{\pi}{4})}=\sqrt{3}$

$\therefore \quad \tan(2x-\frac{\pi}{4})=\frac{1}{\sqrt{3}}$

There are two points on the unit circle with tangent $\frac{1}{\sqrt{3}}$.

They correspond to angles $\frac{\pi}{6}$ and $\frac{7\pi}{6}$.



$0\leqslant x\leqslant 2\pi$

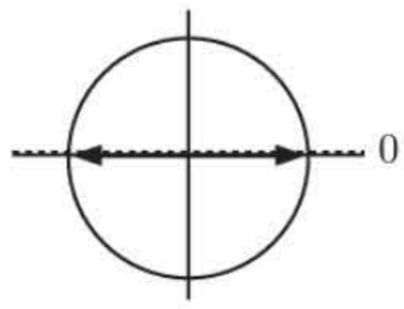
$\therefore \quad -\frac{\pi}{4}\leqslant 2x-\frac{\pi}{4}\leqslant\frac{15\pi}{4}$

So, $2x-\frac{\pi}{4}=\frac{\pi}{6},\ \frac{7\pi}{6},\ \frac{13\pi}{6},\ \frac{19\pi}{6}$

$\therefore \quad 2x=\frac{5\pi}{12},\ \frac{17\pi}{12},\ \frac{29\pi}{12},\ \frac{41\pi}{12}$

$\therefore \quad x=\frac{5\pi}{24},\ \frac{17\pi}{24},\ \frac{29\pi}{24},\ \frac{41\pi}{24}$

10 **a** The zeros of $y = \sin 2x$ are the solutions of $\sin 2x = 0$ $\{0° \leqslant x \leqslant 180°\}$

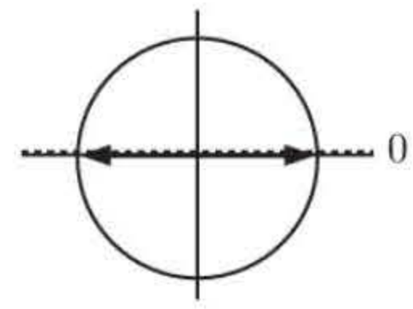


$0° \leqslant x \leqslant 180°$ $\therefore$ $0 \leqslant 2x \leqslant 360°$

So, $2x = 0°, 180°, 360°$

$\therefore$ $x = 0°, 90°, 180°$

b The zeros of $y = \sin(x - \frac{\pi}{4})$ are the solutions of $\sin(x - \frac{\pi}{4}) = 0$ $\{0 \leqslant x \leqslant 3\pi\}$

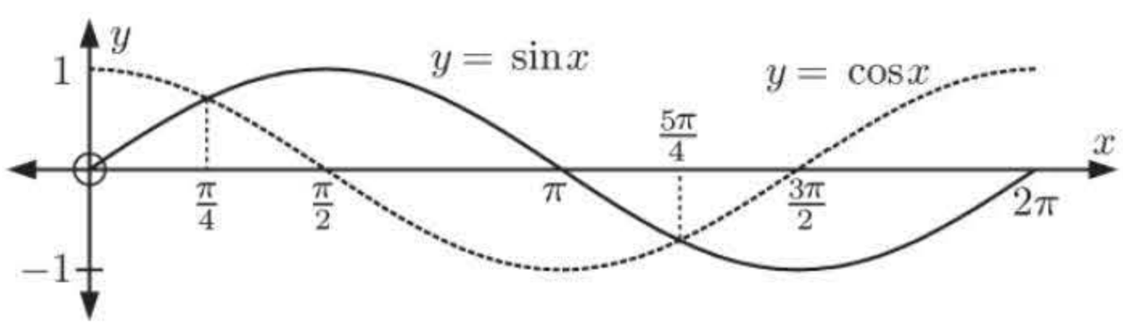


$0 \leqslant x \leqslant 3\pi$ $\therefore$ $-\frac{\pi}{4} \leqslant x - \frac{\pi}{4} \leqslant \frac{11\pi}{4}$

So, $x - \frac{\pi}{4} = 0, \pi, 2\pi$

$\therefore$ $x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}$

11 **a**



b $x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$

12 **a** $\sin x = -\cos x, \quad 0 \leqslant x \leqslant 2\pi$

$\therefore$ $\dfrac{\sin x}{\cos x} = -1$

$\therefore$ $\tan x = -1$

On the domain $0 \leqslant x \leqslant 2\pi$,

$\therefore$ $x = \frac{3\pi}{4}, \frac{7\pi}{4}$

b $\sin(3x) = \cos(3x), \quad 0 \leqslant x \leqslant 2\pi$

$\therefore$ $\dfrac{\sin(3x)}{\cos(3x)} = 1$

$\therefore$ $\tan(3x) = 1$

The two points on the unit circle with tangent 1 correspond to angles $\frac{\pi}{4}$ and $\frac{5\pi}{4}$.

$0 \leqslant x \leqslant 2\pi$ $\therefore$ $0 \leqslant 3x \leqslant 6\pi$

So, $3x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}, \frac{17\pi}{4}, \frac{21\pi}{4}$

$\therefore$ $x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{7\pi}{4}$

c $\sin(2x) = \sqrt{3}\cos(2x), \quad 0 \leqslant x \leqslant 2\pi$

$\therefore$ $\dfrac{\sin(2x)}{\cos(2x)} = \sqrt{3}$

$\therefore$ $\tan(2x) = \sqrt{3}$

The two points on the unit circle with tangent $\sqrt{3}$ correspond to angles $\frac{\pi}{3}$ and $\frac{4\pi}{3}$.

$0 \leqslant x \leqslant 2\pi$ $\therefore$ $0 \leqslant 2x \leqslant 4\pi$

So, $2x = \frac{\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{3}, \frac{10\pi}{3}$

$\therefore$ $x = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$

13 **a** The range of $y = \arctan x$ is $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

$\frac{\pi}{4}$ is within the range.

$\therefore$ $x = \tan(\frac{\pi}{4}) = 1$

b The range of $y = \arcsin x$ is $-\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$.

$-\frac{\pi}{3}$ is within the range.

$\therefore$ $x = \sin(-\frac{\pi}{3}) = -\frac{\sqrt{3}}{2}$

c The range of $y = \arccos x$ is $0 \leqslant y \leqslant \pi$.

$\frac{3\pi}{4}$ is within the range.

$\therefore$ $x = \cos(\frac{3\pi}{4}) = -\frac{1}{\sqrt{2}}$

d The range of $y = \arcsin(x + 1)$ is $-\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$.

$\frac{\pi}{6}$ is within the range.

$\therefore$ $x + 1 = \sin(\frac{\pi}{6})$

$\therefore$ $x + 1 = \frac{1}{2}$

$\therefore$ $x = -\frac{1}{2}$

e The range of $y = \arccos x$ is $0 \leqslant y \leqslant \pi$.

$-\frac{\pi}{4}$ is outside this range.

$\therefore$ no solution exists, even though we can find $\cos(-\frac{\pi}{4})$.

f The range of $y = \arctan(x - \sqrt{3})$ is $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

$-\frac{\pi}{3}$ is within the range.

$\therefore \quad x - \sqrt{3} = \tan(-\frac{\pi}{3})$

$\therefore \quad x - \sqrt{3} = -\sqrt{3}$

$\therefore \quad x = 0$

EXERCISE 13B

1 **a** $H(t) = 10\sin\left(\frac{\pi}{50}(t-25)\right) + 12$ metres

After 50 seconds, $t = 50$ and $H(50) = 10\sin\left(\frac{\pi}{50}(50-25)\right) + 12 = 22$ metres.

So, the green light will be 22 metres high.

b A full circle is completed in one period.

$$\text{period} = \frac{2\pi}{\frac{\pi}{50}} = 2\pi \times \frac{50}{\pi} = 100 \text{ seconds}$$

So, it takes 100 seconds for the wheel to complete a full circle.

c We need to solve $H(t) = 16$ so $10\sin\left(\frac{\pi}{50}(t-25)\right) + 12 = 16$.

3 minutes $= 3 \times 60 = 180$ seconds

Using technology to find the intersection points of $Y_1 = 10\sin\left(\frac{\pi}{50}(t-25)\right)+12$ and $Y_2 = 16$ for $0 \leqslant t \leqslant 180$, $t \approx 31.5, 68.5, 132, 168$

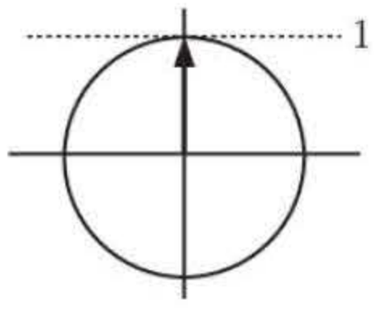
So, in the first three minutes, the green light is 16 metres above the ground at 31.5, 68.5, 132, and 168 seconds.

2 **a** $P(t) = 7500 + 3000\sin(\frac{\pi t}{8}), \quad 0 \leqslant t \leqslant 12$

i $P(0) = 7500 + 3000\sin 0$
$= 7500 + 0$
$= 7500$ grasshoppers

ii $P(5) = 7500 + 3000\sin(\frac{5\pi}{8})$
$\approx 10\,271.63$
$\approx 10\,300$ grasshoppers

b The greatest value of $P(t)$ occurs when $\sin\left(\frac{\pi t}{8}\right) = 1$, so the greatest population is $7500 + 3000 = 10\,500$ grasshoppers.



The point on the unit circle with sine 1 corresponds to angle $\frac{\pi}{2}$.

$0 \leqslant t \leqslant 12 \quad \therefore \quad 0 \leqslant \frac{\pi t}{8} \leqslant \frac{3\pi}{2}$

So $\frac{\pi t}{8} = \frac{\pi}{2}$

$\therefore \quad t = 4$

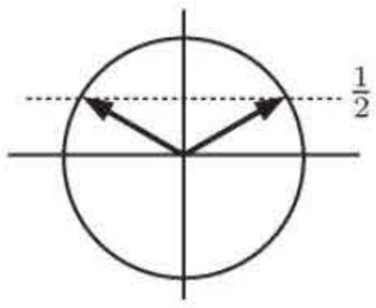
So the greatest population occurs after 4 weeks.

c **i** When $P(t) = 9000$,

$7500 + 3000\sin\left(\frac{\pi t}{8}\right) = 9000$

$\therefore \quad 3000\sin\left(\frac{\pi t}{8}\right) = 1500$

$\therefore \quad \sin\left(\frac{\pi t}{8}\right) = \frac{1}{2}$



The points on the unit circle with sine $\frac{1}{2}$ correspond to angles $\frac{\pi}{6}$ and $\frac{5\pi}{6}$.

$0 \leqslant t \leqslant 12 \quad \therefore \quad 0 \leqslant \frac{\pi t}{8} \leqslant \frac{3\pi}{2}$

So $\frac{\pi t}{8} = \frac{\pi}{6}, \frac{5\pi}{6}$

$\therefore \quad t = \frac{8}{6}, \frac{40}{6}$

$\therefore \quad t = 1\frac{1}{3}$ or $6\frac{2}{3}$

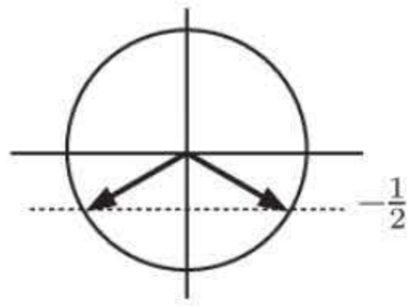
So, the population is 9000 at $1\frac{1}{3}$ weeks and $6\frac{2}{3}$ weeks.

ii When $P(t) = 6000$,

$7500 + 3000\sin\left(\frac{\pi t}{8}\right) = 6000$

$\therefore \quad 3000\sin\left(\frac{\pi t}{8}\right) = -1500$

$\therefore \quad \sin\left(\frac{\pi t}{8}\right) = -\frac{1}{2}$



The points on the unit circle with sine $-\frac{1}{2}$ correspond to angles $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$.

$0 \leqslant t \leqslant 12 \quad \therefore \quad 0 \leqslant \frac{\pi t}{8} \leqslant \frac{3\pi}{2}$

So $\frac{\pi t}{8} = \frac{7\pi}{6}$

$\therefore \quad t = \frac{56}{6}$

$\therefore \quad t = 9\frac{1}{3}$

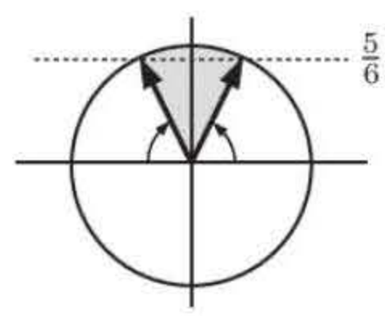
So, the population is 6000 at $9\frac{1}{3}$ weeks.

d If $P(t) > 10\,000$, then

$7500 + 3000\sin\left(\frac{\pi t}{8}\right) > 10\,000$

$\therefore \quad 3000\sin\left(\frac{\pi t}{8}\right) > 2500$

$\therefore \quad \sin\left(\frac{\pi t}{8}\right) > \frac{5}{6}$



Solving $\sin\left(\frac{\pi t}{8}\right) = \frac{5}{6}$ using technology

$t \approx 2.51$ or 5.49 So, $2.51 \leqslant t \leqslant 5.49$ weeks.

3 $H(t) = 20 - 19\sin\left(\frac{2\pi t}{3}\right)$

a $H(0) = 20 - 19(0)$
$= 20$ m

So, at time $t = 0$, the light is 20 m above the ground.

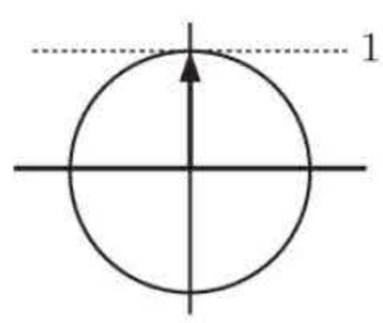
b H is lowest when $\sin\left(\frac{2\pi t}{3}\right) = 1$

$\therefore \quad \frac{2\pi t}{3} = \frac{\pi}{2} + k2\pi$

$\therefore \quad \frac{2t}{3} = \frac{1}{2} + k\,2$

$\therefore \quad t = \frac{3}{4} + k\,3$

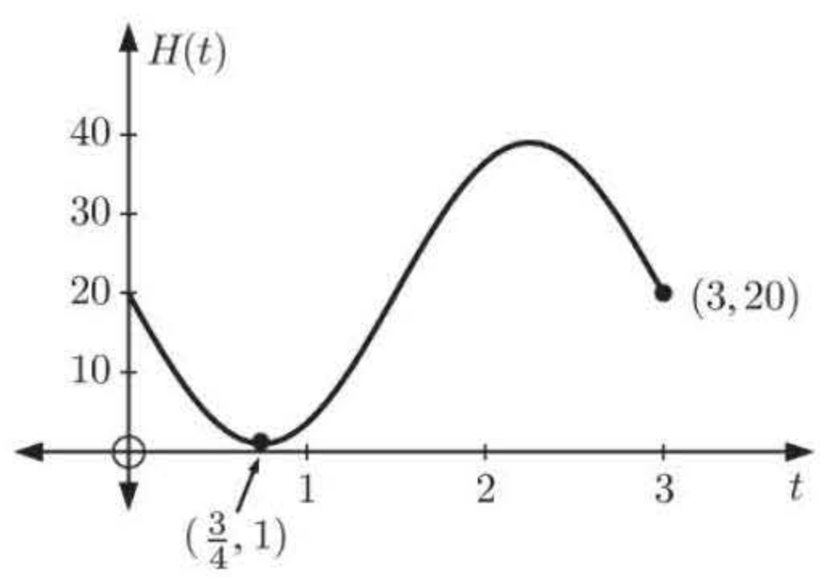
$\therefore \quad t = \frac{3}{4}$ min $\{$as $k = 0\}$



c period $= \dfrac{2\pi}{\frac{2\pi}{3}} = 3$ min

$\therefore$ one revolution takes 3 minutes

d



4 $P(t) = 400 + 250\sin\left(\frac{\pi t}{2}\right)$

a $P(0) = 400 + 250(0)$
$= 400$ water buffalo

c $P(1) = 400 + 250\sin(\frac{\pi}{2})$
$= 400 + 250 \times 1$
$= 650$ water buffalo

This is the maximum herd size.

b **i** $P(\frac{1}{2}) = 400 + 250\sin\left(\dfrac{\pi(\frac{1}{2})}{2}\right)$

$= 400 + 250\sin(\frac{\pi}{4})$

$= 400 + 250 \times \frac{1}{\sqrt{2}}$

≈ 577 water buffalo

ii $P(2) = 400 + 250\sin\pi$
$= 400 + 250(0)$
$= 400$ water buffalo

d $P(t)$ is smallest when $\sin\left(\frac{\pi t}{2}\right) = -1$ and is $400 - 250 = 150$ water buffalo.

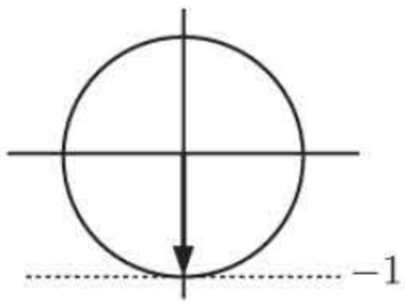
It occurs when

$$\frac{\pi t}{2} = \frac{3\pi}{2} + k2\pi$$

$$\therefore \quad \frac{t}{2} = \tfrac{3}{2} + k\,2$$

$$\therefore \quad t = 3 + 4k$$

So, the first time is after 3 years.

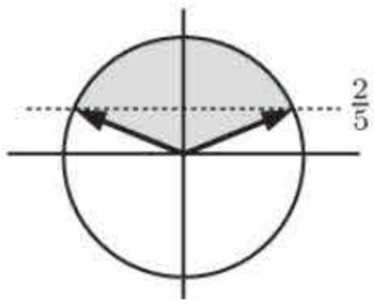


e If $P(t) > 500$ then

$$400 + 250\sin\left(\tfrac{\pi t}{2}\right) > 500$$

$$\therefore \quad 250\sin\left(\tfrac{\pi t}{2}\right) > 100$$

$$\therefore \quad \sin\left(\tfrac{\pi t}{2}\right) > \tfrac{2}{5}$$



Using technology:

$\sin\left(\frac{\pi t}{2}\right) = \frac{2}{5}$ when

$$\frac{\pi t}{2} \approx 0.4115 \text{ or } \pi - 0.4115$$

$$\therefore \quad t \approx 0.262 \text{ or } 1.74$$

So, for $\sin\left(\frac{\pi t}{2}\right) > \frac{2}{5}$, $\quad 0.262 < t < 1.74$

$\therefore$ the herd first exceeded 500 when $t \approx 0.262$ years.

5 **a** The period is 4 seconds.

$$\therefore \quad \frac{2\pi}{b} = 4$$

$$\therefore \quad b = \tfrac{\pi}{2}$$

Amplitude is 3 $\quad \therefore \quad a = 3$

$d = 1 + 3 = 4, \quad c = 0$

$\therefore \quad H(t) = 3\cos(\frac{\pi}{2}(t - 0)) + 4$ metres

$\therefore \quad H(t) = 3\cos(\frac{\pi}{2}t) + 4$ metres

Check: When $t = 0$, $H(0) = 3\cos 0 + 4 = 7$ ✓

b X enters the water when $H(t) = 2$

$$\therefore \quad 3\cos\left(\tfrac{\pi t}{2}\right) + 4 = 2$$

$$\therefore \quad \cos\left(\tfrac{\pi t}{2}\right) = -\tfrac{2}{3}$$

Using technology, $\quad t \approx 1.46$ seconds

6 $C(t) = 9.2\sin(\frac{\pi}{7}(t - 4)) + 107.8$ cents L^{-1}

a **i** 107.8 is the median value. Values are between $107.8 - 9.2$ and $107.8 + 9.2$

$= 98.6$ cents L^{-1} (↑ min.) and 117.0 cents L^{-1} (↑ max.)

$\therefore$ the statement is true.

ii period $= \dfrac{2\pi}{\frac{\pi}{7}} = 14$ days $\quad \therefore \quad$ true

b $C(7) = 9.2\sin(\frac{\pi}{7}(3)) + 107.8 \approx 116.8$ cents L^{-1}

c When $C(t) = \$1.10\ \mathrm{L}^{-1}$ then $9.2\sin(\frac{\pi}{7}(t - 4)) + 107.8 = 110$

$$\therefore \quad \sin(\tfrac{\pi}{7}(t - 4)) = \frac{2.2}{9.2} \approx 0.239\,13$$

$$\therefore \quad \tfrac{\pi}{7}(t - 4) \approx 0.2415,\ \pi - 0.2415,\ 2\pi + 0.2415,\ 3\pi - 0.2415$$

$$\approx 0.2415,\ 2.9001,\ 6.5247,\ 9.1833$$

$$\therefore \quad t - 4 \approx 0.538,\ 6.462,\ 14.538,\ 20.462$$

$$\therefore \quad t \approx 4.538,\ 10.462,\ 18.538,\ 24.462$$

So, the price is \$1.10 per litre on the 5th, 11th, 19th, and 25th days.

d The minimum cost per litre is $-9.2 + 107.8 = 98.6$ cents L^{-1}

when $\sin(\frac{\pi}{7}(t - 4)) = -1$

$$\therefore \quad \tfrac{\pi}{7}(t - 4) = \tfrac{3\pi}{2}$$

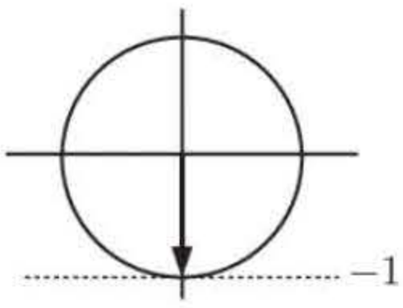
$$\therefore \quad \frac{t - 4}{7} = \tfrac{3}{2}$$

$$\therefore \quad 2t - 8 = 21$$

$$\therefore \quad 2t = 29$$

$$\therefore \quad t = 14.5 \pm 14k$$

{period is 14 days}



So, the minimum occurred on the 1st day and the 15th day.

EXERCISE 13C.1

1 **a** $\sin\theta + \sin\theta$
$= 2\sin\theta$

b $2\cos\theta + \cos\theta$
$= 3\cos\theta$

c $3\sin\theta - \sin\theta$
$= 2\sin\theta$

d $3\sin\theta - 2\sin\theta$
$= \sin\theta$

e $\tan\theta - 3\tan\theta$
$= -2\tan\theta$

f $2\cos^2\theta - 5\cos^2\theta$
$= -3\cos^2\theta$

2 **a** $3\sin^2\theta + 3\cos^2\theta$
$= 3(\sin^2\theta + \cos^2\theta)$
$= 3(1)$
$= 3$

b $-2\sin^2\theta - 2\cos^2\theta$
$= -2(\sin^2\theta + \cos^2\theta)$
$= -2(1)$
$= -2$

c $-\cos^2\theta - \sin^2\theta$
$= -(\cos^2\theta + \sin^2\theta)$
$= -(1)$
$= -1$

d $3 - 3\sin^2\theta$
$= 3(1 - \sin^2\theta)$
$= 3\cos^2\theta$

e $4 - 4\cos^2\theta$
$= 4(1 - \cos^2\theta)$
$= 4\sin^2\theta$

f $\cos^3\theta + \cos\theta\sin^2\theta$
$= \cos\theta(\cos^2\theta + \sin^2\theta)$
$= \cos\theta(1)$
$= \cos\theta$

g $\cos^2\theta - 1$
$= 1 - \sin^2\theta - 1$
$= -\sin^2\theta$

h $\sin^2\theta - 1$
$= 1 - \cos^2\theta - 1$
$= -\cos^2\theta$

i $2\cos^2\theta - 2$
$= -2(1 - \cos^2\theta)$
$= -2\sin^2\theta$

j $\dfrac{1 - \sin^2\theta}{\cos^2\theta}$
$= \dfrac{\cos^2\theta}{\cos^2\theta}$
$= 1$

k $\dfrac{1 - \cos^2\theta}{\sin\theta}$
$= \dfrac{\sin^2\theta}{\sin\theta}$
$= \sin\theta$

l $\dfrac{\cos^2\theta - 1}{-\sin\theta}$
$= \dfrac{1 - \sin^2\theta - 1}{-\sin\theta}$
$= \dfrac{-\sin^2\theta}{-\sin\theta}$
$= \sin\theta$

3 **a** $3\tan x - \dfrac{\sin x}{\cos x}$
$= 3\tan x - \tan x$
$= 2\tan x$

b $\dfrac{\sin^2 x}{\cos^2 x}$
$= \left(\dfrac{\sin x}{\cos x}\right)^2$
$= \tan^2 x$

c $\tan x\cos x$
$= \dfrac{\sin x}{\cos x} \times \cos x$
$= \sin x$

d $\dfrac{\sin x}{\tan x}$
$= \sin x \div \dfrac{\sin x}{\cos x}$
$= \sin x \times \dfrac{\cos x}{\sin x}$
$= \cos x$

e $3\sin x + 2\cos x\tan x$
$= 3\sin x + 2\cos x\,\dfrac{\sin x}{\cos x}$
$= 3\sin x + 2\sin x$
$= 5\sin x$

f $\dfrac{2\tan x}{\sin x}$
$= 2\left(\dfrac{\cancel{\sin x}}{\cos x}\right) \times \dfrac{1}{\cancel{\sin x}_1}$
$= \dfrac{2}{\cos x}$

4 **a** $(1 + \sin\theta)^2$
$= 1 + 2\sin\theta + \sin^2\theta$

b $(\sin\alpha - 2)^2$
$= \sin^2\alpha - 4\sin\alpha + 4$

c $(\tan\alpha - 1)^2$
$= \tan^2\alpha - 2\tan\alpha + 1$

d $(\sin\alpha + \cos\alpha)^2$
$= \sin^2\alpha + 2\sin\alpha\cos\alpha + \cos^2\alpha$
$= 1 + 2\sin\alpha\cos\alpha$

e $(\sin\beta - \cos\beta)^2$
$= \sin^2\beta - 2\sin\beta\cos\beta + \cos^2\beta$
$= 1 - 2\sin\beta\cos\beta$

f $-(2 - \cos\alpha)^2$
$= -[4 - 4\cos\alpha + \cos^2\alpha]$
$= -4 + 4\cos\alpha - \cos^2\alpha$

5 **a**
$$\begin{aligned} & 1-\sec^2\beta \\ &= 1-\frac{1}{\cos^2\beta} \\ &= \frac{\cos^2\beta-1}{\cos^2\beta} \\ &= \frac{-\sin^2\beta}{\cos^2\beta} \\ &= -\tan^2\beta \end{aligned}$$

b
$$\begin{aligned} & \frac{\tan^2\theta(\cot^2\theta+1)}{\tan^2\theta+1} \\ &= \frac{\tan^2\theta\cot^2\theta+\tan^2\theta}{\tan^2\theta+1} \\ &= \frac{1+\tan^2\theta}{\tan^2\theta+1} \\ &= 1 \end{aligned}$$

c
$$\begin{aligned} & \cos^2\alpha(\sec^2\alpha-1) \\ &= \cos^2\alpha\sec^2\alpha-\cos^2\alpha \\ &= 1-\cos^2\alpha \\ &= \sin^2\alpha \end{aligned}$$

d
$$\begin{aligned} & (\sin x+\tan x)(\sin x-\tan x) \\ &= \sin^2 x-\tan^2 x \quad \{\text{or } -\sin^2 x\tan^2 x\} \end{aligned}$$

e
$$\begin{aligned} & (2\sin\theta+3\cos\theta)^2+(3\sin\theta-2\cos\theta)^2 \\ &= 4\sin^2\theta+12\sin\theta\cos\theta+9\cos^2\theta \\ &\quad +9\sin^2\theta-12\sin\theta\cos\theta+4\cos^2\theta \\ &= 13\sin^2\theta+13\cos^2\theta \\ &= 13(\sin^2\theta+\cos^2\theta) \\ &= 13 \end{aligned}$$

f
$$\begin{aligned} & (1+\csc\theta)(\sin\theta-\sin^2\theta) \\ &= \sin\theta-\sin^2\theta+\csc\theta\sin\theta-\csc\theta\sin^2\theta \\ &= \sin\theta-\sin^2\theta+1-\sin\theta \\ &= 1-\sin^2\theta \\ &= \cos^2\theta \end{aligned}$$

g
$$\begin{aligned} & \sec A-\sin A\tan A-\cos A \\ &= \frac{1}{\cos A}-\frac{\sin A\sin A}{\cos A}-\frac{\cos^2 A}{\cos A} \\ &= \frac{1-\sin^2 A-\cos^2 A}{\cos A} \\ &= \frac{1-(\sin^2 A+\cos^2 A)}{\cos A} \\ &= 0 \end{aligned}$$

EXERCISE 13C.2

1 **a**
$$\begin{aligned} & 1-\sin^2\theta \\ &= (1+\sin\theta)(1-\sin\theta) \end{aligned}$$

b
$$\begin{aligned} & \sin^2\alpha-\cos^2\alpha \\ &= (\sin\alpha+\cos\alpha)(\sin\alpha-\cos\alpha) \end{aligned}$$

c
$$\begin{aligned} & \tan^2\alpha-1 \\ &= (\tan\alpha+1)(\tan\alpha-1) \end{aligned}$$

d
$$\begin{aligned} & 2\sin^2\beta-\sin\beta \\ &= \sin\beta(2\sin\beta-1) \end{aligned}$$

e
$$\begin{aligned} & 2\cos\phi+3\cos^2\phi \\ &= \cos\phi(2+3\cos\phi) \end{aligned}$$

f
$$\begin{aligned} & 3\sin^2\theta-6\sin\theta \\ &= 3\sin\theta(\sin\theta-2) \end{aligned}$$

g
$$\begin{aligned} & \tan^2\theta+5\tan\theta+6 \\ &= (\tan\theta+2)(\tan\theta+3) \end{aligned}$$

h
$$\begin{aligned} & 2\cos^2\theta+7\cos\theta+3 \\ &= (2\cos\theta+1)(\cos\theta+3) \end{aligned}$$

i
$$\begin{aligned} & 6\cos^2\alpha-\cos\alpha-1 \\ &= (3\cos\alpha+1)(2\cos\alpha-1) \end{aligned}$$

j
$$\begin{aligned} & 3\tan^2\alpha-2\tan\alpha \\ &= \tan\alpha(3\tan\alpha-2) \end{aligned}$$

k
$$\begin{aligned} & \sec^2\beta-\csc^2\beta \\ &= (\sec\beta+\csc\beta)(\sec\beta-\csc\beta) \end{aligned}$$

l
$$\begin{aligned} & 2\cot^2 x-3\cot x+1 \\ &= (2\cot x-1)(\cot x-1) \end{aligned}$$

m
$$\begin{aligned} & 2\sin^2 x+7\sin x\cos x+3\cos^2 x \\ &= (2\sin x+\cos x)(\sin x+3\cos x) \end{aligned}$$

2 **a**
$$\begin{aligned} & \frac{1-\sin^2\alpha}{1-\sin\alpha} \\ &= \frac{(1+\sin\alpha)\cancel{(1-\sin\alpha)}}{\cancel{1-\sin\alpha}_{\,1}} \\ &= 1+\sin\alpha \end{aligned}$$

b
$$\begin{aligned} & \frac{\tan^2\beta-1}{\tan\beta+1} \\ &= \frac{\cancel{(\tan\beta+1)}(\tan\beta-1)}{\cancel{\tan\beta+1}_{\,1}} \\ &= \tan\beta-1 \end{aligned}$$

c
$$\frac{\cos^2\phi - \sin^2\phi}{\cos\phi + \sin\phi}$$
$$= \frac{\cancel{(\cos\phi + \sin\phi)}(\cos\phi - \sin\phi)}{\cancel{\cos\phi + \sin\phi}_{\,1}}$$
$$= \cos\phi - \sin\phi$$

d
$$\frac{\cos^2\phi - \sin^2\phi}{\cos\phi - \sin\phi}$$
$$= \frac{(\cos\phi + \sin\phi)\cancel{(\cos\phi - \sin\phi)}}{\cancel{\cos\phi - \sin\phi}_{\,1}}$$
$$= \cos\phi + \sin\phi$$

e
$$\frac{\sin\alpha + \cos\alpha}{\sin^2\alpha - \cos^2\alpha}$$
$$= \frac{\cancel{\sin\alpha + \cos\alpha}^{\,1}}{\cancel{(\sin\alpha + \cos\alpha)}(\sin\alpha - \cos\alpha)}$$
$$= \frac{1}{\sin\alpha - \cos\alpha}$$

f
$$\frac{3 - 3\sin^2\theta}{6\cos\theta} = \frac{3(1 - \sin^2\theta)}{6\cos\theta}$$
$$= \frac{{}^{1}\cancel{3}\cos^{\cancel{2}}\theta}{{}_{2}\cancel{6}\,\cancel{\cos\theta}}$$
$$= \frac{\cos\theta}{2}$$

g
$$1 - \frac{\cos^2\theta}{1 + \sin\theta}$$
$$= \frac{(1 + \sin\theta) - (1 - \sin^2\theta)}{1 + \sin\theta}$$
$$= \frac{1 + \sin\theta - 1 + \sin^2\theta}{1 + \sin\theta}$$
$$= \frac{\sin\theta\cancel{(1 + \sin\theta)}}{\cancel{1 + \sin\theta}_{\,1}}$$
$$= \sin\theta$$

h
$$\frac{1 + \cot\theta}{\csc\theta} - \frac{\sec\theta}{\tan\theta + \cot\theta}$$
$$= \sin\theta\left(1 + \frac{\cos\theta}{\sin\theta}\right) - \frac{1}{\cos\theta\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)}$$
$$= \sin\theta + \cos\theta - \frac{1}{\sin\theta + \frac{\cos^2\theta}{\sin\theta}}$$
$$= \sin\theta + \cos\theta - \frac{\sin\theta}{\sin^2\theta + \cos^2\theta}$$
$$= \sin\theta + \cos\theta - \sin\theta$$
$$= \cos\theta$$

i
$$\frac{\tan^2\theta}{\sec\theta - 1} = \frac{\sec^2\theta - 1}{\sec\theta - 1}$$
$$= \frac{(\sec\theta + 1)\cancel{(\sec\theta - 1)}}{\cancel{\sec\theta - 1}_{\,1}}$$
$$= \sec\theta + 1$$

3 **a**
$$(\cos\theta + \sin\theta)^2 + (\cos\theta - \sin\theta)^2$$
$$= \cos^2\theta + \cancel{2\cos\theta\sin\theta} + \sin^2\theta$$
$$+ \cos^2\theta - \cancel{2\cos\theta\sin\theta} + \sin^2\theta$$
$$= 2\cos^2\theta + 2\sin^2\theta$$
$$= 2(\cos^2\theta + \sin^2\theta)$$
$$= 2$$

b
$$(2\sin\theta + 3\cos\theta)^2 + (3\sin\theta - 2\cos\theta)^2$$
$$= 4\sin^2\theta + \cancel{12\sin\theta\cos\theta} + 9\cos^2\theta$$
$$+ 9\sin^2\theta - \cancel{12\sin\theta\cos\theta} + 4\cos^2\theta$$
$$= 13\sin^2\theta + 13\cos^2\theta$$
$$= 13(\sin^2\theta + \cos^2\theta)$$
$$= 13$$

c
$$(1 - \cos\theta)\left(1 + \frac{1}{\cos\theta}\right)$$
$$= 1 + \frac{1}{\cos\theta} - \cos\theta - 1$$
$$= \frac{1}{\cos\theta} - \cos\theta$$
$$= \frac{1 - \cos^2\theta}{\cos\theta}$$
$$= \frac{\sin^2\theta}{\cos\theta}$$
$$= \tan\theta\sin\theta$$

d
$$\left(1 + \frac{1}{\sin\theta}\right)(\sin\theta - \sin^2\theta)$$
$$= \cancel{\sin\theta} - \sin^2\theta + 1 - \cancel{\sin\theta}$$
$$= 1 - \sin^2\theta$$
$$= \cos^2\theta$$

e $\sec A - \cos A = \dfrac{1}{\cos A} - \cos A$
$= \dfrac{1-\cos^2 A}{\cos A}$
$= \dfrac{\sin^2 A}{\cos A}$
$= \tan A \sin A$

f $\dfrac{\cos\theta}{1-\sin\theta} = \dfrac{\cos\theta(1+\sin\theta)}{(1-\sin\theta)(1+\sin\theta)}$
$= \dfrac{\cos\theta + \cos\theta\sin\theta}{1-\sin^2\theta}$
$= \dfrac{\cos\theta + \cos\theta\sin\theta}{\cos^2\theta}$
$= \dfrac{1}{\cos\theta} + \dfrac{\sin\theta}{\cos\theta}$
$= \sec\theta + \tan\theta$

g $\dfrac{\cos\alpha}{1-\tan\alpha} + \dfrac{\sin\alpha}{1-\cot\alpha}$
$= \dfrac{\cos\alpha}{1-\frac{\sin\alpha}{\cos\alpha}} + \dfrac{\sin\alpha}{1-\frac{\cos\alpha}{\sin\alpha}}$
$= \dfrac{\cos\alpha}{\frac{\cos\alpha-\sin\alpha}{\cos\alpha}} + \dfrac{\sin\alpha}{\frac{\sin\alpha-\cos\alpha}{\sin\alpha}}$
$= \dfrac{\cos^2\alpha}{\cos\alpha-\sin\alpha} + \dfrac{\sin^2\alpha}{\sin\alpha-\cos\alpha}$
$= \dfrac{\cos^2\alpha-\sin^2\alpha}{\cos\alpha-\sin\alpha}$
$= \dfrac{(\cos\alpha+\sin\alpha)\cancel{(\cos\alpha-\sin\alpha)}}{\cancel{\cos\alpha-\sin\alpha}_{\,1}}$
$= \sin\alpha + \cos\alpha$

h $\dfrac{\sin\theta}{1+\cos\theta} + \dfrac{1+\cos\theta}{\sin\theta}$
$= \dfrac{\sin^2\theta + (1+\cos\theta)(1+\cos\theta)}{\sin\theta(1+\cos\theta)}$
$= \dfrac{\sin^2\theta + 1 + 2\cos\theta + \cos^2\theta}{\sin\theta(1+\cos\theta)}$
$= \dfrac{1+1+2\cos\theta}{\sin\theta(1+\cos\theta)}$
$= \dfrac{2\cancel{(1+\cos\theta)}}{\sin\theta\cancel{(1+\cos\theta)}}$
$= \dfrac{2}{\sin\theta}$
$= 2\csc\theta$

i $\dfrac{\sin\theta}{1-\cos\theta} - \dfrac{\sin\theta}{1+\cos\theta}$
$= \dfrac{\sin\theta(1+\cos\theta) - \sin\theta(1-\cos\theta)}{(1-\cos\theta)(1+\cos\theta)}$
$= \dfrac{\cancel{\sin\theta} + \sin\theta\cos\theta - \cancel{\sin\theta} + \sin\theta\cos\theta}{1-\cos^2\theta}$
$= \dfrac{2\cancel{\sin\theta}\cos\theta}{\sin^{\cancel{2}}\theta}$
$= \dfrac{2\cos\theta}{\sin\theta} = 2\cot\theta$

j $\dfrac{1}{1-\sin\theta} + \dfrac{1}{1+\sin\theta}$
$= \dfrac{1+\sin\theta+1-\sin\theta}{(1-\sin\theta)(1+\sin\theta)}$
$= \dfrac{2}{1-\sin^2\theta}$
$= \dfrac{2}{\cos^2\theta}$
$= 2\sec^2\theta$

EXERCISE 13D

1 **a** $\sin 2\theta = 2\sin\theta\cos\theta$
$= 2(\frac{4}{5})(\frac{3}{5})$
$= \frac{24}{25}$

b $\cos 2\theta = \cos^2\theta - \sin^2\theta$
$= \frac{9}{25} - \frac{16}{25}$
$= -\frac{7}{25}$

c $\tan 2\theta = \dfrac{\sin 2\theta}{\cos 2\theta}$
$= \dfrac{\frac{24}{25}}{-\frac{7}{25}}$
$= -\frac{24}{7}$

2 **a** $\cos 2A = 2\cos^2 A - 1$
$= 2(\frac{1}{3})^2 - 1$
$= 2 \times \frac{1}{9} - 1$
$= \frac{2}{9} - 1$
$= -\frac{7}{9}$

b $\cos 2\phi = 1 - 2\sin^2\phi$
$= 1 - 2(-\frac{2}{3})^2$
$= 1 - 2(\frac{4}{9})$
$= 1 - \frac{8}{9}$
$= \frac{1}{9}$

3 **a** $\sin\alpha = -\frac{2}{3}$

α is in Q3

$\therefore\ \cos\alpha < 0$

$\cos^2\alpha + \sin^2\alpha = 1$

$\therefore\ \cos^2\alpha + \frac{4}{9} = 1$

$\therefore\ \cos^2\alpha = \frac{5}{9}$

$\therefore\ \cos\alpha = -\frac{\sqrt{5}}{3}$

$\{\text{since } \cos\alpha < 0\}$

b $\sin 2\alpha = 2\sin\alpha\cos\alpha$

$= 2\left(-\frac{2}{3}\right)\left(-\frac{\sqrt{5}}{3}\right)$

$= \frac{4\sqrt{5}}{9}$

4 **a** $\cos\beta = \frac{2}{5}$

β is in Q4

$\therefore\ \sin\beta < 0$

$\cos^2\beta + \sin^2\beta = 1$

$\therefore\ \frac{4}{25} + \sin^2\beta = 1$

$\therefore\ \sin^2\beta = \frac{21}{25}$

$\therefore\ \sin\beta = -\frac{\sqrt{21}}{5}$

$\{\text{since } \sin\beta < 0\}$

b $\sin 2\beta = 2\sin\beta\cos\beta$

$= 2\left(-\frac{\sqrt{21}}{5}\right)\left(\frac{2}{5}\right)$

$= -\frac{4\sqrt{21}}{25}$

5 α is acute $\therefore$ $\cos\alpha$ and $\sin\alpha$ are positive

a $\cos 2\alpha = 2\cos^2\alpha - 1$

$\therefore\ -\frac{7}{9} = 2\cos^2\alpha - 1$

$\therefore\ 2\cos^2\alpha = \frac{2}{9}$

$\therefore\ \cos^2\alpha = \frac{1}{9}$

$\therefore\ \cos\alpha = \frac{1}{3}$ $\{\text{since } \cos\alpha > 0\}$

b $\sin^2\alpha = 1 - \cos^2\alpha$

$\therefore\ \sin\alpha = \sqrt{1-\cos^2\alpha}$ $\{\text{since } \sin\alpha > 0\}$

$= \sqrt{1 - \frac{1}{9}}$

$= \sqrt{\frac{8}{9}}$

$= \frac{2\sqrt{2}}{3}$

6 **a** $\tan 2A = \frac{21}{20}$

$\therefore\ \dfrac{2\tan A}{1-\tan^2 A} = \frac{21}{20}$

$\therefore\ 40\tan A = 21 - 21\tan^2 A$

$\therefore\ 21\tan^2 A + 40\tan A - 21 = 0$

$\therefore\ (7\tan A - 3)(3\tan A + 7) = 0$

$\therefore\ \tan A = \frac{3}{7}$ or $-\frac{7}{3}$

but A is obtuse $\therefore$ $\tan A$ is negative

$\therefore\ \tan A = -\frac{7}{3}$

b $\tan 2A = -\frac{12}{5}$

$\therefore\ \dfrac{2\tan A}{1-\tan^2 A} = -\frac{12}{5}$

$\therefore\ 10\tan A = -12 + 12\tan^2 A$

$\therefore\ 12\tan^2 A - 10\tan A - 12 = 0$

$\therefore\ 2(6\tan^2 A - 5\tan A - 6) = 0$

$\therefore\ 2(3\tan A + 2)(2\tan A - 3) = 0$

$\therefore\ \tan A = -\frac{2}{3}$ or $\frac{3}{2}$

but A is acute $\therefore$ $\tan A$ is positive

$\therefore\ \tan A = \frac{3}{2}$

7 $\tan(\frac{\pi}{4}) = 1$

$\therefore\ \tan(2\times\frac{\pi}{8}) = 1$

$\therefore\ \dfrac{2\tan\frac{\pi}{8}}{1-\tan^2(\frac{\pi}{8})} = 1$

$\therefore\ 2\tan(\frac{\pi}{8}) = 1 - \tan^2(\frac{\pi}{8})$

$\therefore\ \tan^2(\frac{\pi}{8}) + 2\tan(\frac{\pi}{8}) - 1 = 0$

$\therefore\ \tan(\frac{\pi}{8}) = \dfrac{-2 \pm \sqrt{2^2 - 4(1)(-1)}}{2}$

$= \dfrac{-2 \pm 2\sqrt{2}}{2}$

$= -1 \pm \sqrt{2}$

but $\frac{\pi}{8}$ is in Q1 $\therefore$ $\tan(\frac{\pi}{8})$ is positive

$\therefore\ \tan(\frac{\pi}{8}) = \sqrt{2} - 1$

8 $\left[\cos(\frac{\pi}{12}) + \sin(\frac{\pi}{12})\right]^2$

$= \cos^2(\frac{\pi}{12}) + 2\cos(\frac{\pi}{12})\sin(\frac{\pi}{12}) + \sin^2(\frac{\pi}{12})$

$= 1 + 2\cos(\frac{\pi}{12})\sin(\frac{\pi}{12})$

$= 1 + \sin(\frac{\pi}{6})$ $\{\sin 2A = 2\cos A\sin A\}$

$= 1 + \frac{1}{2}$

$= \frac{3}{2}$

9 **a** $2\sin\alpha\cos\alpha$
$= \sin 2\alpha$

b $4\cos\alpha\sin\alpha$
$= 2(2\sin\alpha\cos\alpha)$
$= 2\sin 2\alpha$

c $\sin\alpha\cos\alpha$
$= \frac{1}{2}(2\sin\alpha\cos\alpha)$
$= \frac{1}{2}\sin 2\alpha$

d $2\cos^2\beta - 1$
$= \cos 2\beta$

e $1 - 2\cos^2\phi$
$= -(2\cos^2\phi - 1)$
$= -\cos 2\phi$

f $1 - 2\sin^2 N$
$= \cos 2N$

g $2\sin^2 M - 1$
$= -(1 - 2\sin^2 M)$
$= -\cos 2M$

h $\cos^2\alpha - \sin^2\alpha$
$= \cos 2\alpha$

i $\sin^2\alpha - \cos^2\alpha$
$= -(\cos^2\alpha - \sin^2\alpha)$
$= -\cos 2\alpha$

j $2\sin 2A\cos 2A$
$= \sin 2(2A)$
$= \sin 4A$

k $2\cos 3\alpha\sin 3\alpha$
$= \sin 2(3\alpha)$
$= \sin 6\alpha$

l $2\cos^2 4\theta - 1$
$= \cos 2(4\theta)$
$= \cos 8\theta$

m $1 - 2\cos^2 3\beta$
$= -(2\cos^2 3\beta - 1)$
$= -\cos 2(3\beta)$
$= -\cos 6\beta$

n $1 - 2\sin^2 5\alpha$
$= \cos 2(5\alpha)$
$= \cos 10\alpha$

o $2\sin^2 3D - 1$
$= -(1 - 2\sin^2 3D)$
$= -\cos 2(3D)$
$= -\cos 6D$

p $\cos^2 2A - \sin^2 2A$
$= \cos 2(2A)$
$= \cos 4A$

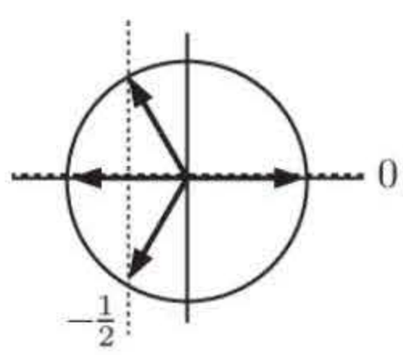
q $\cos^2(\frac{\alpha}{2}) - \sin^2(\frac{\alpha}{2})$
$= \cos 2(\frac{\alpha}{2})$
$= \cos\alpha$

r $2\sin^2 3P - 2\cos^2 3P$
$= -2[\cos^2 3P - \sin^2 3P]$
$= -2\cos 2(3P)$
$= -2\cos 6P$

10 **a** $(\sin\theta + \cos\theta)^2$
$= \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta$
$= \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta$
$= 1 + \sin 2\theta$

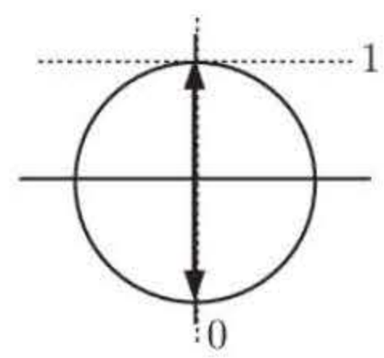
b $\cos^4\theta - \sin^4\theta$
$= (\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta)$
$= 1 \times \cos 2\theta$
$= \cos 2\theta$

11 **a** $\sin 2x + \sin x = 0$
$\therefore\ 2\sin x\cos x + \sin x = 0$
$\therefore\ \sin x(2\cos x + 1) = 0$
$\therefore\ \sin x = 0$ or $\cos x = -\frac{1}{2}$



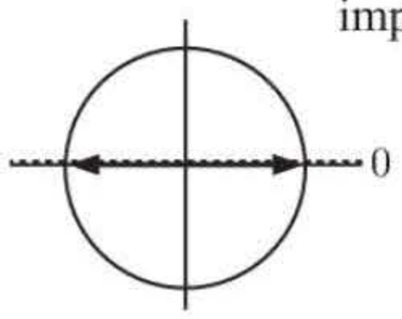
$\therefore\ x = 0, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, 2\pi$

b $\sin 2x - 2\cos x = 0$
$\therefore\ 2\sin x\cos x - 2\cos x = 0$
$\therefore\ 2\cos x(\sin x - 1) = 0$
$\therefore\ \cos x = 0$ or $\sin x = 1$



$\therefore\ x = \frac{\pi}{2}, \frac{3\pi}{2}$

c $\sin 2x + 3\sin x = 0$
$\therefore\ 2\sin x\cos x + 3\sin x = 0$
$\therefore\ \sin x(2\cos x + 3) = 0$
$\therefore\ \sin x = 0$ or $\cos x = -\frac{3}{2}$
impossible



$\therefore\ x = 0, \pi, 2\pi$

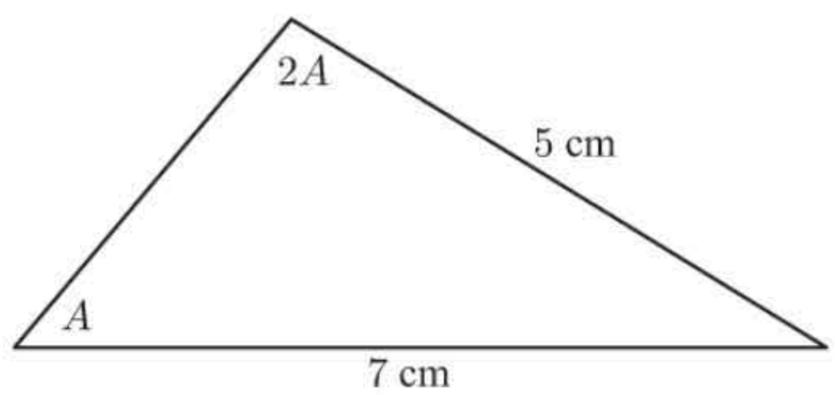
12 **a**

$$\begin{aligned}&\tfrac{1}{2}-\tfrac{1}{2}\cos 2\theta\\ &=\tfrac{1}{2}-\tfrac{1}{2}(1-2\sin^2\theta)\\ &=\tfrac{1}{2}-\tfrac{1}{2}+\sin^2\theta\\ &=\sin^2\theta\end{aligned}$$

b

$$\begin{aligned}&\tfrac{1}{2}+\tfrac{1}{2}\cos 2\theta\\ &=\tfrac{1}{2}+\tfrac{1}{2}(2\cos^2\theta-1)\\ &=\tfrac{1}{2}+\cos^2\theta-\tfrac{1}{2}\\ &=\cos^2\theta\end{aligned}$$

13 **a**



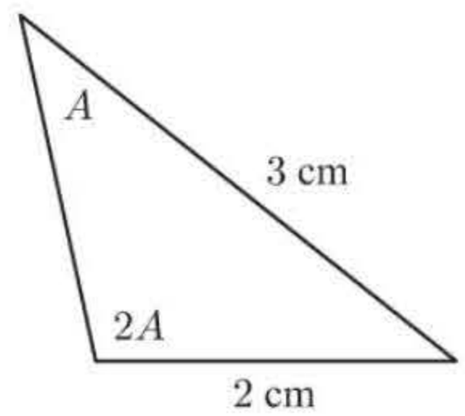
Using the sine rule,

$$\frac{\sin 2A}{7}=\frac{\sin A}{5}$$

$$\therefore\quad \frac{2\cancel{\sin A}\cos A}{7}=\frac{\cancel{\sin A}}{5}$$

$$\therefore\quad \cos A=\tfrac{7}{10}$$

b



Using the sine rule,

$$\frac{\sin 2A}{3}=\frac{\sin A}{2}$$

$$\therefore\quad \frac{2\cancel{\sin A}\cos A}{3}=\frac{\cancel{\sin A}}{2}$$

$$\therefore\quad \cos A=\tfrac{3}{4}$$

14 **a**

$$\begin{aligned}\frac{\sin 2\theta}{1-\cos 2\theta}&=\frac{2\sin\theta\cos\theta}{1-(1-2\sin^2\theta)}\\ &=\frac{2\sin\theta\cos\theta}{1-1+2\sin^2\theta}\\ &=\frac{\cancel{2}\cancel{\sin\theta}\cos\theta}{\cancel{2}\sin^{\cancel{2}}\theta}\\ &=\frac{\cos\theta}{\sin\theta}\\ &=\cot\theta\end{aligned}$$

b

$$\begin{aligned}&\frac{\sin\theta+\sin 2\theta}{1+\cos\theta+\cos 2\theta}\\ &=\frac{\sin\theta+2\sin\theta\cos\theta}{\cancel{1}+\cos\theta+2\cos^2\theta-\cancel{1}}\\ &=\frac{\sin\theta\cancel{(1+2\cos\theta)}}{\cos\theta\cancel{(1+2\cos\theta)}}\\ &=\frac{\sin\theta}{\cos\theta}\\ &=\tan\theta\end{aligned}$$

15 **a**

$$\begin{aligned}&\frac{\sin 2\theta}{1+\cos 2\theta}-\tan\theta\\ &=\frac{2\sin\theta\cos\theta}{\cancel{1}+(2\cos^2\theta-\cancel{1})}-\frac{\sin\theta}{\cos\theta}\\ &=\frac{\cancel{2}\sin\theta\cancel{\cos\theta}}{\cancel{2}\cos^{\cancel{2}}\theta}-\frac{\sin\theta}{\cos\theta}\\ &=\frac{\sin\theta}{\cos\theta}-\frac{\sin\theta}{\cos\theta}\\ &=0\end{aligned}$$

b

$$\begin{aligned}\tan\theta+\cot 2\theta&=\frac{\sin\theta}{\cos\theta}+\frac{\cos 2\theta}{\sin 2\theta}\\ &=\frac{\sin\theta}{\cos\theta}+\frac{1-2\sin^2\theta}{2\sin\theta\cos\theta}\\ &=\frac{\cancel{2\sin^2\theta}+1-\cancel{2\sin^2\theta}}{2\sin\theta\cos\theta}\\ &=\frac{1}{2\sin\theta\cos\theta}\\ &=\frac{1}{\sin 2\theta}\\ &=\csc 2\theta\end{aligned}$$

c

$$\begin{aligned}\frac{\sin 2\theta}{\sin\theta}-\frac{\cos 2\theta}{\cos\theta}&=\frac{2\cancel{\sin\theta}\cos\theta}{\cancel{\sin\theta}_1}-\frac{2\cos^2\theta-1}{\cos\theta}\\ &=2\cos\theta-\frac{2\cos^2\theta-1}{\cos\theta}\\ &=\frac{\cancel{2\cos^2\theta}-\cancel{2\cos^2\theta}+1}{\cos\theta}\\ &=\frac{1}{\cos\theta}=\sec\theta\end{aligned}$$

EXERCISE 13E

1 **a**
$$\begin{aligned}&\sin(90^\circ+\theta)\\&=\sin 90^\circ\cos\theta+\cos 90^\circ\sin\theta\\&=(1)\cos\theta+(0)\sin\theta\\&=\cos\theta\end{aligned}$$

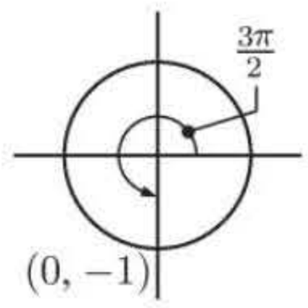
b
$$\begin{aligned}&\cos(90^\circ+\theta)\\&=\cos 90^\circ\cos\theta-\sin 90^\circ\sin\theta\\&=(0)\cos\theta-(1)\sin\theta\\&=-\sin\theta\end{aligned}$$

c
$$\begin{aligned}&\sin(180^\circ-\theta)\\&=\sin 180^\circ\cos\theta-\cos 180^\circ\sin\theta\\&=(0)\cos\theta-(-1)\sin\theta\\&=\sin\theta\end{aligned}$$

d
$$\begin{aligned}&\cos(\pi+\alpha)\\&=\cos\pi\cos\alpha-\sin\pi\sin\alpha\\&=(-1)\cos\alpha-(0)\sin\alpha\\&=-\cos\alpha\end{aligned}$$

e
$$\begin{aligned}&\sin(2\pi-A)\\&=\sin 2\pi\cos A-\cos 2\pi\sin A\\&=(0)\cos A-(1)\sin A\\&=-\sin A\end{aligned}$$

f
$$\begin{aligned}&\cos\left(\tfrac{3\pi}{2}-\theta\right)\\&=\cos\left(\tfrac{3\pi}{2}\right)\cos\theta+\sin\left(\tfrac{3\pi}{2}\right)\sin\theta\\&=(0)\cos\theta+(-1)\sin\theta\\&=-\sin\theta\end{aligned}$$

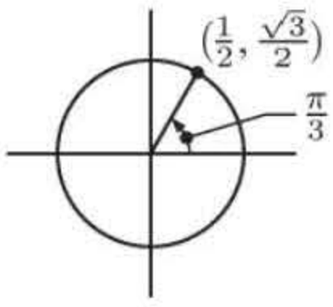


g
$$\begin{aligned}&\tan\left(\tfrac{\pi}{4}+\theta\right)\\&=\frac{\tan\frac{\pi}{4}+\tan\theta}{1-\tan\frac{\pi}{4}\tan\theta}\\&=\frac{1+\tan\theta}{1-\tan\theta}\end{aligned}$$

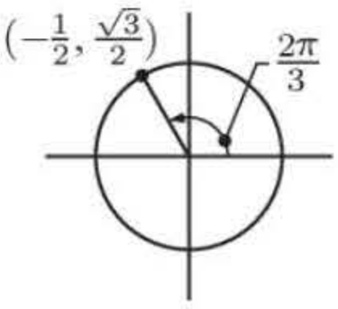
h
$$\begin{aligned}&\tan\left(\theta-\tfrac{3\pi}{4}\right)\\&=\frac{\tan\theta-\tan\frac{3\pi}{4}}{1+\tan\theta\tan\frac{3\pi}{4}}\\&=\frac{\tan\theta-(-1)}{1+\tan\theta(-1)}\\&=\frac{1+\tan\theta}{1-\tan\theta}\end{aligned}$$

i
$$\begin{aligned}&\tan(\pi+\theta)\\&=\frac{\tan\pi+\tan\theta}{1-\tan\pi\tan\theta}\\&=\frac{0+\tan\theta}{1-(0)\tan\theta}\\&=\tan\theta\end{aligned}$$

2 **a**
$$\begin{aligned}&\sin\left(\theta+\tfrac{\pi}{3}\right)\\&=\sin\theta\cos\left(\tfrac{\pi}{3}\right)+\cos\theta\sin\left(\tfrac{\pi}{3}\right)\\&=\sin\theta\times\left(\tfrac{1}{2}\right)+\cos\theta\times\left(\tfrac{\sqrt{3}}{2}\right)\\&=\tfrac{1}{2}\sin\theta+\tfrac{\sqrt{3}}{2}\cos\theta\end{aligned}$$



b
$$\begin{aligned}&\cos\left(\tfrac{2\pi}{3}-\theta\right)\\&=\cos\left(\tfrac{2\pi}{3}\right)\cos\theta+\sin\left(\tfrac{2\pi}{3}\right)\sin\theta\\&=\left(-\tfrac{1}{2}\right)\cos\theta+\left(\tfrac{\sqrt{3}}{2}\right)\sin\theta\\&=-\tfrac{1}{2}\cos\theta+\tfrac{\sqrt{3}}{2}\sin\theta\\&=\tfrac{\sqrt{3}}{2}\sin\theta-\tfrac{1}{2}\cos\theta\end{aligned}$$



c
$$\begin{aligned}&\cos\left(\theta+\tfrac{\pi}{4}\right)\\&=\cos\theta\cos\left(\tfrac{\pi}{4}\right)-\sin\theta\sin\left(\tfrac{\pi}{4}\right)\\&=\tfrac{1}{\sqrt{2}}\cos\theta-\tfrac{1}{\sqrt{2}}\sin\theta\\&=-\tfrac{1}{\sqrt{2}}\sin\theta+\tfrac{1}{\sqrt{2}}\cos\theta\end{aligned}$$

d
$$\begin{aligned}&\sin\left(\tfrac{\pi}{6}-\theta\right)\\&=\sin\left(\tfrac{\pi}{6}\right)\cos\theta-\cos\left(\tfrac{\pi}{6}\right)\sin\theta\\&=\tfrac{1}{2}\cos\theta-\tfrac{\sqrt{3}}{2}\sin\theta\\&=-\tfrac{\sqrt{3}}{2}\sin\theta+\tfrac{1}{2}\cos\theta\end{aligned}$$

3 **a**
$$\begin{aligned}&\cos 2\theta\cos\theta+\sin 2\theta\sin\theta\\&=\cos(2\theta-\theta)\\&=\cos\theta\end{aligned}$$

b
$$\begin{aligned}&\sin 2A\cos A+\cos 2A\sin A\\&=\sin(2A+A)\\&=\sin 3A\end{aligned}$$

c
$$\begin{aligned}&\cos A\sin B-\sin A\cos B\\&=\sin B\cos A-\cos B\sin A\\&=\sin(B-A)\end{aligned}$$

d
$$\begin{aligned}&\sin\alpha\sin\beta+\cos\alpha\cos\beta\\&=\cos\alpha\cos\beta+\sin\alpha\sin\beta\\&=\cos(\alpha-\beta)\end{aligned}$$

e
$$\begin{aligned}&\sin\phi\sin\theta-\cos\phi\cos\theta\\&=-[\cos\phi\cos\theta-\sin\phi\sin\theta]\\&=-\cos(\phi+\theta)\end{aligned}$$

f
$$\begin{aligned}&2\sin\alpha\cos\beta-2\cos\alpha\sin\beta\\&=2\,[\sin\alpha\cos\beta-\cos\alpha\sin\beta]\\&=2\sin(\alpha-\beta)\end{aligned}$$

g $\dfrac{\tan 2\theta - \tan\theta}{1+\tan 2\theta\tan\theta} = \tan(2\theta-\theta)$
$= \tan\theta$

h $\dfrac{\tan 2A + \tan A}{1-\tan 2A\tan A} = \tan(2A+A)$
$= \tan 3A$

4 **a** $\sin 2\theta = \sin(\theta+\theta)$
$= \sin\theta\cos\theta + \cos\theta\sin\theta$
$= 2\sin\theta\cos\theta$

b $\cos 2\theta = \cos(\theta+\theta)$
$= \cos\theta\cos\theta - \sin\theta\sin\theta$
$= \cos^2\theta - \sin^2\theta$

c $\tan 2\theta = \tan(\theta+\theta)$
$= \dfrac{\tan\theta+\tan\theta}{1-\tan\theta\tan\theta}$
$= \dfrac{2\tan\theta}{1-\tan^2\theta}$

5 **a** $\cos(\alpha+\beta)\cos(\alpha-\beta) - \sin(\alpha+\beta)\sin(\alpha-\beta)$
$= \cos[(\alpha+\beta)+(\alpha-\beta)]$
$= \cos 2\alpha$

b $\sin(\theta-2\phi)\cos(\theta+\phi) - \cos(\theta-2\phi)\sin(\theta+\phi)$
$= \sin[(\theta-2\phi)-(\theta+\phi)]$
$= \sin(-3\phi)$
$= -\sin 3\phi$

c $\cos\alpha\cos(\beta-\alpha)$
$- \sin\alpha\sin(\beta-\alpha)$
$= \cos[\alpha+(\beta-\alpha)]$
$= \cos\beta$

6 **a** $\cos 75^\circ$
$= \cos(45^\circ+30^\circ)$
$= \cos 45^\circ\cos 30^\circ - \sin 45^\circ\sin 30^\circ$
$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right)$
$= \left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)\frac{\sqrt{2}}{\sqrt{2}}$
$= \frac{\sqrt{6}-\sqrt{2}}{4}$

b $\sin 105^\circ$
$= \sin(60^\circ+45^\circ)$
$= \sin 60^\circ\cos 45^\circ + \cos 60^\circ\sin 45^\circ$
$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right)$
$= \left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)\frac{\sqrt{2}}{\sqrt{2}}$
$= \frac{\sqrt{6}+\sqrt{2}}{4}$

c $\cos\left(\frac{13\pi}{12}\right)$
$= \cos\left(\frac{10\pi}{12}+\frac{3\pi}{12}\right)$
$= \cos\left(\frac{5\pi}{6}+\frac{\pi}{4}\right)$
$= \cos\left(\frac{5\pi}{6}\right)\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{5\pi}{6}\right)\sin\left(\frac{\pi}{4}\right)$
$= \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right)$
$= \left(\frac{-\sqrt{3}-1}{2\sqrt{2}}\right)\frac{\sqrt{2}}{\sqrt{2}}$
$= \frac{-\sqrt{6}-\sqrt{2}}{4}$

7 **a** $\tan\left(\frac{5\pi}{12}\right)$
$= \tan\left(\frac{5\times 180^\circ}{12}\right)$
$= \tan 75^\circ$
$= \tan(45^\circ+30^\circ)$
$= \dfrac{\tan 45^\circ + \tan 30^\circ}{1-\tan 45^\circ\tan 30^\circ}$
$= \dfrac{1+\frac{1}{\sqrt{3}}}{1-(1)\left(\frac{1}{\sqrt{3}}\right)}$
$= \left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)\left(\frac{\sqrt{3}+1}{\sqrt{3}+1}\right)$
$= \dfrac{3+\sqrt{3}+\sqrt{3}+1}{3-1}$
$= \dfrac{4+2\sqrt{3}}{2}$
$= 2+\sqrt{3}$

b $\tan 105^\circ$
$= \tan(180^\circ - 75^\circ)$
$= \dfrac{\tan 180^\circ - \tan 75^\circ}{1+\tan 180^\circ\tan 75^\circ}$
$= \dfrac{0-(2+\sqrt{3})}{1+(0)(2+\sqrt{3})}$ {using **a**}
$= -2-\sqrt{3}$

8 $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

$= \dfrac{\frac{2}{3} - \frac{1}{5}}{1 - (\frac{2}{3})(-\frac{1}{5})}$

$= \dfrac{\frac{10}{15} - \frac{3}{15}}{1 + \frac{2}{15}} = \dfrac{\frac{7}{15}}{\frac{17}{15}} = \frac{7}{17}$

9 $\tan(A + \frac{\pi}{4}) = \dfrac{\tan A + \tan \frac{\pi}{4}}{1 - \tan A \tan \frac{\pi}{4}}$

$= \dfrac{\frac{3}{4} + 1}{1 - (\frac{3}{4})(1)}$

$= \dfrac{\frac{7}{4}}{\frac{1}{4}} = 7$

10 **a** $\tan(A + \frac{\pi}{4}) \tan(A - \frac{\pi}{4})$

$= \dfrac{\tan A + \tan \frac{\pi}{4}}{1 - \tan A \tan \frac{\pi}{4}} \times \dfrac{\tan A - \tan \frac{\pi}{4}}{1 + \tan A \tan \frac{\pi}{4}}$

$= \left(\dfrac{\tan A + 1}{1 - \tan A}\right)\left(\dfrac{\tan A - 1}{1 + \tan A}\right)$

$= \dfrac{\tan^2 A - 1}{1 - \tan^2 A}$

$= -1$

b $\dfrac{\tan(A+B) + \tan(A-B)}{1 - \tan(A+B)\tan(A-B)}$

$= \tan[(A+B) + (A-B)]$

$= \tan 2A$

11 $\sin A = -\frac{1}{3}$

A is in Q3

$\therefore \cos A < 0$

$\cos^2 A + \sin^2 A = 1$

$\therefore \cos^2 A + \frac{1}{9} = 1$

$\therefore \cos^2 A = \frac{8}{9}$

$\therefore \cos A = -\frac{2\sqrt{2}}{3}$

$\therefore \tan A = \dfrac{\sin A}{\cos A}$

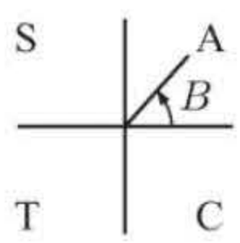
$= \dfrac{-\frac{1}{3}}{-\frac{2\sqrt{2}}{3}}$

$= \frac{1}{2\sqrt{2}}$

$\cos B = \frac{1}{\sqrt{5}}$

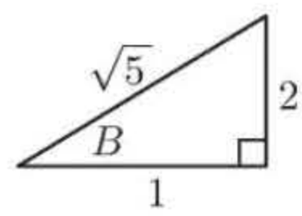
B is in Q1

$\therefore \sin B > 0$



$\therefore \sin B = \frac{2}{\sqrt{5}}$

$\tan B = 2$



a $\tan(A+B)$

$= \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

$= \dfrac{\frac{1}{2\sqrt{2}} + 2}{1 - (\frac{1}{2\sqrt{2}})(2)} \times \left(\dfrac{2\sqrt{2}}{2\sqrt{2}}\right)$

$= \dfrac{1 + 4\sqrt{2}}{2\sqrt{2} - 2} \times \left(\dfrac{2\sqrt{2} + 2}{2\sqrt{2} + 2}\right)$

$= \dfrac{2\sqrt{2} + 2 + 16 + 8\sqrt{2}}{8 - 4}$

$= \dfrac{18 + 10\sqrt{2}}{4}$

$= \dfrac{9 + 5\sqrt{2}}{2}$

b $\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$

$= \dfrac{2(\frac{1}{2\sqrt{2}})}{1 - (\frac{1}{2\sqrt{2}})^2}$

$= \dfrac{\frac{1}{\sqrt{2}}}{1 - \frac{1}{8}}$

$= \dfrac{\frac{1}{\sqrt{2}}}{\frac{7}{8}}$

$= \frac{8}{7\sqrt{2}} \times \left(\frac{\sqrt{2}}{\sqrt{2}}\right)$

$= \frac{8\sqrt{2}}{14}$

$= \frac{4\sqrt{2}}{7}$

12 $\dfrac{\tan 80^\circ - \tan 20^\circ}{1 + \tan 80^\circ \tan 20^\circ} = \tan(80^\circ - 20^\circ)$

$= \tan 60^\circ$

$= \sqrt{3}$

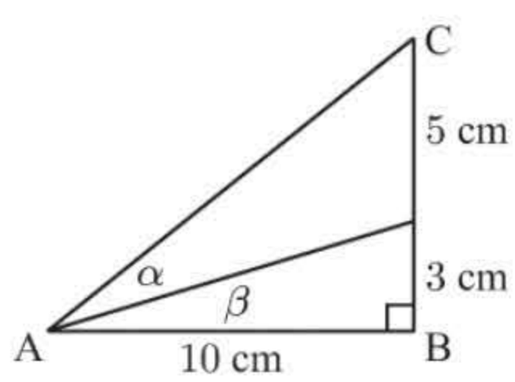
13

$$\tan(A+B) = \tfrac{3}{5}$$
$$\therefore \quad \frac{\tan A + \tan B}{1 - \tan A \tan B} = \tfrac{3}{5}$$
$$\therefore \quad \frac{\tan A + \frac{2}{3}}{1 - \tan A(\frac{2}{3})} = \tfrac{3}{5}$$
$$\therefore \quad 5\tan A + \tfrac{10}{3} = 3 - 2\tan A$$
$$\therefore \quad 7\tan A = -\tfrac{1}{3}$$
$$\therefore \quad \tan A = -\tfrac{1}{21}$$

14

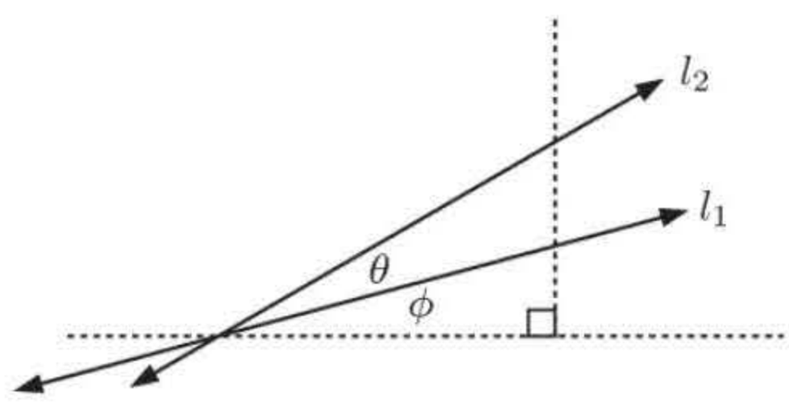
$$\tan(A-B)\tan(A+B) = 1$$
$$\therefore \quad \left(\frac{\tan A - \tan B}{1 + \tan A \tan B}\right)\left(\frac{\tan A + \tan B}{1 - \tan A \tan B}\right) = 1$$
$$\therefore \quad \frac{\tan^2 A - \tan^2 B}{1 - \tan^2 A \tan^2 B} = 1$$
$$\therefore \quad \tan^2 A - \tan^2 B = 1 - \tan^2 A \tan^2 B$$
$$\therefore \quad \tan^2 A(\tan^2 B + 1) = 1 + \tan^2 B$$
$$\therefore \quad \tan^2 A = 1$$
$$\therefore \quad \tan A = \pm 1$$

15



$$\tan\beta = \tfrac{3}{10}$$
$$\tan(\alpha+\beta) = \tfrac{8}{10}$$
$$\therefore \quad \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \tfrac{8}{10}$$
$$\therefore \quad \frac{\tan\alpha + \frac{3}{10}}{1 - \tan\alpha(\frac{3}{10})} = \tfrac{4}{5}$$
$$\therefore \quad 5\tan\alpha + \tfrac{15}{10} = 4 - \tfrac{12}{10}\tan\alpha$$
$$\therefore \quad \tfrac{31}{5}\tan\alpha = \tfrac{5}{2}$$
$$\therefore \quad \tan\alpha = \tfrac{25}{62}$$

16



θ is the acute angle between the lines l_1 and l_2.

$$\tan\phi = \tfrac{1}{2} \quad \{\text{gradient of } l_1\}$$
$$\tan(\theta+\phi) = \tfrac{2}{3} \quad \{\text{gradient of } l_2\}$$
$$\therefore \quad \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi} = \tfrac{2}{3}$$
$$\therefore \quad \frac{\tan\theta + \frac{1}{2}}{1 - \tan\theta(\frac{1}{2})} = \tfrac{2}{3}$$
$$\therefore \quad 3\tan\theta + \tfrac{3}{2} = 2 - \tan\theta$$
$$\therefore \quad 4\tan\theta = \tfrac{1}{2}$$
$$\therefore \quad \tan\theta = \tfrac{1}{8}$$

$\therefore$ the tangent of the acute angle is $\frac{1}{8}$.

17

$$\begin{aligned}
\tan(A+B+C) &= \tan[(A+B)+C] \\
&= \frac{\tan(A+B) + \tan C}{1 - \tan(A+B)\tan C} \\
&= \frac{\frac{\tan A + \tan B}{1 - \tan A \tan B} + \tan C}{1 - \frac{\tan A + \tan B}{1 - \tan A \tan B} \times \tan C} \\
&= \frac{\tan A + \tan B + \tan C(1 - \tan A \tan B)}{1 - \tan A \tan B - (\tan A + \tan B) \times \tan C} \qquad \{\times \text{ top and bottom by } 1 - \tan A \tan B\} \\
&= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan A \tan C - \tan B \tan C}
\end{aligned}$$

If A, B, and C are the angles of a triangle then $A + B + C = 180°$

$$\therefore \quad \tan(A+B+C) = 0$$
$$\therefore \quad \tan A + \tan B + \tan C - \tan A \tan B \tan C = 0$$
$$\therefore \quad \tan A + \tan B + \tan C = \tan A \tan B \tan C$$

18 **a**

$$\begin{aligned}&\sqrt{2}\cos\left(\theta+\tfrac{\pi}{4}\right)\\&=\sqrt{2}\left[\cos\theta\cos\left(\tfrac{\pi}{4}\right)-\sin\theta\sin\left(\tfrac{\pi}{4}\right)\right]\\&=\sqrt{2}\left[\tfrac{1}{\sqrt{2}}\cos\theta-\tfrac{1}{\sqrt{2}}\sin\theta\right]\\&=\cos\theta-\sin\theta\end{aligned}$$

b

$$\begin{aligned}&2\cos\left(\theta-\tfrac{\pi}{3}\right)\\&=2\left[\cos\theta\cos\left(\tfrac{\pi}{3}\right)+\sin\theta\sin\left(\tfrac{\pi}{3}\right)\right]\\&=2\left[\tfrac{1}{2}\cos\theta+\tfrac{\sqrt{3}}{2}\sin\theta\right]\\&=\cos\theta+\sqrt{3}\sin\theta\end{aligned}$$

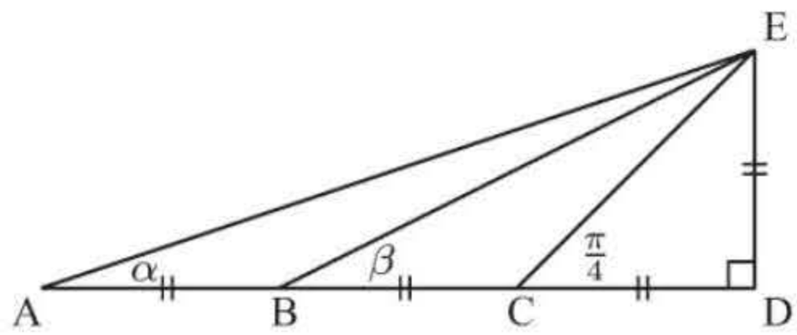
c

$$\begin{aligned}&\cos(\alpha+\beta)-\cos(\alpha-\beta)\\&=\cos\alpha\cos\beta-\sin\alpha\sin\beta-[\cos\alpha\cos\beta+\sin\alpha\sin\beta]\\&=\cancel{\cos\alpha\cos\beta}-\sin\alpha\sin\beta-\cancel{\cos\alpha\cos\beta}-\sin\alpha\sin\beta\\&=-2\sin\alpha\sin\beta\end{aligned}$$

d

$$\begin{aligned}&\cos(\alpha+\beta)\cos(\alpha-\beta)\\&=[\cos\alpha\cos\beta-\sin\alpha\sin\beta]\,[\cos\alpha\cos\beta+\sin\alpha\sin\beta]\\&=\cos^2\alpha\cos^2\beta-\sin^2\alpha\sin^2\beta\\&=\cos^2\alpha\left[1-\sin^2\beta\right]-\left[1-\cos^2\alpha\right]\sin^2\beta\\&=\cos^2\alpha-\cancel{\cos^2\alpha\sin^2\beta}-\sin^2\beta+\cancel{\cos^2\alpha\sin^2\beta}\\&=\cos^2\alpha-\sin^2\beta\end{aligned}$$

19 $\tan\alpha=\frac{1}{3}$ and $\tan\beta=\frac{1}{2}$



$$\begin{aligned}\therefore\quad\tan(\alpha+\beta)&=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\\&=\frac{\frac{1}{3}+\frac{1}{2}}{1-(\frac{1}{3})(\frac{1}{2})}\\&=\frac{\frac{5}{6}}{\frac{5}{6}}=1\end{aligned}$$

$\therefore\quad\alpha+\beta=\frac{\pi}{4}+k\pi,\ k\in\mathbb{Z}\qquad\{\text{since}\quad\tan\frac{\pi}{4}=1\}$

But clearly both α and $\beta<\frac{\pi}{2}$

$\therefore\quad\alpha+\beta=\frac{\pi}{4}$

20

$$\begin{aligned}\sqrt{3}\sin x+\cos x&=k\sin(x+b)\\&=k[\sin x\cos b+\cos x\sin b]\\&=k\cos b\sin x+k\sin b\cos x\end{aligned}$$

Equating coefficients of $\sin x$ and $\cos x$,

$k\cos b=\sqrt{3}$ and $k\sin b=1$ ..... (1)

$\therefore\quad k^2\cos^2 b=3$ and $k^2\sin^2 b=1$ {squaring both sides}

$\therefore\quad k^2(\cos^2 b+\sin^2 b)=4$ {adding the 2 equations}

$\therefore\quad k^2=4$

$\therefore\quad k=2\quad\{k>0\}$

Substituting $k=2$ into (1) gives

$\cos b=\frac{\sqrt{3}}{2}$ and $\sin b=\frac{1}{2}$

$\therefore\quad b=\frac{\pi}{6}$

21 **a**

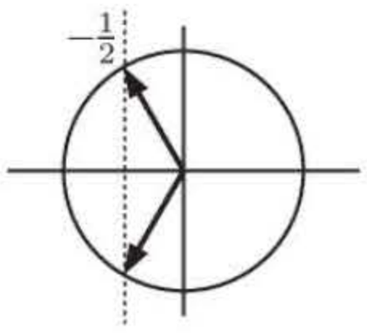
$$\begin{aligned}&\cos 3\theta\\&=\cos(2\theta+\theta)\\&=\cos 2\theta\cos\theta-\sin 2\theta\sin\theta\\&=(2\cos^2\theta-1)\cos\theta-(2\sin\theta\cos\theta)\sin\theta\\&=2\cos^3\theta-\cos\theta-2\sin^2\theta\cos\theta\\&=2\cos^3\theta-\cos\theta-2(1-\cos^2\theta)\cos\theta\\&=2\cos^3\theta-\cos\theta-2\cos\theta+2\cos^3\theta\\&=4\cos^3\theta-3\cos\theta\end{aligned}$$

b $8\cos^3\theta-6\cos\theta+1=0$

$\therefore\quad 8\cos^3\theta-6\cos\theta=-1$

$\therefore\quad 4\cos^3\theta-3\cos\theta=-\frac{1}{2}$

$\therefore\quad\cos 3\theta=-\frac{1}{2}$



$\therefore\quad 3\theta=-\frac{8\pi}{3},-\frac{4\pi}{3},-\frac{2\pi}{3},\ \frac{2\pi}{3},\ \frac{4\pi}{3},\ \frac{8\pi}{3}$

$\{3\theta\in[-3\pi,3\pi]\}$

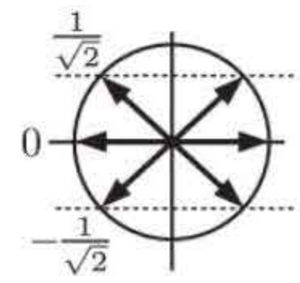
$\therefore\quad\theta=-\frac{8\pi}{9},-\frac{4\pi}{9},-\frac{2\pi}{9},\ \frac{2\pi}{9},\ \frac{4\pi}{9},\ \frac{8\pi}{9}$

22 **a**

$$\begin{aligned}
&\sin 3\theta \\
&= \sin(2\theta + \theta) \\
&= \sin 2\theta \cos\theta + \cos 2\theta \sin\theta \\
&= (2\sin\theta\cos\theta)\cos\theta + (1 - 2\sin^2\theta)\sin\theta \\
&= 2\sin\theta\cos^2\theta + \sin\theta - 2\sin^3\theta \\
&= 2\sin\theta(1 - \sin^2\theta) + \sin\theta - 2\sin^3\theta \\
&= 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta \\
&= -4\sin^3\theta + 3\sin\theta
\end{aligned}$$

b

$$\begin{aligned}
\sin 3\theta &= \sin\theta \\
\therefore\ -4\sin^3\theta + 3\sin\theta &= \sin\theta \\
\therefore\ 4\sin^3\theta - 2\sin\theta &= 0 \\
\therefore\ 2\sin\theta(2\sin^2\theta - 1) &= 0 \\
\therefore\ \sin\theta = 0 \ \text{ or } \ \sin^2\theta &= \tfrac{1}{2} \\
\therefore\ \sin\theta = 0 \ \text{ or } \ \sin\theta &= \pm\tfrac{1}{\sqrt{2}} \\
\therefore\ \theta = 0,\ \tfrac{\pi}{4},\ \tfrac{3\pi}{4},\ \pi,\ \tfrac{5\pi}{4},\ \tfrac{7\pi}{4},\ 2\pi,\ \tfrac{9\pi}{4},\ \tfrac{11\pi}{4},\ 3\pi
\end{aligned}$$



23 The period of a function $f(x)$ is the smallest $p > 0$ such that

$$f(x+p) = f(x) \quad \text{for all } x$$

$$\therefore\ \sin[n(x+p)] = \sin(nx)$$

$$\therefore\ \sin(nx + np) = \sin(nx)$$

$$\therefore\ \sin(nx)\cos(np) + \cos(nx)\sin(np) = \sin(nx)$$

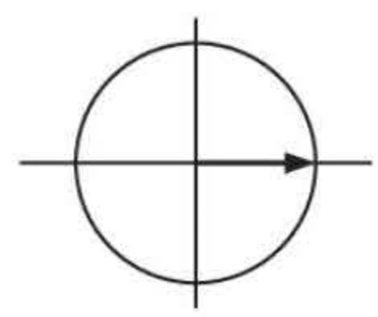
Equating coefficients of $\sin(nx)$ and $\cos(nx)$,

$$\cos(np) = 1 \quad \text{and} \quad \sin(np) = 0$$

$$\therefore\ np = 2k\pi, \quad k \in \mathbb{Z}$$

$$\therefore\ p = \frac{2k\pi}{n}, \quad k \in \mathbb{Z}$$

The smallest $p > 0$ occurs when $k = 1$ $\quad\therefore\ p = \dfrac{2\pi}{n}$



24 **a**

$$\begin{aligned}
2\cos x - 5\sin x &= k\cos(x+b) \\
&= k[\cos x\cos b - \sin x\sin b] \\
&= k\cos b\cos x - k\sin b\sin x
\end{aligned}$$

Equating coefficients of $\cos x$ and $\sin x$,

$$k\cos b = 2 \quad \text{and} \quad k\sin b = 5 \quad \text{..... (1)}$$

$$\therefore\ k^2\cos^2 b = 4 \quad \text{and} \quad k^2\sin^2 b = 25 \quad \{\text{squaring both sides}\}$$

$$\therefore\ k^2(\cos^2 b + \sin^2 b) = 29 \quad \{\text{adding the two equations}\}$$

$$\therefore\ k^2 = 29 \quad \therefore\ k = \sqrt{29} \quad \{k > 0\}$$

Substituting $k = \sqrt{29}$ into (1) gives

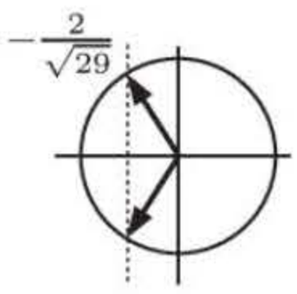
$$\cos b = \tfrac{2}{\sqrt{29}} \quad \text{and} \quad \sin b = \tfrac{5}{\sqrt{29}}$$

$$\therefore\ b = \cos^{-1}\left(\tfrac{2}{\sqrt{29}}\right) \approx 1.19$$

$$\therefore\ 2\cos x - 5\sin x \approx \sqrt{29}\cos(x + 1.19)$$

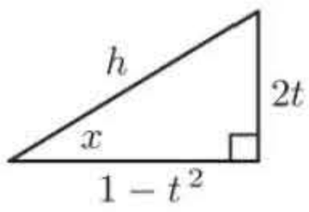
b

$$\begin{aligned}
2\cos x - 5\sin x &= -2 \\
\therefore\ \sqrt{29}\cos(x + 1.19) &\approx -2 \\
\therefore\ \cos(x + 1.19) &\approx -\tfrac{2}{\sqrt{29}} \\
\therefore\ x + 1.19 &\approx 1.951,\ 4.33 \\
\therefore\ x &\approx 0.761,\ 3.14 \leftarrow \text{this solution is exactly } \pi
\end{aligned}$$



c

$$\begin{aligned}
\tan x &= \tan\left(2 \times \tfrac{x}{2}\right) \\
&= \frac{2\tan\left(\frac{x}{2}\right)}{1 - \tan^2\left(\frac{x}{2}\right)} \\
&= \frac{2t}{1 - t^2}
\end{aligned}$$



Now

$$\begin{aligned}
h^2 &= (1 - t^2)^2 + (2t)^2 \\
&= 1 - 2t^2 + t^4 + 4t^2 \\
&= 1 + 2t^2 + t^4 \\
&= (1 + t^2)^2
\end{aligned}$$

$$\therefore\ h = 1 + t^2 \quad \{\text{since } h > 0\}$$

$$\therefore\ \sin x = \frac{2t}{1 + t^2} \quad \text{and} \quad \cos x = \frac{1 - t^2}{1 + t^2}$$

d
$$2\cos x - 5\sin x = -2$$
$$\therefore\ 2\left(\frac{1-t^2}{1+t^2}\right) - 5\left(\frac{2t}{1+t^2}\right) = -2$$
$$\therefore\ \frac{2-2t^2-10t}{1+t^2} = -2$$
$$\therefore\ 2 - \cancel{2t^2} - 10t = -2 - \cancel{2t^2}$$
$$\therefore\ 4 = 10t$$
$$\therefore\ t = \tfrac{2}{5}$$

So $\tan\left(\frac{x}{2}\right) = \frac{2}{5}$
$$\therefore\ \frac{x}{2} \approx 0.3805 \quad \left\{0 \leqslant \frac{x}{2} \leqslant \tfrac{\pi}{2}\right\}$$
$$\therefore\ x \approx 0.761$$

The $x = \pi$ solution has been lost since t is undefined when $x = \pi$.

25 Let $\arctan(5) = \theta$ and $\arctan(\frac{2}{3}) = \phi$

$\therefore\ 5 = \tan\theta$ $\qquad$ $\therefore\ \frac{2}{3} = \tan\phi$

Now $\tan(\theta - \phi) = \dfrac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi}$
$$\therefore\ \tan(\theta-\phi) = \frac{5 - \frac{2}{3}}{1 + 5 \times \frac{2}{3}}$$
$$= \frac{\frac{13}{3}}{\frac{13}{3}}$$
$$= 1$$
$$\therefore\ \theta - \phi = \tfrac{\pi}{4} + k\pi, \quad k \in \mathbb{Z}$$
$$\therefore\ \theta - \phi = \tfrac{\pi}{4} \qquad \{\text{since } 0 < \theta,\ \phi < \tfrac{\pi}{2}\}$$
$$\therefore\ \arctan(5) - \arctan(\tfrac{2}{3}) = \tfrac{\pi}{4}$$

26 **a** Let $\arctan(\frac{1}{5}) = \theta$ and $\arctan(\frac{2}{3}) = \phi$

$\therefore\ \frac{1}{5} = \tan\theta$ $\qquad$ $\therefore\ \frac{2}{3} = \tan\phi$

Now $\tan(\theta + \phi) = \dfrac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$
$$\therefore\ \tan(\theta+\phi) = \frac{\frac{1}{5} + \frac{2}{3}}{1 - \frac{1}{5} \times \frac{2}{3}}$$
$$= \frac{\frac{13}{15}}{\frac{13}{15}}$$
$$= 1$$
$$\therefore\ \theta + \phi = \tfrac{\pi}{4} + k\pi, \quad k \in \mathbb{Z}$$
$$\therefore\ \theta + \phi = \tfrac{\pi}{4} \qquad \{\text{since } 0 < \theta,\ \phi < \tfrac{\pi}{2}\}$$
$$\therefore\ \arctan(\tfrac{1}{5}) + \arctan(\tfrac{2}{3}) = \tfrac{\pi}{4}$$

b Let $\arctan(\frac{4}{3}) = \theta$ and $\arctan(\frac{1}{2}) = \phi$

$\therefore\ \frac{4}{3} = \tan\theta$ $\qquad$ $\therefore\ \frac{1}{2} = \tan\phi$
$$\therefore\ \tan 2\phi = \frac{2(\frac{1}{2})}{1 - (\frac{1}{2})^2} = \frac{1}{\frac{3}{4}} = \tfrac{4}{3}$$

Now $\tan(\theta - 2\phi) = \dfrac{\tan\theta - \tan 2\phi}{1 + \tan\theta\tan 2\phi}$

$= \dfrac{\frac{4}{3} - \frac{4}{3}}{1 + (\frac{4}{3})(\frac{4}{3})}$

$= 0$

$\therefore\ \theta - 2\phi = 0 + k\pi, \quad k \in \mathbb{Z}$

$\therefore\ \theta - 2\phi = 0 \quad \{\text{since } 0 < \theta,\ \phi < \frac{\pi}{2}\}$

$\therefore\ \theta = 2\phi$

$\therefore\ \arctan(\frac{4}{3}) = 2\arctan(\frac{1}{2})$

27 Let $\arctan(\frac{1}{5}) = \theta$ and $\arctan(\frac{1}{239}) = \phi$

$\therefore\ \frac{1}{5} = \tan\theta$ $\qquad$ $\therefore\ \frac{1}{239} = \tan\phi$

$\therefore\ \tan 2\theta = \dfrac{2(\frac{1}{5})}{1 - (\frac{1}{5})^2}$ and $\tan 4\theta = \dfrac{2(\frac{5}{\cancel{12}})_6}{1 - (\frac{5}{12})^2}$

$= \dfrac{\frac{2}{5}}{\frac{24}{25}}$ $\qquad$ $= \dfrac{\frac{5}{6}}{1 - \frac{25}{144}}$

$= \frac{5}{12}$ $\qquad$ $= \dfrac{\frac{5}{6}}{\frac{119}{144}}$

$= \frac{120}{119}$

$\therefore\ \tan(4\theta - \phi) = \dfrac{\tan 4\theta - \tan\phi}{1 + \tan 4\theta\tan\phi}$

$= \dfrac{\frac{120}{119} - \frac{1}{239}}{1 + \frac{120}{119} \times \frac{1}{239}}$

$= \dfrac{\frac{28\,680 - 119}{28\,441}}{\frac{28\,441 + 120}{28\,441}}$

$= \cancel{\frac{28\,561}{28\,441}} \times \cancel{\frac{28\,441}{28\,561}}$

$= 1$

$\therefore\ 4\theta - \phi = \frac{\pi}{4} + k\pi, \quad k \in \mathbb{Z}$

$\therefore\ 4\theta - \phi = \frac{\pi}{4} \quad \{\text{since } 0 < \theta,\ \phi < \frac{\pi}{4}\}$

$\therefore\ 4\arctan(\frac{1}{5}) - \arctan(\frac{1}{239}) = \frac{\pi}{4}$

28 **a** $\sin(A + B) + \sin(A - B) = \sin A\cos B + \cancel{\cos A\sin B} + \sin A\cos B - \cancel{\cos A\sin B}$

$= 2\sin A\cos B$

b $2\sin A\cos B = \sin(A + B) + \sin(A - B)$

$\therefore\ \sin A\cos B = \frac{1}{2}\sin(A + B) + \frac{1}{2}\sin(A - B)$ $\quad$ {dividing both sides by 2}

c **i** $\sin 3\theta\cos\theta$

$= \frac{1}{2}\sin(3\theta + \theta) + \frac{1}{2}\sin(3\theta - \theta)$

$= \frac{1}{2}\sin 4\theta + \frac{1}{2}\sin 2\theta$

ii $\sin 6\alpha\cos\alpha$

$= \frac{1}{2}\sin(6\alpha + \alpha) + \frac{1}{2}\sin(6\alpha - \alpha)$

$= \frac{1}{2}\sin 7\alpha + \frac{1}{2}\sin 5\alpha$

iii $2\sin 5\beta\cos\beta$

$= 2\left[\frac{1}{2}\sin(5\beta + \beta) + \frac{1}{2}\sin(5\beta - \beta)\right]$

$= \sin 6\beta + \sin 4\beta$

iv $4\cos\theta\sin 4\theta$

$= 4\left[\sin 4\theta\cos\theta\right]$

$= 4\left[\frac{1}{2}\sin 5\theta + \frac{1}{2}\sin 3\theta\right]$

$= 2\sin 5\theta + 2\sin 3\theta$

v $6\cos 4\alpha \sin 3\alpha$
$= 6\sin 3\alpha \cos 4\alpha$
$= 6\left[\frac{1}{2}\sin 7\alpha + \frac{1}{2}\sin(-\alpha)\right]$
$= 3\sin 7\alpha + 3\sin(-\alpha)$
$= 3\sin 7\alpha - 3\sin\alpha$

vi $\frac{1}{3}\cos 5A \sin 3A$
$= \frac{1}{3}\sin 3A \cos 5A$
$= \frac{1}{3}\left[\frac{1}{2}\sin 8A + \frac{1}{2}\sin(-2A)\right]$
$= \frac{1}{6}\sin 8A - \frac{1}{6}\sin 2A$

29 **a** $\cos(A+B) + \cos(A-B)$
$= \cos A\cos B - \cancel{\sin A \sin B} + \cos A \cos B + \cancel{\sin A \sin B}$
$= 2\cos A \cos B$

b $2\cos A\cos B = \cos(A+B) + \cos(A-B)$
$\therefore\ \cos A \cos B = \frac{1}{2}\cos(A+B) + \frac{1}{2}\cos(A-B)$ {dividing both sides by 2}

c **i** $\cos 4\theta \cos\theta$
$= \frac{1}{2}\cos 5\theta + \frac{1}{2}\cos 3\theta$

ii $\cos 7\alpha \cos\alpha$
$= \frac{1}{2}\cos 8\alpha + \frac{1}{2}\cos 6\alpha$

iii $2\cos 3\beta \cos\beta$
$= 2\left[\frac{1}{2}\cos 4\beta + \frac{1}{2}\cos 2\beta\right]$
$= \cos 4\beta + \cos 2\beta$

iv $6\cos x \cos 7x$
$= 6\cos 7x \cos x$
$= 6\left[\frac{1}{2}\cos 8x + \frac{1}{2}\cos 6x\right]$
$= 3\cos 8x + 3\cos 6x$

v $3\cos P \cos 4P$
$= 3\cos 4P \cos P$
$= 3\left[\frac{1}{2}\cos 5P + \frac{1}{2}\cos 3P\right]$
$= \frac{3}{2}\cos 5P + \frac{3}{2}\cos 3P$

vi $\frac{1}{4}\cos 4x \cos 2x$
$= \frac{1}{4}\left[\frac{1}{2}\cos 6x + \frac{1}{2}\cos 2x\right]$
$= \frac{1}{8}\cos 6x + \frac{1}{8}\cos 2x$

30 **a** $\cos(A-B) - \cos(A+B)$
$= \cancel{\cos A \cos B} + \sin A \sin B - [\cancel{\cos A \cos B} - \sin A \sin B]$
$= \sin A \sin B + \sin A \sin B$
$= 2\sin A \sin B$

b $2\sin A \sin B = \cos(A-B) - \cos(A+B)$
$\therefore\ \sin A \sin B = \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B)$ {dividing both sides by 2}

c **i** $\sin 3\theta \sin\theta$
$= \frac{1}{2}\cos 2\theta - \frac{1}{2}\cos 4\theta$

ii $\sin 6\alpha \sin\alpha$
$= \frac{1}{2}\cos 5\alpha - \frac{1}{2}\cos 7\alpha$

iii $2\sin 5\beta \sin\beta$
$= 2\left[\frac{1}{2}\cos 4\beta - \frac{1}{2}\cos 6\beta\right]$
$= \cos 4\beta - \cos 6\beta$

iv $4\sin\theta \sin 4\theta$
$= 4\sin 4\theta \sin\theta$
$= 4\left[\frac{1}{2}\cos 3\theta - \frac{1}{2}\cos 5\theta\right]$
$= 2\cos 3\theta - 2\cos 5\theta$

v $10\sin 2A \sin 8A$
$= 10\sin 8A \sin 2A$
$= 10\left[\frac{1}{2}\cos 6A - \frac{1}{2}\cos 10A\right]$
$= 5\cos 6A - 5\cos 10A$

vi $\frac{1}{5}\sin 3M \sin 7M$
$= \frac{1}{5}\sin 7M \sin 3M$
$= \frac{1}{5}\left[\frac{1}{2}\cos 4M - \frac{1}{2}\cos 10M\right]$
$= \frac{1}{10}\cos 4M - \frac{1}{10}\cos 10M$

31 **a** (1) becomes $\sin A \cos A = \frac{1}{2}\sin(A+A) + \frac{1}{2}\sin(A-A)$
$= \frac{1}{2}\sin(2A) + \frac{1}{2}\sin(0)$
$= \frac{1}{2}\sin 2A$

(2) becomes $\cos A \cos A = \frac{1}{2}\cos(A+A) + \frac{1}{2}\cos(A-A)$
$\therefore\ \cos^2 A = \frac{1}{2}\cos(2A) + \frac{1}{2}\cos(0)$
$= \frac{1}{2}\cos 2A + \frac{1}{2}$

(3) becomes $\sin A \sin A = \frac{1}{2}\cos(A-A) - \frac{1}{2}\cos(A+A)$
$\therefore\ \sin^2 A = \frac{1}{2}\cos(0) - \frac{1}{2}\cos(2A)$
$= \frac{1}{2} - \frac{1}{2}\cos 2A$

b **i** $A+B=S$ $\quad\therefore\quad 2A=S+D \quad\therefore\quad A=\frac{S+D}{2}$

$A-B=D$ $\quad$ and $\quad B=S-A=S-\left(\frac{S+D}{2}\right)=\frac{2S}{2}-\left(\frac{S+D}{2}\right)=\frac{2S-S-D}{2}=\frac{S-D}{2}$

ii $\sin A\cos B=\frac{1}{2}\sin(A+B)+\frac{1}{2}\sin(A-B)$ becomes

$\sin\left(\frac{S+D}{2}\right)\cos\left(\frac{S-D}{2}\right)=\frac{1}{2}\sin S+\frac{1}{2}\sin D$

or $\quad \sin S+\sin D=2\sin\left(\frac{S+D}{2}\right)\cos\left(\frac{S-D}{2}\right)$ .... (4)

iii Replacing D by $(-D)$ in (4) gives

$\sin S+\sin(-D)=2\sin\left(\frac{S-D}{2}\right)\cos\left(\frac{S+D}{2}\right)$

or $\quad \sin S-\sin D=2\cos\left(\frac{S+D}{2}\right)\sin\left(\frac{S-D}{2}\right)$

c $\cos A\cos B=\frac{1}{2}\cos(A+B)+\frac{1}{2}\cos(A-B)$ becomes

$\cos\left(\frac{S+D}{2}\right)\cos\left(\frac{S-D}{2}\right)=\frac{1}{2}\cos S+\frac{1}{2}\cos D$

or $\quad \cos S+\cos D=2\cos\left(\frac{S+D}{2}\right)\cos\left(\frac{S-D}{2}\right)$

d $\sin A\sin B=\frac{1}{2}\cos(A-B)-\frac{1}{2}\cos(A+B)$ becomes

$\sin\left(\frac{S+D}{2}\right)\sin\left(\frac{S-D}{2}\right)=\frac{1}{2}\cos D-\frac{1}{2}\cos S$

or $\quad \cos D-\cos S=2\sin\left(\frac{S+D}{2}\right)\sin\left(\frac{S-D}{2}\right)$

or $\quad \cos S-\cos D=-2\sin\left(\frac{S+D}{2}\right)\sin\left(\frac{S-D}{2}\right)$

32 **a** $\sin 5x+\sin x$

$=2\sin\left(\frac{5x+x}{2}\right)\cos\left(\frac{5x-x}{2}\right)$

$=2\sin 3x\cos 2x$

b $\cos 8A+\cos 2A$

$=2\cos\left(\frac{8A+2A}{2}\right)\cos\left(\frac{8A-2A}{2}\right)$

$=2\cos 5A\cos 3A$

c $\cos 3\alpha-\cos\alpha$

$=-2\sin\left(\frac{3\alpha+\alpha}{2}\right)\sin\left(\frac{3\alpha-\alpha}{2}\right)$

$=-2\sin 2\alpha\sin\alpha$

d $\sin 5\theta-\sin 3\theta$

$=2\cos\left(\frac{5\theta+3\theta}{2}\right)\sin\left(\frac{5\theta-3\theta}{2}\right)$

$=2\cos 4\theta\sin\theta$

e $\cos 7\alpha-\cos\alpha$

$=-2\sin\left(\frac{7\alpha+\alpha}{2}\right)\sin\left(\frac{7\alpha-\alpha}{2}\right)$

$=-2\sin 4\alpha\sin 3\alpha$

f $\sin 3\alpha+\sin 7\alpha$

$=\sin 7\alpha+\sin 3\alpha$

$=2\sin\left(\frac{7\alpha+3\alpha}{2}\right)\cos\left(\frac{7\alpha-3\alpha}{2}\right)$

$=2\sin 5\alpha\cos 2\alpha$

g $\cos 2B-\cos 4B$

$=-\left[\cos 4B-\cos 2B\right]$

$=-\left[-2\sin\left(\frac{4B+2B}{2}\right)\sin\left(\frac{4B-2B}{2}\right)\right]$

$=2\sin 3B\sin B$

h $\sin(x+h)-\sin x$

$=2\cos\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right)$

$=2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)$

$=2\cos\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)$

i $\cos(x+h)-\cos x$

$=-2\sin\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right)$

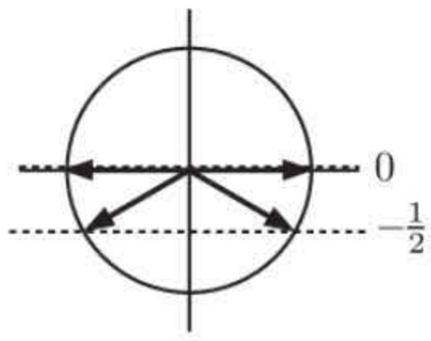
$=-2\sin\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)$

$=-2\sin\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)$

EXERCISE 13F

1 **a**

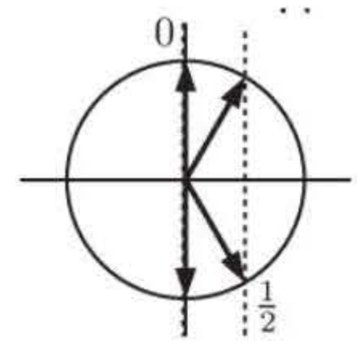
$$2\sin^2 x + \sin x = 0$$
$$\therefore \quad \sin x(2\sin x + 1) = 0$$
$$\therefore \quad \sin x = 0 \text{ or } -\tfrac{1}{2}$$



$$\therefore \quad x = 0,\ \pi,\ \tfrac{7\pi}{6},\ \tfrac{11\pi}{6},\ 2\pi$$

b

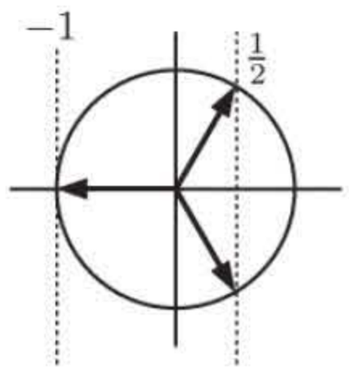
$$2\cos^2 x = \cos x$$
$$\therefore \quad 2\cos^2 x - \cos x = 0$$
$$\therefore \quad \cos x(2\cos x - 1) = 0$$
$$\therefore \quad \cos x = 0 \text{ or } \tfrac{1}{2}$$



$$\therefore \quad x = \tfrac{\pi}{3},\ \tfrac{\pi}{2},\ \tfrac{3\pi}{2},\ \tfrac{5\pi}{3}$$

c

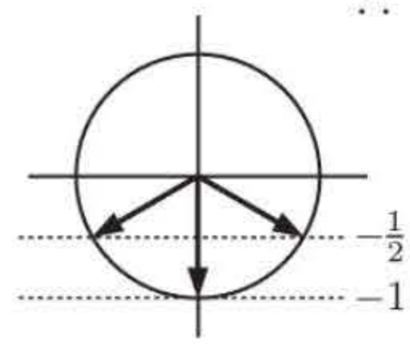
$$2\cos^2 x + \cos x - 1 = 0$$
$$\therefore \quad (2\cos x - 1)(\cos x + 1) = 0$$
$$\therefore \quad \cos x = \tfrac{1}{2} \text{ or } -1$$



$$\therefore \quad x = \tfrac{\pi}{3},\ \pi,\ \tfrac{5\pi}{3}$$

d

$$2\sin^2 x + 3\sin x + 1 = 0$$
$$\therefore \quad (2\sin x + 1)(\sin x + 1) = 0$$
$$\therefore \quad \sin x = -\tfrac{1}{2} \text{ or } -1$$



$$\therefore \quad x = \tfrac{7\pi}{6},\ \tfrac{3\pi}{2},\ \tfrac{11\pi}{6}$$

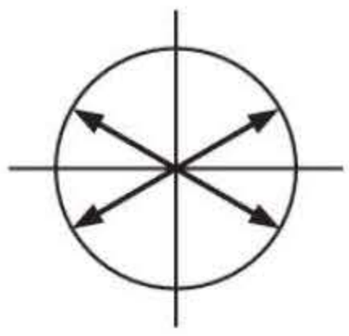
e

$$\sin^2 x = 2 - \cos x$$
$$\therefore \quad 1 - \cos^2 x = 2 - \cos x$$
$$\therefore \quad \cos^2 x - \cos x + 1 = 0$$
$$\text{where} \quad \Delta = (-1)^2 - 4(1)(1)$$
$$= 1 - 4$$
$$= -3$$

$\therefore$ no real solutions exist

f

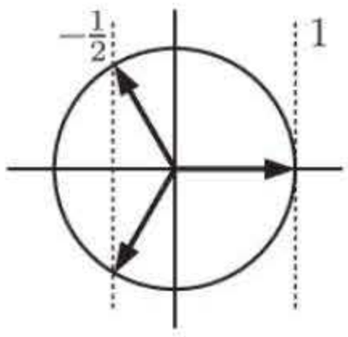
$$3\tan x = \cot x$$
$$\therefore \quad 3\tan x = \frac{1}{\tan x}$$
$$\therefore \quad 3\tan^2 x = 1$$
$$\therefore \quad \tan^2 x = \tfrac{1}{3}$$
$$\therefore \quad \tan x = \pm\tfrac{1}{\sqrt{3}}$$



$$\therefore \quad x = \tfrac{\pi}{6},\ \tfrac{5\pi}{6},\ \tfrac{7\pi}{6},\ \tfrac{11\pi}{6}$$

2 **a**

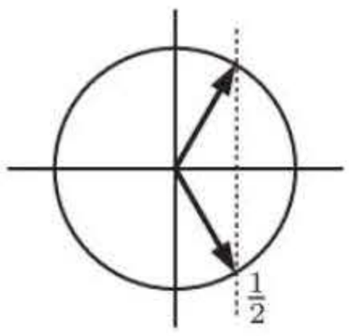
$$\cos 2x - \cos x = 0$$
$$\therefore \quad (2\cos^2 x - 1) - \cos x = 0$$
$$\therefore \quad 2\cos^2 x - \cos x - 1 = 0$$
$$\therefore \quad (2\cos x + 1)(\cos x - 1) = 0$$
$$\therefore \quad \cos x = -\tfrac{1}{2} \text{ or } 1$$



$$\therefore \quad x = 0,\ \tfrac{2\pi}{3},\ \tfrac{4\pi}{3},\ 2\pi$$

b

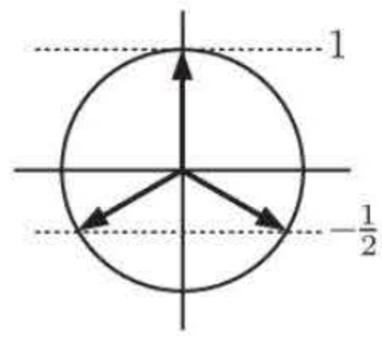
$$\cos 2x + 3\cos x = 1$$
$$\therefore \quad (2\cos^2 x - 1) + 3\cos x = 1$$
$$\therefore \quad 2\cos^2 x + 3\cos x - 2 = 0$$
$$\therefore \quad (2\cos x - 1)(\cos x + 2) = 0$$
$$\therefore \quad \cos x = \tfrac{1}{2}$$
$$\{-1 \leqslant \cos x \leqslant 1\}$$



$$\therefore \quad x = \tfrac{\pi}{3},\ \tfrac{5\pi}{3}$$

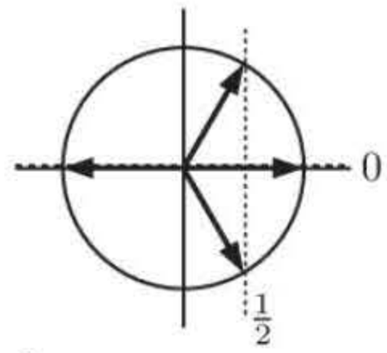
c

$$\cos 2x + \sin x = 0$$
$$\therefore\ (1 - 2\sin^2 x) + \sin x = 0$$
$$\therefore\ -2\sin^2 x + \sin x + 1 = 0$$
$$\therefore\ 2\sin^2 x - \sin x - 1 = 0$$
$$\therefore\ (2\sin x + 1)(\sin x - 1) = 0$$
$$\therefore\ \sin x = -\tfrac{1}{2} \text{ or } 1$$
$$\therefore\ x = \tfrac{\pi}{2}, \tfrac{7\pi}{6}, \tfrac{11\pi}{6}$$



d

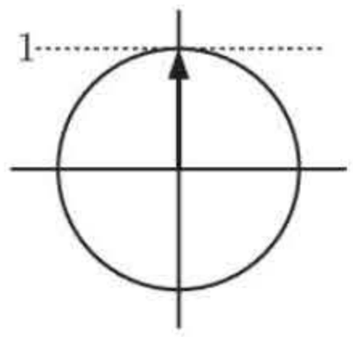
$$\sin 4x = \sin 2x$$
$$\therefore\ 2\sin 2x\cos 2x = \sin 2x$$
$$\therefore\ 2\sin 2x\cos 2x - \sin 2x = 0$$
$$\therefore\ \sin 2x(2\cos 2x - 1) = 0$$
$$\therefore\ \sin 2x = 0 \quad \text{or} \quad \cos 2x = \tfrac{1}{2}$$
$$\therefore\ 2x = 0, \tfrac{\pi}{3}, \pi, \tfrac{5\pi}{3}, 2\pi, \tfrac{7\pi}{3}, 3\pi, \tfrac{11\pi}{3}, 4\pi \quad \{0 \leqslant 2x \leqslant 4\pi\}$$
$$\therefore\ x = 0, \tfrac{\pi}{6}, \tfrac{\pi}{2}, \tfrac{5\pi}{6}, \pi, \tfrac{7\pi}{6}, \tfrac{3\pi}{2}, \tfrac{11\pi}{6}, 2\pi$$



e $\sin x + \cos x = \sqrt{2}$

Squaring both sides we get:

$$\sin^2 x + 2\sin x\cos x + \cos^2 x = 2$$
$$\therefore\ \sin 2x + 1 = 2$$
$$\therefore\ \sin 2x = 1$$
$$\therefore\ 2x = \tfrac{\pi}{2}, \tfrac{5\pi}{2} \quad \{0 \leqslant 2x \leqslant 4\pi\}$$
$$\therefore\ x = \tfrac{\pi}{4}, \tfrac{5\pi}{4}$$

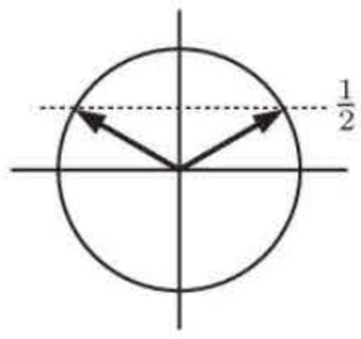


Since we squared the original equation, we must check our answers.

$$\sin\tfrac{\pi}{4} + \cos\tfrac{\pi}{4} = \tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}} = \tfrac{2}{\sqrt{2}} = \sqrt{2} \quad ✓$$
$$\sin\tfrac{5\pi}{4} + \cos\tfrac{5\pi}{4} = -\tfrac{1}{\sqrt{2}} + (-\tfrac{1}{\sqrt{2}}) = -\tfrac{2}{\sqrt{2}} = -\sqrt{2} \quad ✗$$
$$\therefore\ x = \tfrac{\pi}{4} \quad \text{is the only solution}$$

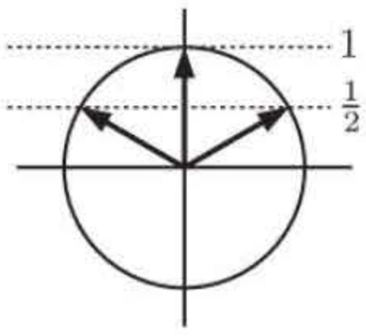
f

$$2\cos^2 x = 3\sin x$$
$$\therefore\ 2(1 - \sin^2 x) = 3\sin x$$
$$\therefore\ 2\sin^2 x + 3\sin x - 2 = 0$$
$$\therefore\ (2\sin x - 1)(\sin x + 2) = 0$$
$$\therefore\ \sin x = \tfrac{1}{2} \quad \{-1 \leqslant \sin x \leqslant 1\}$$
$$\therefore\ x = \tfrac{\pi}{6}, \tfrac{5\pi}{6}$$



3 **a**

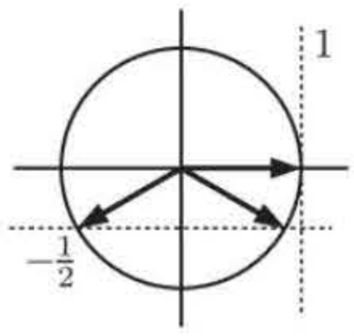
$$2\sin x + \csc x = 3$$
$$\therefore\ 2\sin^2 x + 1 = 3\sin x$$
$$\therefore\ 2\sin^2 x - 3\sin x + 1 = 0$$
$$\therefore\ (2\sin x - 1)(\sin x - 1) = 0$$
$$\therefore\ \sin x = \tfrac{1}{2} \text{ or } 1$$



$$\therefore\ x = \tfrac{\pi}{6}, \tfrac{\pi}{2}, \tfrac{5\pi}{6}$$

b

$$\sin 2x + \cos x - 2\sin x - 1 = 0$$
$$\therefore\ 2\sin x\cos x + \cos x - 2\sin x - 1 = 0$$
$$\therefore\ (2\sin x + 1)(\cos x - 1) = 0$$
$$\therefore\ \sin x = -\tfrac{1}{2} \quad \text{or} \quad \cos x = 1$$



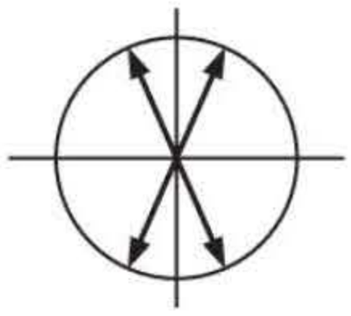
$$\therefore\ x = -\tfrac{5\pi}{6}, -\tfrac{\pi}{6}, 0$$

c $\tan^4 x - 2\tan^2 x - 3 = 0$

$\therefore\ (\tan^2 x - 3)(\tan^2 x + 1) = 0$

$\therefore\ \tan^2 x = 3$ or -1

$\therefore\ \tan^2 x = \pm\sqrt{3}$



$\therefore\ x = -\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}$

4 **a** $2\cos^2 x = \sin x$

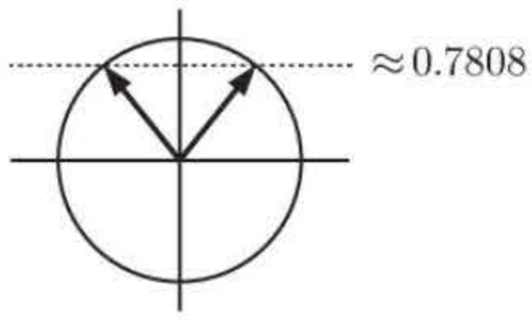
$\therefore\ 2(1 - \sin^2 x) - \sin x = 0$

$\therefore\ 2 - 2\sin^2 x - \sin x = 0$

$\therefore\ 2\sin^2 x + \sin x - 2 = 0$

$\therefore\ \sin x = \dfrac{-1 \pm \sqrt{1 - 4(2)(-2)}}{2(2)}$

$\therefore\ \sin x = \dfrac{-1 \pm \sqrt{17}}{4} \approx 0.7808$ or -1.281



$\therefore\ \sin x \approx 0.7808$ as $-1 \leqslant \sin x \leqslant 1$

$\therefore\ x \approx \arcsin(0.7808)$ or $\pi - \arcsin(0.7808)$

$\therefore\ x \approx 0.896$ or 2.25

b $\cos 2x + 5\sin x = 0$

$\therefore\ 1 - 2\sin^2 x + 5\sin x = 0$

$\therefore\ 2\sin^2 x - 5\sin x - 1 = 0$

$\therefore\ \sin x = \dfrac{5 \pm \sqrt{25 - 4(2)(-1)}}{2(2)}$

$= \dfrac{5 \pm \sqrt{33}}{4} \approx 2.6861$ or -0.1861

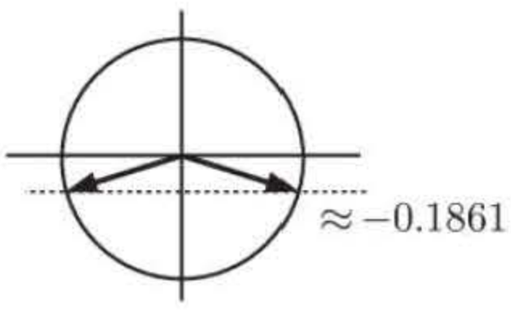
$\therefore\ \sin x \approx -0.1861$ $\{$as $-1 \leqslant \sin x \leqslant 1\}$

Now $\arcsin(-0.1861) \approx -0.1872$

$\therefore\ x \approx \pi + 0.1872$

or $2\pi - 0.1872$

$\therefore\ x \approx 3.33$ or 6.10



c $2\tan^2 x + 3\sec^2 x = 7$

$\therefore\ 2\tan^2 x + 3(\tan^2 x + 1) = 7$

$\therefore\ 5\tan^2 x + 3 = 7$

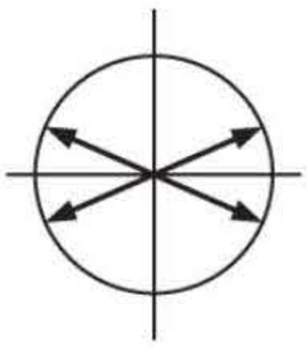
$\therefore\ 5\tan^2 x = 4$

$\therefore\ \tan^2 x = \frac{4}{5}$

$\therefore\ \tan x = \pm\frac{2}{\sqrt{5}}$

$\therefore\ x \approx 0.730, \pi - 0.730,$
$\pi + 0.730,$
$2\pi - 0.730$

$\therefore\ x \approx 0.730, 2.41,$
$3.87, 5.55$



EXERCISE 13G

1 **a** $1 + \sin x + \sin^2 x + \sin^3 x + ... + \sin^{n-1} x$ is a geometric series with

$u_1 = 1, \quad r = \sin x$

$\therefore\ \text{sum} = \dfrac{u_1(1 - r^n)}{1 - r}$

$= \dfrac{1(1 - \sin^n x)}{1 - \sin x}$

$= \dfrac{1 - \sin^n x}{1 - \sin x}$

b $S = \dfrac{u_1}{1 - r} = \dfrac{1}{1 - \sin x}$

for $-1 < \sin x < 1$,
not convergent for $\sin x = \pm 1$

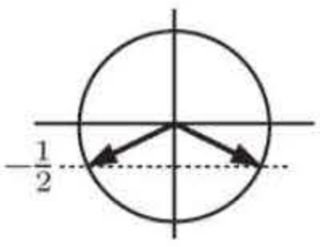
c If $S = \frac{2}{3}$, $\dfrac{1}{1 - \sin x} = \frac{2}{3}$

$\therefore\ 3 = 2 - 2\sin x$

$\therefore\ 2\sin x = -1$

$\therefore\ \sin x = -\frac{1}{2}$

$\therefore\ x = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$



2 **a** **i** $2\sin x(\cos x + \cos 3x)$

$= 2\sin x\cos x + 2\sin x\cos 3x$

$= \sin 2x + \sin 4x + \sin(-2x)$

$\{2\sin A\cos B = \sin(A + B) + \sin(A - B)\}$

$= \cancel{\sin 2x} + \sin 4x - \cancel{\sin 2x}$

$= \sin 4x$

ii $2\sin x(\cos x + \cos 3x + \cos 5x)$

$= 2\sin x(\cos x + \cos 3x) + 2\sin x\cos 5x$

$= \sin 4x + \sin 6x + \sin(-4x)$ $\{$from **i**$\}$

$= \cancel{\sin 4x} + \sin 6x - \cancel{\sin 4x}$

$= \sin 6x$

b **i** $2\sin x(\cos x + \cos 3x + \cos 5x + \cos 7x) = \sin 8x$

ii $2\sin x(\cos x + \cos 3x + \cos 5x + ... + \cos 19x) = \sin 20x$

$$\therefore \quad \cos x + \cos 3x + \cos 5x + ... + \cos 19x = \frac{\sin 20x}{2\sin x}$$

iii In general, $\cos x + \cos 3x + \cos 5x + ... + \cos(2n-1)x = \dfrac{\sin 2nx}{2\sin x}$

c P_n is "$\cos\theta + \cos 3\theta + \cos 5\theta + ... + \cos(2n-1)\theta = \dfrac{\sin 2n\theta}{2\sin\theta}$", $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= \cos\theta$ and RHS $= \dfrac{\sin 2\theta}{2\sin\theta} = \dfrac{2\sin\theta\cos\theta}{2\sin\theta} = \cos\theta$ $\therefore$ P_1 is true.

(2) If P_k is true, then

$$\cos\theta + \cos 3\theta + \cos 5\theta + ... + \cos(2k-1)\theta = \frac{\sin 2k\theta}{2\sin\theta}$$

$$\begin{aligned}
\therefore \quad & \cos\theta + \cos 3\theta + \cos 5\theta + ... + \cos(2k-1)\theta + \cos(2k+1)\theta \\
&= \frac{\sin 2k\theta}{2\sin\theta} + \cos(2k+1)\theta \\
&= \frac{\sin 2k\theta + 2\sin\theta\cos(2k+1)\theta}{2\sin\theta} \\
&= \frac{\sin 2k\theta + \sin[\theta + (2k+1)\theta] + \sin[\theta - (2k+1)\theta]}{2\sin\theta} \\
&= \frac{\sin 2k\theta + \sin(\theta + 2k\theta + \theta) + \sin(\theta - 2k\theta - \theta)}{2\sin\theta} \\
&= \frac{\sin 2k\theta + \sin(2k\theta + 2\theta) + \sin(-2k\theta)}{2\sin\theta} \\
&= \frac{\cancel{\sin 2k\theta} + \sin 2(k+1)\theta - \cancel{\sin 2k\theta}}{2\sin\theta} \\
&= \frac{\sin 2(k+1)\theta}{2\sin\theta}
\end{aligned}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true.

$\therefore$ P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

3 **a** **i**

$$\begin{aligned}
& \sin x\cos x\cos 2x \\
&= \tfrac{1}{2}(2\sin x\cos x)\cos 2x \\
&= \tfrac{1}{2}\sin 2x\cos 2x \\
&= \tfrac{1}{4}(2\sin 2x\cos 2x) \\
&= \tfrac{1}{4}\sin 4x \quad .... (1) \\
&= \frac{\sin(2^2x)}{2^2}
\end{aligned}$$

ii

$$\begin{aligned}
& (\sin x\cos x\cos 2x)\cos 4x \\
&= \tfrac{1}{4}\sin 4x\cos 4x \quad \{\text{from (1)}\} \\
&= \tfrac{1}{8}(2\sin 4x\cos 4x) \\
&= \frac{\sin 8x}{8} \\
&= \frac{\sin(2^3x)}{2^3}
\end{aligned}$$

b **i** $\dfrac{\sin(2^4x)}{2^4}$

ii $\dfrac{\sin(2^6x)}{2^6}$

c $\sin x\cos x\cos 2x\cos 4x....\cos(2^nx) = \dfrac{\sin(2^{n+1}x)}{2^{n+1}}$

P_n is "$\sin x\cos x\cos 2x\cos 4x....\cos(2^nx) = \dfrac{\sin(2^{n+1}x)}{2^{n+1}}$", $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= \sin x\cos x\cos 2x = \dfrac{\sin(2^2x)}{2^2}$ {from part **a i**}

and RHS $= \dfrac{\sin(2^{1+1}x)}{2^{1+1}} = \dfrac{\sin(2^2x)}{2^2}$

$\therefore$ P_1 is true.

(2) If P_k is true, then $\sin x \cos x \cos 2x \cos 4x .... \cos(2^k x) = \dfrac{\sin(2^{k+1}x)}{2^{k+1}}$

$$\therefore \quad \sin x \cos x \cos 2x \cos 4x .... \cos(2^k x)\cos(2^{k+1}x) = \frac{\sin(2^{k+1}x)}{2^{k+1}} \cos(2^{k+1}x)$$
$$= \tfrac{1}{2}\,\frac{2\sin(2^{k+1}x)\cos(2^{k+1}x)}{2^{k+1}}$$
$$= \frac{\sin 2(2^{k+1}x)}{2 \times 2^{k+1}}$$
$$= \frac{\sin(2^{k+2}x)}{2^{k+2}}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true.

$\therefore$ P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

4 **a** P_n is "$\sin\theta + \sin 3\theta + \sin 5\theta + ... + \sin(2n-1)\theta = \dfrac{1-\cos 2n\theta}{2\sin\theta}$", $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= \sin\theta$ and RHS $= \dfrac{1-\cos 2\theta}{2\sin\theta} = \dfrac{1-(1-2\sin^2\theta)}{2\sin\theta} = \dfrac{2\sin^2\theta}{2\sin\theta} = \sin\theta$

$\therefore$ P_1 is true.

(2) If P_k is true, then

$$\sin\theta + \sin 3\theta + \sin 5\theta + ... + \sin(2k-1)\theta = \frac{1-\cos 2k\theta}{2\sin\theta}$$

$$\therefore \quad \sin\theta + \sin 3\theta + \sin 5\theta + ... + \sin(2k-1)\theta + \sin(2k+1)\theta$$
$$= \frac{1-\cos 2k\theta}{2\sin\theta} + \sin(2k+1)\theta$$
$$= \frac{1-\cos 2k\theta + 2\sin(2k+1)\theta\sin\theta}{2\sin\theta}$$
$$= \frac{1-\cos 2k\theta + \cos[(2k+1)\theta-\theta] - \cos[(2k+1)\theta+\theta]}{2\sin\theta}$$
$$= \frac{1-\cancel{\cos 2k\theta} + \cancel{\cos 2k\theta} - \cos[(2k+2)\theta]}{2\sin\theta}$$
$$= \frac{1-\cos 2(k+1)\theta}{2\sin\theta}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true.

$\therefore$ P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

b Thus $\sin\frac{\pi}{7} + \sin\frac{3\pi}{7} + \sin\frac{5\pi}{7} + ... + \sin\frac{13\pi}{7}$ has $2n-1 = 13$ and $\theta = \frac{\pi}{7}$

$\therefore$ $n = 7$ and $\theta = \frac{\pi}{7}$

$\therefore$ the sum is $\dfrac{1-\cos(2\times 7\times\frac{\pi}{7})}{2\sin\frac{\pi}{7}} = \dfrac{1-\cos 2\pi}{2\sin\frac{\pi}{7}} = \dfrac{1-1}{2\sin\frac{\pi}{7}} = 0$

5 P_n is "$\cos x \times \cos 2x \times \cos 4x \times \cos 8x \times .... \times \cos(2^{n-1}x) = \dfrac{\sin(2^n x)}{2^n \times \sin x}$", $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= \cos x$, RHS $= \dfrac{\sin 2x}{2\sin x} = \dfrac{2\sin x\cos x}{2\sin x} = \cos x$ $\therefore$ P_1 is true.

(2) If P_k is true, then $\cos x \times \cos 2x \times \cos 4x \times .... \times \cos(2^{k-1}x) = \dfrac{\sin(2^k x)}{2^k \sin x}$

$\therefore \quad \cos x \times \cos 2x \times \cos 4x \times .... \times \cos(2^{k-1}x) \times \cos(2^k x)$

$= \dfrac{\sin(2^k x)}{2^k \sin x} \times \cos(2^k x)$

$= \dfrac{2\sin(2^k x)\cos(2^k x)}{2 \times 2^k \sin x}$

$= \dfrac{\sin(2 \times 2^k x)}{2^{k+1}\sin x} \qquad \{2\sin\theta\cos\theta = \sin 2\theta\}$

$= \dfrac{\sin(2^{k+1}x)}{2^{k+1}\sin x}$

Thus P_{k+1} is true whenever P_k is true and P_1 is true.

$\therefore \quad P_n$ is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

6 P_n is "$\cos^2\theta + \cos^2 2\theta + \cos^2 3\theta + ... + \cos^2(n\theta) = \frac{1}{2}\left[n + \dfrac{\cos[(n+1)\theta]\sin n\theta}{\sin\theta}\right]$", $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= \cos^2\theta$, RHS $= \frac{1}{2}\left[1 + \dfrac{\cos 2\theta \cancel{\sin\theta}}{\cancel{\sin\theta}}\right]$

$= \frac{1}{2} + \frac{1}{2}(2\cos^2\theta - 1)$

$= \cos^2\theta \qquad \therefore \quad P_1$ is true.

(2) If P_k is true then $\cos^2\theta + \cos^2 2\theta + \cos^2 3\theta + ... + \cos^2(k\theta) = \frac{1}{2}\left[k + \dfrac{\cos[(k+1)\theta]\sin k\theta}{\sin\theta}\right]$

$\therefore \quad \cos^2\theta + \cos^2 2\theta + \cos^2 3\theta + ... + \cos^2(k\theta) + \cos^2[(k+1)\theta]$

$= \frac{1}{2}\left[k + \dfrac{\cos[(k+1)\theta]\sin k\theta}{\sin\theta}\right] + \cos^2[(k+1)\theta]$

$= \frac{1}{2}\left[k + \dfrac{\cos[(k+1)\theta]\sin k\theta}{\sin\theta}\right] + \frac{1}{2} + \frac{1}{2}\cos[2(k+1)\theta] \qquad \{\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta\}$

$= \frac{1}{2}(k+1) + \dfrac{\cos[(k+1)\theta]\sin k\theta + \cos[2(k+1)\theta]\sin\theta}{2\sin\theta}$

$= \frac{1}{2}(k+1) +$

$\dfrac{\frac{1}{2}\sin[k\theta + (k+1)\theta] + \frac{1}{2}\sin[k\theta - (k+1)\theta] + \frac{1}{2}\sin[\theta + 2(k+1)\theta] + \frac{1}{2}\sin[\theta - 2(k+1)\theta]}{2\sin\theta}$

{products to sums formula}

$= \frac{1}{2}(k+1) + \dfrac{\frac{1}{2}\sin[(2k+1)\theta] + \frac{1}{2}\sin(-\theta) + \frac{1}{2}\sin[(2k+3)\theta] + \frac{1}{2}\sin[(-2k-1)\theta]}{2\sin\theta}$

$= \frac{1}{2}(k+1) + \dfrac{\cancel{\frac{1}{2}\sin[(2k+1)\theta]} - \frac{1}{2}\sin\theta + \frac{1}{2}\sin[(2k+3)\theta] - \cancel{\frac{1}{2}\sin[(2k+1)\theta]}}{2\sin\theta}$

$= \frac{1}{2}(k+1) + \dfrac{\frac{1}{2}\sin[(2k+3)\theta] - \frac{1}{2}\sin\theta}{2\sin\theta}$

$= \frac{1}{2}(k+1) + \dfrac{\cos\left[\frac{(2k+3)\theta+\theta}{2}\right]\sin\left[\frac{(2k+3)\theta-\theta}{2}\right]}{2\sin\theta} \qquad$ {factor formula}

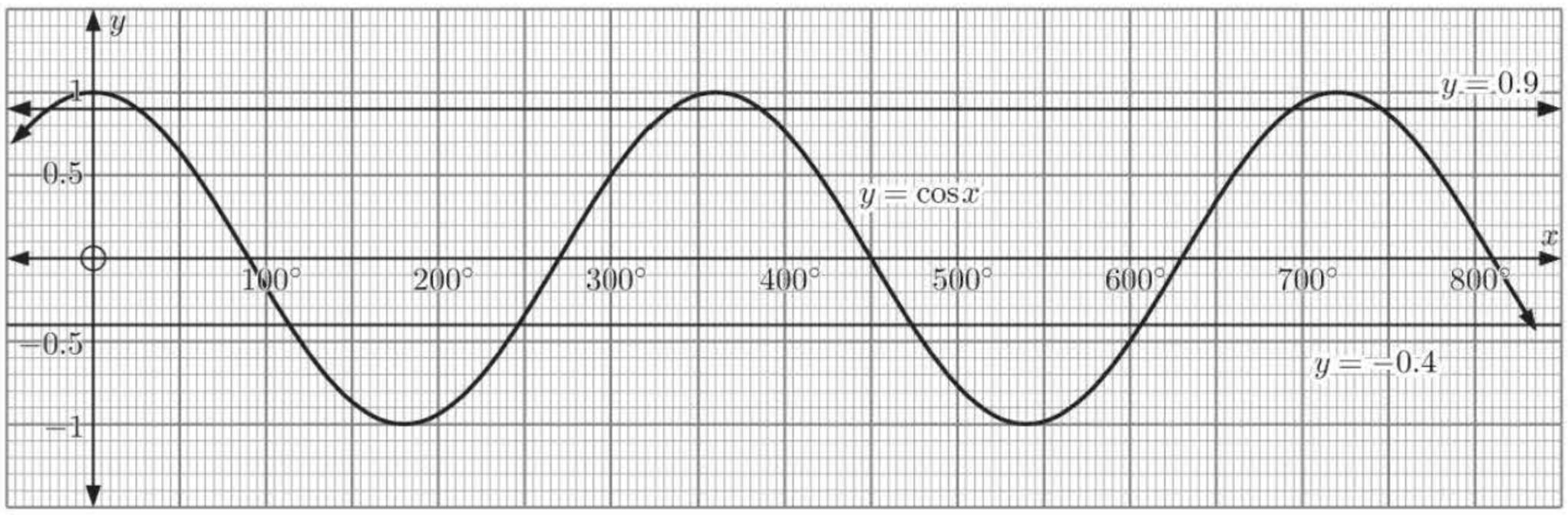
$= \frac{1}{2}(k+1) + \dfrac{\cos[(k+2)\theta]\sin[(k+1)\theta]}{2\sin\theta} = \frac{1}{2}\left[(k+1) + \dfrac{\cos[(k+2)\theta]\sin[(k+1)\theta]}{\sin\theta}\right]$

Thus P_{k+1} is true whenever P_k is true and P_1 is true.

$\therefore \quad P_n$ is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

REVIEW SET 13A

1

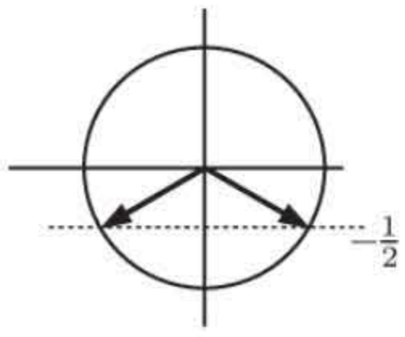


a When $\cos x = -0.4$, $0° \leqslant x \leqslant 800°$,
$x \approx 115°, 245°, 475°, 605°$

b When $\cos x = 0.9$, $0° \leqslant x \leqslant 600°$,
$x \approx 25°, 335°, 385°$

2 **a** $2\sin x = -1$, $0 \leqslant x \leqslant 4\pi$

$\therefore \quad \sin x = -\frac{1}{2}$

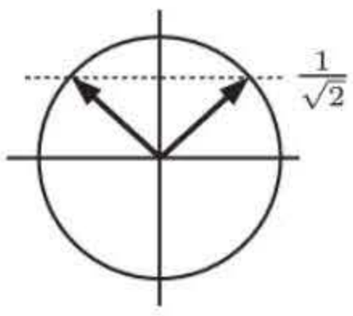


The points on the unit circle with sine $-\frac{1}{2}$ correspond to angles $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$.

$\therefore \quad x = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$

b $\sqrt{2}\sin x - 1 = 0$, $-2\pi \leqslant x \leqslant 2\pi$

$\therefore \quad \sin x = \frac{1}{\sqrt{2}}$

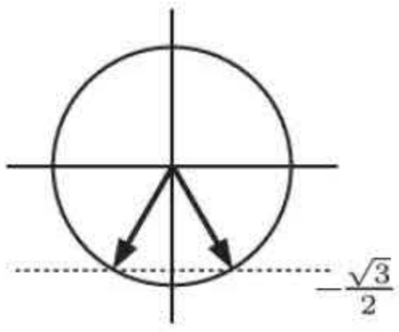


The points on the unit circle with sine $\frac{1}{\sqrt{2}}$ correspond to angles $\frac{\pi}{4}$ and $\frac{3\pi}{4}$.

$\therefore \quad x = -\frac{7\pi}{4}, -\frac{5\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$

3 **a** $2\sin 3x + \sqrt{3} = 0$, $0 \leqslant x \leqslant 2\pi$

$\therefore \quad \sin 3x = -\frac{\sqrt{3}}{2}$



The points on the unit circle with sine $-\frac{\sqrt{3}}{2}$ correspond to angles $\frac{4\pi}{3}$ and $\frac{5\pi}{3}$.

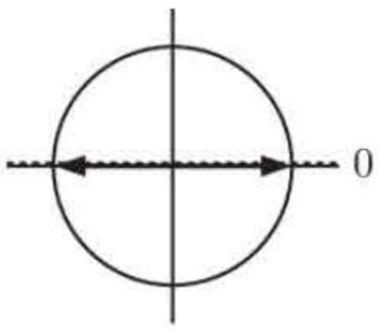
$0 \leqslant x \leqslant 2\pi \quad \therefore \quad 0 \leqslant 3x \leqslant 6\pi$

So $\quad 3x = \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{10\pi}{3}, \frac{11\pi}{3}, \frac{16\pi}{3}, \frac{17\pi}{3}$

So, the x-intercepts are
$\frac{4\pi}{9}, \frac{5\pi}{9}, \frac{10\pi}{9}, \frac{11\pi}{9}, \frac{16\pi}{9}, \frac{17\pi}{9}$.

b $\sqrt{2}\sin(x + \frac{\pi}{4}) = 0$, $0 \leqslant x \leqslant 3\pi$

$\therefore \quad \sin(x + \frac{\pi}{4}) = 0$



The points on the unit circle with sine 0 correspond to angles 0 and π.

$0 \leqslant x \leqslant 3\pi \quad \therefore \quad \frac{\pi}{4} \leqslant x + \frac{\pi}{4} \leqslant \frac{13\pi}{4}$

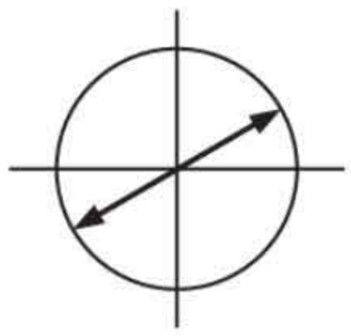
So $\quad x + \frac{\pi}{4} = \pi, 2\pi, 3\pi$

So, the x-intercepts are $\frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}$.

4 **a** $\cot x = \sqrt{3}, \quad x \in [0, 2\pi]$

$\therefore \quad \dfrac{1}{\tan x} = \sqrt{3}$

$\therefore \quad \tan x = \frac{1}{\sqrt{3}}$



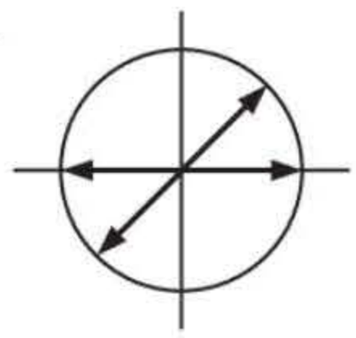
$\therefore \quad x = \frac{\pi}{6}, \frac{7\pi}{6}$

b $\sec^2 x = \tan x + 1, \quad x \in [0, 2\pi]$

$\therefore \quad \tan^2 x \cancel{+1} = \tan x \cancel{+1}$

$\therefore \quad \tan x(\tan x - 1) = 0$

$\therefore \quad \tan x = 0 \text{ or } 1$



$\therefore \quad x = 0, \frac{\pi}{4}, \pi, \frac{5\pi}{4}, 2\pi$

5 **a** $\cos\left(\frac{3\pi}{2} - \theta\right)$

$= \cos\left(\frac{3\pi}{2}\right)\cos\theta + \sin\left(\frac{3\pi}{2}\right)\sin\theta$

$= (0)\cos\theta + (-1)\sin\theta$

$= -\sin\theta$

b $\sin\left(\theta + \frac{\pi}{2}\right)$

$= \sin\theta\cos\left(\frac{\pi}{2}\right) + \cos\theta\sin\left(\frac{\pi}{2}\right)$

$= \sin\theta(0) + \cos\theta(1)$

$= \cos\theta$

6 **a** $\dfrac{1 - \cos^2\theta}{1 + \cos\theta}$

$= \dfrac{\cancel{(1 + \cos\theta)}(1 - \cos\theta)}{\cancel{1 + \cos\theta}\ 1}$

$= 1 - \cos\theta$

b $\dfrac{\sin\alpha - \cos\alpha}{\sin^2\alpha - \cos^2\alpha}$

$= \dfrac{\cancel{\sin\alpha - \cos\alpha}\ 1}{(\sin\alpha + \cos\alpha)\cancel{(\sin\alpha - \cos\alpha)}}$

$= \dfrac{1}{\sin\alpha + \cos\alpha}$

c $\dfrac{4\sin^2\alpha - 4}{8\cos\alpha}$

$= \dfrac{-4(1 - \sin^2\alpha)}{8\cos\alpha}$

$= \dfrac{-4\cos^2\alpha}{8\cos\alpha}$

$= \dfrac{-\cos\alpha}{2}$

7 **a**

$\cos^2\alpha + \sin^2\alpha = 1$

$\therefore \quad \cos^2\alpha + \frac{9}{16} = 1$

$\therefore \quad \cos^2\alpha = \frac{7}{16}$

$\therefore \quad \cos\alpha = \pm\frac{\sqrt{7}}{4}$

But in Q3, $\quad \cos\alpha < 0$

$\therefore \quad \cos\alpha = -\frac{\sqrt{7}}{4}$

b $\sin 2\alpha = 2\sin\alpha\cos\alpha$

$= 2(-\frac{3}{4})(-\frac{\sqrt{7}}{4})$

$= \frac{3\sqrt{7}}{8}$

c $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

$= \left(-\frac{\sqrt{7}}{4}\right)^2 - \left(-\frac{3}{4}\right)^2$

$= \frac{7}{16} - \frac{9}{16}$

$= -\frac{2}{16}$

$= -\frac{1}{8}$

d $\tan 2\alpha = \dfrac{\sin 2\alpha}{\cos 2\alpha}$

$= \dfrac{\frac{3\sqrt{7}}{8}}{-\frac{1}{8}}$

$= -3\sqrt{7}$

8 $\dfrac{\sin 2\alpha - \sin\alpha}{\cos 2\alpha - \cos\alpha + 1} = \dfrac{2\sin\alpha\cos\alpha - \sin\alpha}{2\cos^2\alpha - 1 - \cos\alpha + 1}$

$= \dfrac{\sin\alpha\cancel{(2\cos\alpha - 1)}}{\cos\alpha\cancel{(2\cos\alpha - 1)}}$

$= \dfrac{\sin\alpha}{\cos\alpha}$

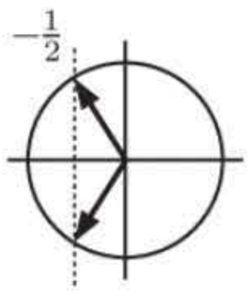
$= \tan\alpha$

9 **a**
$$\begin{aligned}&\cos(165^\circ)\\ &= \cos(120^\circ + 45^\circ)\\ &= \cos 120^\circ \cos 45^\circ - \sin 120^\circ \sin 45^\circ\\ &= \left(-\tfrac{1}{2}\right)\left(\tfrac{1}{\sqrt{2}}\right) - \left(\tfrac{\sqrt{3}}{2}\right)\left(\tfrac{1}{\sqrt{2}}\right)\\ &= \frac{-1-\sqrt{3}}{2\sqrt{2}}\end{aligned}$$

b
$$\begin{aligned}\tan\left(\tfrac{\pi}{12}\right) &= \tan\left(\tfrac{3\pi}{12} - \tfrac{2\pi}{12}\right)\\ &= \frac{\tan\left(\frac{\pi}{4}\right) - \tan\left(\frac{\pi}{6}\right)}{1 + \tan\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{6}\right)}\\ &= \frac{1 - \frac{1}{\sqrt{3}}}{1 + (1)\left(\frac{1}{\sqrt{3}}\right)} \times \left(\frac{\sqrt{3}}{\sqrt{3}}\right)\\ &= \frac{\sqrt{3}-1}{\sqrt{3}+1} \times \left(\frac{\sqrt{3}-1}{\sqrt{3}-1}\right)\\ &= \frac{3 - 2\sqrt{3} + 1}{3-1}\\ &= \frac{4 - 2\sqrt{3}}{2}\\ &= 2 - \sqrt{3}\end{aligned}$$

10 **a** $2\cos 2x + 1 = 0$

$\therefore\ \cos 2x = -\frac{1}{2}$

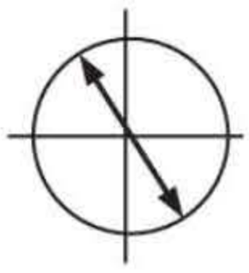


$\therefore\ 2x = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}$

$\{0 \leqslant 2x \leqslant 4\pi\}$

$\therefore\ x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

b $\sin 2x = -\sqrt{3}\cos 2x$



$\therefore\ \dfrac{\sin 2x}{\cos 2x} = -\sqrt{3}$

$\therefore\ \tan 2x = -\sqrt{3}$

$\therefore\ 2x = \frac{2\pi}{3}, \frac{5\pi}{3}, \frac{8\pi}{3}, \frac{11\pi}{3}$

$\{0 \leqslant 2x \leqslant 4\pi\}$

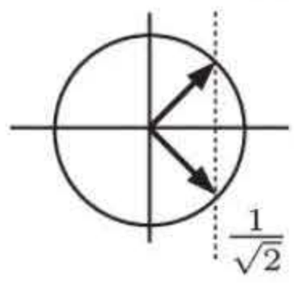
$\therefore\ x = \frac{\pi}{3}, \frac{5\pi}{6}, \frac{4\pi}{3}, \frac{11\pi}{6}$

11 **a**
$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta\\ &= 2\left(\frac{b}{c}\right)\left(\frac{a}{c}\right)\\ &= \frac{2ab}{c^2}\end{aligned}$$

b
$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta\\ &= \left(\frac{a}{c}\right)^2 - \left(\frac{b}{c}\right)^2\\ &= \frac{a^2 - b^2}{c^2}\end{aligned}$$

12 **a** $\sqrt{2}\cos\left(x + \frac{\pi}{4}\right) - 1 = 0, \quad x \in [0, 4\pi]$

$\therefore\ \cos\left(x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$



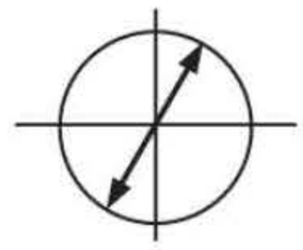
$\therefore\ x + \frac{\pi}{4} = \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}, \frac{17\pi}{4}$

$\{\frac{\pi}{4} \leqslant x + \frac{\pi}{4} \leqslant \frac{17\pi}{4}\}$

$\therefore\ x = 0, \frac{3\pi}{2}, 2\pi, \frac{7\pi}{2}, 4\pi$

b $\tan 2x - \sqrt{3} = 0, \quad x \in [0, 2\pi]$

$\therefore\ \tan 2x = \sqrt{3}$



$\therefore\ 2x = \frac{\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{3}, \frac{10\pi}{3}$

$\{0 \leqslant 2x \leqslant 4\pi\}$

$\therefore\ x = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$

13 **a** By the sine rule, $\dfrac{\sin 2\alpha}{5} = \dfrac{\sin\alpha}{3}$

$\therefore\ \dfrac{2\sin\alpha\cos\alpha}{\sin\alpha} = \frac{5}{3}$

$\therefore\ 2\cos\alpha = \frac{5}{3} \quad \{\sin\alpha \neq 0\}$

$\therefore\ \cos\alpha = \frac{5}{6}$

b Using the cosine rule

$3^2 = x^2 + 5^2 - 2 \times x \times 5 \times \cos\alpha$

$\therefore\ 9 = x^2 + 25 - 10x\left(\frac{5}{6}\right)$

$\therefore\ x^2 - \frac{25}{3}x + 16 = 0$

$\therefore\ 3x^2 - 25x + 48 = 0$

c $(3x - 16)(x - 3) = 0$

$\therefore\ x = \frac{16}{3}$ or 3

REVIEW SET 13B

1 **a** $\sin x = 0.382$

$\sin^{-1}(0.382) \approx 0.392$ {using technology}

$\therefore$ on $0 \leqslant x \leqslant 8$ the solutions are

$x \approx 0.392,\ \pi - 0.392,\ 2\pi + 0.392$

$\therefore\ x \approx 0.392, 2.75, 6.68$

b $\tan\left(\frac{x}{2}\right) = -0.458$

$\tan^{-1}(-0.458) \approx -0.429$

{using technology}

$\therefore$ on $0 \leqslant \dfrac{x}{2} \leqslant 4$ the solutions are

$\dfrac{x}{2} \approx -0.429 + \pi$

$\therefore\ \dfrac{x}{2} \approx 2.71$

$\therefore\ x \approx 5.42$

2 **a** $\cos x = 0.4379$

$\cos^{-1}(0.4379) \approx 1.12$ {using technology}

$\therefore$ on $0 \leqslant x \leqslant 10$ the solutions are

$x \approx 1.12,\ 2\pi - 1.12,\ 2\pi + 1.12$

$\therefore\ x \approx 1.12, 5.17, 7.40$

b $\cos(x - 2.4) = -0.6014$

$\cos^{-1}(-0.6014) \approx 2.216$

{using technology}

$\therefore$ on $-2.4 \leqslant x - 2.4 \leqslant 3.6$ the solutions are $x - 2.4 \approx -2.216, 2.216$

$\therefore\ x \approx 0.184, 4.62$

3 **a** $\sin 2A = 2\sin A\cos A$

$= 2(\frac{5}{13})(\frac{12}{13})$

$= \frac{120}{169}$

b $\cos 2A = \cos^2 A - \sin^2 A$

$= (\frac{12}{13})^2 - (\frac{5}{13})^2$

$= \frac{144-25}{169}$

$= \frac{119}{169}$

c $\tan 2A = \dfrac{\sin 2A}{\cos 2A}$

$= \dfrac{\frac{120}{169}}{\frac{119}{169}}$

$= \frac{120}{119}$

4 **a** **i** $\tan x = 4$

$\tan^{-1}(4) \approx 1.33$ {using technology}

$\therefore$ on $0 \leqslant x \leqslant 10$ the solutions are

$x \approx 1.33,\ 1.33 + \pi,\ 1.33 + 2\pi$

$\therefore\ x \approx 1.33, 4.47, 7.61$

ii $\tan\left(\frac{x}{4}\right) = 4$

$\tan^{-1}(4) \approx 1.33$ {using technology}

$\therefore$ on $0 \leqslant \dfrac{x}{4} \leqslant \frac{5}{2}$ the solutions are

$\dfrac{x}{4} \approx 1.33$

$\therefore\ x \approx 5.30$

iii $\tan(x - 1.5) = 4$

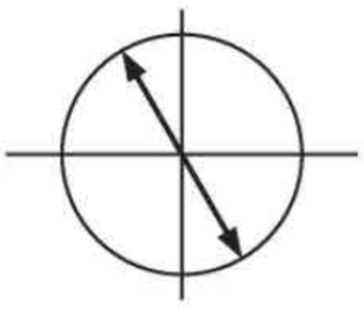
$\tan^{-1}(4) \approx 1.33$ {using technology}

$\therefore$ on $-1.5 \leqslant x - 1.5 \leqslant 8.5$ the solutions are $x - 1.5 \approx 1.33,\ 1.33 + \pi,\ 1.33 + 2\pi$

$\therefore\ x - 1.5 \approx 1.33, 4.47, 7.61$

$\therefore\ x \approx 2.83, 5.97, 9.11$

b **i** $\tan(x + \frac{\pi}{6}) = -\sqrt{3},\quad -\pi \leqslant x \leqslant \pi$



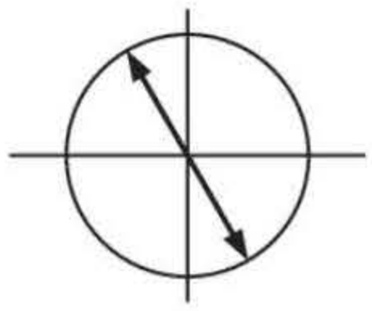
The points on the unit circle with tangent $-\sqrt{3}$ correspond to angles $\frac{2\pi}{3}$ and $\frac{5\pi}{3}$.

$-\pi \leqslant x \leqslant \pi \quad \therefore\ -\frac{5\pi}{6} \leqslant x + \frac{\pi}{6} \leqslant \frac{7\pi}{6}$

So $x + \frac{\pi}{6} = -\frac{\pi}{3}, \frac{2\pi}{3}$

$\therefore\ x = -\frac{\pi}{2}, \frac{\pi}{2}$

ii $\tan 2x = -\sqrt{3},\quad -\pi \leqslant x \leqslant \pi$



The points on the unit circle with tangent $-\sqrt{3}$ correspond to angles $\frac{2\pi}{3}$ and $\frac{5\pi}{3}$.

$-\pi \leqslant x \leqslant \pi \quad \therefore\ -2\pi \leqslant 2x \leqslant 2\pi$

So $2x = -\frac{4\pi}{3}, -\frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{3}$

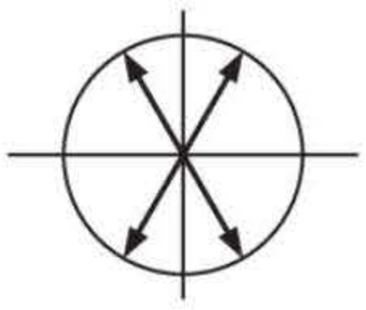
$\therefore\ x = -\frac{2\pi}{3}, -\frac{\pi}{6}, \frac{\pi}{3}, \frac{5\pi}{6}$

iii $\tan^2 x - 3 = 0, \quad -\pi \leqslant x \leqslant \pi$

$\therefore \quad \tan x = \pm\sqrt{3}$

The points on the unit circle with tangent $\pm\sqrt{3}$ correspond to angles $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$.

$\therefore \quad x = -\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}$



c $3\tan(x - 1.2) = -2$

$\therefore \quad \tan(x - 1.2) = -\frac{2}{3}$

$\tan^{-1}(-\frac{2}{3}) \approx -0.588$ {using technology}

$\therefore$ on $-1.2 \leqslant x - 1.2 \leqslant 8.8$ the solutions are

$x - 1.2 \approx -0.588, \ -0.588 + \pi, \ -0.588 + 2\pi$

$\therefore \quad x - 1.2 \approx -0.588, 2.55, 5.70$

$\therefore \quad x \approx 0.612, 3.75, 6.90$

5 **a** $\sqrt{2}\cos\left(\theta + \frac{\pi}{4}\right)$

$= \sqrt{2}\left[\cos\theta\cos\left(\frac{\pi}{4}\right) - \sin\theta\sin\left(\frac{\pi}{4}\right)\right]$

$= \sqrt{2}\left[\cos\theta \times \frac{1}{\sqrt{2}} - \sin\theta \times \frac{1}{\sqrt{2}}\right]$

$= \cos\theta - \sin\theta$

b $\cos\alpha\cos(\beta - \alpha) - \sin\alpha\sin(\beta - \alpha)$

$= \cos[\alpha + (\beta - \alpha)]$

$= \cos\beta$

6 $\cos 2A = 1 - 2\sin^2 A$

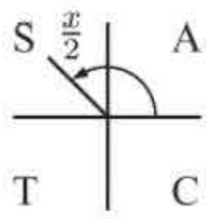
$\therefore \quad \cos x = 1 - 2\sin^2\left(\frac{x}{2}\right)$ {letting $2A = x, \ A = \frac{x}{2}$}

$\therefore \quad -\frac{3}{4} = 1 - 2\sin^2\left(\frac{x}{2}\right)$

$\therefore \quad 2\sin^2\left(\frac{x}{2}\right) = \frac{7}{4}$

$\therefore \quad \sin^2\left(\frac{x}{2}\right) = \frac{7}{8}$

$\therefore \quad \sin\left(\frac{x}{2}\right) = \pm\frac{\sqrt{7}}{2\sqrt{2}}$



But $\frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4}$ (in Q2) $\quad \therefore \quad \sin\left(\frac{x}{2}\right) = \frac{\sqrt{7}}{2\sqrt{2}}$

7 **a** $\cos x = 0.3$

$\cos^{-1}(0.3) \approx 1.27$ {using technology}

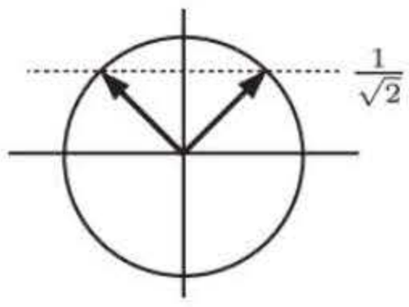
$\therefore$ on $0 \leqslant x \leqslant 2\pi$ the solutions are

$x \approx 1.27, \ 2\pi - 1.27$

$\therefore \quad x \approx 1.27, 5.02$

b $2\sin(3x) = \sqrt{2}$

$\therefore \quad \sin(3x) = \frac{1}{\sqrt{2}}$



on $0 \leqslant x \leqslant 2\pi \quad \therefore \quad 0 \leqslant 3x \leqslant 6\pi$

so $3x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}, \frac{17\pi}{4}, \frac{19\pi}{4}$

$\therefore \quad x = \frac{\pi}{12}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{11\pi}{12}, \frac{17\pi}{12}, \frac{19\pi}{12}$

c $43 + 8\sin x = 50.1$

$\therefore \quad 8\sin x = 7.1$

$\therefore \quad \sin x = \frac{7.1}{8}$

$\sin^{-1}\left(\frac{7.1}{8}\right) \approx 1.09$

$\therefore$ on $0 \leqslant x \leqslant 2\pi$ the solutions are $x \approx 1.09, \ \pi - 1.09$

$\therefore \quad x \approx 1.09, 2.05$

8 $P(t) = 5 + 2\sin\left(\frac{\pi t}{3}\right), \quad 0 \leqslant t \leqslant 8,$ where $P(t)$ is in thousands of water beetles.

a $P(0) = 5 + 2\sin 0$

$= 5$

So, 5000 water beetles.

b Smallest $P = 5 + 2(-1) = 3$

Largest $P = 5 + 2(1) = 7$

$\therefore$ smallest is 3000 water beetles

largest is 7000 water beetles

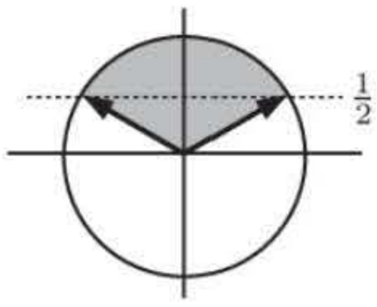
c If population is > 6000,

then $P(t) > 6$

$\therefore\ 5 + 2\sin\left(\frac{\pi t}{3}\right) > 6$

$\therefore\ 2\sin\left(\frac{\pi t}{3}\right) > 1$

$\therefore\ \sin\left(\frac{\pi t}{3}\right) > \frac{1}{2}$



The points on the unit circle with sine $\frac{1}{2}$ correspond to angles $\frac{\pi}{6}$ and $\frac{5\pi}{6}$.

$0 \leqslant t \leqslant 8 \quad \therefore\ 0 \leqslant \frac{\pi t}{3} \leqslant \frac{8\pi}{3}$

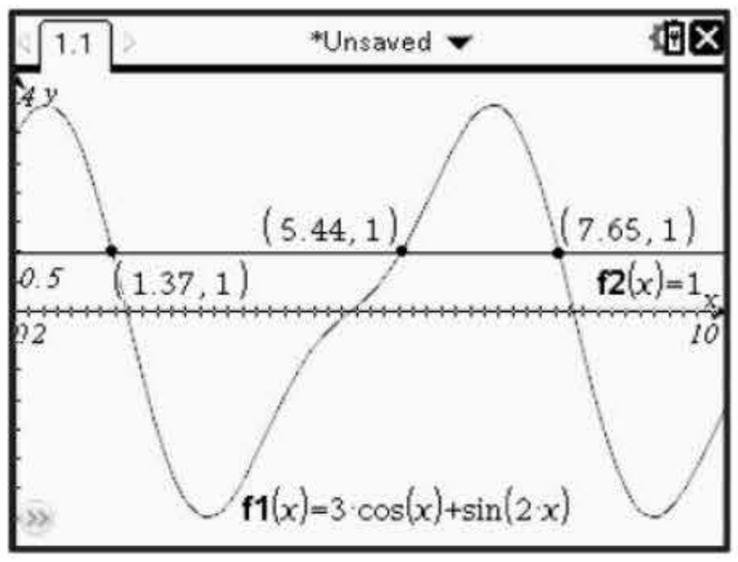
So $\frac{\pi}{6} < \frac{\pi t}{3} < \frac{5\pi}{6}, \quad \frac{13\pi}{6} < \frac{\pi t}{3} \leqslant \frac{8\pi}{3}$

$\therefore\ \frac{1}{2} < t < \frac{5}{2}, \quad \frac{13}{2} < t \leqslant 8$

$\therefore\ 0.5 < t < 2.5, \quad 6.5 < t \leqslant 8$

9 $3\cos x + \sin 2x = 1, \quad 0 \leqslant x \leqslant 10$

We graph $\mathrm{Y}_1 = 3\cos x + \sin 2x$ and $\mathrm{Y}_2 = 1$ on the same set of axes.



$\therefore\ x \approx 1.37, 5.44, 7.65$

10 $3\sin x + 4\cos x = k\cos(x + b)$

$= k[\cos x \cos b - \sin x \sin b]$

$= k\cos x \cos b - k\sin x \sin b$

Equating coefficients of $\cos x$ and $\sin x$,

$k\cos b = 4$ .... (1) and $-k\sin b = 3$ .... (2)

$\therefore\ k^2\cos^2 b = 16$ and $k^2\sin^2 b = 9$ {squaring both sides}

$\therefore\ k^2(\cos^2 b + \sin^2 b) = 25$ {adding the 2 equations}

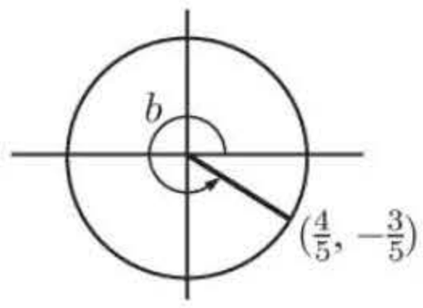
$\therefore\ k^2 = 25$

$\therefore\ k = 5$ {since $k > 0$}

Substituting $k = 5$ into (1) gives $5\cos b = 4 \quad \therefore\ \cos b = \frac{4}{5}$

and into (2) gives $-5\sin b = 3 \quad \therefore\ \sin b = -\frac{3}{5}$

$\therefore\ b \approx 5.64$



So, $3\sin x + 4\cos x \approx 5\cos(x + 5.64)$

11 Let the shooter be x m from the wall.

$\therefore\ \tan\alpha = \dfrac{20}{x}, \quad \tan 2\alpha = \dfrac{45}{x}$

$\therefore\ \dfrac{2\tan\alpha}{1 - \tan^2\alpha} = \dfrac{45}{x}$

$\therefore\ 2x\tan\alpha = 45 - 45\tan^2\alpha$

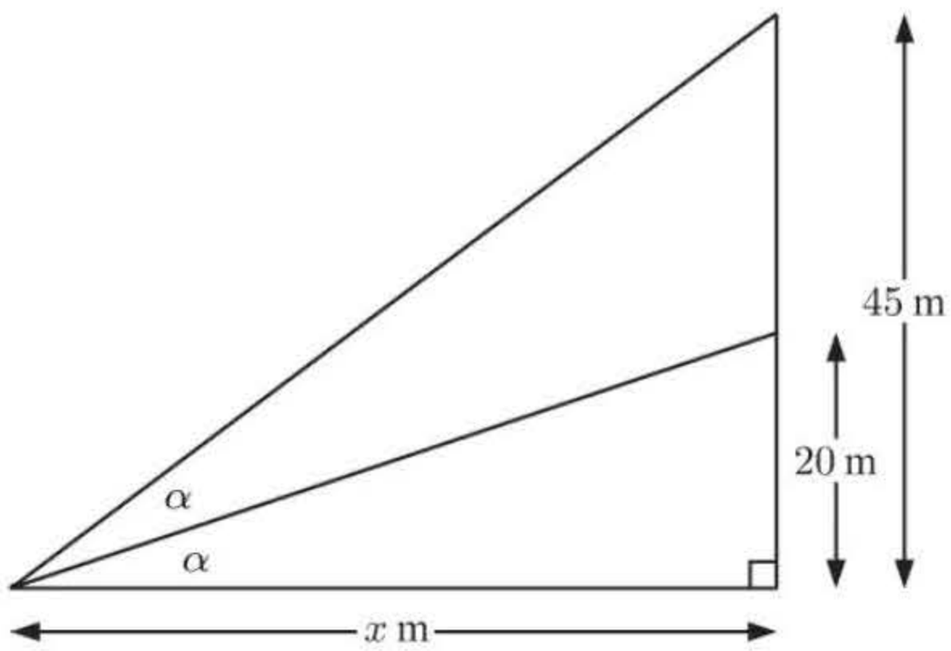
$\therefore\ 2x\left(\dfrac{20}{x}\right) = 45 - 45\left(\dfrac{20}{x}\right)^2$

$\therefore\ 40 = 45 - \dfrac{18\,000}{x^2}$

$\therefore\ \dfrac{18\,000}{x^2} = 5$

$\therefore\ x^2 = 3600$

$\therefore\ x = 60$ So, the shooter is 60 m from the wall.



12

$$\tan 2\theta = \tfrac{4}{3}$$
$$\therefore \quad \frac{2\tan\theta}{1-\tan^2\theta} = \tfrac{4}{3}$$
$$\therefore \quad 6\tan\theta = 4 - 4\tan^2\theta$$
$$\therefore \quad 2(2\tan^2\theta + 3\tan\theta - 2) = 0$$
$$\therefore \quad 2(2\tan\theta - 1)(\tan\theta + 2) = 0$$
$$\therefore \quad \tan\theta = \tfrac{1}{2} \text{ or } -2$$

But θ is clearly acute,

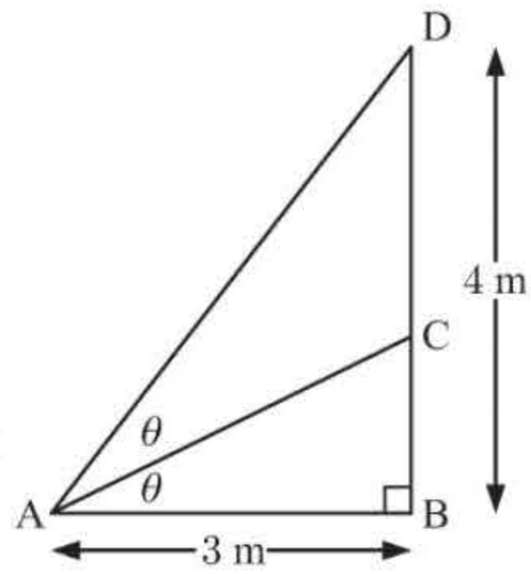
so $\tan\theta > 0$

$\therefore \quad \tan\theta = \frac{1}{2}$

$\therefore \quad \frac{\text{BC}}{3} = \frac{1}{2}$

$\therefore \quad \text{BC} = 1.5$

$\therefore$ [BC] is 1.5 m long.



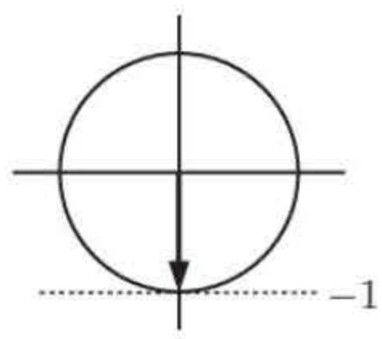
13 $P(t) = 40 + 12\sin\frac{2\pi}{7}\left(t - \frac{37}{12}\right)$ mg

a $P(t)$ has a minimum of $40 + 12(-1)$

$= 28$ mg per m^3

b When $\sin\frac{2\pi}{7}(t - \frac{37}{12}) = -1$

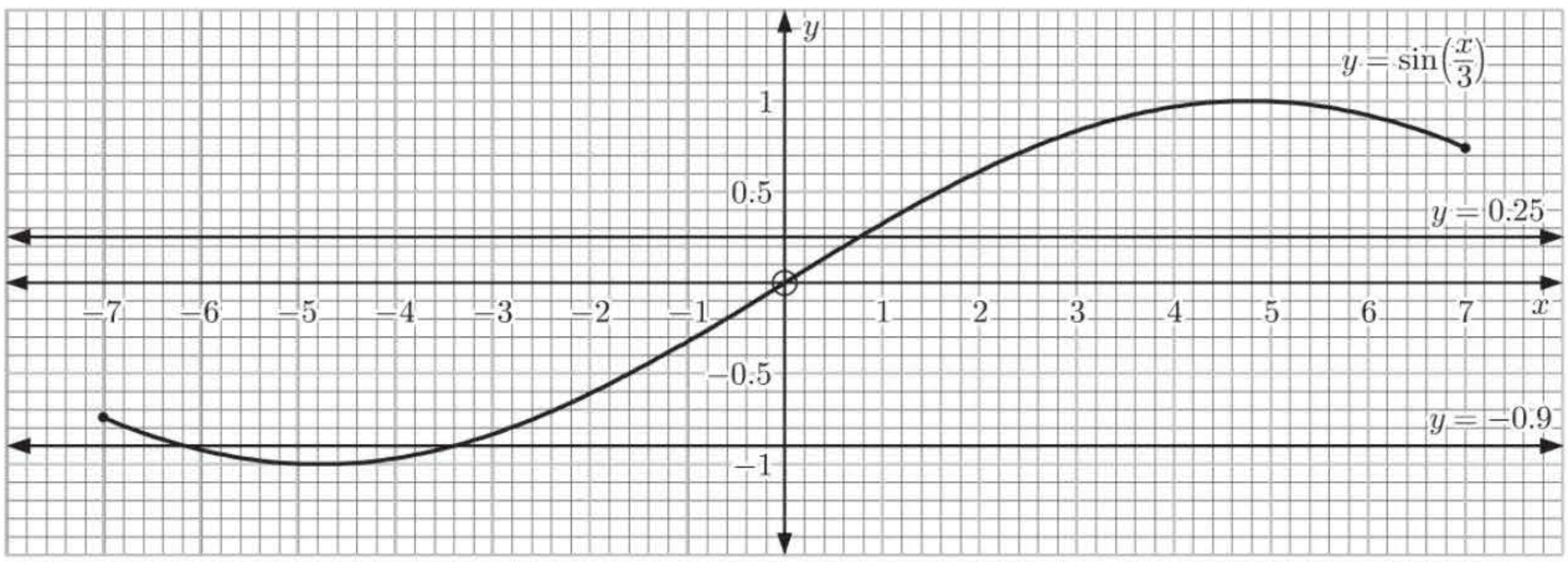
$$\therefore \quad \tfrac{2\pi}{7}(t - \tfrac{37}{12}) = \tfrac{3\pi}{2} + k2\pi$$
$$\therefore \quad \tfrac{2}{7}(t - \tfrac{37}{12}) = \tfrac{3}{2} + k\,2$$
$$\text{So,} \quad t - \tfrac{37}{12} = \tfrac{21}{4} + k\,7$$
$$\therefore \quad t = 8\tfrac{1}{3} + k\,7$$
$$\therefore \quad t = 1\tfrac{1}{3},\ 8\tfrac{1}{3},\ 15\tfrac{1}{3}, \text{ and so on.}$$



$\therefore$ on Mondays at 8:00 am $\{1\frac{1}{3}$ days after midnight Saturday$\}$

REVIEW SET 13C

1



a $\sin\left(\frac{x}{3}\right) = -0.9, \quad -7 \leqslant x \leqslant 7$

$\therefore \quad x \approx -6.1, -3.4$

b $\sin\left(\frac{x}{3}\right) = \frac{1}{4}, \quad -7 \leqslant x \leqslant 7$

$\therefore \quad x \approx 0.8$

2 **a** $\sin^2 x - \sin x - 2 = 0, \ 0 \leqslant x \leqslant 2\pi$

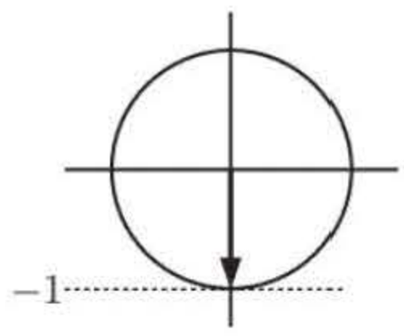
$\therefore \quad (\sin x - 2)(\sin x + 1) = 0$

$\therefore \quad \sin x = 2$ or -1

But $\sin x$ values lie between -1 and 1 inclusive.

$\therefore \quad \sin x = -1$

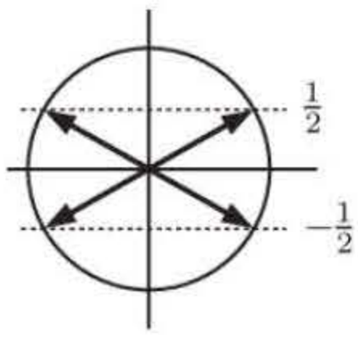
$\therefore \quad x = \frac{3\pi}{2}$



b $4\sin^2 x = 1, \quad 0 \leqslant x \leqslant 2\pi$

$\therefore \quad \sin^2 x = \frac{1}{4}$

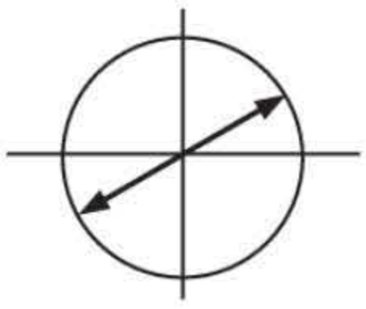
$\therefore \quad \sin x = \pm\frac{1}{2}$



$\therefore \quad x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

3 **a** $\tan(x - \frac{\pi}{3}) = \frac{1}{\sqrt{3}}, \quad 0 \leqslant x \leqslant 4\pi$

The points on the unit circle with tangent $\frac{1}{\sqrt{3}}$ correspond to angles $\frac{\pi}{6}$ and $\frac{7\pi}{6}$.



$0 \leqslant x \leqslant 4\pi$

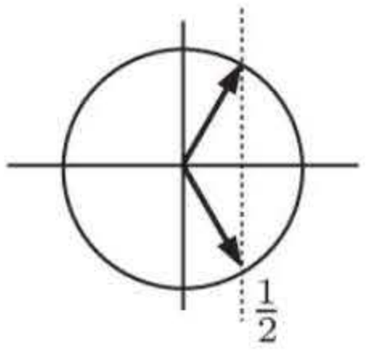
$\therefore \quad -\frac{\pi}{3} \leqslant x - \frac{\pi}{3} \leqslant \frac{11\pi}{3}$

So $\quad x - \frac{\pi}{3} = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}, \frac{19\pi}{6}$

$\therefore \quad x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

b $\cos(x + \frac{2\pi}{3}) = \frac{1}{2}, \quad -2\pi \leqslant x \leqslant 2\pi$

The points on the unit circle with cosine $\frac{1}{2}$ correspond to angles $\frac{\pi}{3}$ and $\frac{5\pi}{3}$.



$-2\pi \leqslant x \leqslant 2\pi$

$\therefore \quad -\frac{4\pi}{3} \leqslant x + \frac{2\pi}{3} \leqslant \frac{8\pi}{3}$

So $\quad x + \frac{2\pi}{3} = -\frac{\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$

$\therefore \quad x = -\pi, -\frac{\pi}{3}, \pi, \frac{5\pi}{3}$

4 **a**
$$\begin{aligned} & \cos^3\theta + \sin^2\theta\cos\theta \\ &= \cos\theta(\cos^2\theta + \sin^2\theta) \\ &= \cos\theta \end{aligned}$$

b
$$\begin{aligned} \frac{\cos^2\theta - 1}{\sin\theta} &= \frac{-(1-\cos^2\theta)}{\sin\theta} \\ &= -\frac{\sin^2\theta}{\sin\theta} \\ &= -\sin\theta \end{aligned}$$

c
$$\begin{aligned} & 5 - 5\sin^2\theta \\ &= 5(1 - \sin^2\theta) \\ &= 5\cos^2\theta \end{aligned}$$

d
$$\begin{aligned} \frac{\sin^2\theta - 1}{\cos\theta} &= -\frac{(1-\sin^2\theta)}{\cos\theta} \\ &= -\frac{\cos^2\theta}{\cos\theta} \\ &= -\cos\theta \end{aligned}$$

e
$$\begin{aligned} & \frac{\tan\theta + \cot\theta}{\sec\theta} \\ &= \frac{\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}}{\frac{1}{\cos\theta}} \\ &= \cos\theta\left(\frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta}\right) \\ &= \frac{\cancel{\cos\theta}}{\cancel{\cos\theta}\sin\theta} \qquad \{\text{as} \quad \sin^2\theta + \cos^2\theta = 1\} \\ &= \frac{1}{\sin\theta} \\ &= \csc\theta \end{aligned}$$

f
$$\begin{aligned} & \cos^2\theta(\tan\theta + 1)^2 - 1 \\ &= \cos^2\theta(\tan^2\theta + 2\tan\theta + 1) - 1 \\ &= \cos^2\theta\left(\frac{\sin^2\theta}{\cos^2\theta} + \frac{2\sin\theta}{\cos\theta} + 1\right) - 1 \\ &= \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta - 1 \\ &= 2\sin\theta\cos\theta \quad \{\text{as} \quad \sin^2\theta + \cos^2\theta = 1\} \\ &= \sin 2\theta \end{aligned}$$

5 $\tan 2\alpha = \frac{4}{3}, \quad 0 < \alpha < \frac{\pi}{2}$

$\therefore \quad \dfrac{2\tan\alpha}{1 - \tan^2\alpha} = \frac{4}{3}$

$\therefore \quad 6\tan\alpha = 4 - 4\tan^2\alpha$

$\therefore \quad 4\tan^2\alpha + 6\tan\alpha - 4 = 0$

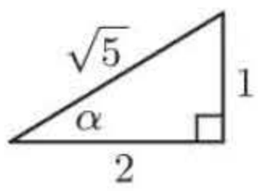
$\therefore \quad 2(2\tan^2\alpha + 3\tan\alpha - 2) = 0$

$\therefore \quad 2(2\tan\alpha - 1)(\tan\alpha + 2) = 0$

$\therefore \quad \tan\alpha = \frac{1}{2}$ or -2

But α is in quadrant 1.

$\therefore \quad \tan\alpha = \frac{1}{2}$



$\therefore \quad \sin\alpha = \frac{1}{\sqrt{5}}$

6 **a**
$$\begin{aligned} & (2\sin\alpha - 1)^2 \\ &= 4\sin^2\alpha - 4\sin\alpha + 1 \end{aligned}$$

b
$$\begin{aligned} & (\cos\alpha - \sin\alpha)^2 \\ &= \cos^2\alpha - 2\sin\alpha\cos\alpha + \sin^2\alpha \\ &= \cos^2\alpha + \sin^2\alpha - 2\sin\alpha\cos\alpha \\ &= 1 - \sin 2\alpha \end{aligned}$$

7 **a**

$$\frac{\cos\theta}{1+\sin\theta}+\frac{1+\sin\theta}{\cos\theta}$$
$$=\frac{\cos^2\theta+(1+\sin\theta)^2}{(1+\sin\theta)\cos\theta}$$
$$=\frac{\cos^2\theta+1+2\sin\theta+\sin^2\theta}{(1+\sin\theta)\cos\theta}$$
$$=\frac{2+2\sin\theta}{(1+\sin\theta)\cos\theta}\quad\{\cos^2\theta+\sin^2\theta=1\}$$
$$=\frac{2\cancel{(1+\sin\theta)}}{\cancel{(1+\sin\theta)}\cos\theta}$$
$$=\frac{2}{\cos\theta}$$
$$=2\sec\theta$$

b

$$\left(1+\frac{1}{\cos\theta}\right)\left(\cos\theta-\cos^2\theta\right)$$
$$=\cancel{\cos\theta}-\cos^2\theta+1-\cancel{\cos\theta}$$
$$=1-\cos^2\theta$$
$$=\sin^2\theta$$

8 **a**

$$\arcsin x=\tfrac{\pi}{3}$$
$$x=\sin\tfrac{\pi}{3}$$
$$=\tfrac{\sqrt{3}}{2}$$

b

$$\arctan(x-2)=\tfrac{\pi}{6}$$
$$x-2=\tan\tfrac{\pi}{6}$$
$$x-2=\tfrac{1}{\sqrt{3}}$$
$$x=2+\tfrac{1}{\sqrt{3}}$$

9

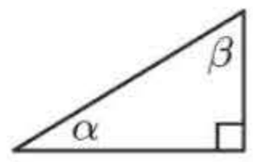
$$\cos 2\theta=1-2\sin^2\theta$$
$$\therefore\ \cos\left(\tfrac{\pi}{4}\right)=1-2\sin^2\left(\tfrac{\pi}{8}\right)\quad\{\text{letting }\ \theta=\tfrac{\pi}{8}\}$$
$$\therefore\ \tfrac{1}{\sqrt{2}}=1-2\sin^2\left(\tfrac{\pi}{8}\right)$$
$$\therefore\ 2\sin^2\left(\tfrac{\pi}{8}\right)=1-\tfrac{1}{\sqrt{2}}=\tfrac{\sqrt{2}-1}{\sqrt{2}}$$
$$\therefore\ \sin^2\left(\tfrac{\pi}{8}\right)=\tfrac{\sqrt{2}-1}{2\sqrt{2}}\times\tfrac{\sqrt{2}}{\sqrt{2}}$$
$$\therefore\ \sin^2\left(\tfrac{\pi}{8}\right)=\tfrac{2-\sqrt{2}}{4}$$
$$\therefore\ \sin\left(\tfrac{\pi}{8}\right)=\pm\frac{\sqrt{2-\sqrt{2}}}{2}$$

But $\sin\left(\tfrac{\pi}{8}\right)$ is positive as $\tfrac{\pi}{8}$ is in quadrant 1.

$$\therefore\ \sin\left(\tfrac{\pi}{8}\right)=\tfrac{1}{2}\sqrt{2-\sqrt{2}}$$

10

$$\alpha+\beta=\tfrac{\pi}{2}\quad\{\text{angles of a }\triangle\}$$
$$\therefore\ \beta=\tfrac{\pi}{2}-\alpha$$



So, $\sin 2\beta=\sin(\pi-2\alpha)$
$$=\sin\pi\cos 2\alpha-\cos\pi\sin 2\alpha$$
$$=(0)\cos 2\alpha-(-1)\sin 2\alpha$$
$$=\sin 2\alpha$$

11 **a**

$$(\sin\theta+\cos\theta)^2$$
$$=\sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta$$
$$=\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta$$
$$=1+\sin 2\theta$$

b

$$\csc(2x)+\cot(2x)$$
$$=\frac{1}{\sin 2x}+\frac{\cos 2x}{\sin 2x}$$
$$=\frac{1+\cos 2x}{\sin 2x}$$
$$=\frac{\cancel{1}+2\cos^2 x-\cancel{1}}{2\sin x\cos x}$$
$$=\frac{\cancel{2}\cos x\,\cancel{\cos x}}{\cancel{2}\sin x\,\cancel{\cos x}}$$
$$=\cot x$$

12 P_n is "$\sin^2\theta+\sin^2 2\theta+\sin^2 3\theta+\ldots.+\sin^2(n\theta)=\tfrac{1}{2}\left[n-\dfrac{\cos[(n+1)\theta]\sin n\theta}{\sin\theta}\right]$", $n\in\mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n=1$, LHS $=\sin^2\theta$, RHS $=\tfrac{1}{2}\left[1-\dfrac{\cos 2\theta\,\cancel{\sin\theta}}{\cancel{\sin\theta}}\right]$
$$=\tfrac{1}{2}-\tfrac{1}{2}(1-2\sin^2\theta)$$
$$=\sin^2\theta\qquad\therefore\ P_1\text{ is true.}$$

(2) If P_k is true then

$$\sin^2\theta + \sin^2 2\theta + \sin^2 3\theta + ... + \sin^2(k\theta) = \tfrac{1}{2}\left[k - \frac{\cos(k+1)\theta\sin k\theta}{\sin\theta}\right]$$

$$\therefore \quad \sin^2\theta + \sin^2 2\theta + \sin^2 3\theta + ... + \sin^2(k\theta) + \sin^2(k+1)\theta$$

$$= \tfrac{1}{2}\left[k - \frac{\cos(k+1)\theta\sin k\theta}{\sin\theta}\right] + \sin^2(k+1)\theta$$

$$= \tfrac{1}{2}\left[k - \frac{\cos(k+1)\theta\sin k\theta}{\sin\theta}\right] + \tfrac{1}{2} - \tfrac{1}{2}\cos 2(k+1)\theta \qquad \{\sin^2\theta = \tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta\}$$

$$= \tfrac{1}{2}(k+1) - \frac{\cos(k+1)\theta\sin k\theta + \cos 2(k+1)\theta\sin\theta}{2\sin\theta}$$

$$= \tfrac{1}{2}(k+1) - \frac{\tfrac{1}{2}\sin[k\theta + (k+1)\theta] + \tfrac{1}{2}\sin[k\theta - (k+1)\theta] + \tfrac{1}{2}\sin[\theta + 2(k+1)\theta] + \tfrac{1}{2}\sin[\theta - 2(k+1)\theta]}{2\sin\theta}$$

{products to sums formula}

$$= \tfrac{1}{2}(k+1) - \frac{\tfrac{1}{2}\sin(2k+1)\theta + \tfrac{1}{2}\sin(-\theta) + \tfrac{1}{2}\sin(2k+3)\theta + \tfrac{1}{2}\sin(-2k-1)\theta}{2\sin\theta}$$

$$= \tfrac{1}{2}(k+1) - \frac{\cancel{\tfrac{1}{2}\sin(2k+1)\theta} - \tfrac{1}{2}\sin\theta + \tfrac{1}{2}\sin(2k+3)\theta - \cancel{\tfrac{1}{2}\sin(2k+1)\theta}}{2\sin\theta}$$

$$= \tfrac{1}{2}(k+1) - \frac{\tfrac{1}{2}\sin(2k+3)\theta - \tfrac{1}{2}\sin\theta}{2\sin\theta}$$

$$= \tfrac{1}{2}(k+1) - \frac{\cos\left[\frac{(2k+3)\theta+\theta}{2}\right]\sin\left[\frac{(2k+3)\theta-\theta}{2}\right]}{2\sin\theta} \qquad \{\text{factor formula}\}$$

$$= \tfrac{1}{2}(k+1) - \frac{\cos(k+2)\theta\sin(k+1)\theta}{2\sin\theta}$$

$$= \tfrac{1}{2}\left[(k+1) - \frac{\cos(k+2)\theta\sin(k+1)\theta}{\sin\theta}\right]$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true.

$\therefore$ P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

13 **a** $\cos(\alpha-\beta) - \cos(\alpha+\beta)$

$= [\cancel{\cos\alpha\cos\beta} + \sin\alpha\sin\beta] - [\cancel{\cos\alpha\cos\beta} - \sin\alpha\sin\beta]$ {compound angle formulae}

$= \sin\alpha\sin\beta + \sin\alpha\sin\beta$

$= 2\sin\alpha\sin\beta$

b $2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$

$\therefore \quad \sin\alpha\sin\beta = \tfrac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta))$

c $\sin[(k+1)\theta]\sin\frac{\theta}{2}+\sin\frac{k\theta}{2}\sin\frac{(k+1)\theta}{2}$

$$= \tfrac{1}{2}\left[\cos\left((k+1)\theta-\frac{\theta}{2}\right)-\cos\left((k+1)\theta+\frac{\theta}{2}\right)\right]$$
$$+\tfrac{1}{2}\left[\cos\left(\frac{k\theta}{2}-\frac{(k+1)\theta}{2}\right)-\cos\left(\frac{k\theta}{2}+\frac{(k+1)\theta}{2}\right)\right]$$
$$= \tfrac{1}{2}\left[\cancel{\cos\left(k\theta+\frac{\theta}{2}\right)}-\cos\left((k+1)\theta+\frac{\theta}{2}\right)+\cos\left(-\frac{\theta}{2}\right)-\cancel{\cos\left(k\theta+\frac{\theta}{2}\right)}\right]$$
$$= \tfrac{1}{2}\left[\cos\left(\frac{\theta}{2}\right)-\cos\left((k+1)\theta+\frac{\theta}{2}\right)\right]$$
$$= \tfrac{1}{2}\left[-2\sin\left(\frac{\frac{\theta}{2}+(k+1)\theta+\frac{\theta}{2}}{2}\right)\sin\left(\frac{\frac{\theta}{2}-(k+1)\theta-\frac{\theta}{2}}{2}\right)\right] \quad \{\text{factor formulae}\}$$
$$= -\sin\left(\frac{(k+2)\theta}{2}\right)\sin\left(-\frac{(k+1)\theta}{2}\right)$$
$$= \sin\left(\frac{\theta(k+1)}{2}\right)\sin\left(\frac{\theta(k+2)}{2}\right)$$

d P_n is "$\sin\theta+\sin 2\theta+\sin 3\theta+....+\sin(n\theta)=\dfrac{\sin\left[\frac{1}{2}(n+1)\theta\right]\sin\left(\frac{1}{2}n\theta\right)}{\sin\left(\frac{1}{2}\theta\right)}$", $n\in\mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n=1$, LHS $=\sin\theta$, RHS$=\dfrac{\sin\left[\frac{1}{2}(2)\theta\right]\cancel{\sin\left(\frac{1}{2}\theta\right)}}{\cancel{\sin\left(\frac{1}{2}\theta\right)}}=\sin\theta$ $\quad\therefore$ P_1 is true.

(2) If P_k is true, then

$$\sin\theta+\sin 2\theta+\sin 3\theta+....+\sin(k\theta)=\frac{\sin\left[\frac{1}{2}(k+1)\theta\right]\sin\left(\frac{1}{2}k\theta\right)}{\sin\left(\frac{1}{2}\theta\right)}$$

$\therefore$ $\sin\theta+\sin 2\theta+\sin 3\theta+....+\sin(k\theta)+\sin(k+1)\theta$

$$=\frac{\sin\left[\frac{1}{2}(k+1)\theta\right]\sin\left(\frac{1}{2}k\theta\right)}{\sin\left(\frac{1}{2}\theta\right)}+\sin(k+1)\theta$$
$$=\frac{1}{\sin\left(\frac{1}{2}\theta\right)}\left(\sin\left[\tfrac{1}{2}(k+1)\theta\right]\sin\left(\tfrac{1}{2}k\theta\right)+\sin(k+1)\theta\sin\left(\tfrac{1}{2}\theta\right)\right)$$
$$=\frac{1}{\sin\left(\frac{1}{2}\theta\right)}\left(\sin\frac{(k+1)\theta}{2}\sin\frac{(k+2)\theta}{2}\right) \quad \{\text{using part } \mathbf{c}\}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true.

$\therefore$ P_n is true for all $n\in\mathbb{Z}^+$ {Principle of mathematical induction}